

Course IMSc, Chennai, India

January-March 2018



The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 1

RSK

The Robinson-Schensted-correspondence (Ch1c)

IMSc, Chennai

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From Ch 1a, 1b:

The Robinson-Schensted correspondence

- Schensted's insertions
- geometric version with "shadow lines »
- Fomin "local rules" or "growth diagrams »

$$\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10)$$

$$(3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7)$$



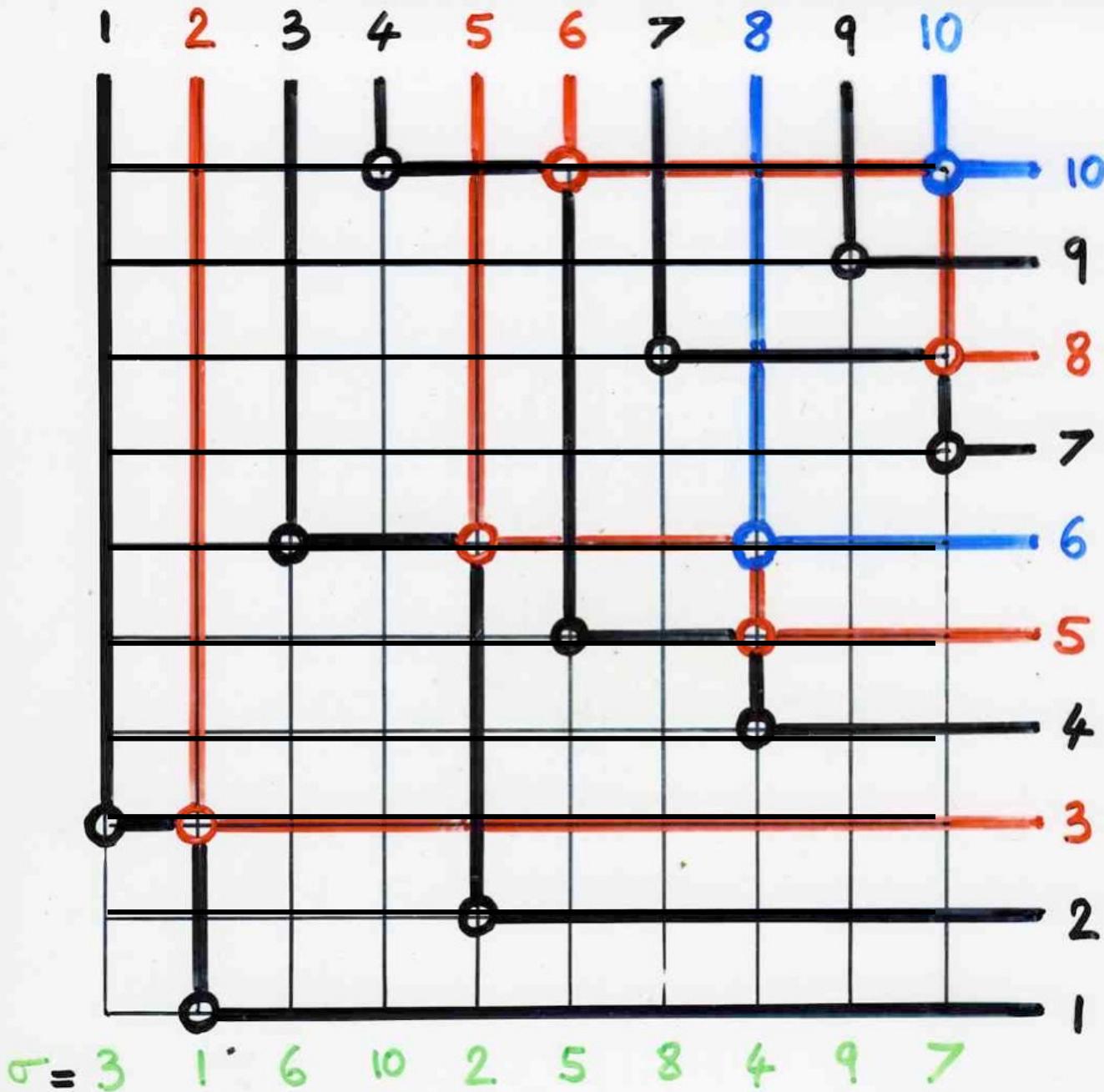
6	10			
3	5	8		
1	2	4	7	9

P

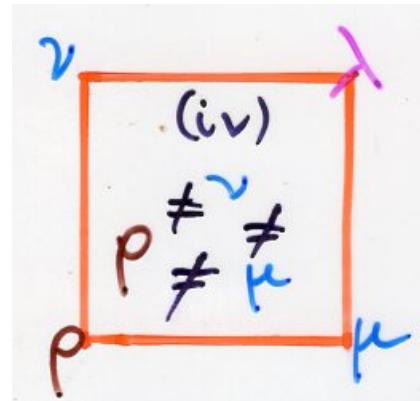
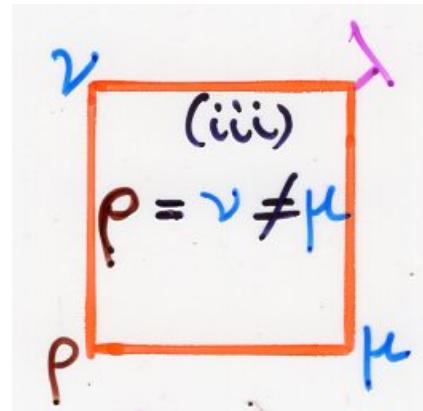
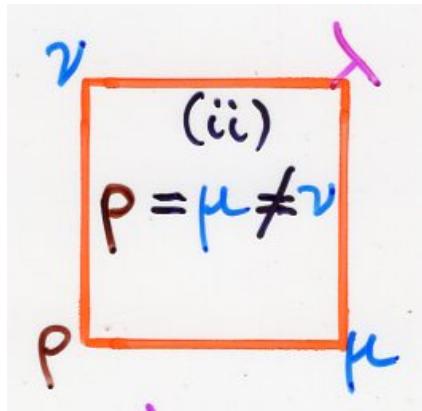
8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape



"local rules"

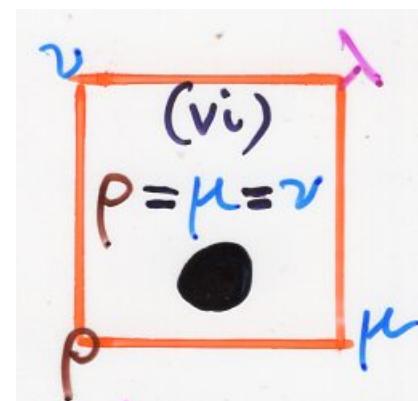
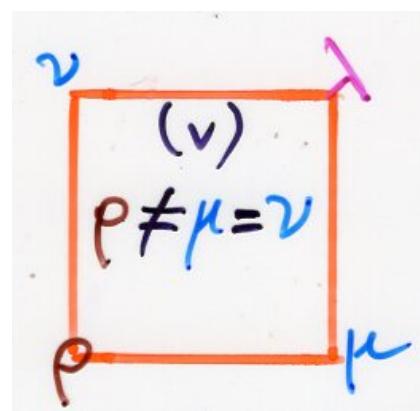
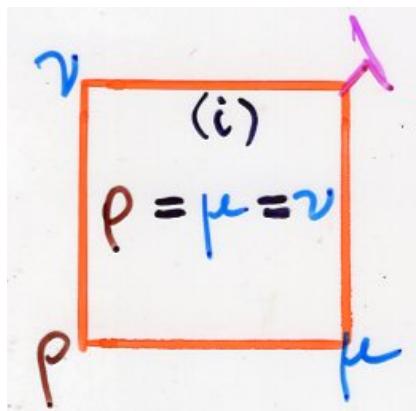


$$\mu \neq \nu$$

$$\lambda = \nu$$

$$\lambda = \mu$$

$$\lambda = \mu \cup \nu$$

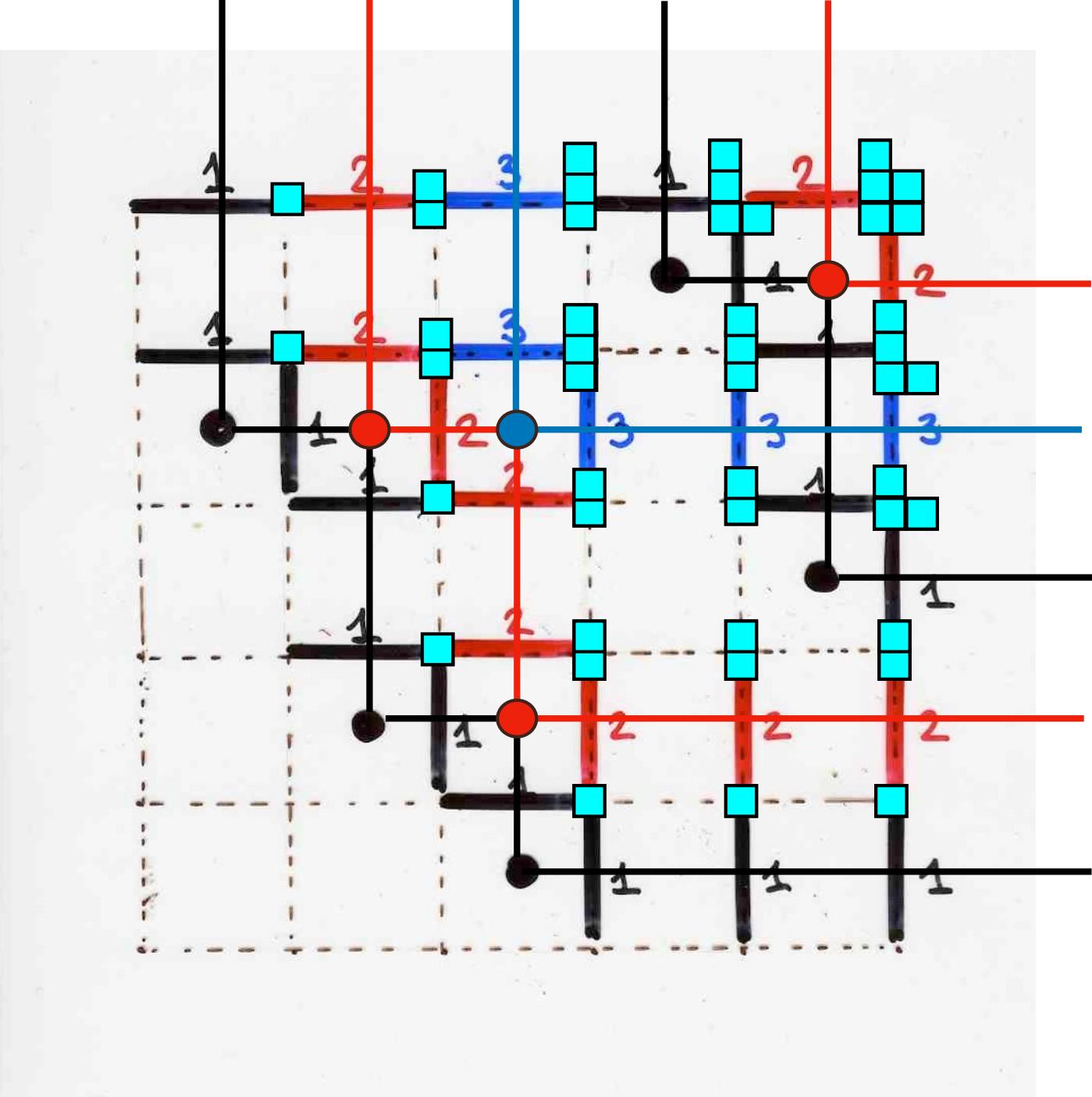


$$\mu = \nu$$

$$\lambda = \rho$$

$$\lambda = \begin{cases} \mu & \text{if } i=0 \\ \nu + (i+1) & \text{otherwise} \end{cases}$$

$$\lambda = \begin{cases} \rho & \text{if } i=0 \\ \nu + (i+1) & \text{otherwise} \end{cases}$$



Schützenberger

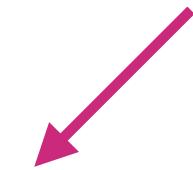
Duality!

5		
3	4	
1	2	

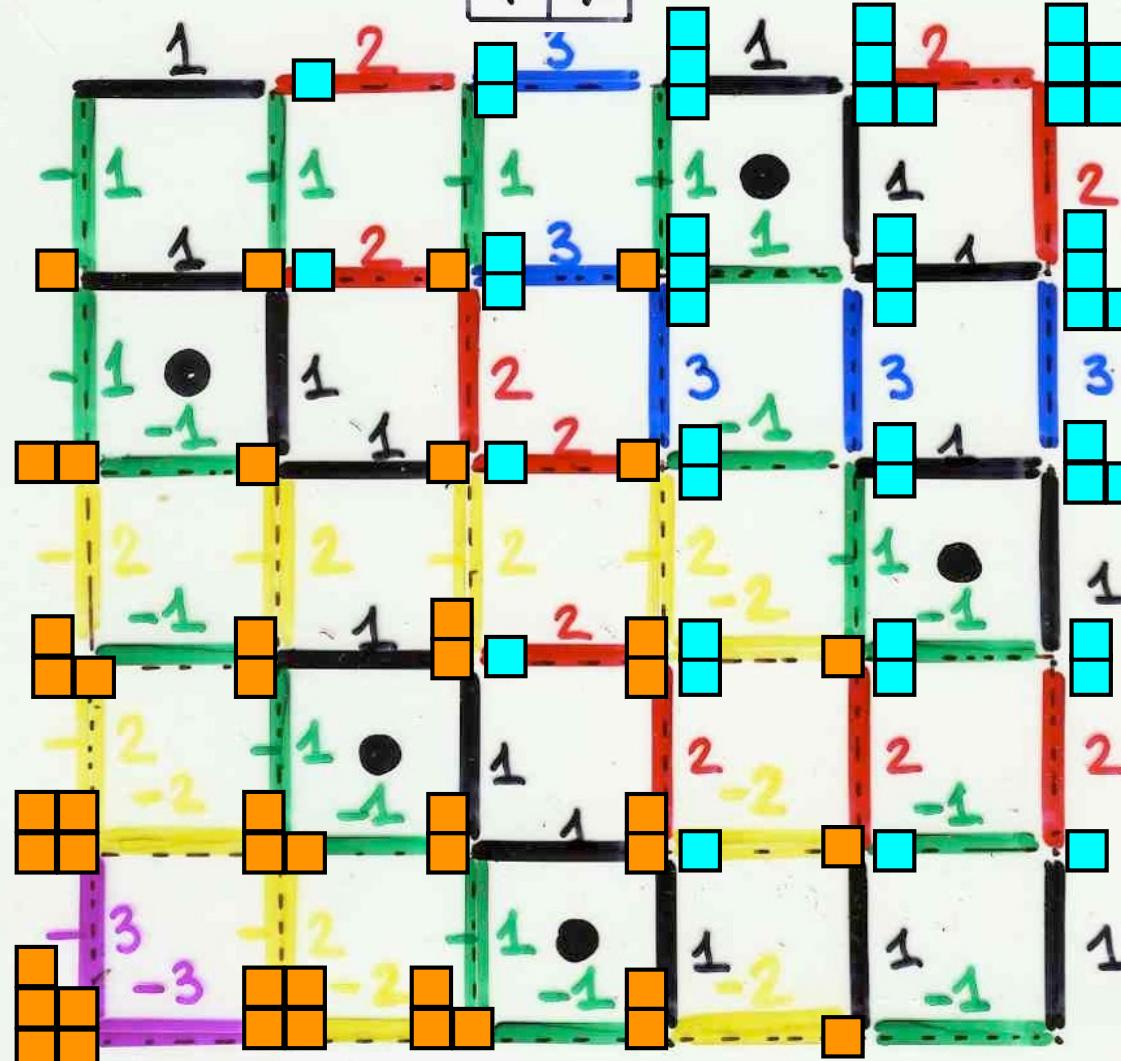


3		
2	5	
1	4	

4		
2	5	
1	3	



5		
2	4	
1	3	



"The **cellular** ansatz."

quadratic
algebra **Q**

$$UD = DU + \text{Id}$$

Q-tableaux

combinatorial objects
on a 2D lattice

permutations

towers placements

representation of **Q**
by combinatorial
operators

bijections

RSK

pairs of
Young tableaux

(i) first step

(ii) second step

commutations

rewriting rules

planarization

an example

Heisenberg operators U, D

creation and annihilation operators
quantum mechanics

$$UD = D U + Id$$

commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering
in physics

The monomials $\{D^i U^j\}_{i,j \geq 0}$
form a basis of the
Weyl-Heisenberg algebra

$$Q = \mathbb{C}\langle U, D \rangle / J$$

non-commutative polynomials
in variable U and D
(free associative algebra)

J ideal generated by
the relation $UD = DU + I$

$$UD = DU + Id$$

commutations

$$UD \rightarrow DU \quad UD \rightarrow Id$$

rewriting rules

UUDD

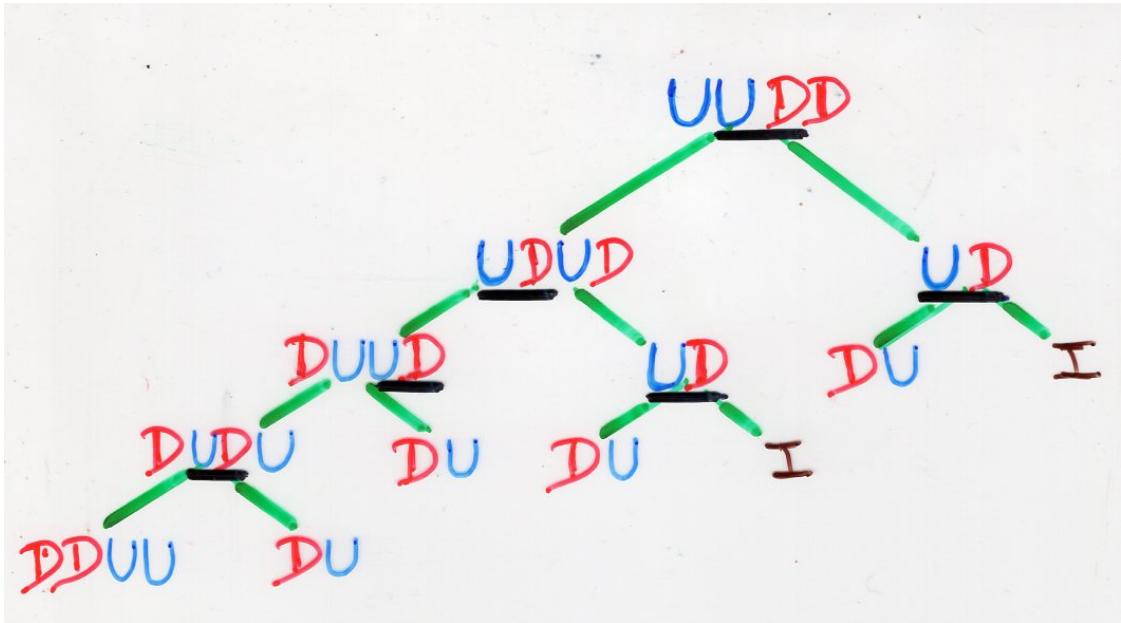
$$UUDD = UDUD + UD$$

$$= D U U D + 2 U D$$

$$= (DUDU + DU) + 2(DU + Id)$$

$$= (DDUU + 2DU) + 2(DU + Id)$$

$$= DDUU + 4DU + 2 Id$$



$$U^2 D^2 = D^2 U^2 + 4 D U + 2 I$$

this polynomial is independant
of the order of the substitutions.

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

Planarization of the rewriting rules

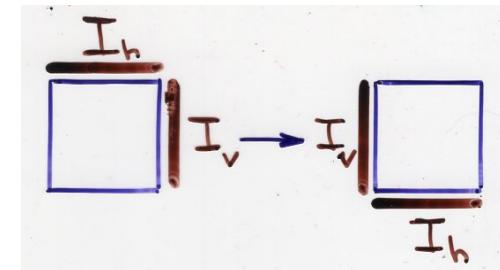
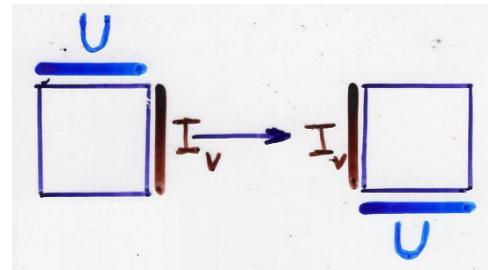
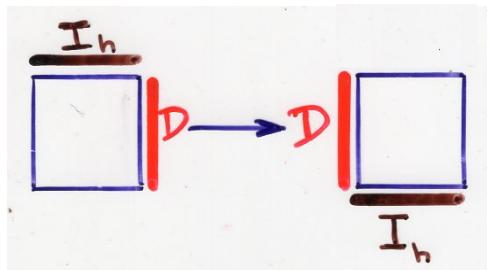
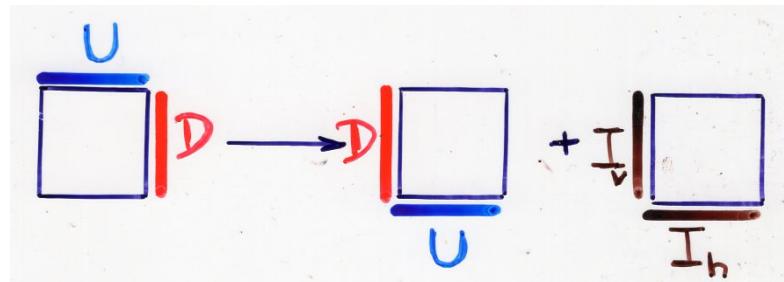
$$UD = DU + Id$$

commutations

$$UD \rightarrow DU \quad UD \rightarrow Id$$

rewriting rules

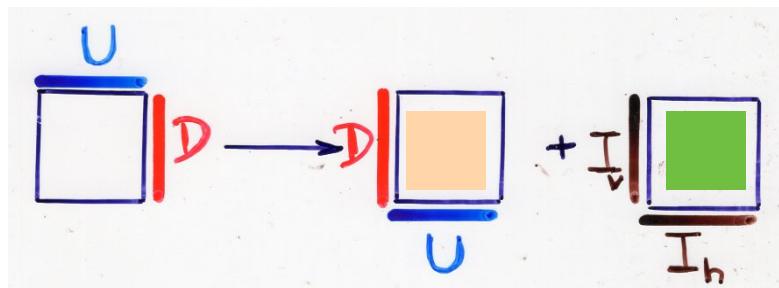
planarization of the rewriting rules



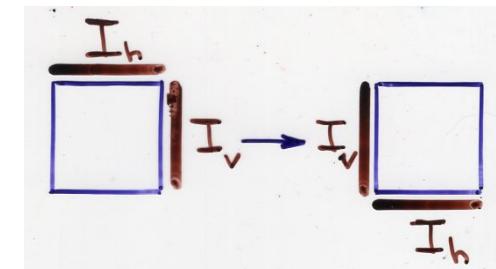
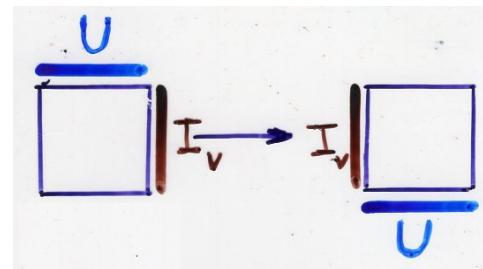
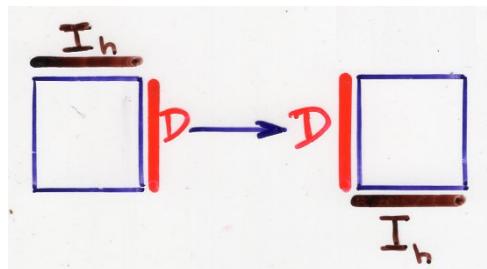
homogenization
of the system
of commutations
relations

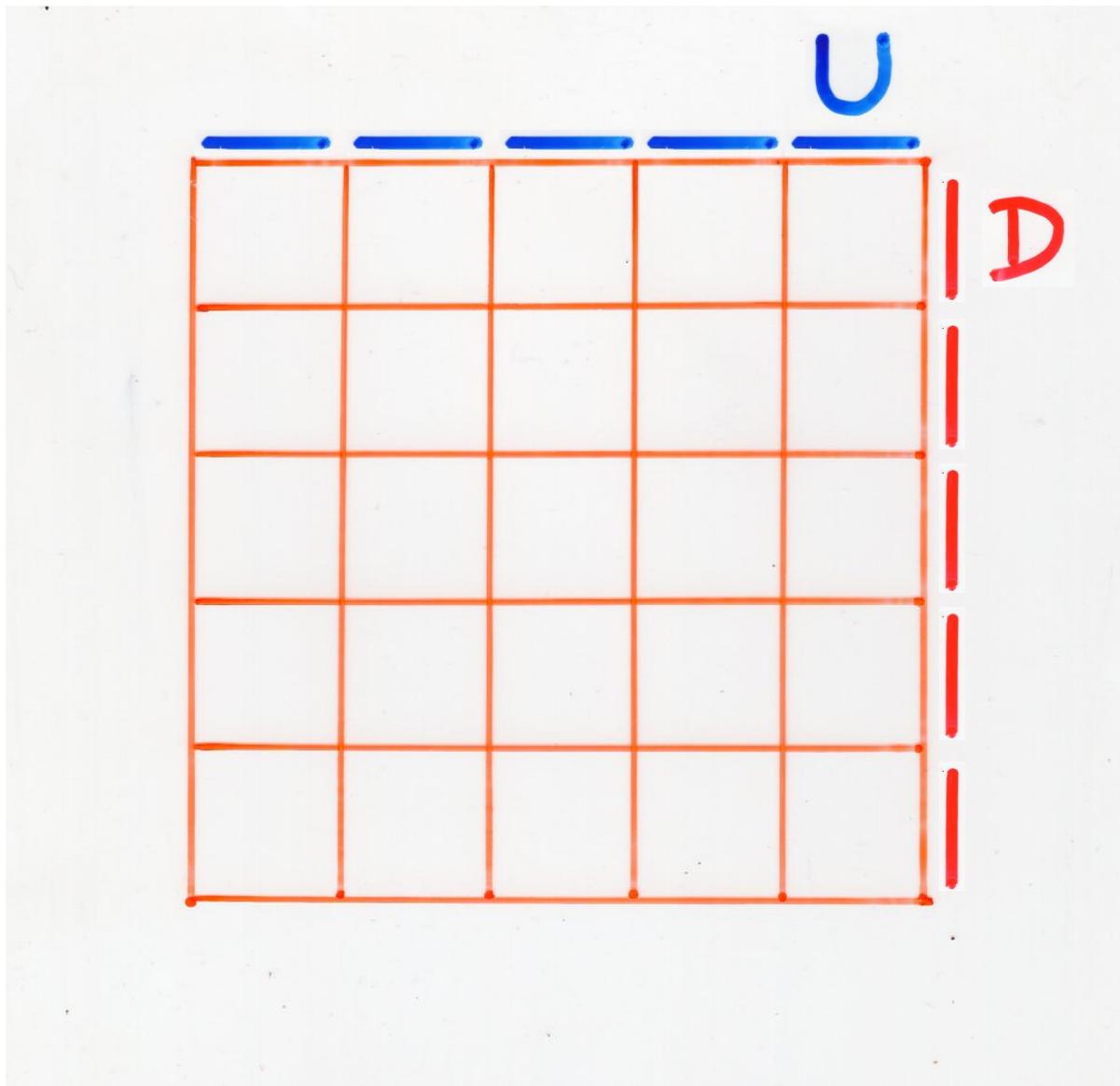
$$\left\{ \begin{array}{l} UD = DU + I_v I_h \\ UI_v = I_v U \\ I_h D = DI_h \\ I_h I_v = I_v I_h \end{array} \right.$$

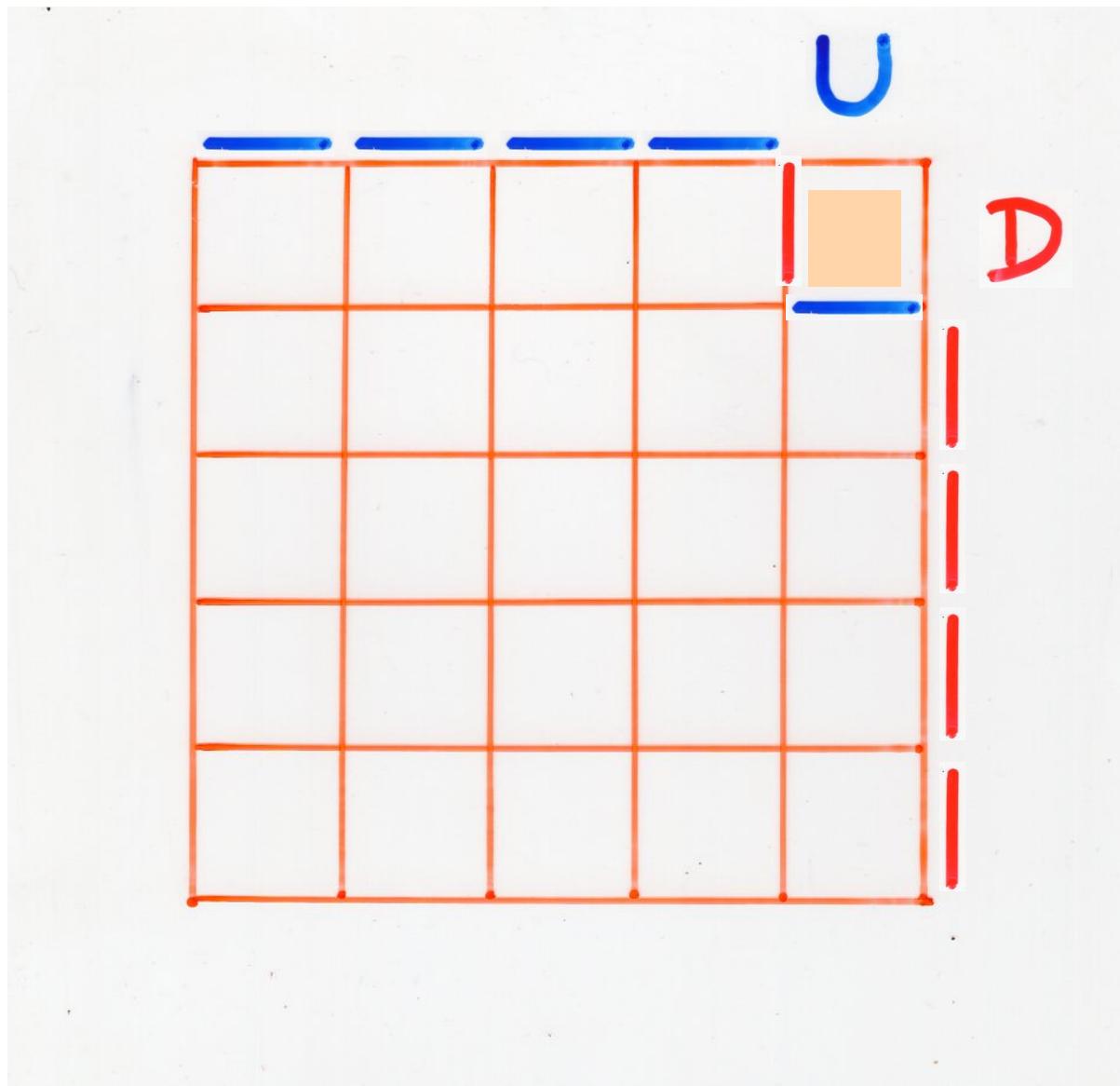
$$\left\{ \begin{array}{l} UD \rightarrow DU \\ UI_v \rightarrow I_v U \\ I_h D \rightarrow DI_h \\ I_h I_v \rightarrow I_v I_h \end{array} \right. \quad \begin{array}{l} UD \rightarrow I_v I_h \\ \text{rewriting rules} \end{array}$$

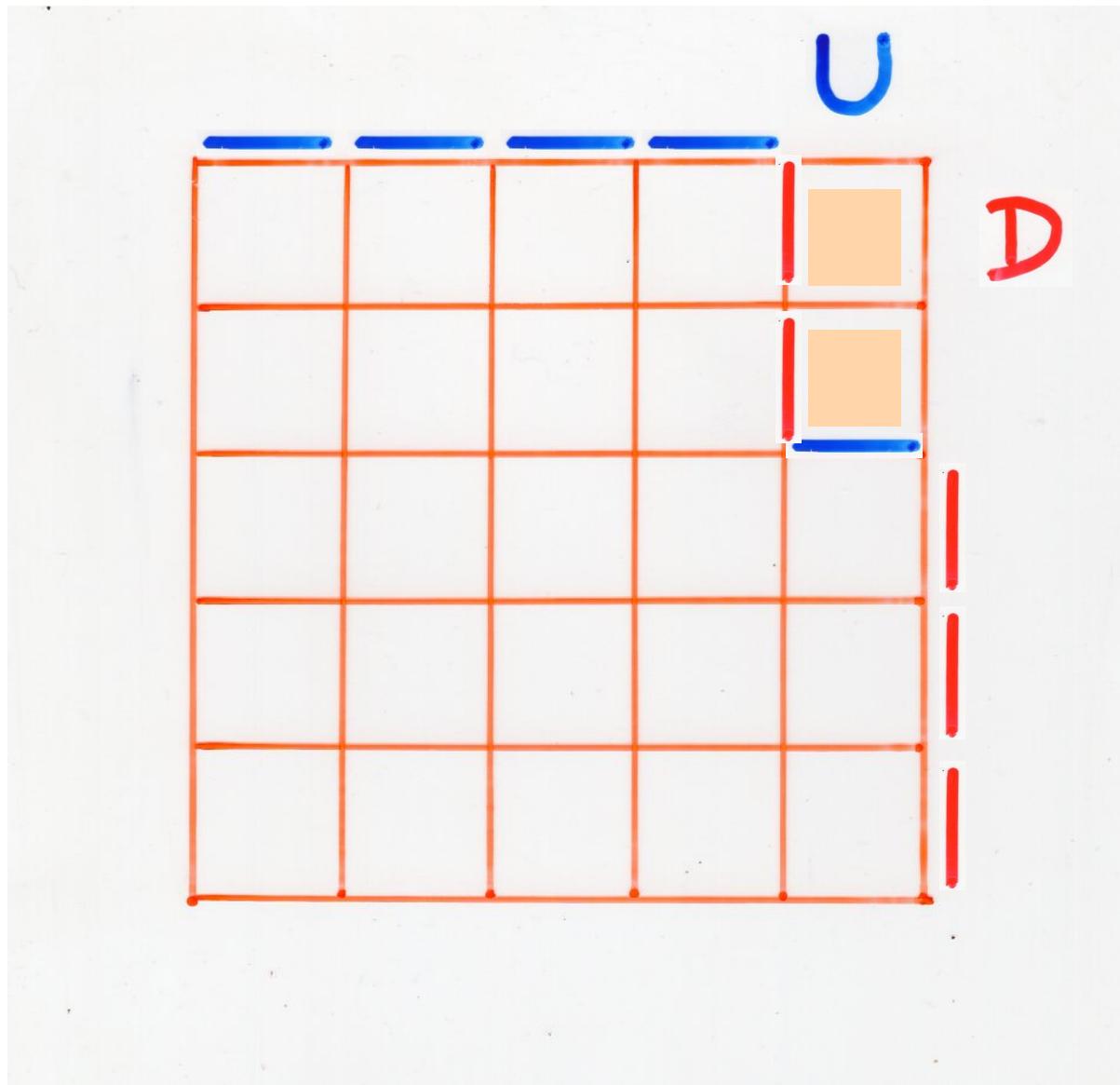


"planarization" of the "rewriting rules"

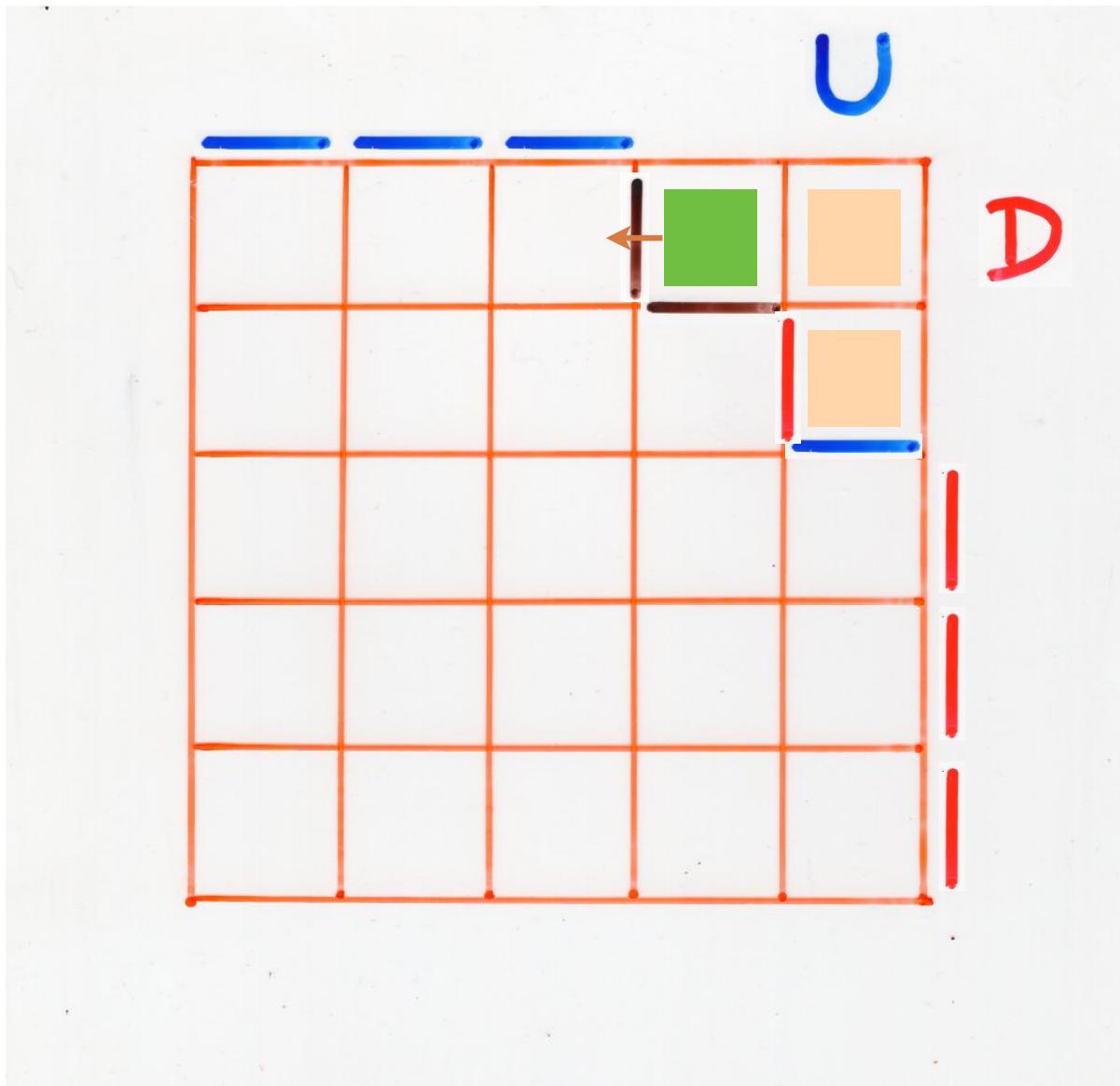


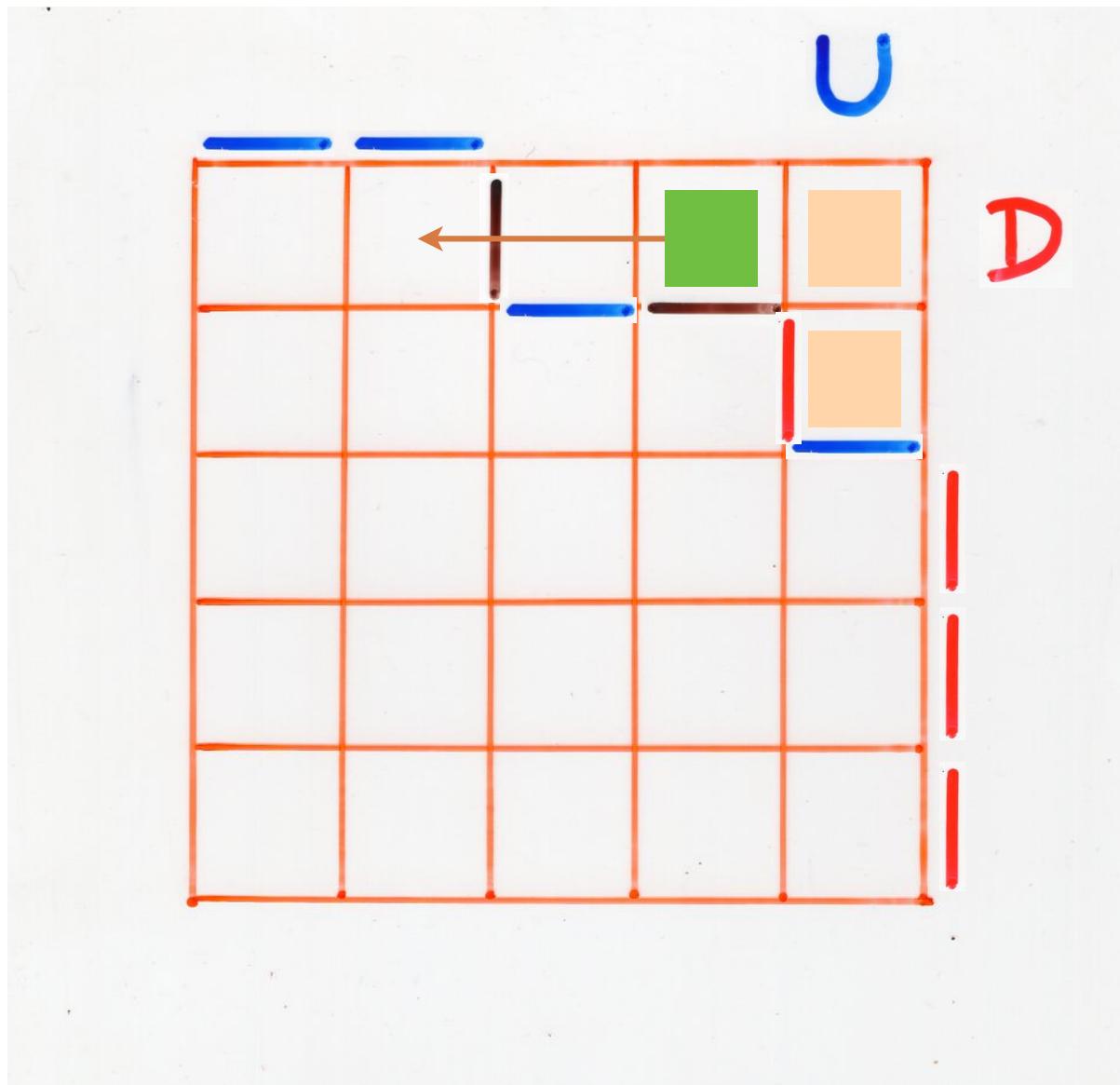


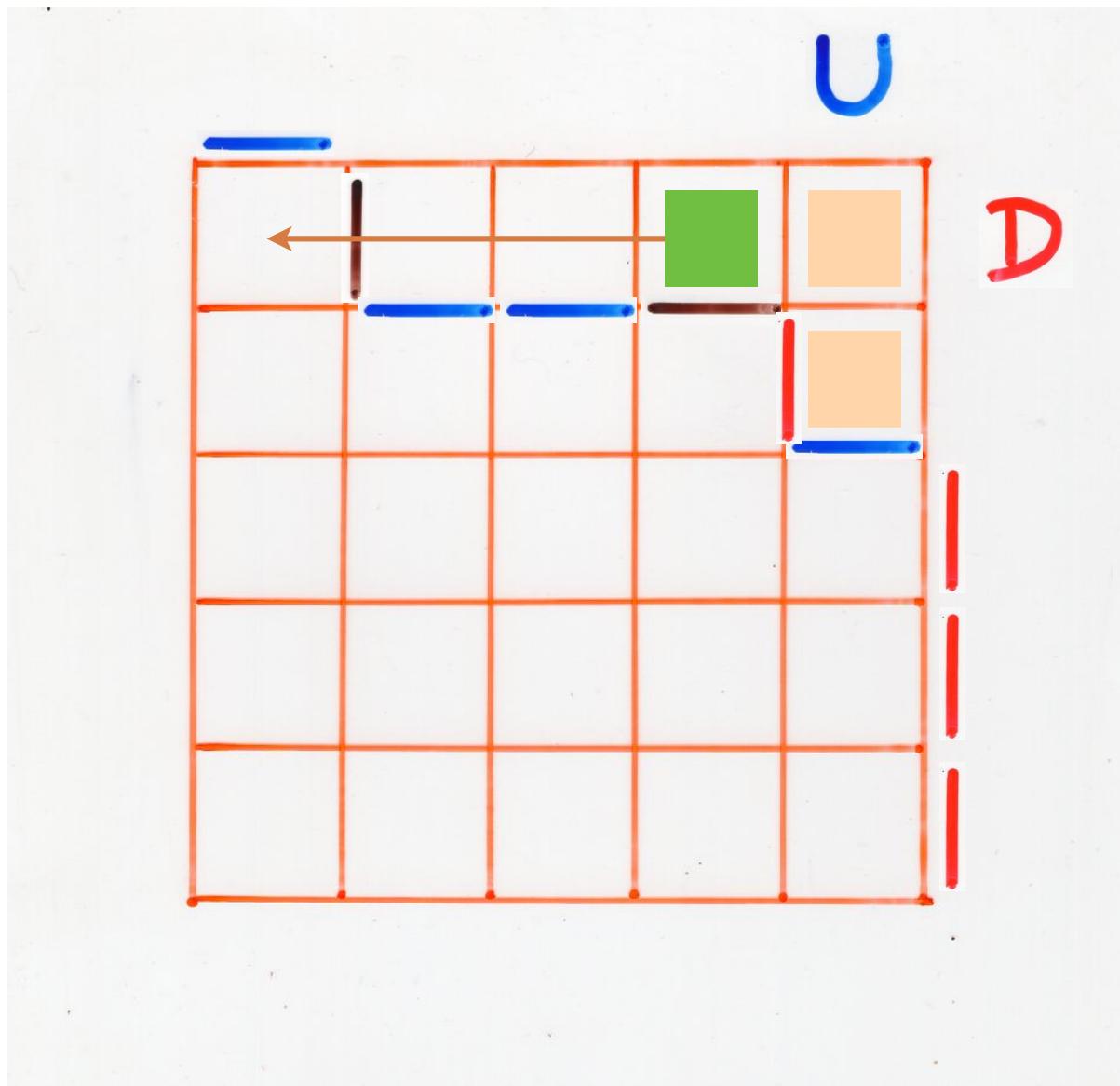


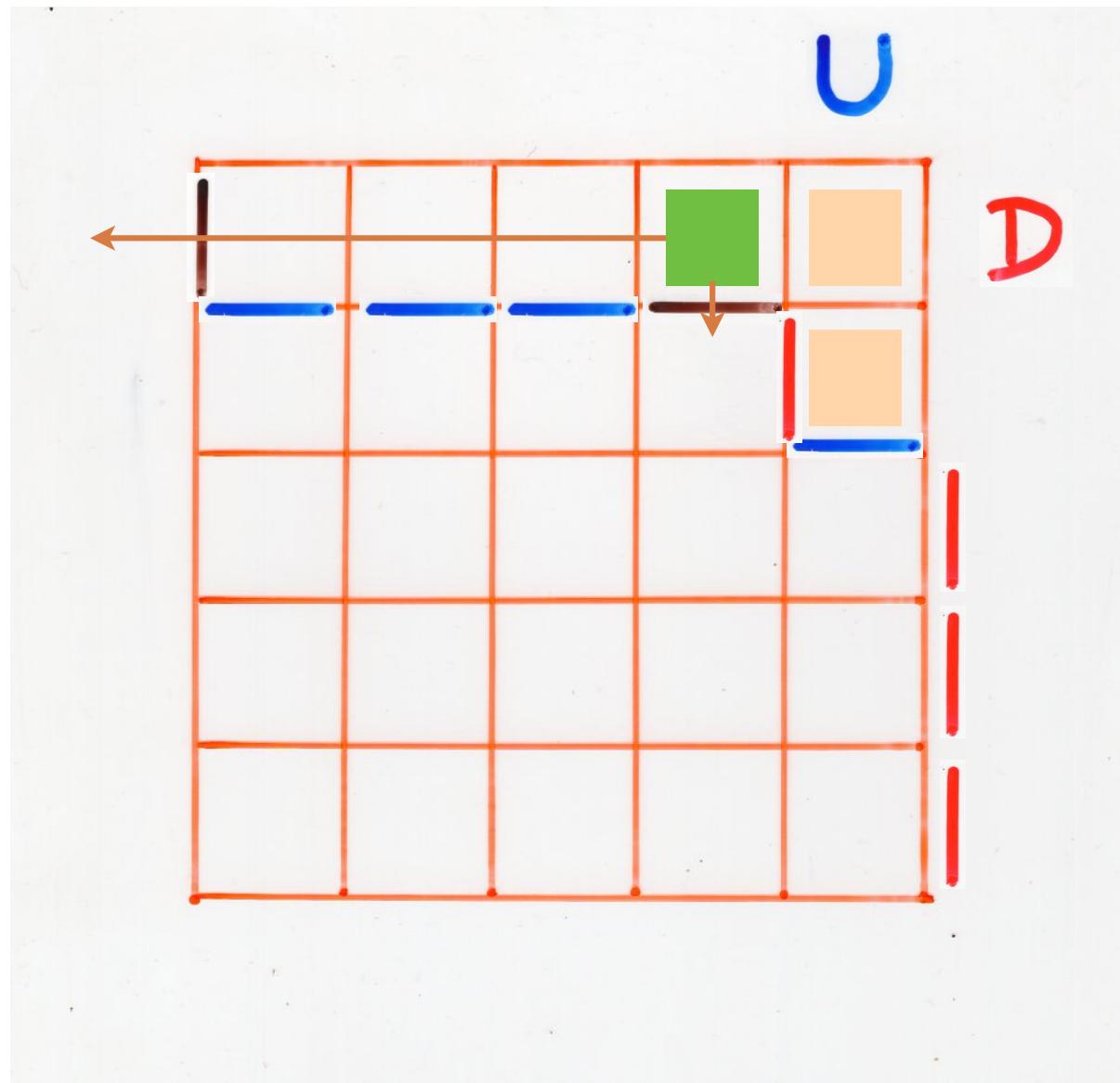


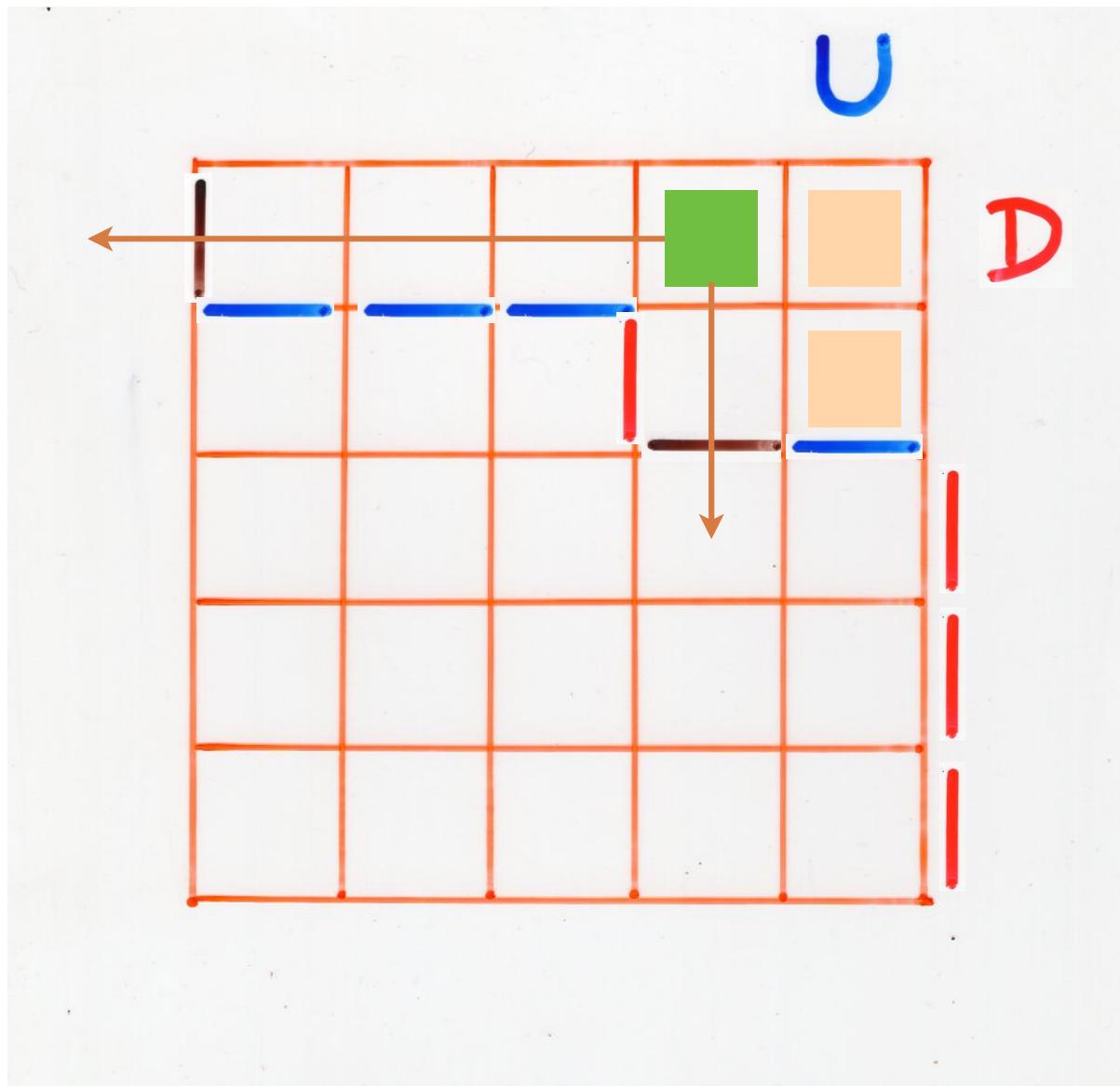
$\text{---} \quad I_h$

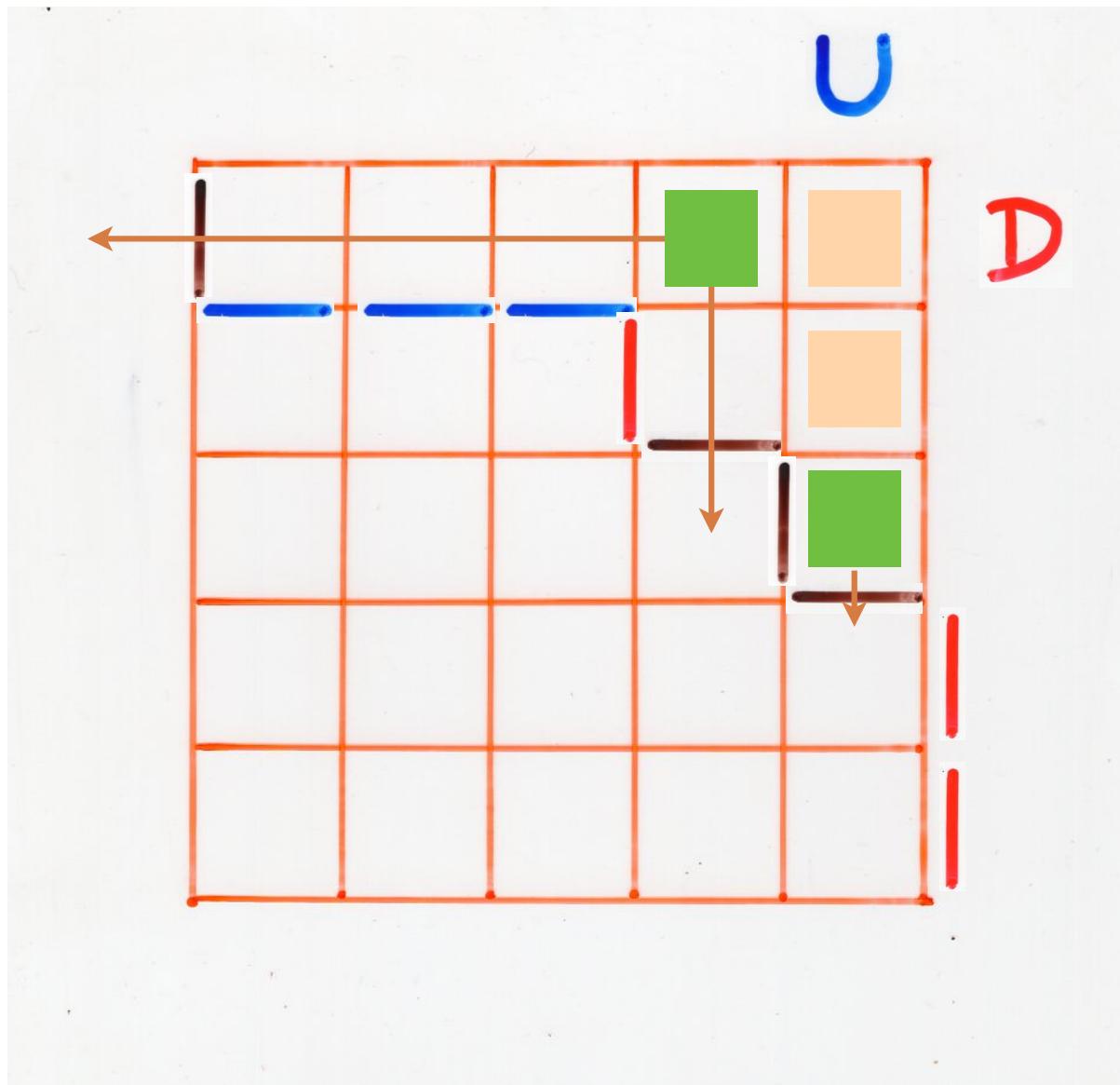


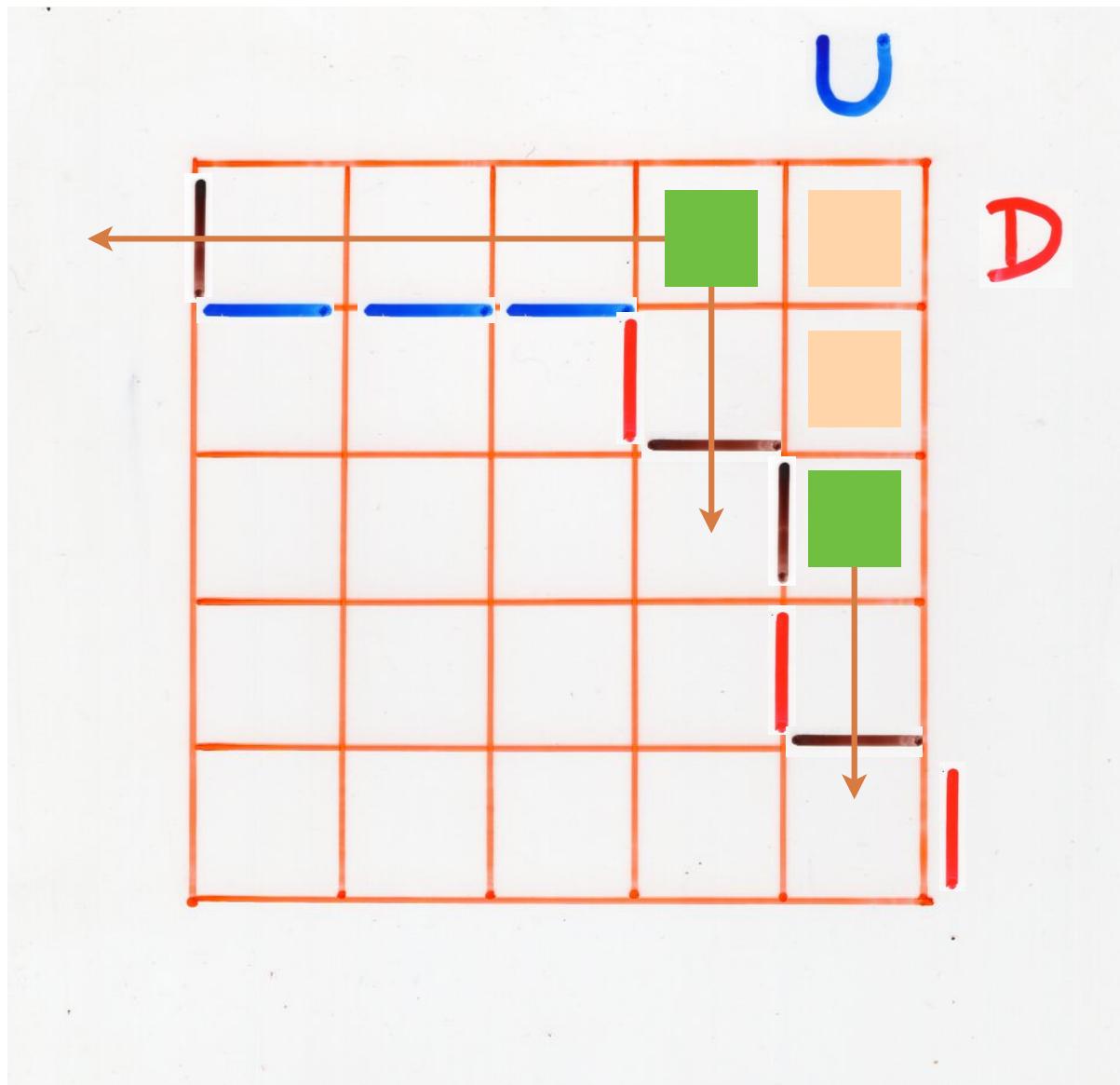


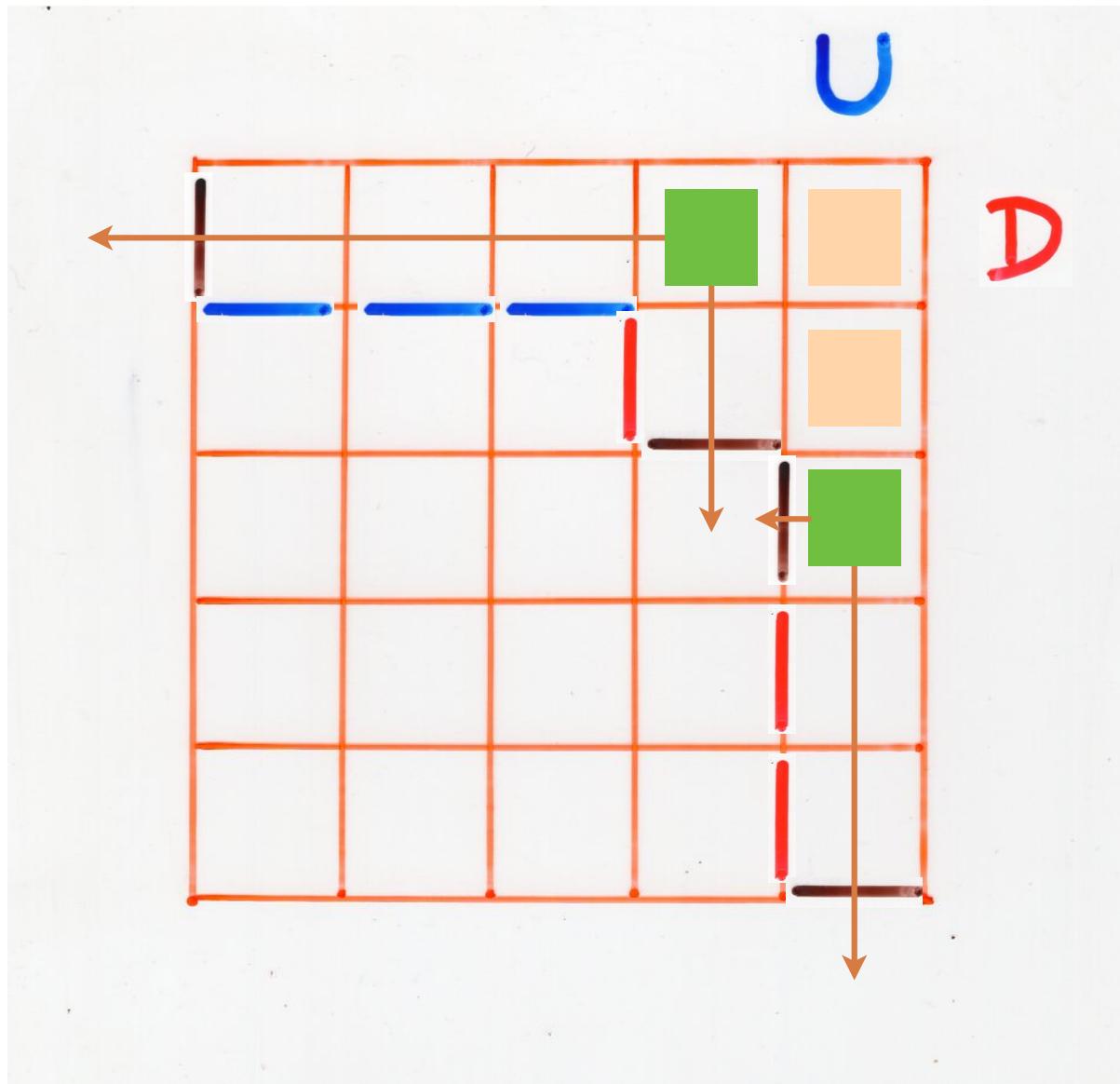


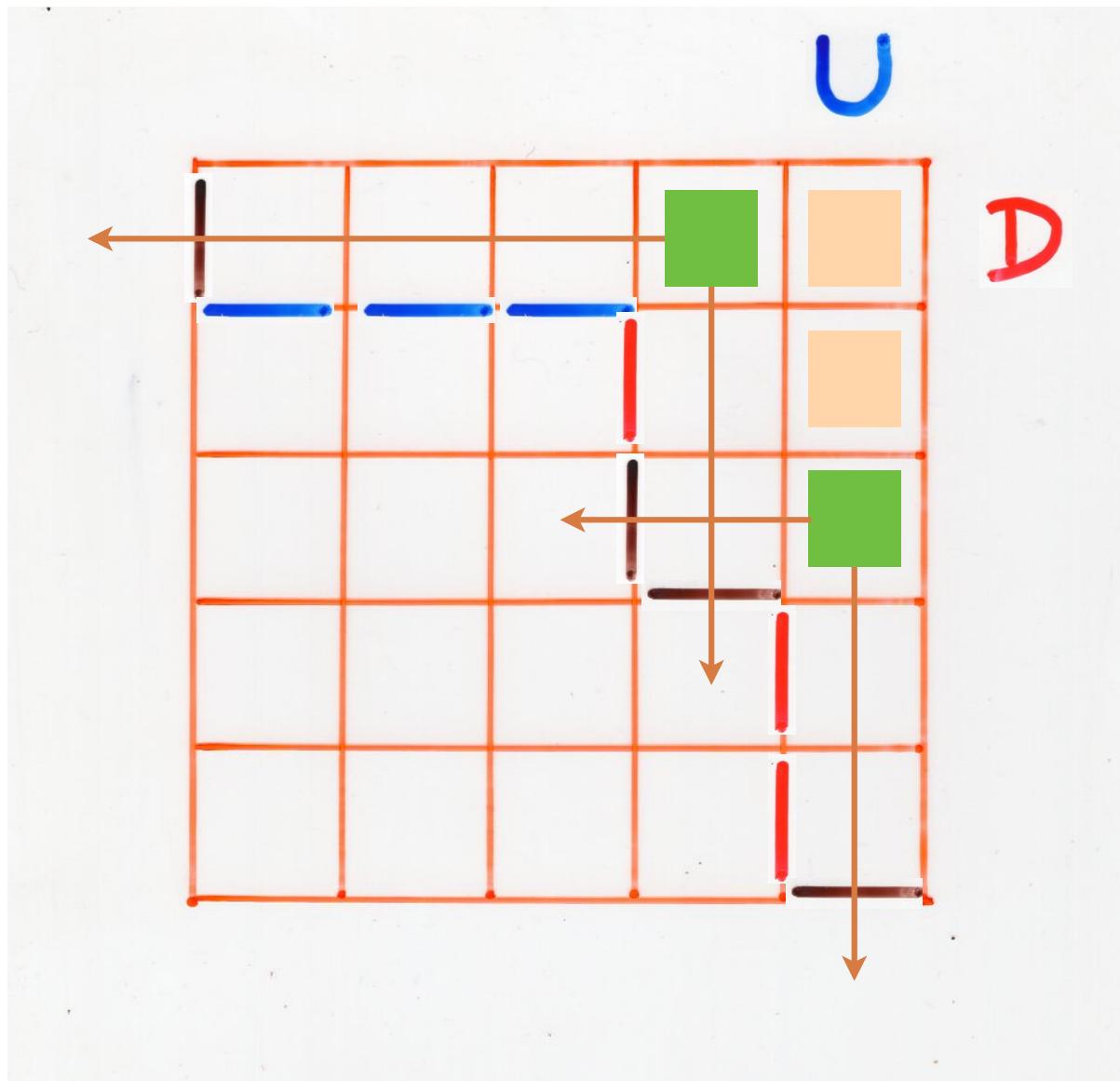


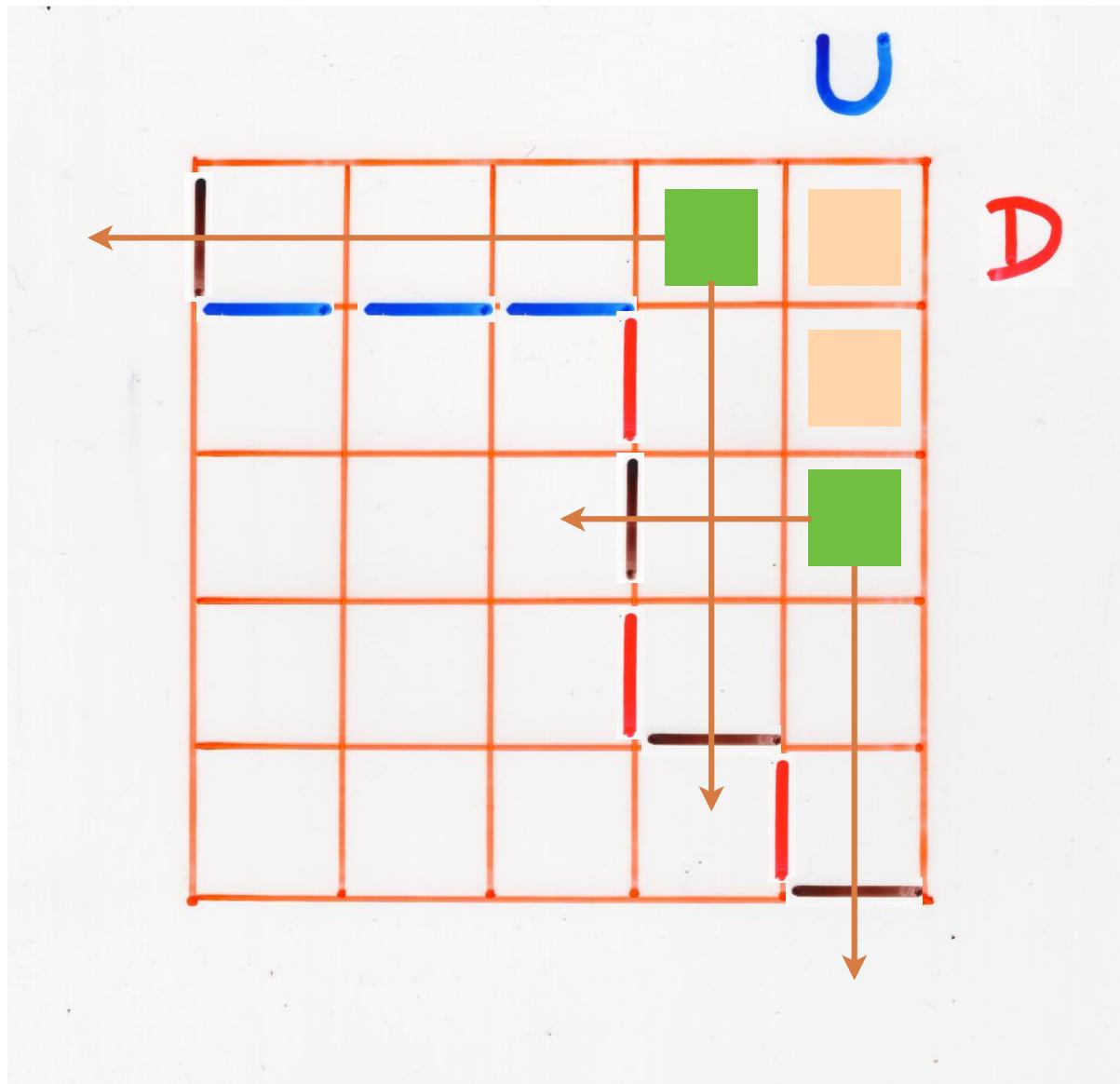


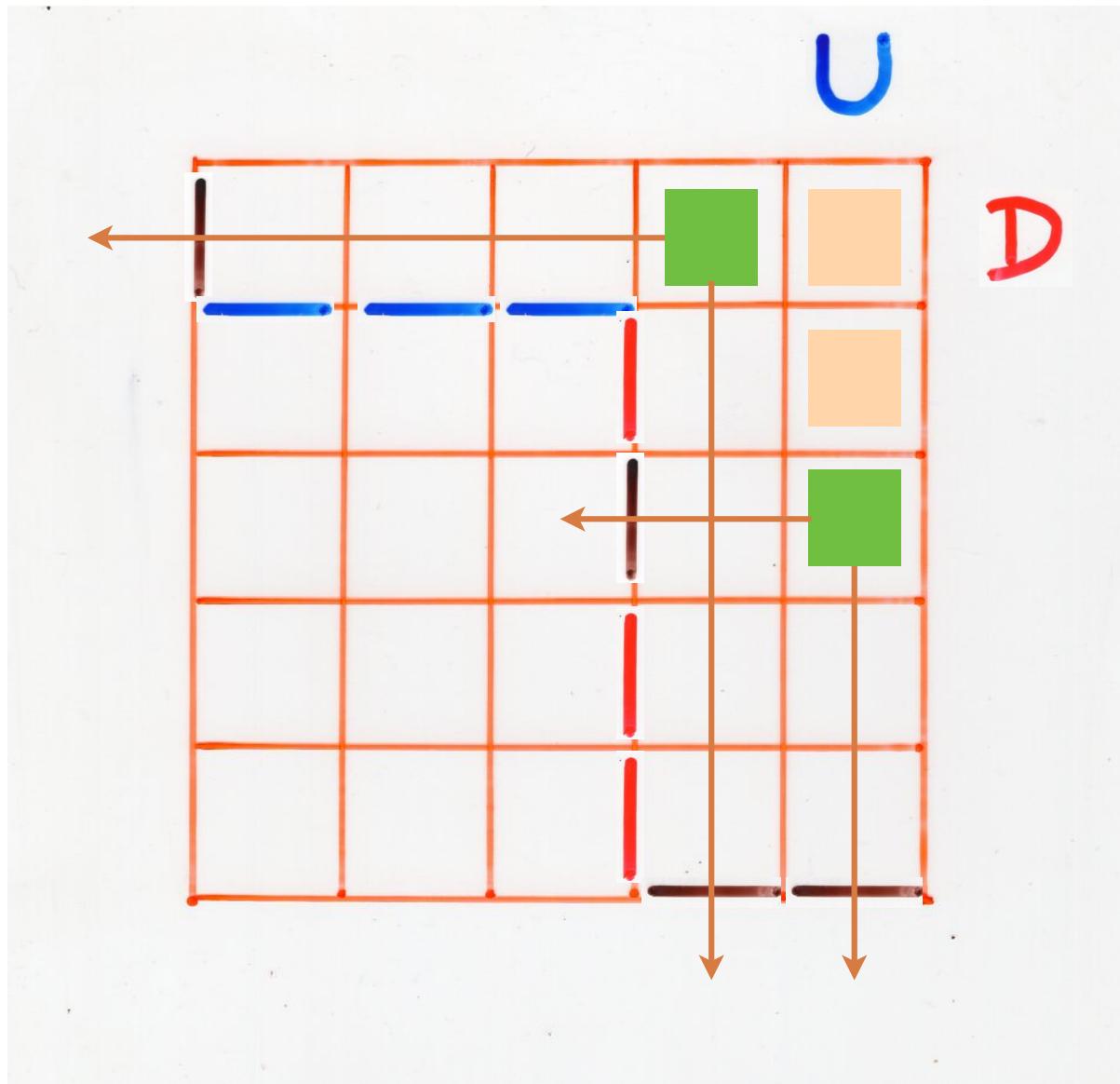


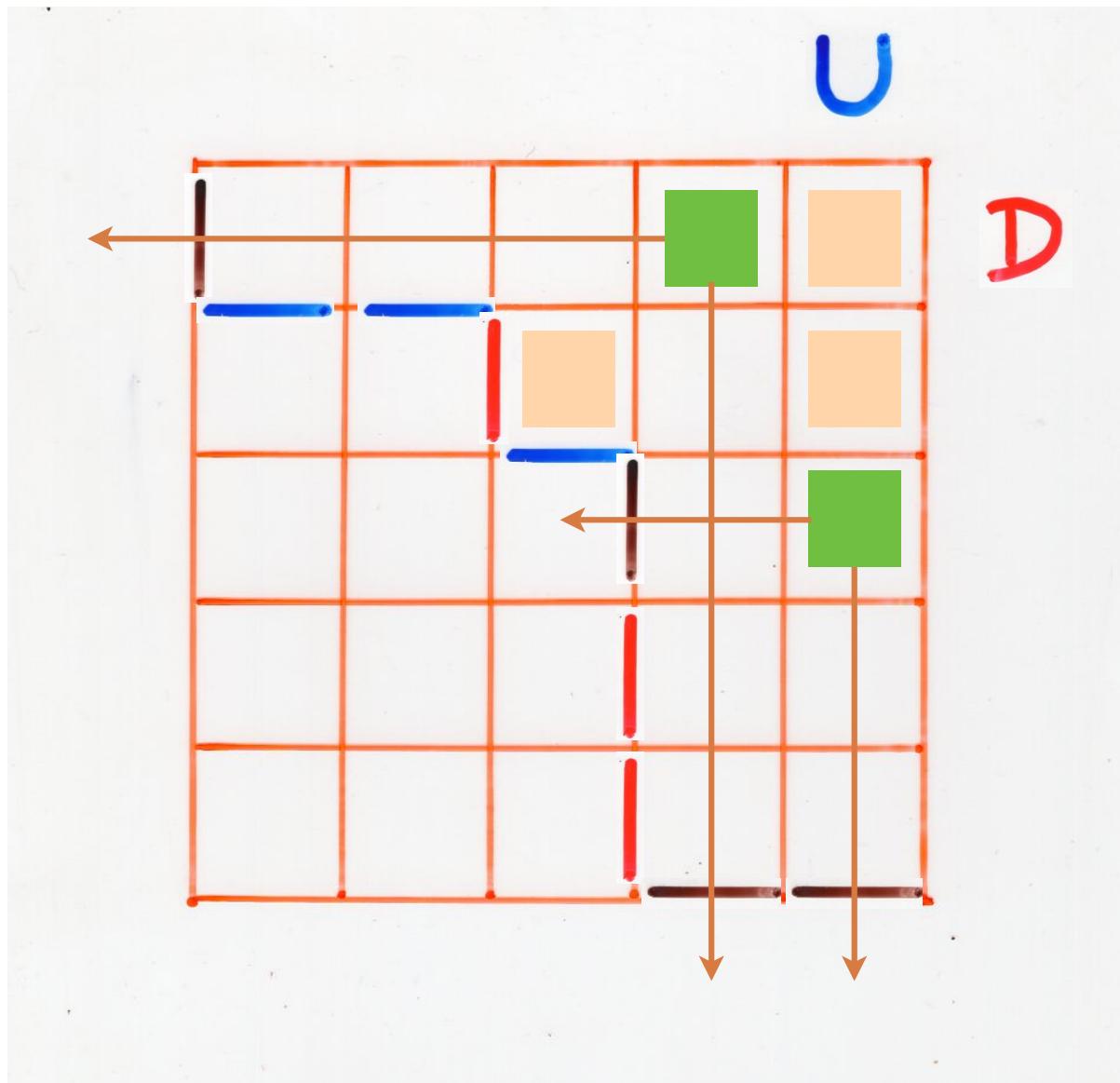


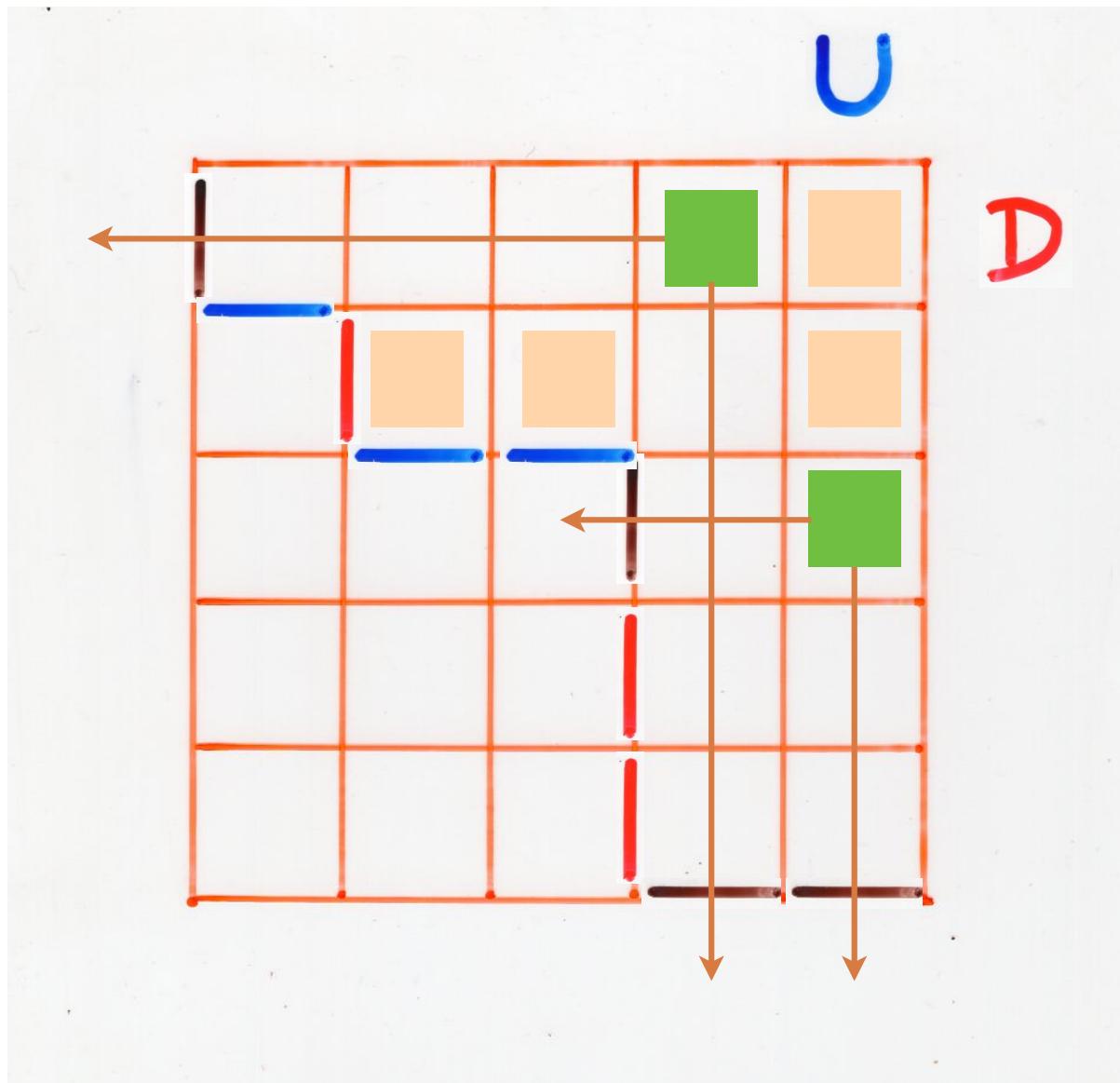


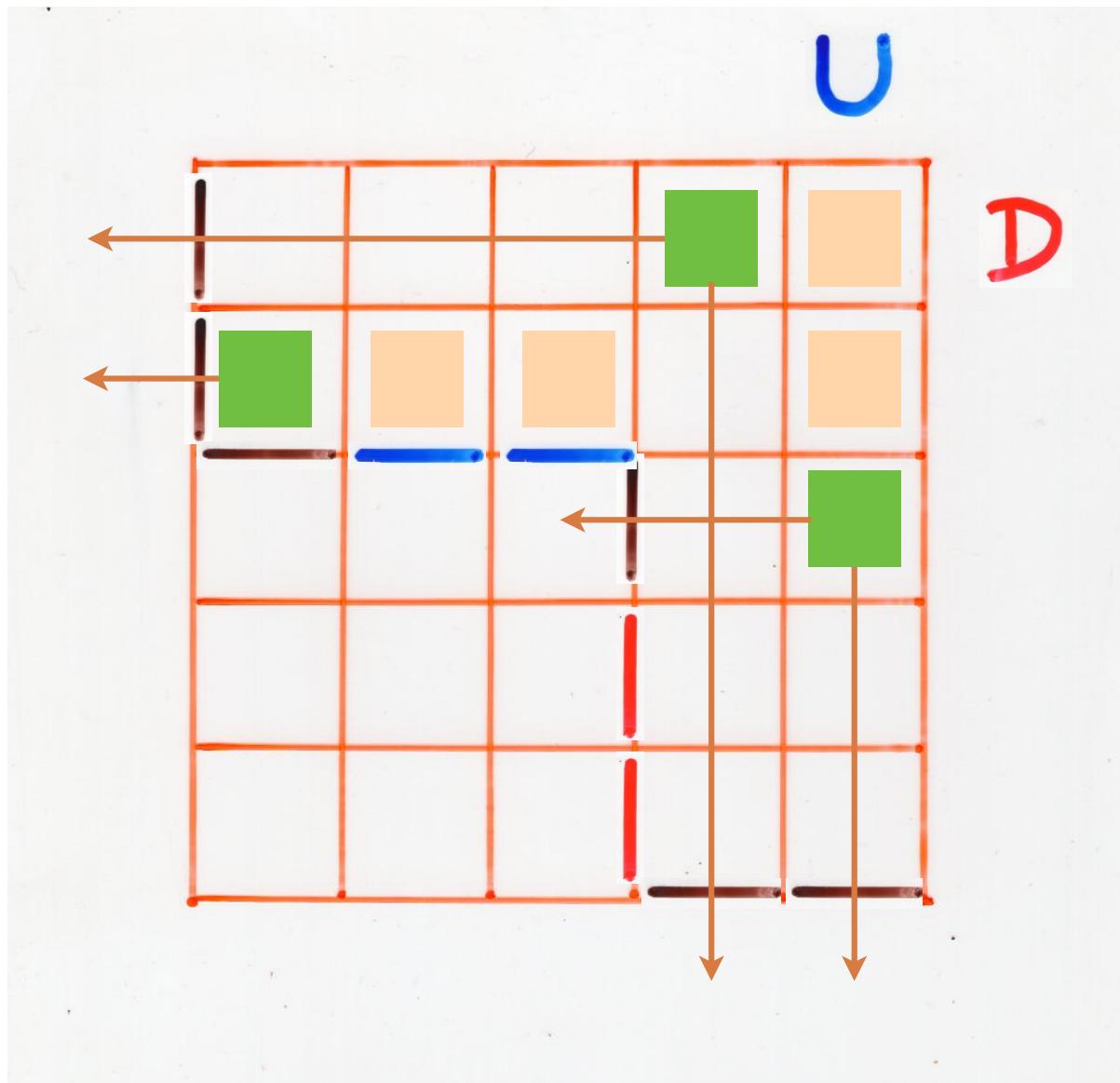


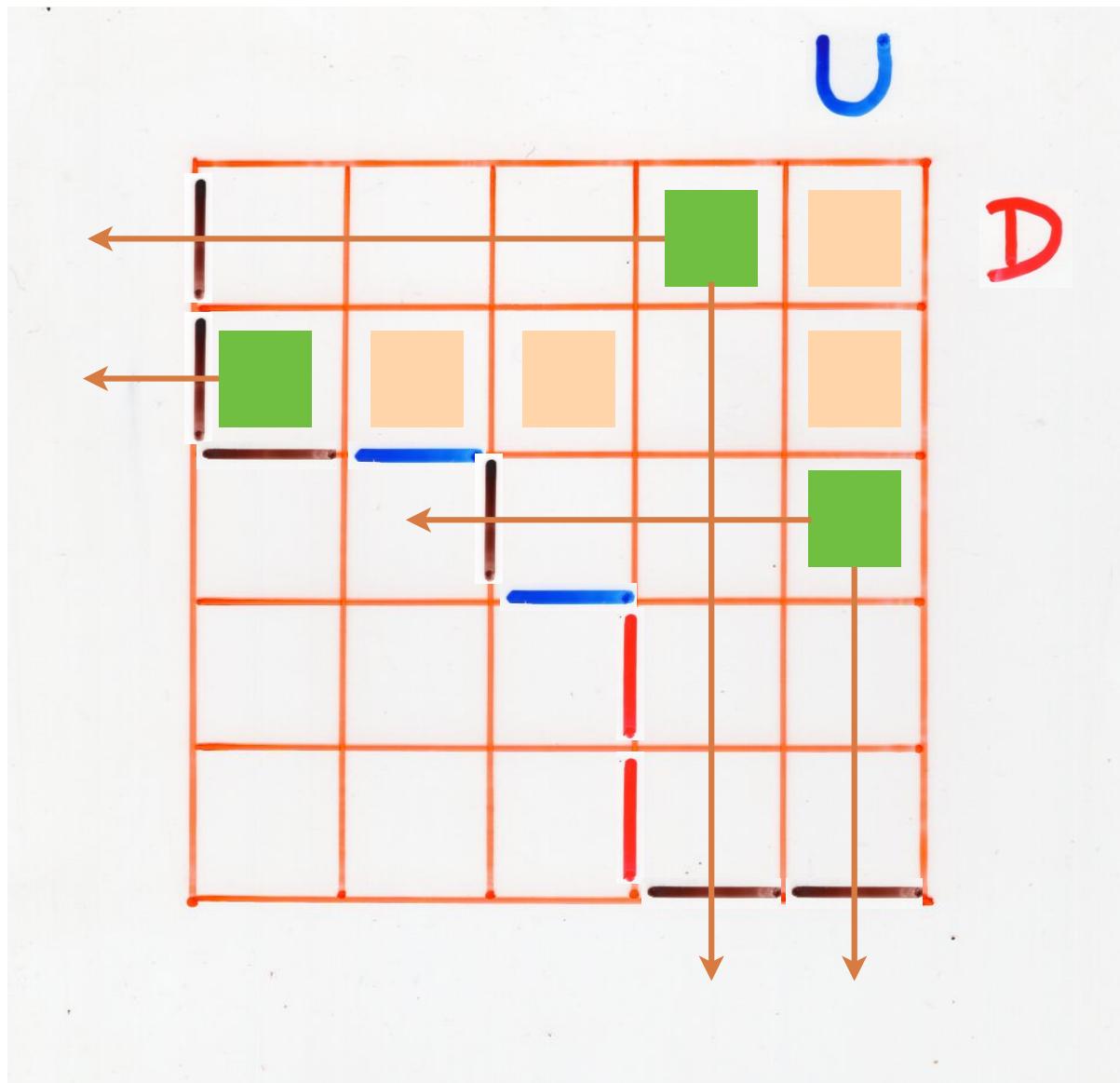


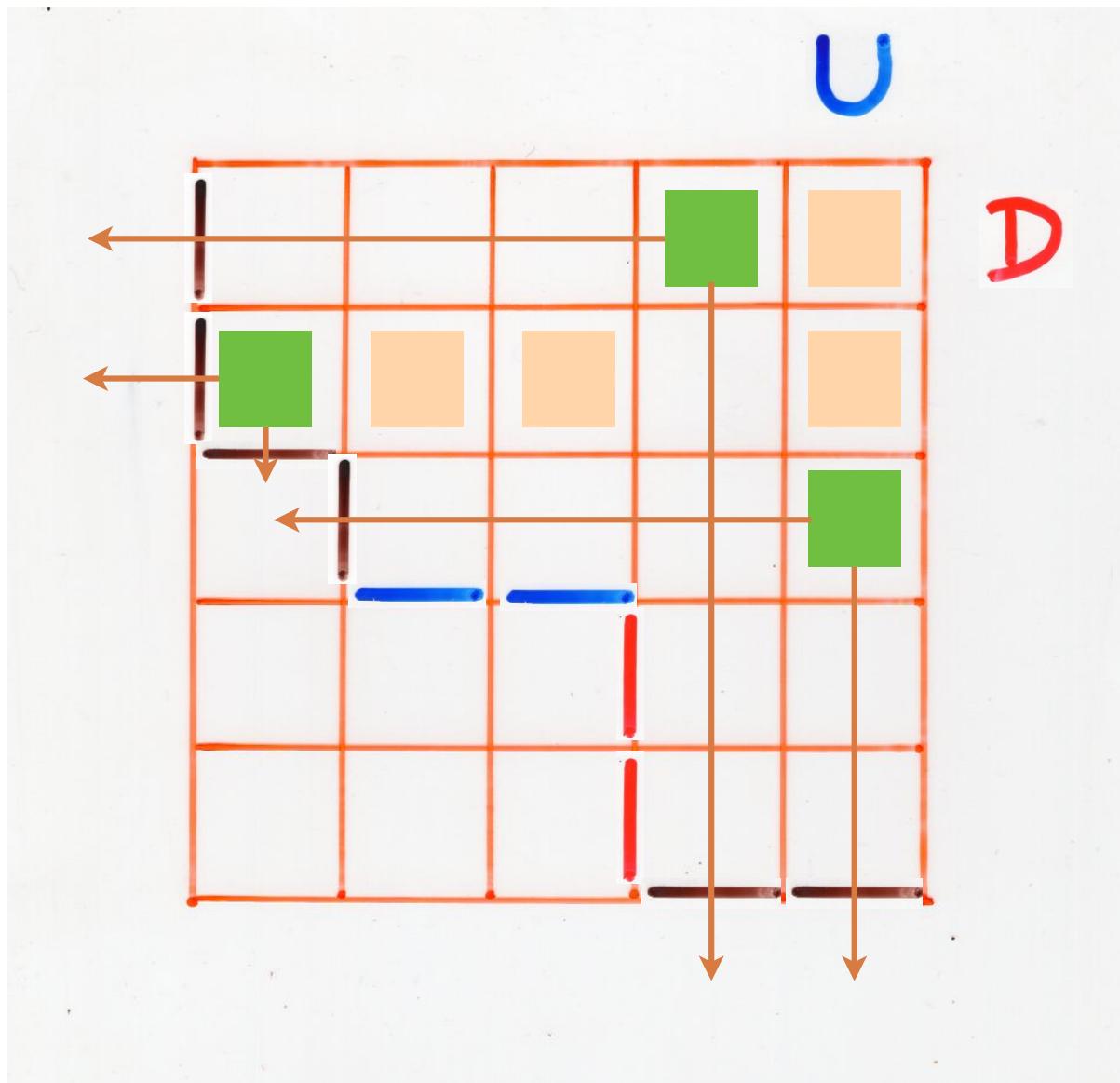


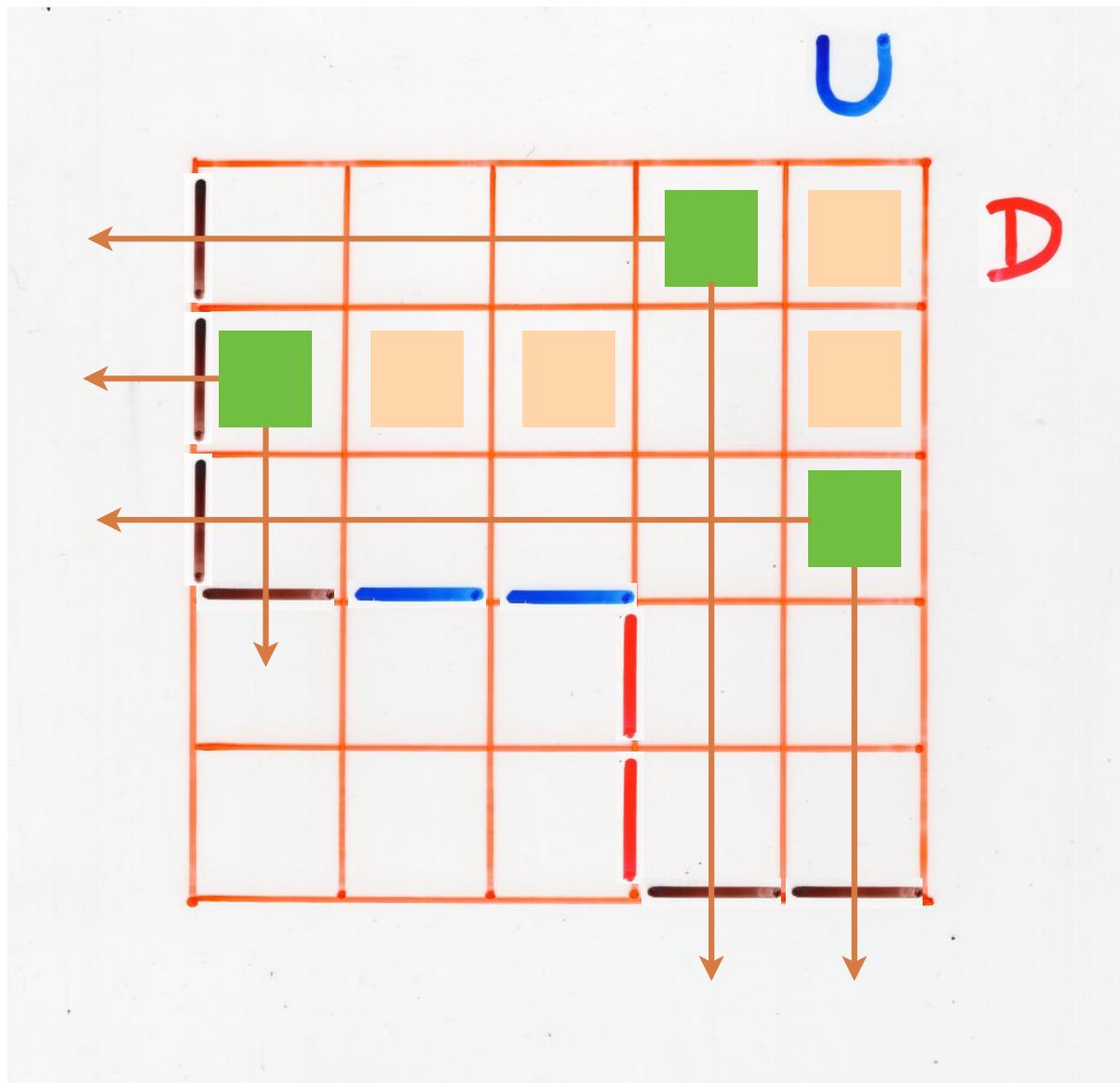


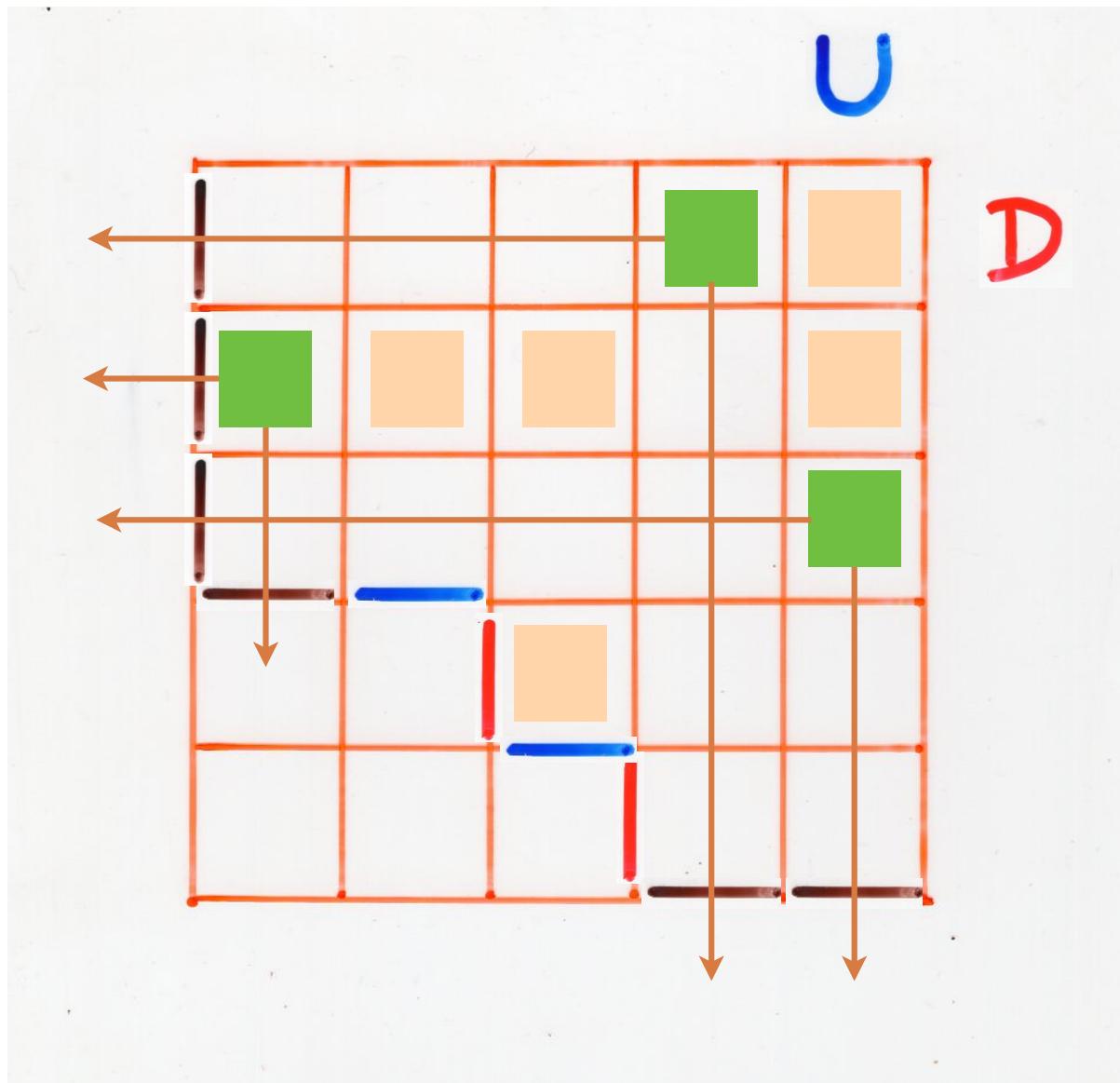


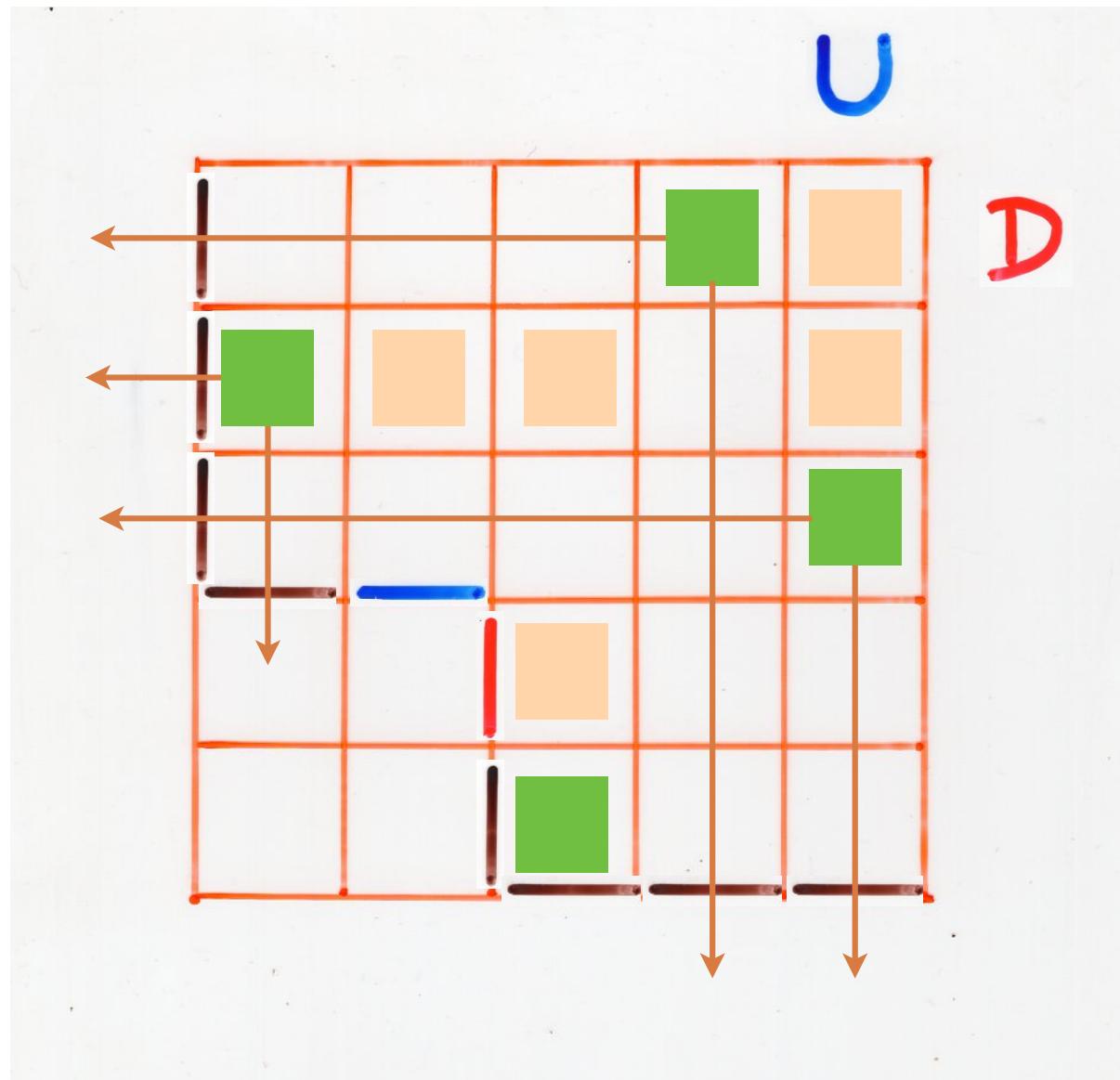


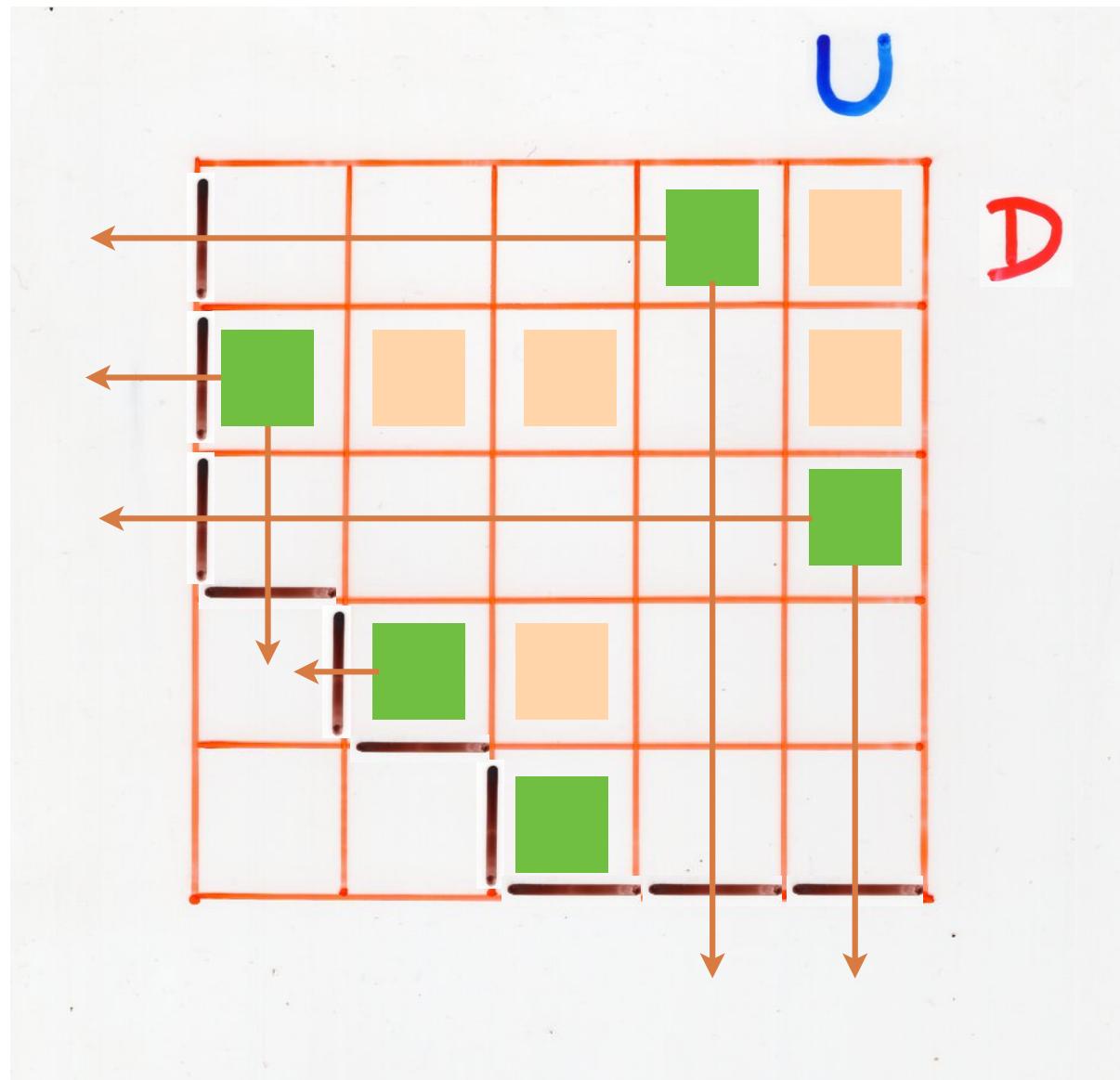


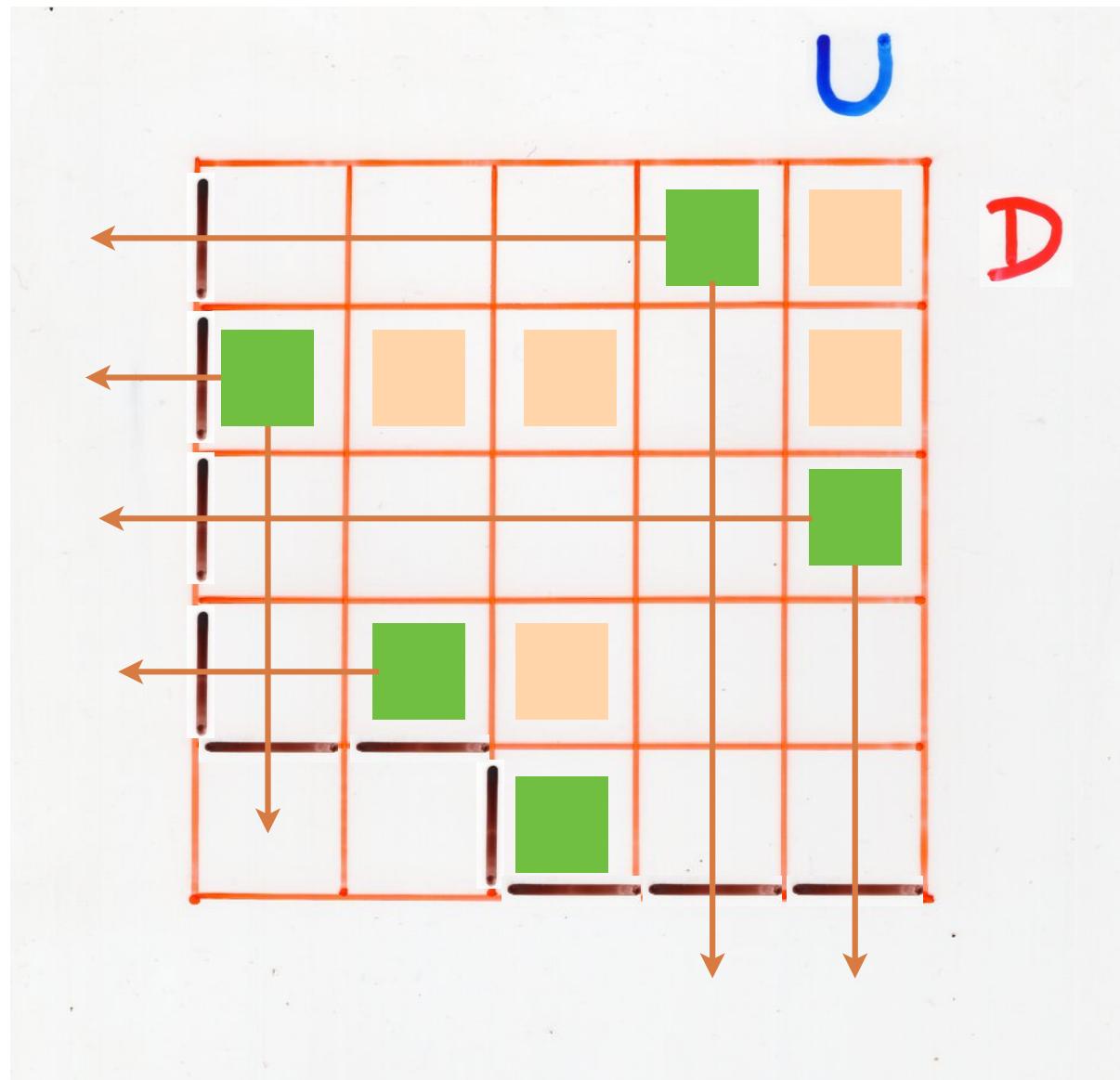


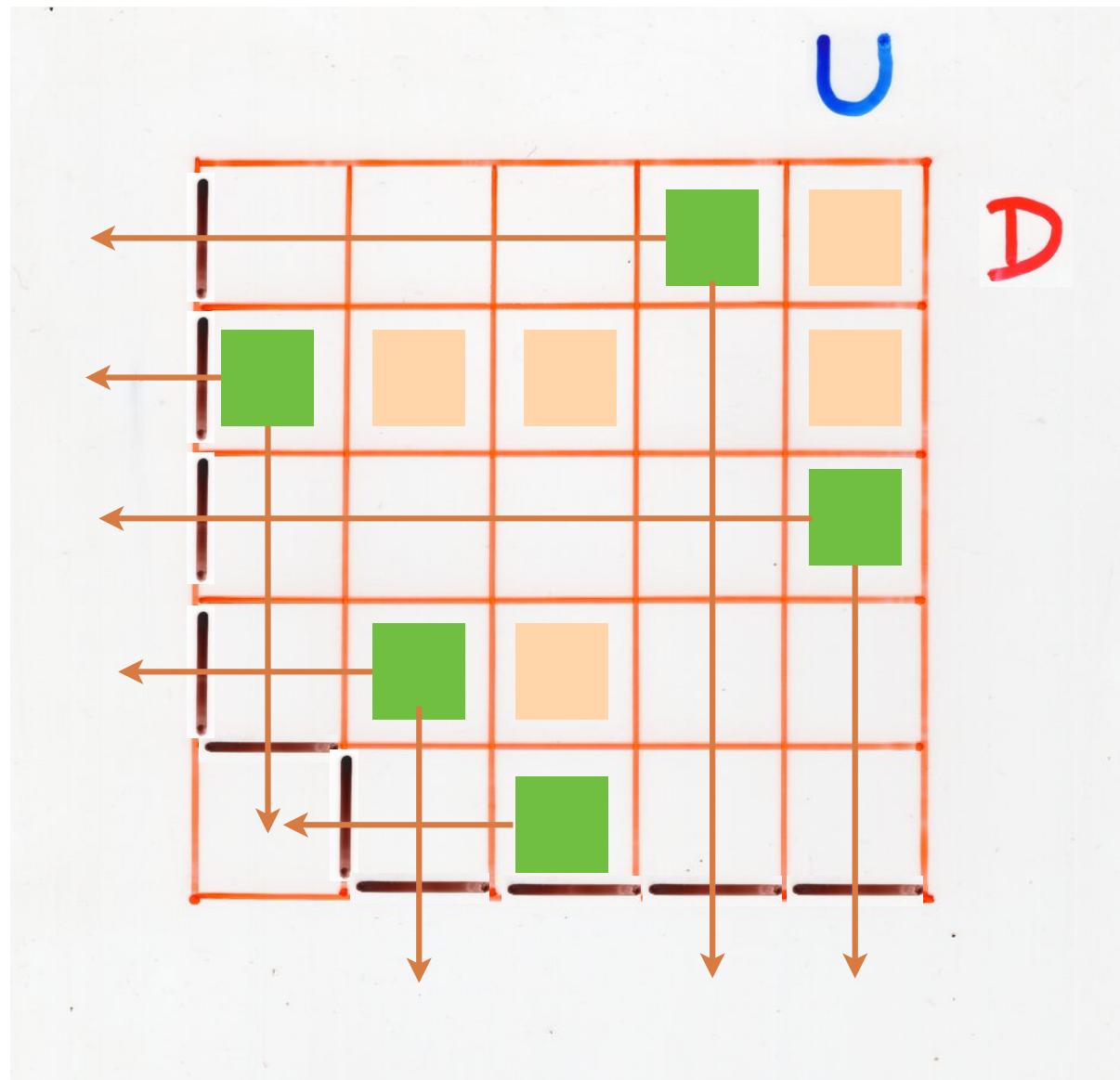


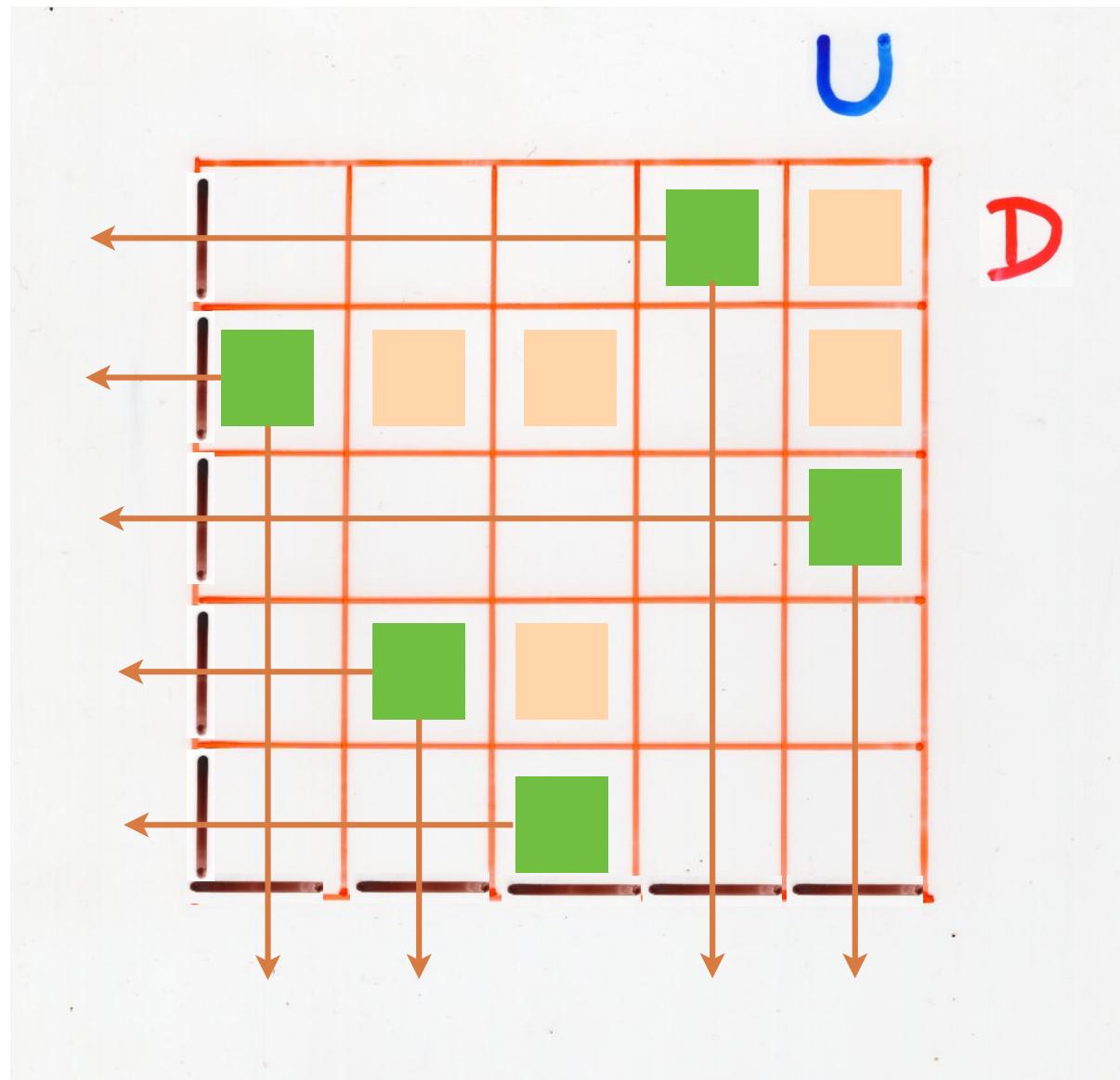


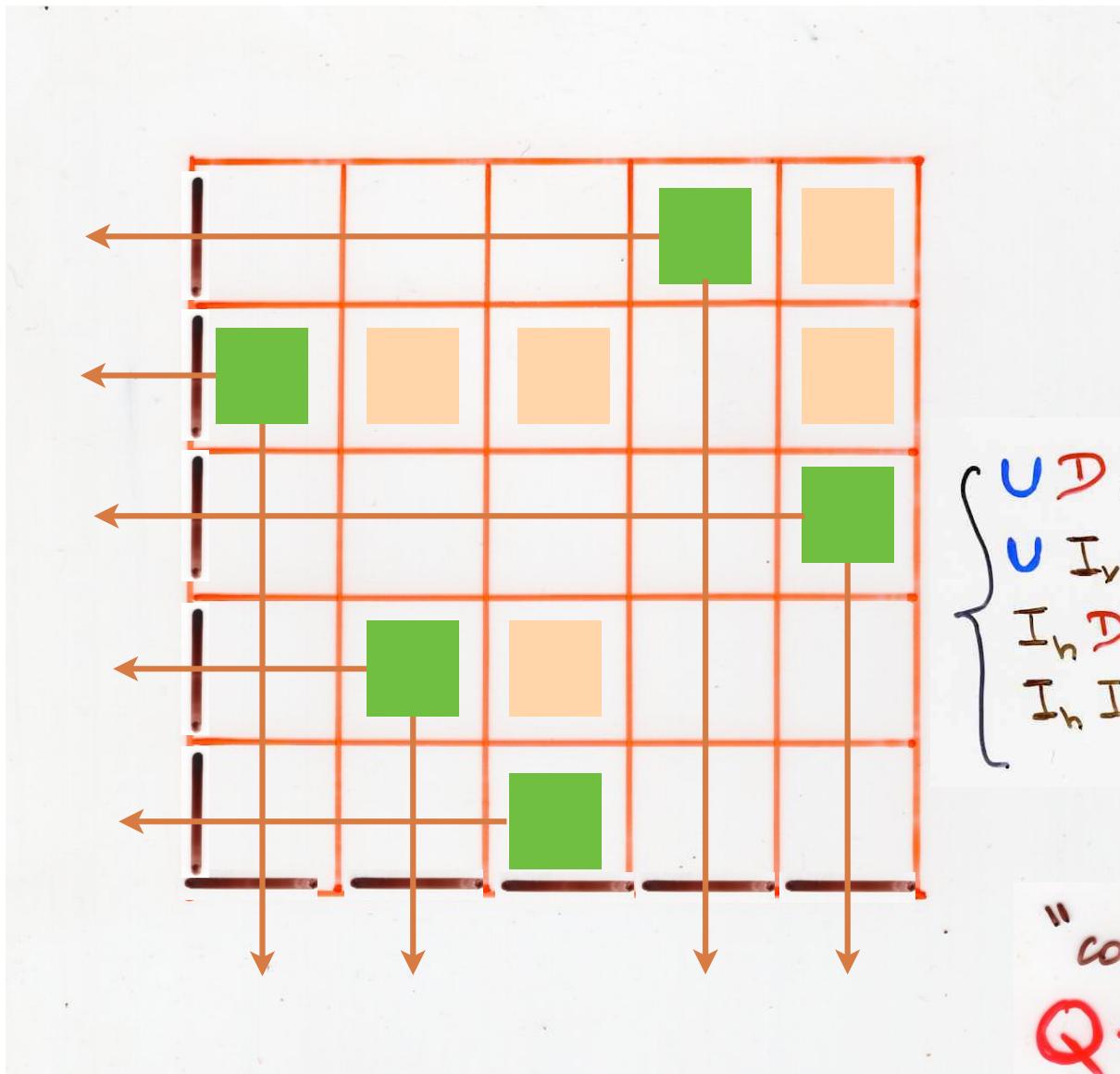










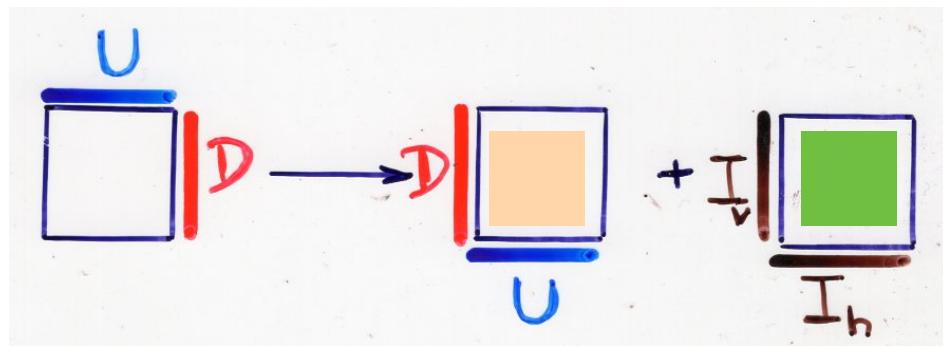


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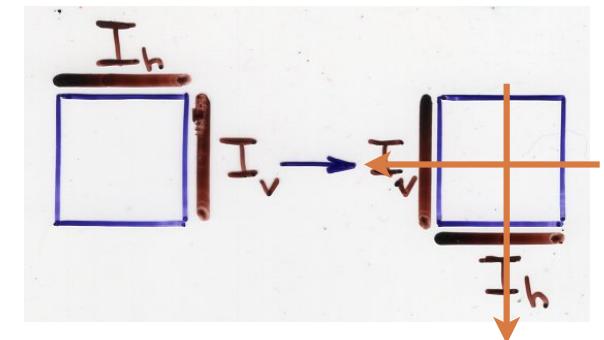
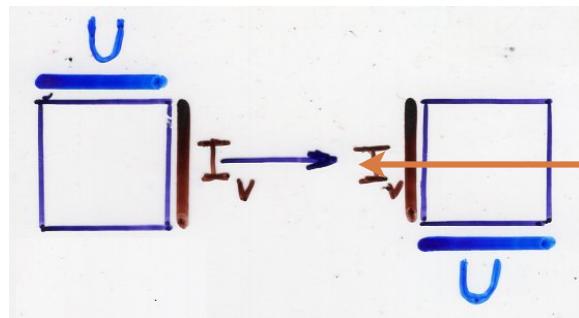
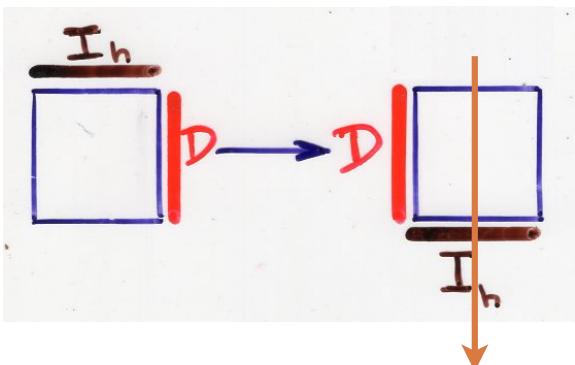
"complete"
Q-tableau

$$\left\{ \begin{array}{l} U D = D U + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$

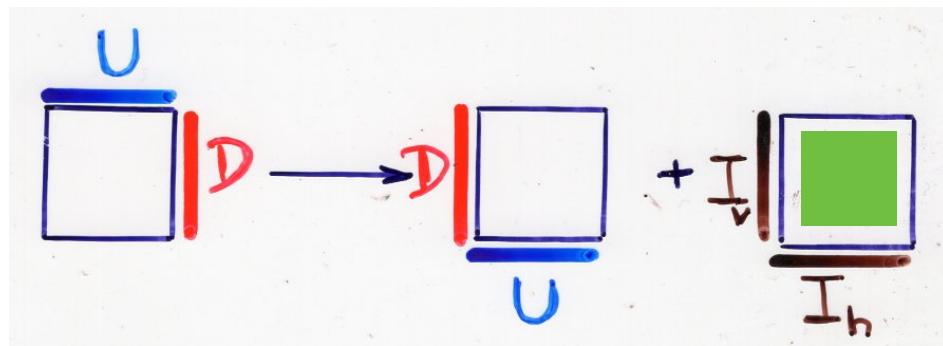
$$UD = qDU + I$$



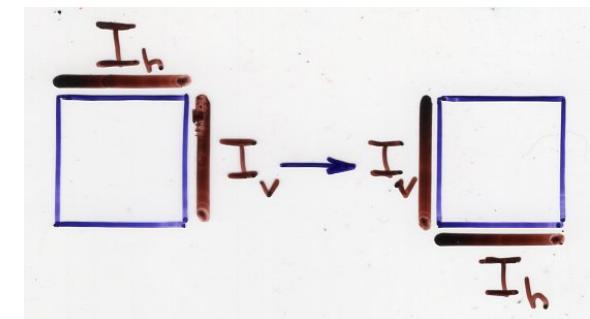
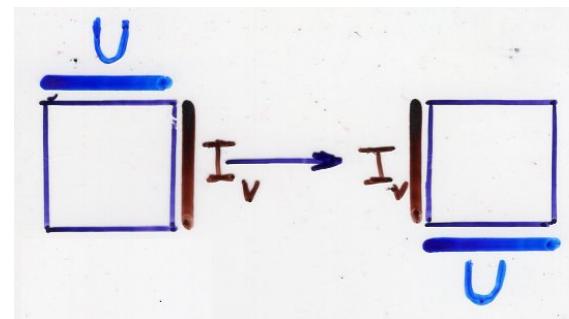
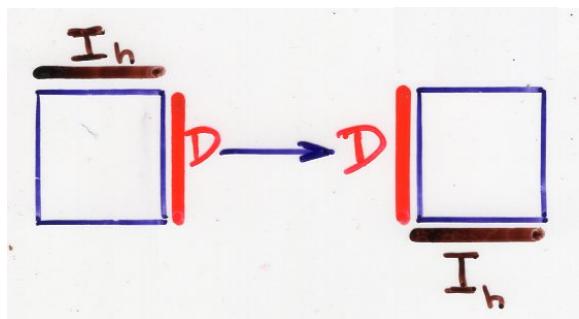
"complete"
Q-tableau

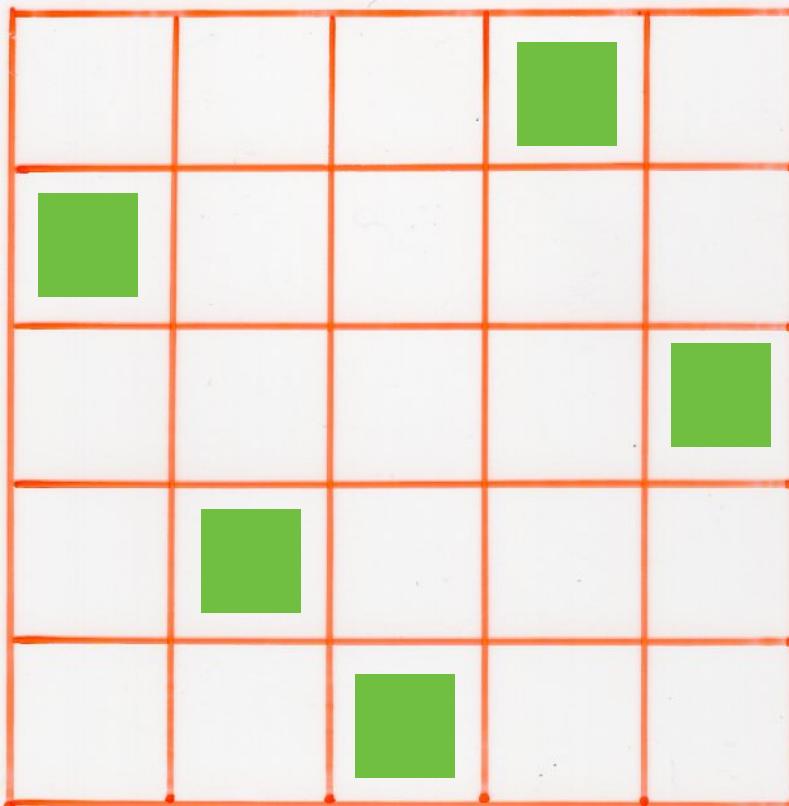


$$\left\{ \begin{array}{l} U D = D U + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$



Q-tableau



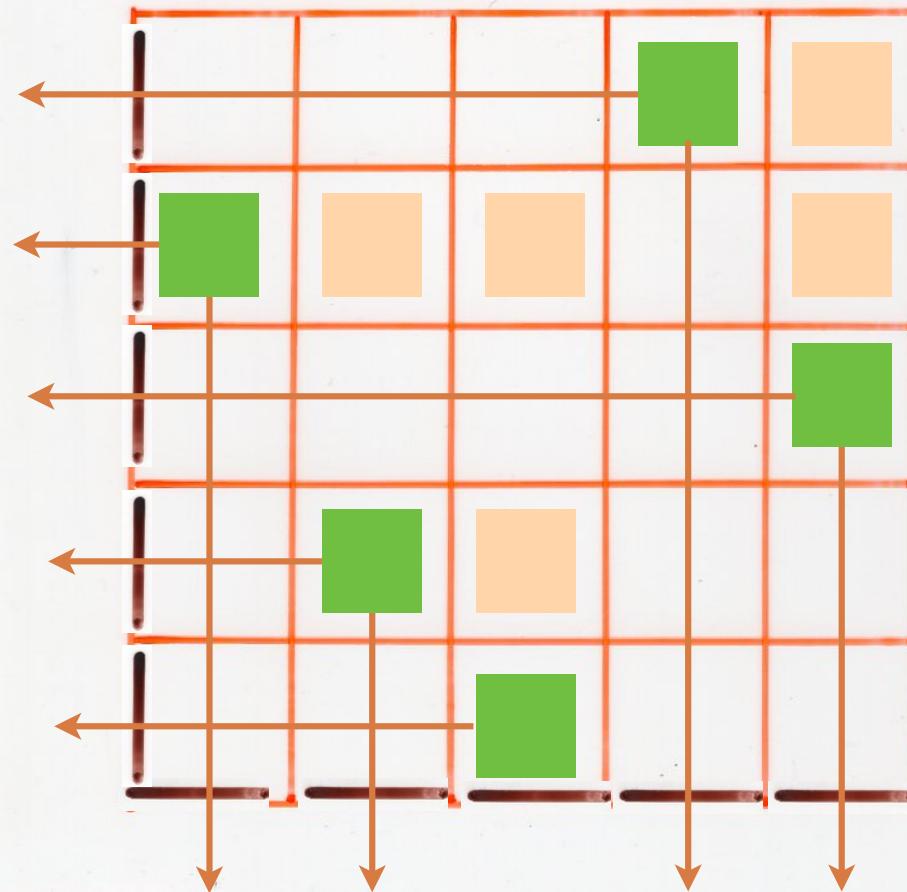


permutation
as a Q-tableau

q

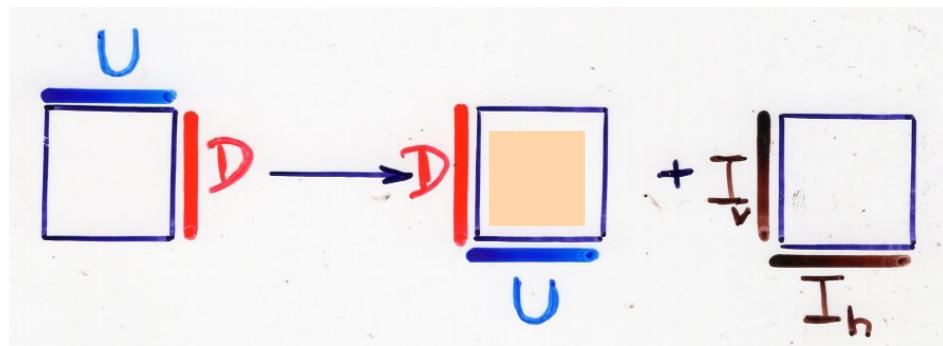
q-analog

number of
inversions
of a permutation σ

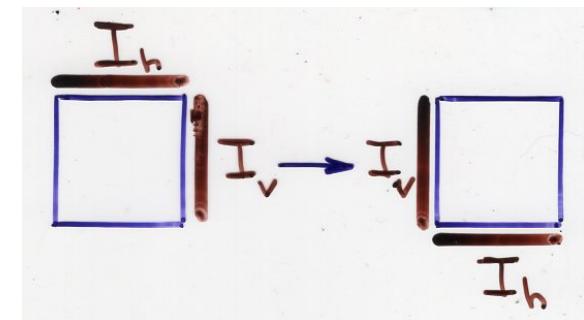
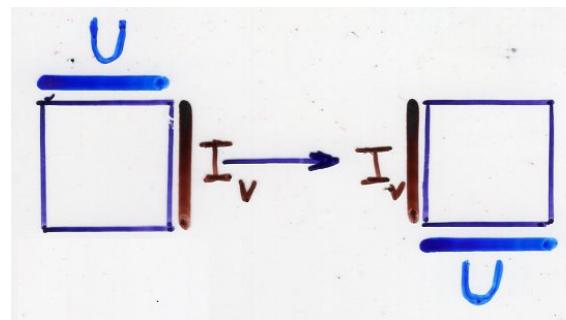
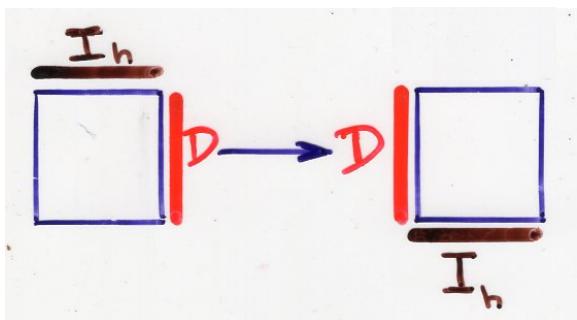


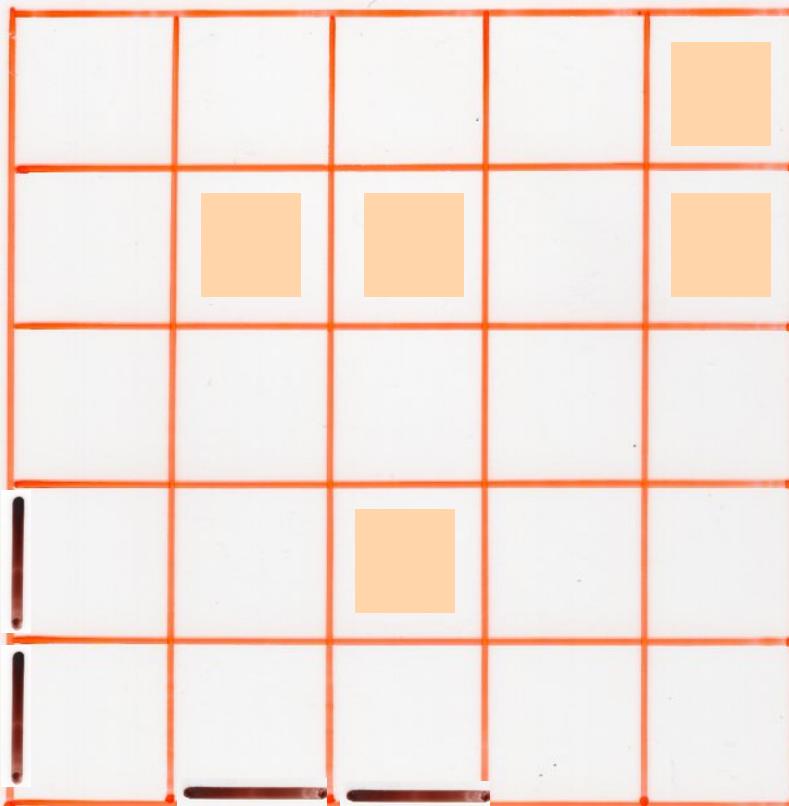
"complete"
Q-tableau

$$\left\{ \begin{array}{l} U D = D U + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$



Q-tableau





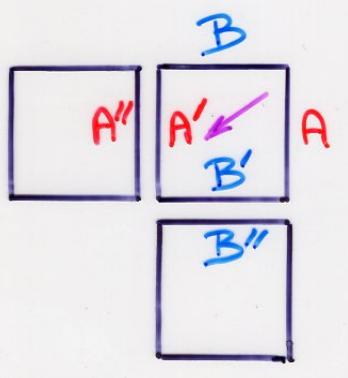
another **Q**-tableau
Rothe diagram
of a permutation

Definition $w(U, D)$ word of $\{U, D\}^*$

complete Q -tableau

- labeling of the cells of a Ferrers diagram
 $F = F(w)$ by the set R of rewriting rules
- with "compatibility" adjacent cells

i.e.



$$\begin{matrix} A, A', B, B' \\ A'', B'' \end{matrix} \in \{U, D, I_v, I_h\}$$

$$BA \rightarrow A'B'$$

$$\text{then } A' = A'', B' = B''$$

- if the cell A is at the NE border of F , then $B = U$, $A = D$

$$\varphi: R \longrightarrow L$$

map

L a set of "labels"
 (for the cell of $[n] \times [n]$)

R = set of rewriting rules
 of the homogenous system
 associated to Q

here 5 terms

examples

$$L = \{\boxed{}, \boxed{}\}$$

examples

$$L = \{\boxed{}, \boxed{}\}$$

φ satisfies $(*)$:

$(*)$ if $\varphi(\alpha \rightarrow \beta) = \varphi(\alpha' \rightarrow \beta')$
then $\alpha \neq \alpha'$

i.e. in a single commutation equation

$$\alpha = \beta_1 + \dots + \beta_r$$

all elements $\varphi(\alpha \rightarrow \beta_i) \in L$ are \neq
set of labels

$$\left\{ \begin{array}{l} U D = D U + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$

if $\varphi : R \rightarrow L$
satisfies $(*)$
then

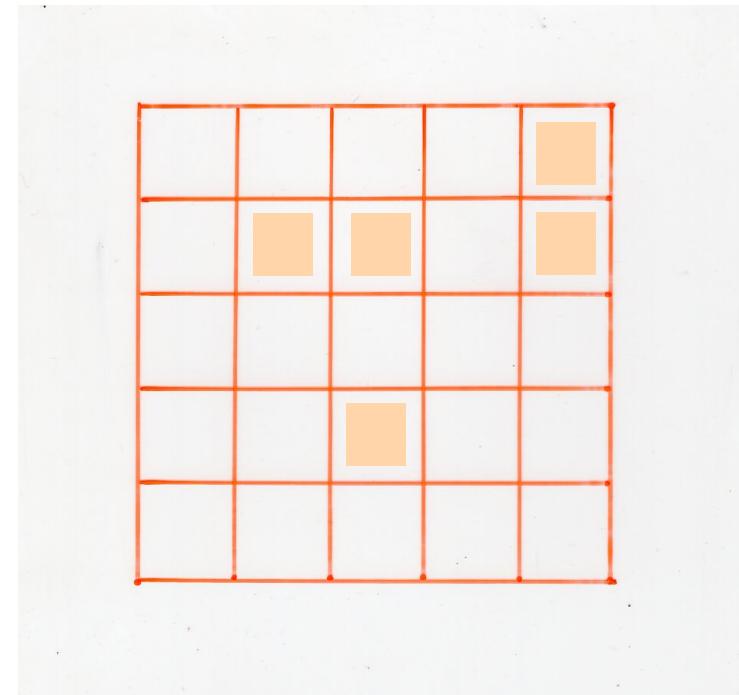
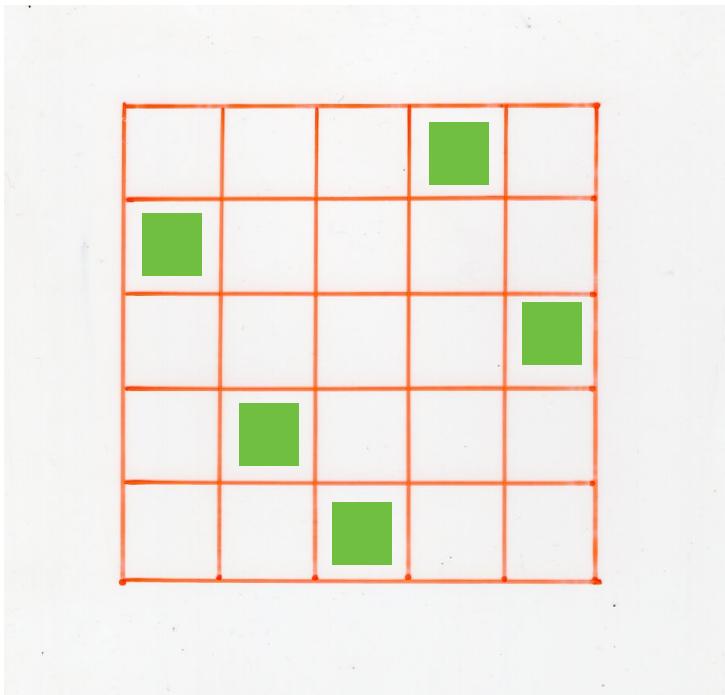
bijection
complete Q -tableaux \longleftrightarrow Q -tableaux

$$\varphi: \begin{array}{l} UD \rightarrow DU, UI_v \rightarrow I_v U, \\ I_h D \rightarrow DI_h, I_h I_v \rightarrow I_v I_h \end{array} \rightarrow \boxed{\text{empty cell}}$$

$$\varphi: \begin{array}{l} UD \rightarrow I_v I_h, UI_v \rightarrow I_v U, \\ I_h D \rightarrow DI_h, I_h I_v \rightarrow I_v I_h \end{array} \rightarrow \boxed{\text{empty cell}}$$

$$\varphi(UD \rightarrow I_v I_h) = \boxed{\text{green}}$$

$$\varphi(U D \rightarrow D U) = \boxed{\text{orange}}$$



"The **cellular** ansatz."

quadratic
algebra **Q**

$$UD = DU + \text{Id}$$

Q-tableaux

combinatorial objects
on a 2D lattice

permutations

towers placements

(i) first step

commutations

rewriting rules

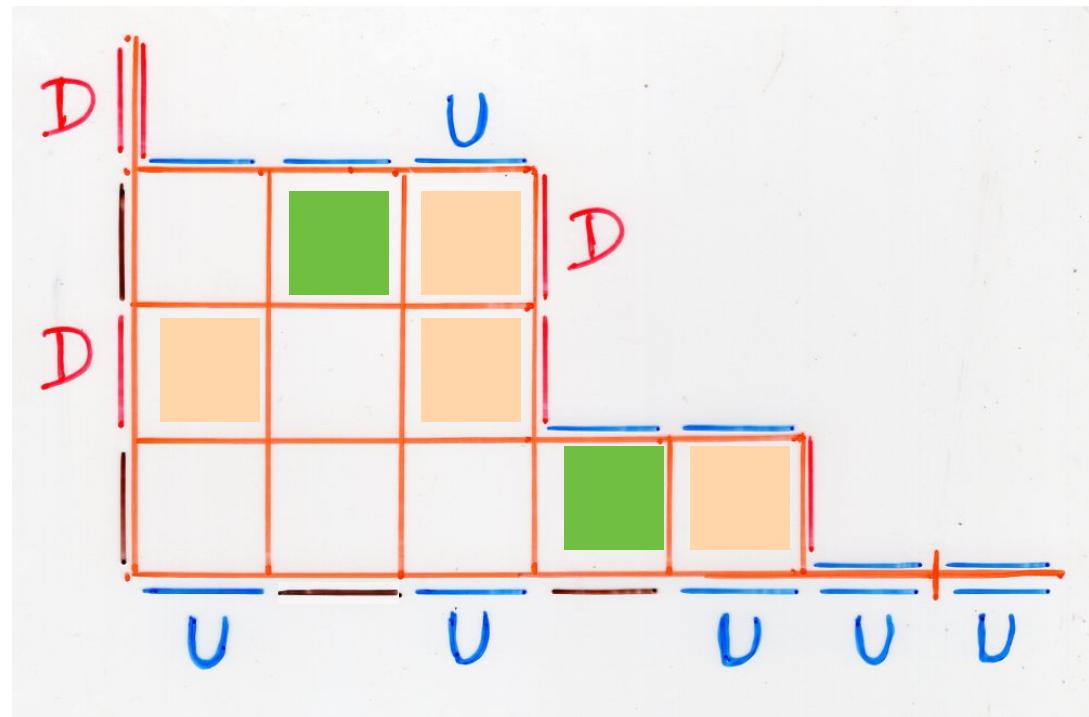
planarization

$$w = D U^3 D^2 U^2 D U^2$$

$$w \longrightarrow F = F(w)$$

F Ferrers diagram

Rooks
placement



The cellular ansatz
second part:

guided construction
of a bijection
from the representation of U and D

$$UD = DU + I$$

"The **cellular** ansatz."

quadratic
algebra **Q**

$$UD = DU + \text{Id}$$

Q-tableaux

combinatorial objects
on a 2D lattice

permutations

towers placements

representation of **Q**
by combinatorial
operators

bijections

RSK

pairs of
Young tableaux

(i) first step

(ii) second step

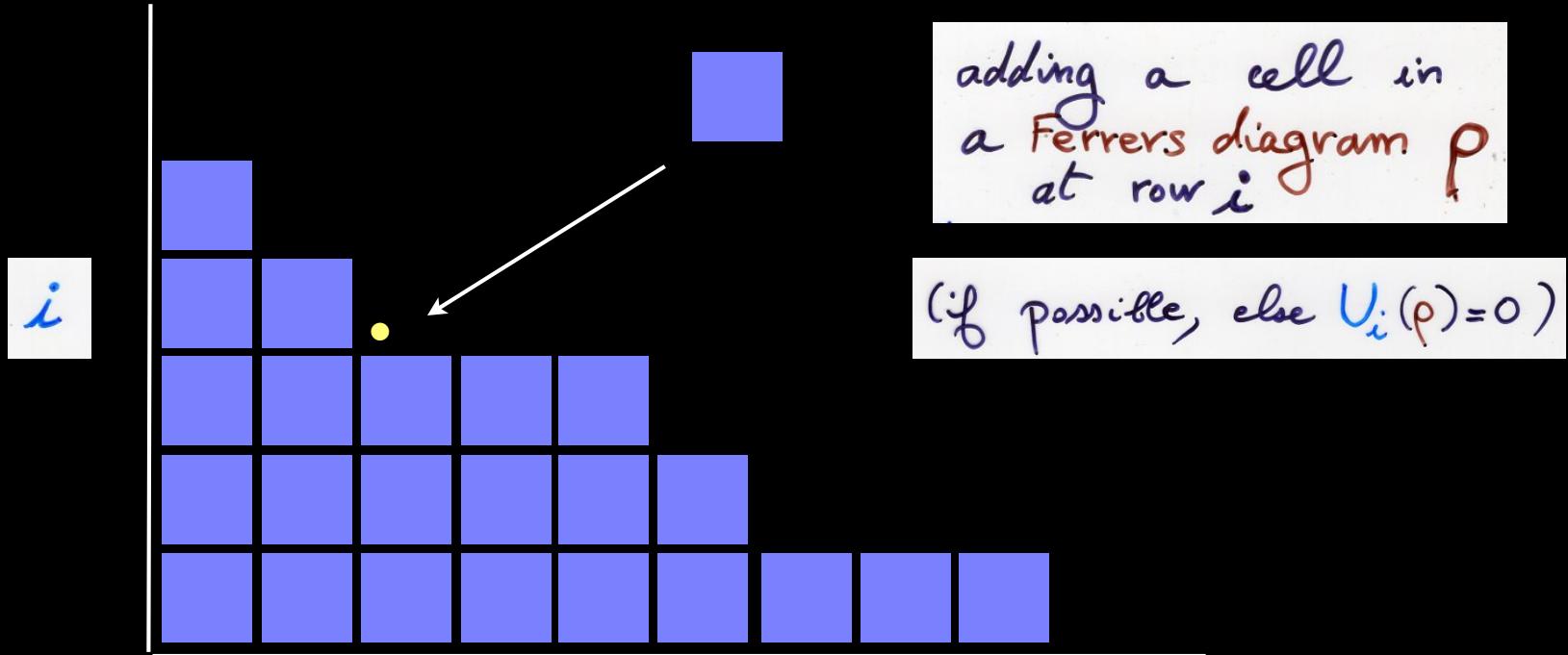
commutations

rewriting rules

planarization

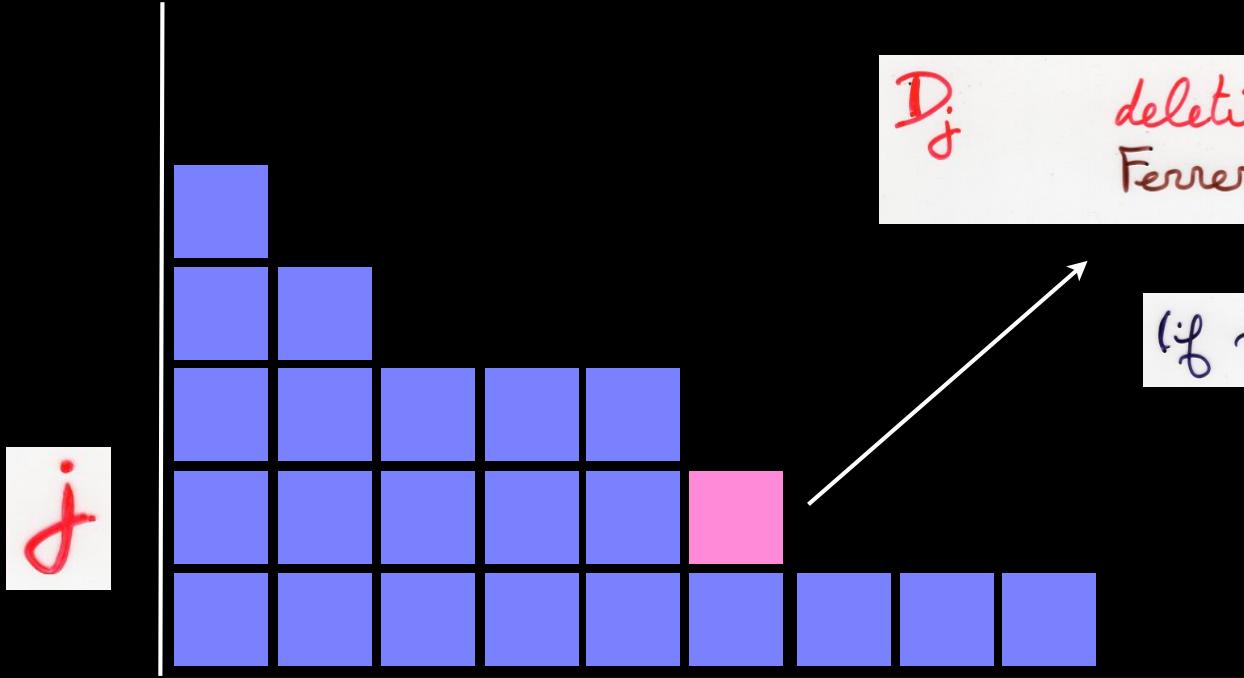
notations

operator U_i



$$U_i(\rho) = \rho + (i)$$

$$D_j(\rho) = \rho - (j)$$



D_j deleting a cell in a
Ferrers diagram ρ at row j

(if possible, else $D_j(\rho)=0$)

$$U = \sum_{i \geq 1} U_i$$

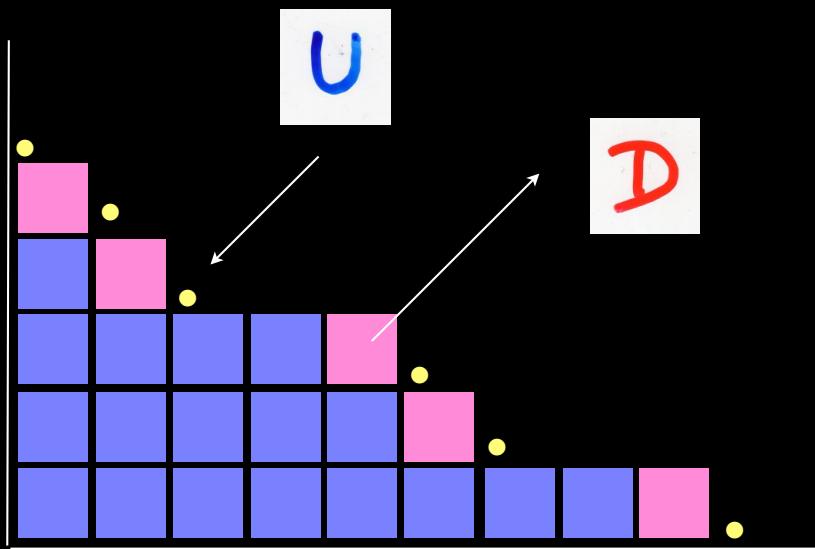
$$D = \sum_{i \geq 1} D_i$$

U and D are operators acting on
the vector space generated by Ferrers
diagrams.

$$U \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \end{array} = \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \color{blue}{\blacksquare} \\ \hline \end{array} \end{array} + \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \color{blue}{\blacksquare} \\ \hline \end{array} \end{array} + \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \color{blue}{\blacksquare} \\ \hline \end{array} \end{array}$$

$$D \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \end{array} = \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \color{red}{\bullet} \\ \hline \end{array} \end{array} + \begin{array}{c} \\ \text{Ferrers diagram: } \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \end{array} \color{red}{\bullet}$$

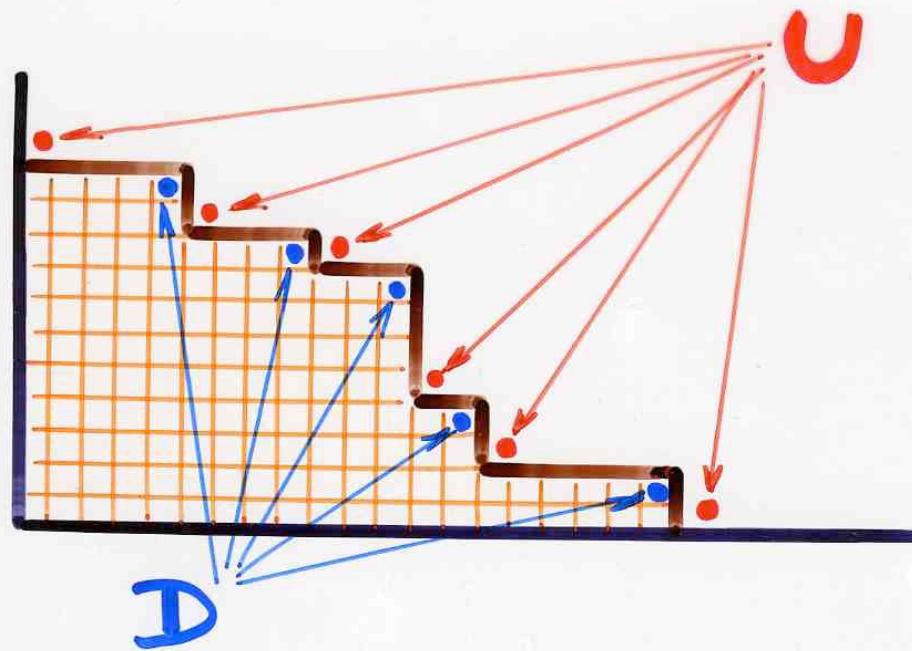
operators
 U and D



Young lattice

{ U adding
 D deleting a cell in a Ferrers diagram

$$UD = DU + I$$



In this course, product of operators
are written from left to right

$$\xrightarrow{A \rightarrow B} (\mu) = B(A(\mu))$$

should be written $(\mu) A B$
or $\langle \mu | A B$

with operators written from right to left

$$\xleftarrow{B \leftarrow A} (\mu) = B(A(\mu))$$

$$UD - DU = I \quad \text{becomes}$$

$$DU - UD = I$$

$$\begin{array}{ccccccc} \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} & = & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} & + & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} & + & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} & + & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} & + & \begin{array}{c} \text{U} \\ \text{D} \\ \text{UD} \end{array} \\ & & & & & & \end{array}$$

The diagram illustrates the decomposition of a 3x3 matrix into its upper triangular (U), lower triangular (D), and diagonal (UD) components. The matrices are represented as 3x3 grids of squares.

- U:** The first row has 3 squares, the second row has 2 squares, and the third row has 1 square.
- D:** The first column has 3 squares, the second column has 2 squares, and the third column has 1 square.
- UD:** This term is highlighted in red. It consists of two parts:
 - The first part is the difference between U and D, represented by a grid where the top-left square is empty (0) and the rest are 1s.
 - The second part is the product of D and U, represented by a grid where the top-left square is 1 and the rest are 0s.

Red arrows point from the terms in the first equation to the corresponding terms in the second equation, showing how the decomposition is derived.

$$\begin{array}{l} \begin{array}{c} \text{U} \\ = \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last column removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first column removed} \end{array} \\ \\ \begin{array}{c} \text{D} \\ = \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first two columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last two columns removed} \end{array} \\ \\ \begin{array}{c} \text{UD} \\ = \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last three columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first column removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first two columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last two columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first three columns removed} \end{array} \\ \\ \begin{array}{c} \text{DU} \\ = \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first column removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with first two columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last three columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last two columns removed} \\ + \end{array} \begin{array}{c} \text{Diagram: } 3 \times 3 \text{ grid with last column removed} \end{array} \end{array} \end{array}$$

The diagram illustrates the decomposition of a 3x3 grid into smaller components. The top row shows the decomposition of the grid into three 3x3 grids: one with the last column removed, one with all columns, and one with the first column removed. The second row shows the decomposition of the grid into three 3x3 grids: one with the first two columns removed, one with the last two columns removed, and one with the first three columns removed. The third row shows the decomposition of the grid into five 3x3 grids: one with the last three columns removed, one with the first column removed, one with the first two columns removed, one with the last two columns removed, and one with the last column removed. The bottom row shows the decomposition of the grid into five 3x3 grids: one with the first column removed, one with the first two columns removed, one with the last three columns removed, one with the last two columns removed, and one with the last column removed. Red arrows point from the original grid to the first three terms of the UD equation. Blue arrows point from the original grid to the first three terms of the DU equation.

$$\begin{array}{c} \text{U} \\ \text{U} \end{array} = \begin{array}{c} \text{U} \\ \text{U} \end{array} + \begin{array}{c} \text{U} \\ \text{U} \end{array} + \begin{array}{c} \text{U} \\ \text{U} \end{array}$$

$$\begin{array}{c} \text{D} \\ \text{D} \end{array} = \begin{array}{c} \text{D} \\ \text{D} \end{array} + \begin{array}{c} \text{D} \\ \text{D} \end{array}$$

$$\begin{array}{c} \text{UD} \\ \text{UD} \end{array} = \begin{array}{c} \text{UD} \\ \text{UD} \end{array} + \begin{array}{c} \text{UD} \\ \text{UD} \end{array}$$

$$\begin{array}{c} \text{DU} \\ \text{DU} \end{array} = \begin{array}{c} \text{DU} \\ \text{DU} \end{array} + \begin{array}{c} \text{DU} \\ \text{DU} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{U} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{D} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{UD} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{DU} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ (\text{UD-DU}) = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{U} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{D} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{UD} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{DU} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ (\text{UD}-\text{DU}) = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{U} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

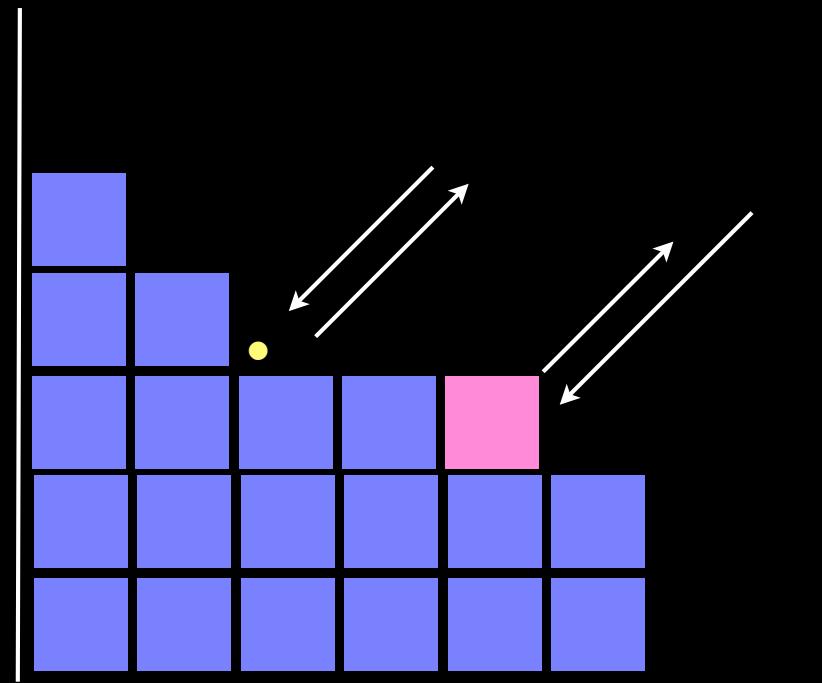
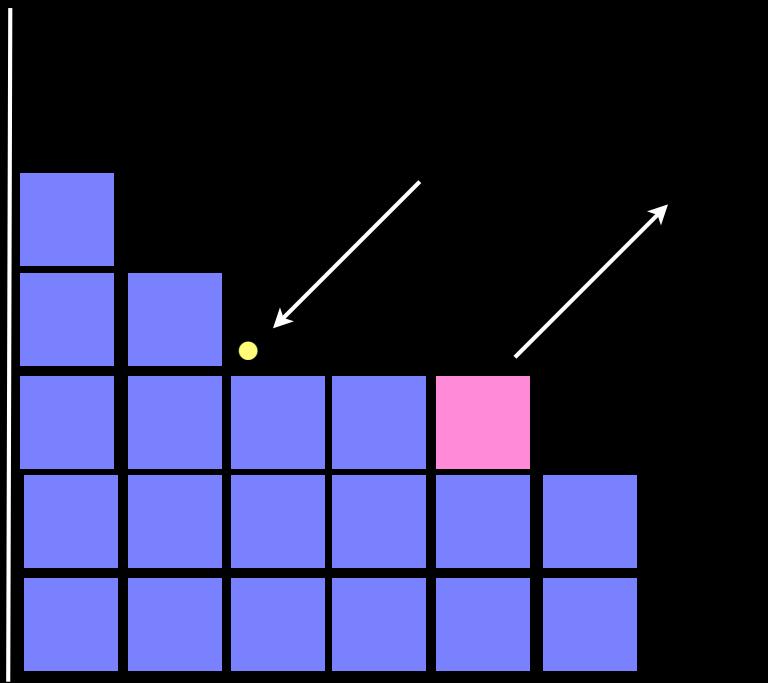
$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{D} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 4 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{UD} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

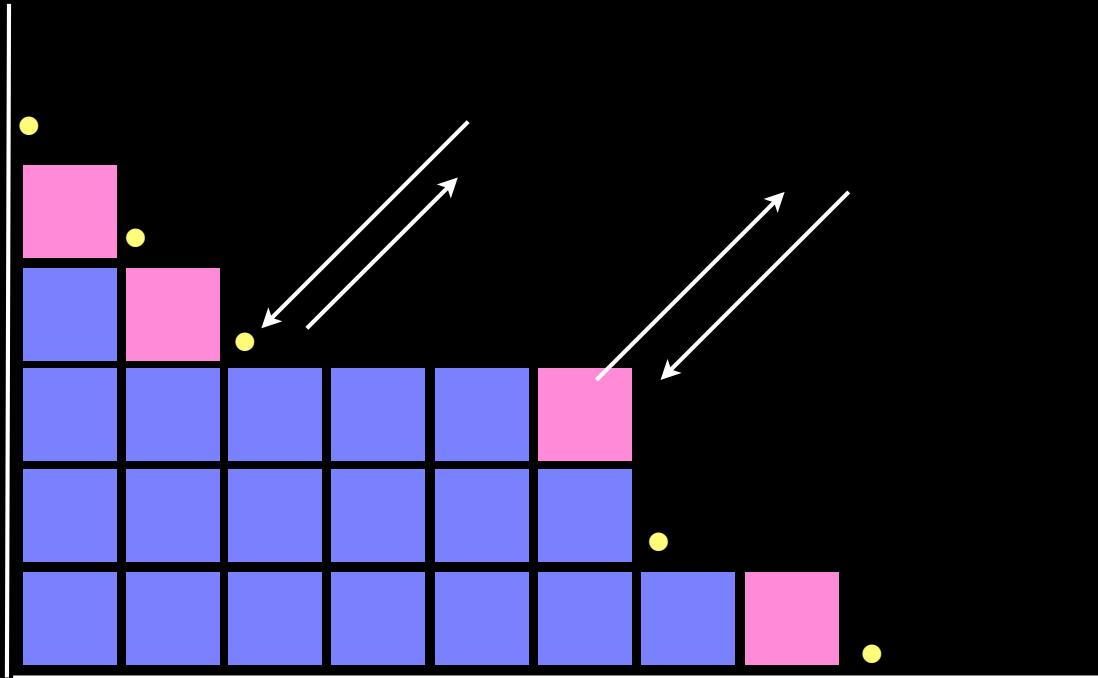
$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ \text{DU} = \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array} + \begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, middle, and bottom-right columns]} \\ + \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with 5 black squares in the top-left, bottom-right, and middle columns]} \\ (\text{UD}-\text{DU}) = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with all 9 squares black]} \end{array}$$

$$U \mathcal{D} = \mathcal{D} U + I$$

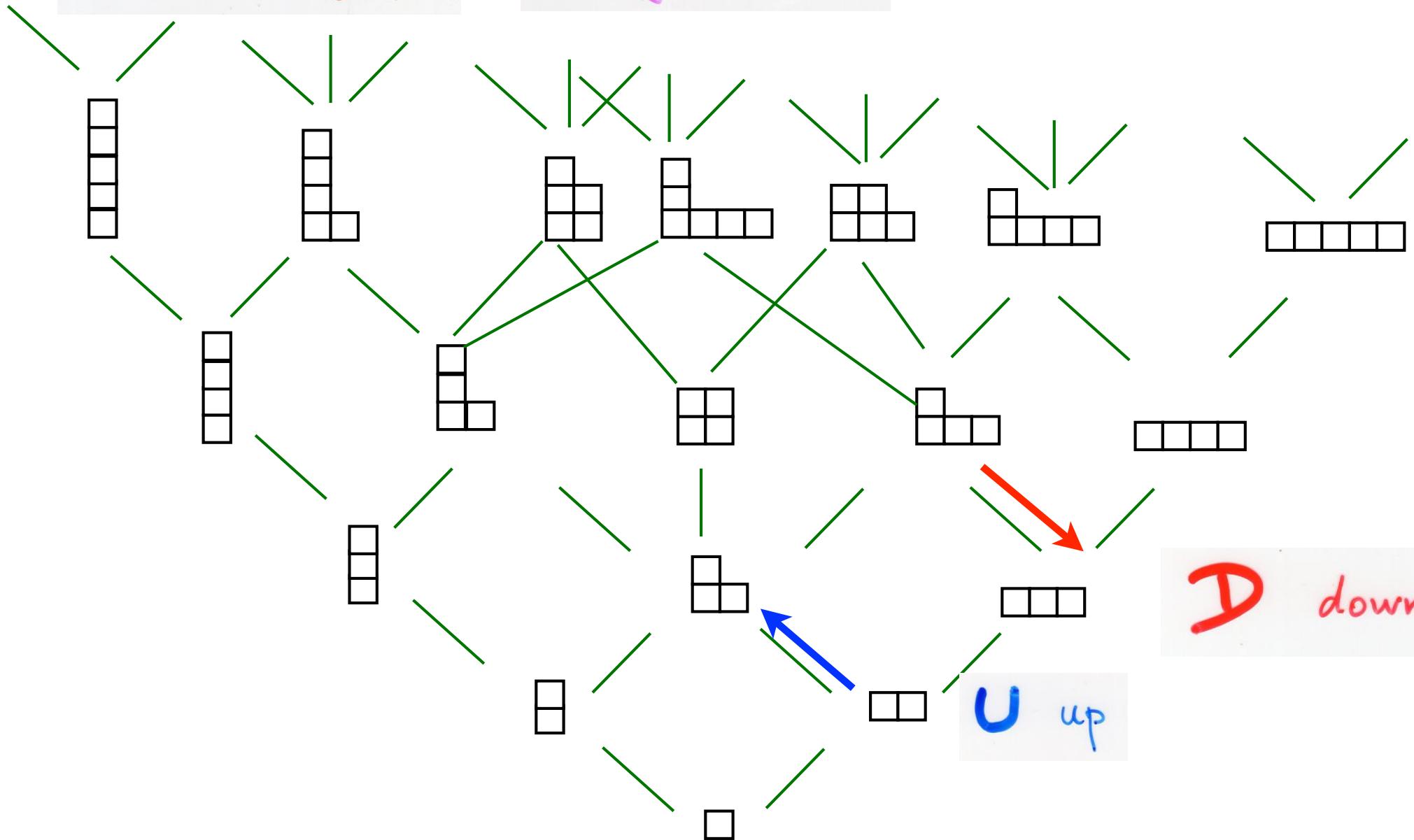


$$UD = DU + I$$



Hasse diagram

Young lattice



differential poset

Fomin (1992, 1995)

Stanley (1988, 1990)

Roby (1991)

$$UD = DU + I$$

U up

D down

direct proof of the identity

permutations

pairs of Young tableaux,
same shape

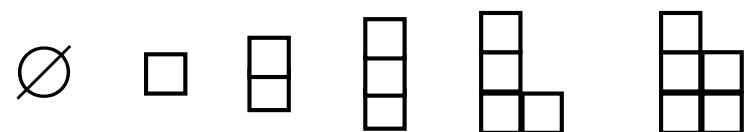
$$n! = \sum_{\lambda} (\mathfrak{f}_\lambda)^2$$

partition
of n

$$U D = D U + I$$

$$\langle \emptyset | U^n D^n | \emptyset \rangle$$

\emptyset empty Ferrers diagram
in

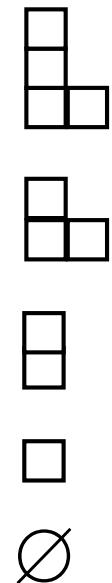


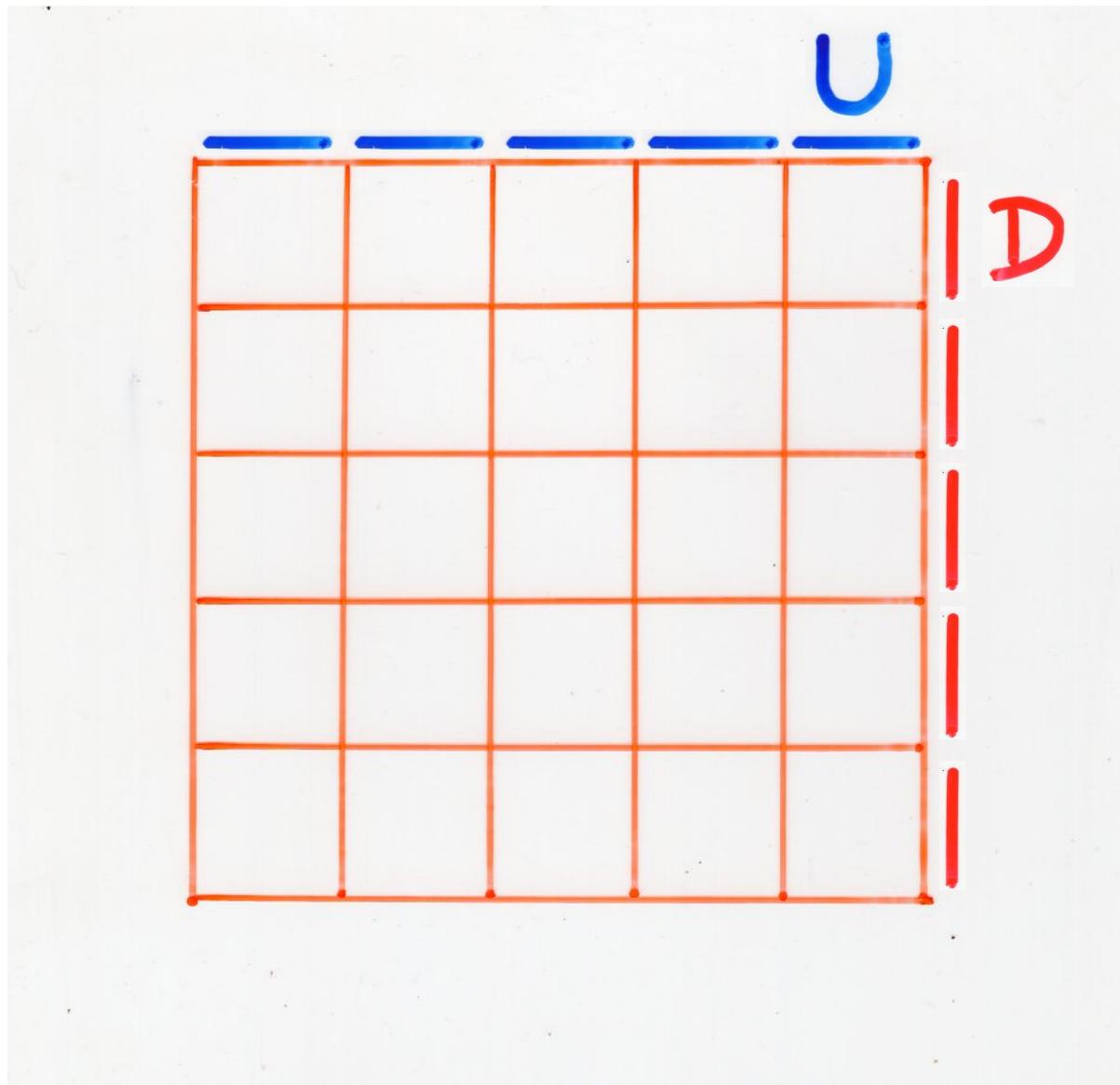
$$= \sum_{\lambda} (\mathcal{f}_{\lambda})^2$$

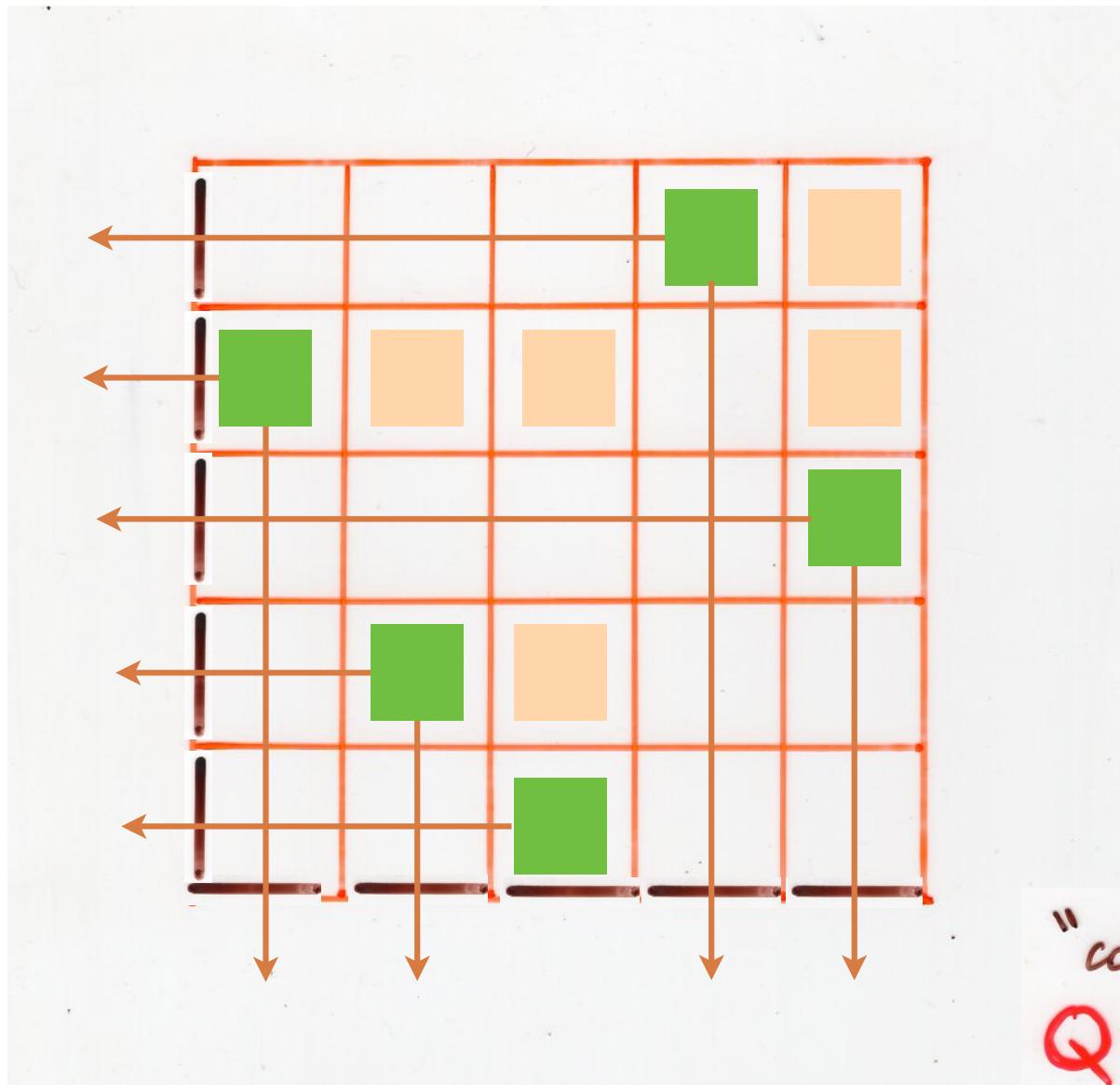
partition
of n

$$= \sum_{i \geq 0} c_{n,i} \langle \emptyset | D^i U^i | \emptyset \rangle$$

$$= c_{n,0} \quad = n!$$







"complete"
Q-tableau

$$c_{n,0} = n!$$

$$= \sum_{\substack{\lambda \\ \text{partition} \\ \text{of } n}} \left(f_\lambda \right)^2$$

permutation
as a **Q-tableau**

construction of the RSK correspondence
by «propagation» on the grid
of an elementary «diagram bijection»
related to each cell of the grid

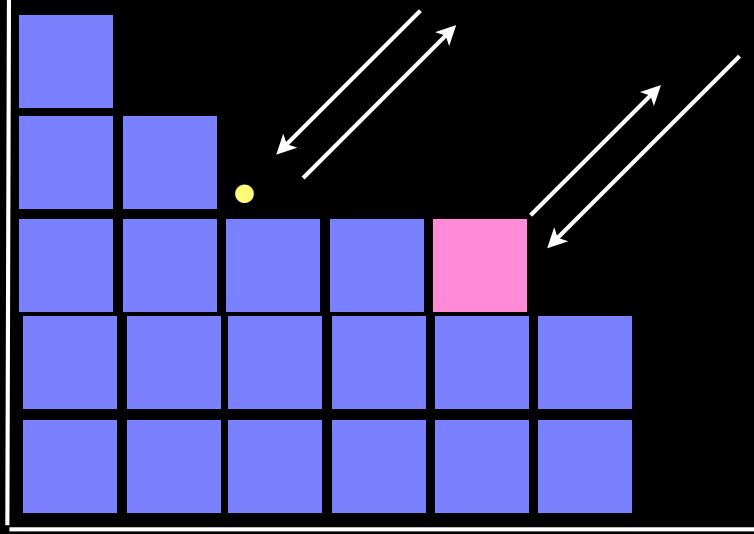
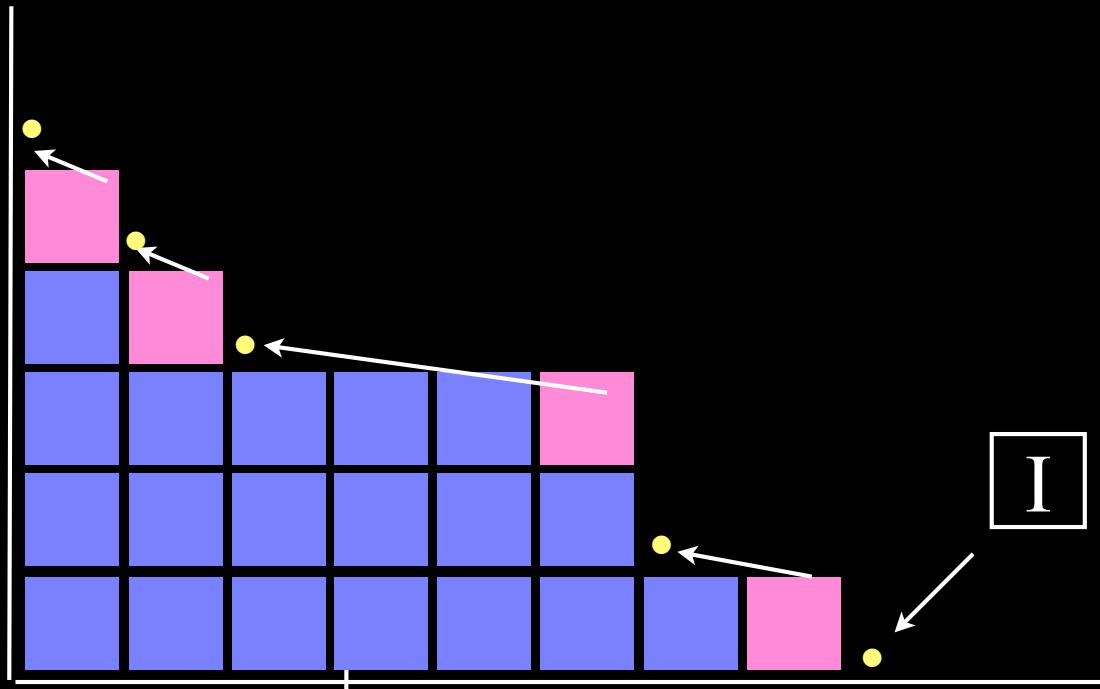
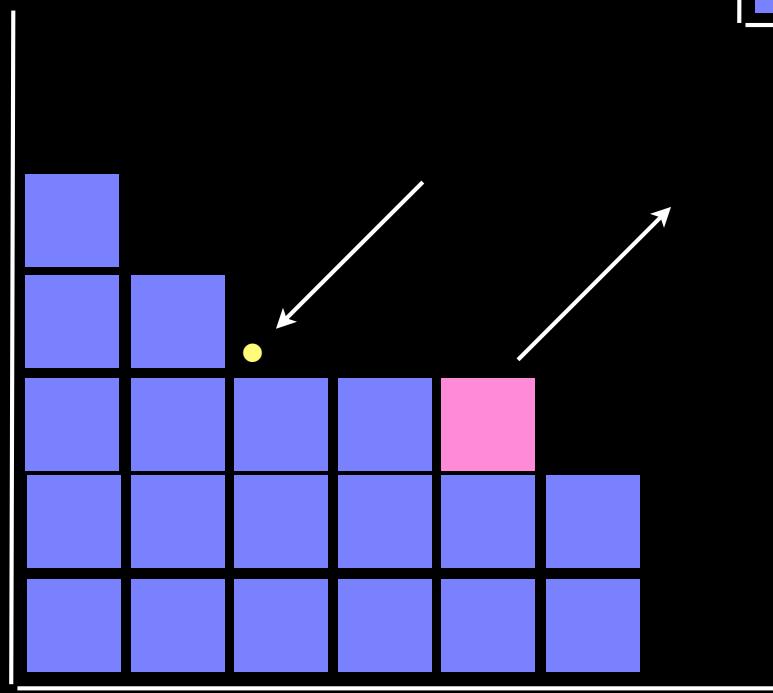
$$\begin{array}{c} \text{Diagram A} \\ \text{Diagram B} \\ \text{Diagram C} \end{array} = \begin{array}{c} \text{Diagram D} \\ \text{Diagram E} \end{array} + \begin{array}{c} \text{Diagram F} \\ \text{Diagram G} \end{array} + \begin{array}{c} \text{Diagram H} \\ \text{Diagram I} \end{array}$$

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array} \quad \text{D} \quad = \quad \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array} \quad + \quad \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}$$

$$\begin{array}{c} \text{Diagram A} \\ \text{UD} \end{array} = \begin{array}{c} \text{Diagram B} \\ + \end{array} + \begin{array}{c} \text{Diagram C} \\ + \end{array} + \begin{array}{c} \text{Diagram D} \\ + \end{array} + \begin{array}{c} \text{Diagram E} \\ + \end{array} + \begin{array}{c} \text{Diagram F} \end{array}$$

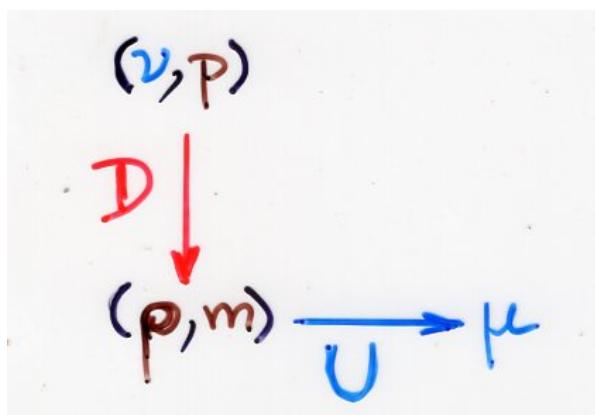
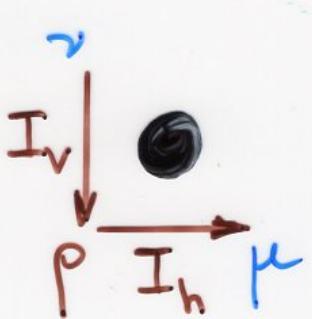
$$\begin{array}{c|c}
 \text{Diagram} & \text{DU} = \\
 \hline
 \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} & \begin{array}{c} \text{Diagram} \\ + \end{array} \\
 \begin{array}{c} \text{Diagram} \\ + \end{array} & \begin{array}{c} \text{Diagram} \\ + \end{array} \\
 \begin{array}{c} \text{Diagram} \\ + \end{array} & \begin{array}{c} \text{Diagram} \\ + \end{array}
 \end{array}$$

$$\begin{array}{c} \text{Diagram A} \\ \text{Diagram B} \end{array} \quad (\text{UD-DU}) \quad = \quad \begin{array}{c} \text{Diagram C} \\ \text{Diagram D} \end{array}$$

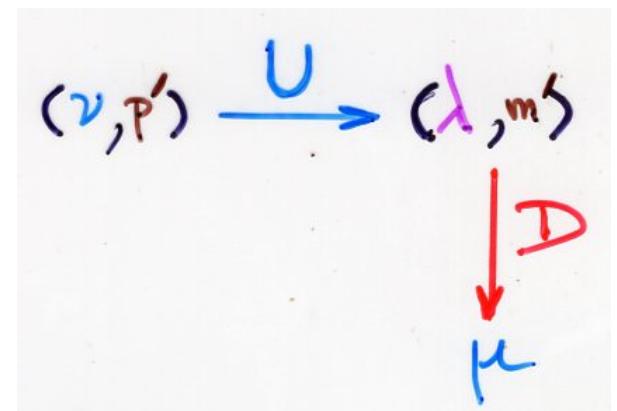


$$UD = DU + I_v I_h$$

"commutation diagrams"

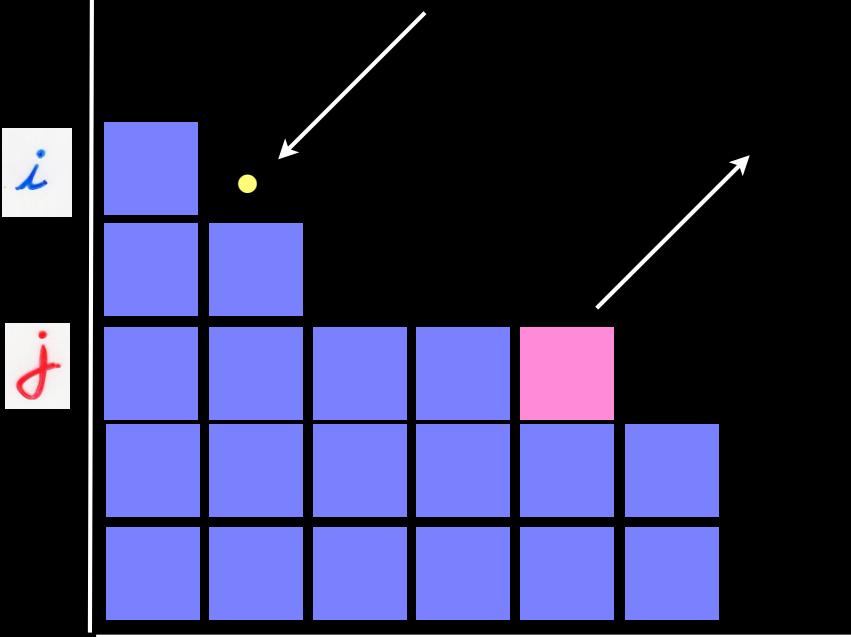
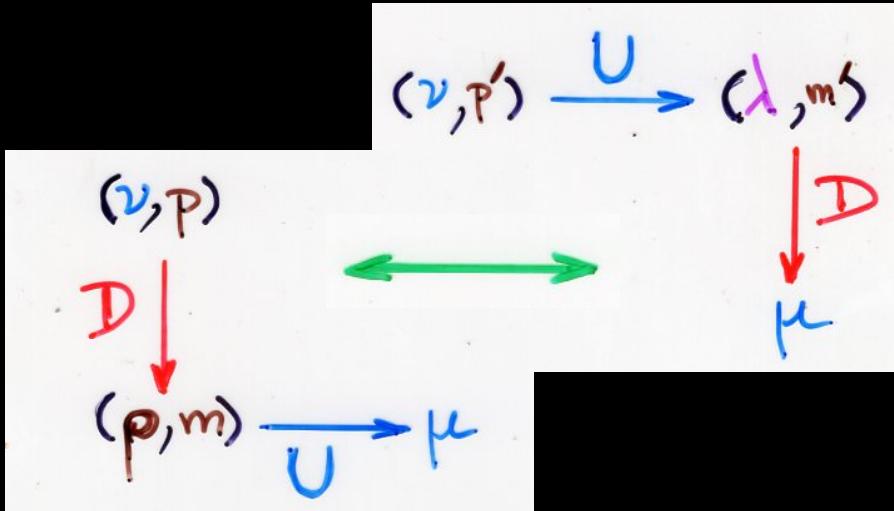


bijection



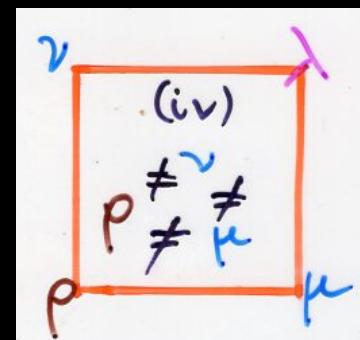
p, m, p', m' are "positions"

in v, p, v, λ respectively

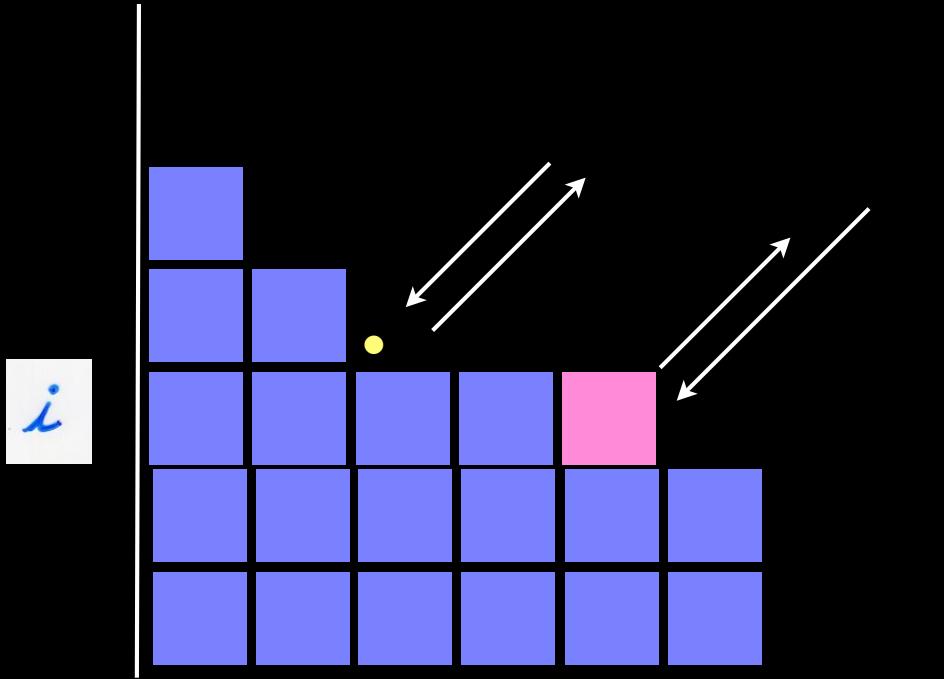
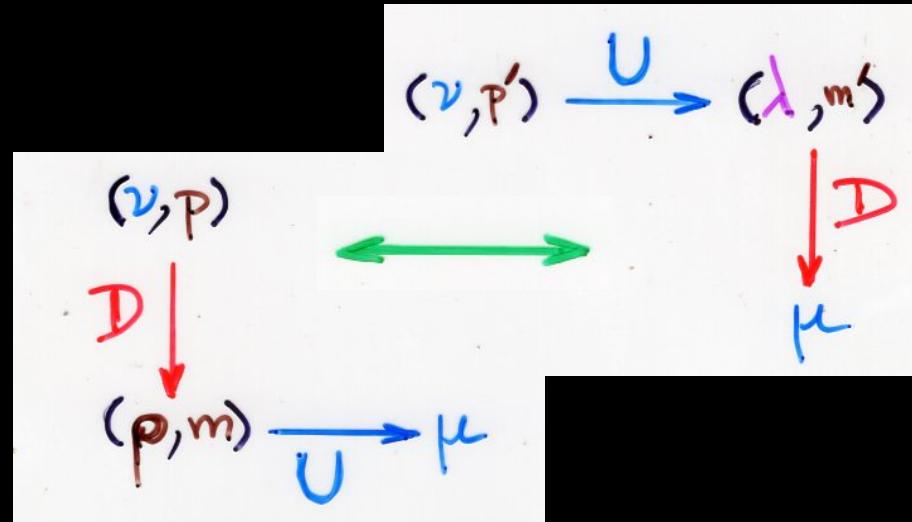


$$\begin{aligned}
 p &= j \\
 m &= i
 \end{aligned}$$

$$\begin{aligned}
 p' &= i \\
 m' &= j
 \end{aligned}$$

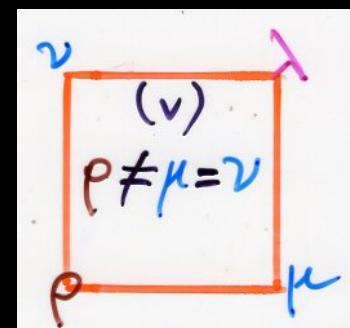


$$\begin{aligned}
 \nu &= p + (j) \\
 \mu &= p + (i) \\
 \lambda &= p + (i) + (j)
 \end{aligned}$$

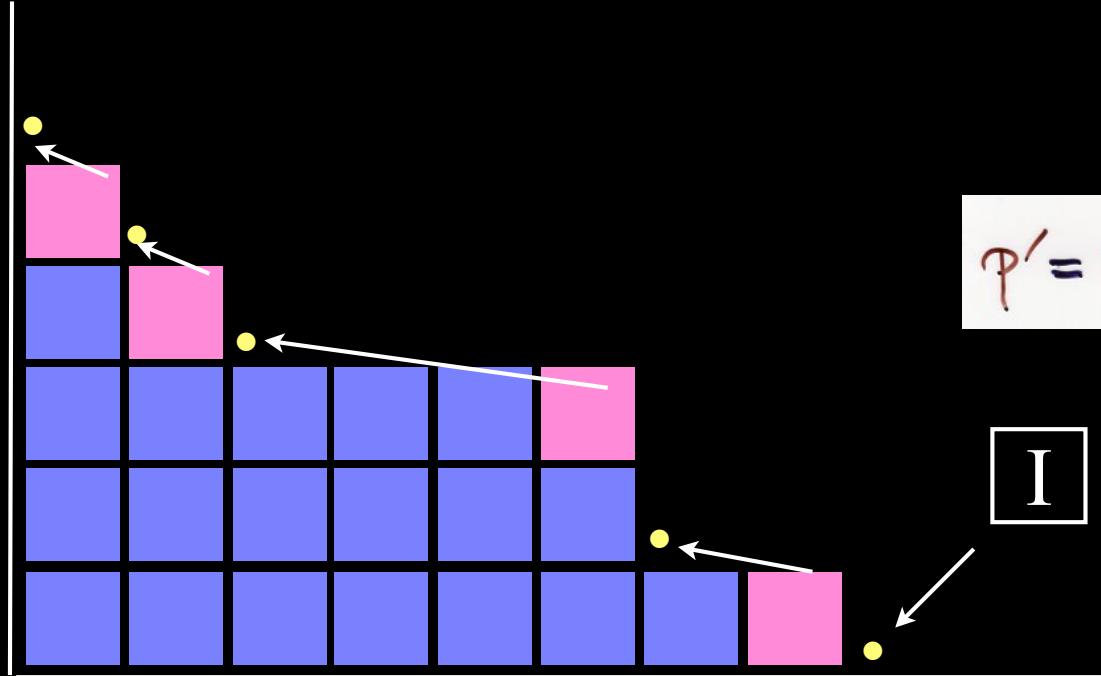
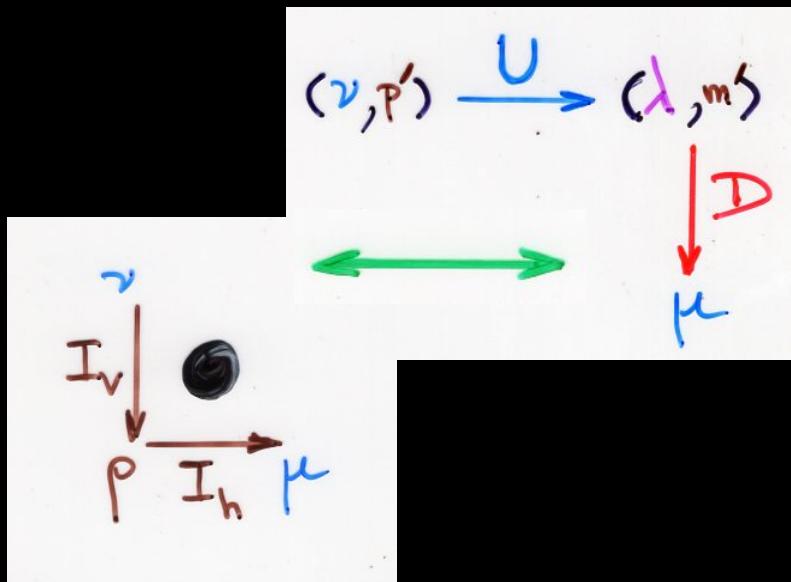


$$\begin{aligned}
 p &= i \\
 m &= i
 \end{aligned}$$

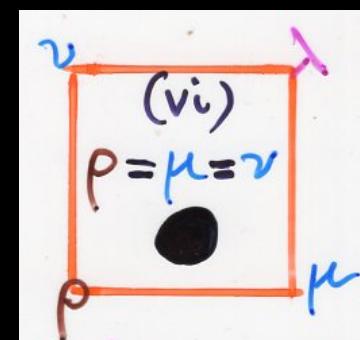
$$\begin{aligned}
 p' &= i+1 \\
 m' &= i+1
 \end{aligned}$$



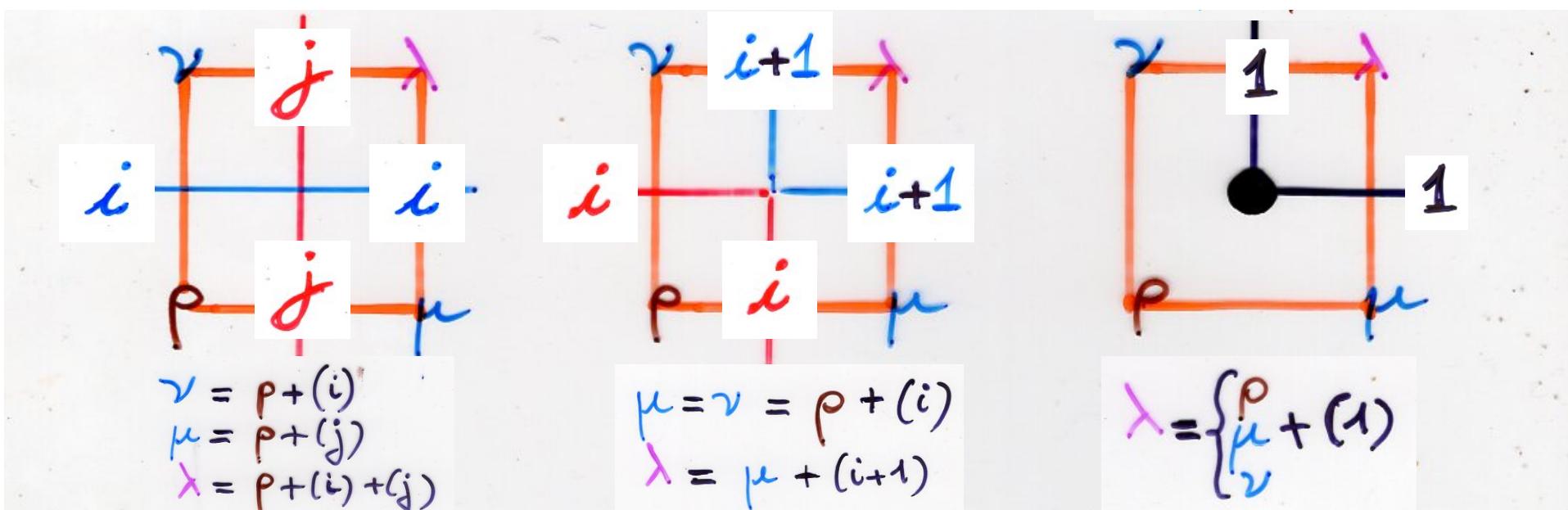
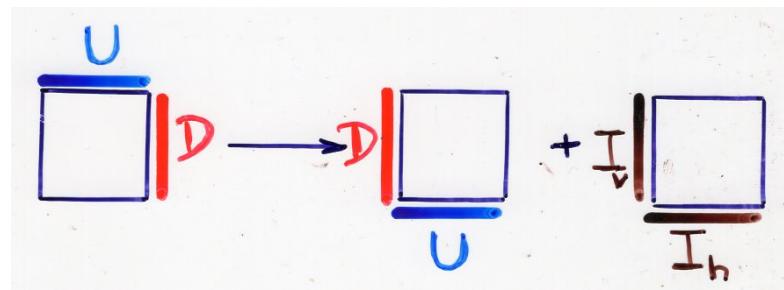
$$\begin{aligned}
 \mu &= v = p + (i) \\
 \lambda &= \mu + (i+1)
 \end{aligned}$$



$$p' = m' = 1$$



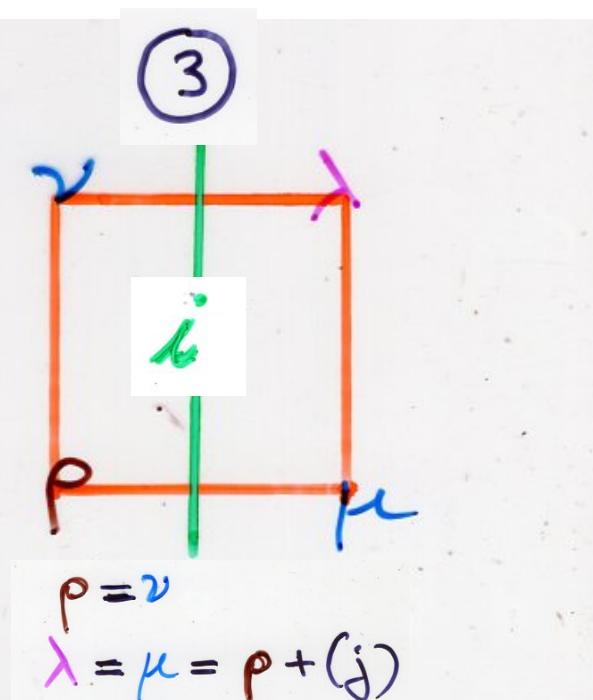
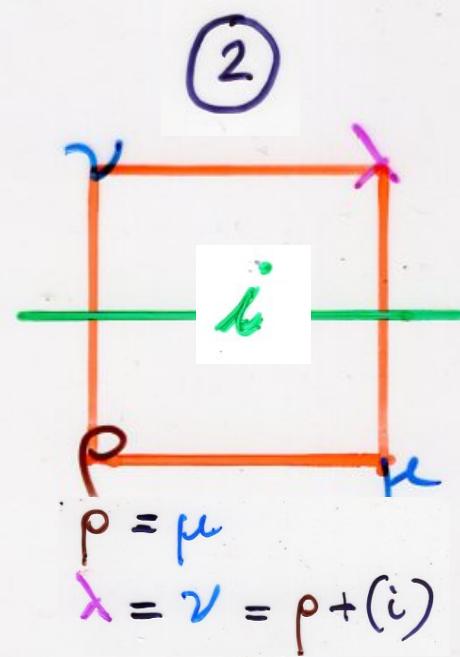
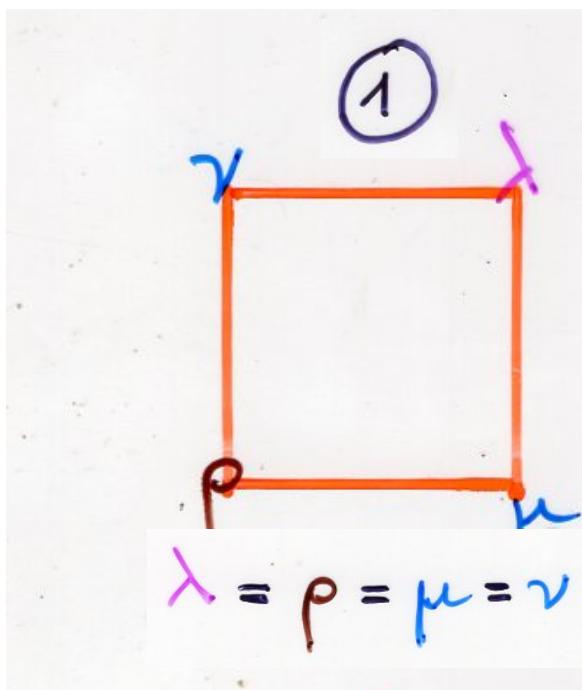
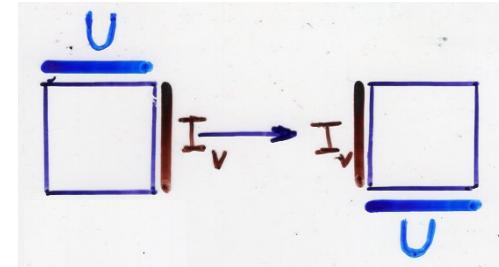
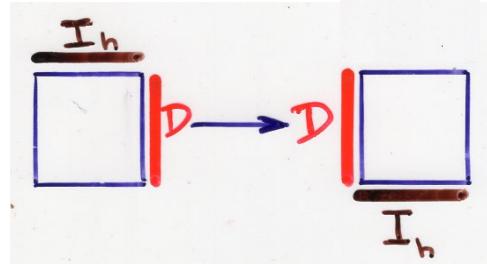
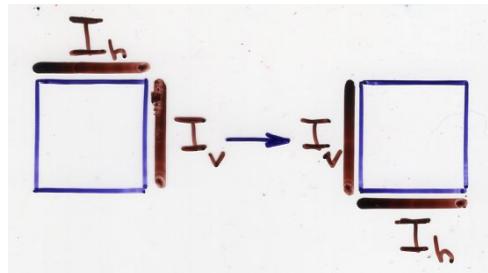
$$\lambda = \begin{cases} p \\ \mu + (1) \\ v \end{cases}$$

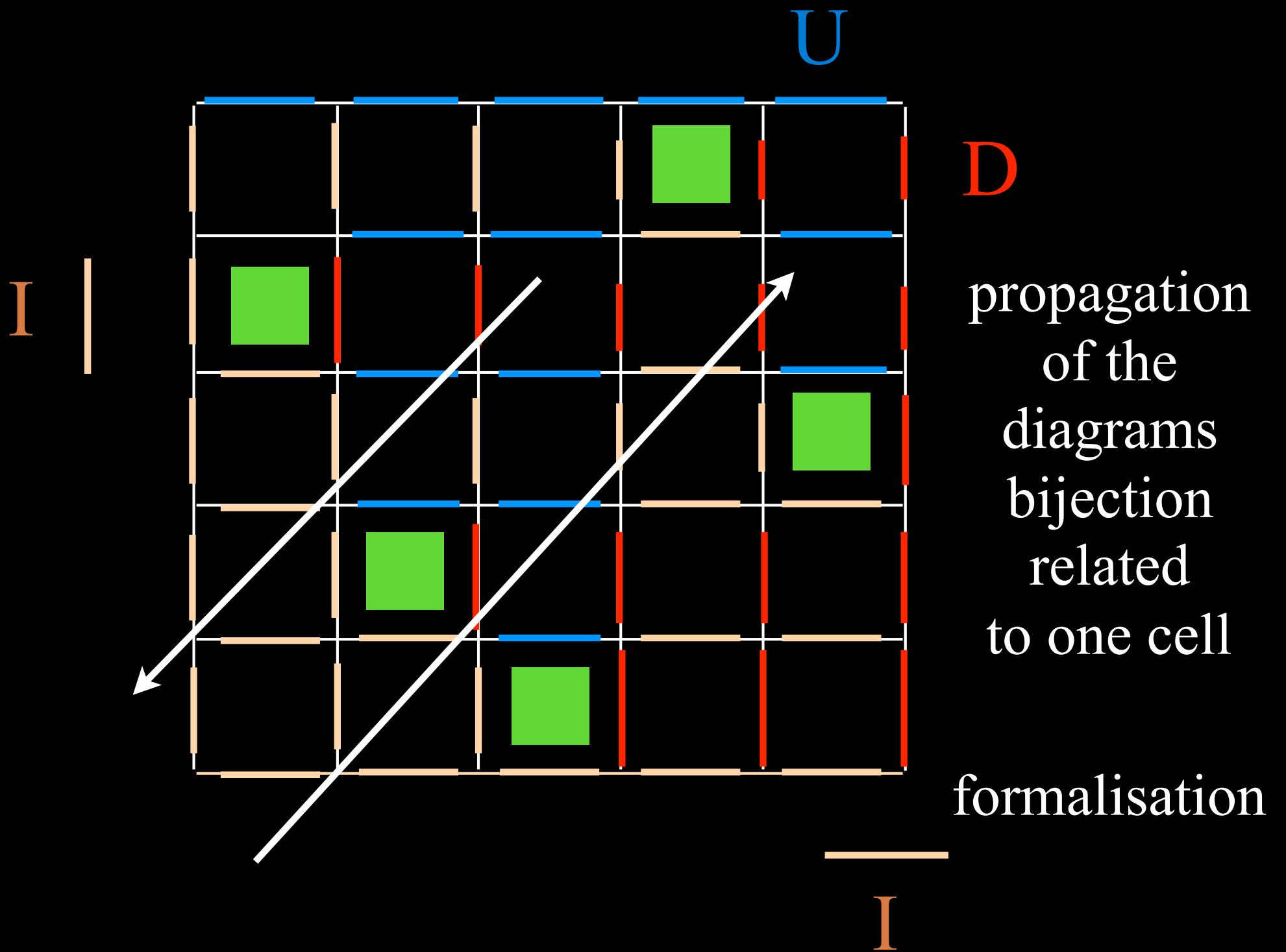


④

⑤

⑥





For word $w = w_1 \dots w_n$ of $\{U, D, I_h, I_v\}^*$

we consider sequences ~~of~~ h

$$h = ((\mu_1, p_1), \dots, (\mu_n, p_n), \mu_{n+1})$$

where $\mu_i, i=1, \dots, n+1$ are partitions (Ferrers diagrams)

and for $i=1, \dots, n$ μ_{i+1} is obtained from μ_i by applying the operator w_i at position p_i

$$w = w(h)$$

If $w_i = I_h$ or I_v , then $(\mu_{i+1}, p_{i+1}) = (\mu_i, p_i)$

h "histories"

admissible sequence

2-colored vacillating tableaux

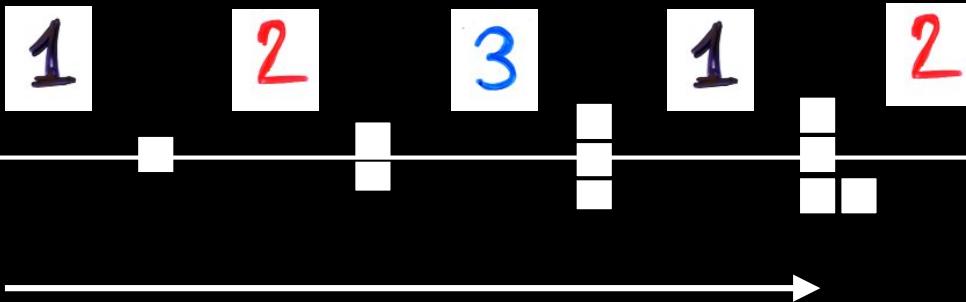
$(P, Q) \rightarrow (\alpha, \beta) \rightarrow$ sequence \mathfrak{h}
 Young tableaux
 same shape λ pair of maximal chains
 $\emptyset \rightarrow \lambda$

$$\mathfrak{h} = ((\mu_1, \rho_1), \dots, (\mu_{2n}, \rho_{2n}), \mu_{2n+1})$$

with $\mu_1 = \mu_{2n+1} = \emptyset$ and

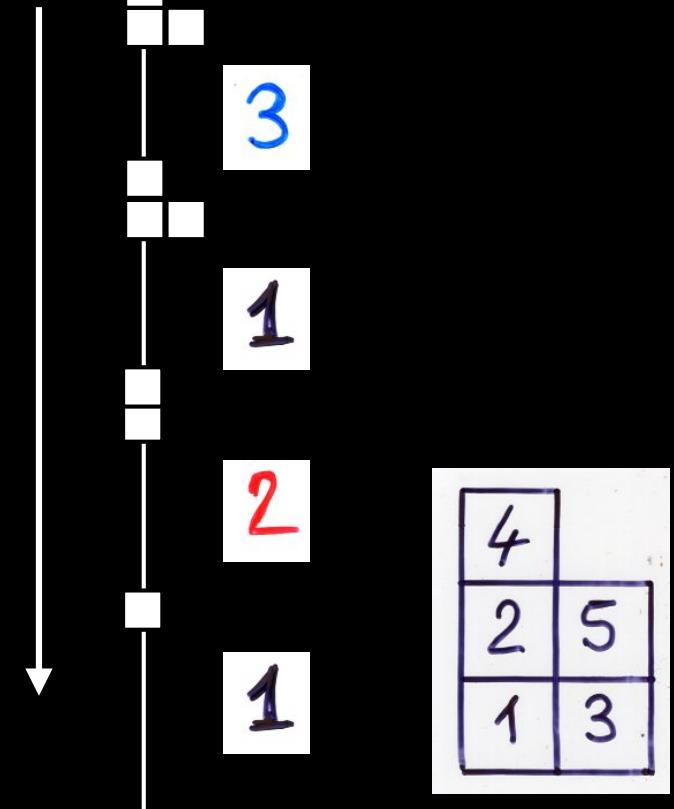
 $w(\mathfrak{h}) = U^n D^n$

3	
2	5
1	4



$$h = ((\mu_1, p_1), \dots, (\mu_{2n}, p_{2n}), \mu_{2n+1})$$

with $\mu_1 = \mu_{2n+1} = \emptyset$ and
 $w(h) = U^n D^n$

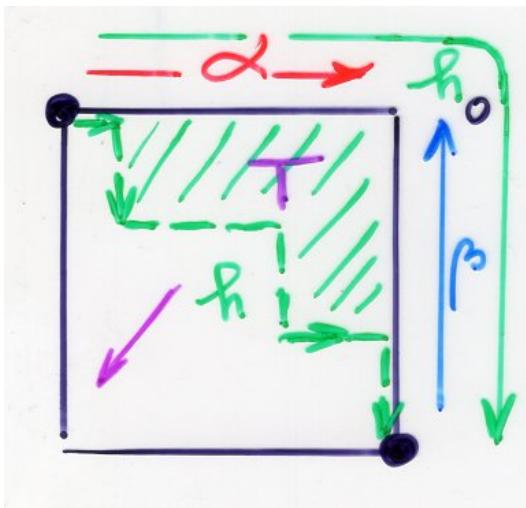


4	
2	5
1	3

Starting from $h_0(\alpha, \beta) = h_0(P, Q)$ $T = \emptyset$

we "propagate" the "commutation diagrams" through the lattice $[n] \times [n]$.

At any step, we have a pair



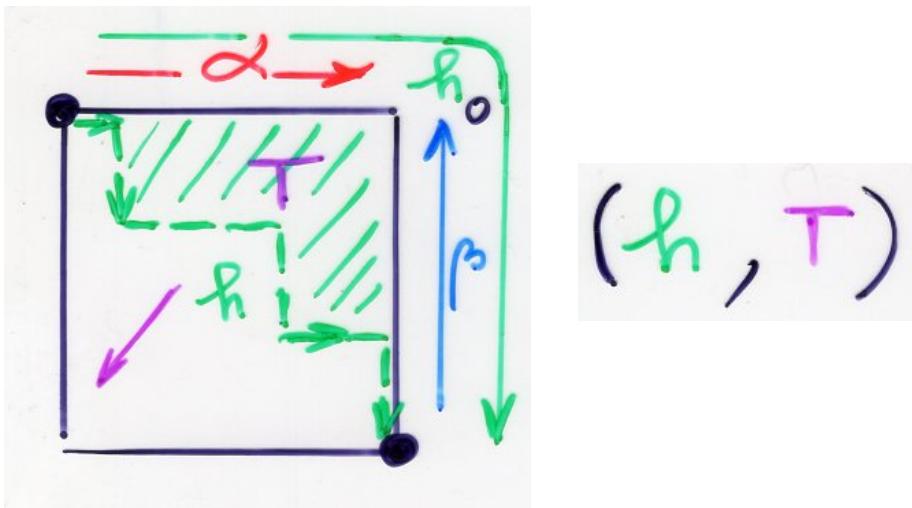
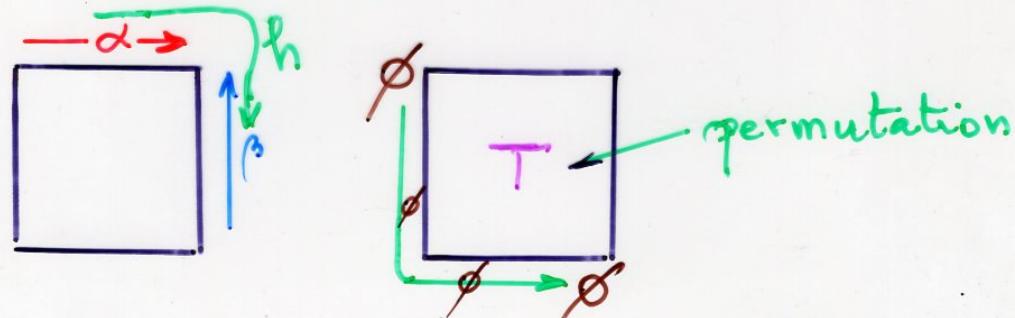
(h, T)

T tableau above the path
associated to $w(h)$
with cells labeled
by \square \bullet

$(h, T) \longleftrightarrow h_0 = h(\alpha, \beta)$
are in bijection

By recurrence

Thus $h(\alpha, \beta)$ in bijection with



(h, T)

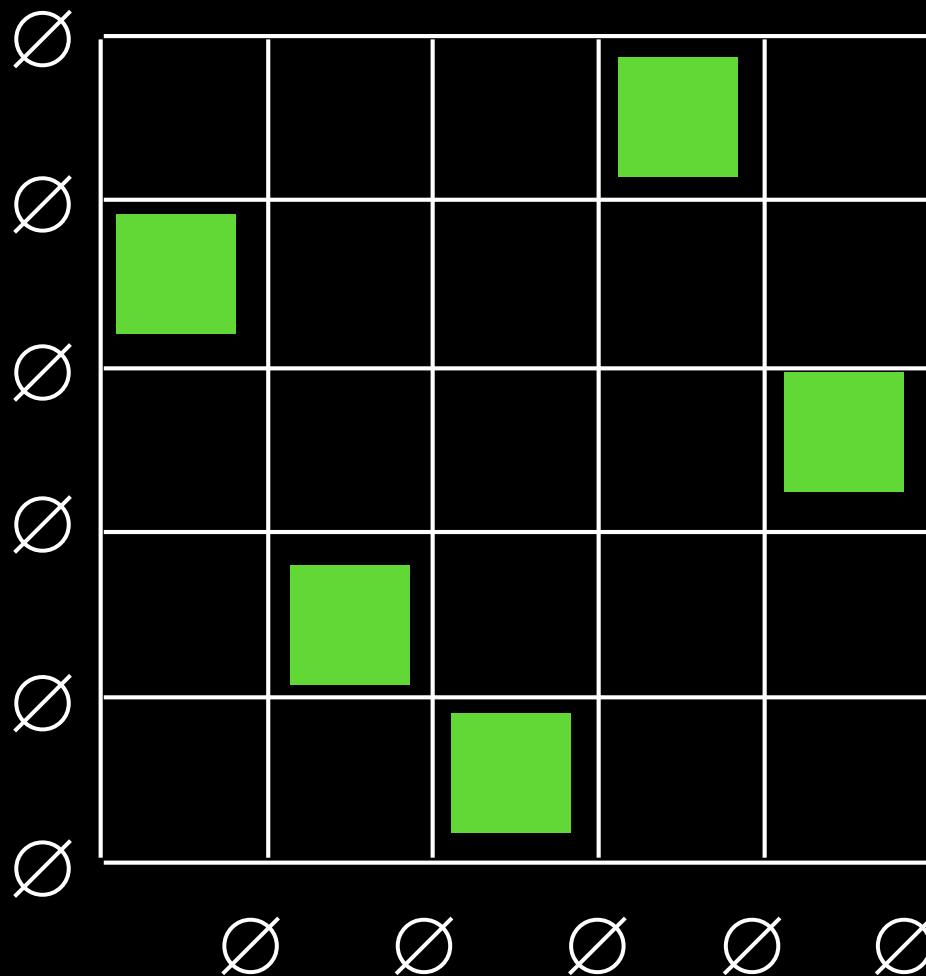
T tableau above the path
associated to $w(h)$
with cells labeled
by \square \bullet

$(h, T) \longleftrightarrow h_0 = h(\alpha, \beta)$
are in bijection

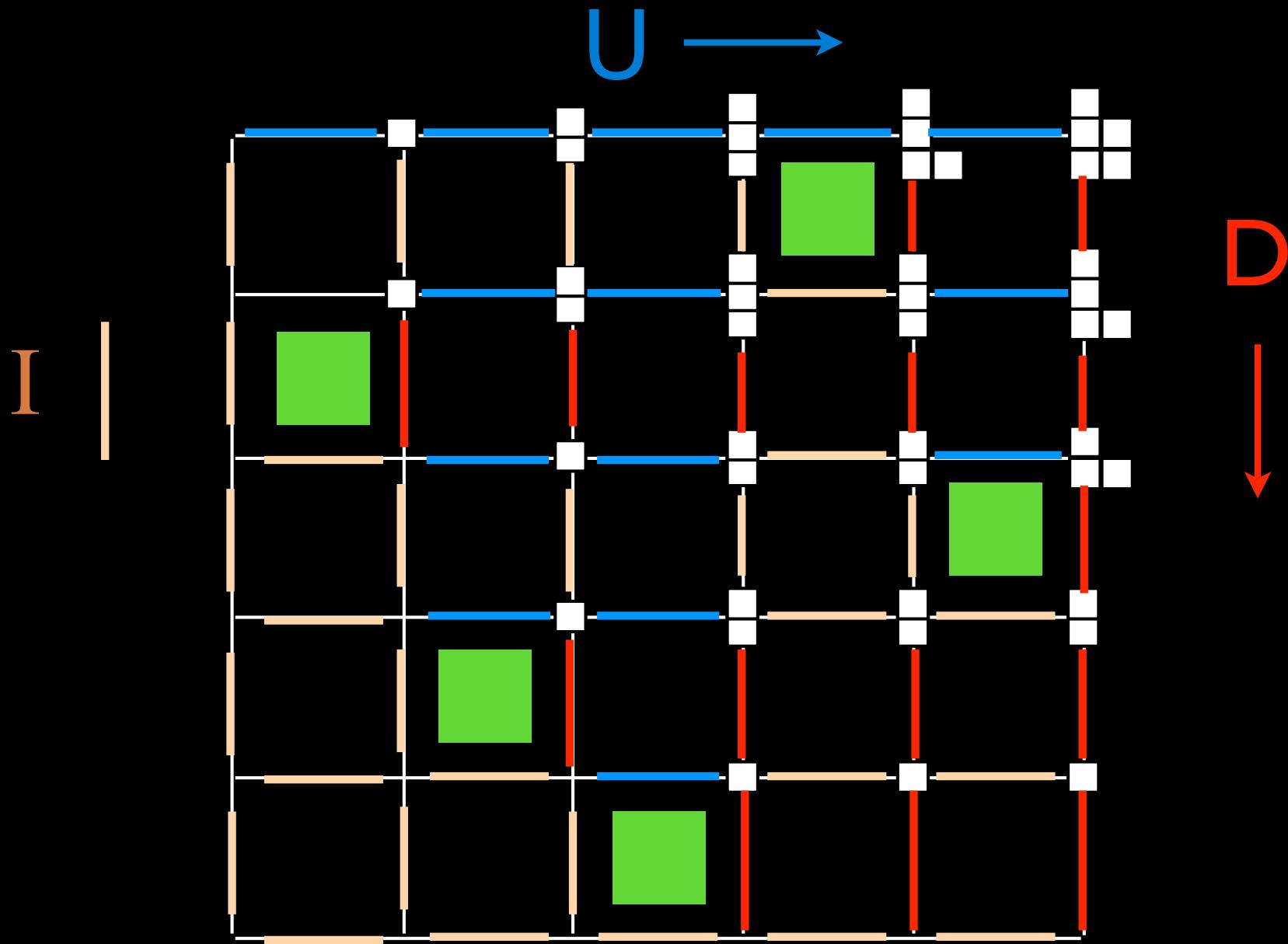
By recurrence

3		
2	5	
1	4	

1 2 3 1 2

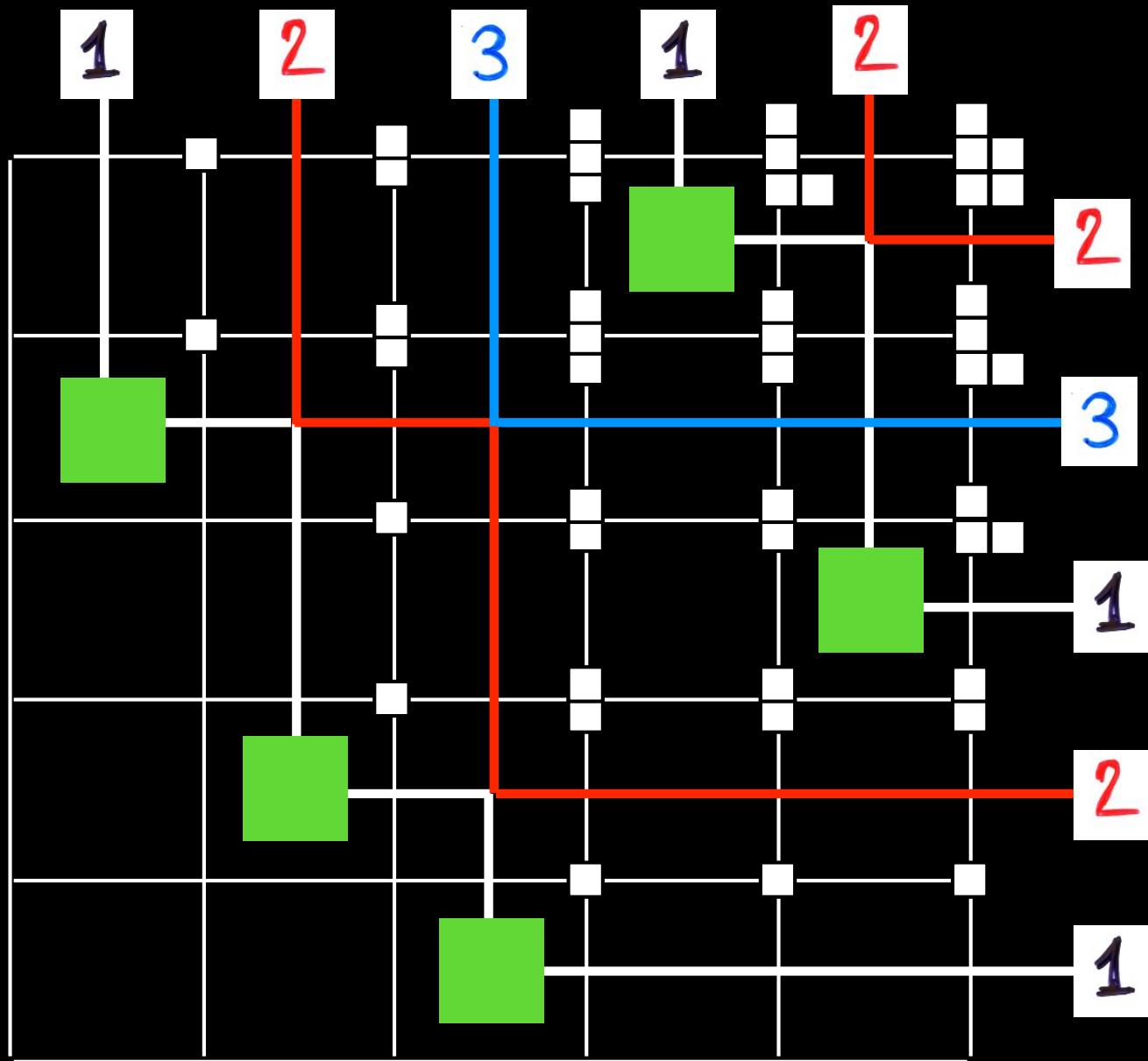


4		
2	5	
1	3	

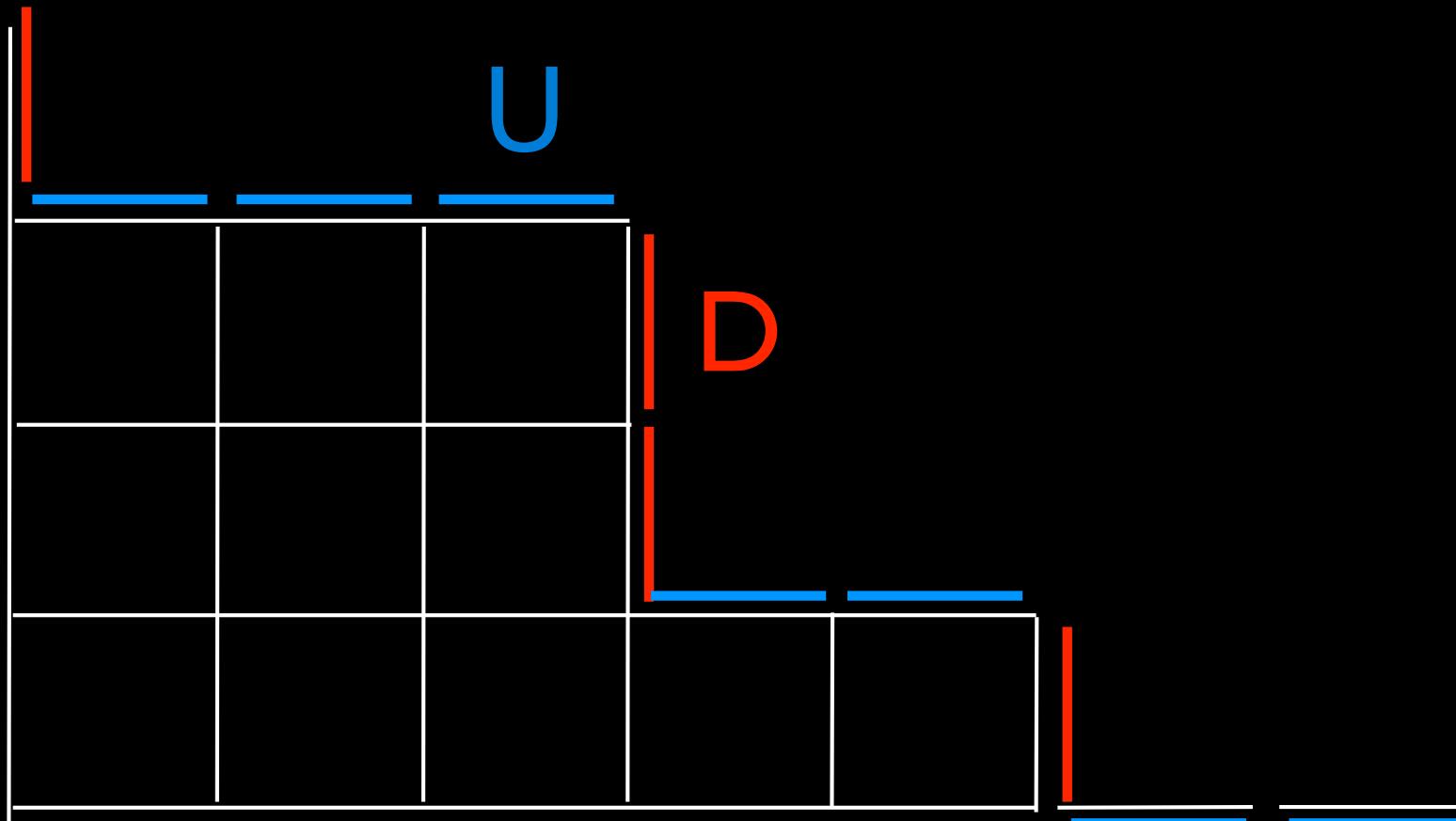


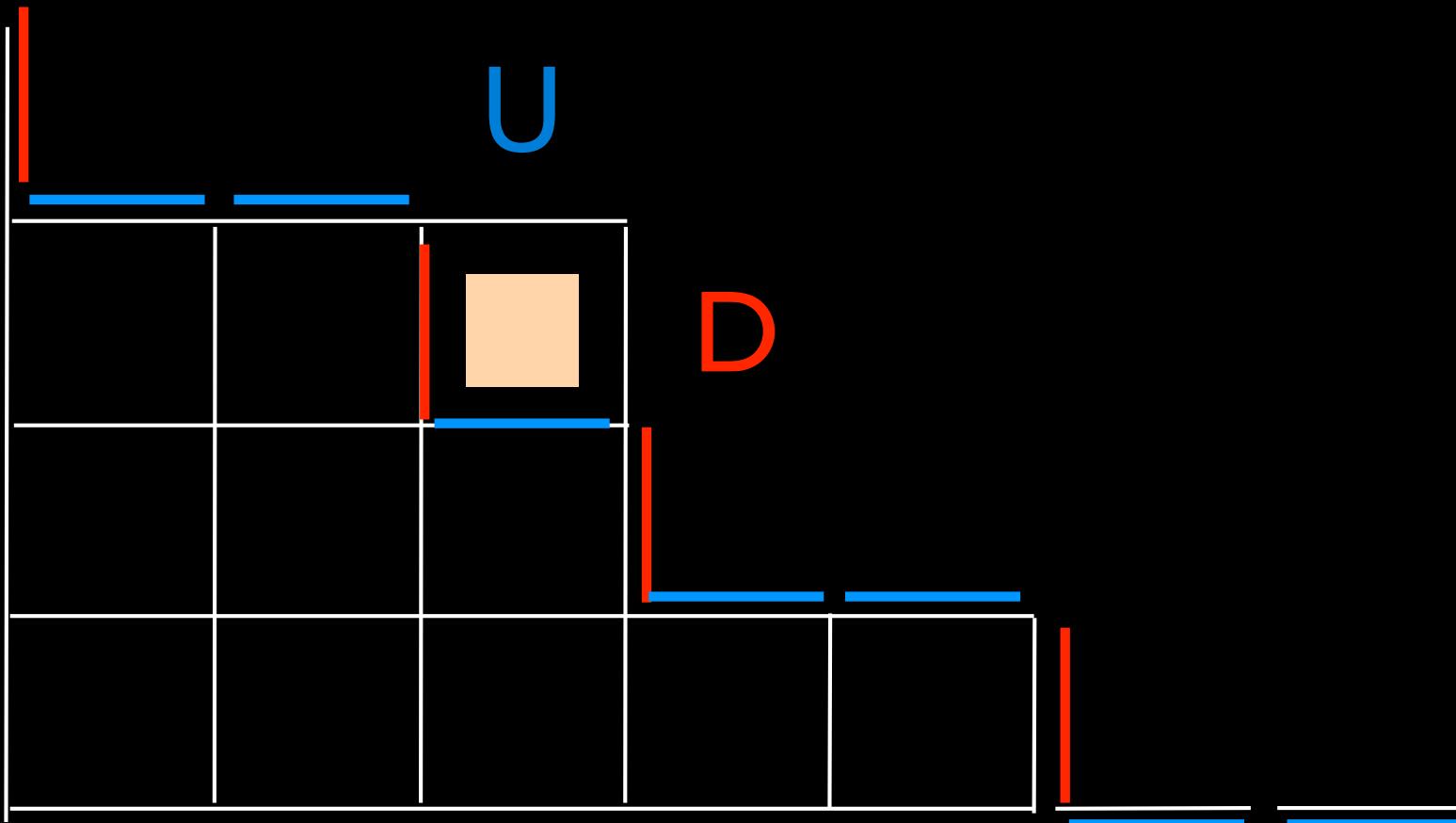
This "propagation" algorithm is
exactly the reverse of Fomin's "growth
diagrams"

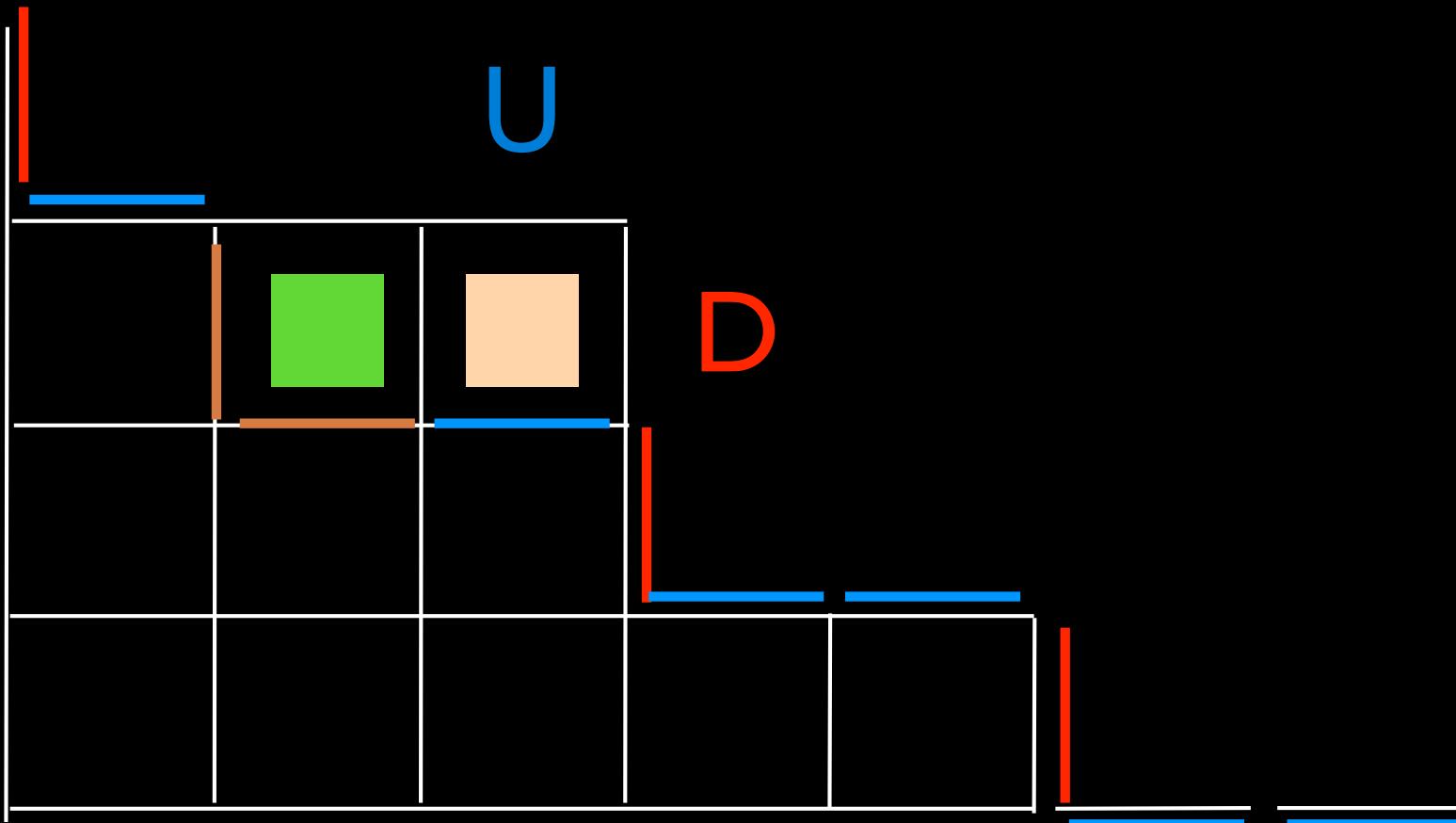
I

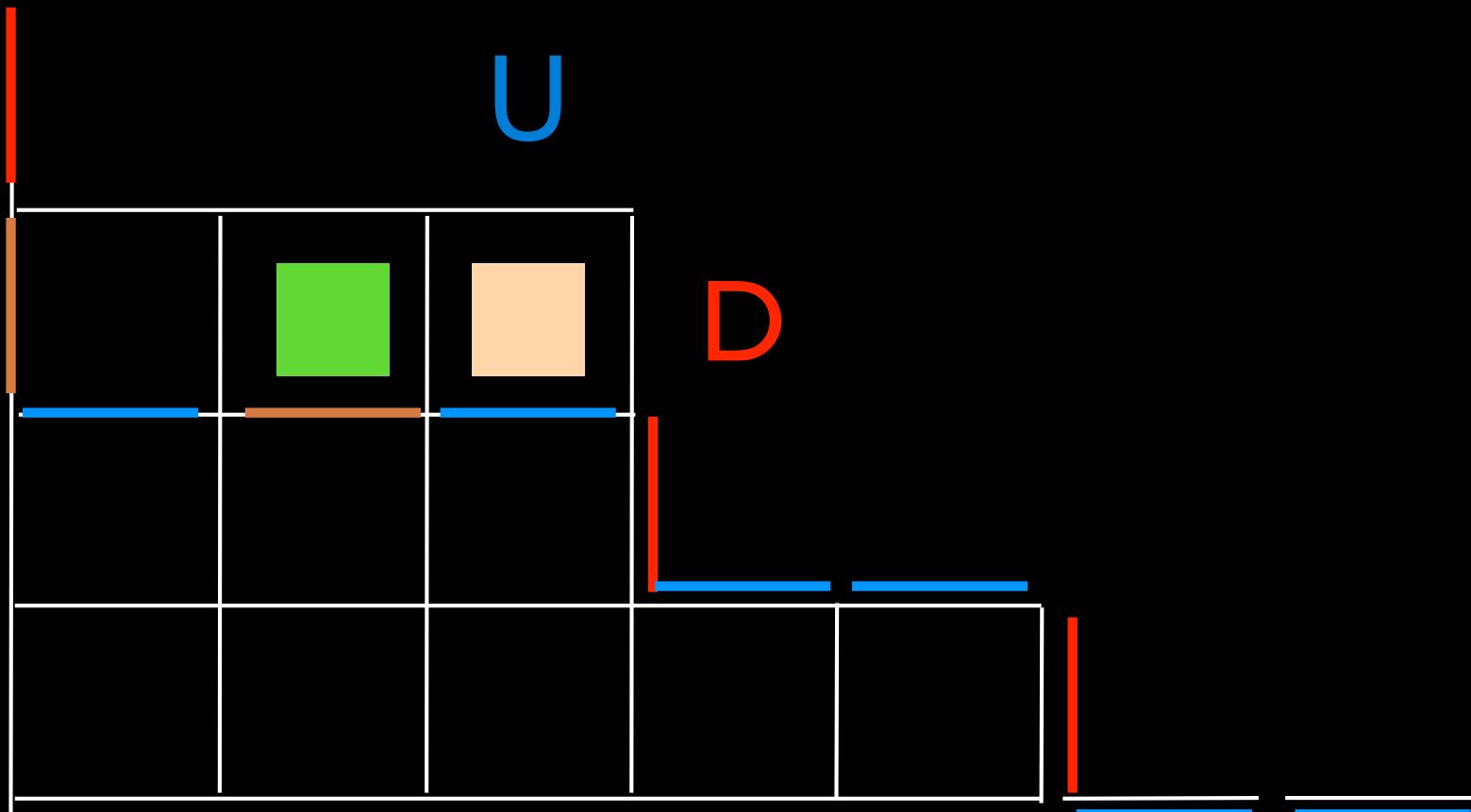


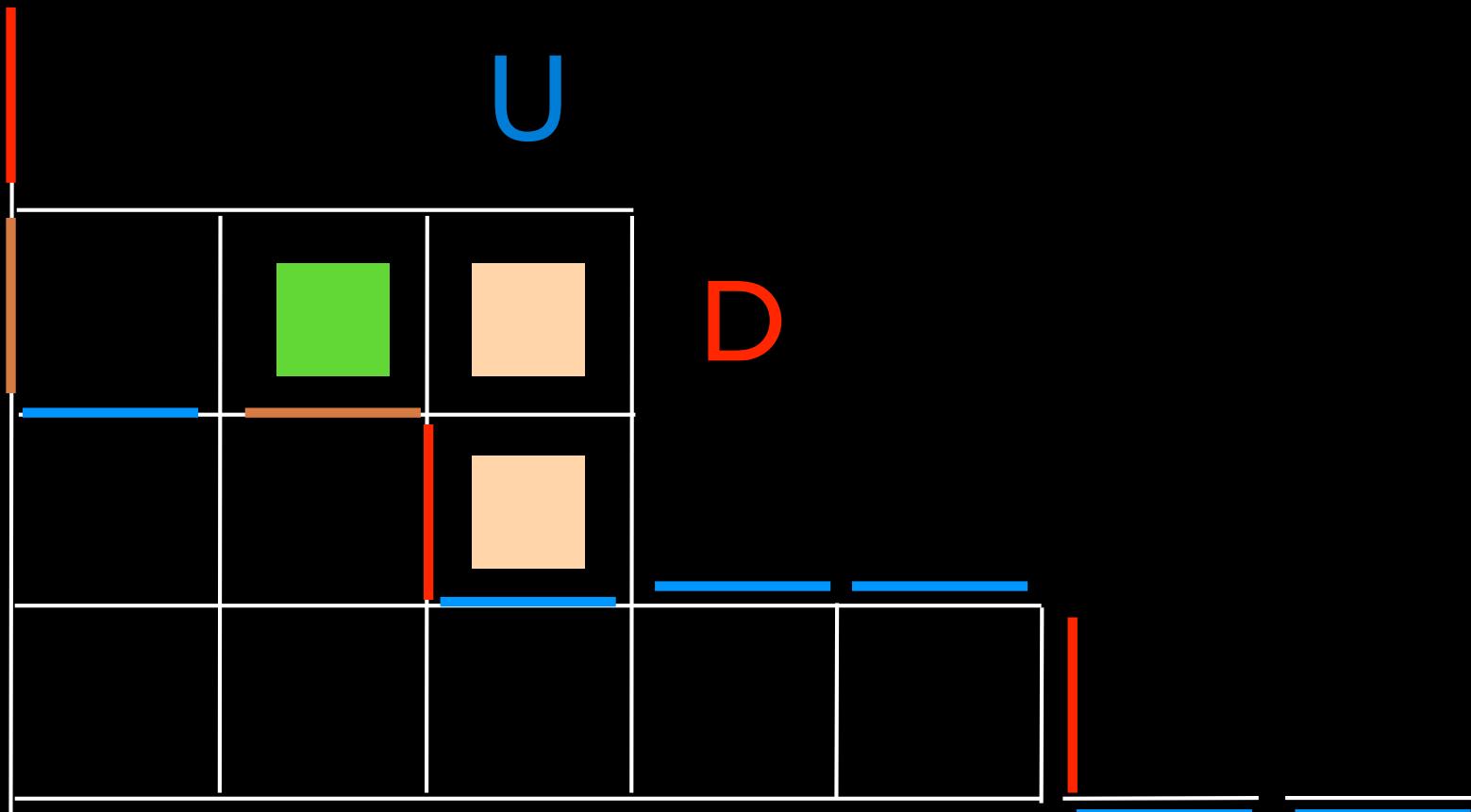
rook placements

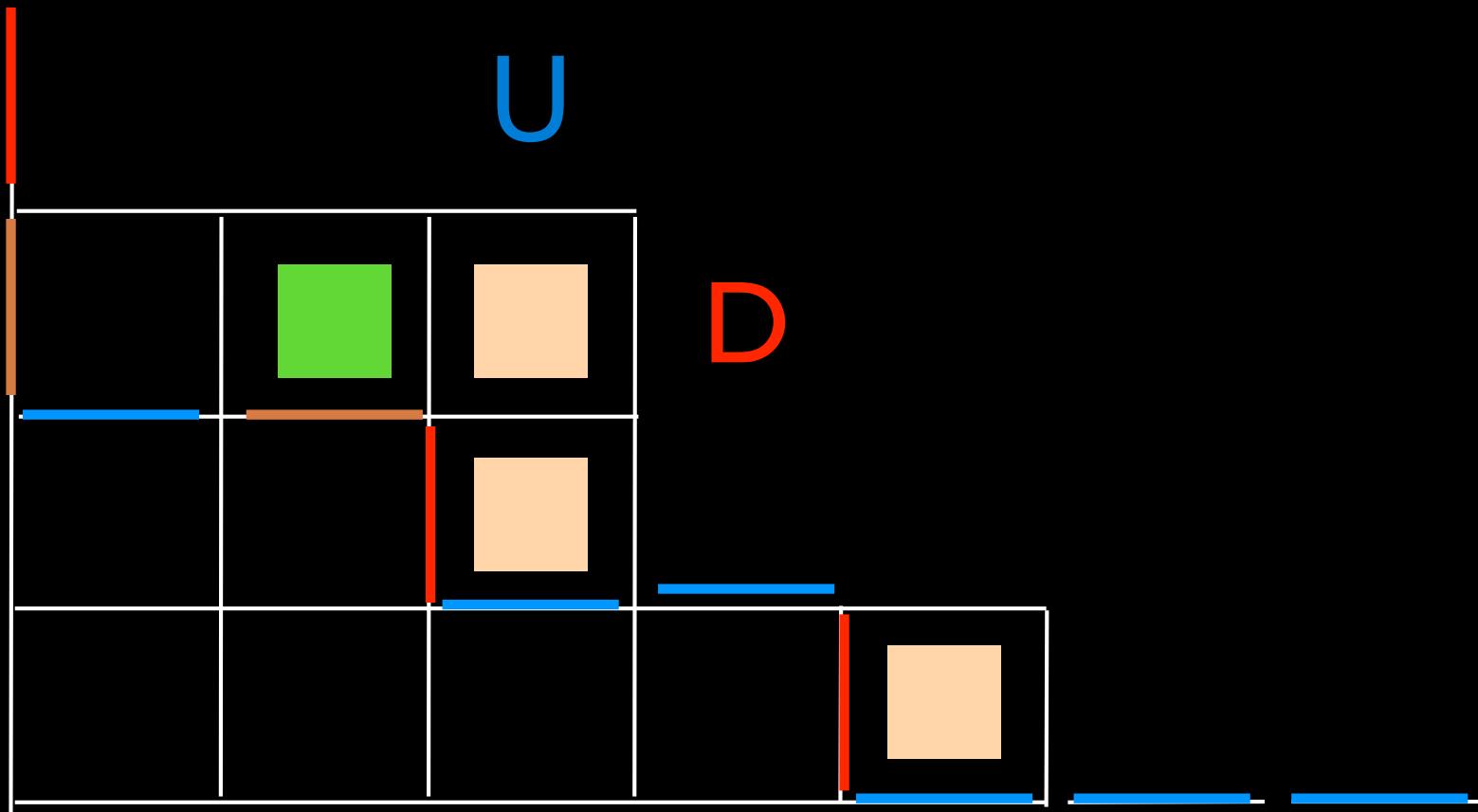


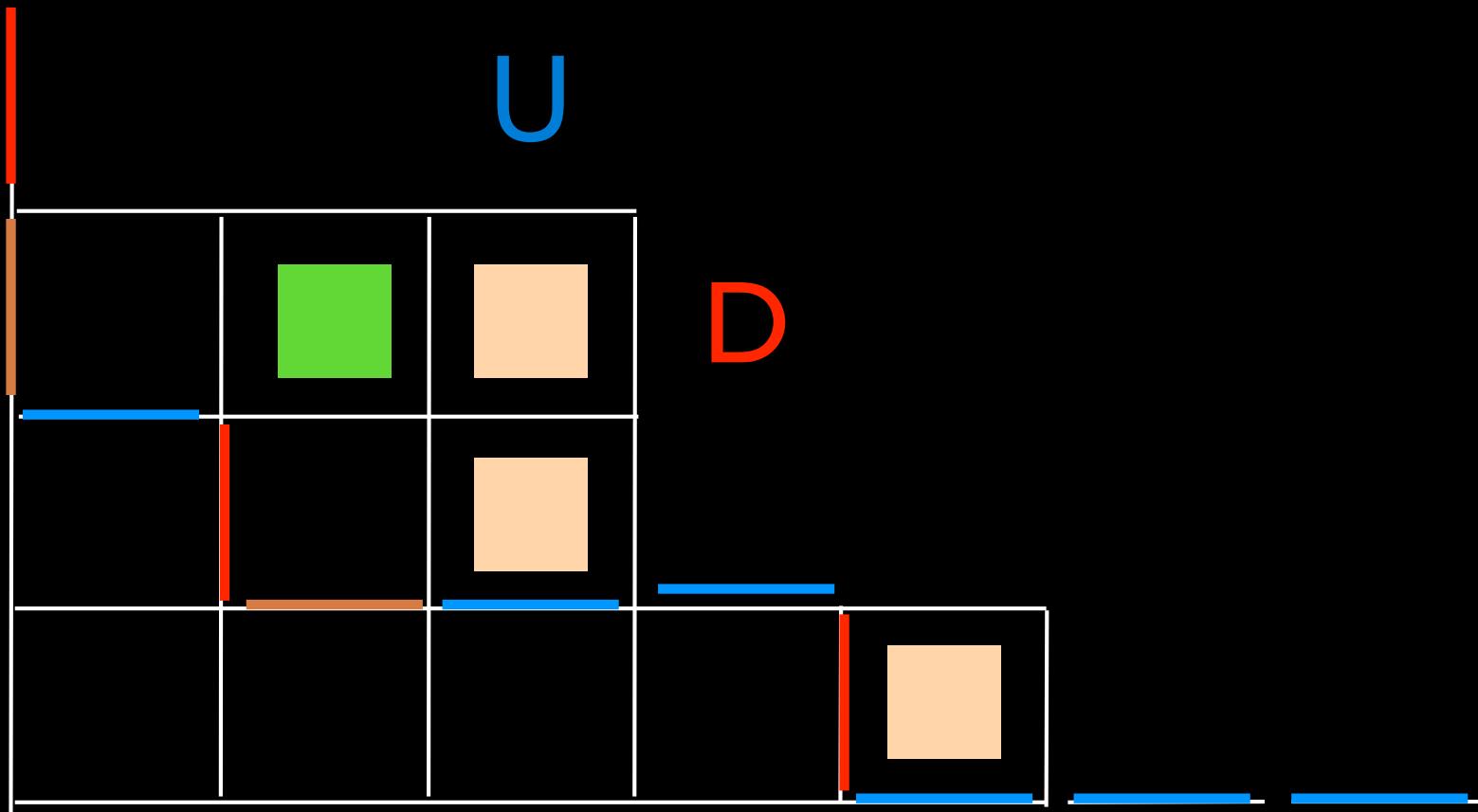


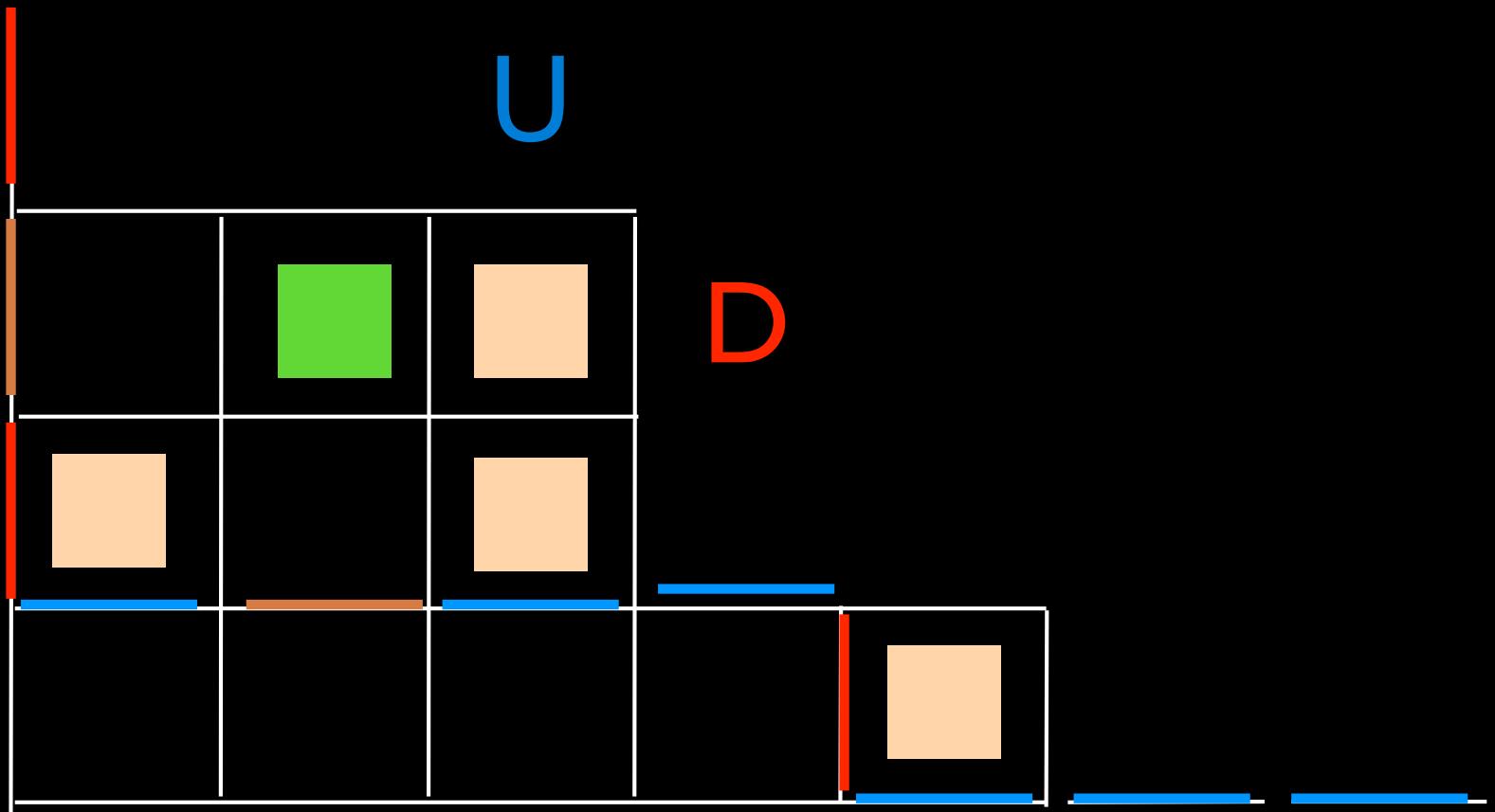


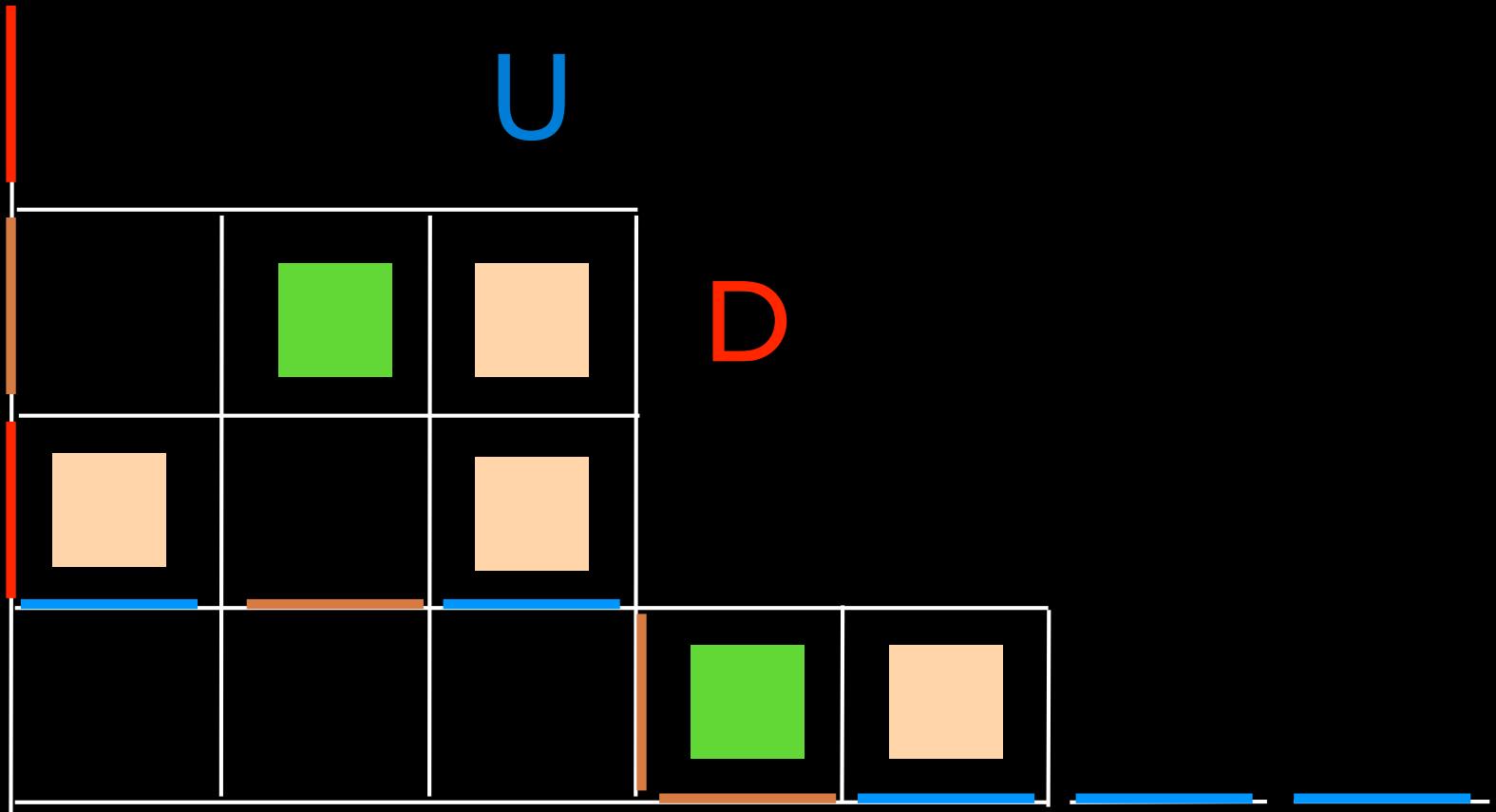


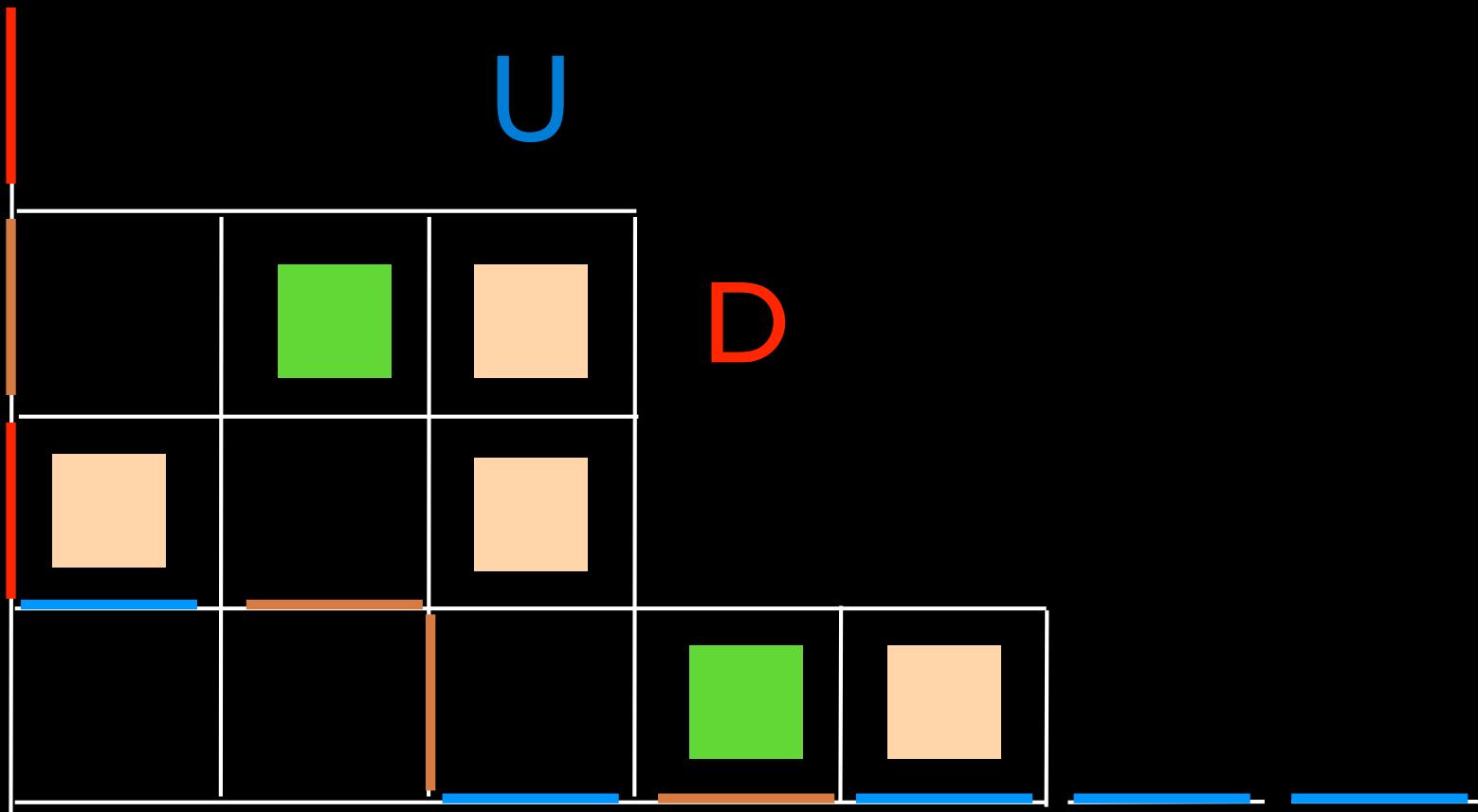


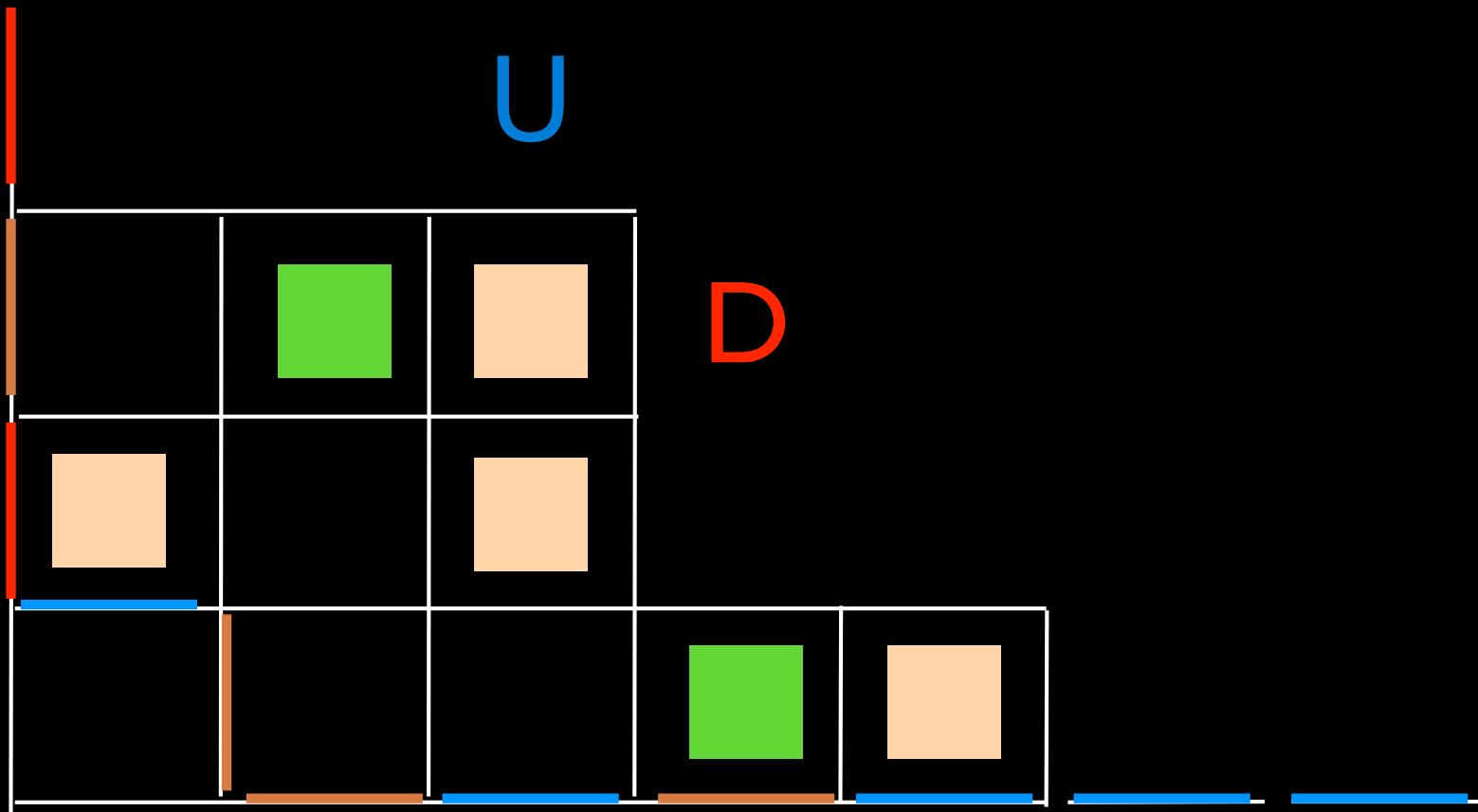




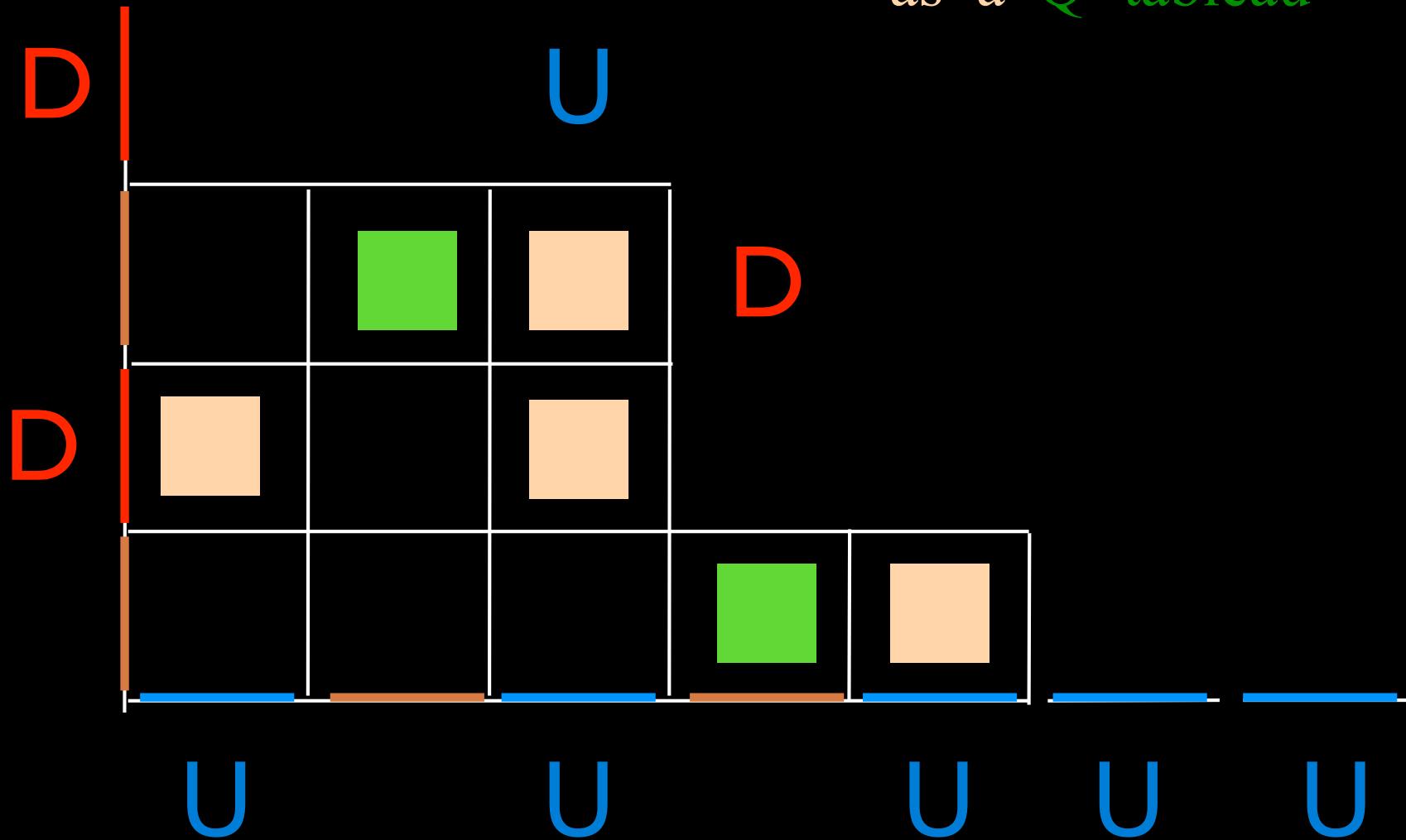






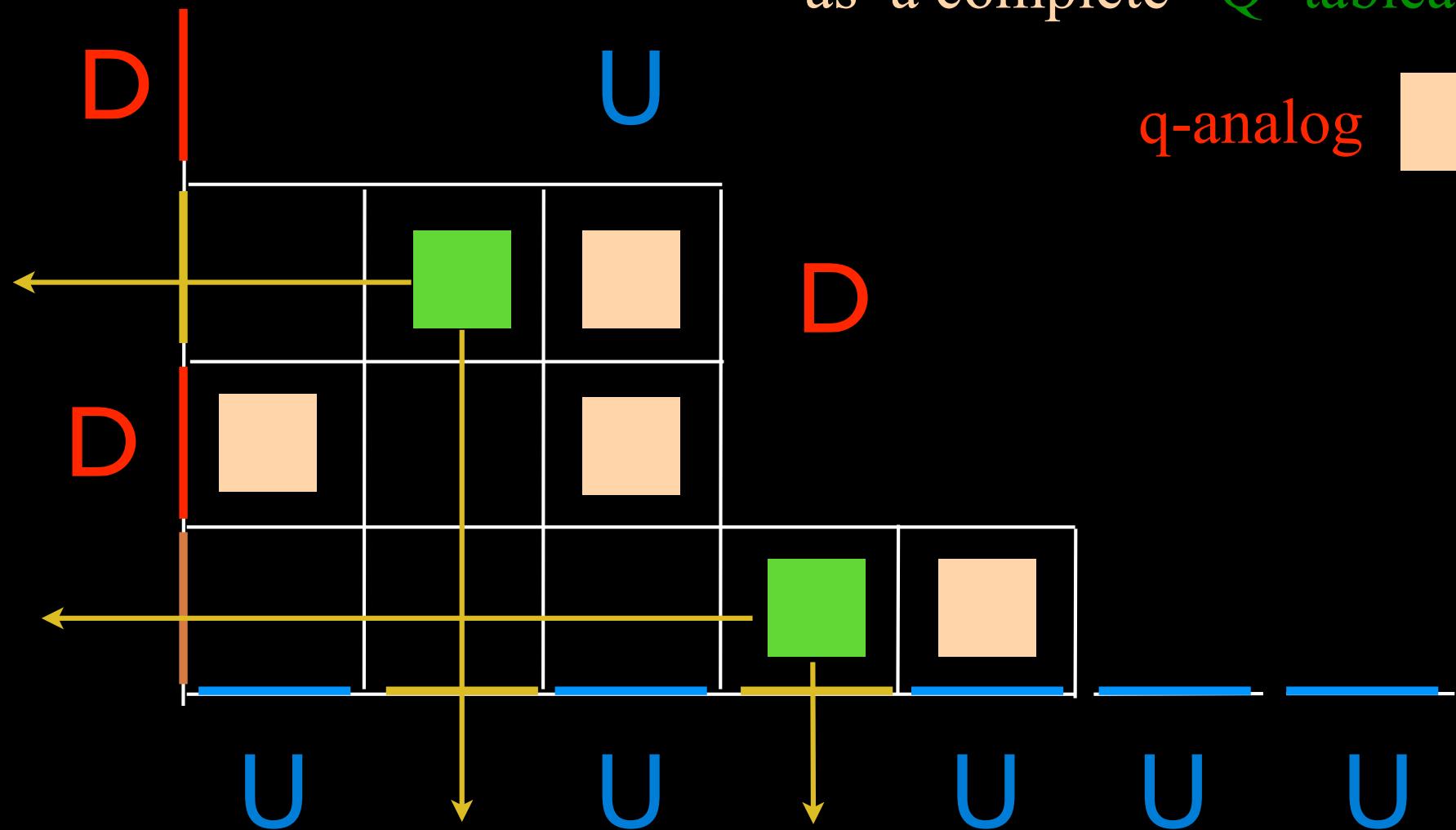


rook placement
as a Q-tableau



rook placement
as a complete Q -tableau

q-analog



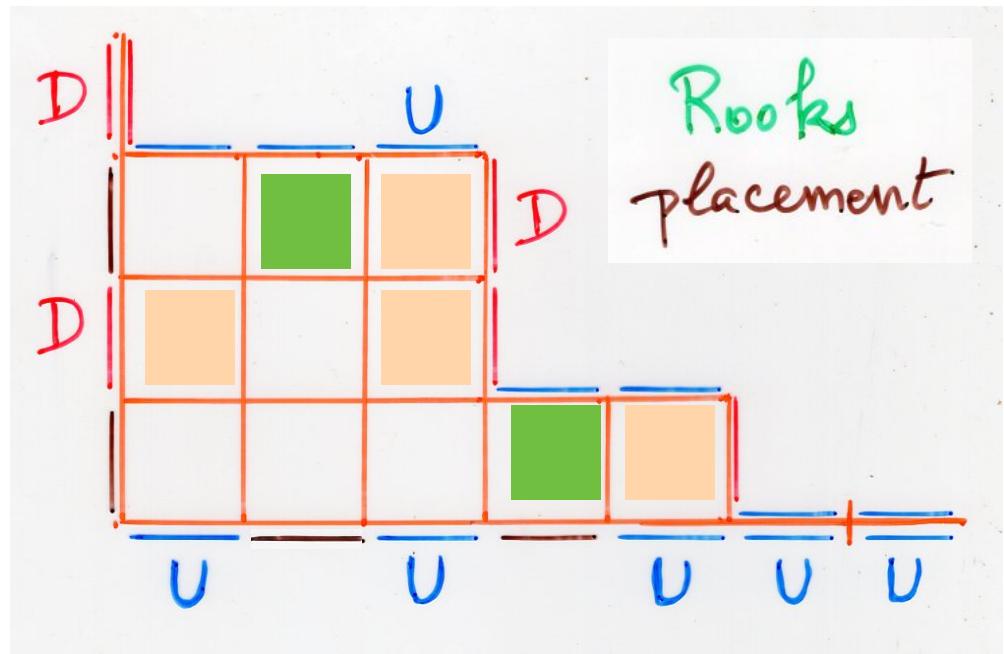
$$w = D U^3 D^2 U^2 D U^2$$

$$w \rightarrow F = F(w)$$

F Ferrers diagram

Proposition

$$w(U, D) = \sum_T D^{i(T)} U^{j(T)}$$



$\begin{cases} i(T) = \text{number of rows with no cell labeled} \\ j(T) = \text{number of columns } UD \rightarrow I_v I_h \end{cases}$

$$q^{k(T)}$$

Lemma Every word w with letters U and D can be written in a unique way

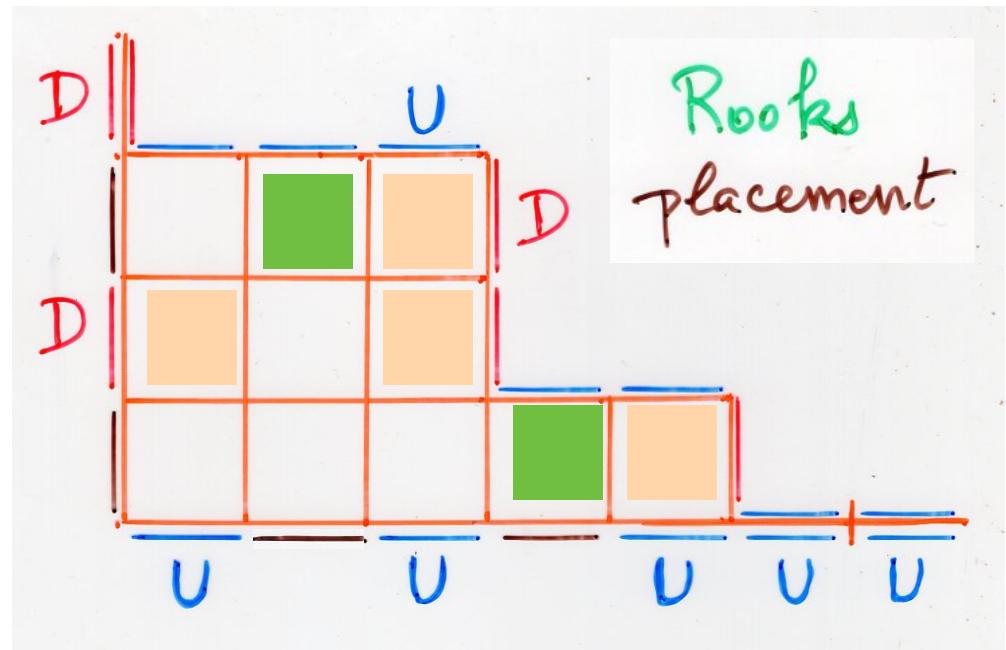
$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

Proposition

$c_{ij}(w)$ = number of placements of k rooks on the Ferrers "board" F

$$\text{with } i = |w|_D - k$$

$$j = |w|_U - k$$



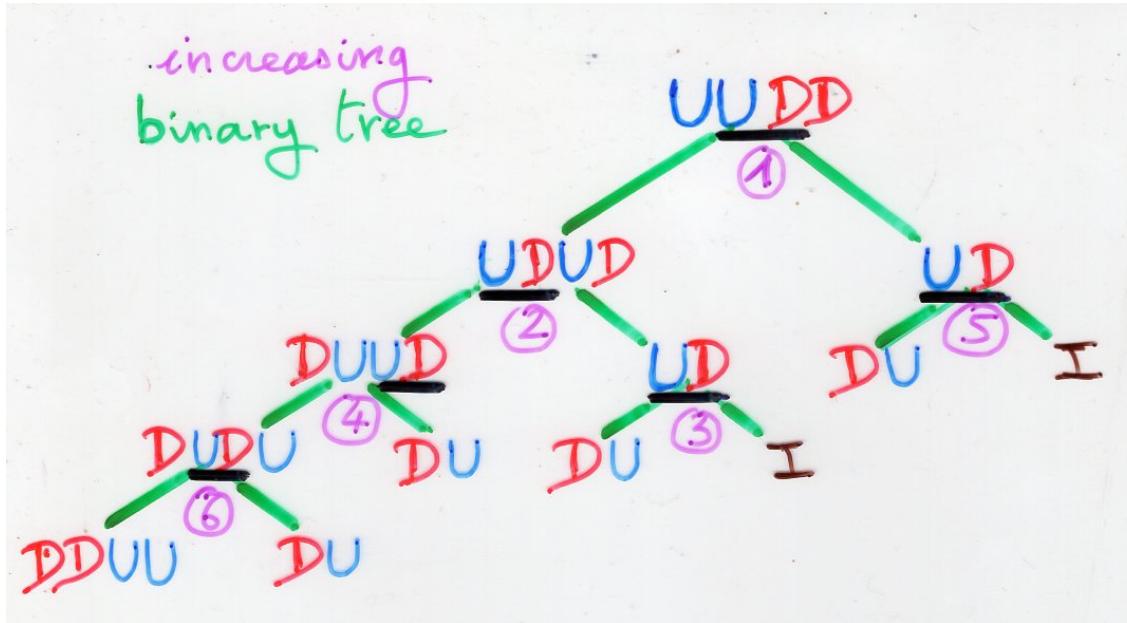
$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

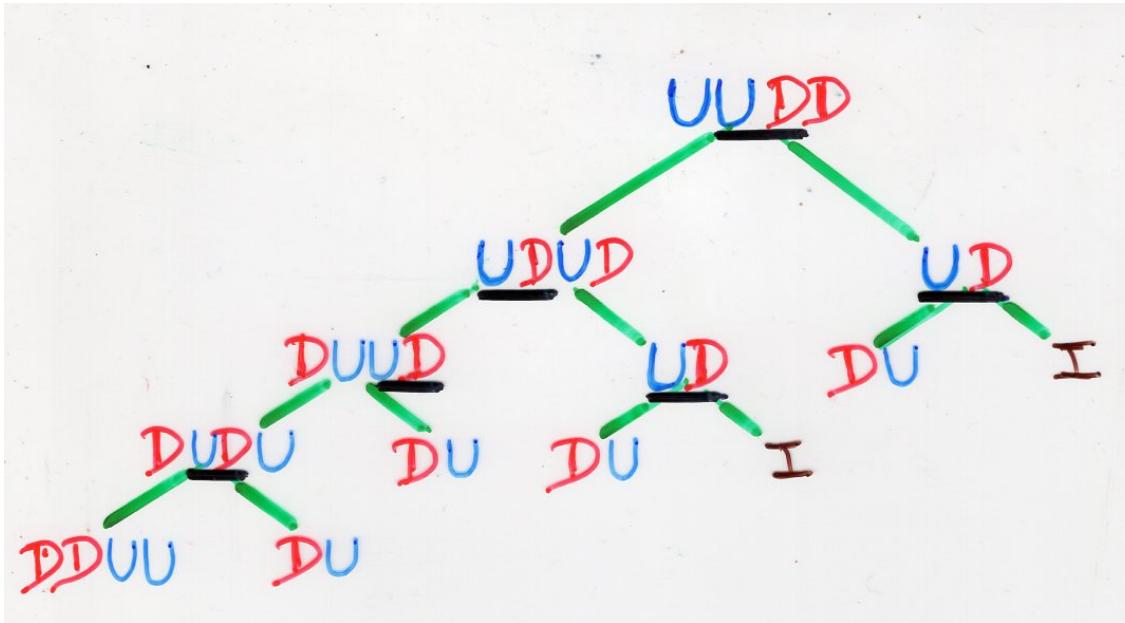
choice of the underlying grid $\binom{n}{i}^2$



binary tree T
associated
to a possible
rewriting process

$$U^2 D^2 = D^2 U^2 + 4 D U + 2 I$$

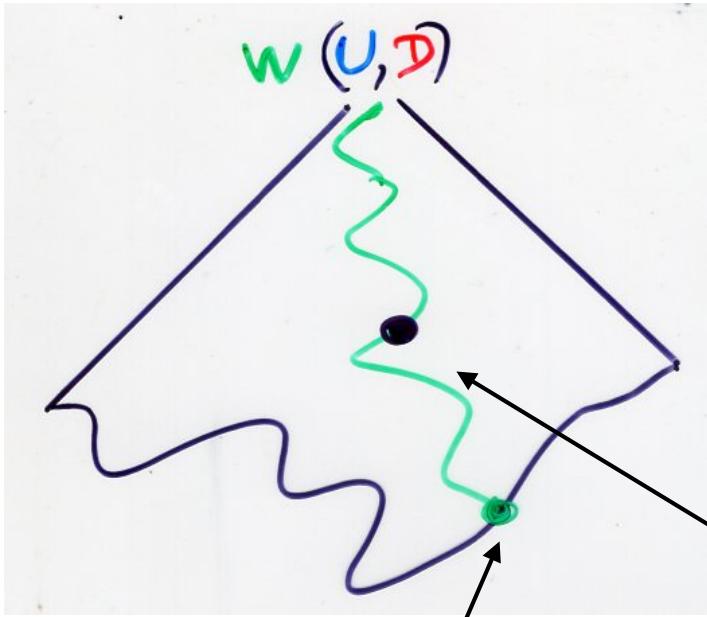
this polynomial is independant
of the order of the substitutions



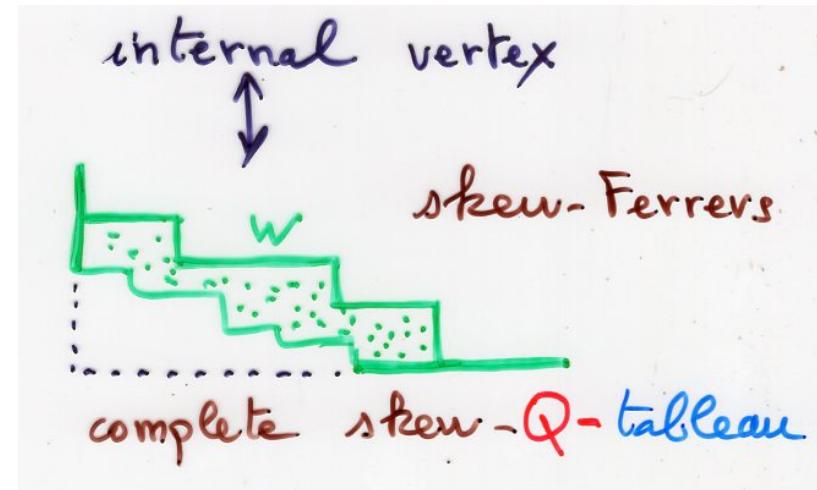
binary tree T
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$$U^2 D^2 = D^2 U^2 + 4 D U + 2 I$$

this polynomial is independant
of the order of the substitutions



binary tree T
associated
to a possible
rewriting process



leaves of T

bijection

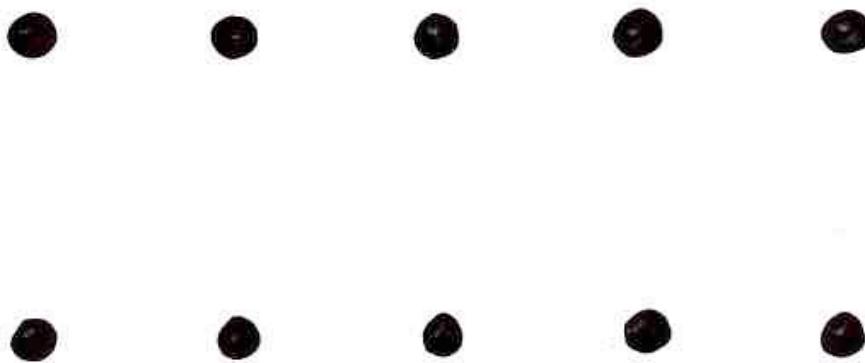
complete
 Q -tableaux
shape λ

$\lambda = F(w)$

Another representation of the algebra

$$UD = DU + I$$

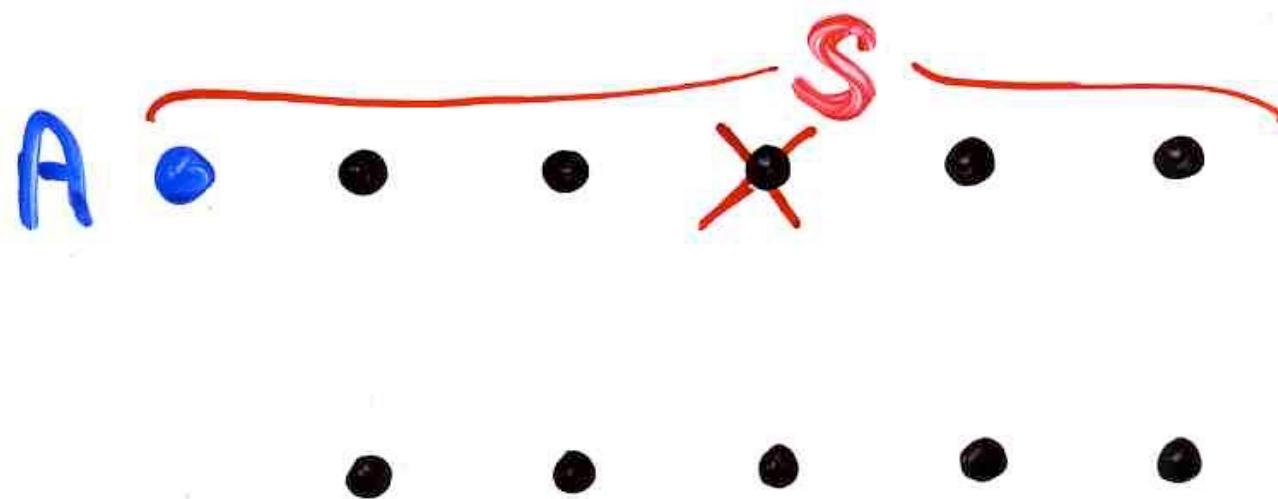
Polya urn

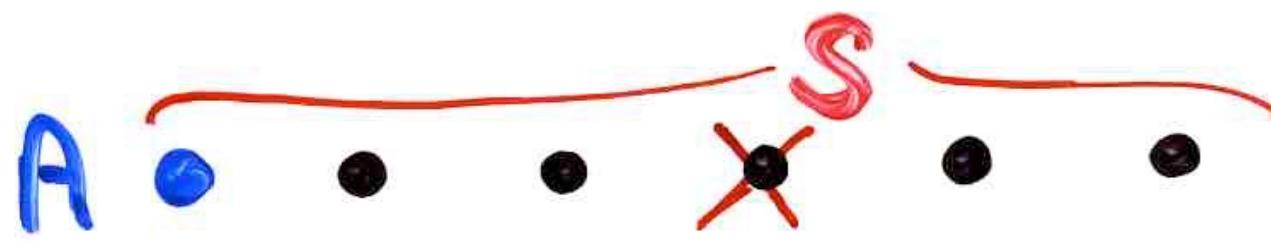


A .

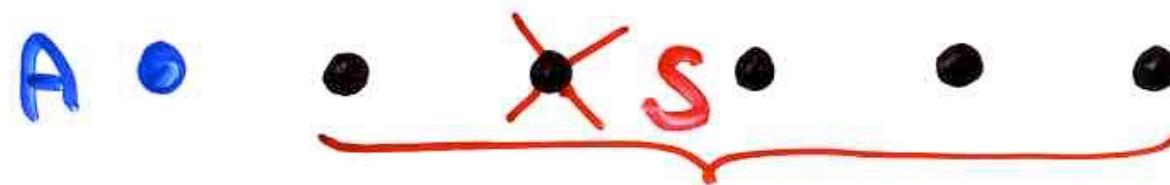
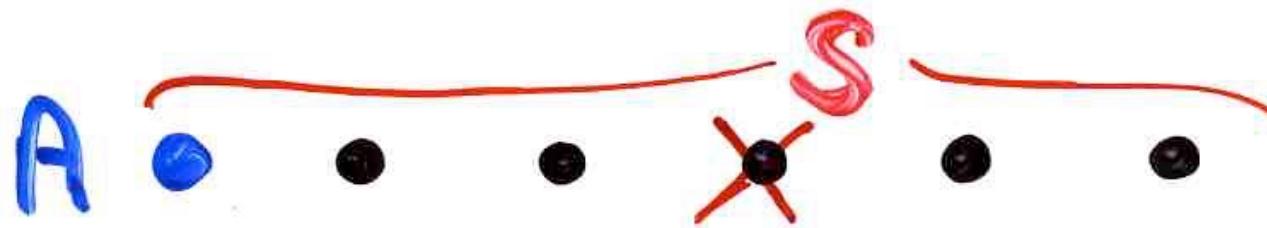
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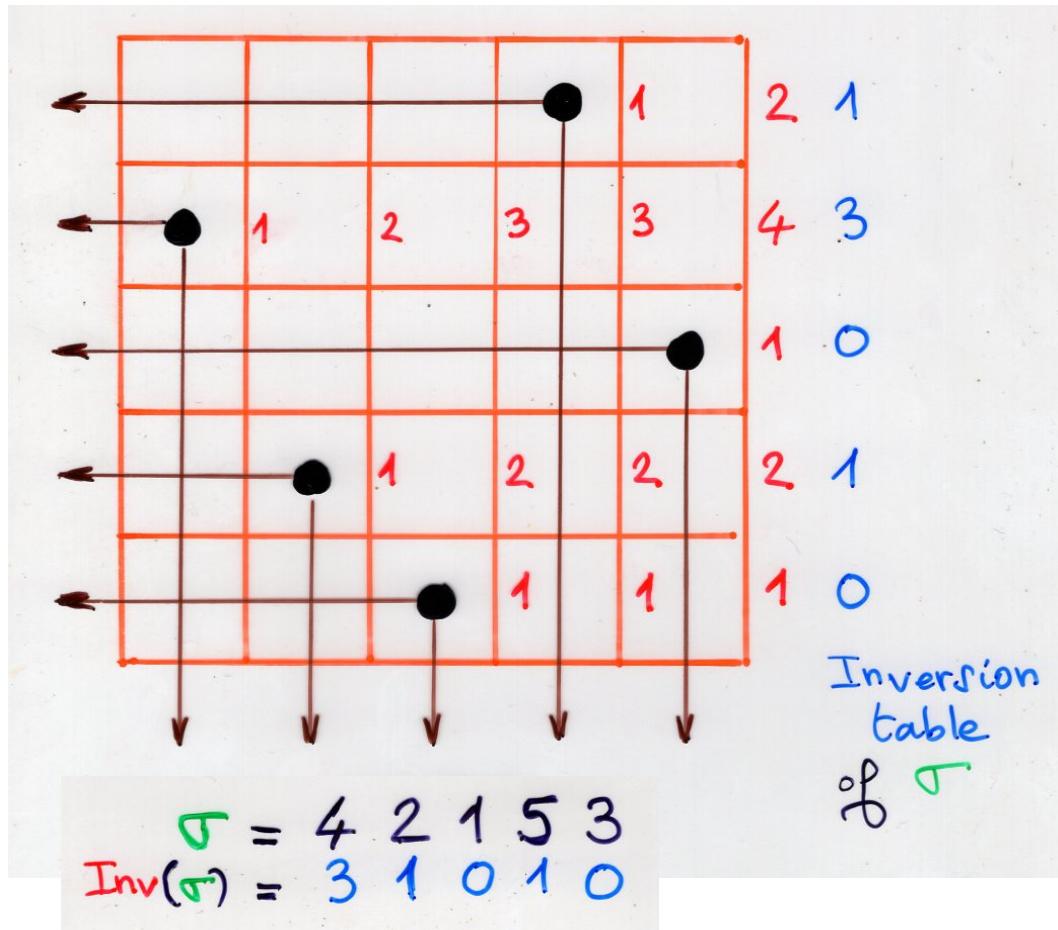
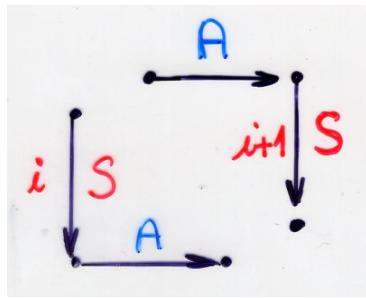
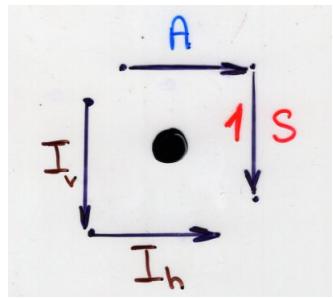
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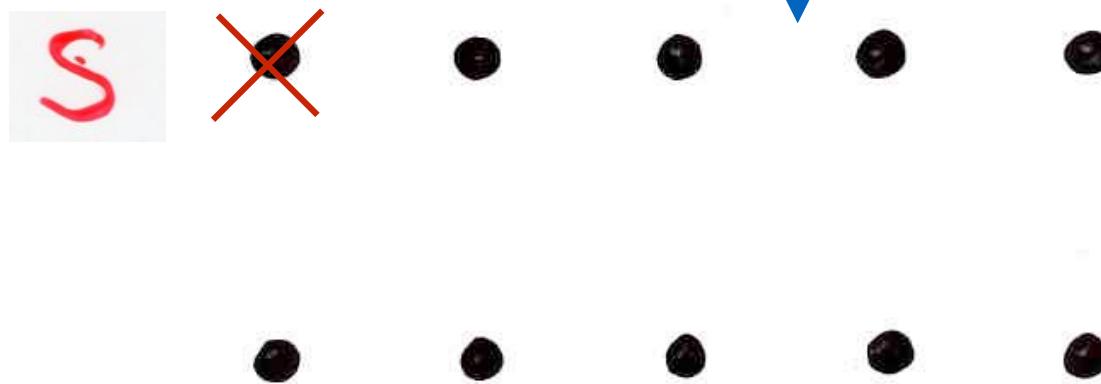
$$A S - S A = I$$





Priority queue

$$A S - S A = I$$



data structures

Computer Science

