

Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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mirror website

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Chapter 3

Tableaux for the PASEP quadratic algebra

Ch3c

(first part)

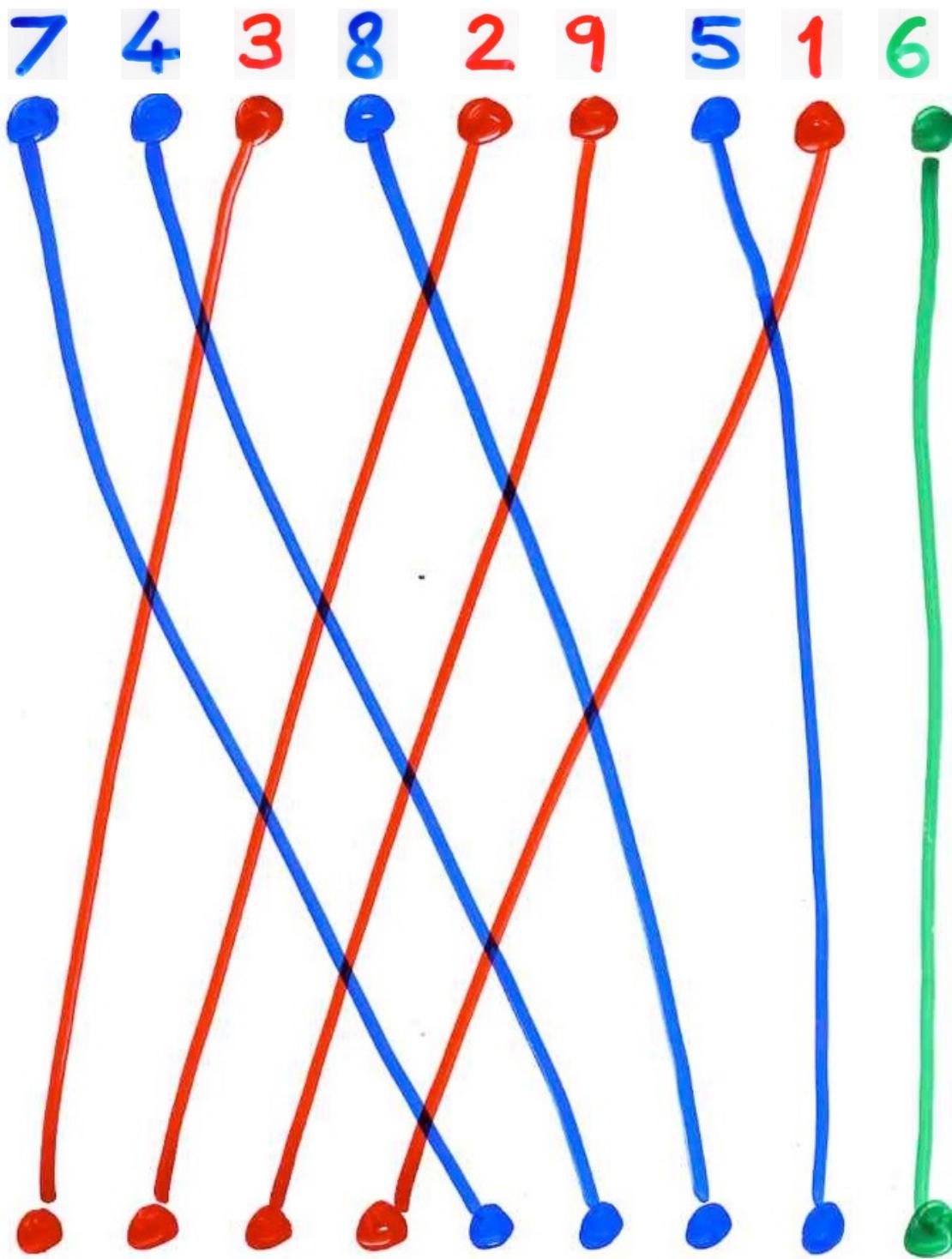
IMSc, Chennai
February 22, 2018

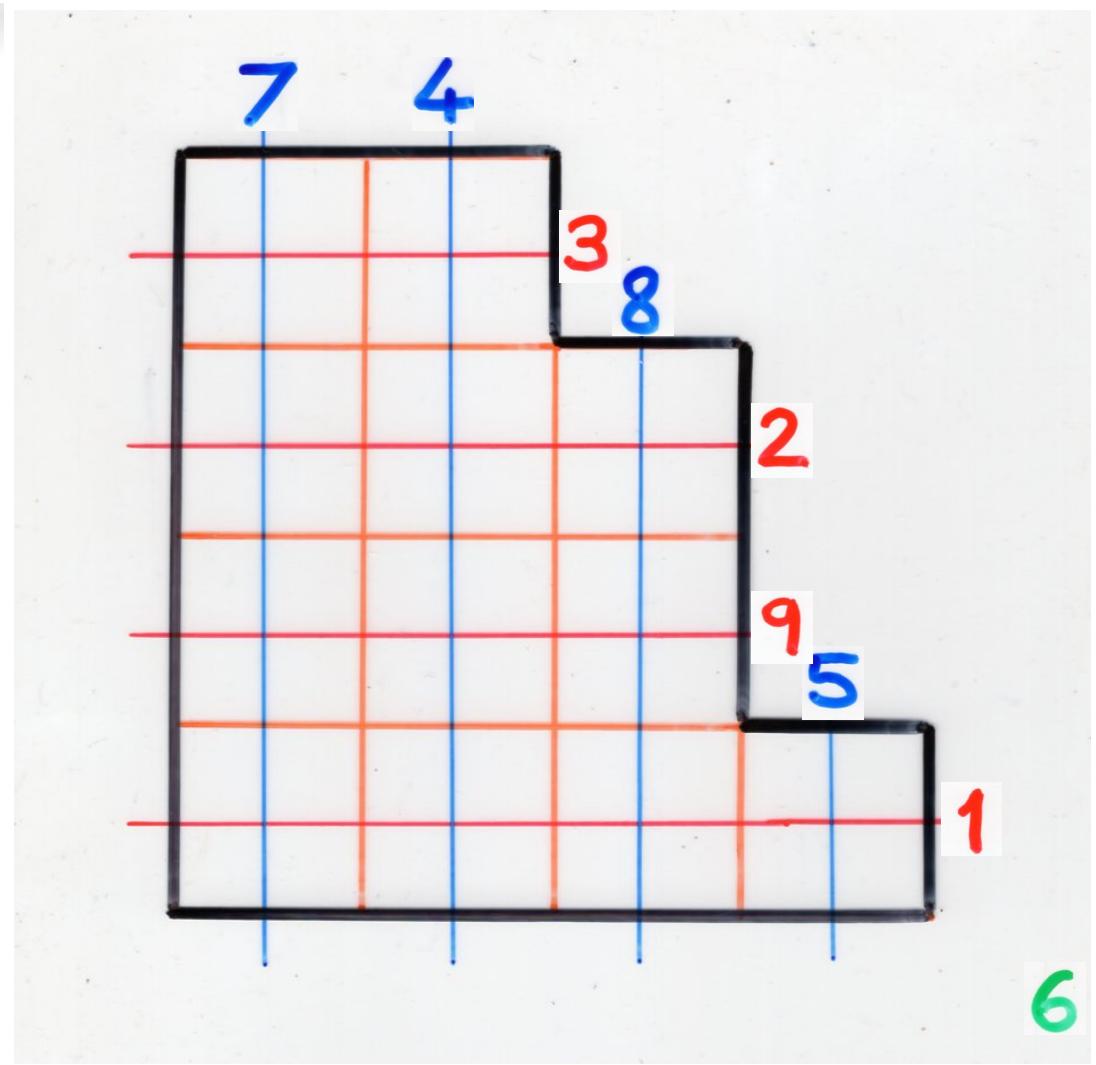
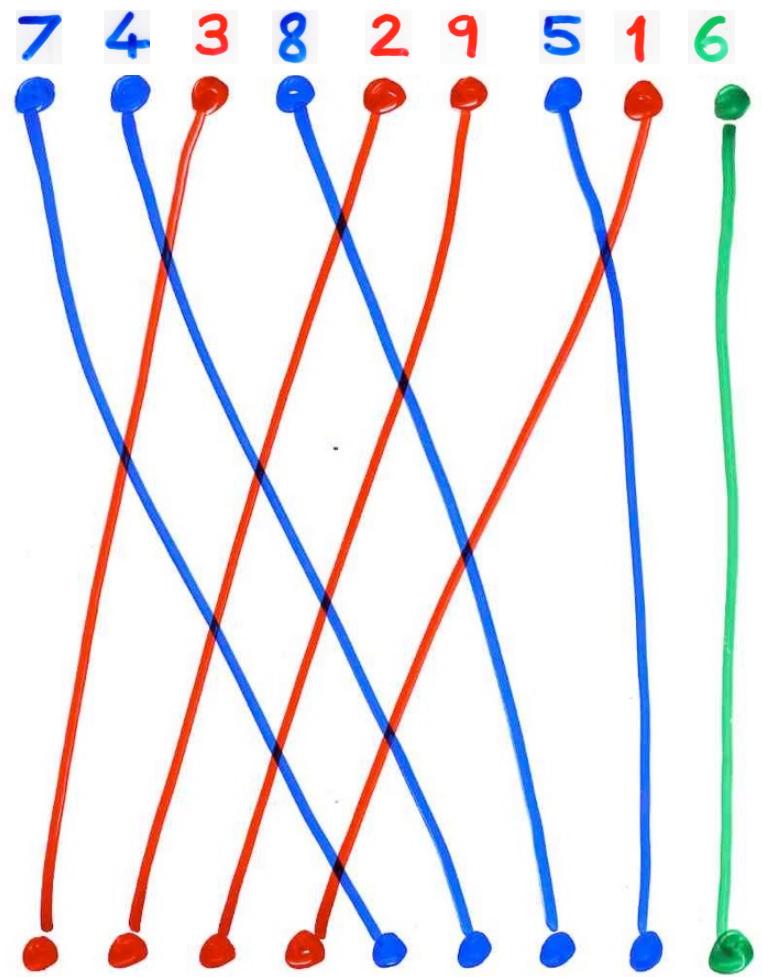
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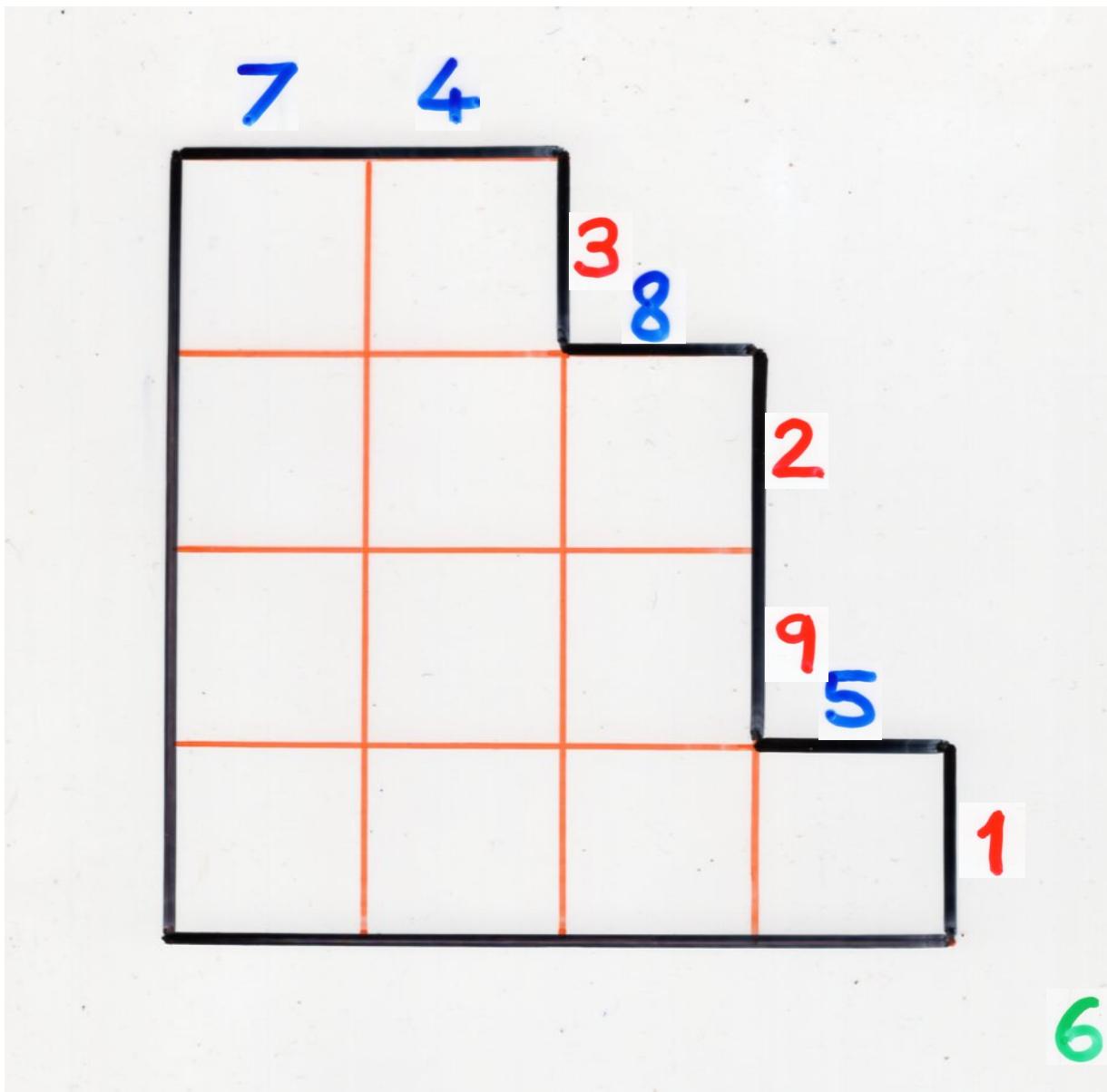
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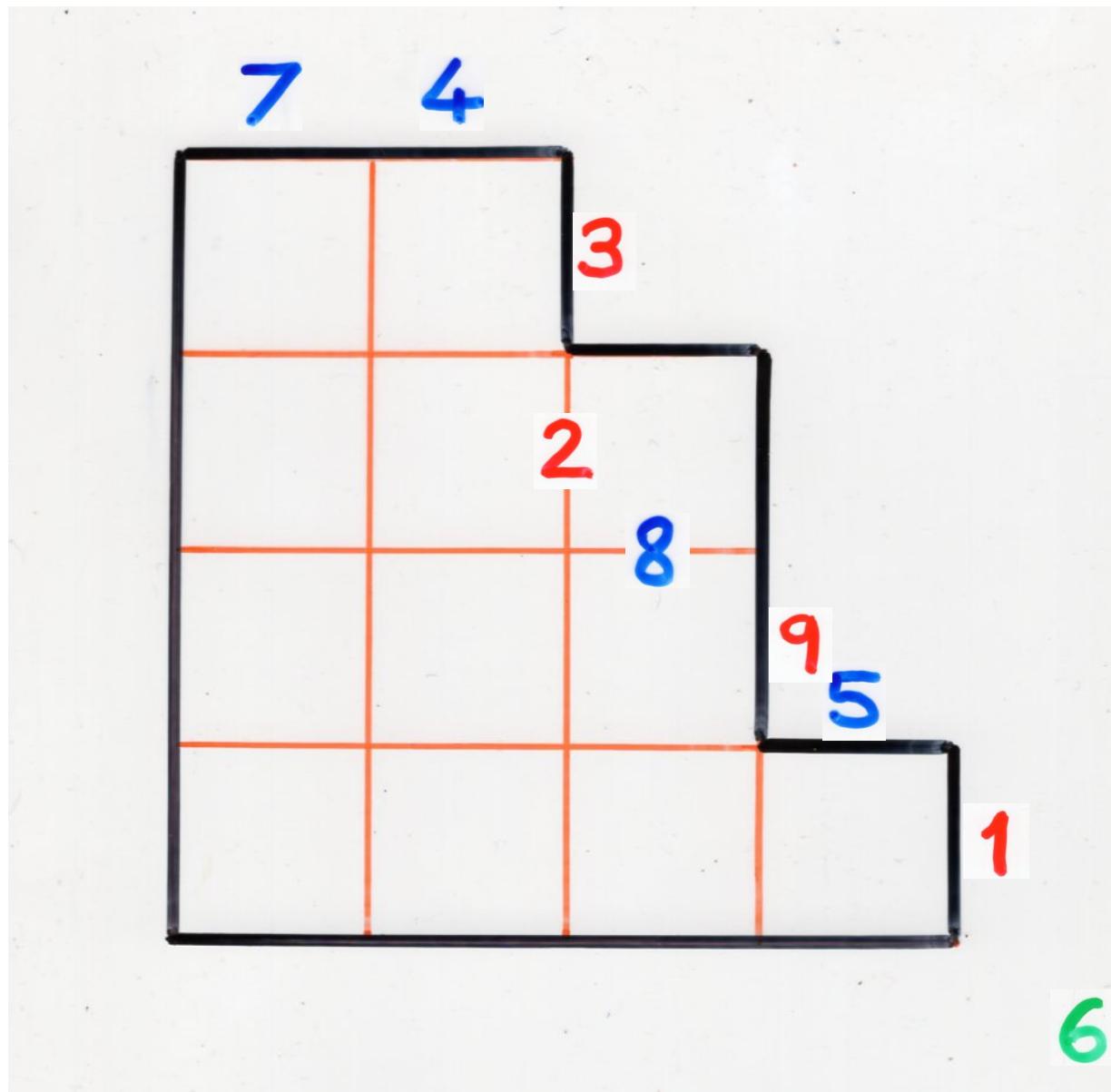
Reminding Ch3b

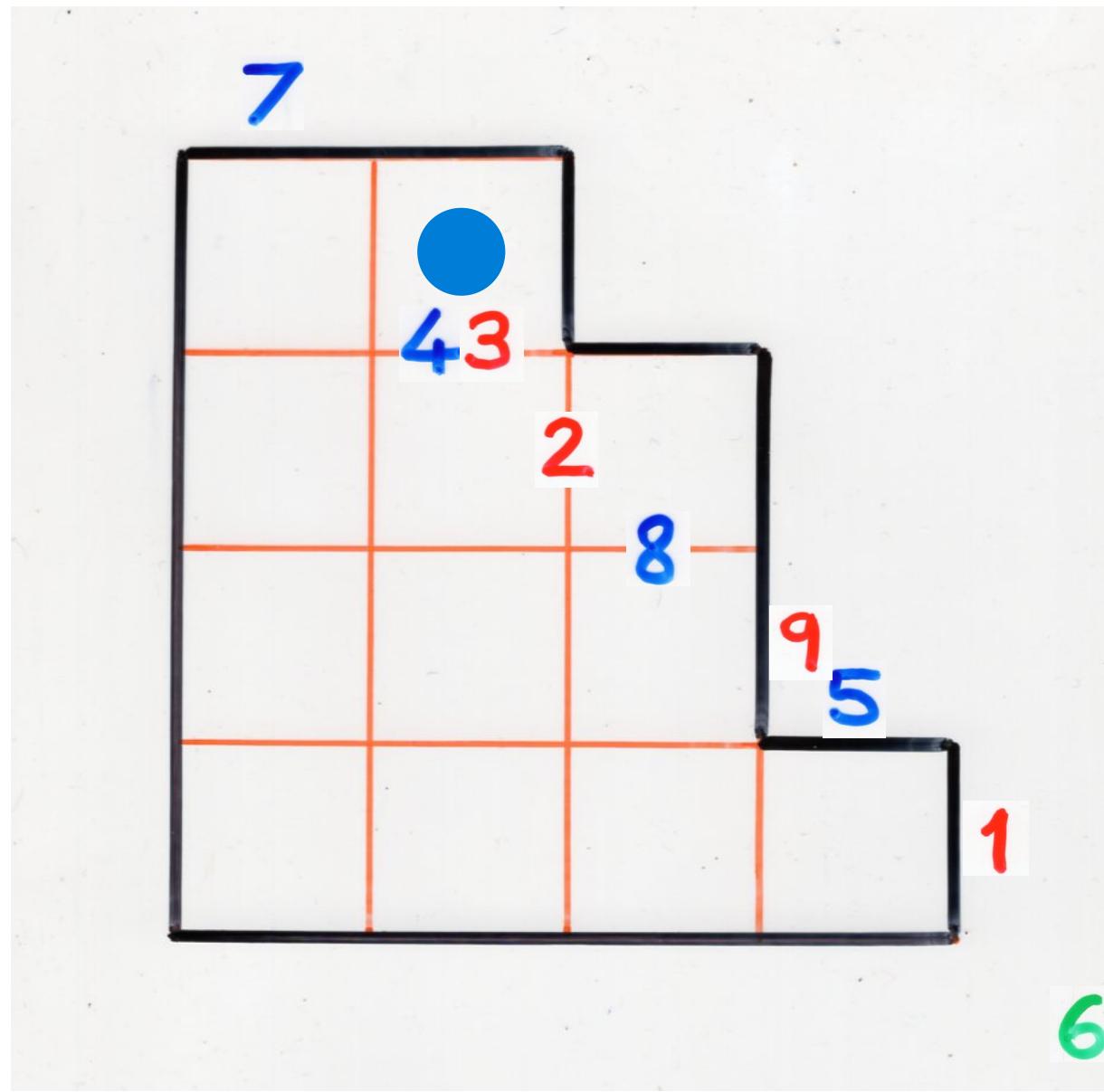
The “exchange-fusion” algorithm

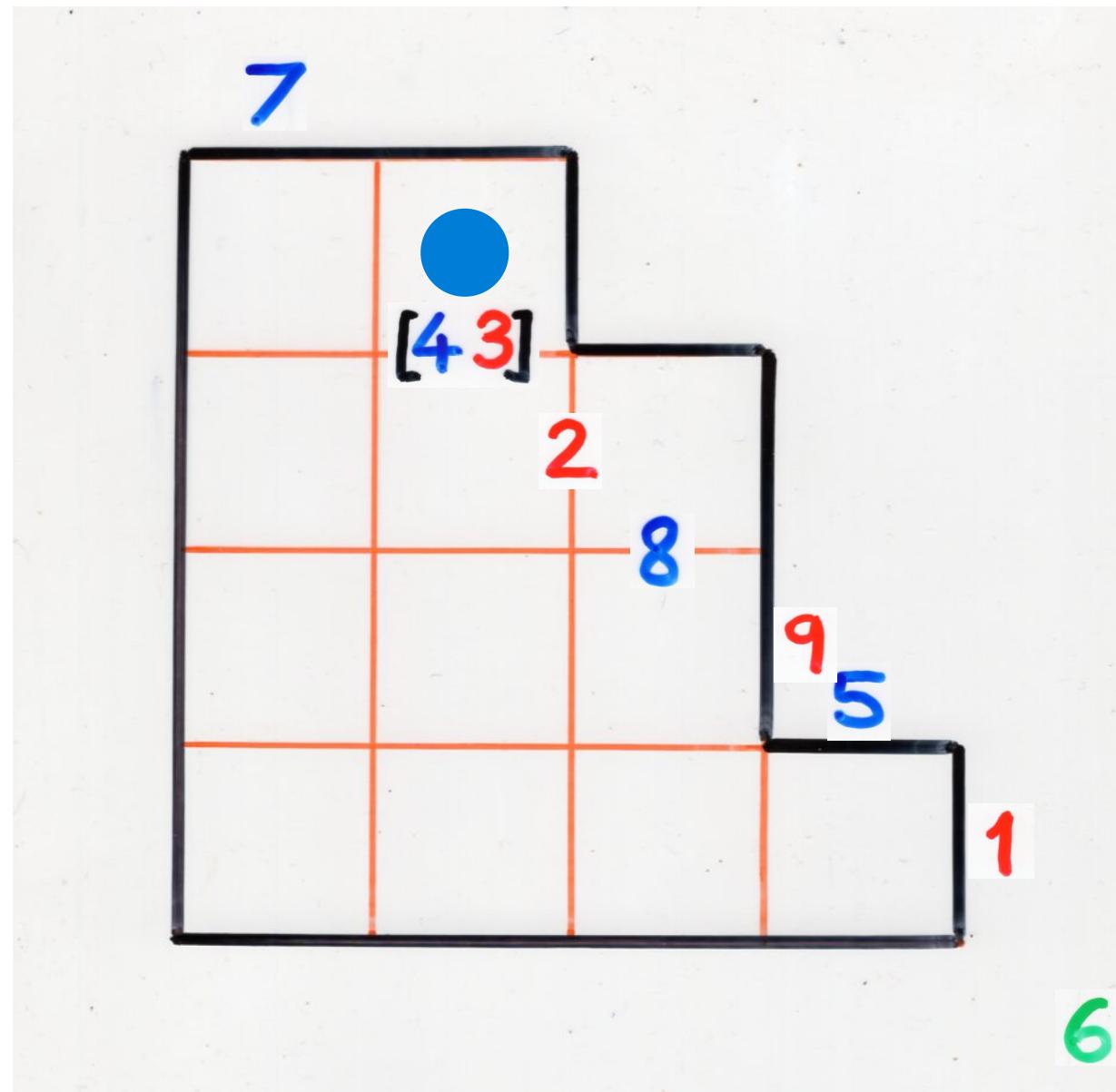


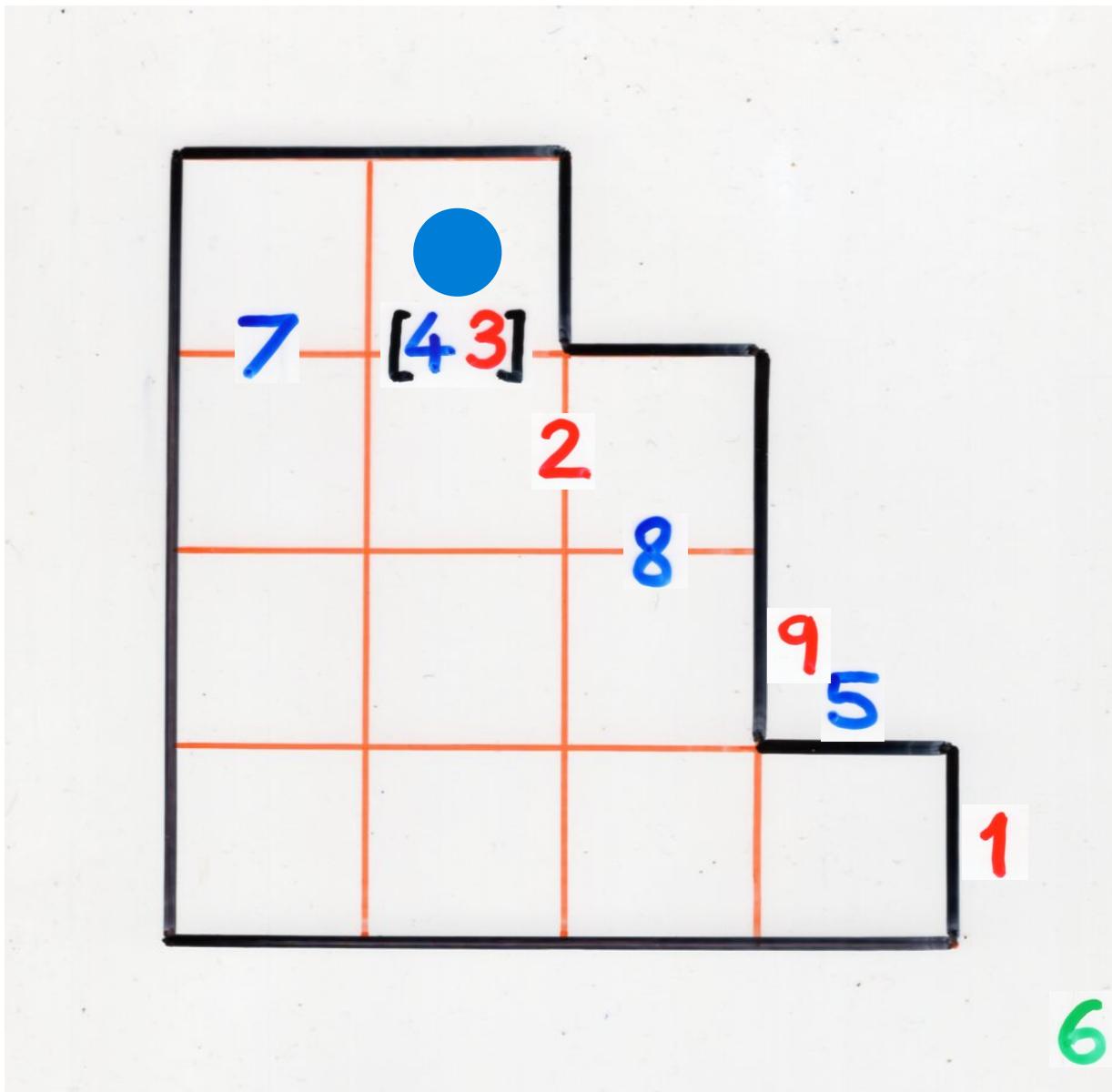


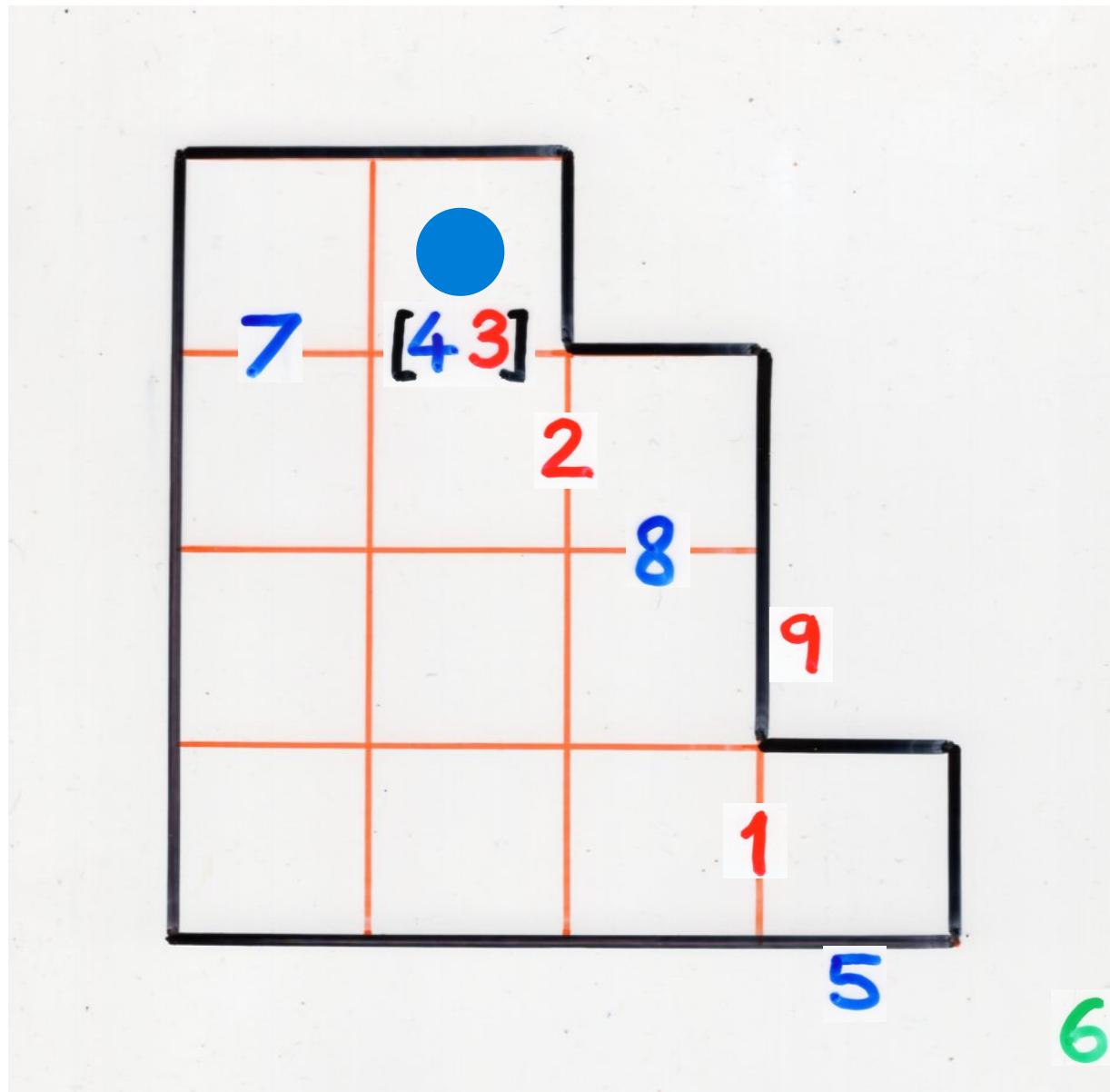


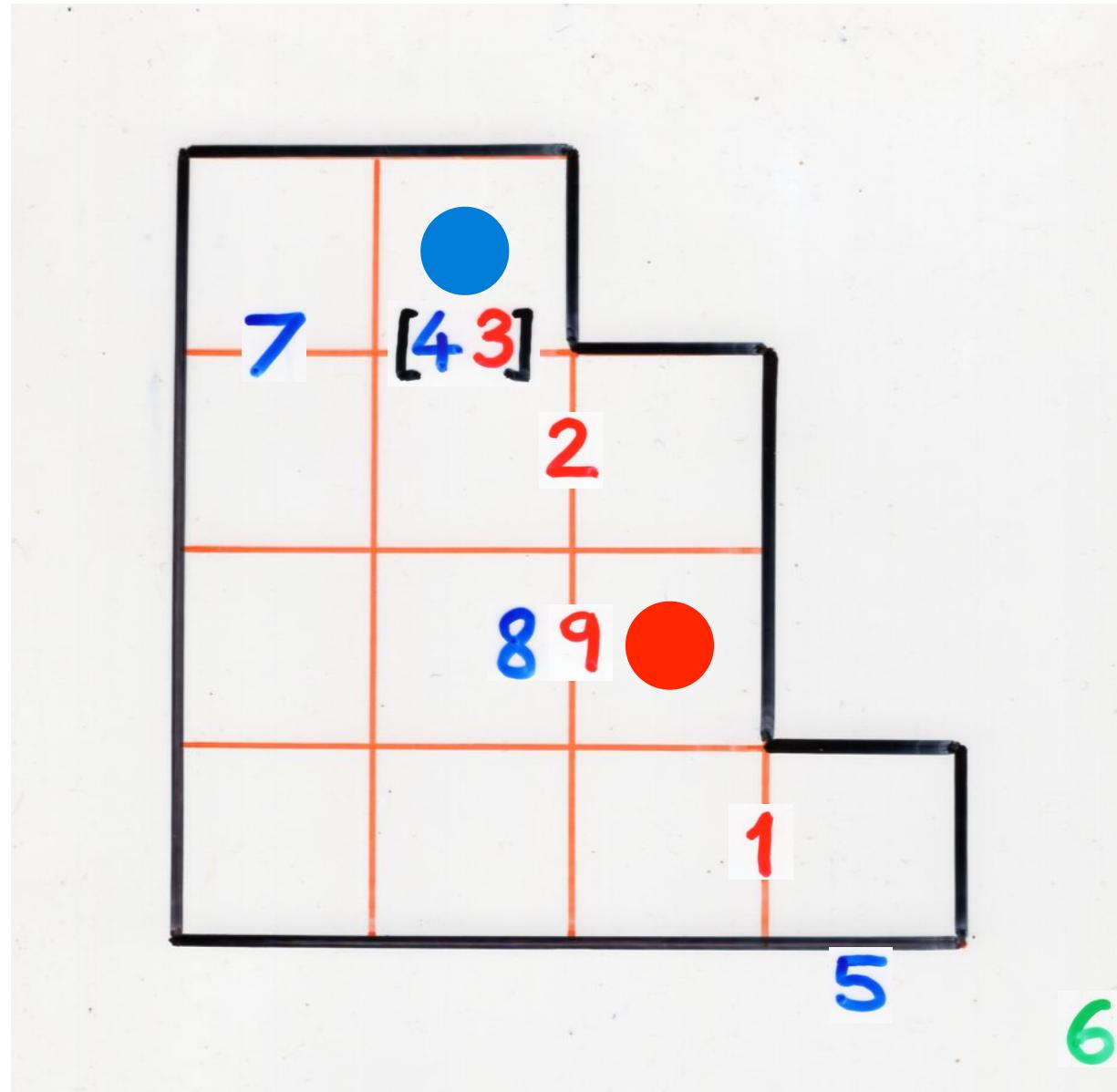


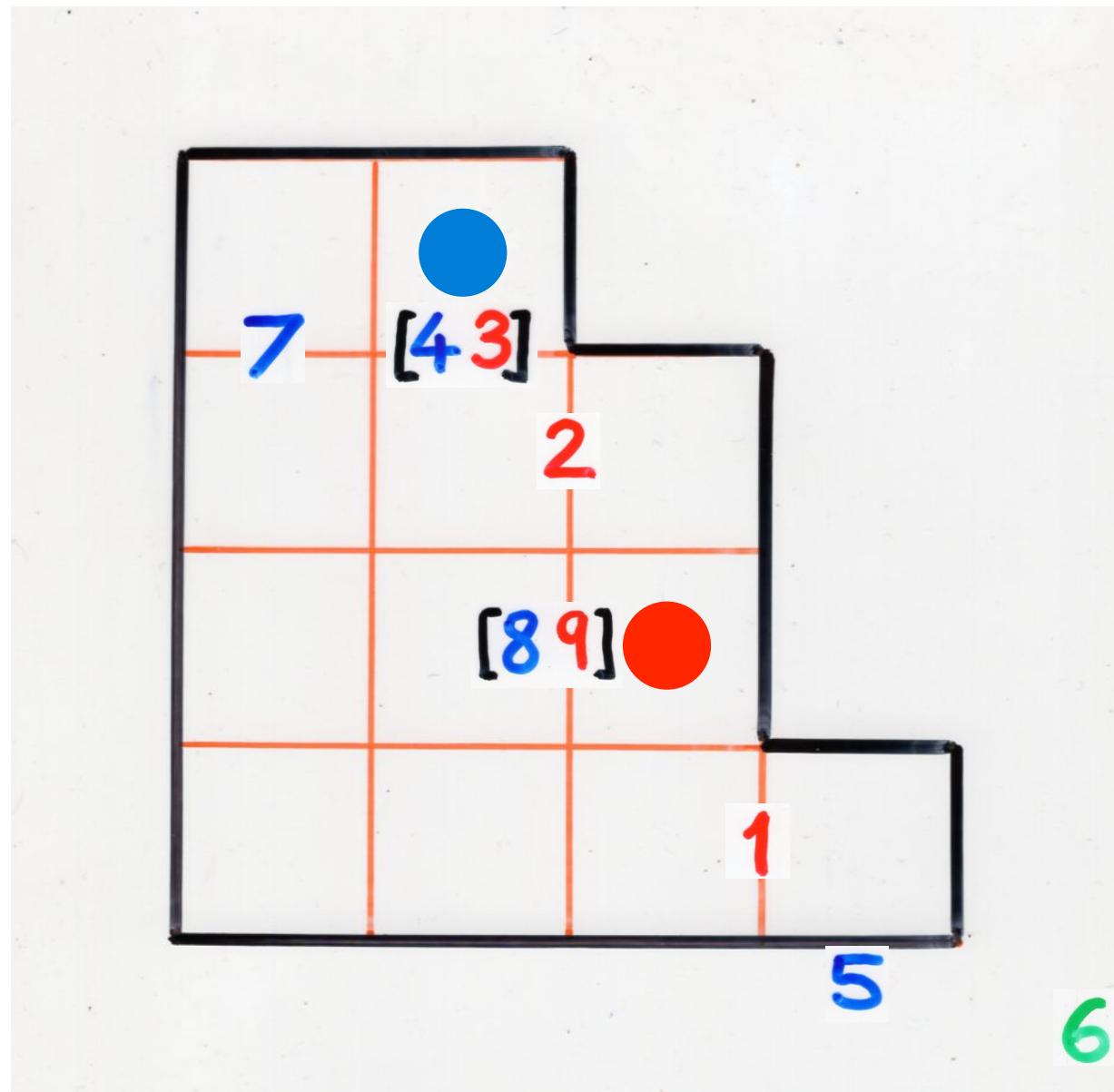


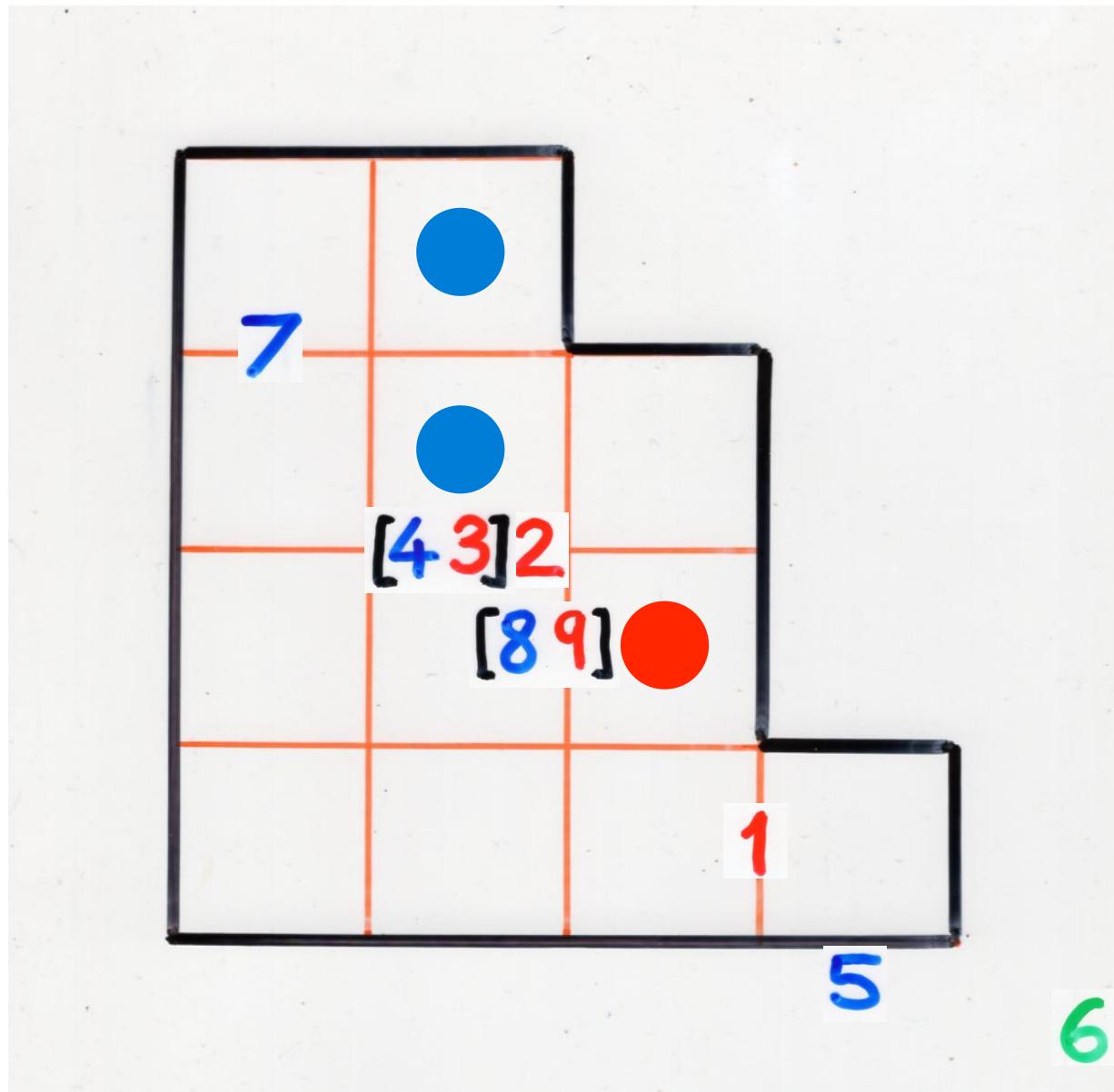


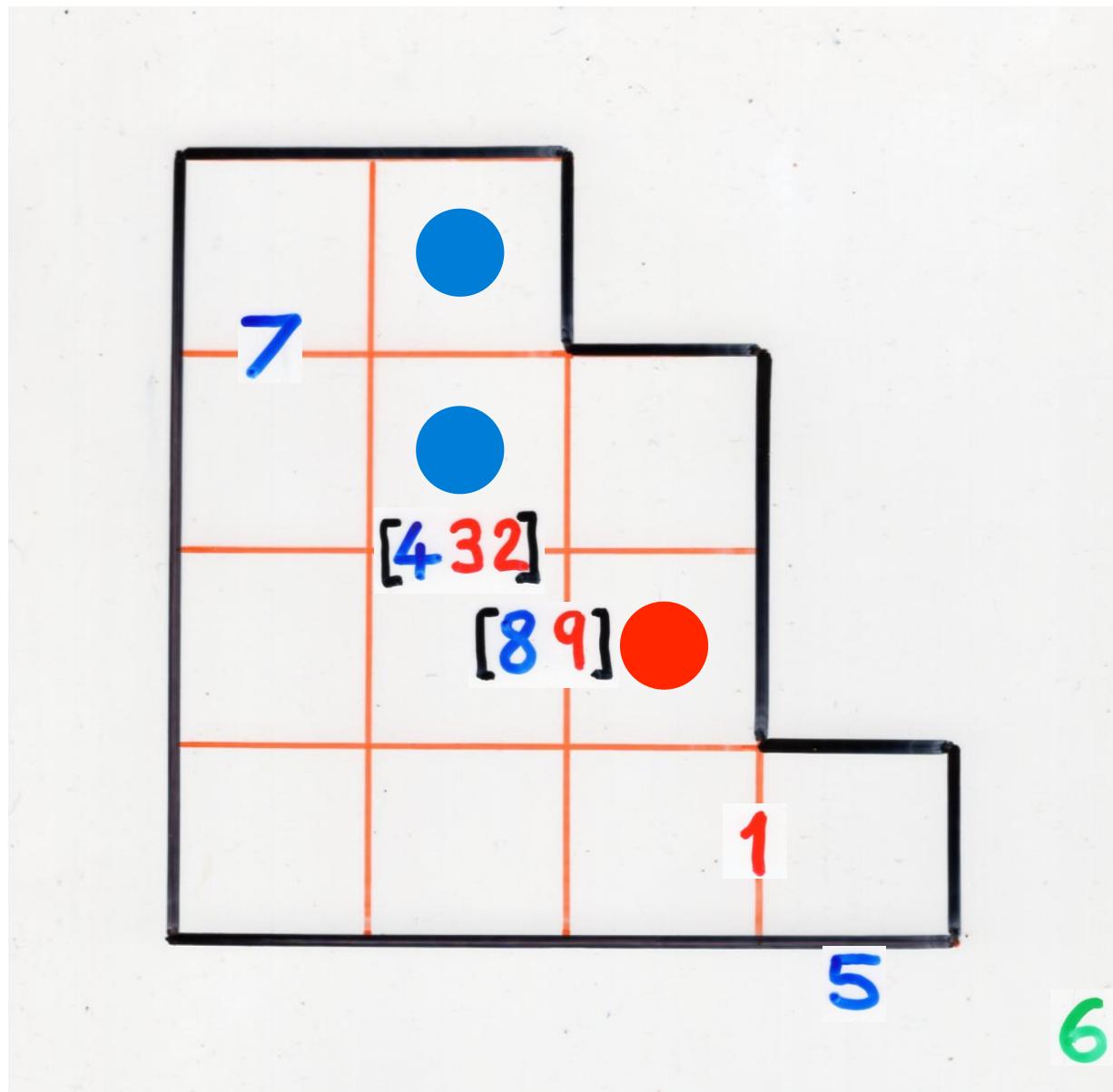


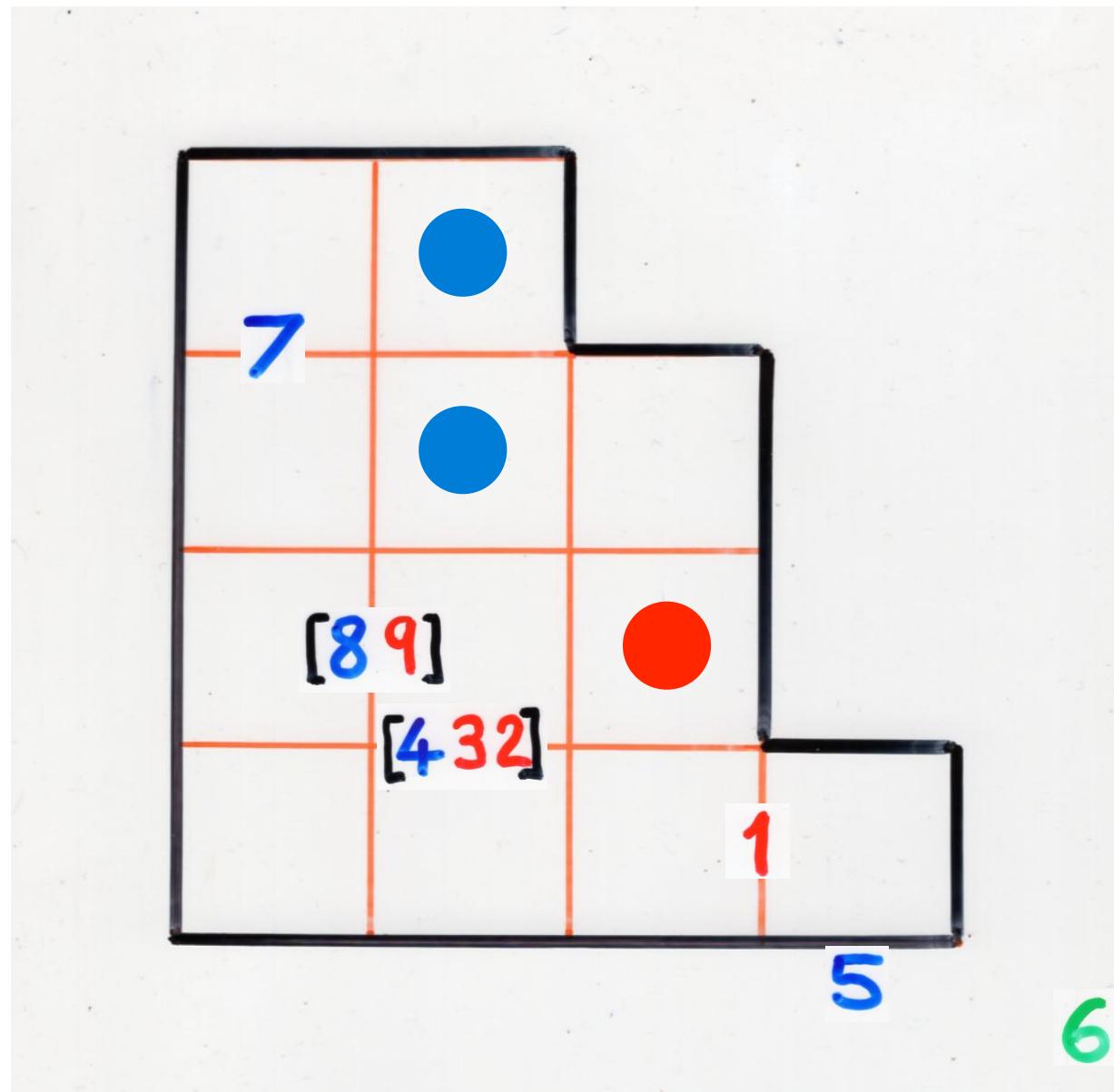


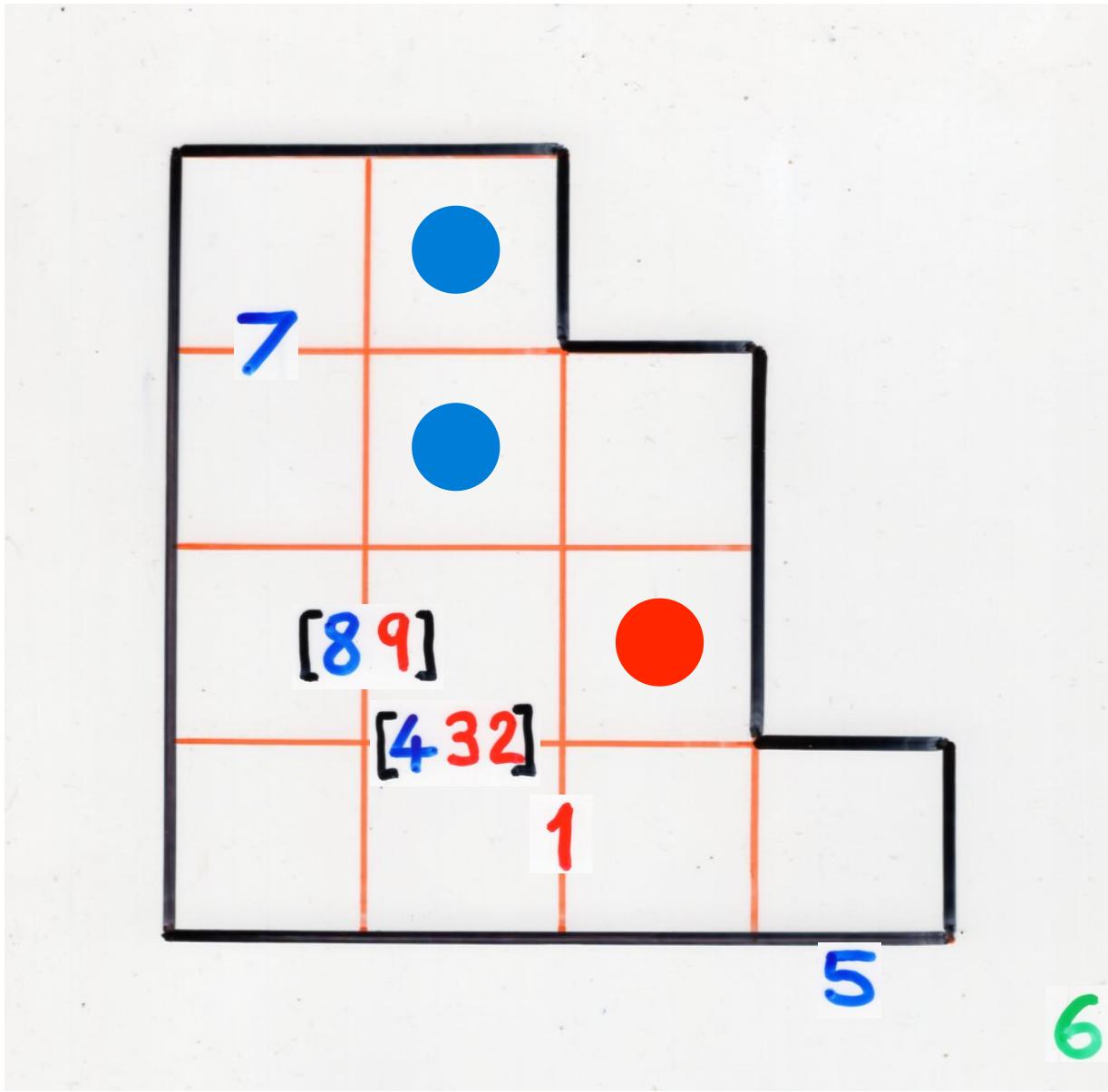


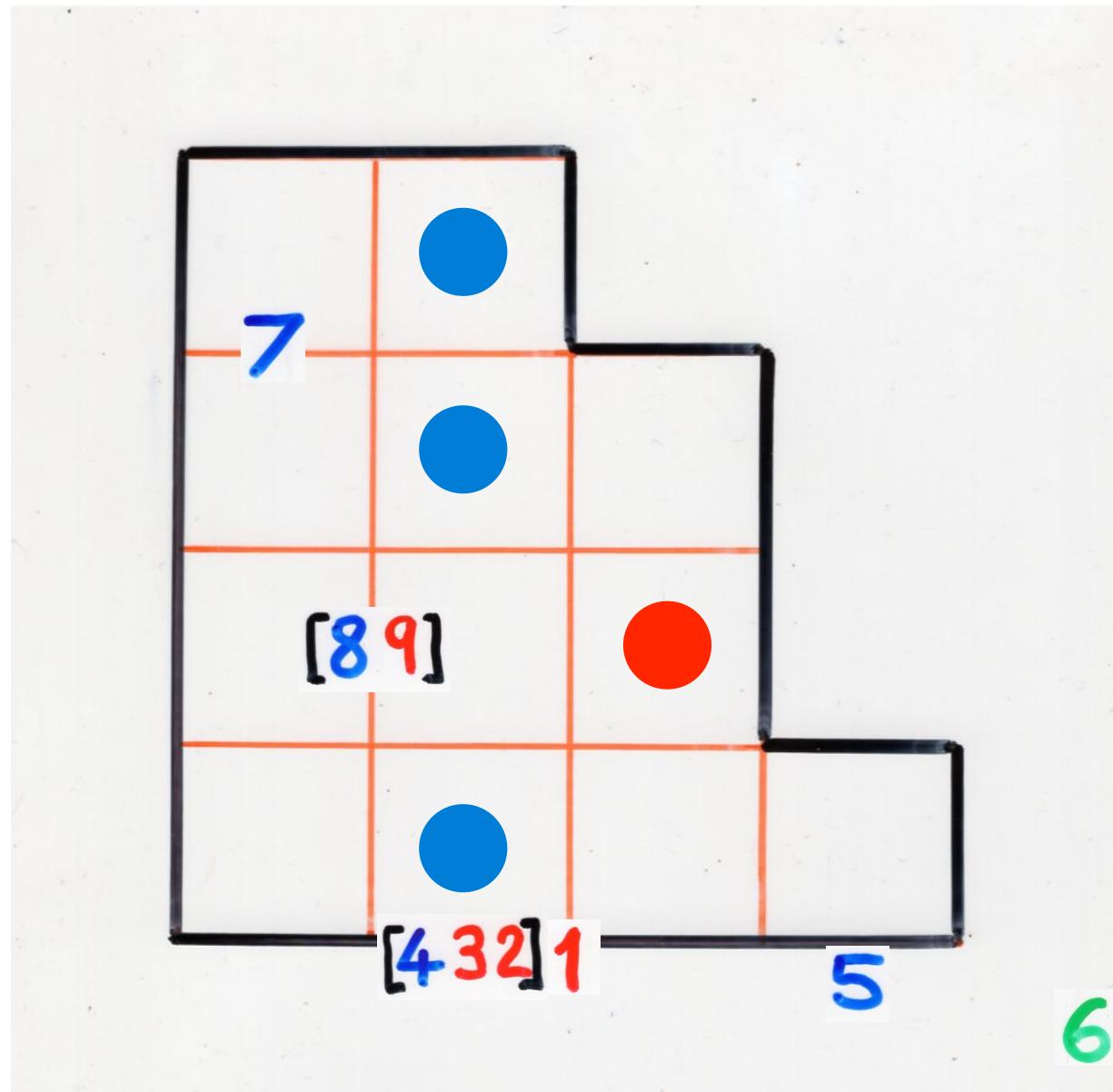


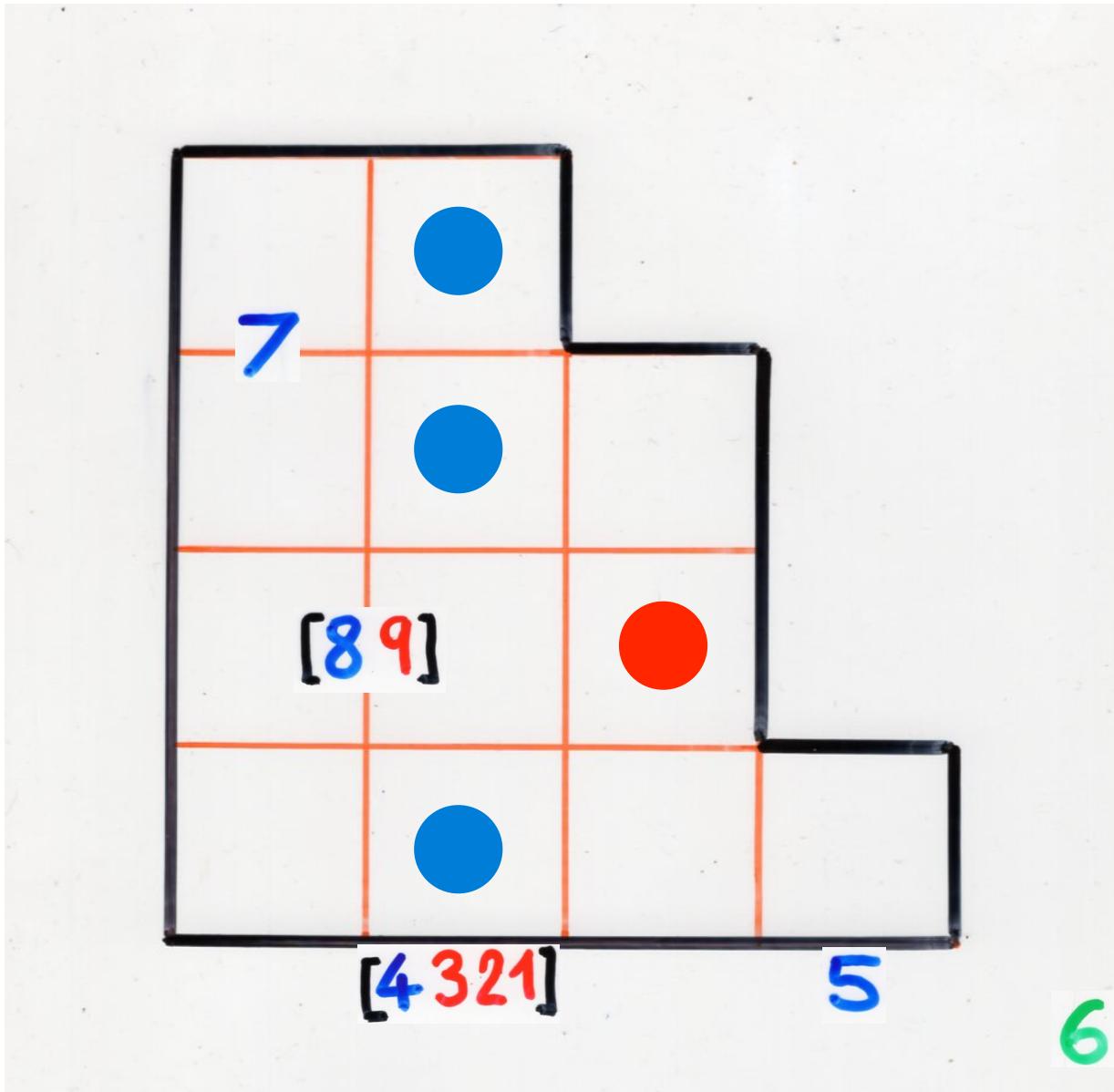


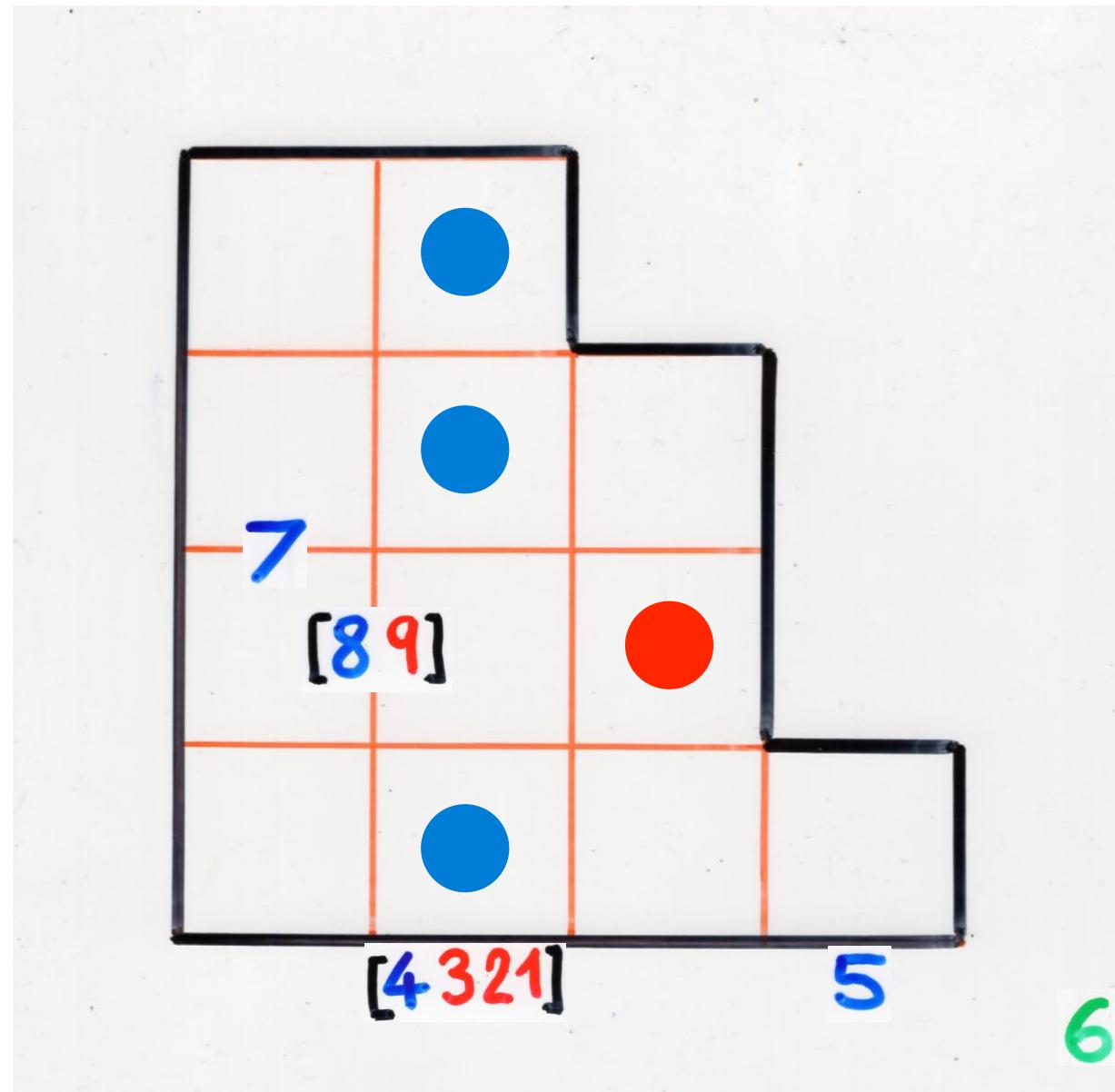


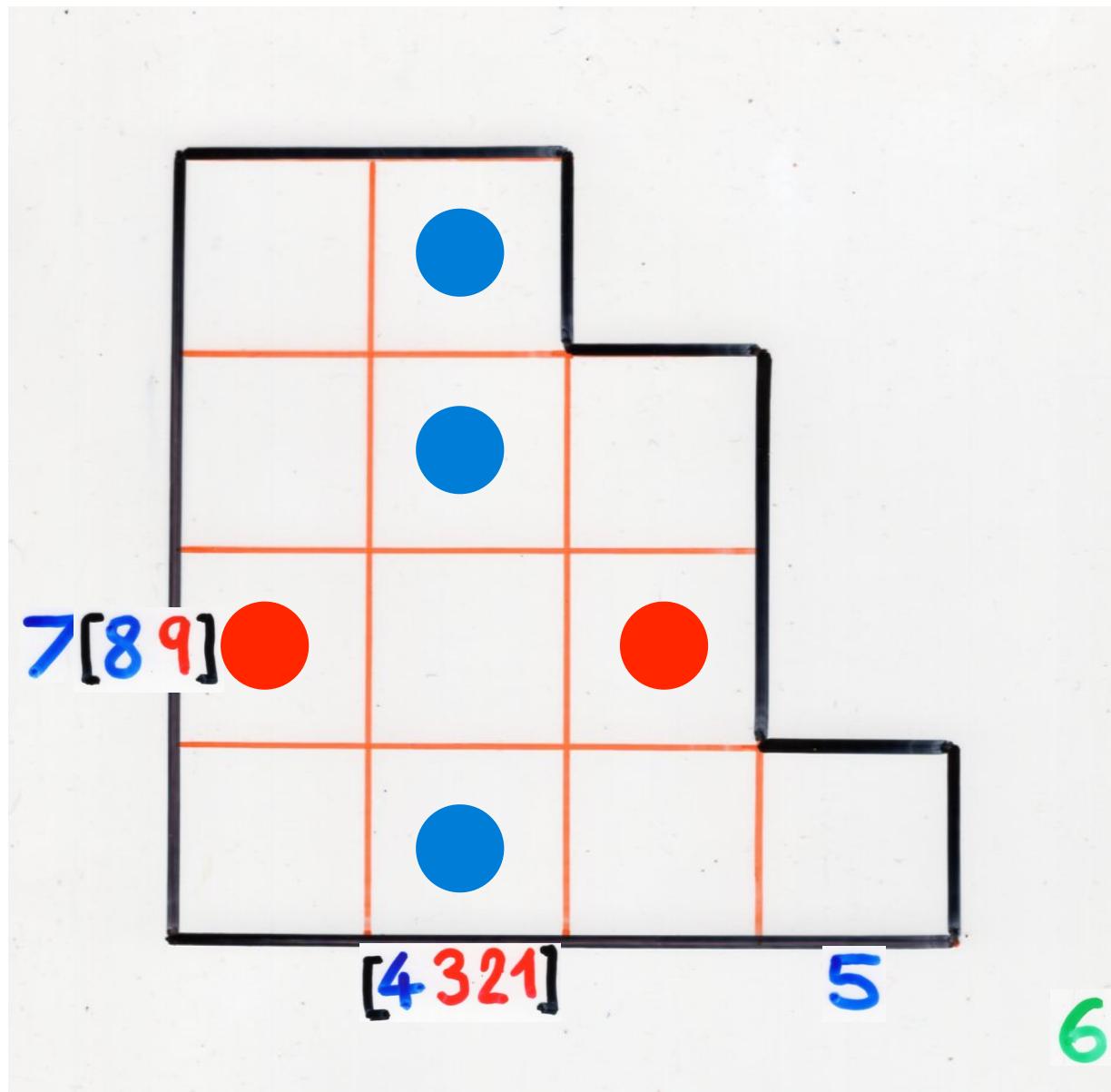


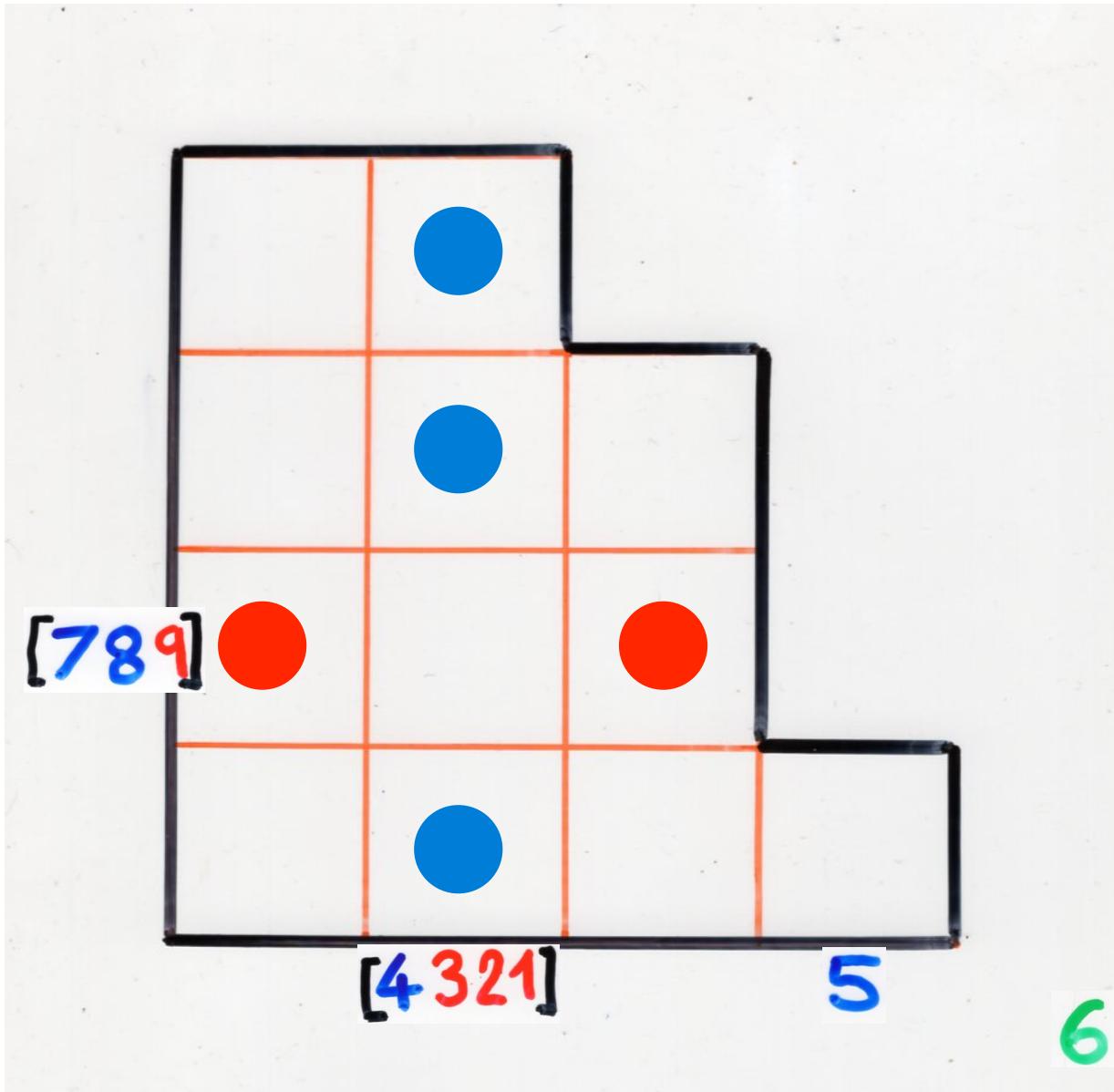


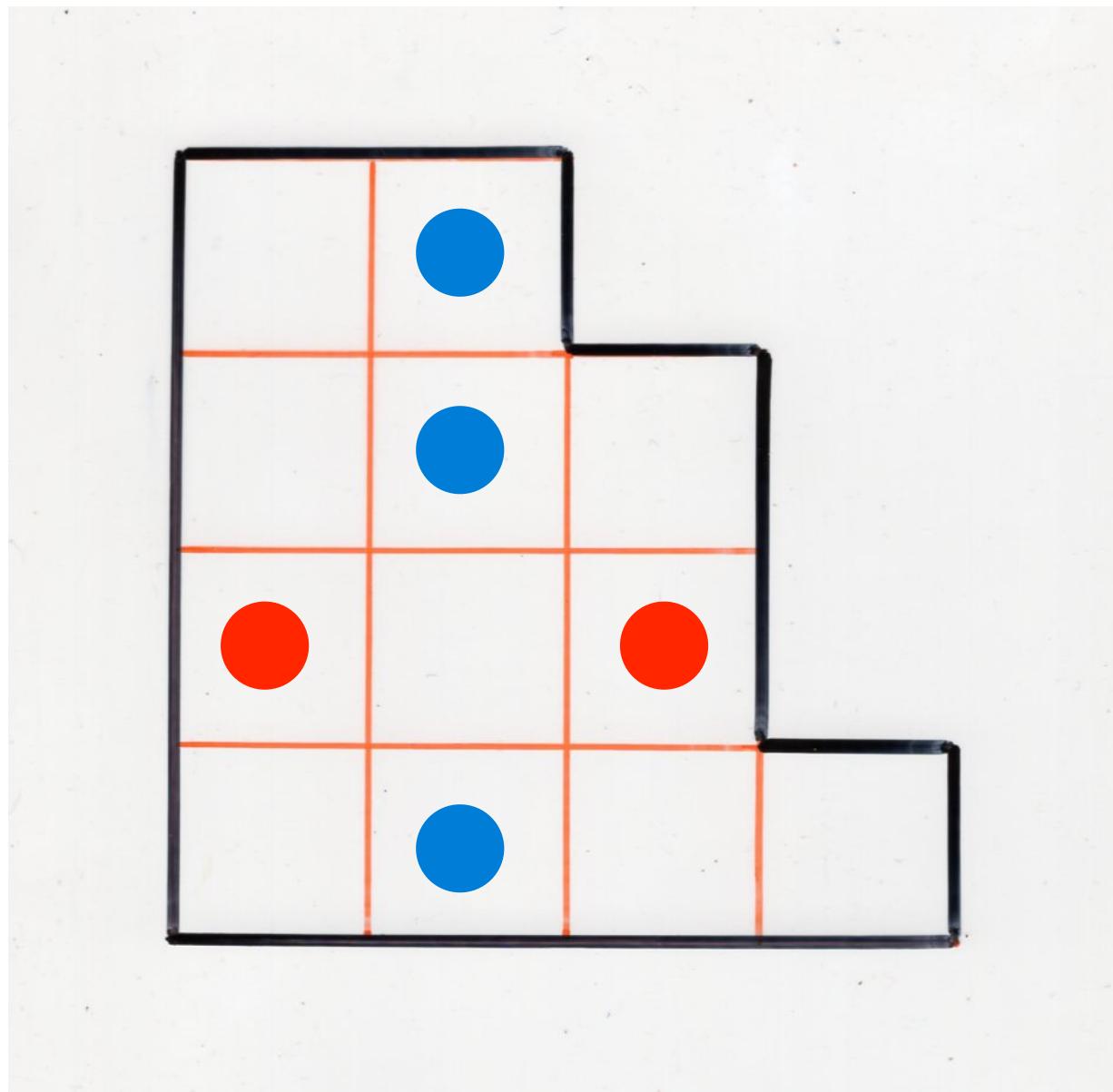








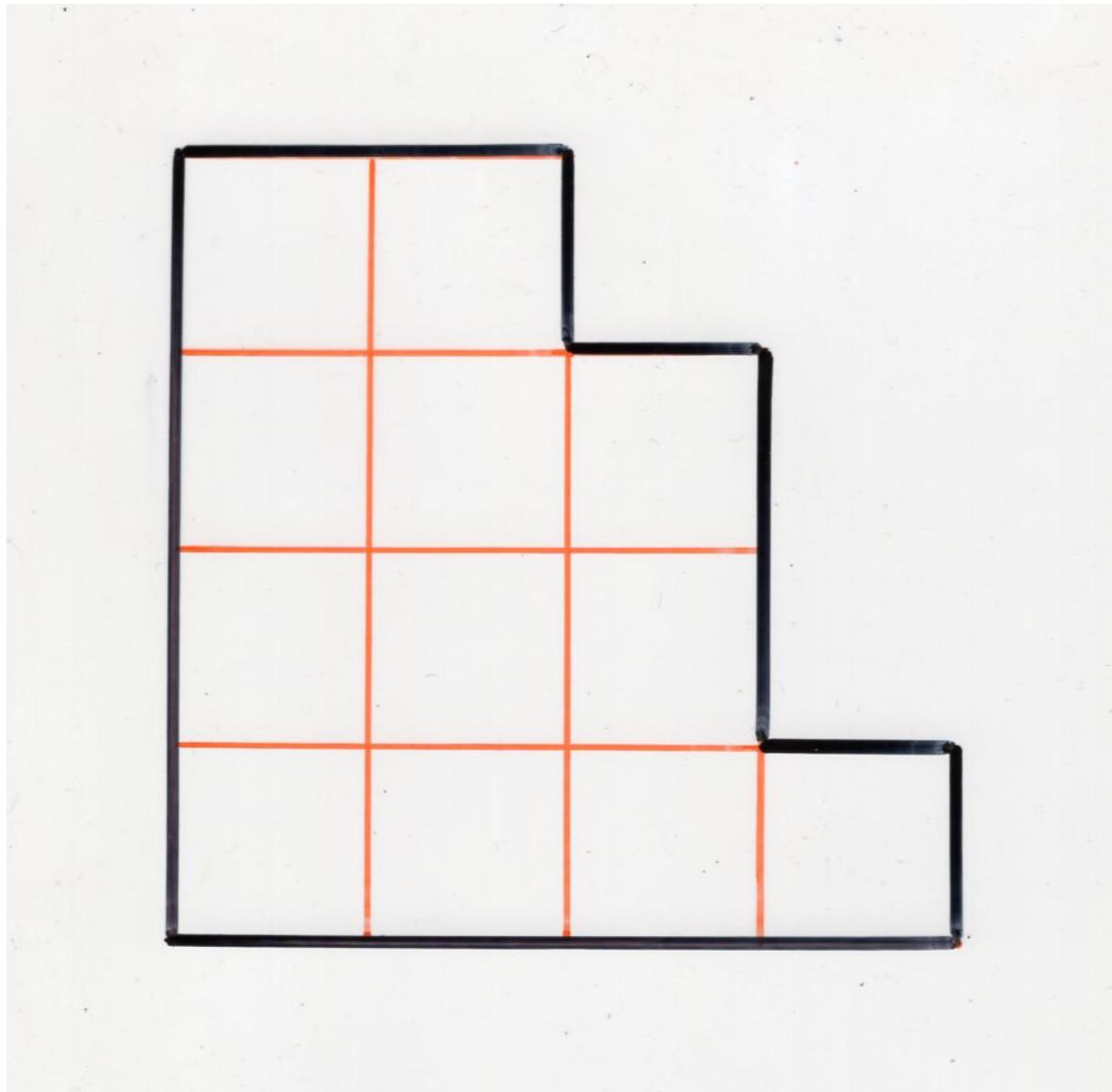


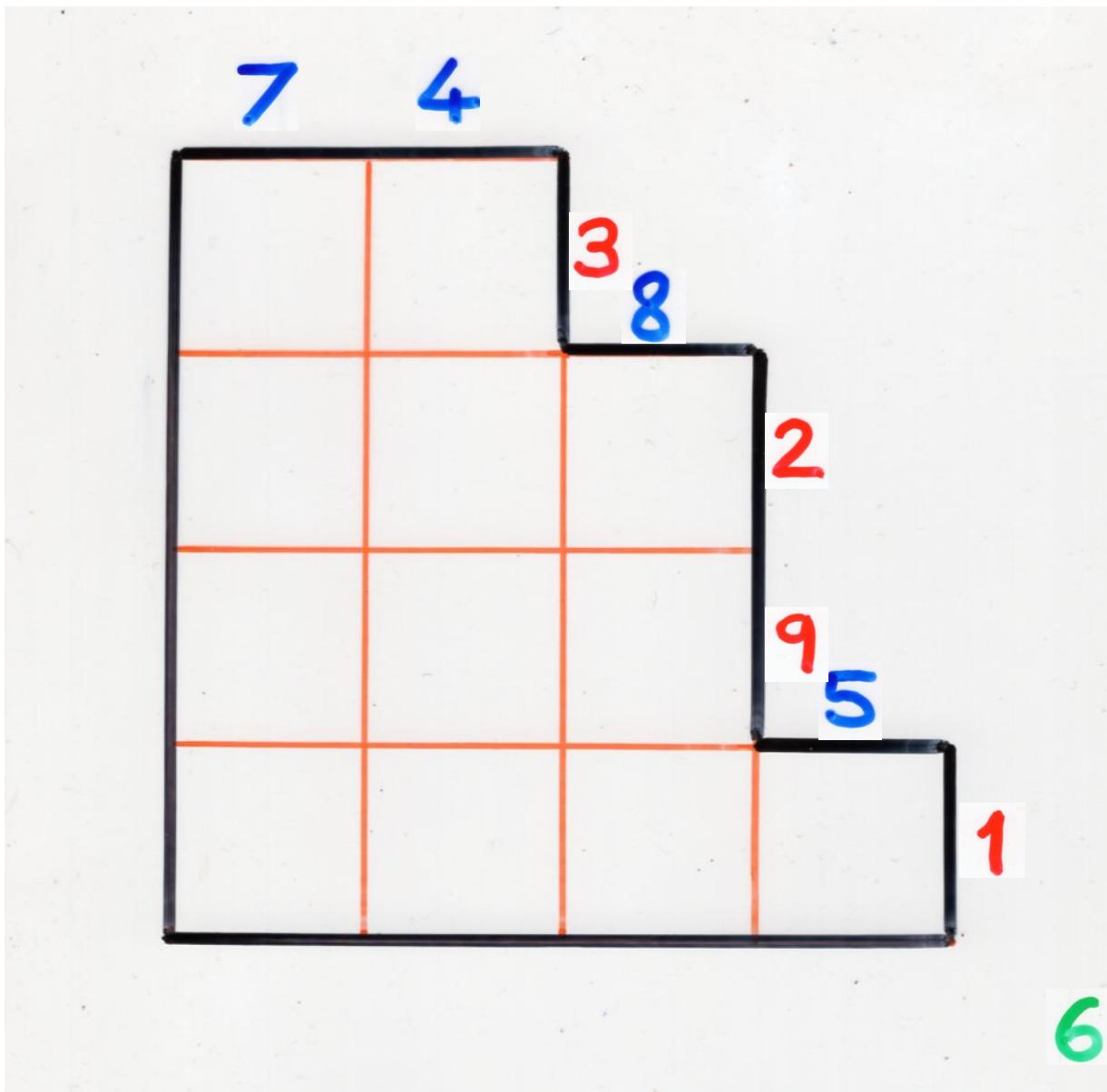


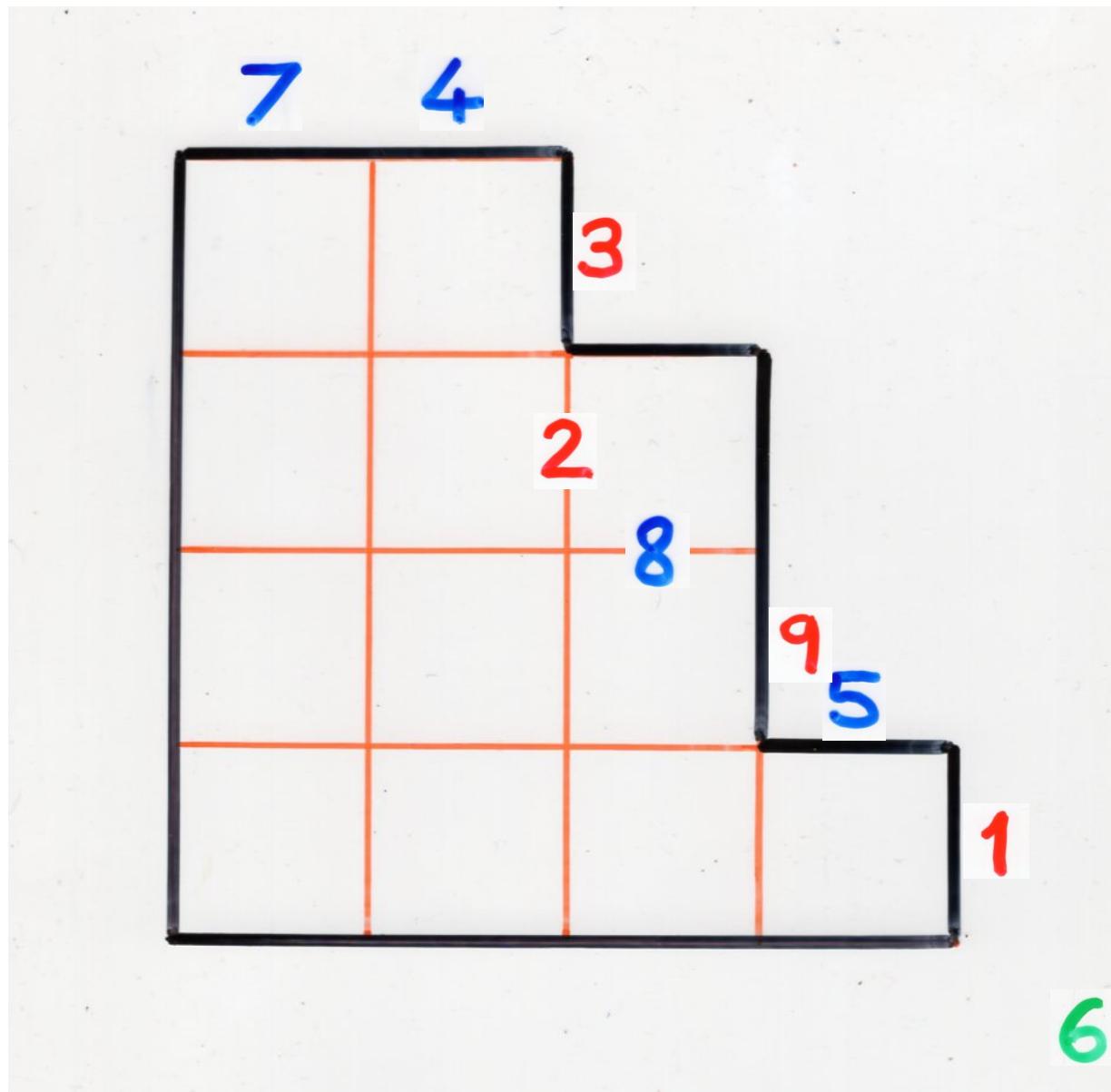
Reminding Ch3b

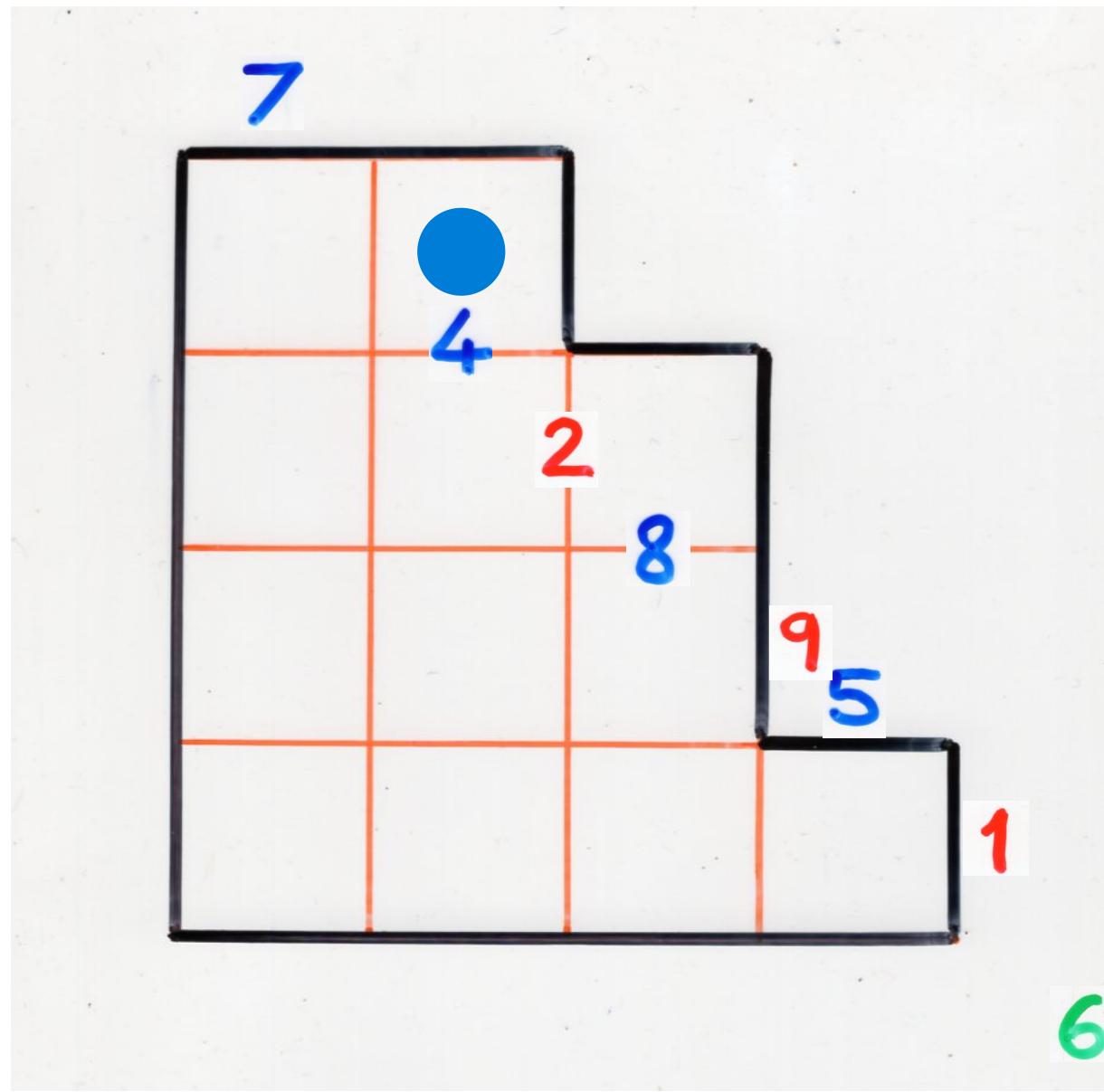
The “exchange-delete” algorithm

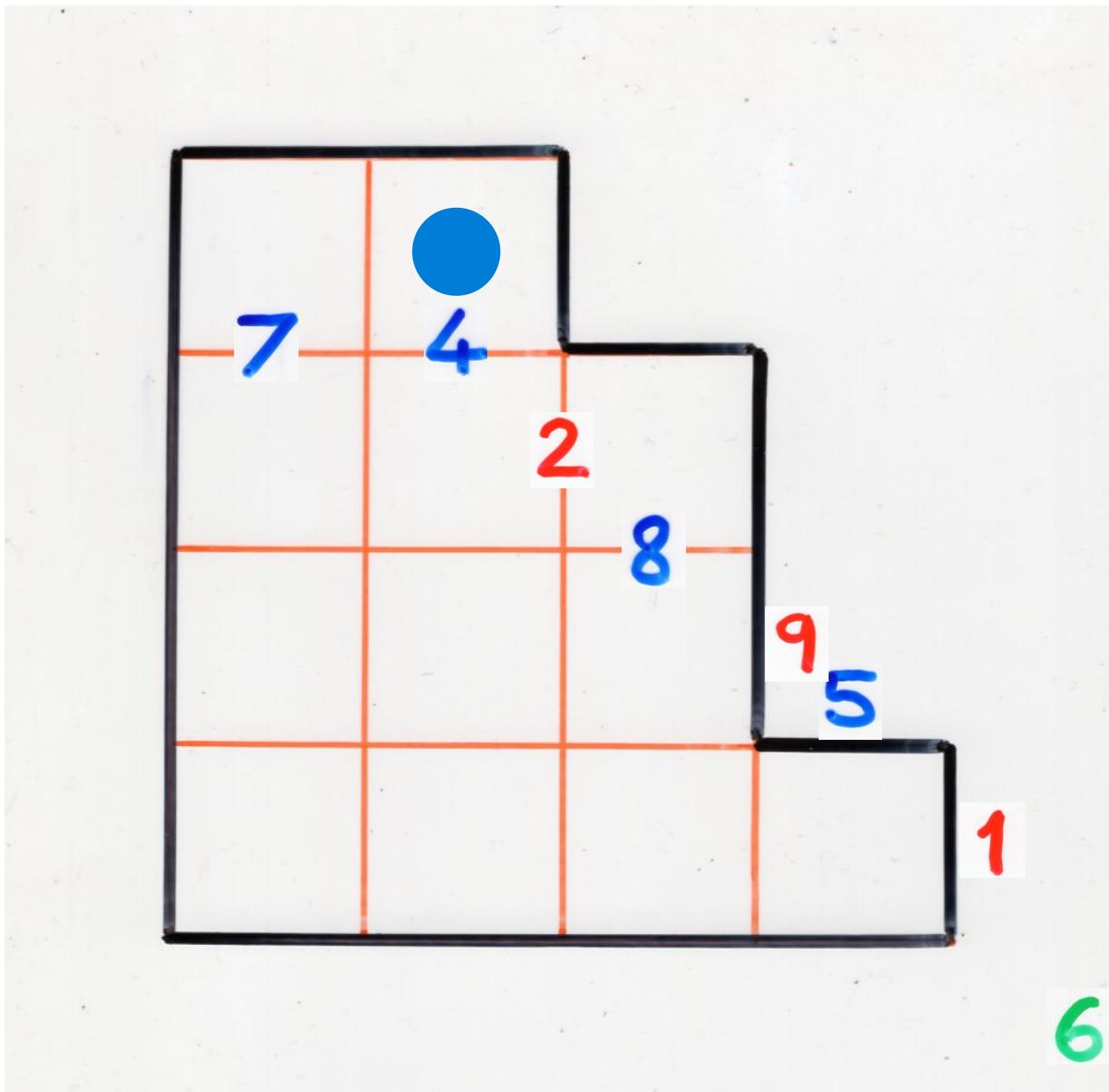
7 4 3 8 2 9 5 1 6

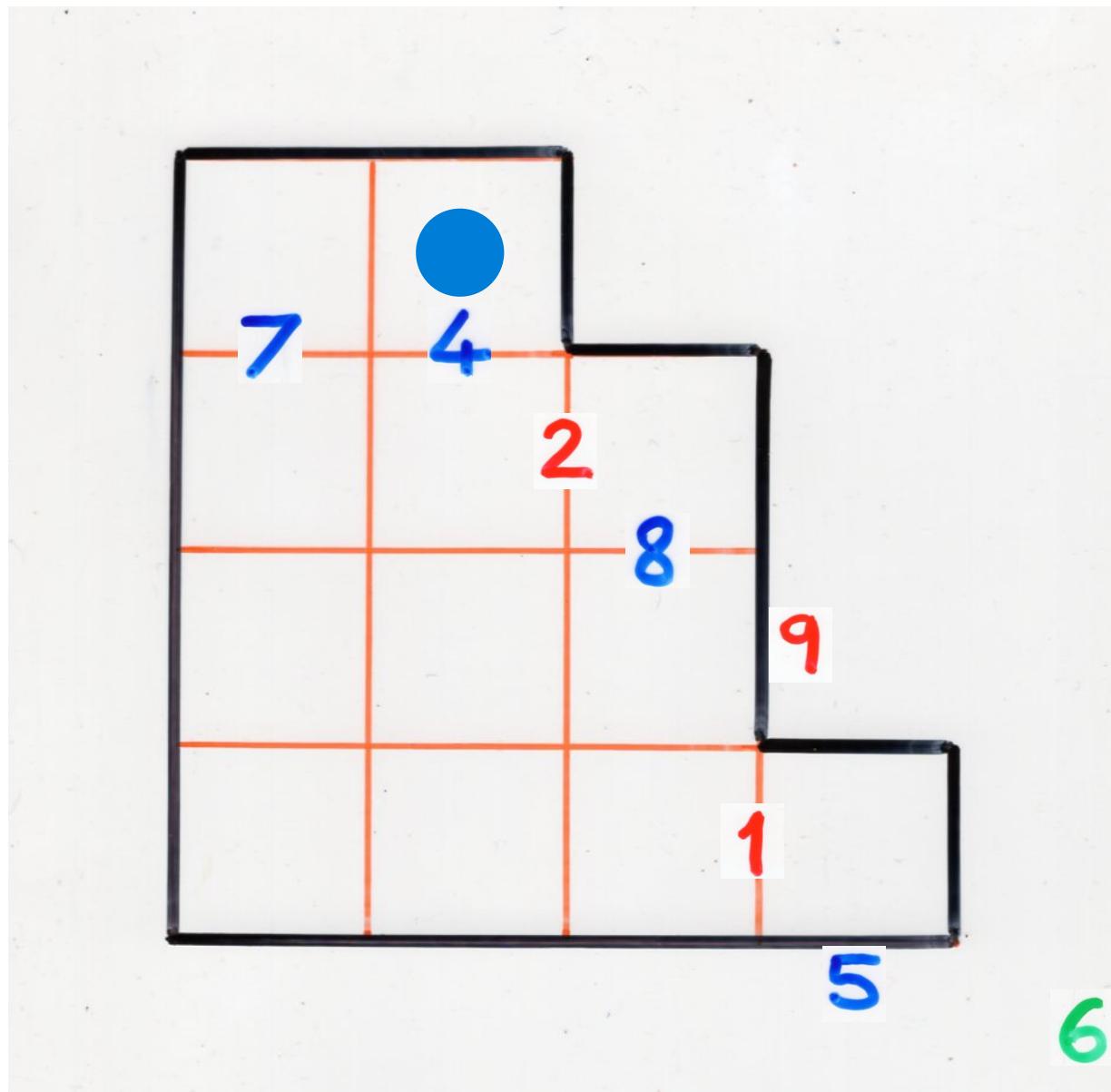


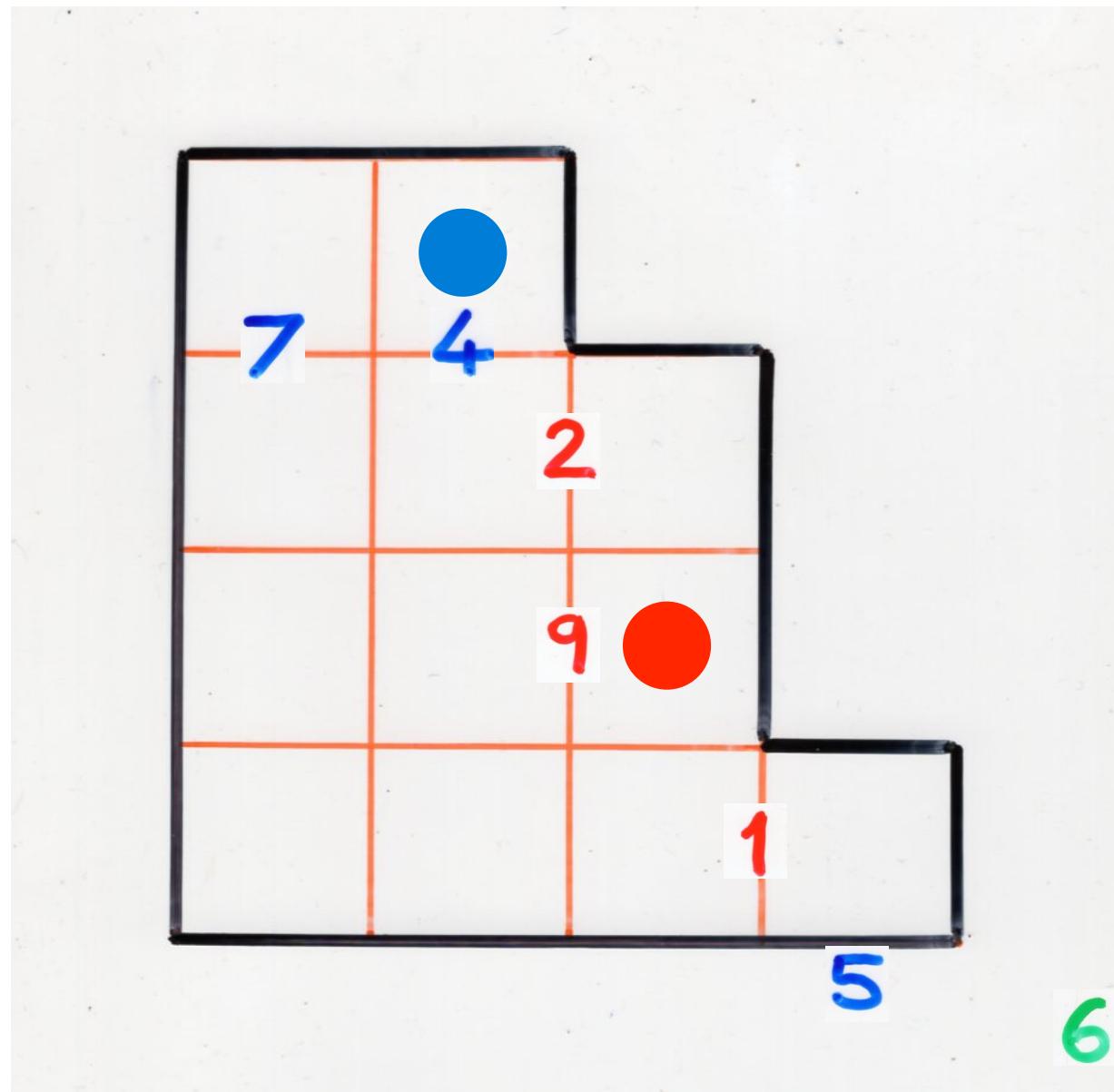


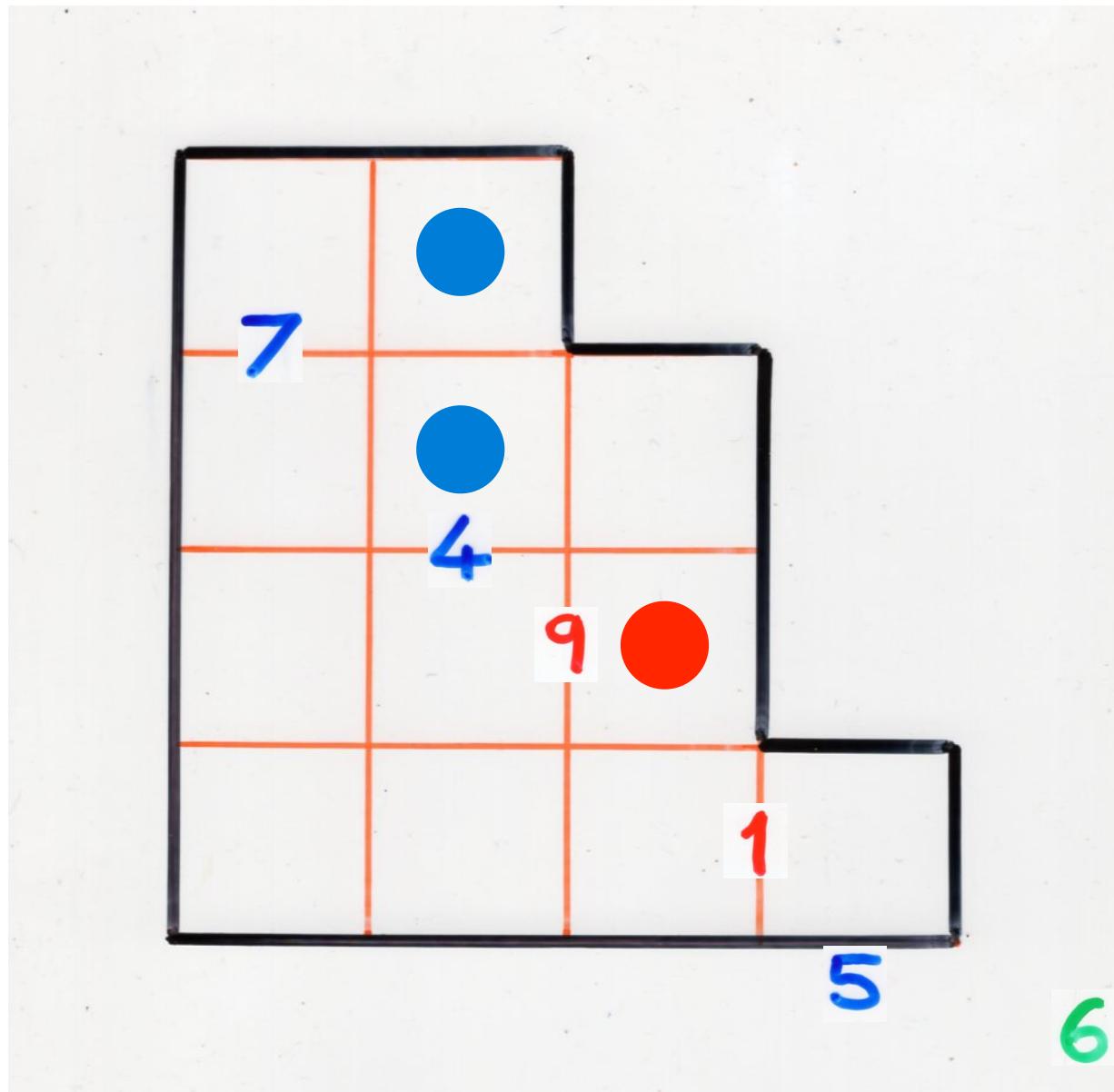


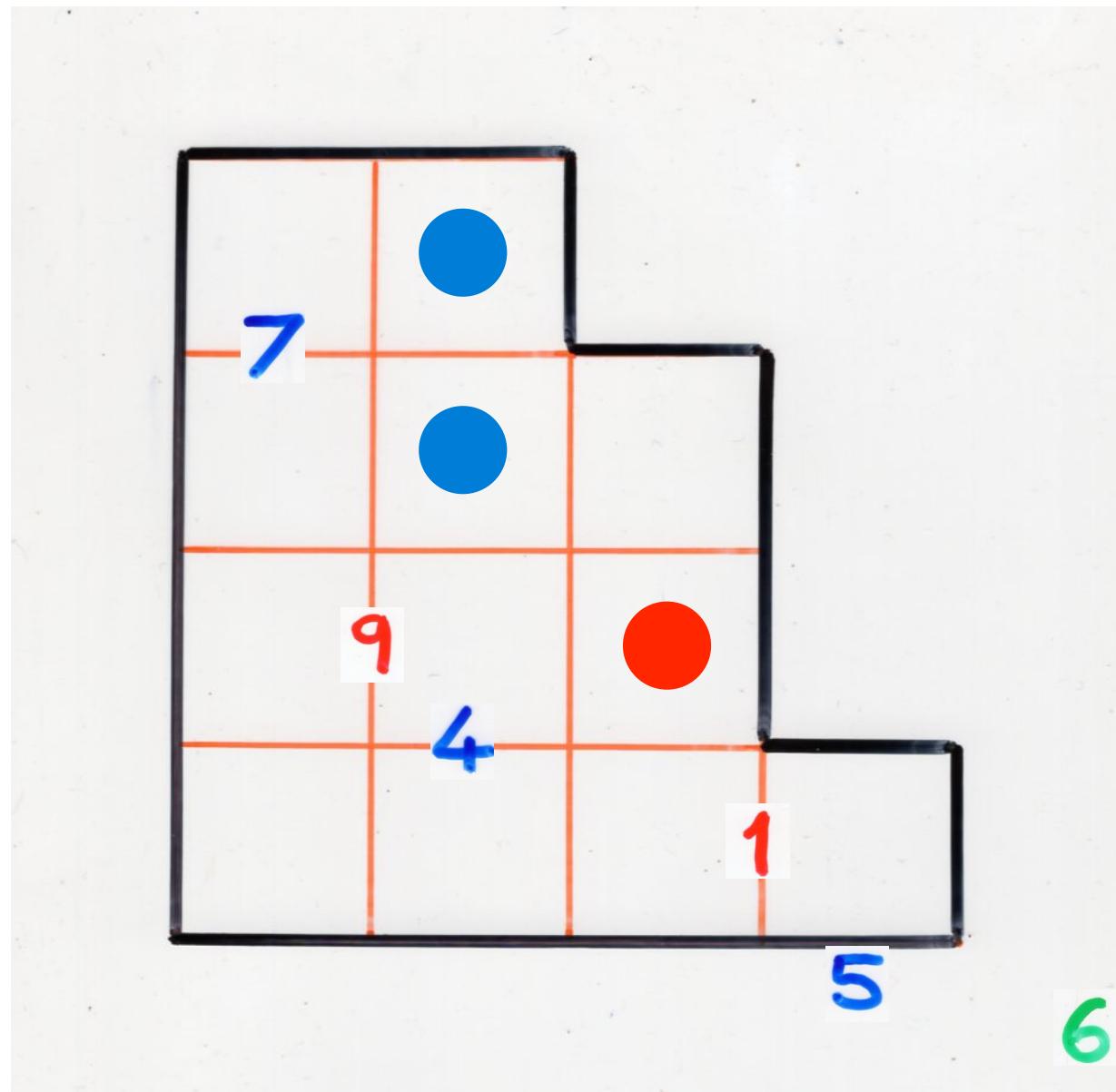


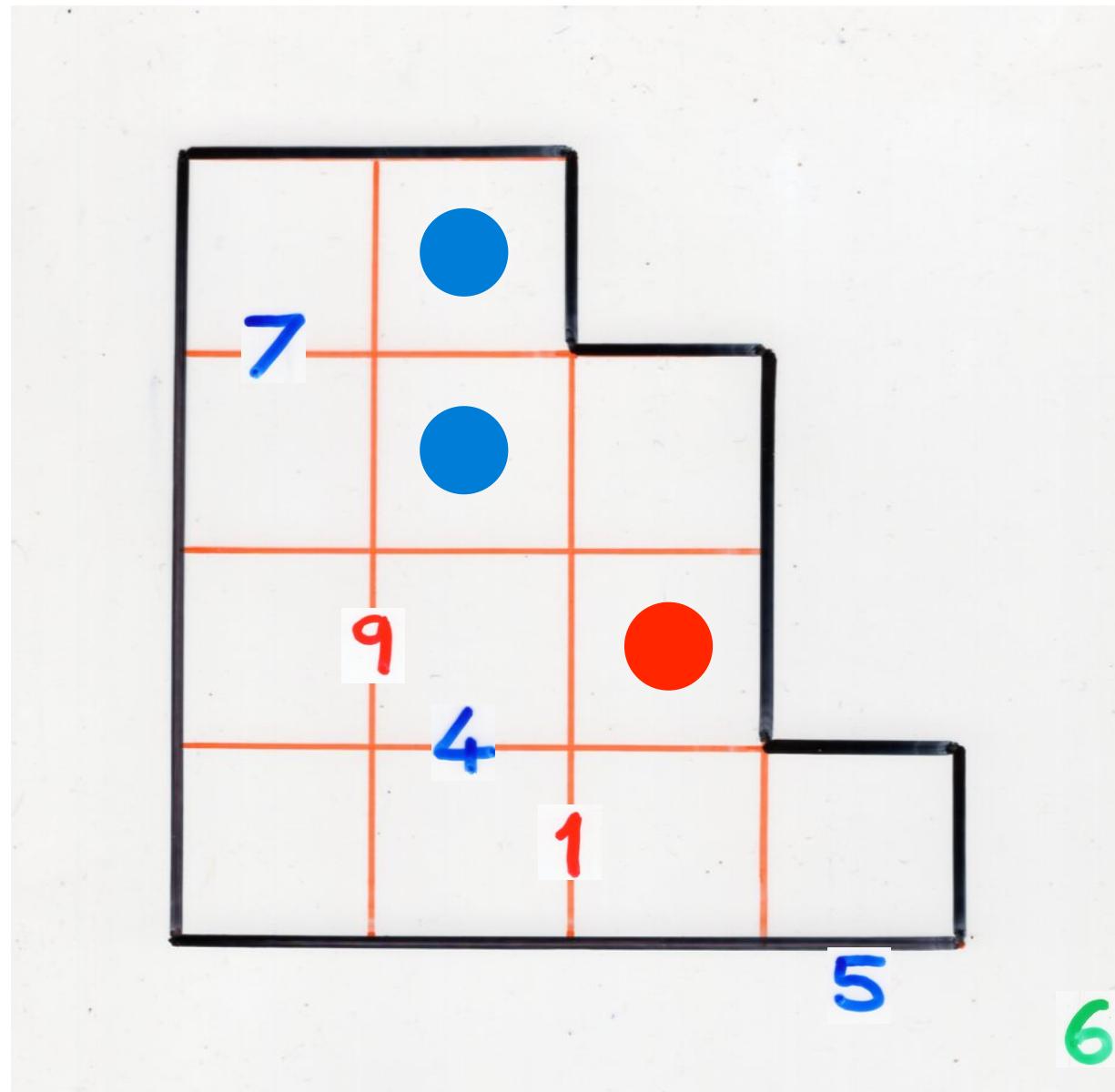


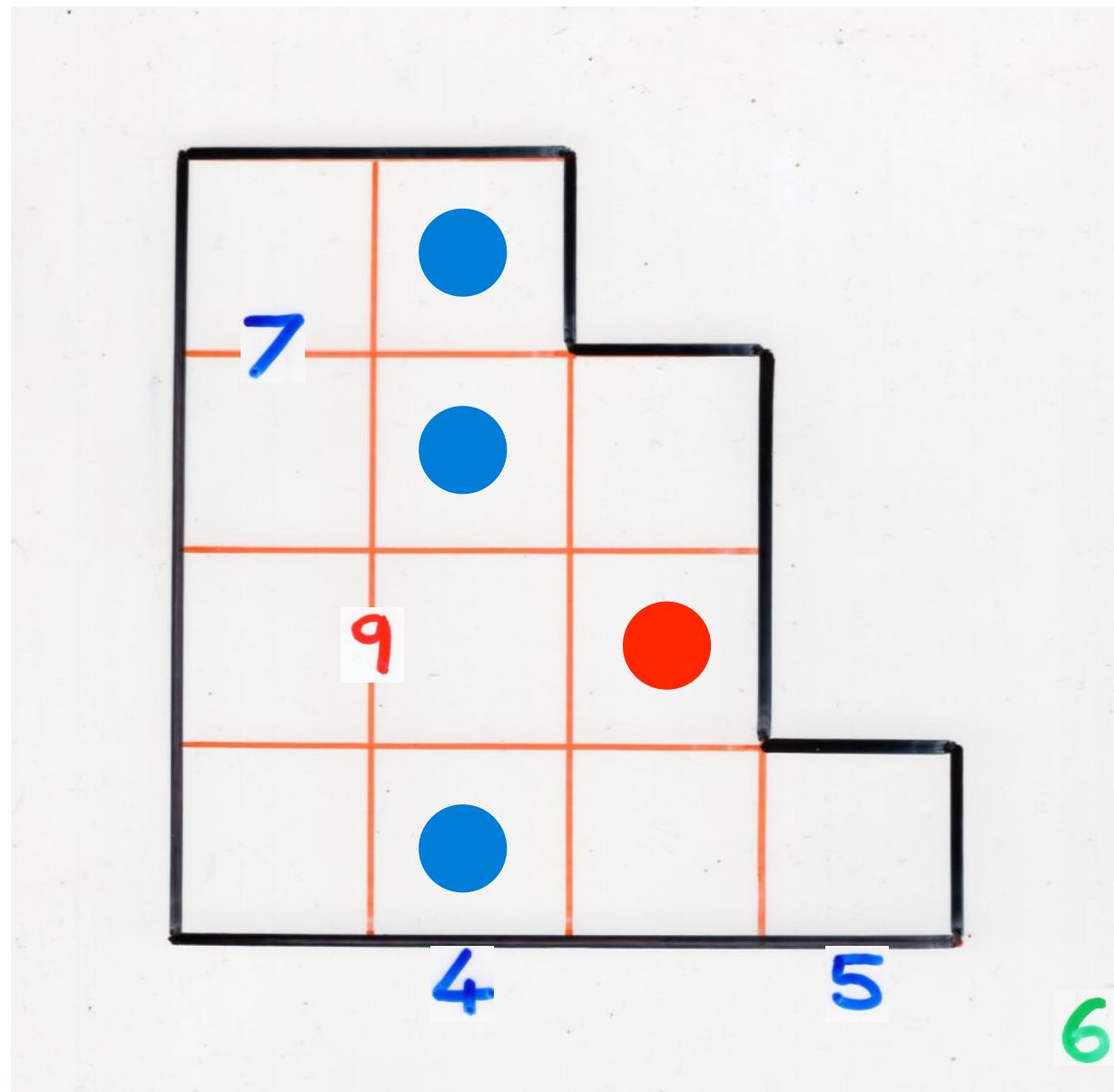


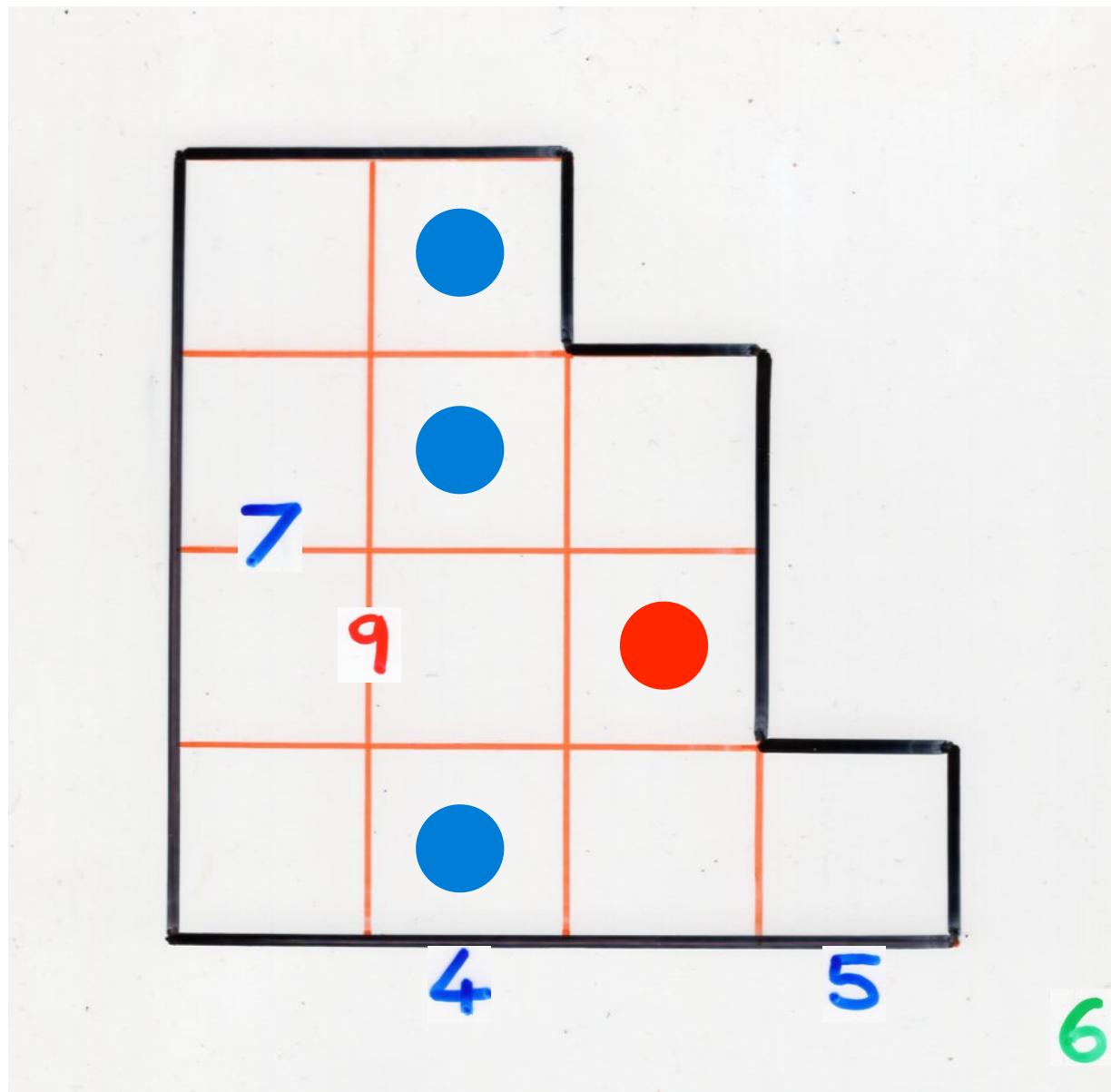


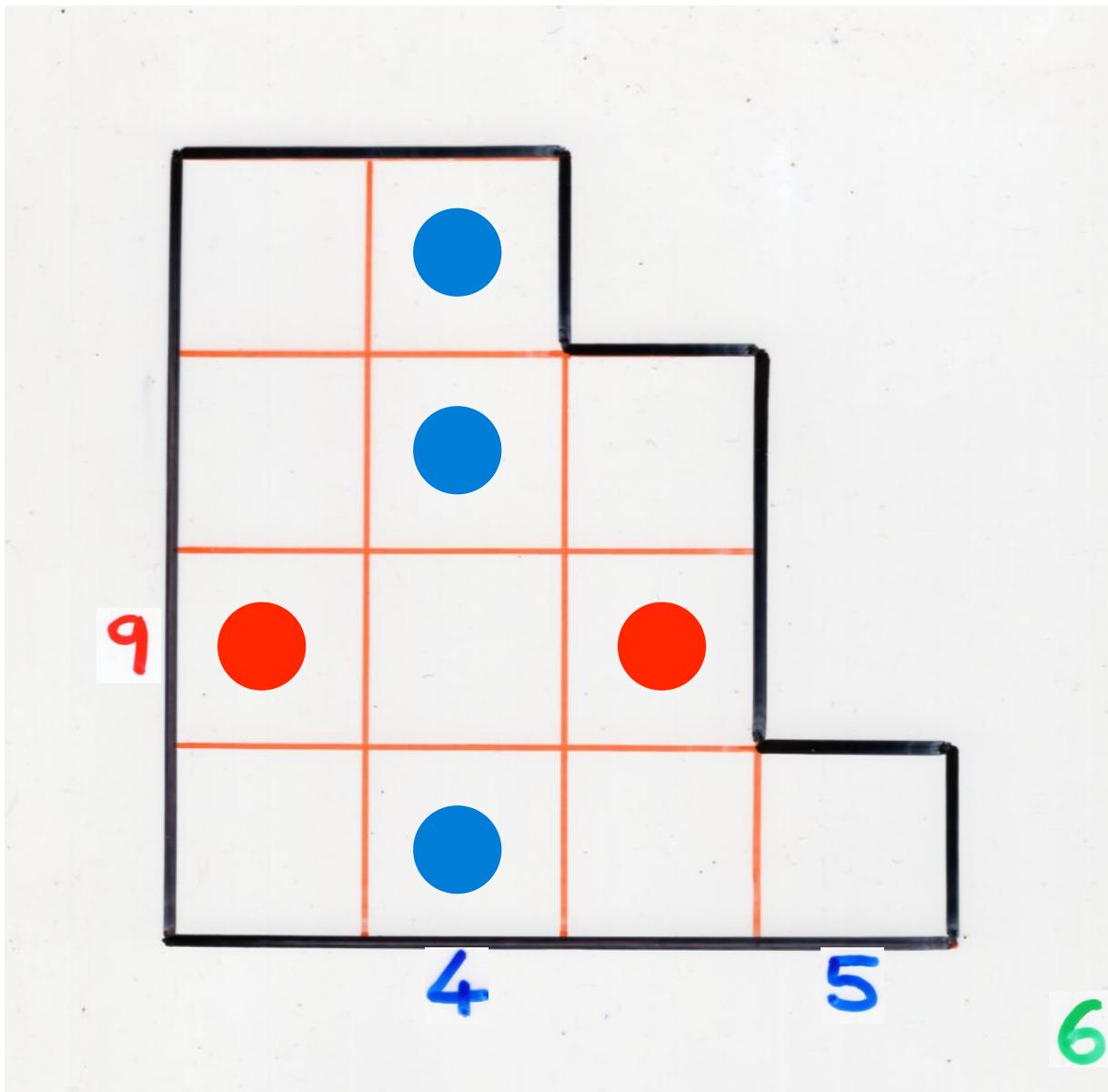


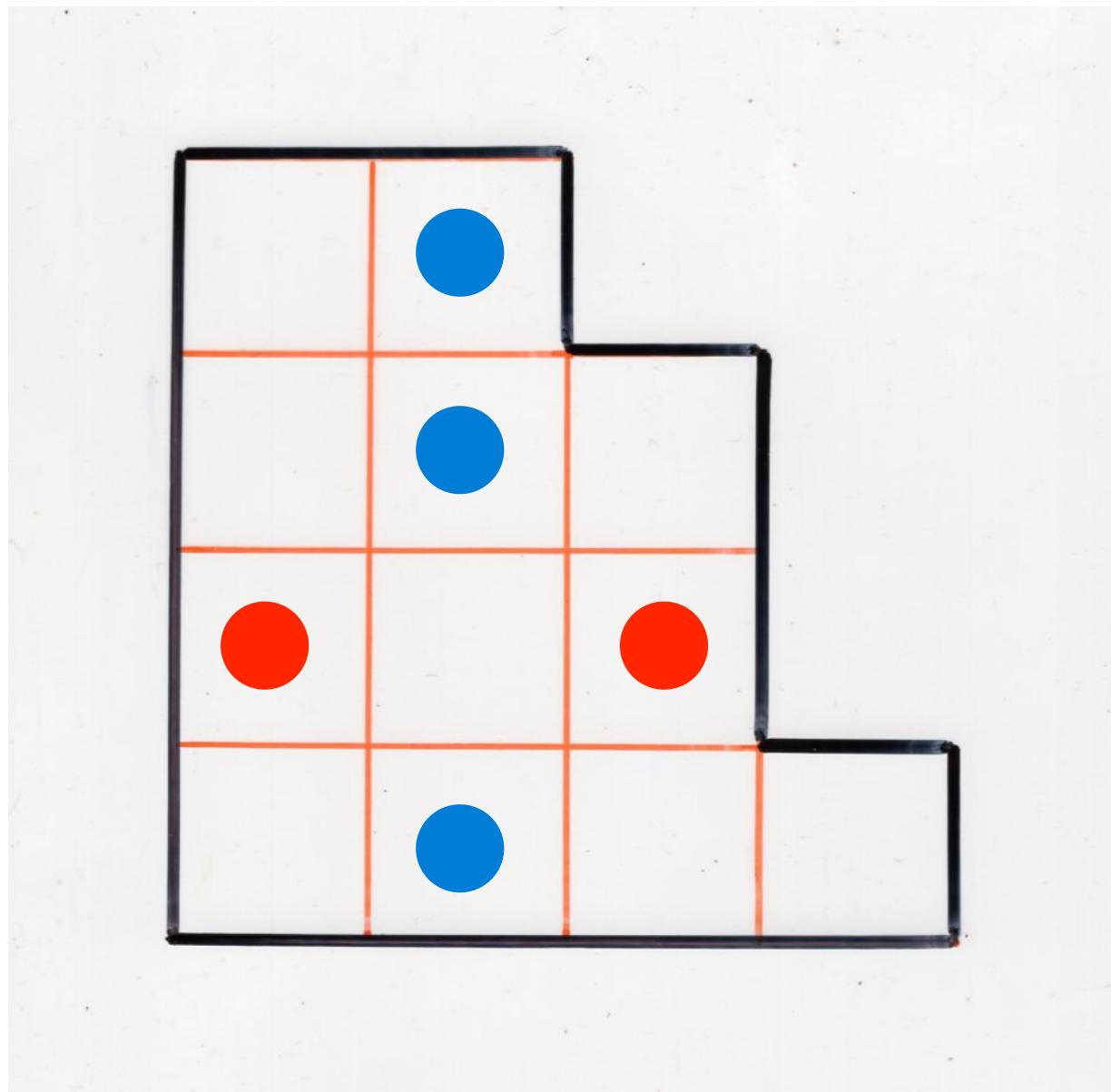










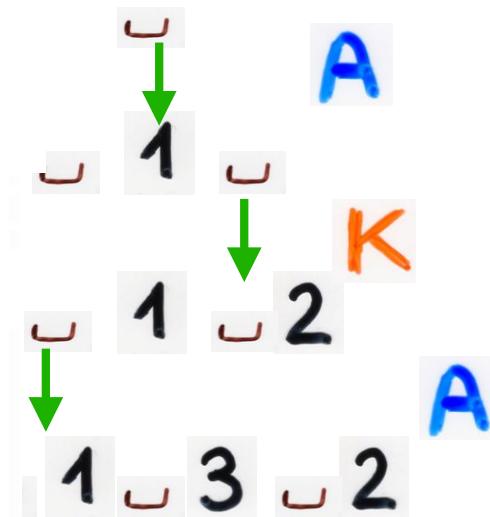
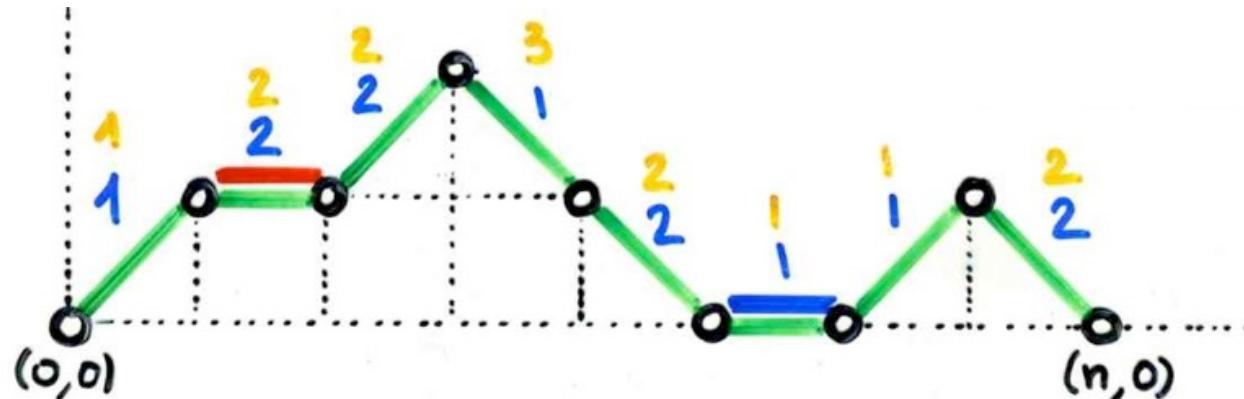


Reminding Ch3b

bijection

Laguerre histories \longrightarrow permutations

description with operators



1 \sqcup 3 \sqcup 2

4 1 \sqcup 3 5 2

4 1 6 \sqcup 3 5 2

4 1 6 \sqcup 7 \sqcup 3 5 2

4 1 6 \sqcup 7 8 3 5 2

4 1 6 9 7 8 3 5 2

S

S

J

A

S

Reminding Ch3b

equivalence

local rules
(commutation diagrams)
and Laguerre histories

exchange-fusion
(or exchange-delete)
algorithm

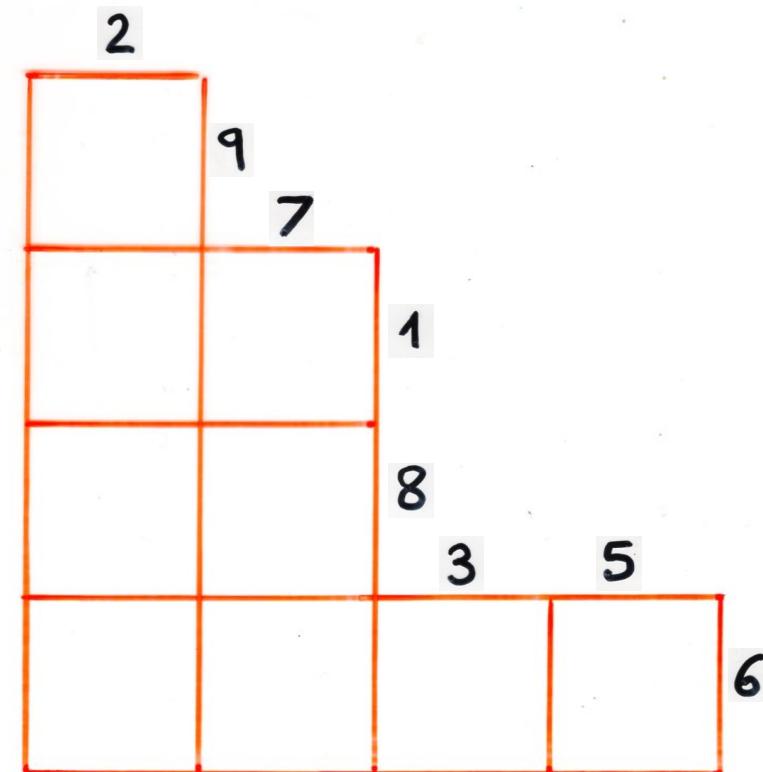
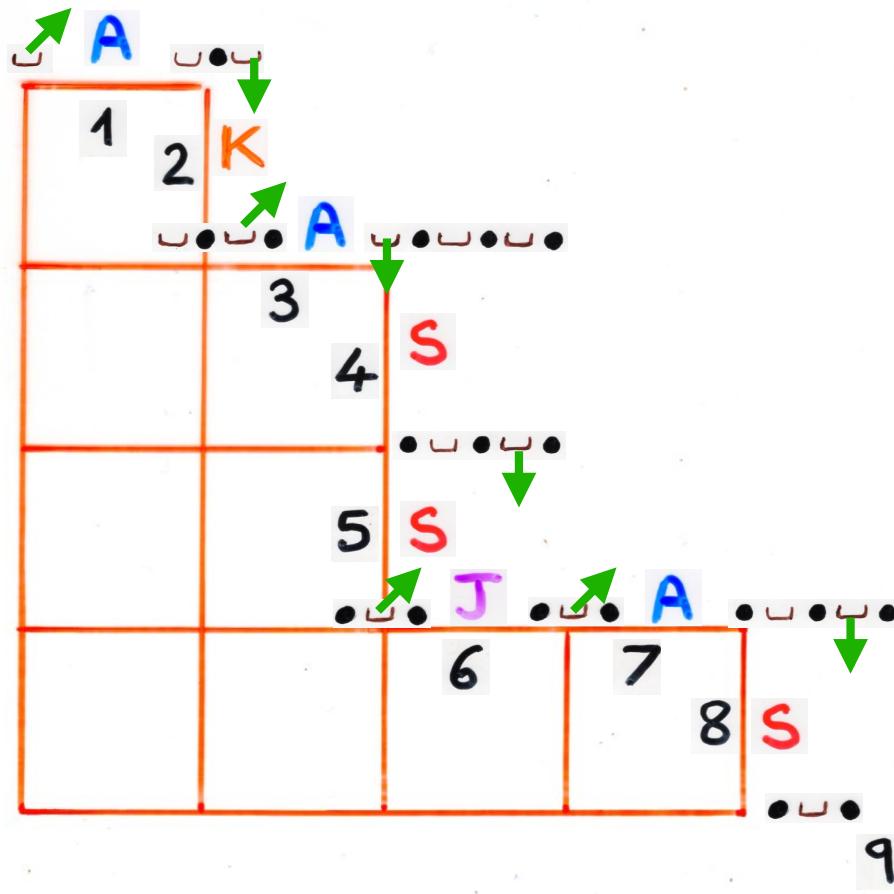
4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4

$$\sigma = \tau^{-1}$$

4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4



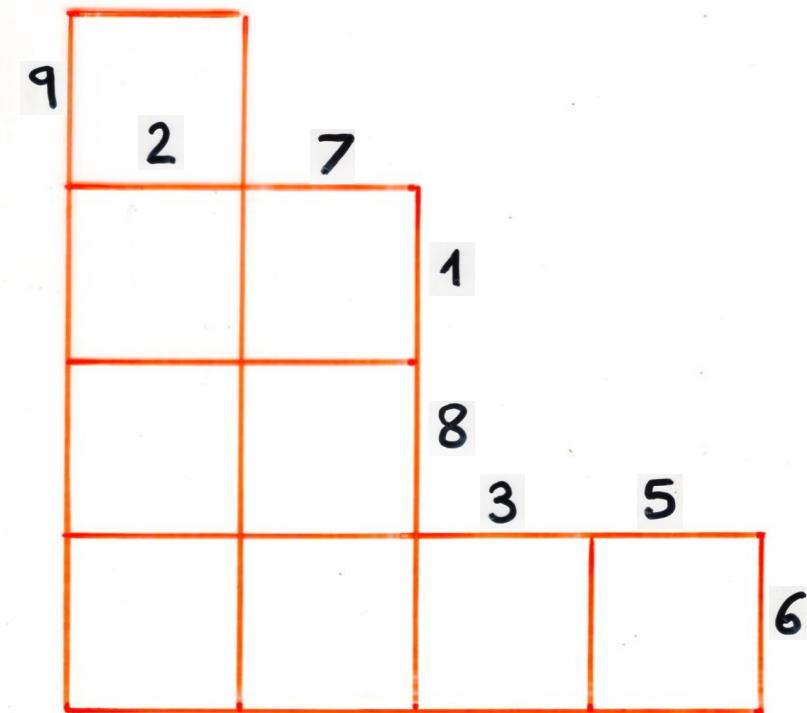
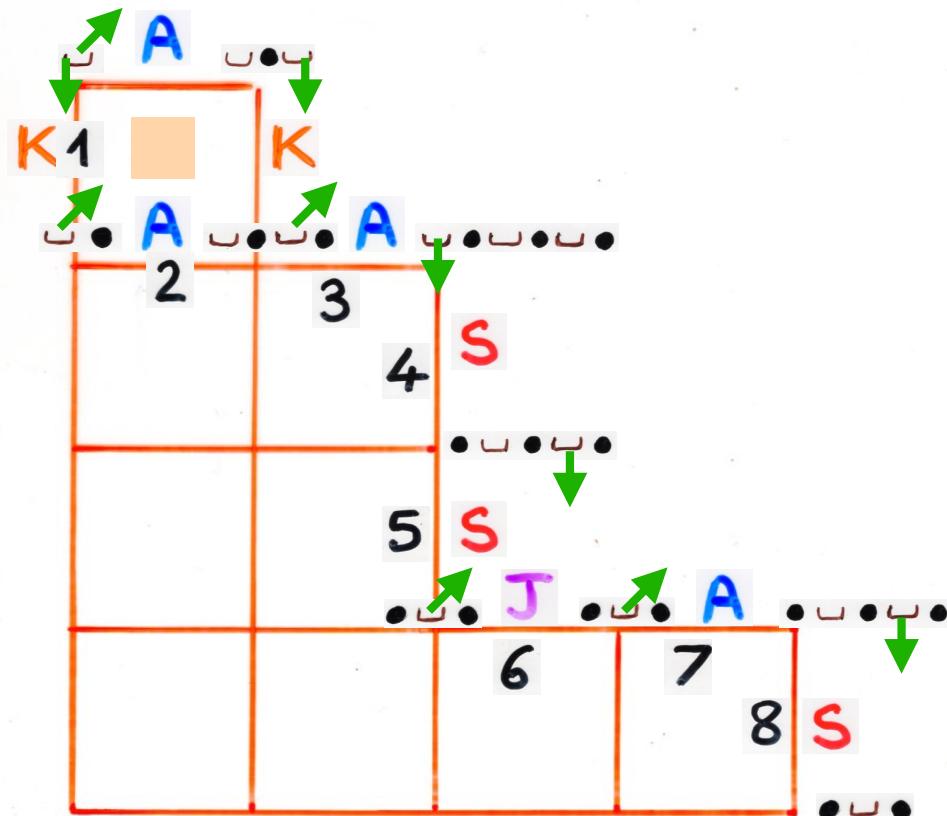
4

4 1 6 9 7 8 3 5 2

4 2 6 9 7 8 3 5 1

2 9 7 1 8 3 5 6 4

9 2 7 1 8 3 5 6 4



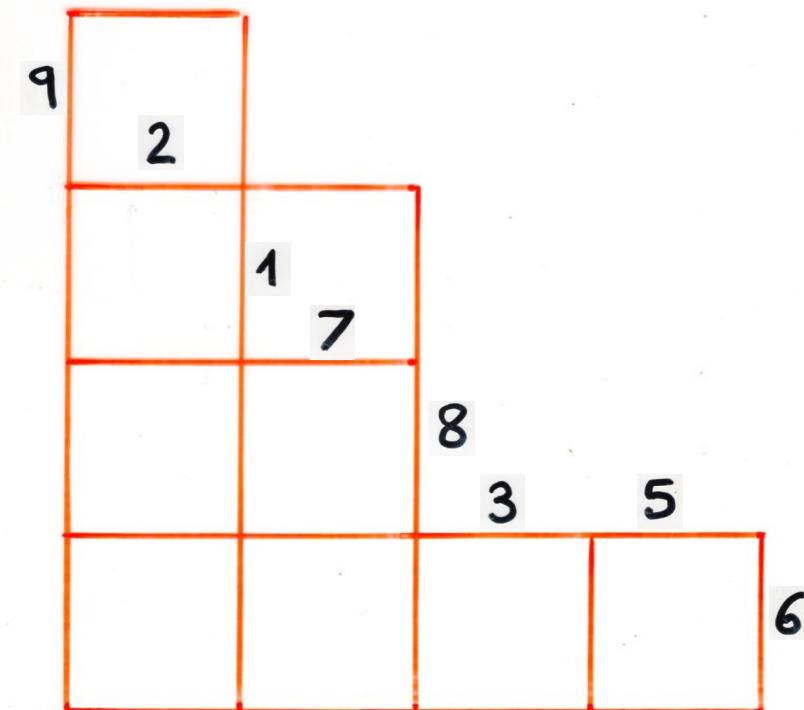
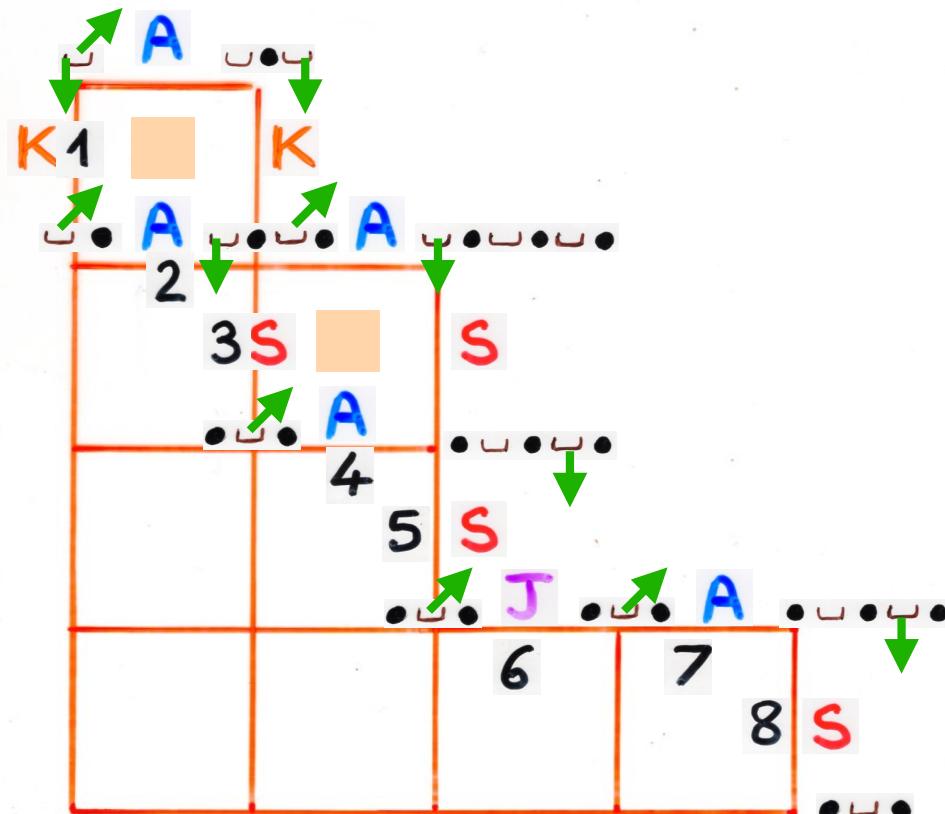
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4 2 6 9 7 8 3 5 1

3 2 6 9 7 8 4 5 1

9 2 7 1 8 3 5 6 4

9 2 1 7 8 3 5 6 4



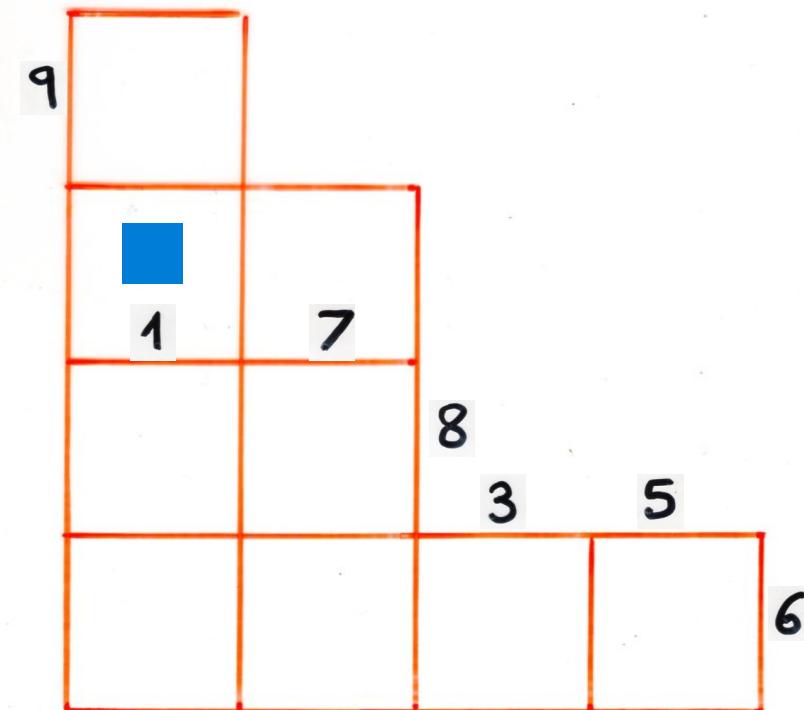
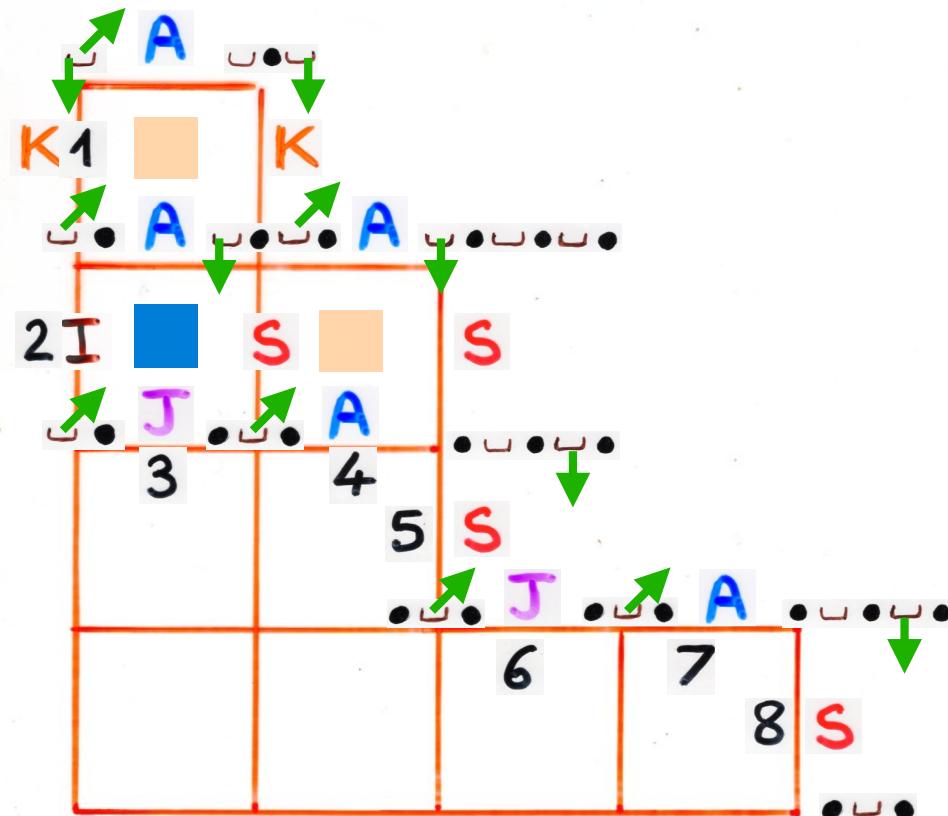
4

326978451

921783564

3 6978451

9 1783564

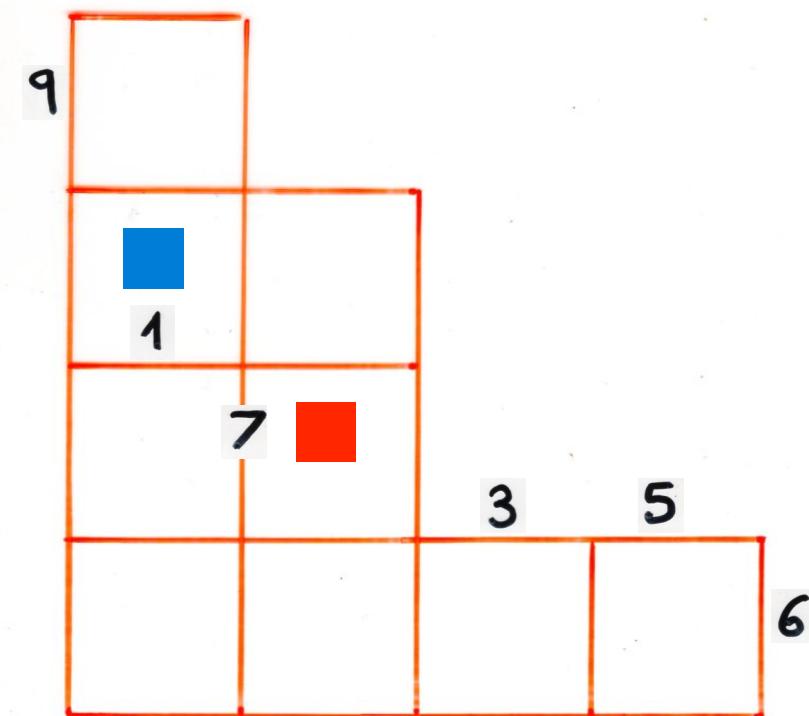
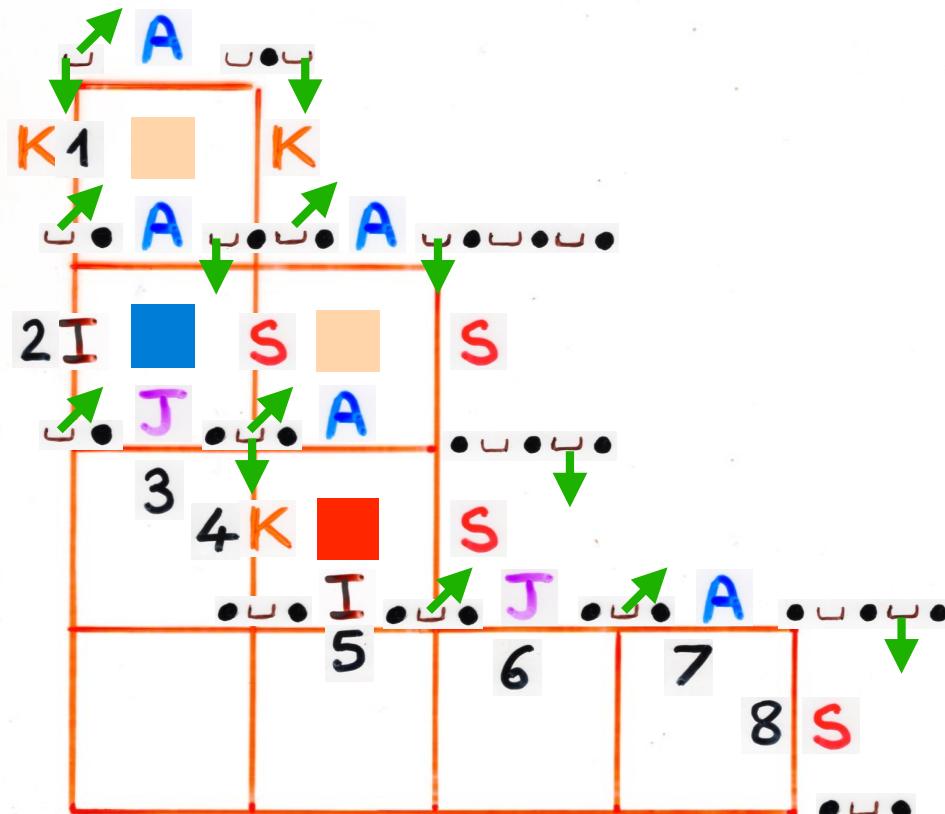


3 6 9 7 8 4 5 1

3 6 9 7 8 4 1

9 1 7 8 3 5 6 4

9 1 7 3 5 6 4

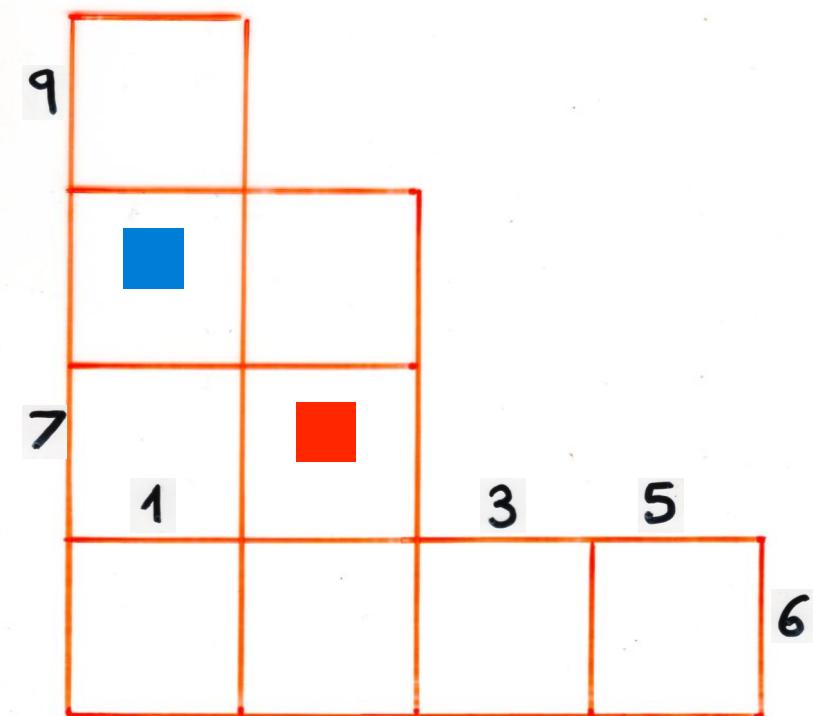
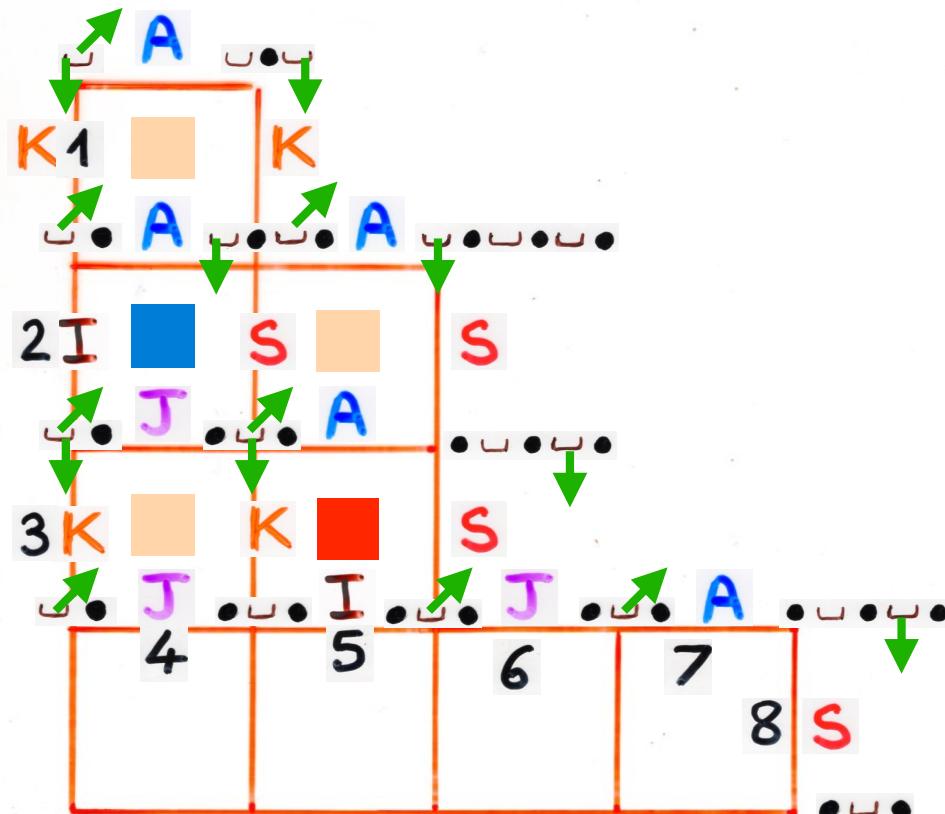


3 6 9 7 8 4 5 1

4 6 9 7 8 3 1

9 1 7 8 3 5 6 4

9 7 1 3 5 6 4



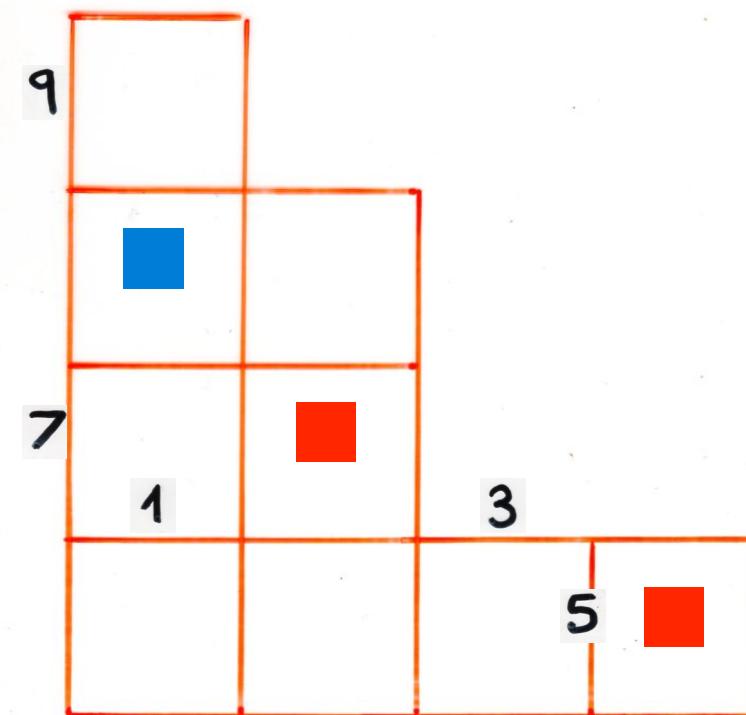
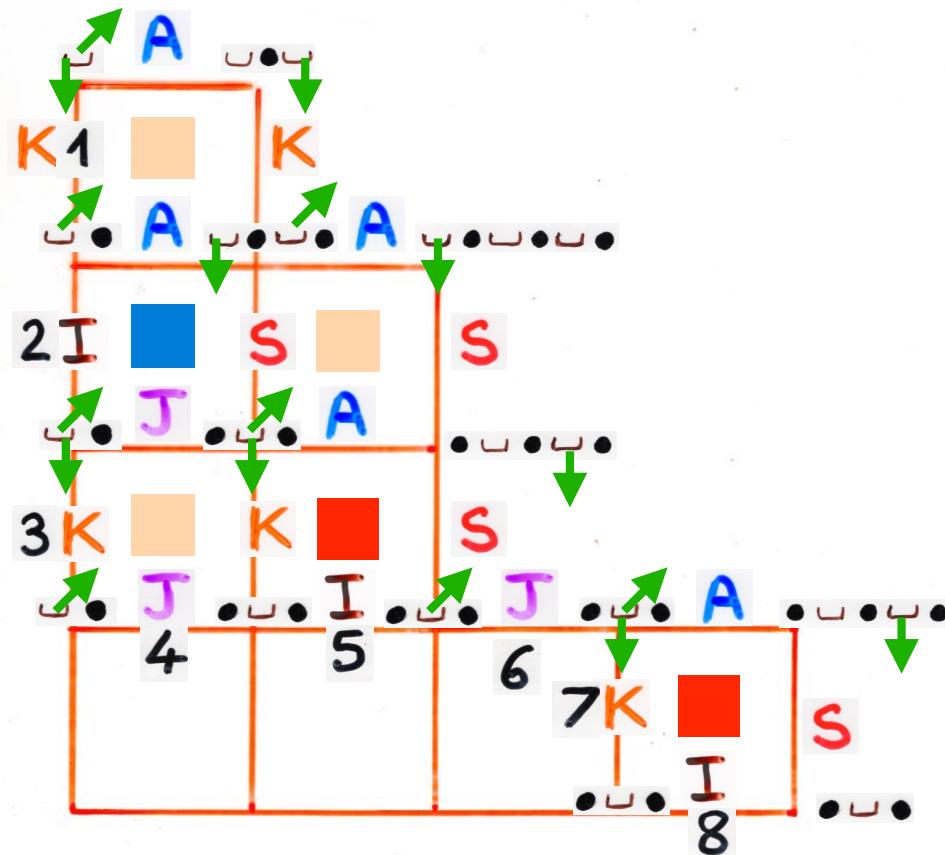
4

4 6 9 7 8 3 1

9 7 1 3 5 6 4

4 6 9 7 3 1

9 7 1 3 5 4

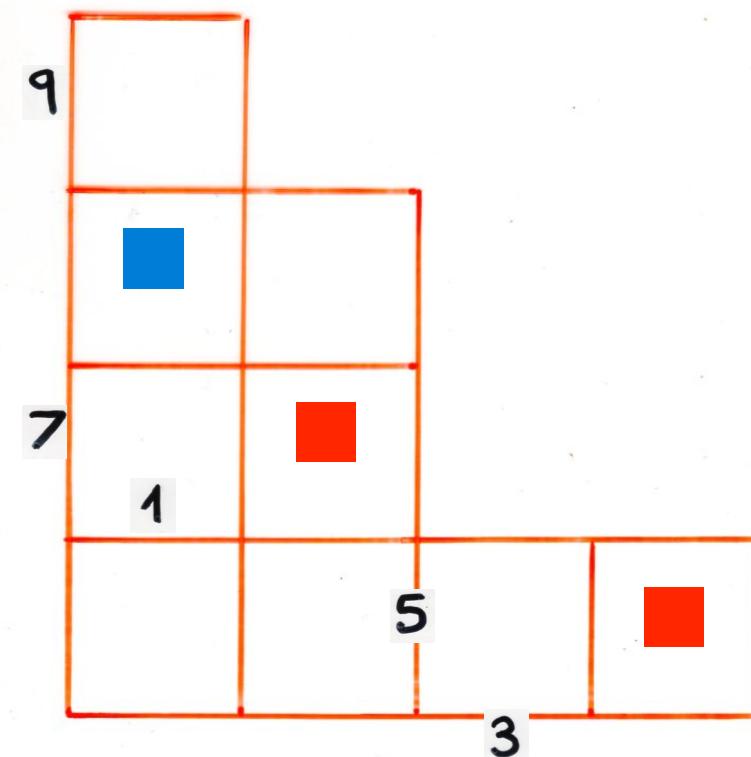
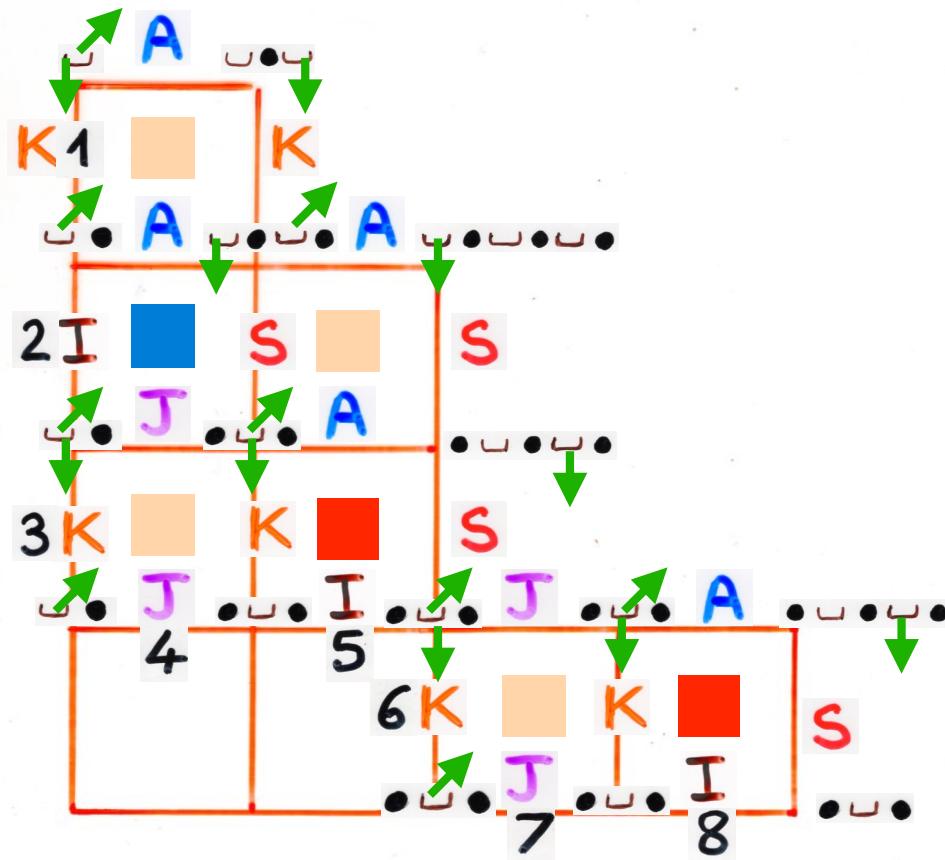


4 697 3 1

9 71 35 4

4 796 3 1

9 71 53 4



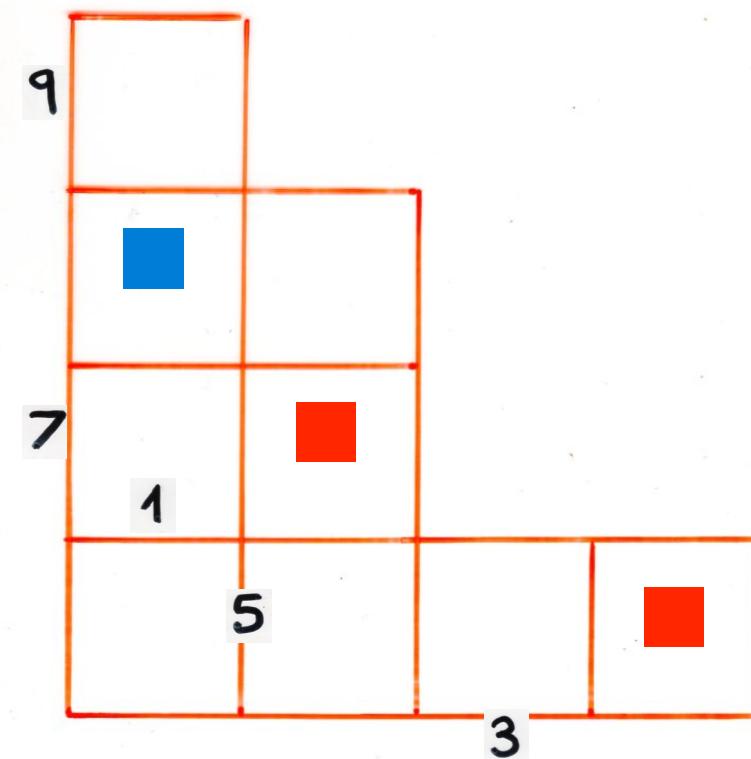
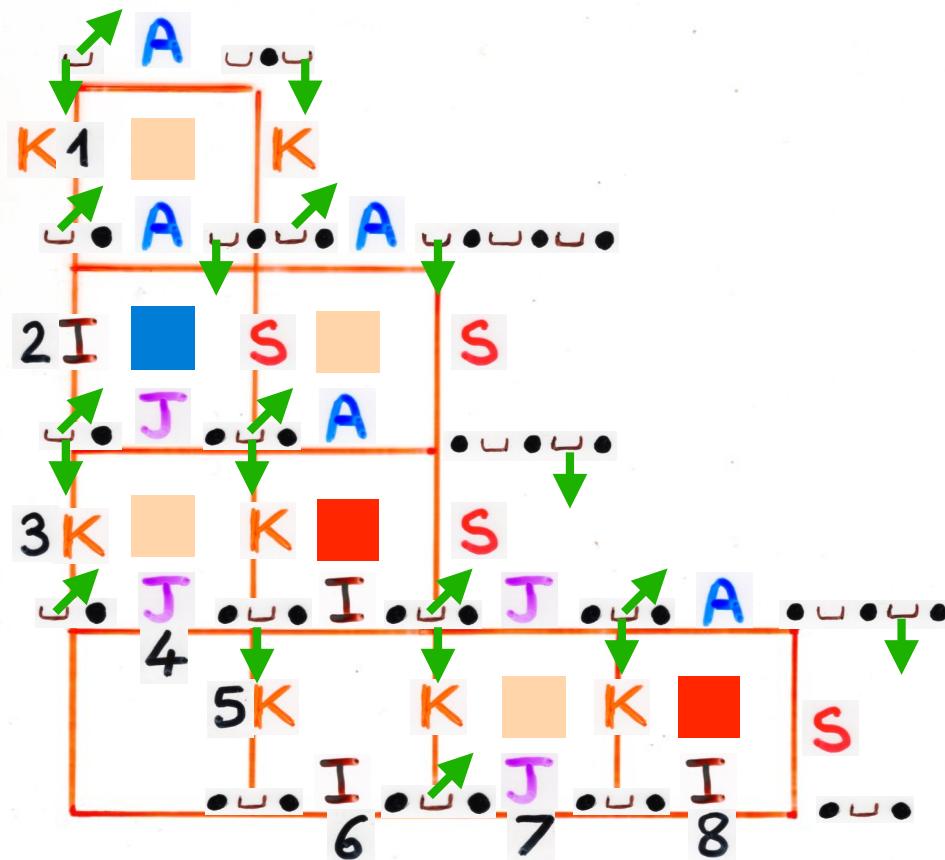
4

4 796 3 1

9 71 53 4

4 795 3 1

9 715 3 4



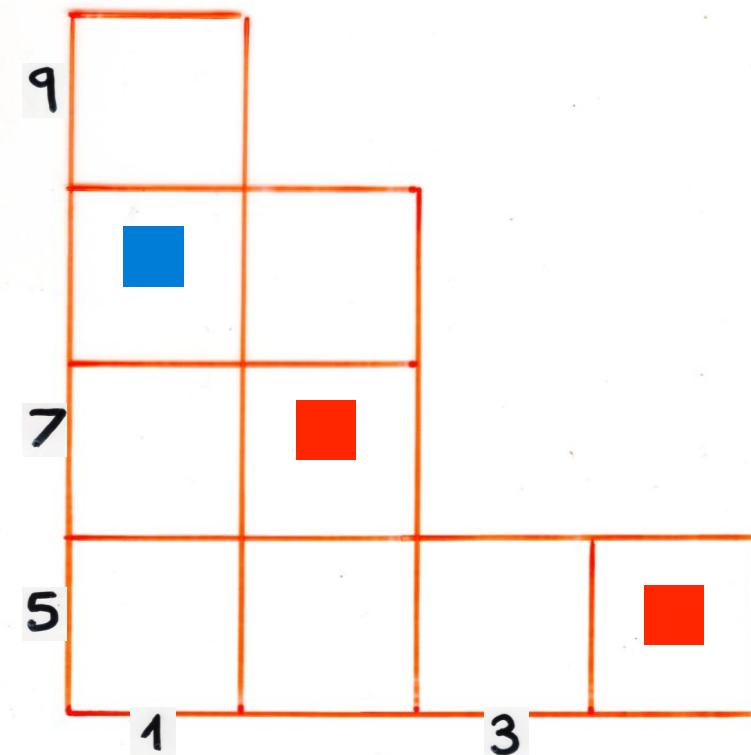
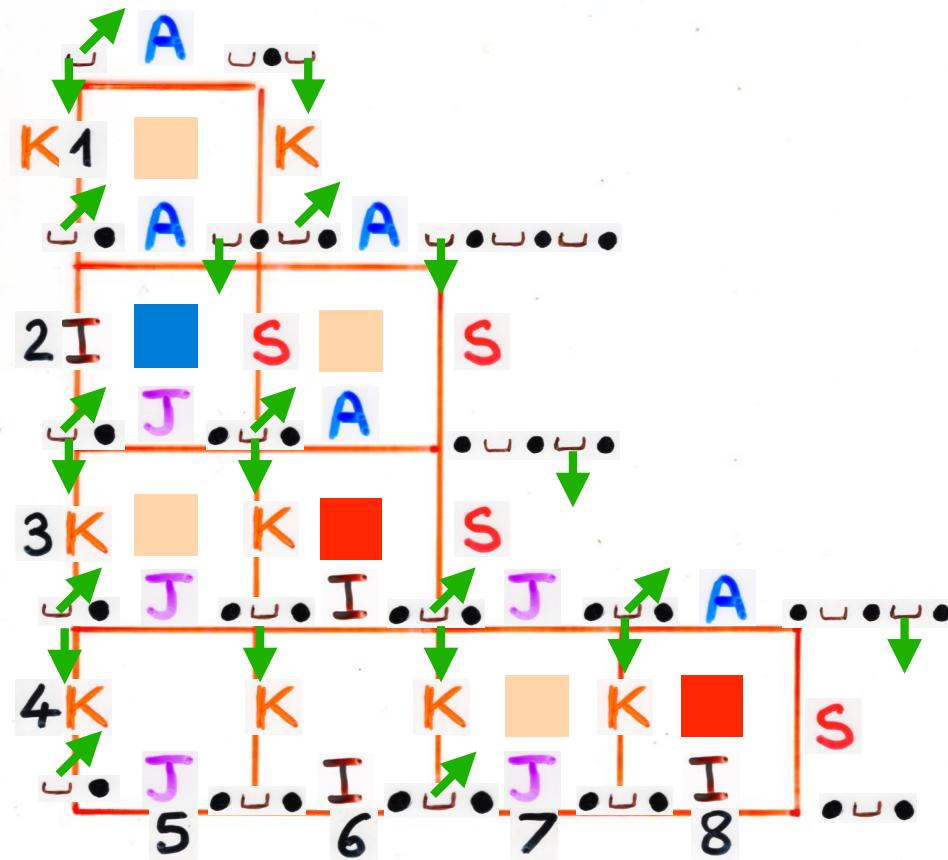
4

4 795 3 1

5 794 3 1

9 715 3 4

9 751 3 4



4

Two bijections

- from a combinatorial representation
of the PASEP algebra (X.V., 2008)

equivalent to a bijection
Corteel, Nadeau (2007)

(with permutation tableaux)

Steingrimsson, Williams
(2005, 2007)

Postnikov

- direct bijection (with tree-like tableaux)
Aval, Boussicault, Nadeau (2011)

tableaux
size $(n+1)$ \leftrightarrow (tableaux
size n , $1 \leq i \leq n+1$)

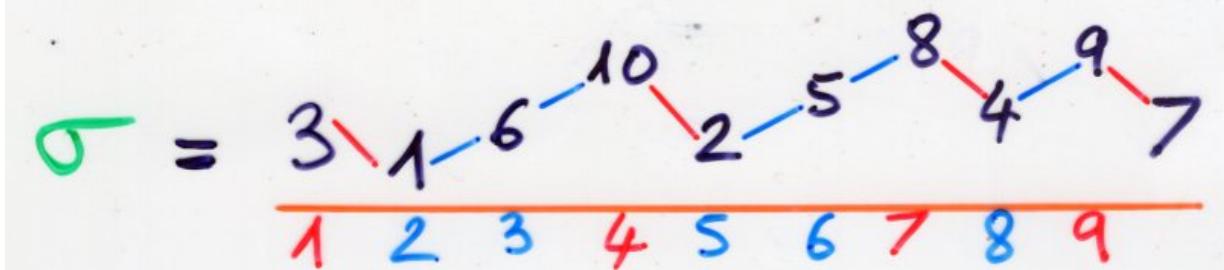
$(n+1)!$

Genocchi sequence
of a permutation

$\sigma \in S_n$

i is a descent of σ iff
 $\sigma(i) > \sigma(i+1)$

rise $\sigma(i) < \sigma(i+1)$ (or ascent)



up-down sequence
of a permutation

$w \in \{-, +\}^*$

$|w| = n-1$

$\{-, +\}^*$

D. André (1880)

6 \searrow 2 \nearrow 9 \searrow 7 \nearrow 8 \searrow 4 \nearrow 5 \searrow 1 \nearrow 3

alternating sequence

alternating permutations

up-down sequence
of a permutation

T_{2n+1} = number of alternating permutations
on $[1, 2n+1]$

BJC 1

(bijective course, Part I)

Ch 3b, p 63-65

$$\tan t = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

Definition Genocchi sequence
of a permutation

$$\sigma = \sigma(1) \dots \sigma(n)$$

$$G(\sigma) = z_1 \dots z_{n-1} \in \{a, d\}^*$$

$$\begin{aligned} z_x &= \begin{cases} a & \text{iff } \sigma(i) < \sigma(i+1) \\ d & \text{otherwise} \end{cases} \\ &\text{"value"} \\ &\downarrow \\ &\text{"index"} \end{aligned}$$

Proposition $\sigma \in S_{n+1} \xrightarrow{ExF} T$ alternative tableau size n

"Exchange-Fusion" algorithm

$$G(\sigma^{-1}) = w \in \{a, d\}^* \quad |w| = n$$

word

The profile of T is the word $f(w)$ obtained from w by replacing

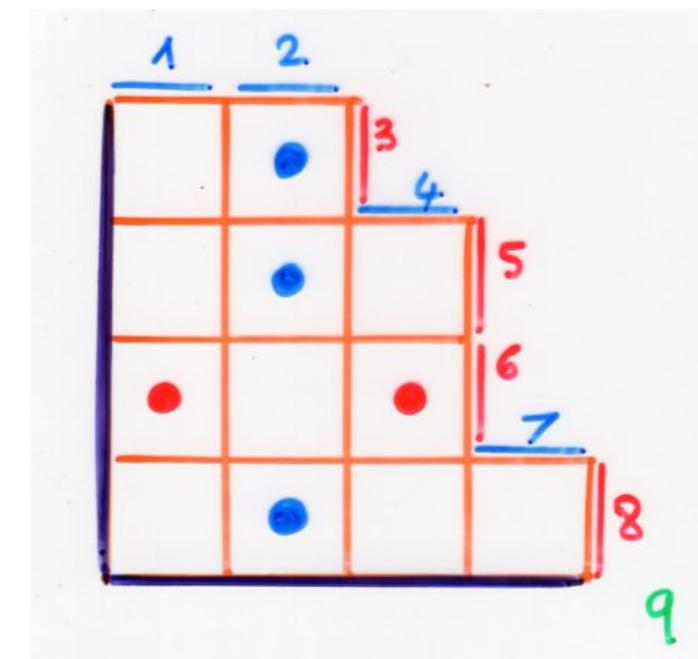
$a \rightarrow D$ (East step) 

$d \rightarrow E$ (South step) 

example

$$G(\sigma^{-1}) = a a d a d d a d \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

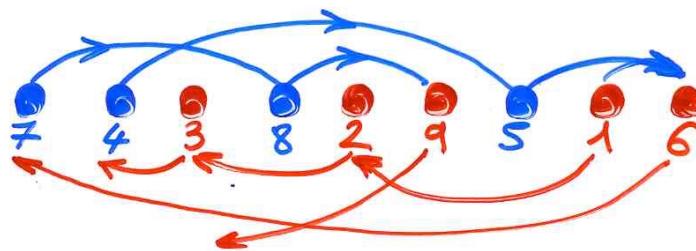


(value) $x \in \begin{cases} \text{advance in } \sigma \text{ ("avance")} \\ \text{backward in } \sigma^{-1} \text{ ("recul")} \end{cases}$

iff (the index) x is $\begin{cases} \text{rise} \\ \text{descent} \end{cases}$ in σ^{-1}

$$\begin{aligned}\sigma^{-1}(x) &< \sigma^{-1}(x+1) \\ \sigma^{-1}(x) &> \sigma^{-1}(x+1)\end{aligned}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$



$$\sigma^{-1} = \begin{pmatrix} 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

\\ \\ \\ / / / / /

Remark The Genocchi sequence of the permutation σ^{-1} is also the word $w = z_1 \dots z_{n-1}$ defined by

$z_i = \begin{cases} a & \text{iff there is an advance ("avancé")} \\ d & \text{iff there is a backward ("recul")} \end{cases}$
 "at the index i "

$$\begin{aligned}\sigma(i) &< \sigma(i+1) \\ \sigma(i) &> \sigma(i+1)\end{aligned}$$

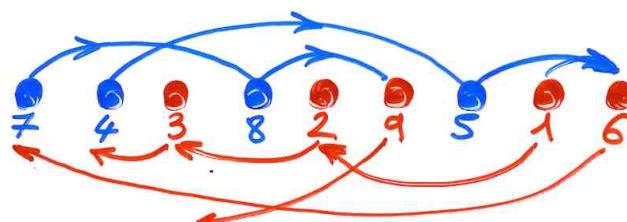
$$\sigma^{-1} = (8\ 5\ 3\ 2\ 7\ 9\ 1\ 4\ 6) \\ (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$$

example

$$G(\sigma^{-1}) = \underset{1\ 2\ 3}{a} \underset{4\ 5\ 6}{a} \underset{7\ 8}{d} \underset{9}{d} \underset{1\ 2\ 3}{a} \underset{4\ 5\ 6}{d} \underset{7\ 8}{d} \underset{9}{d}$$

$$\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9) \\ (7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6)$$

convention $\sigma(n+1) = 0$
 $(\sigma(n) \text{ is a descent})$



alternating
sequence

dada...ada

Proposition Dumont (1974)

The number of permutations on $[1, 2n]$
having an alternating Genocchi sequence
is the Genocchi number G_{2n+2}

$$G_{2n} = 2(2^{2n}-1) B_{2n}$$

Bernoulli
numbers

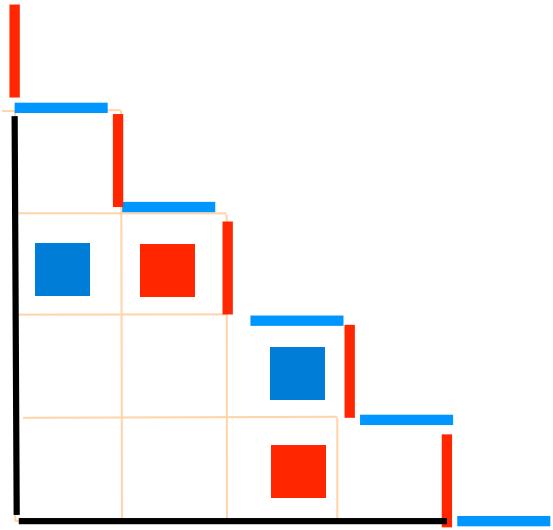
$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

Tangent
numbers

Genocchi
numbers

Angelo Genocchi
(1817-1889)





Genocchi numbers

$$G_{2n+2}$$

alternating shape

Josuat-Vergès
(2010)

Hinc igitur calculo instituto reperietur :

$$A = 1$$

$$B = 1$$

$$C = 3$$

$$D = 17$$

$$E = 155 = 5 \cdot 31$$

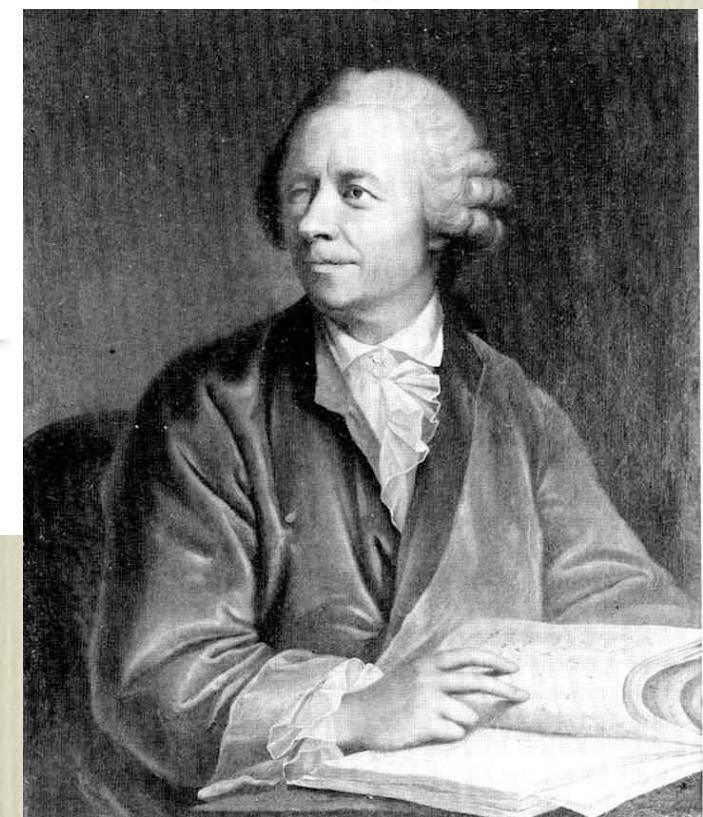
$$F = 2073 = 691 \cdot 3$$

$$G = 38227 = 7 \cdot 5461 =$$

$$H = 929569 = 3617 \cdot 257$$

$$I = 28820619 = 43867 \cdot 9 \cdot 73$$

Leonhard Euler
(1707- 1783)



E_{2n} = number of alternating permutations
on $[1, 2n]$

$$\frac{1}{\cos t} = 1 + 5 \frac{t^2}{2!} + 61 \frac{t^4}{4!} + 1385 \frac{t^6}{6!} + \dots$$

BJC 1

(bijective course, Part I)

Ch 3b, p 63-65

BORDEAUX 1. Le professeur Donald Knuth consacre sa vie à la programmation informatique, considérée comme un art. Il vient d'être sacré docteur honoris causa à Bordeaux

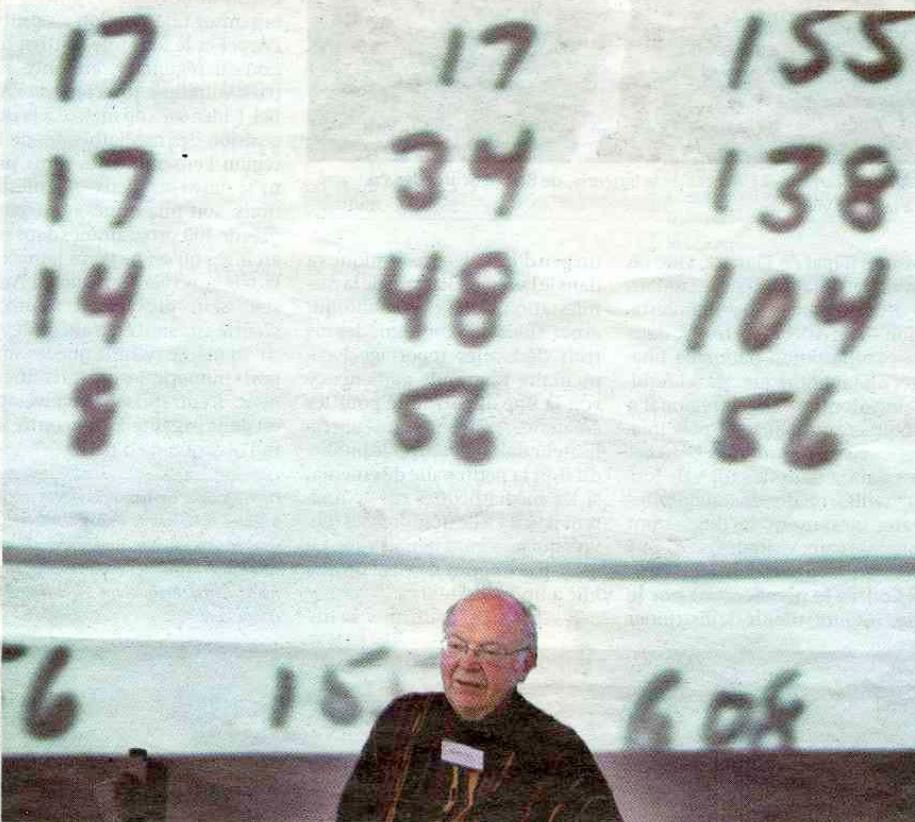
L'ermite de l'informatique

par Bernard Broustet

Une sommité de l'informatique mondiale a séjourné en Gironde ces derniers jours. Donald Knuth, 69 ans, a été sacré mardi docteur honoris causa de l'université Bordeaux 1, après avoir été lundi au centre d'une journée d'échanges qui réunissait une bonne partie du gratin français et européen de la recherche en informatique (1).

Depuis son premier contact, il y a un demi-siècle, avec un monumental et dinosaure IBM 650, Donald Knuth n'a cessé d'être habité par la passion de l'informatique. Physicien, puis mathématicien de formation, ce géant affable et modeste a voué sa vie à ce qu'il appelle « l'art de la programmation informatique ». Car, à ses yeux, plus qu'une technique, c'est une forme d'activité qui requiert à la fois rigueur, intuition et sens esthétique. Les programmes informatiques réussis ont une sorte de beauté à laquelle même les non-spécialistes peuvent être sensibles.

Une encyclopédie. Au long de sa carrière académique (pour l'essentiel à l'université californienne de Stanford), Donald Knuth a fait preuve d'une grande fécondité, en jouant notamment un rôle essentiel dans le développement de langages toujours utilisés par la communauté des mathématiciens. Mais, à 55 ans, le professeur Knuth a décidé de prendre sa retraite de Stanford. Il trouve que les fonctions administratives sont trop absorbantes pour lui permettre de mener à bien l'œuvre entamée à la fin des années 60 sous le titre de « Art of computer programming », sorte d'encyclopédie de l'algorithme et de la programmation informatique.



Donald Knuth, à Bordeaux, le 29 octobre. À 69 ans, il animait une journée d'échanges avec le gratin européen de la recherche en informatique

PHOTO LAURENT THEILLET

que. Donald Knuth a publié, il y a quelque temps déjà, les trois premiers volumes de cette gigantesque somme, traduite en russe, en japonais, en polonais, etc. mais pas en français. Le quatrième tome est pour bientôt. Et Donald Knuth se dit décidé à poursuivre sa tâche tant qu'il en aura la force. Ses ouvrages, dont les ventes cumulées au fil des ans approchent le million d'exemplaires, visent essentiellement les informaticiens et créateurs de programmes. Une communauté cer-

tes minoritaire à travers le monde, mais qui se trouve investie d'une mission considérable. En quelques décennies, l'écriture informatique a aidé à résoudre d'innombrables problèmes. « Mais il y en a tant d'autres qui attendent des solutions, notamment dans le domaine médical », affirme le professeur émérite de Stanford.

Un chèque de 2,56 dollars. Pour mener à bien sa tâche, Donald Knuth s'est imposé une vie

d'ermite. D'ordinaire, sa journée débute par la bibliothèque ou la piscine. Après quoi, il passe tout le reste de son temps à sa table de travail, dimanche compris. Il n'a plus d'e-mail depuis le début des années 90, considérant que le courrier électronique représente une perte de temps, dès lors qu'on veut aller au fond des choses et non pas rester à leur surface. Une secrétaire lui fait passer les messages considérés comme les plus urgents. Pour le reste, Donald Knuth demande qu'on lui

écrive par courrier ordinaire ou par fax, dont il prend parfois connaissance avec des mois de retard. Il s'oblige, en revanche, à tenir aussi scrupuleusement que possible sa promesse d'envoyer un chèque de 2,56 dollars à tout lecteur ayant détecté une erreur dans un de ses livres. Par ailleurs, pour se détendre, il pratique l'orgue, appris dans sa prime jeunesse auprès de son père qui partagea sa vie entre la musique et l'enseignement.

L'orgue de Sainte-Croix. Donald Knuth n'est pas fermé aux choses de ce monde. Sur son site Internet, à la rubrique « Questions qui ne me sont pas fréquemment posées », il demande entre autres : « Pourquoi mon pays a-t-il le droit d'occuper l'Irak ? », « Pourquoi mon pays ne soutient-il pas une Cour internationale de justice ? » Mais cet homme de conscience ne se veut pas militant, pas plus qu'il n'aspire au vedettariat et à la richesse. « Beaucoup de gens, dit-il, ont tendance à considérer que l'informatique, c'est surtout des histoires de business, d'entreprise. Ce n'est pas mon cas. » Sortant de sa semi-reclusion, Donald Knuth s'est donc laissé convaincre d'accepter les hommages de l'université de Bordeaux, après celles de Harvard, d'Oxford, de Tübingen. Il a eu le coup de foudre pour la beauté et l'agrément de la ville. Et il n'oubliera sans doute pas de sitôt l'orgue illustre de l'église Sainte-Croix (2), sur lequel il a eu le bonheur d'exercer son talent.

(1) Ces journées étaient organisées par le Laboratoire bordelais de recherche en informatique (Labri).

(2) Thierry Semenoux, professeur d'orgue au conservatoire de Bordeaux, a joué dans ce domaine un rôle de cicéronne auprès de Donald Knuth.

some parameters

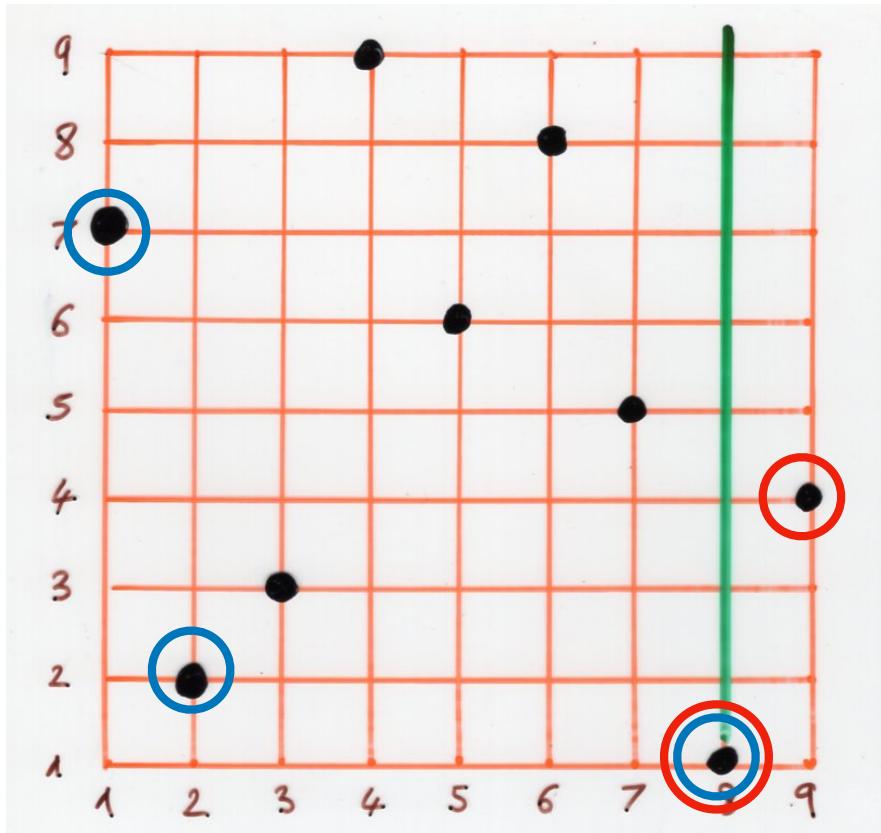
$$\tau = \textcolor{red}{6} \ 8 \ \textcolor{blue}{5} \ \textcolor{red}{2} \ 3 \ 9 \ \textcolor{red}{1} \ 4 \ 7$$

$w = x_1 x_2 \dots x_n$
word with distinct letters

lr-min

left to right minimum element

$$x_i = \min(x_1, x_2, \dots, x_i)$$



σ permutation = 7 2 3 9 6 8 5 1 4 word

left-to-right
right-to-left

minimum

elements

Definition

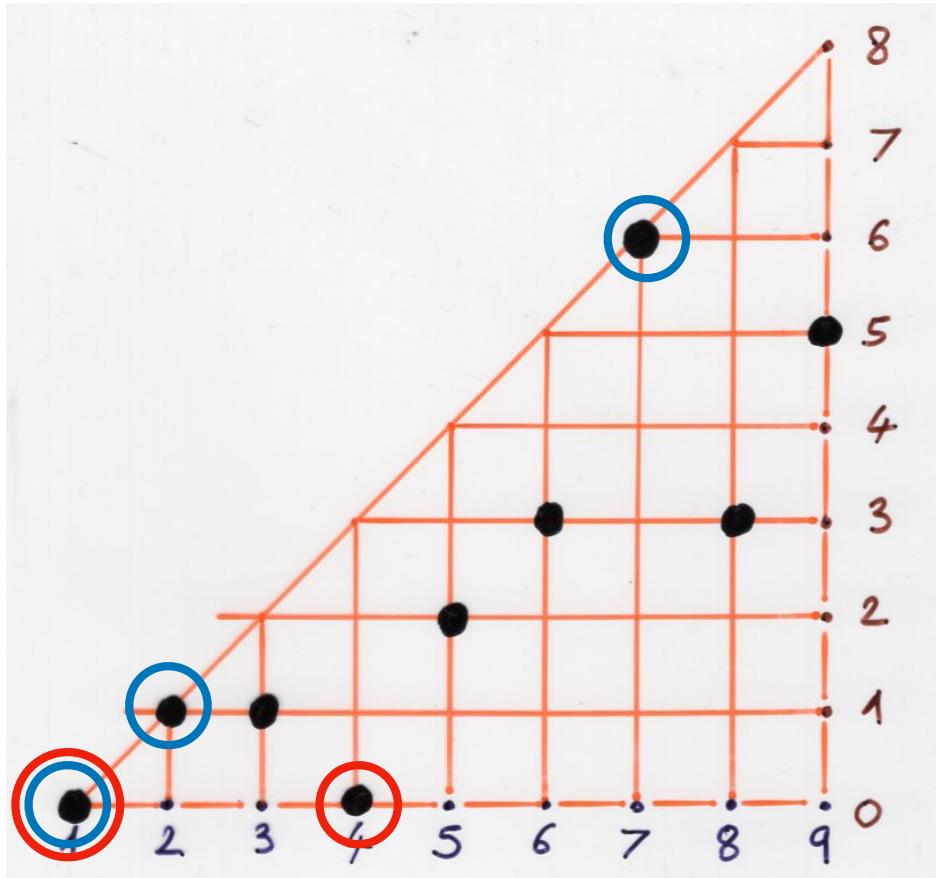
sub-excedant functions

$$f: [1, n] \rightarrow [0, n-1]$$

pour tout $1 \leq i \leq n$, $0 \leq f(i) < i$

Ω_n set of sub-excedante functions

$$|\Omega_n| = n!$$



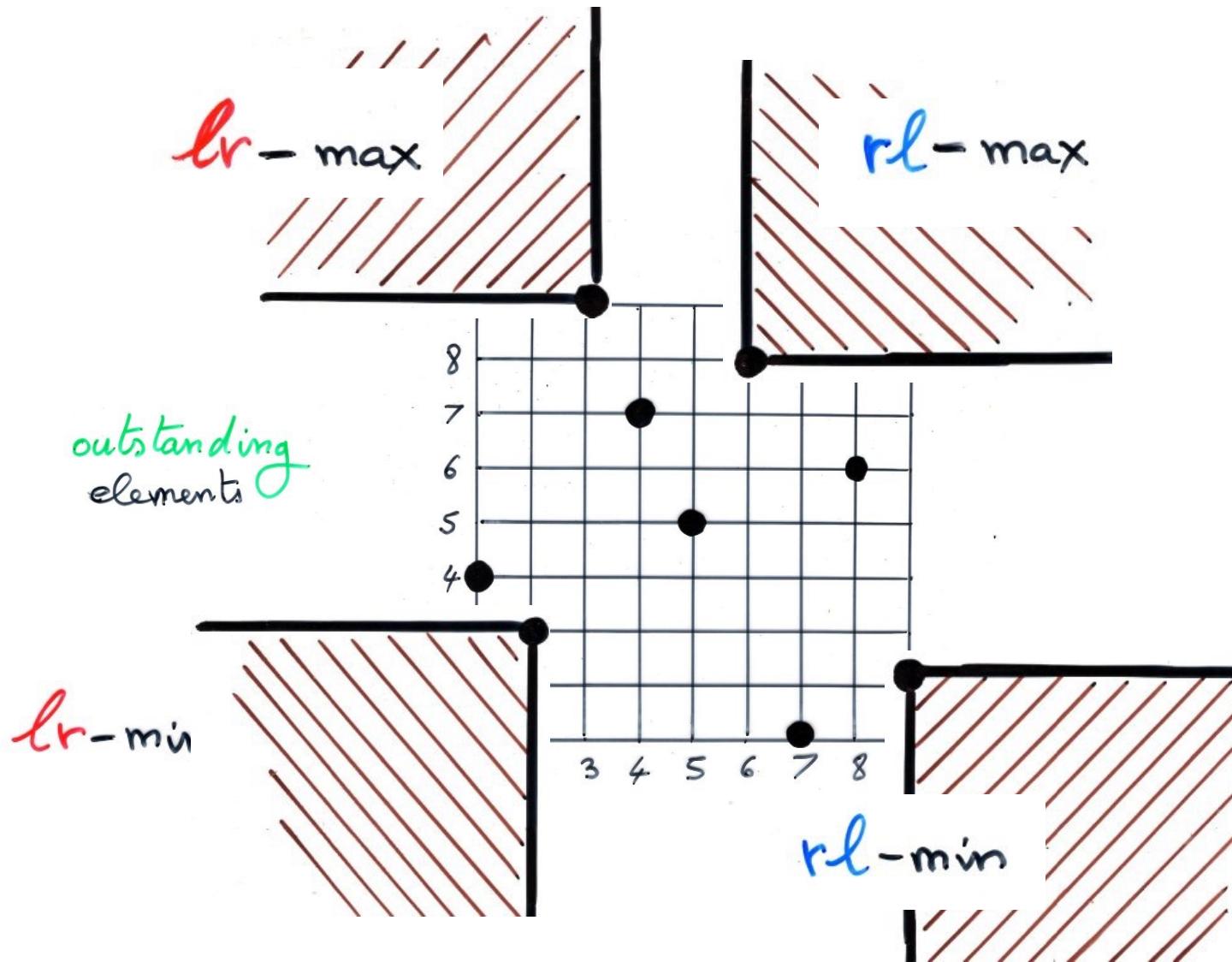
left-to-right
right-to-left

minimum
maximal

generating - polynomial:
 $xy(x+y)(x+1+y) \dots (x+n-1+y)$

σ = 7 2 3 9 6 8 5 1 4
 permutation word

elements



left-to-right
right-to-left

minimum
maximal

elements

Proposition

The **maximum** letter of the blocks of letters reaching the ground level are:

- for the **columns** of T , the left-to-right **maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) < \sigma(n+1)$
- for the **rows** of T , the right-to-left **maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) > \sigma(i+1)$

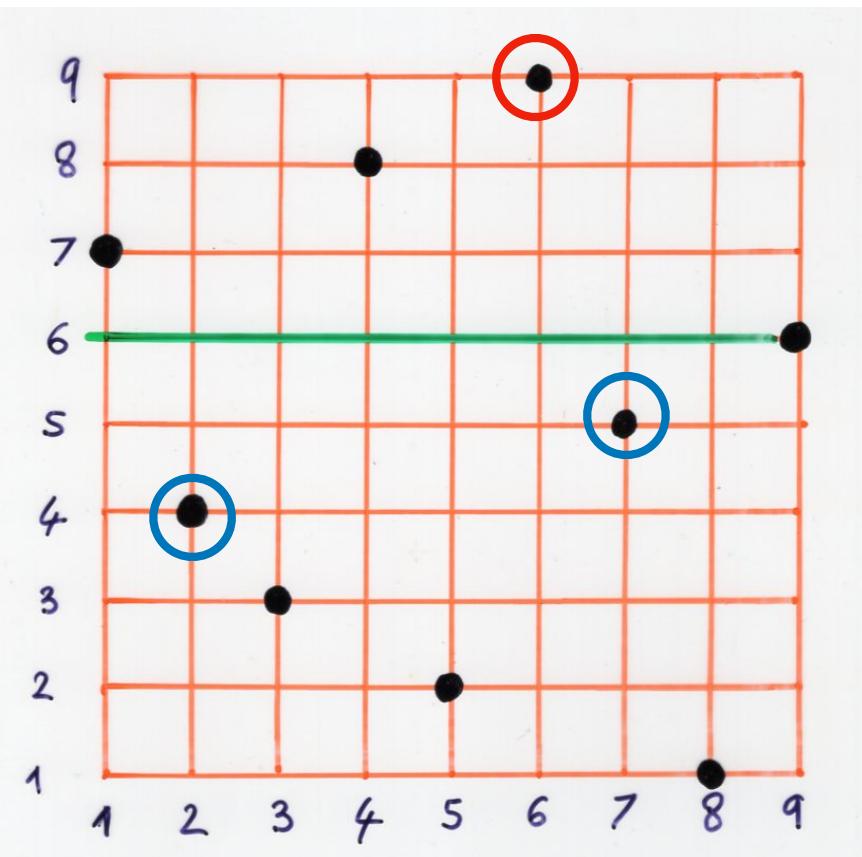
example

$$\sigma = 7 4 3 8 2 9 5 1 6$$

$$7 \quad 8 \quad 9 \quad \leftarrow 6$$

$$\rightarrow 4 3 2 \underline{5} 1 6$$

7 4 3 8 2 9 5 1 6



- for the columns of T , the left-to-right maximum elements of the subsequence of the permutation σ formed with the values $\sigma(i) < \sigma(n+1)$

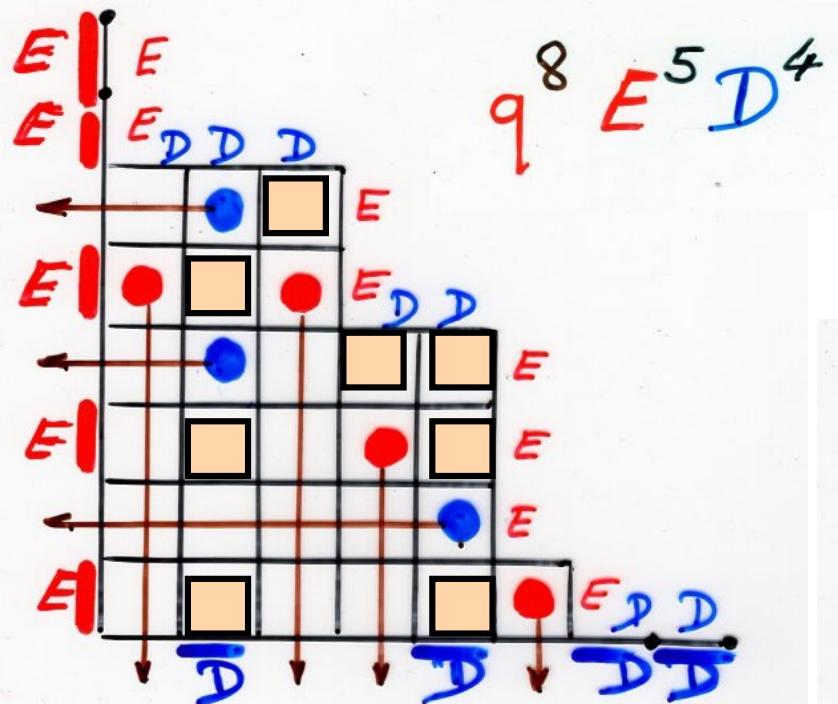
- for the rows of T , the right-to-left maximum elements of the subsequence of the permutation σ formed with the values $\sigma(i) > \sigma(i+1)$

example

$$\sigma = 7 4 3 8 2 9 5 1 6$$

$$7 \quad 8 \quad 9 \leftarrow 6$$

$$\rightarrow 4 \quad 3 \quad 2 \quad 5 \quad 1 \quad 6$$



$q^8 E^5 D^4$

- interpretation of the two parameters on alternative tableaux
- number of "open" columns
(i.e. columns without a red cell)
 - number of "open" rows
(i.e. rows without a blue cell)

$$i(T) = \text{nb of rows without } \bullet$$

$$j(T) = \text{nb of columns without } \circ$$

Corollary for T alternative tableau (size n)

The double distribution $(i(T), j(T))$
of parameters

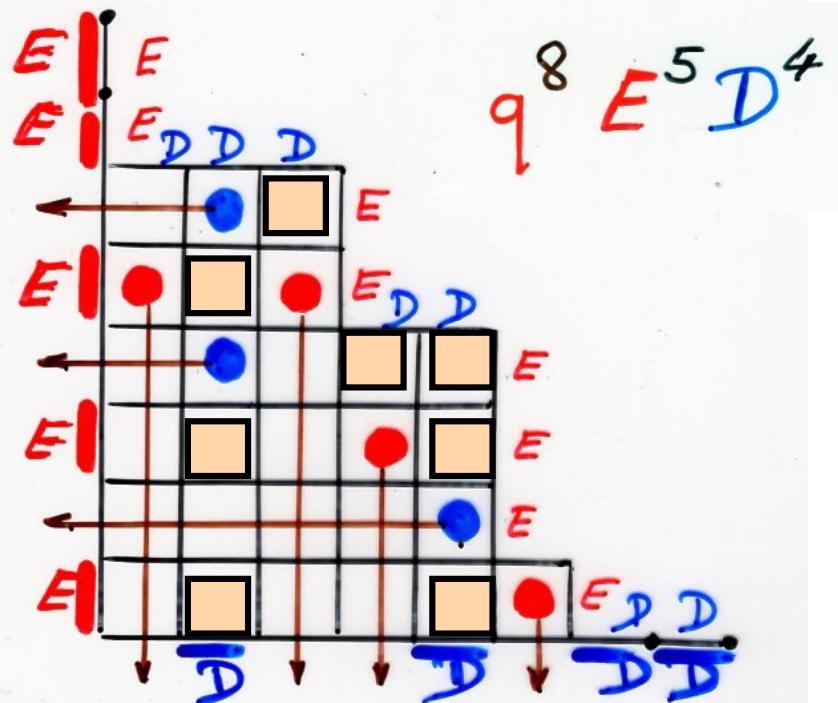
↑
number of
open rows ↑
number of
open column

is the same as the double distribution
of permutations of \mathfrak{S}_{n+1} according
to the parameters

(left-to-right minimum elements of T
right-to-left)

generating - polynomial:

$$xy(x+y)(x+1+y) \dots (x+n-1+y)$$



cell

crossing of T

exchange in the "Ex-Fu" algorithm

$k(T) = \text{nb of cells } \square$

"exchange-fusion"
algorithm

the parameter q

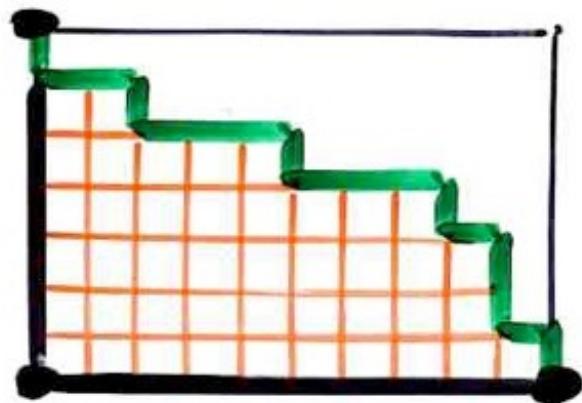
this is another story !

.... related to q -analogue of Laguerre polynomials

permutation
tableaux

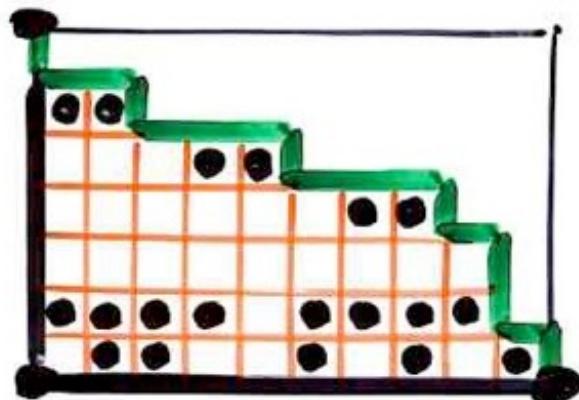
Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells
with 0 and 1

(i)

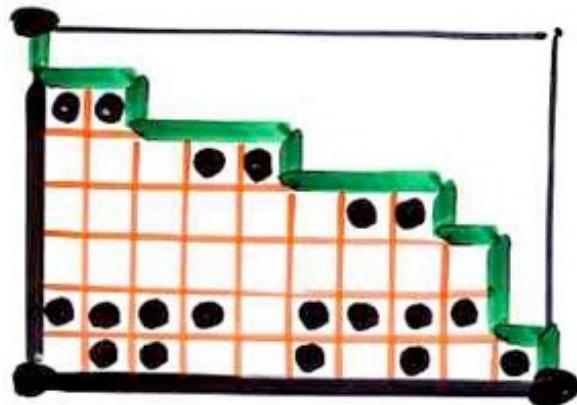
$\square = 0$

$\bullet = 1$

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells
with 0 and 1

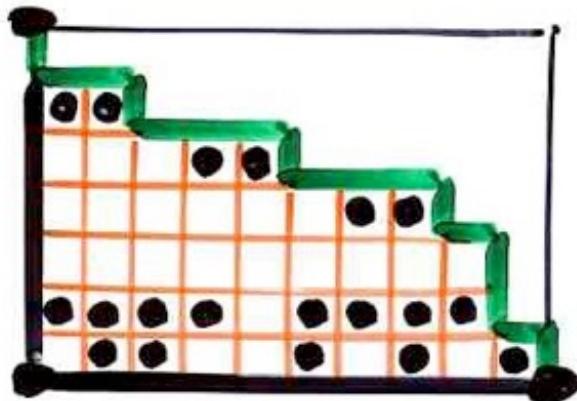
(i) in each column :
at least one 1

$$\square = 0 \quad \bullet = 1$$

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle

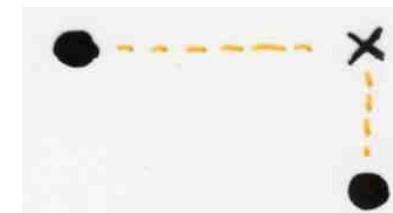


filling of the cells
with 0 and 1

(i) in each column :
at least one 1

$$\square = 0 \quad \bullet = 1$$

(ii)



forbidden

permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

The total number of permutation tableaux (n fixed, $1 \leq k \leq n$) is

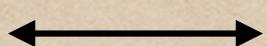
$$n!$$

bijection
permutations \longleftrightarrow permutation
tableaux

- Postnikov , Steingrímsson , Williams (2005)
- Corteel (2006)
- Corteel, Nadeau (2007)

Bijection

permutation
tableaux



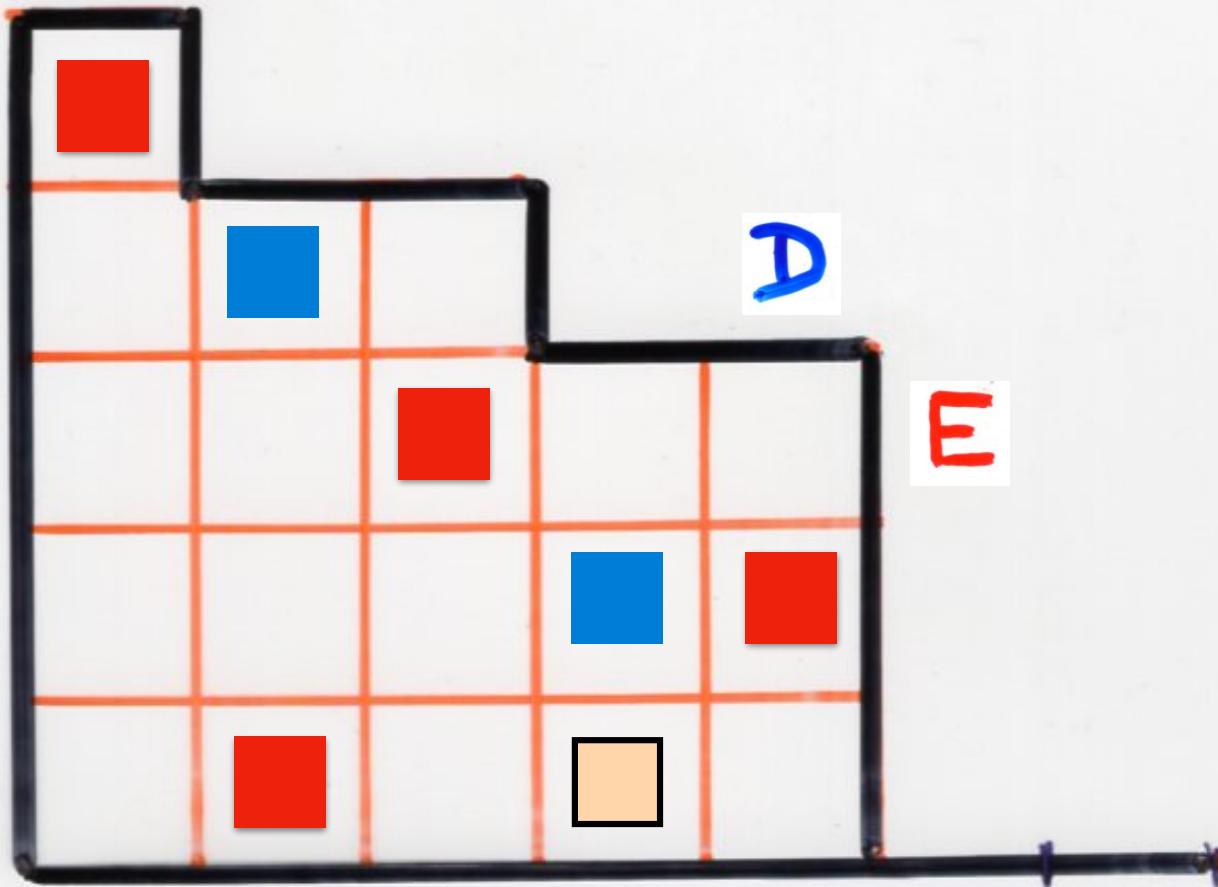
alternative tableaux

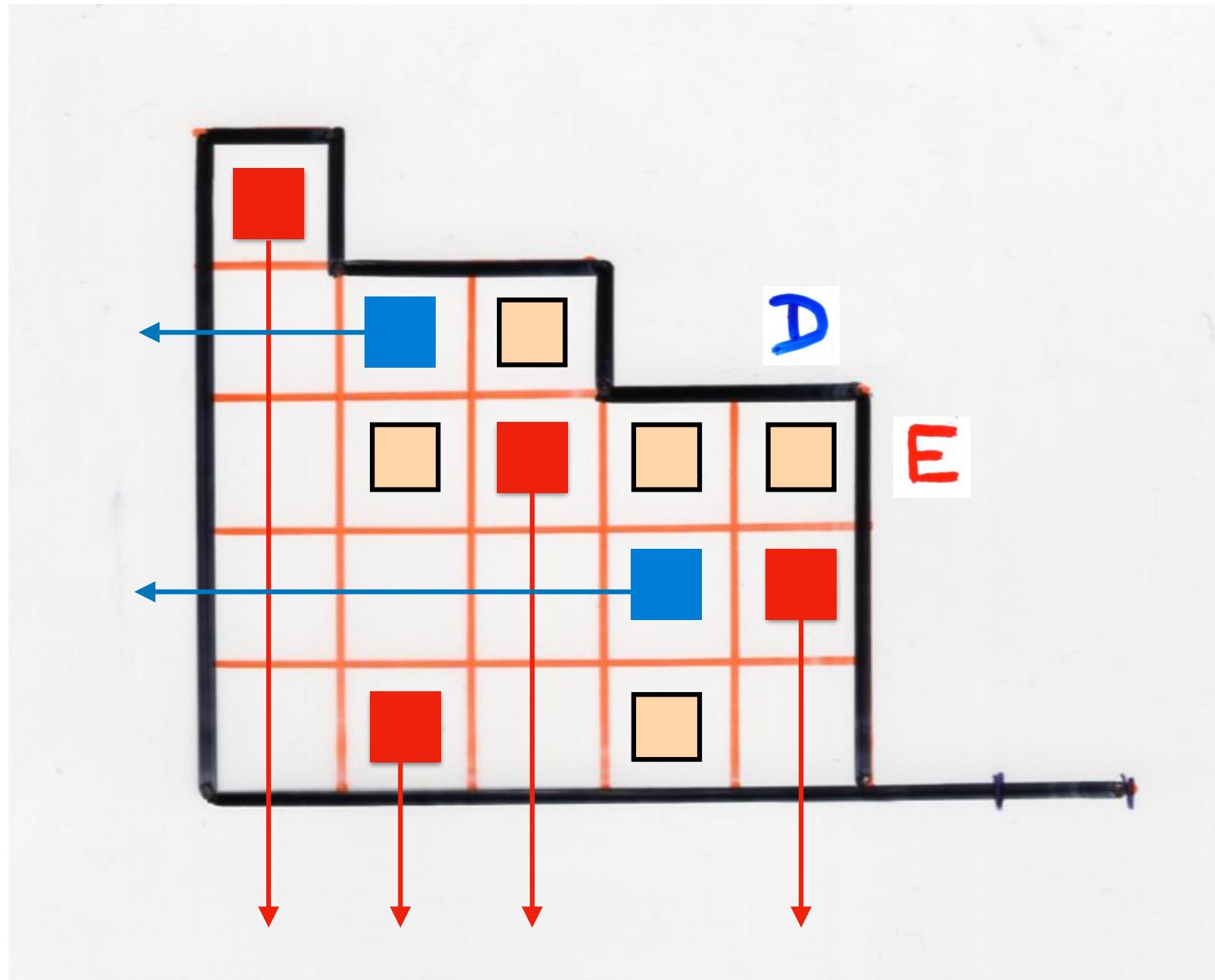
bijection

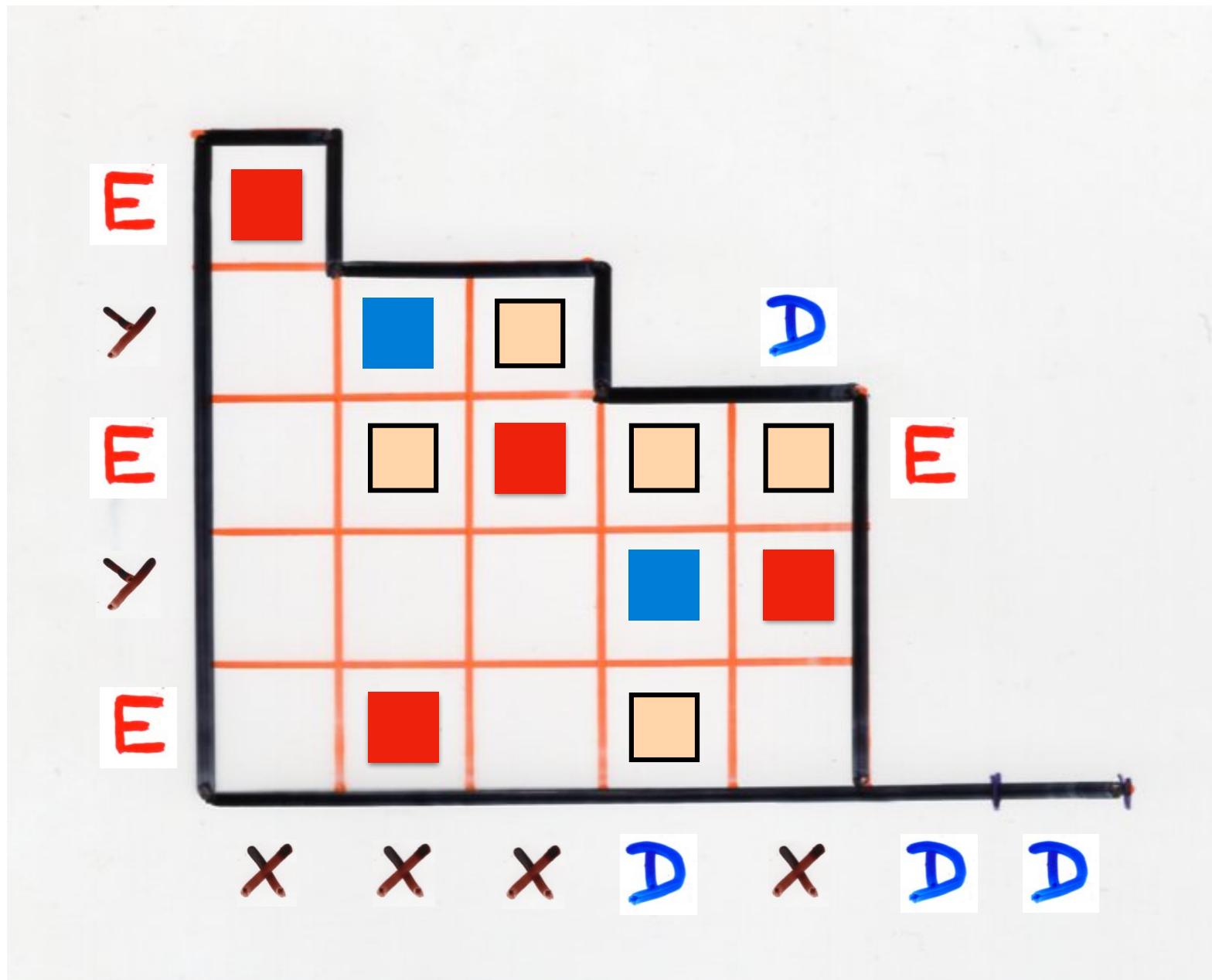


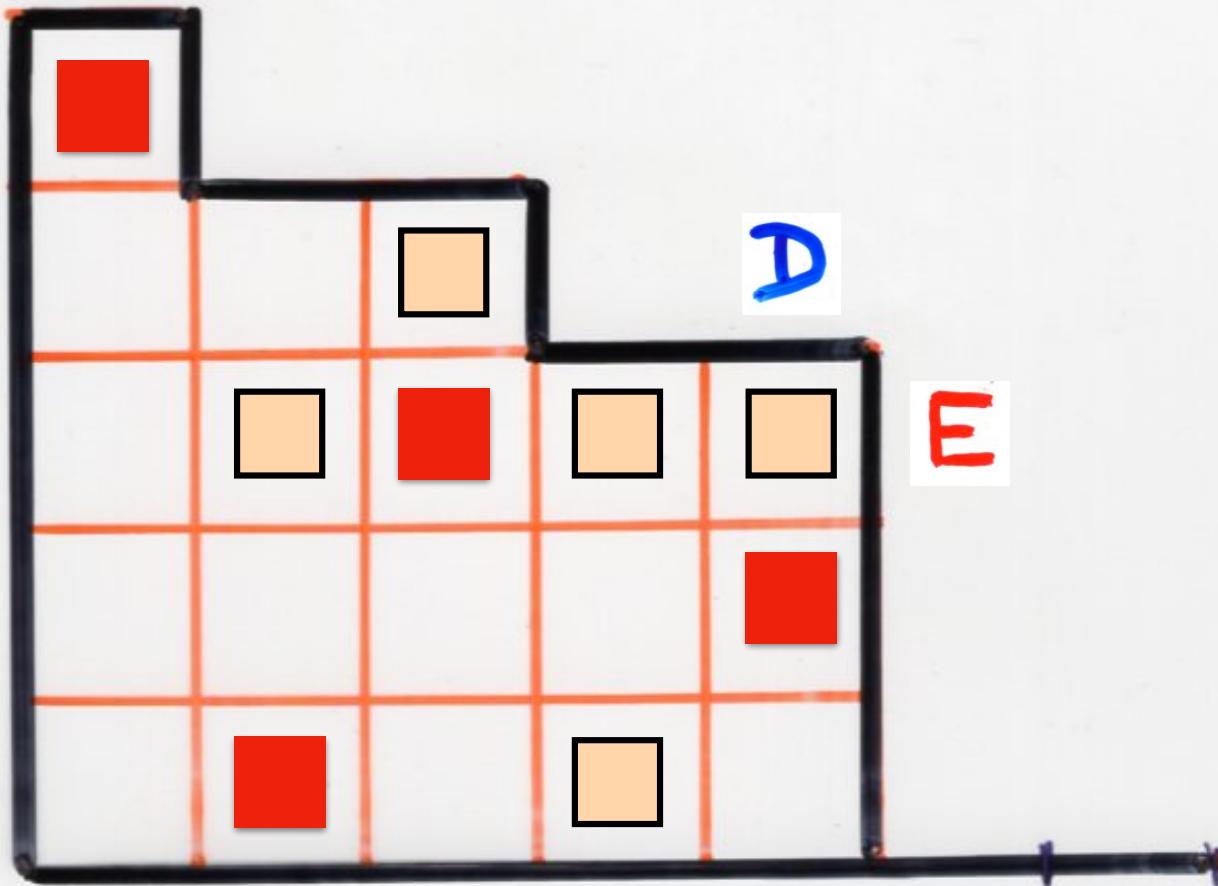
alternative tableaux size n

permutation tableaux size $(n+1)$

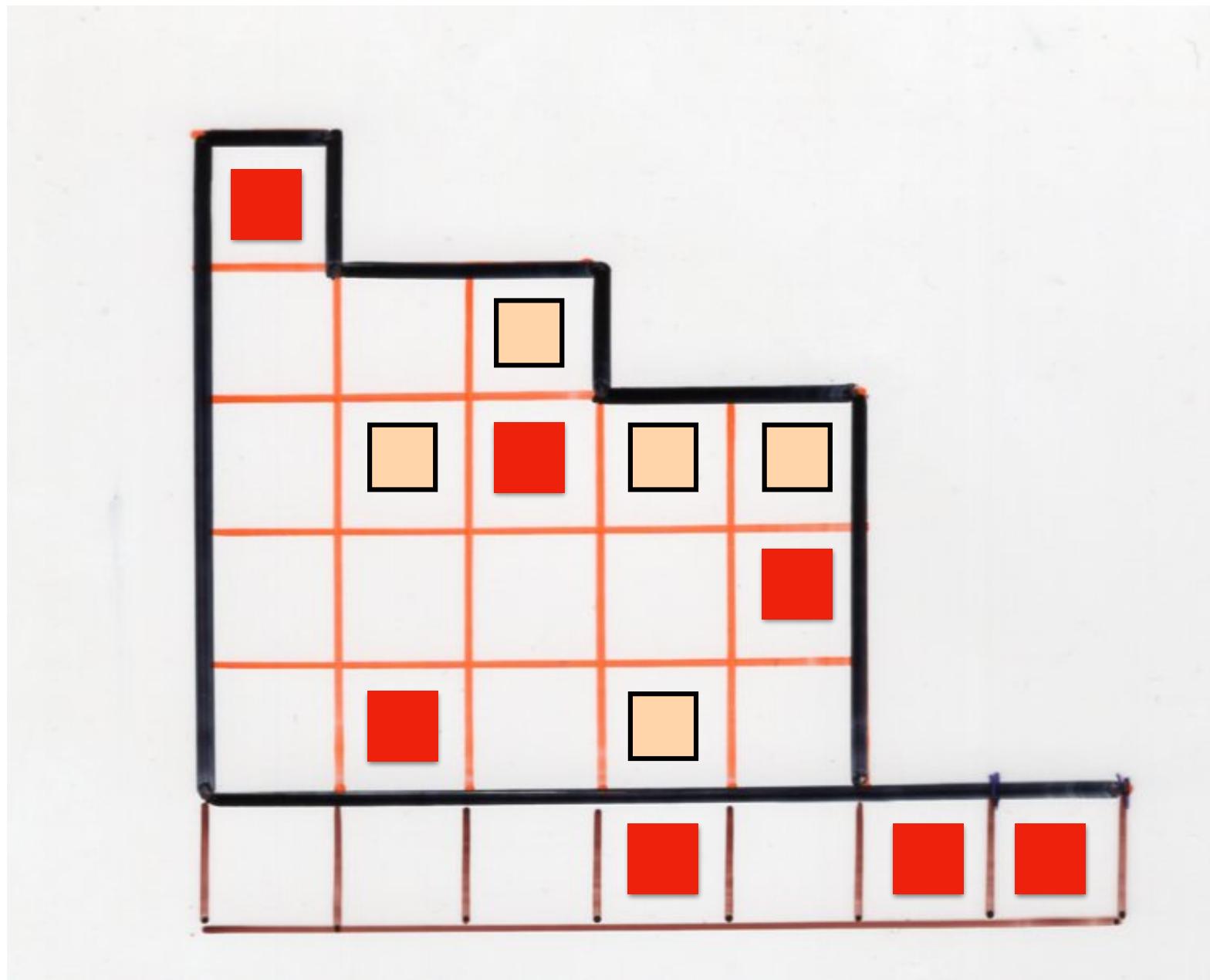


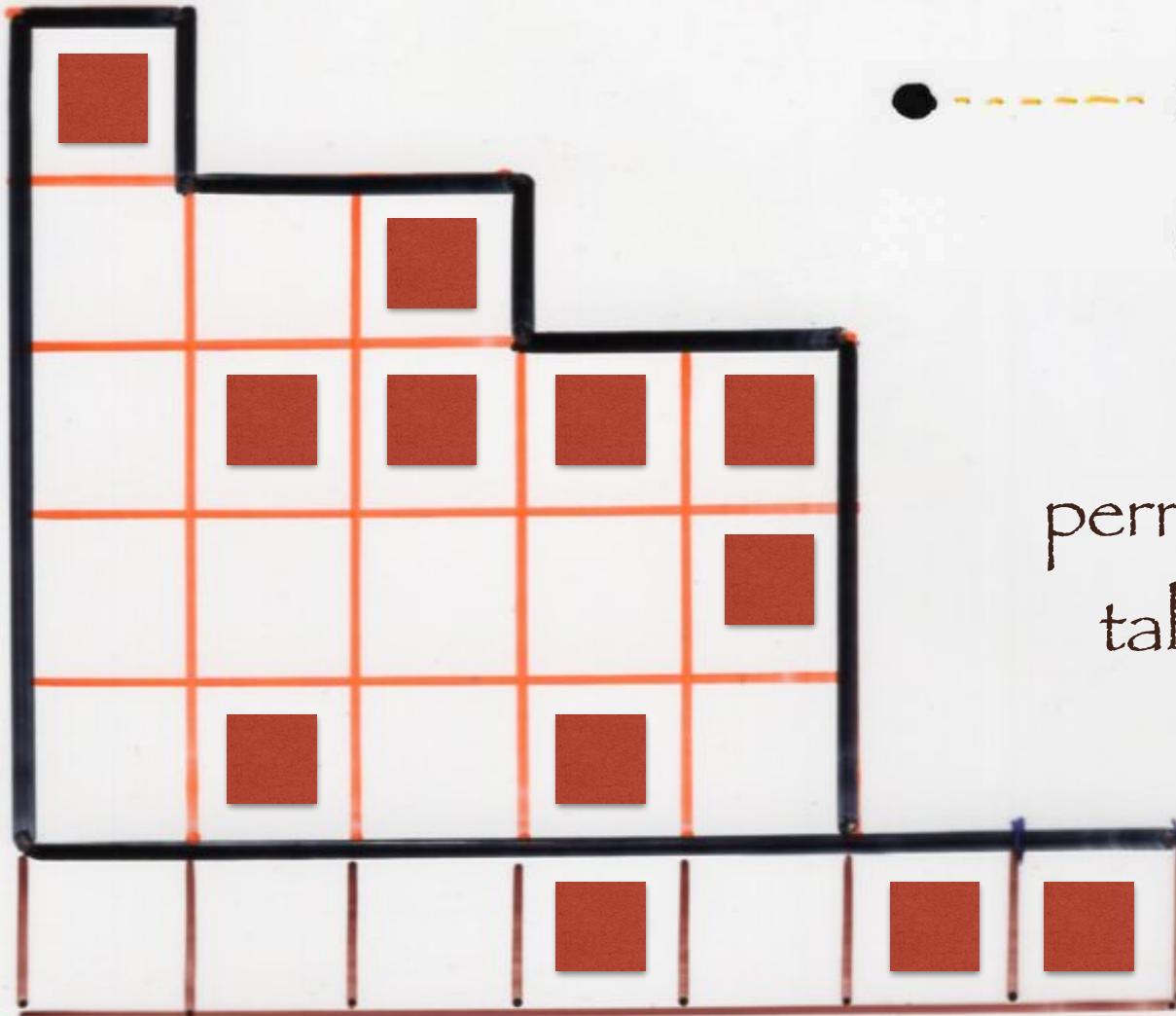




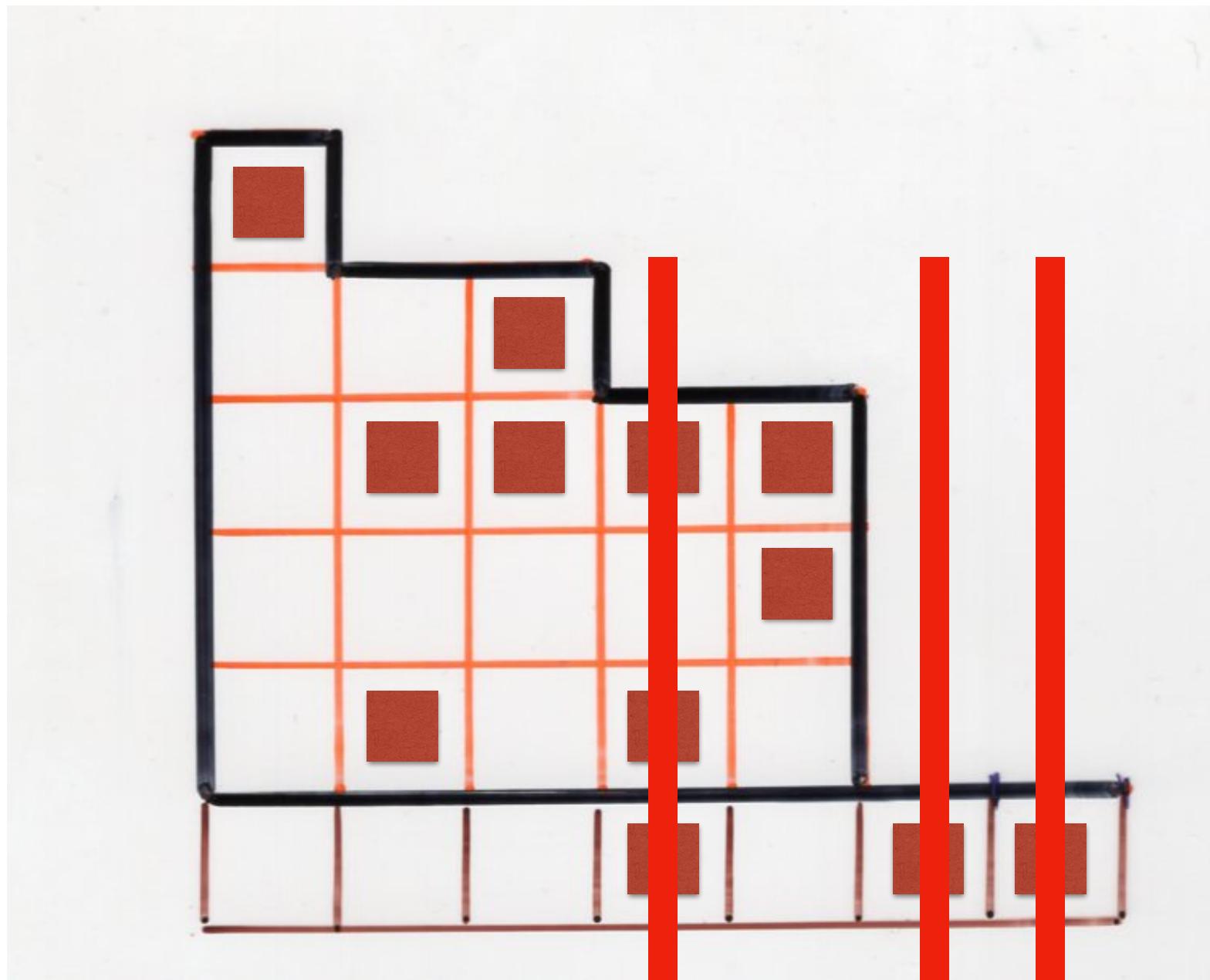


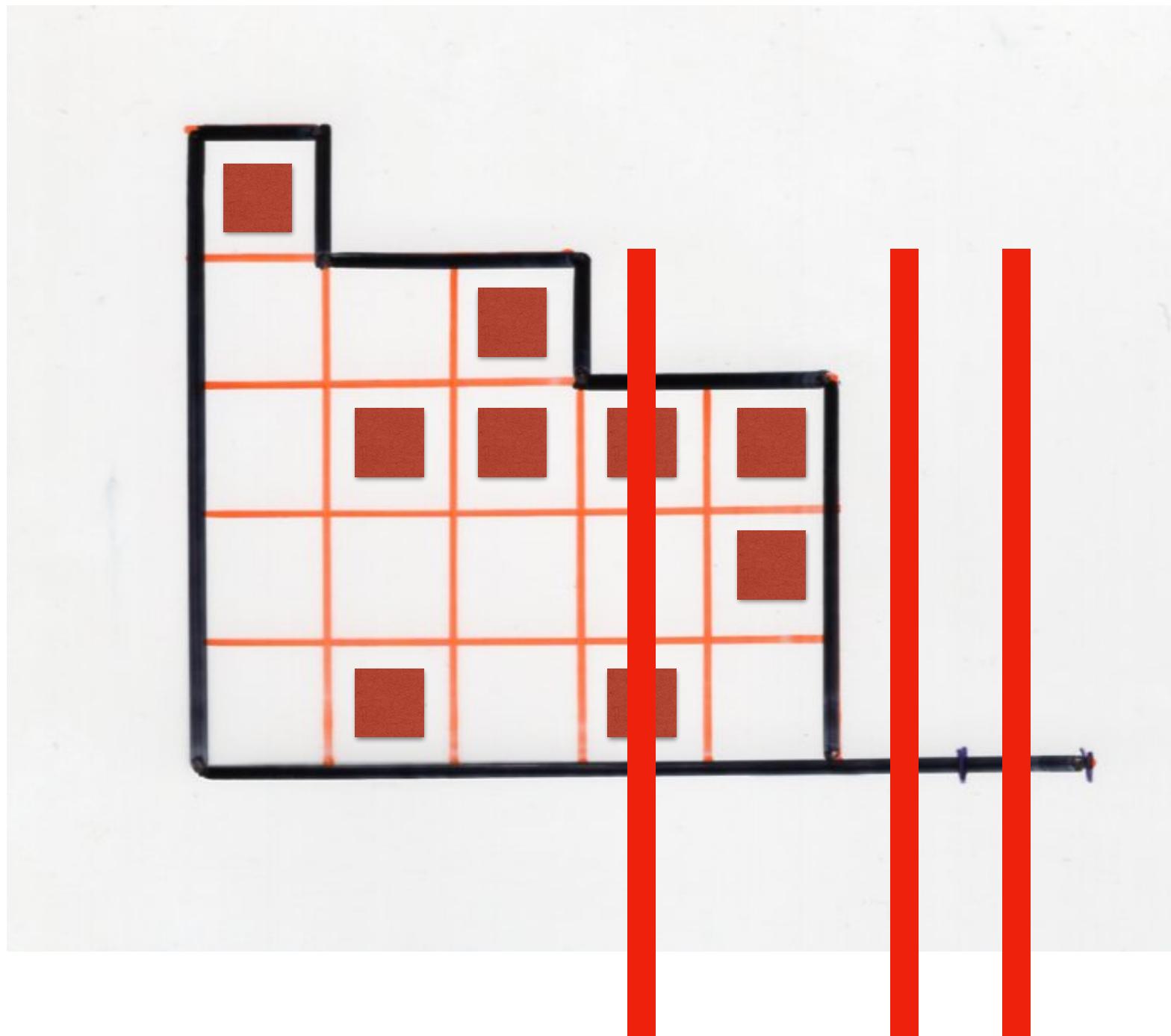
X X X D X D D

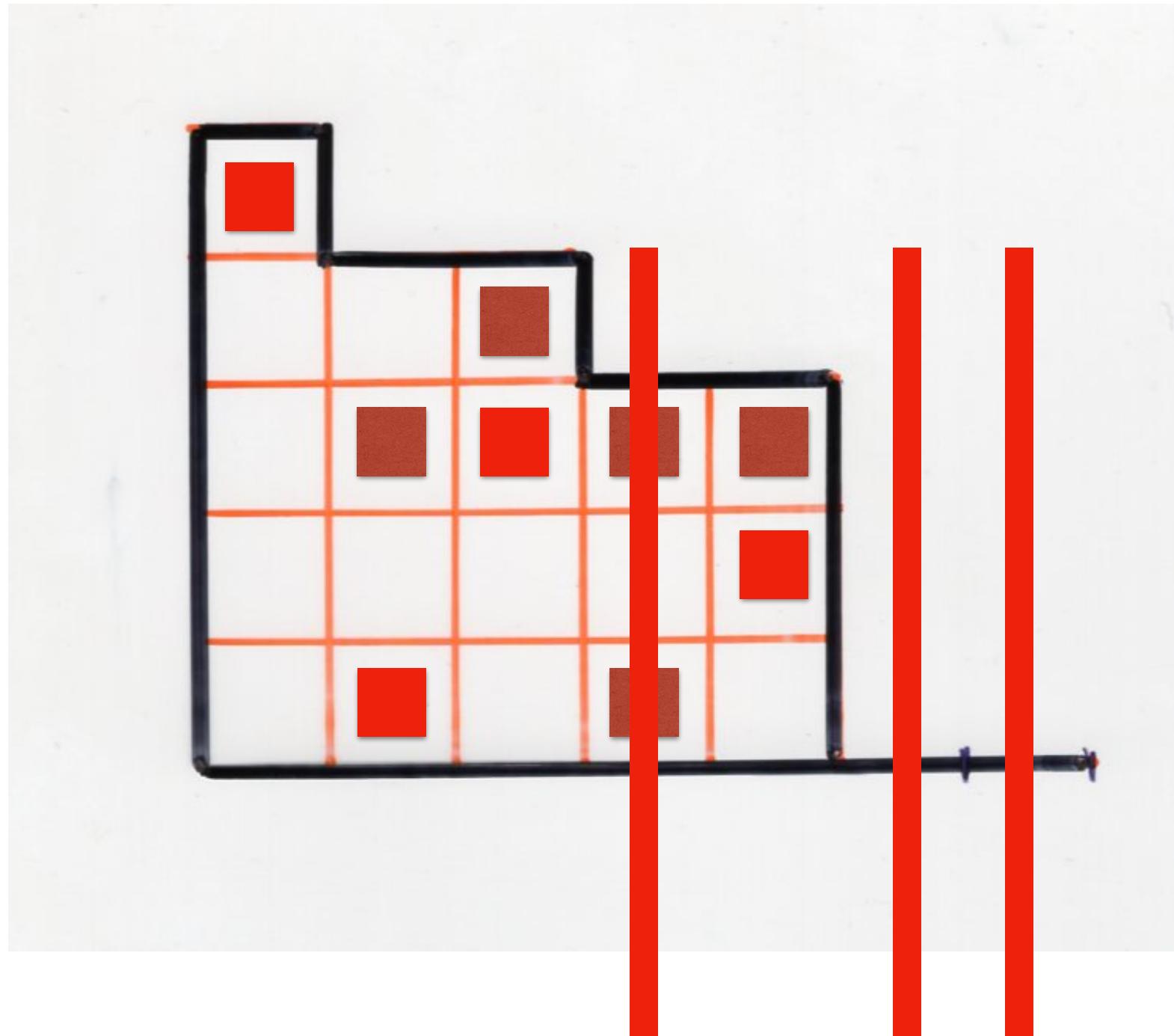


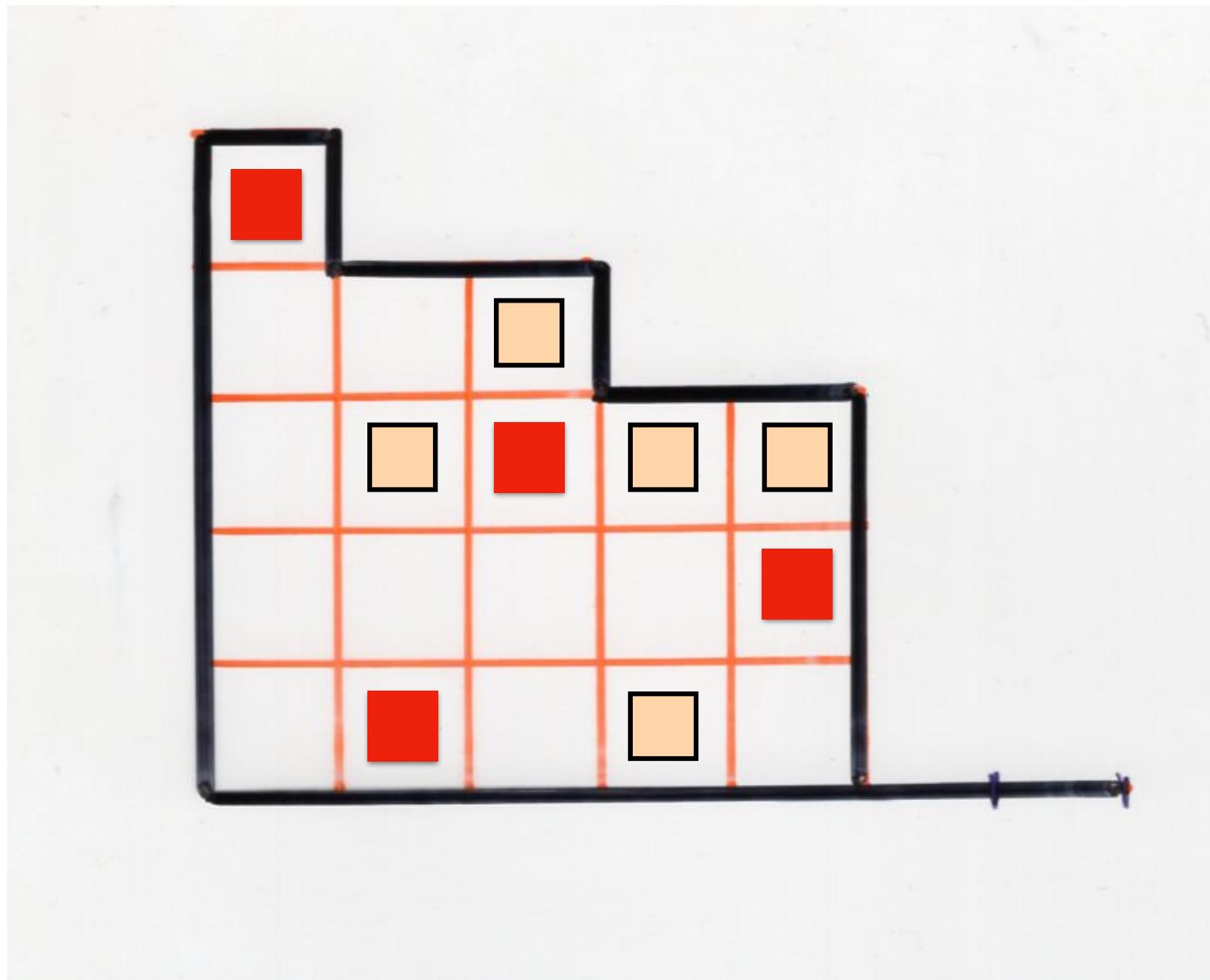


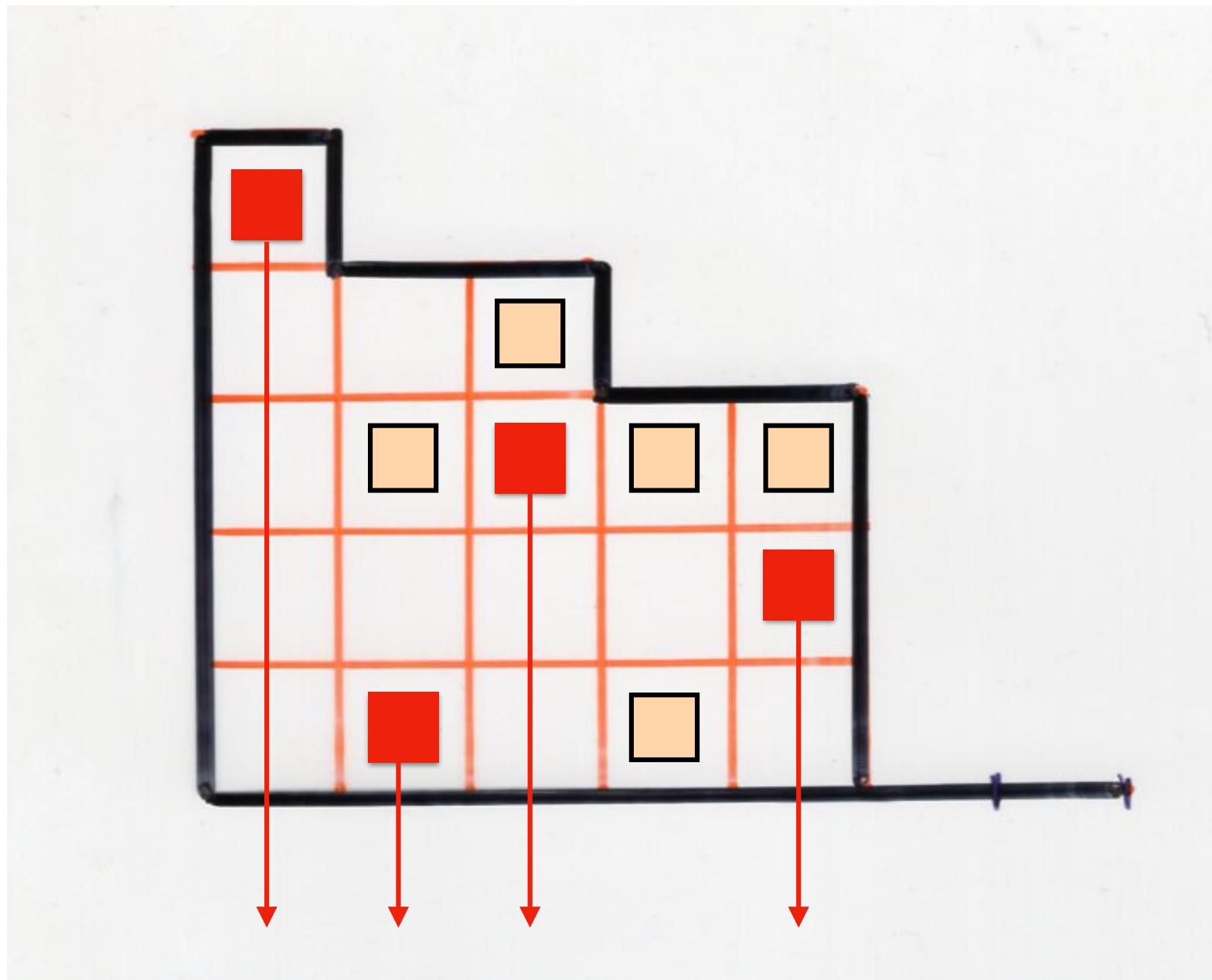
permutation
tableaux

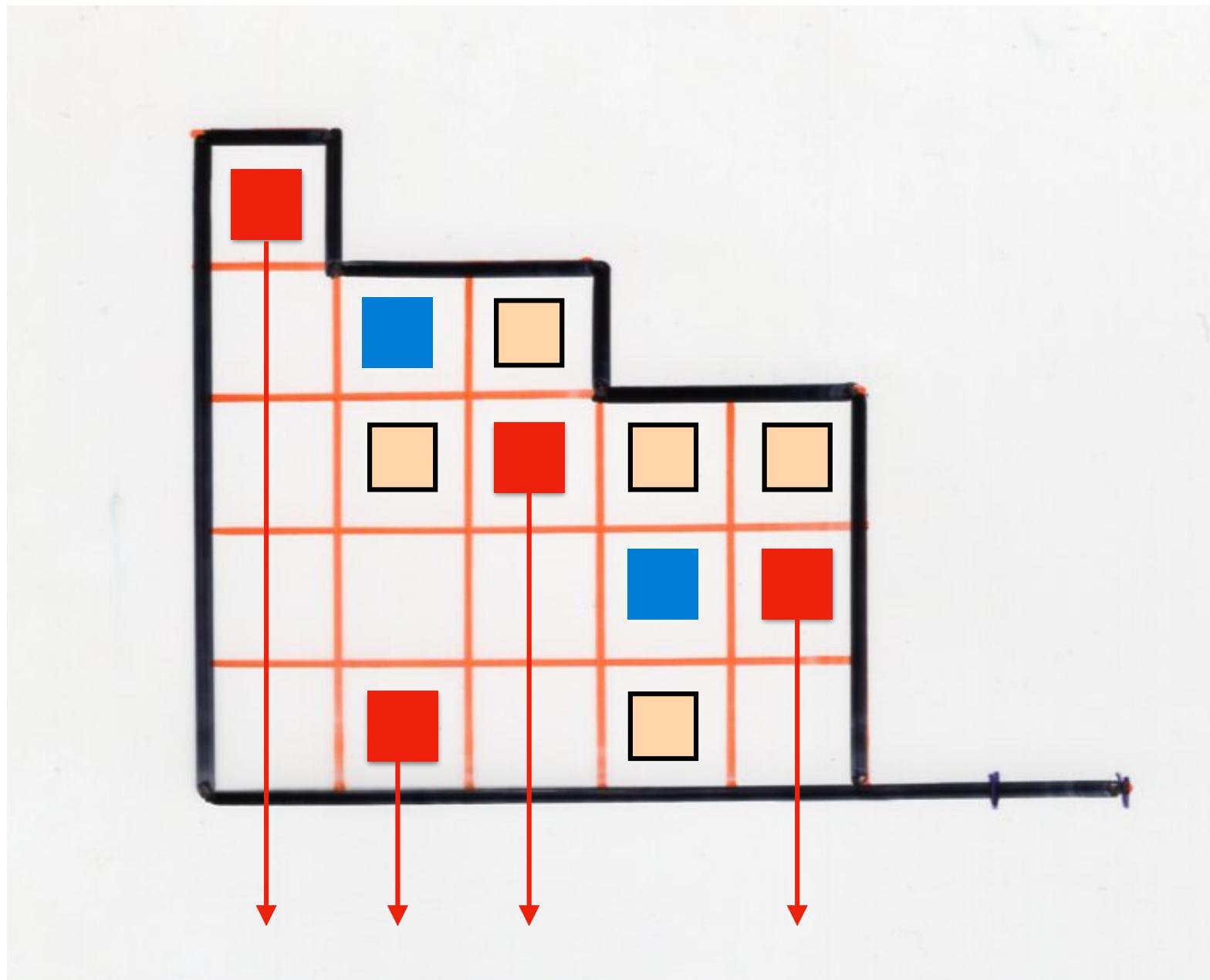


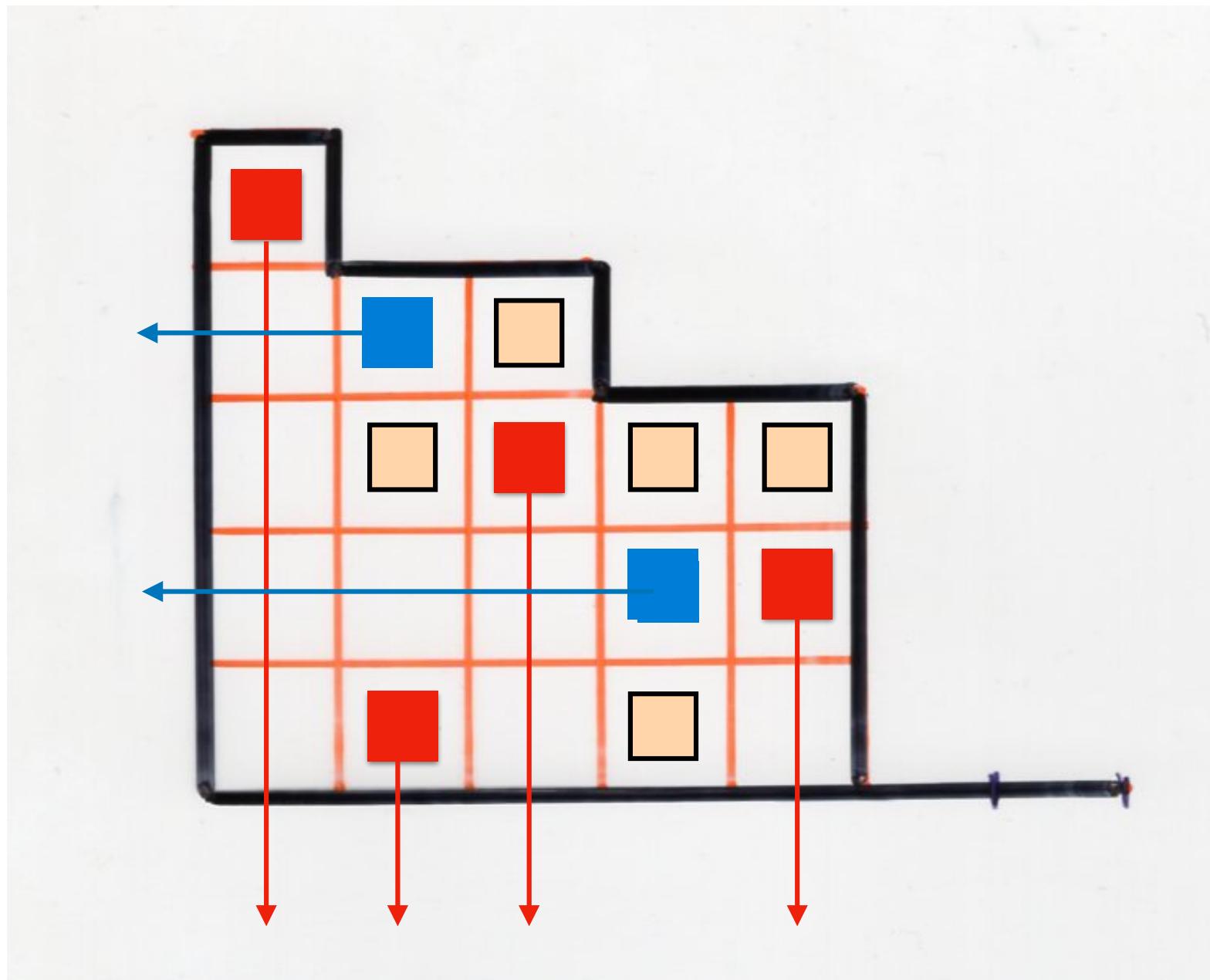












$$Q \left\{ \begin{array}{l} DE = \boxed{\textcolor{red}{E}} D + \boxed{\textcolor{red}{E}} X + \boxed{\textcolor{blue}{Y}} D \\ XE = \boxed{\textcolor{blue}{D}} E X \\ DY = \boxed{\textcolor{blue}{X}} Y D \\ XY = \boxed{\textcolor{blue}{X}} Y X \end{array} \right.$$

\xrightarrow{D} $\downarrow Y$ E

Permutation tableaux

$$Q^+ \left\{ \begin{array}{l} ED = \boxed{\textcolor{blue}{D}} E \\ EX = \boxed{\textcolor{blue}{X}} X E + \boxed{\textcolor{blue}{D}} E \\ DY = \boxed{\textcolor{blue}{D}} Y D - \boxed{\textcolor{blue}{D}} E \\ XY = \boxed{\textcolor{blue}{X}} X Y \end{array} \right.$$

$$Q \left\{ \begin{array}{l} DE = \square ED - EX + YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$

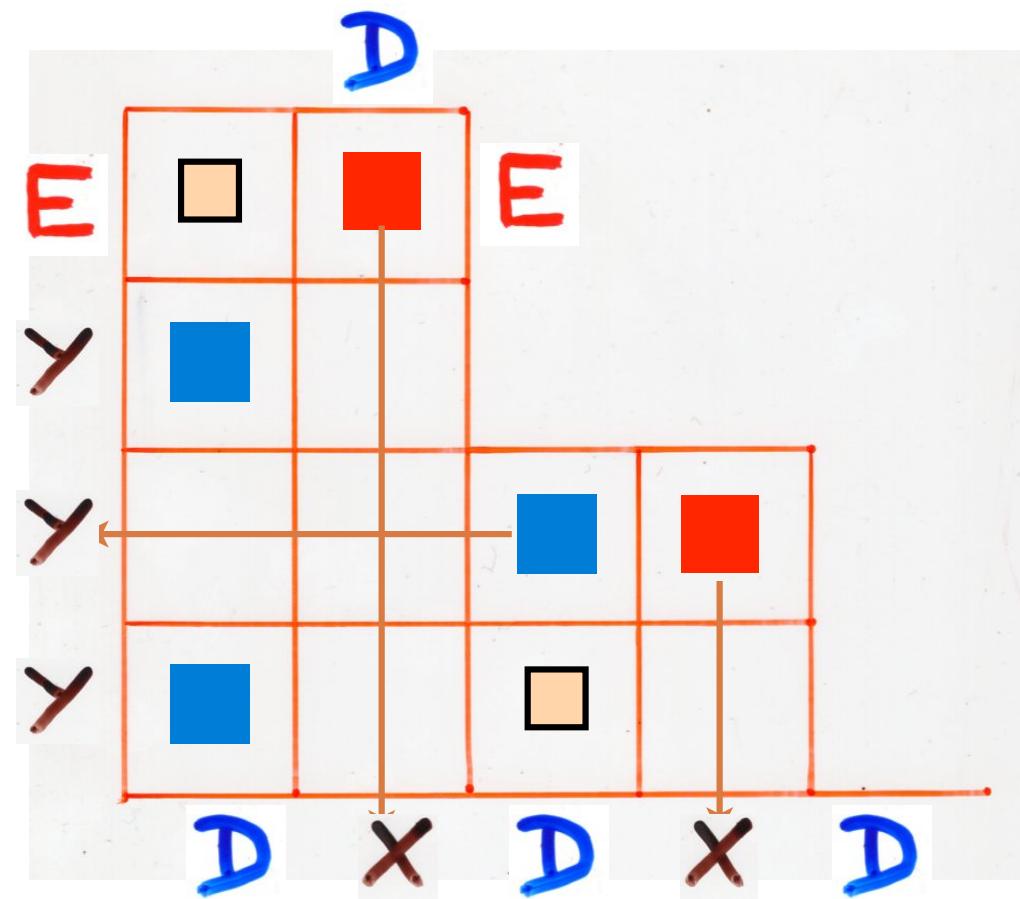
\overbrace{D}^{\square} $\overbrace{Y}^{\square} \overbrace{E}^{\square}$

alternative

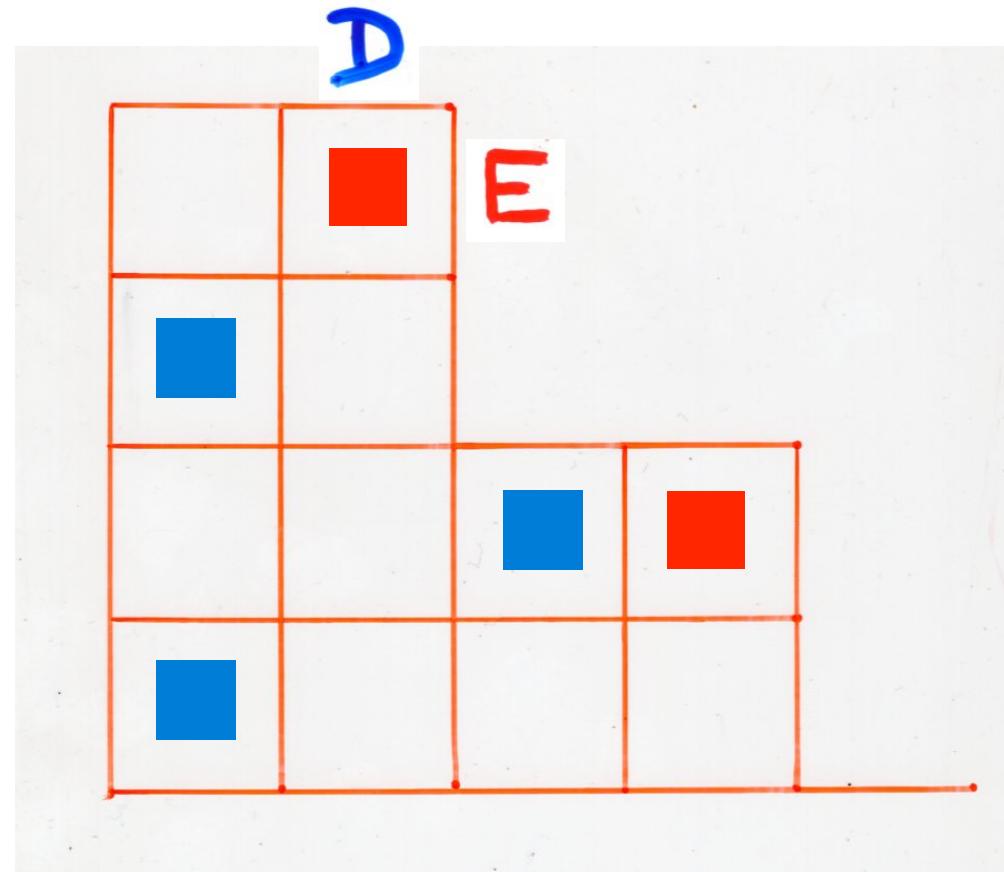
tableaux

$$Q^+ \left\{ \begin{array}{l} ED = \square DE \\ EX = \square XE + \square DE \\ DY = \square DY - \square DE \\ YX = \square XY \end{array} \right.$$

alternative
tableaux



alternative
tableaux



$$Q \left\{ \begin{array}{l} DE = \square ED - \blacksquare EX + \blacksquare YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$

\xrightarrow{D} $\downarrow E$

tree-like tableaux

$$Q^+ \left\{ \begin{array}{l} ED = \square DE \\ EX = \square XE - \blacksquare DE \\ YD = \square DY - \blacksquare DE \\ YX = \square XY \end{array} \right.$$

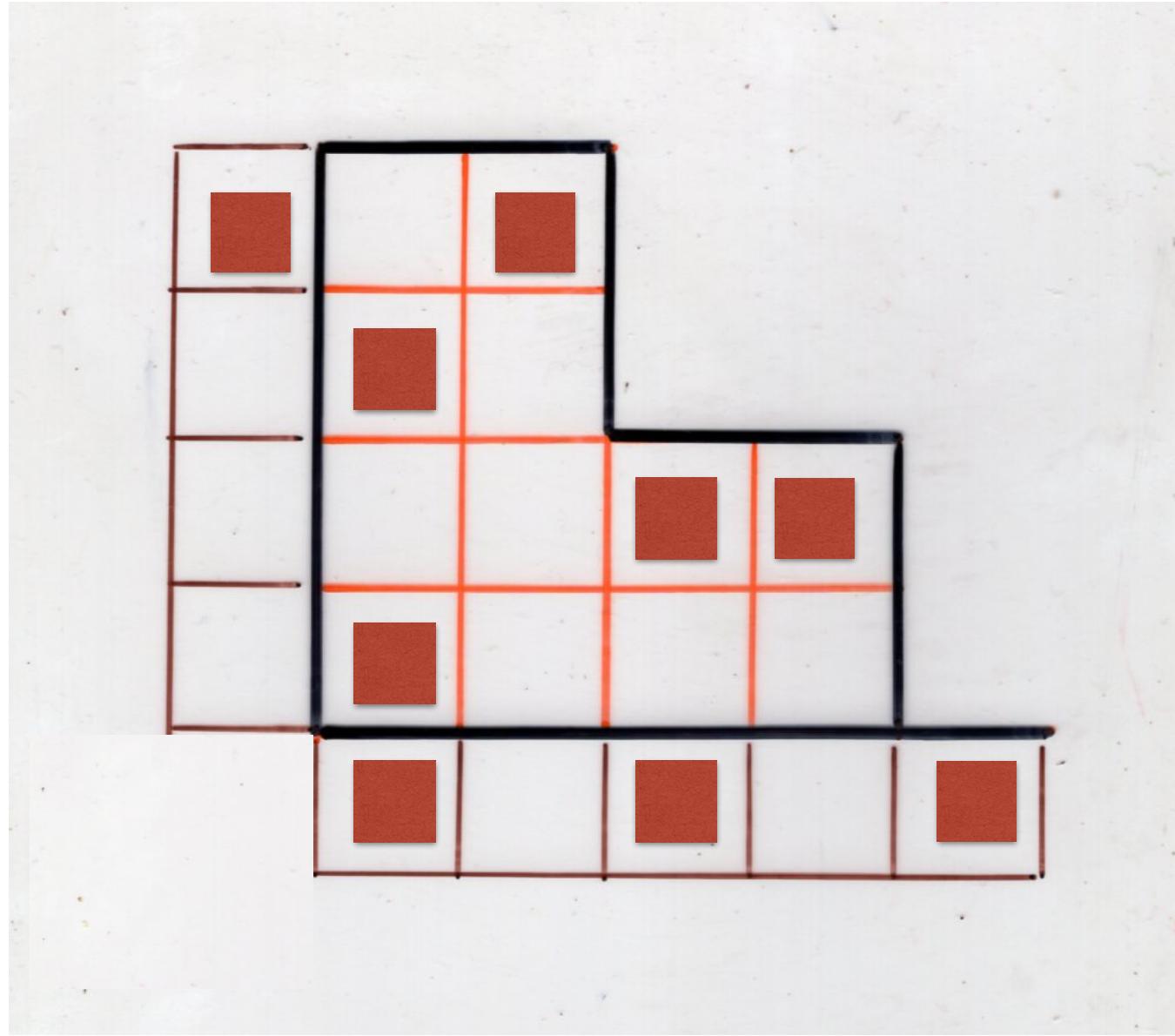
alternative
tableaux

			D		
E		■	■	E	
Y		■			
Y			■	■	
Y		■		■	
D	X	D	X	D	

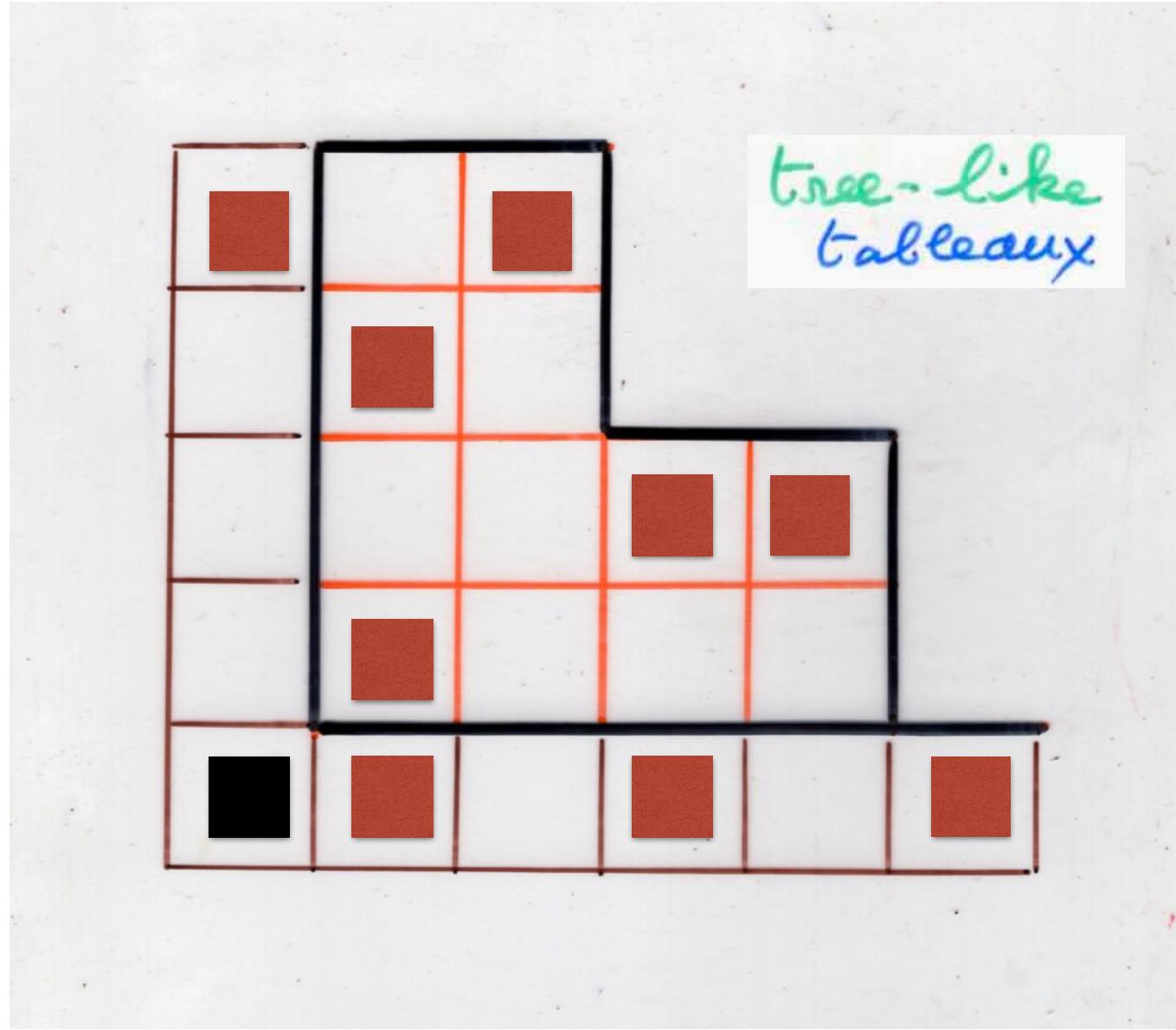
= reverse Q-tableau

E			
Y			
Y			
Y			
D	X	D	X
			D

= reverse Q-tableau



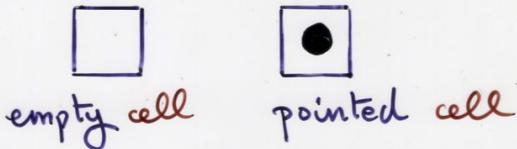
= reverse Q-tableau



Definition Tree-like tableaux

Aval, Boussicault, Nadeau (2011)

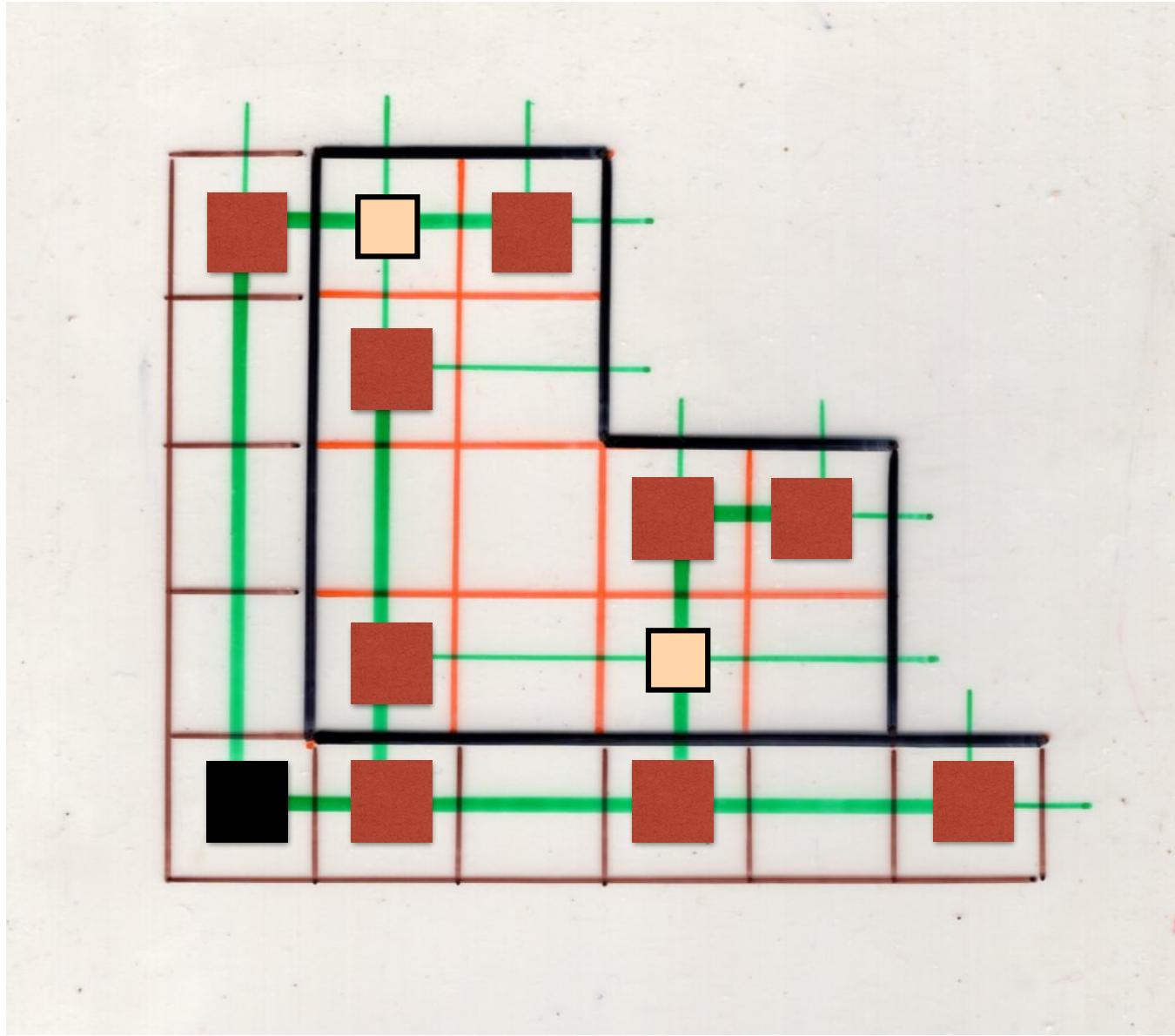
Ferrers diagram F with cell



(i) the bottom left cell is pointed
(called the root cell)

(ii) for every non-root pointed cell c ,
there exist a pointed cell below c
in the same column, or a pointed cell
to its left in the same row,
but not both

(iii) every column and every row
possesses at least one pointed cell



tree-like
tableaux

$$Q \left\{ \begin{array}{l} DE = \square ED - \blacksquare EX + \blacksquare YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$



alternative

tableaux

$$Q \left\{ \begin{array}{l} DE = \square ED - \blacksquare EX + \blacksquare YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$



tree-like

tableaux

$$Q \left\{ \begin{array}{l} DE = \blacksquare ED - \blacksquare EX + \square YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$



permutation tableaux



permutation
tableaux

$n!$

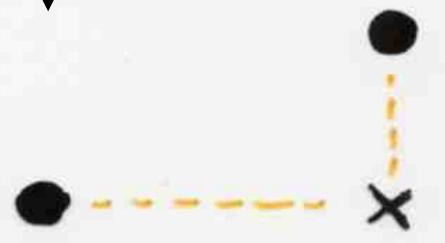
permutations

alternative
tableaux

X.V. (2008)

tree-like
tableaux

Aval, Boussicault, Nadeau (2013)



J-diagrams

decorated
permutations

Steingrímsson, Williams (2007)

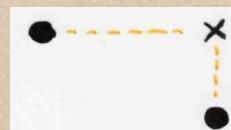
Postnikov (2006)

totally non-negative part
of the type A Grassmannian

Complements

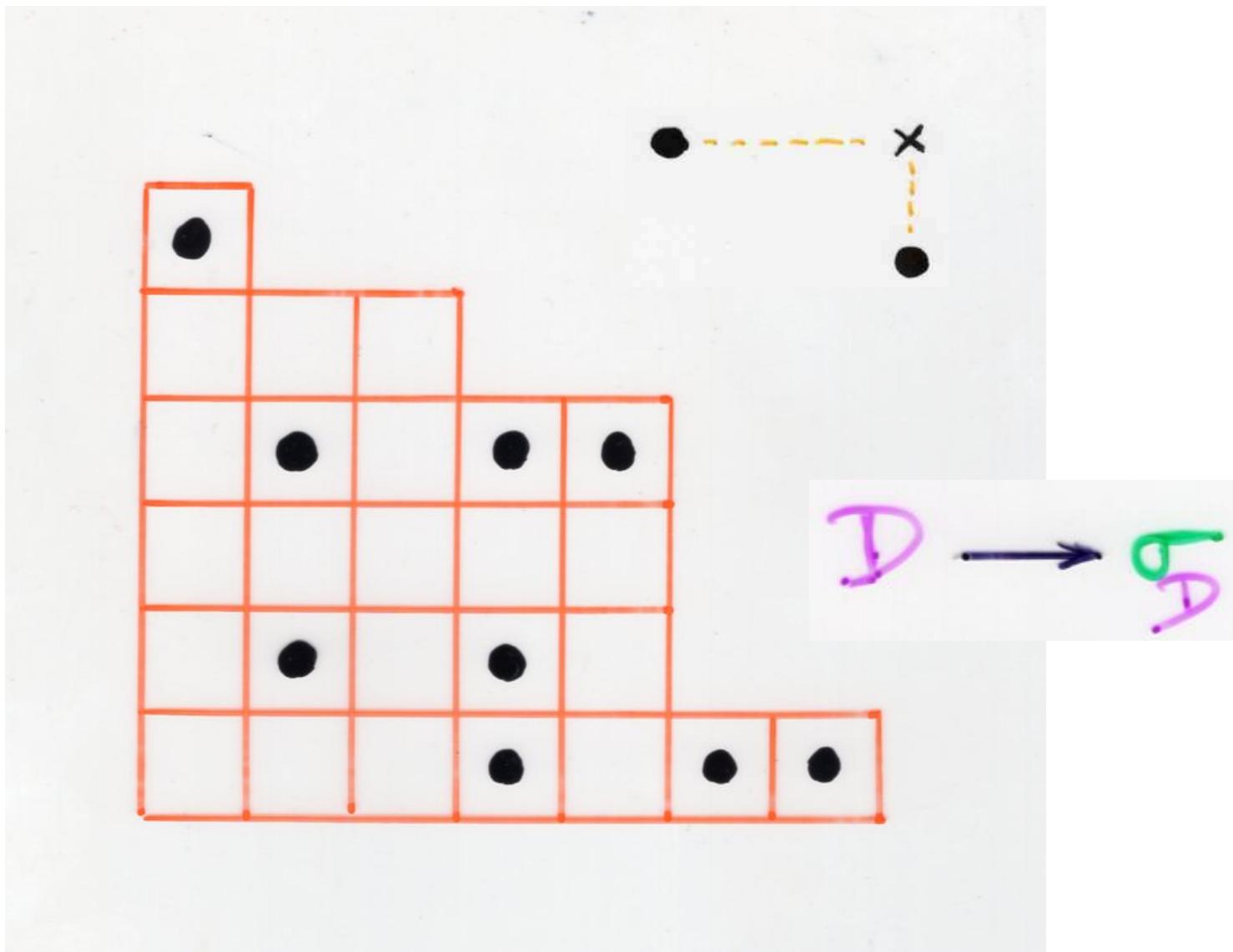
Postnikov bijection

pipe dreams and

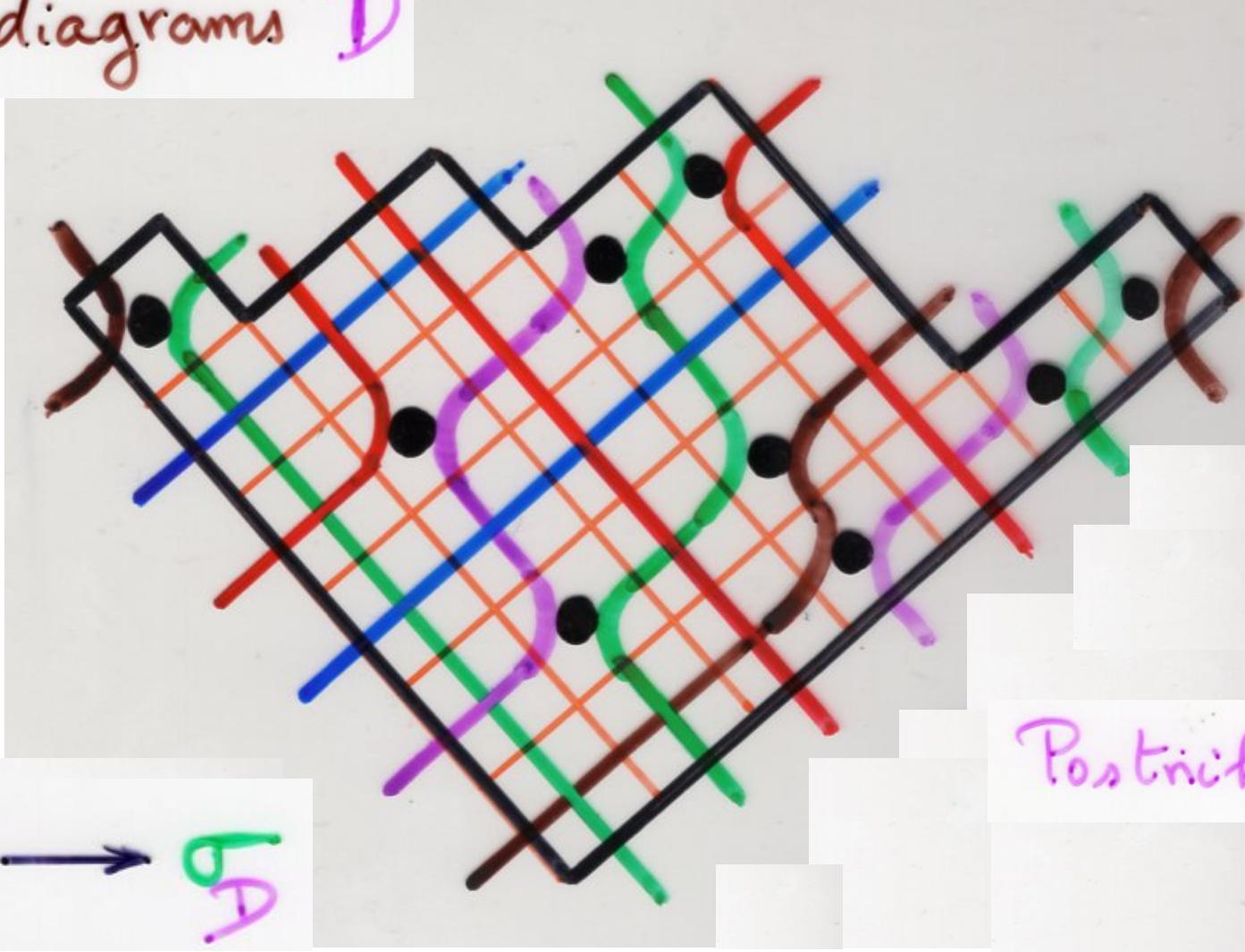


-diagrams

$\text{7-diagrams } \mathcal{D}$



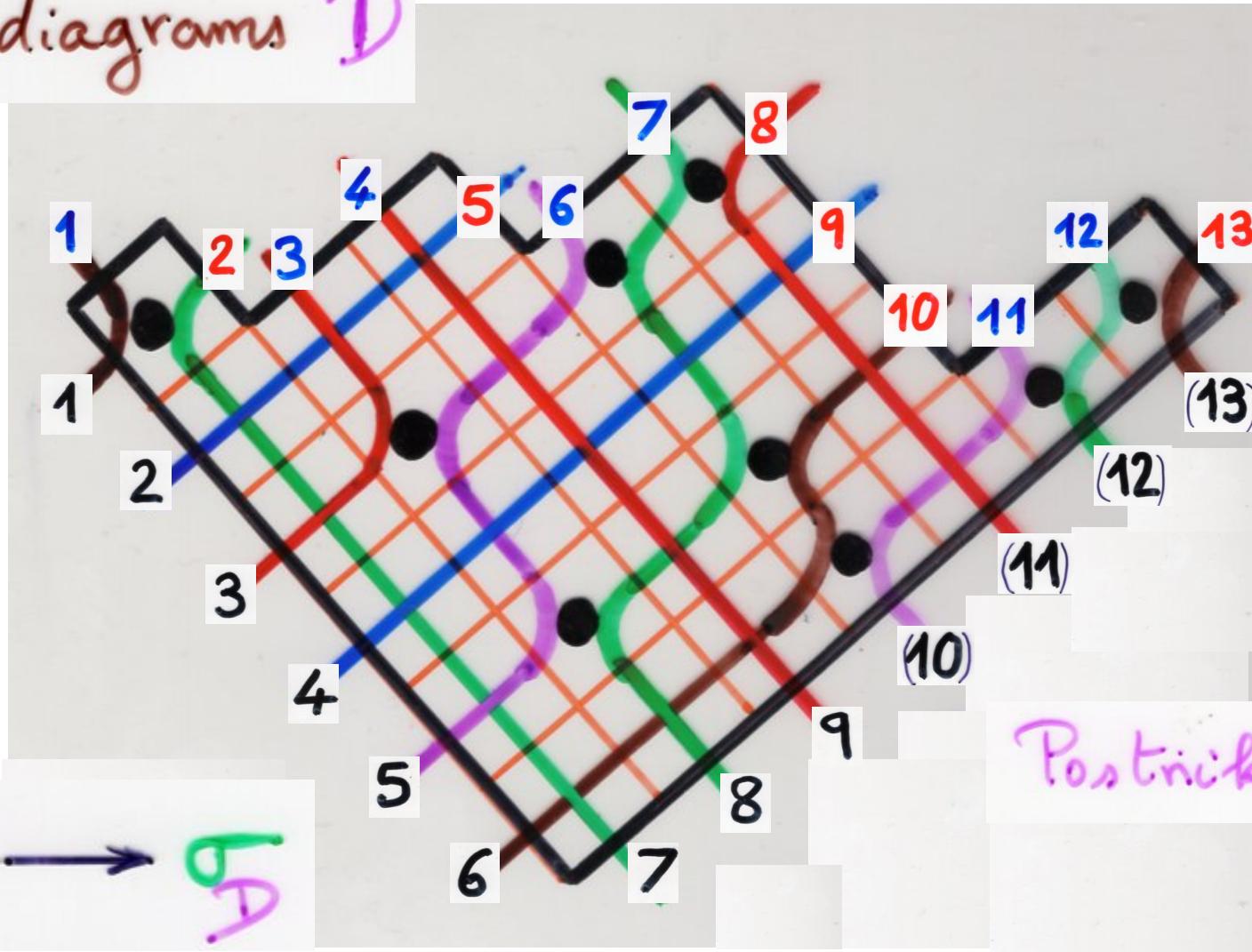
\mathbb{T} -diagrams D



$D \rightarrow \mathfrak{D}_D$

Postnikov (2006)

T-diagrams \mathcal{D}



1 2 3 4 5 6 7 8 9 (10) (11) (12) (13)

1 5 3 9 6 10 2 7 4 11 8 12 13

Postnikov (2006)

bijection

$$\mathcal{D} \rightarrow \mathfrak{S}_{\mathcal{D}}$$

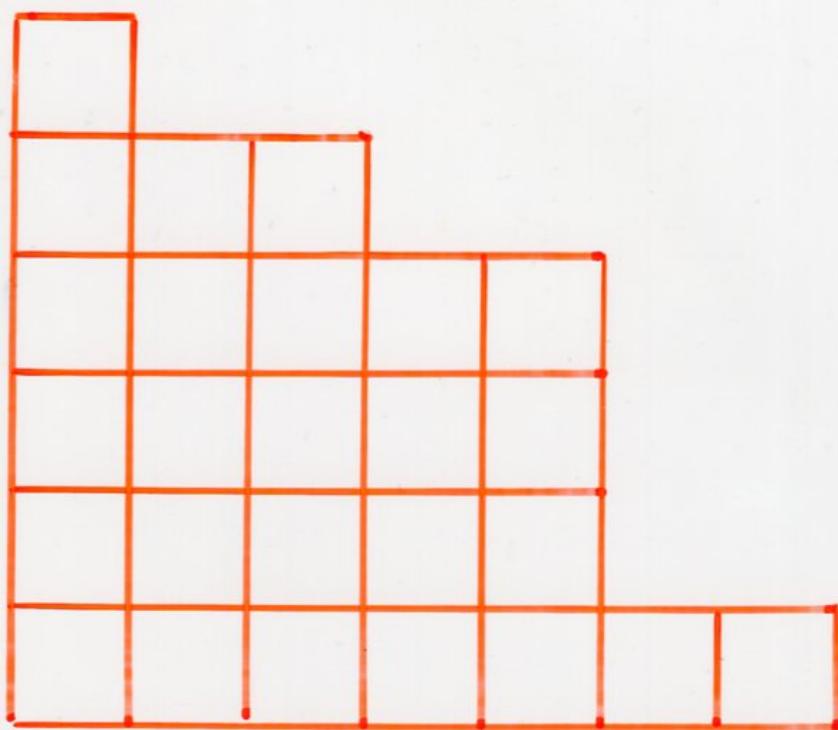
7-diagrams \mathcal{D}



shape λ

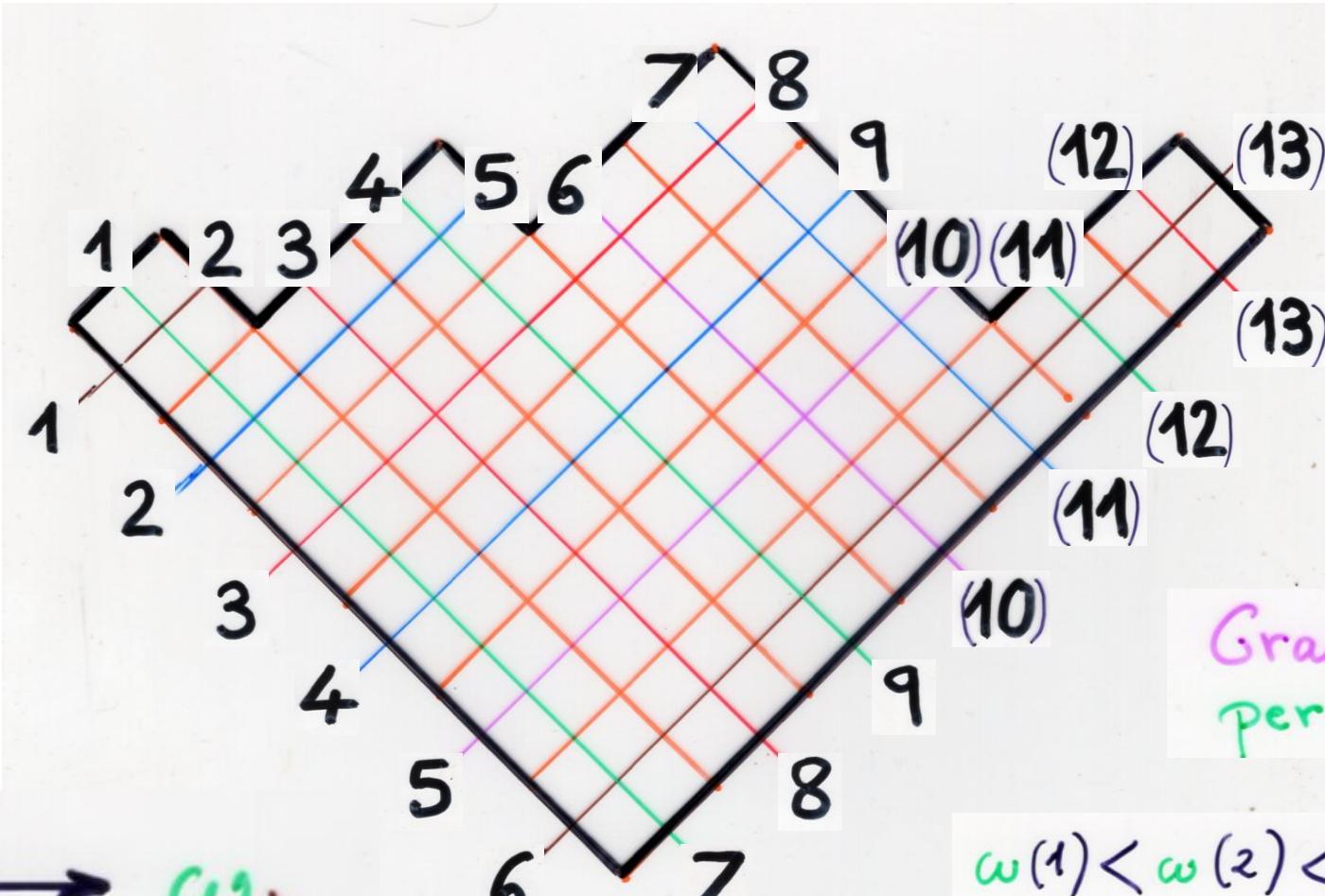
permutation $\sigma \in \mathfrak{S}_n$
 $\sigma \leqslant \omega_{\lambda}$
(Bruhat order)

$$(k, n) \quad \lambda \subseteq (n-k)^k$$



$$\lambda \rightarrow \omega_\lambda$$

Grassmannian
permutation



Grassmannian
permutation

$$\omega(1) < \omega(2) < \dots < \omega(k)$$

$$\omega(k+1) < \omega(k+2) < \dots < \omega(n)$$

$\lambda \rightarrow \omega_\lambda$

1 2 3 4 5 6 7 8 9 (10) (11) (12) (13)

2 5 8 9 (10)(13) 1 3 4 6 7 (11) (12)

Postnikov (2006)

bijection

$$\mathcal{D} \rightarrow \mathfrak{S}_{\mathcal{D}}$$

7-diagrams \mathcal{D}

shape λ



permutation $\sigma \in \mathfrak{S}_n$
 $\sigma \leqslant \omega_{\lambda}$
(Bruhat order)

$$\text{number of } \bullet = \ell(\omega_{\lambda}) - \ell(\sigma_{\mathcal{D}})$$

Complements

reminding Ch6a, BJC2

Heaps of dimers

and the symmetric group

Symmetric group S_n

$n!$ permutations

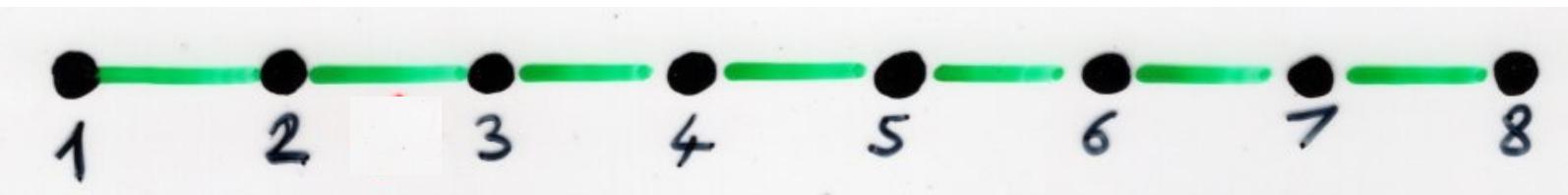
$$\sigma_i = (i, i+1) \quad i=1, 2, \dots, n-1$$

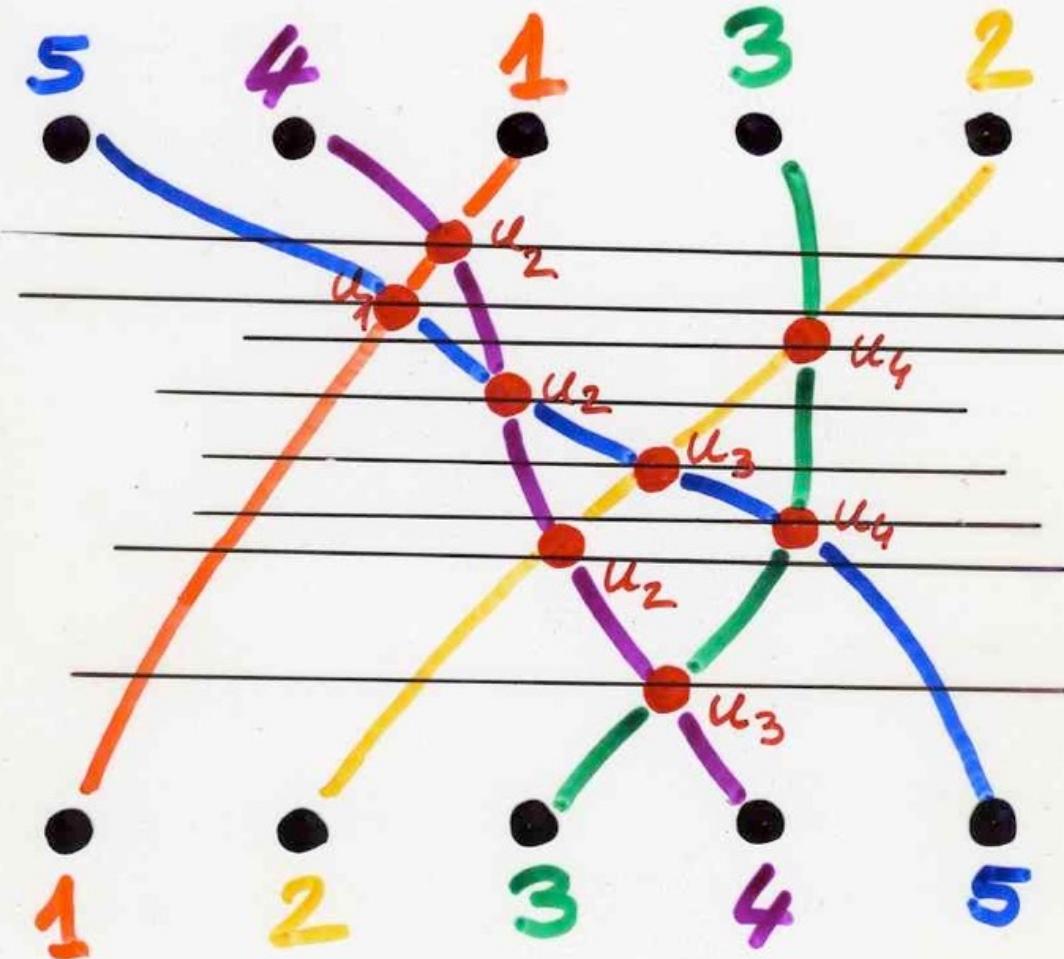
transposition of two consecutive elements

- (i) $\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2$
- (ii) $\sigma_i^2 = 1,$
- (iii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$

Moore-Coxeter
Yang-Baxter

Coxeter graph



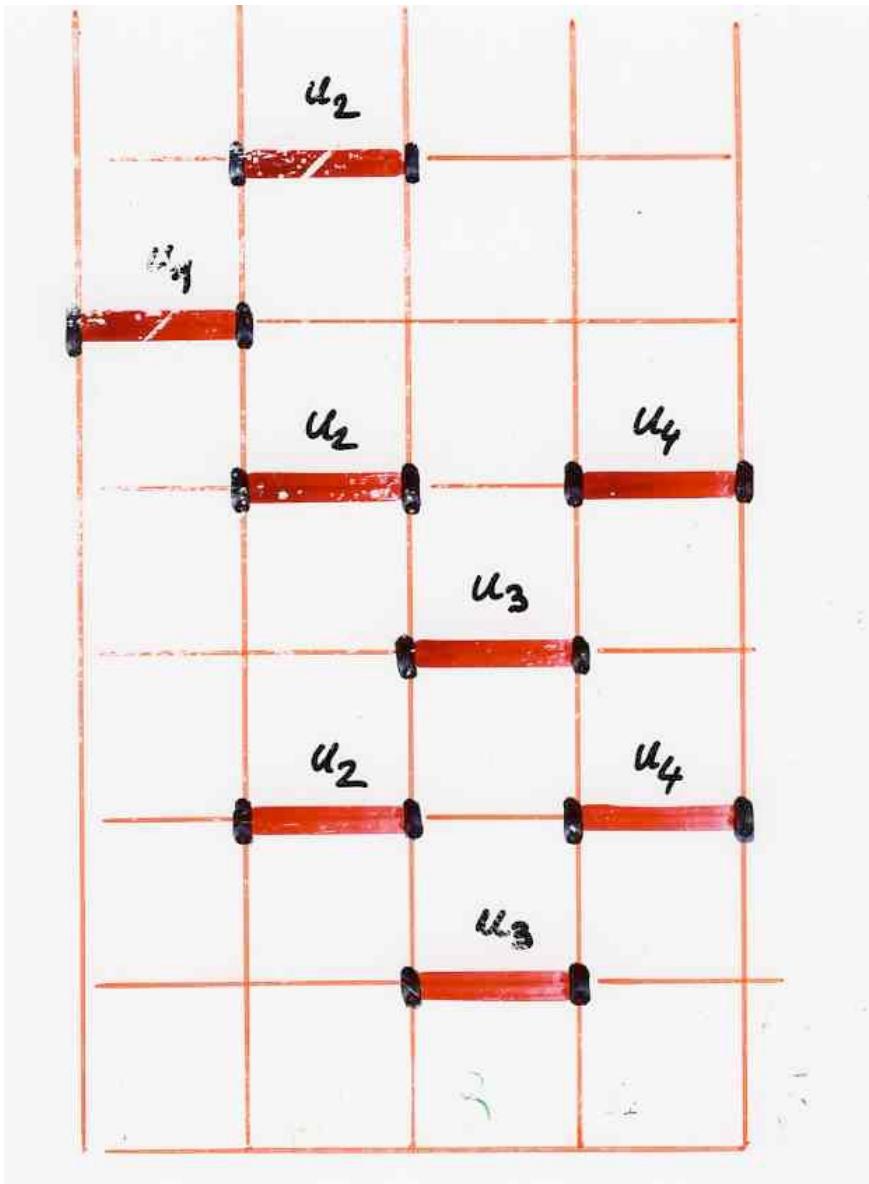


reduced decomposition
of a permutation

$$\sigma = u_{i_1} \dots u_{i_k}$$

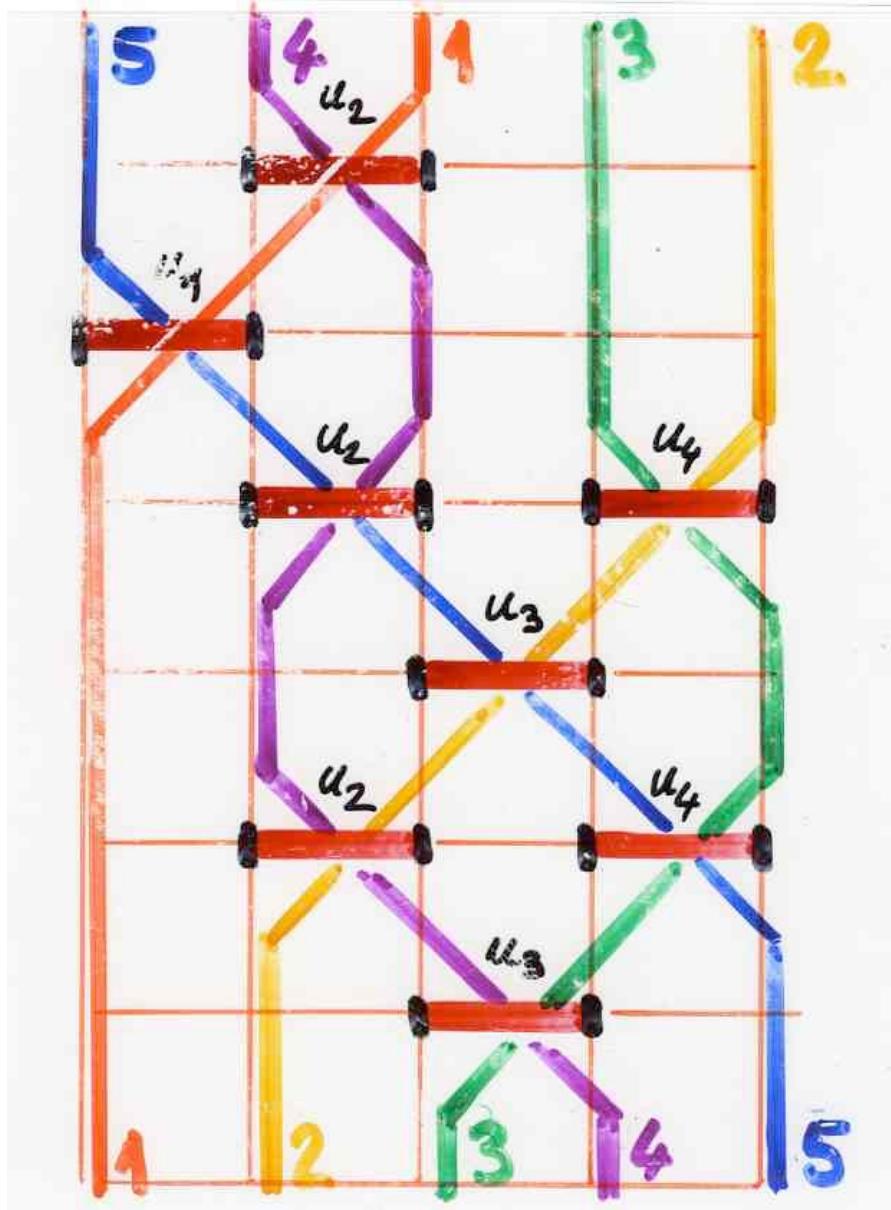
k minimum

(nb of inversion)

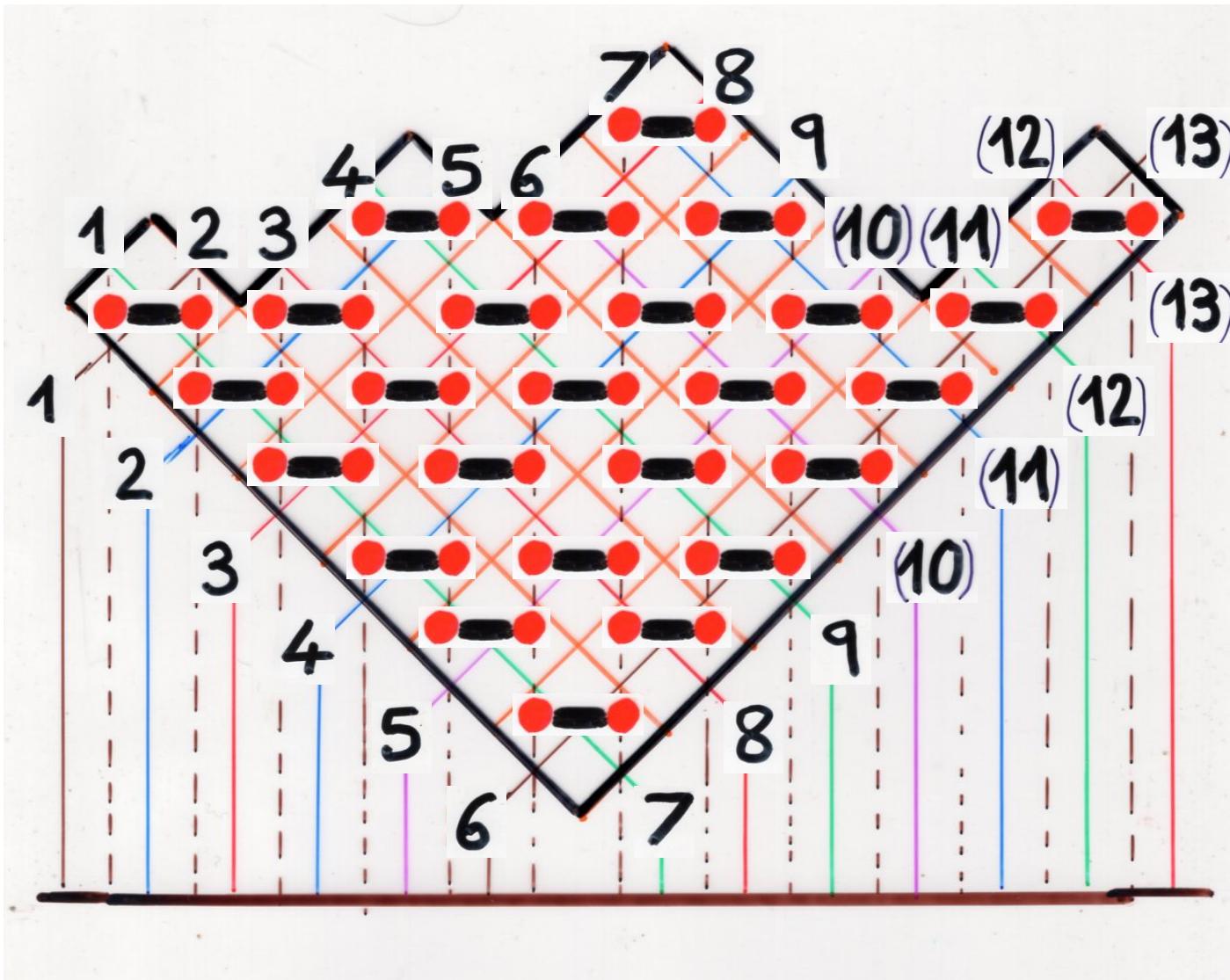


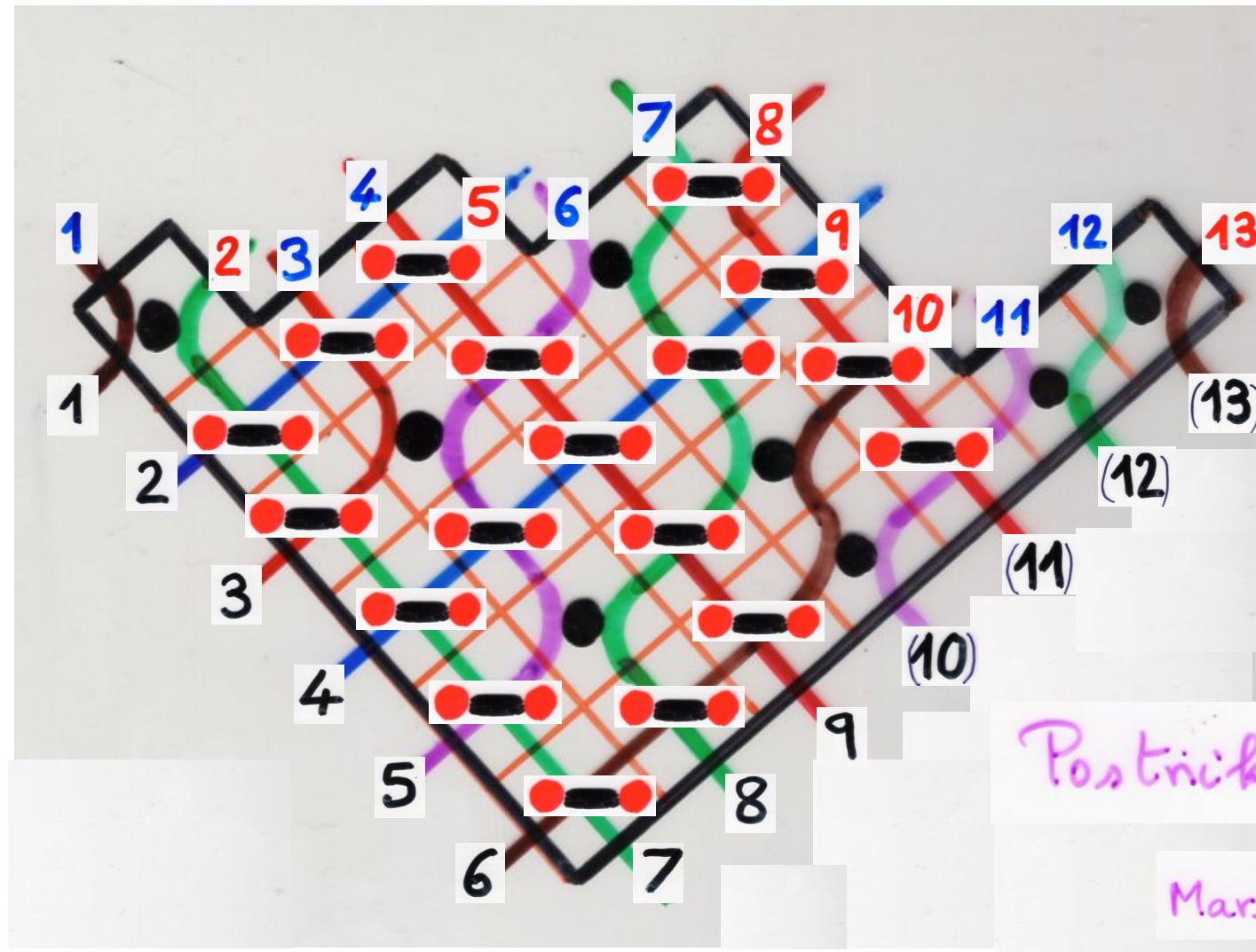
heaps of dimers
 $(i, i+1)$
 on $\{0, 1, \dots, n-1\}$
 generators $\{\tau_0, \tau_1, \dots, \tau_{n-1}\}$
 $\tau_i \tau_j = \tau_j \tau_i$
 iff $|i-j| \geq 2$

heap
 of
 dimers $[1, n]$ \rightarrow permutation S_n



heap
of
dimers $[1, n]$ \longrightarrow permutation S_n





Marsh-Rietzsch (2003)

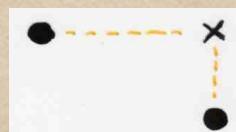
1 2 3 4 5 6 7 8 9 (10) (11) (12) (13)

1 5 3 9 6 10 2 7 4 11 8 12 13

Complements

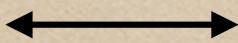
Postnikov bijections

decorated
permutations

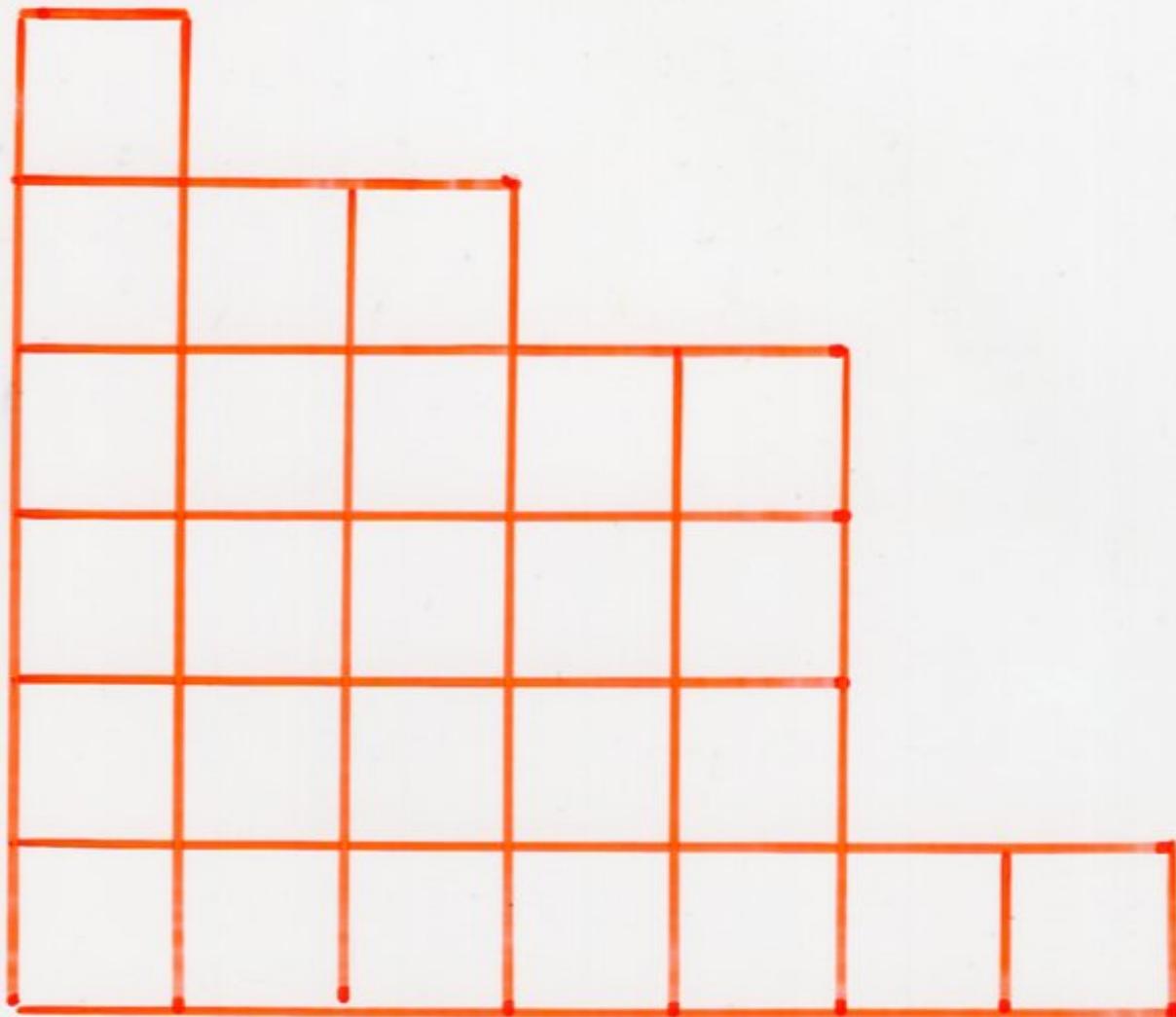


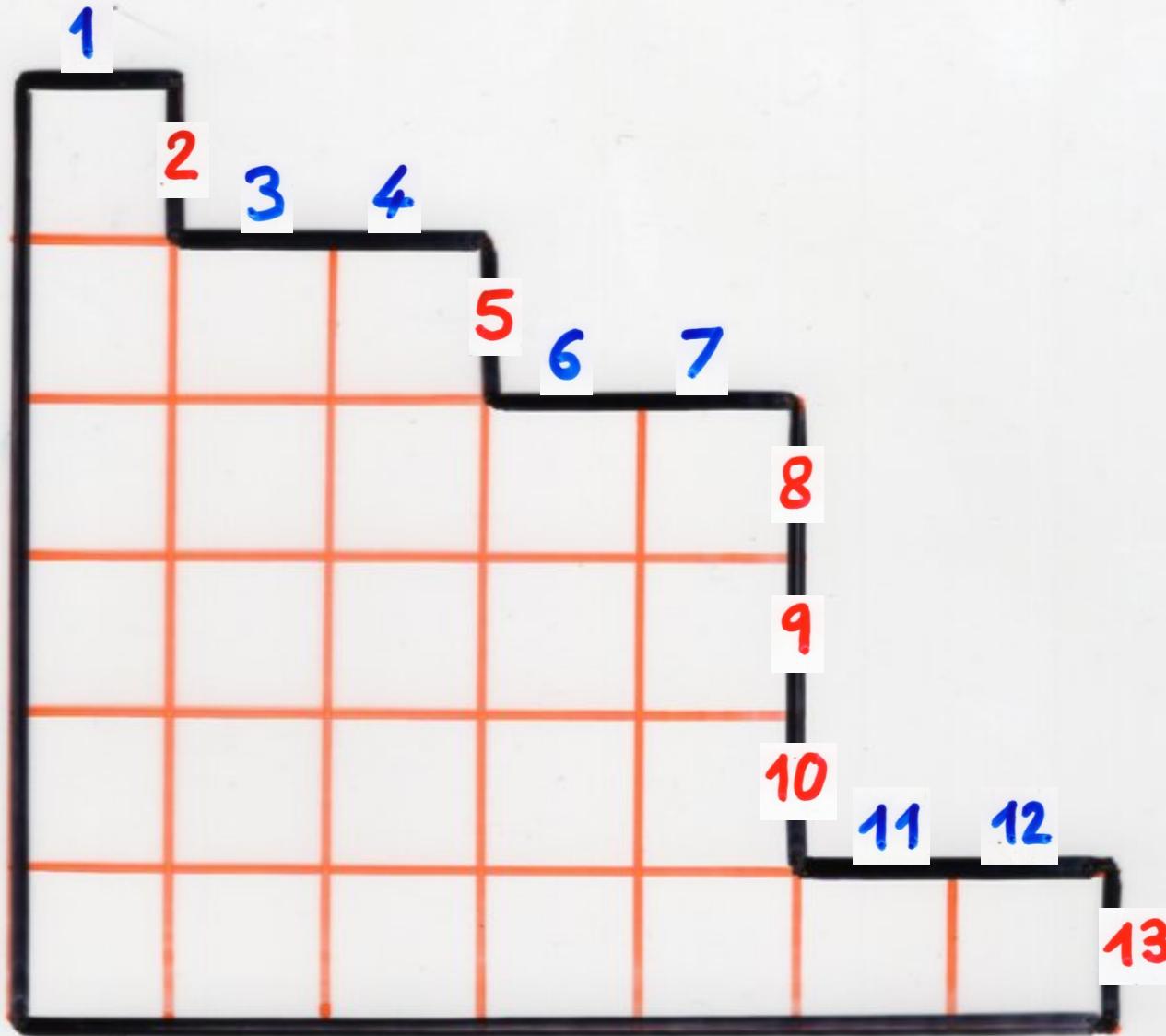
-diagrams

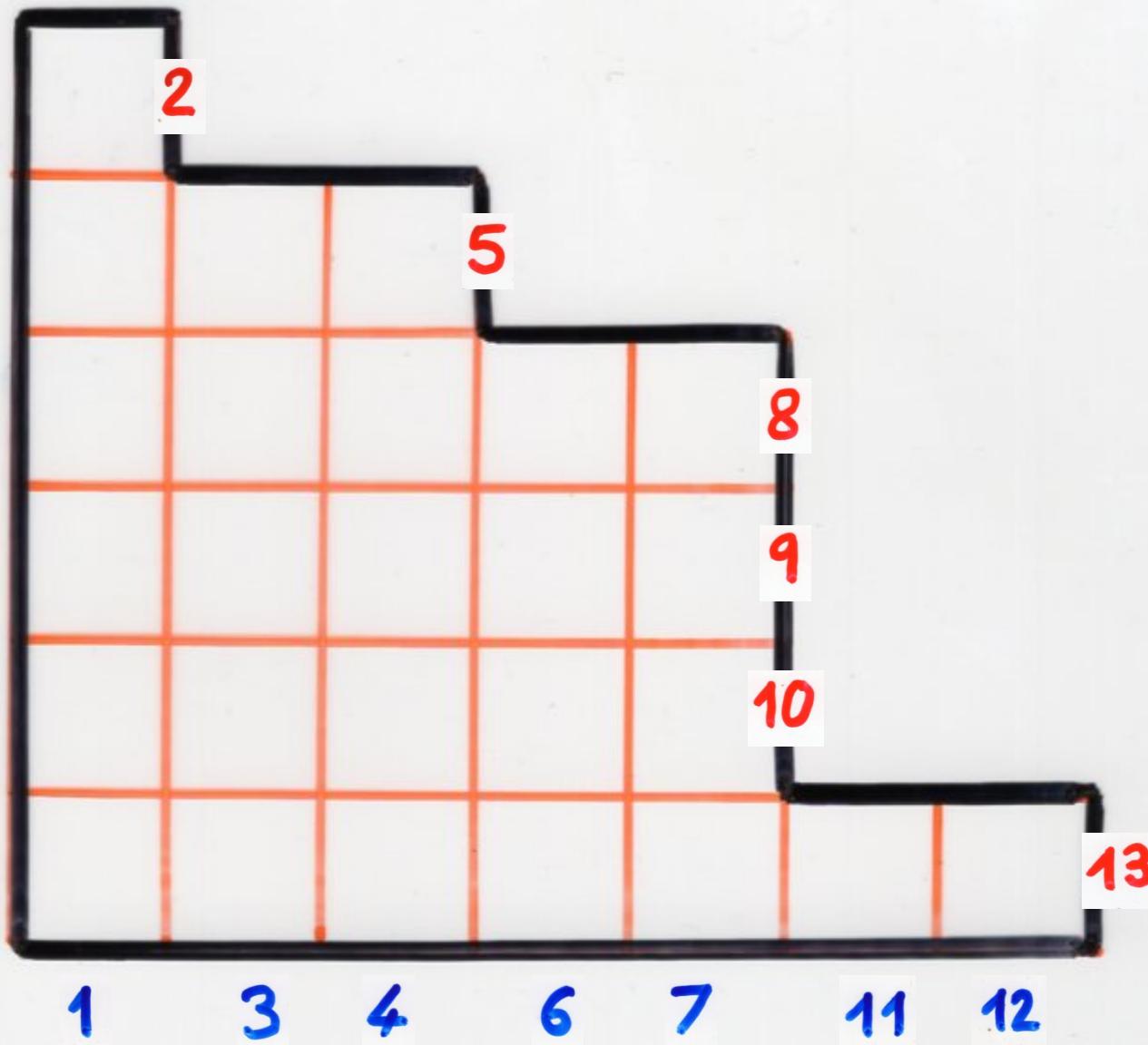
permutations



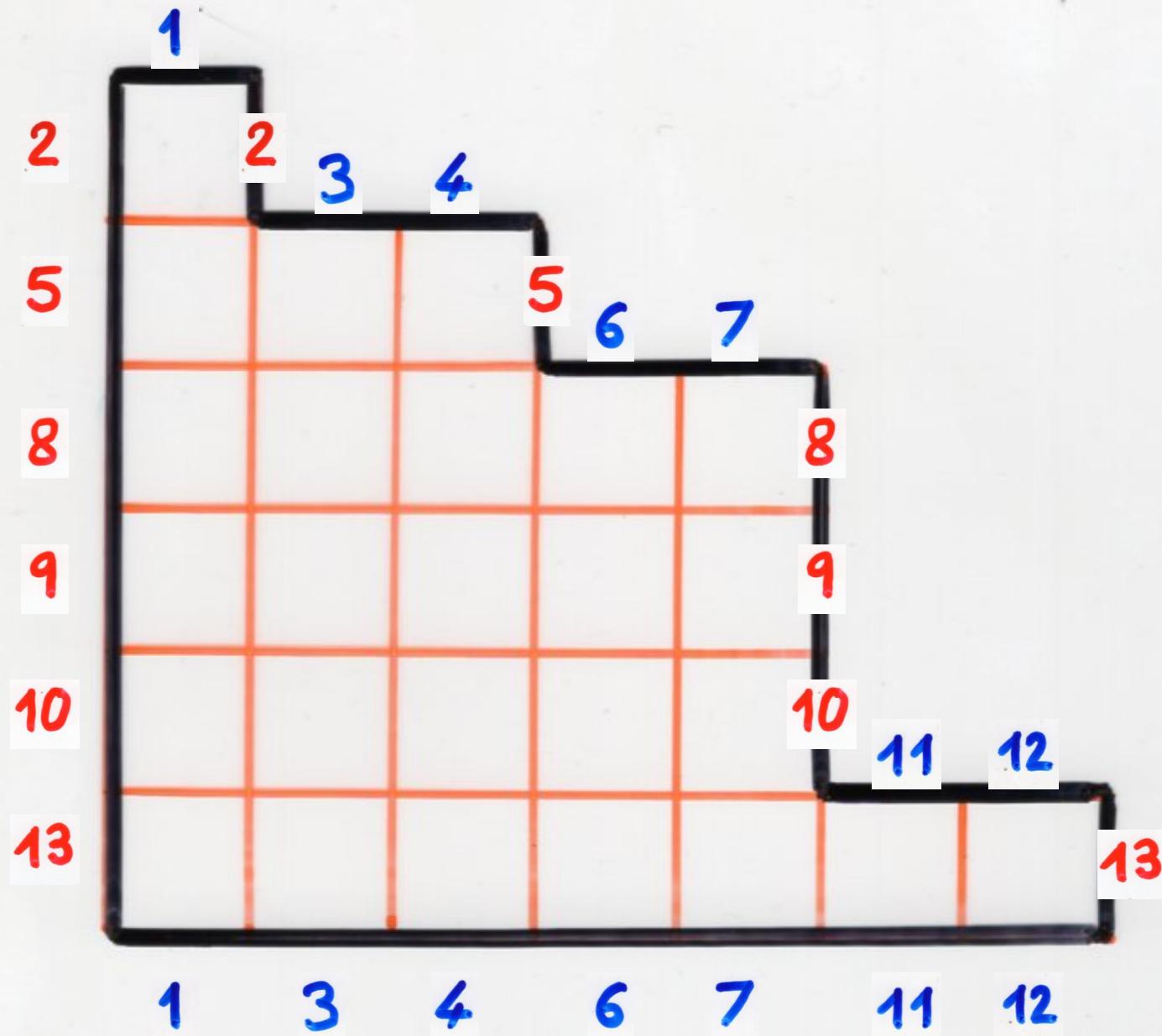
permutation
tableaux

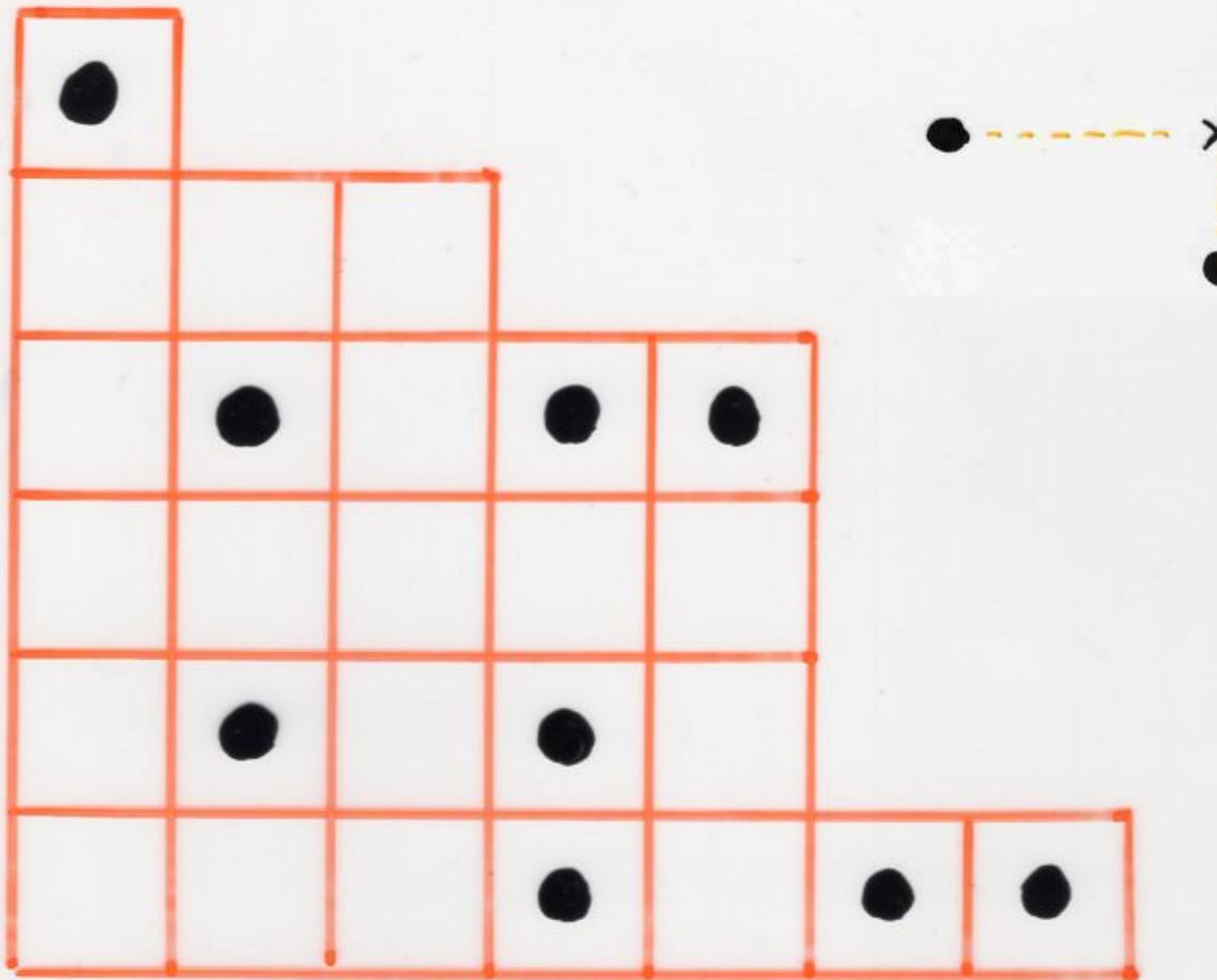




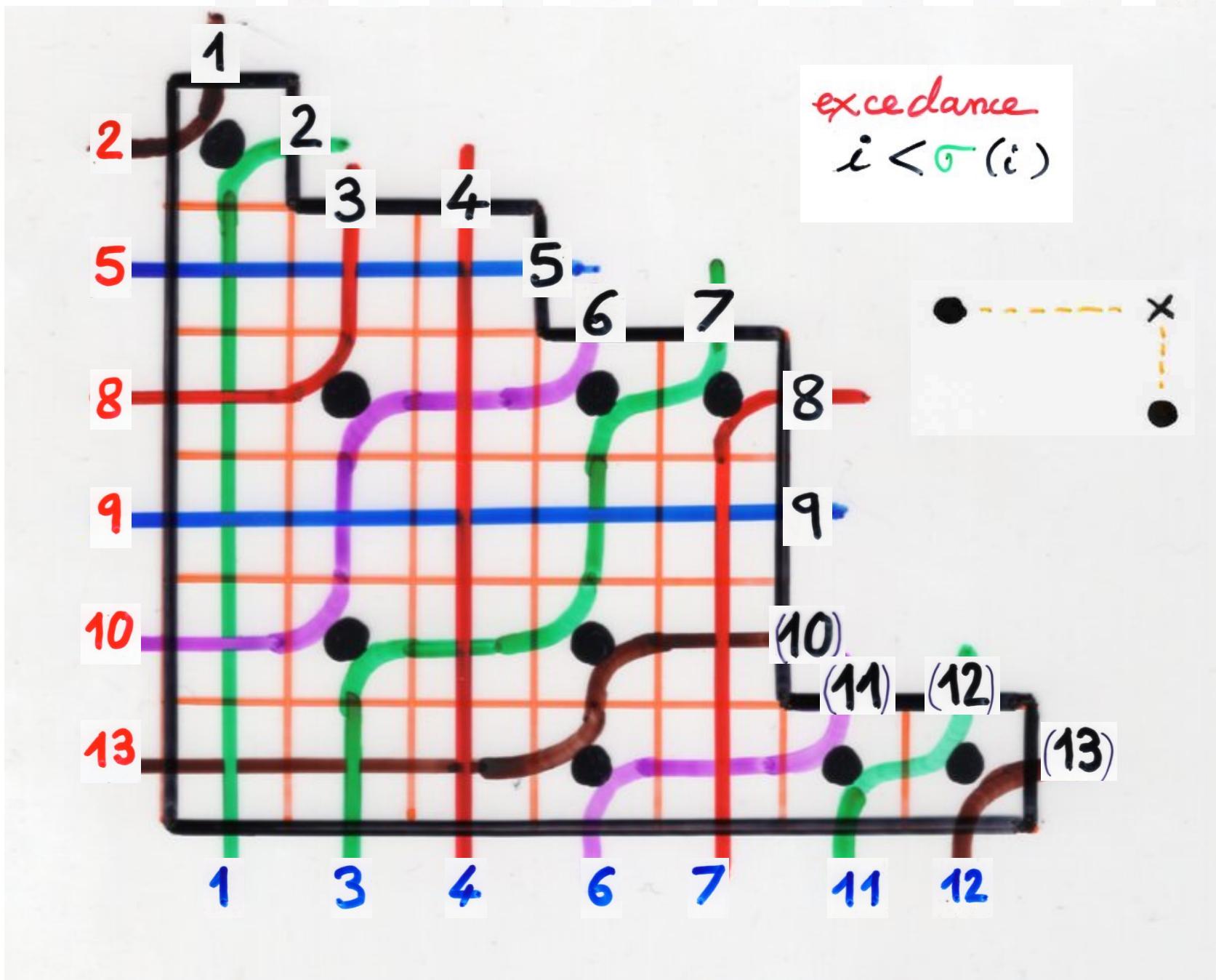








1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
2	1	8	4	5	10	3	7	9	13	6	11	12



1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
2	1	8	4	5	10	3	7	9	13	6	11	12

D \longrightarrow
 7-diagrams decorated permutations
 exceedance set
 $I(\lambda)$

exceedance
 $i < \sigma(i)$

bijection

permutations \longleftrightarrow permutation tableaux

Postnikov (2006)

Steingrimsson, Williams
(2005, 2007)

exceedances

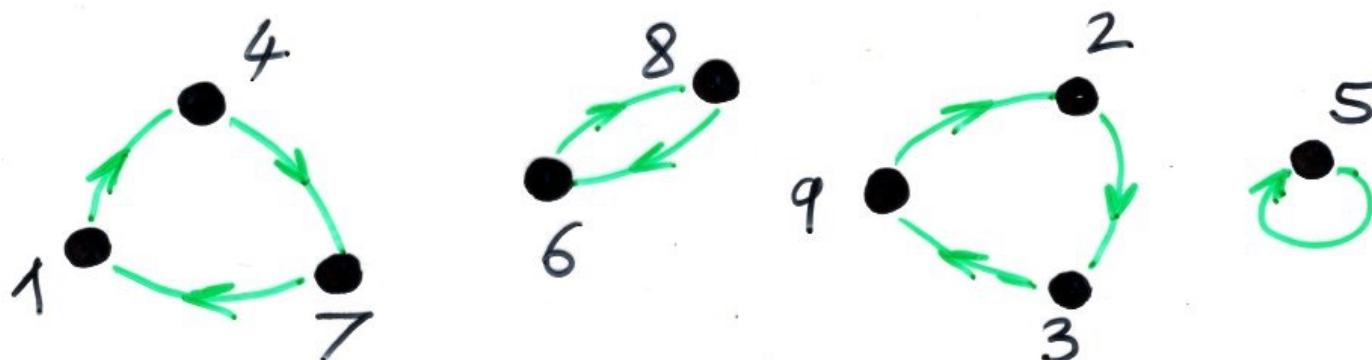
and

Genocchi shape of a permutation

a classical bijection

very classic !

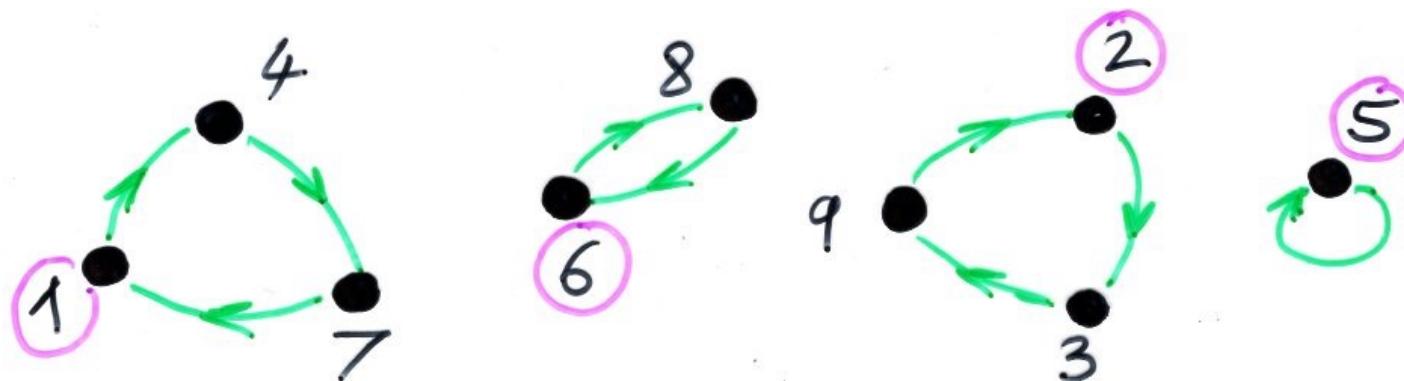
σ cycles \xrightarrow{f} word $\tau = f(\sigma)$
no notation



a classical bijection

very classic !

σ cycles \xrightarrow{f} word $\tau = f(\sigma)$
no notation



$$\tau = /6 \ 8 / 5 / 2 \ 3 \ 9 / 1 \ 4 \ 7$$

excedance

$$i < \sigma(i)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

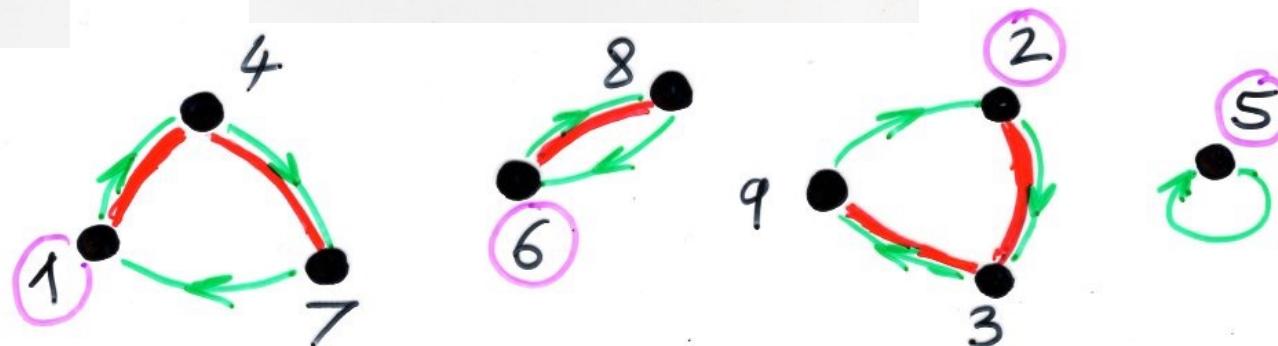
excedance

$$i < \sigma(i)$$

$$\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9) \\ | | | | | | | | | \\ 4 \ 3 \ 9 \ 7 \ 5 \ 8 \ 1 \ 6 \ 2$$

"excedance"
sequence
of σ

1 2 3 4 5 6 7 8 (9)
a a a a d a d d



Genocchi sequence
of τ

$$\tau = /6-8/5/2-3-9/1-4-7$$

The direct bijection

subexceedant
functions



Tree-like tableaux

Two bijections

- from a combinatorial representation
of the PASEP algebra (X.V., 2008)

equivalent to a bijection
Corteel, Nadeau (2007)

(with permutation tableaux)

Steingrimsson, Williams
(2005, 2007)

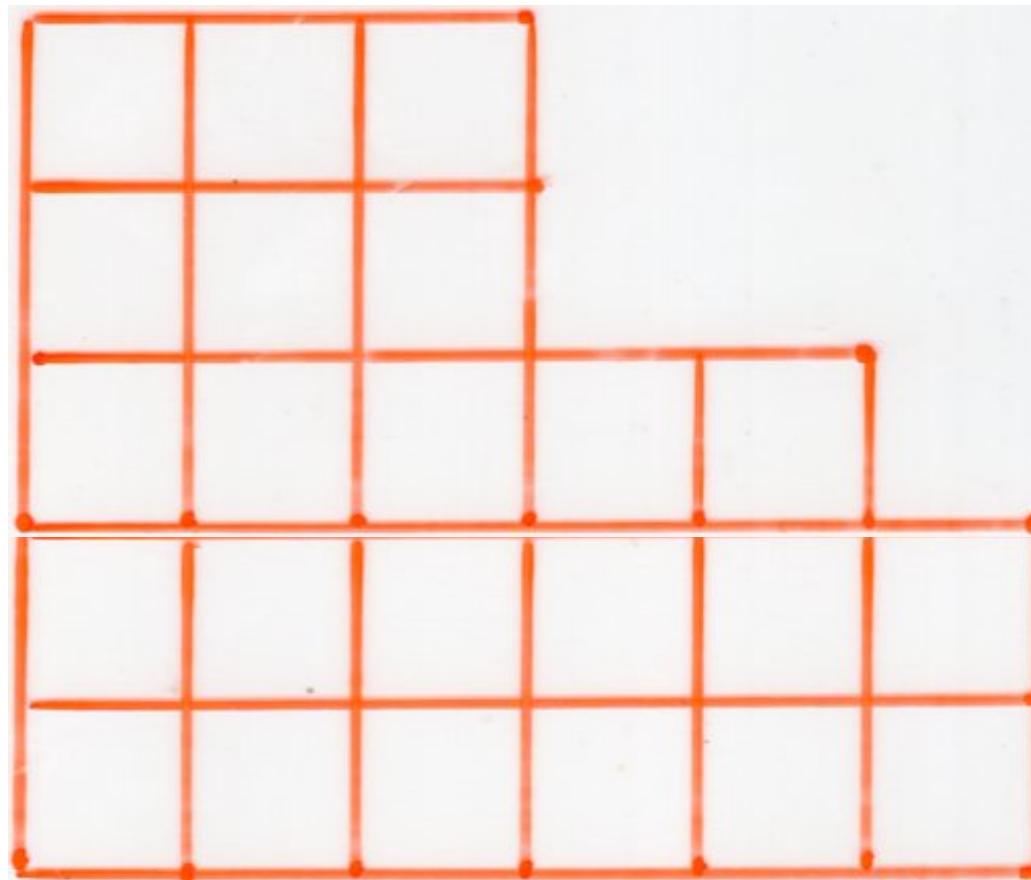
Postnikov

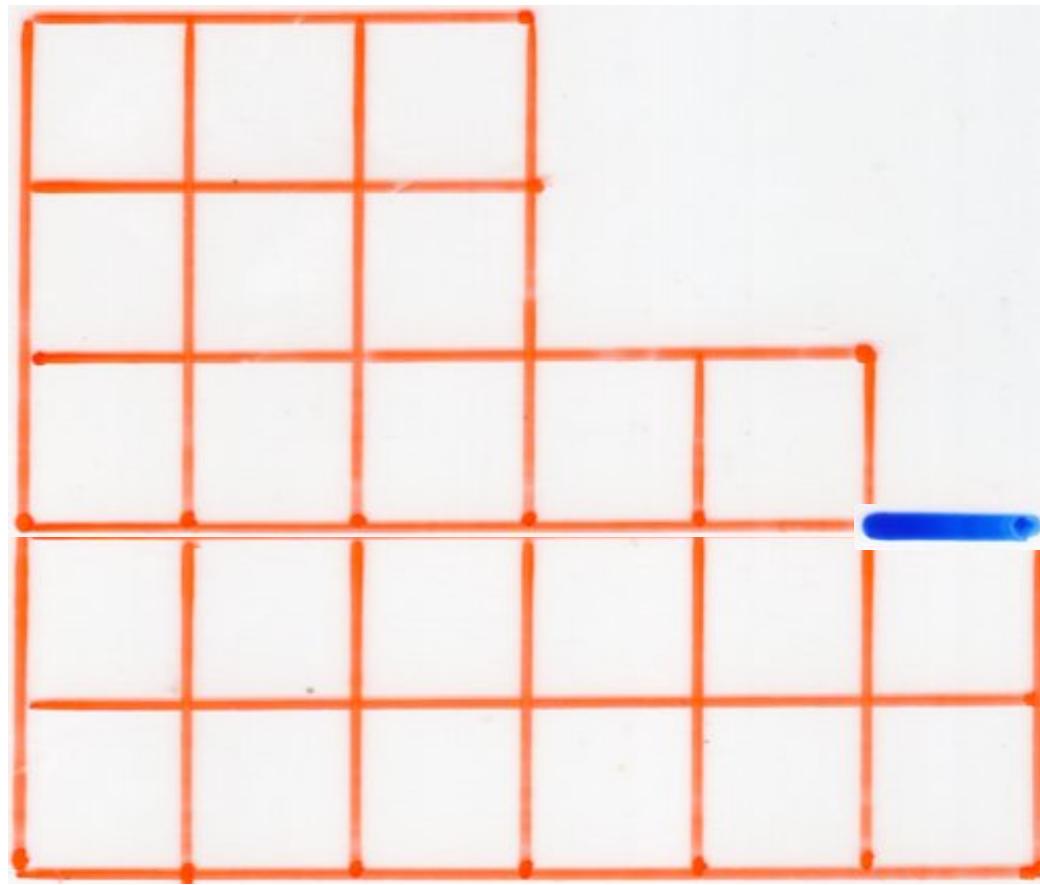
- direct bijection (with tree-like tableaux)
Aval, Boussicault, Nadeau (2011)

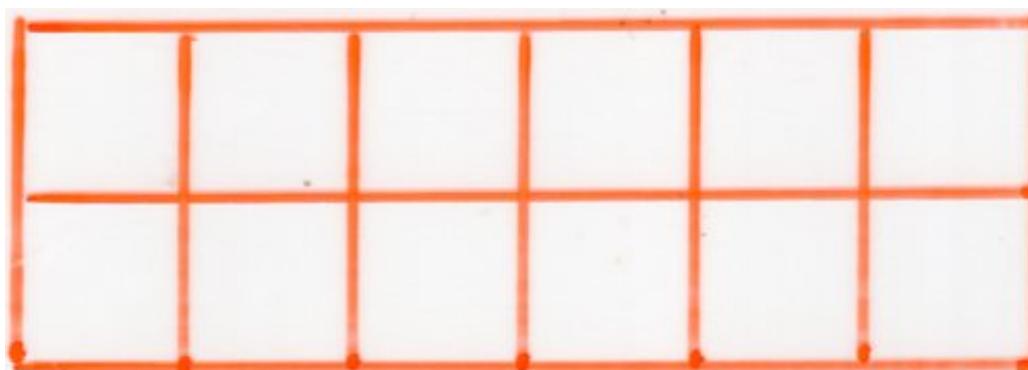
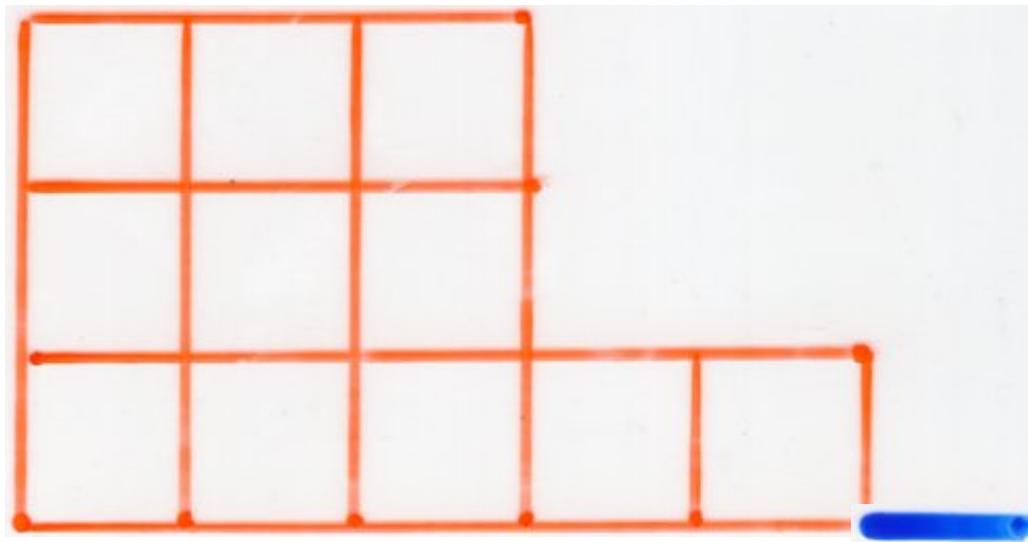
tableaux
size $(n+1)$ \leftrightarrow (tableaux
size n , $1 \leq i \leq n+1$)

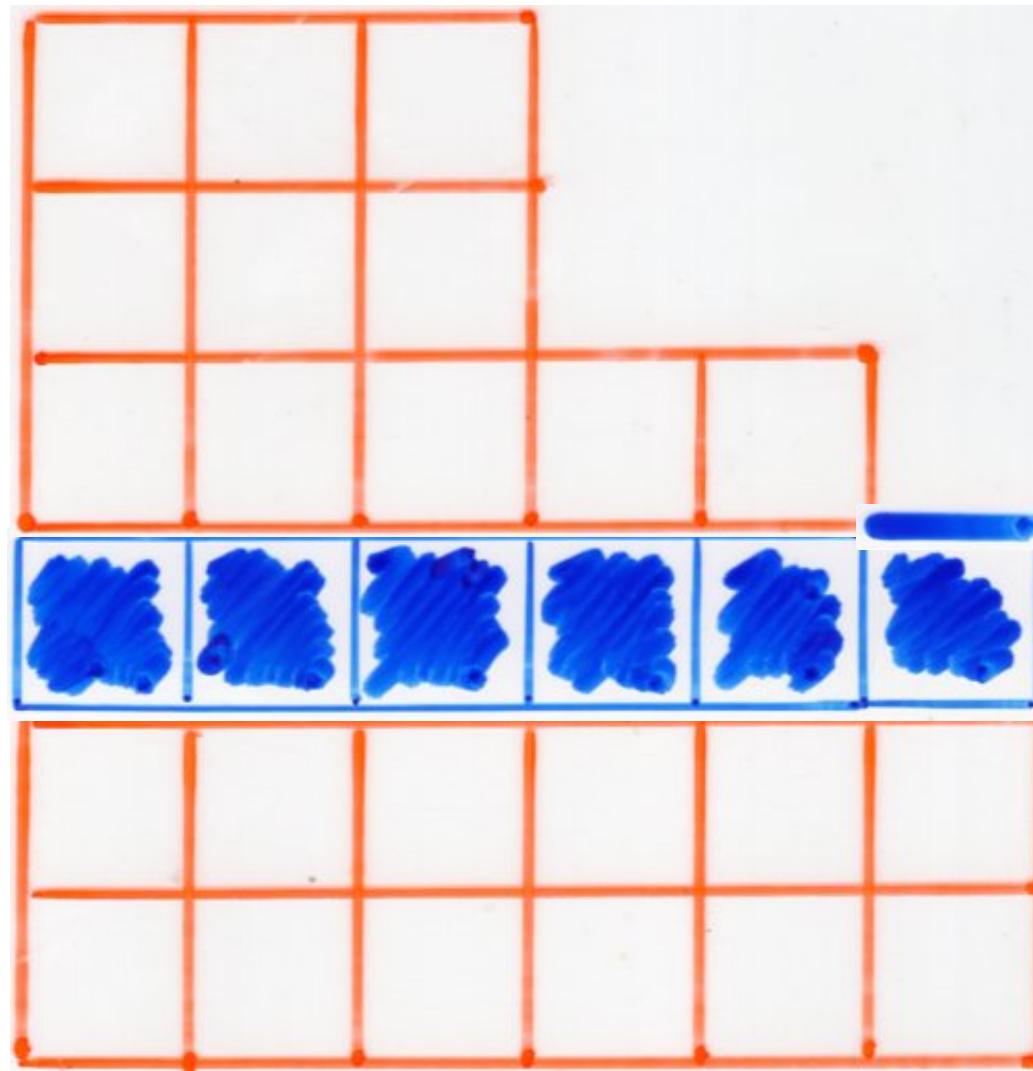
$(n+1)!$

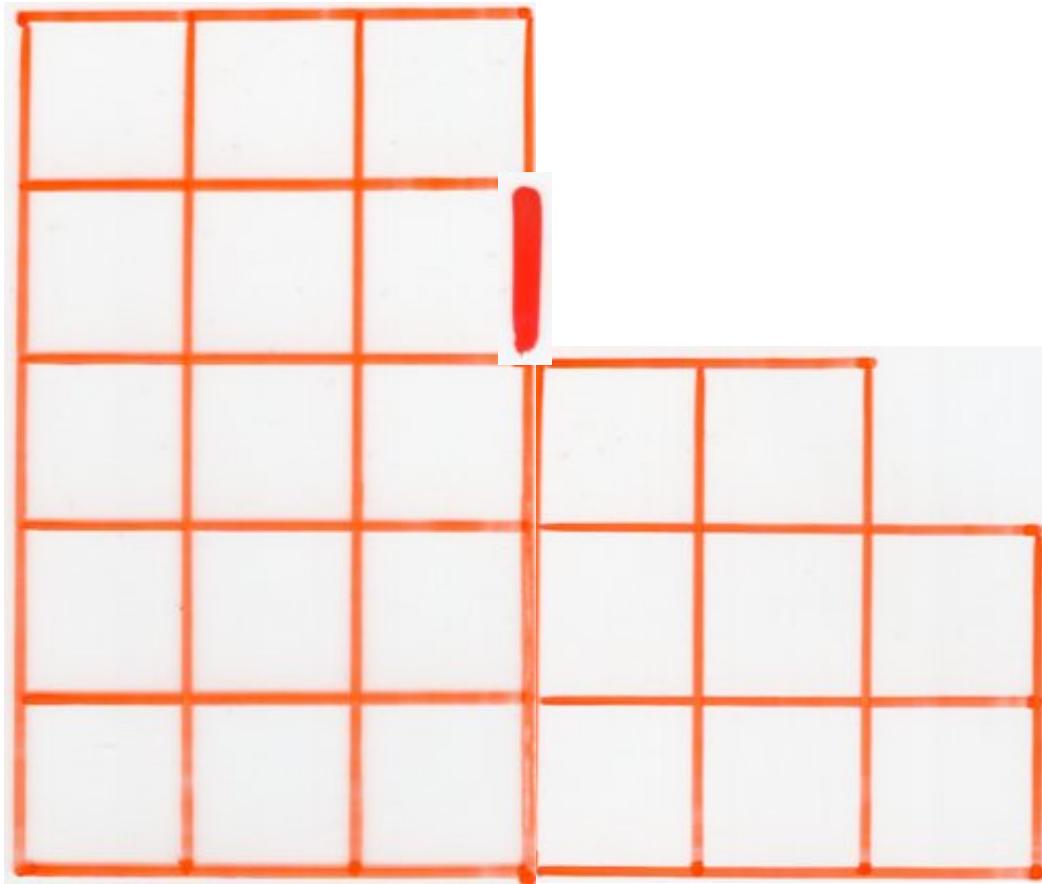
row/column insertion
in a Ferrers diagram

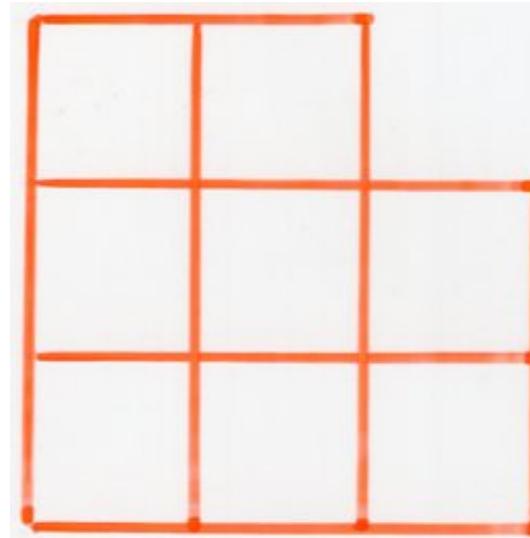
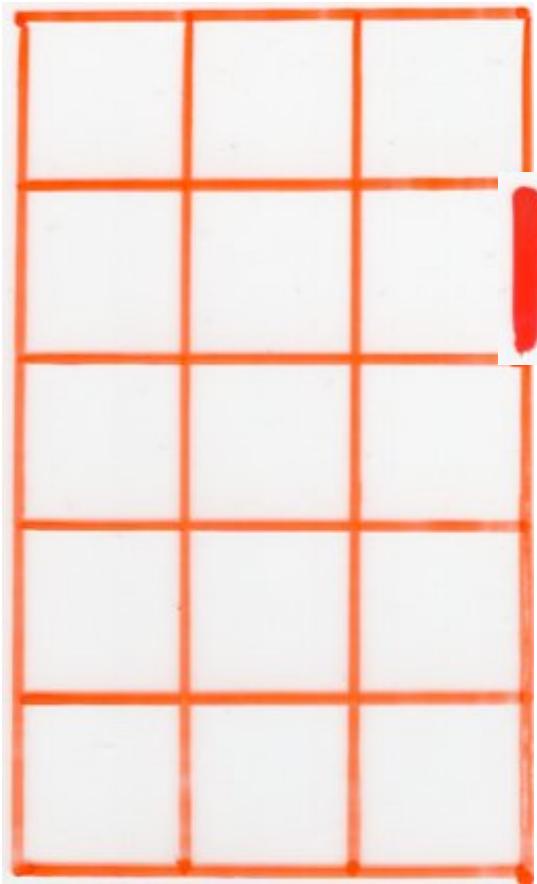


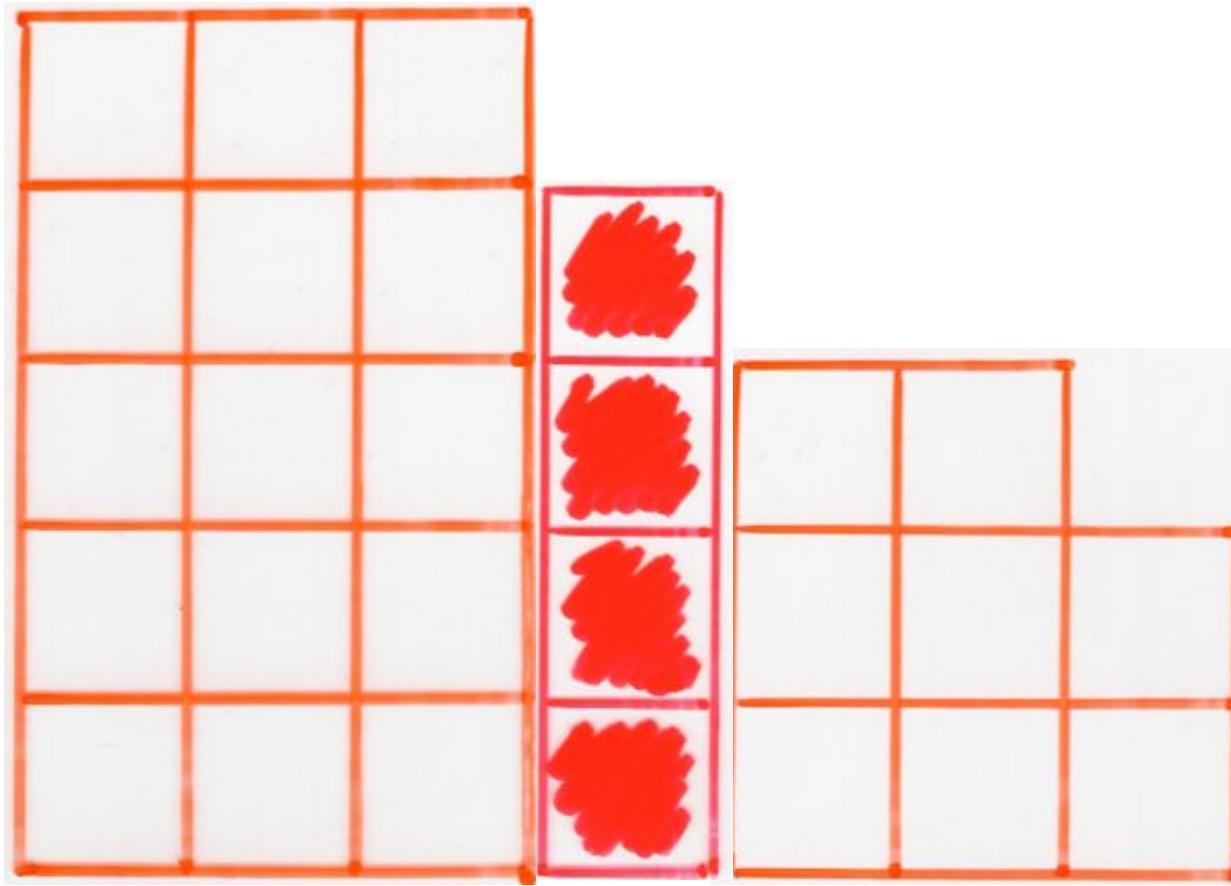




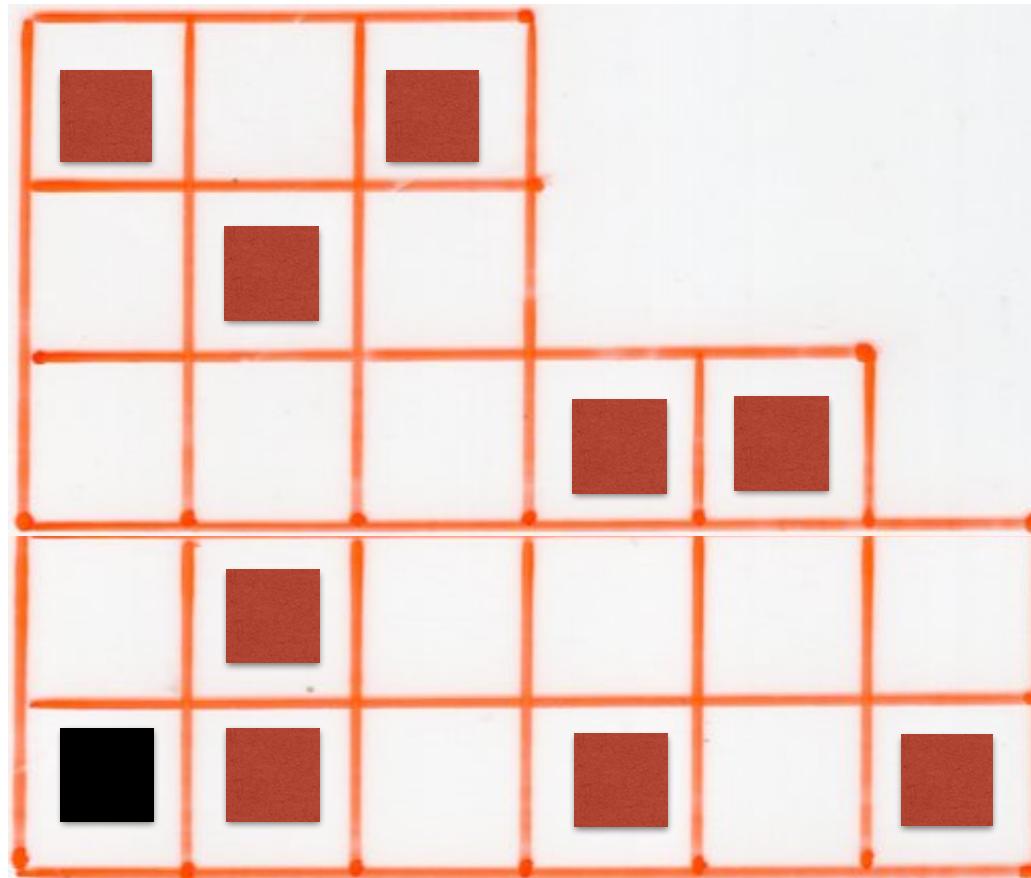




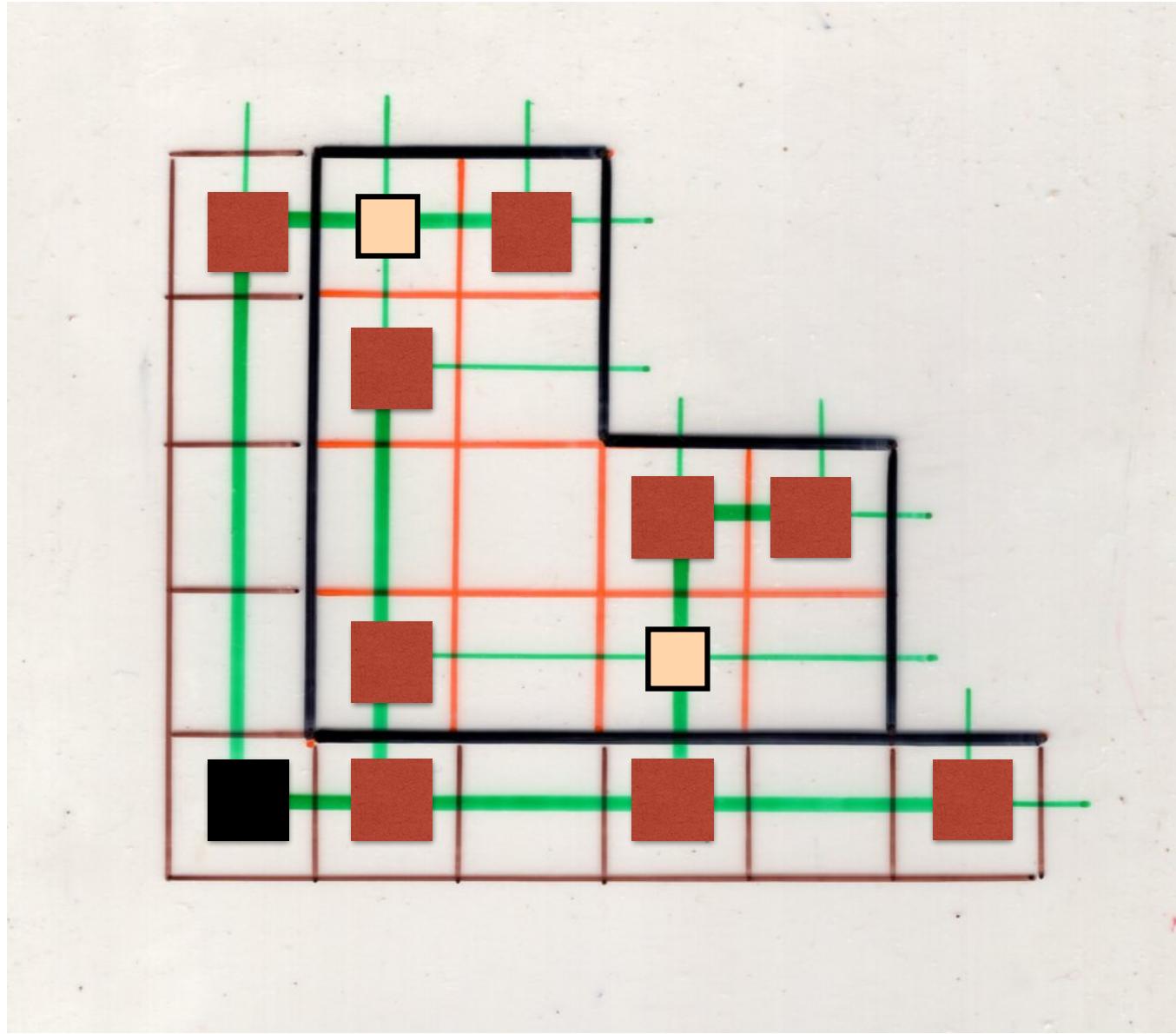




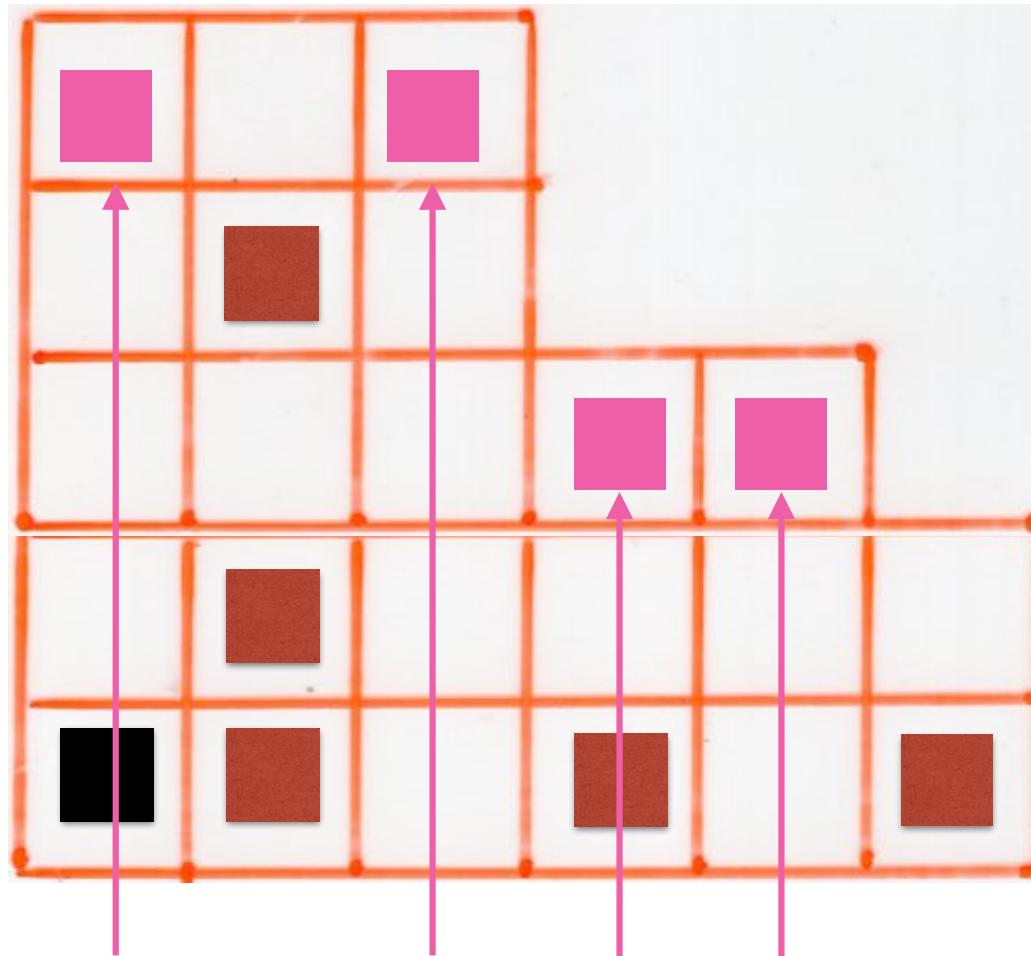
special point
of a tree-like tableau



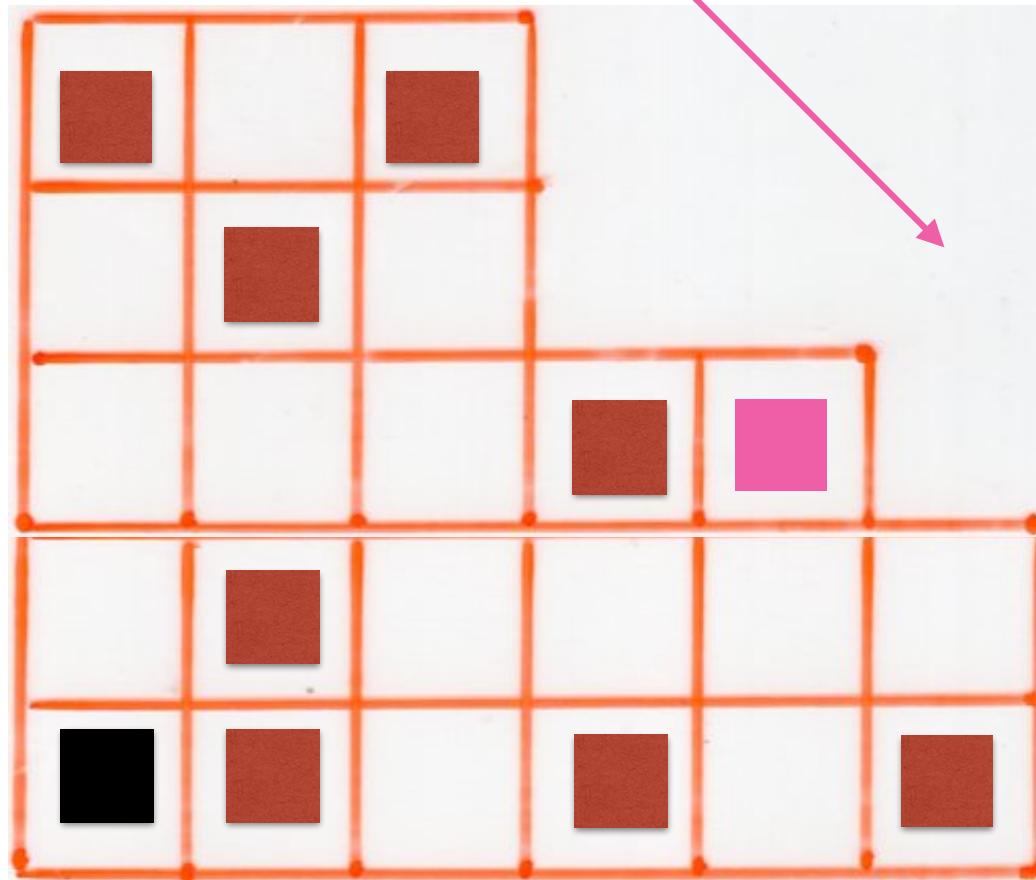
tree-like
tableaux



tree-like
tableaux



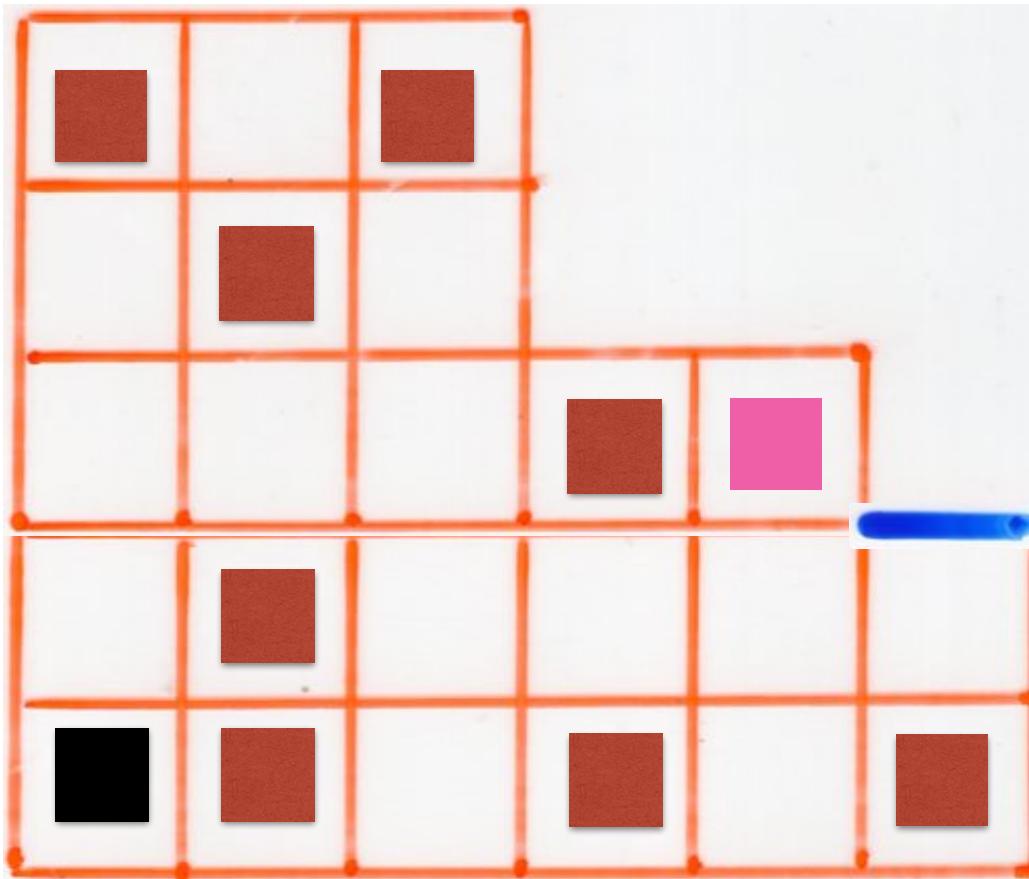
« end point »
in each column



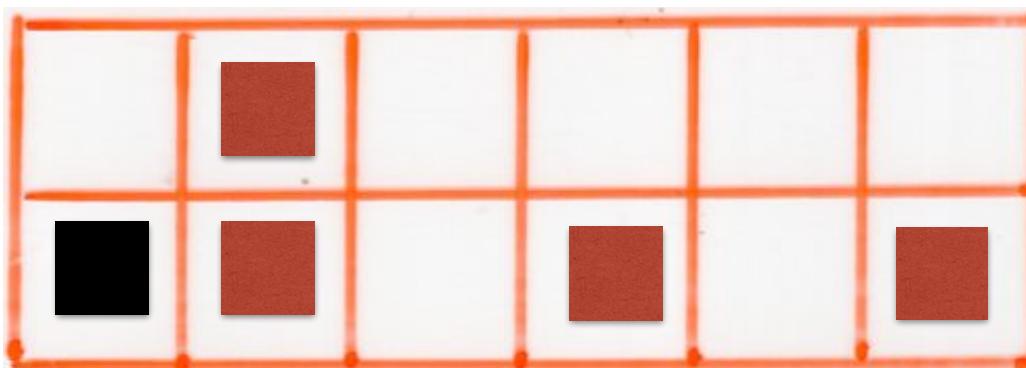
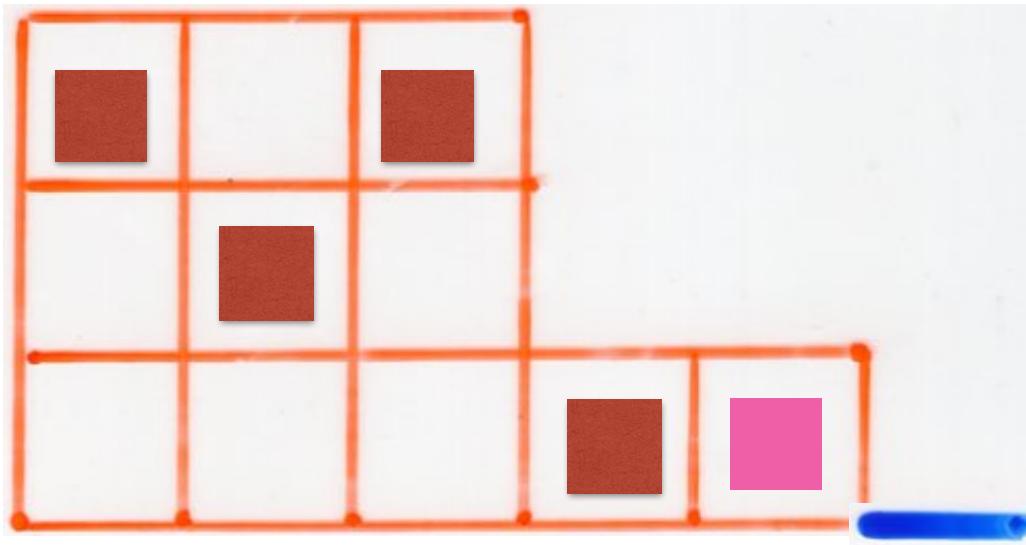
special point
of a tree-like
tableau

Insertion algorithm
in a tree-like tableau

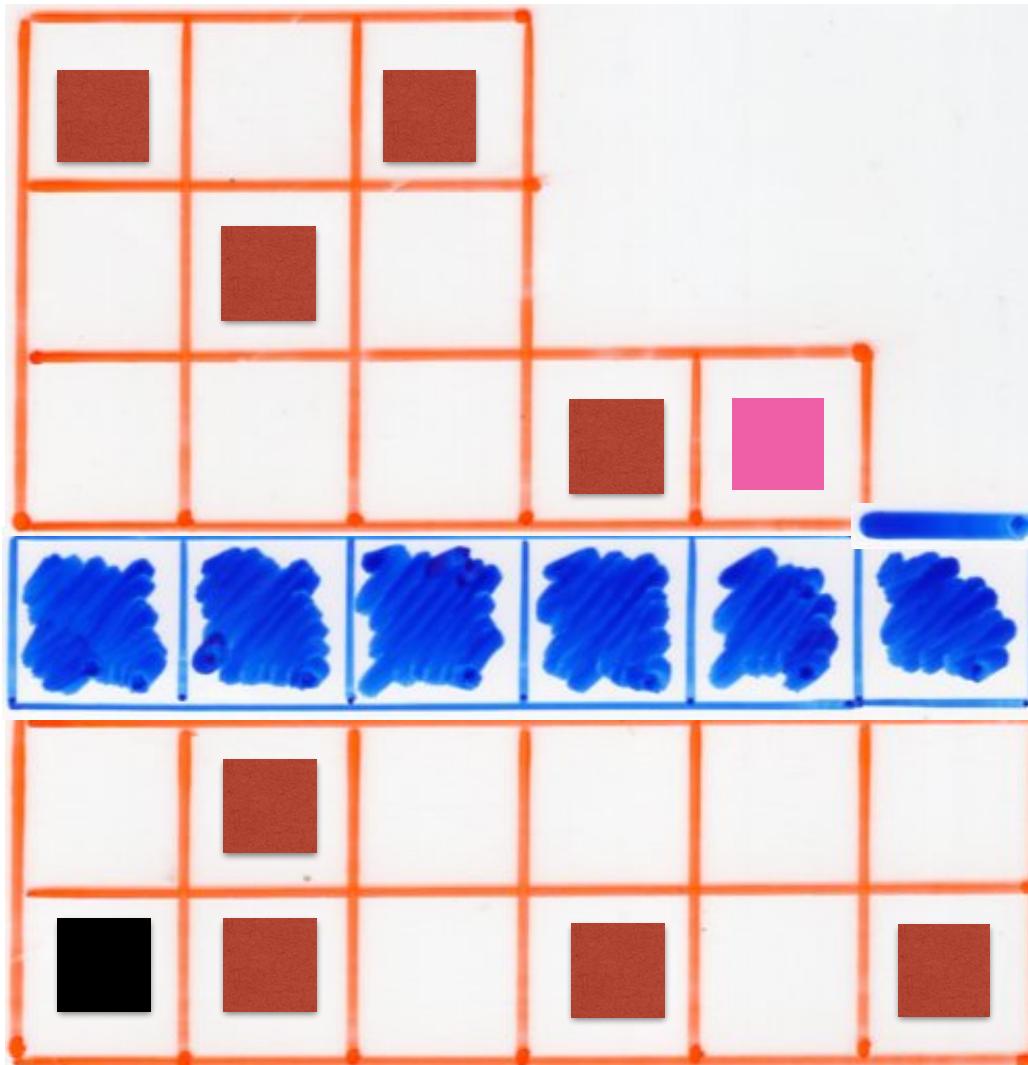
Example 1



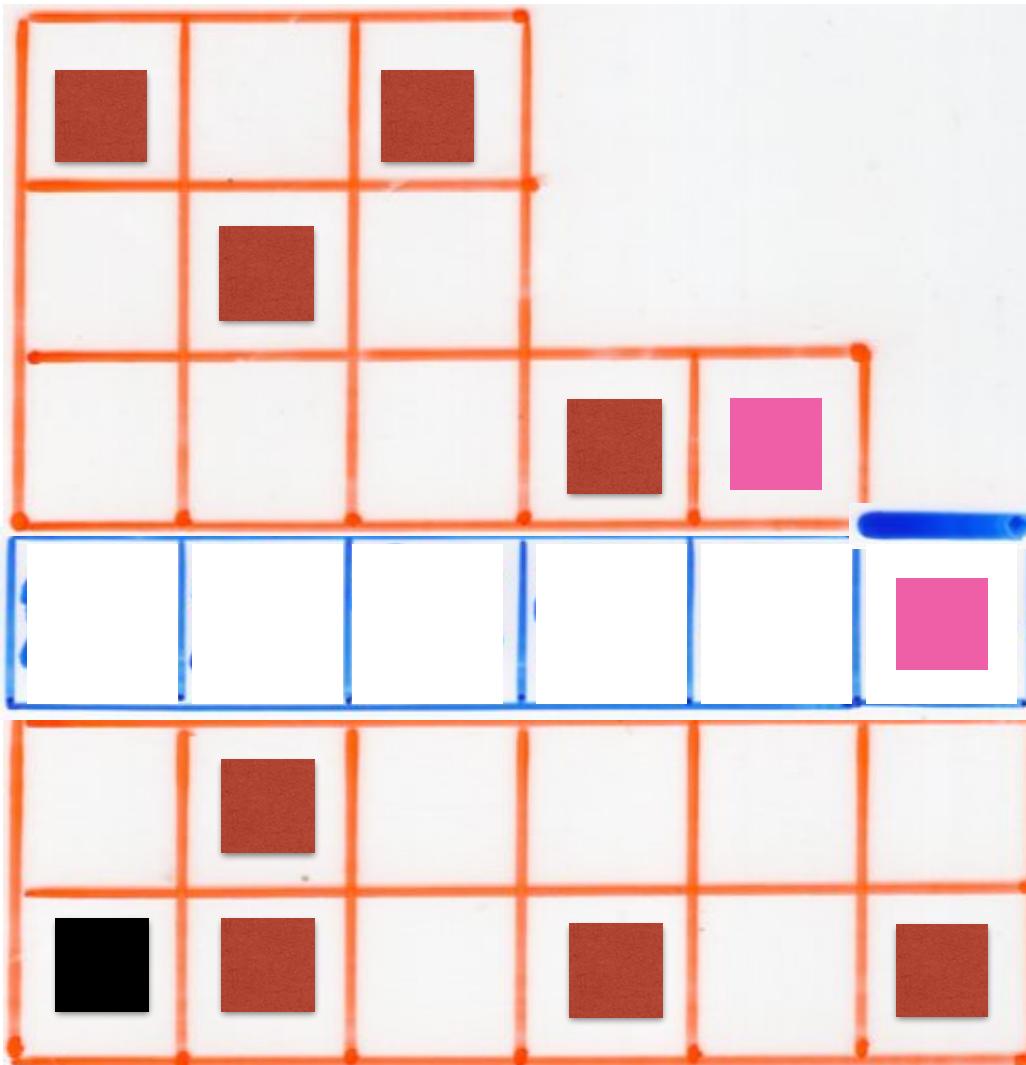
Example 1



Example 1

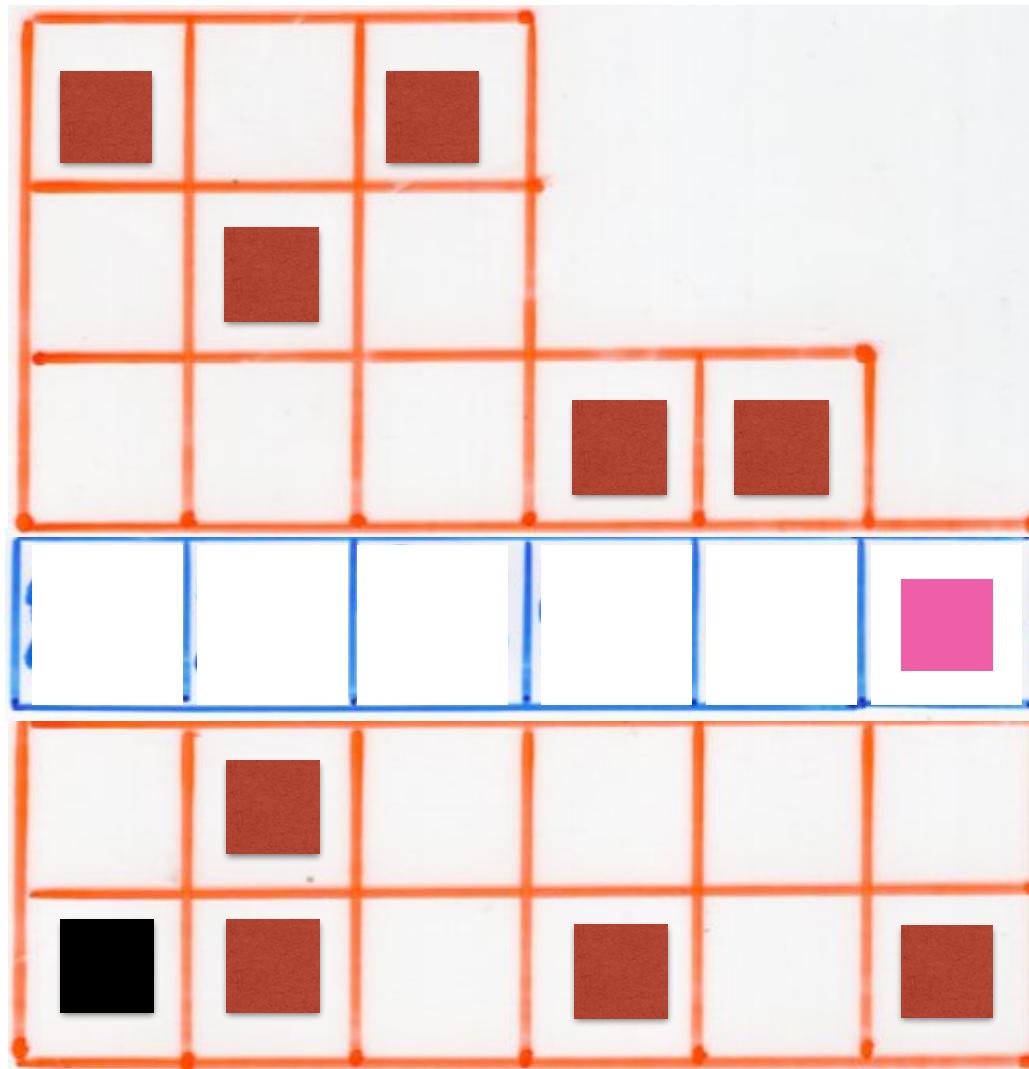


Example 1

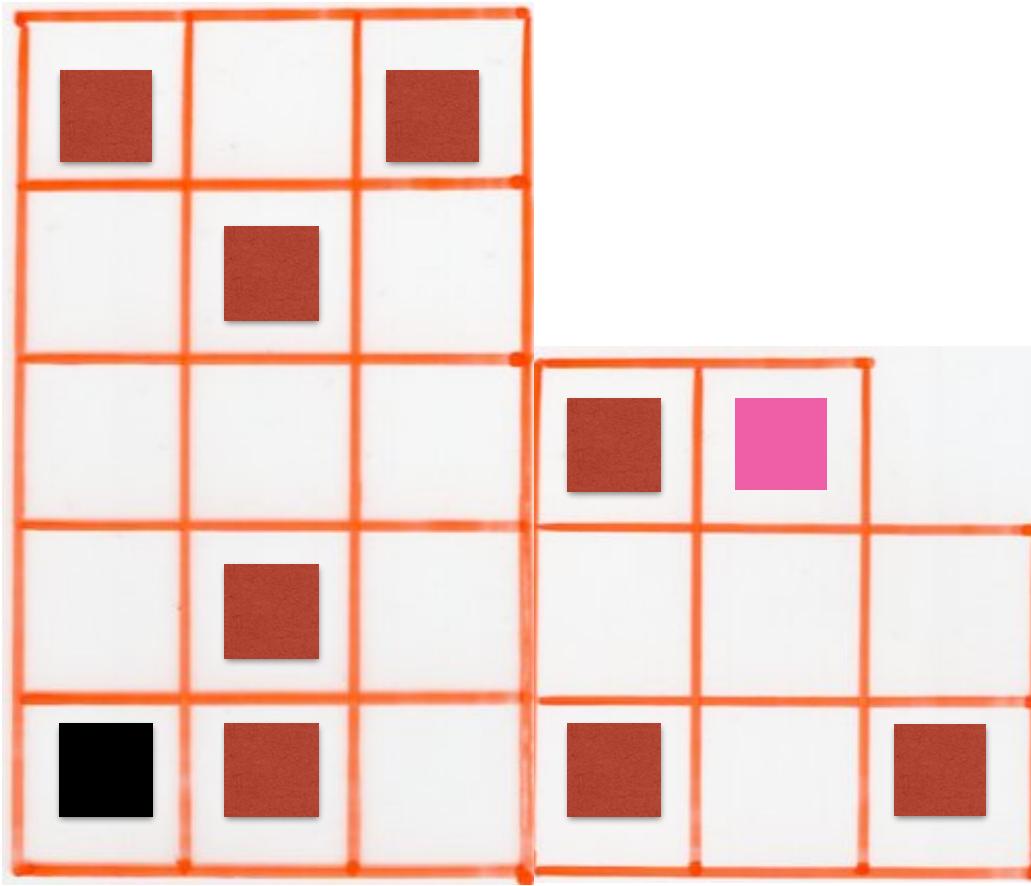


Case (i)

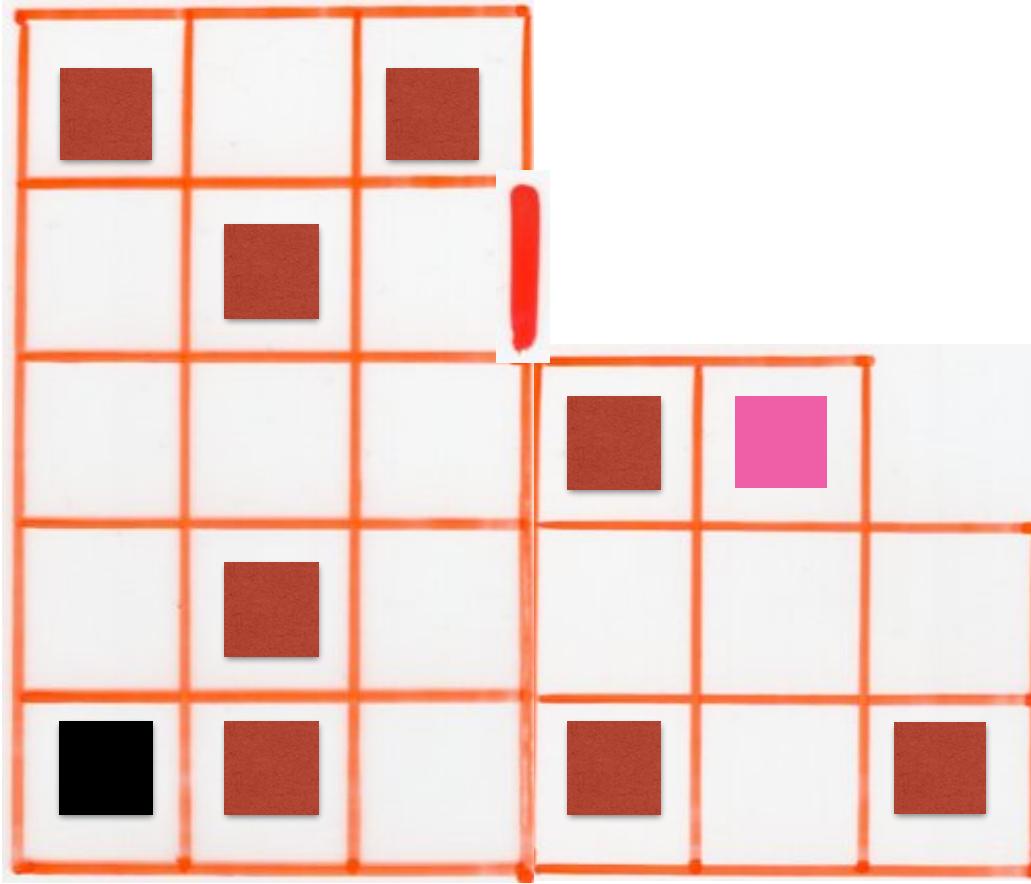
Example 1



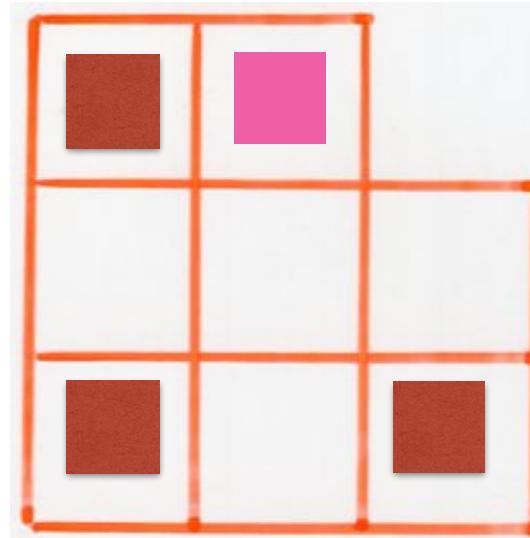
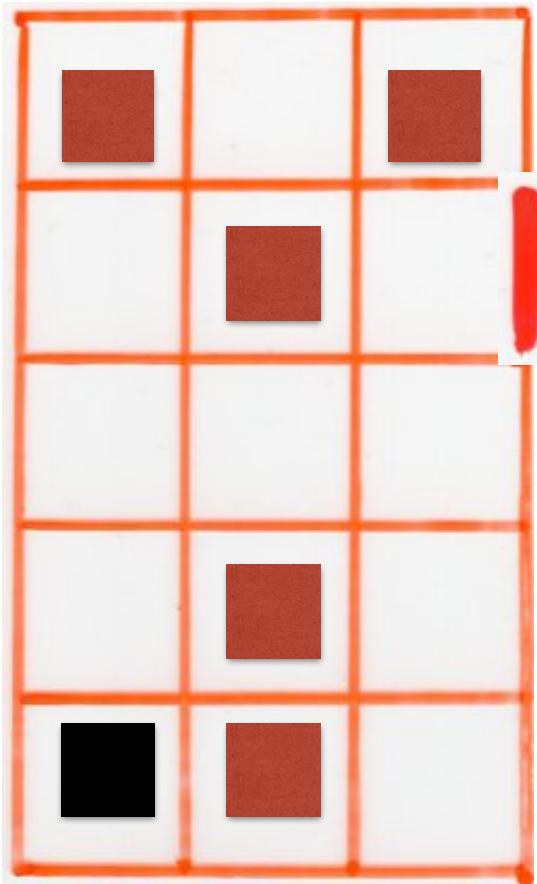
Example 2



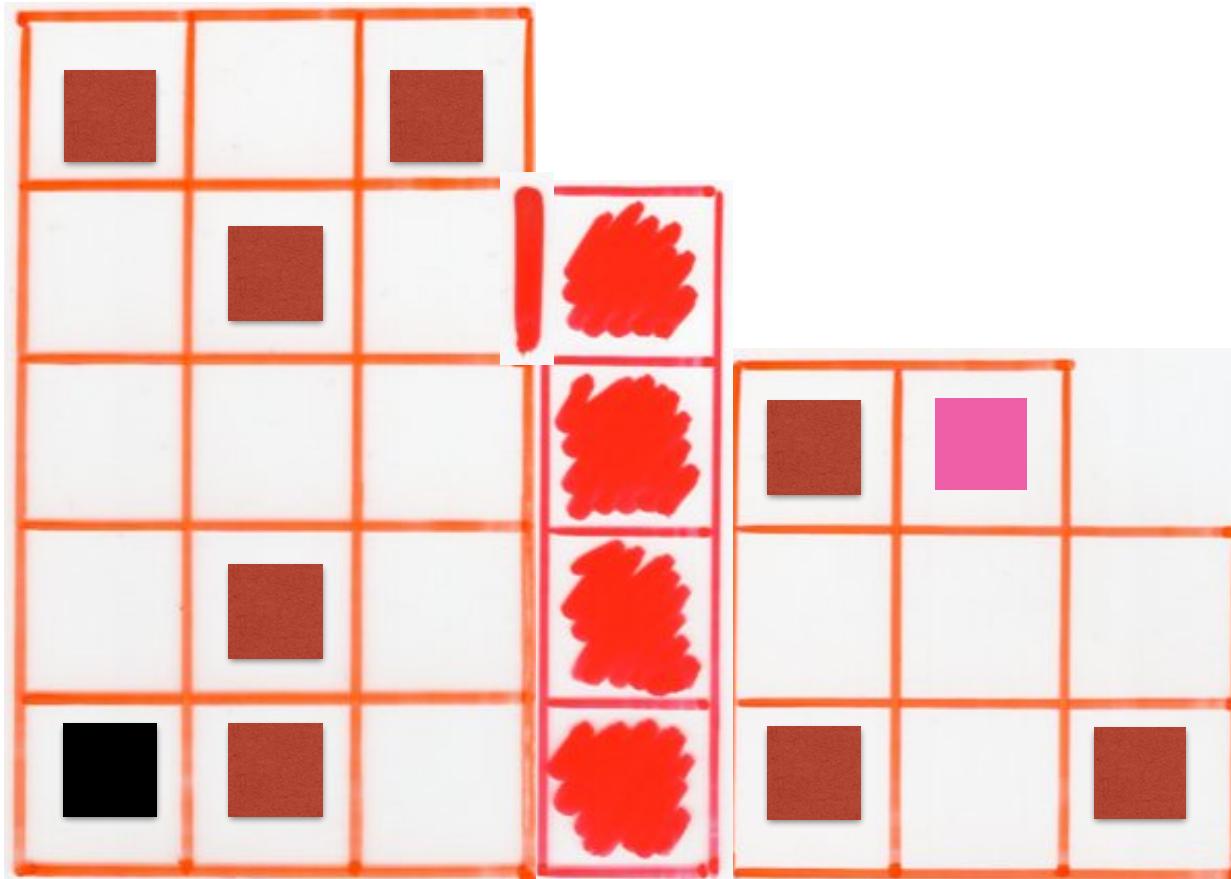
Example 2



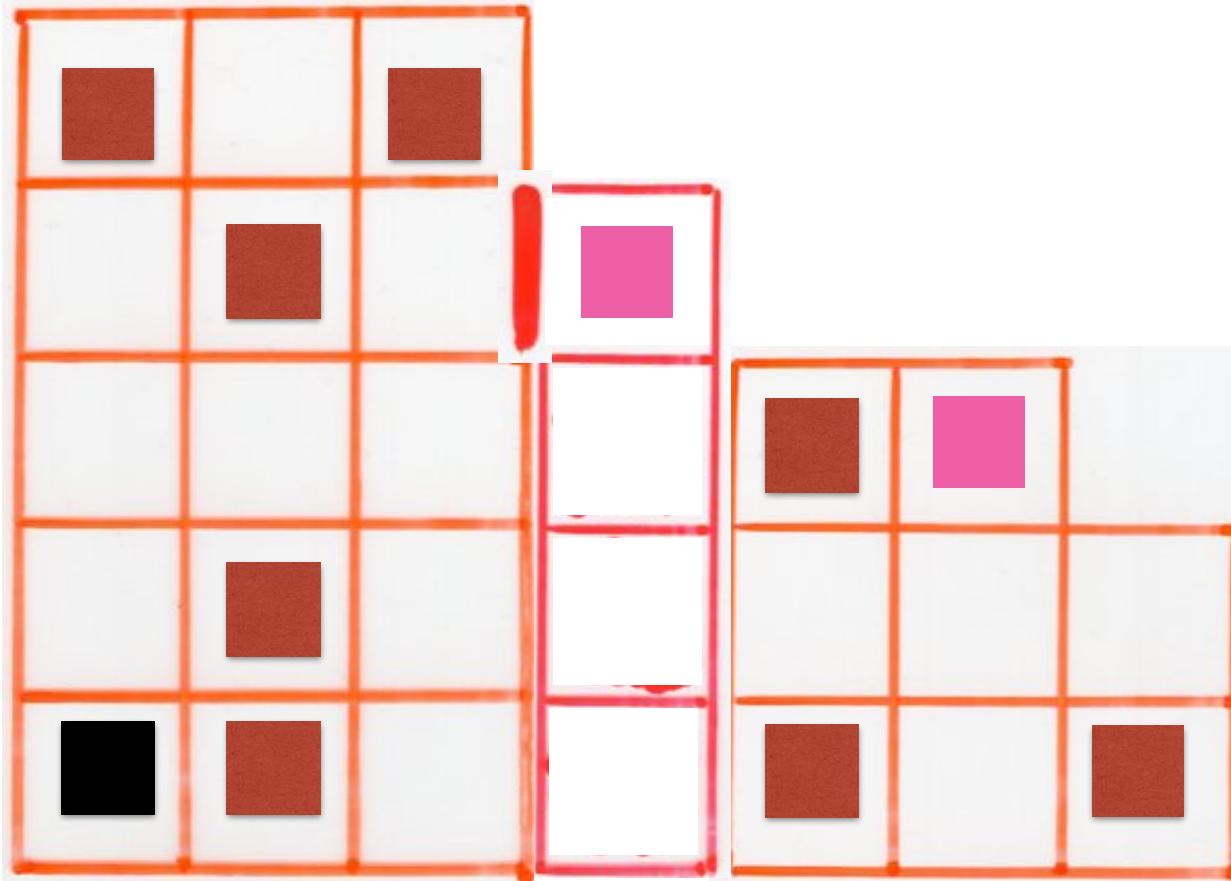
Example 2



Example 2

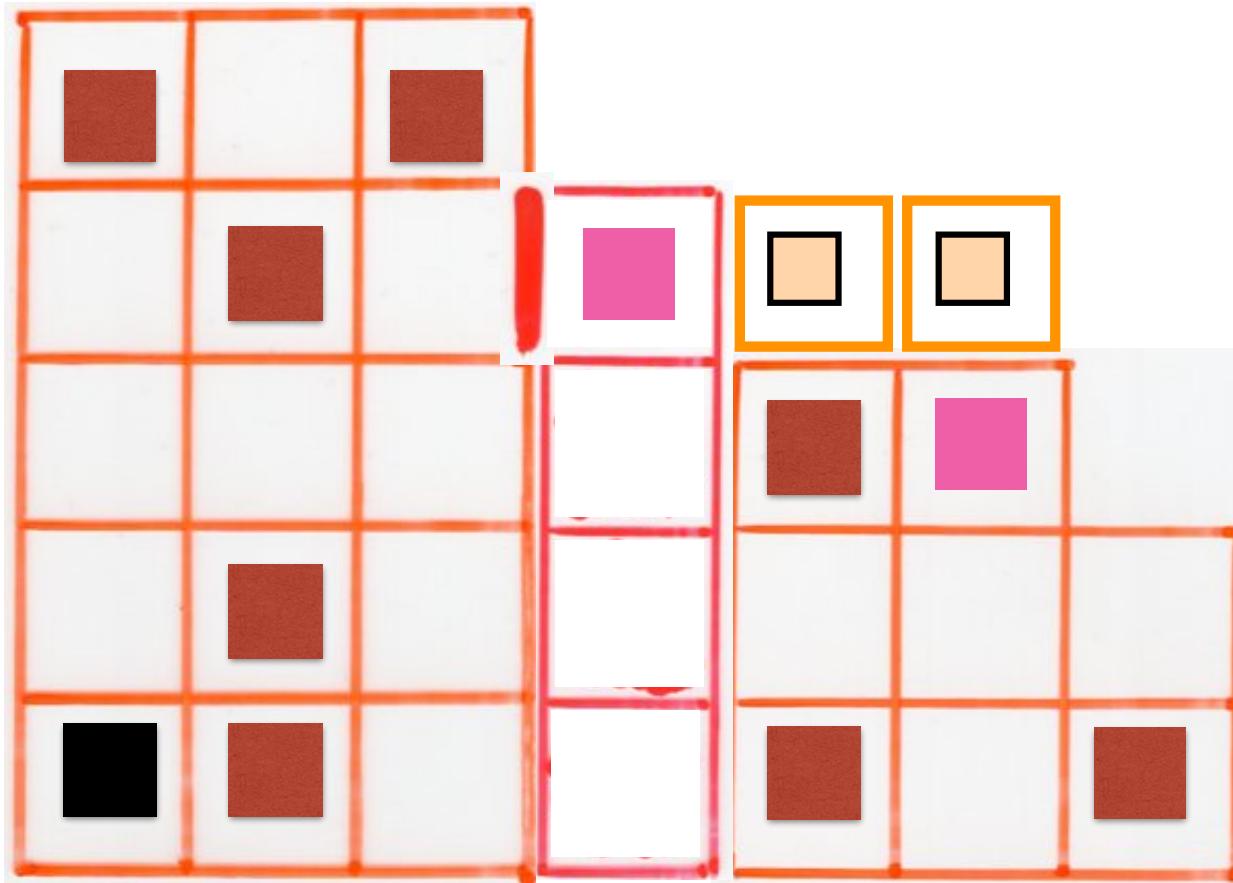


Example 2

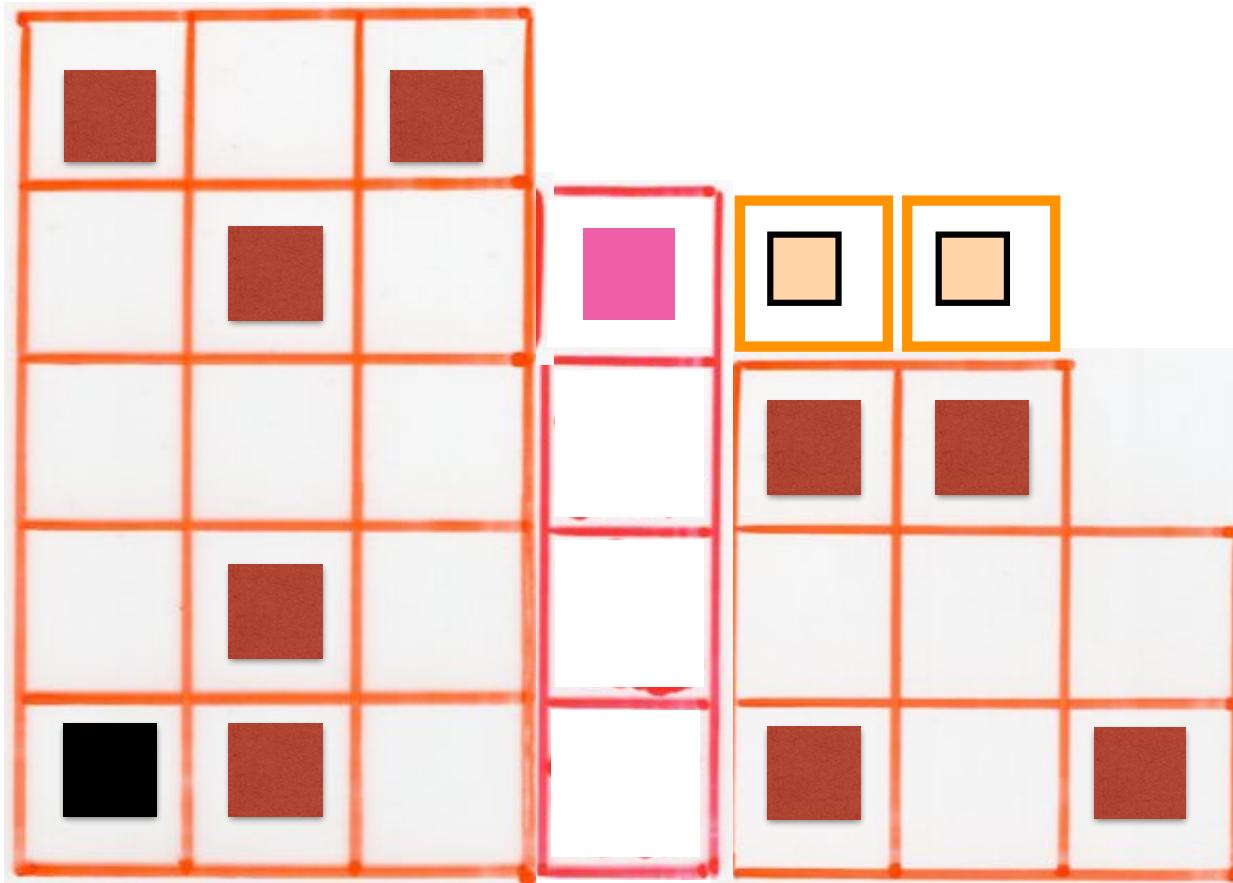


Case (ii)

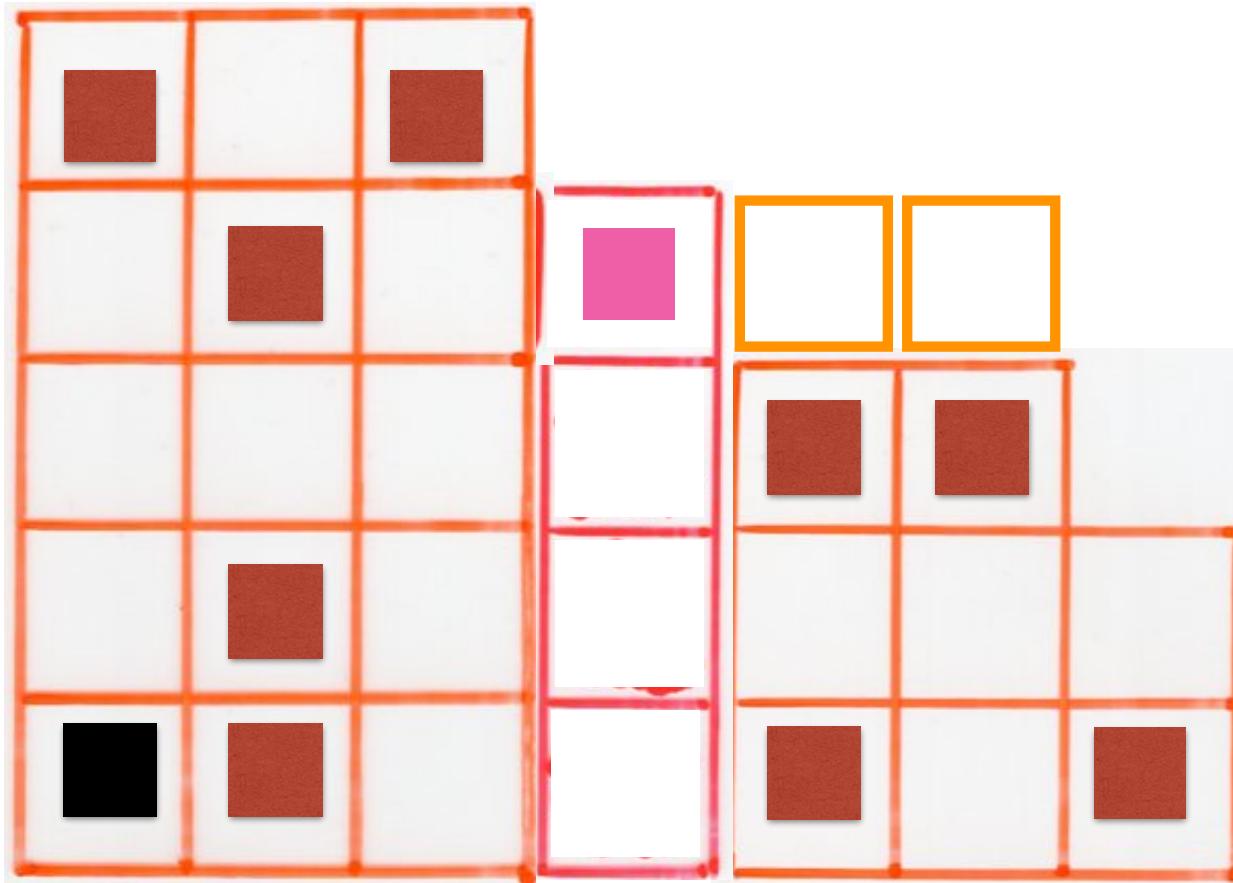
Example 2



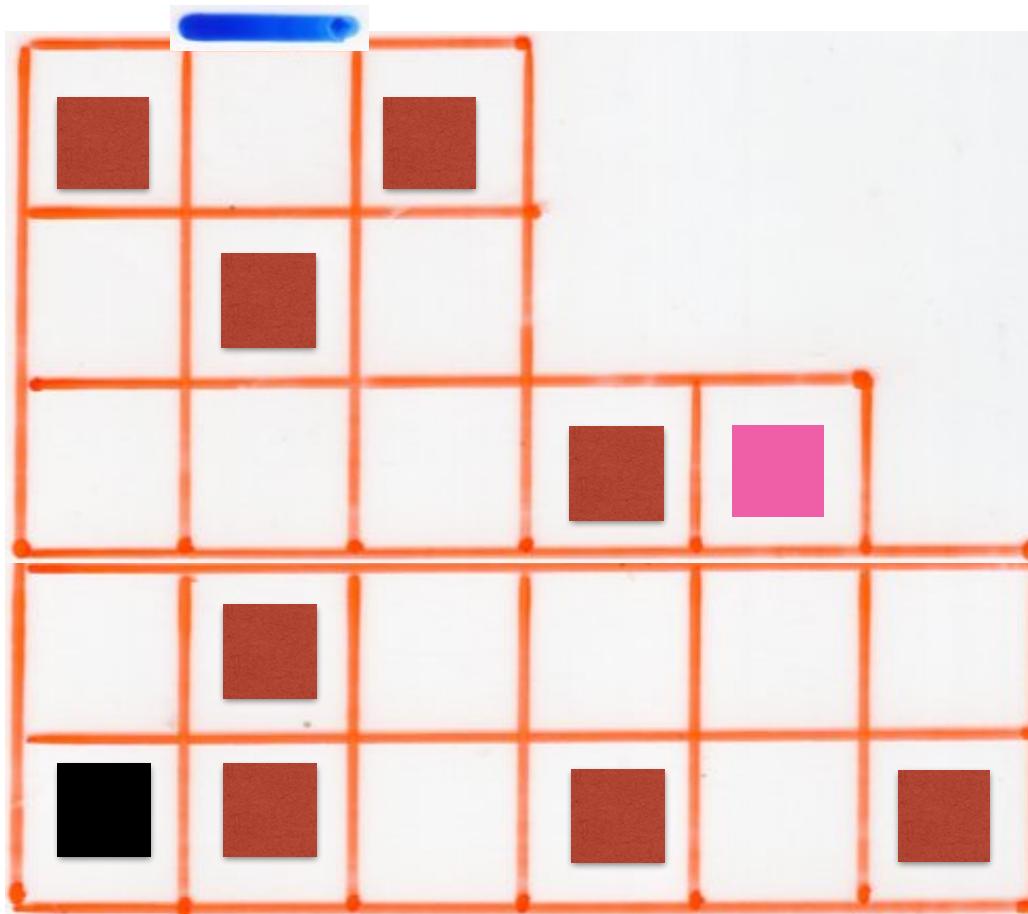
Example 2

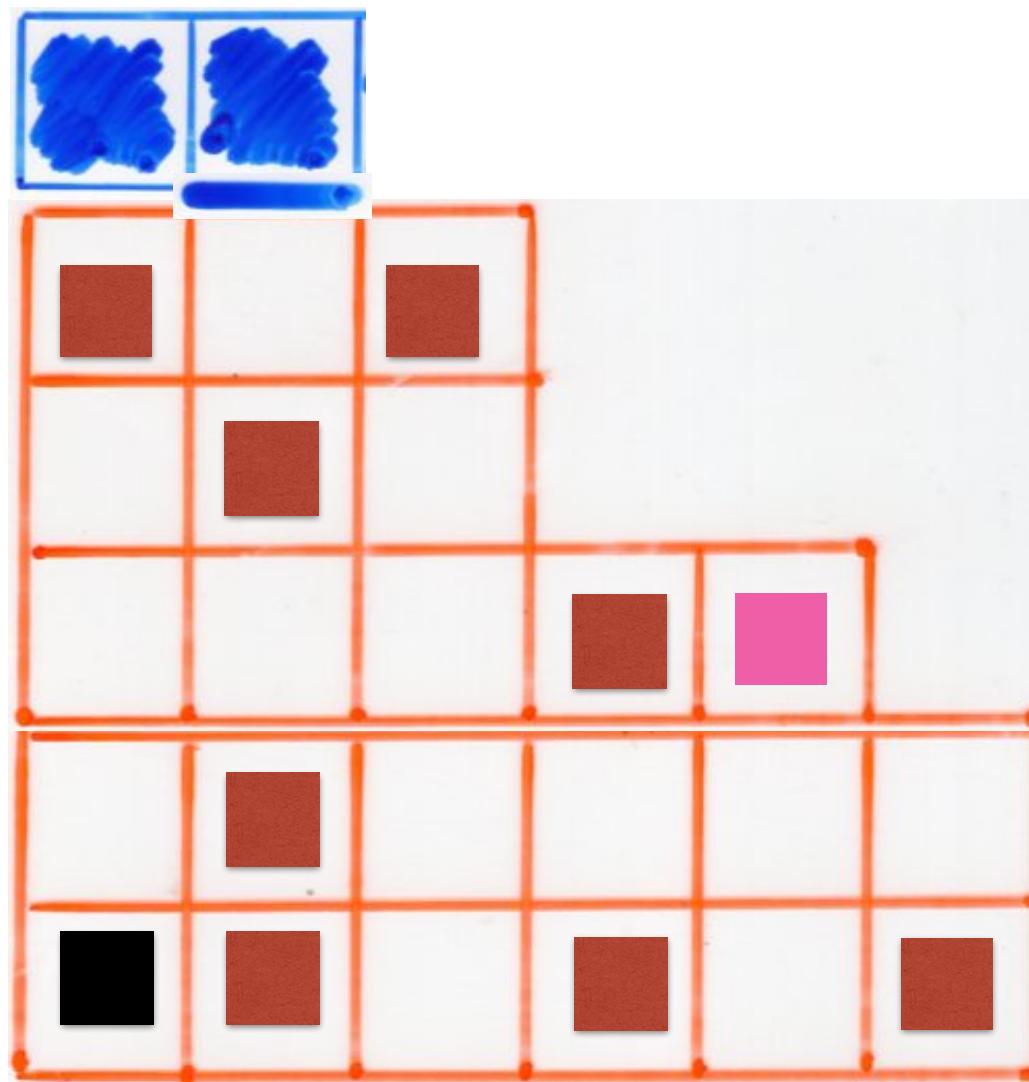


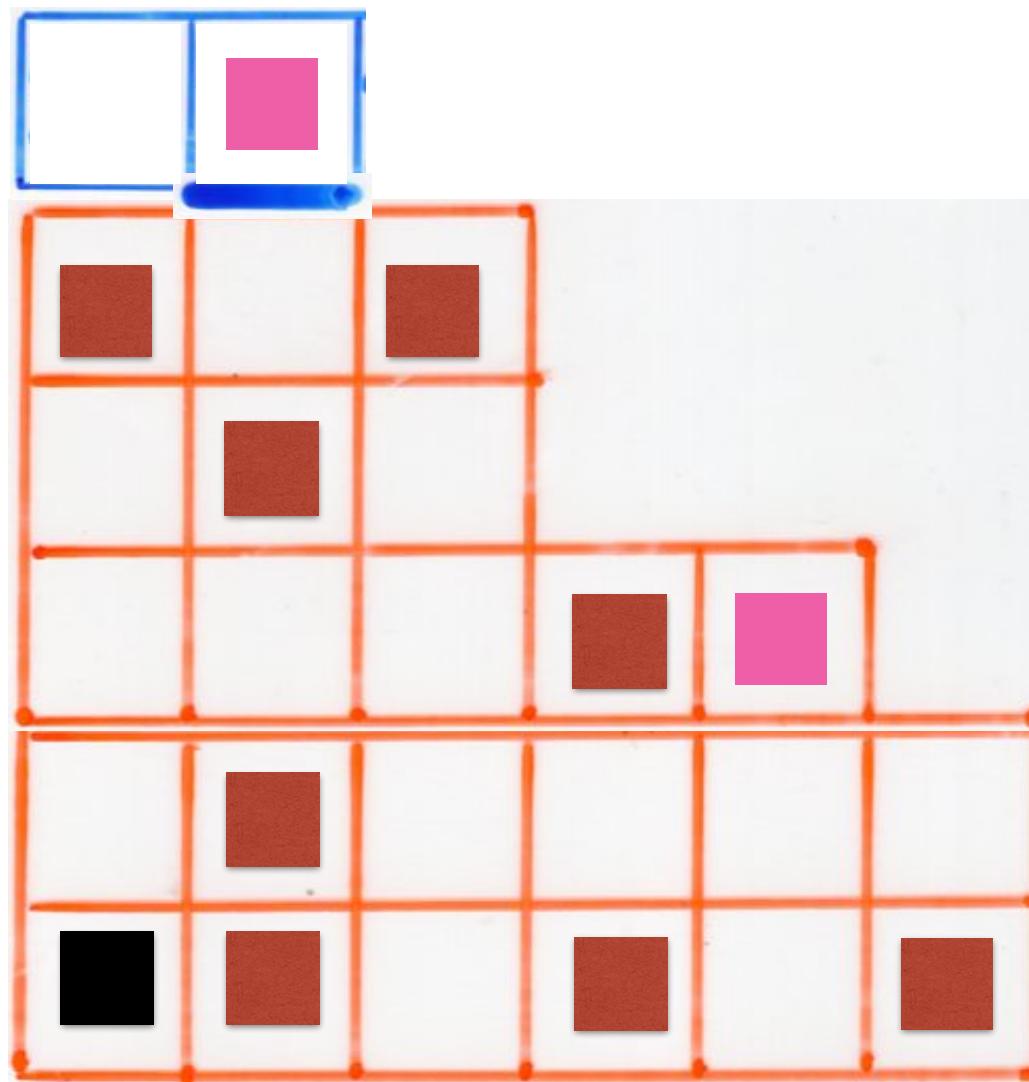
Example 2



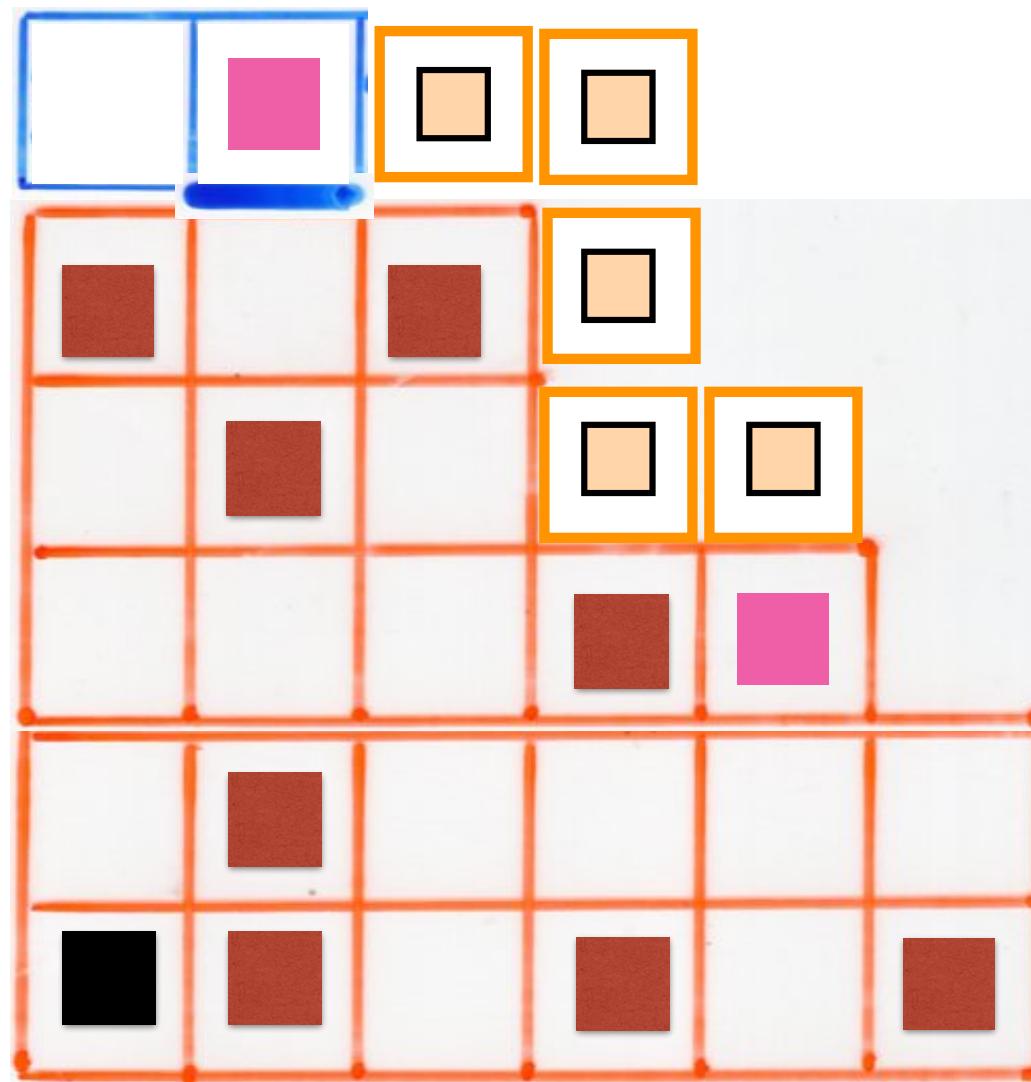
Example 3

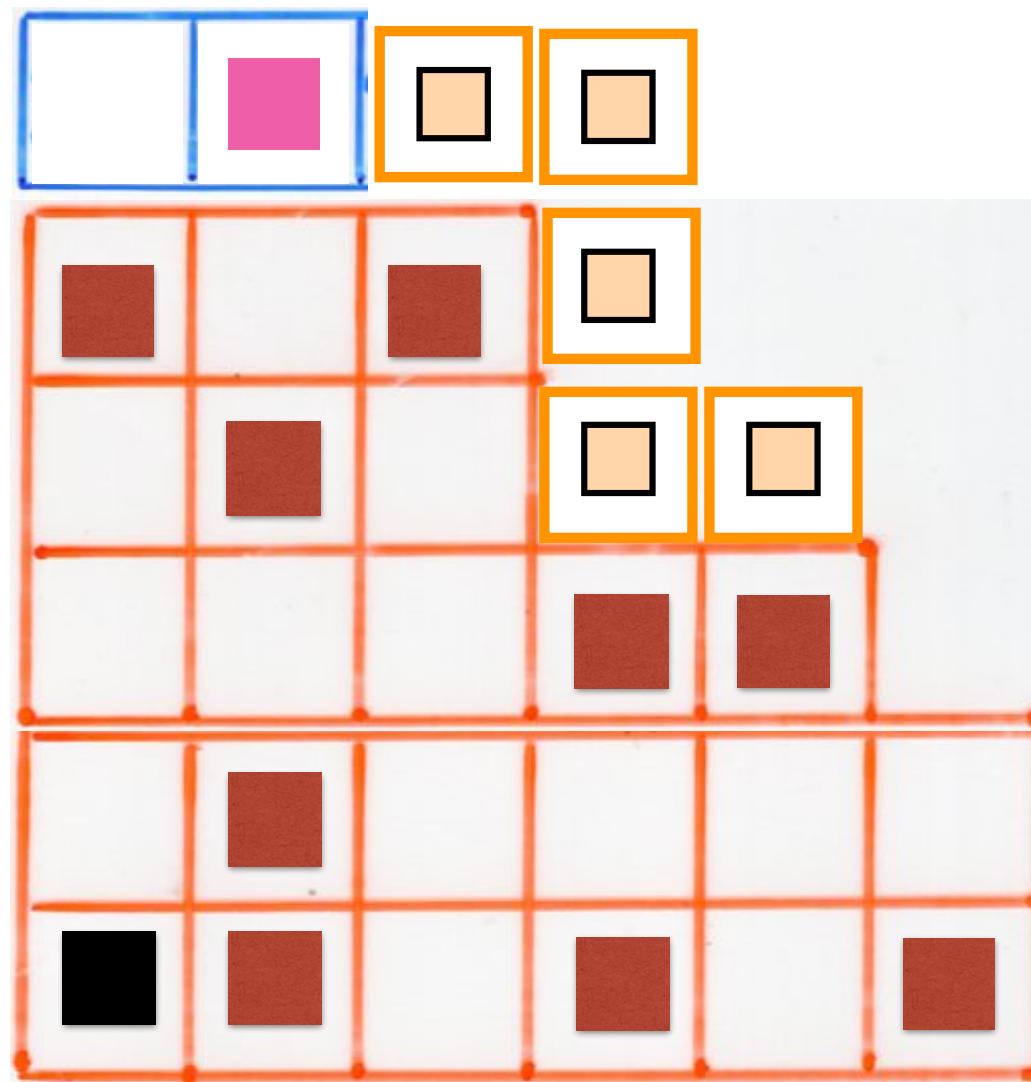


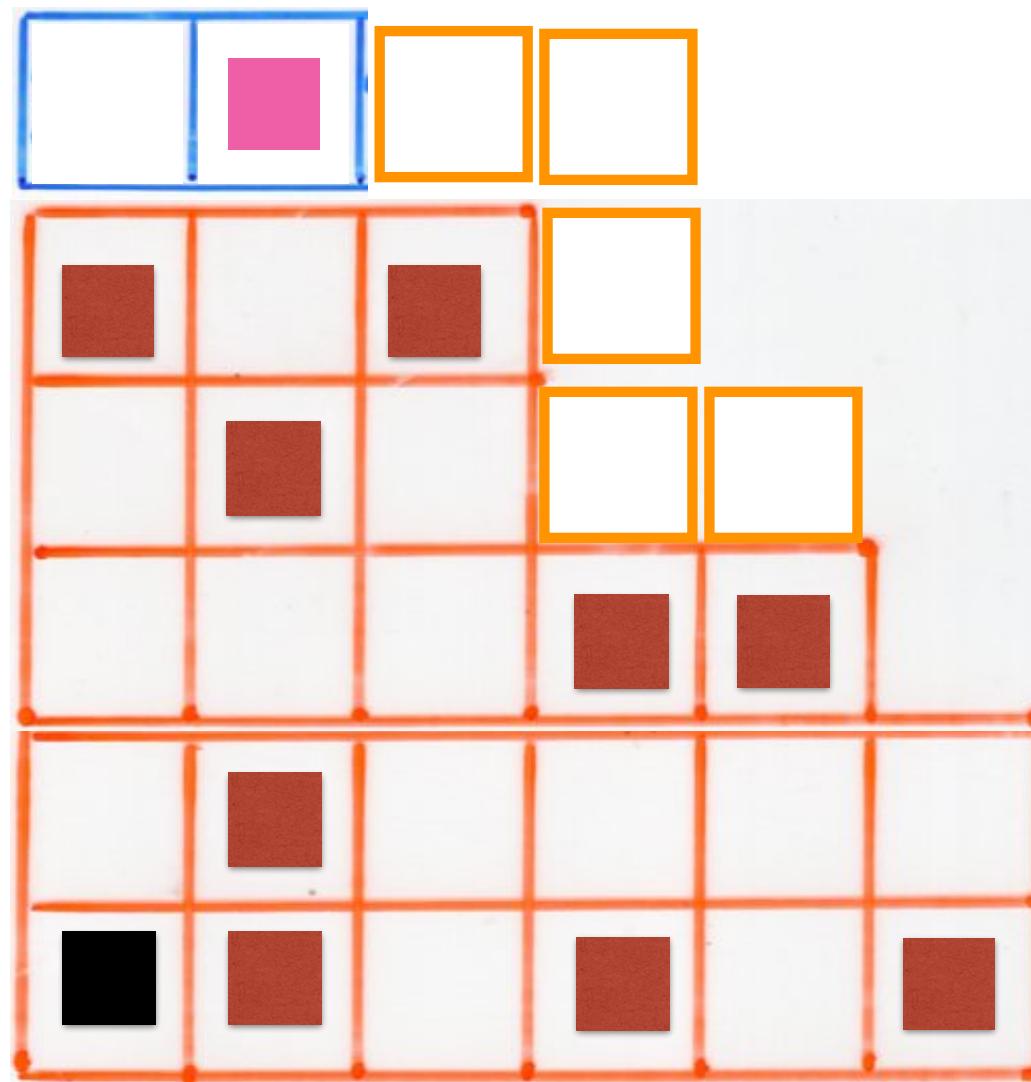




Case (ii)





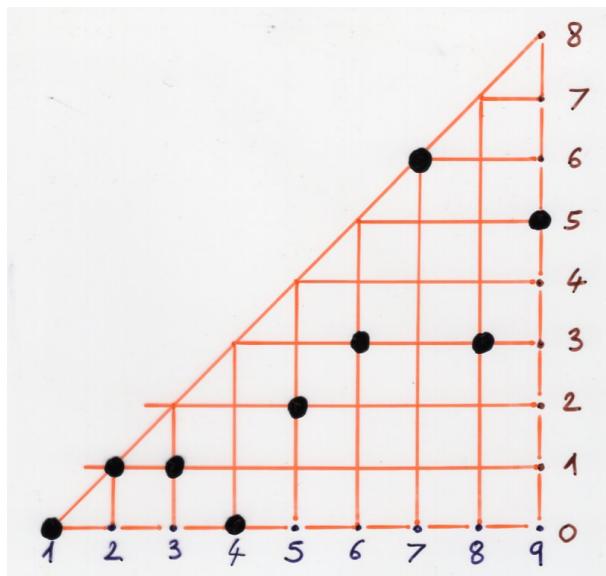


bijection

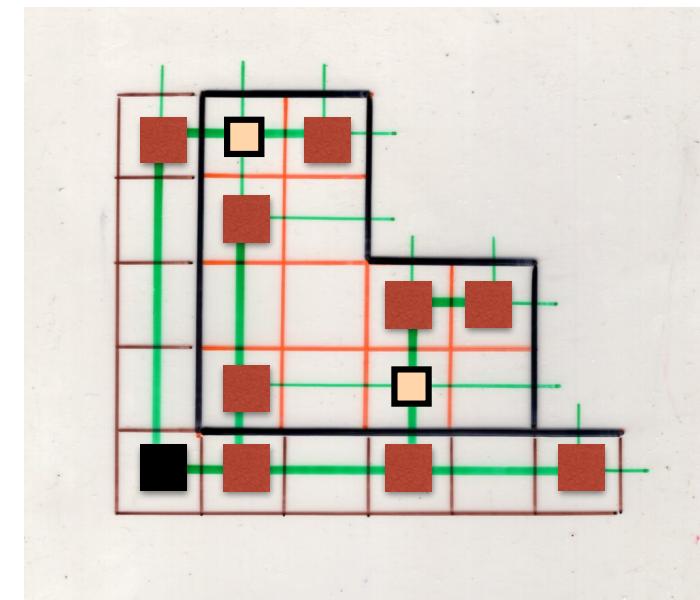
tree-like
tableaux
size $(n+1)$

↔

tree-like
tableaux
size n $\rightarrow 1 \leq i \leq (n+1)$



bijection



sub-excedant
functions
on $[n]$

$$0 \leq f(i) \leq (i-1)$$

tree-like
tableaux
size n

parameters

We have seen (first bijection):

Corollary for T alternative tableau (size n)
The double distribution $(i(T), j(T))$
of parameters
 $\begin{array}{ccc} \text{number of open rows} & & \text{number of open column} \\ \uparrow & & \uparrow \end{array}$

is the same as the double distribution
of permutations of S_{n+1} according
to the parameters

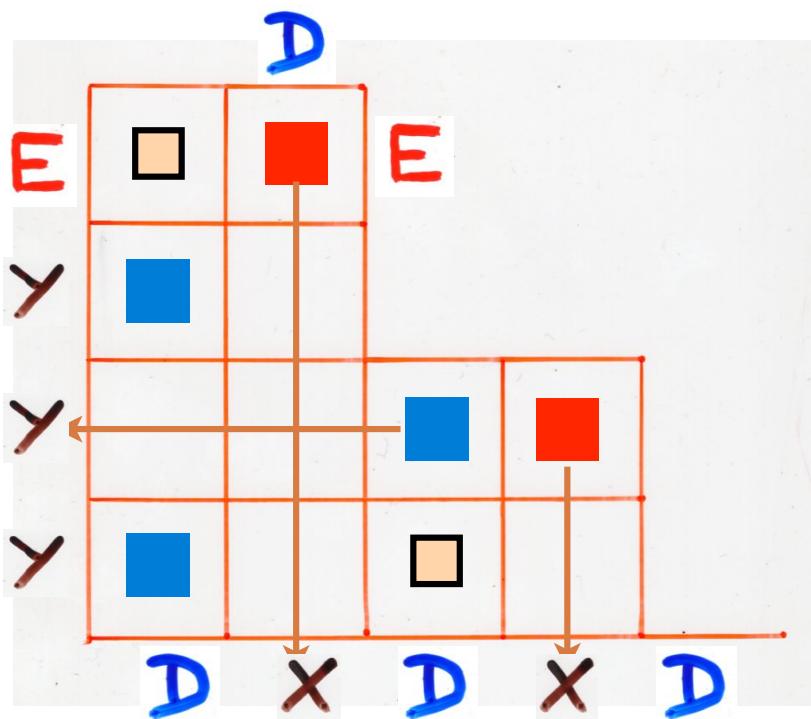
(left-to-right minimum elements of T
right-to-left

generating polynomial:

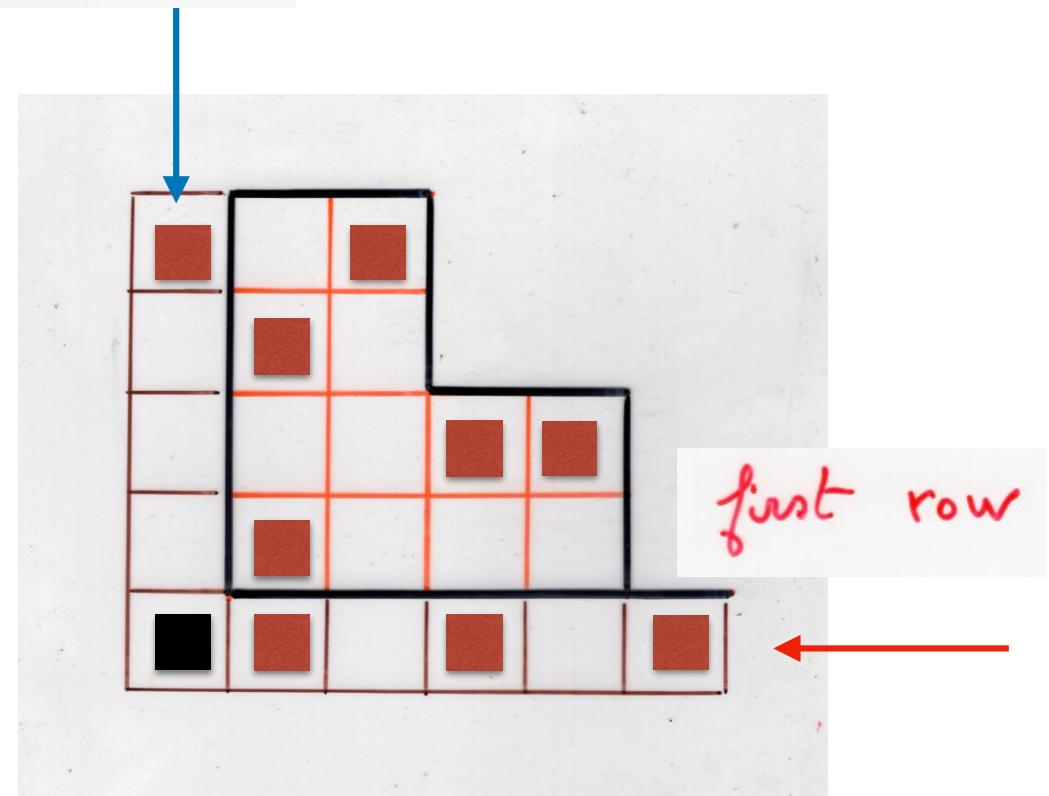
$$xy(x+y)(x+1+y) \dots (x+n-1+y)$$

$i(T) = \text{nb of rows without } \bullet$
 $j(T) = \text{nb of columns without } \bullet$

number of elements
in the first column



alternative
tableaux



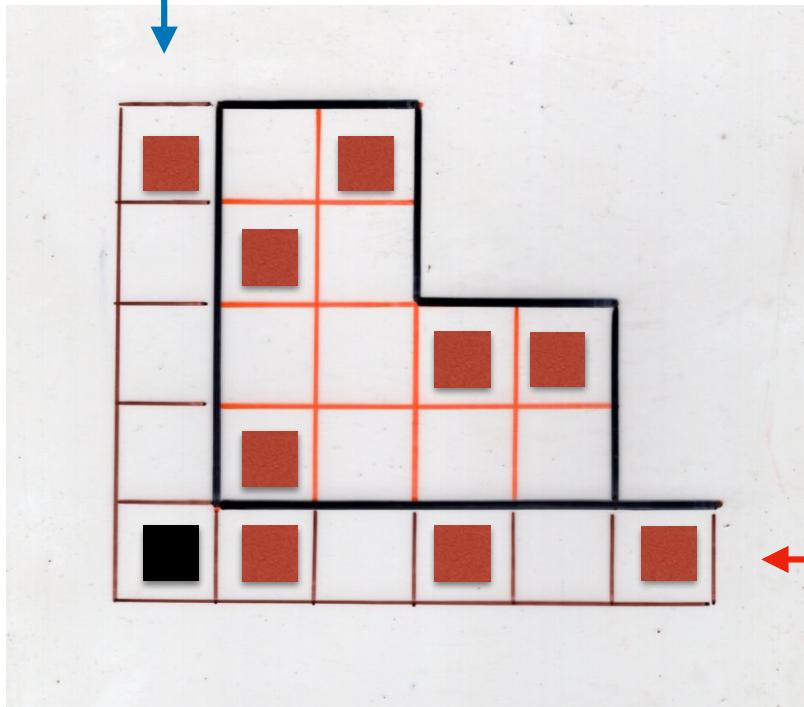
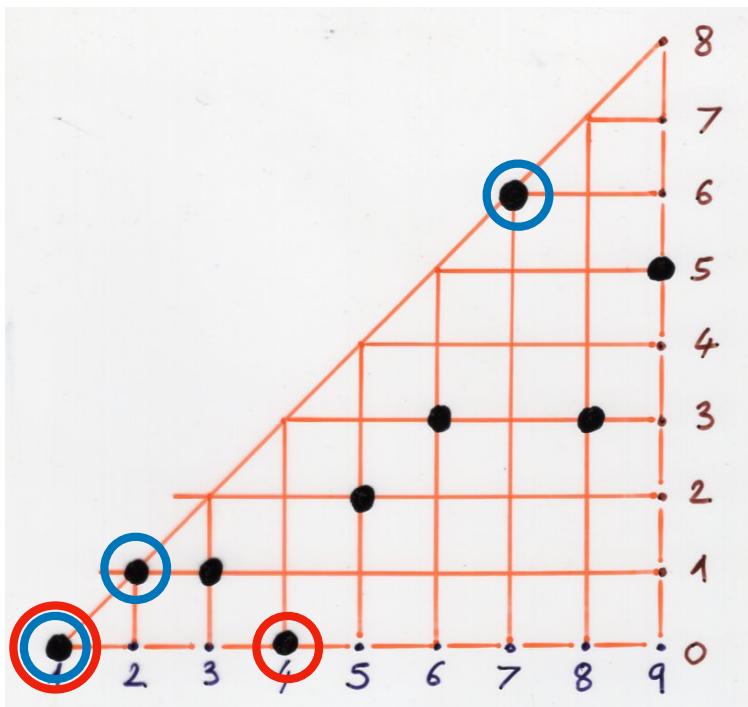
tree-like
tableaux

number of elements
in the first column

$$\text{number of } i\text{'s} \\ f(i) = (i-1)$$

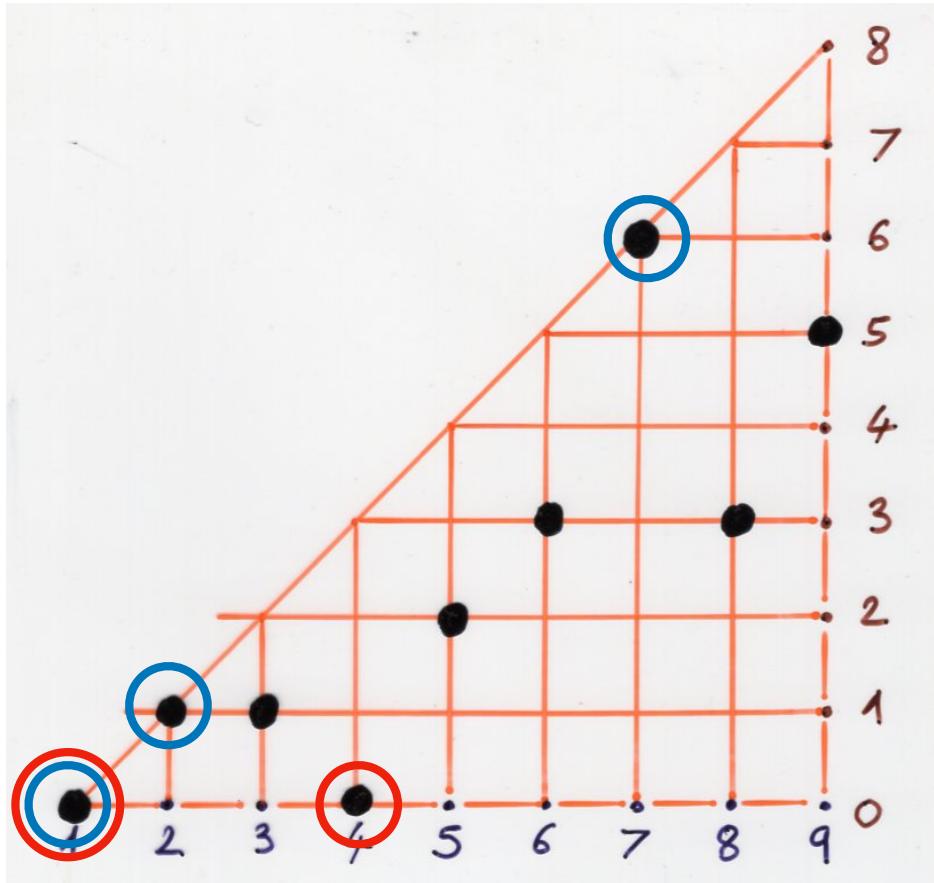
first row

$$f(i) = 0$$



$$f \rightarrow T$$

tree-like
tableaux



generating - polynomial:

$$xy(x+y)(x+1+y) \dots (x+n-1+y)$$

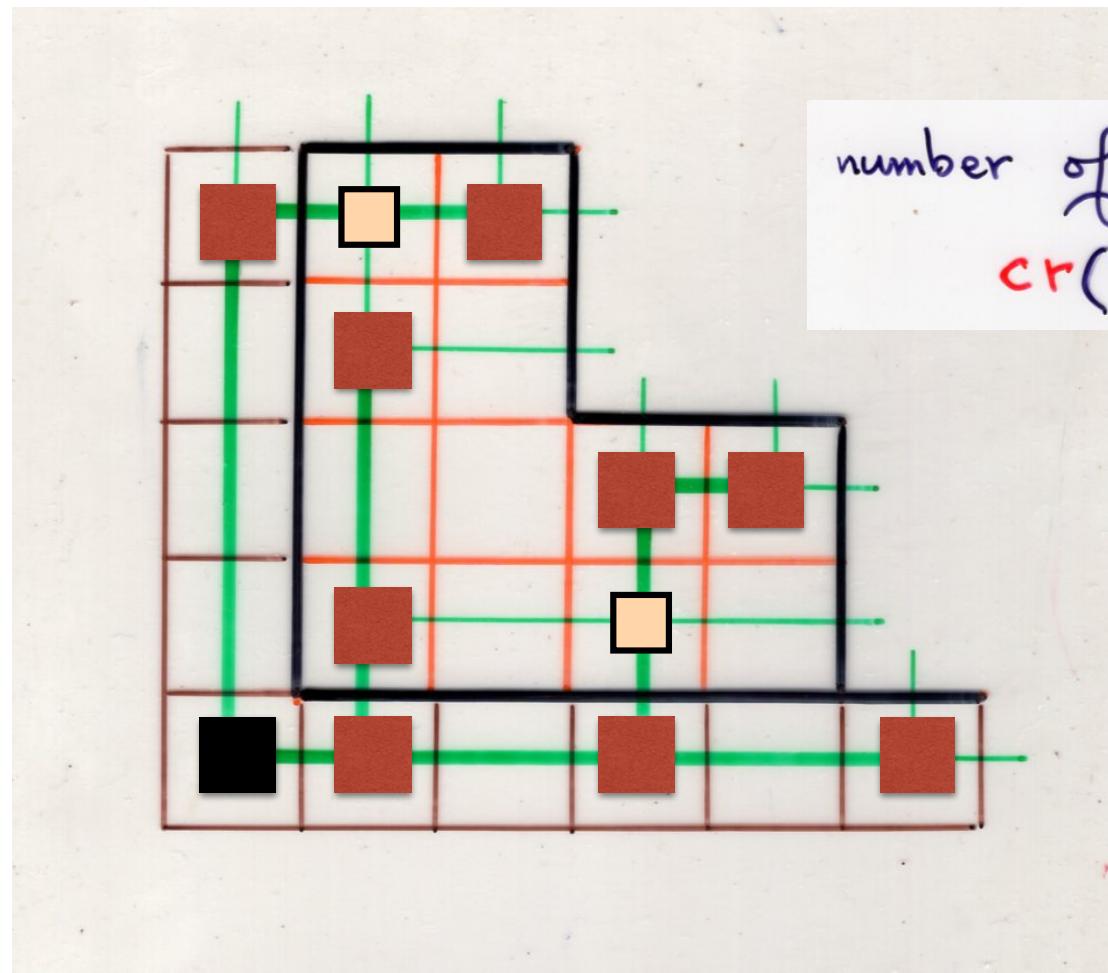
σ = 7 2 3 9 6 8 5 1 4
 permutation word

left-to-right
right-to-left

minimum
maximal

elements

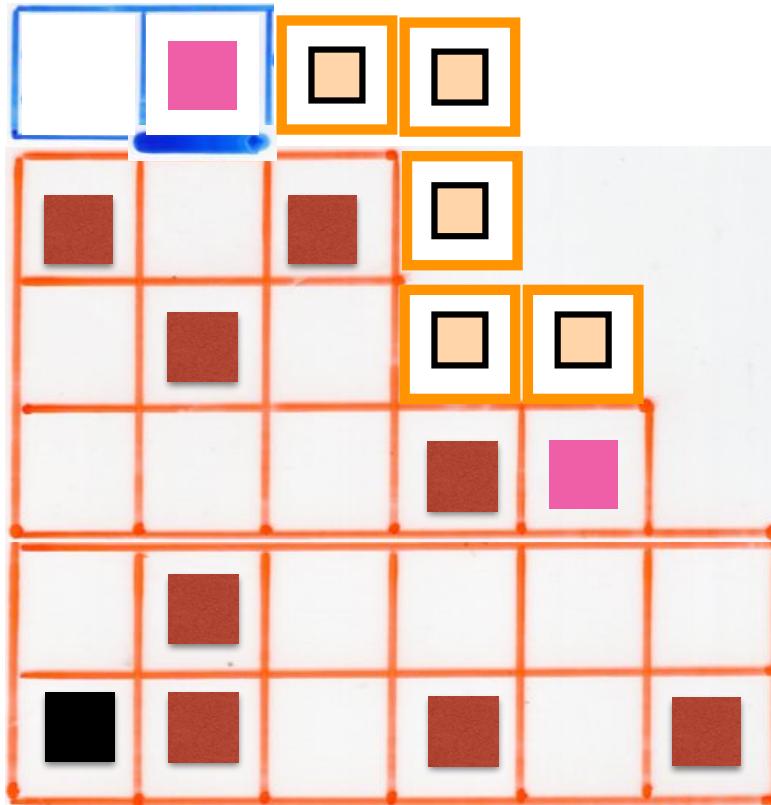
the parameter q :



number of crossings
 $cr(T)$

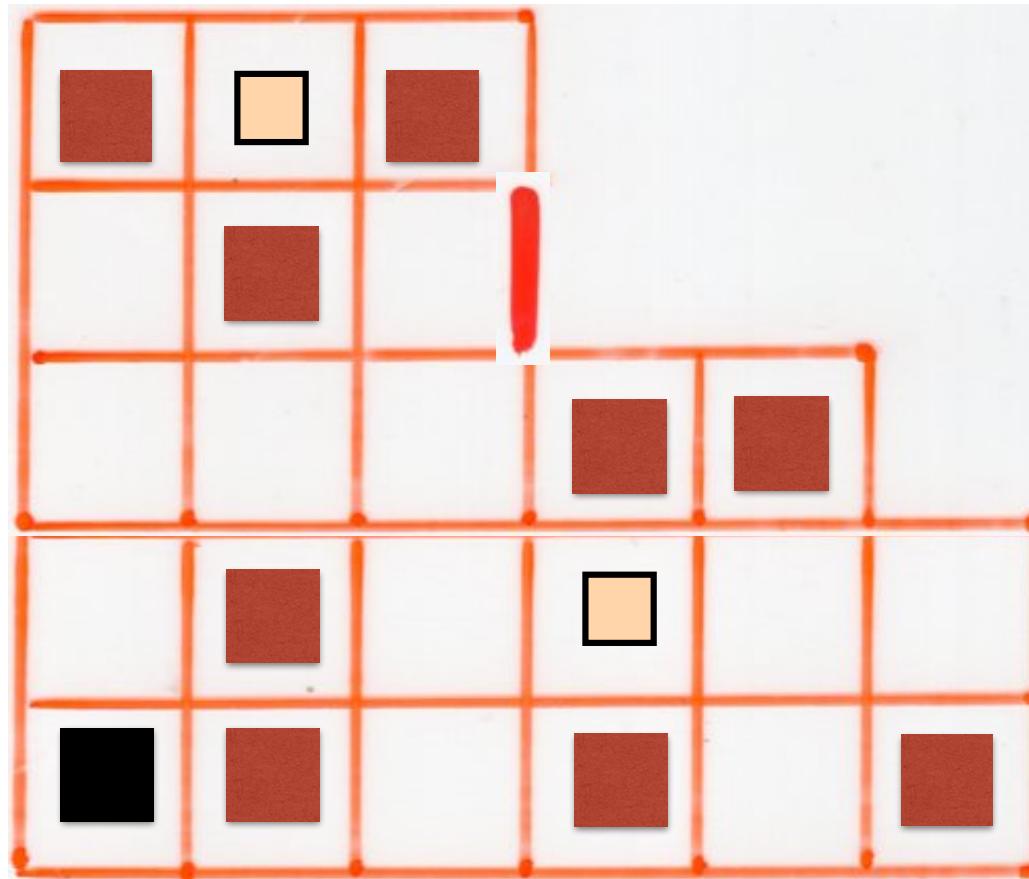
tree-like
tableaux

the parameter q :



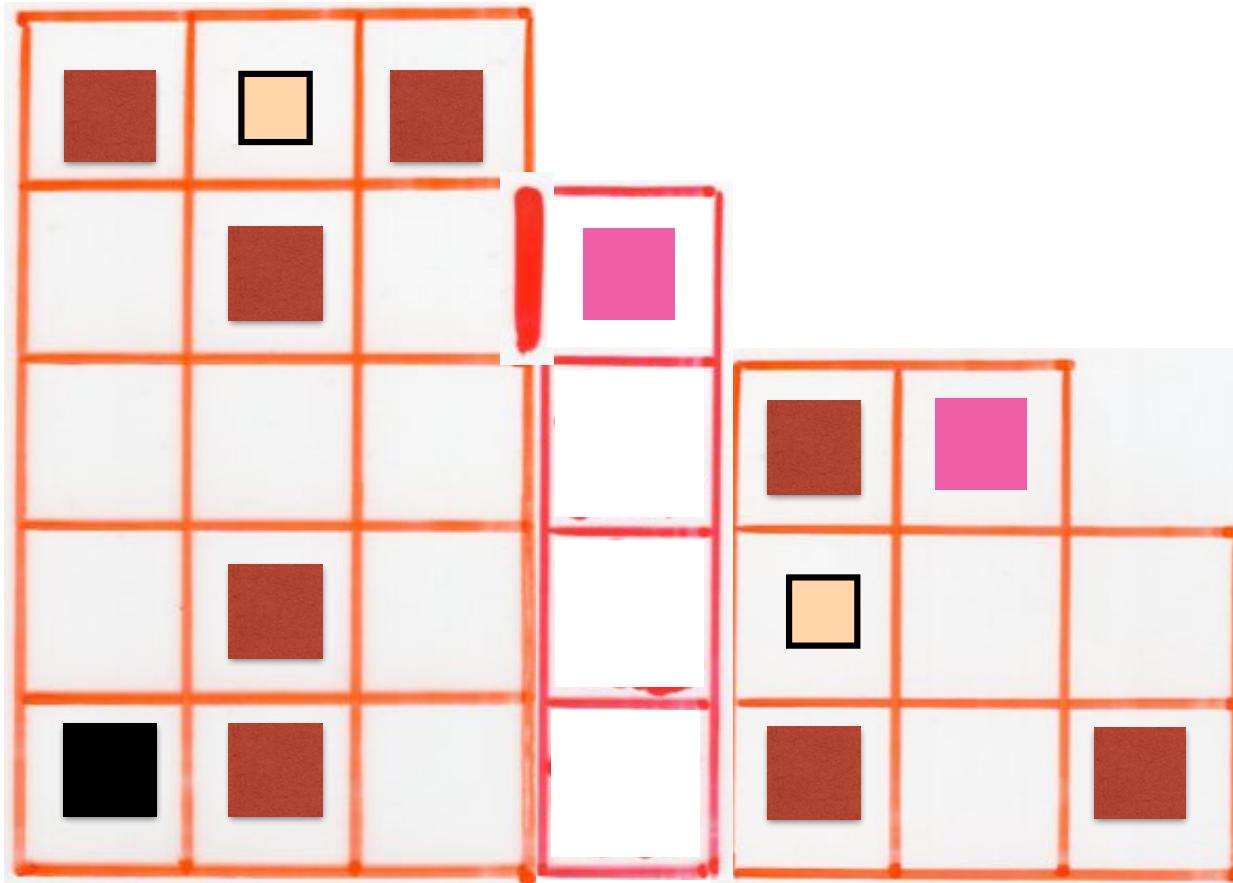
number of crossings
 $\text{cr}(T)$

= sum of the length of all rim-hooks
added in the algorithm

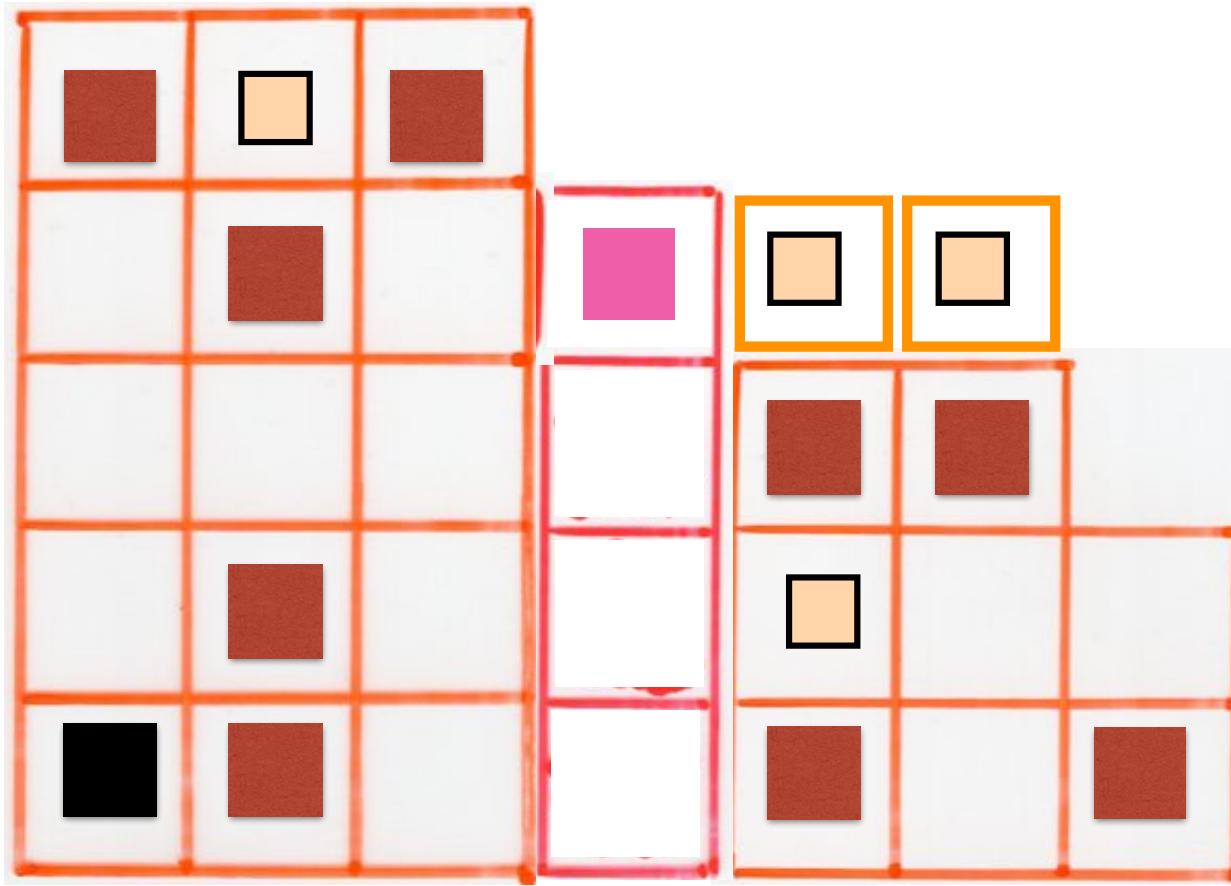


tree-like
tableaux

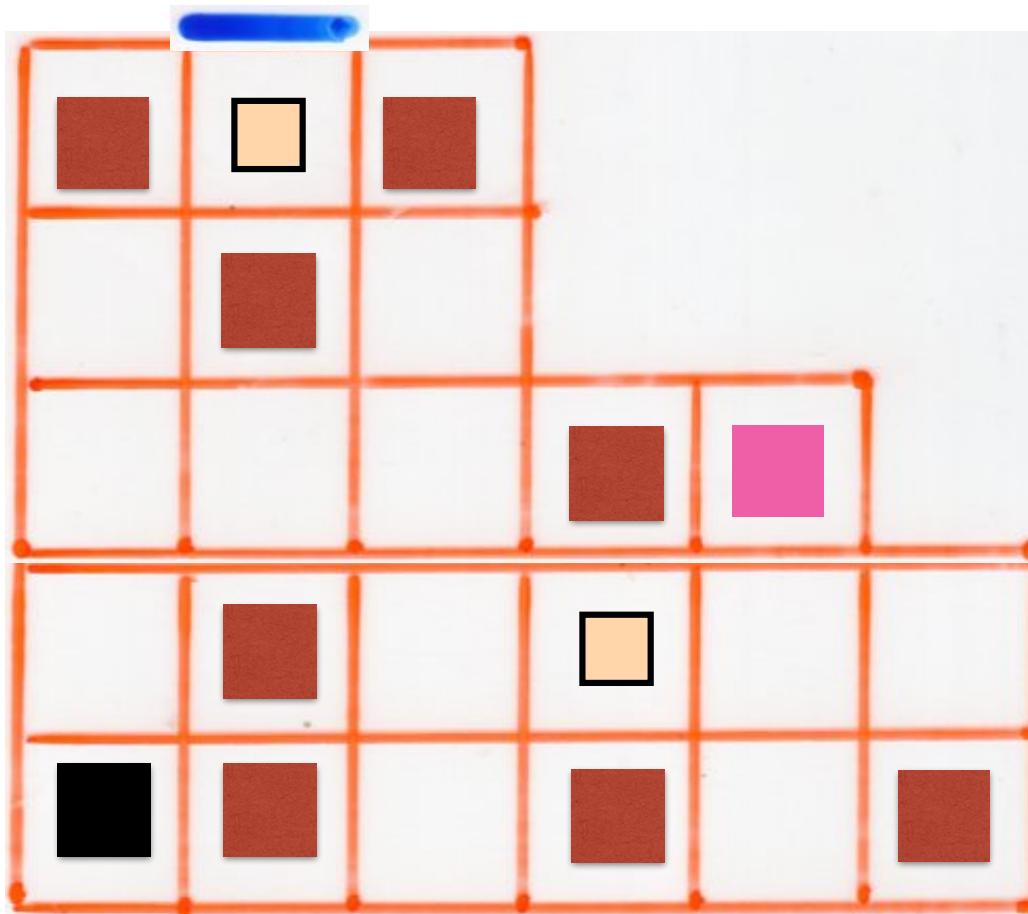
Example 2

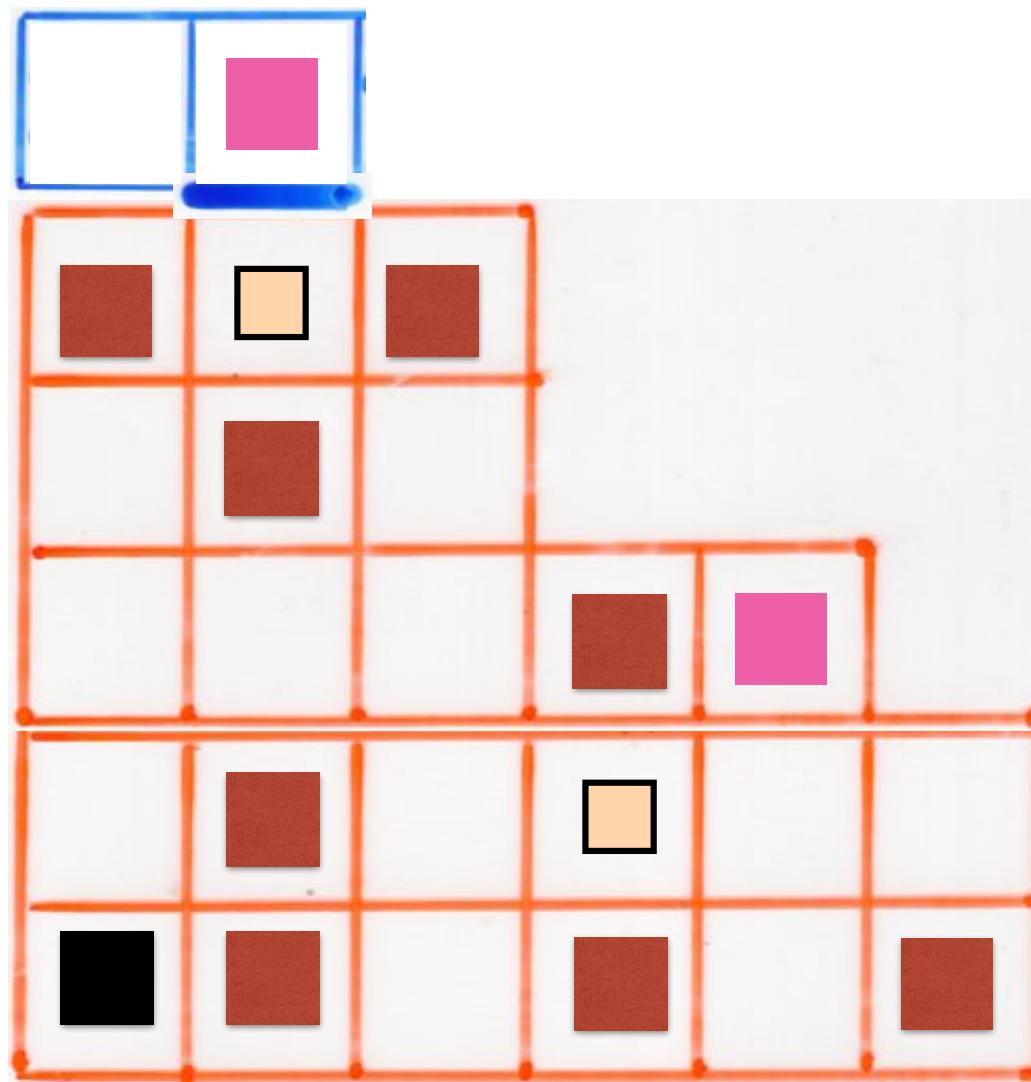


Example 2

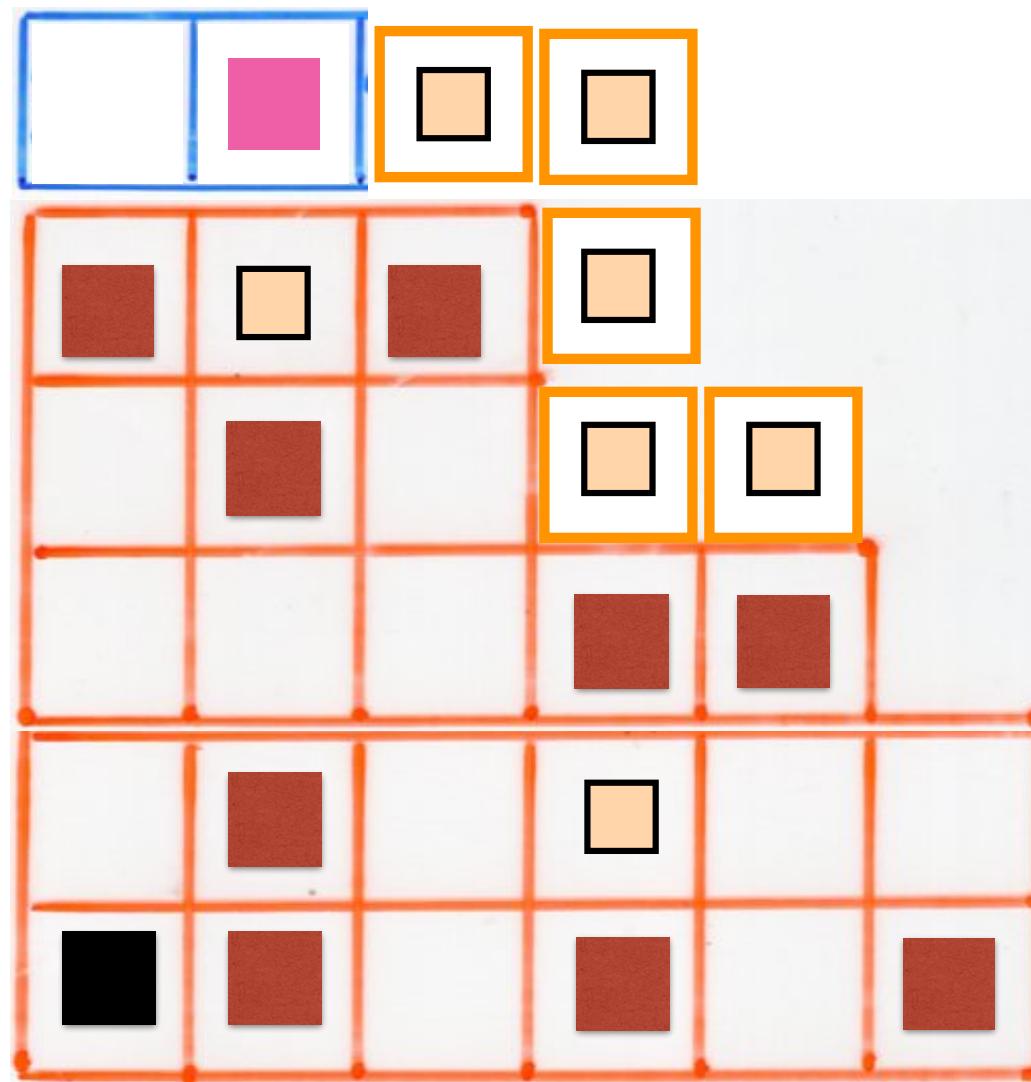


Example 3



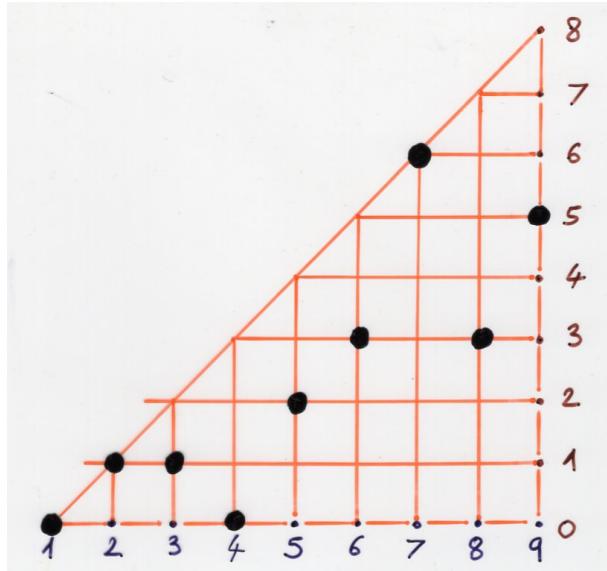


Case (ii)



bijection

$$f \rightarrow T$$



number of crossings
 $cr(T)$

= sum of the length of all rim-hooks
added in the algorithm

$$= \sum_{1 \leq i \leq (n-1)} \max \left((f(i+1) - f(i)), 0 \right)$$

the parameter q :

this is another story !

.... related to q -analogue of Laguerre polynomials

weighted histories

q-Laguerre

continuous

q -Laguerre
Polynomials

discrete

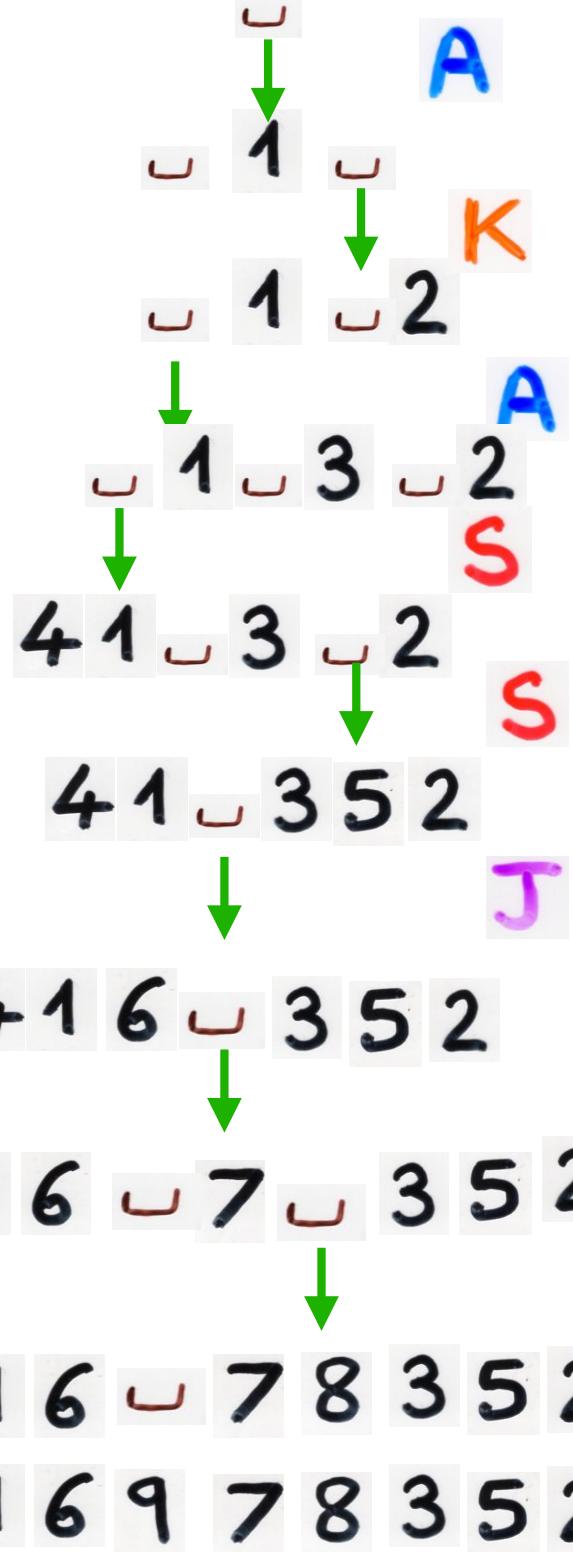
$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_R = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_R = [k+1]_q \\ b''_R = [k+1]_q \\ a_R = [k+1]_q \\ c_R = [k+1]_q \end{cases}$$

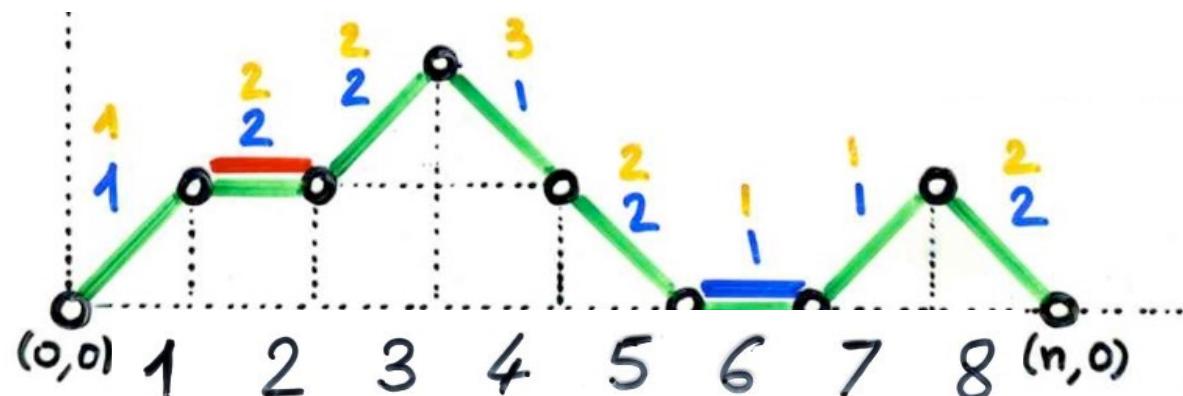
weighted
 q -Laguerre
histories

$$q^{\left[\sum_{i=1}^n (p_i - 1) \right]}$$

choice function



"q-analogue"
of
Laguerre
histories



choice function

$i =$	1 2 3 4 5 6 7 8
$p_i =$	1 2 2 1 2 1 1 2
$p_{i-1} =$	0 1 1 0 1 0 0 1

weighted
q-Laguerre
histories

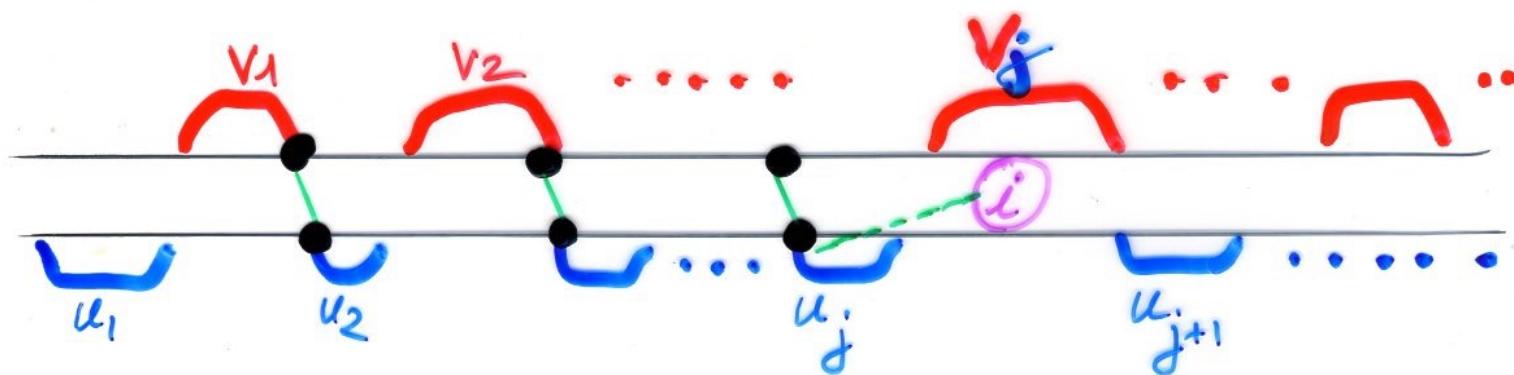
q^4

◻
◻ 1 ◻
◻ 1 ◻ 2
◻ 1 ◻ 3 ◻ 2
4 1 ◻ 3 ◻ 2
4 1 ◻ 3 5 2
4 1 6 ◻ 3 5 2
4 1 6 ◻ 7 ◻ 3 5 2
4 1 6 ◻ 7 8 3 5 2
4 1 6 9 7 8 3 5 2 = $\frac{G}{EG^{n+1}}$

weighted
q-Laguerre
histories

$$q^{\left[\sum_{i=1}^n (p_i - 1) \right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern $(31-2)$



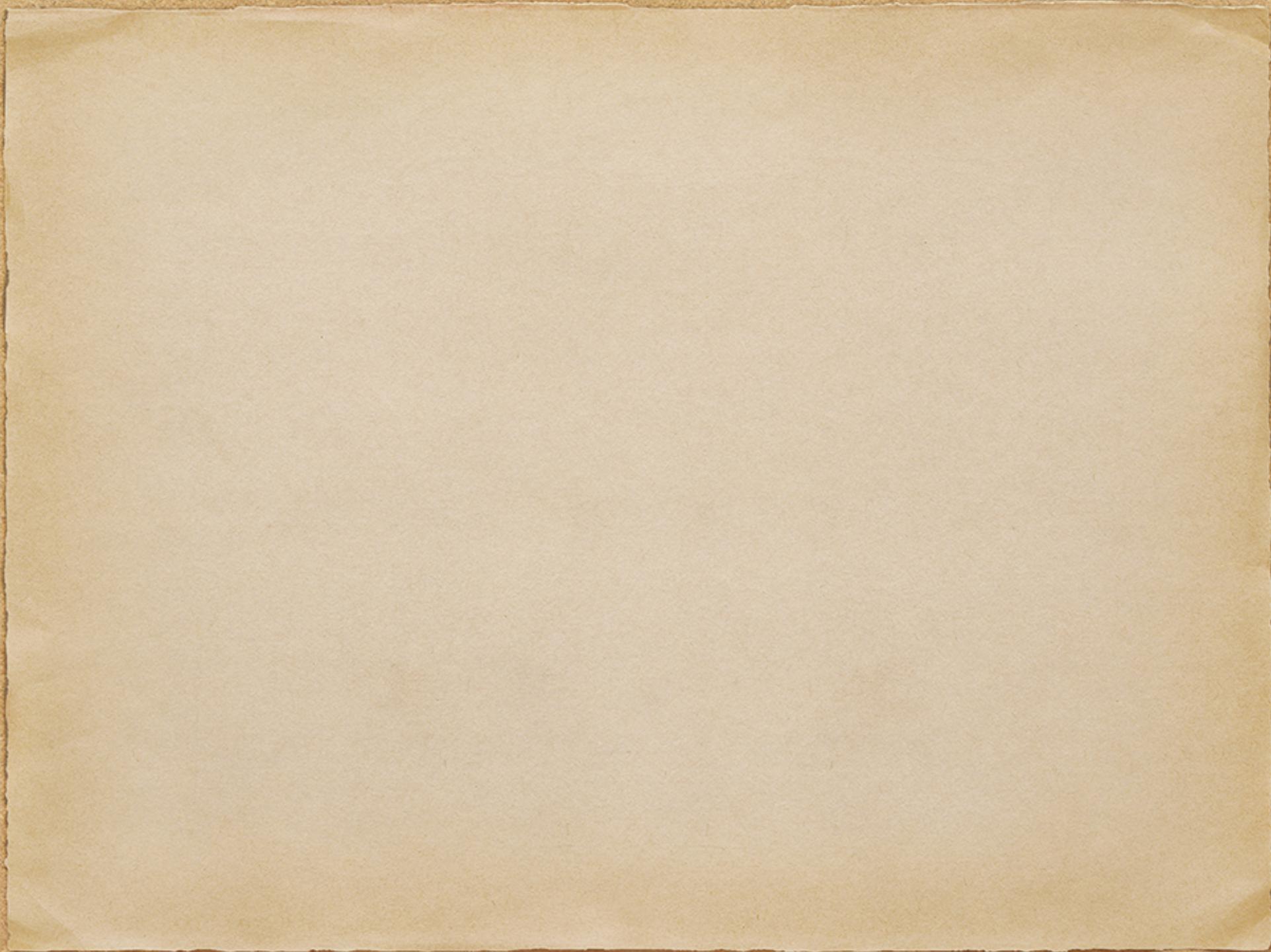
The parameter

« number of crossings »

in alternating tableaux
or in tree-like tableaux

has the same distribution as the parameter

« q-Laguerre » for permutations



σ \longrightarrow T

permutation

G_{n+1}

alternative
tableau
size n

D. Bernardi (2008)

The number of crossings of T is
the number of pairs (x, y)
 $x = \sigma(i), y = \sigma(j), 1 \leq i < j \leq n+1$

such that there exist two integers $k, l \geq 0$

such that the set of values

$x+1, x+2, \dots, x+k, y+1, y+2, \dots, y+l$

are located between x and y (in σ)

and $x+k+1$ is located (in T) at the right of y

and $y+l+1$ is located (in T) at the left of x
(convention: $(n+2)$ at the left of all values)

$$\sigma = (y+l+1) \quad (x+1) \quad (y+l) \quad (x+k+1)$$
$$\sigma = \dots \cdot \underset{\substack{\uparrow \\ \text{||} \\ \sigma(i)}}{x} \cdot \underset{\substack{\dots \\ (y+1) \\ (x+k)}}{\dots} \cdot \underset{\substack{\dots \\ \dots \\ \text{||} \\ \sigma(j)}}{y} \cdot \underset{\substack{\uparrow \\ \text{||} \\ \sigma(j)}}{x} \cdot \dots$$

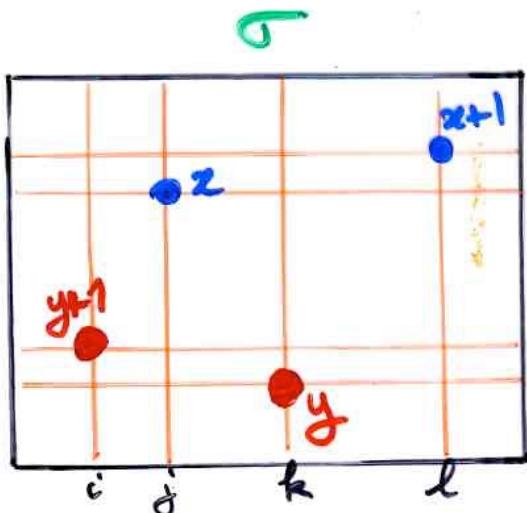
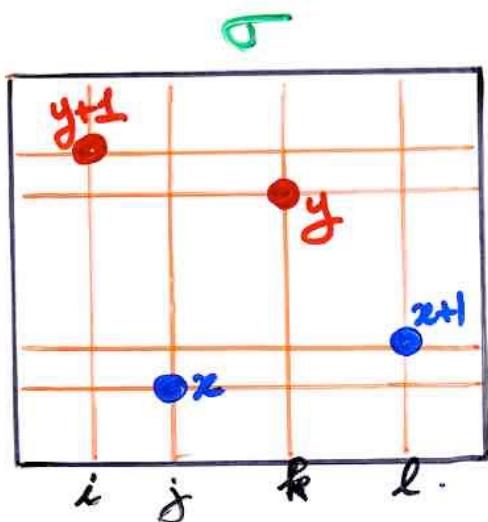
Proposition Bernardi (2008)

The number of permutations of G_n with no subsequences of the type

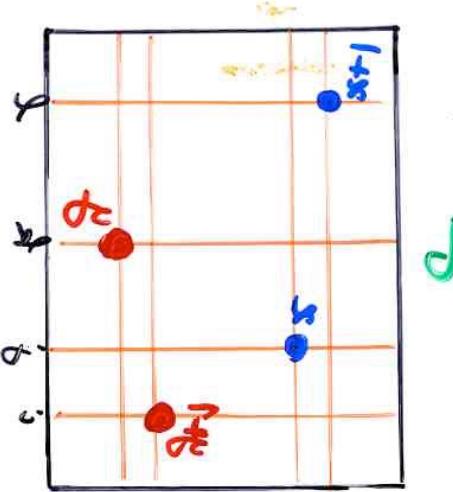
$\dots (y+1) \dots x \dots y \dots (x+1) \dots$

is the Catalan number C_n

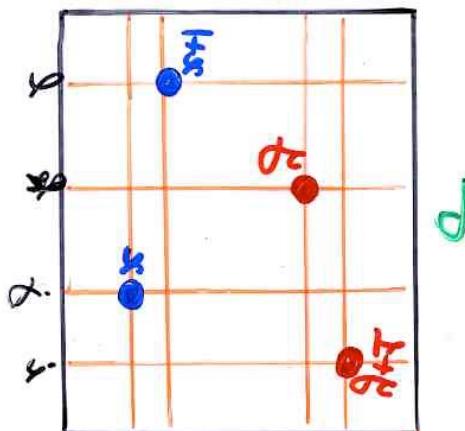
example $\sigma = 6 4 5 3 9 7 8 (10) 1 2$



inverse permutation σ^{-1}



permutation with "forbidden patterns":



31-24 and 24-31

Bernardi permutations

From work of Corteeel, Nadeau,
Steingrimsson, Williams
we know that parameter
"number of crossing" in alternating
tableaux :
same distribution as
"q-analog of Laguerre histories"

