Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 5
Orthogonal polynomials and exponential structures

Ch5c

IMSc, Chennai
March 13, 2019

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Orthogonal Sheffer polynomials

Sheffer polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\exp\left(x f(t)\right)$$

binomial type polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \qquad \exp\left(x f(t)\right)$$

$\{P_n(x)\}_{n\geq 0}$ orthogonal polynomials are Sheffer polynomials

Meixner (1934)

positive-definite OPS Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak+b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a,b,c,d \in \mathbb{R} \\ c \geq 0,\ c+d > 0 \end{cases}$

$\{P_n(x)\}_{n \geqslant 0}$ orthogonal polynomials

Meixner (1934)

are Sheffer polynomials

$\Longleftrightarrow$ $\{P_n(x)\}_{n \geqslant 0}$ are one of the 5 possible types:

Hermite

Laguerre

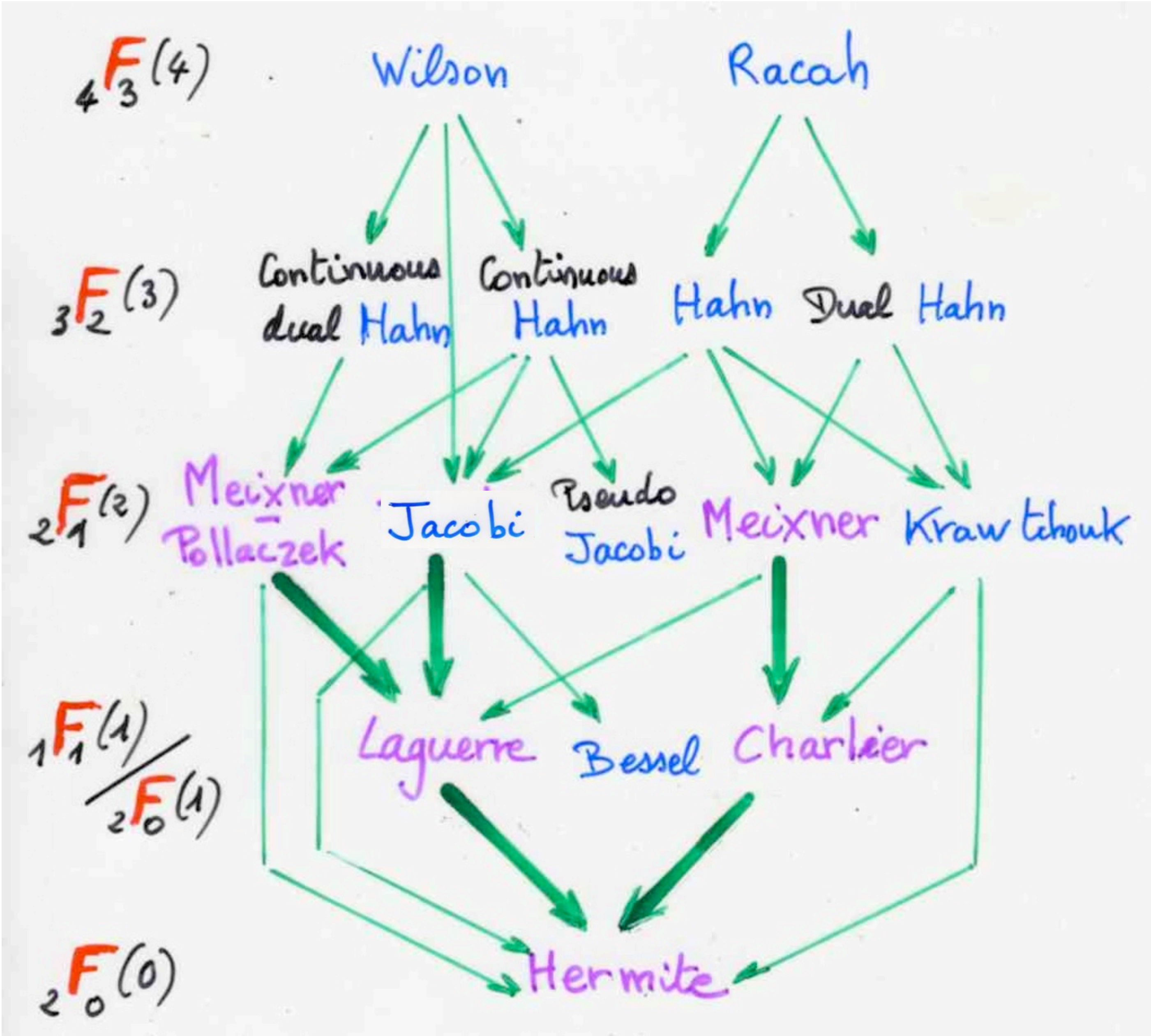
Charlier

Meixner

Meixner-Pollaczek

Askey scheme of hypergeometric orthogonal polynomials

orthogonal
Sheffer
polynomials



Sheffer polynomials

(S, Q)

delta operator Q

Rota (1973)
"Finite operator calculus"

umbral calculus

umbral calculus

Blissard (1861)

Lucas, Sylvester

Bell (1930, 1940)... Riordan 60's

umbral calculus

example

Bernoulli polynomials $B_n(x)$

$$B_n(x) = \sum_{0 \leq k \leq n} \binom{n}{k} B_{n-k}\, x^k$$

B_k Bernoulli numbers

$$\sum_{n \geq 0} B_n(x) \frac{t^n}{n!} = \frac{t e^{xt}}{e^t - 1}$$

umbral calculus

$$B_n(x) = \sum_{0 \le k \le n} \binom{n}{k} B_{n-k}\, x^k$$

$$B_n(x) = \sum_{0 \le k \le n} \binom{n}{k} B^{n-k} x^k \qquad = (B + x)^n$$

$$\frac{d}{dx} B_n(x) \qquad = n\,(B + x)^{n-1}$$

$$= n\, B_{n-1}(x)$$

umbral calculus

70's G-C. Rota S. Roman (1984)
book

linear functionals

Sheffer polynomials

delta operator Q

Gian-Carlo Rota

1932-1999

Prof. applied mathematics (M.I.T.)
and philosophy

$\simeq$ 50 students

W. W. Chen, M. Haiman, R. Stanley,
C. Yan, ...

R. Stanley $\simeq$ 60 students

A. Garsia (born 1928)

$\{P_n(x)\}_{n \geq 0}$ Sheffer $\rightarrow \begin{cases} S \\ Q \end{cases}$

delta operator

combinatorial interpretation of the operator Q and S for the 5 classes of Sheffer orthogonal polynomials

Sheffer polynomials

definition with delta operators

Rota (1973)

"Finite operator calculus"

$\mathbb{K}$ field $\mathbb{K}[x]$

polynomial sequence

$\{P_n(x)\}_{n\geq 0}$

$\deg(P_n(x)) = n$

Definition binomial type

$$P_n(x+y) = \sum_{0\leq k\leq n} \binom{n}{k} P_k(x) P_{n-k}(y)$$

Σ algebra of shift-invariant operators

$Tp(x) \quad (Tp)(x)$

$$E^a p(x) = p(x+a)$$

<u>Definition</u> shift-invariant operators

$$T \circ E^a = E^a \circ T$$

Definition delta operator Q

shift-invariant operators

- Qx non-zero constant

Lemma for Q delta operator

$p(x)$ degree n

then $Qp(x)$ is of degree $n-1$

Definition Q delta operator

$\{p_n(x)\}_{n\geq 0}$ basic for Q

(i) $p_0(x) = 1$

(ii) $p_n(0) = 0$ for every $n \geqslant 1$

(iii) $Q\, p_n(x) = n\, p_{n-1}(x)$

Proposition

- every delta operator has a unique sequence of basic polynomials

Proposition

- $\{p_n(x)\}_{n\geq 0}$ binomial type

$\Updownarrow$

- basic sequence for some delta operator Q

isomorphism

Σ
algebra
shift-invariant
operators

$\longleftrightarrow$

$\mathbb{K}[[t]]$

$$T = \sum_{k \geq 0} \frac{a_k}{k!} D^k$$

$$D x^n = n x^{n-1}$$

Q delta operator

$Q = q(D)$

$q(t)$

Lemma

T delta operator iff $a_0 = 0$ and $a_1 \neq 0$

Q delta operator

$\{P_n(x)\}$ basic for Q

Proposition

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \exp\left(x\, q^{\langle -1\rangle}(t)\right)$$

reciprocal power series

Definition $\{P_n(x)\}$ Sheffer sequence of polynomials for Q delta operator

(i) $P_0(x) = c \neq 0$

(iii) $Q\,P_n(x) = n\,P_{n-1}(x)$

Proposition Q delta operator

$\{q_n(x)\}$ basic sequence for Q

- $\{P_n(x)\}$ Sheffer for Q
  iff $\exists\ S \in \Sigma$ invertible such that

$$P_n(x) = S^{-1} q_n(x)$$

$\Updownarrow$

$$P_n(x+y) = \sum_{0 \leq k \leq n} \binom{n}{k} P_k(x)\, q_{n-k}(x)$$

Proposition $\{P_n(x)\}_{n\geq 0}$ Sheffer $\rightarrow \begin{cases} S = s(D) \\ Q = q(D) \end{cases}$

$S = s(D)$ $s(t)$ $Q = q(D)$ $q(t)$

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{1}{s(q^{\langle -1\rangle}(t))} \exp\left(x\, q^{\langle -1\rangle}(t)\right)$$

$g(t)$ $f(t)$

$q^{\langle -1\rangle}(t)$

reciprocal of $q(t)$ $q^{\langle -1\rangle}(q(t)) = t$

Inverse polynomials

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

See Ch 1d

$$Q_n(x) = \sum_{i=0}^{n} q_{n,i} x^i$$

inverse sequence

$$\{Q_n(x)\}_{n\geq 0}$$

Proposition

$$\{P_n(x)\}_{n\geq 0} \text{ Sheffer} \iff \{Q_n(x)\}_{n\geq 0} \text{ Sheffer}$$

inverse sequence

Proposition

$$\{P_n(x)\}_{n\geq 0} \text{ Sheffer} \rightarrow \begin{cases} S = s(D) \\ Q = q(D) \end{cases}$$

$$\{Q_n(x)\}_{n\geq 0} \text{ Sheffer} \rightarrow \begin{cases} T = \dfrac{1}{s\left(q^{\langle -1\rangle}(D)\right)} \\ P = q^{\langle -1\rangle}(D) \end{cases}$$

$$\{P_n(x)\}_{n\geqslant 0} \quad \text{Sheffer} \quad \longrightarrow \begin{cases} S = s(D) \\ Q = q(D) \end{cases}$$

$S = s(D) \qquad s(t)$ $\qquad$ $Q = q(D) \qquad q(t)$

$$\sum_{n\geqslant 0} P_n(x)\frac{t^n}{n!} = \frac{1}{s\left(q^{<-1>}(t)\right)} \exp\left(x\, q^{<-1>}(t)\right)$$

$$\sum_{n\geqslant 0} Q_n(x)\frac{t^n}{n!} = s(t)\exp\left(x\, q(t)\right)$$

inverse sequence

→ Riordan arrays

$$Q = (q_{n,i})_{i,n}$$

$$P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i} x^i$$

$$\begin{cases} p_{n,i} = 0, & i > n \\ q_{n,i} = 0, & i > n \end{cases}$$

triangular matrices

(1 on the diagonal)

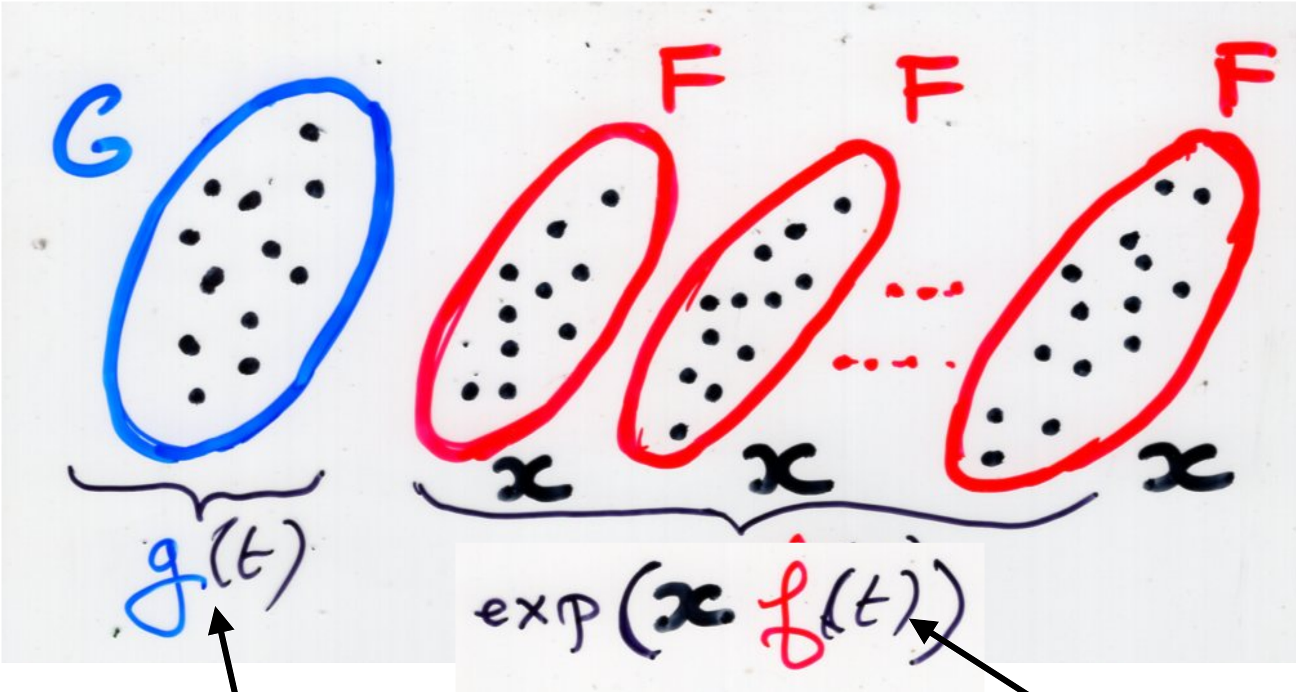
$$Q = P^{-1}$$

species G-structure

F-structure

"assemblée" of F-structures

$g(t)$



$S = s(D)$ $\quad s(t)$

$Q = q(D)$ $\quad q(t)$

inverse sequence $P_n(x)$

Appell sequence $\{P_n(x)\}$

(S, Q) Sheffer polynomial with $Q = D$

$\Longleftrightarrow$

$$\frac{d}{dx} P_n(x) = n P_{n-1}(x)$$

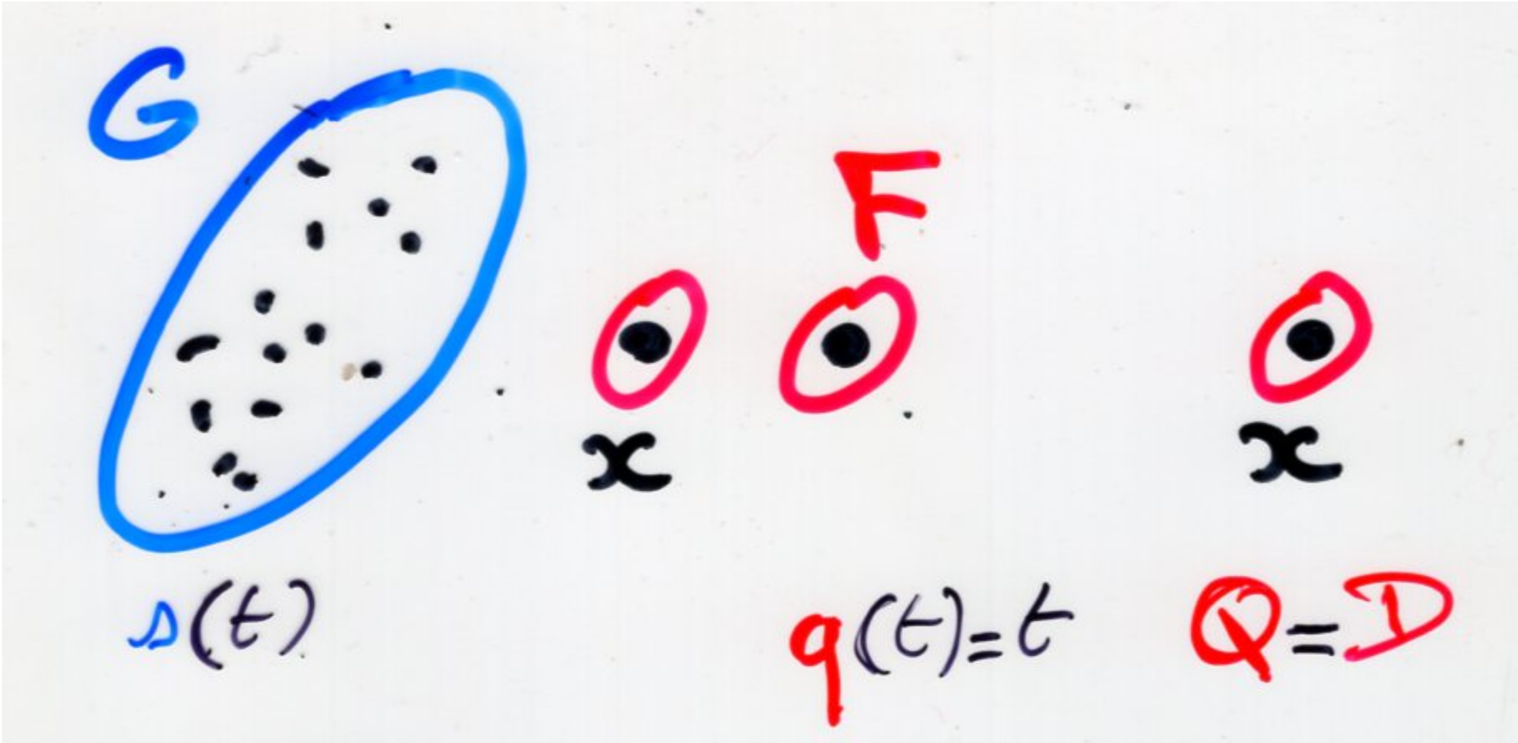
for every $n \geq 1$

and $P_0(x)$ is non-zero constant

<u>example</u>

Hermite polynomials $H_n(x)$

Bernoulli polynomials $B_n(x)$


G
F
x
x
s(t)
q(t)=t
Q=D

Inverse polynomials

From Ch1d

$\{P_n(x)\}_{n\geq 0}$ $\qquad$ $P_n \in \mathbb{K}[x]$

monic $\qquad$ $\deg(P_n) = n$

$P_n(x) = x^n + \dots$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ $\qquad$ $b_k, \lambda_k \in \mathbb{K}$

ring

Definition Inverse polynomials

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

$$Q_n(x) = \sum_{i=0}^{n} q_{n,i} x^i$$

inverse sequence $\{Q_n(x)\}_{n\geq 0}$

$$Q = (q_{n,i})_{i,n} \qquad P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i}\, x^i$$

$$\begin{cases} p_{n,i} = 0, & i > n \\ q_{n,i} = 0, & i > n \end{cases}$$

triangular
matrices
(1 on the diagonal)

$$Q = P^{-1}$$

Definition vertical polynomials

$$V_n(x) = \sum_{i=0}^{n} \mu_{n,i}\, x^i$$

$$\mu_{n,i} = \sum_{\substack{\omega \\ \text{"Motzkin" path} \\ |\omega|=n,\ 0 \to i}} v(\omega)$$

Motzkin path going from level 0 to level i

$$V = (\mu_{n,i})_{n,i \geq 0}$$

Proposition

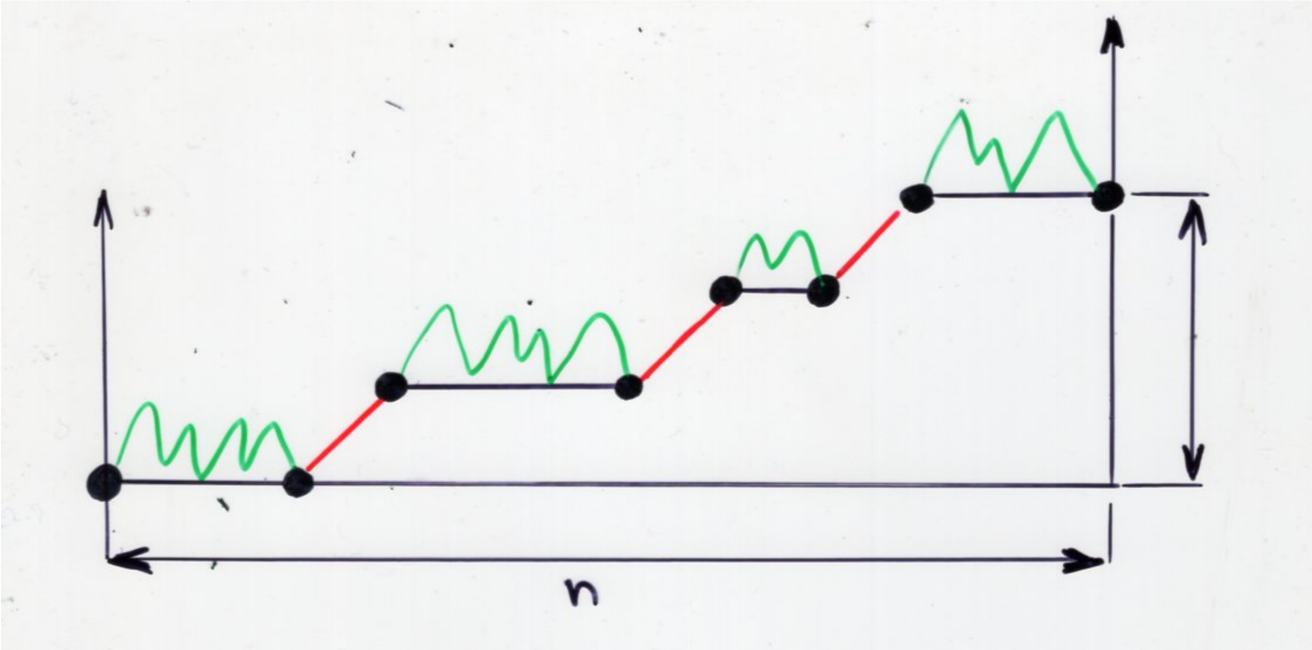
$\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 1}$ $\quad b_k, \lambda_k \in \mathbb{K}$

$\{V_n(x)\}_{n\geq 0}$ is the inverse sequence of $\{P_n(x)\}_{n\geq 0}$

defined by the 3-terms reccurence relation

in other words $V = Q = P^{-1}$

$\mu_{n,i}$

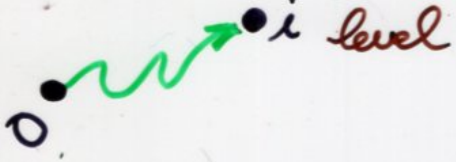


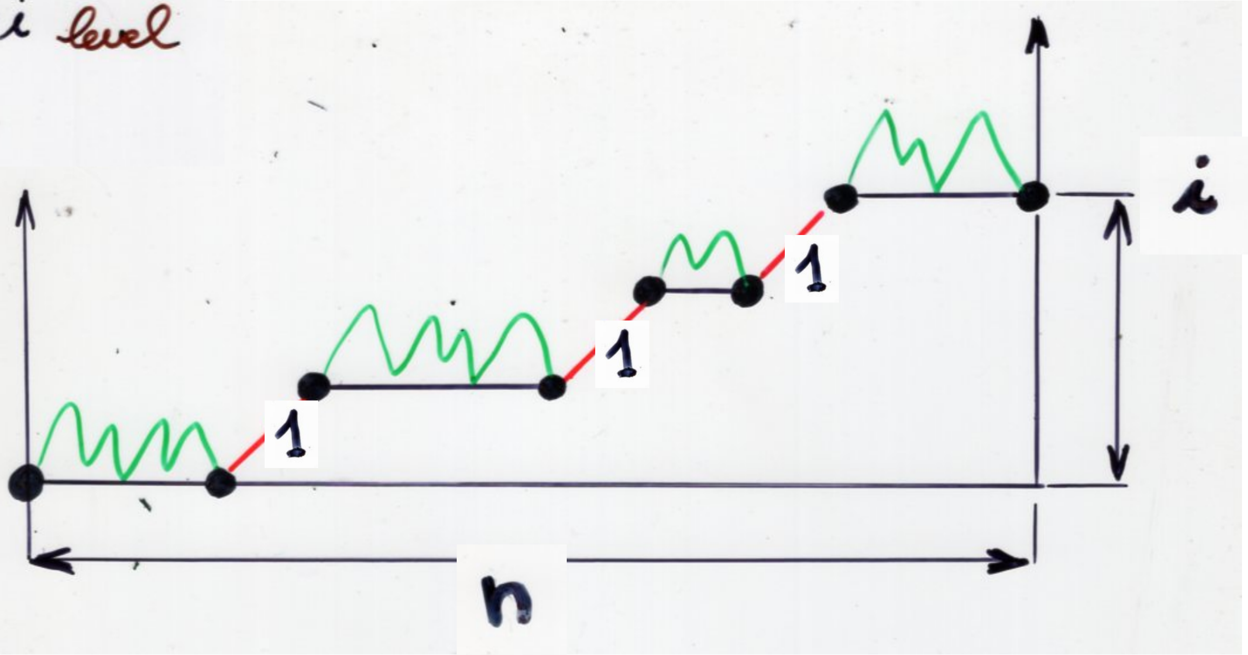
$$\left(b_k \;, \lambda_k \right)_{\substack{k\geq 0 \\ k \geq 1}}$$

Modification for $\mu_{n,i}$

$$\sum_{|\omega|=n} v(\omega) = \mu_{n,i}$$

"Motzkin path"

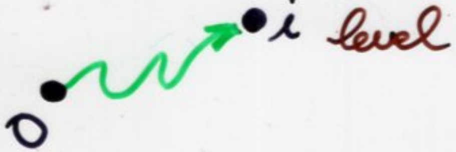


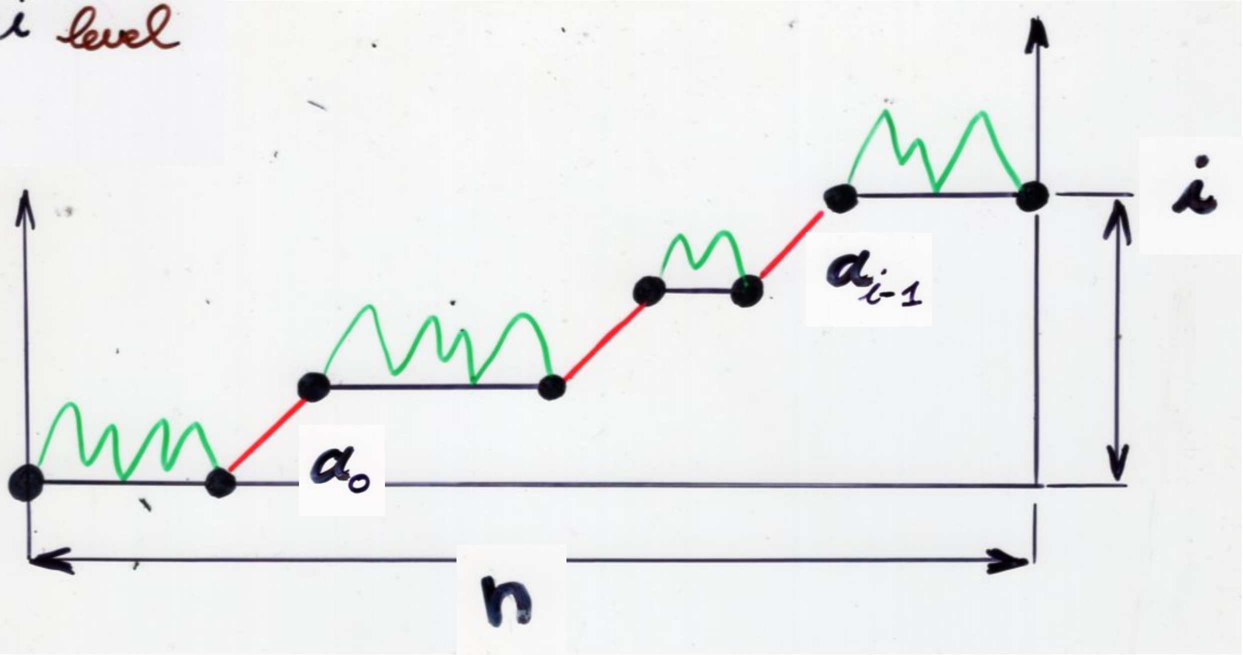


$$\left(\underset{k\geq 0}{b_k} \,,\, \underset{k\geq 1}{\lambda_k} \right)$$

$$\sum_{|\omega|=n} v^{*}(\omega) = a_0 a_1 \cdots a_{i-1} \mu_{n,i}$$

"Motzkin path"





$$\left(a_k, b'_k, b''_k, c_k\right)_{k\geq 0,\ k\geq 1}$$

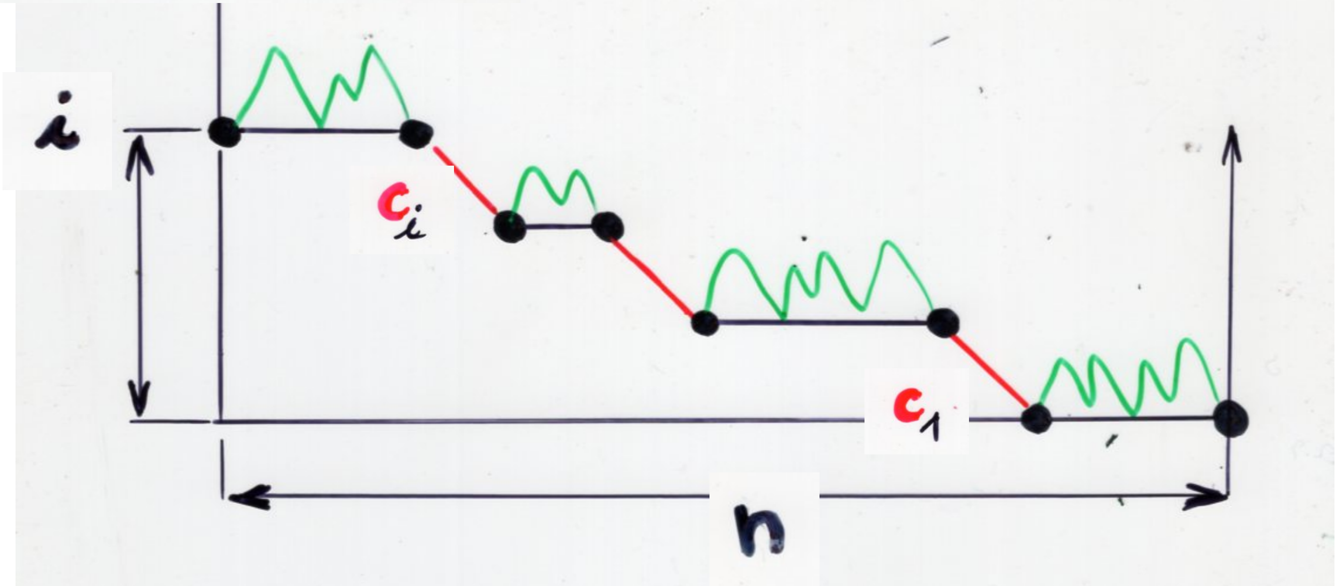
$$\begin{cases} \lambda_k = a_{k-1} c_k \\ b_k = b'_k + b''_k \end{cases}$$

$$\bar{\mu}_{n,i} = \sum_{|\omega|=n} v^*(\omega)$$

"Motzkin path"

level i • ⇝ • 0

$$\bar{\mu}_{n,i} = c_1 \cdots c_i \, \mu_{n,i}$$



$$(a_k, b'_k, b''_k, c_k)$$
$k \geq 0$ $\quad$ $k \geq 1$

Hermite

$$\overline{\mu}_{n,i} = c_1 \cdots c_i \, \mu_{n,i}$$

Laguerre

$$c_k = k$$

Charlier

$$\mu_{n,i} = \frac{1}{i!} \overline{\mu}_{n,i}$$

Meixner

Meixner – Pollaczek

except $c_k = \dfrac{k}{1-c}$

$$\mu_{n,i} = (1-c)^i \frac{1}{i!} \mu_{n,i}$$

Laguerre histories

Restricted Laguerre histories

Ch2b, 19-23

positive-definite OPS

Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a, b, c, d \in \mathbb{R} \\ c \geq 0, \quad c+d > 0 \end{cases}$

Hermite

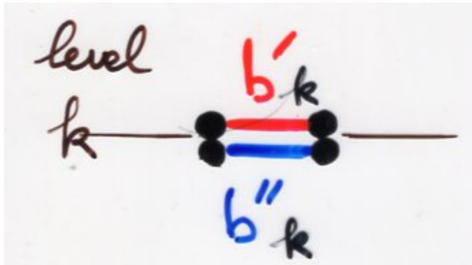
Laguerre

Charlier

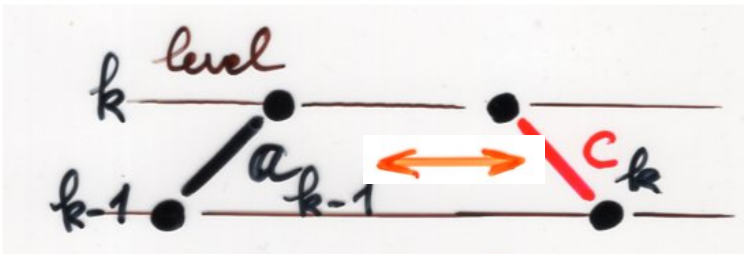
Meixner

Meixner-Pollaczek

$$b_k = b'_k + b''_k$$



$$a_{k-1} c_k = \lambda_k$$



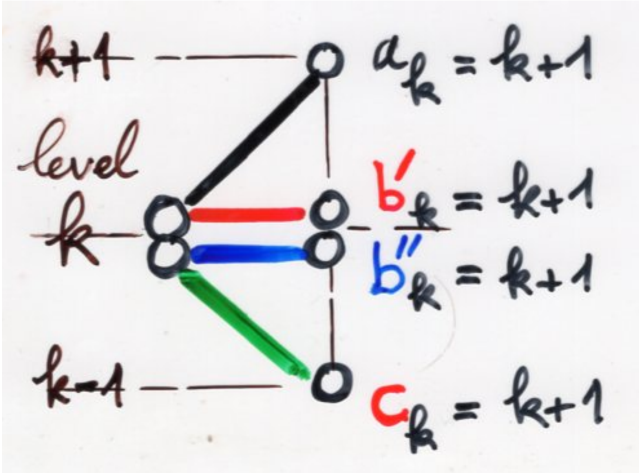
$$\sum_{\substack{|\omega|=n \\ \text{Motzkin} \\ \text{path}}} v(\omega) = \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin} \\ \text{path}}} v^*(\omega) = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases} \qquad \begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

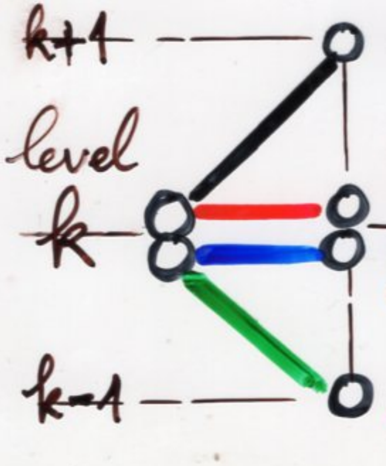
$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$

$$v^*(\omega)$$



Laguerre histories



$a_k = k+1$

$b'_k = k+1$

$b''_k = k+1$

$(k \geq 0)$

$c_k = k+1$ $(k \geq 1)$

$b_k = (2k+2)$

$\lambda_k = k(k+1)$

$$\begin{cases} \lambda_k = a_{k-1} c_k \\ b_k = b'_k + b''_k \end{cases}$$

$L_n^{(1)}(x)$

moments

$\mu_n = (n+1)!$

restricted Laguerre histories

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases}$$

$(k \geq 0)$

$(k \geq 1)$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

$L_n^{(0)}(x)$

$\mu_n = n!$

Laguerre histories

$$a_k = k+1$$
$$b'_k = k+1$$
$$b''_k = k+\alpha$$
$$c_k = k+\alpha$$

$$\begin{cases} b_k = 2k+\alpha+1 \\ \lambda_k = k(k+\alpha) \end{cases}$$

$$\mu_n = (\alpha+1)\cdots(\alpha+n)$$

$$\beta = \alpha + 1$$

$$\begin{cases} \lambda_k = a_{k-1}c_k \\ b_k = b'_k + b''_k \end{cases}$$

restricted Laguerre histories

$$\begin{cases} a_k = k+\beta \\ b'_k = k+\beta & (k \geq 0) \\ b''_k = k \\ c_k = k & (k \geq 1) \end{cases}$$

$$\begin{cases} b_k = 2k+\beta \\ \lambda_k = (k-1+\beta)k \end{cases}$$

$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$

Delta operators Q and S

for Laguerre polynomials

$$\mu_{n,i} = \frac{1}{i!}\,\overline{\mu}_{n,i}$$

extension of
restricted Laguerre histories

$$\sqcup \quad \underbrace{\sqcup \quad \sqcup \quad \cdots \quad \sqcup}_{i}$$

$$B_0 \quad B_1 \quad \cdots\cdots \quad B_i$$

weighted increasing
binary trees

assemblée of

weighted
permutations

$$\mu_{n,i} = \frac{1}{i!}\,\bar{\mu}_{n,i}$$

extension of
restricted Laguerre histories

only operator A, K

for $\begin{array}{c}\sqcup\\ \downarrow\\ x\end{array}$

$A \quad \begin{array}{c}\sqcup\\ \downarrow\\ \sqcup\; x\; \sqcup\end{array}$

$K \quad \begin{array}{c}\sqcup\\ \downarrow\\ \sqcup\; x\end{array}$

$$\sqcup \quad \underbrace{\sqcup \quad \sqcup \quad \cdots \quad \sqcup}_{i}$$

$$\uparrow \qquad \uparrow \qquad\qquad \uparrow$$

$$B_0 \quad B_1 \quad \cdots\cdots \quad B_i$$

$J^b \quad \begin{array}{c}\sqcup\\ \downarrow\\ x\; \sqcup\end{array}$

$S^b \quad \begin{array}{c}\sqcup\\ \downarrow\\ x\end{array}$

weighted increasing
binary trees

weighted
permutations

$$\mu_{n,i} = \frac{1}{i!}\,\overline{\mu}_{n,i}$$

B_0 is constructed with restricted Laguerre histories

$$\begin{cases} a_k = k + \beta \\ b'_k = k + \beta \\ b''_k = k \\ c_k = k \end{cases}$$

$$\sqcup \quad \underbrace{\sqcup \quad \sqcup \quad \cdots \quad \sqcup}_{i}$$

$$B_0 \quad B_1 \quad \cdots\cdots \quad B_i$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

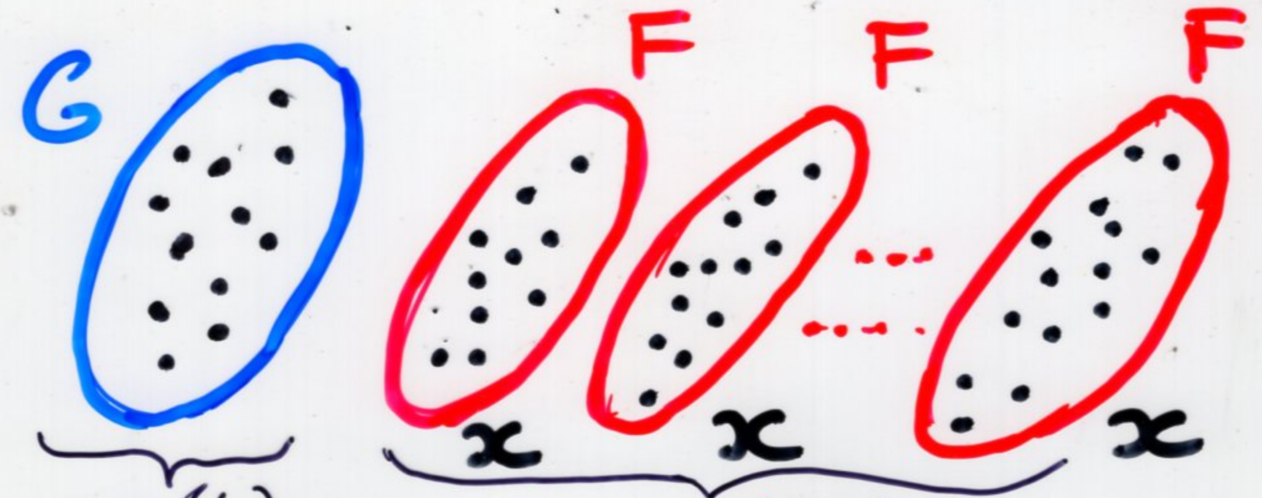
$B_1, \ldots, B_i$ are constructed with Laguerre histories

species G-structure

F-structure

"assemblée" of F-structures

$g(t)$



$g(t)$ $\exp(x f(t))$

$S = s(D)$ $s(t)$

$Q = q(D)$ $q(t)$

	μ_n $(n \geq 0)$	q_n $(n \geq 1)$
Laguerre $L_n^{(\beta)}(x)$	$(\beta)_n$	$n!$

Delta operators Q and S

for general Sheffer polynomials

positive-definite OPS

Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a, b, c, d \in \mathbb{R} \\ c \geqslant 0, \quad c+d > 0 \end{cases}$

Hermite

Laguerre

Charlier

Meixner

Meixner-Pollaczek

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)}\, b^{p(\sigma)}\, c^{dr(\sigma)}\, d^{dd(\sigma)}\, \alpha^{f(\sigma)}\, \beta^{s(\sigma)}$$

a — $v(\sigma)$ = number of valleys of σ

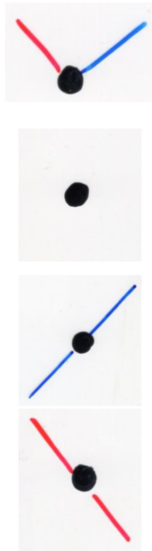
b — $p(\sigma)$ = number of peaks of σ

c — $dr(\sigma)$ = number of double rises of σ

d — $dd(\sigma)$ = number of double descents of σ

α — $f(\sigma)$ = number of lr-min elements which are a descent of σ

β — $s(\sigma)$ = number of lr-min elements of σ



A

S^b

J^b

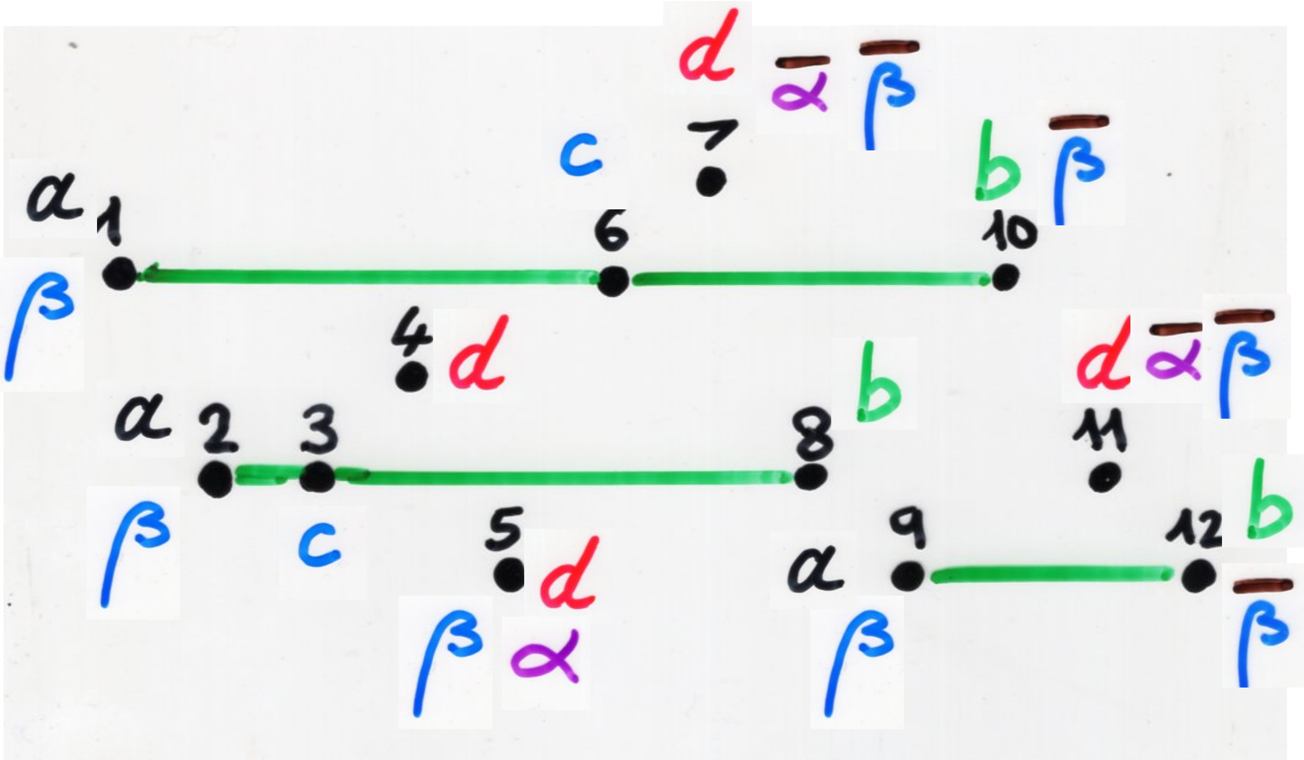
K

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

a	cycle valley	$\sigma^{-1}(x) > x < \sigma(x)$	$cv(\sigma)$	A
b	cycle peak	$\sigma^{-1}(x) < x > \sigma(x)$	$cp(\sigma)$	S^b
c	cycle double rise	$\sigma^{-1}(x) < x < \sigma(x)$	$cdr(\sigma)$	J^b
d	cycle double descent	$\sigma^{-1}(x) > x > \sigma(x)$	$cdd(\sigma)$	K
α	fixed point	$\sigma(x) = x$		
β	number of cycles		$cyc(\sigma)$	

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{(\ )}\, b^{(\ )}\, c^{(\ )}\, d^{(\ )}\, \alpha^{(\ )}\, \beta^{(\ )}\, \bar{\alpha}^{(\ )}\, \bar{\beta}^{(\ )}$$



a	A
b	S^b
c	J^b
d	K
α	$\bar{\alpha}$
β	$\bar{\beta}$

$\sqcup$ 9 (12)(11) 5 2 3 8 4 1 6 (10) 7

rl-max

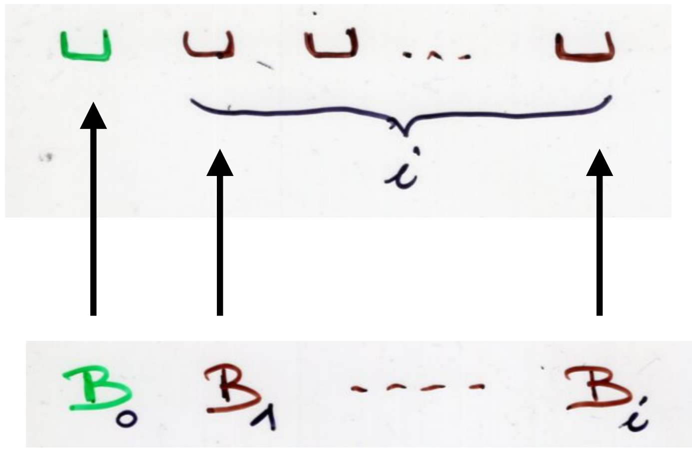
$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$a_k = a(k+\beta)$$

$$c_k = b\,k$$

$$b'_k = kd + \alpha\beta$$

$$b''_k = k\,c$$



$$a_k = a(k+1)$$

$$c_k = b(k+1)$$

$$b'_k = d(k+1)$$

$$b''_k = c(k+1)$$

$$\lambda_k = ab\,k(k+1)$$

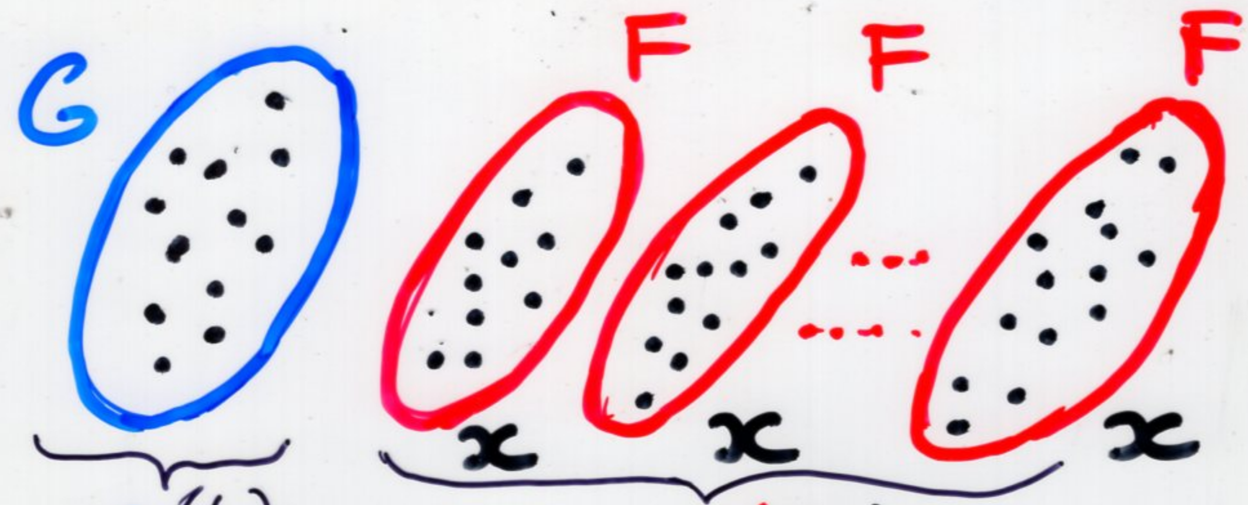
$$b_k = (d+c)(k+1)$$

species G-structure

F-structure

"assemblée" of F-structures

$g(t)$



$g(t)$ $\qquad$ $\exp(x\, f(t))$

$S = s(D)$ $\qquad$ $s(t)$

$Q = q(D)$ $\qquad$ $q(t)$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$\lambda_k = ab\,k(k+1)$

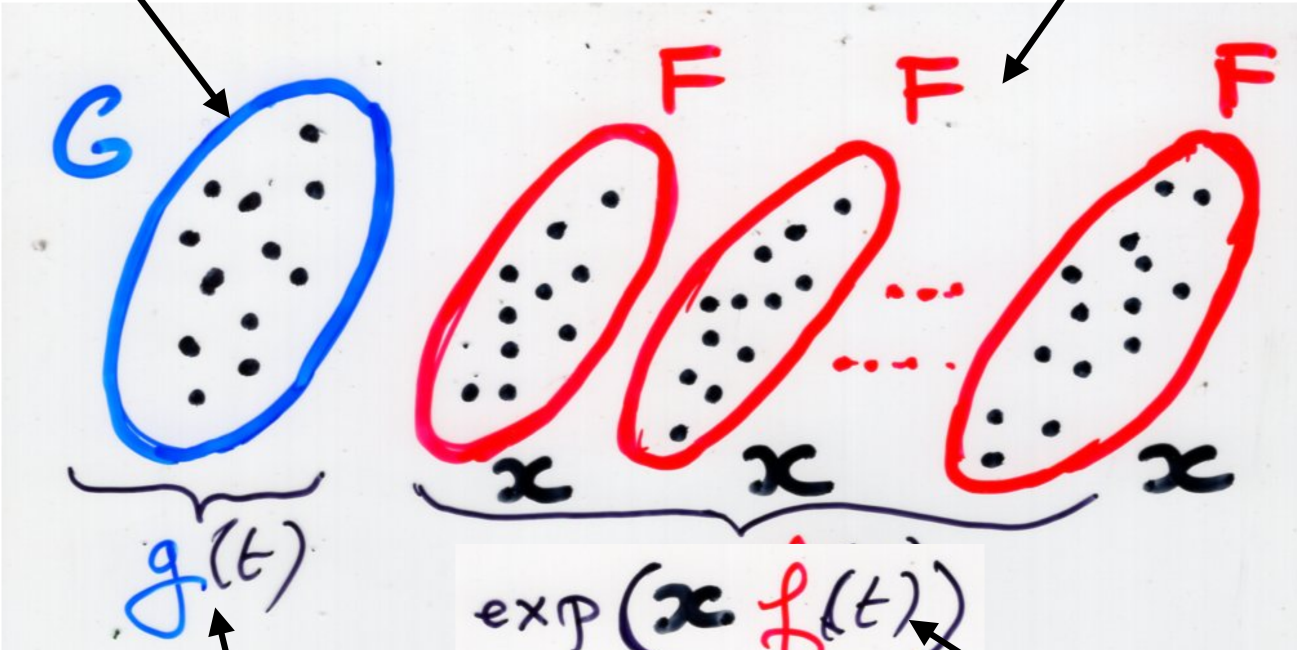
$b_k = (d+c)(k+1)$

$$\sum_{\sigma\in\mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

$$\sum_{\sigma\in\mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)}$$

μ_n

$=$



$\exp(x\, f(t))$

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$Q = q(D)$ $\qquad q(t)$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

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$b_k = (d+c)(k+1)$

Delta operators Q and S

for the 5 Sheffer orthogonal polynomials

B_0 is constructed with restricted Laguerre histories

Sheffer orthogonal polynomials	a_k	c_k	b'_k	b''_k
Laguerre $L_n^{(\beta)}(x)$	$k+\beta$	k	$k+\beta$	k
Hermite $H_n(x)$	1	k	0	0
Charlier $C_n^{(a)}(x)$	a	k	a	k
Meixner $M_n(\beta,c;x)$	$c(k+\beta)$	k	$c(k+\beta)$	k
Meixner-Pollaczek $P_n(\delta,\eta;x)$	$(1+\delta^2)(k+\eta)$	k	$\delta(k+\eta)$	δk
general Sheffer O.P.	$a(k+\beta)$	bk	$dk+\alpha\beta$	ck

$B_1, \dots, B_i$ are constructed with Laguerre histories

Sheffer orthogonal polynomials	a_k	c_k	b'_k	b''_k
Laguerre $L_n^{(\beta)}(x)$	$k+1$	$k+1$	$k+1$	$k+1$
Hermite $H_n(x)$	0	$k+1$	0	0
Charlier $C_n^{(a)}(x)$	0	$k+1$	0	$k+1$
Meixner $M_n(\beta, c; x)$	$c(k+1)$	$k+1$	$c(k+1)$	$k+1$
Meixner-Pollaczek $P_n(\delta, \eta; x)$	$(1+\delta^2)(k+1)$	$k+1$	$\delta(k+1)$	$\delta(k+1)$
general Sheffer O.P.	$a(k+1)$	$b(k+1)$	$d(k+1)$	$c(k+1)$

	operator S	operator Q
	μ_n $(n \geq 0)$	q_n $(n \geq 1)$
Laguerre $L_n^{(\beta)}(x)$	$(\beta)_n$	$n!$
Hermite $H_n(x)$	$\mu_{2n} = 1 \times 3 \times \cdots (2n-1)$ $\mu_{2n+1} = 0$	$\delta_{1,n}$
Charlier $C_n^{(a)}(x)$	$\displaystyle\sum_{k=1}^{n} S(n,k)\, a^k$	1
Meixner $M_n(\beta, c; x)$	$\displaystyle\frac{1}{(1-c)^n} \sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} c^{1+d(\sigma)}$	$\displaystyle\frac{1}{(1-c)^n} \sum_{\sigma \in \mathfrak{S}_n} c^{d(\sigma)}$
Meixner–Pollaczek $P_n(\delta, \eta; x)$	$\displaystyle\delta^n \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \left(1 + \frac{1}{\delta^2}\right)^{c(\sigma)}$	$\displaystyle\delta^n \sum_{\sigma \in \mathfrak{S}} \left(1 + \frac{1}{\delta^2}\right)^{\bar{c}(\sigma)}$

operator S operator Q

	$s(t)$	$q(t)$
Laguerre $L_n^{(\beta)}(x)$	$(1-t)^{-\beta}$	$(1-t)^{-1}$
Hermite $H_n(x)$	$e^{(t^2/2)}$	t
Charlier $C_n^{(a)}(x)$	$\exp\left(a(e^t-1)\right)$	e^t-1
Meixner $M_n(\beta, c; x)$	$\left(\frac{1-c}{1-ce^t}\right)^{\beta}$	$(1-c)\frac{e^t-1}{1-ce^t}$
Meixner-Pollaczek $P_n(\delta, \eta; x)$	$\left[\cos t\,(1-\delta \operatorname{tg} t)\right]^{-\eta}$	$\frac{\operatorname{tg} t}{1-\delta \operatorname{tg} t}$

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{1}{s\left(g^{\langle -1\rangle}(t)\right)} \exp\left(x\, g^{\langle -1\rangle}(t)\right)$$

$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = e^{\left(xt-\frac{t^2}{2}\right)}$$

$$\sum_{n\geq 0} C_n^{(a)}(x)\frac{t^n}{n!} = e^t\left(1-t/a\right)^x$$

$$\sum_{n\geq 0} \tilde{L}_n^{(\alpha)}(x)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^x(1-t)^{-x-\beta}$$

$$\sum_{n\geq 0} P_n(x;\eta,\delta)\frac{t^n}{n!} = \left[(1+\delta t)^2+t^2\right]^{-\eta/2} \exp\left[x \arctan\left(\frac{t}{1+\delta t}\right)\right]$$

<u>in conclusion</u>:

equivalent interpretation
in terms of increasing binary trees

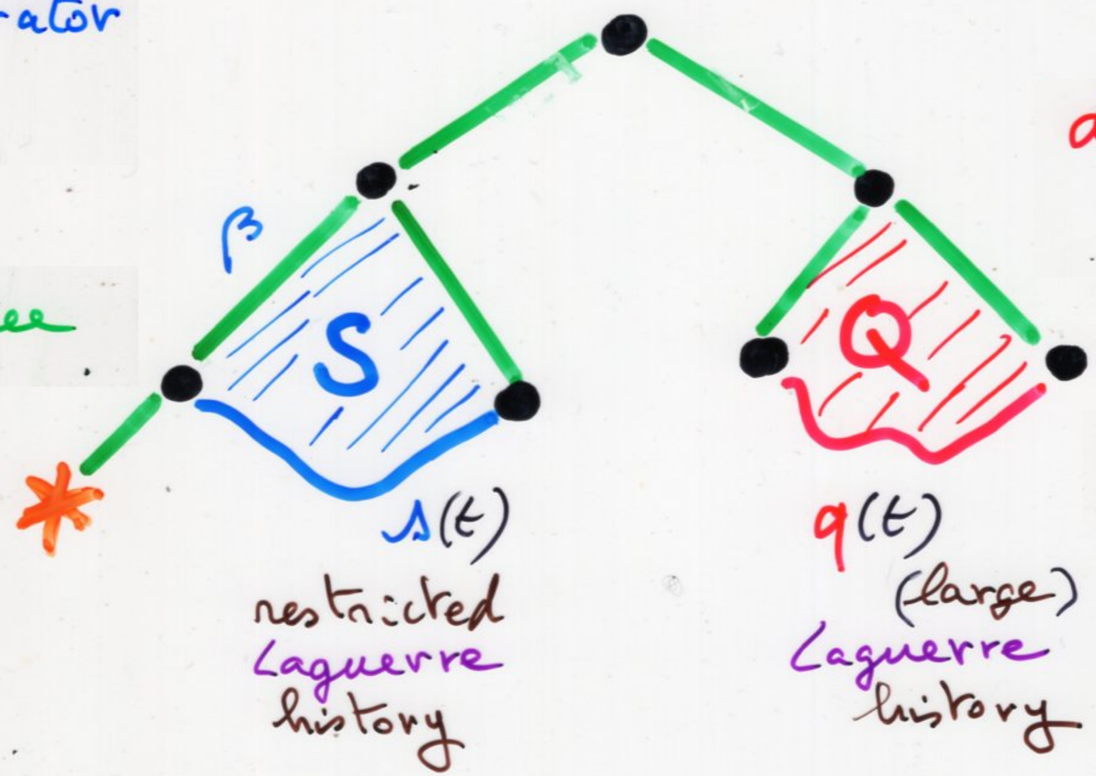
interpretations of $s(t)$ and $q(t)$

delta operator
S

left subtree

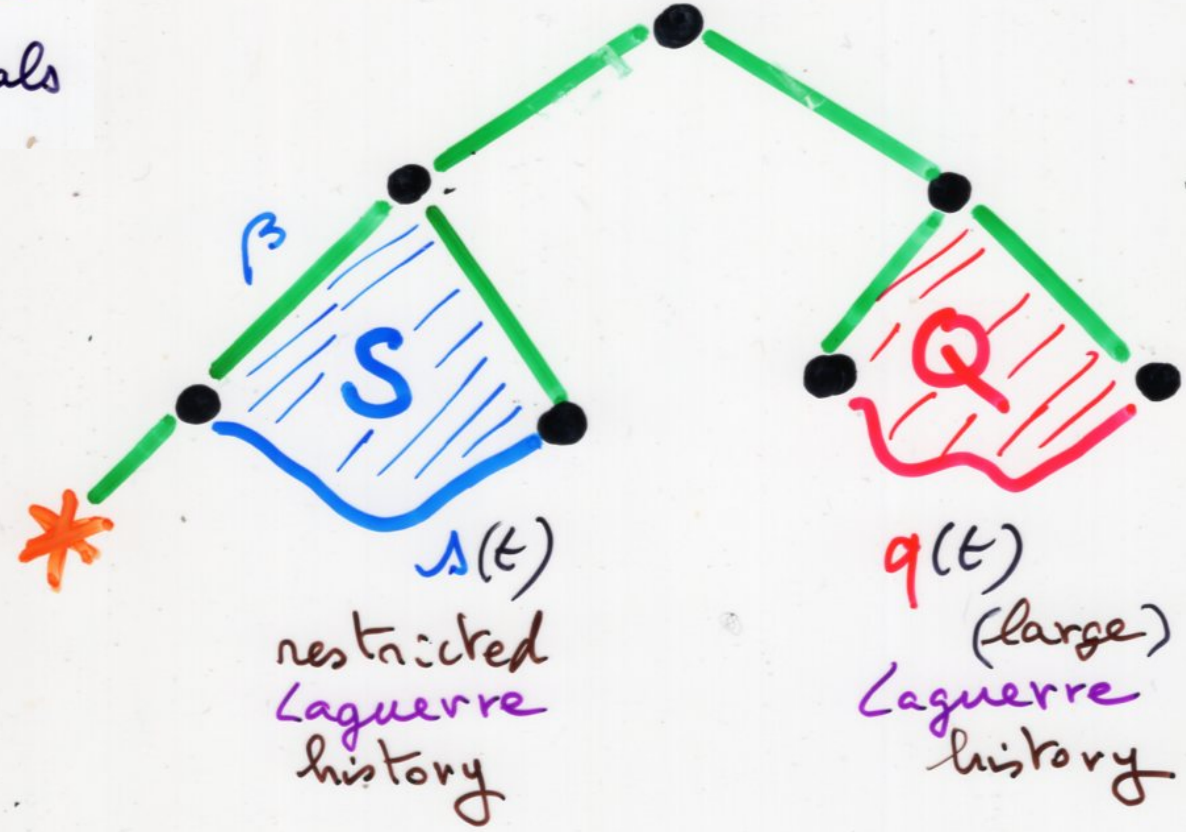
delta operator
Q

right subtree



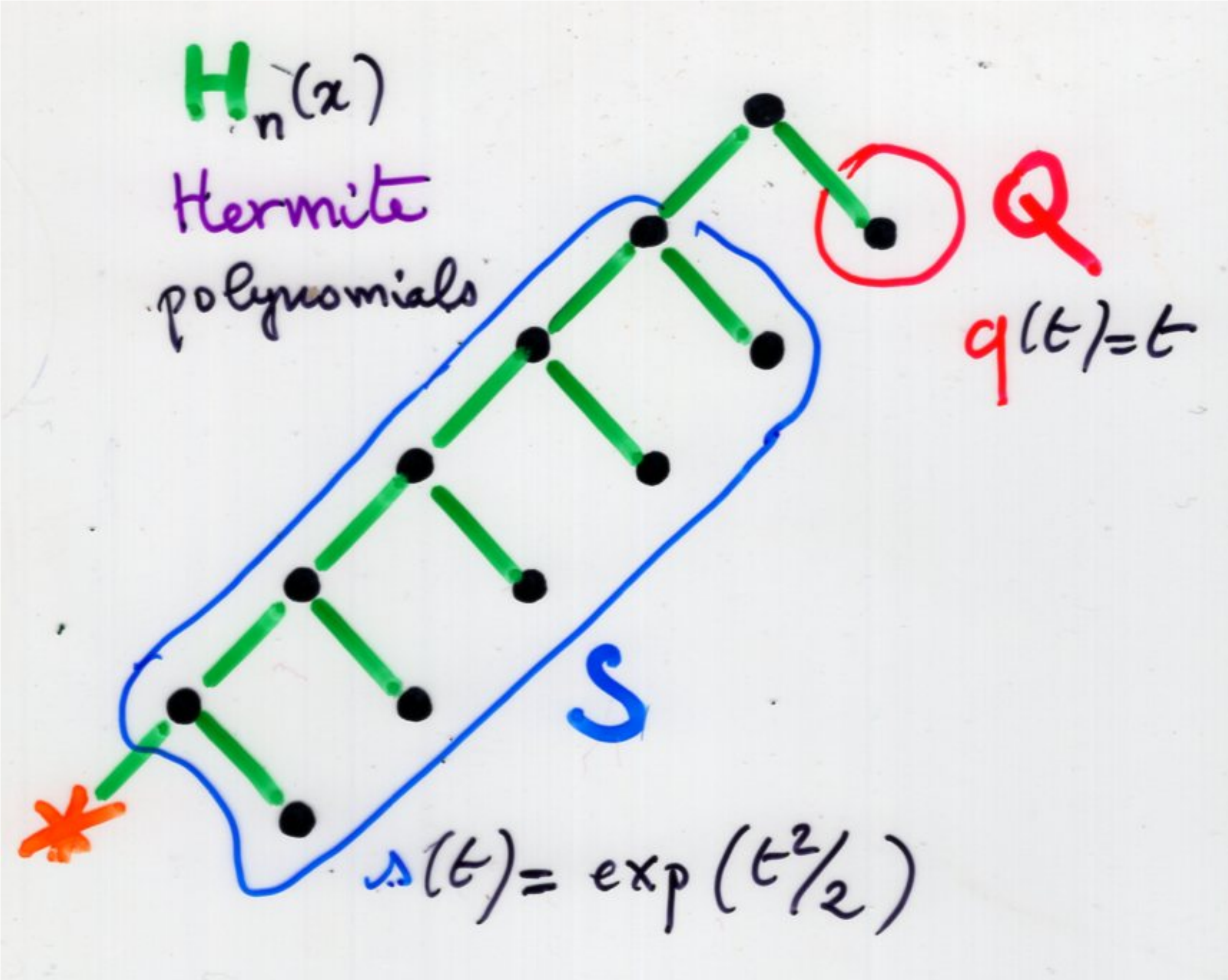
$L_n^{(\beta)}(x)$

Laguerre polynomials

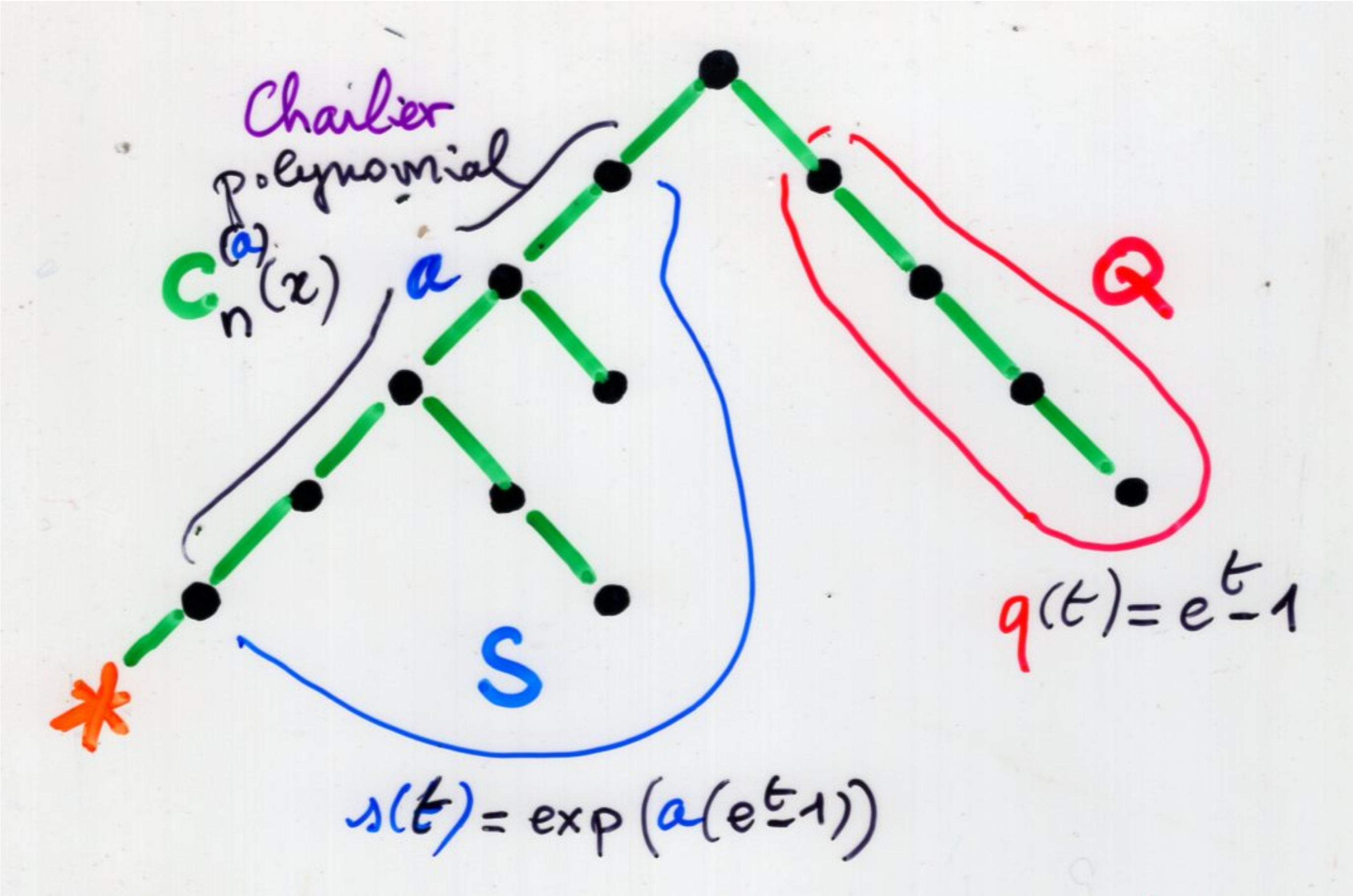


$s(t) = (1-t)^{-\beta}$

$q(t) = (1-t)^{-1}$



$H_n(x)$
Hermite
polynomials
Q
$q(t)=t$
S
$s(t)=\exp(t^2/2)$


Chailier
polynomial
$C_n^{(a)}(x)$
a
Q
S
$q(t) = e^t - 1$
$s(t) = \exp(a(e^t - 1))$

combinatorial interpretation of the operator Q and S for the 5 classes of Sheffer orthogonal polynomials with:

Laguerre histories

$\begin{cases} \text{restricted} & S \\ \text{large} & \rightarrow Q \end{cases}$

duality

orthogonal polynomial $\longleftrightarrow$ moments μ_n

reciprocal of $q(t)$ $\quad q^{<-1>}(t)$