Tianjin-2

An introduction to algebraic combinatorics with RSK

(the Robinson-Schensted-Knuth correspondence)

Tianjin University
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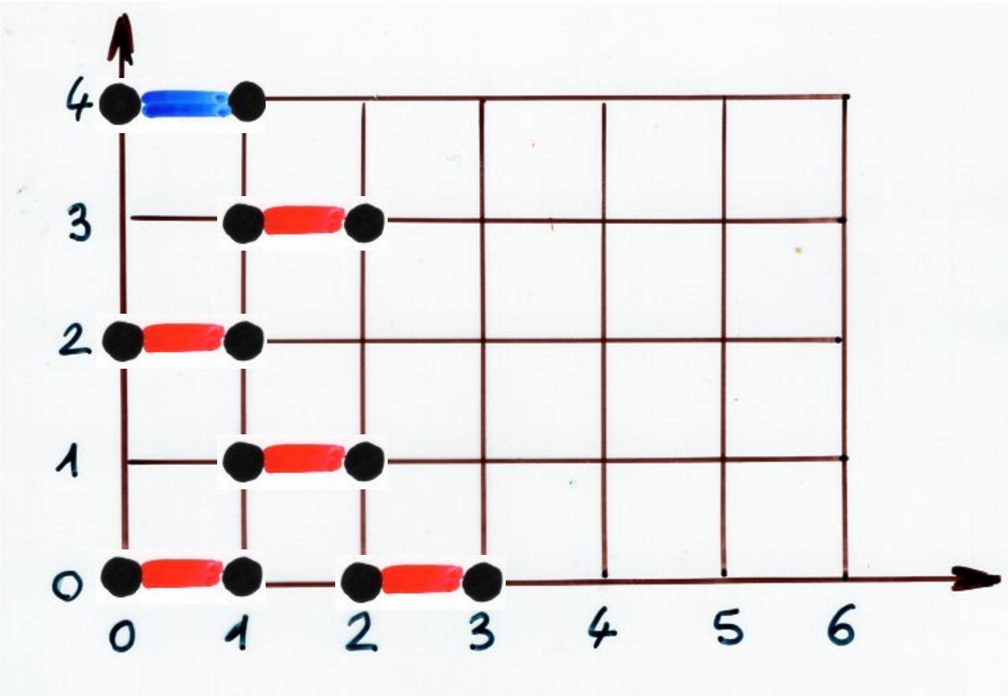
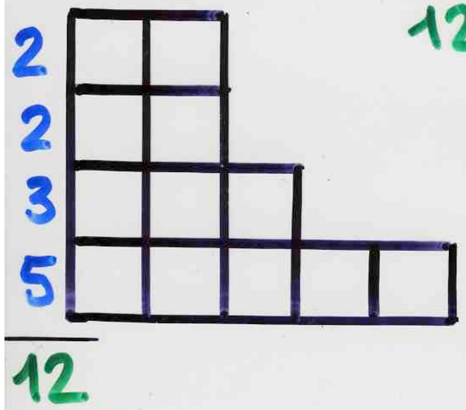
Enumerative combinatorics



The number of semi-pyramids of dimers on $\mathbb{N}$ with n dimers is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



$12 = n = 5 + 3 + 2 + 2$

Ferrers

diagram

Partition of n

generating function for (integer) partitions

$$\sum_{n \geq 0} a_n q^n$$

$$\prod_{i \geq 1} \frac{1}{(1 - q^i)}$$

Tianjin lecture 1

Enumerative combinatorics

Bijective combinatorics

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

Tianjin lecture 1

Enumerative combinatorics

Bijective combinatorics

Algebraic combinatorics

Tianjin lecture 2

Enumerative combinatorics

Bijective combinatorics

Algebraic combinatorics

« Combinatorial physics »

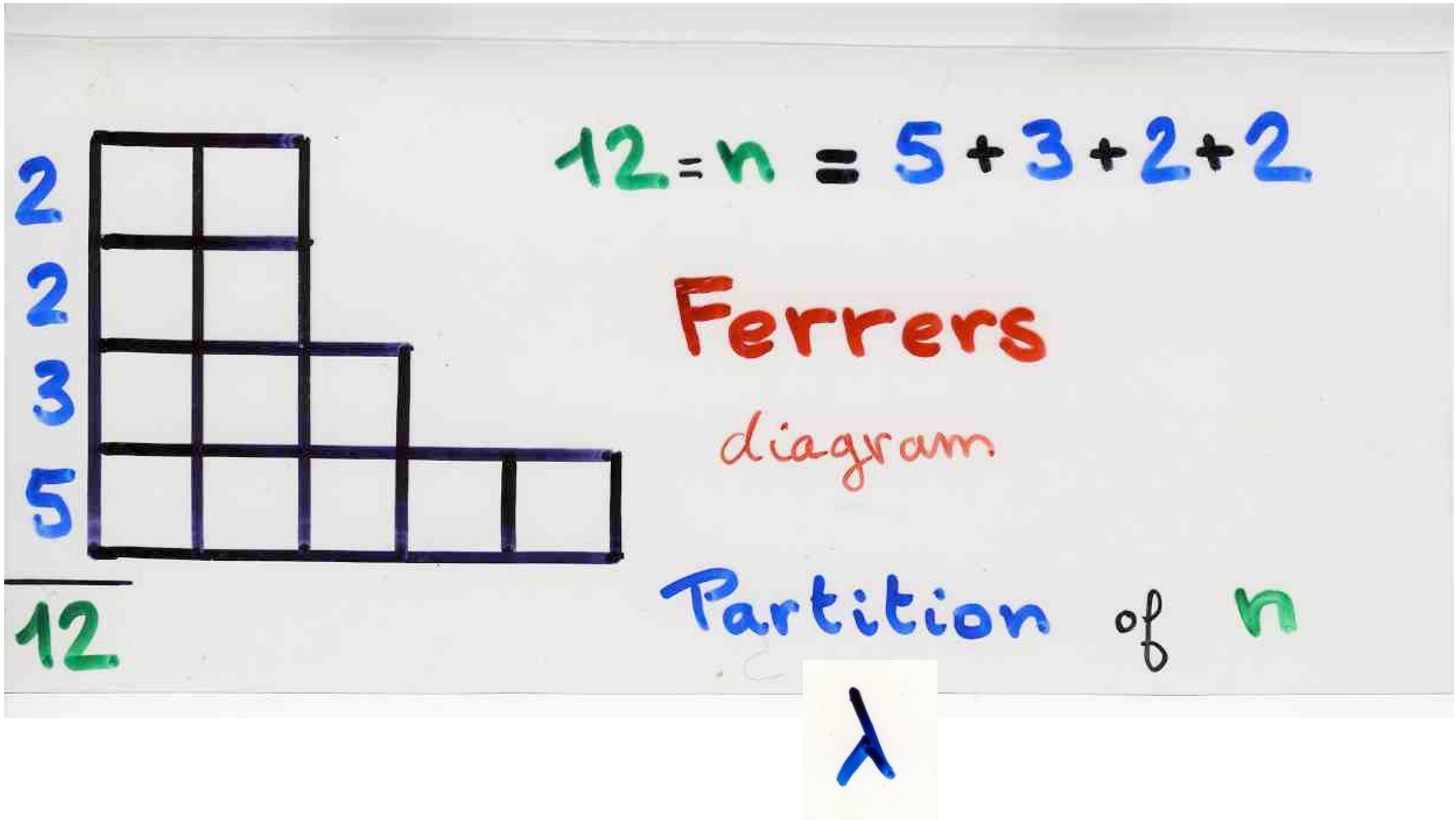
Analytic combinatorics

« Existensialist combinatorics »

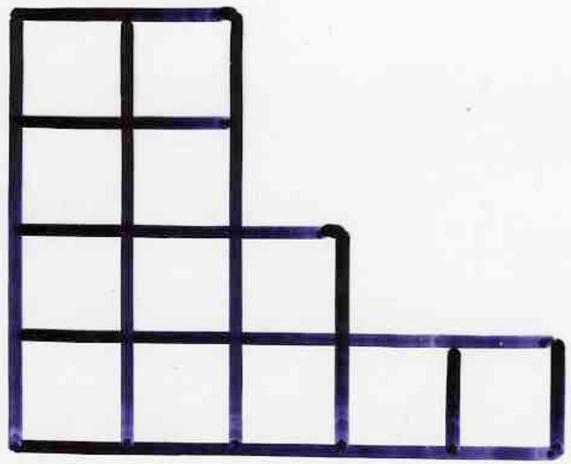
....

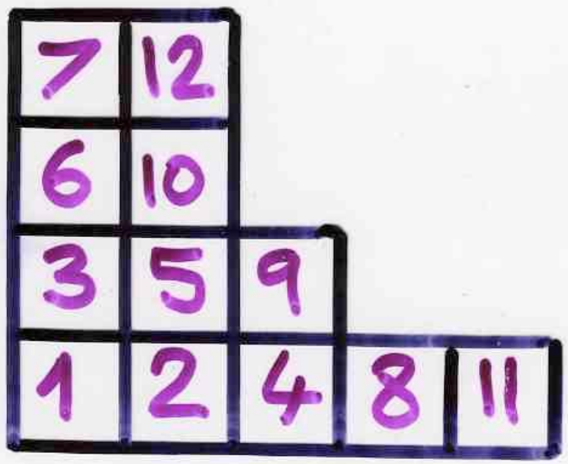
Magic combinatorics !

Young tableaux



12 = n = 5 + 3 + 2 + 2
Ferrers
diagram
Partition of n
λ
2
2
3
5
12





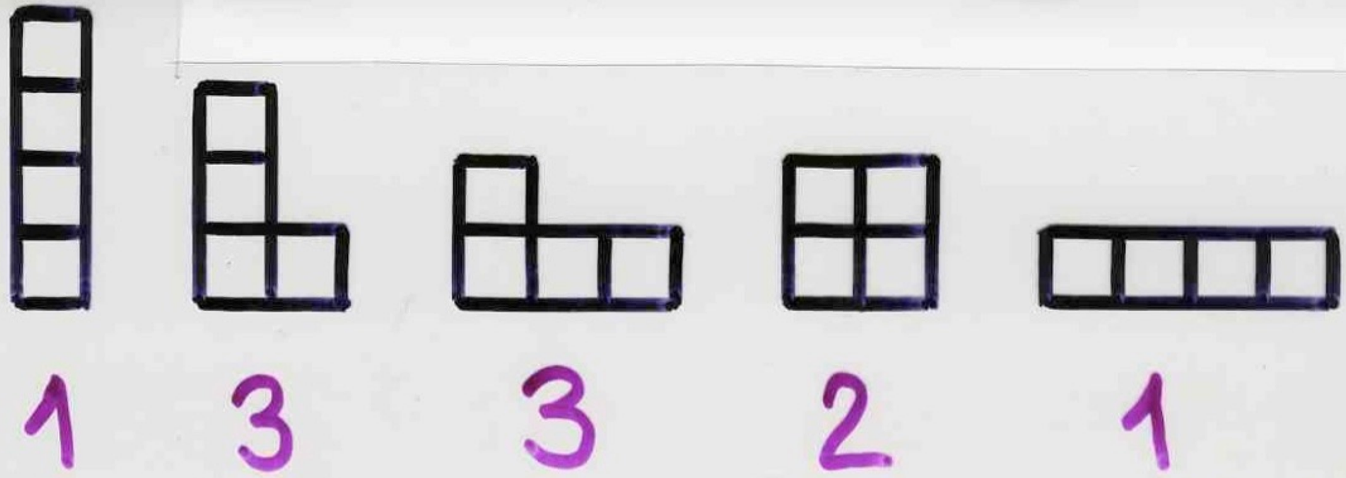
Young tableau

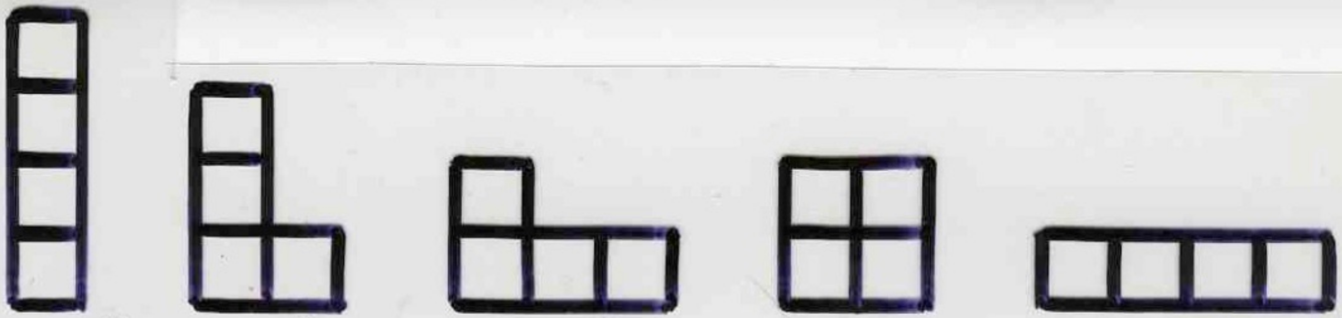
shape λ

f_λ = number of Young tableaux with shape λ

hook length formula

A beautiful Identity


1
3
3
2
1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

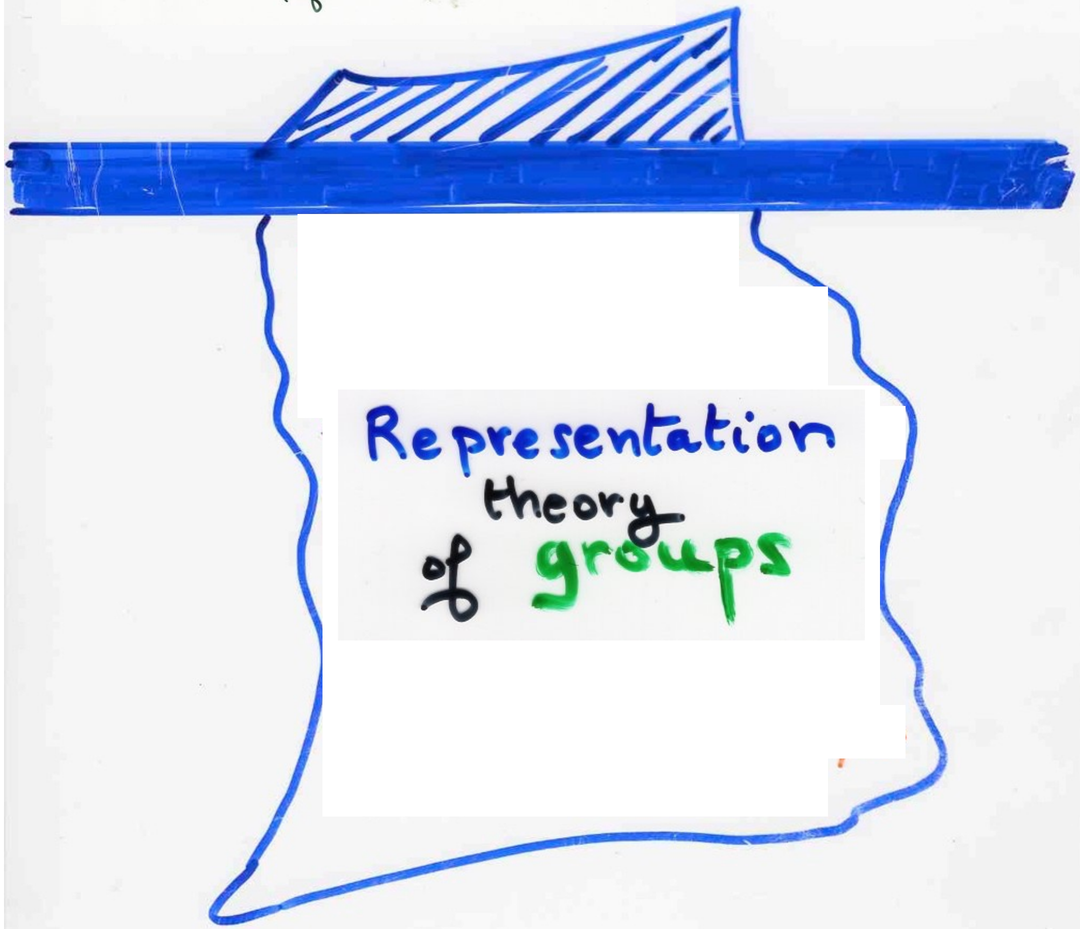
$$n! = \sum_{\substack{\lambda \\ \text{partitions} \\ \text{of } n}} \left(f_{\lambda}\right)^2$$

$$n! = \sum_{\lambda} (f_\lambda)^2$$

partitions of n

$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ: partitions of n



algebraic combinatorics

Representation theory of groups

<u>Case of the group</u> $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) = f_λ number of Young tableaux with shape λ

finite group G

$$|G| = \sum_{R} (\deg R)^2$$

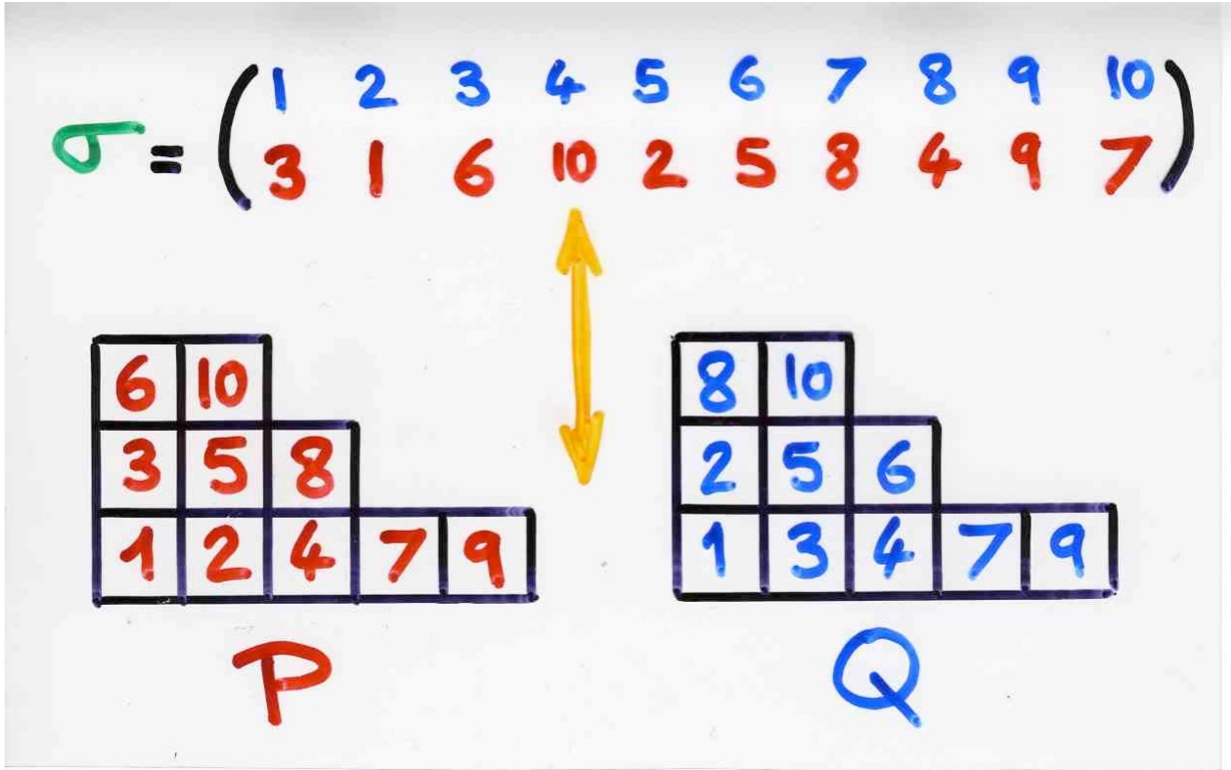
R irreducible representation

for the symmetric group $\mathfrak{S}_n$
(permutations)

$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ partition of n

Bijective combinatorics



The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

An introduction to RS

The Robinson-Schensted correspondence

G. de B. Robinson, 1938

- Schensted insertions algorithm — C. Schensted, 1961
- Geometric version — X.V. 1976
- Growth diagrams — S. Fomin, 1986, 1994

edge local rules

- Combinatorial Representation of a quadratic algebra

$UD = DU + Id$

See. Tianjin lecture 3

The Art of Bijective Combinatorics

www.viennot.org

Part III (2018)

The Cellular ansatz:

bijective combinatorics and quadratic algebra

Ch1 RSK the Robinson-Schensted-Knuth correspondence

RS with Schensted's insertions

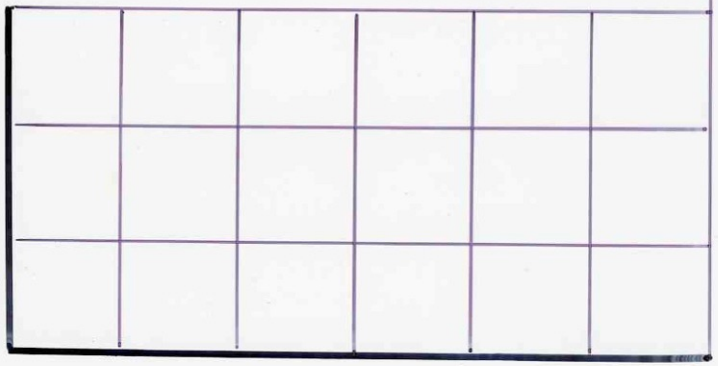
σ =

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q

recording tableau

P



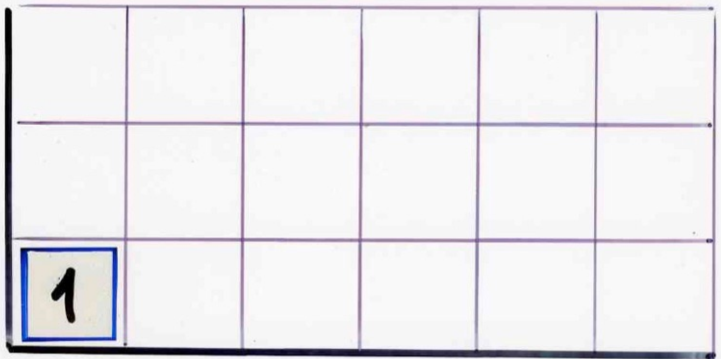
insertion tableau

- read the permutation σ as a word $w = \sigma(1)\sigma(2)..\sigma(n)$ from left to right

σ =

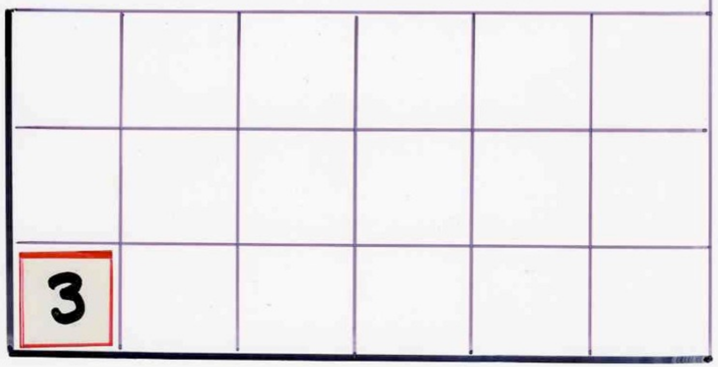
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q



recording tableau

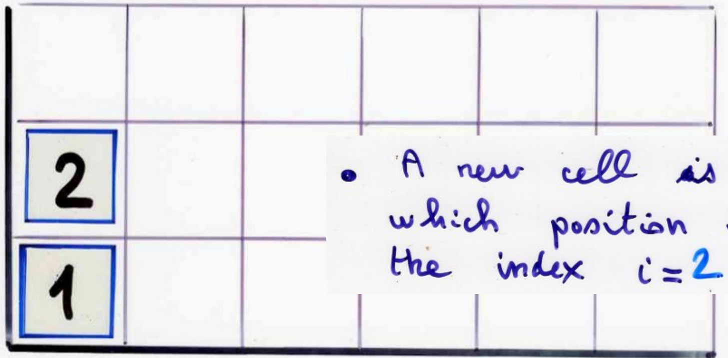
P



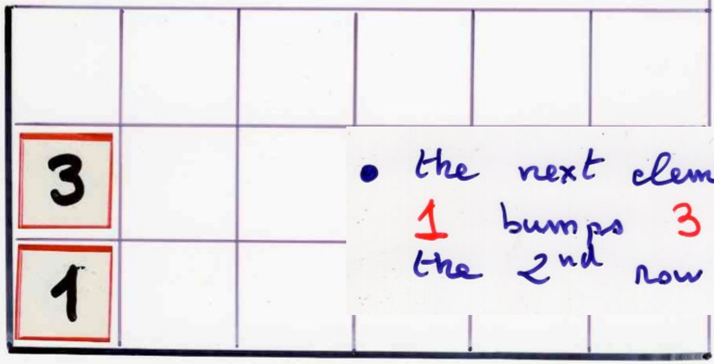
insertion tableau

- insert the first value $3 = \sigma(1)$ in the 1st row of P

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



- A new cell is added in the shape of P, which position is recorded in Q with the index $i=2$

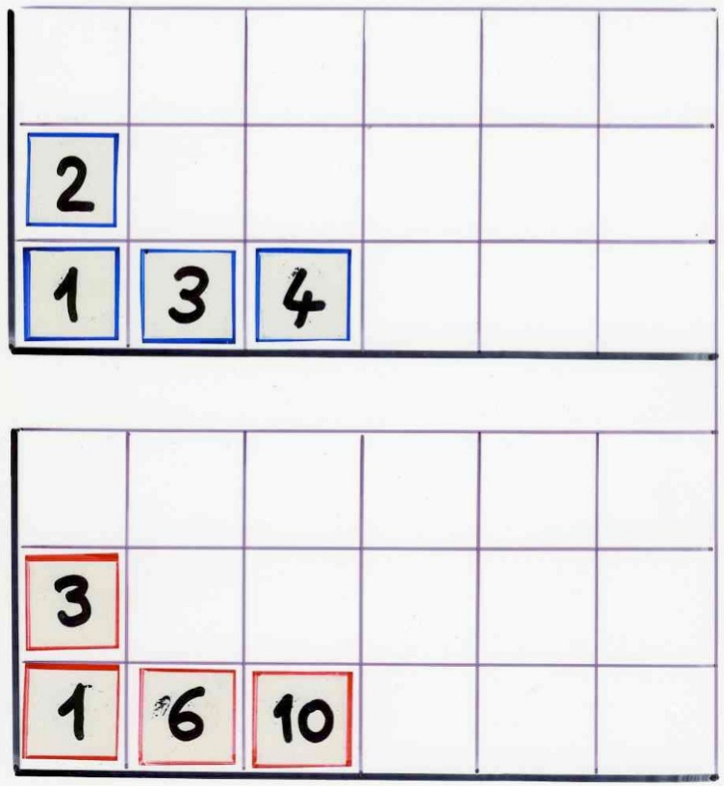


- the next element $1 = \sigma(2)$ is < 3, 1 bumps 3 which is inserted in the 2nd row of P

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

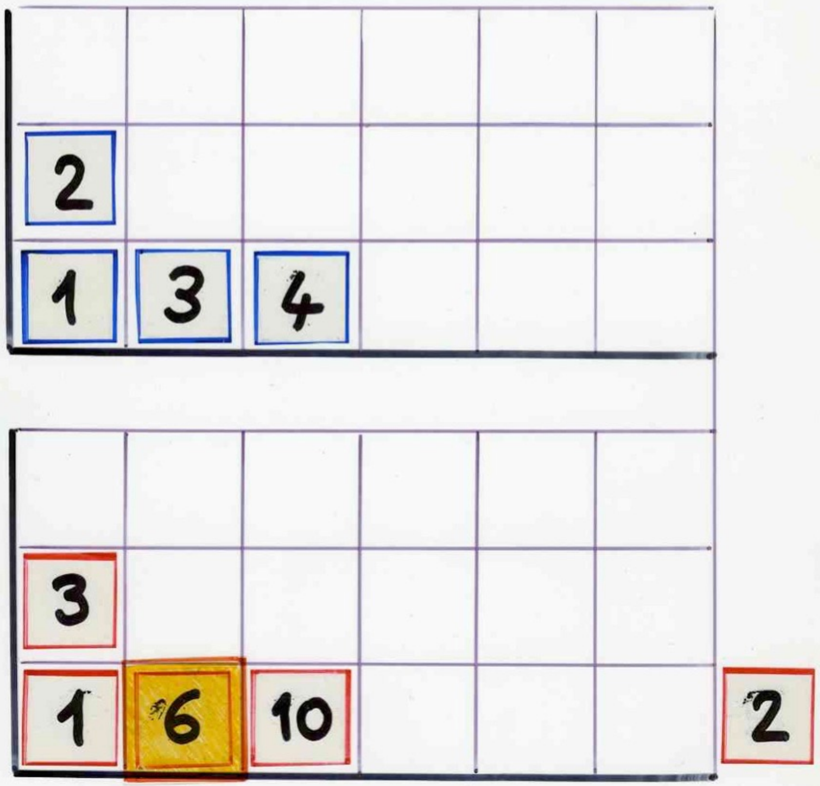
• $6 > 1$ is inserted in the 1st row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



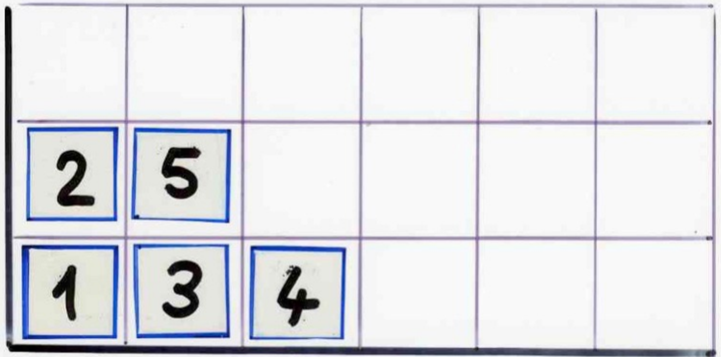
- $\sigma(4) = 10$ is > than all elements of the 1st row, and is added at the end of this 1st row.

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



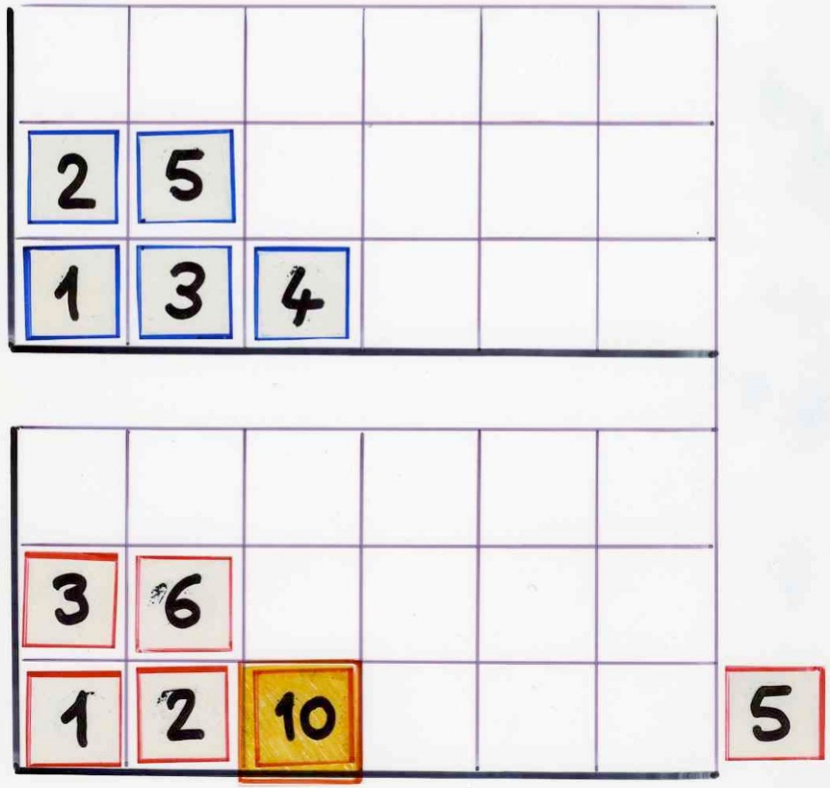
- $\sigma(5) = 2$ cannot be added at the end of the 1st row.
  2 is "bumping" the element 6, which is the smallest element of the 1st row > 2

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

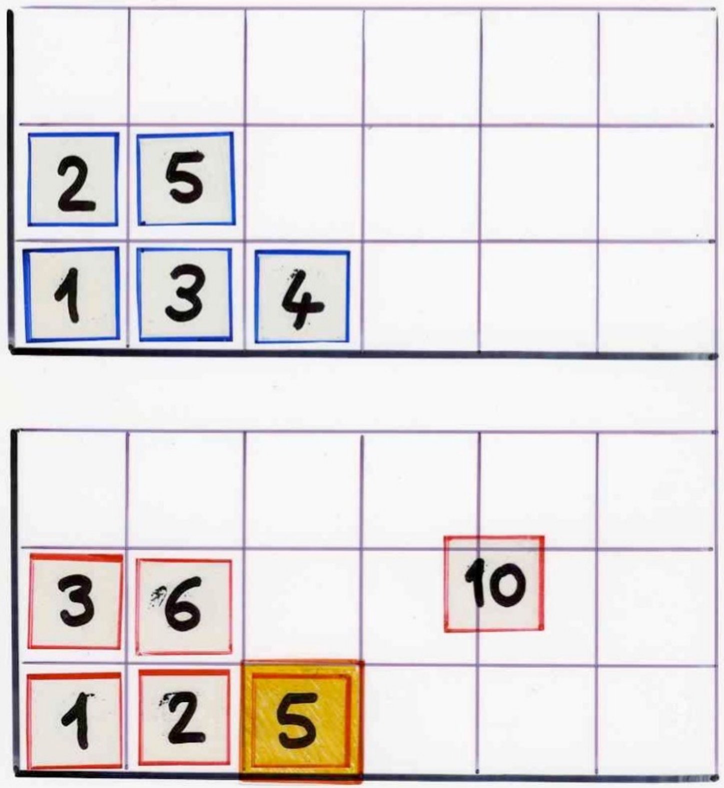


- 2 replaces 6, and 6 is inserted in the second row with the same recursive rule

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

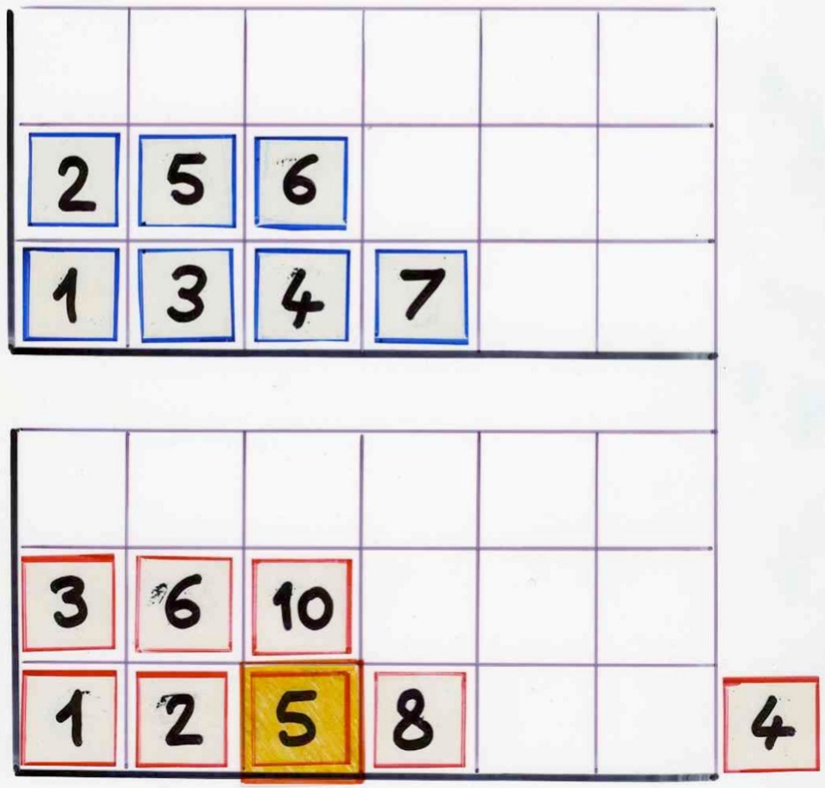
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

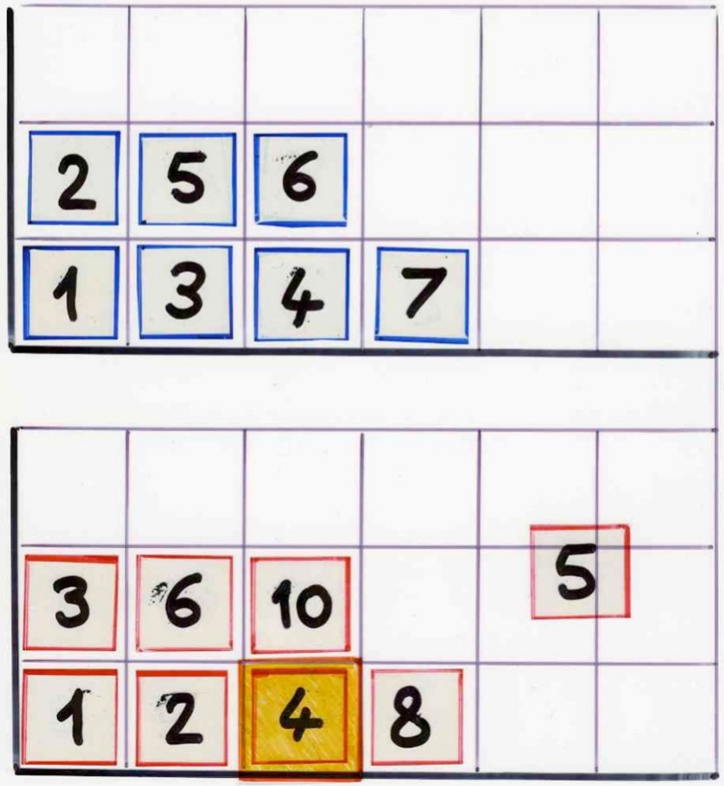
3	6	10			
1	2	5	8		

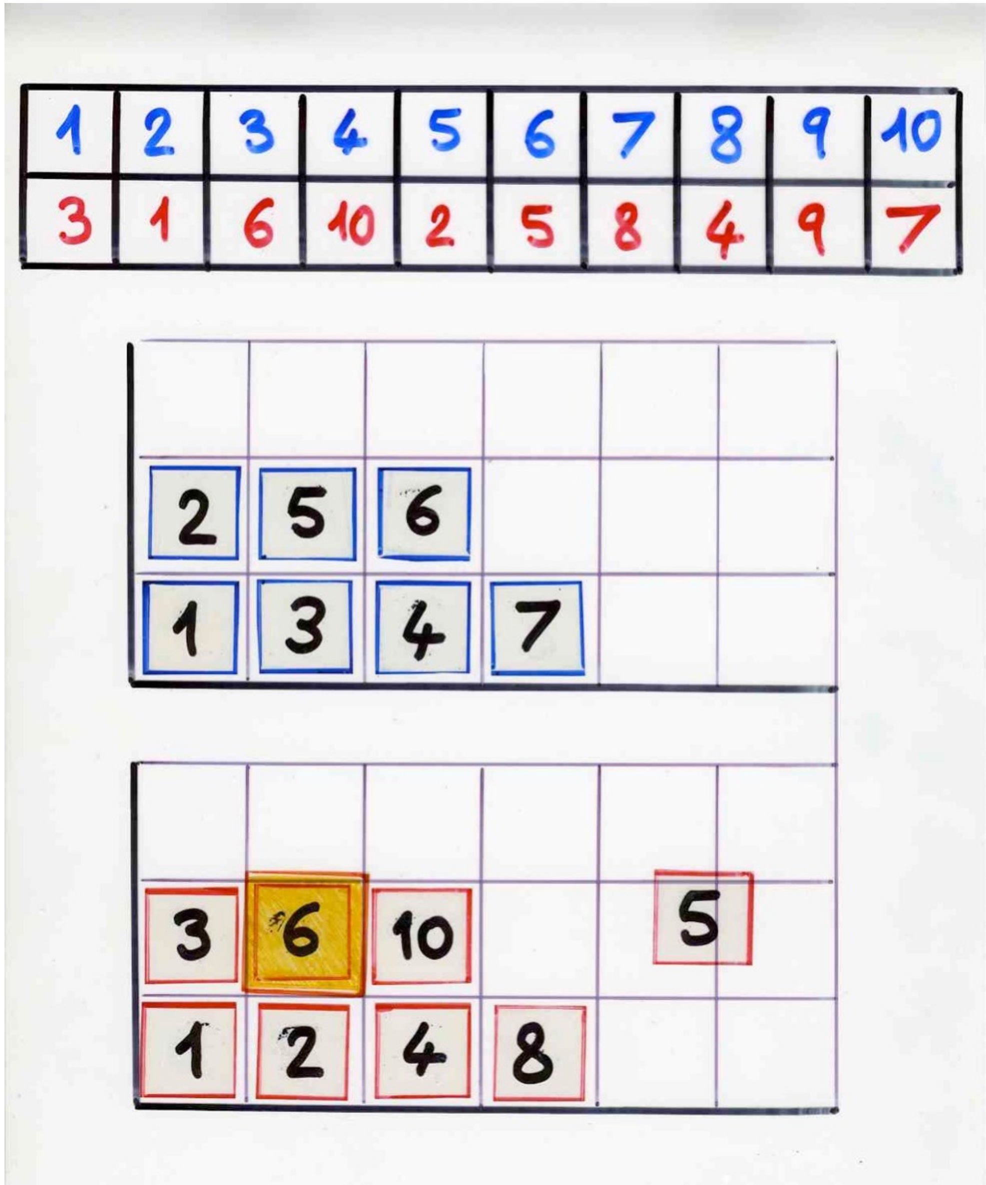
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



• $4 = \sigma(8)$ bumps 5 in the 1st row

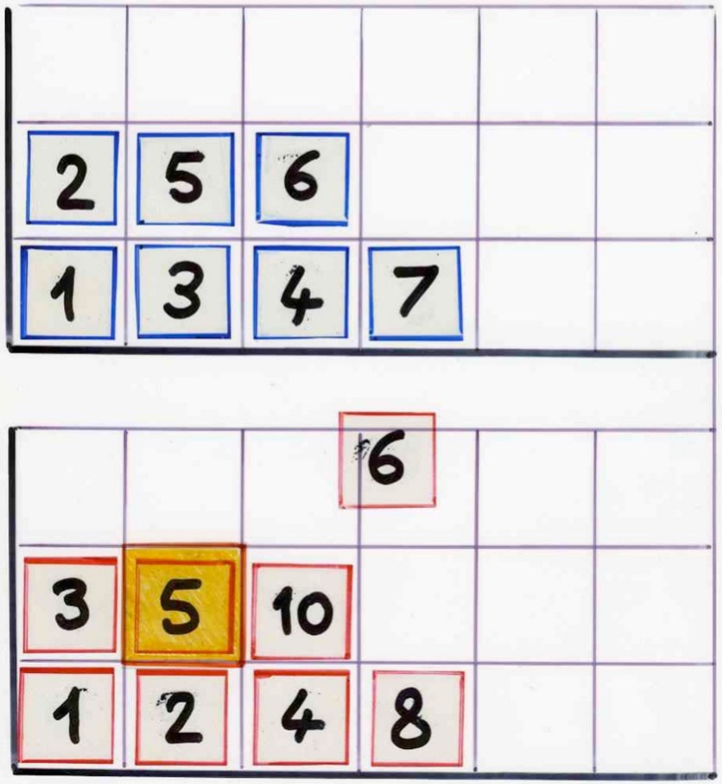
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





• 5 bumps 6 in the 2nd row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



• 6 is inserted in the 3rd row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

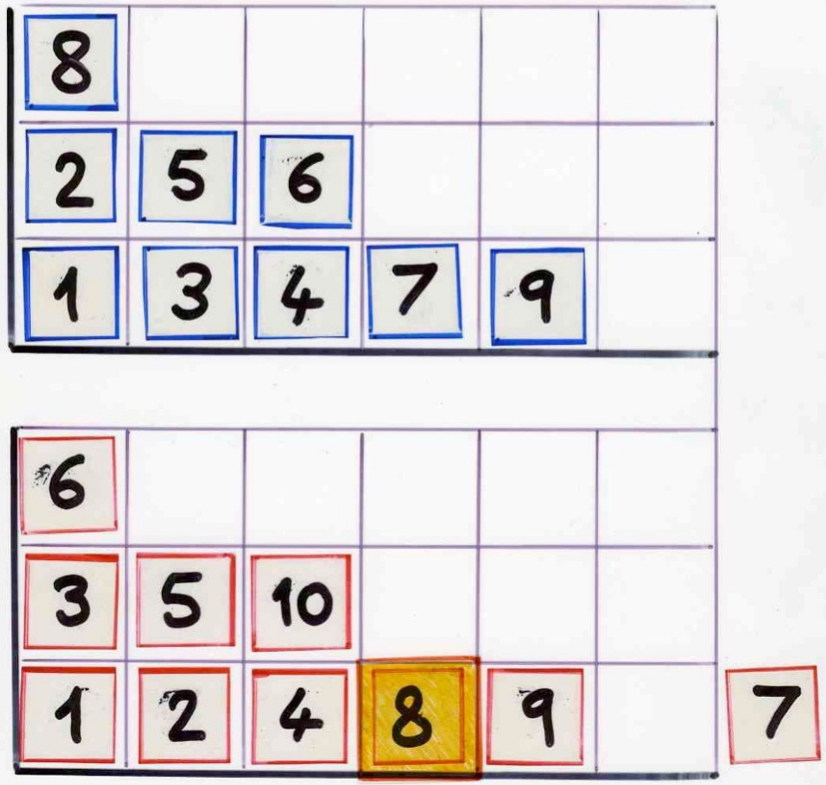
- the new cell added in the common shape of P and Q is recorded in Q with the cell 8

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

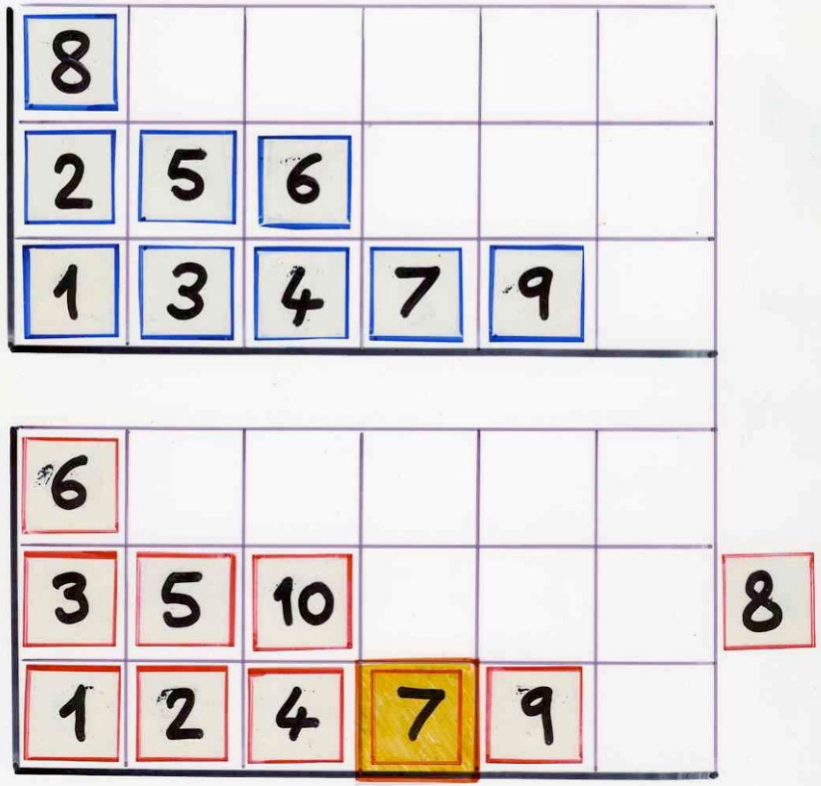
8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

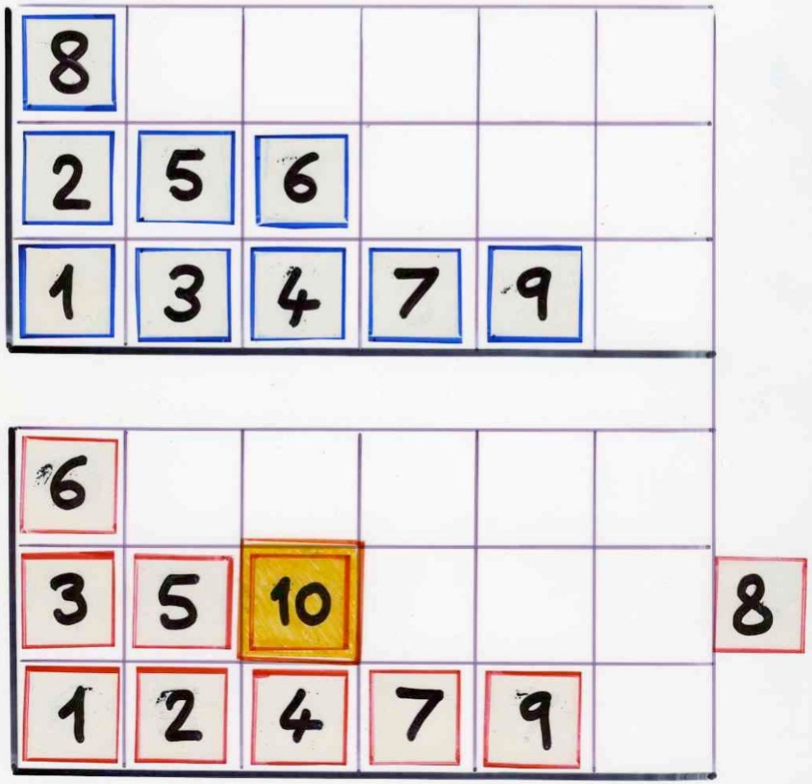
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



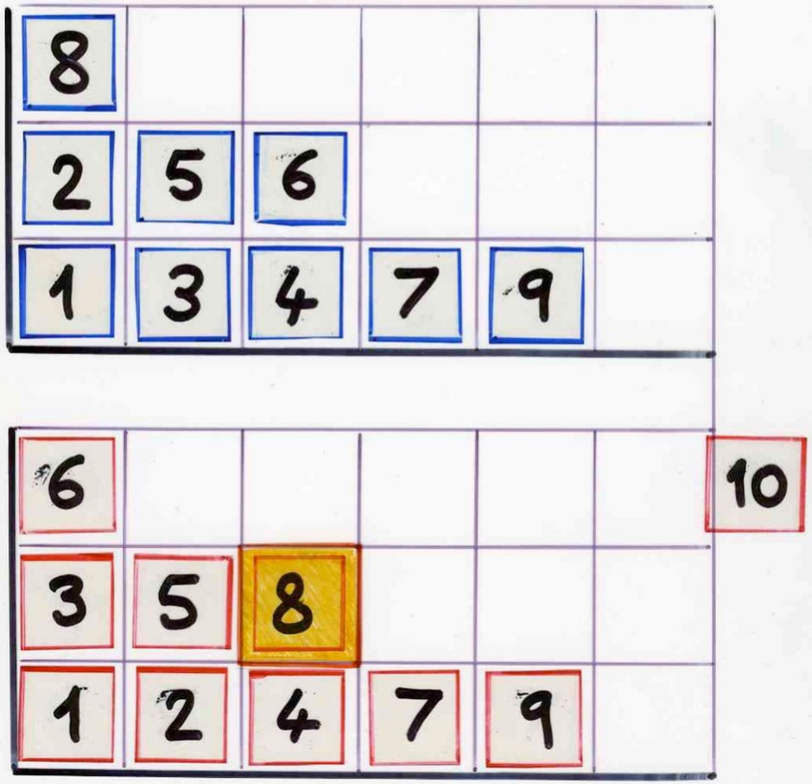
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



σ =

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q =

8	10				
2	5	6			
1	3	4	7	9	

end of the
RS algorithm

P =

6	10				
3	5	8			
1	2	4	7	9	

Reverse algorithm

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

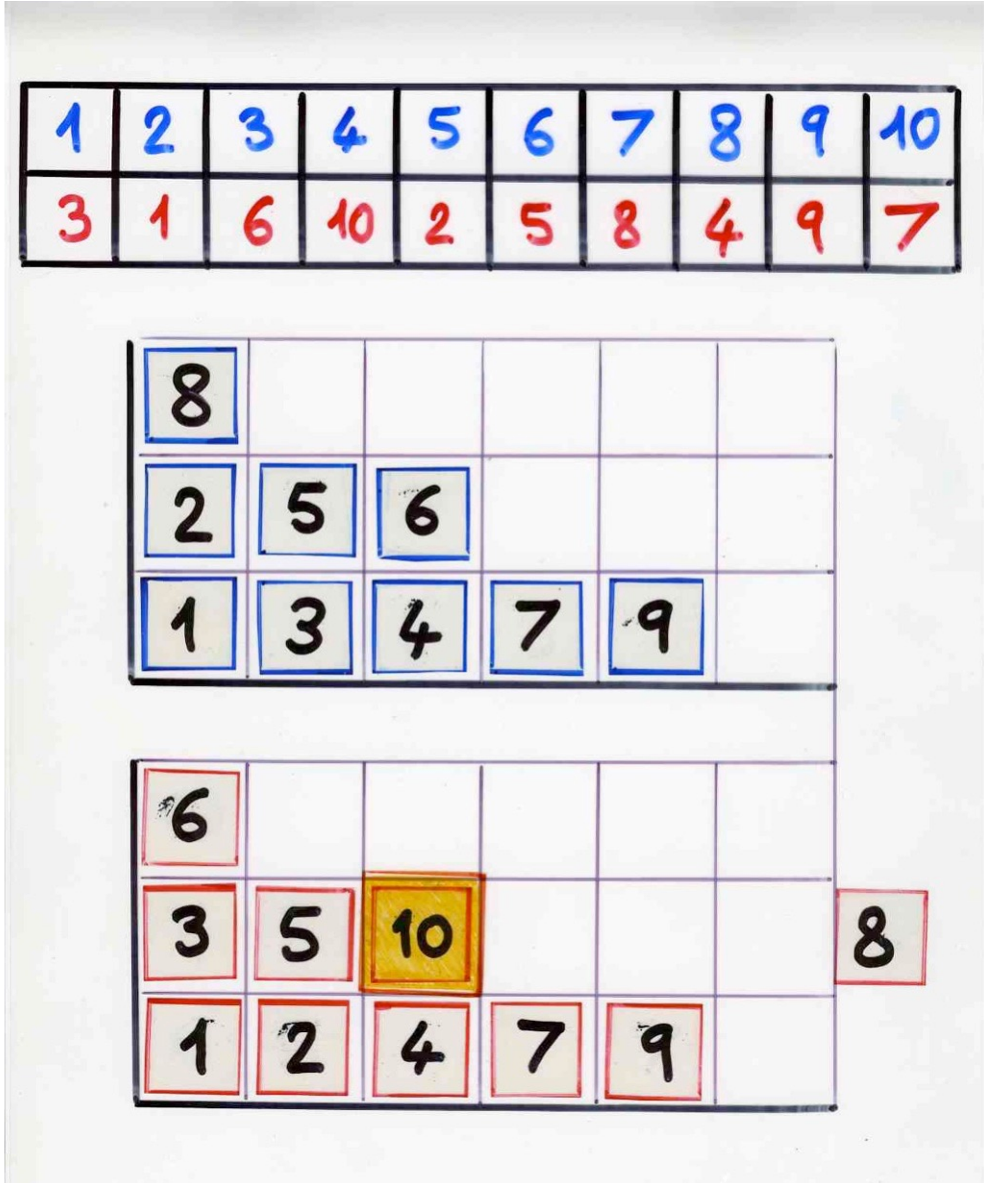
8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

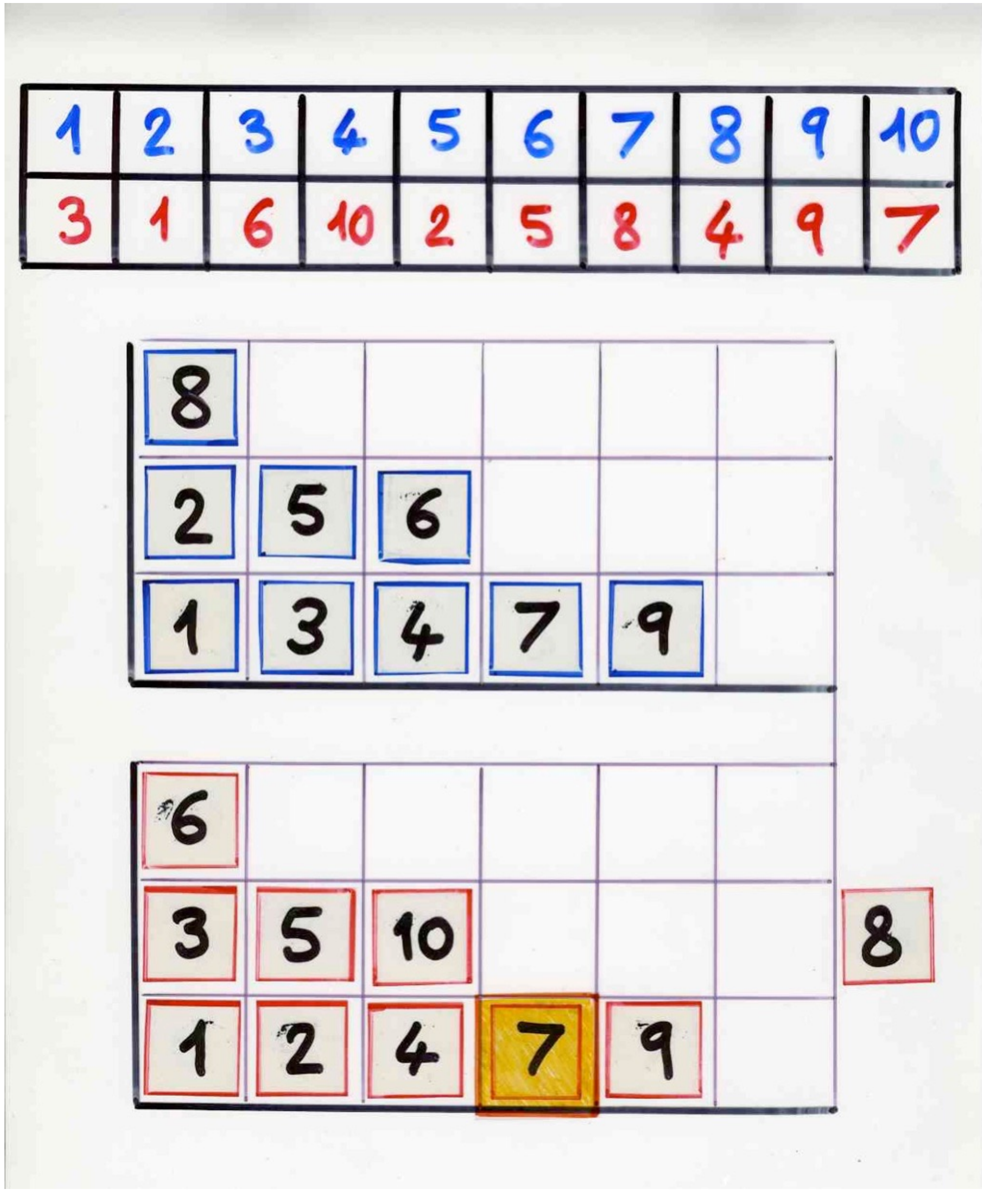
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						10
3	5	8				
1	2	4	7	9		



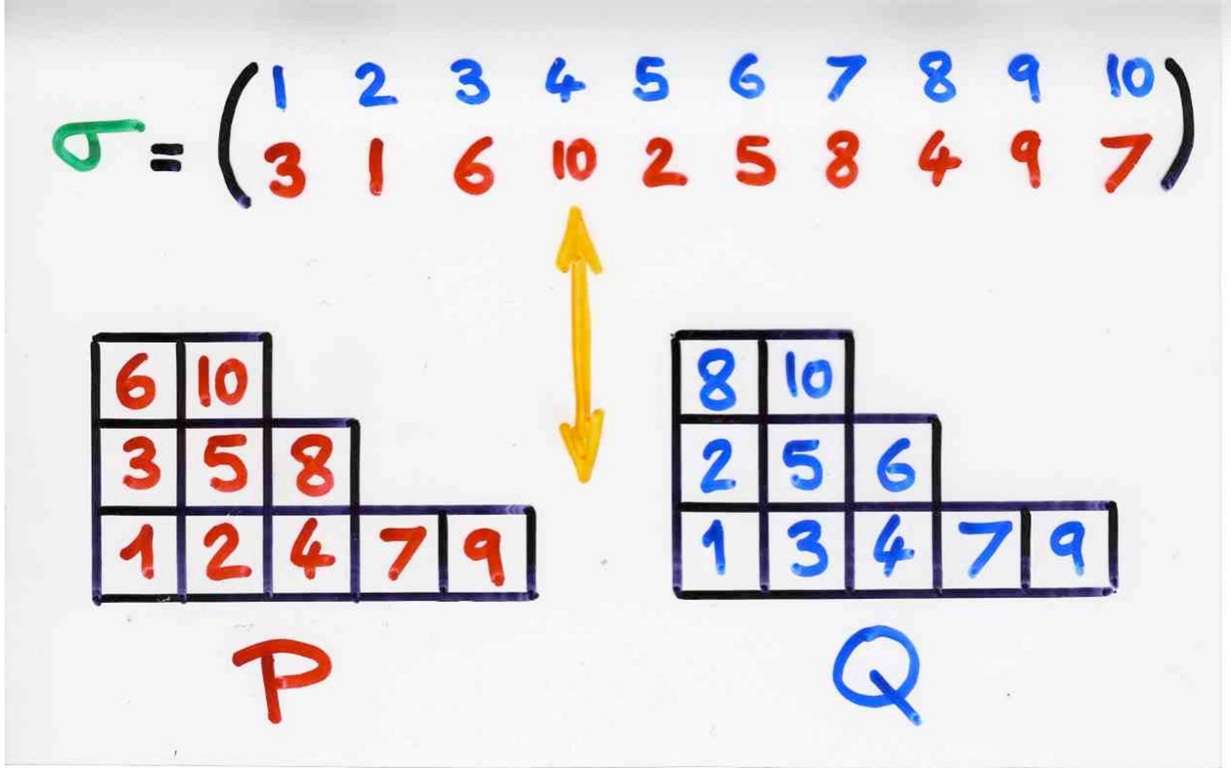
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
8
2 5 6
1 3 4 7 9
6
3 5 10 8
1 2 4 7 9


1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
8
2 5 6
1 3 4 7 9
6
3 5 10
8
1 2 4 7 9

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				
1	2	4	8	9		7



The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

Surprise !

$$\sigma \longleftrightarrow (P, Q)$$

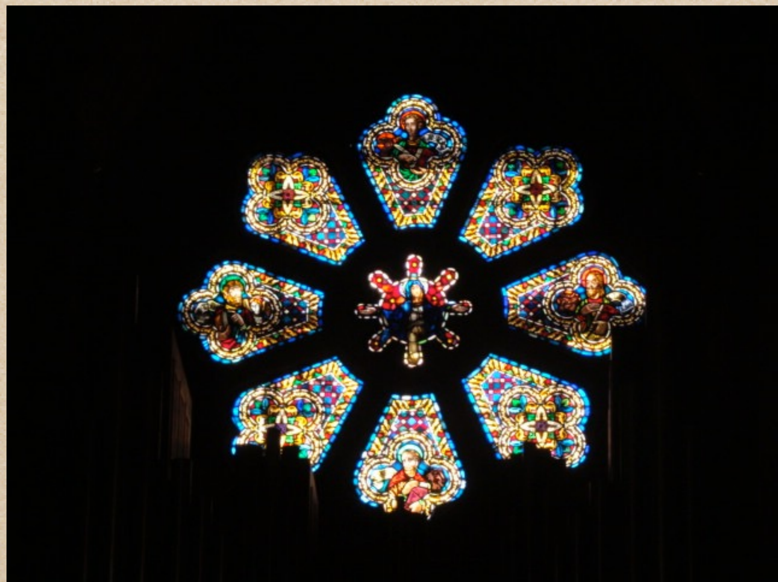
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

" The unusual nature of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scene"

D. Knuth (1972)

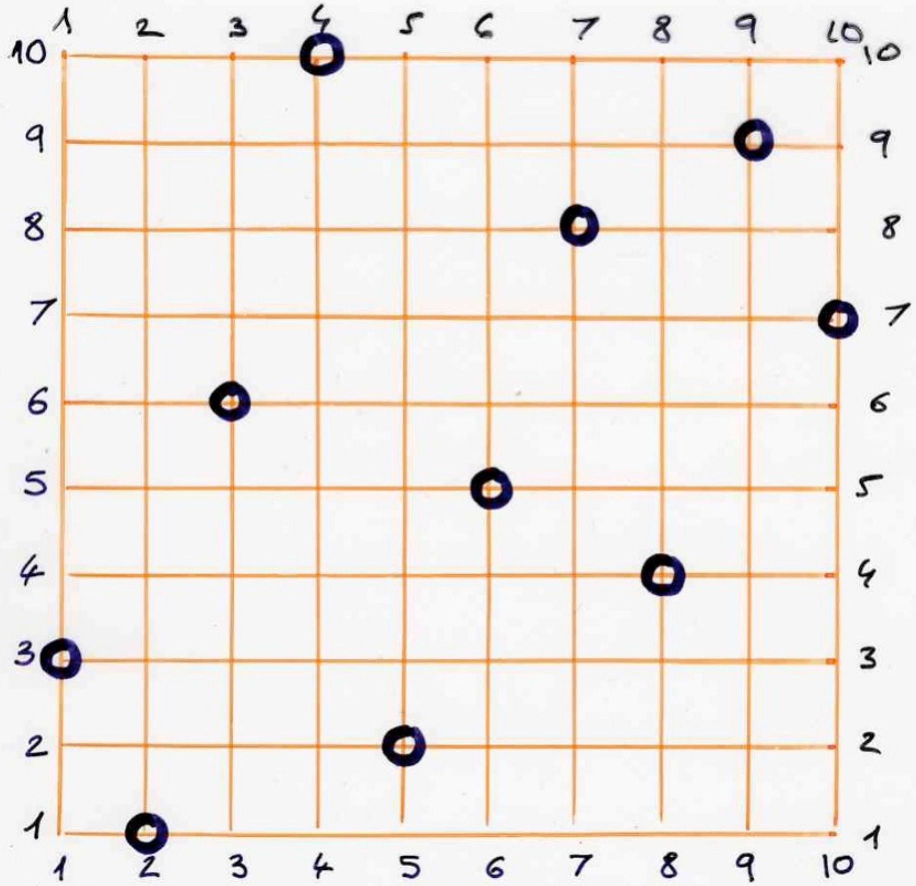
The art of computer programming
Vol. 3

A geometric version of RS with "light" and "shadow lines"



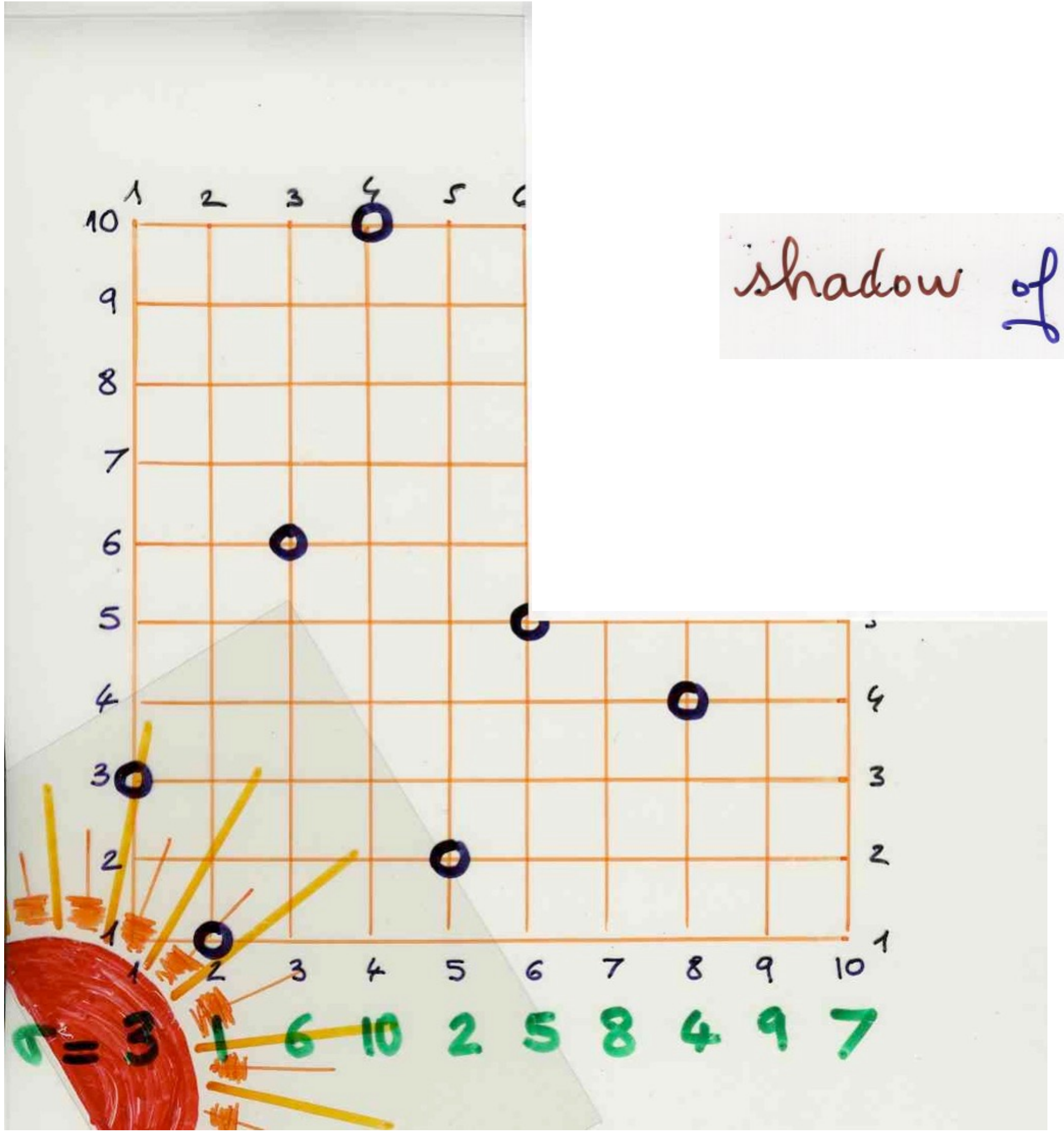
X.V. 1976

$$\{(i,\sigma(i))\}_{i=1,\dots,n} \subseteq [1,n]\times[1,n]$$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

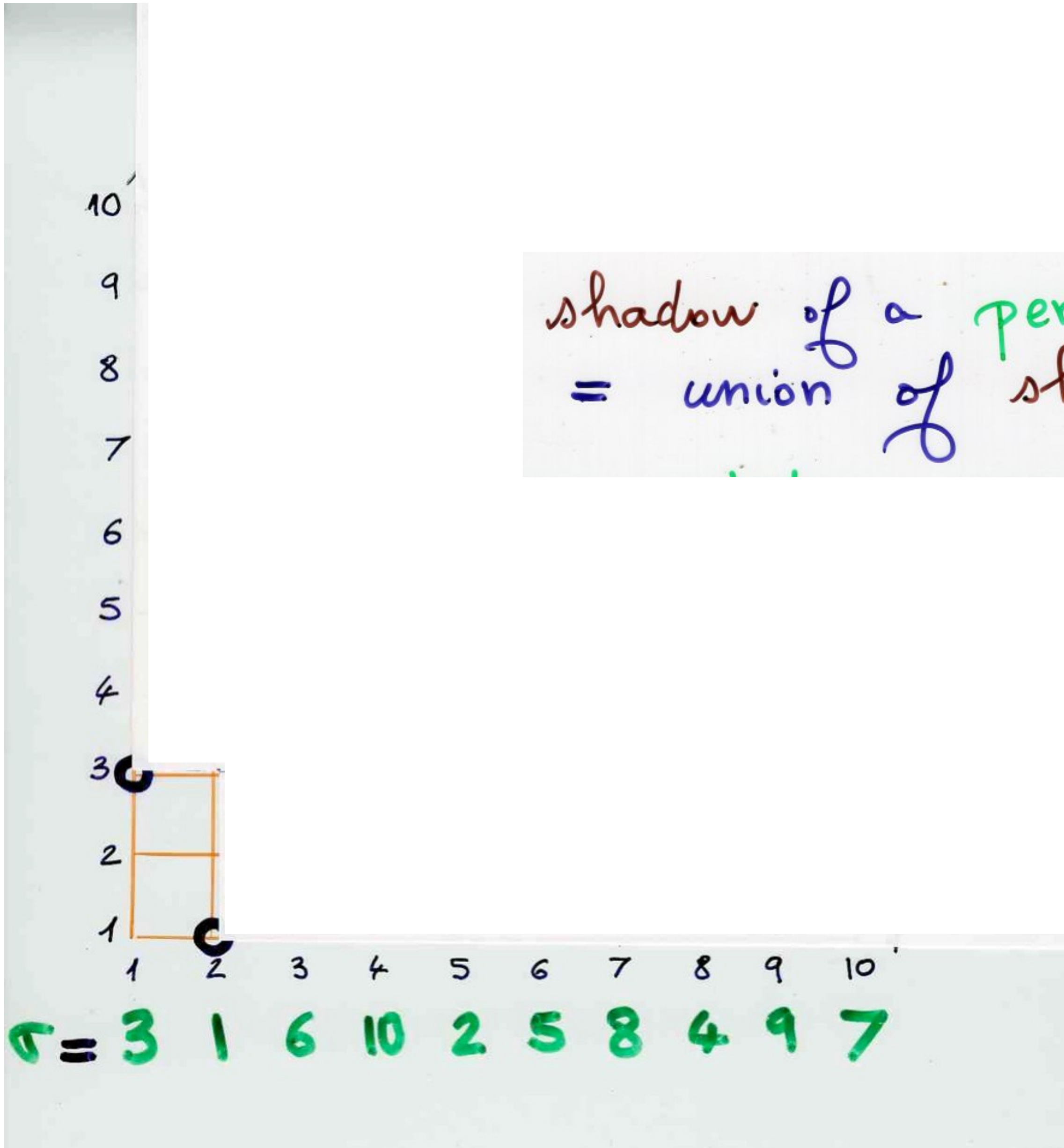
graph of a permutation σ

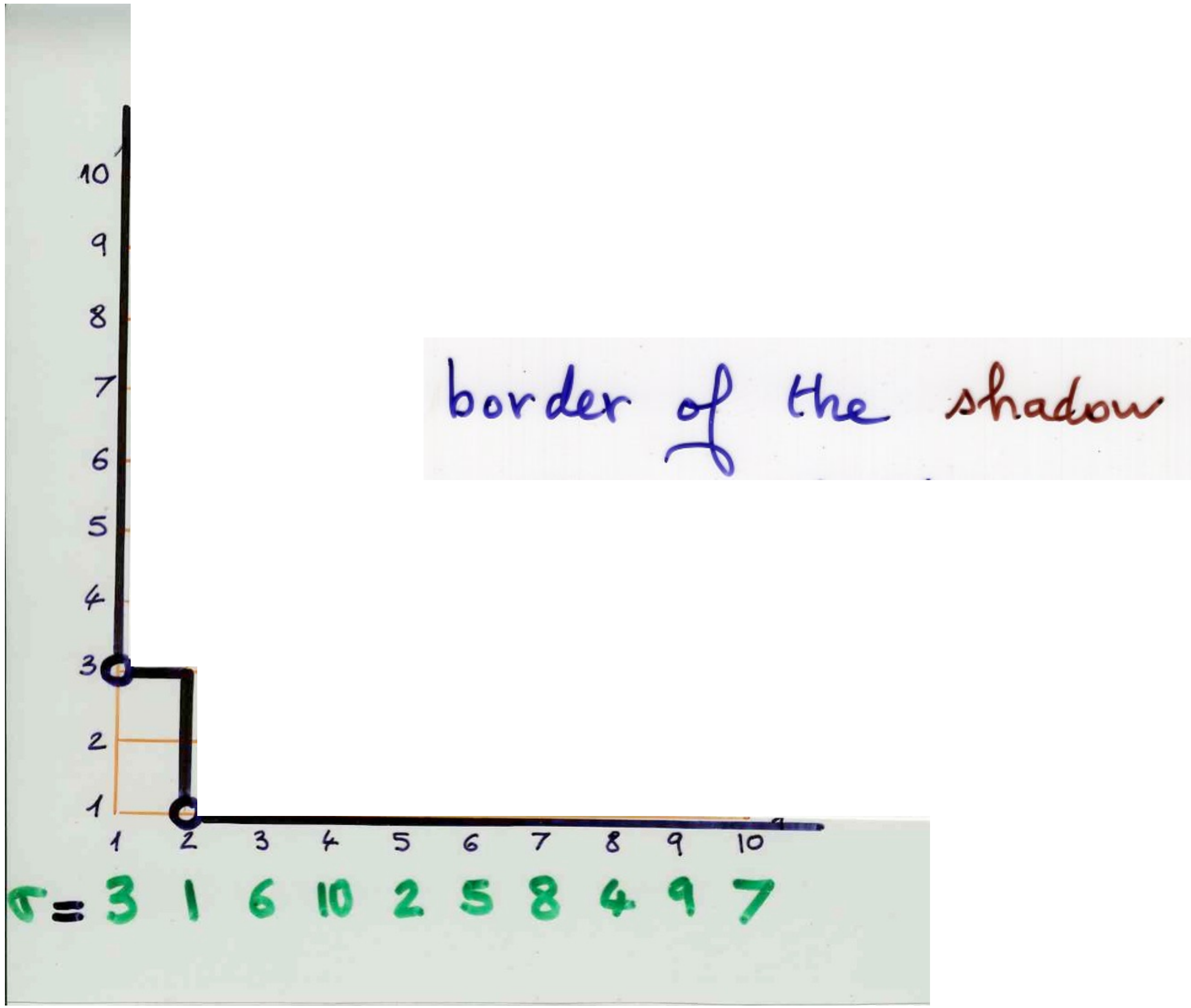


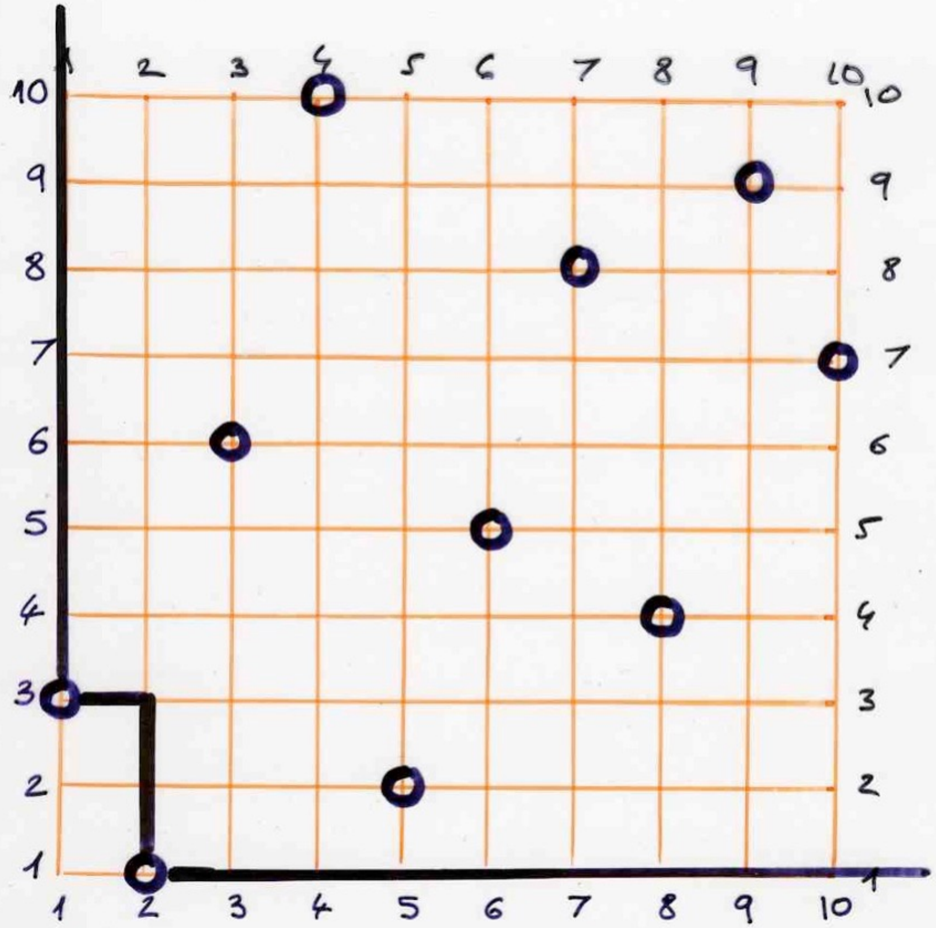
shadow of a point ●

shadow of a permutation

= union of shadows

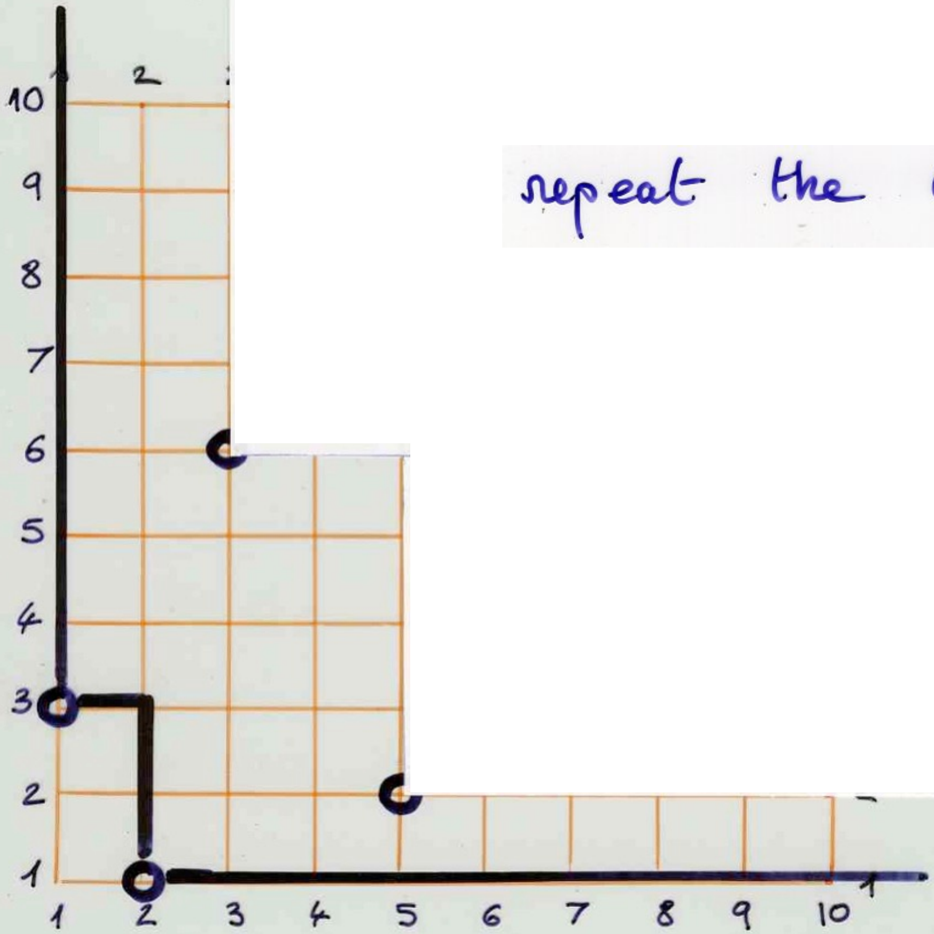



border of the shadow
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7

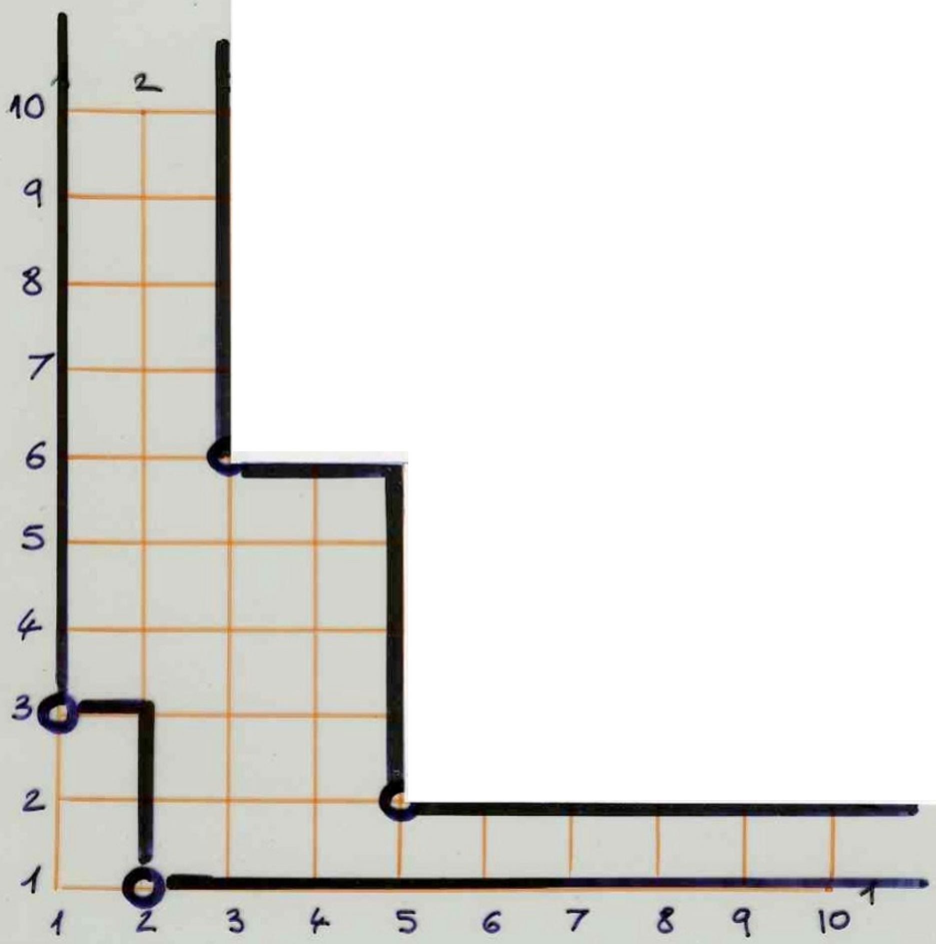


σ = 3 1 6 10 2 5 8 4 9 7

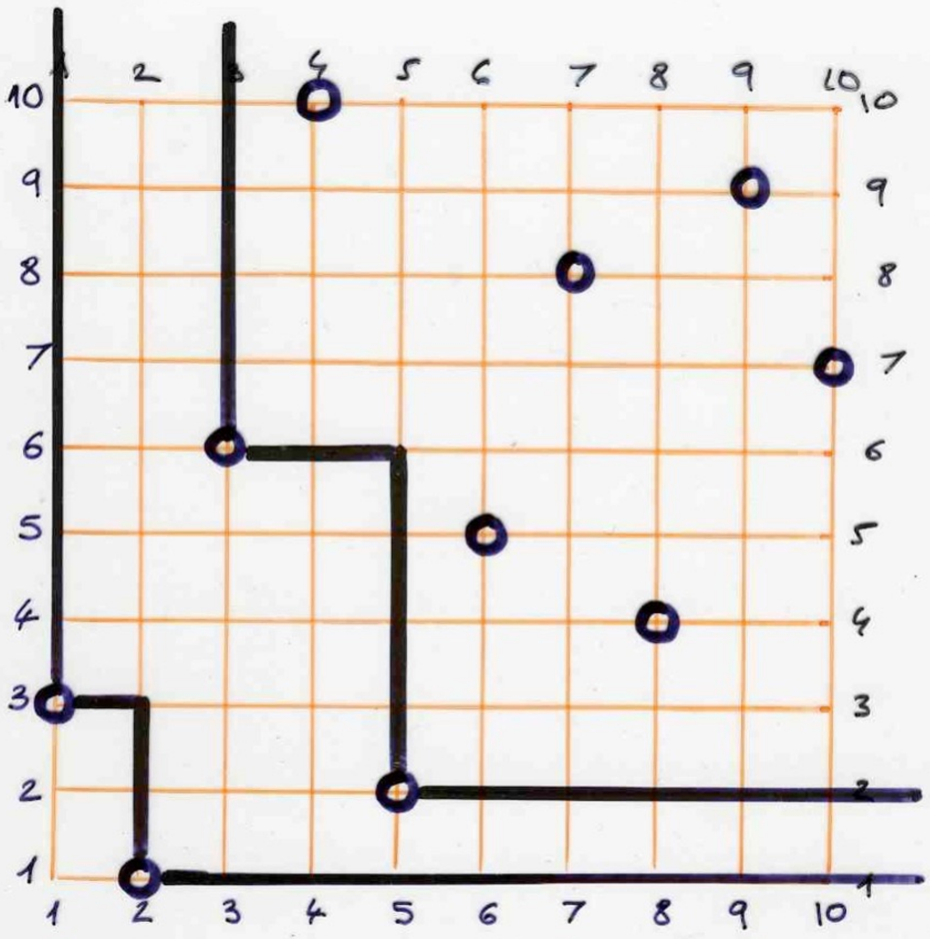
repeat the construction ...



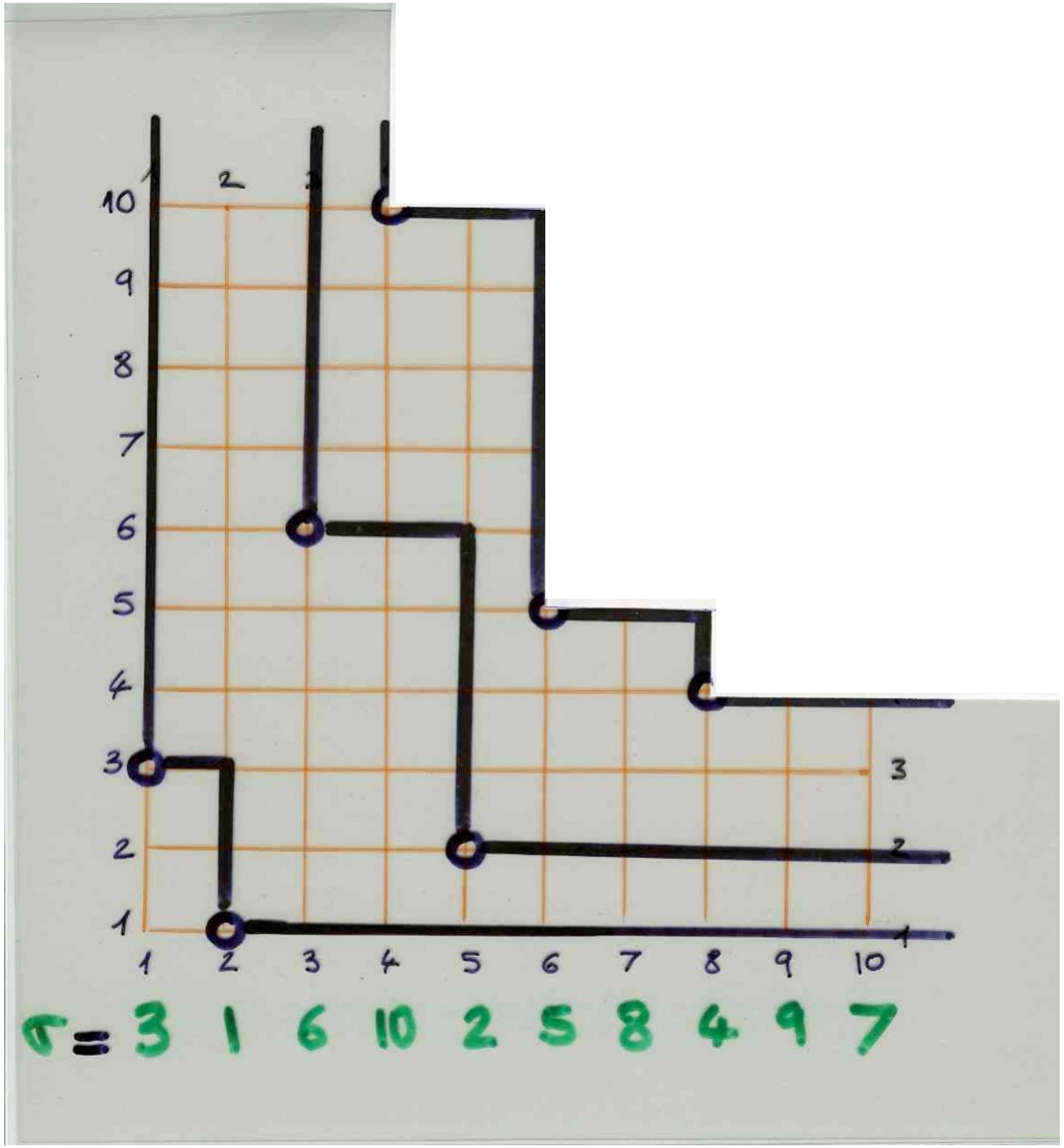
$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

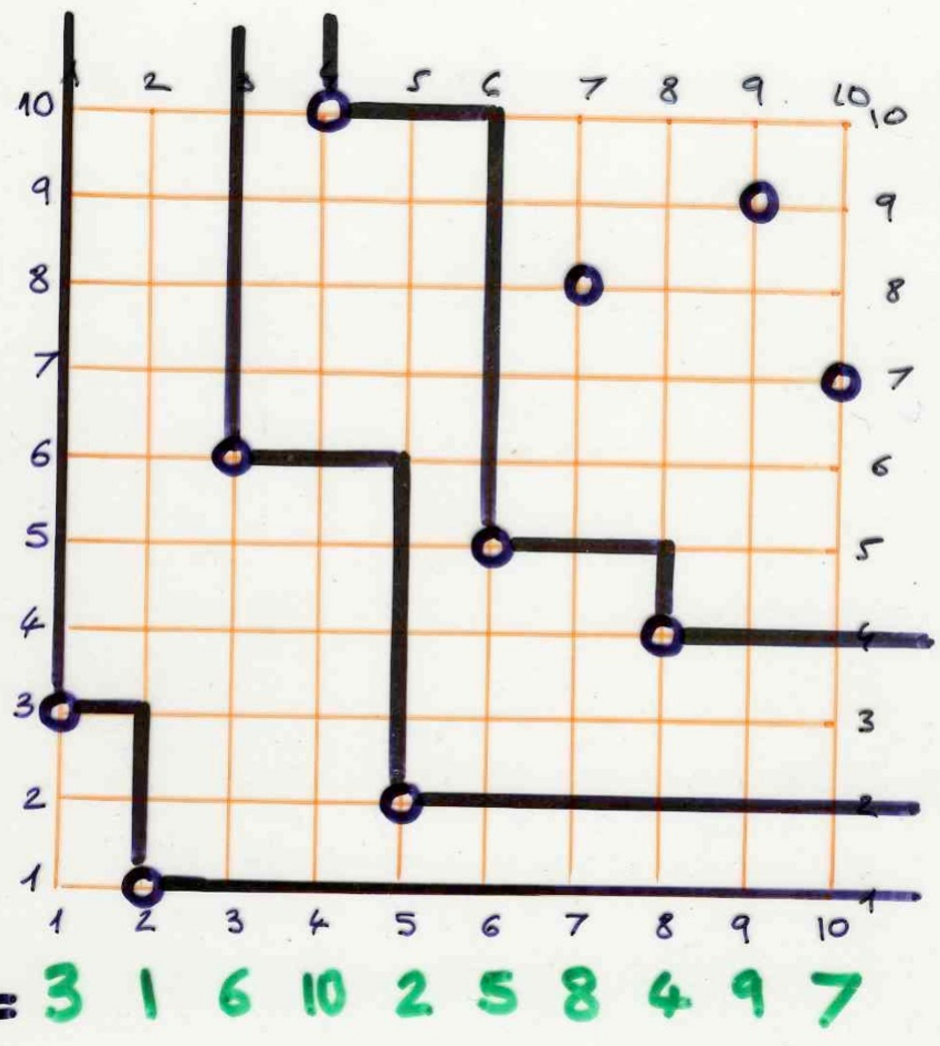


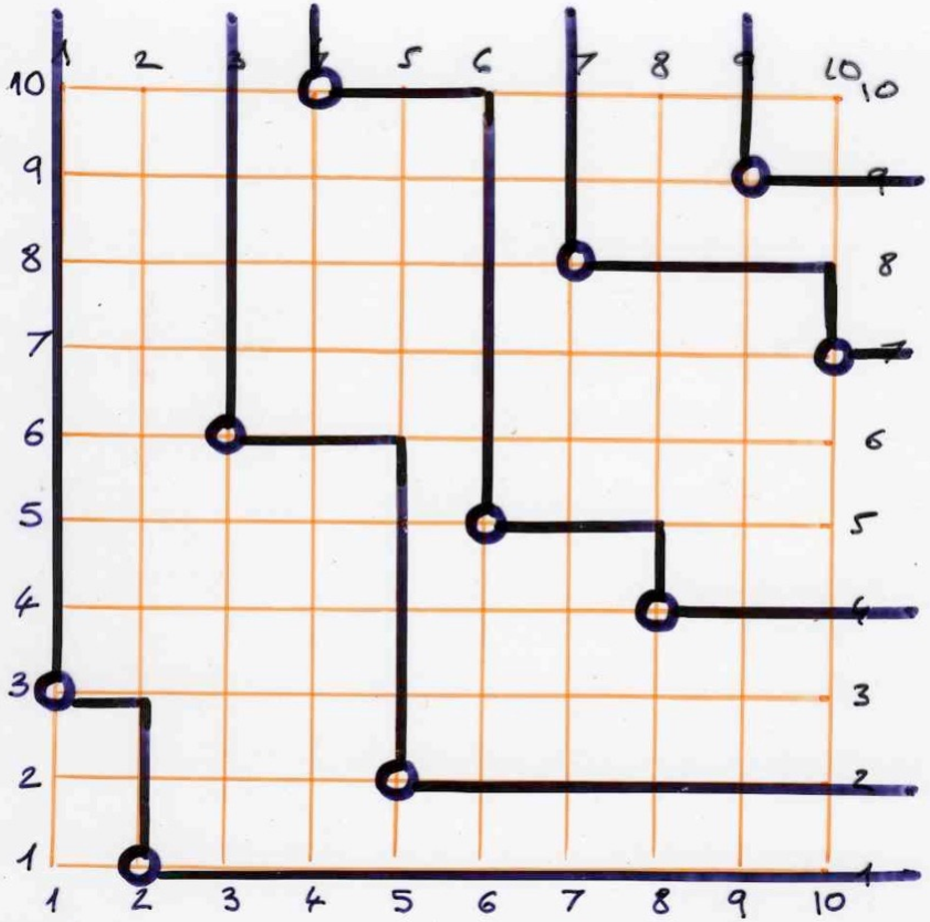
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



σ = 3 1 6 10 2 5 8 4 9 7

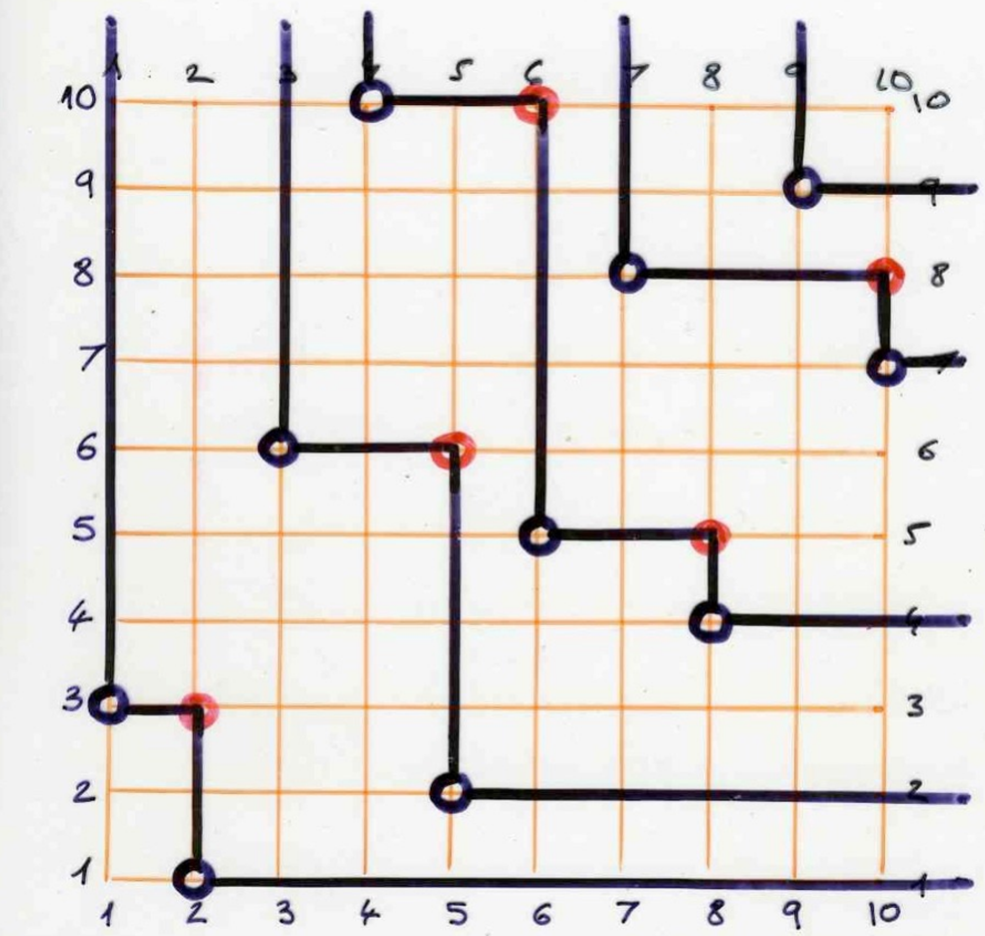

10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7


σ = 3 1 6 10 2 5 8 4 9 7



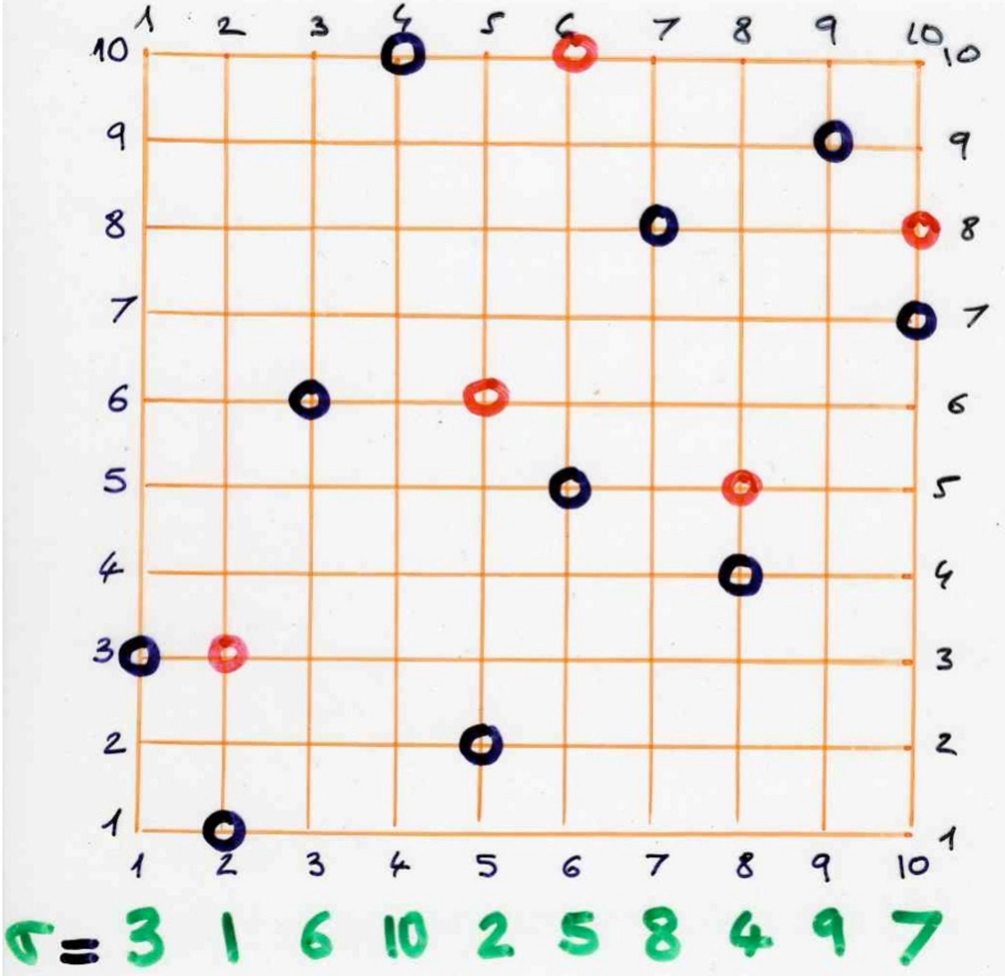
σ = 3 1 6 10 2 5 8 4 9 7

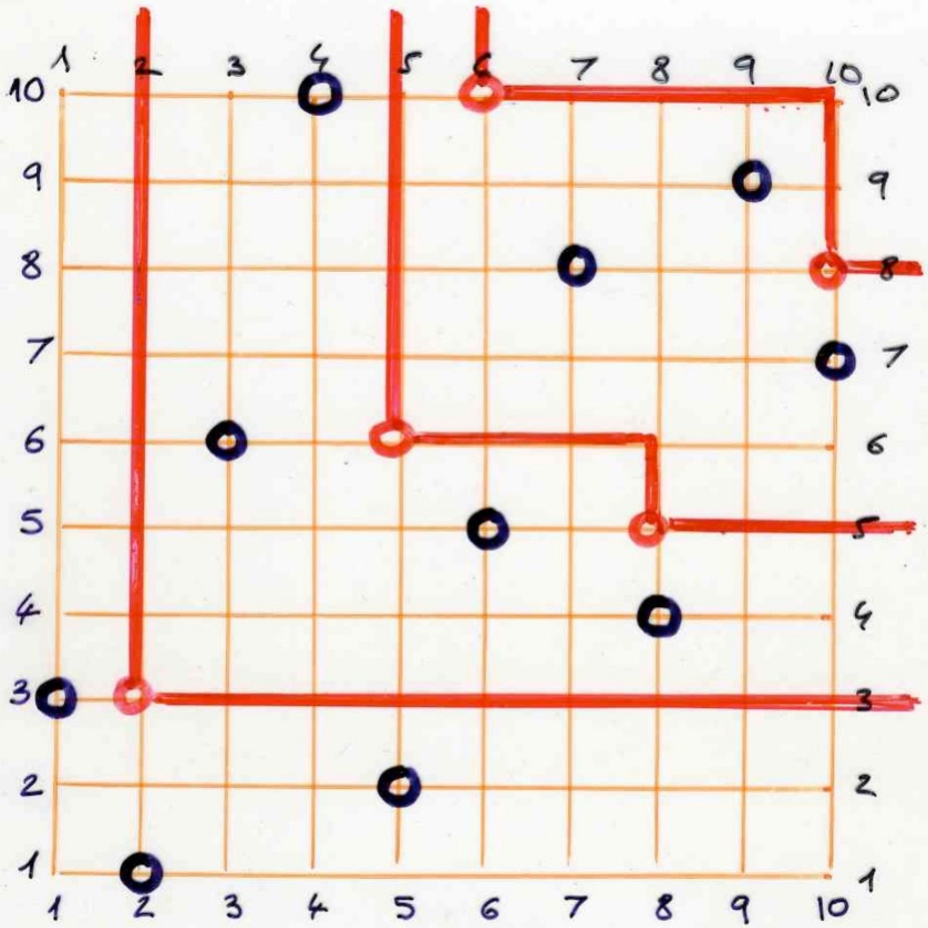
red points



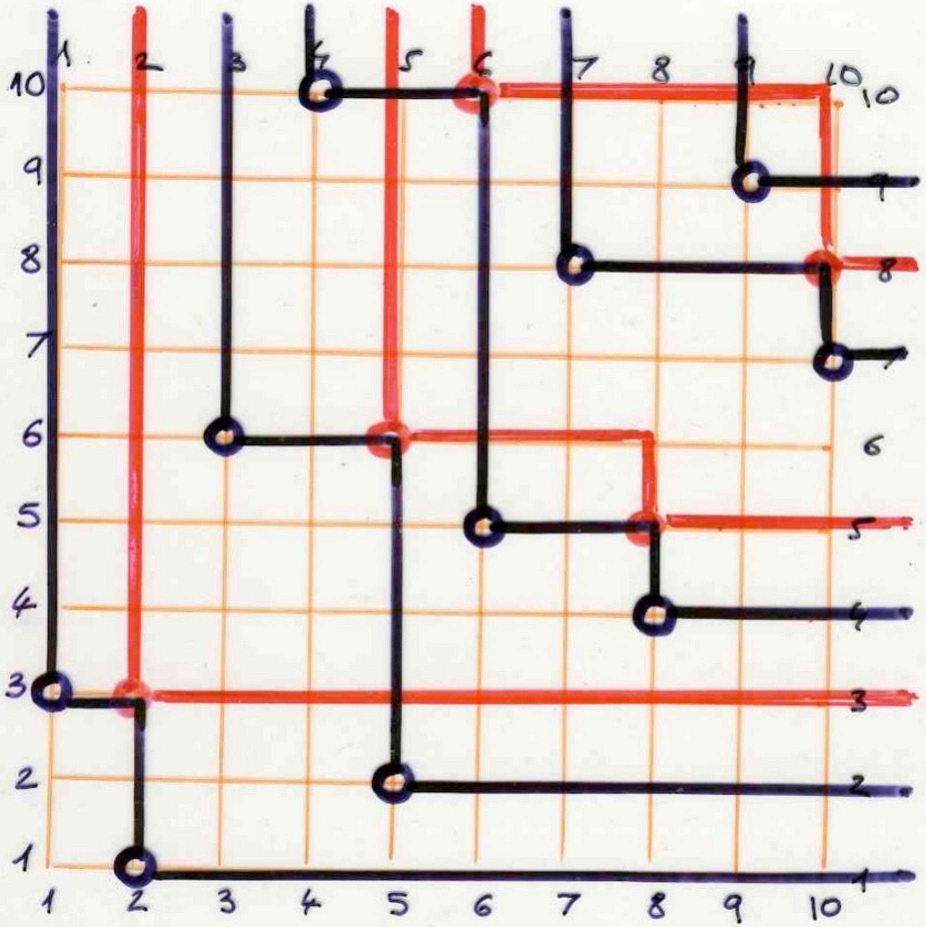
σ = 3 1 6 10 2 5 8 4 9 7

repeat with the red points
the construction of successives shadows



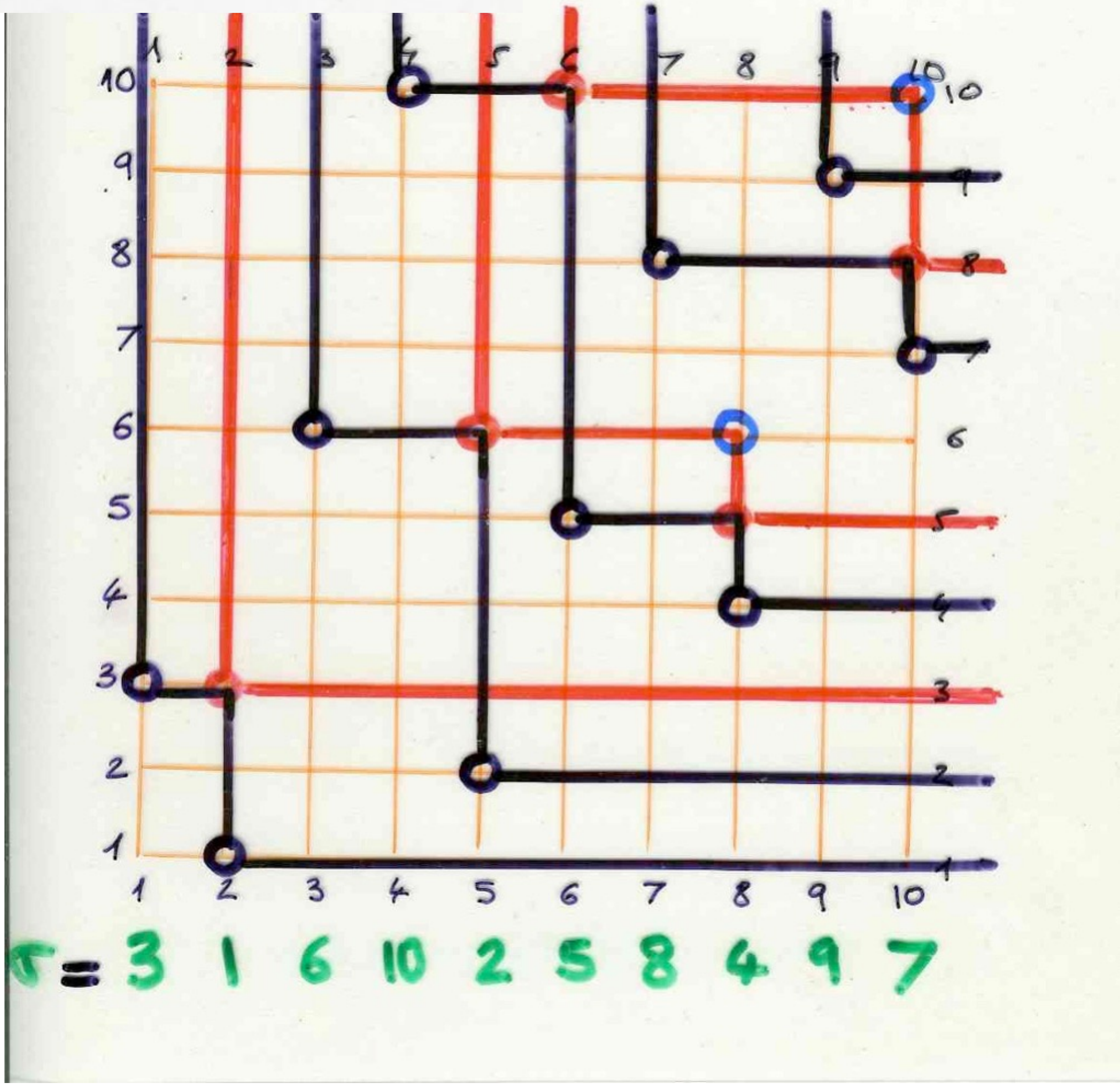


σ = 3 1 6 10 2 5 8 4 9 7

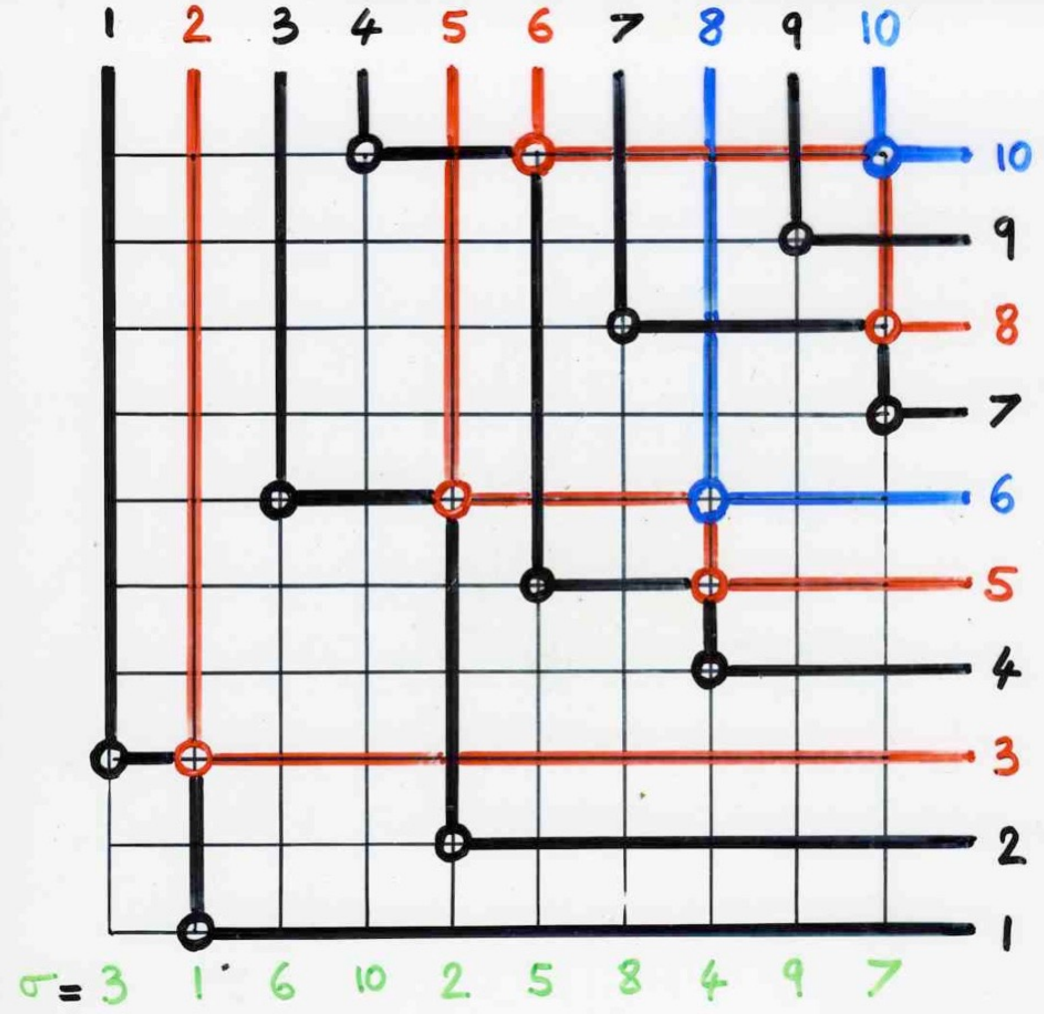


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

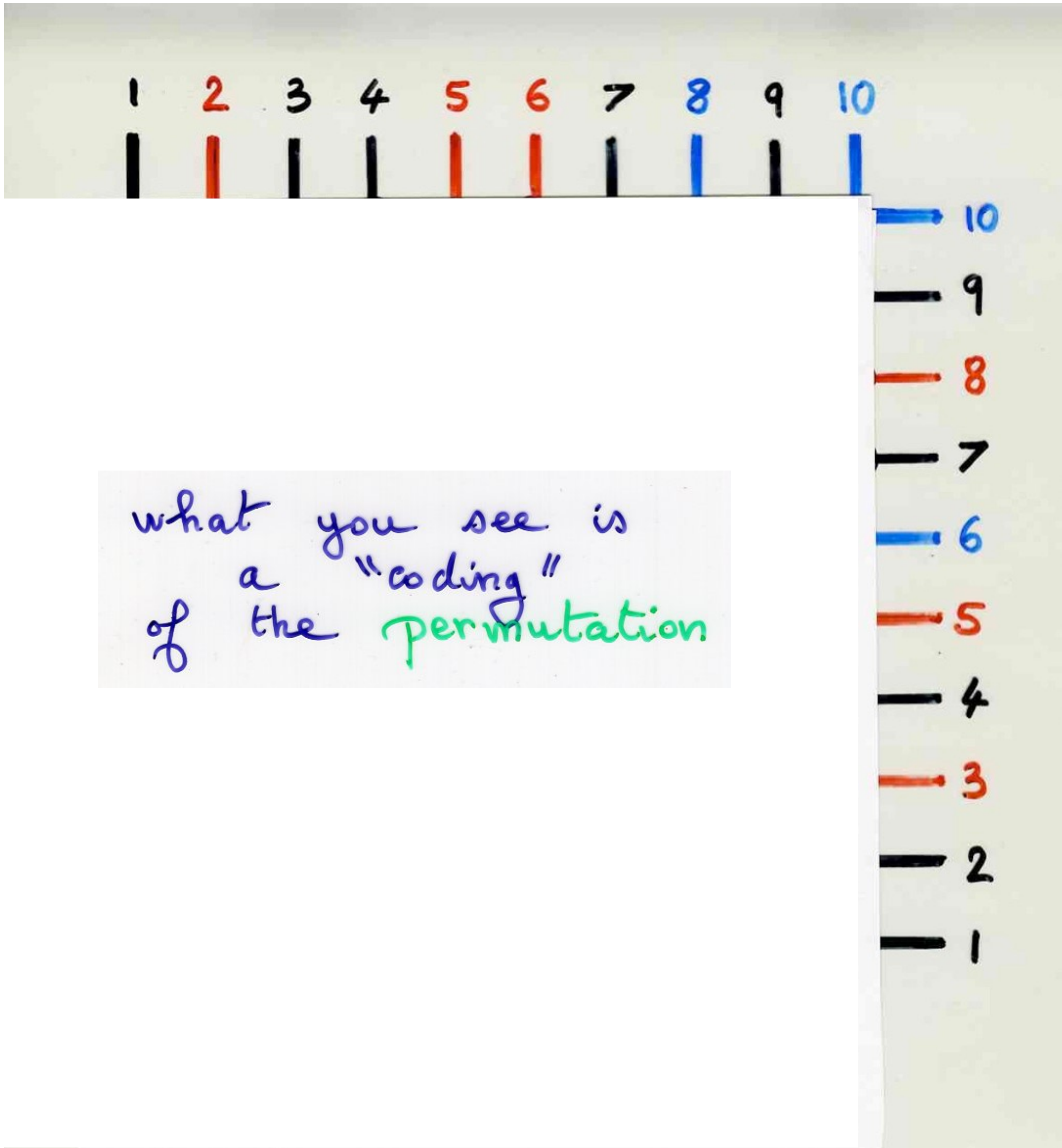
blue points



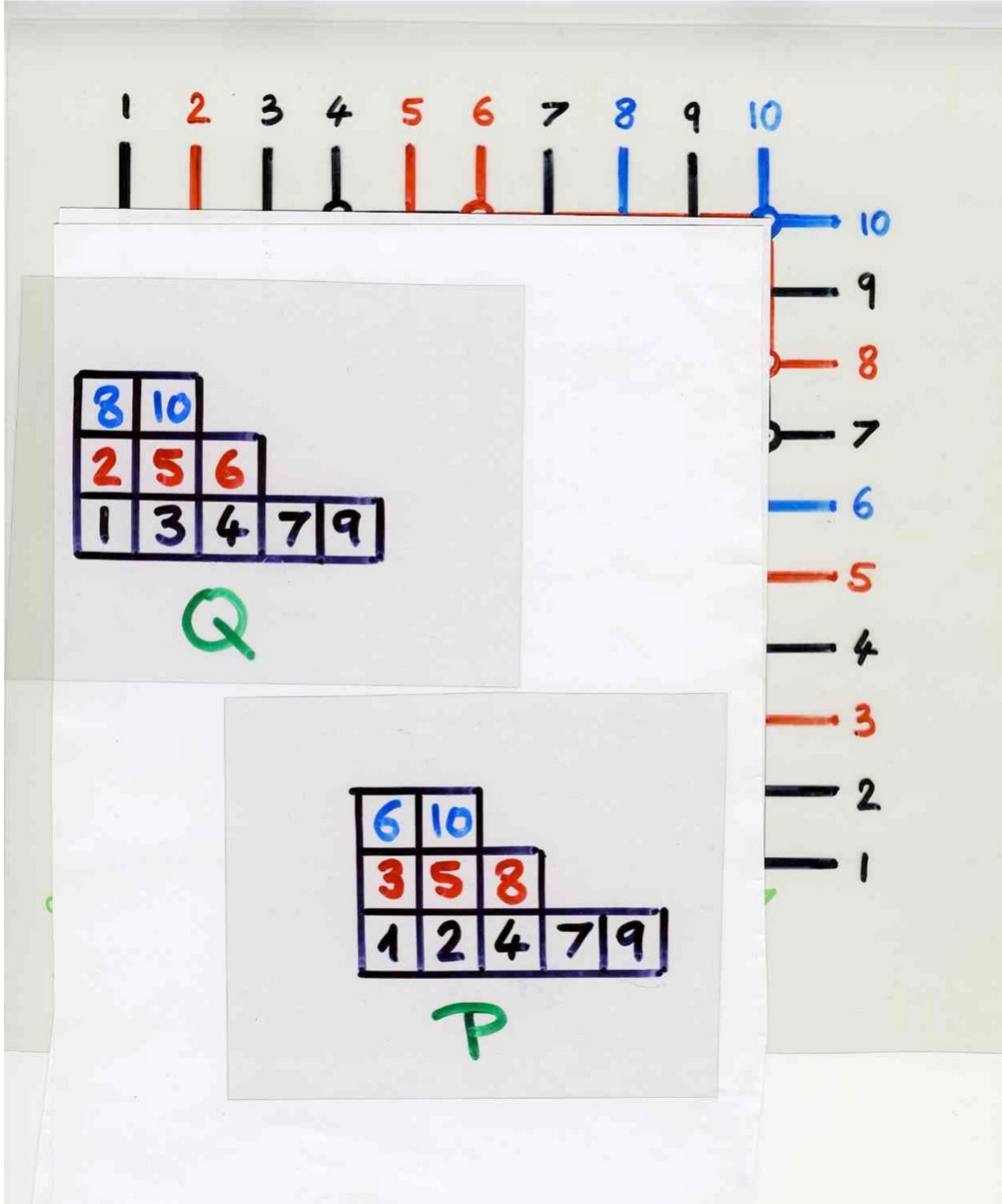
no green points ●



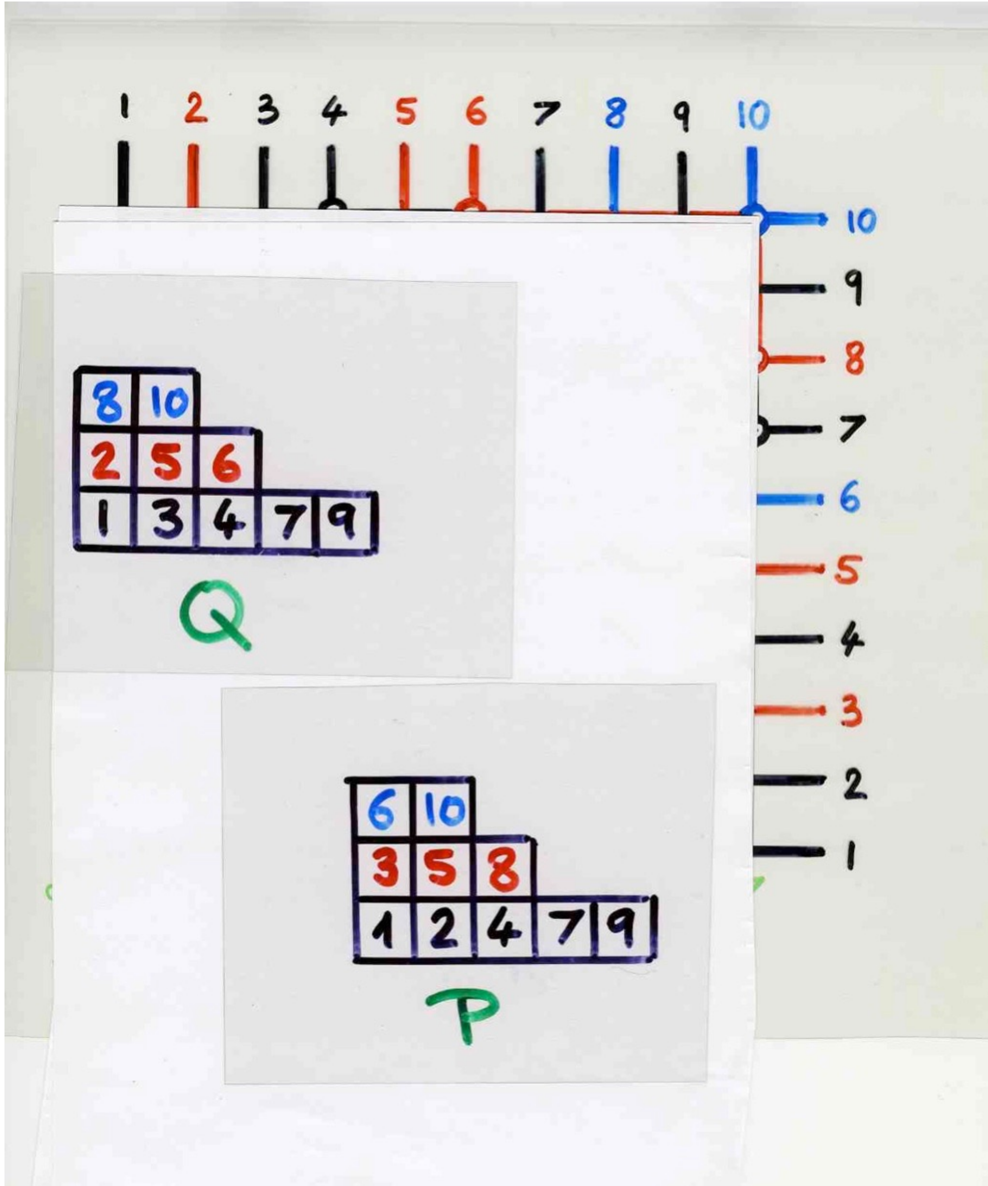
end of the construction



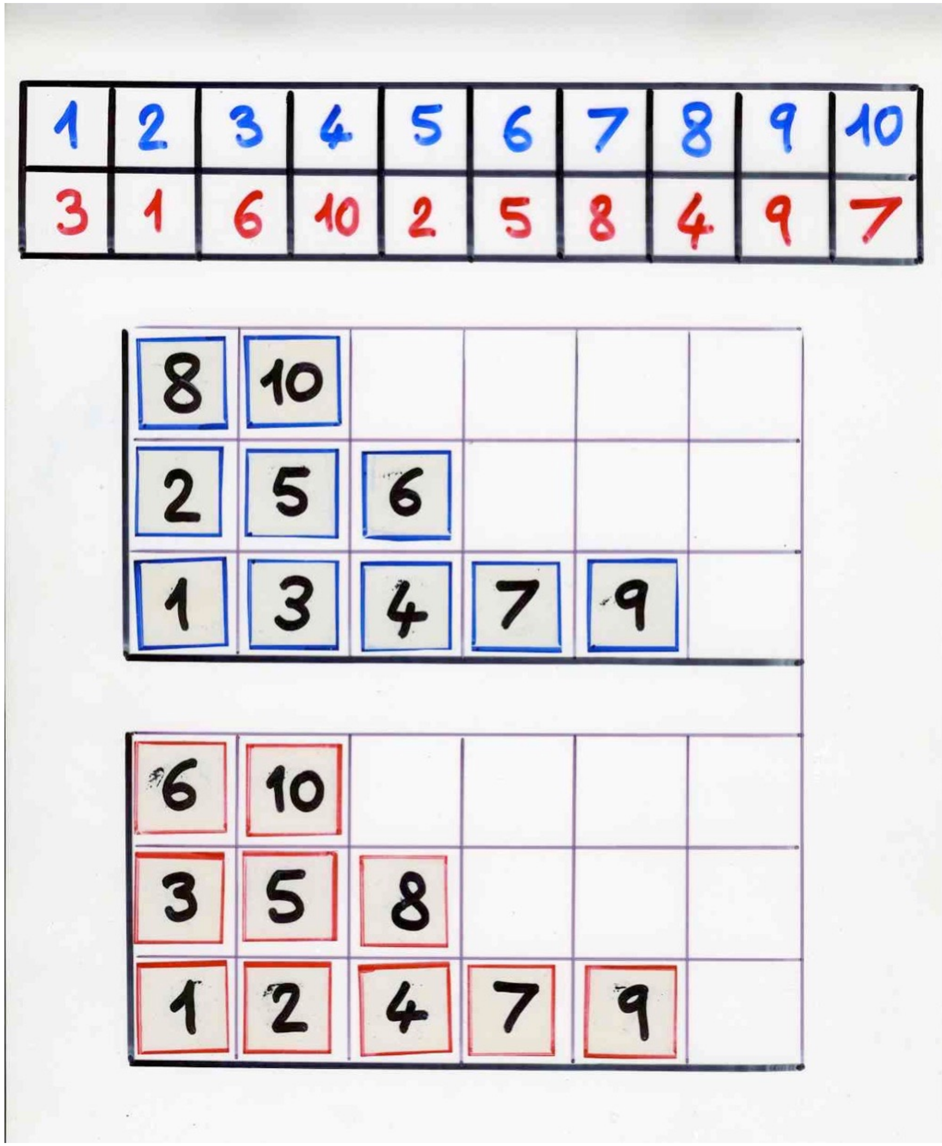
what you see is
a "coding"
of the permutation

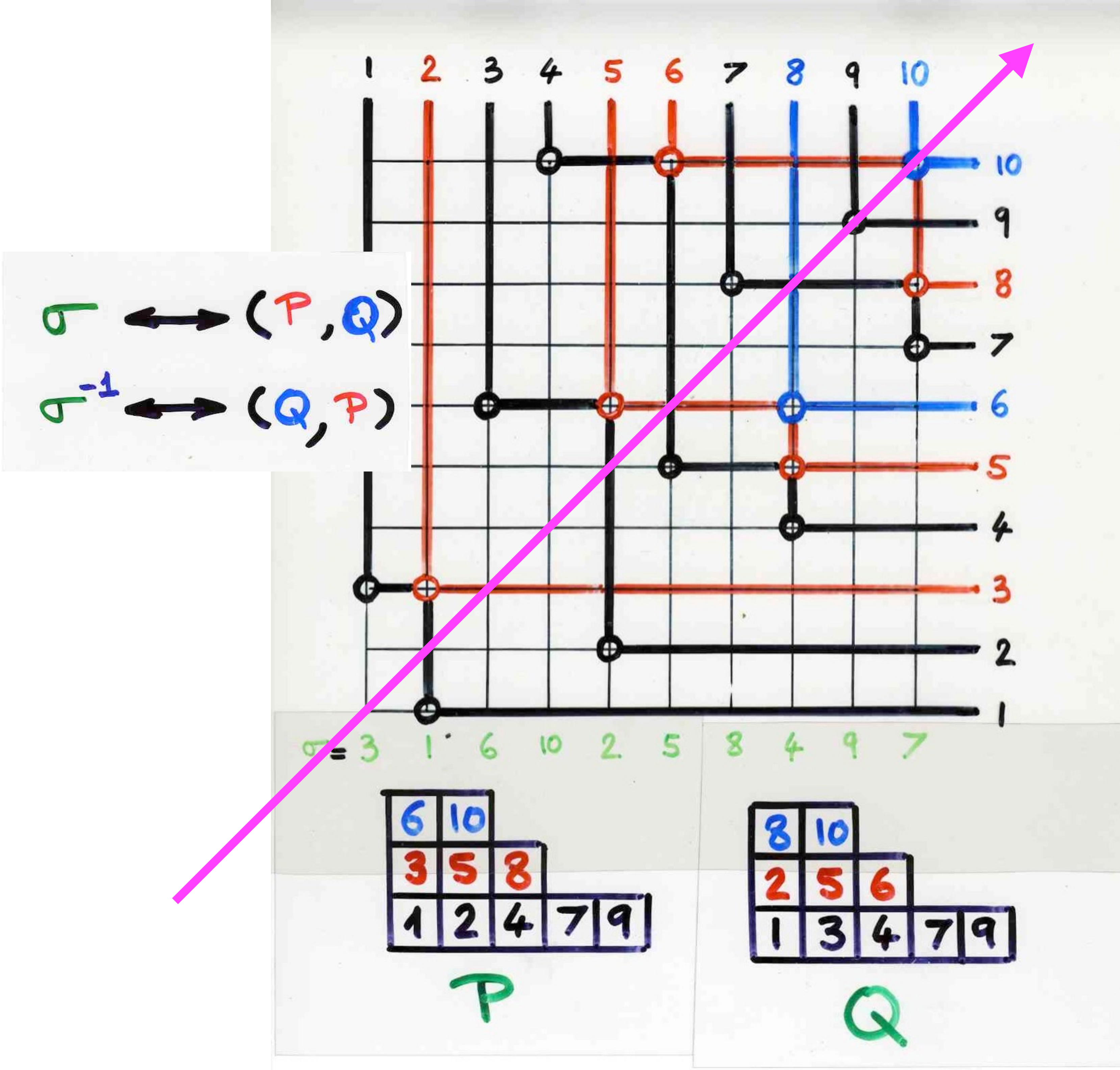

1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P

geometric version with "light" and "shadow"



Schensted's insertions





σ ⟷ (P, Q)
σ⁻¹ ⟷ (Q, P)
1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
P
8 10
2 5 6
1 3 4 7 9
Q

A few things about posets

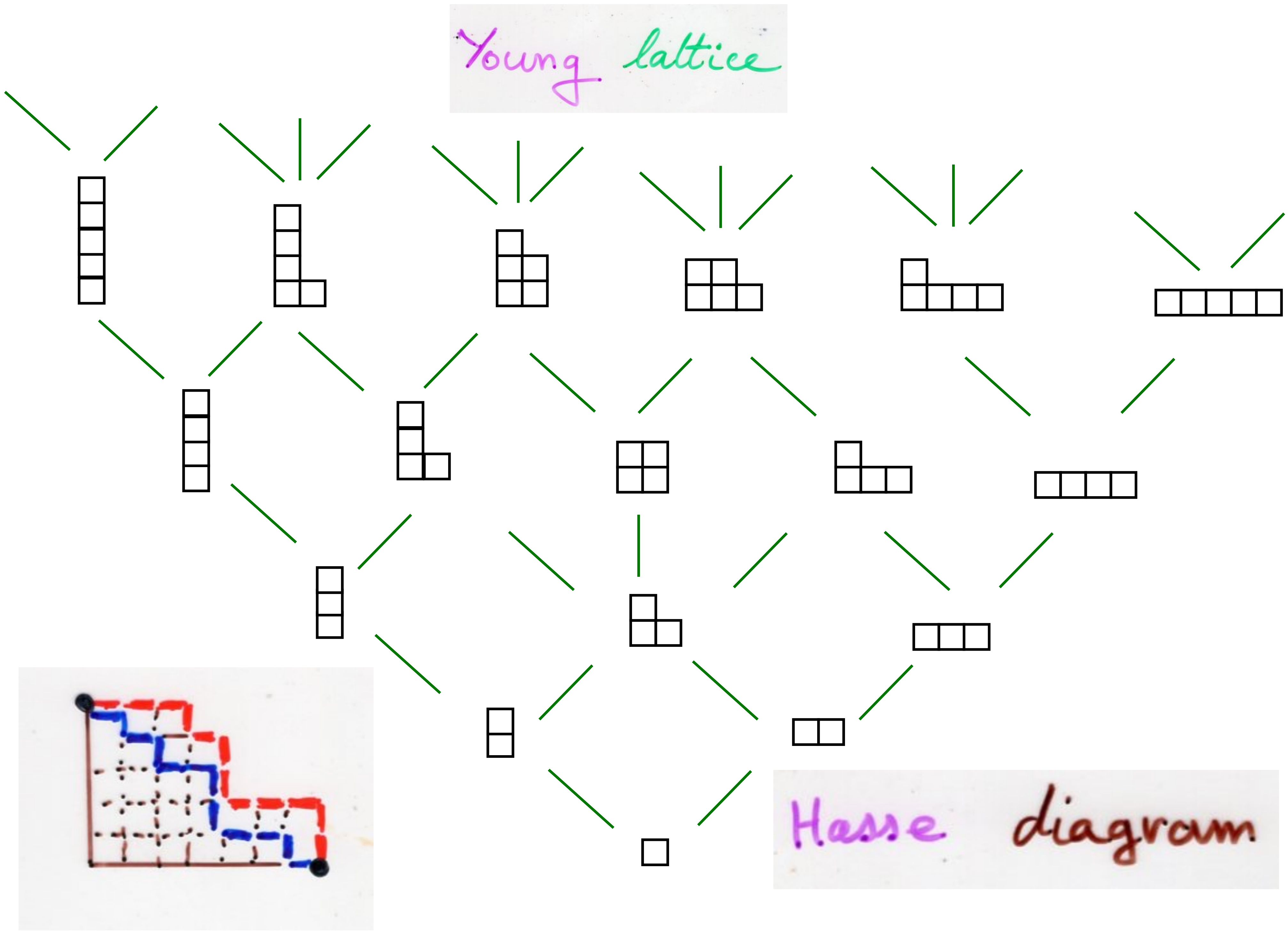
poset $\preccurlyeq$

partially ordered set

covering relation

$\alpha \preccurlyeq \beta$
no γ between α and β

Hasse diagram


Young lattice
Hasse diagram

lattice

every two elements
have a unique
least upper bound (join)

and a unique
greatest lower bound
(meet)

maximal chain
in a poset

$\alpha_1 \lesssim \alpha_2 \lesssim \cdots \lesssim \alpha_k$

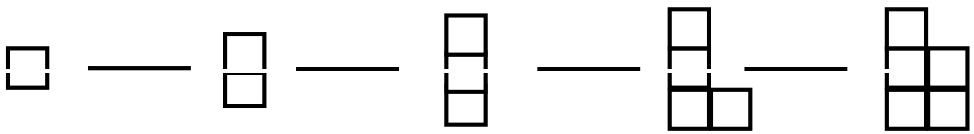
each α_{i+1} is covering α_k

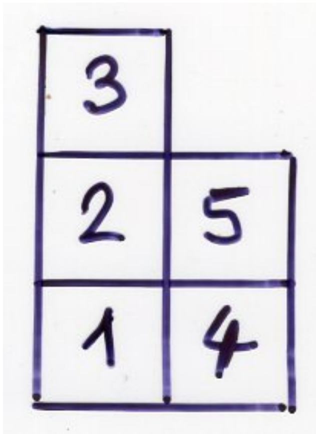
bijection

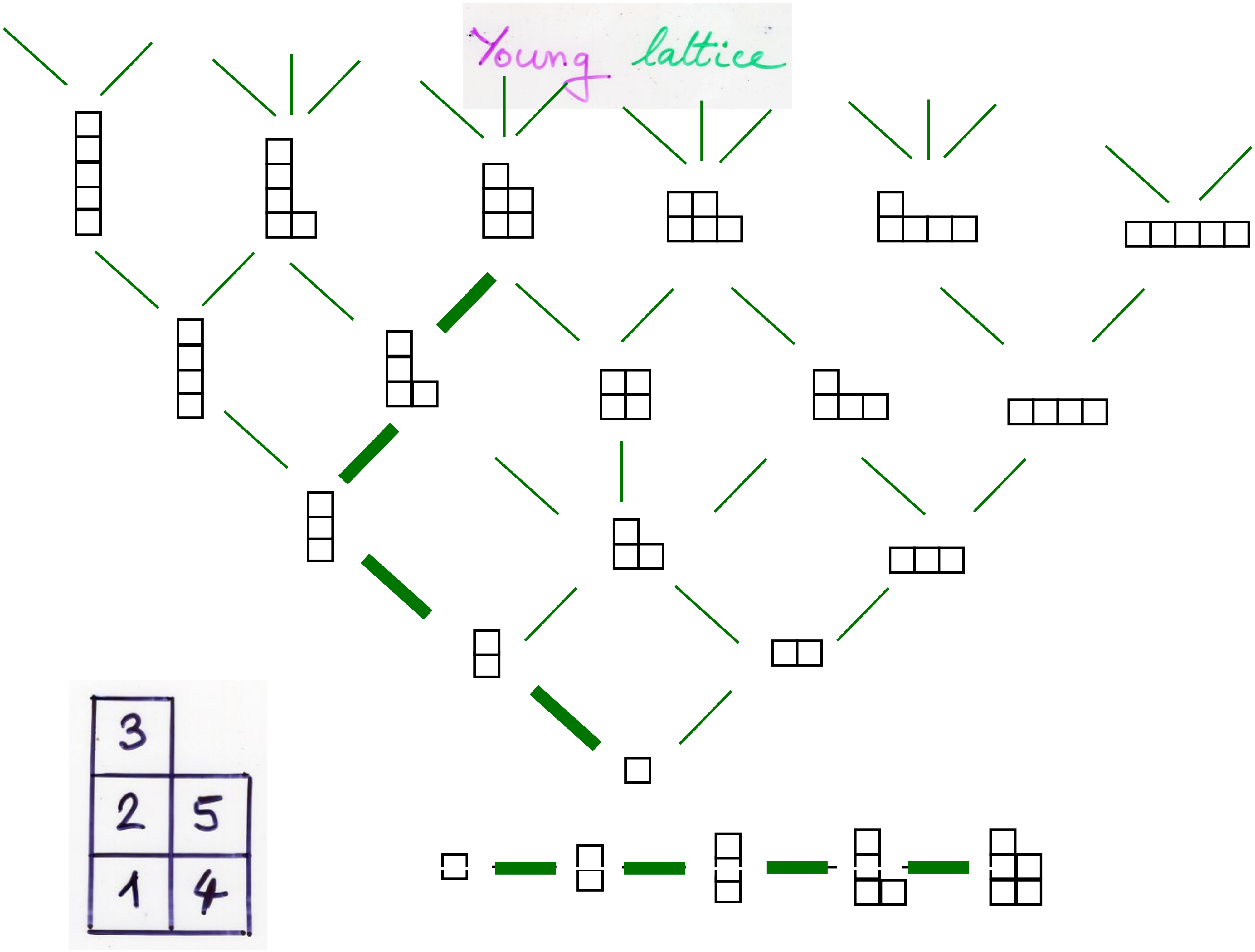
maximal chain
in the Young lattice
$\alpha_1 = \emptyset \lesssim \cdots \lesssim \alpha_k = \lambda$

$\longleftrightarrow$

and Young tableau
with shape λ





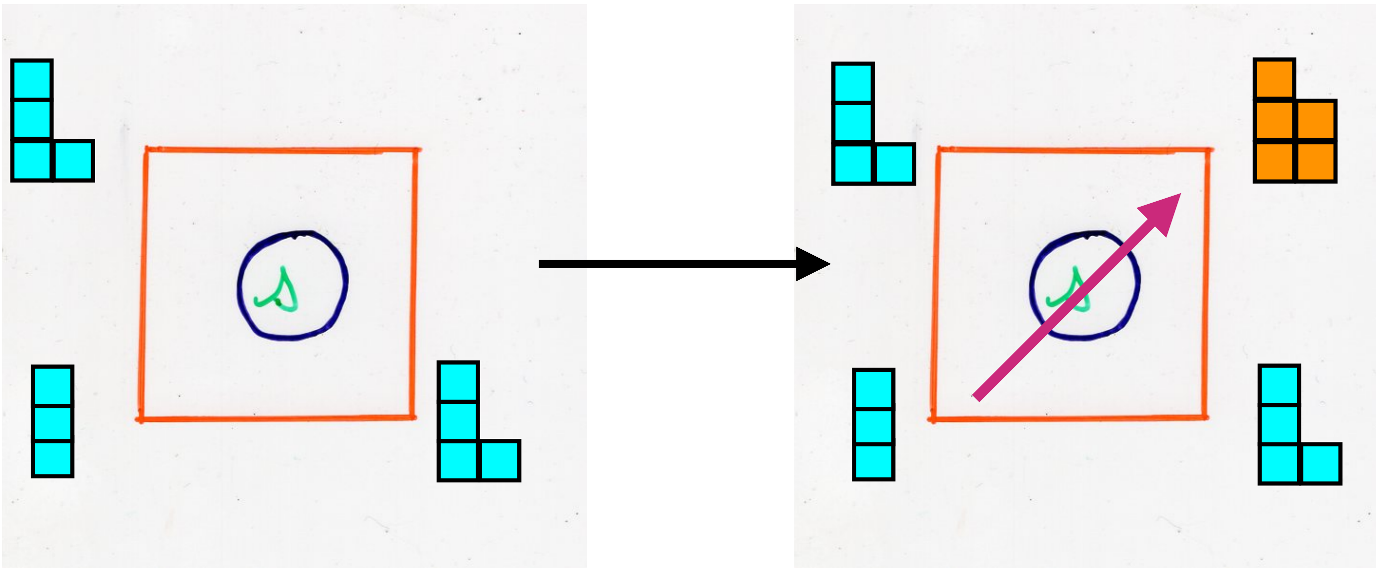

Young lattice
3
2 5
1 4

"local" algorithm on a grid or "growth diagrams"

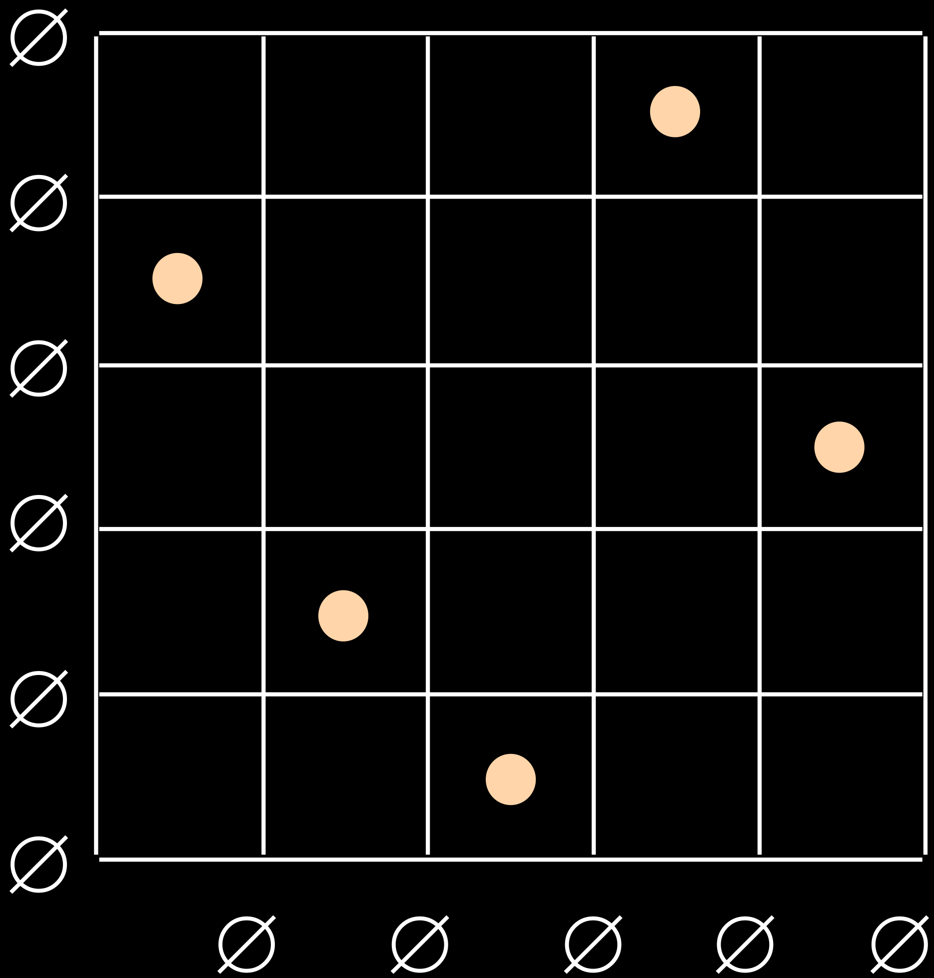
S. Fomin, 1986, 1994

C.Krattenthaler

Fomin's "local rules"
"growth diagrams"

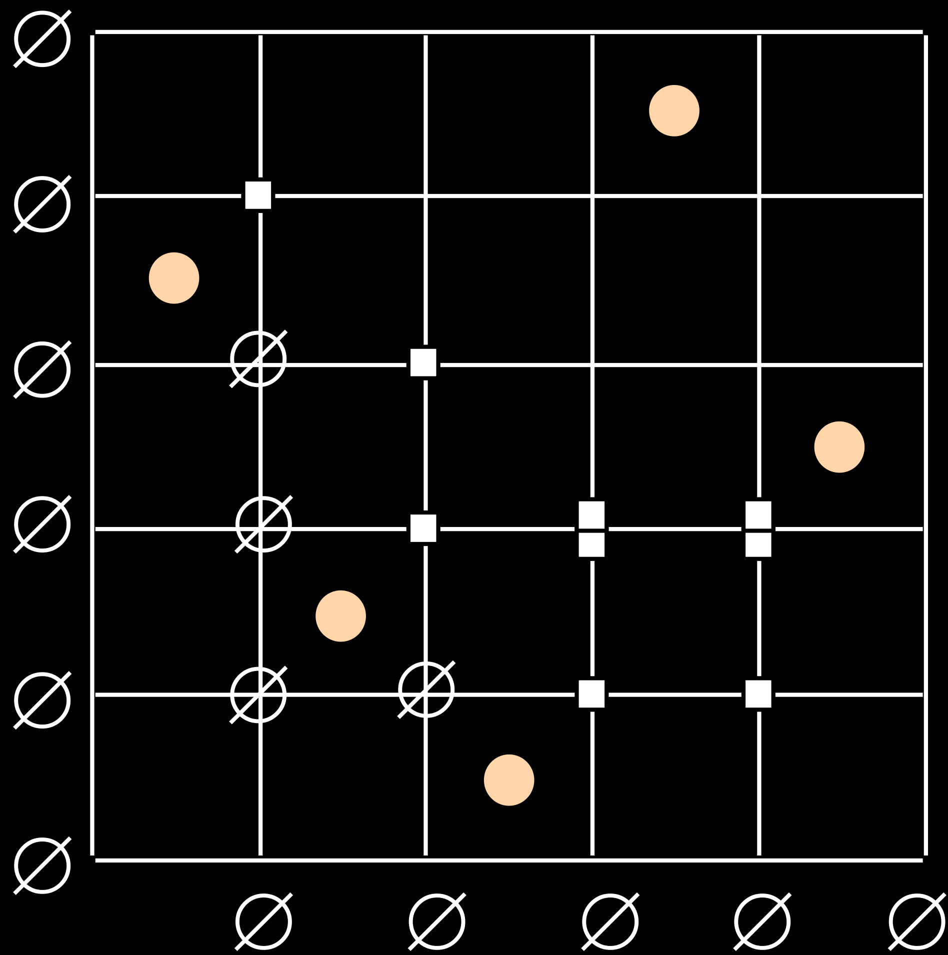


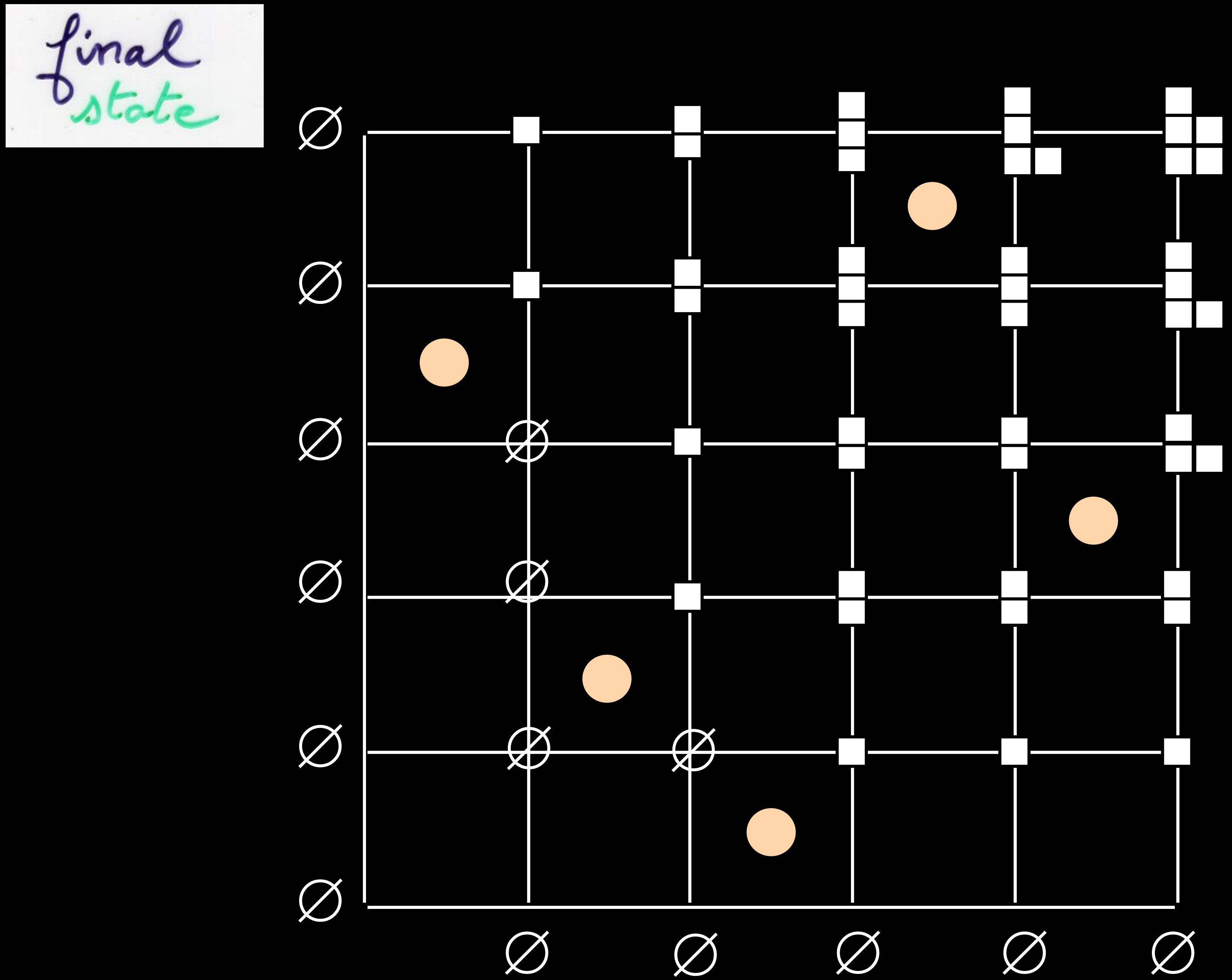
initial state



$\sigma = 4, 2, 1, 5, 3$

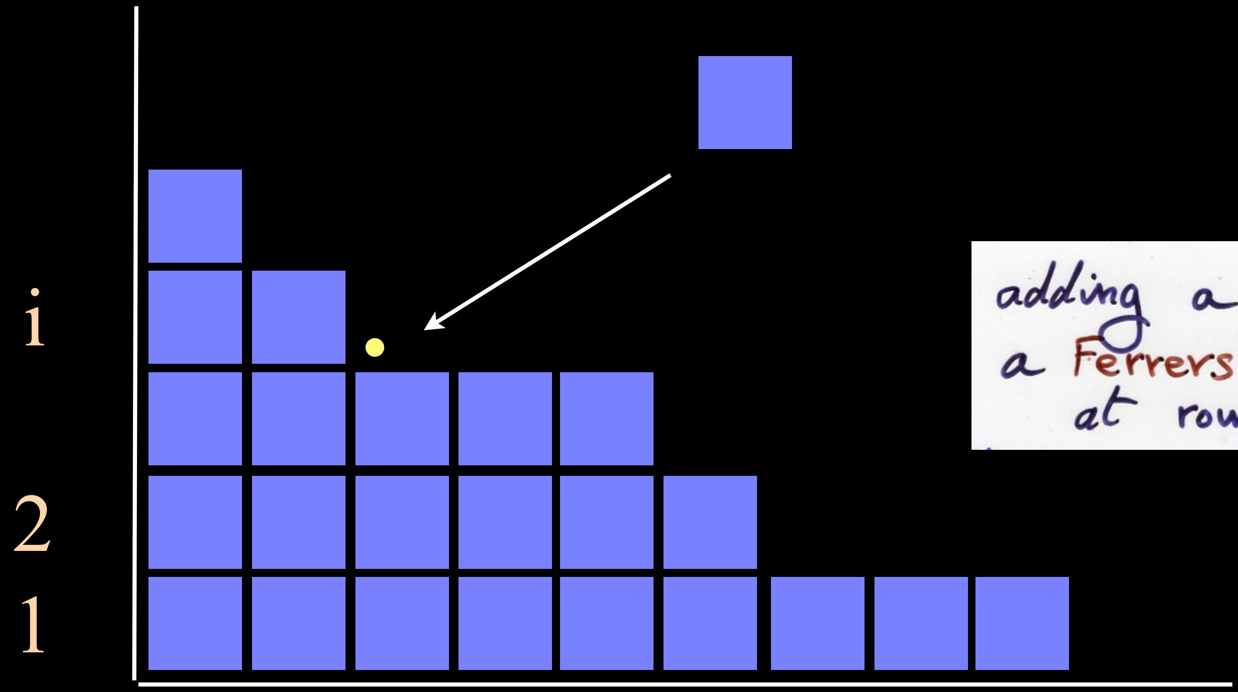
during the labeling process




final state
∅
∅
∅
∅
∅
∅
∅
∅
∅
∅
∅

notations

operator U_i

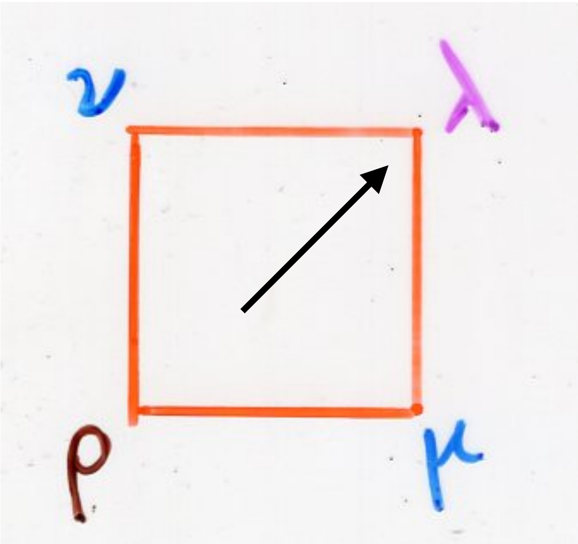


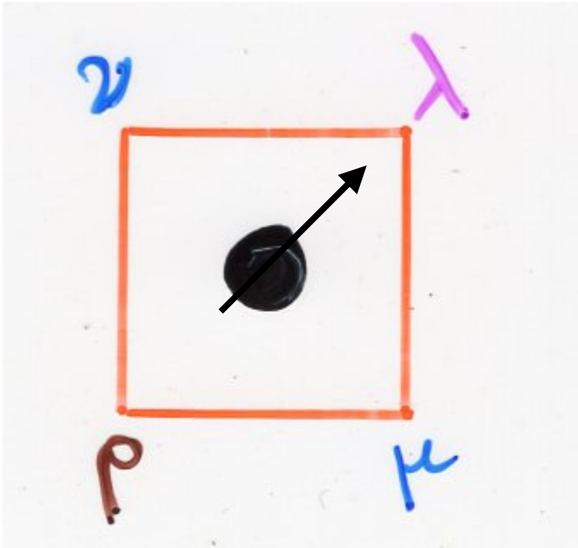
adding a cell in a Ferrers diagram ρ at row i

$$U_i(\rho) = \rho + (i)$$

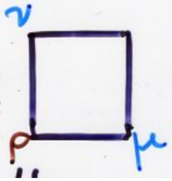
"growth diagrams"

"local rules"





"local rules"

(i) $\rho = \mu = \nu$ and 

then $\lambda = \rho$

(ii) $\rho = \mu \neq \nu$, then $\lambda = \nu$

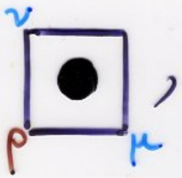
(iii) $\rho = \nu \neq \mu$, then $\lambda = \mu$

(iv) ρ, μ, ν pairwise $\neq$, then $\lambda = \mu \cup \nu$

(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$

given that $\mu = \nu$ and ρ differ in the i-th row

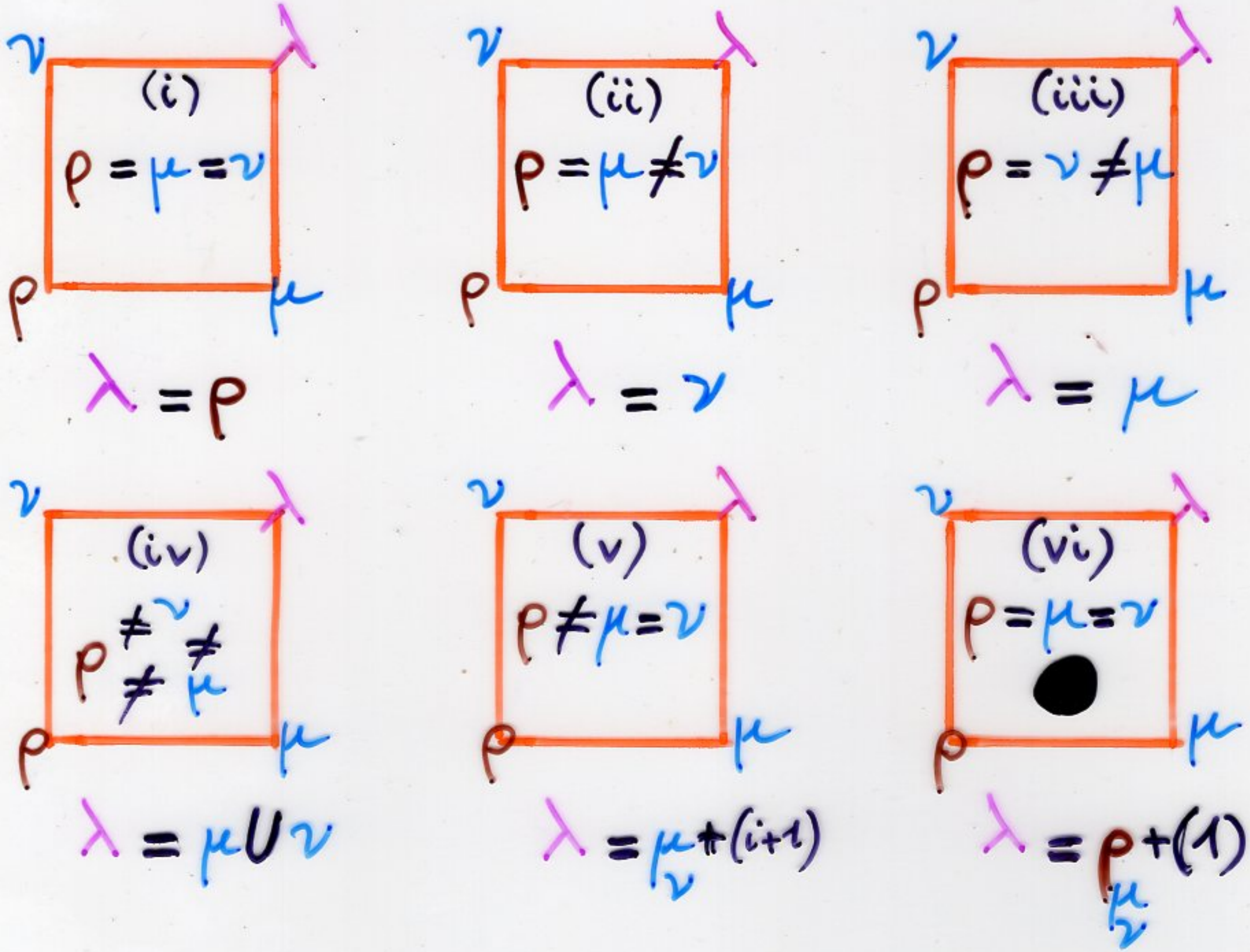
[in fact $\mu = \nu = \rho + (i)$]

(vi) $\rho = \mu = \nu$ and 

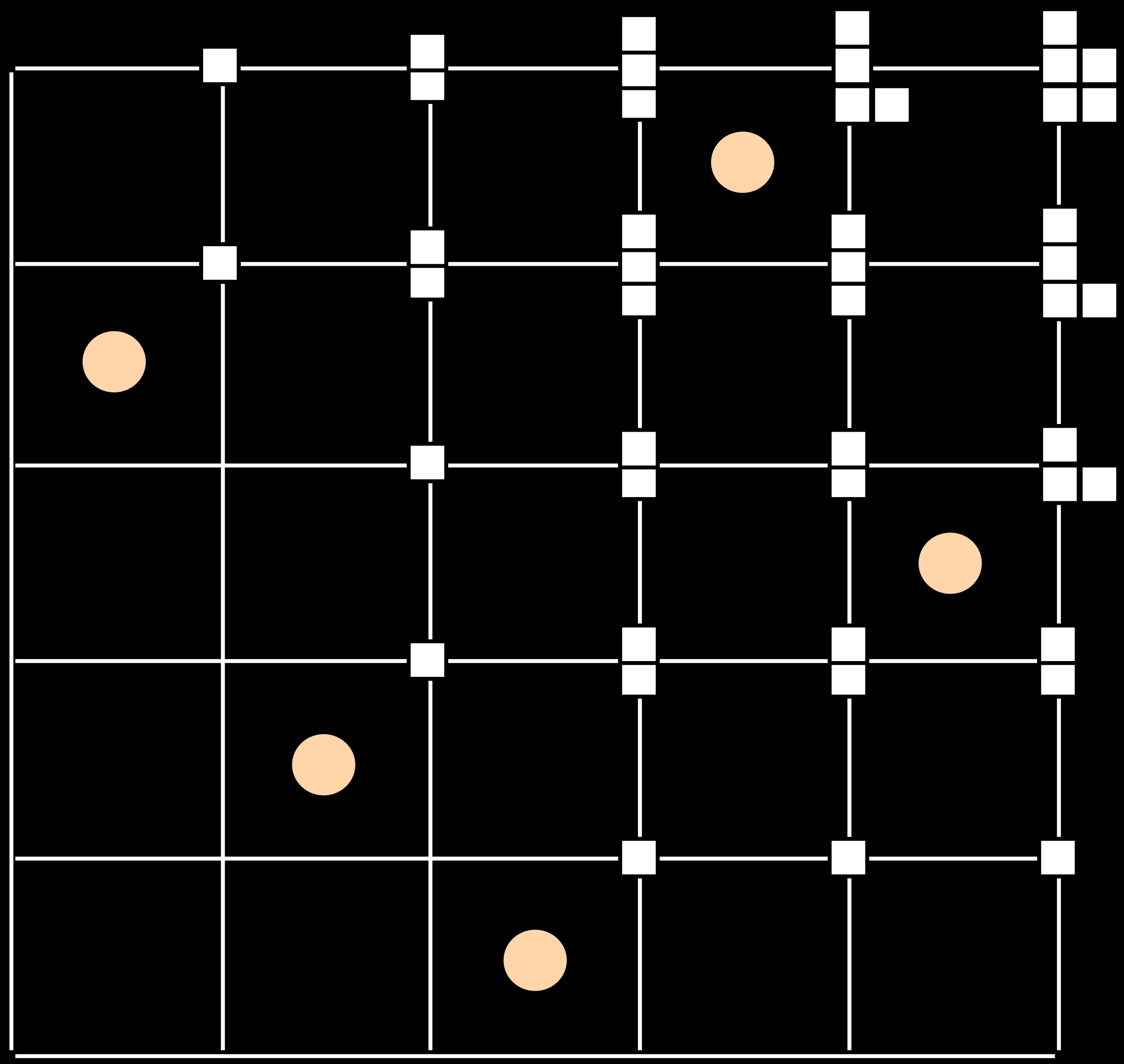
, then $\lambda = \mu + (1)$

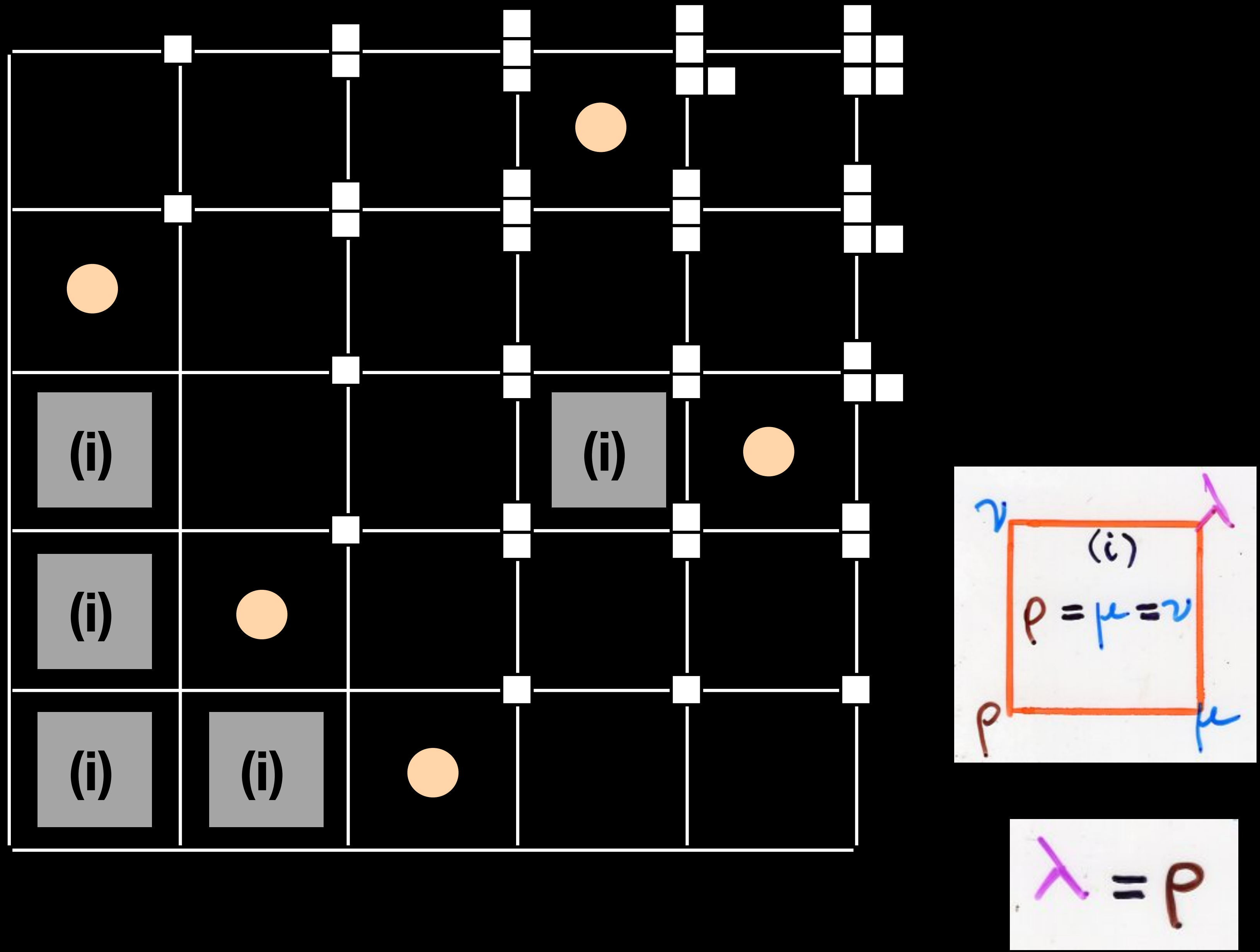
C.Krattentheler, (2006).

GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

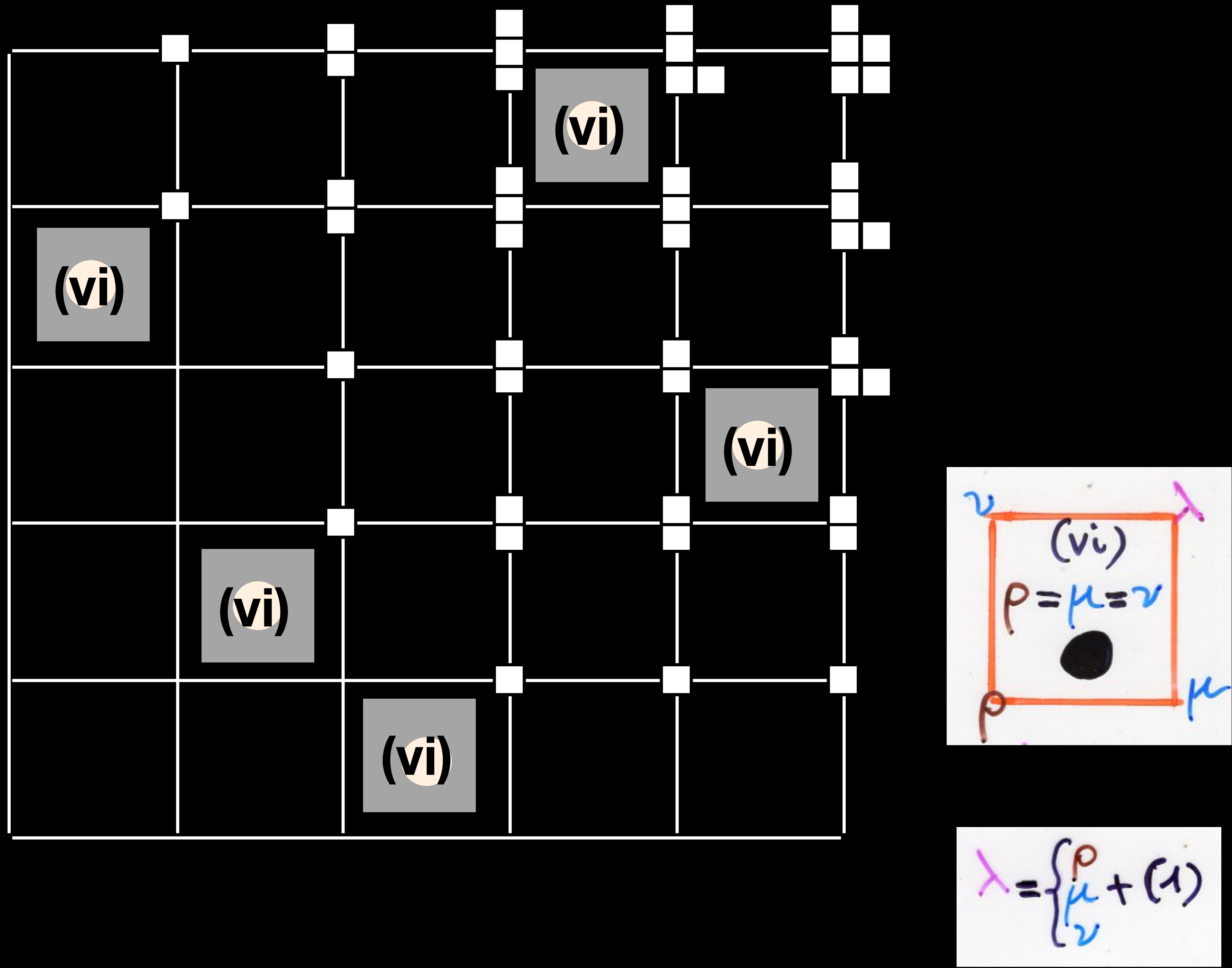


(i)
ρ = μ = ν
λ = ρ
(ii)
ρ = μ ≠ ν
λ = ν
(iii)
ρ = ν ≠ μ
λ = μ
(iv)
λ = μ ∪ ν
(v)
ρ ≠ μ = ν
(vi)
ρ = μ = ν

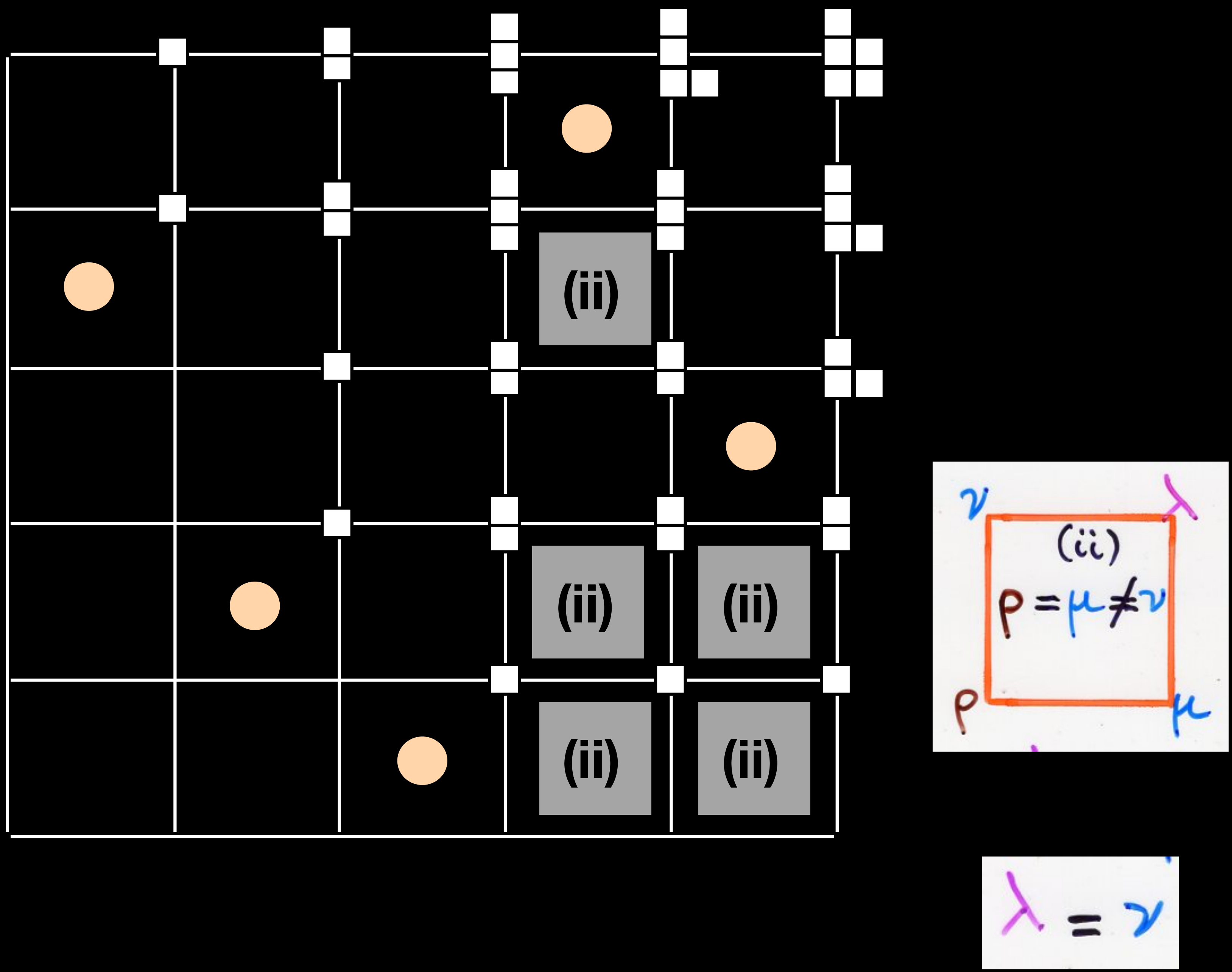


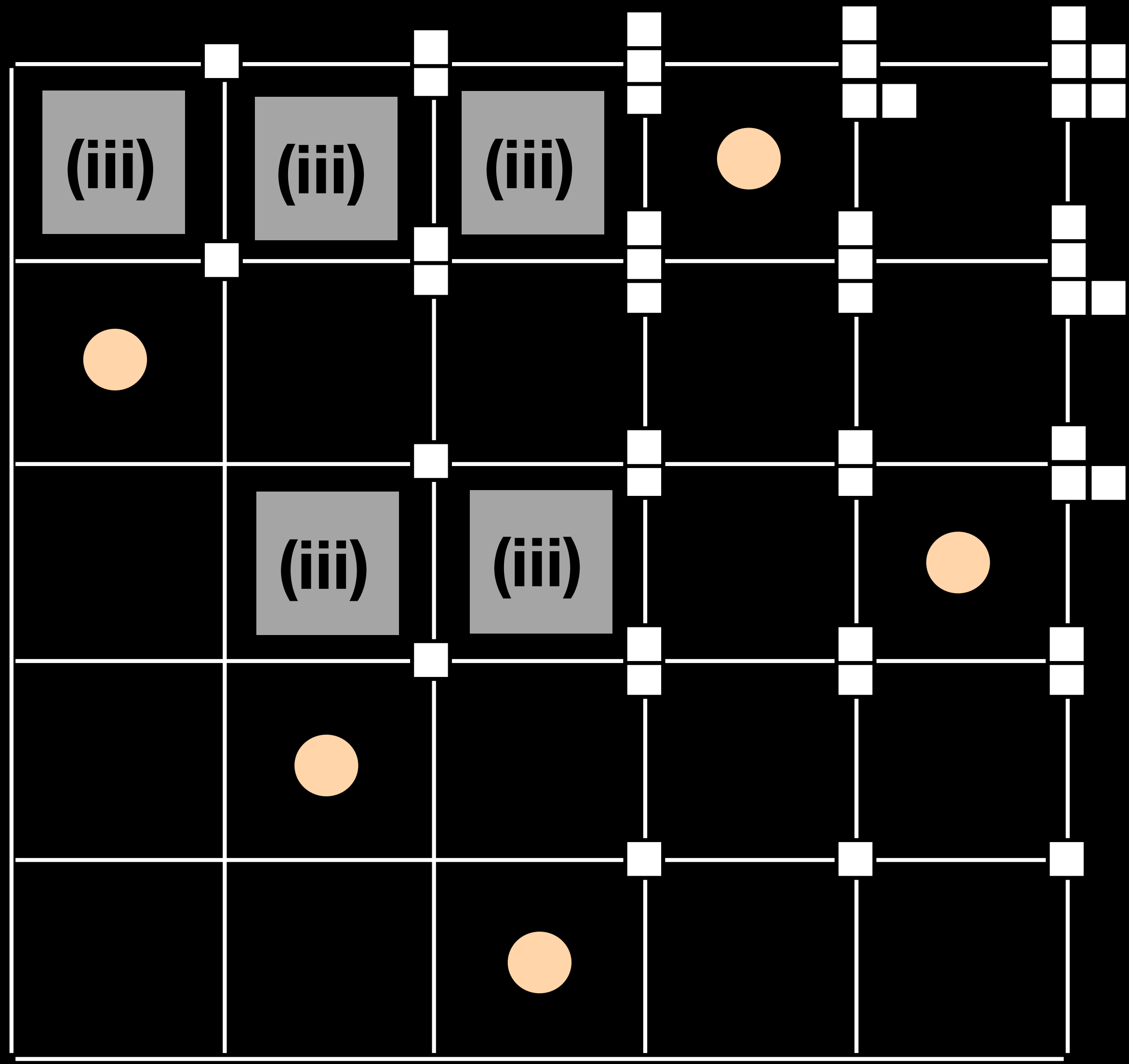


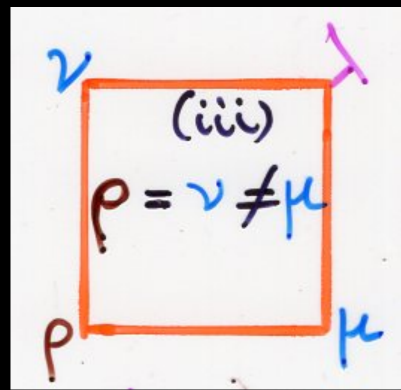
(i)
(i)
(i)
(i)
(i)
(i)
ν
λ
(i)
ρ = μ = ν
ρ
μ
λ = ρ

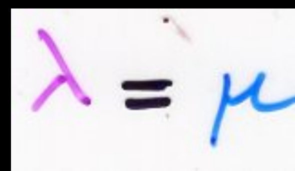


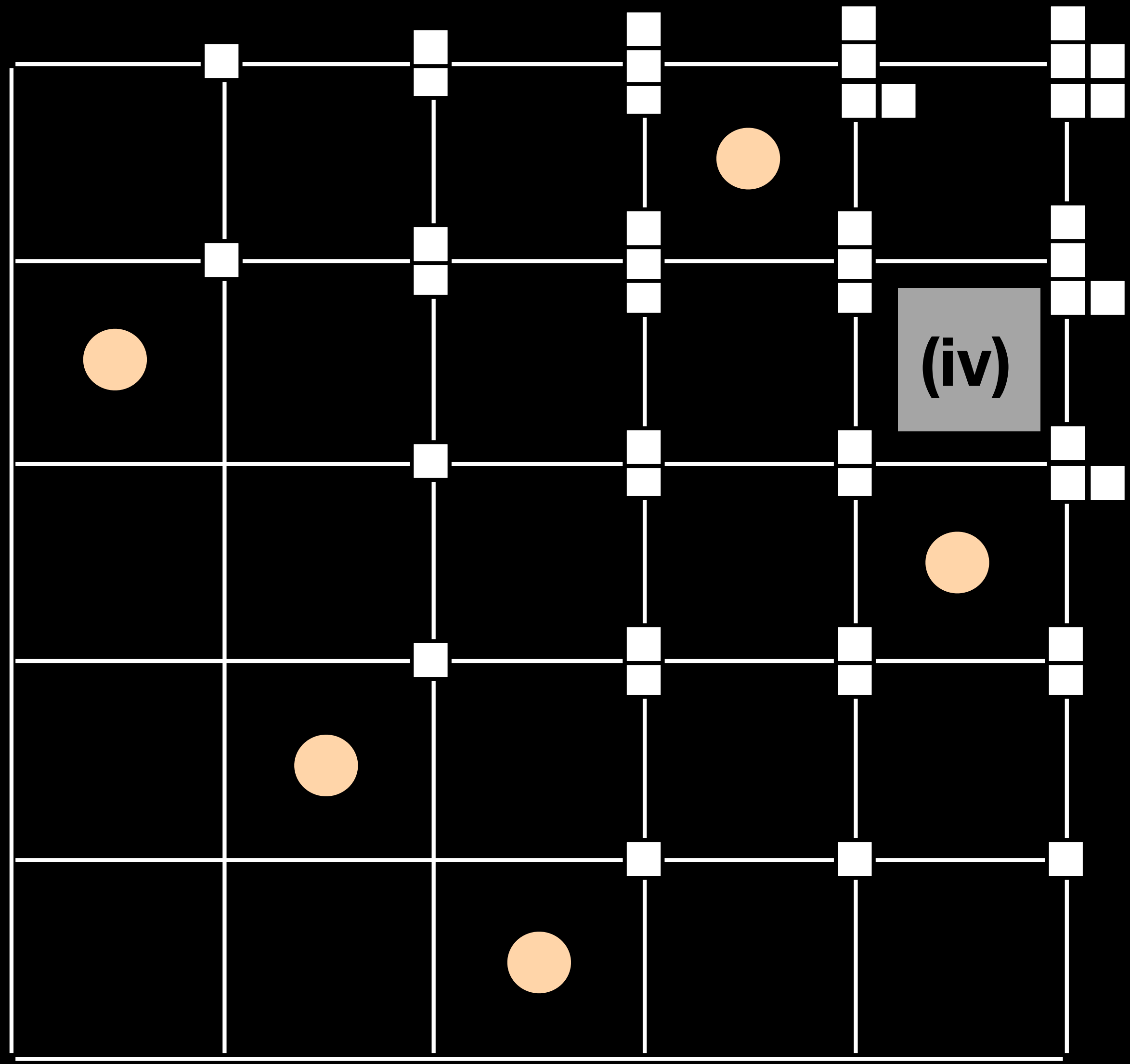
(vi)
(vi)
(vi)
(vi)
(vi)
(vi)
ν
λ
(vi)
ρ=μ=ν
ρ
μ
λ = {ρ μ ν + (1)

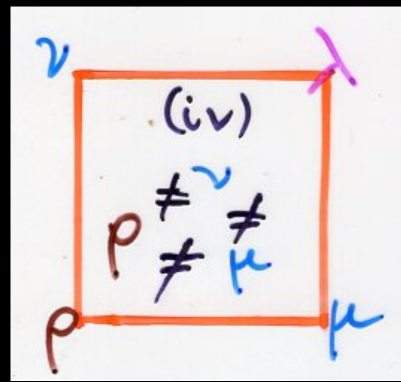

(ii)
(ii)
(ii)
(ii)
(ii)
ν
λ
(ii)
ρ = μ ≠ ν
ρ
μ
λ = ν


(iii)
(iii)
(iii)
(iii)
(iii)

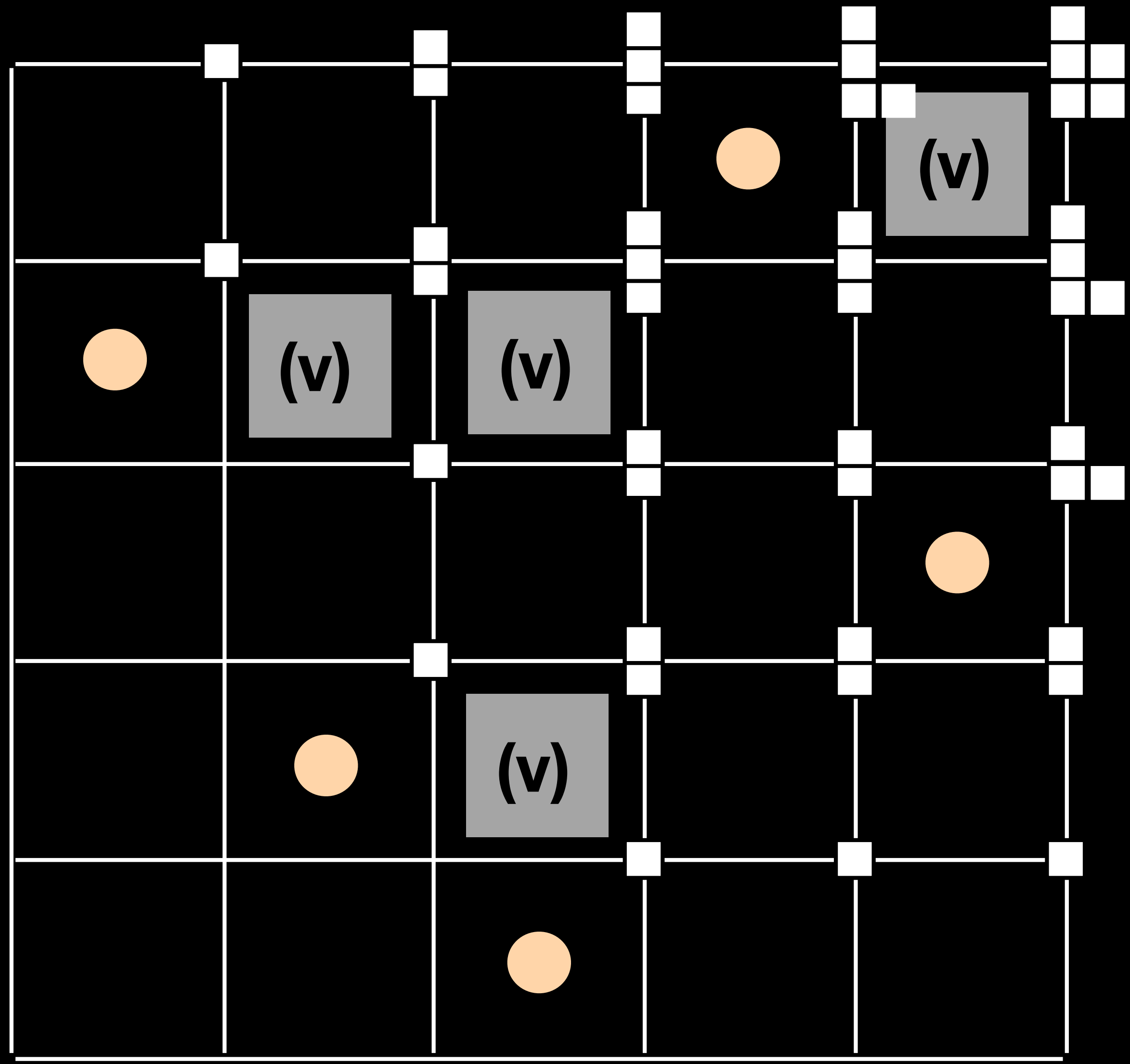

ν
λ
(iii)
ρ = ν ≠ μ
ρ
μ

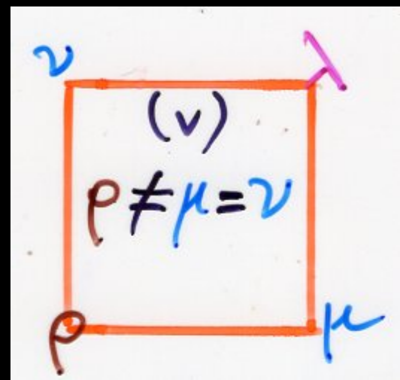

λ = μ



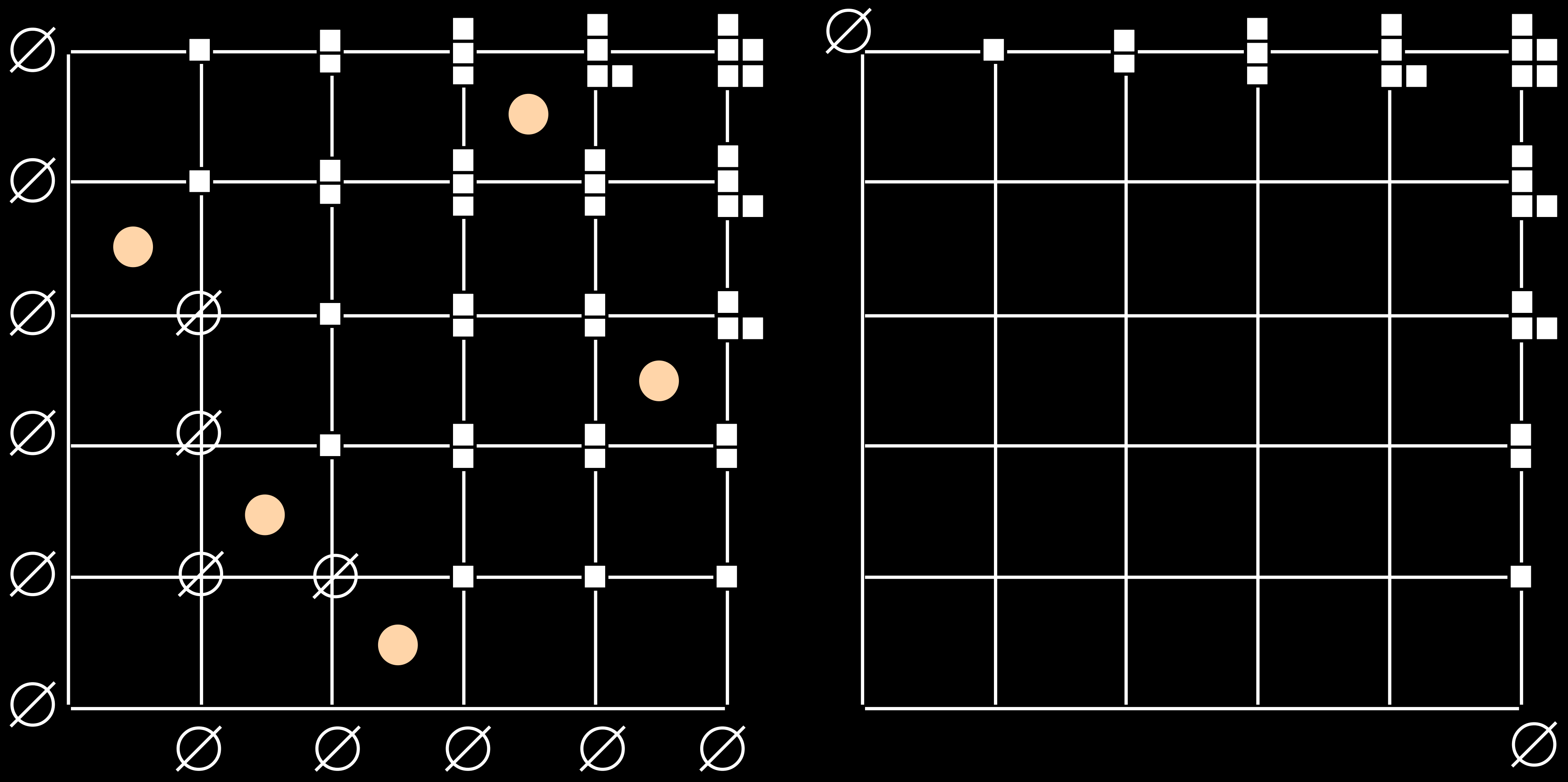


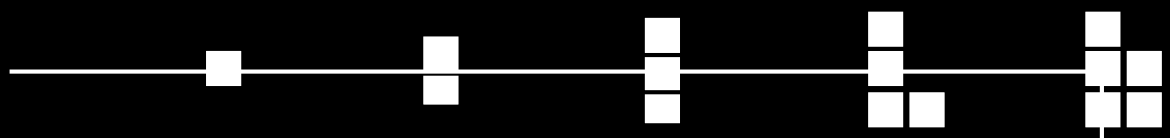
$$\lambda = \mu \cup \nu$$



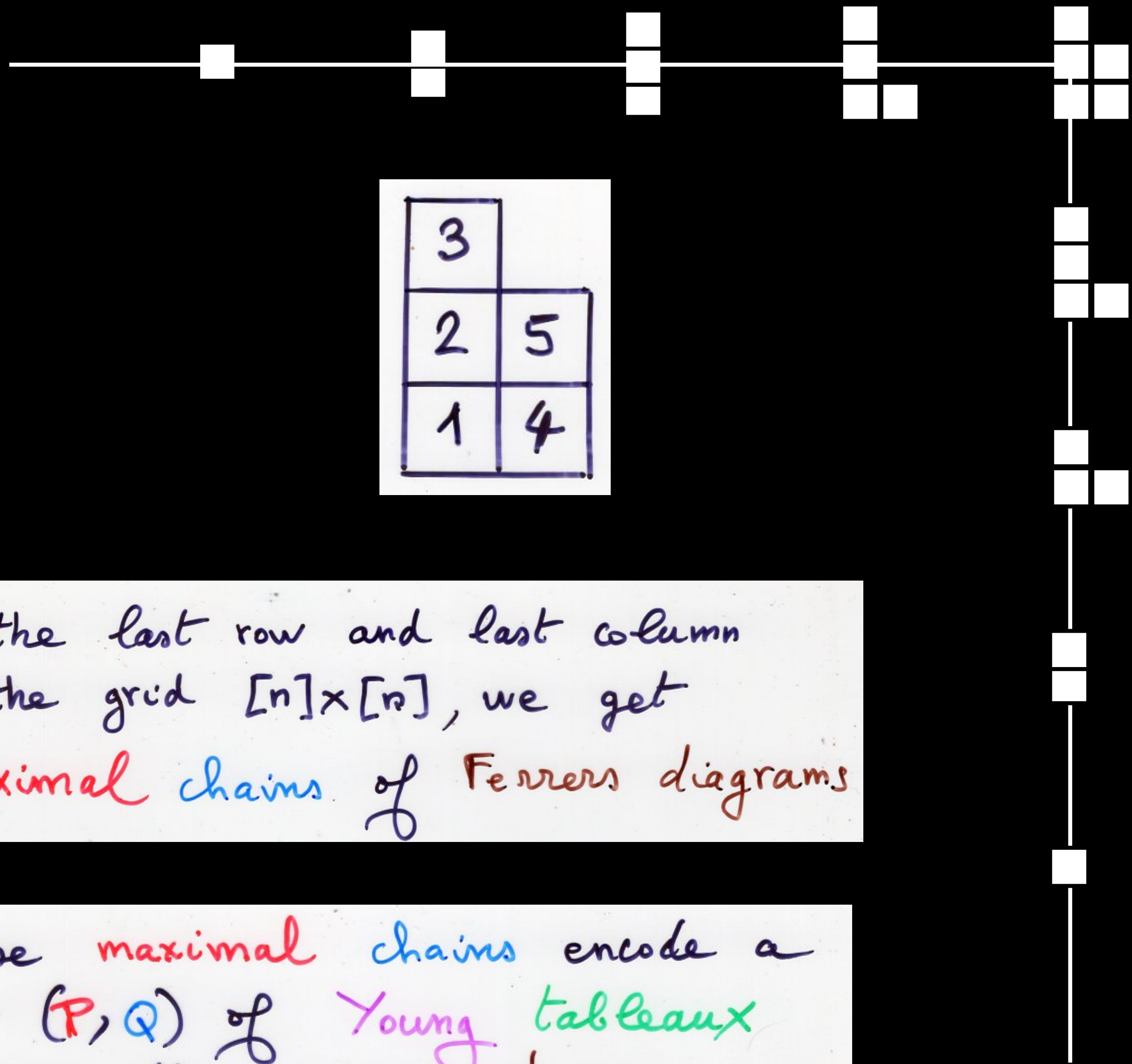


$$\lambda = \begin{cases} \mu \\ \nu \end{cases} + (i+1)$$

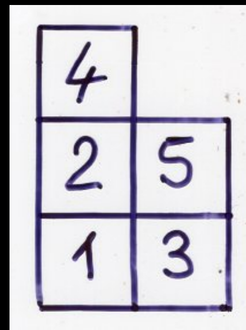




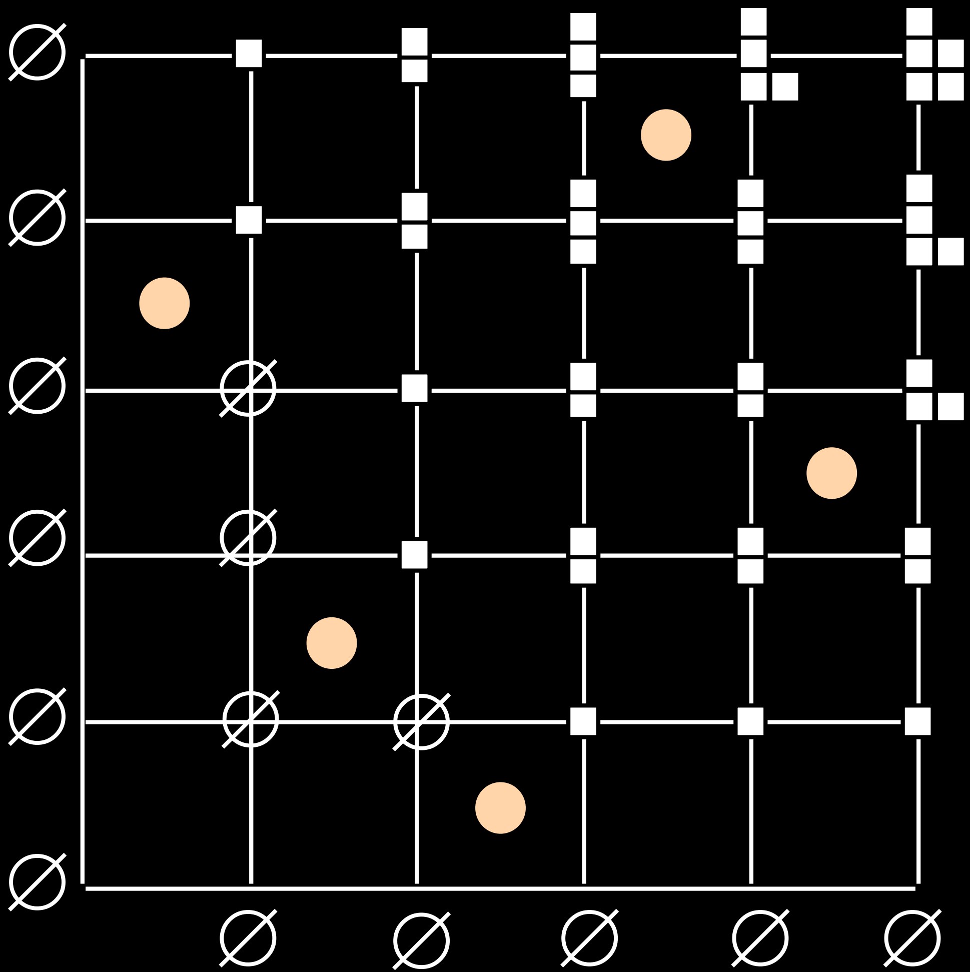
- in the last row and last column of the grid $[n] \times [n]$, we get maximal chains of Ferrers diagrams

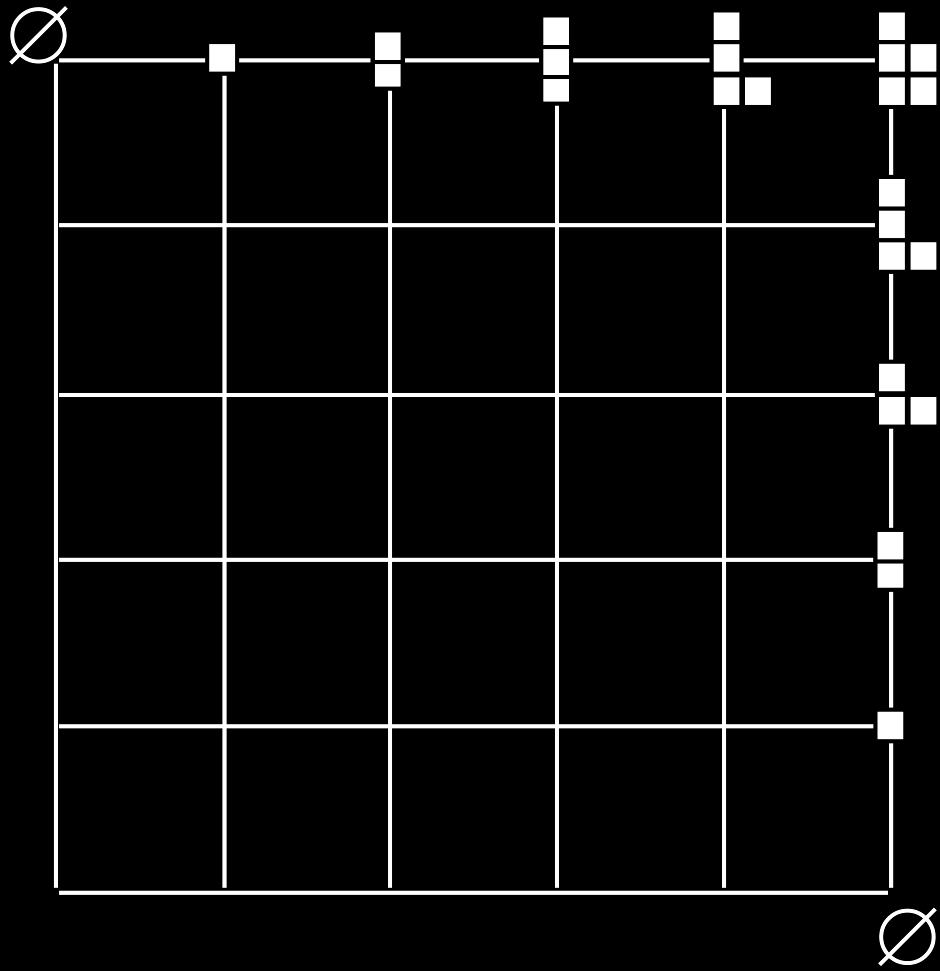


- in the last row and last column of the grid $[n]\times[n]$, we get maximal chains of Ferrers diagrams

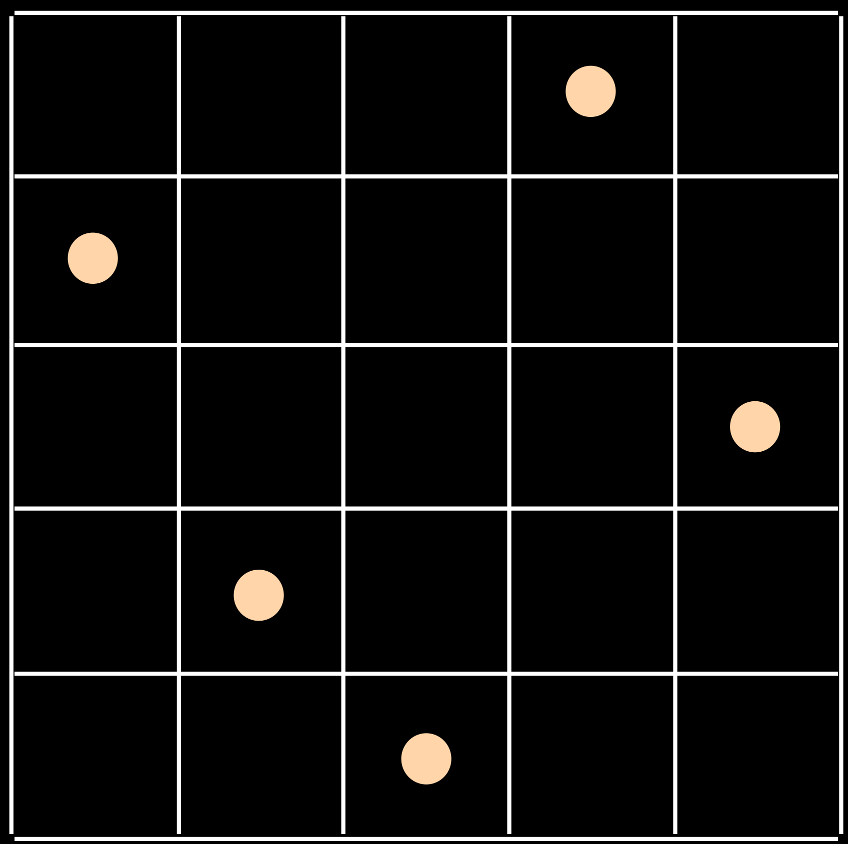


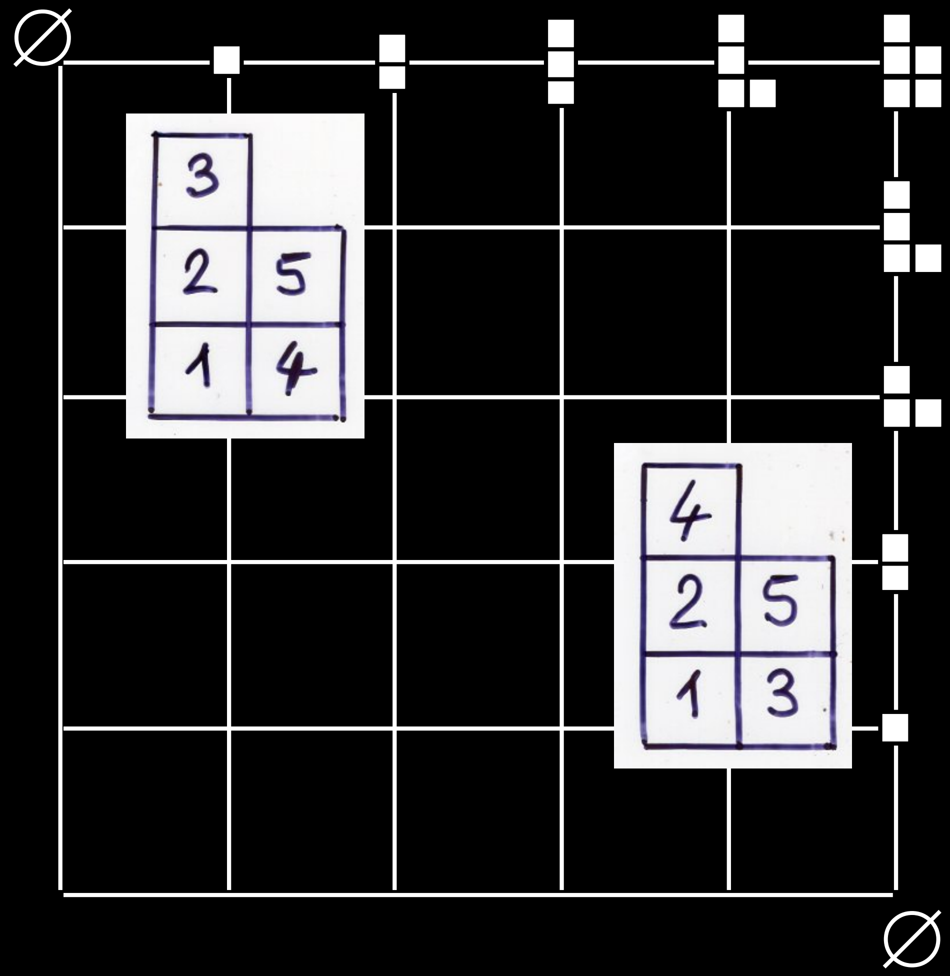
- these maximal chains encode a pair (P, Q) of Young tableaux having the same shape





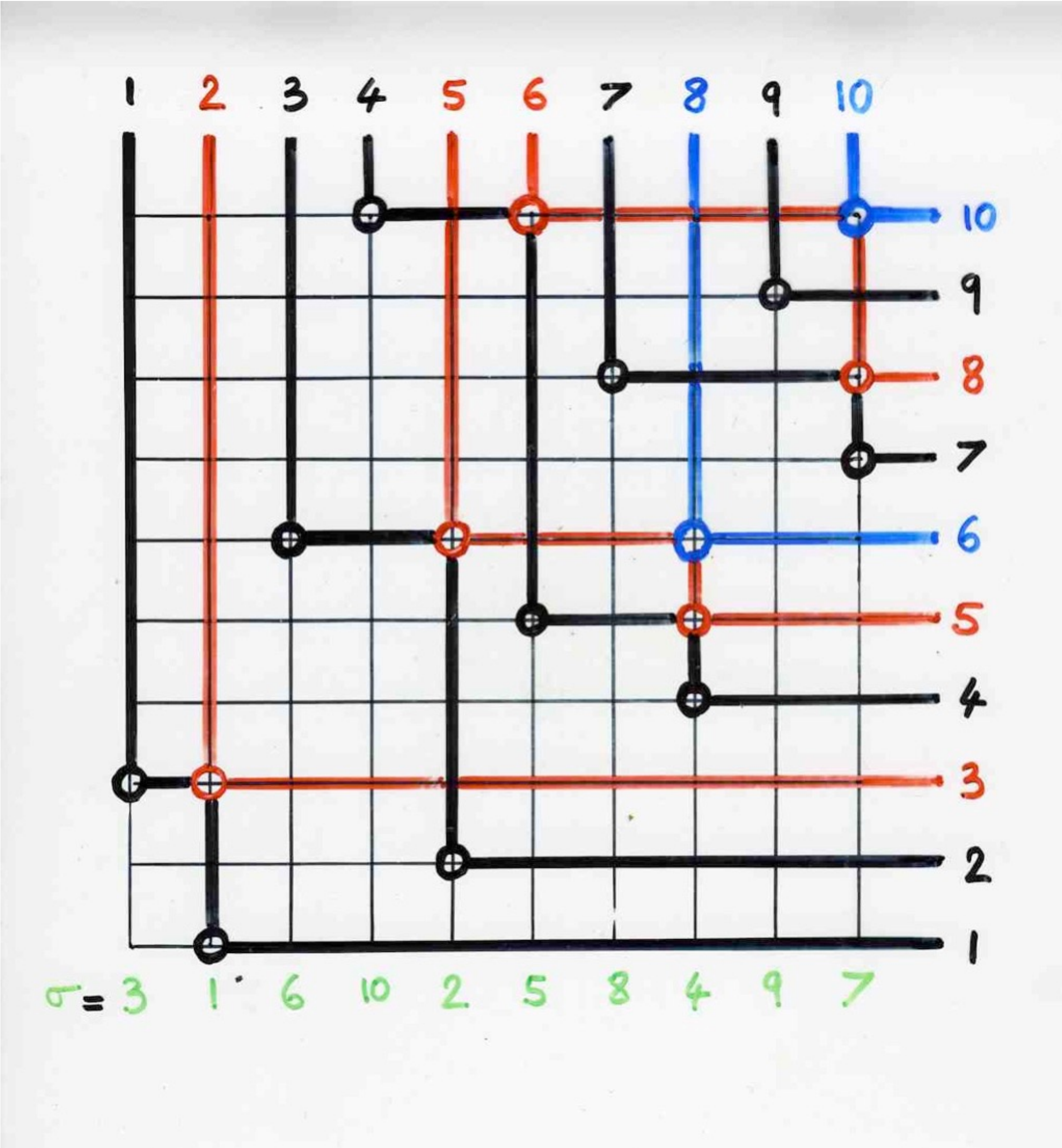
- the algorithm can be reversed: from the pair (P, Q), get back the permutation





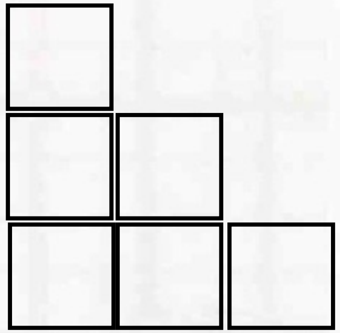
- this bijection is the same as the Robinson-Schensted correspondence

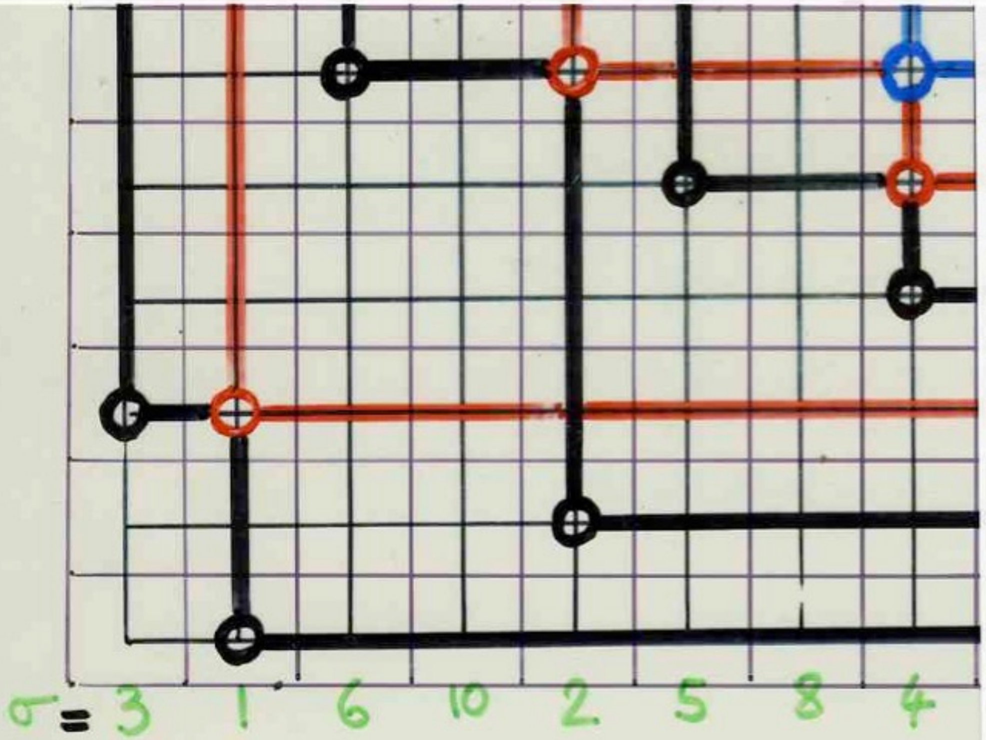
proof of the equivalence
Local RS (growth diagrams) and geometric RS



For any vertex of the grid translated by 1/2 we define a Ferrers diagram in the following Way

We get a tableau of Ferrers diagrams





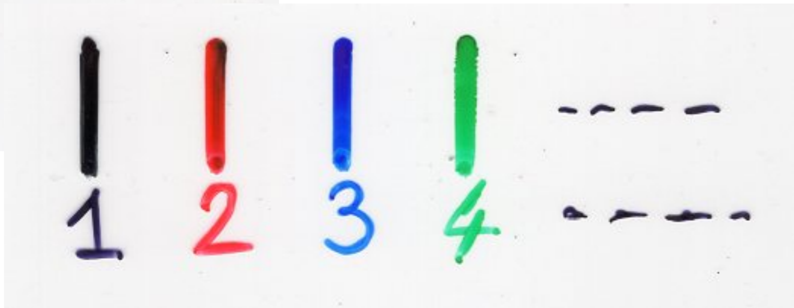
I claim that this tableau is the same as the one we get from Fomin growth diagrams

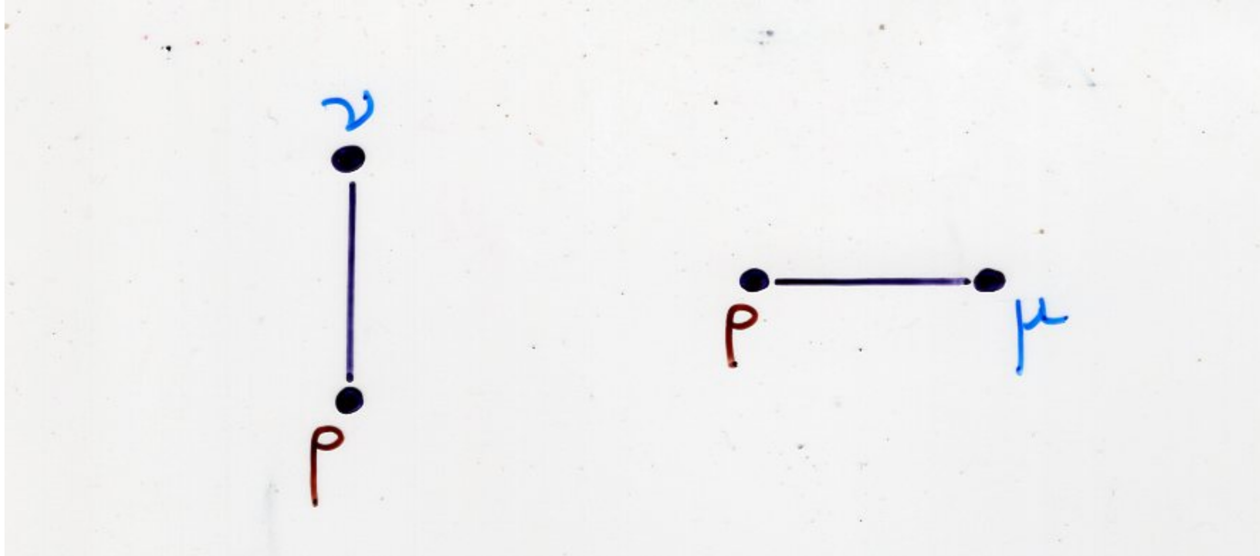
- label the first set of "shadow lines" of the permutation σ by ①
(black lines on the figure)

- then by ② the second set, i.e. the "shadow lines" of the skeleton
$Sq(\sigma)$
(the red lines)

- etc, ~ ③ the blue lines of $Sq(Sq(\sigma))$

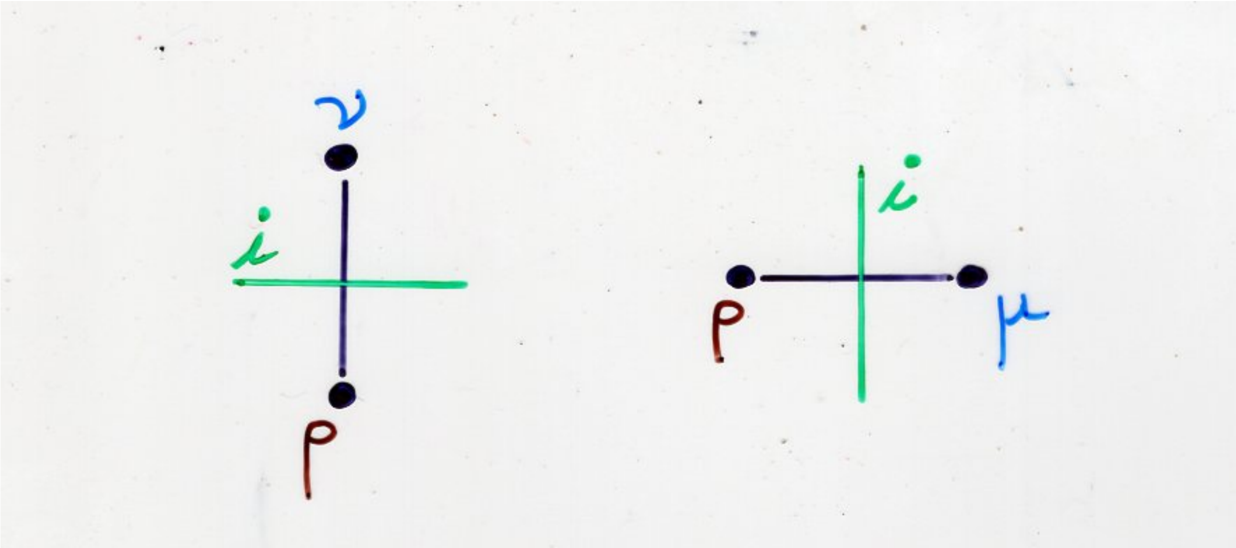
- ...





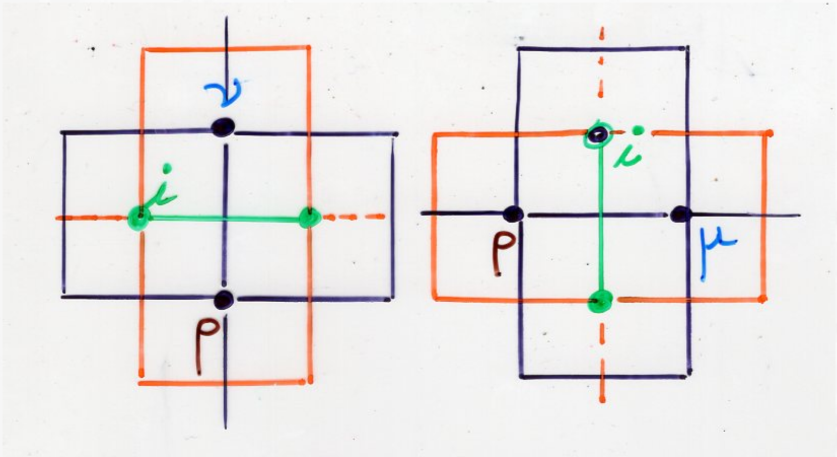
if no shadow lines are crossing, then

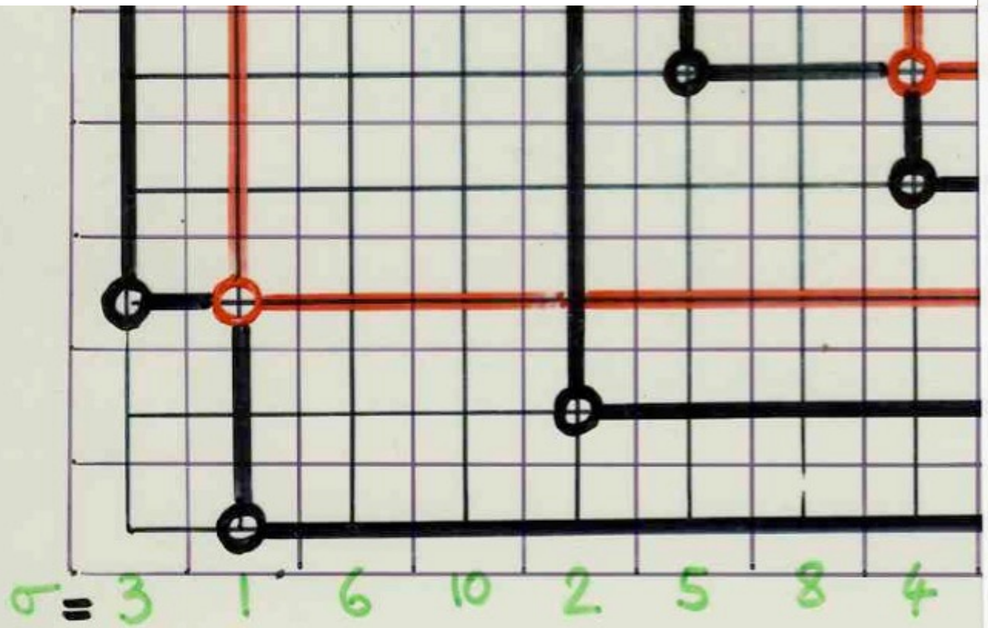
$$\frac{\mu}{\nu} = \rho.$$

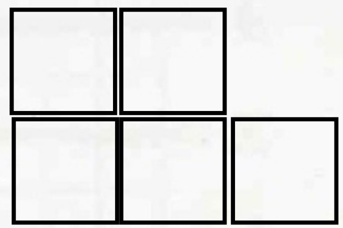


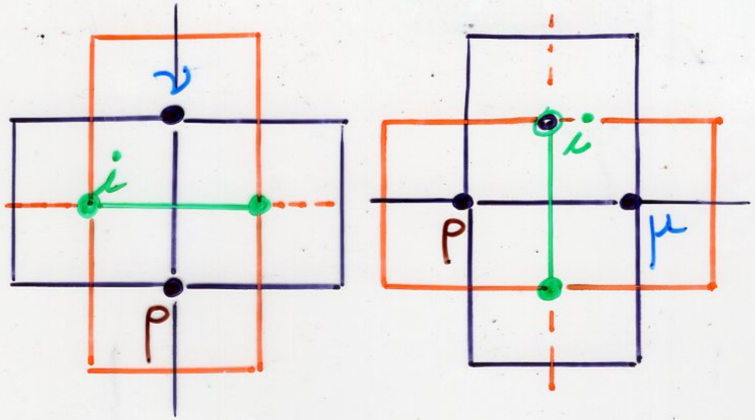
if a shadow line with label i is crossing, then

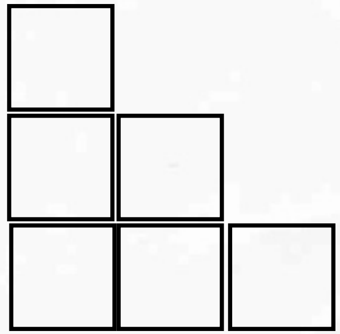
$$\genfrac{}{}{0pt}{}{\mu}{\nu} = \rho + (i)$$

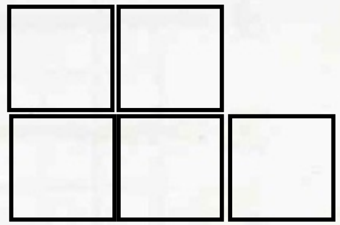

ν
i
ρ
i
ρ
μ

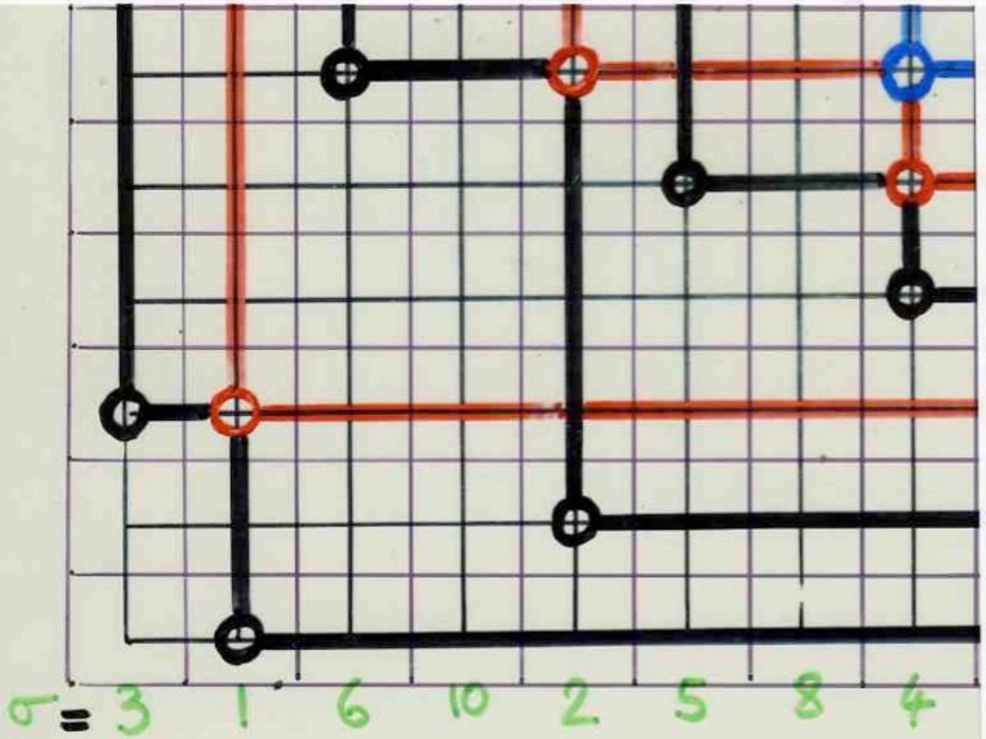

σ = 3 1 6 10 2 5 8 4



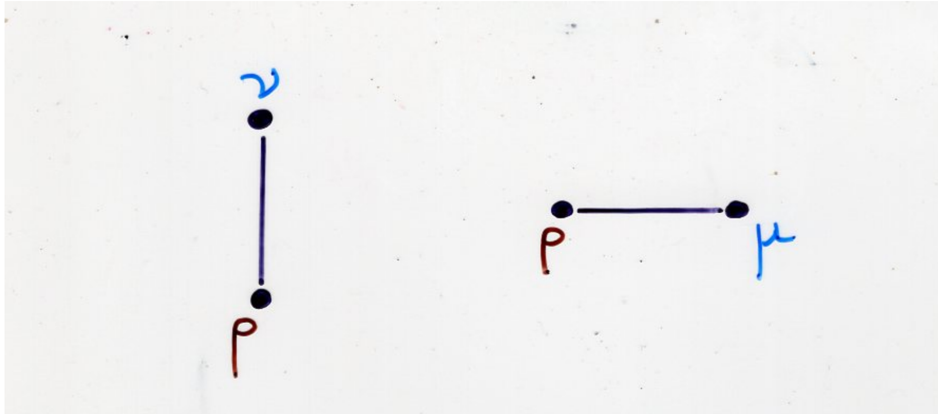


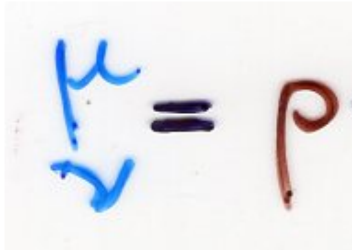


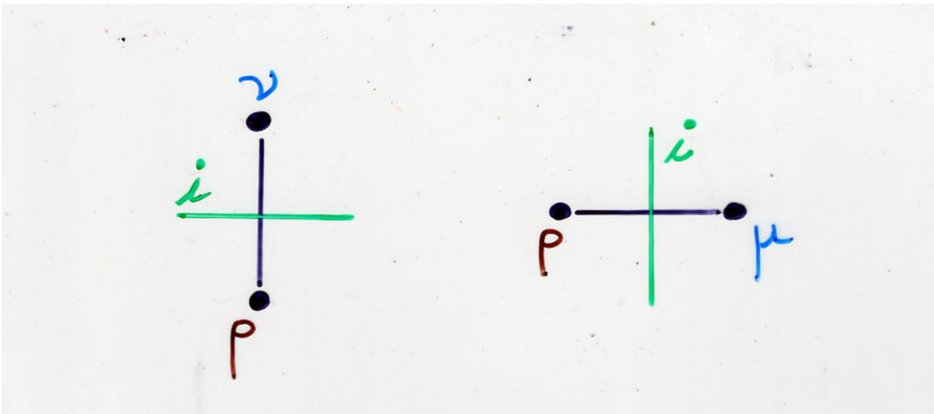


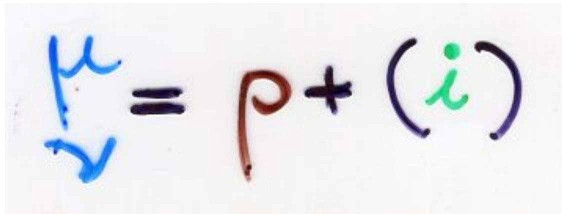


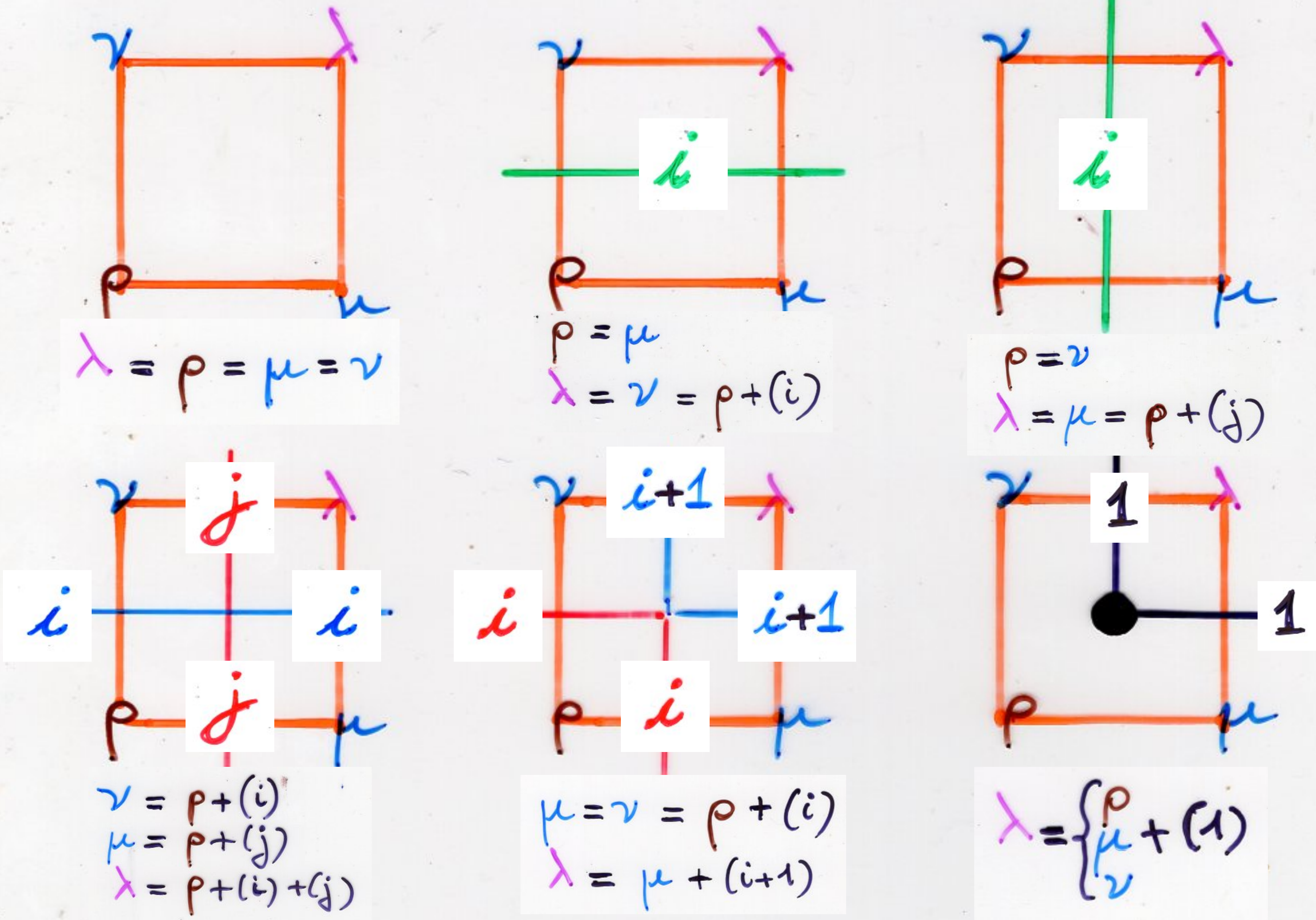
$$\frac{\mu}{\nu} = \rho + (i)$$


ν
ρ
ρ
μ


μ
ν
= ρ

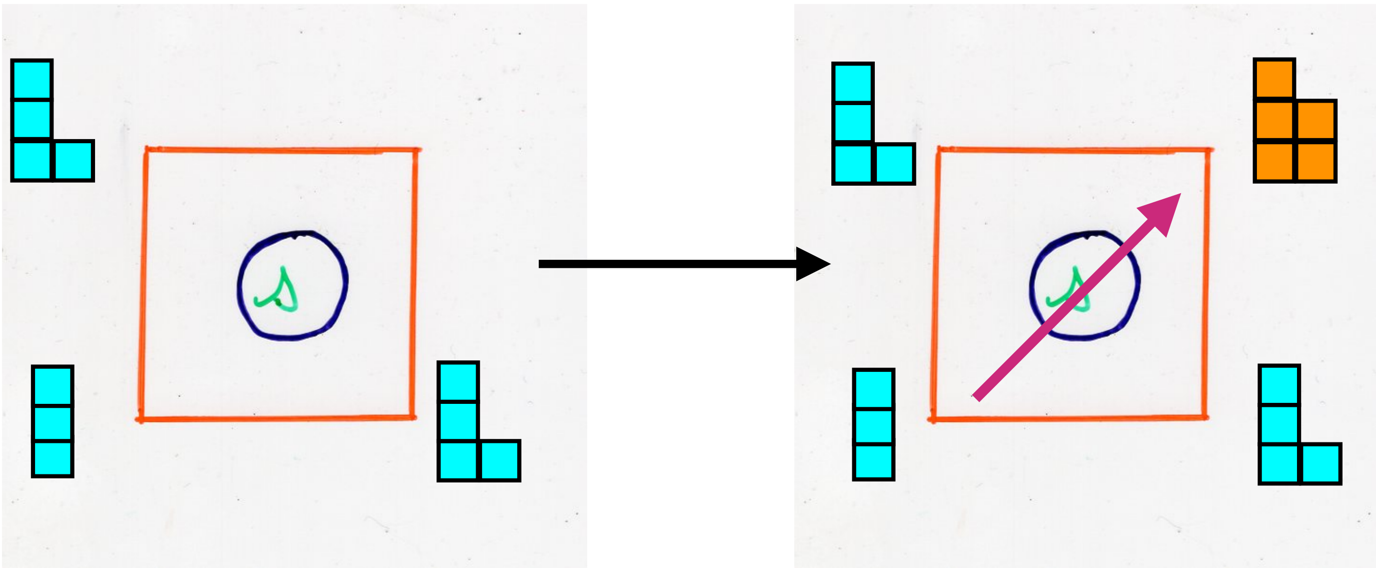

ν
i
ρ
i
ρ
μ


μ
ν
= ρ + (i)


ν λ ρ μ
λ = ρ = μ = ν
ν λ i ρ μ
ρ = μ
λ = ν = ρ + (i)
ν λ i ρ μ
ρ = ν
λ = μ = ρ + (j)
ν j λ i i ρ j μ
ν = ρ + (i)
μ = ρ + (j)
λ = ρ + (i) + (j)
ν i+1 λ i i+1 ρ i μ
μ = ν = ρ + (i)
λ = μ + (i+1)
ν 1 λ 1 ρ μ
λ = {ρ, μ, ν} + (1)

edge local rules

Fomin's "local rules"
"growth diagrams"



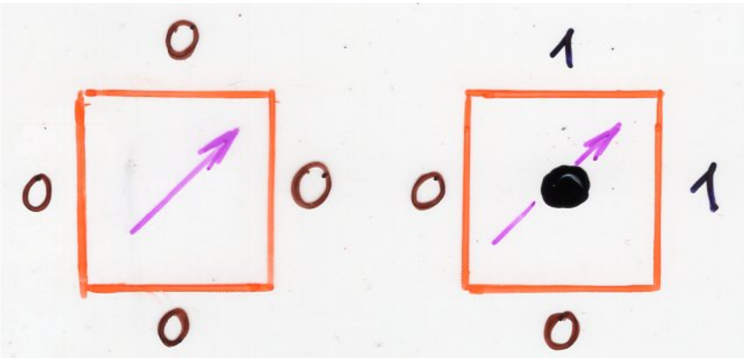
"local rules"
on the vertices

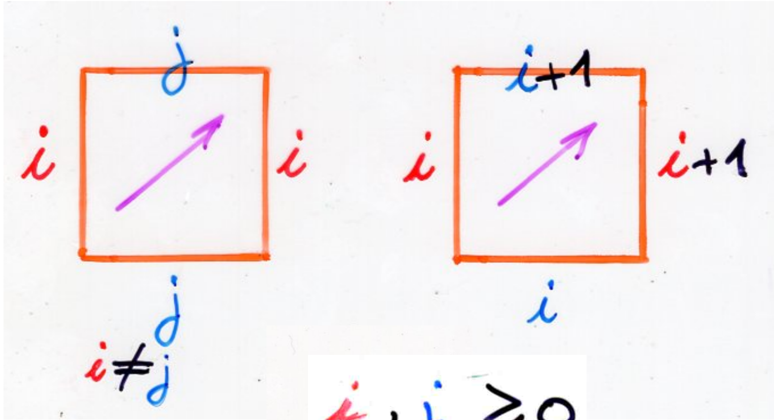
"local rules" on the edges

state {0, 1, 2, ...}

state {0, 1, 2, ...}

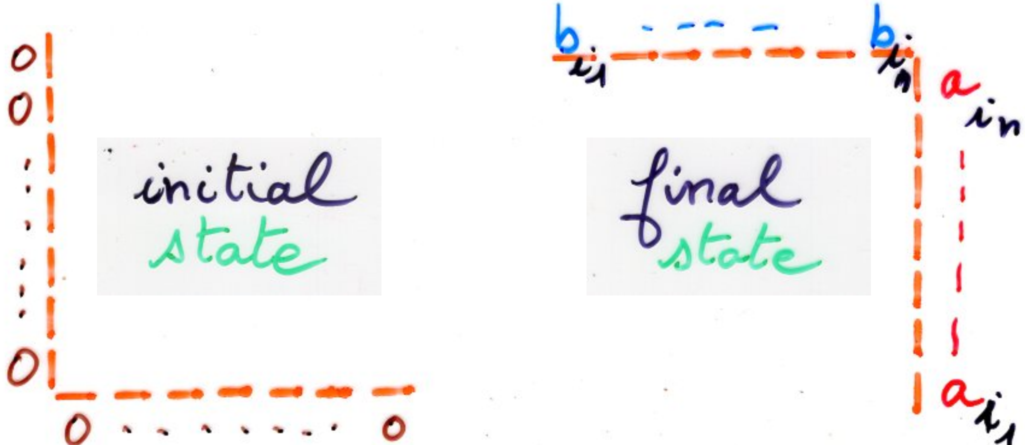
set of labels $L = \{\square, \boxdot\}$

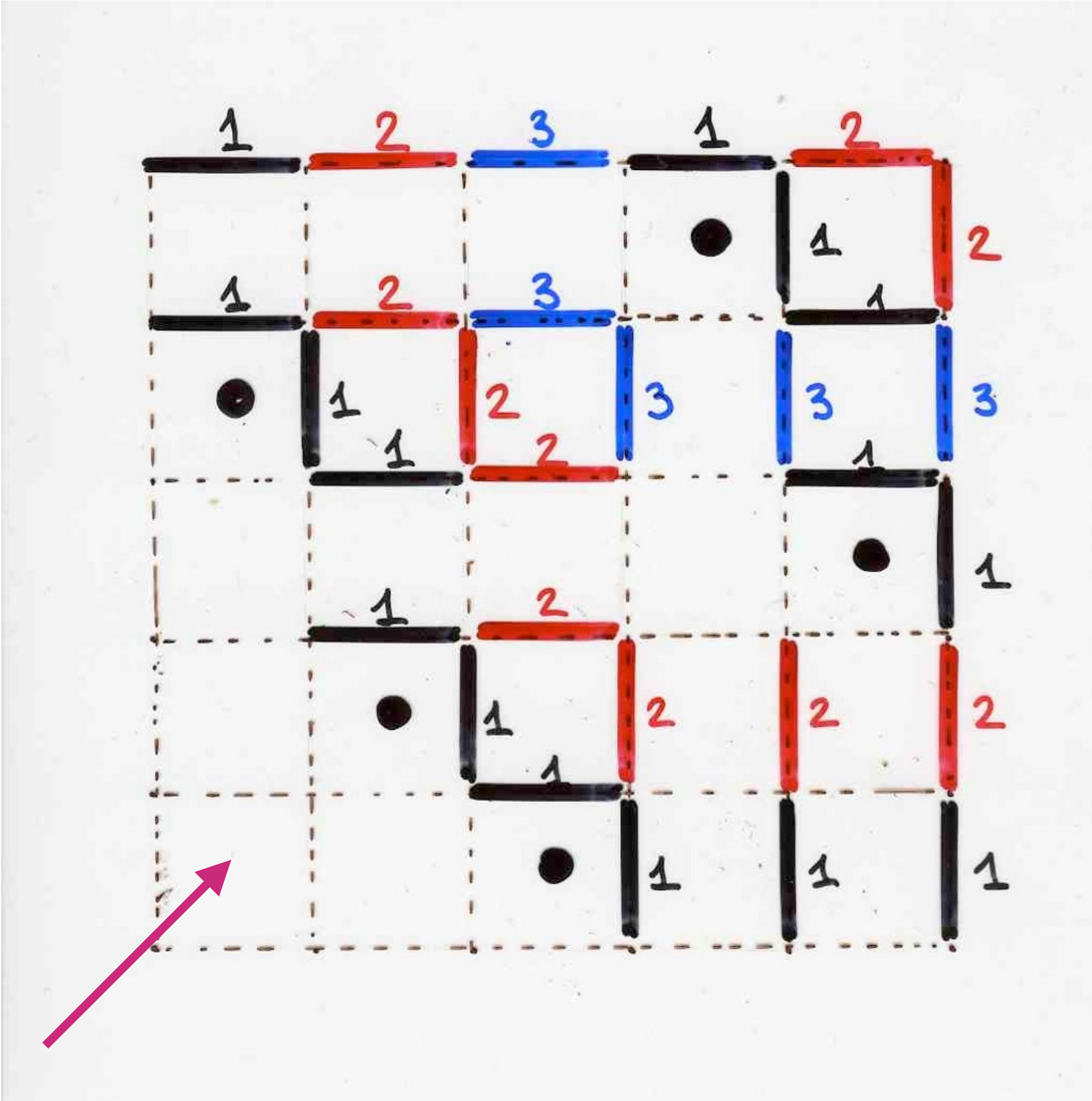




$i, j \geq 0$

"planar rewriting"





1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1

Definition Yamanouchi word w

$w \in \{1, 2, \dots\}^*$

free monoid generated by the alphabet $1, 2, \dots,$

such that:

for every factorization $w = uv$

$$|u|_1 \geq |u|_2 \geq \cdots \geq |u|_i \geq \cdots$$

↑ number of occurrences of the letter i in u

coding of a Young tableau with a Yamanouchi word

(also called lattice permutation)

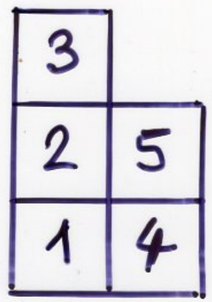
$$= 1\ 2\ 1\ 1\ 2\ 2\ 1\ 3\ 1\ 3$$

$$w = |\ |\ |\ |\ |\ |\ |\ |\ |\ |$$

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10$$

Q =

8	10			
2	5	6		
1	3	4	7	9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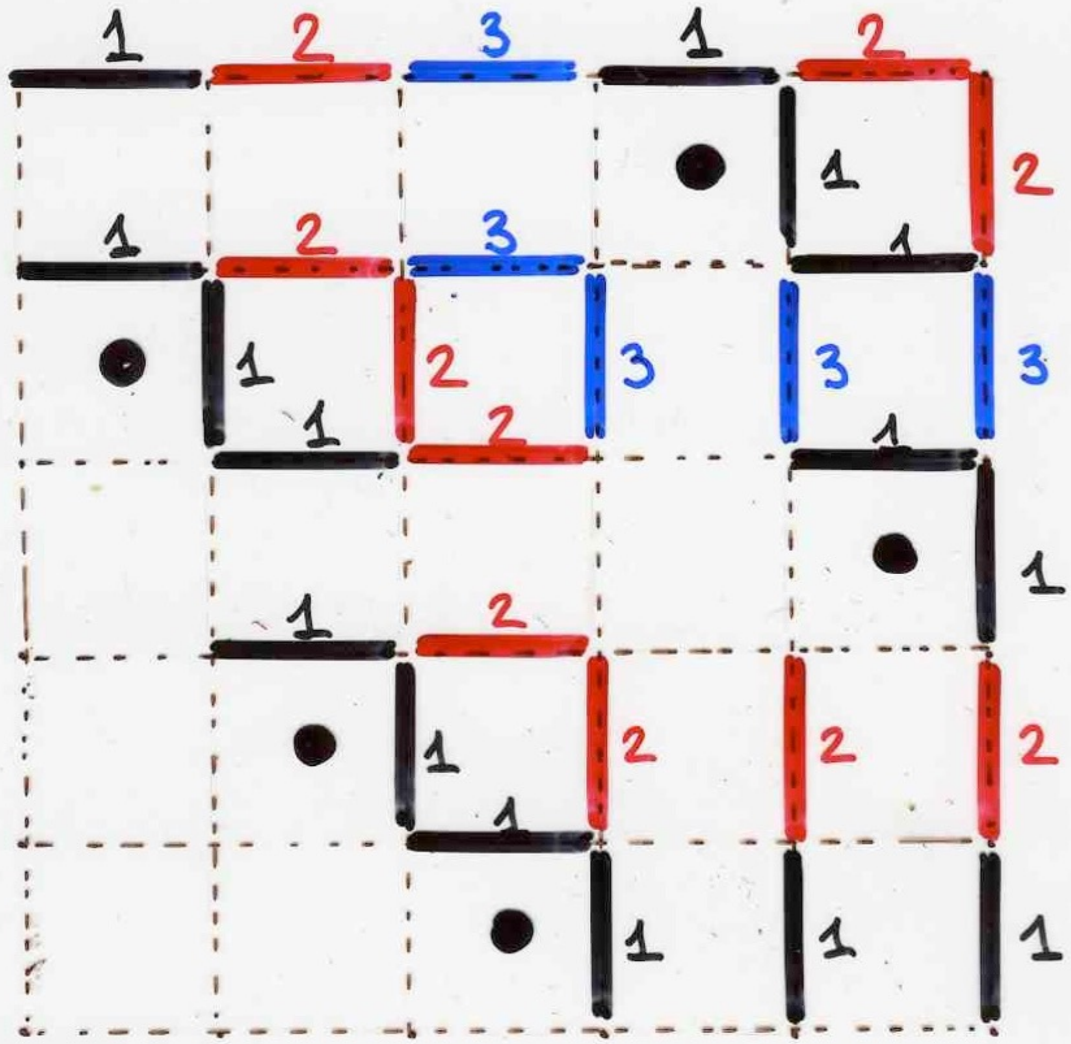
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1 2
1 2 2 2
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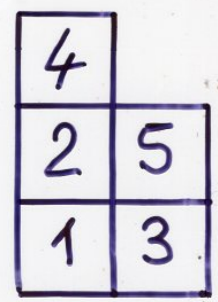
1 1 1


4
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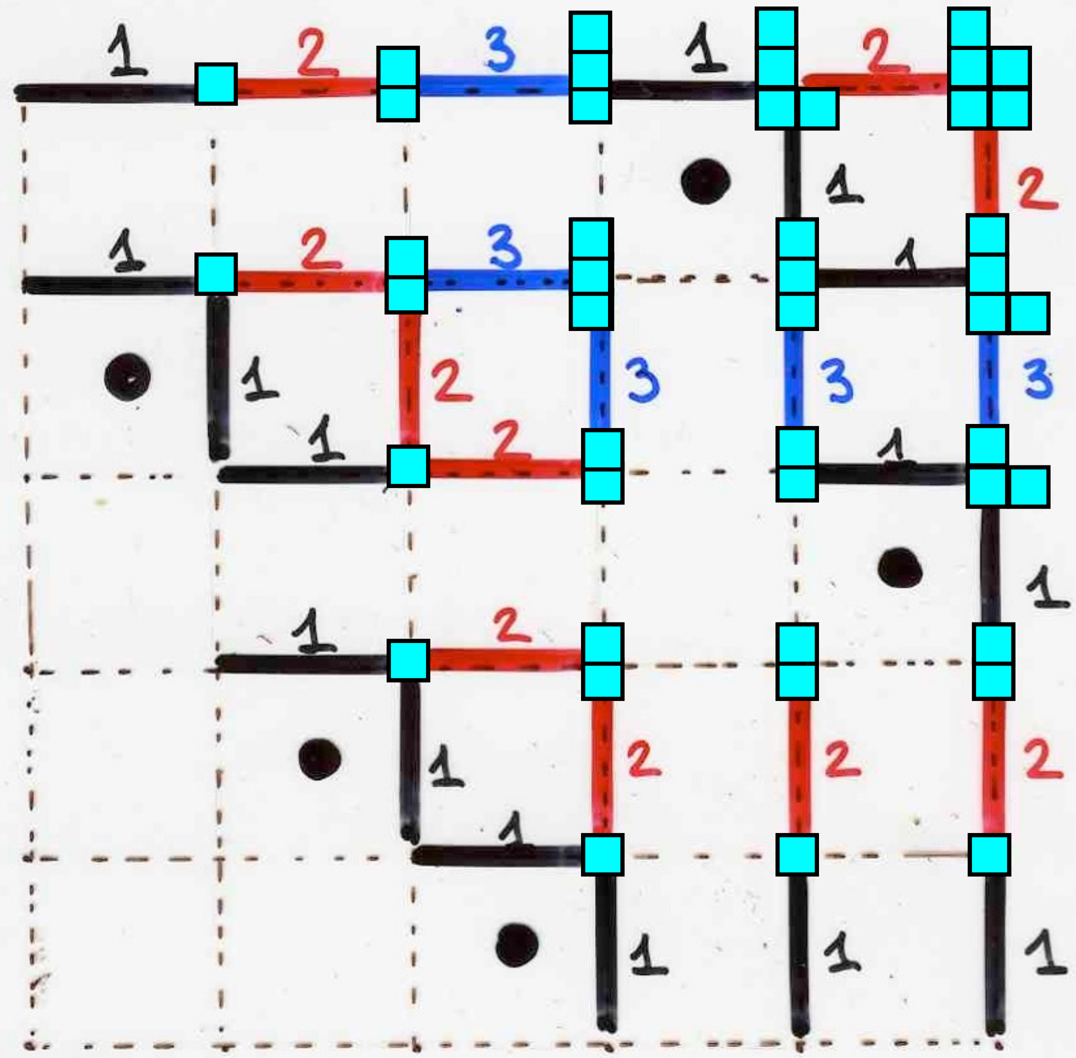
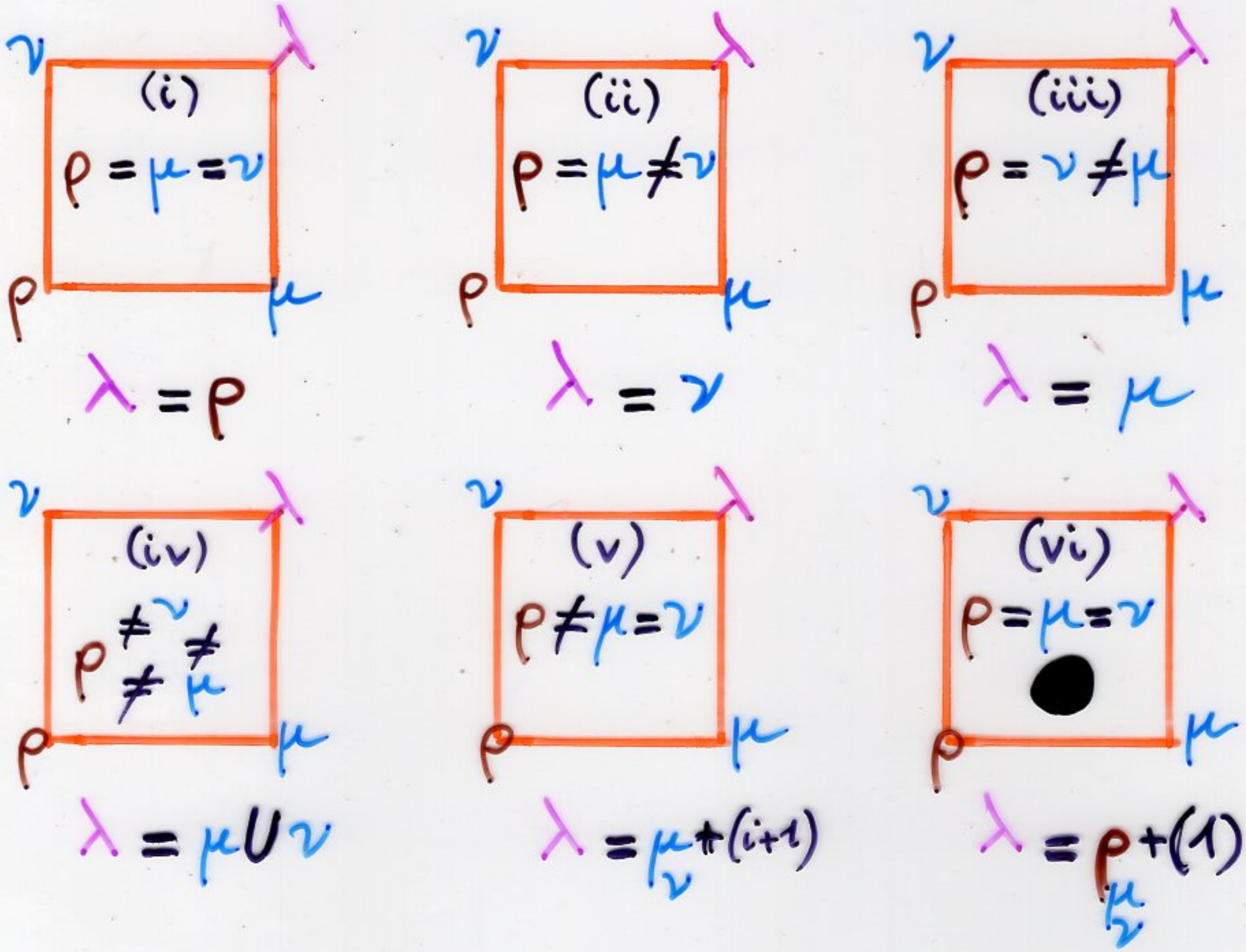
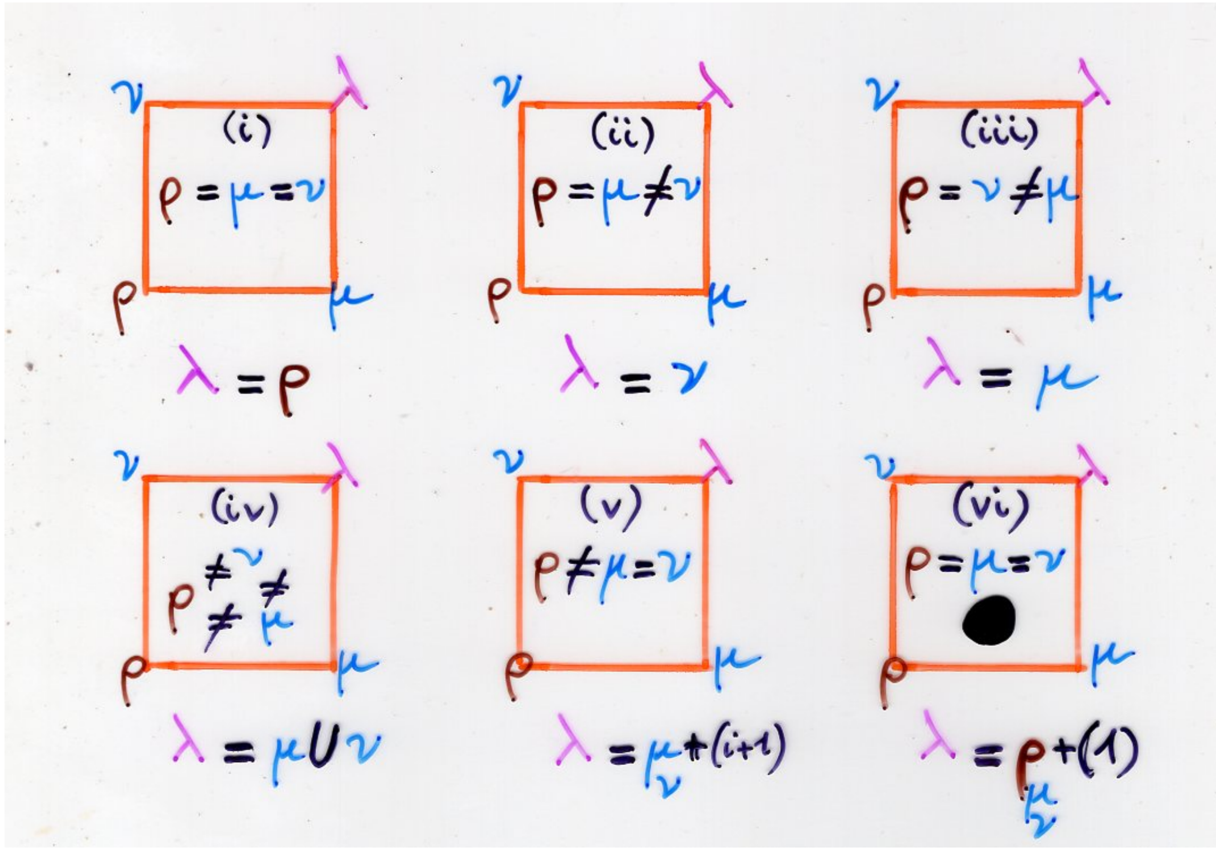
Proposition

The two processes « growth diagrams » and « edge local rules » are equivalent

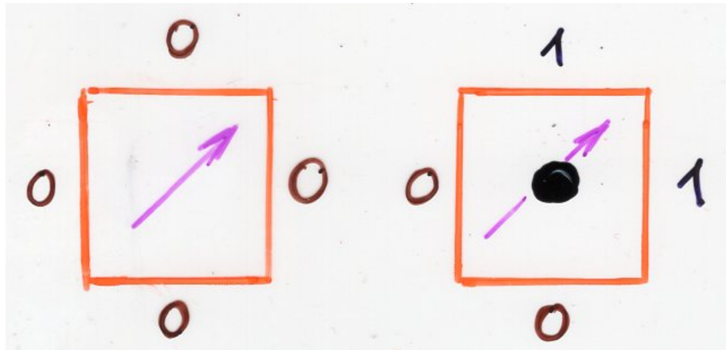


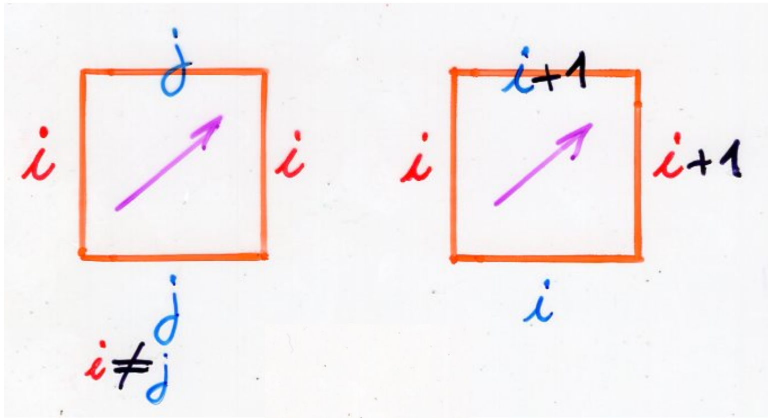

(i)
ρ = μ = ν
λ = ρ
(ii)
ρ = μ ≠ ν
λ = ν
(iii)
ρ = ν ≠ μ
λ = μ
(iv)
ρ ≠ ν ≠ μ ≠ ρ
λ = μ ∪ ν
(v)
ρ ≠ μ = ν
λ = μ/ν + (i+1)
(vi)
ρ = μ = ν
λ = ρ/μ/ν + (1)

"local rules" on the vertices



"local rules" on the edges





« local rules on vertices »

Marc A. A. van Leeuwen (1996)

The Robinson-Schensted and Schützenberger algorithms, an elementary approach

C.Krattentheler, (2006).

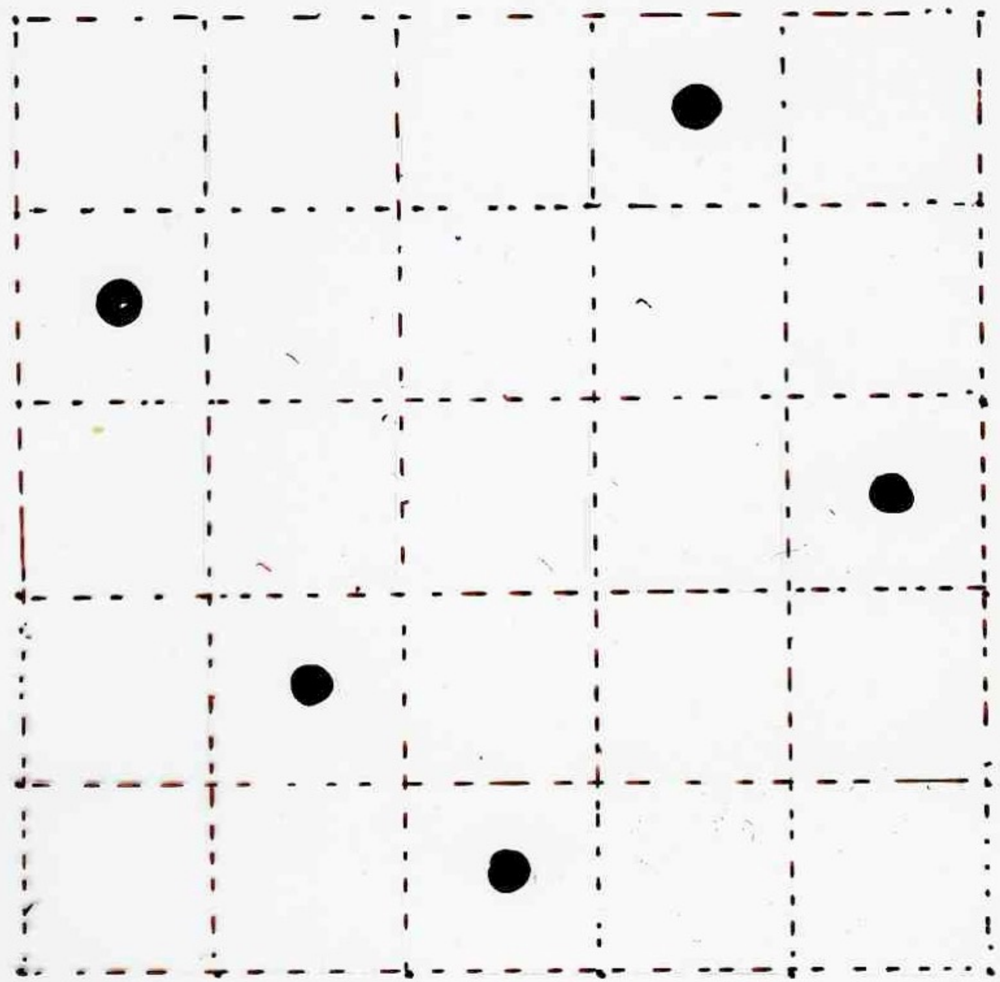
GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

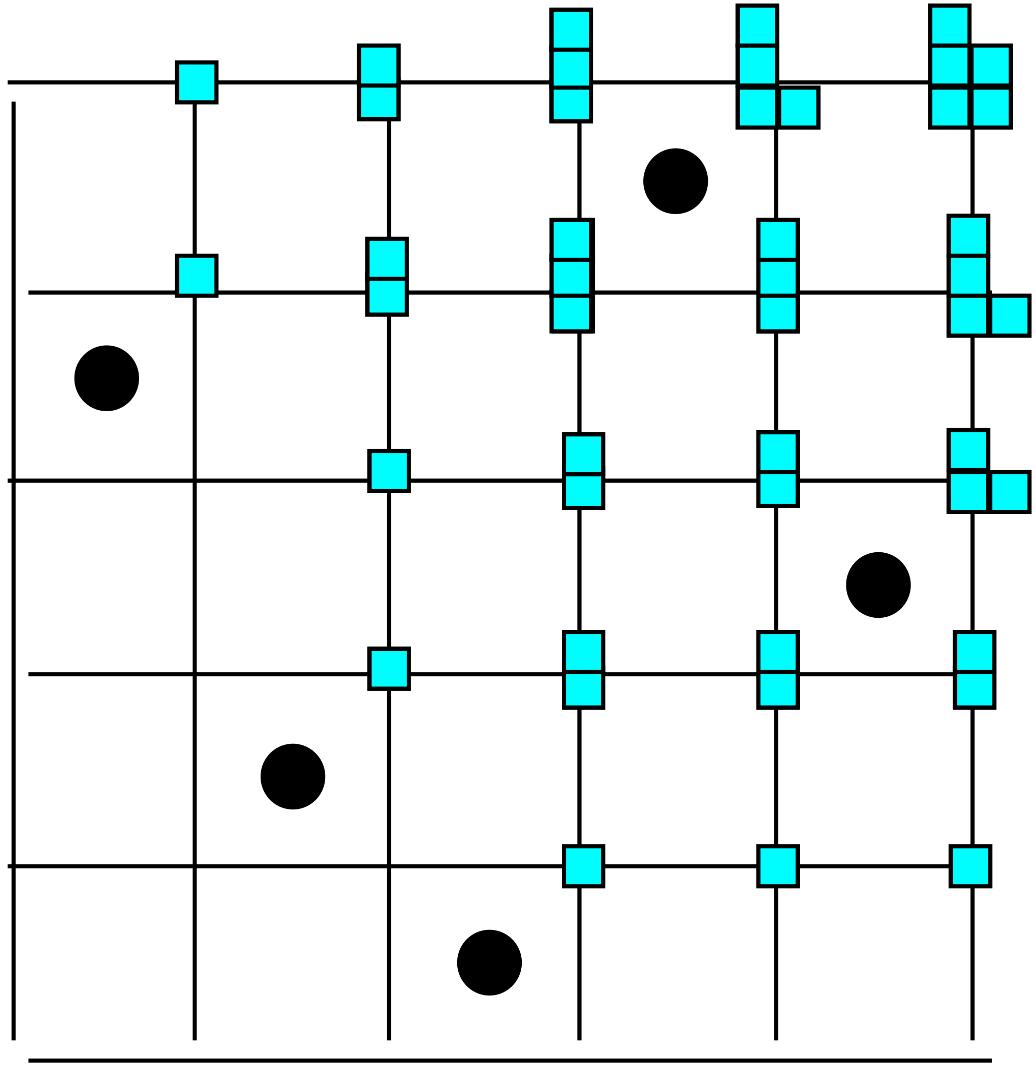
M.Rubey. (2007)

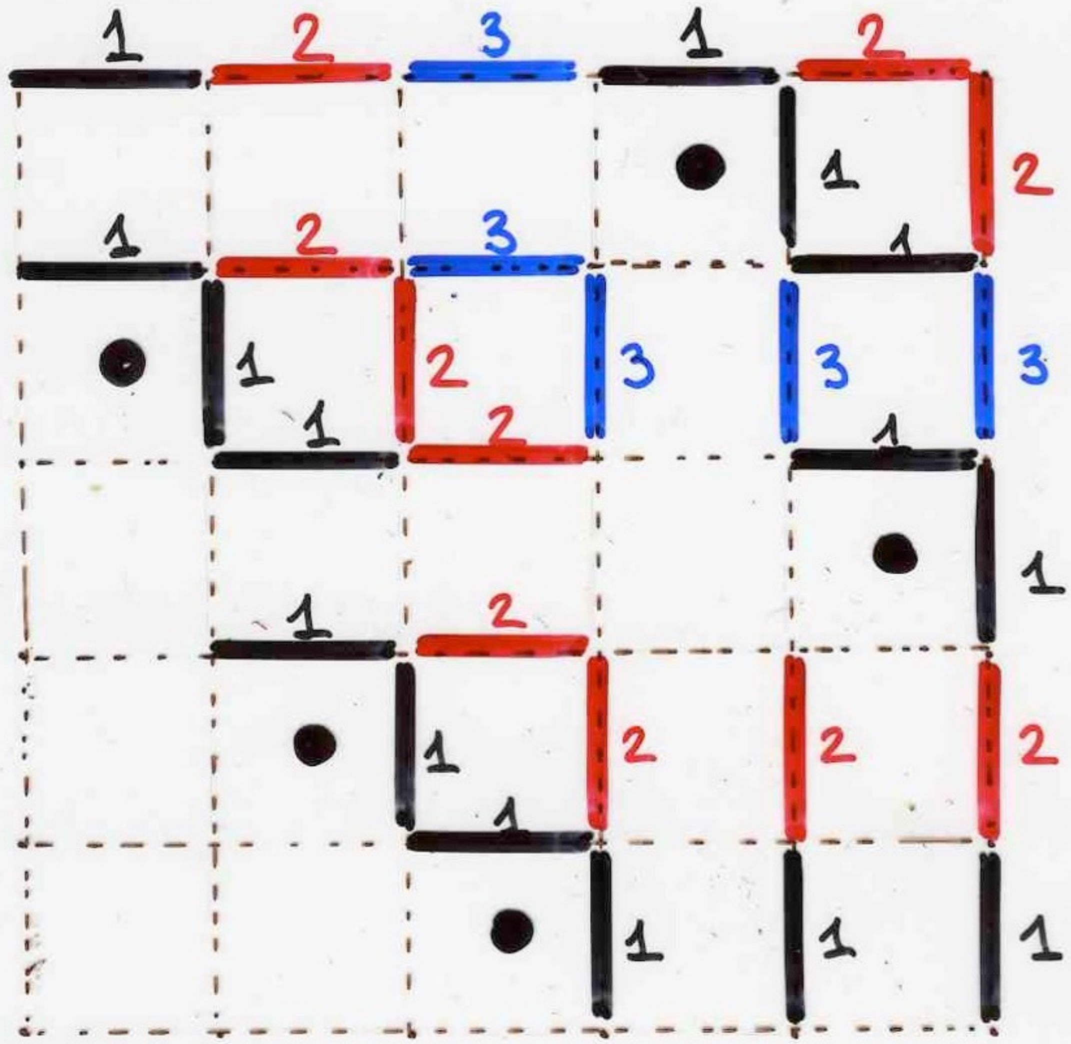
Increasing and Decreasing Sequences in Fillings of Moon Polyominoes

I claim that much attention should be given to the « local rules on edges » rather that « local rules on vertices ».

This is part of the philosophy of the « cellular ansatz »



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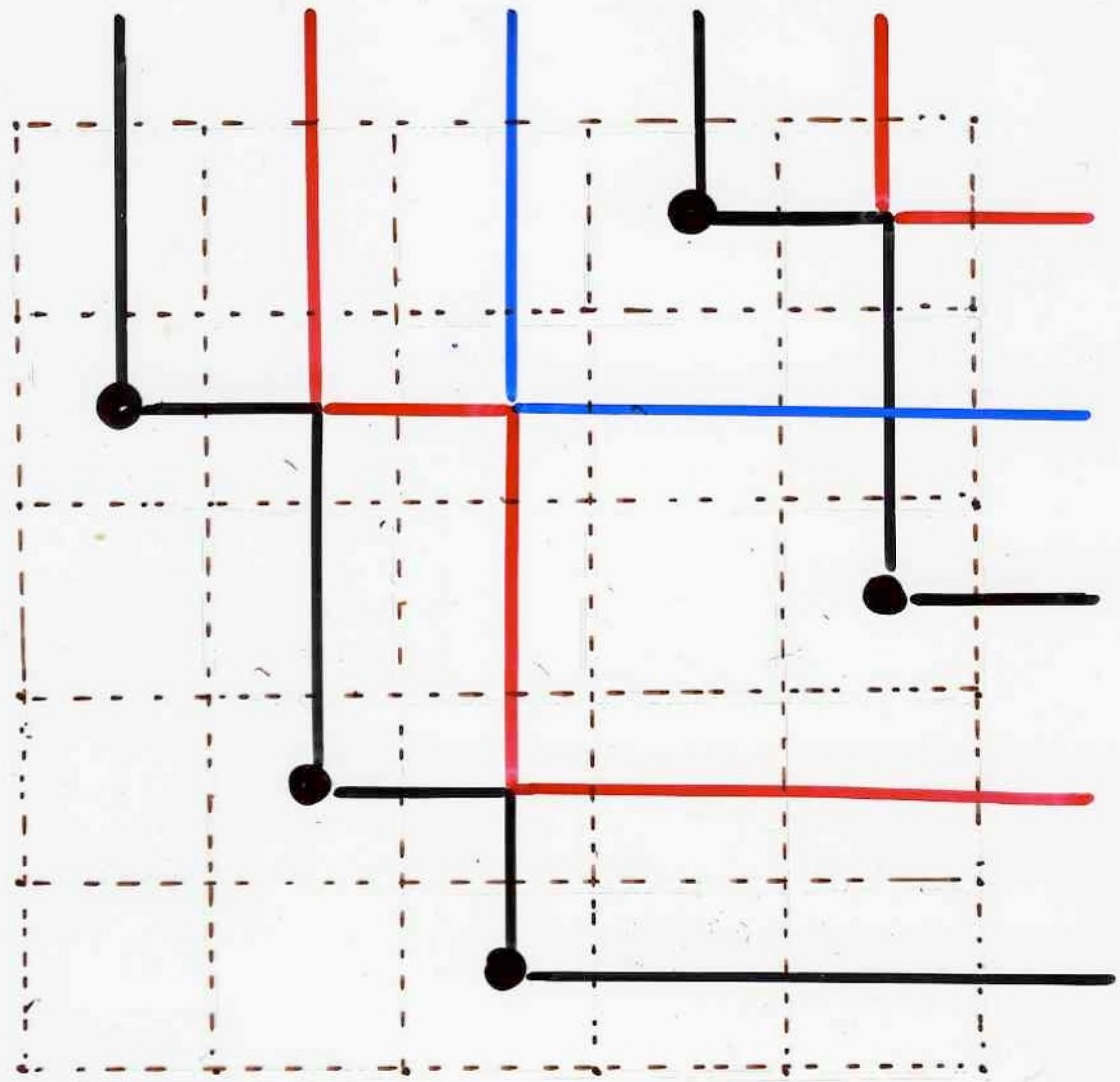
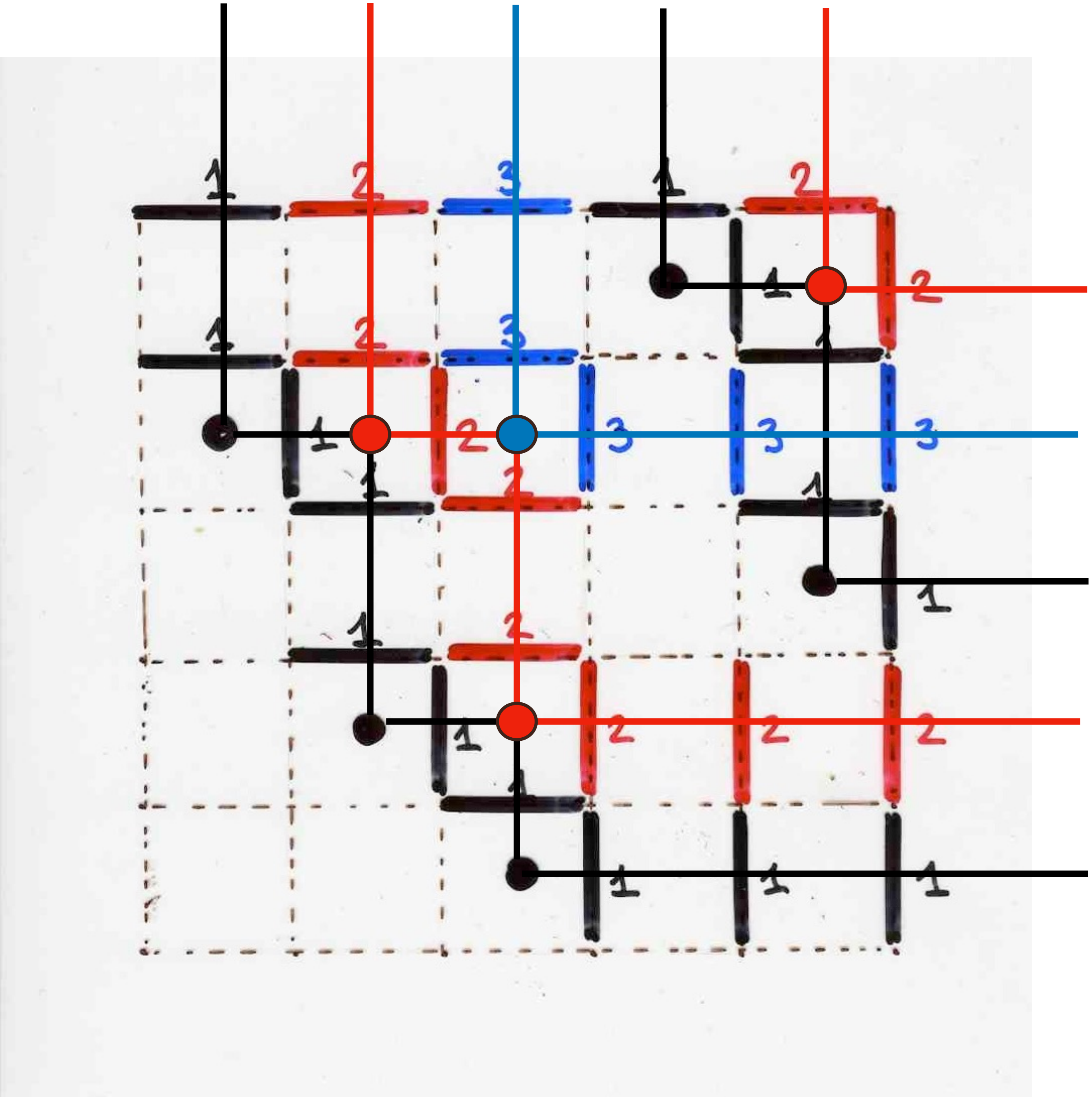
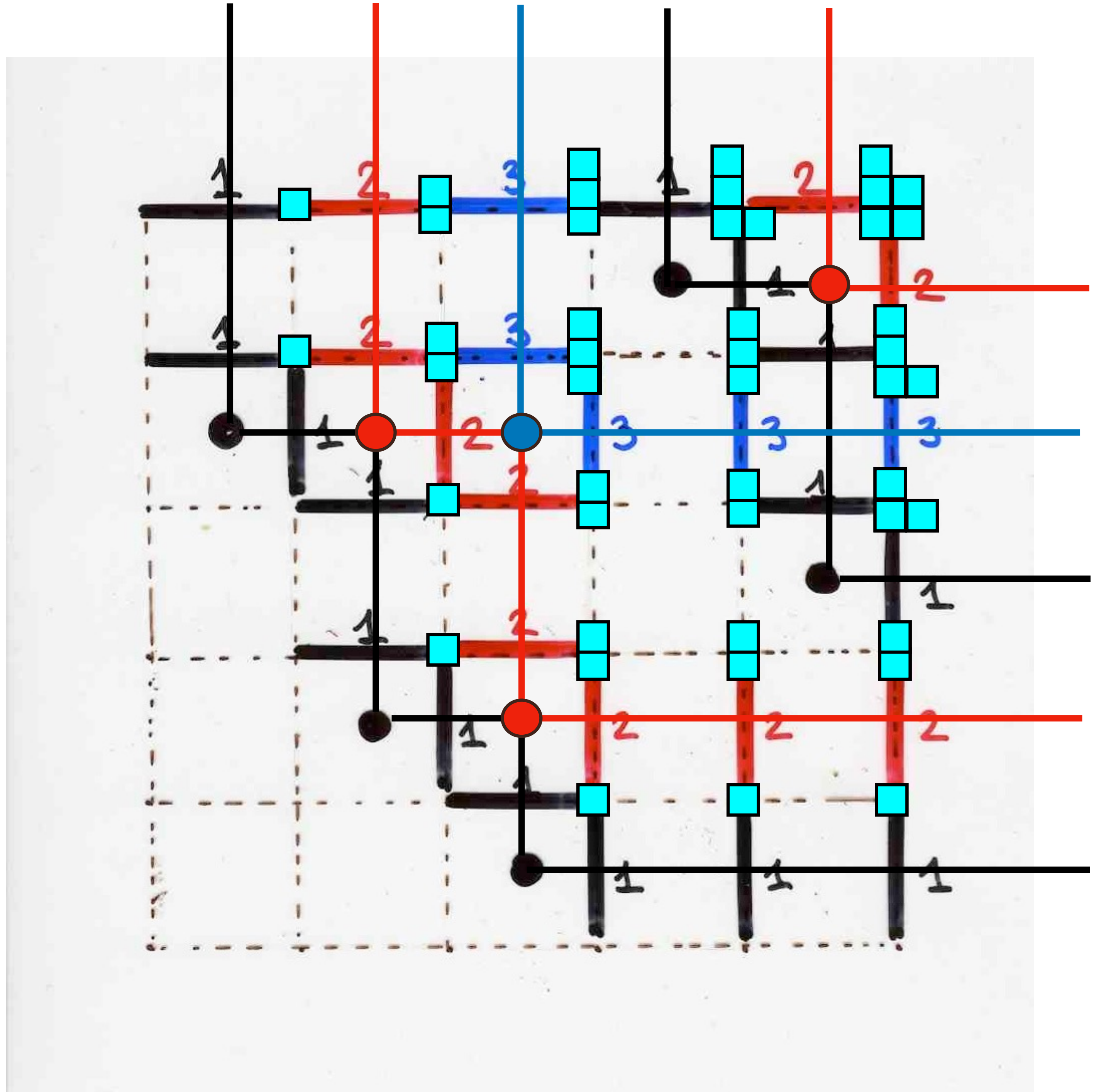
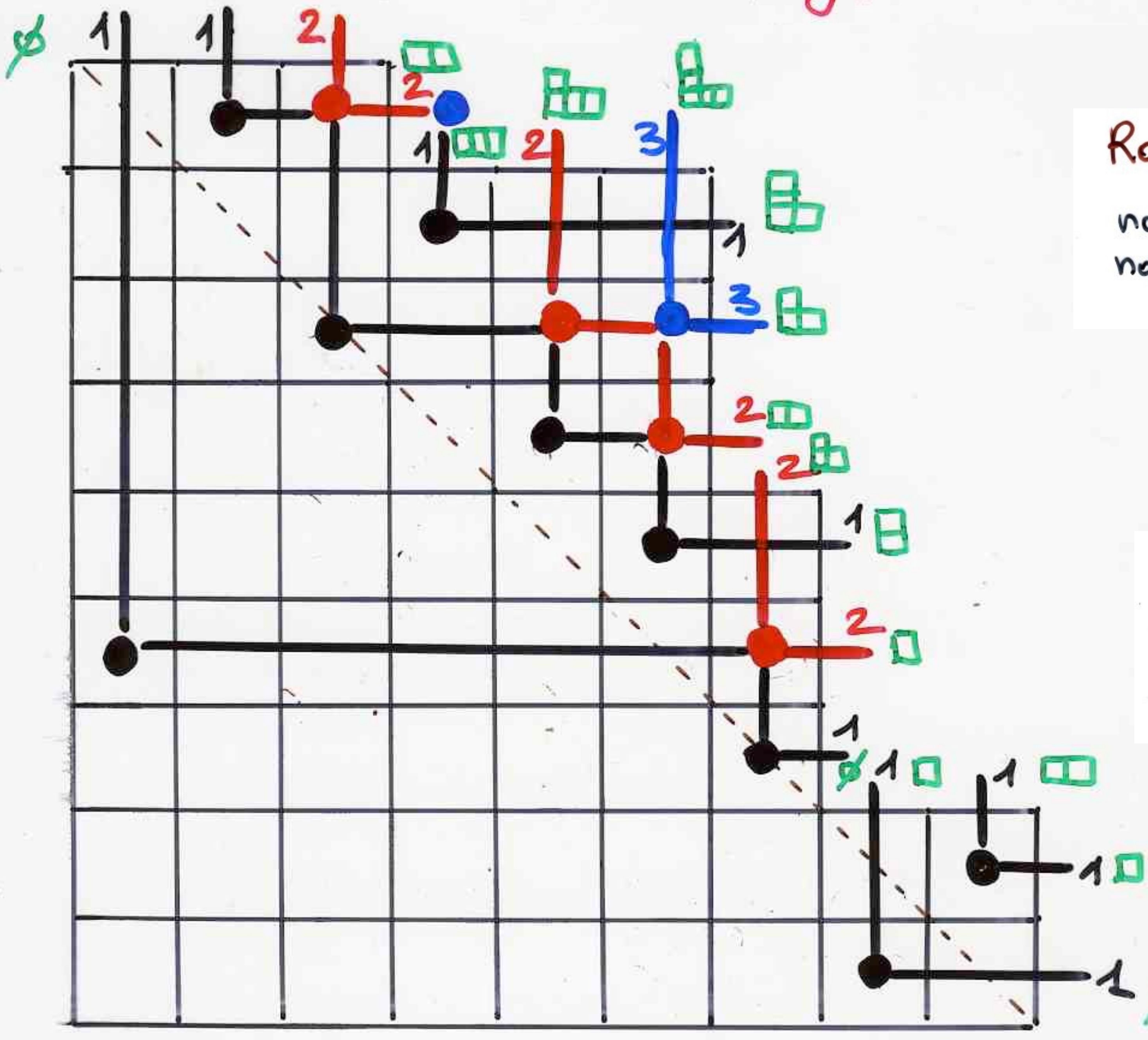






bijections
for rook placements

oscillating tableaux

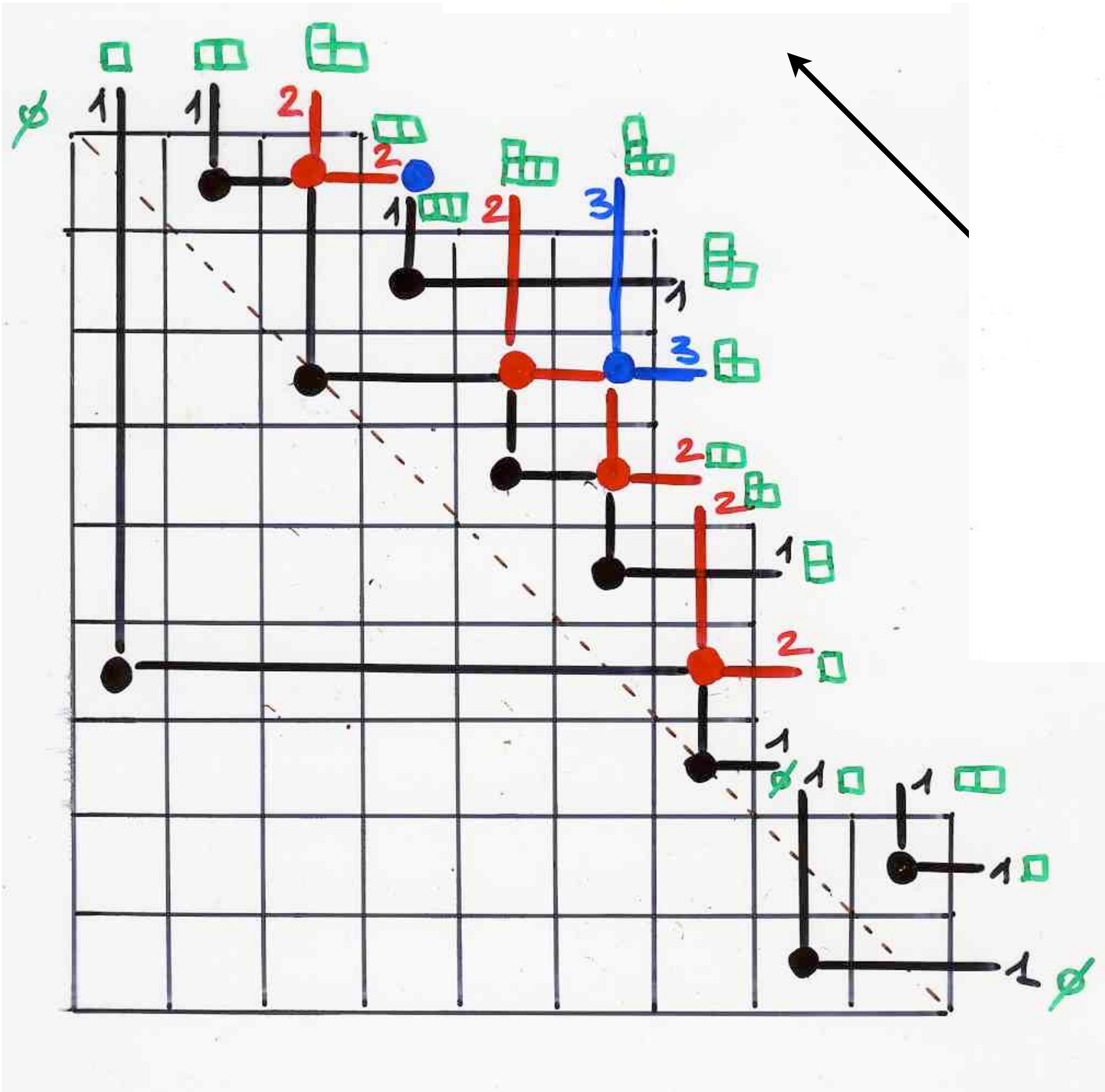


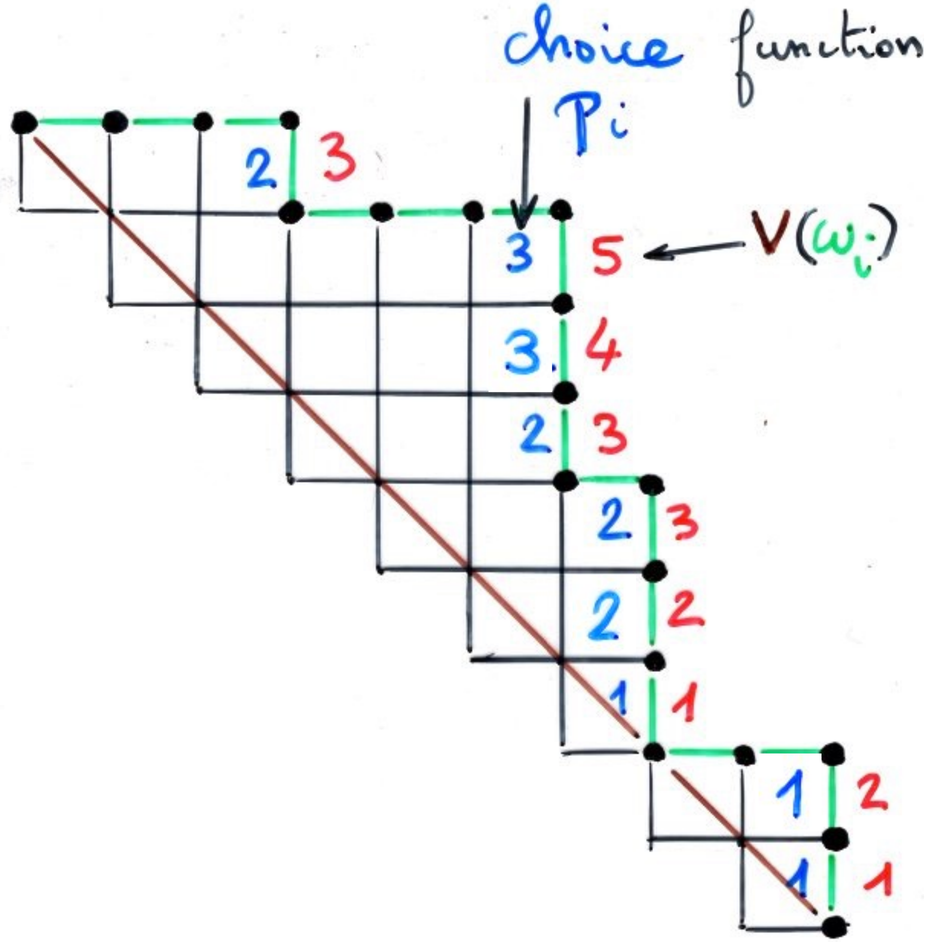
Rook placements with no empty row no empty column

↕

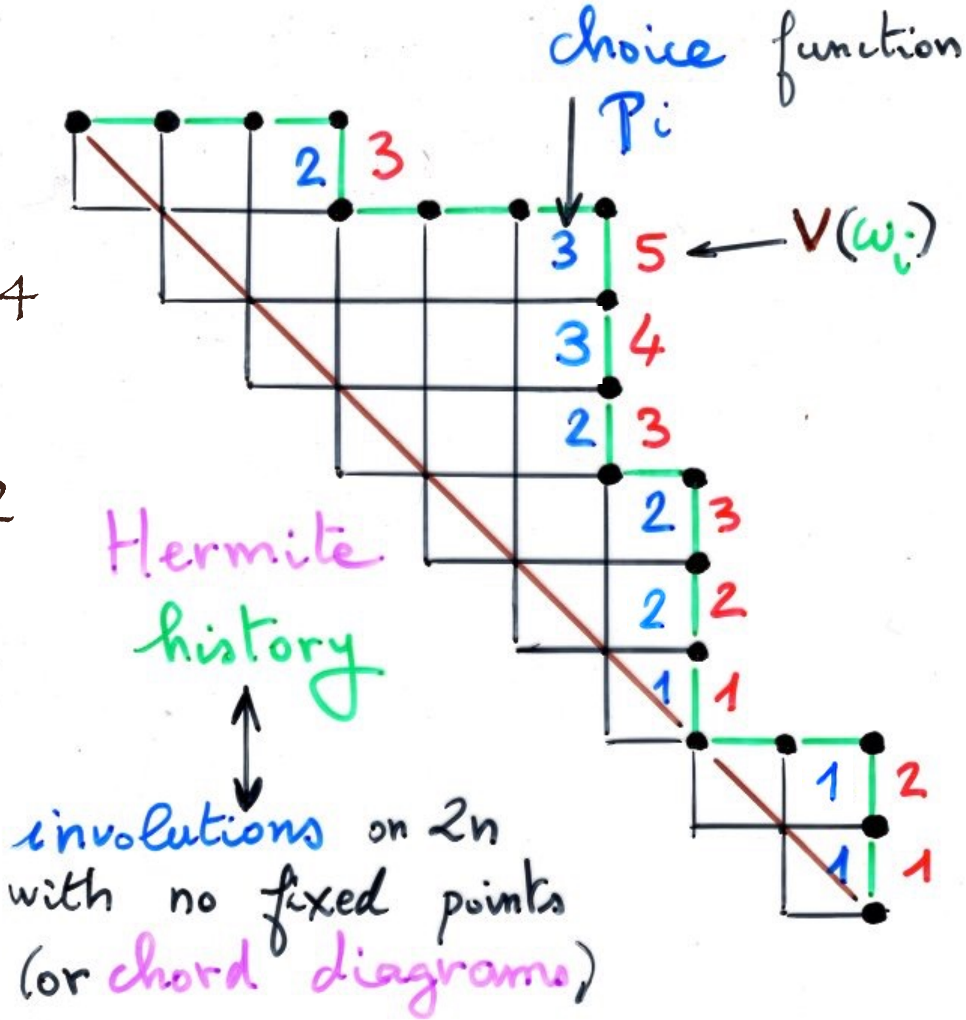
sequences of oscillating tableaux starting and ending at ∅

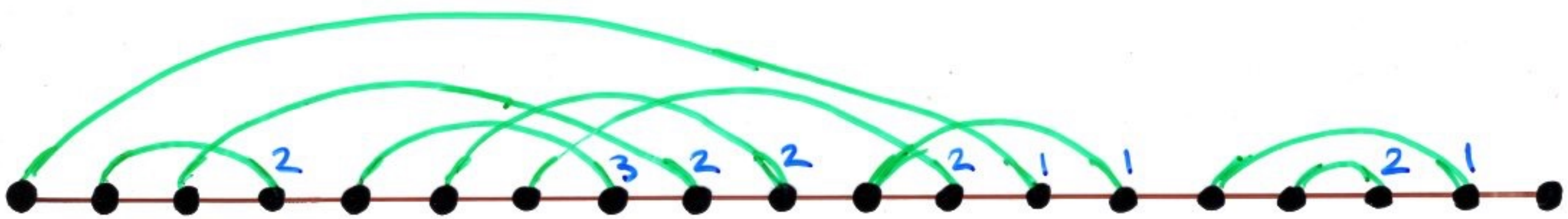
Rook placements
with
no empty row
no empty column





See Tianjin lecture 4
Or
ABjC, Part IV, Ch 2





oscillating tableaux

vacillating tableaux

hesitating tableaux

Chen, Deng, Du, Stanley, Yan (2005)

arXiv:math.CO/0501230. Trans.A.M.S. (2005)

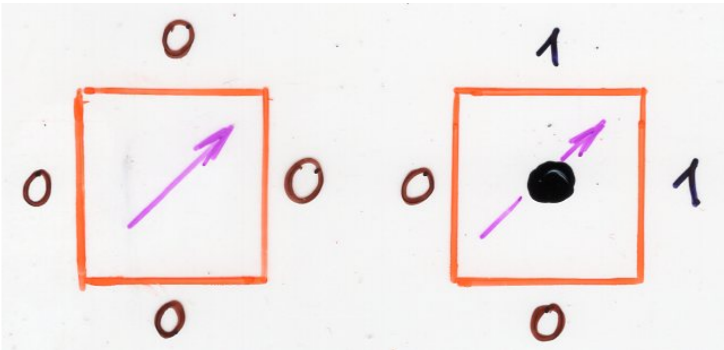
stammering tableaux

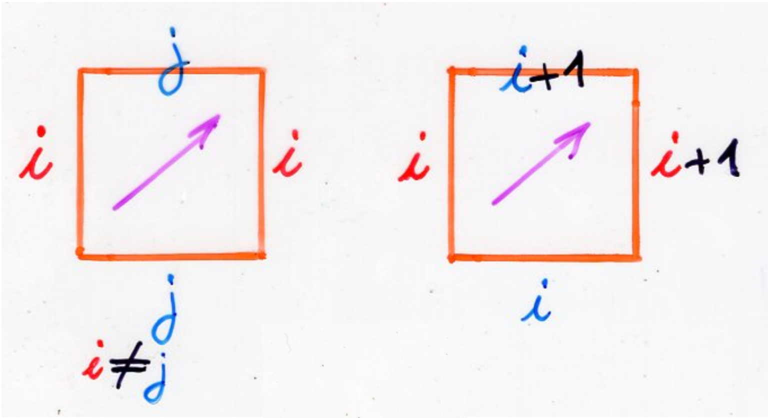
Josuat-Vergès (2012)

Blasiak, Horzela, Penson
Solomon, Duchamp (2007)...

(very !) Quick complements
given at the end of the lecture

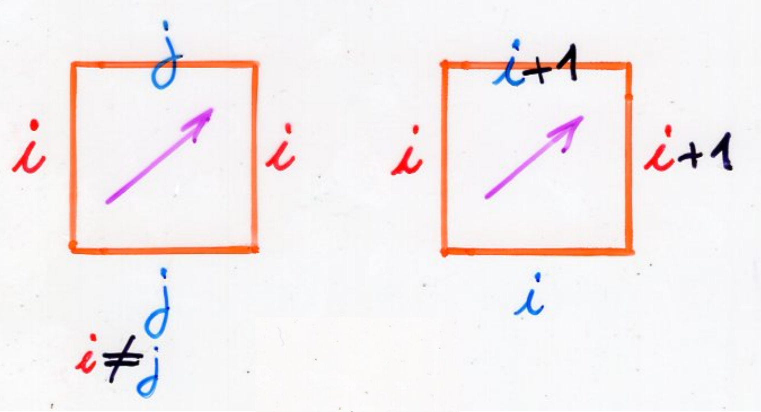
The RSK bilateral edge local rules





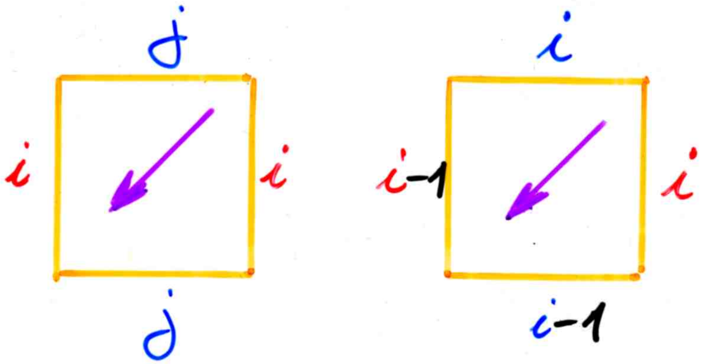
"local rules" on the edges

$i, j \geq 0$



"local rules" on the edges

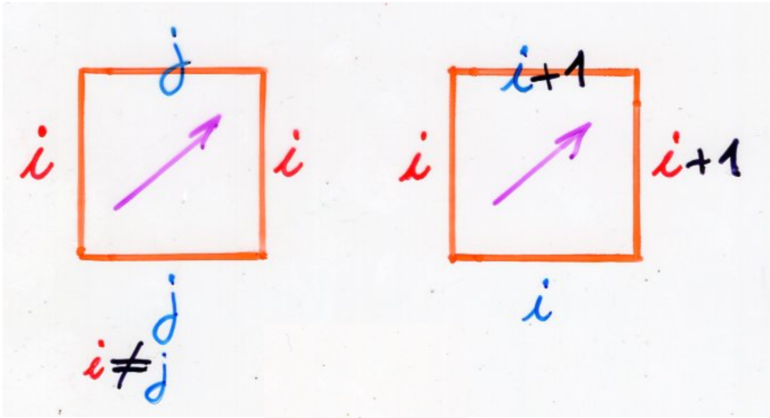
$i, j \geq 0$

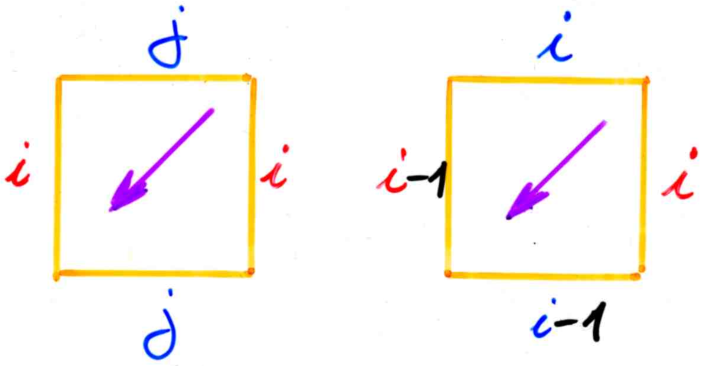


$i \neq j$

$i, j \in \mathbb{Z} - \{0\}$

bilateral
local rules
on edges

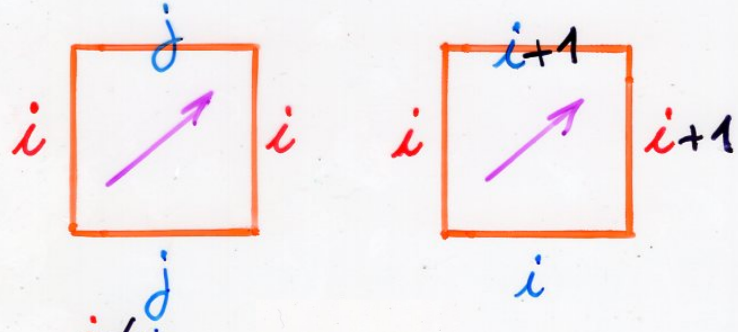




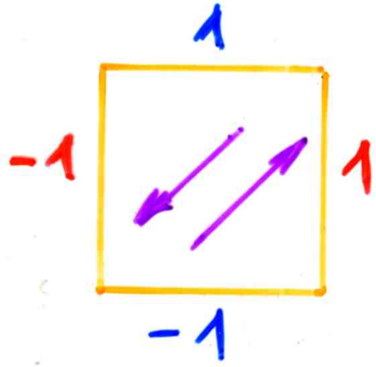
$i \neq j$

bilateral local rules on edges

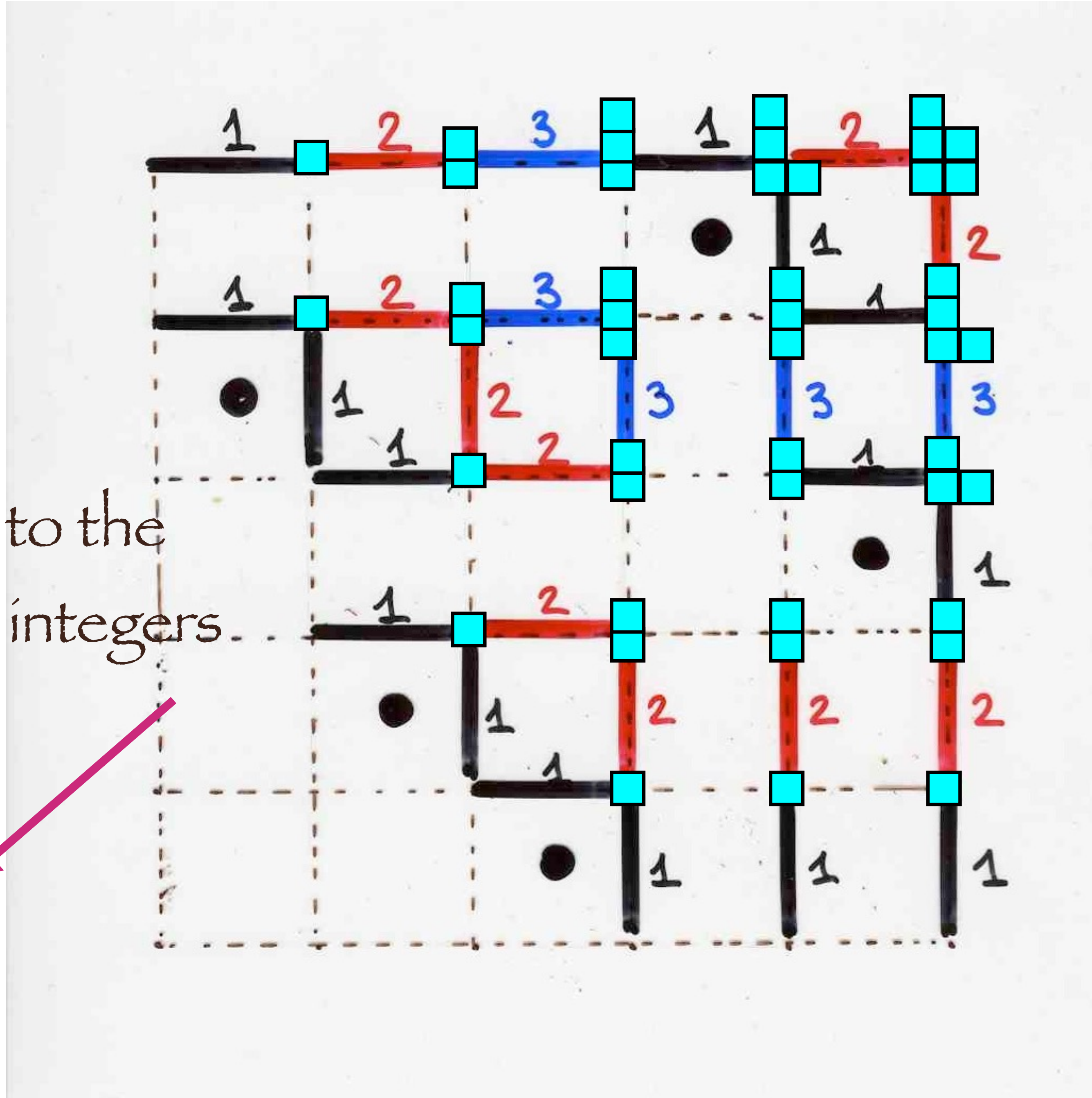
$i, j \in \mathbb{Z} - \{0\}$

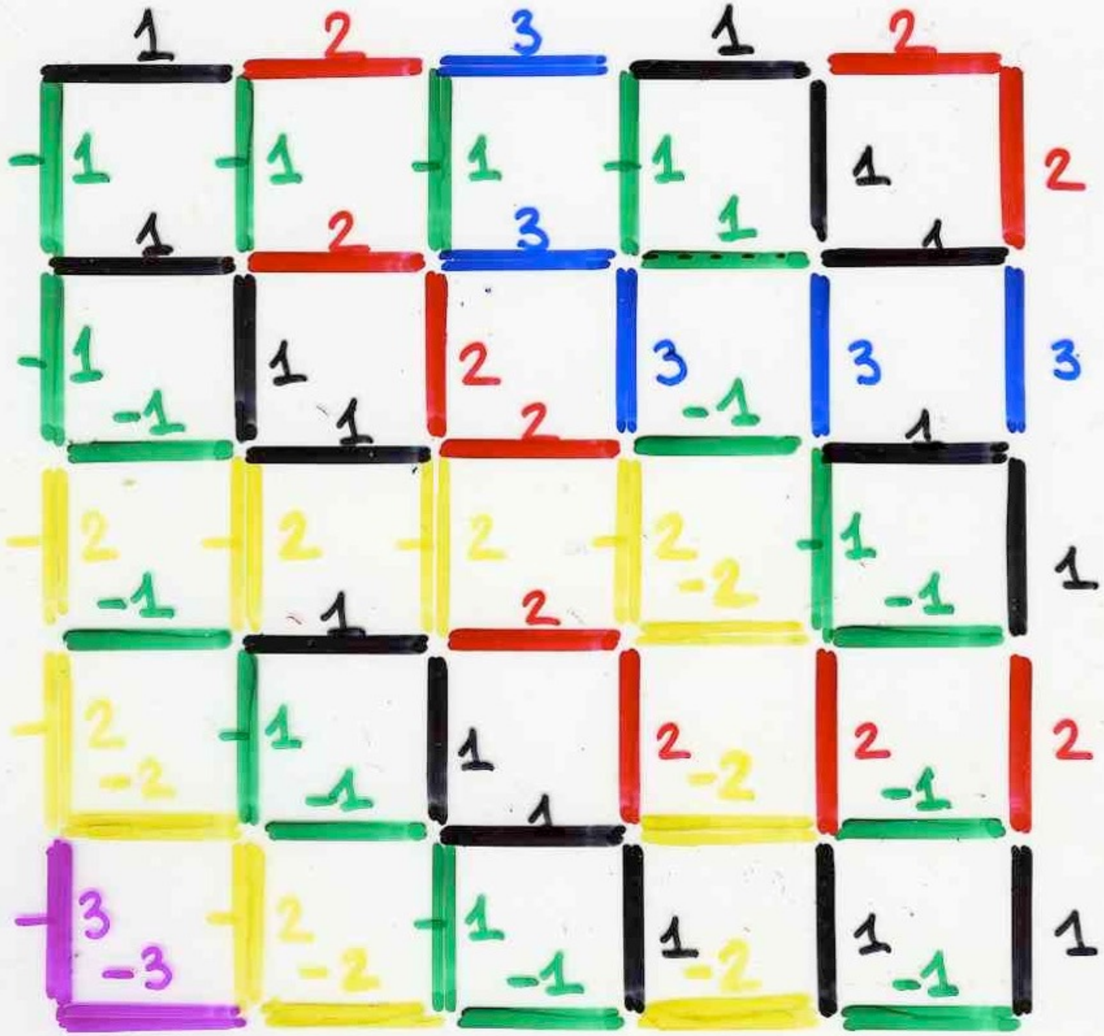


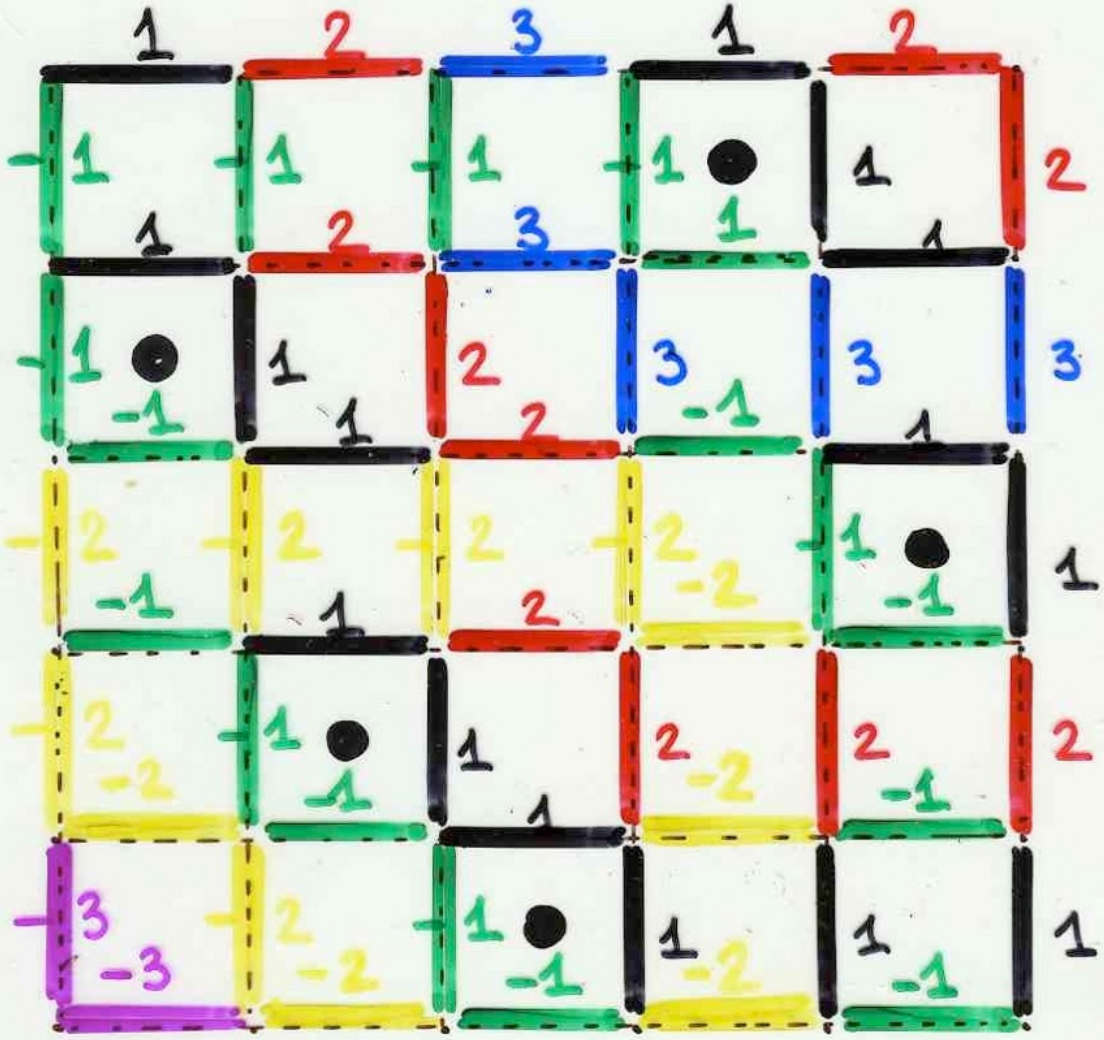
$i \neq j$

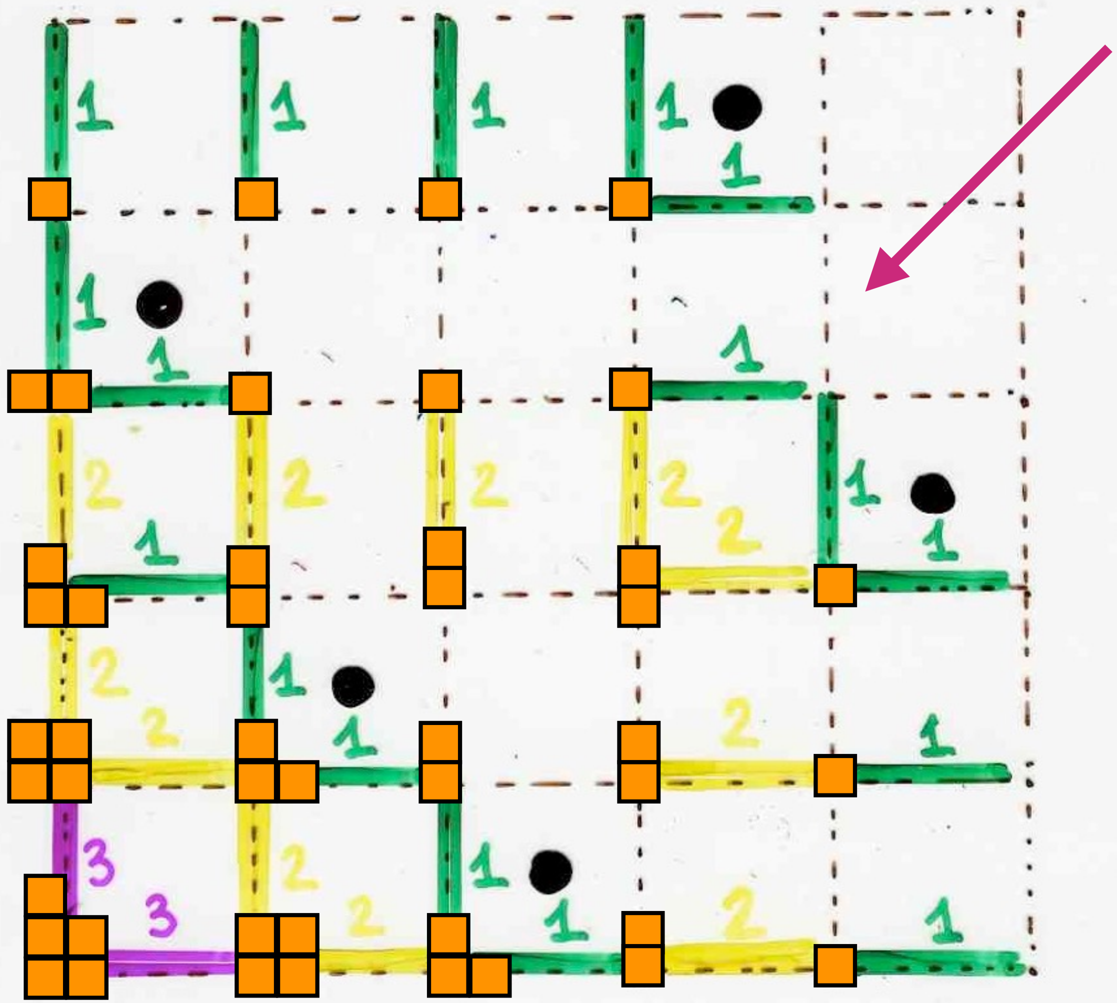


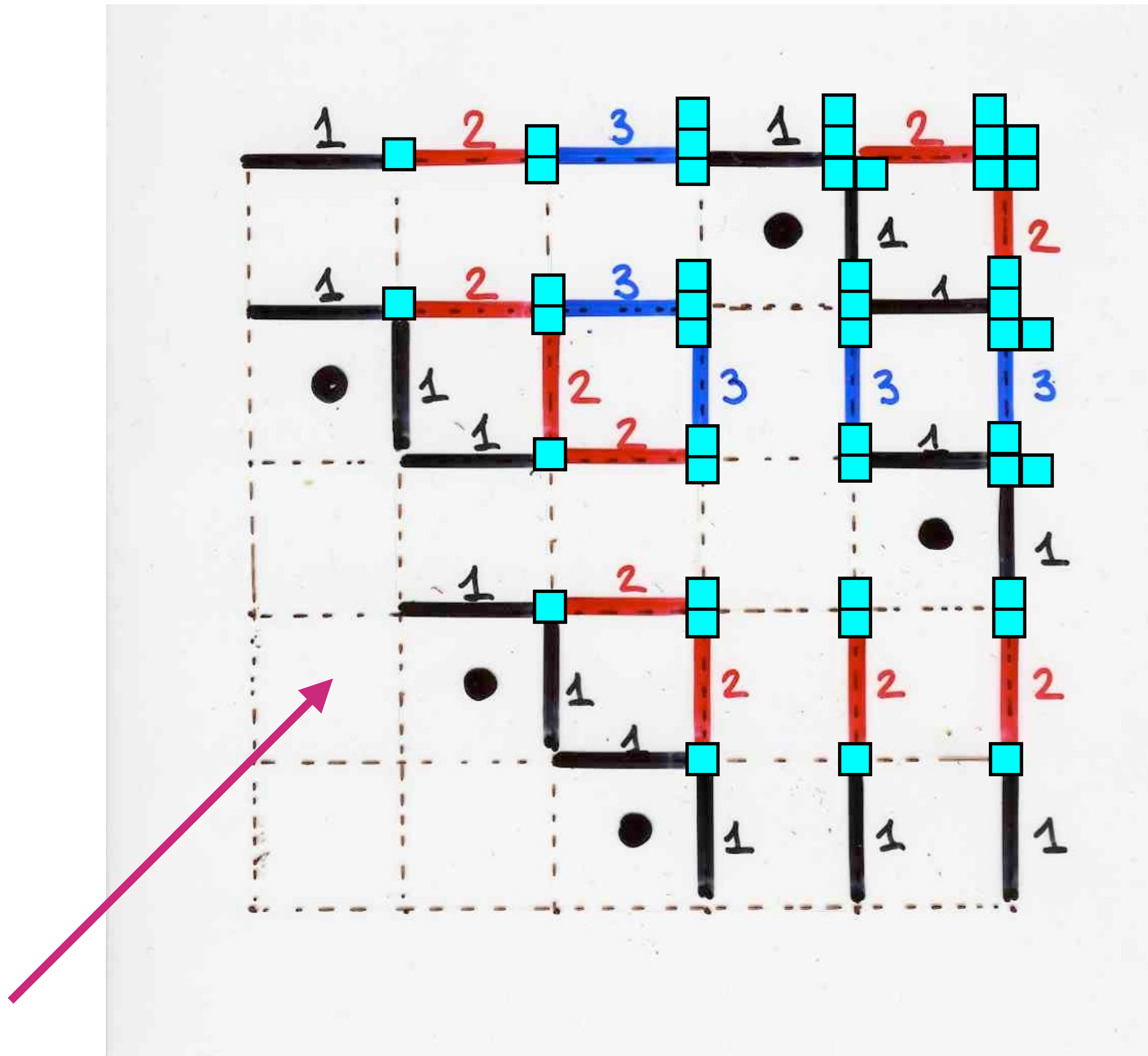
Going to the negative integers







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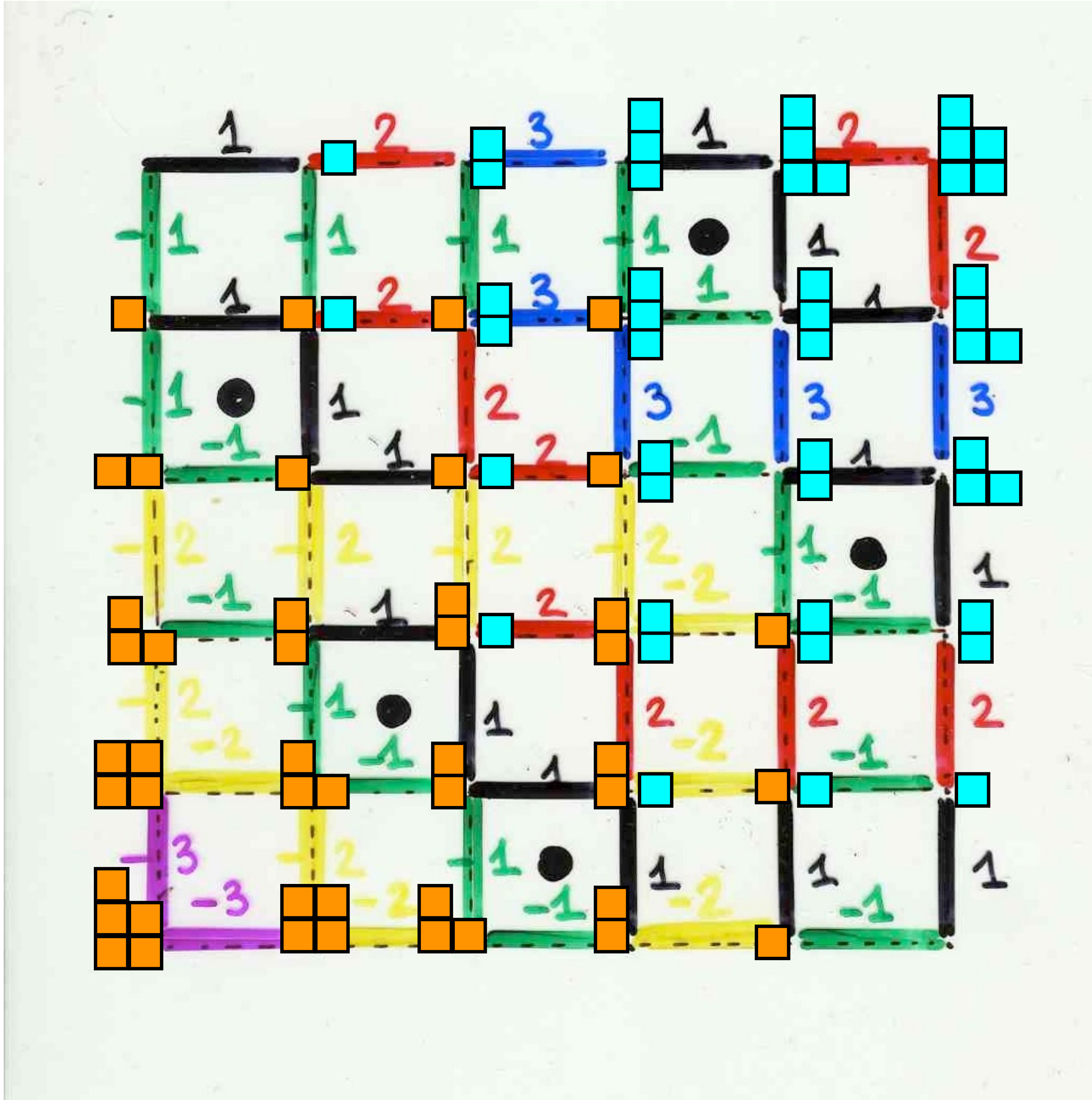
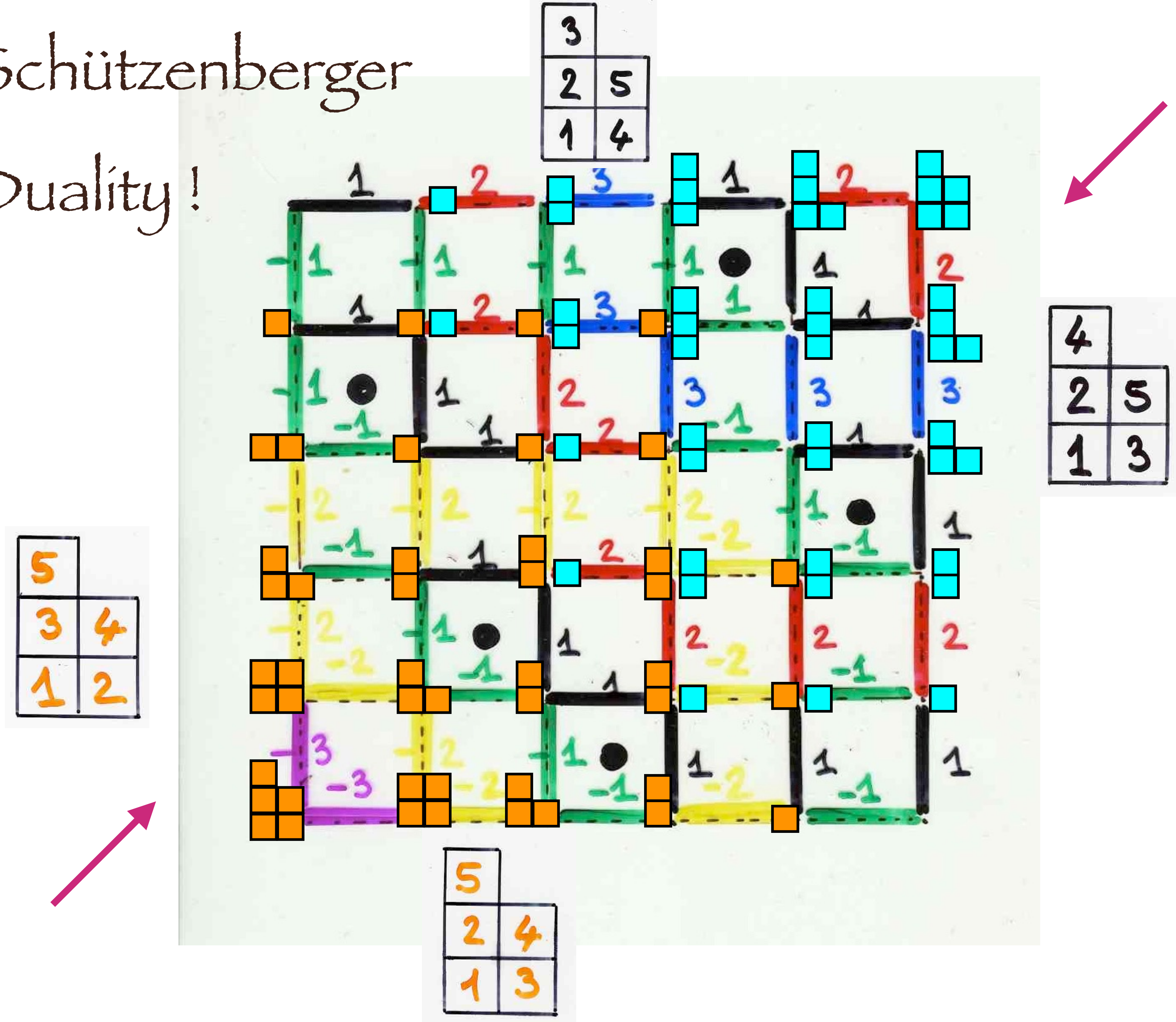
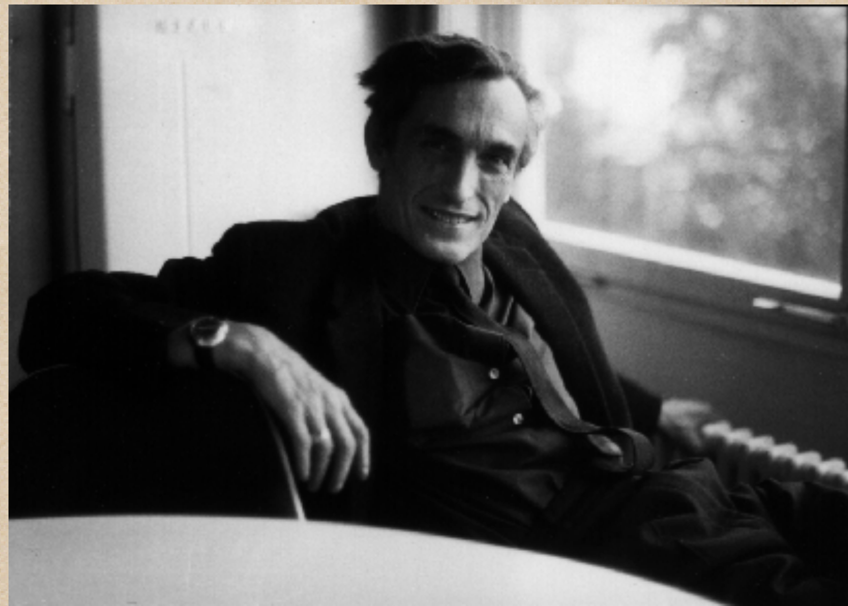


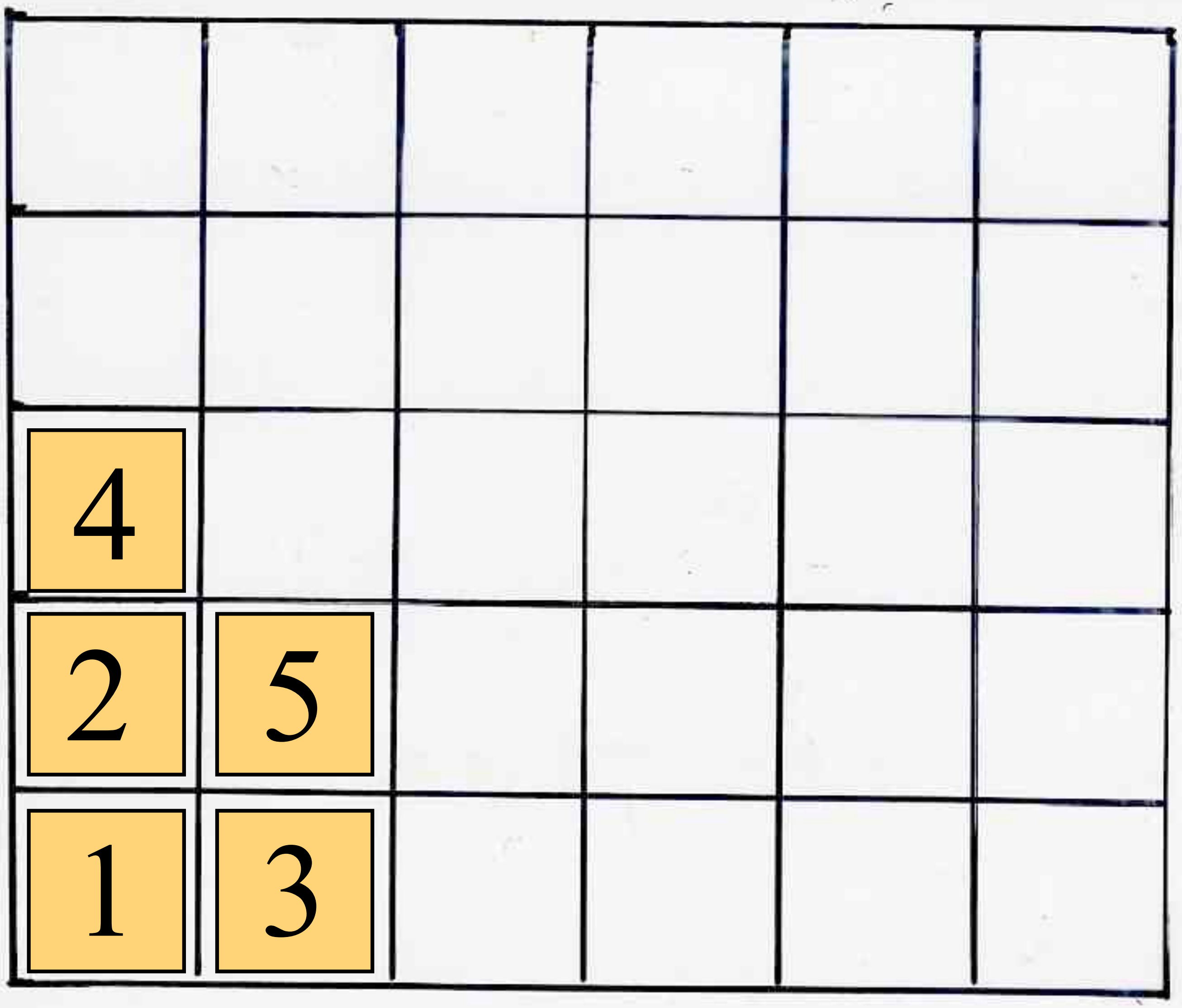
Schützenberger Duality !



dual of a Young tableau



M.P. Schützenberger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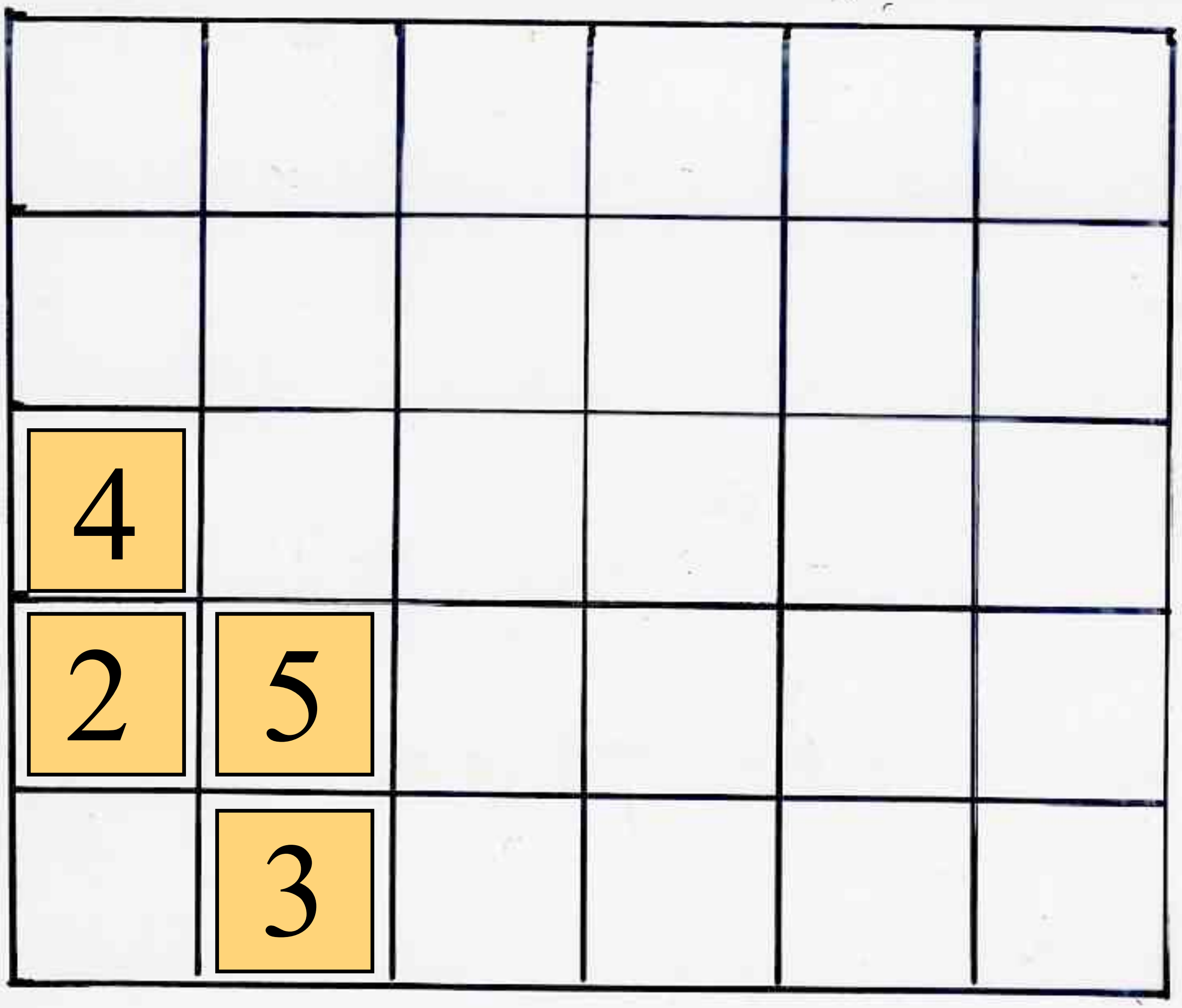
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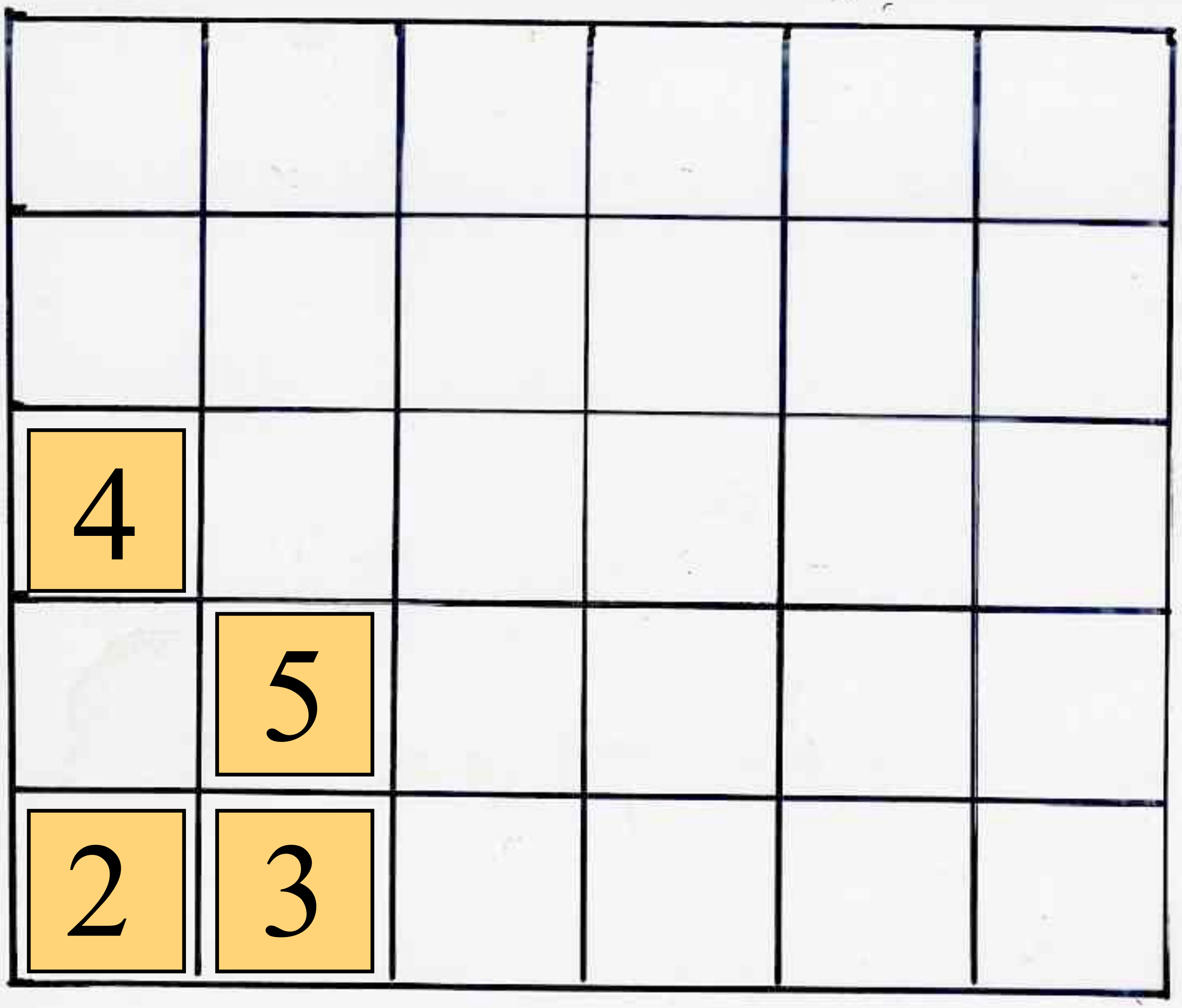
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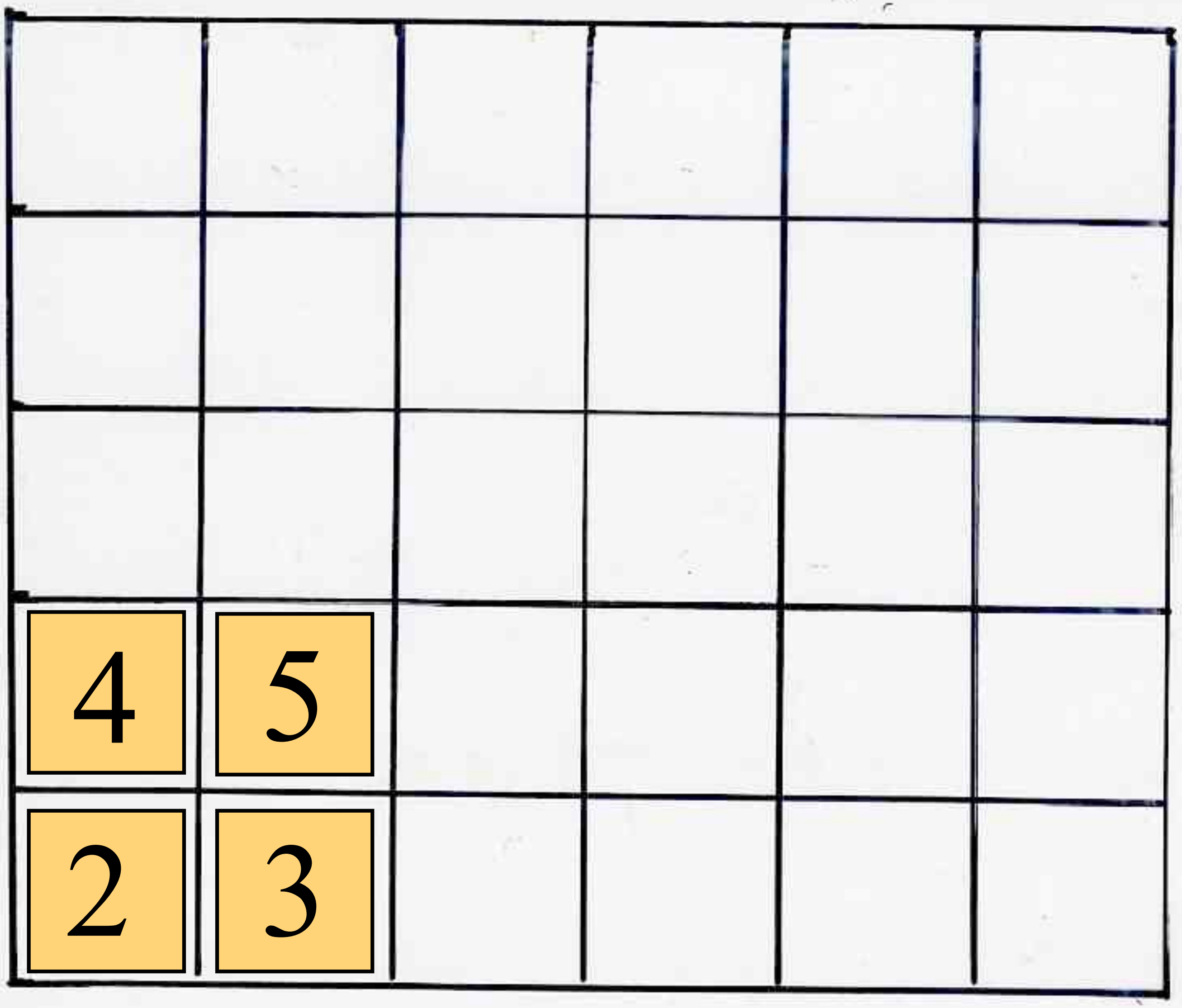
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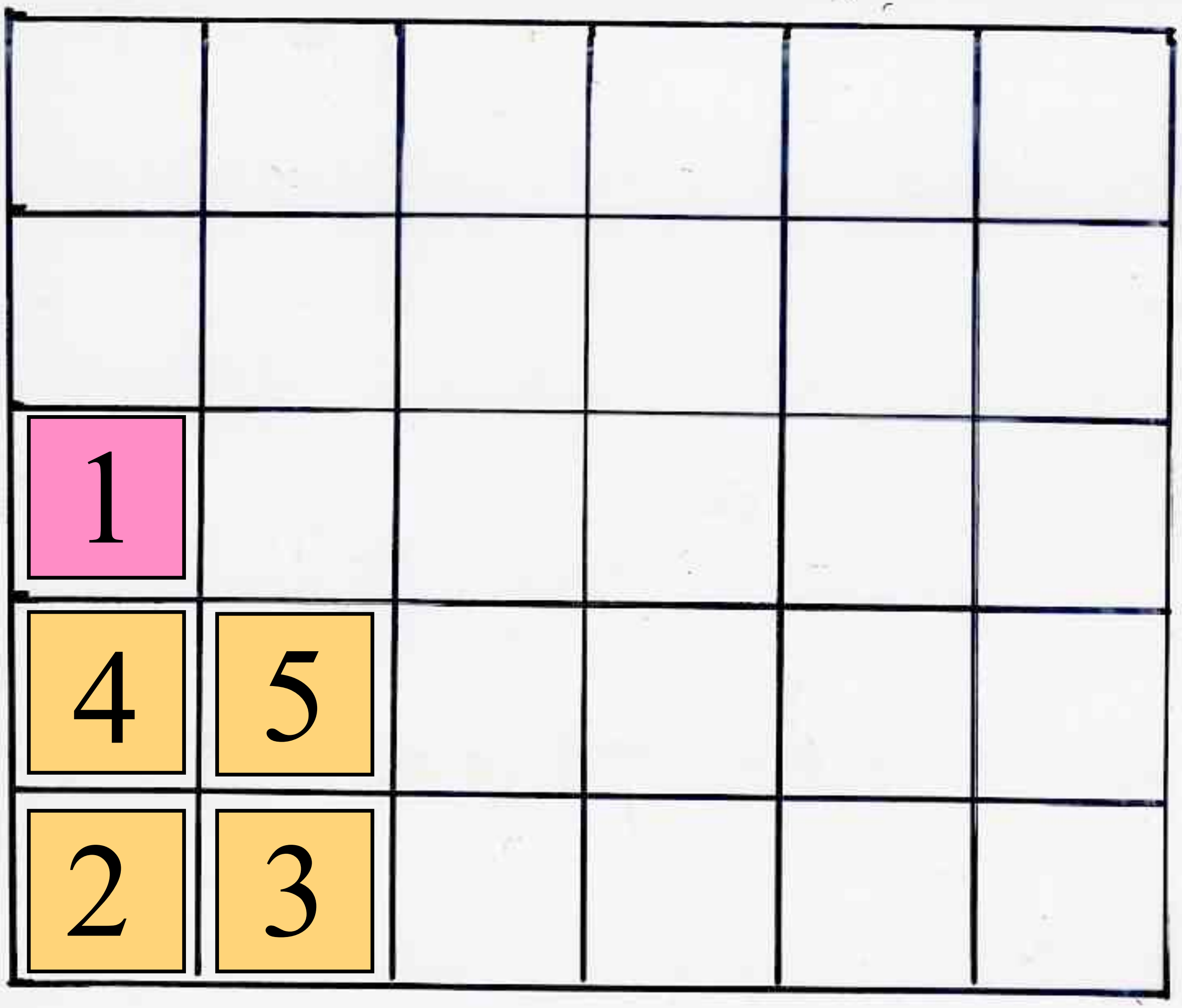
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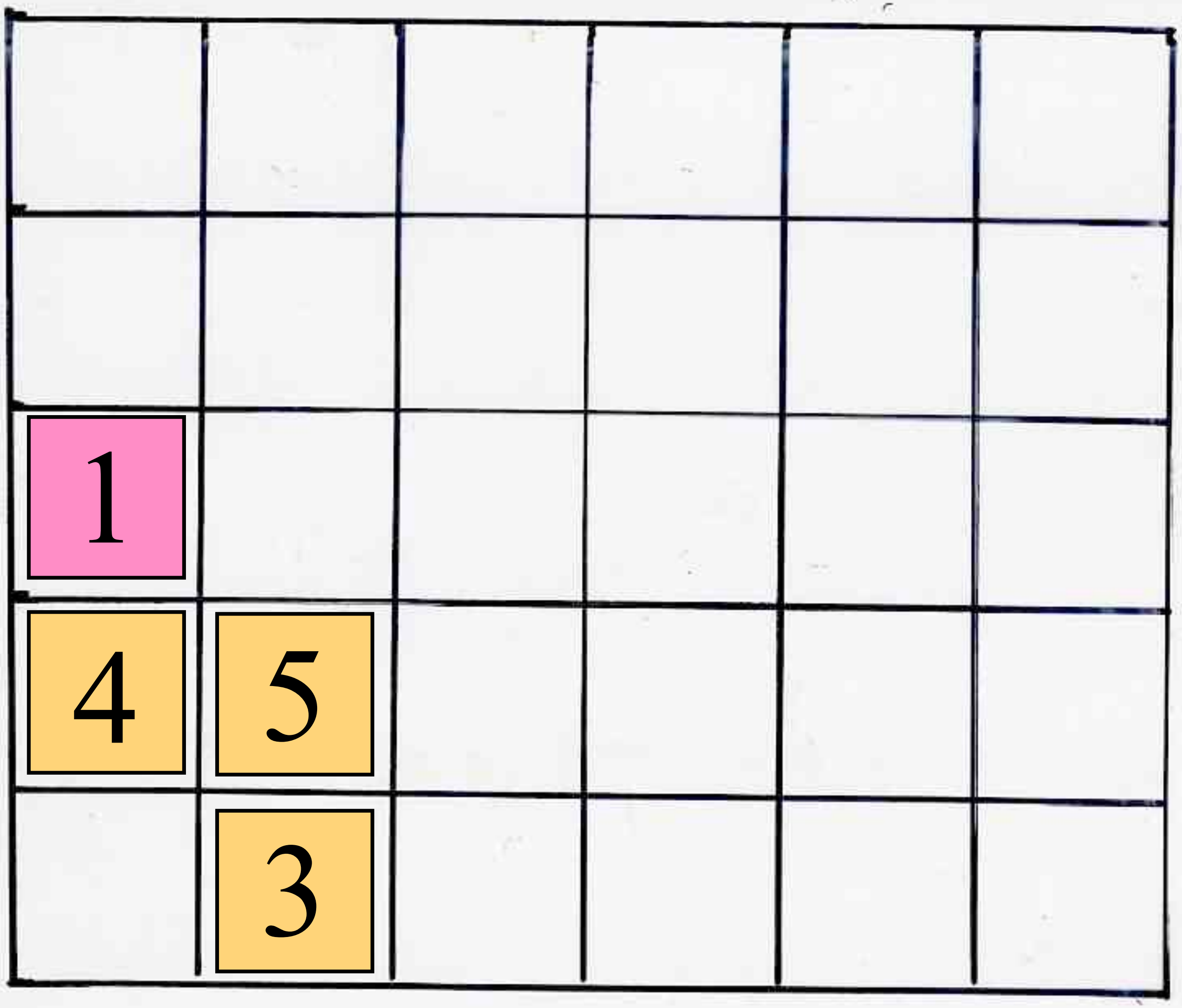
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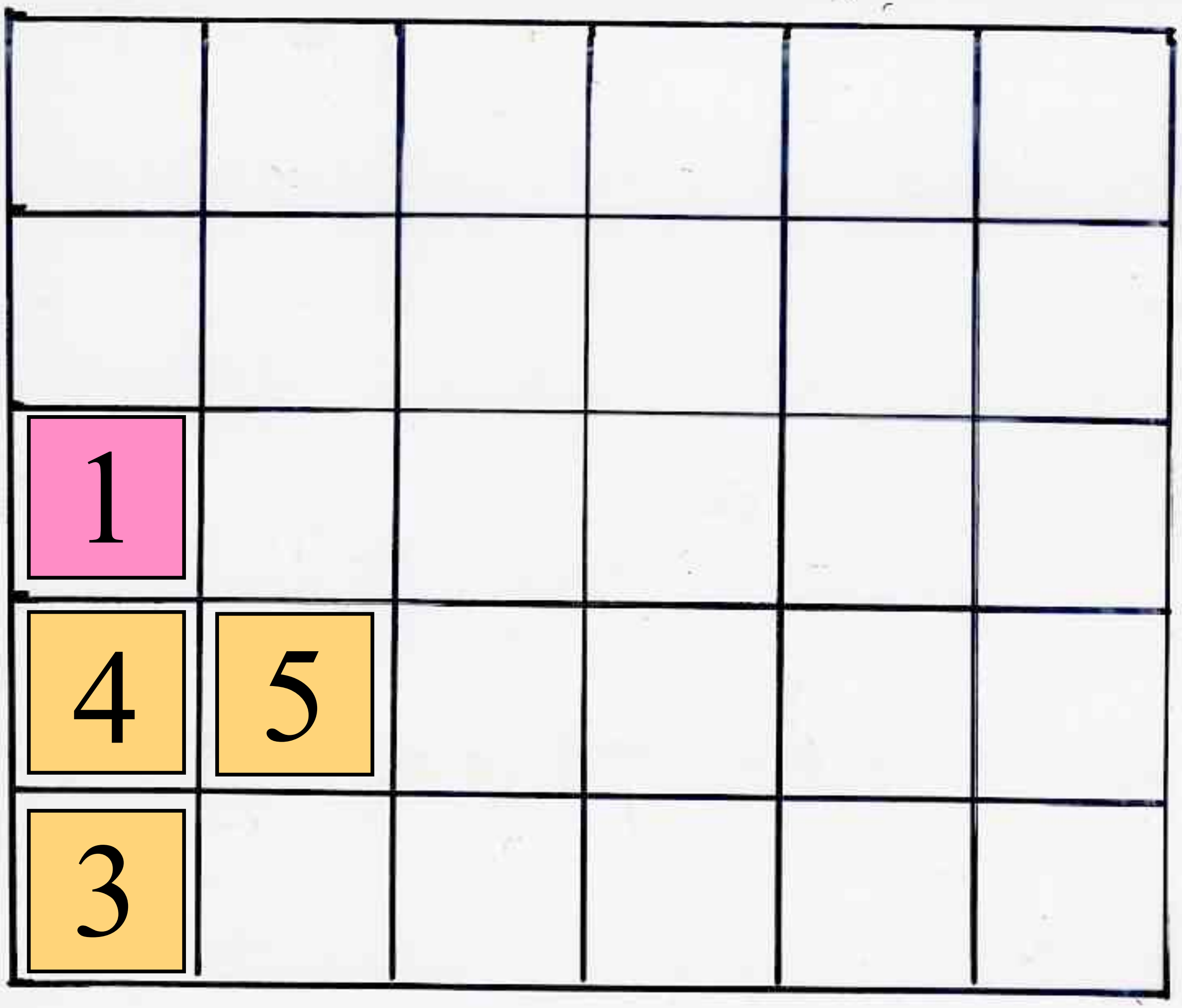
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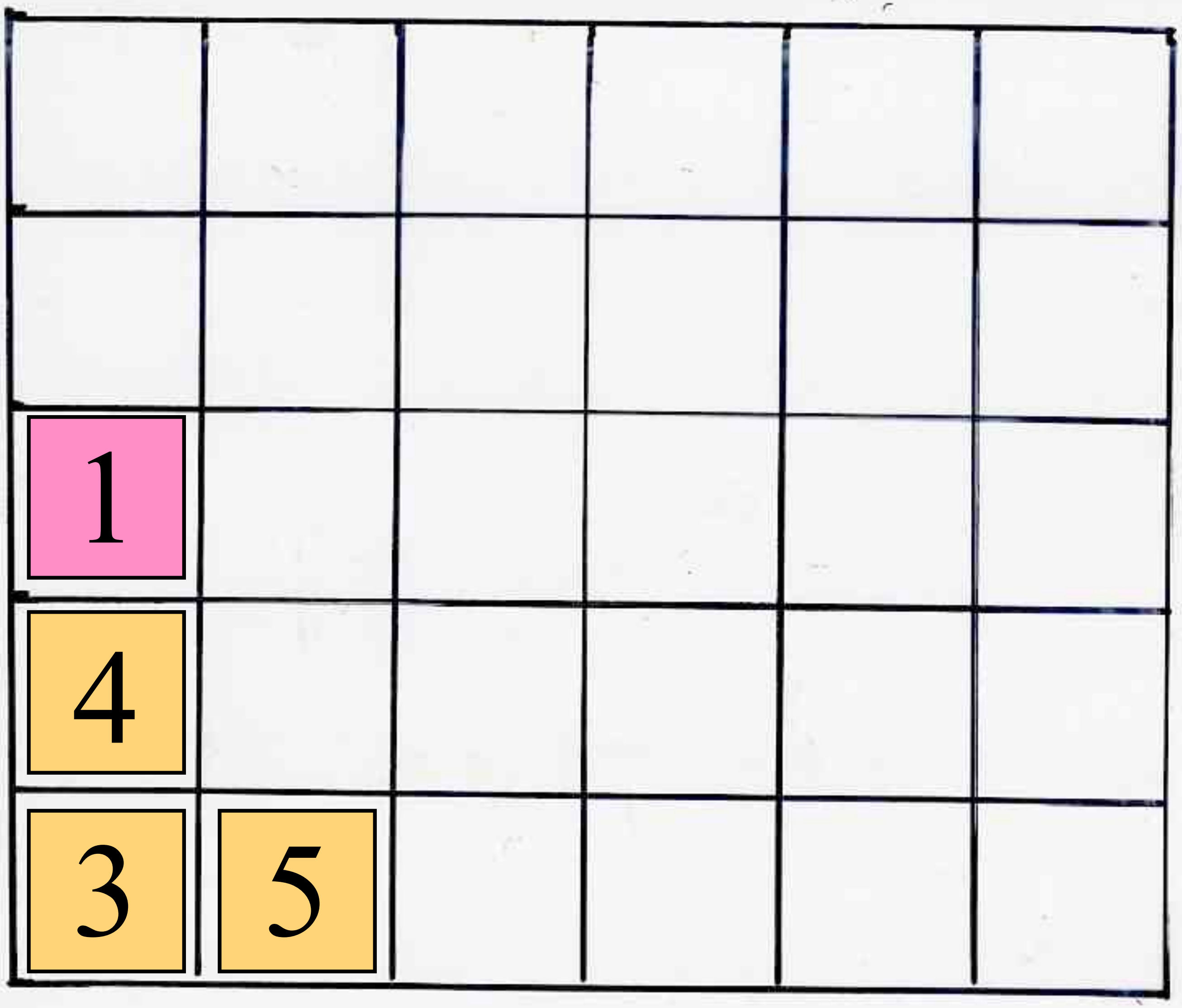
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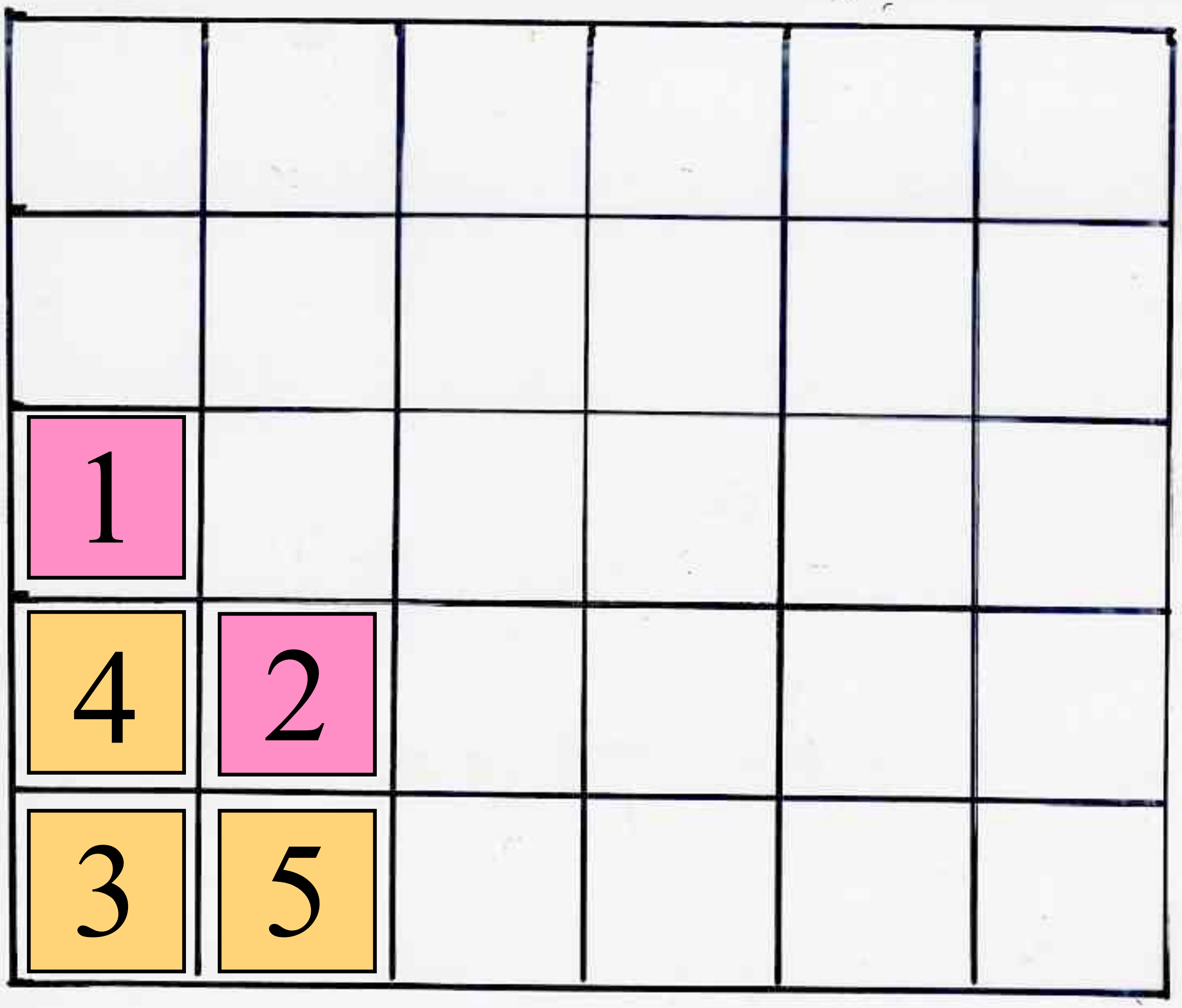
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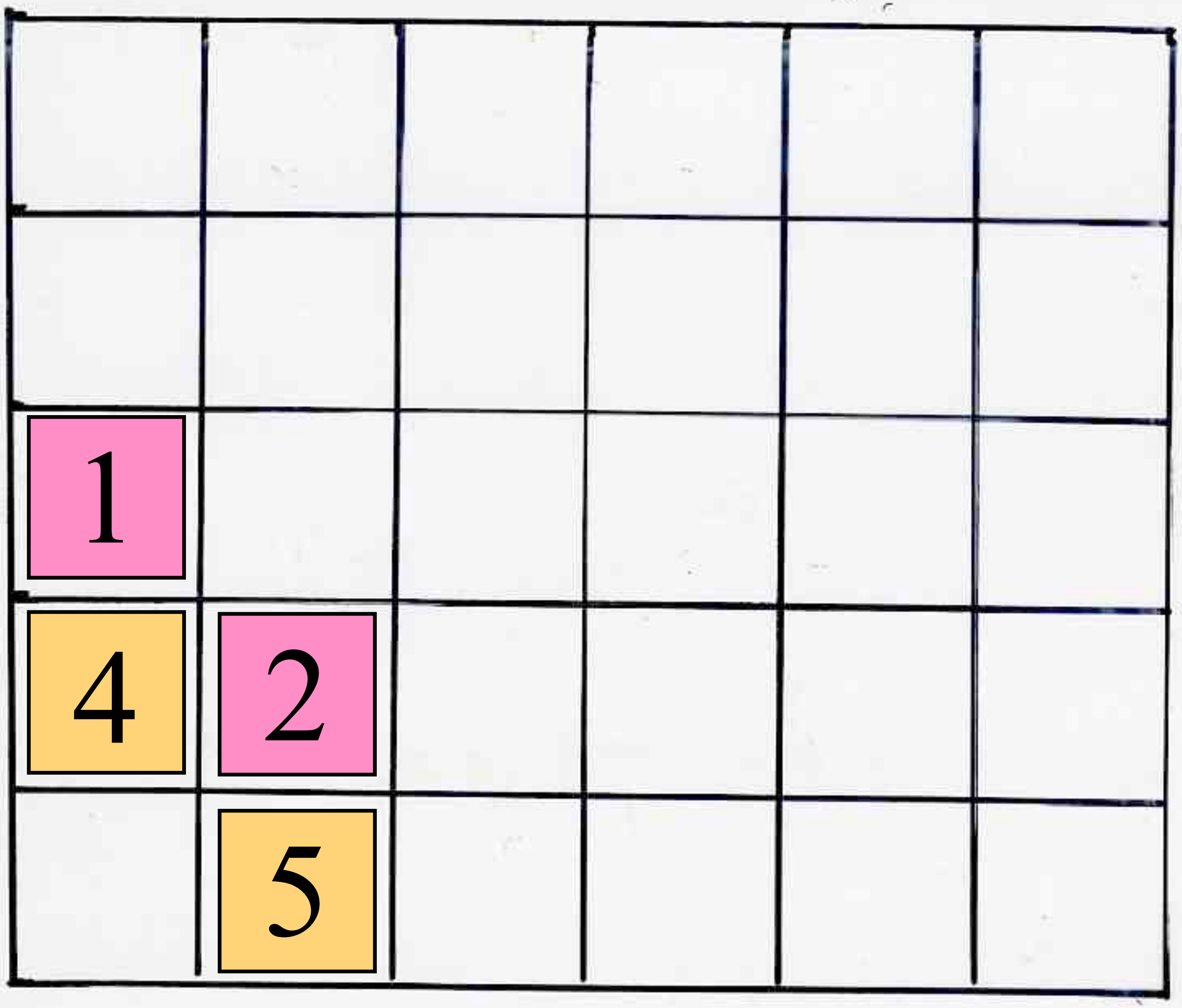
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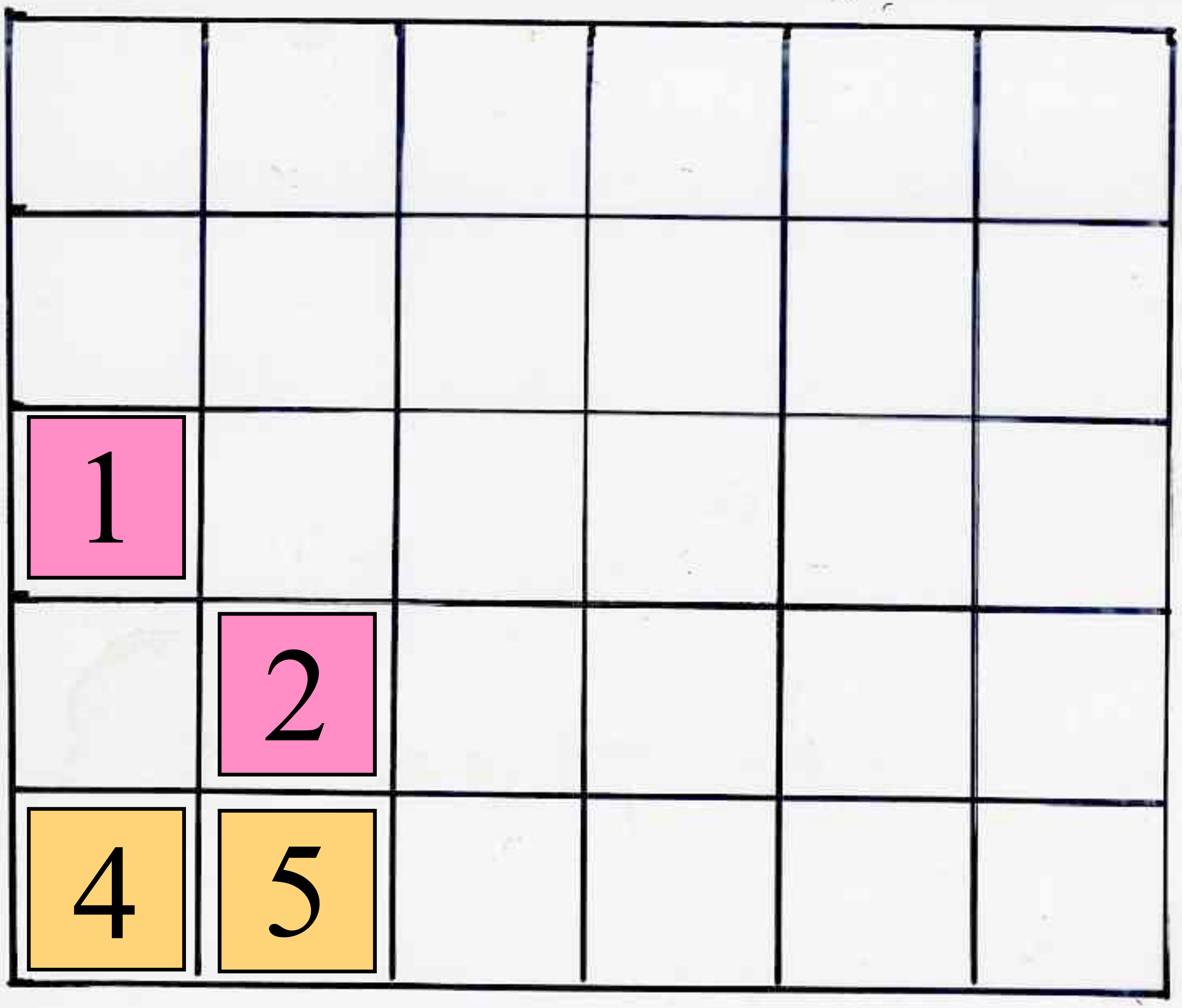
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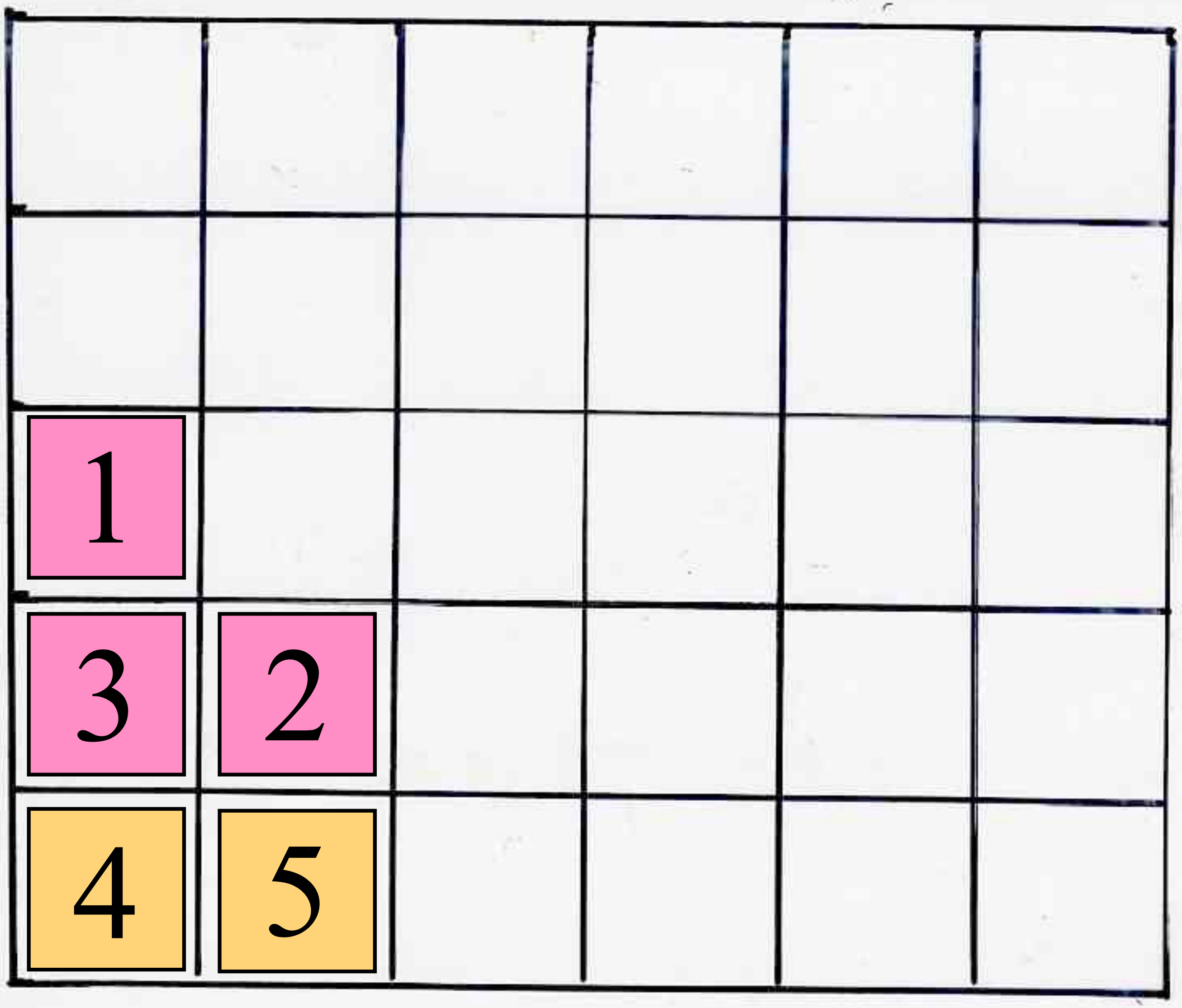
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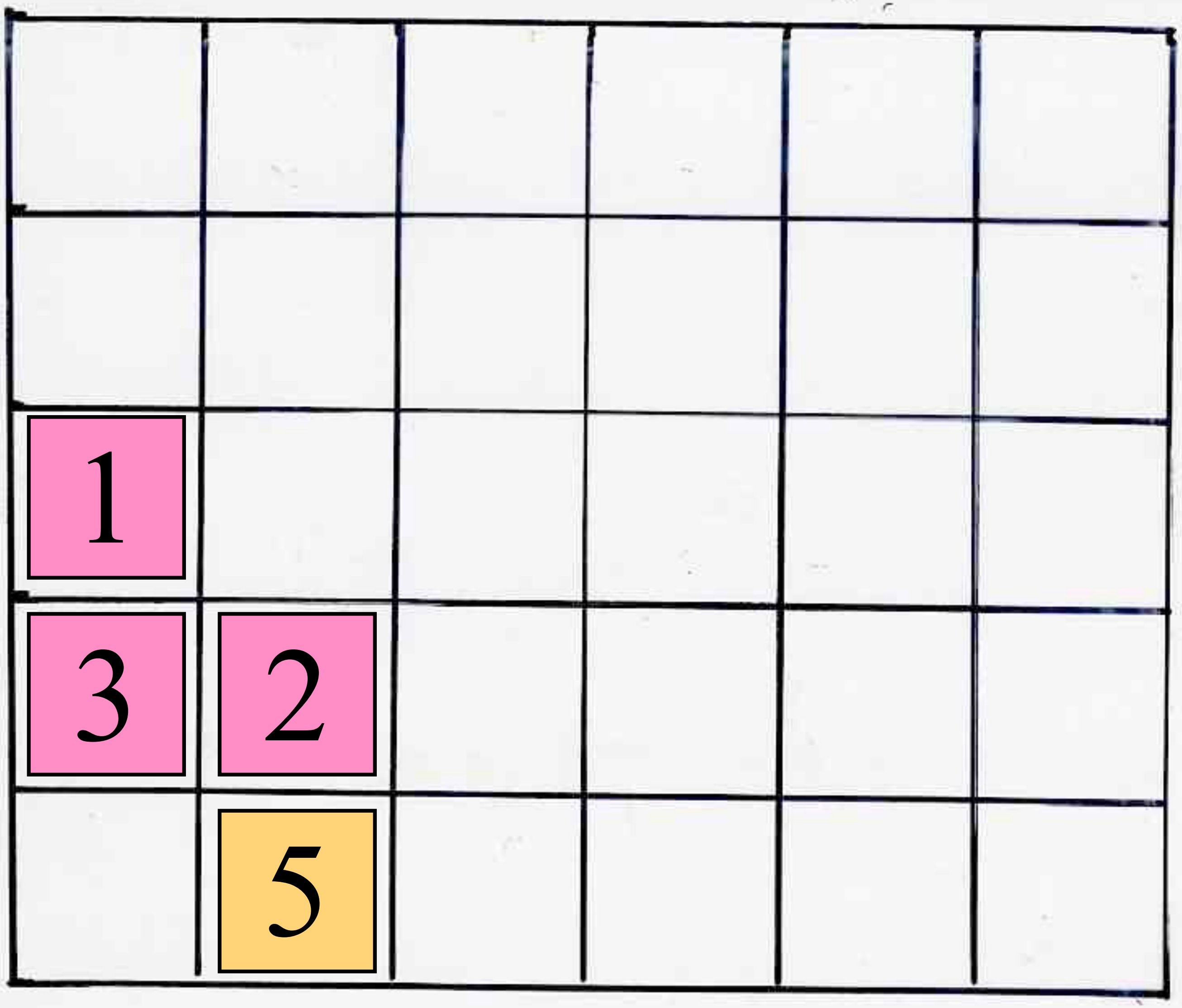
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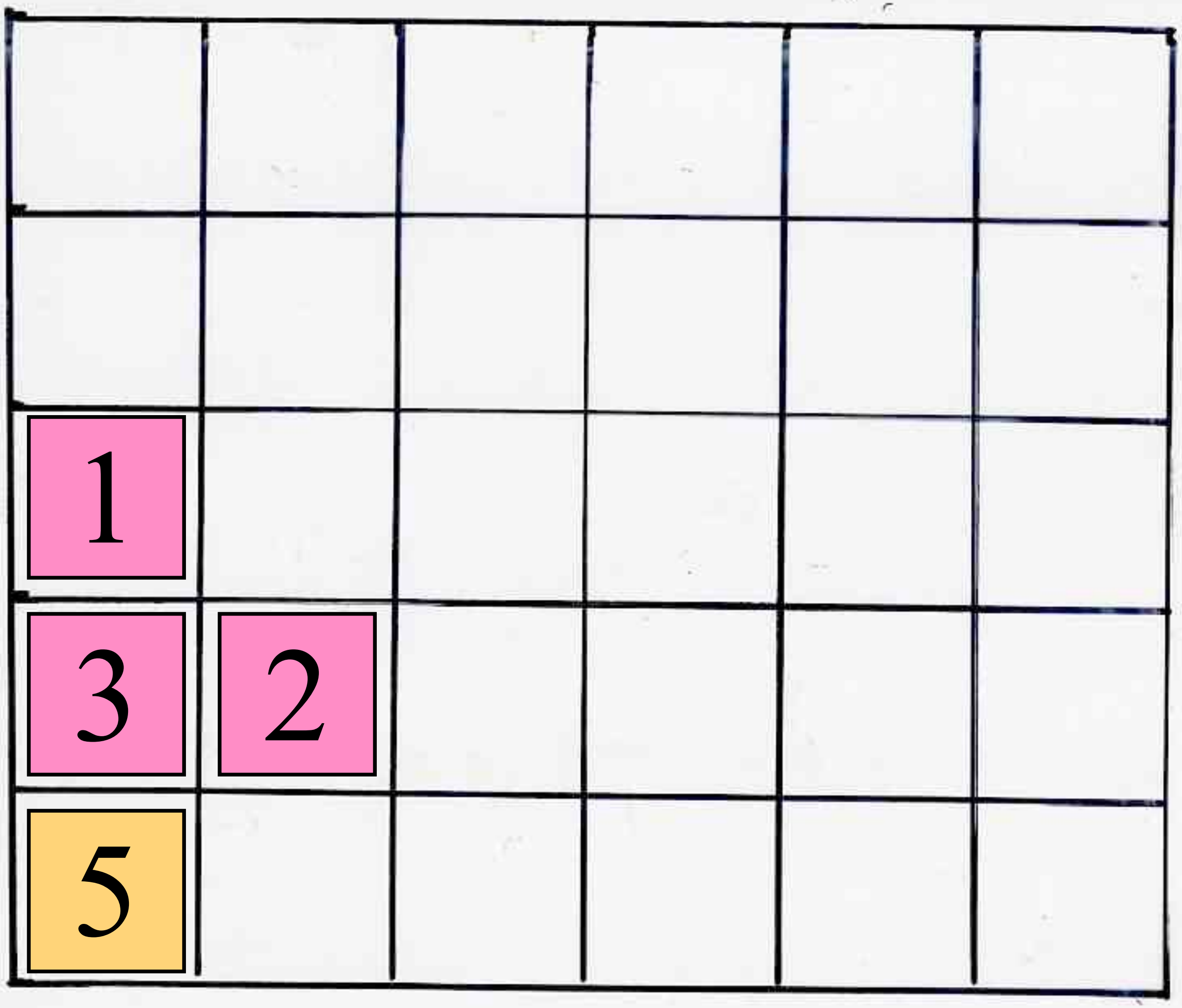
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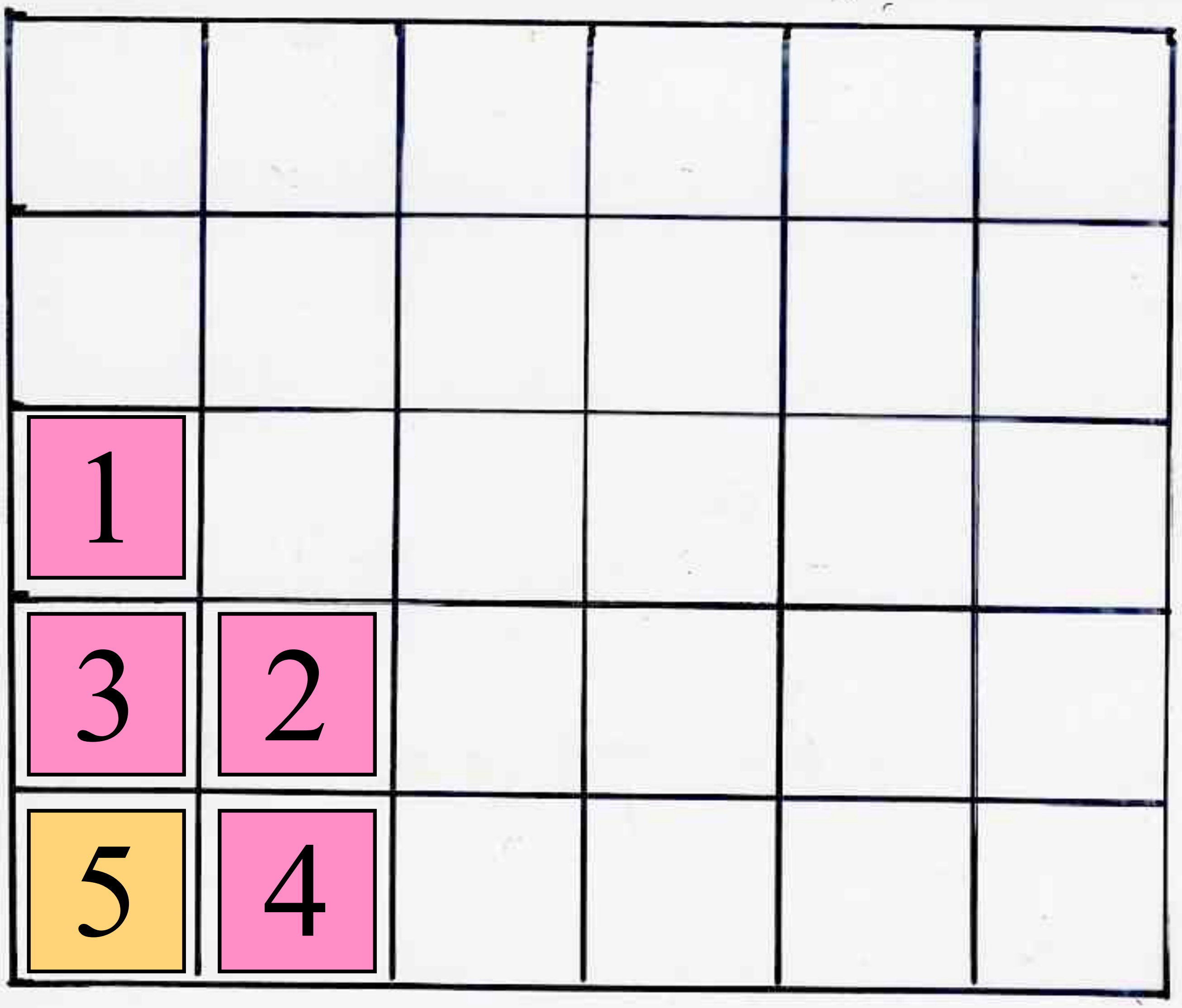
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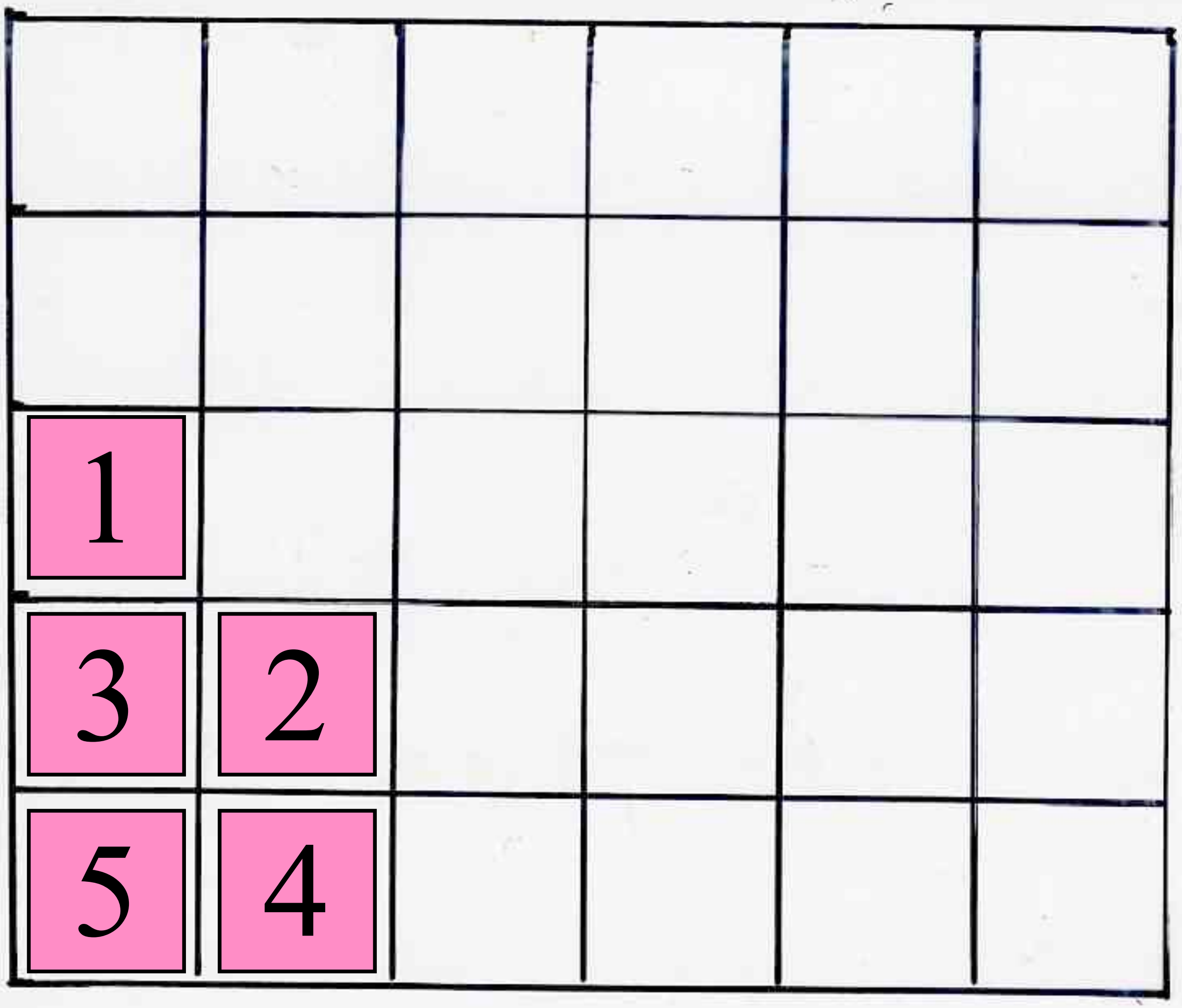
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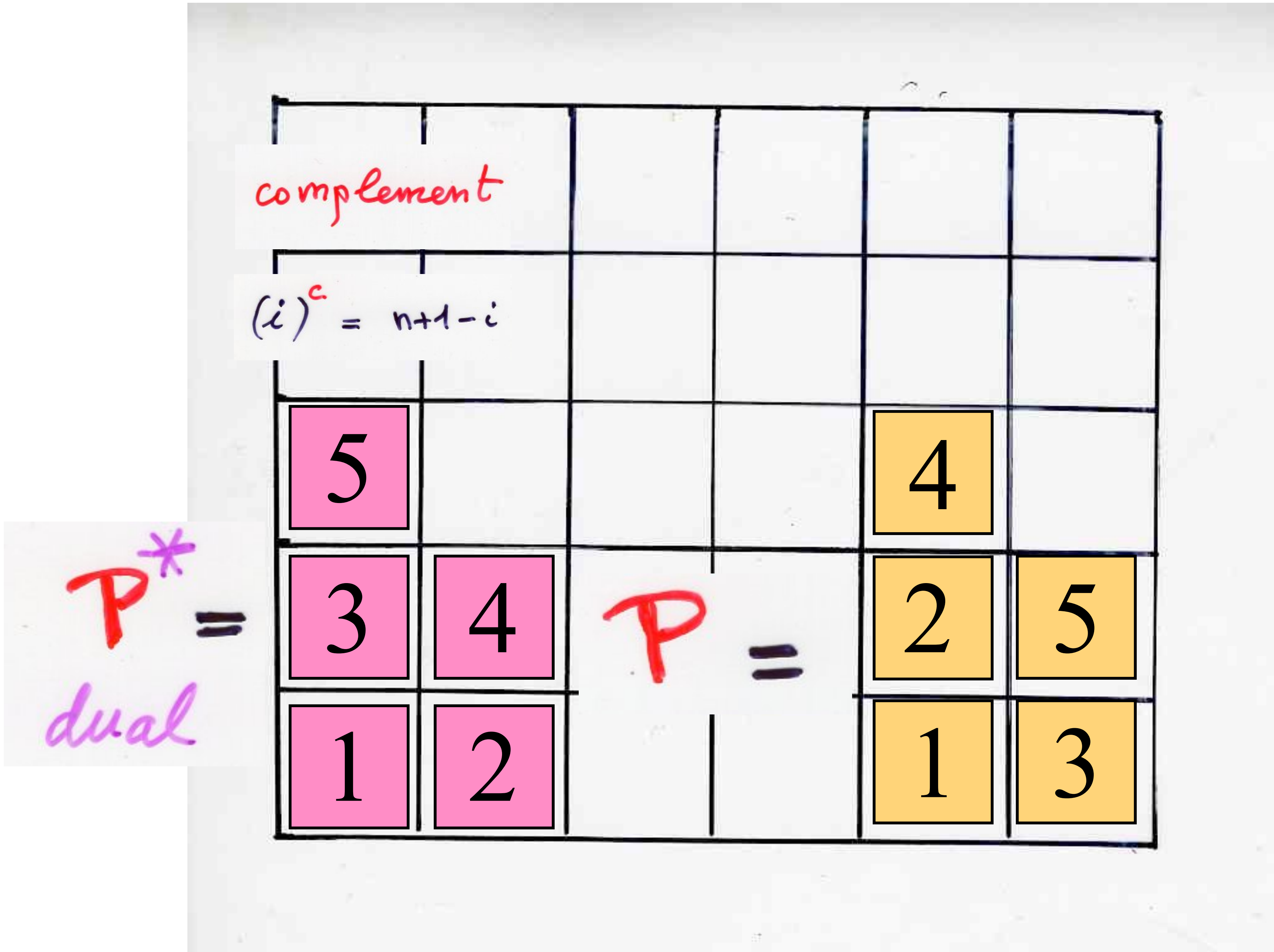
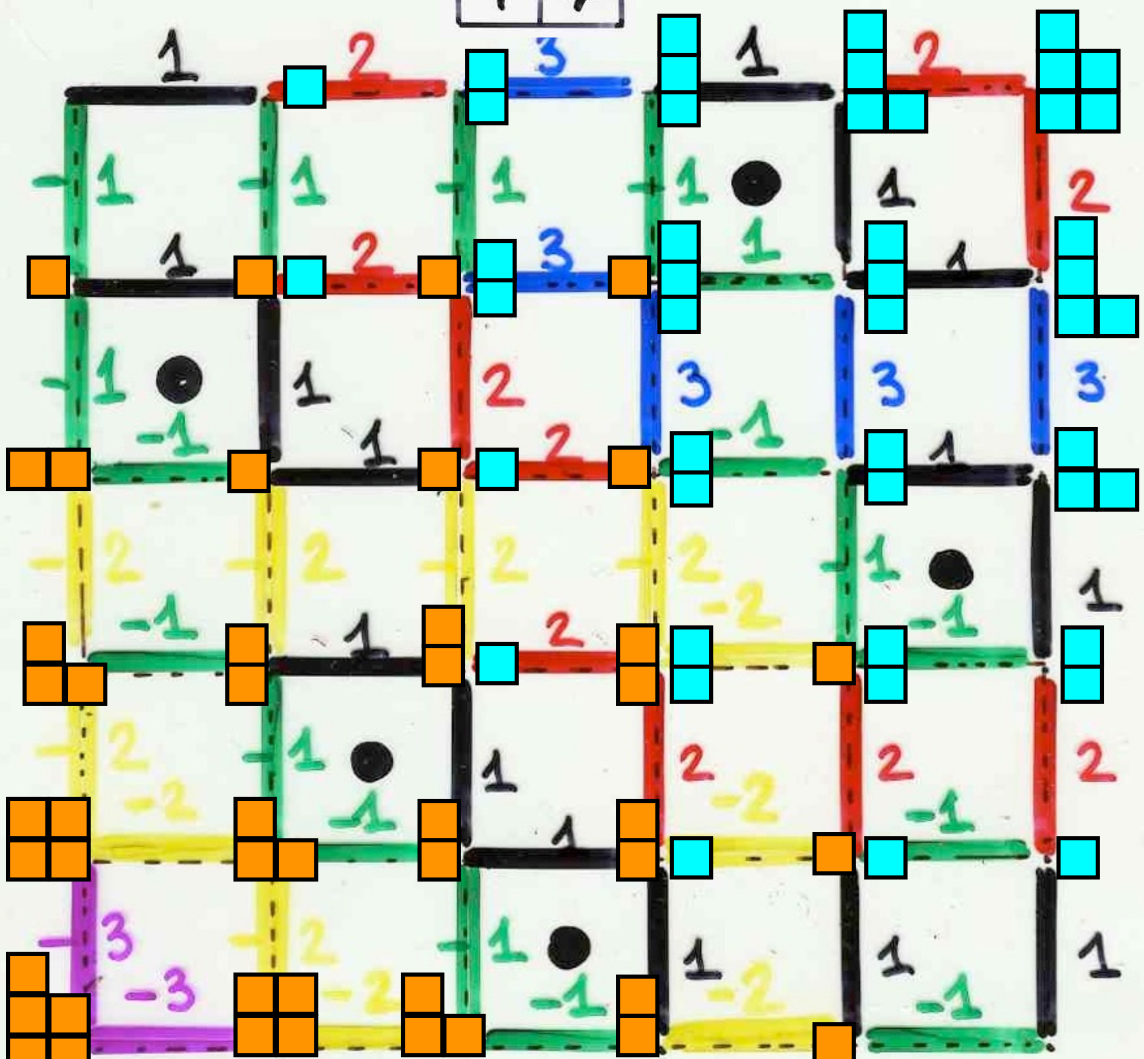

complement
$(i)^c = n+1-i$
P^* =
dual
5
3 4
1 2
P =
4
2 5
1 3

Schützenberger Duality !

3	
2	5
1	4



P^* = dual

5	
3	4
1	2

P =

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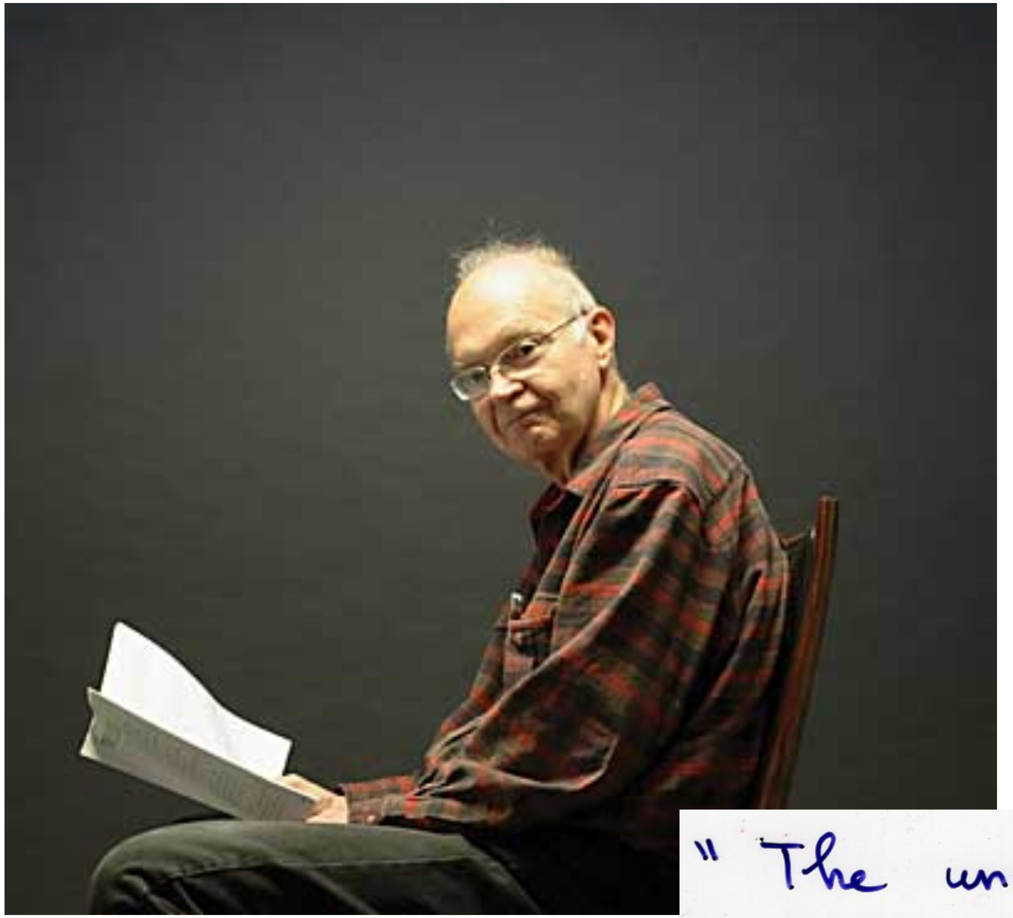


"The unusual nature of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scene"

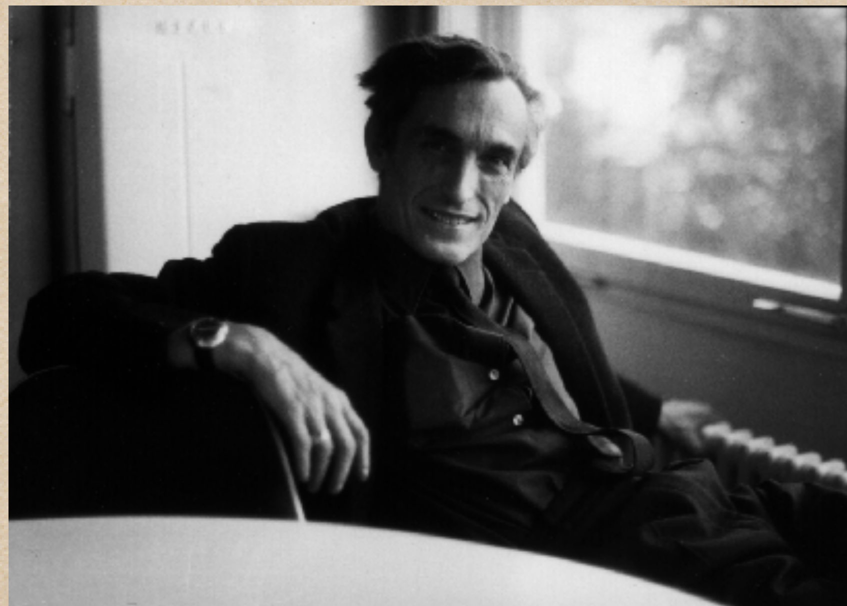
D. Knuth (1972)

The art of computer programming
Vol. 3

Jeu de taquin

M.P. Schützenberger

(1976)



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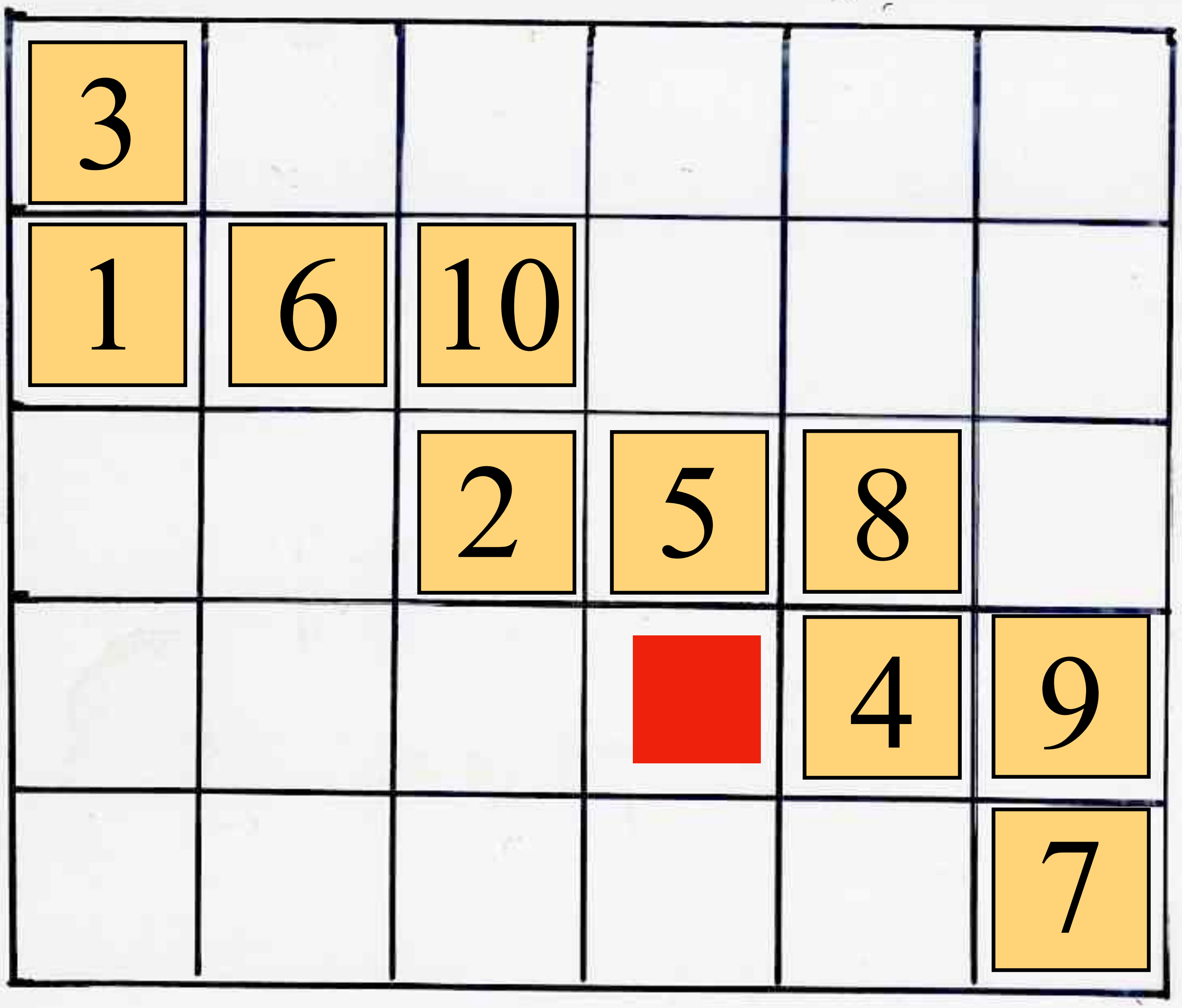
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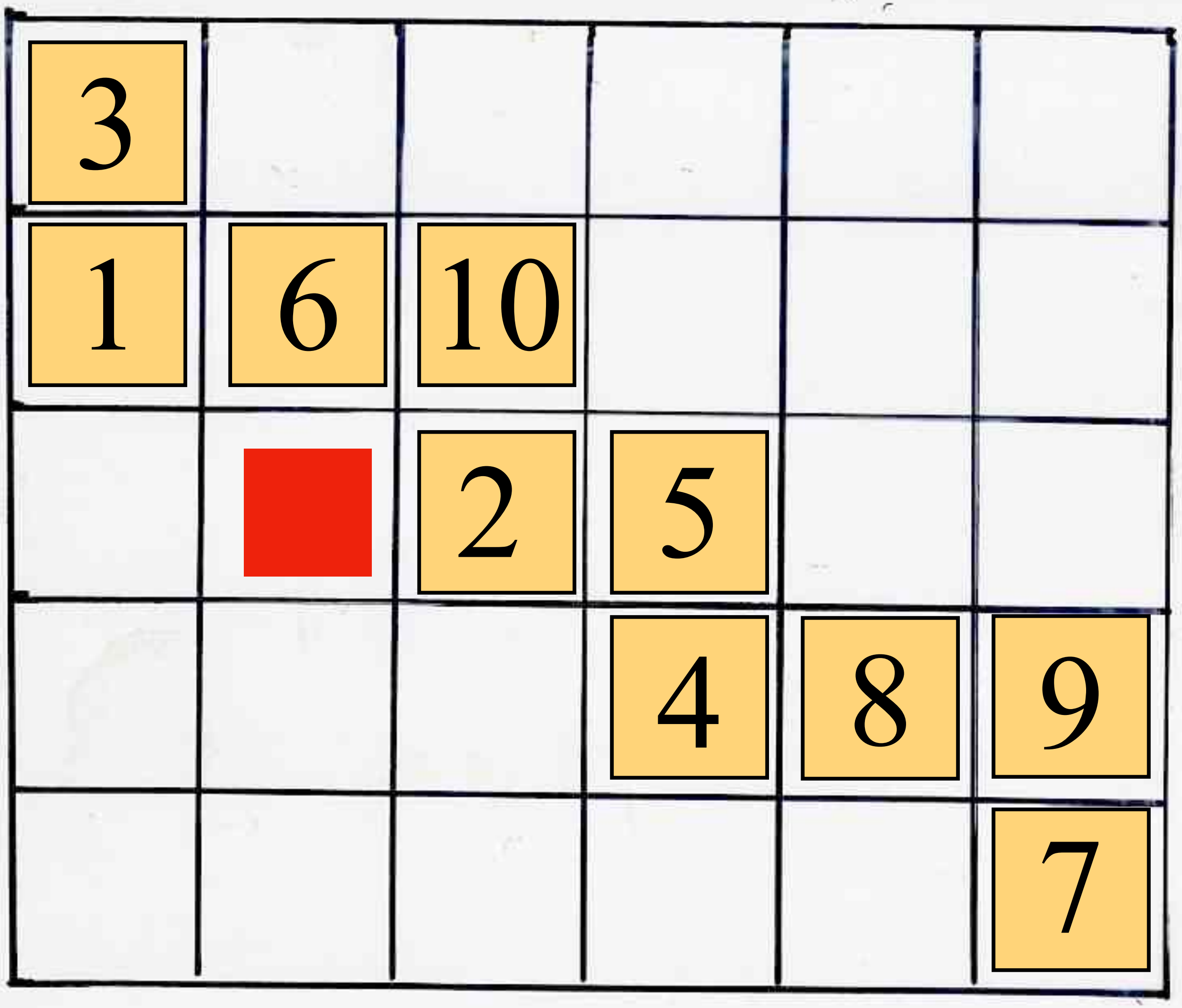
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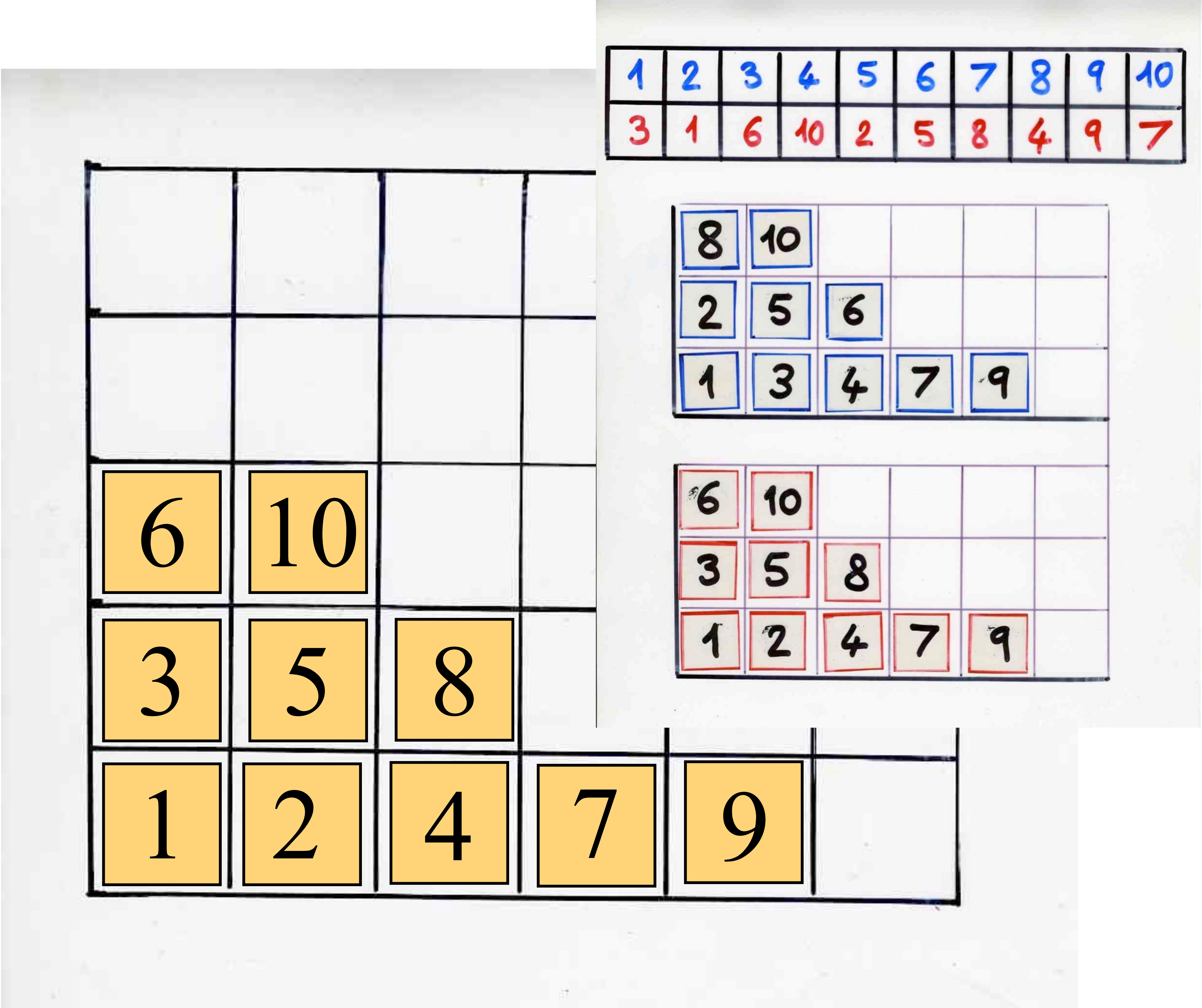


1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
8 10
2 5 6
1 3 4 7 9
6 10
3 5 8
1 2 4 7 9
6 10
3 5 8
1 2 4 7 9

Jeu de taquin
with growth diagrams

S. Fomin, 1986, 1994

appendix, R.Stanley's book
Enumerative Combinatorics, Vol2

edge local rules

X.V. GASCom2018

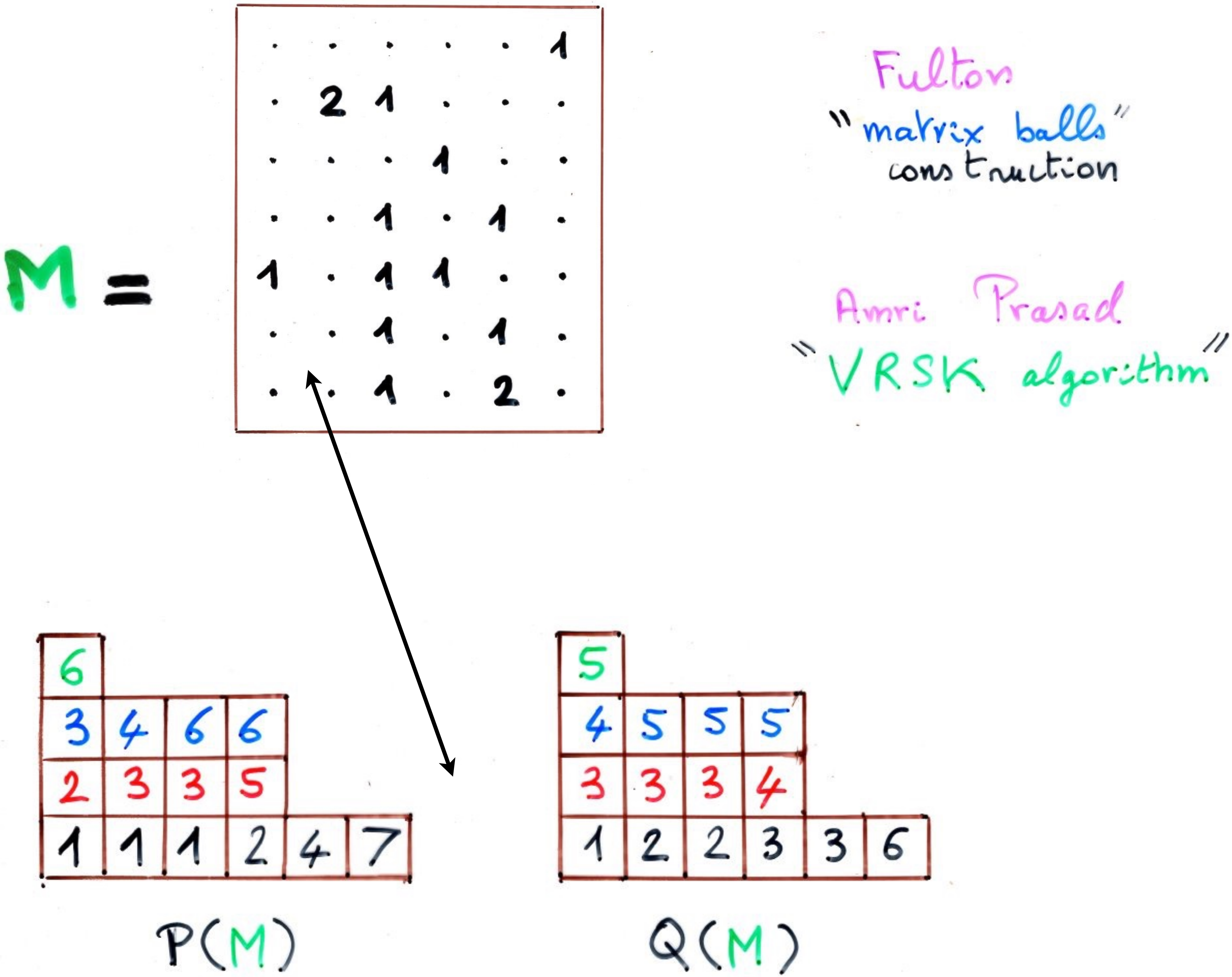
Сергей Владимирович Фомин

Extension to RSK

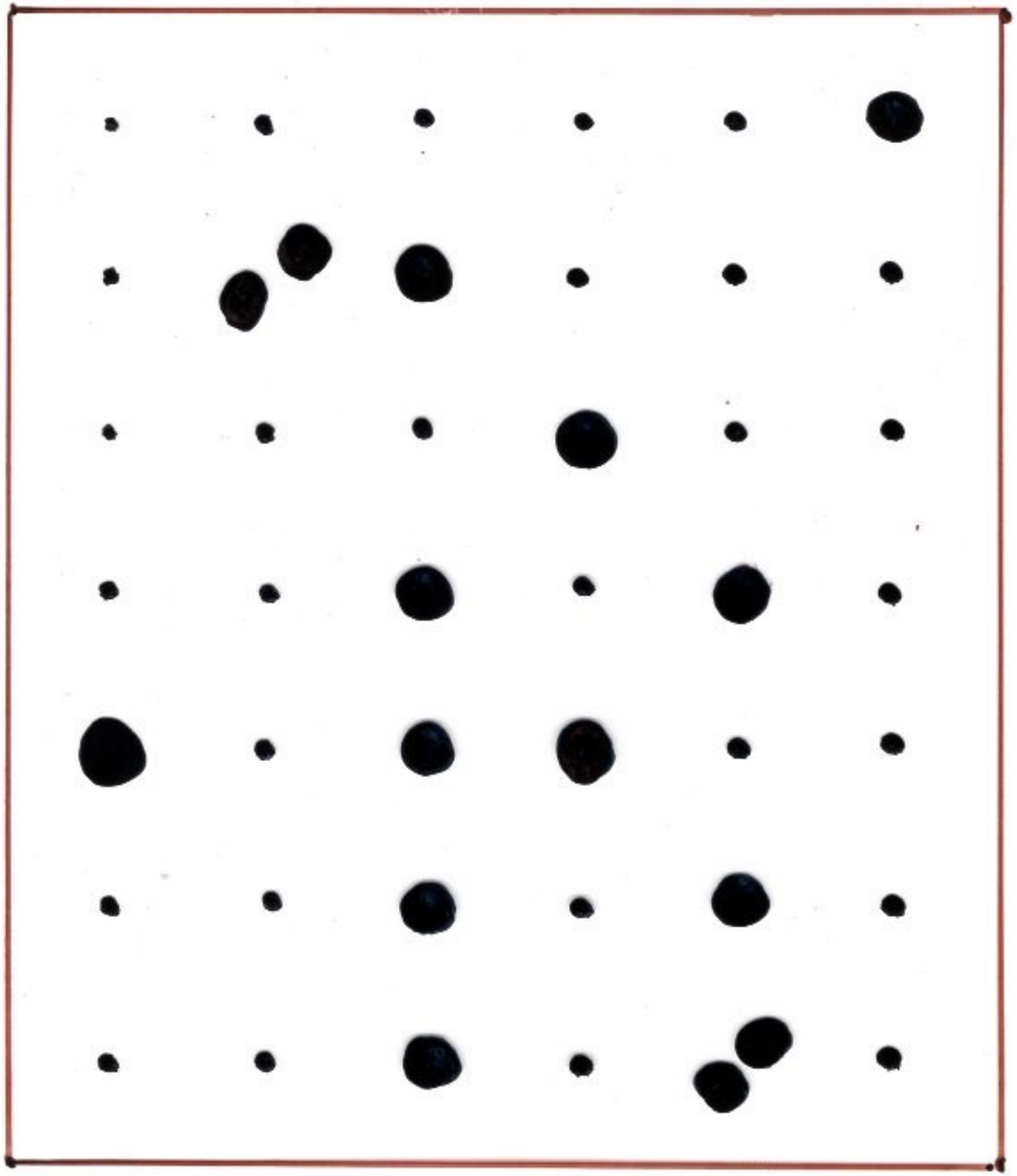
(Robinson-Schensted-Knuth)

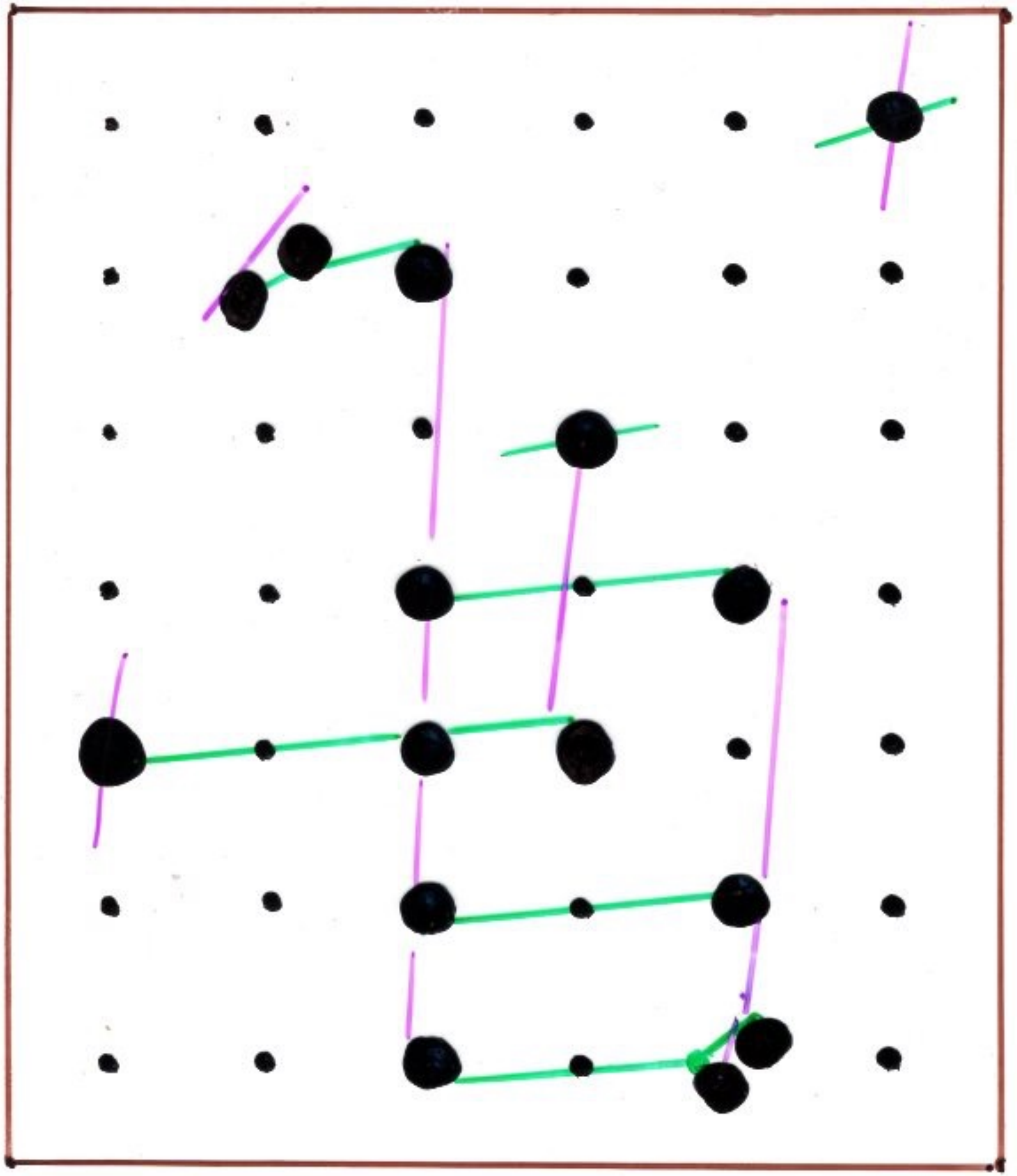
edge local rules for RSK

$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{bmatrix}$$


M =
. 1
. 2 1 . . .
. . . 1 . .
. . 1 . 1 .
1 . 1 1 . .
. . 1 . 1 .
. . 1 . 2 .
Fulton
"matrix balls" construction
Amri Prasad
"VRSK algorithm"
6
3 4 6 6
2 3 3 5
1 1 1 2 4 7
P(M)
5
4 5 5 5
3 3 3 4
1 2 2 3 3 6
Q(M)

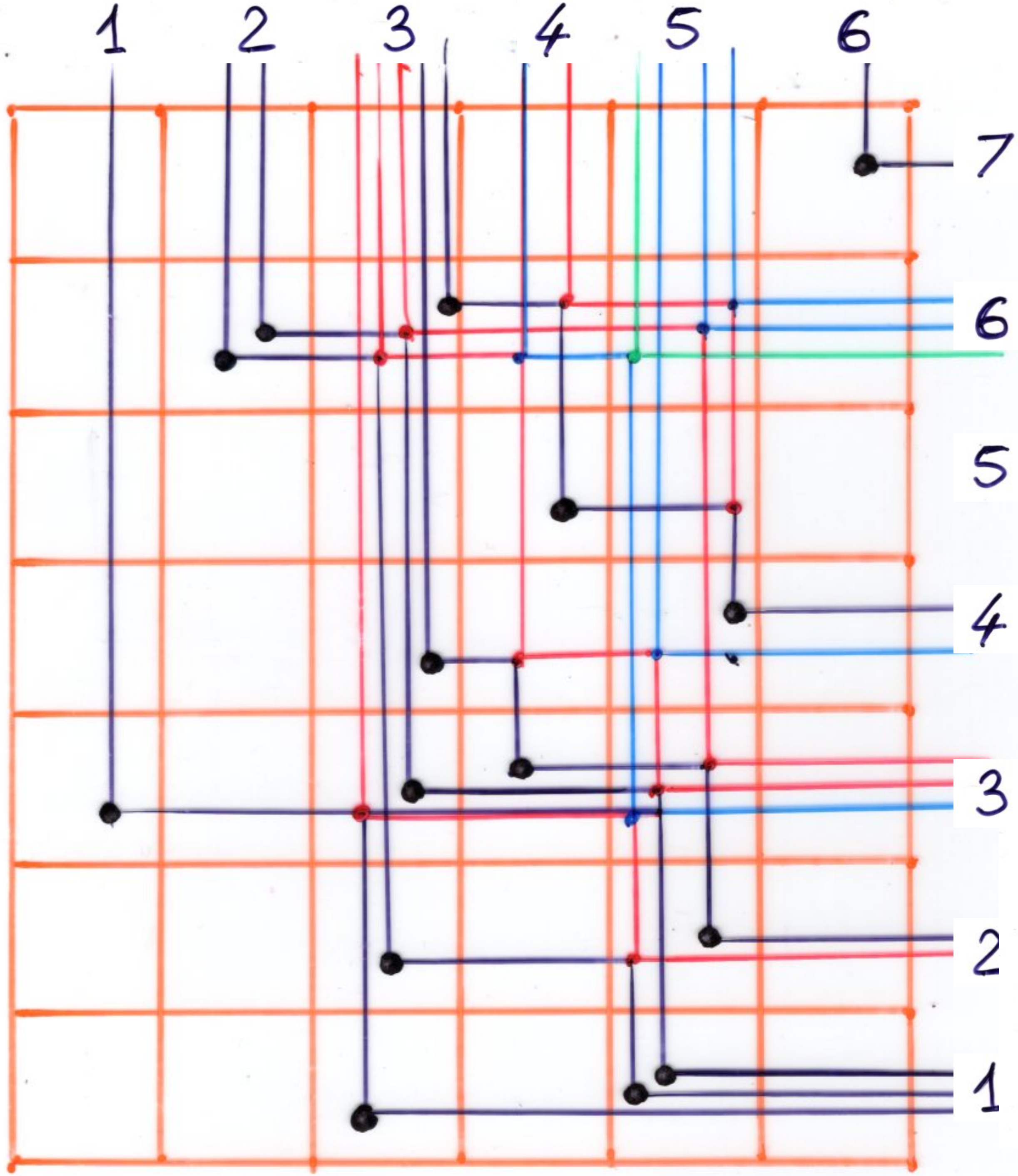
$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{bmatrix}$$





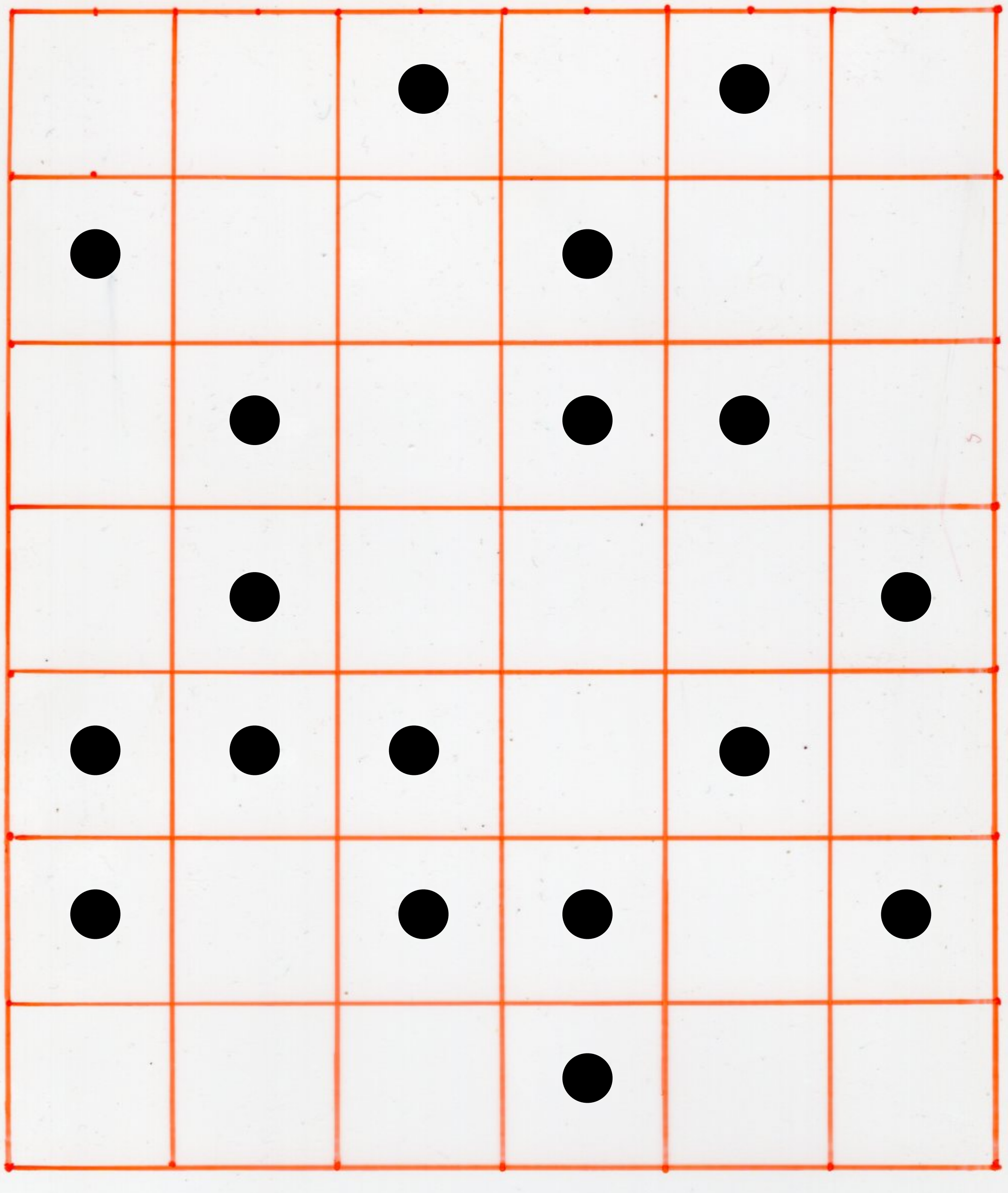
5					
4	5	5	5		
3	3	3	4		
1	2	2	3	3	6

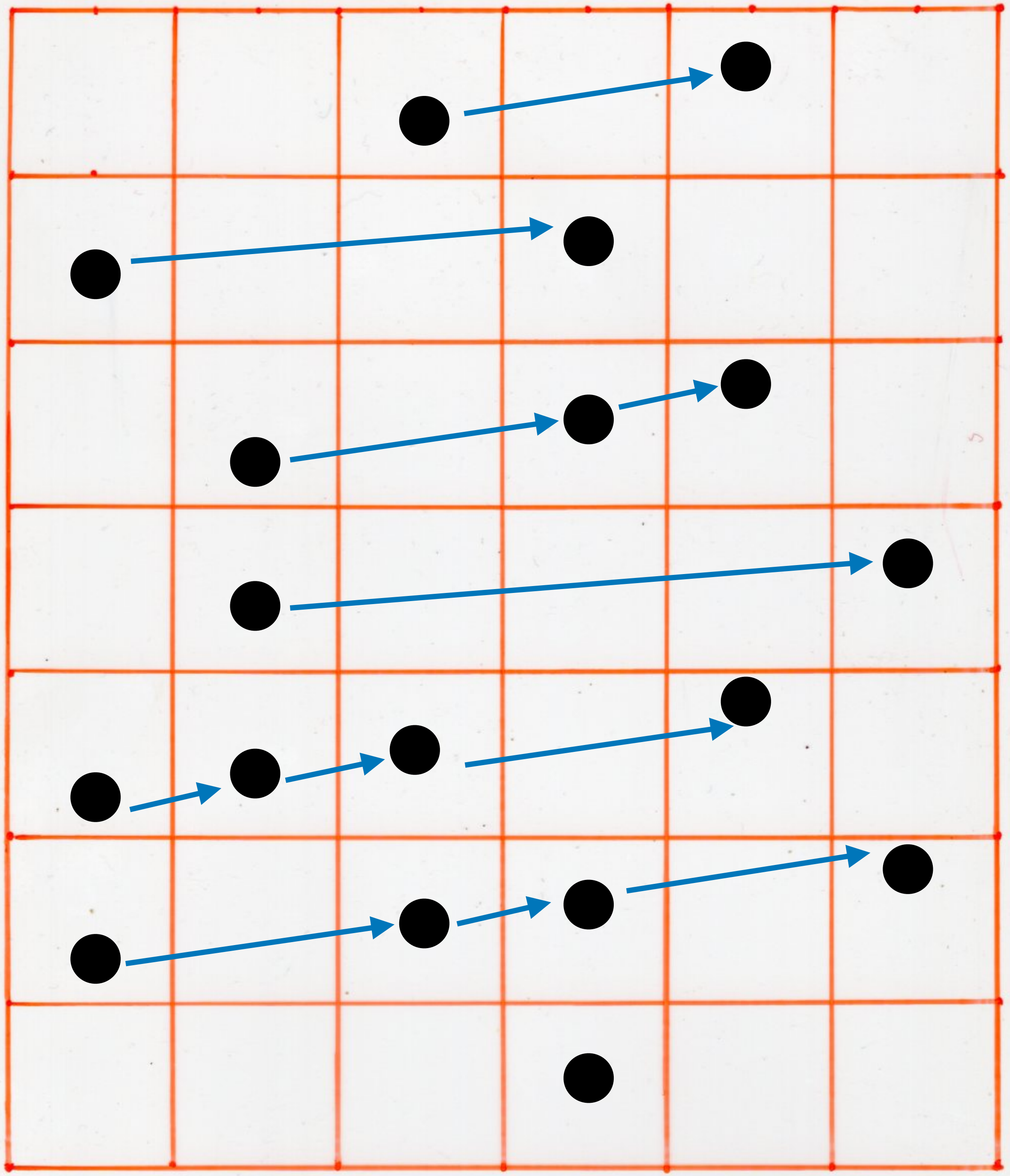
Q(M)

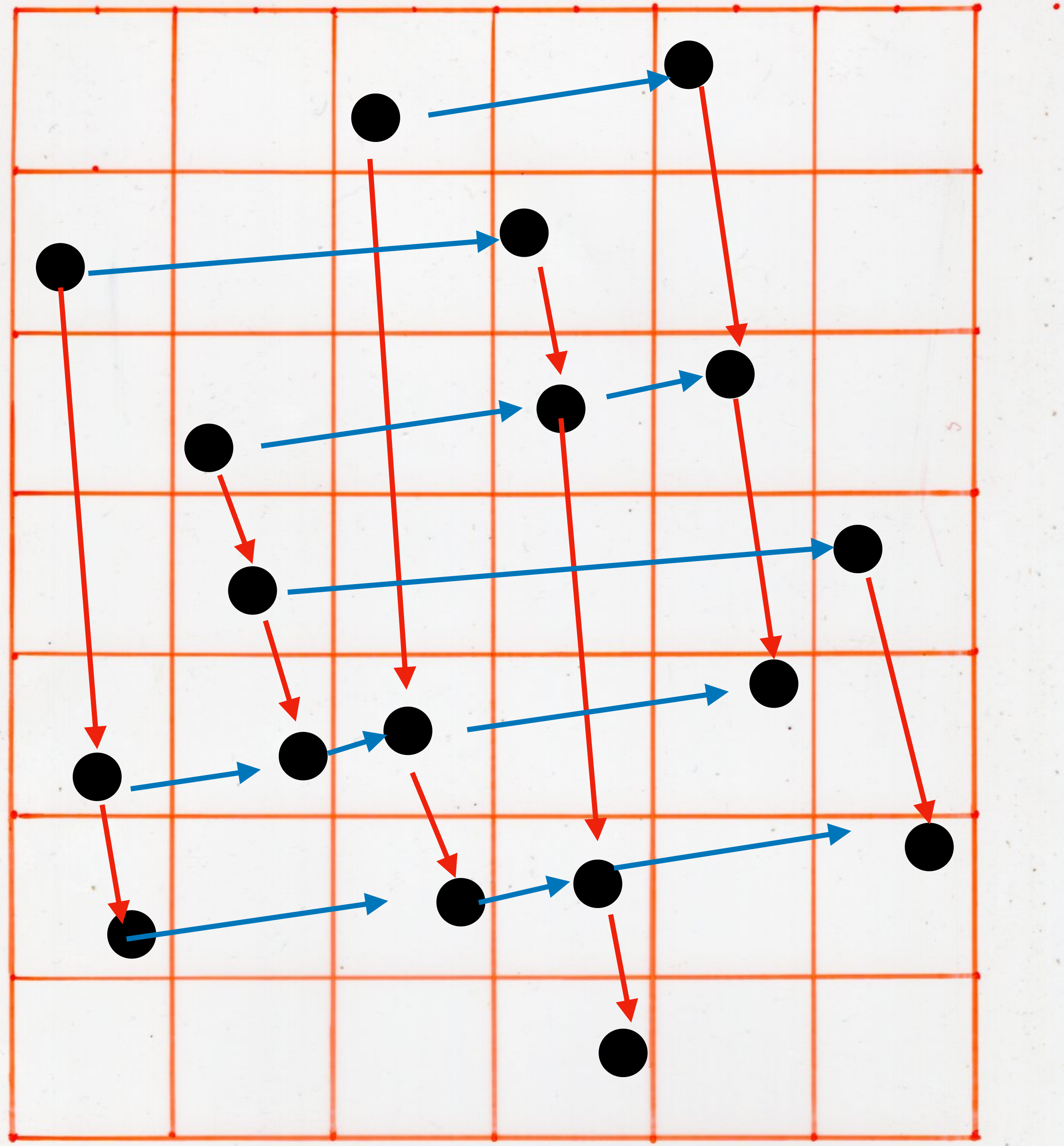


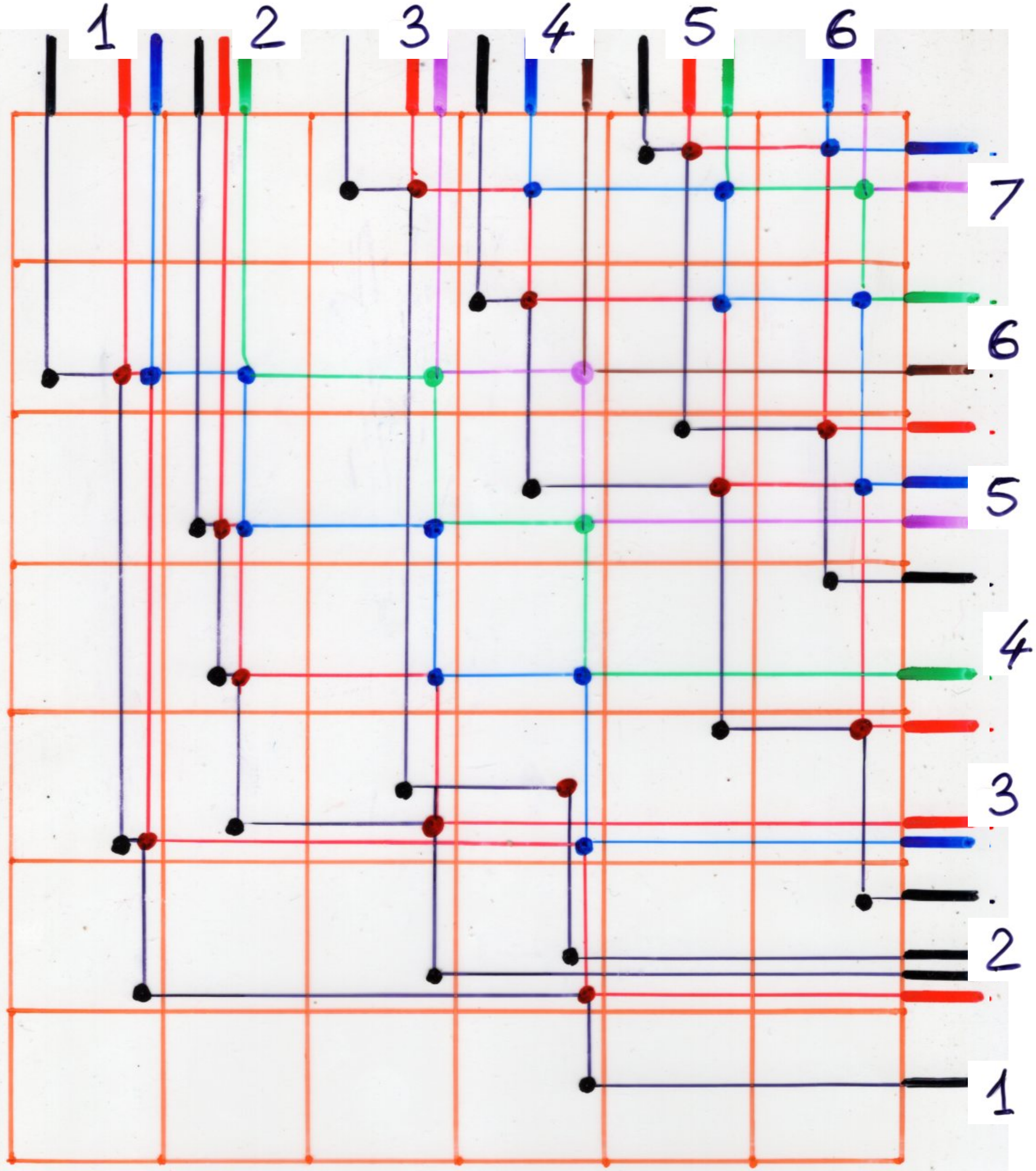
6					
3	4	6	6		
2	3	3	5		
1	1	1	2	4	7

P(M)








1
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1

Some references

see the V-book: www.viennot.org

The Art of Bijective Combinatorics

Part III. The Cellular ansatz:

bijective combinatorics and quadratic algebra

Ch1. RSK the Robinson-Schensted-Knuth correspondence

(5 lectures)

Part III, Lectures related to the course

GASCom 2008, Athens, slides and paper

S. Fomin, appendix to. R.Stanley's book

Enumerative Combinatorics, Vol2

Next talk. Monday 16, September

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

$UD = qDU + Id$

Physics

combinatorial objects on a 2D lattice

permutations

towers placements

bijections

RSK ⟷

pairs of Young tableaux

Next talk, Friday 13th:

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

representation of Q by combinatorial operators

bijections

$UD = qDU + Id$

Physics

permutations

towers placements

RSK

pairs of Young tableaux

$DE = qED + E + D$

alternative tableaux

EXF

"Laguerre histories"

permutations

orthogonal polynomials

Thank you !