An introduction to enumerative and bijective combinatorics with binary trees (part II)

TESSELLATE
STEMS-2021
CMI, Chennai, India
9 January 2021

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analytic proof

$$C_0, C_1, C_2, C_3, C_4, C_5, C_6, \ldots$$

$$1, 1, 2, 5, 14, 42, 132, \ldots$$

$$C_6 = C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0$$

$$132 = (1 \times 42) + (1 \times 14) + (2 \times 5) + (5 \times 2) + (14 \times 1) + (42 \times 1)$$

$$\left(C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4 + C_5 t^5\right)^2$$

$$\left(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5\right)^2$$

$$t$$

$$\begin{array}{ll} c_1 & \\ c_2 & t \\ c_3 & t^2 \\ c_4 & t^3 \\ c_5 & t^4 \\ c_6 & t^5 \end{array}$$

$$\begin{aligned} = {} & c_0 c_0 \\ & + (c_0 c_1 + c_1 c_0)\, t \\ & + (c_0 c_2 + c_1 c_1 + c_2 c_0)\, t^2 \\ & + (c_0 c_3 + c_1 c_2 + c_2 c_1 + c_3 c_0)\, t^3 \\ & + (c_0 c_4 + c_1 c_3 + c_2 c_3 + c_3 c_2 + c_4 c_0)\, t^4 \\ & + (c_0 c_5 + c_1 c_4 + c_2 c_3 + c_3 c_2 + c_4 c_1 + c_5 c_0)\, t^5 \\ & + \cdots \\ & + t^{10} \end{aligned}$$

$$y = 1 + t\, y^2$$

$$y = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots + c_n t^n + \dots$$

$$c_{n+1} = \sum_{i+j=n} c_i c_j \quad \Longleftrightarrow \quad y = 1 + t y^2$$

$$c_0 = 1$$

formal power series

1 1 2 5 14 42

Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynomial

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

$$+ \dots$$

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \dots + C_n t^n + \dots$$

$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

generating function

Formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

a little exercise

$$\frac{1}{1-(t+t^2)} = \ ?$$

$$\frac{1}{1-(t+t^2)} = ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \ldots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \dots$$
$$+ (t^5 \text{---} \text{---} \text{---}$$
$$\ddots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \ldots$$
$$+ (t^5 - - - -$$

$$1 \quad 2 \quad 3 \quad 5 \quad 8$$

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$t + t + t + \ldots\ldots t + \ldots$

$1 + 1 + 1 + \ldots$

$$\xcancel{t + t + t + \ldots\, t + \ldots}$$

$$\xcancel{1 + 1 + 1 + \ldots}$$

analytic proof

with

formal power series

$y = 1 + t y^2 \iff C_{n+1} = \sum_{i+j=n} C_i C_j$

$C_0 = 1$

classical
enumerative
combinatorics

$$ay^2 + by + c = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = 1 + ty^2$$

$$y = \frac{1 - (1-4t)^{1/2}}{2t}$$

$$(1+u)^m = 1 + \binom{m}{1}u + \binom{m}{2} + u^2 + \dots + \binom{m}{m}u^m$$

$$(1+u)^m = 1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 + \dots$$

$$+ \frac{m(m-1)(m-2)\dots(m-n+1)}{n!}u^n + \dots$$

$$m = \frac{1}{2} \qquad u = -4t$$

$$y = \frac{1-(1-4t)^{1/2}}{2t} \qquad m = \frac{1}{2} \qquad u = -4t$$

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}u^n + \cdots$$

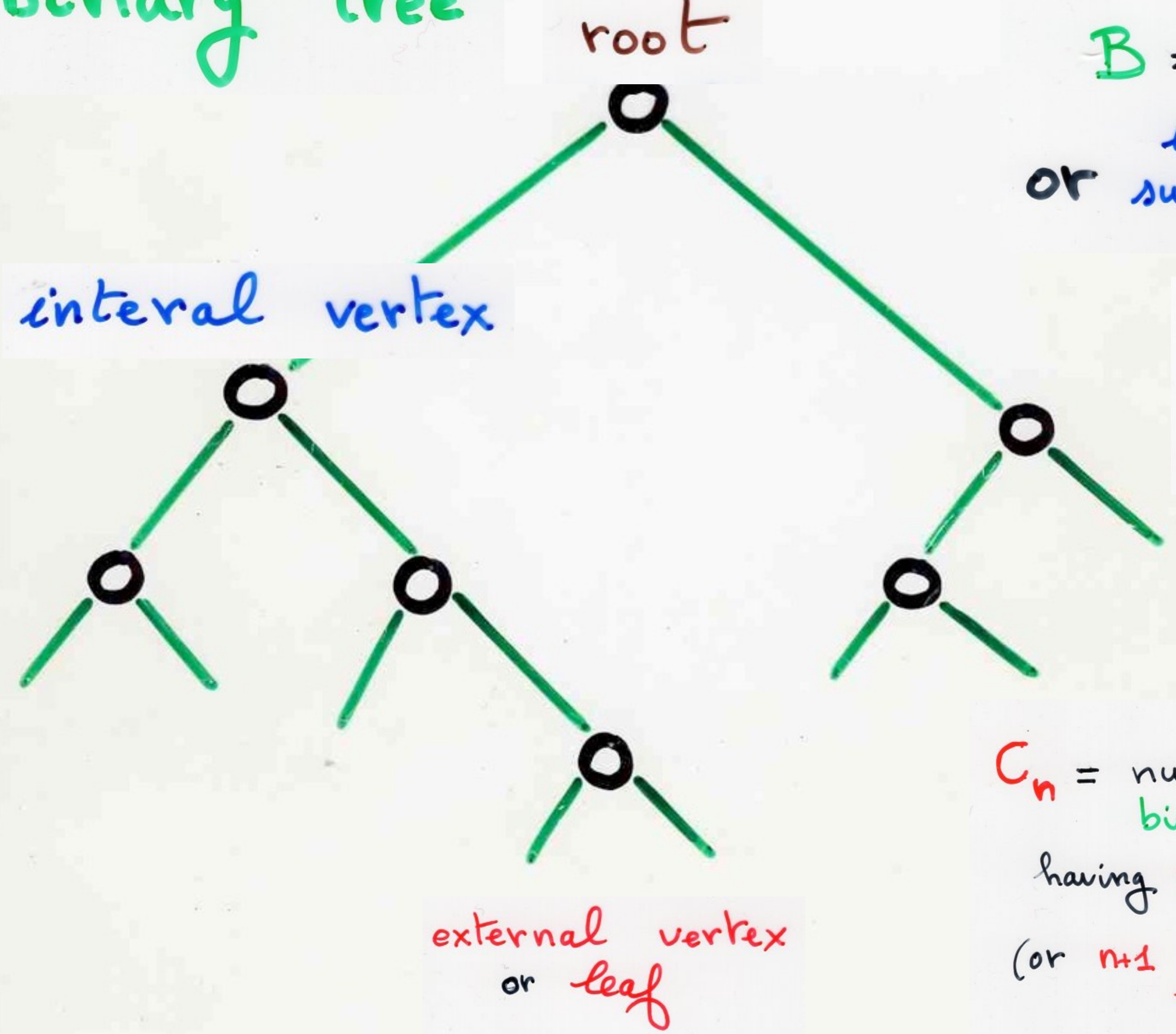
$$y = \sum_{n \geqslant 0} \underbrace{\frac{(-1)}{(n+1)!}\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)\frac{(-4)^{n+1}}{2}}_{\frac{1}{(n+1)!}2n(2n-1)\cdots(2n-n+1)} t^n$$

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

modern combinatorics

operations on combinatorial objects

binary tree



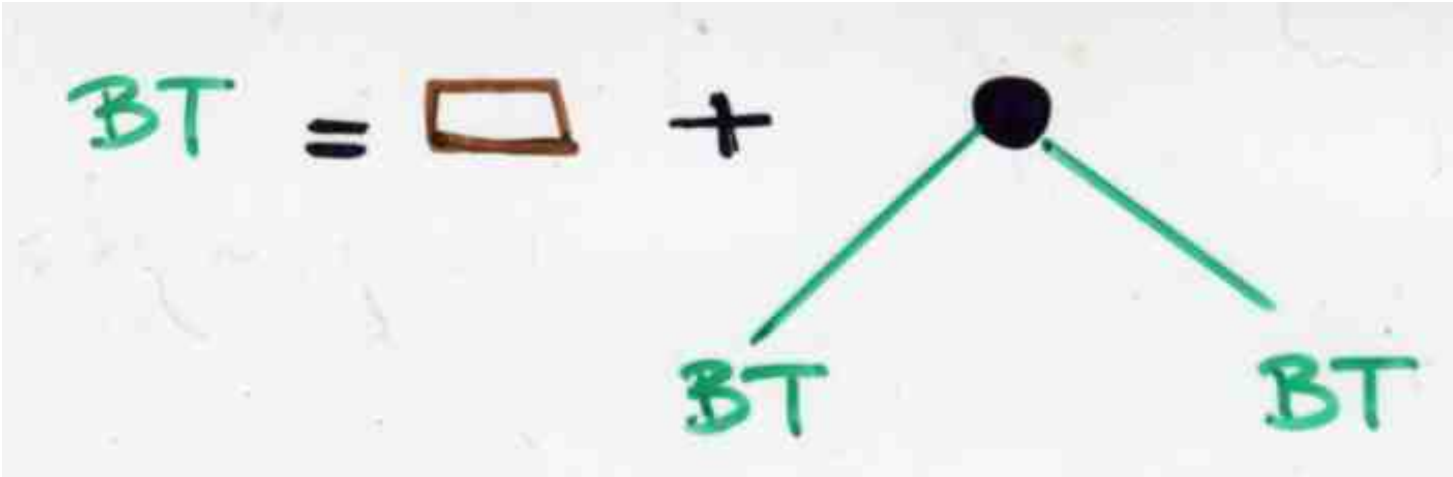
$$B = \langle \underset{\substack{\text{left}\\\text{subtree}}}{L}, \underset{\text{root}}{r}, \underset{\substack{\text{right}\\\text{subtree}}}{R} \rangle$$

or $B = \langle v \rangle$

leaf
or
external
vertex

C_n = number of binary trees
having n internal vertices
(or $n+1$ leaves
= external vertices)

modern
enumerative
combinatorics



$$BT = \square + (BT \times \bullet \times BT)$$

$$y = 1 + t\,y^2$$

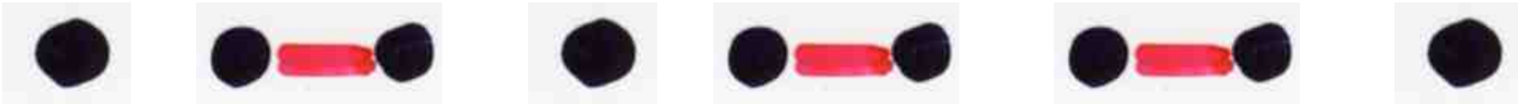
u generating function for objects A

$$\frac{1}{1-u}$$ generating function for sequences of objects A

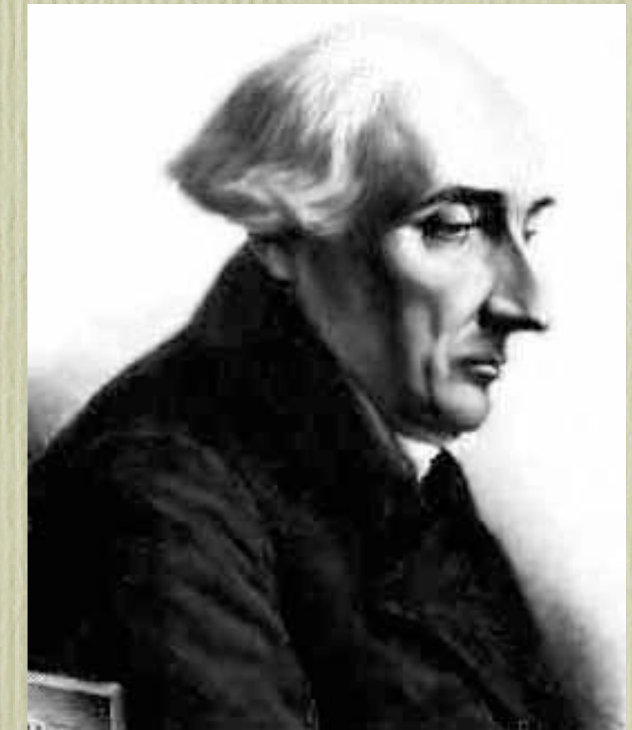
t $\quad\quad$ t^2

$$\frac{1}{1-(t+t^2)} =$$

Fibonacci numbers



the bijective paradigm



Joseph-Louis Lagrange
1736 - 1813

AVERTISSEMENT

DE LA DEUXIÈME ÉDITION.

On a déjà plusieurs Traités de Mécanique, mais le plan de celui-ci est entièrement neuf. Je me suis proposé de réduire la théorie de cette Science, et l'art de résoudre les problèmes qui s'y rapportent, à des formules générales, dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problème.

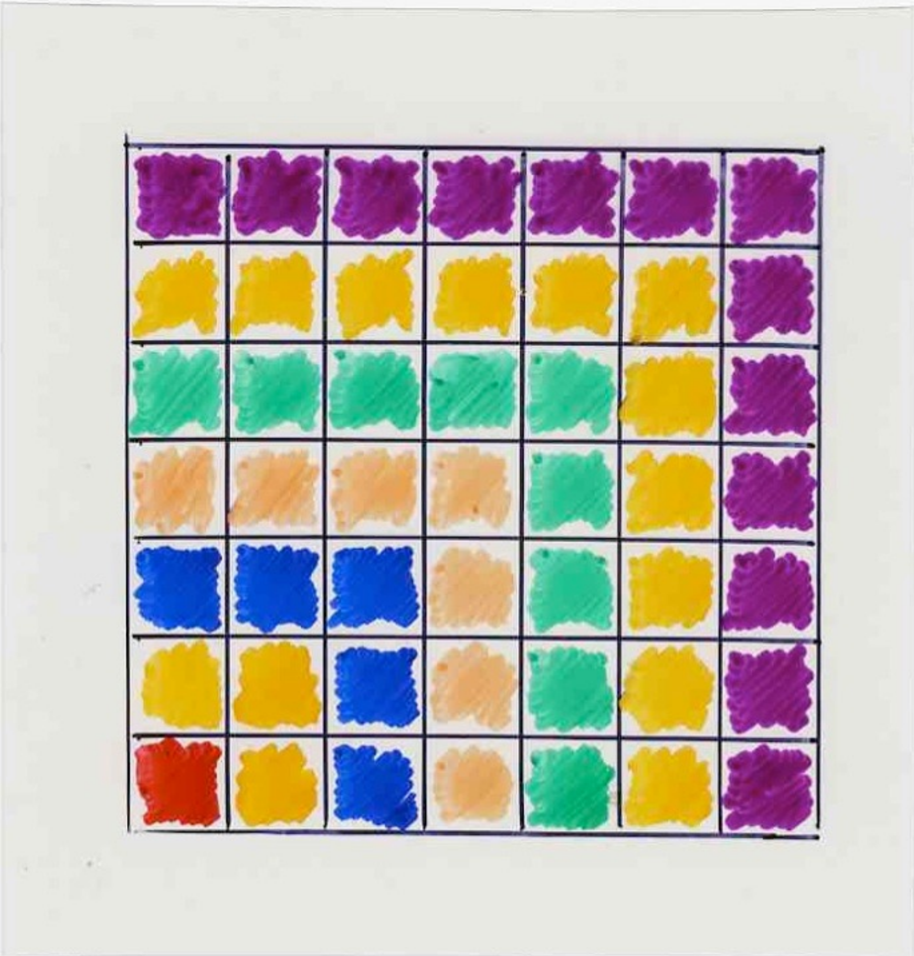
Cet Ouvrage aura d'ailleurs une autre utilité : il réunira et présentera sous un même point de vue les différents principes trouvés jusqu'ici pour faciliter la solution des questions de Mécanique, en montrera la liaison et la dépendance mutuelle, et mettra à portée de juger de leur justesse et de leur étendue.

Je le divise en deux Parties : la Statique ou la Théorie de l'Équilibre, et la Dynamique ou la Théorie du Mouvement; et, dans chacune de ces Parties, je traite séparément des corps solides et des fluides.

On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme. Ceux qui aiment l'Analyse verront avec plaisir la Mécanique en devenir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine.

replacing calculus by figures and bijections

proof without words ....



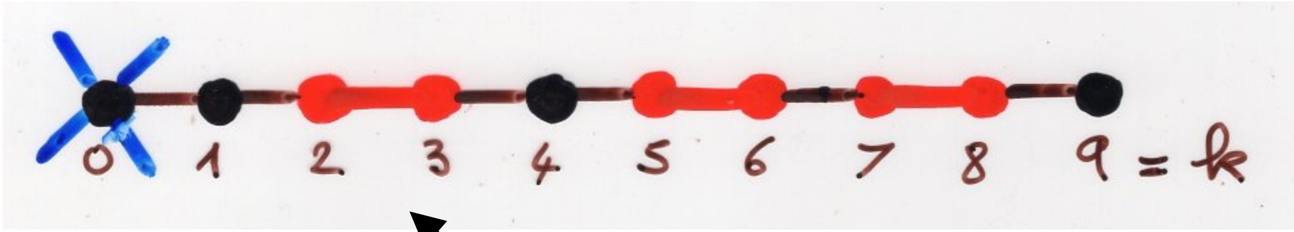
$$n^2 = 1+3+\dots+(2n-1)$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

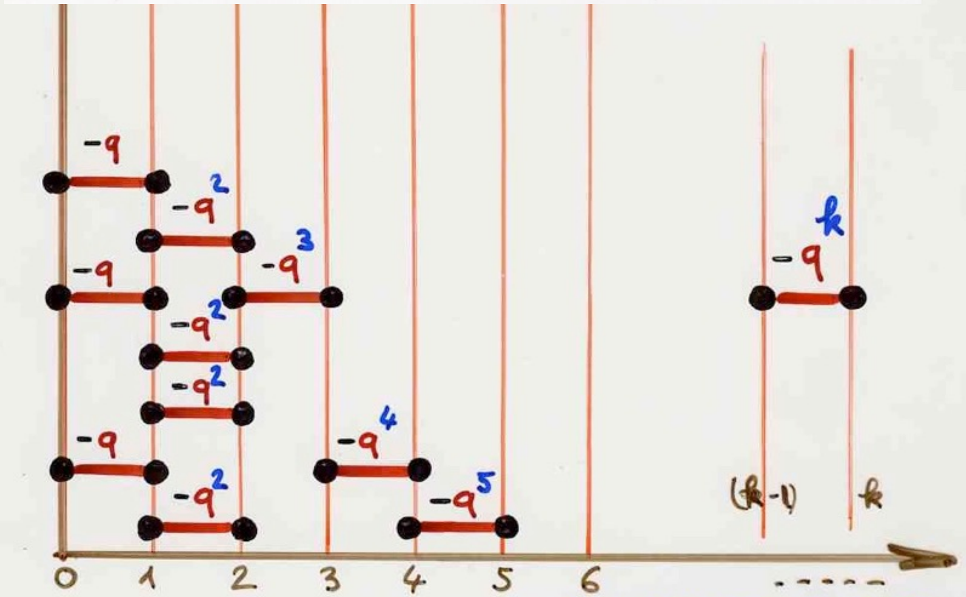
$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cdots\cdots\cfrac{}{1+\cfrac{q^k}{\cdots\cdots}}}}}$$

Ramanujan
continued fraction

N



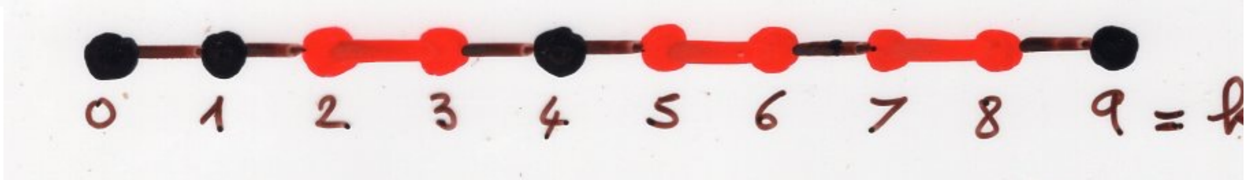
weighted heap $v(E)$



poids total $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

$$= \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

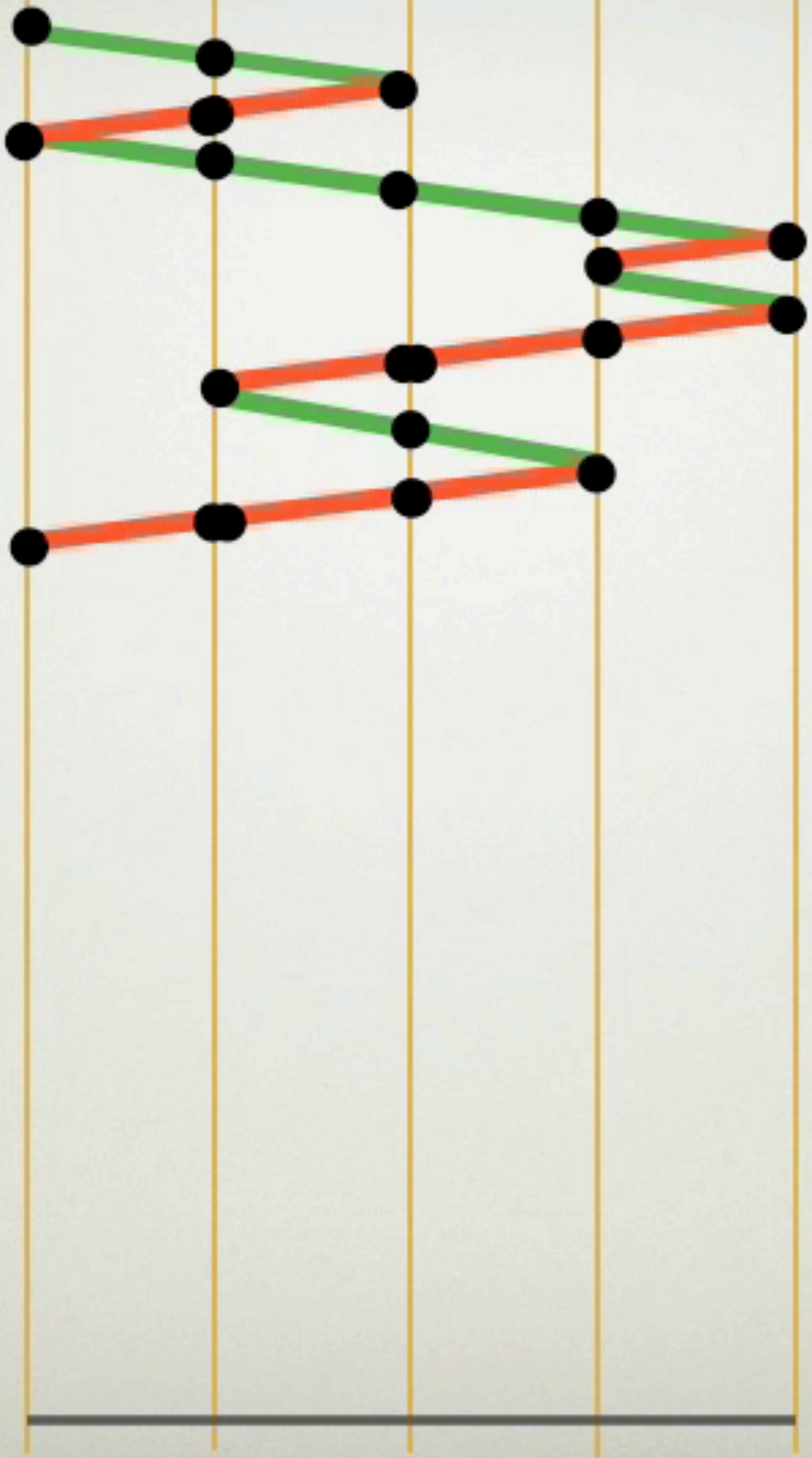
D



bijection

Dyck paths

Heaps of dimers (pyramids)



violins:

Association
Cont'Science

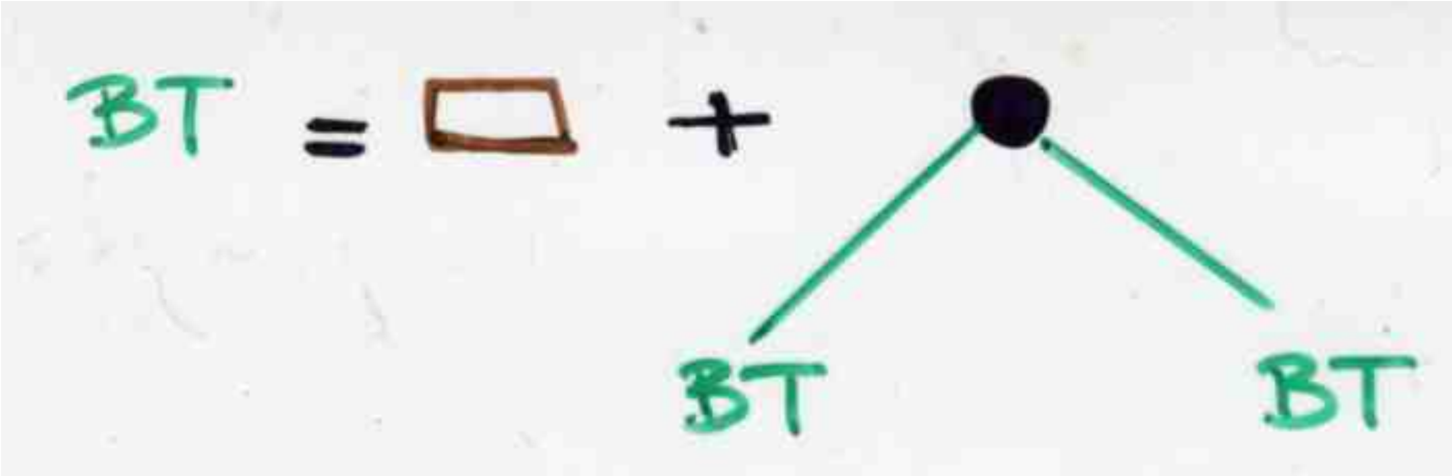
Gérard H.E. Duchamp

replacing calculus by figures and bijections

conversely

making calculus from the visual figures

modern
enumerative
combinatorics



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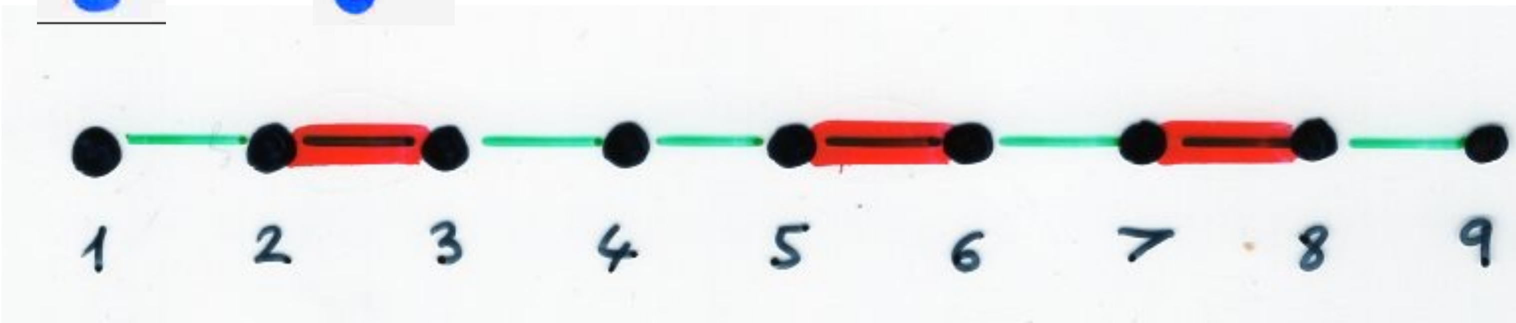
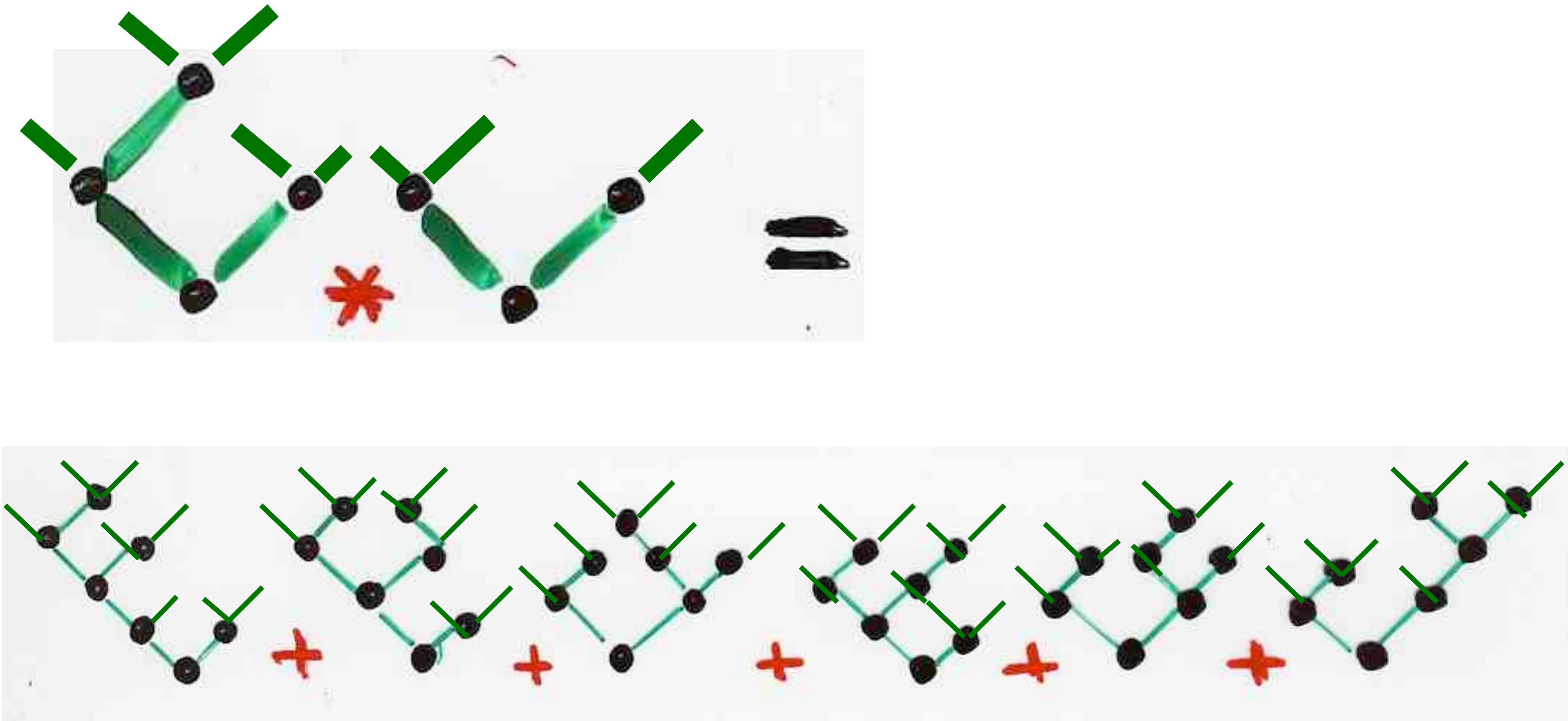
$$\frac{1}{1-(t+t^2)} = \text{Fibonacci numbers}$$

t $\quad$ t^2

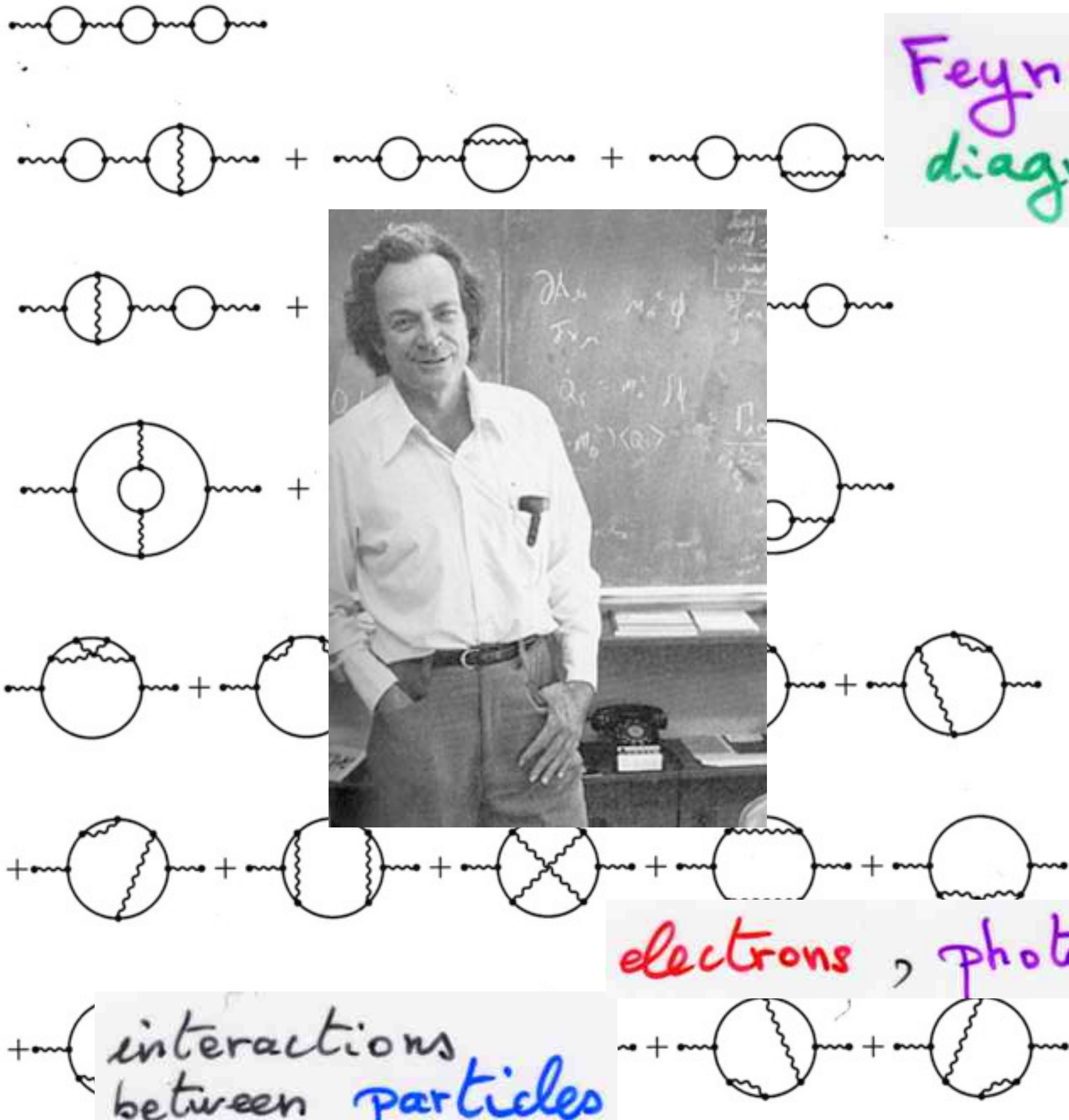




product of binary trees
sum

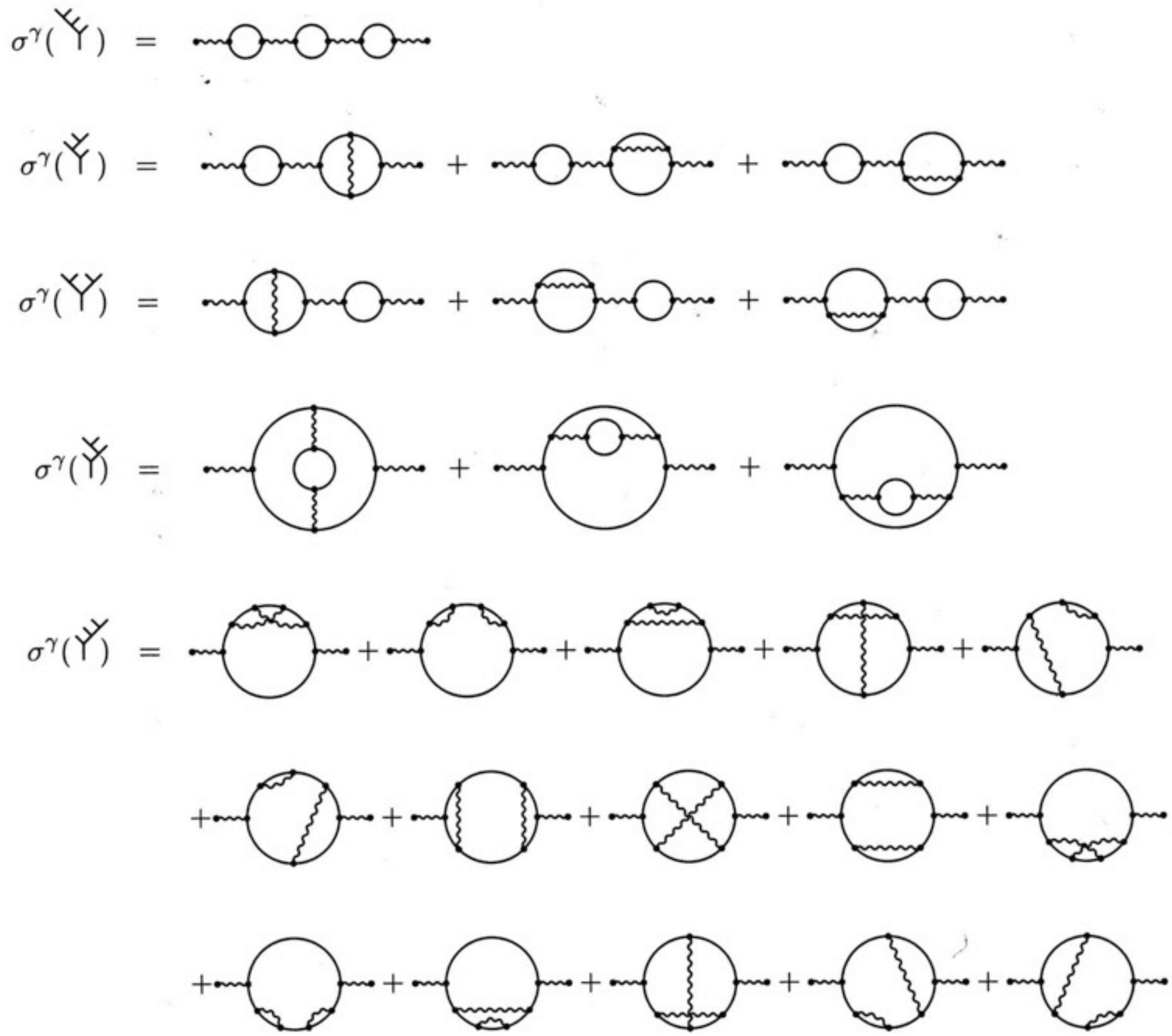
Loday, Ronco (1998)

Feynman diagrams



electrons, photons, ...

interactions between particles



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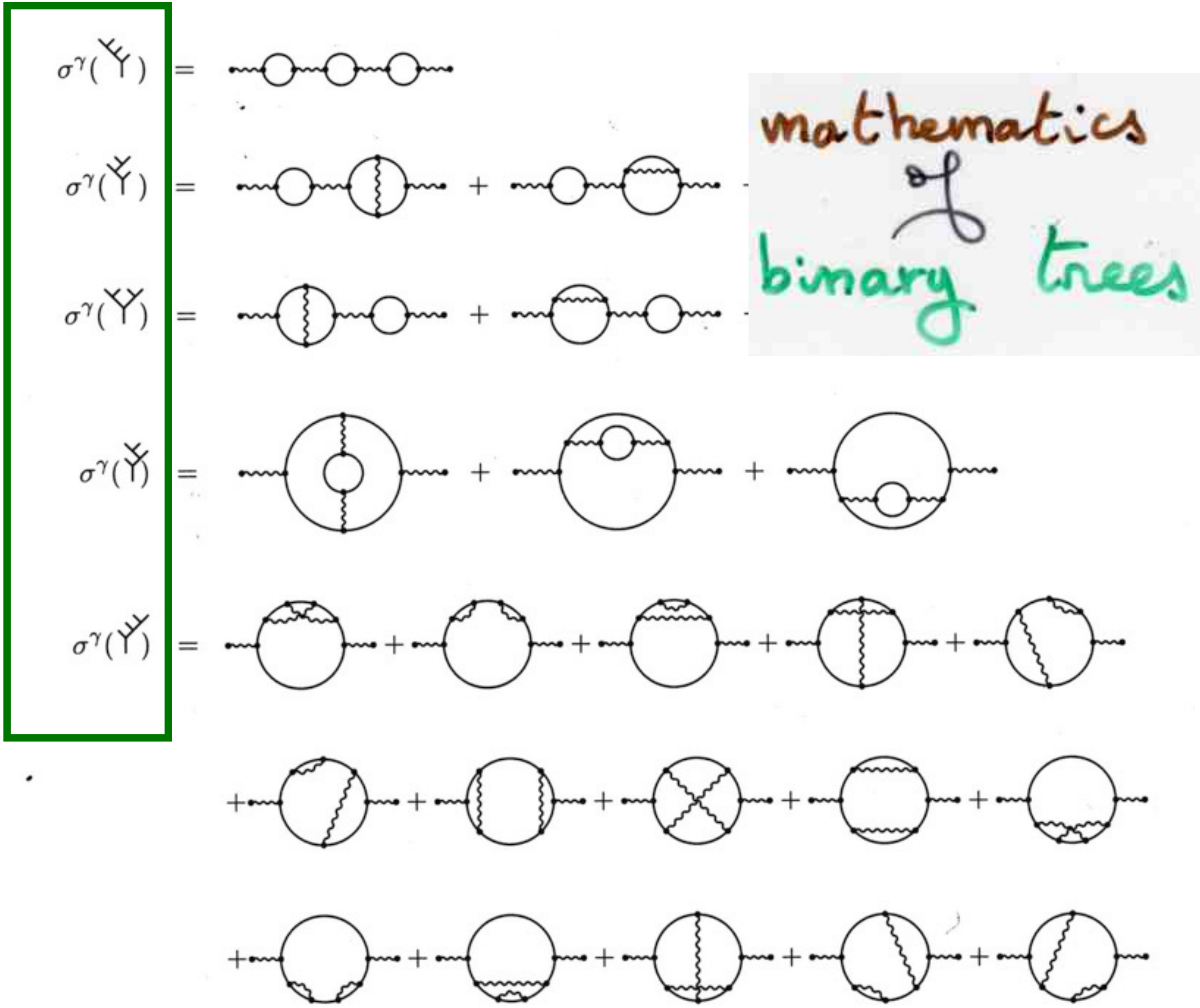

mathematics of binary trees

Triangulations of a convex polygon

(1838)

Note sur une Équation aux différences finies ;

PAR E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \quad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \quad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \dots \frac{4n-6}{n} P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\ 4.\ 5\dots n}. \quad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3)$$
$$= \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \quad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \quad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \quad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned} \frac{1}{n+1}C_{2n,n} &= \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1} \\ &+ \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \dots + \frac{1}{n}C_{2n-2,n-1}. \end{aligned}\right\} \quad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

$$1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924\dots;$$

qu'on les divise respectivement par

on obtiendr

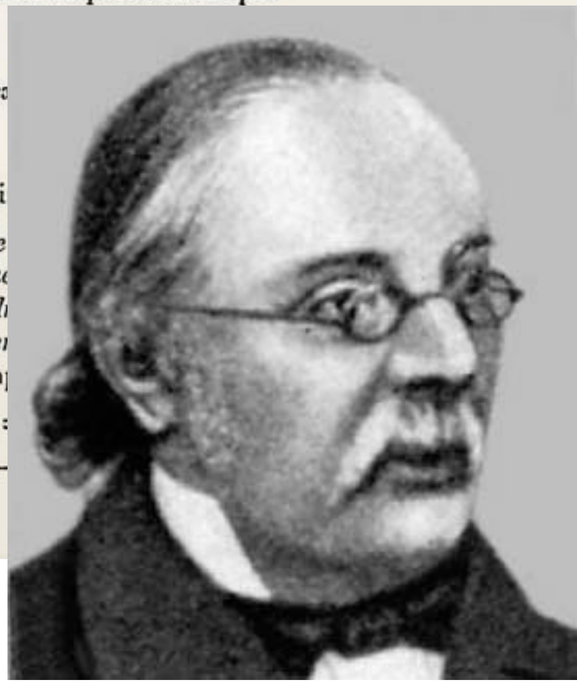
(A)

lesquels joui

Un terme ... *nme des*
produits qu ... *ême, et*
dans un ord ... *n multi-*
pliant les te

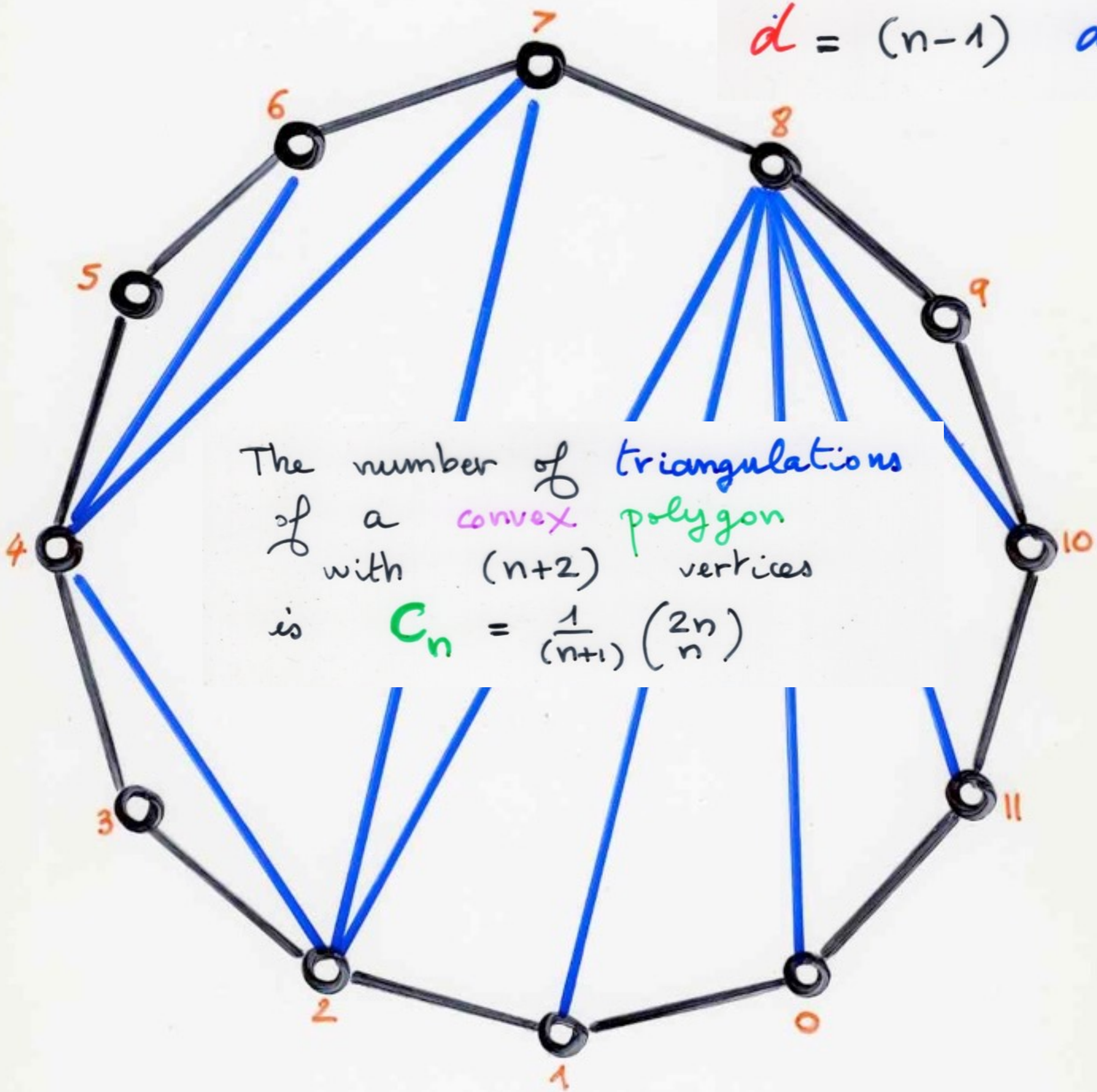
Par exem

132



Eugène Catalan (1814-1894)

$d = (n-1)$ diagonals



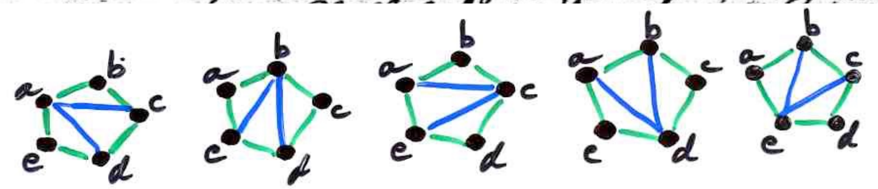
The number of triangulations of a convex polygon with $(n+2)$ vertices is $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

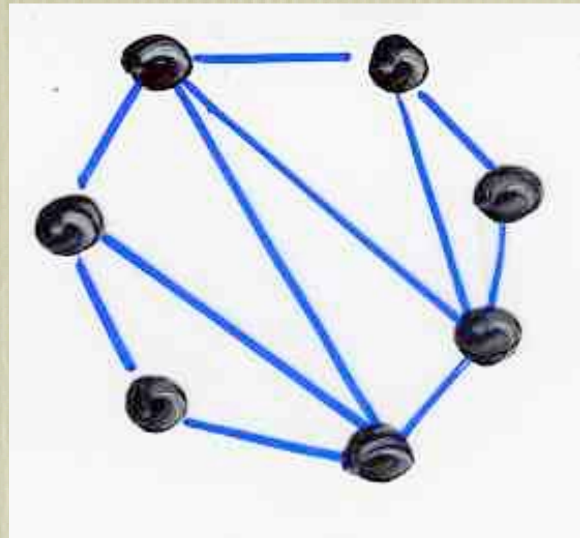
Leonhard
Euler
1707 - 1783

A letter from Leonhard Euler to Christian Goldbach ....

Berlin, 4 September 1751

Diagonalen I. ac, ad; II. bd, be; III. ca, ce; IV. db, da; V. ec, eb





wann $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

$$x = \frac{2.6.10.14.18.22.\ \dots\ (4n-10)}{2.\ 3.\ 4.\ 5.\ 6.\ 7.\ \dots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \frac{2}{2},\ 2 = 1.\frac{6}{3},\ 5 = 2.\frac{10}{4},\ 14 = 5.\frac{14}{5},\ 42 =$

$1, 2, 5, 14, 42, 132,$ etc.

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

wenn $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc} = 4.$

Berlin d. 4ᵗᵉⁿ Sept 1751.

L. Euler

getheilet, und solches kan auf 5 verschiedene Arten geschehen, nehmlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V $\frac{ec}{eb}$.
Ferner kan ein Sechseck durch 3 Diagonales in 4 Triangula zerschnitten und dieses kan auf 14 verschiedene Arten geschehen.
Nun ist die Frage generaliter, da ein Polygonum von n Seiten durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen kann.
Setze ich nun die Anzahl dieser verschiedenen Arten $= x$ ... gefunden

so habe ich

$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

wenn $n =$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$

Hieraus habe ich nun den Schluss gemacht, daß generaliter sey

$$x = \frac{2.6.10.14.18.22.\ \ldots\ (4n-10)}{2.3.4.5.6.7\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$14 = 5.\frac{14}{5}$, $42 = 14.\frac{18}{6}$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a^2}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$1+2a+5a^2+14a^3+42a^4+\text{etc} = 4.$$

also wenn $a=\frac{1}{4}$ so ist $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc}$

[illegible]

Ew. Hochwohlgeb.

[illegible]

Berlin d. 4t Sept 1751.

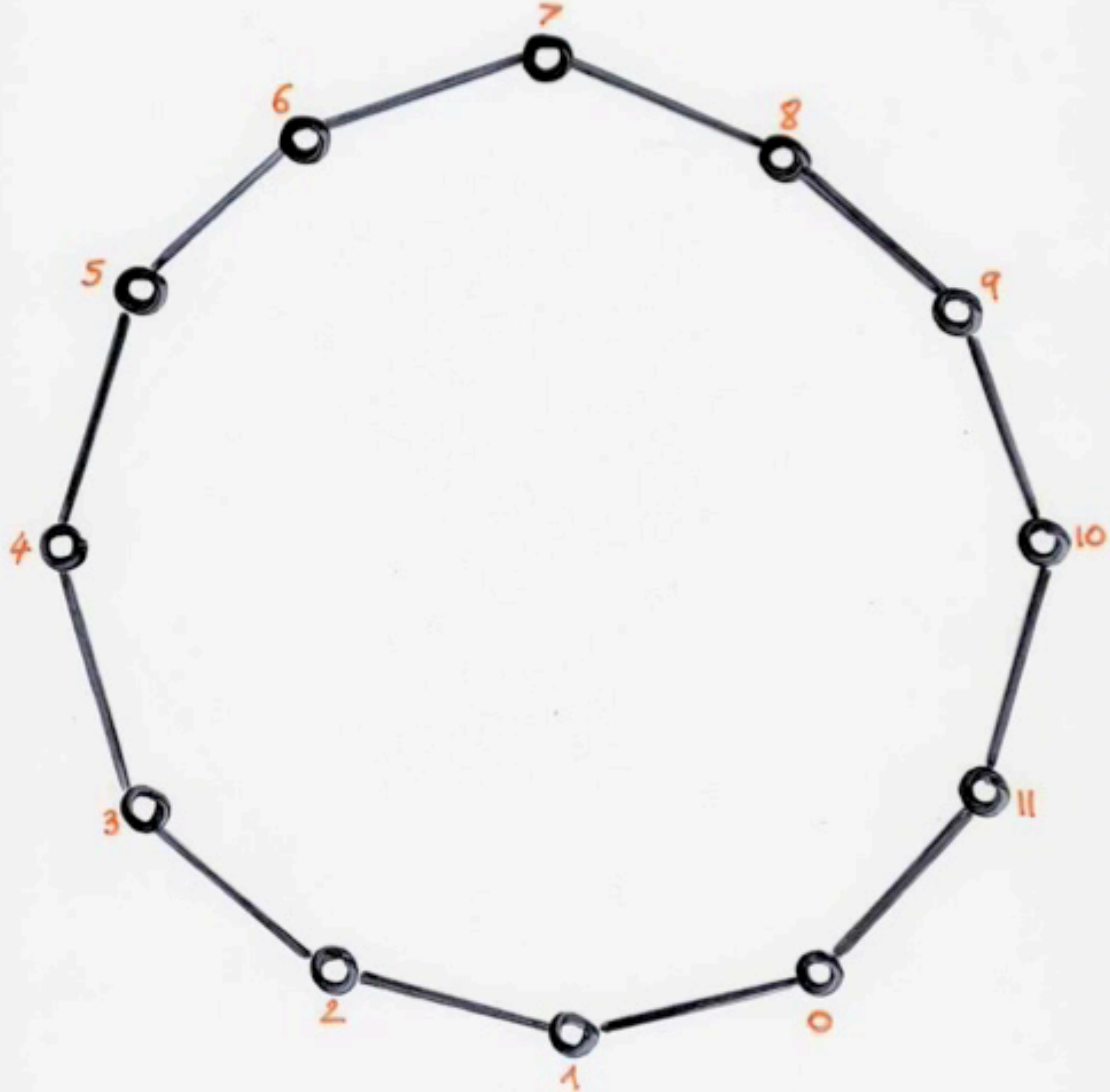
4 Sept 1751
Berlin

L. Euler

bijection

triangulations

binary trees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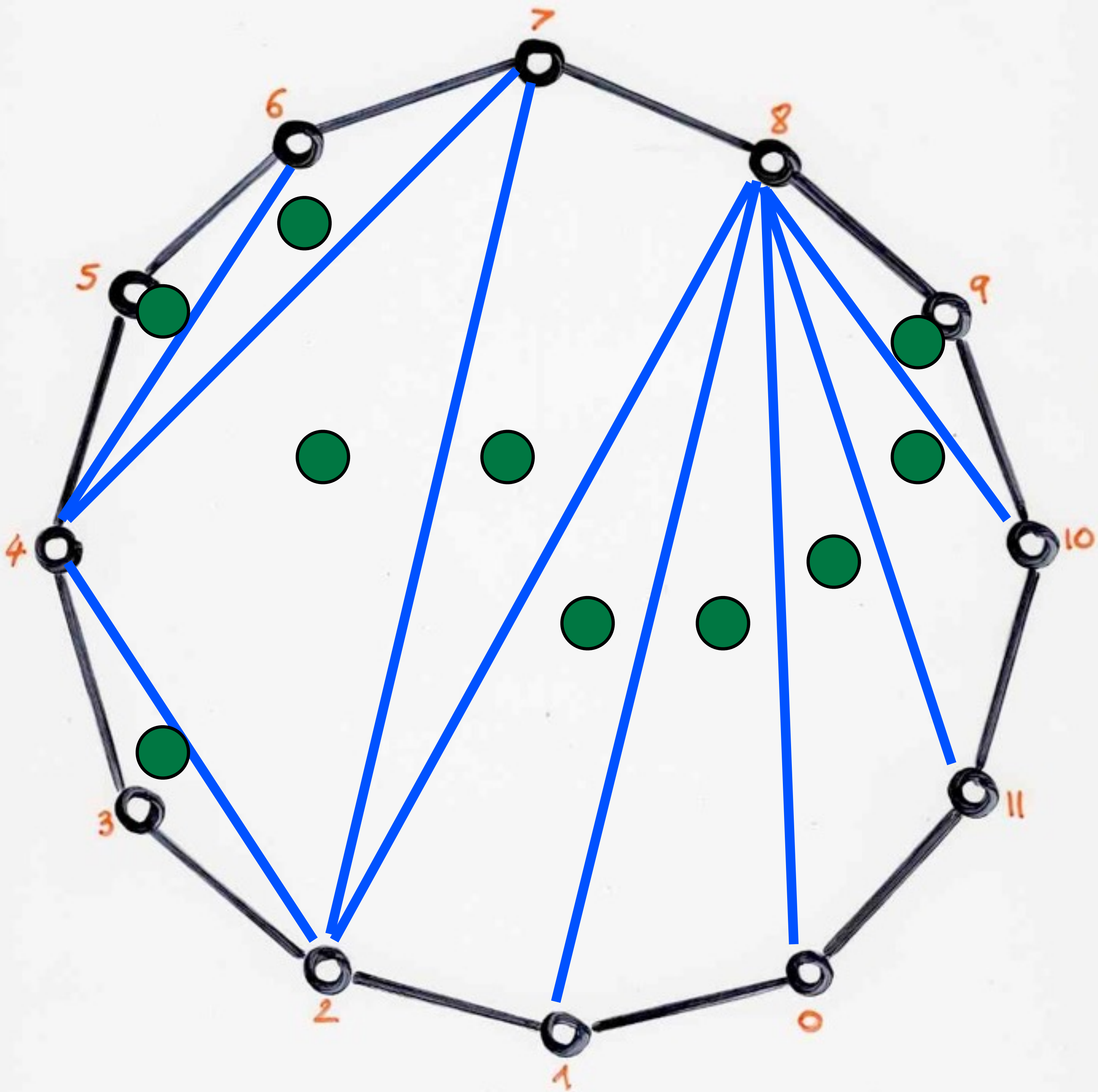
violins:

Mariette Freudentheil

Gérard H.E. Duchamp

Association
Cont'Science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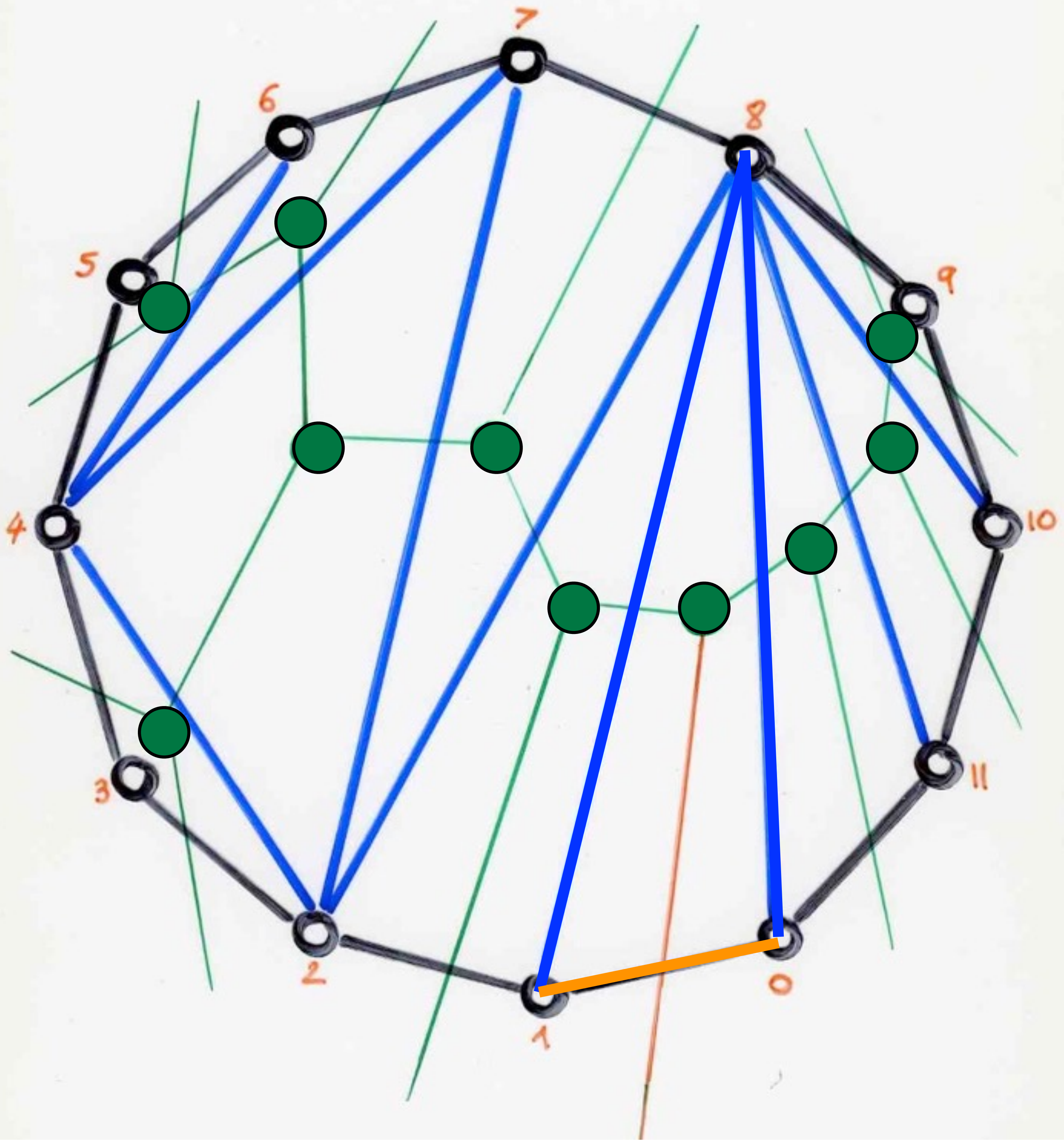
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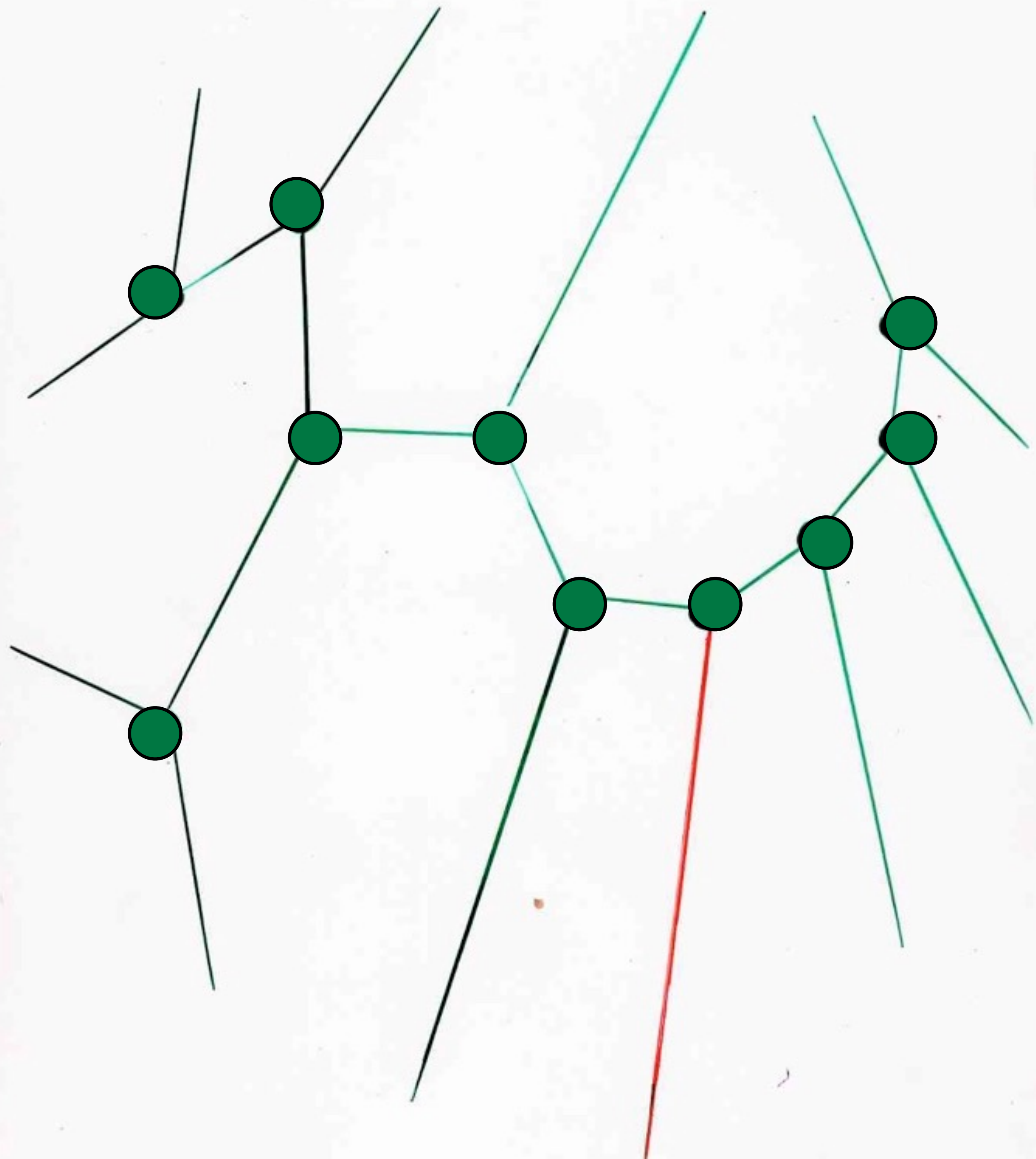
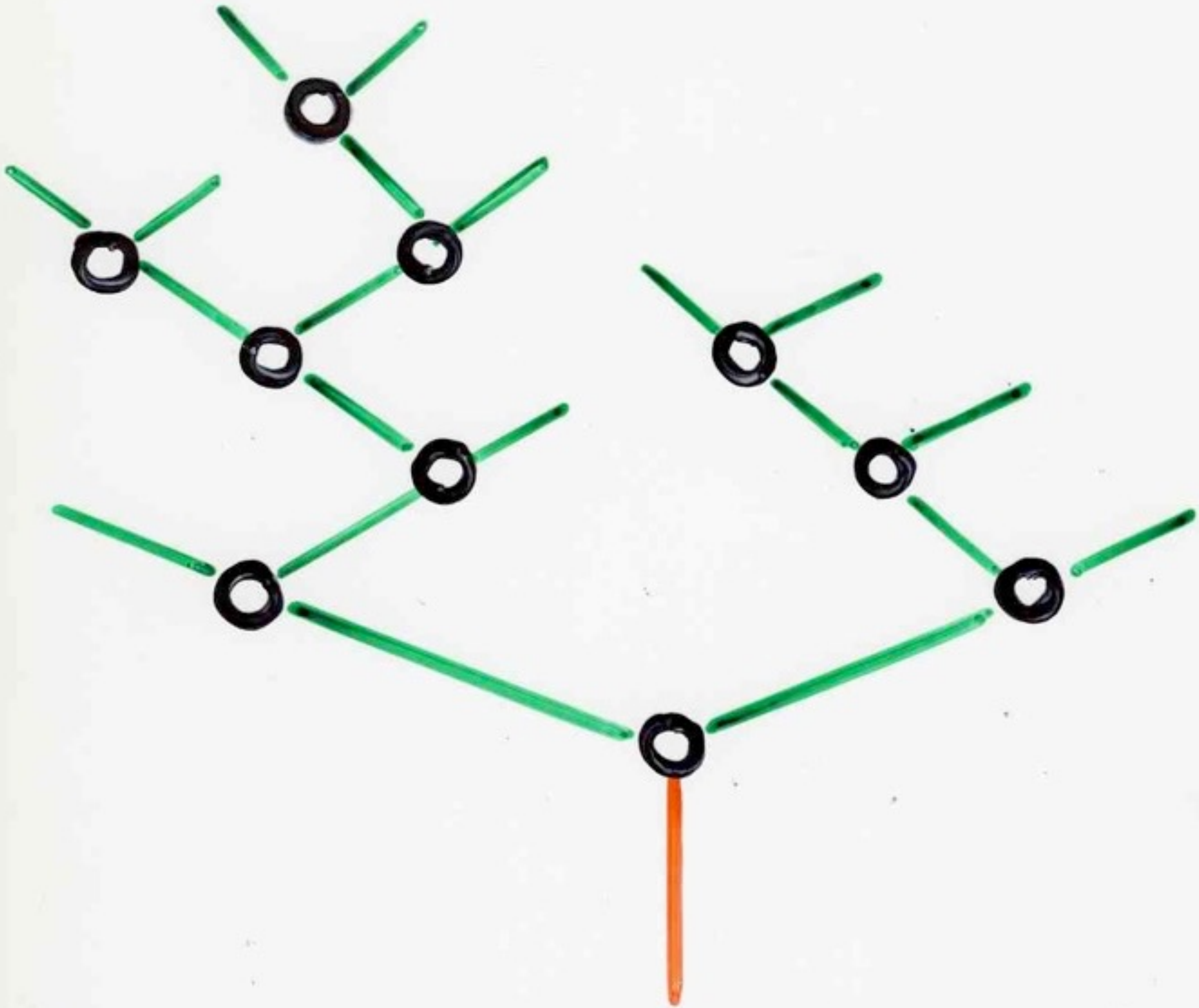
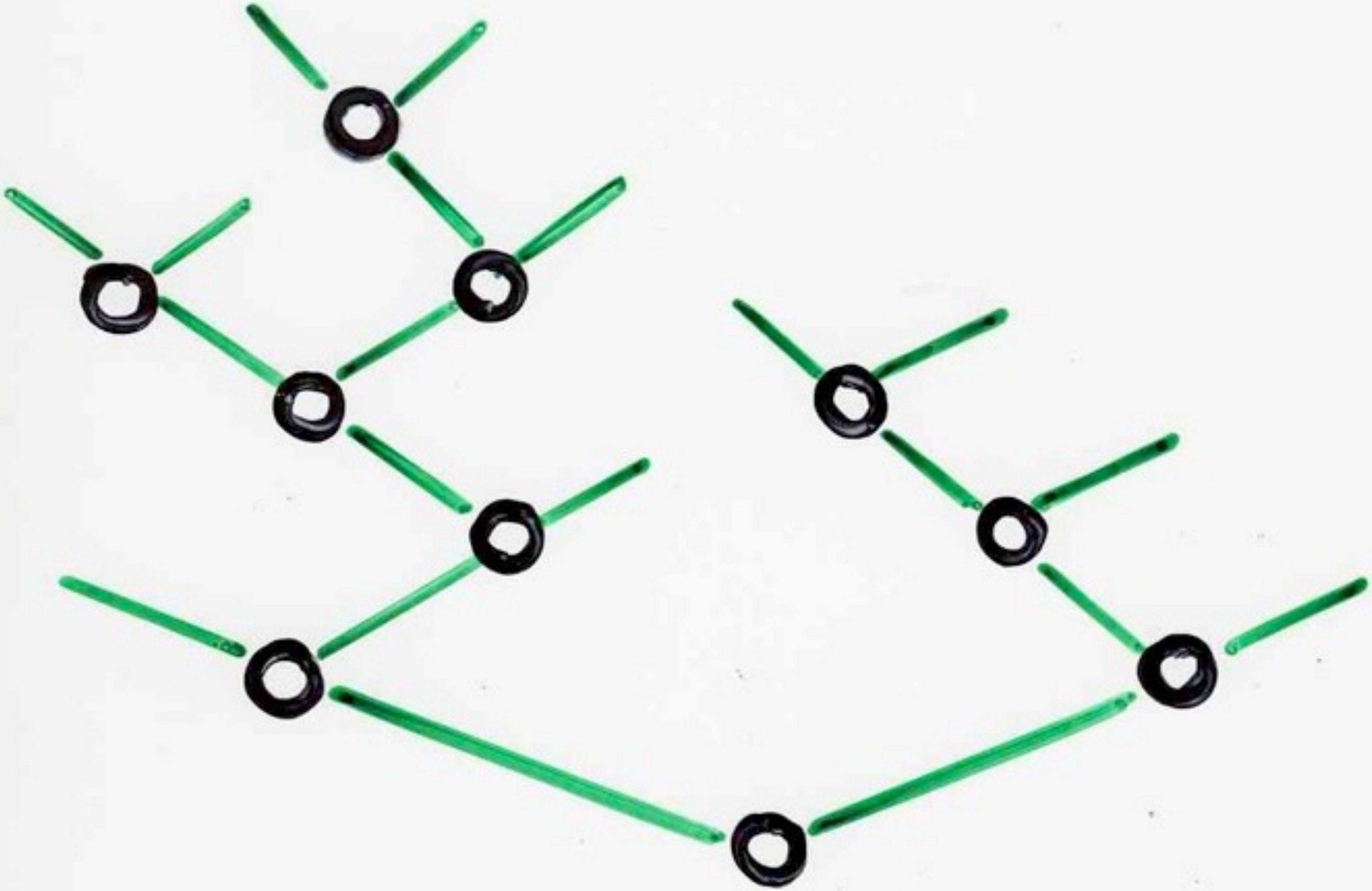






exercise 11

describe the reverse bijection

binary trees $\longrightarrow$ triangulations

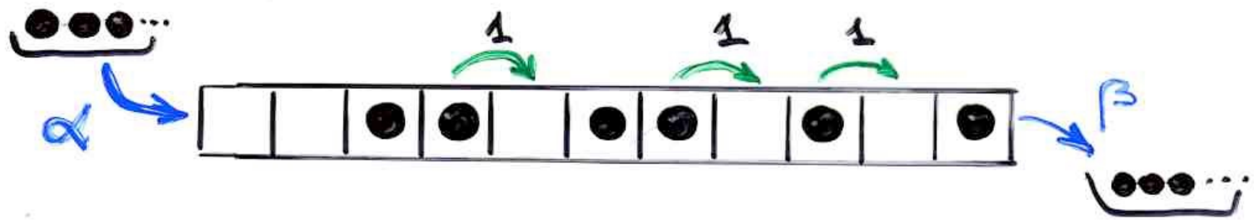
Relation with physics

of dynamical systems

The TASEP

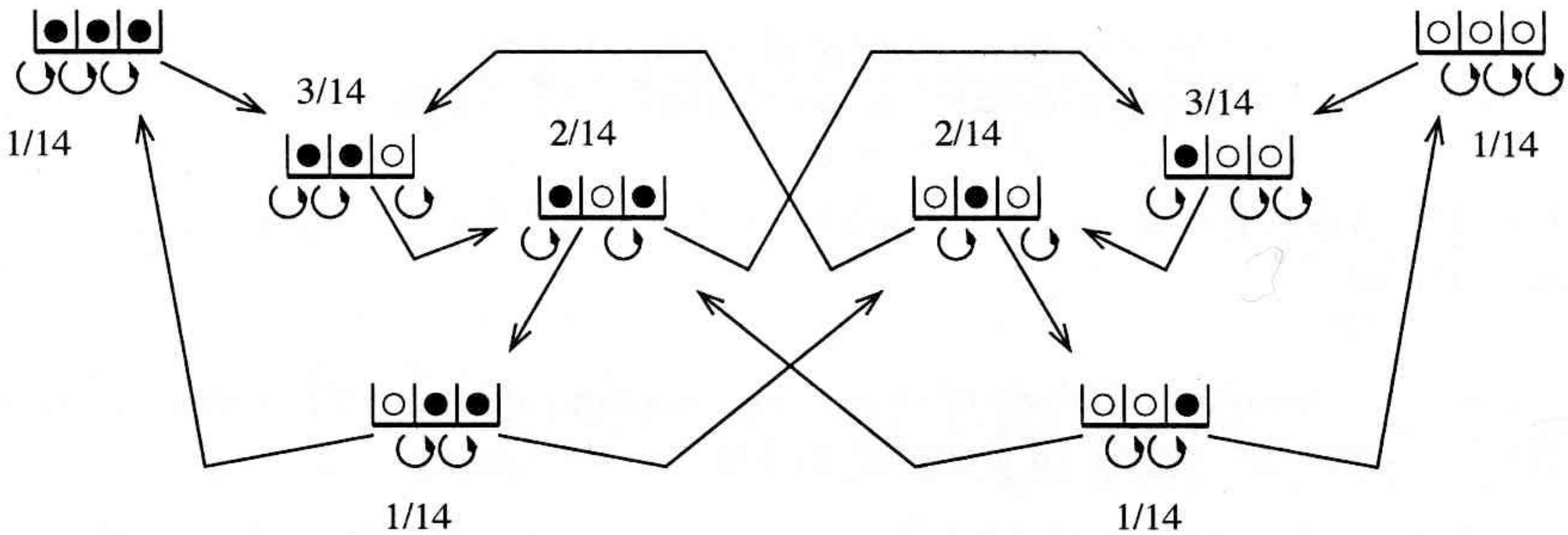
TASEP

"totally asymmetric exclusion process"



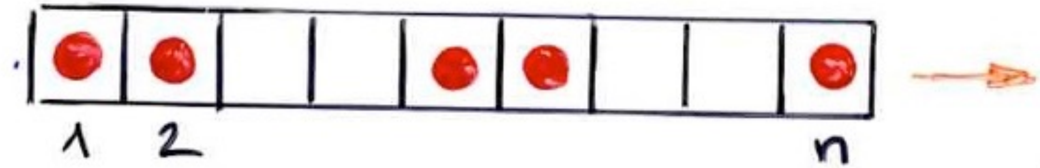
stationnary probabilities

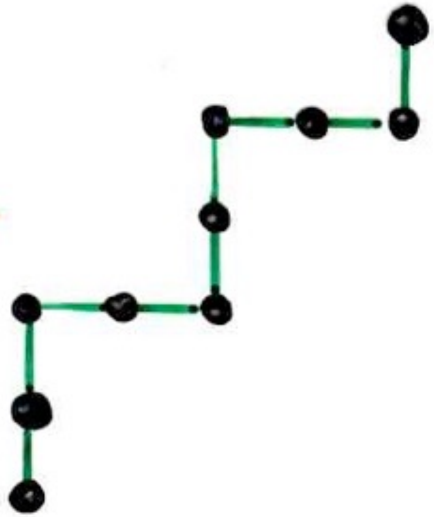
time $\longrightarrow \infty$



1/14
3/14
2/14
2/14
3/14
1/14
1/14
1/14

state $s = (\tau_1, \ldots, \tau_n)$

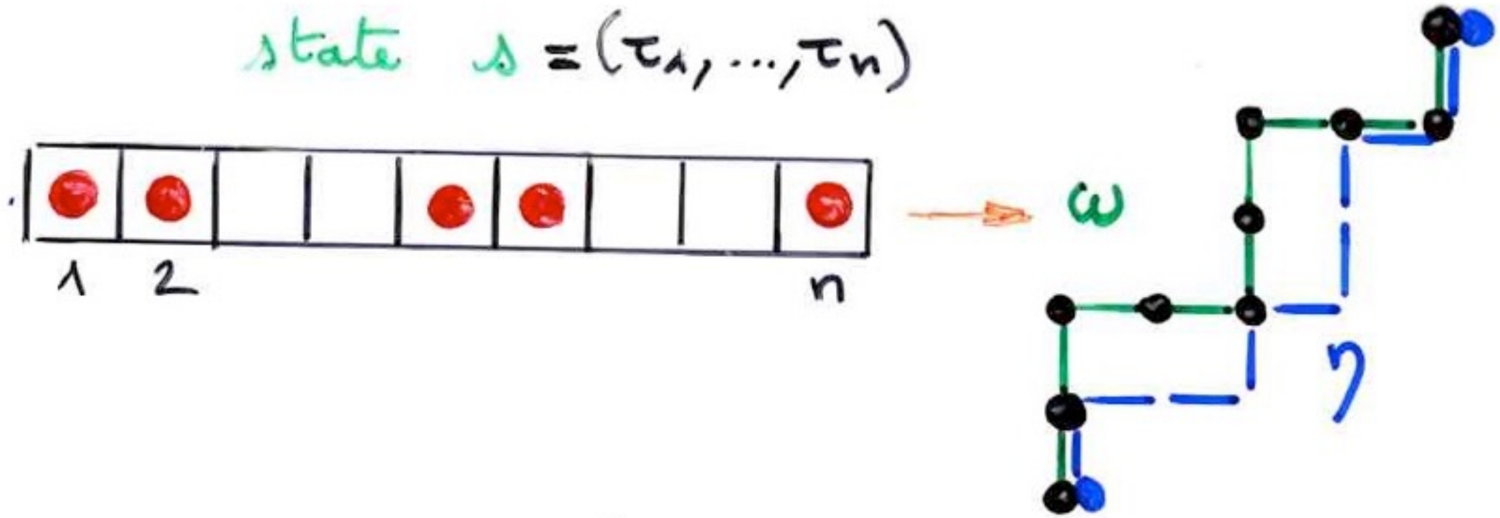




$\mathcal{P}_n(s) =$

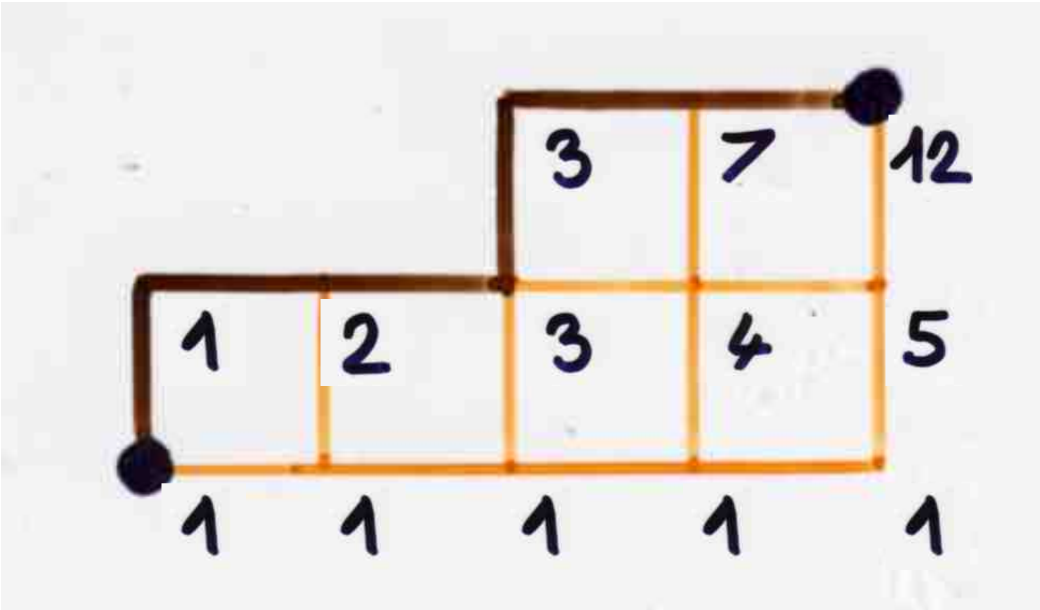
Shapiro, Zeilberger (1982)

state $s = (\tau_1, \dots, \tau_n)$

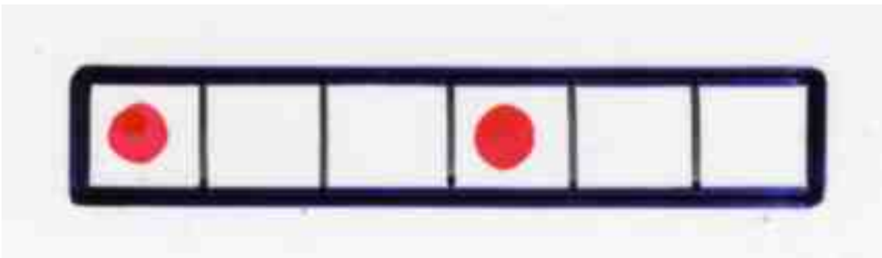


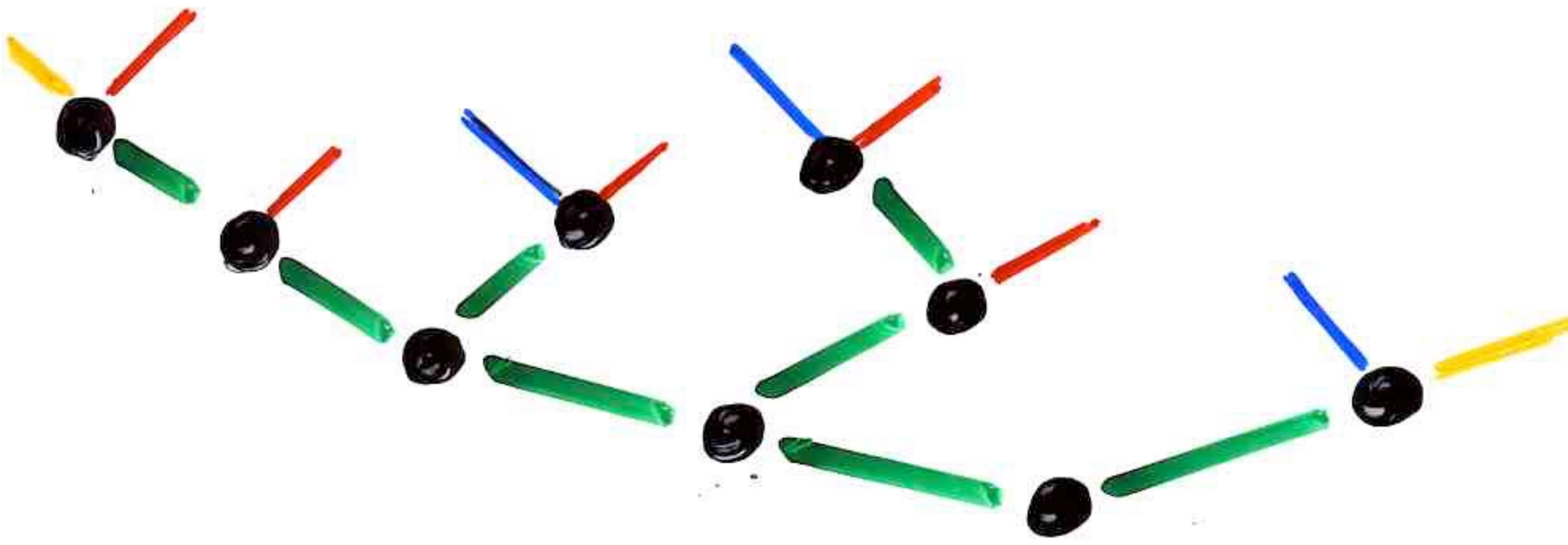
$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger (1982)



$$\frac{12}{429}$$

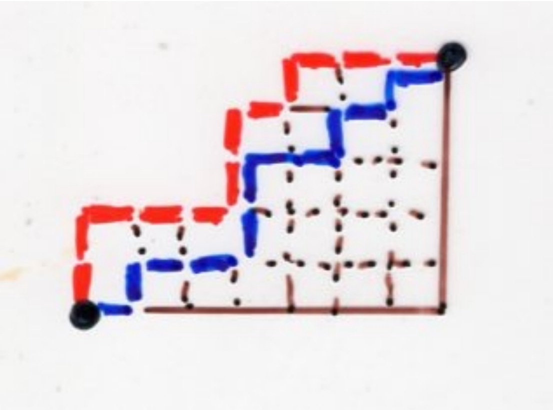


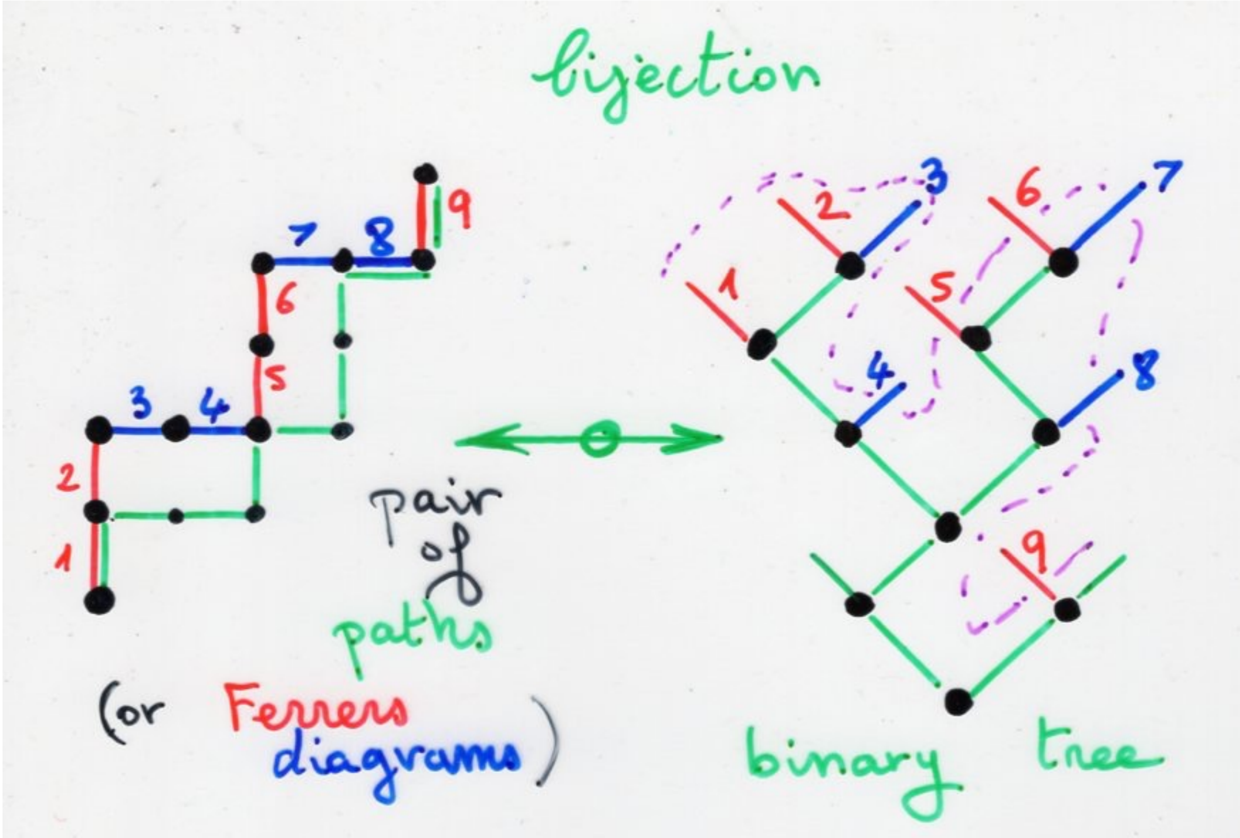


canopy of a binary tree

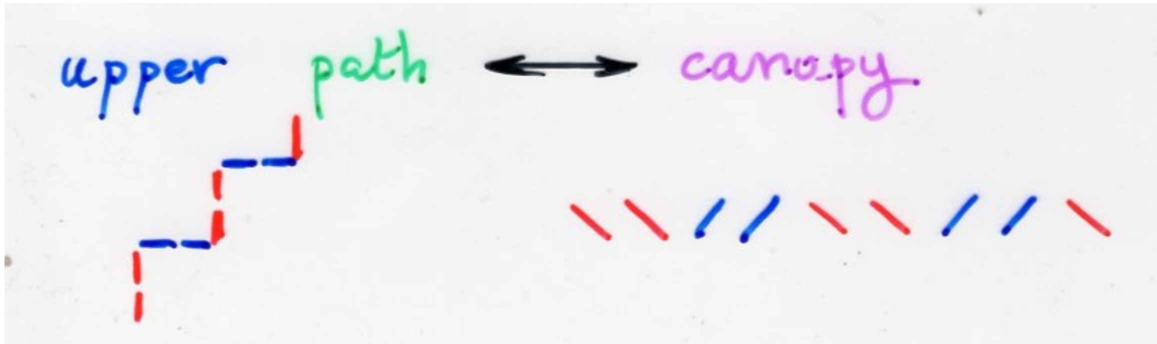
c(B) = / / \ / \ / / \

Loday, Ronco (1998, 2012)





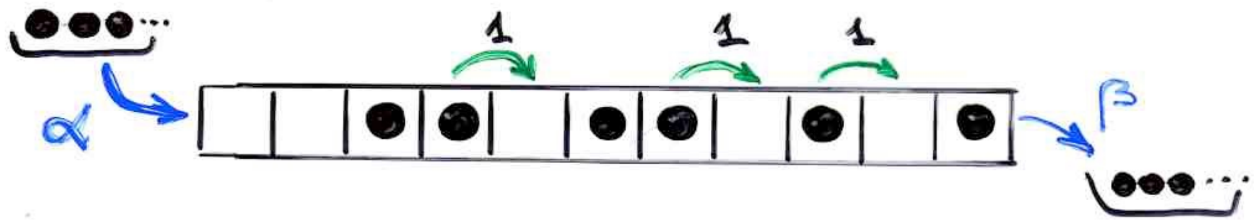
bijection
1
2
3
4
5
6
7
8
9
pair of paths
(or Ferrers diagrams)
1
2
3
4
5
6
7
8
9
binary tree



upper path
canopy

TASEP

"totally asymmetric exclusion process"

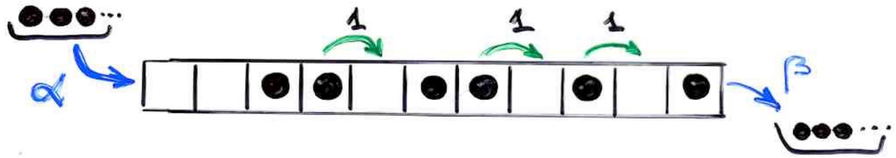


stationnary probabilities

time $\longrightarrow \infty$

TASEP

"totally asymmetric exclusion process"



stationary probabilities

$$\frac{1}{Z_n} \sum_{\substack{\text{binary trees } T \\ c(T) = w}} \bar{\alpha}^{lb(T)} \bar{\beta}^{rb(T)}$$

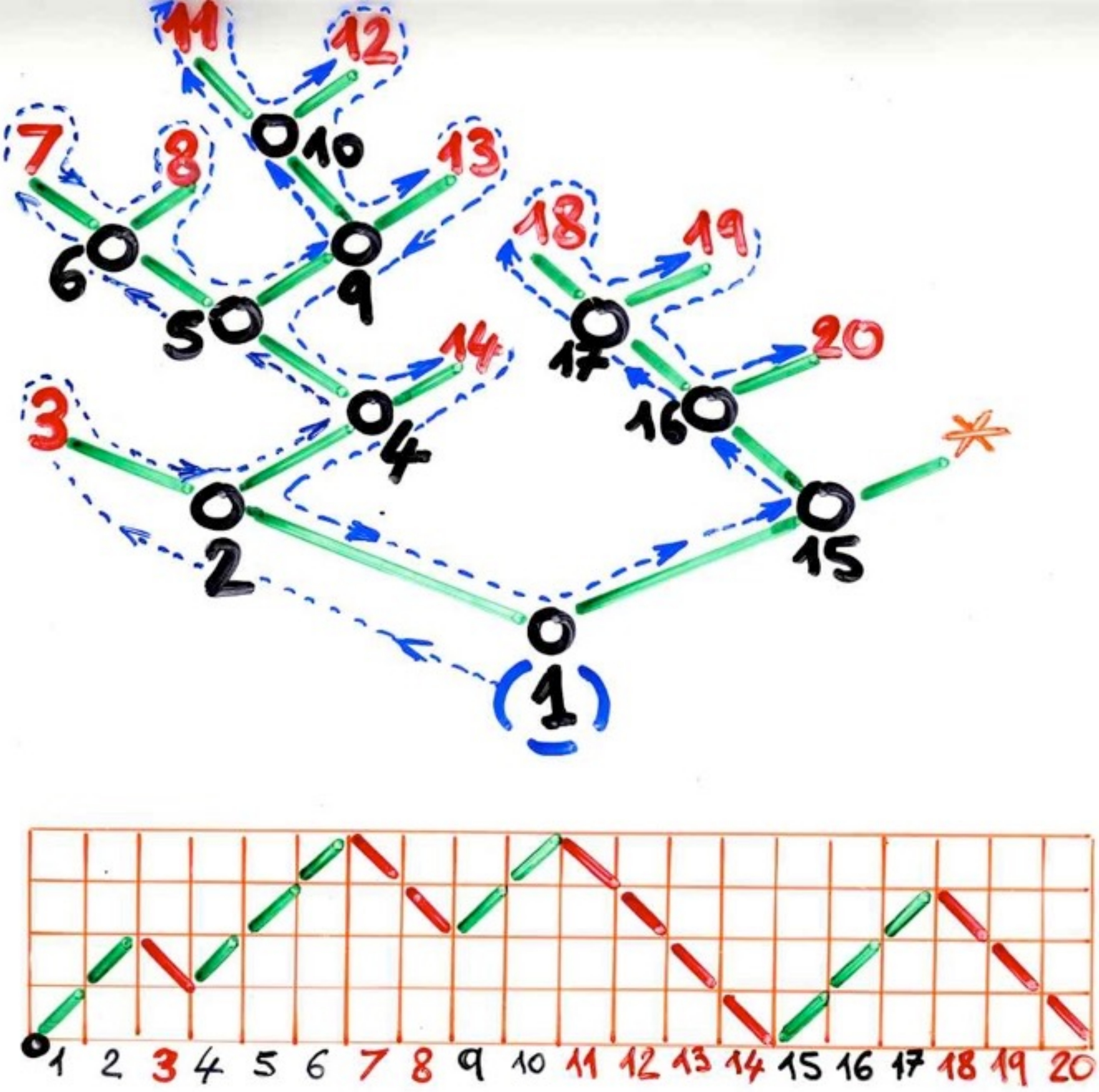
canopy

$$\bar{\alpha} = \alpha^{-1} \qquad \bar{\beta} = \beta^{-1}$$

partition function

$$Z_n = \sum_{T} \bar{\alpha}^{lb(T)} \bar{\beta}^{rb(T)}$$

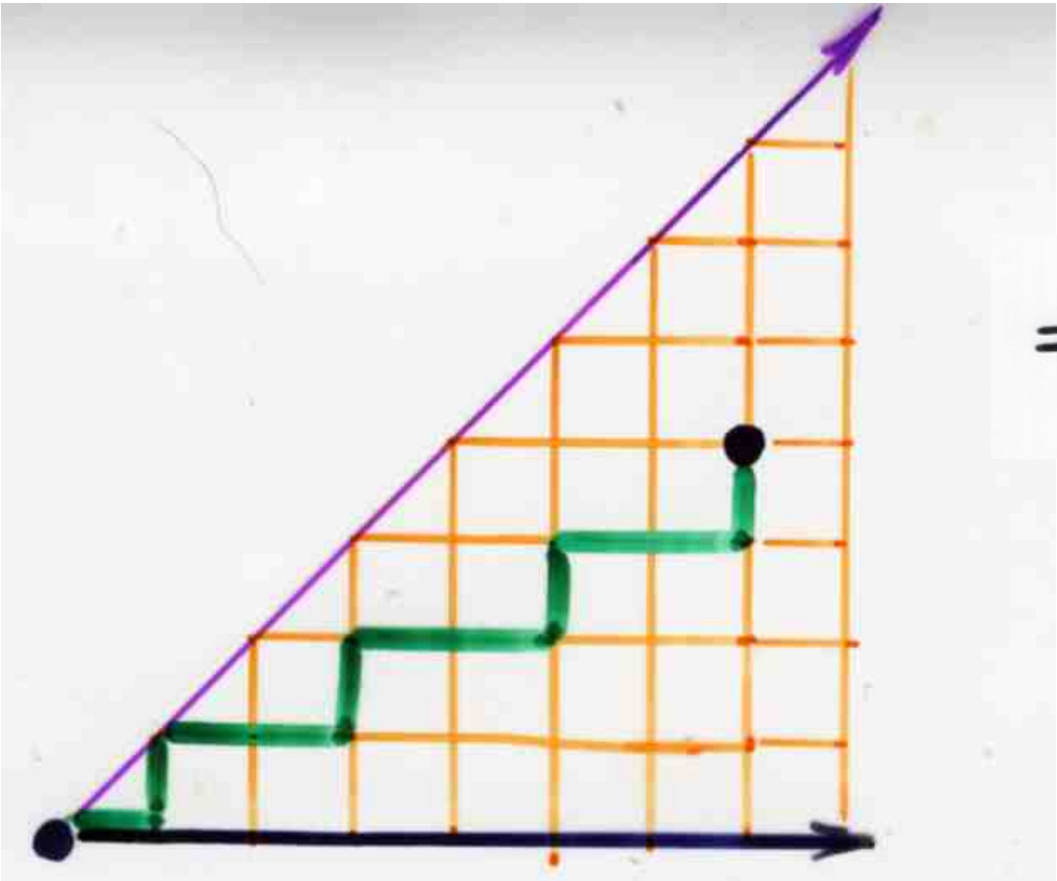
binary trees
n vertices



11
12
7
8
10
13
18
19
6
9
5
17
20
14
3
16
4
2
15
1
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

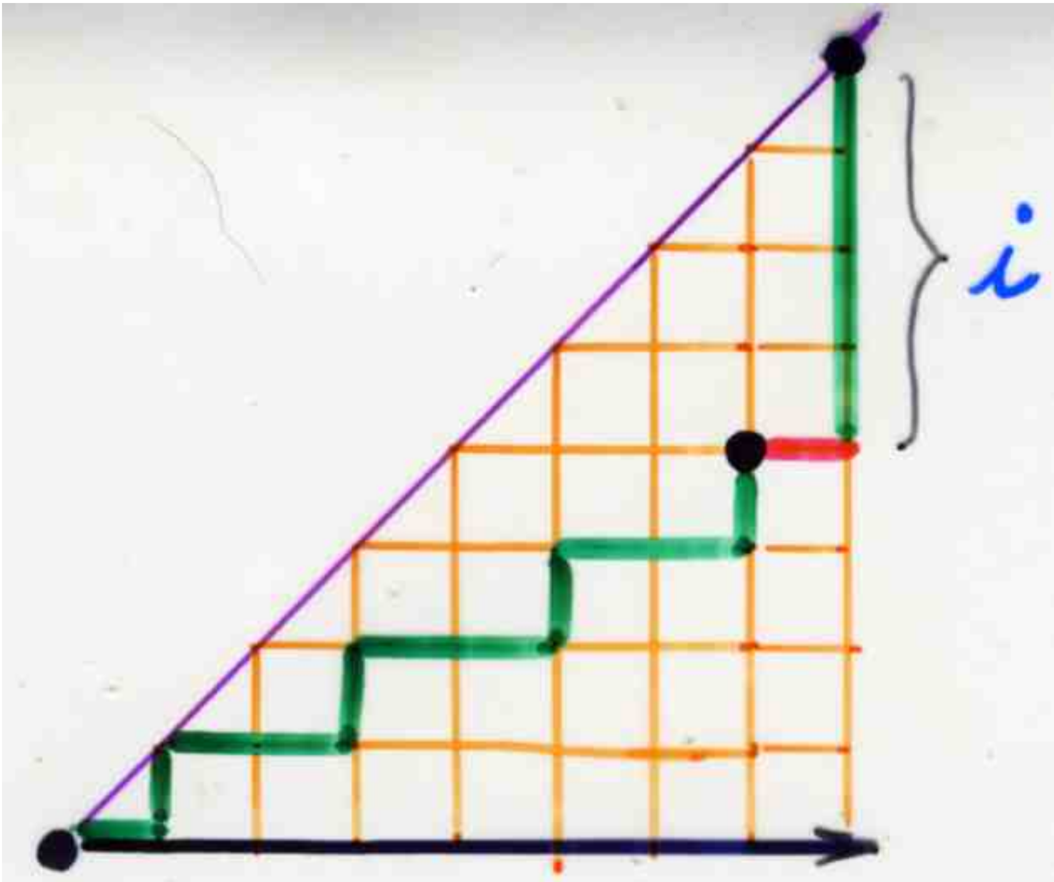
"ballot numbers"

$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$



$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

"ballot numbers"



$$Z_n = \sum_{\substack{T \\ \text{binary trees} \\ n \text{ vertices}}} \bar{\alpha}^{\,lb(T)} \, \bar{\beta}^{\,rb(T)}$$

$$\bar{\alpha} = \alpha^{-1} \qquad \bar{\beta} = \beta^{-1}$$

exercise 12

$$Z_n = \sum \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

$$\left[\bar{\alpha}^{i} + \bar{\alpha}^{(i-1)} \beta + \dots + \bar{\alpha}^{(i-j)} \bar{\beta}^{j} + \dots + \bar{\beta}^{i} \right]$$

A final surprise ...

molecular biology

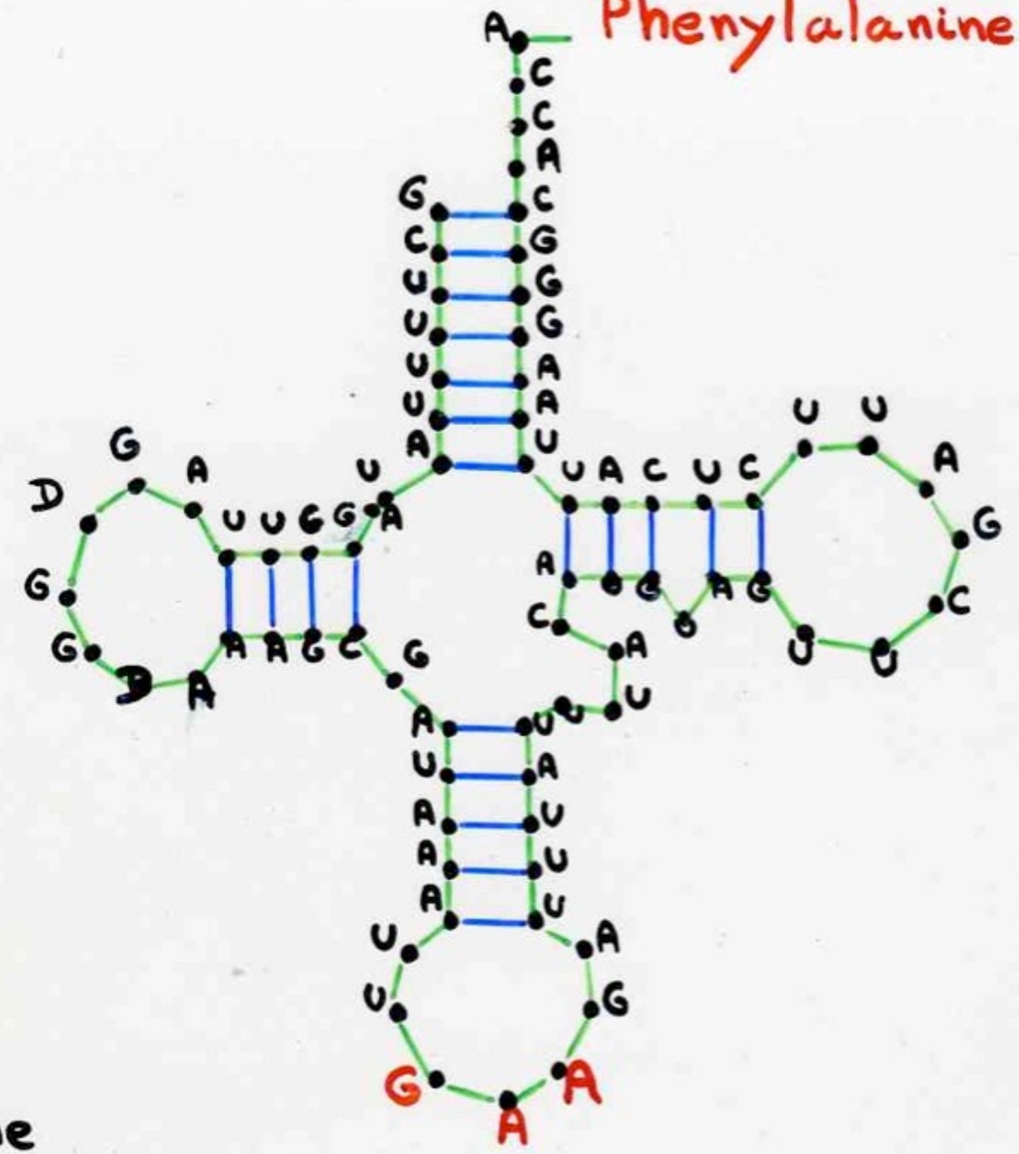
computer science

hydrogeology

Trees everywhere. ...

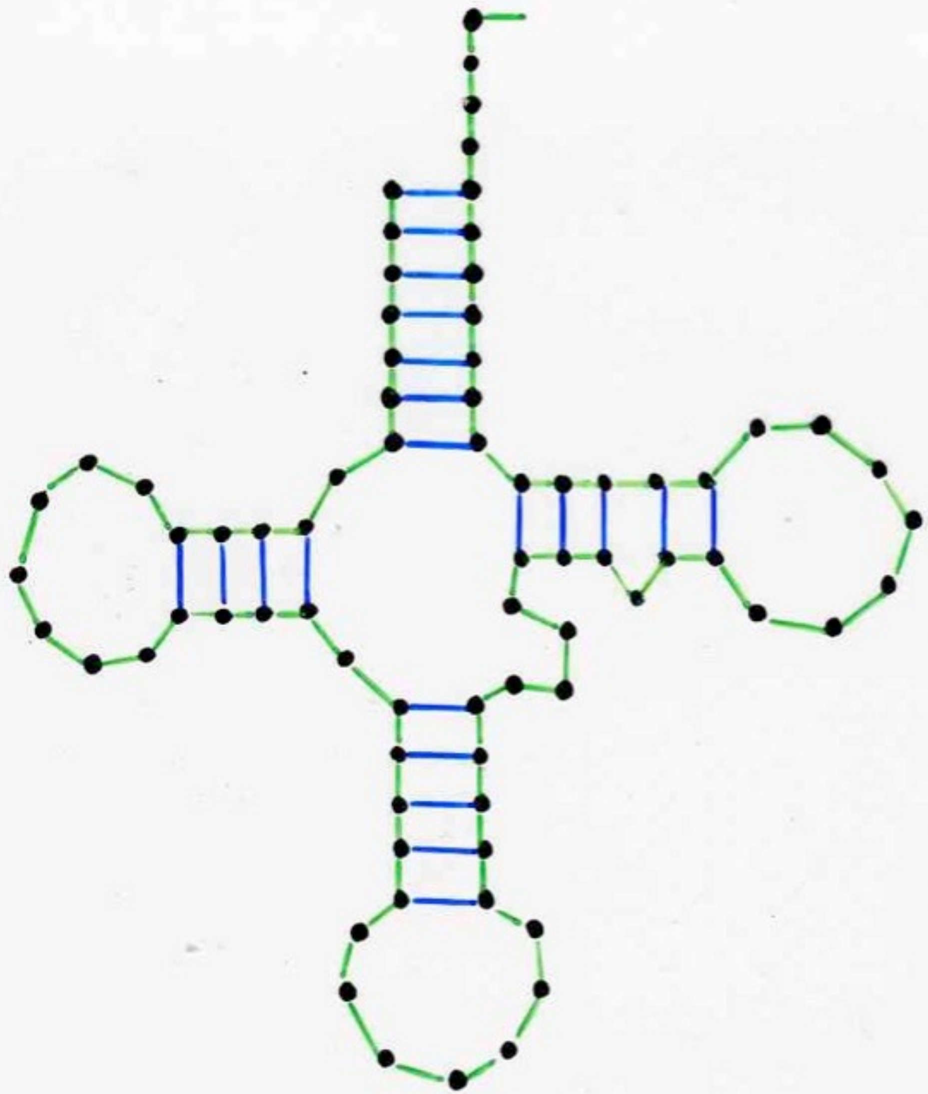
molecular biology

trees
in secondary structures
of RNA

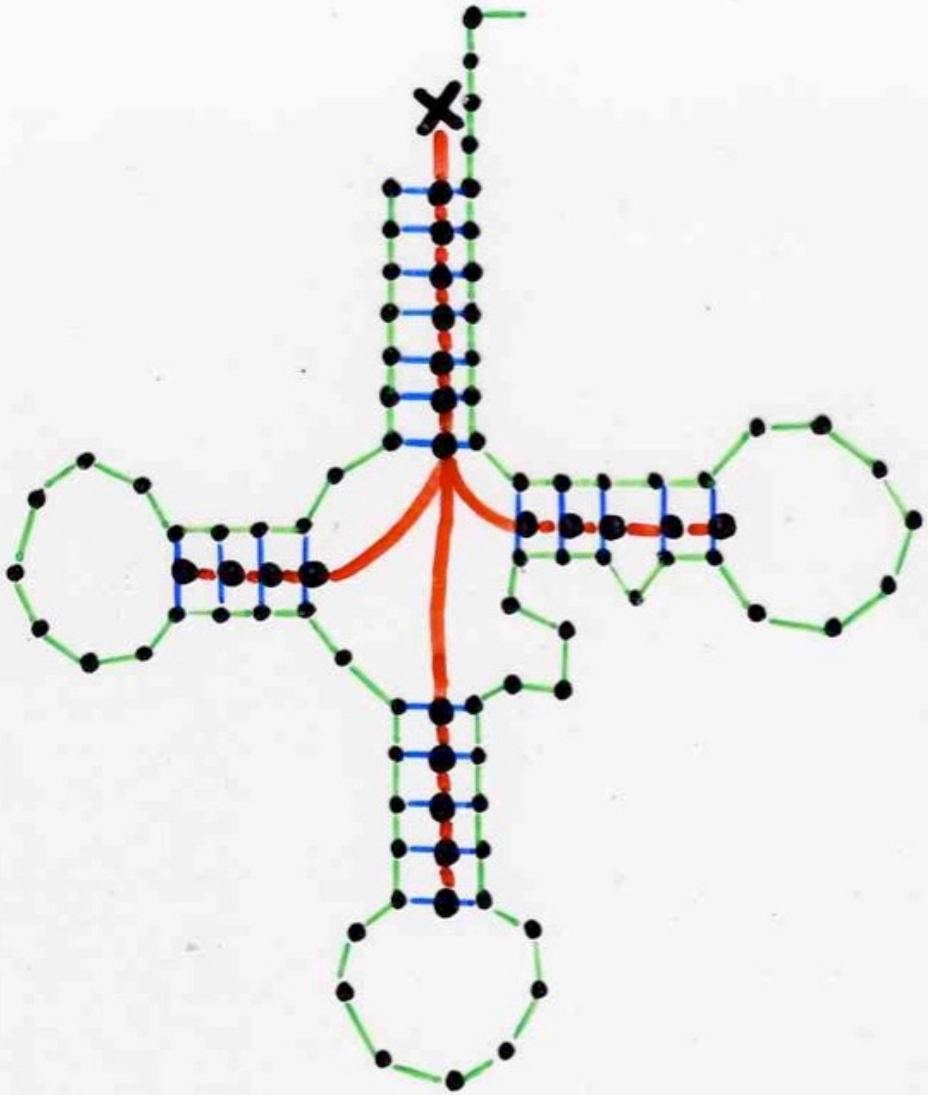


Adenine
Uracyle
Guanine
Cytosine

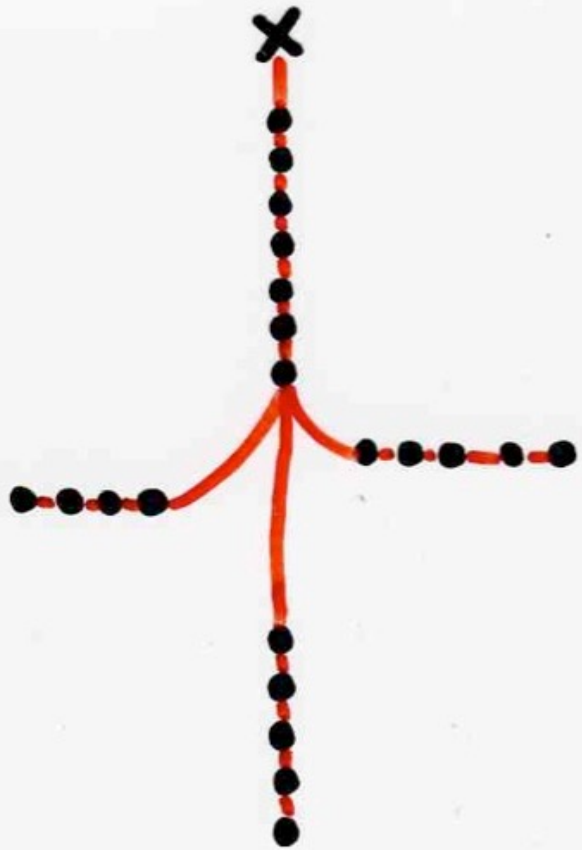
tARN ^{Phe}



tARN ^{Phe}



tARN^{Phe}



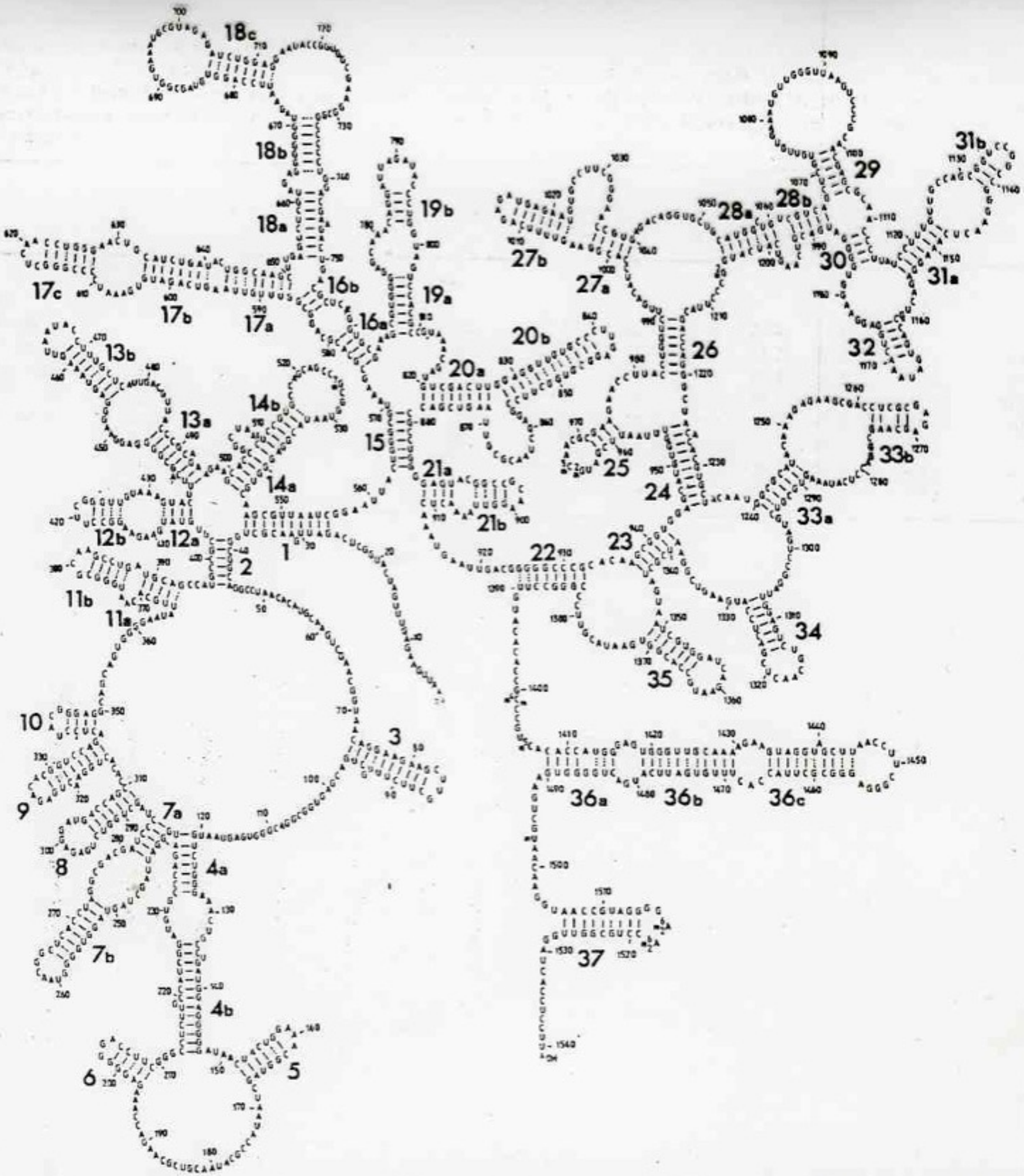
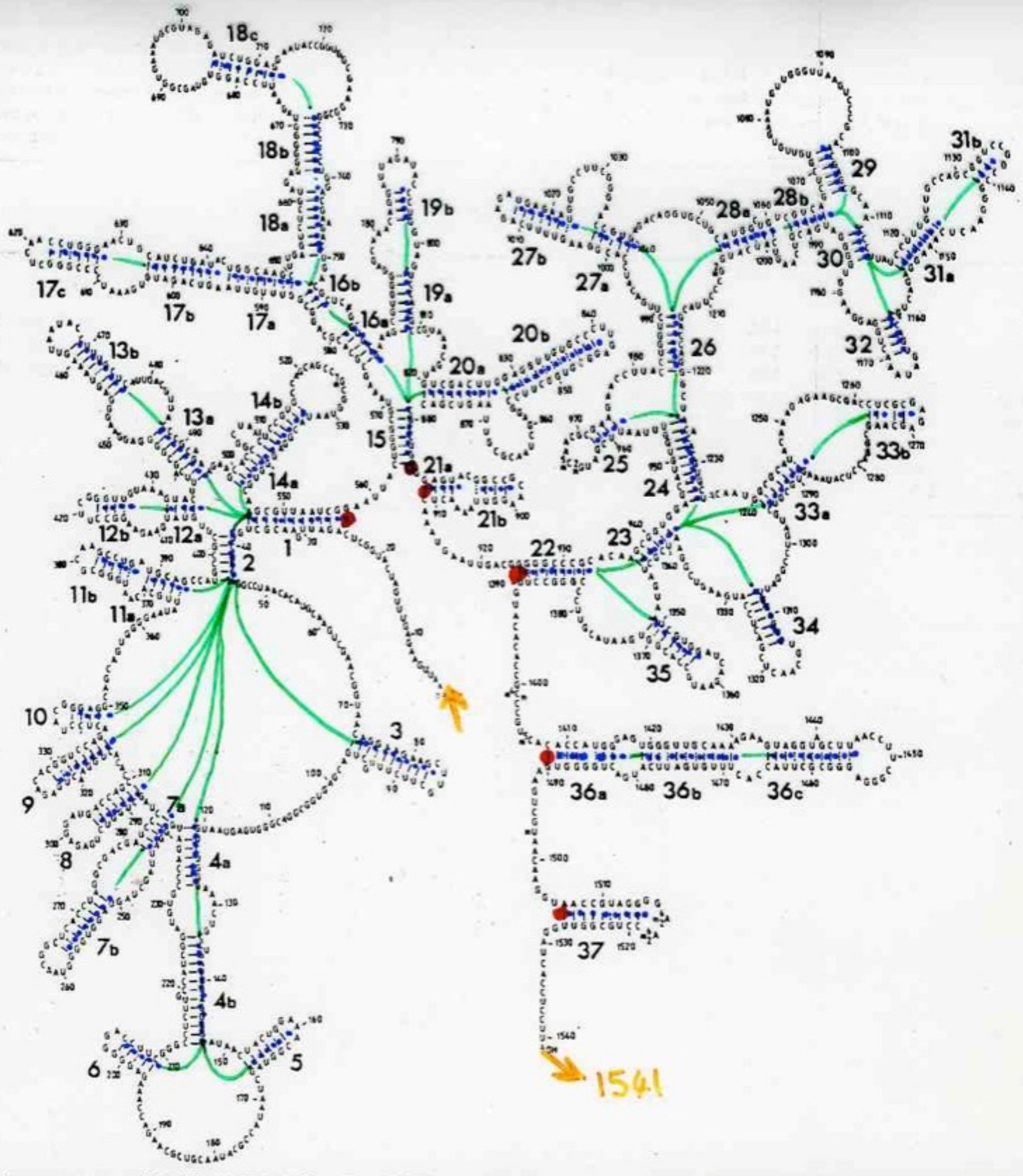


Fig. 1. *Secondary structure model of the 16-S RNA from* E. coli. This model has been fully described elsewhere [18]. The various secondary structure motifs are numbered for reference. Base-pairings 2 and 23 are included in this up-dated scheme and slight modifications have been introduced into helices 18b and 33b



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forest of planar trees = sequence of planar trees

$$y = \frac{1}{1-z}$$

planar tree = ($\bullet$, forest of planar trees)

$$z = ty$$

$$y = \frac{1}{1-ty}$$

$$y - ty^2 = 1$$

$$y = 1 + ty^2$$

Catalan numbers !

bijection

planar trees

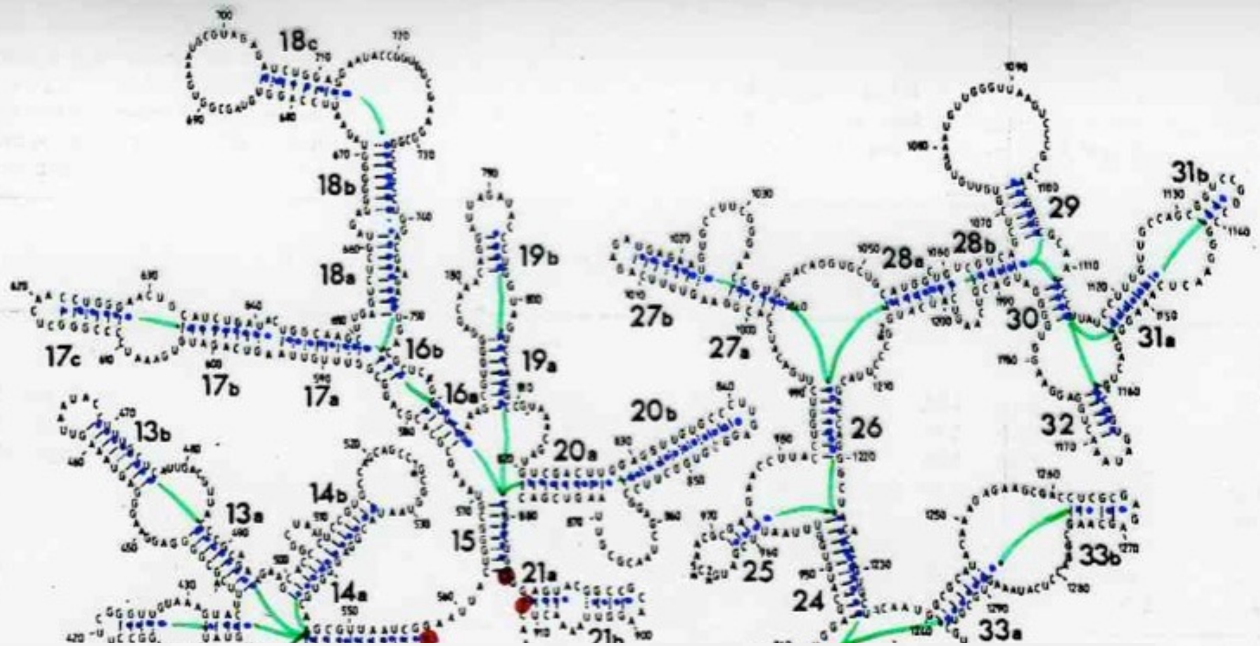
Dyck paths

A final surprise ...

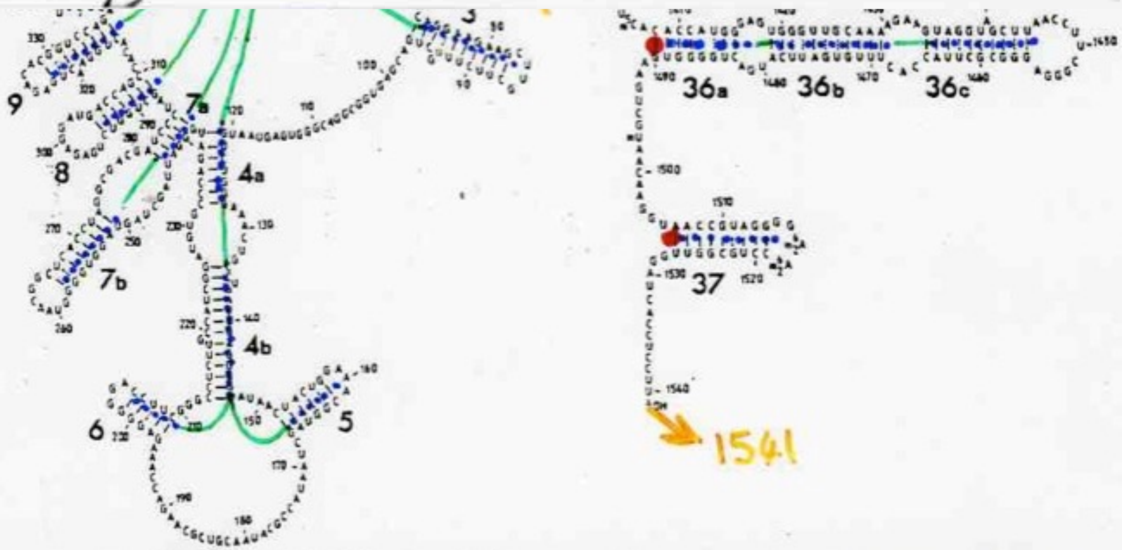
molecular biology

computer science

hydrogeology



- "complexity" of the forest of planar trees underlying the mRNA molecule



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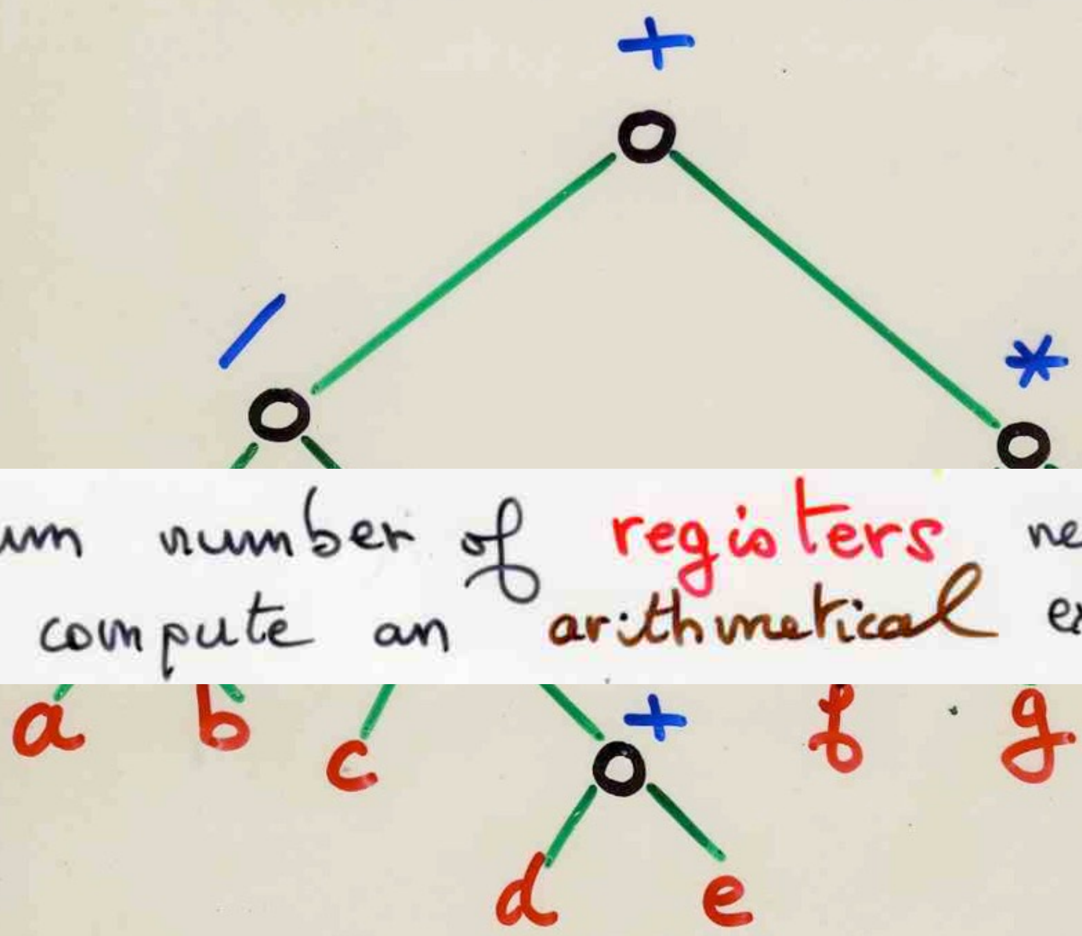
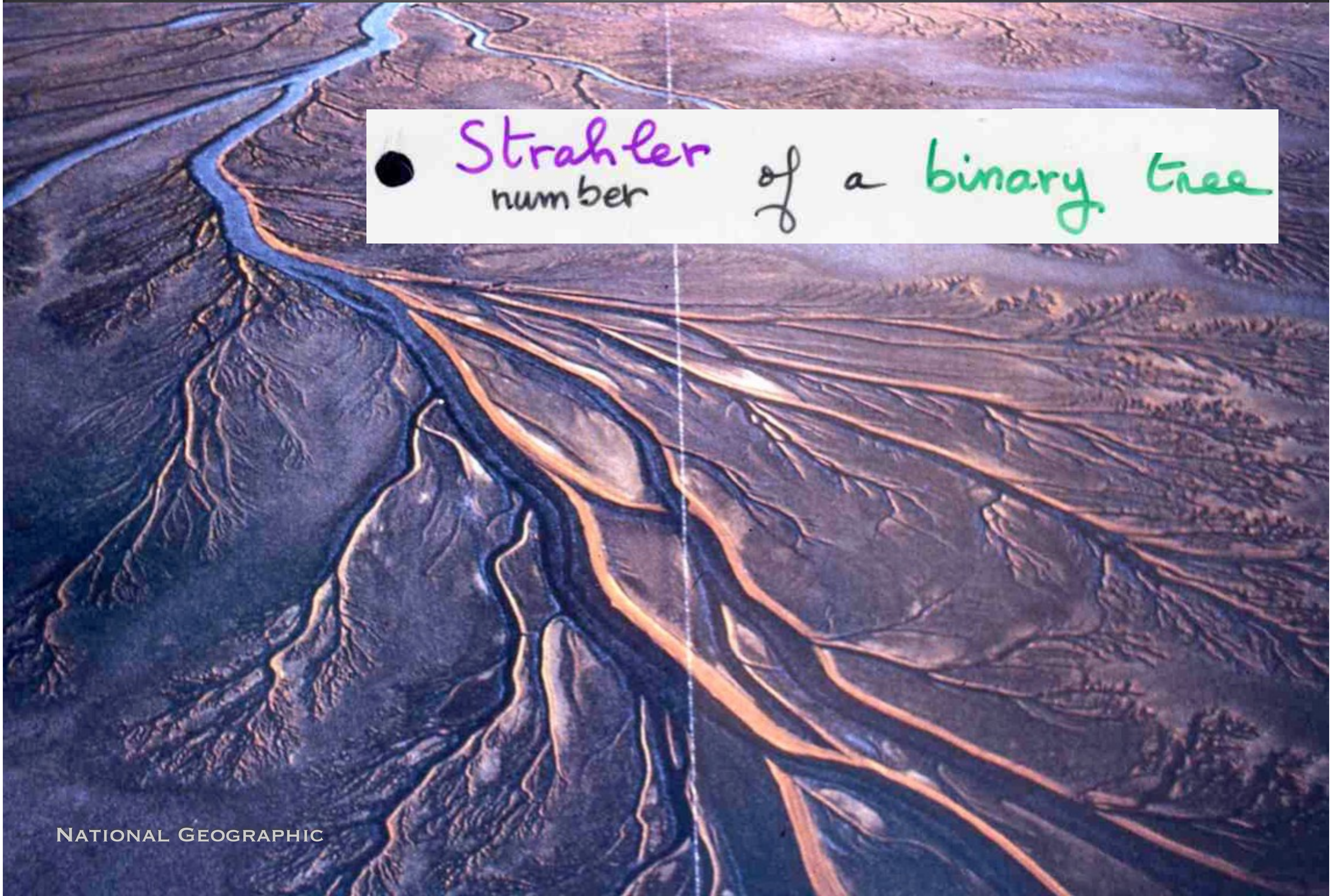


- minimum number of registers needed to compute an arithmetical expression

$$\frac{(a+b)}{c\,(d+e)} + (f-g)\,h$$

• Strahler number of a binary tree



same distribution

- Strahler number of a binary tree

- minimum number of registers needed to compute an arithmetical expression

- "complexity" of the forest of planar trees underlying the mRNA molecule

same generating function

$$S_{n,k} = \left\{ \begin{array}{l} \text{number of binary trees } B \\ \text{with } n \text{ internal vertices} \\ \text{and } \quad St(B) = k \end{array} \right.$$

$$S_k(t) = \sum_{n \geq 0} S_{n,k}\, t^n$$

Fibonacci numbers

$$S_0 = 1 \qquad 1 \qquad F_1$$

$$S_1 = \frac{t}{1-2t} \qquad 3 \qquad F_3$$

$$S_2 = \frac{t^3}{1-6t+10t^2-4t^3} \qquad 21 \qquad F_7$$

$$S_3 = \frac{t^7}{1-14t+78t^2-220t^3+330t^4-252t^5+84t^6-8t^7}$$

$$987 \qquad F_{15}$$

The arithmetical triangle

Fibonacci numbers

$F_{(2^k-1)}$

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4						
1	5	10	10						
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	
1	10	45	120	210	252	210	120	45	10
1	11	55	165	330	462	462	330	165	55
1	12	66	220	495	792	924	792	495	220
1	13	78	286	715	1287	1716	1716	1287	715
1	14	91	364	1001	2002	3003	3432	3003	2002
1	15	105	455	1365	3003	5005	6435	6435	5005

- -

Thank you !

www.viennot.org/abjc-stems21

Video-book « The Art of bijective combinatorics

ॐ सरस्वत्यै नमः।