An introduction to enumerative and bijective combinatorics with binary trees (part I)

TESSELLATE
STEMS-2021
CMI, Chennai, India
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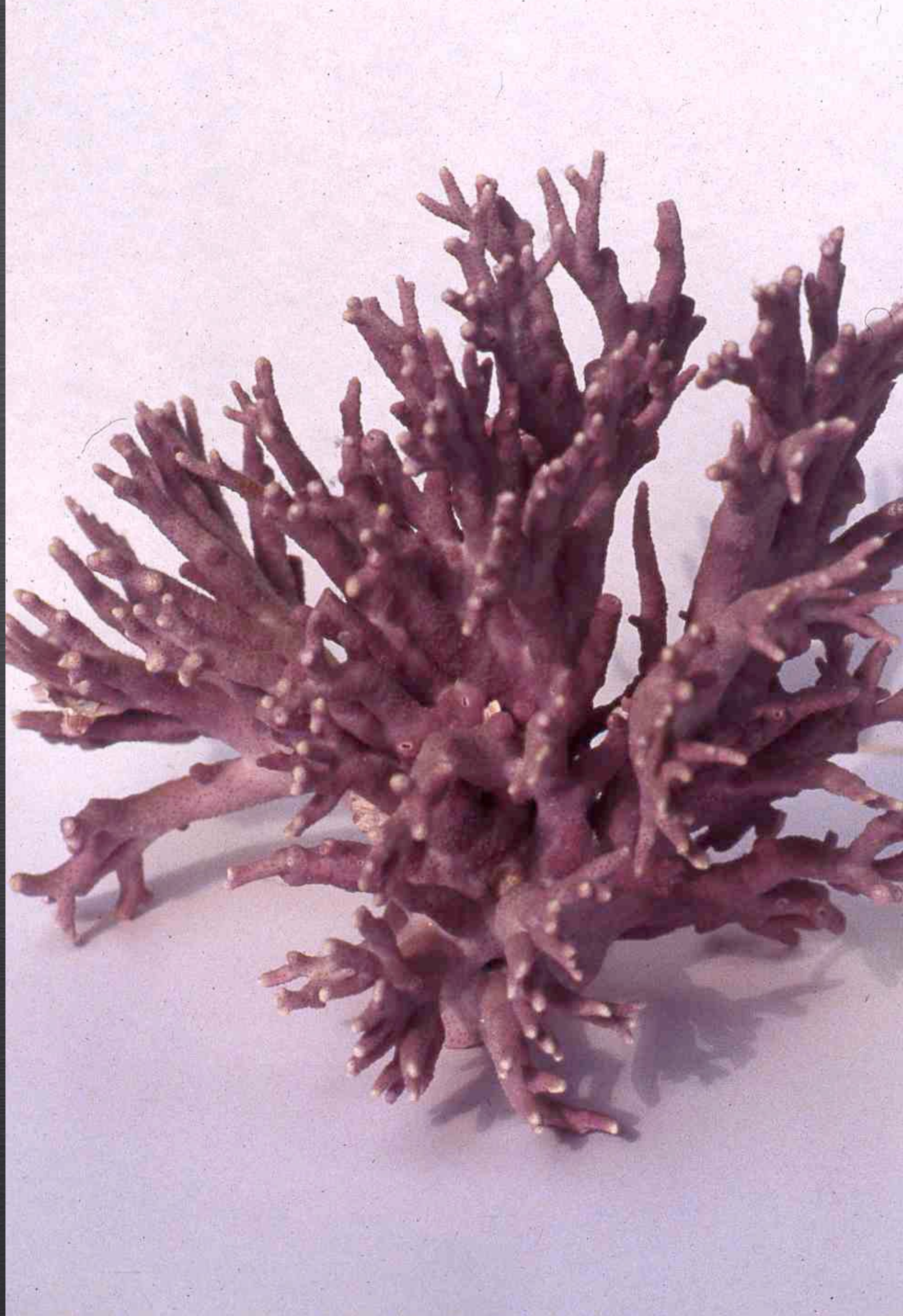
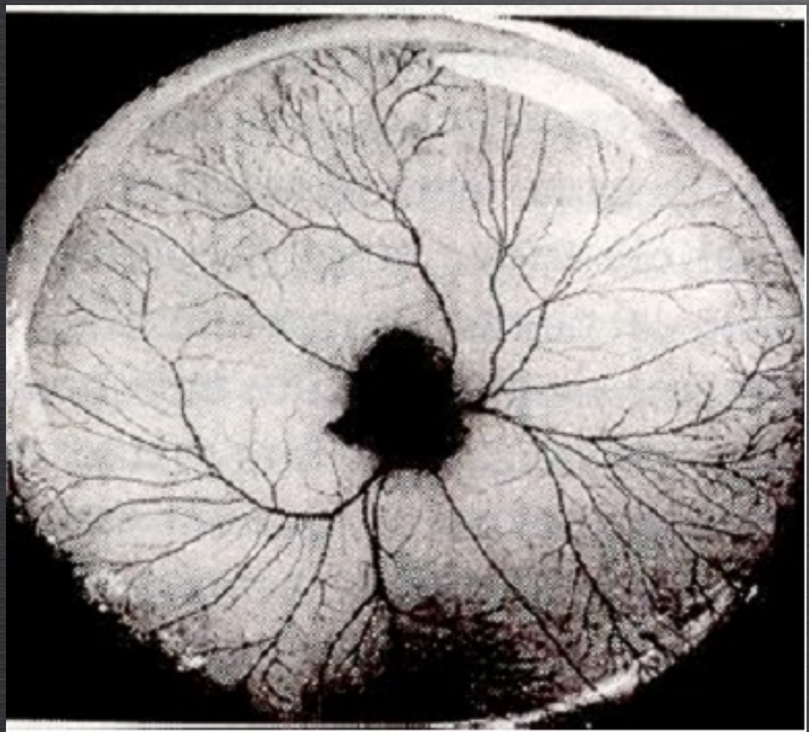
Trees in Nature

Coral




Electrical discharge

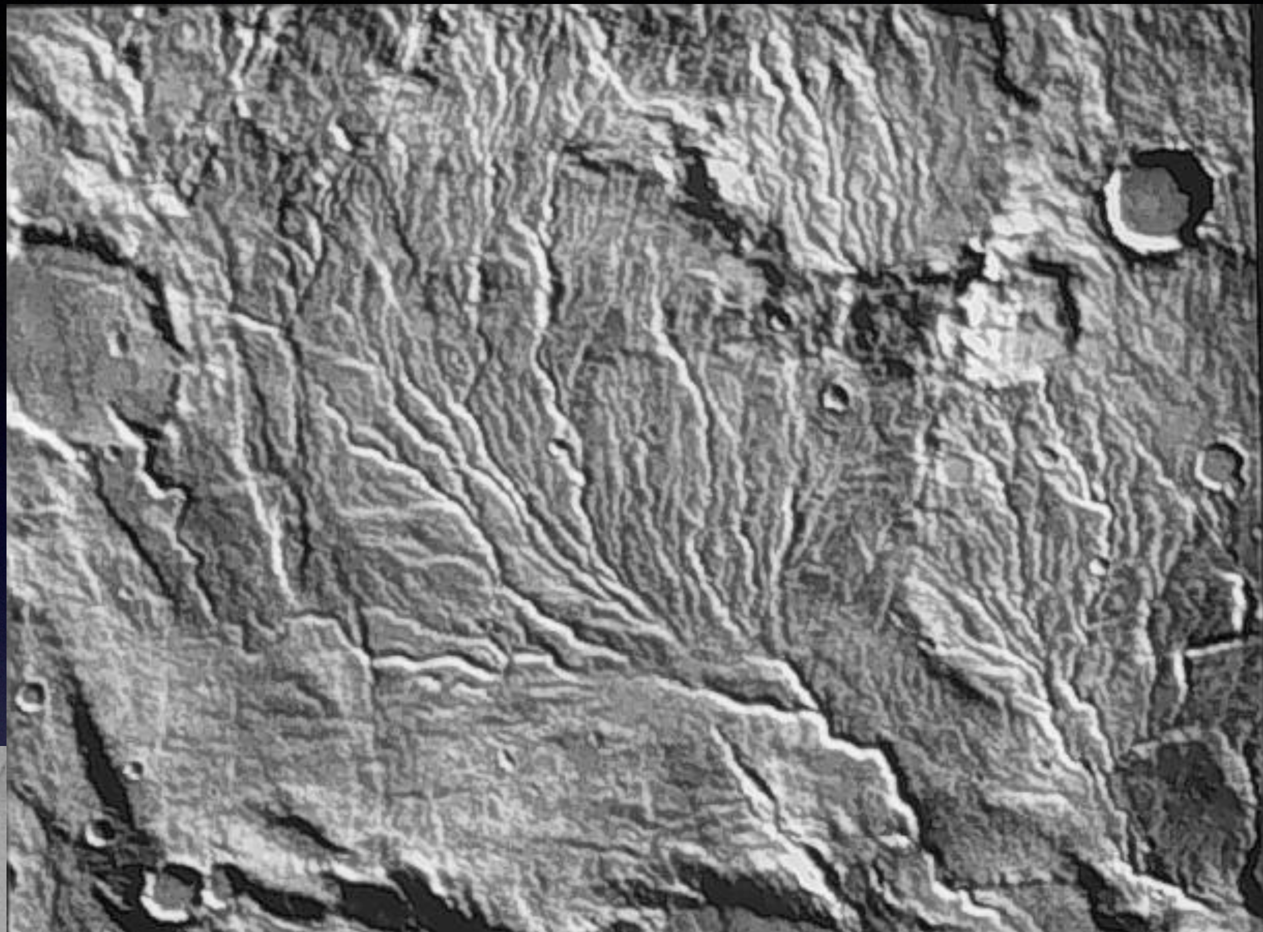
LUNG

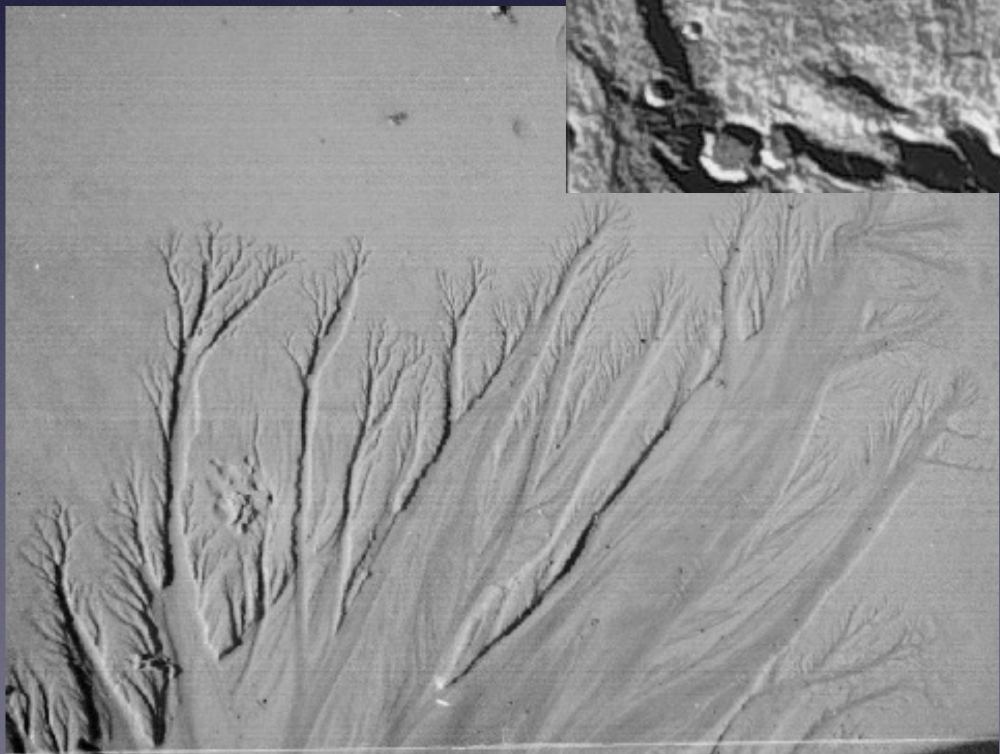


EGG


National Geographic

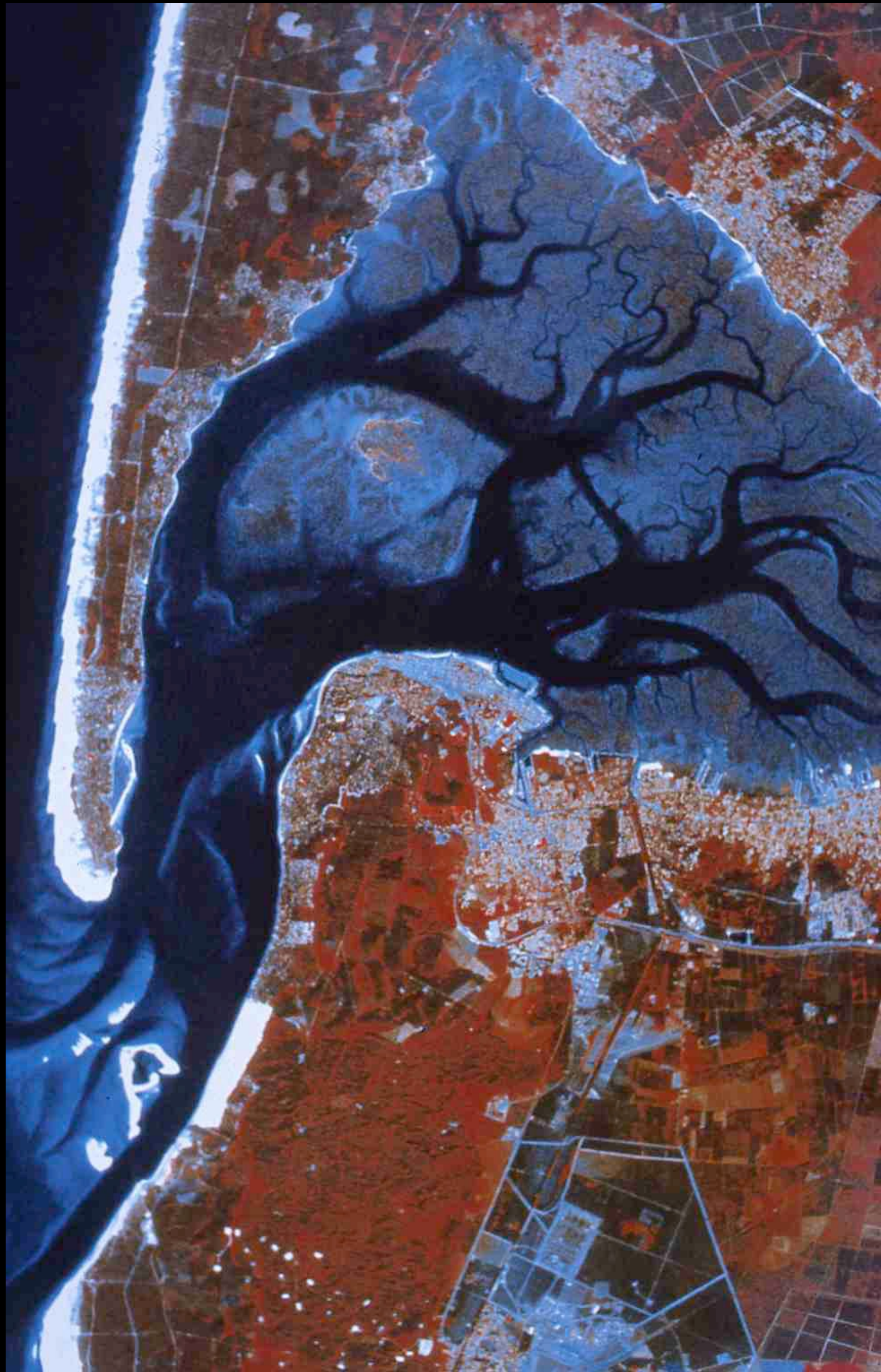
on Mars

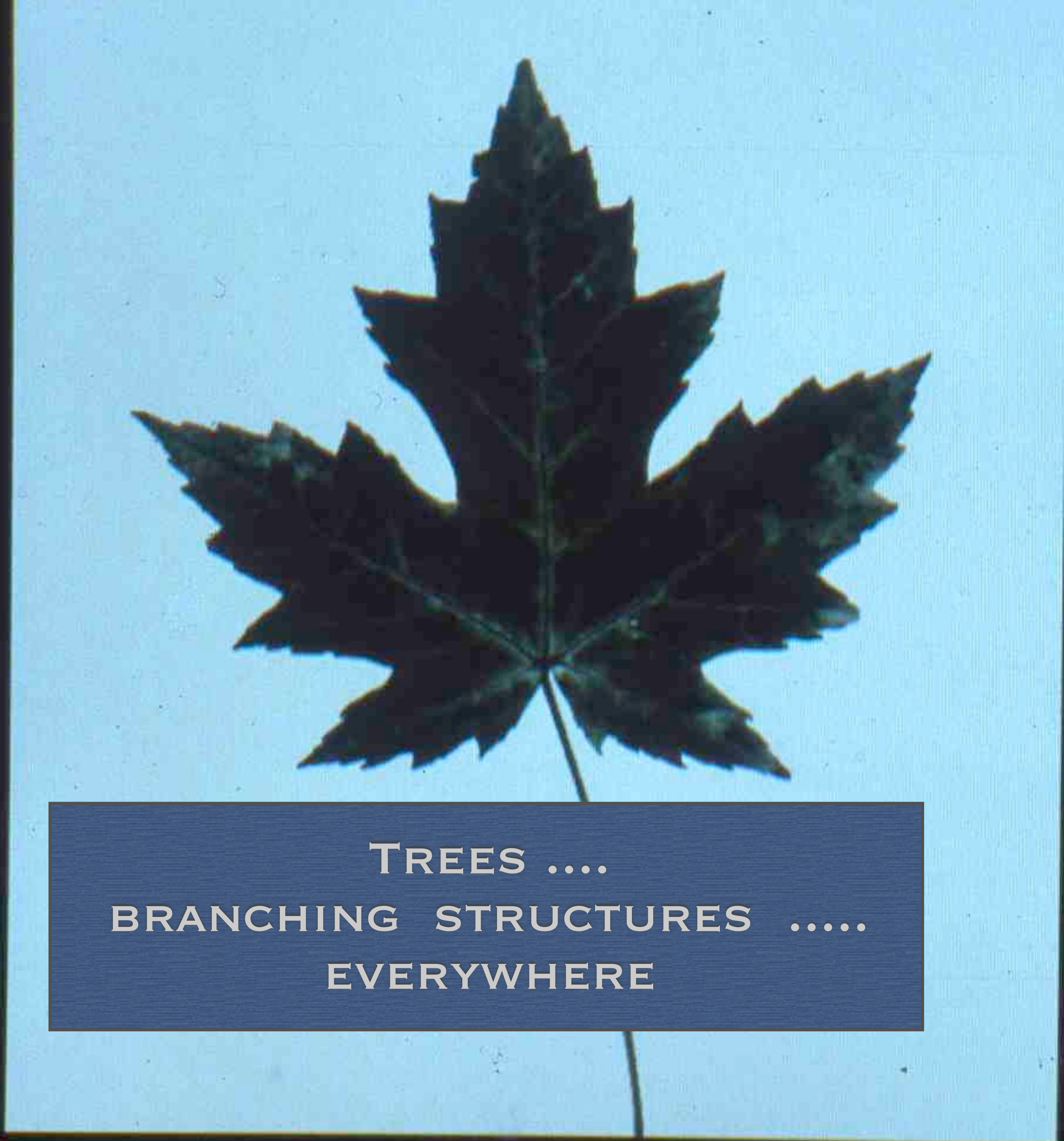




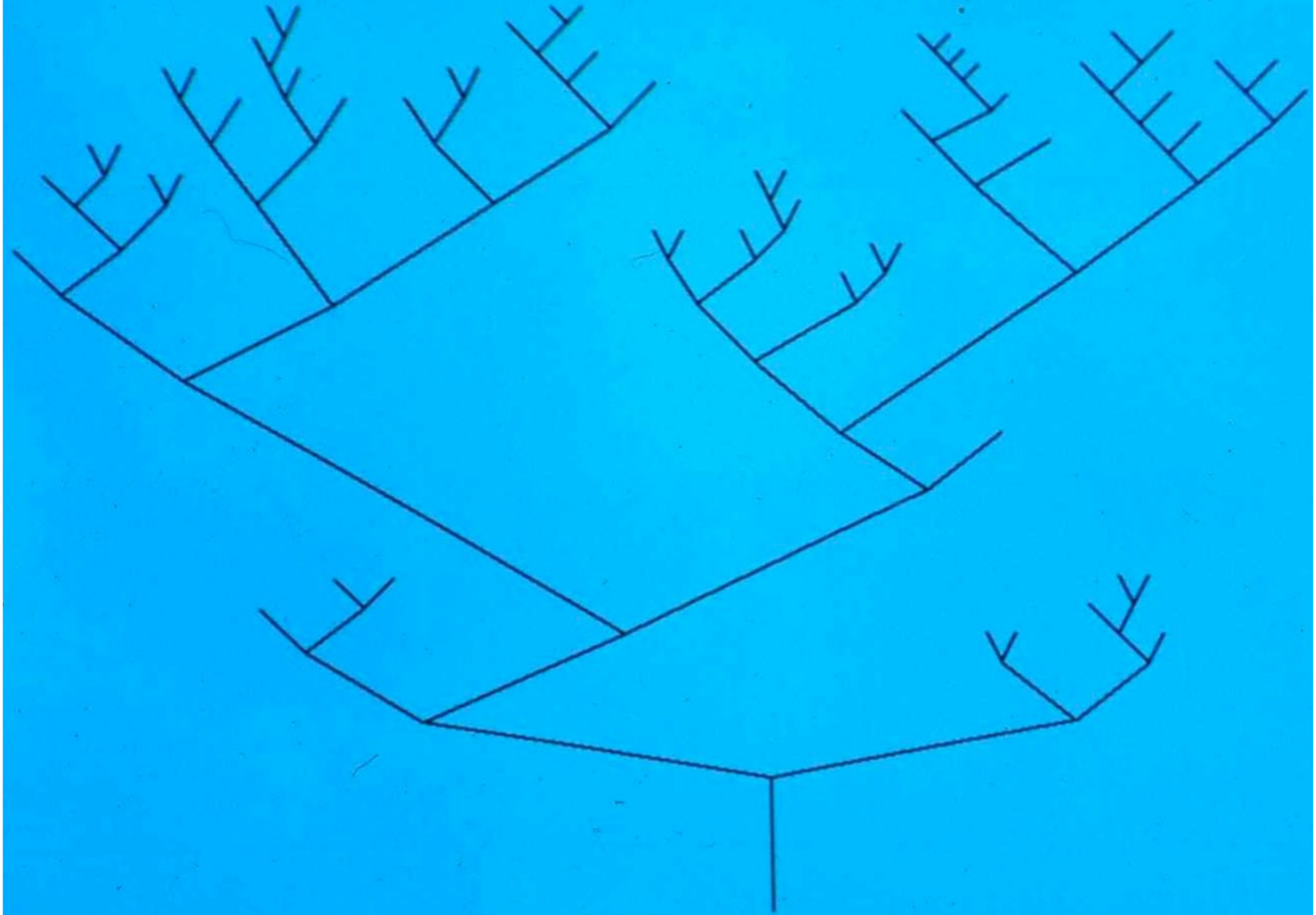
on Earth

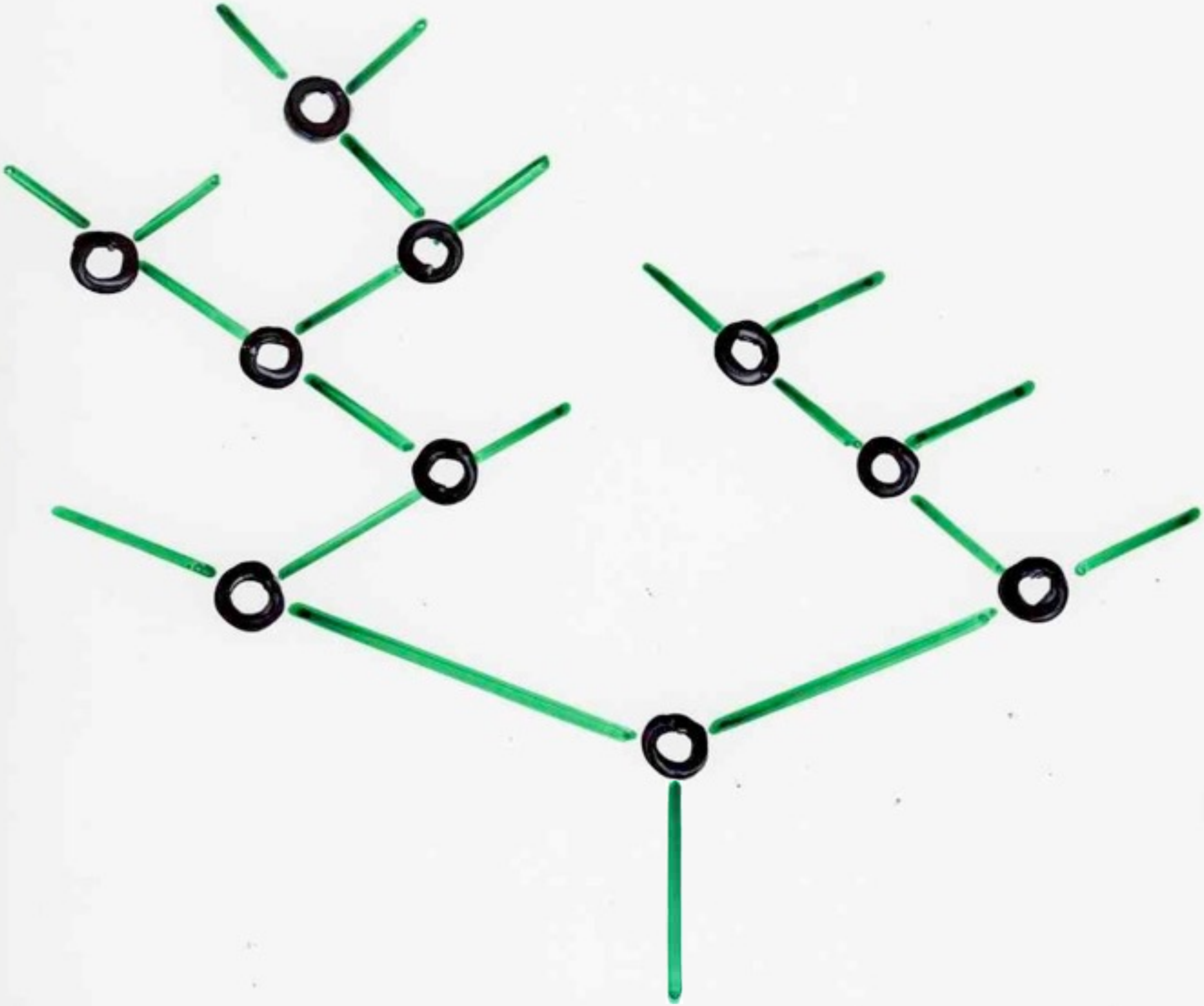
on a beach

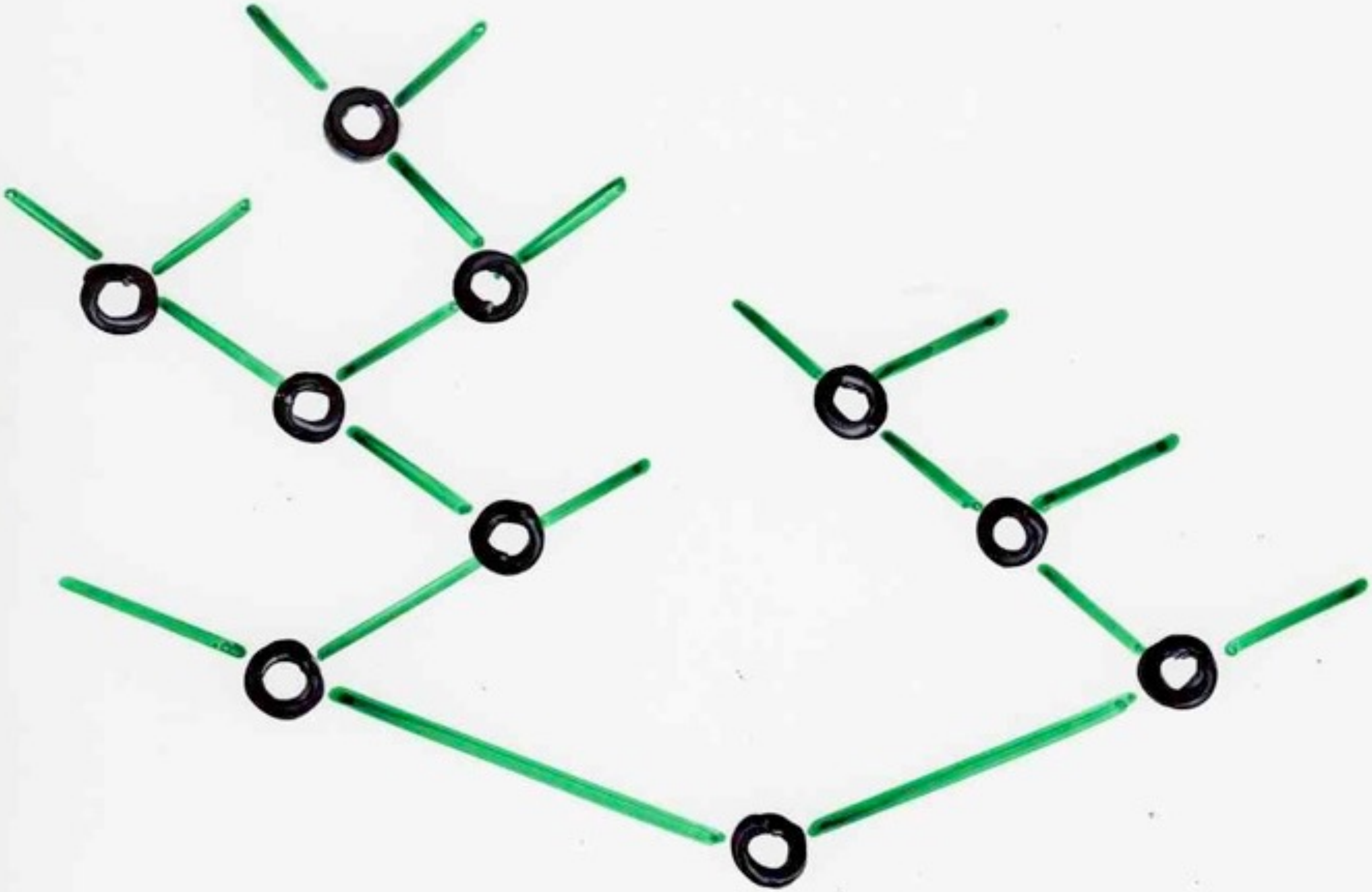



Trees ....
branching structures .....
everywhere

From trees in Nature
to
mathematical trees

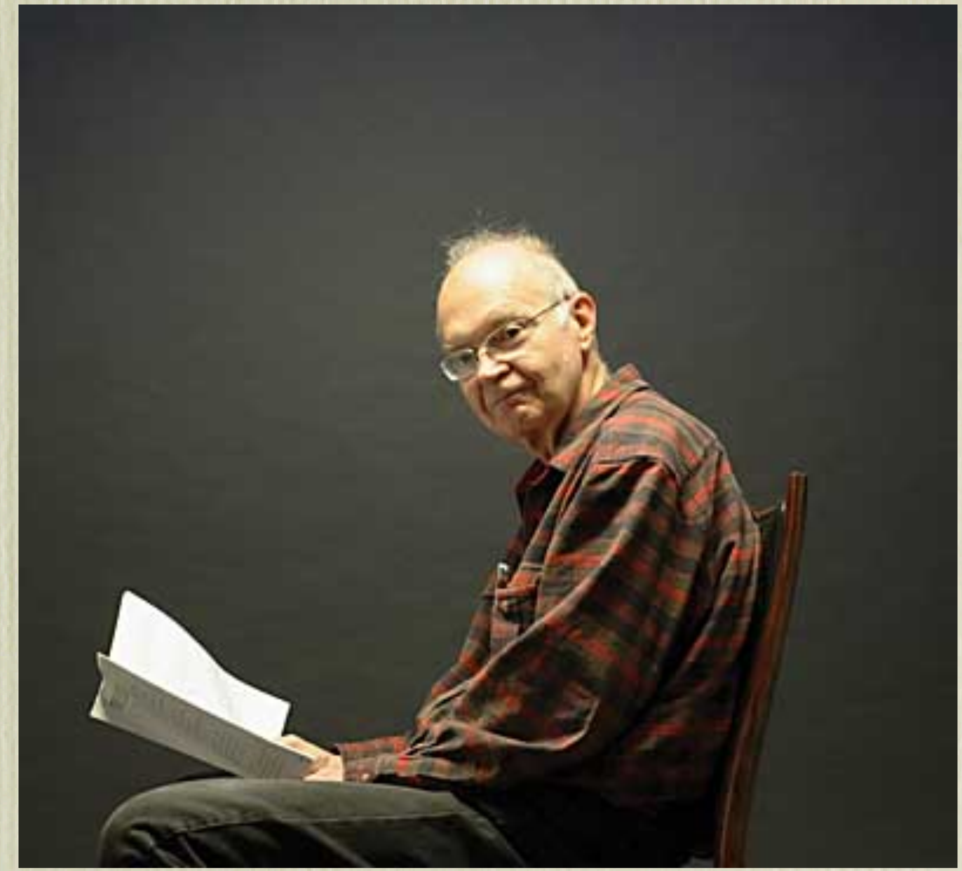






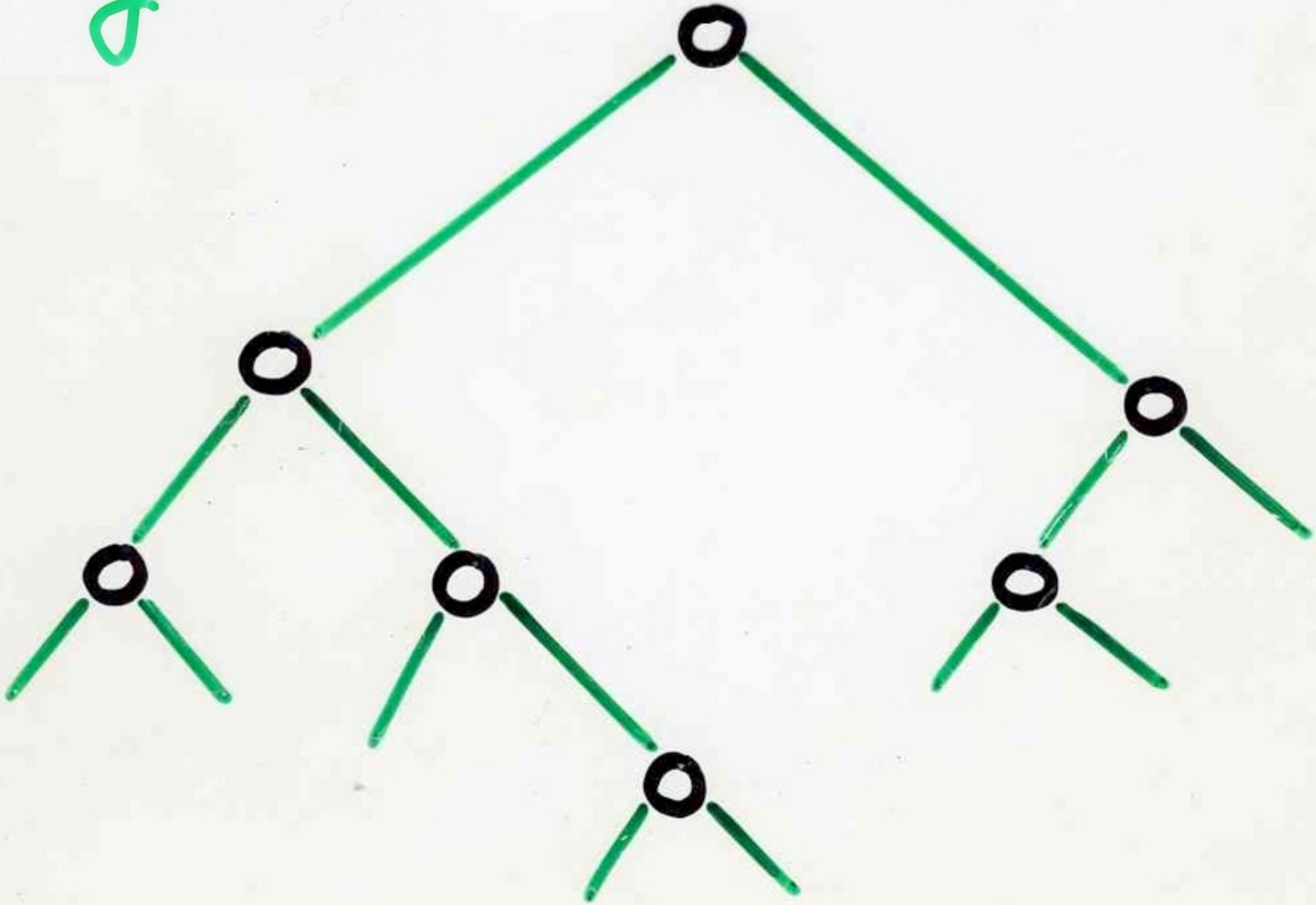
Trees everywhere !

"Trees sprout up just about everywhere in computer science,
in nearly every section of
The Art of Computer Programming"

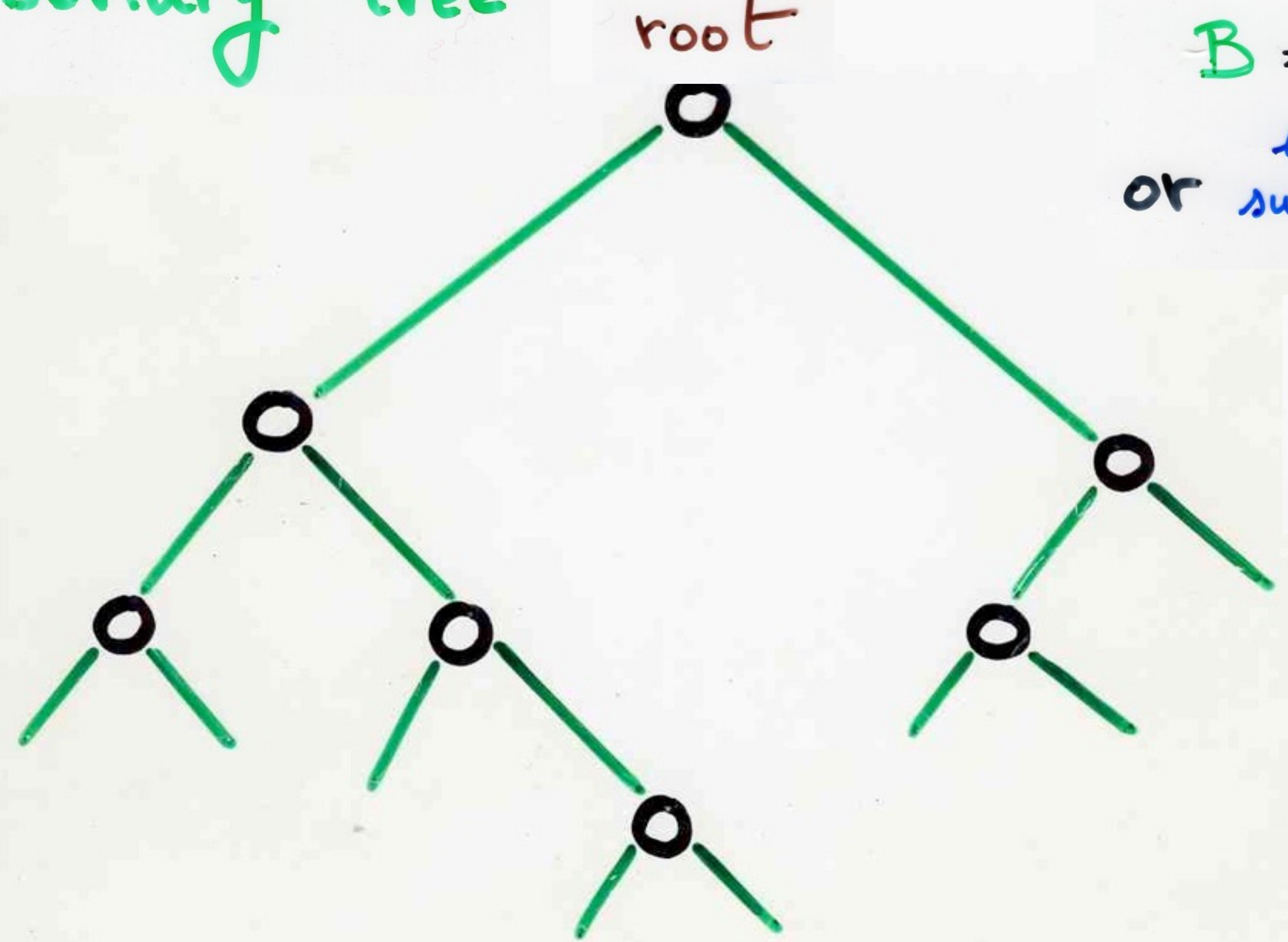


binary tree

binary tree



binary tree



$B = \langle L, r, R \rangle$

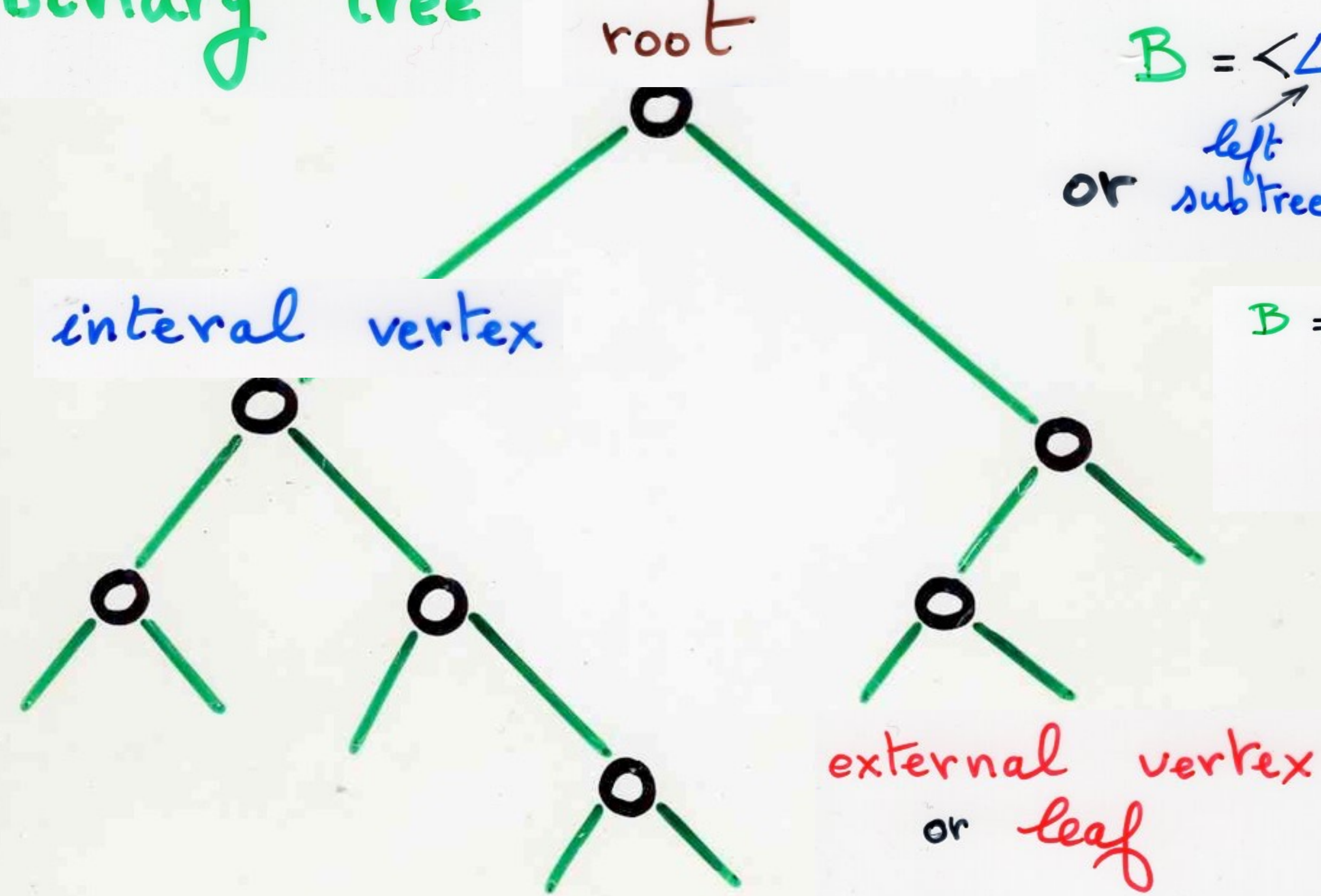
left subtree, root, right subtree

or

$B = \langle v \rangle$

leaf or external vertex

binary tree



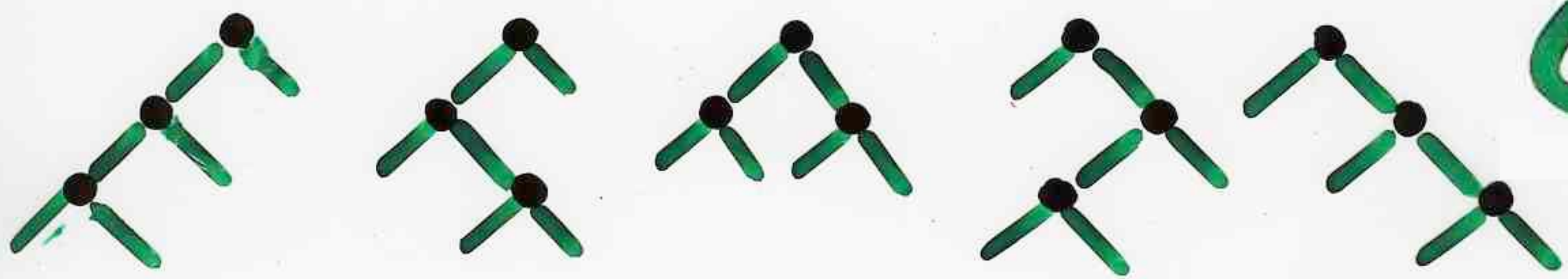
$$B = \langle \underset{\text{left subtree}}{L}, \underset{\text{root}}{r}, \underset{\text{right subtree}}{R} \rangle$$

or

$$B = \langle \underset{\text{leaf or external vertex}}{v} \rangle$$

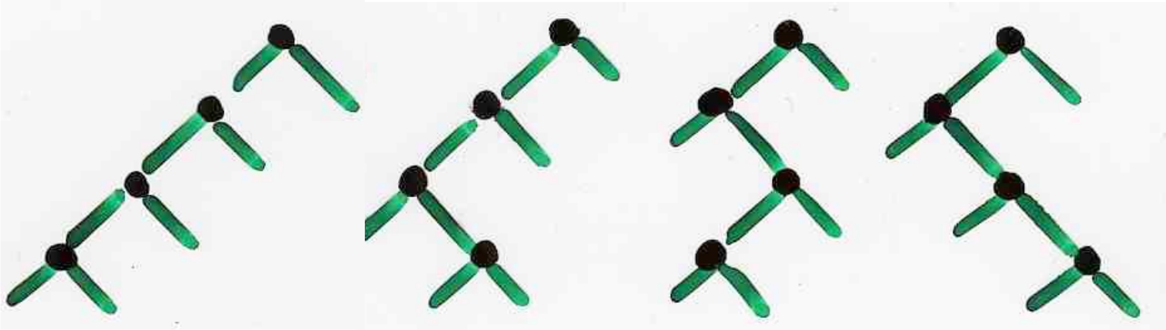
$C_1 = 1$

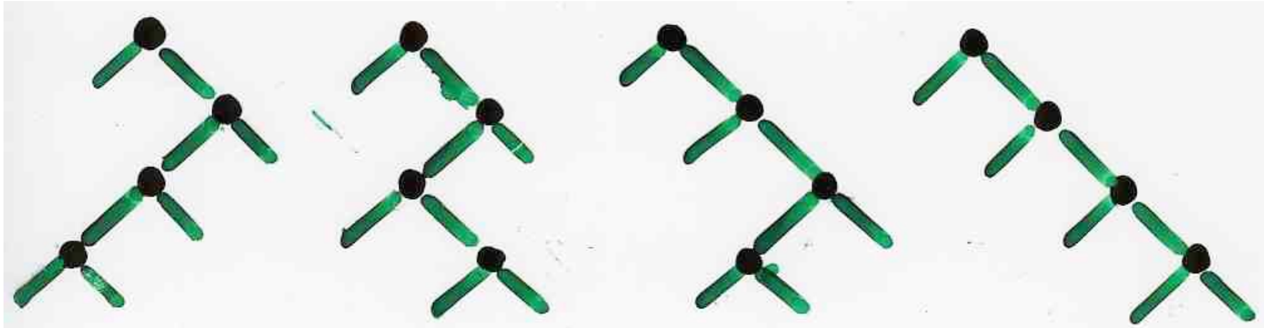
$C_2 = 2$

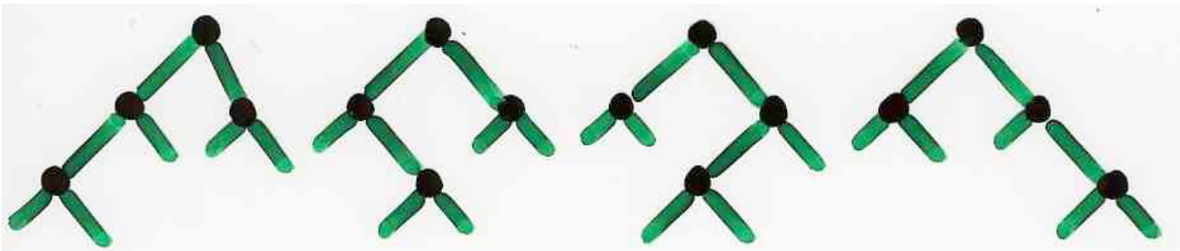


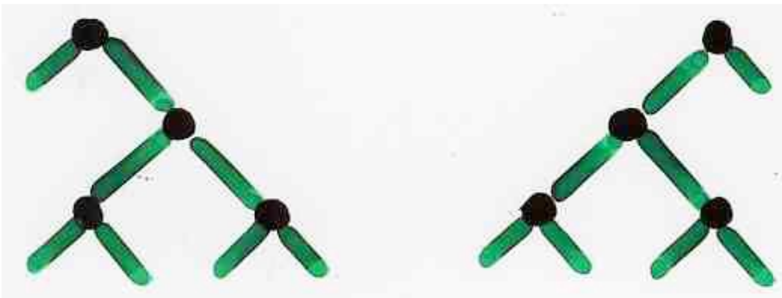
$C_3 = 5$

$C_4 = 14$









exercise 1

B binary tree having n internal vertices

$\Rightarrow$ $(n+1)$ external vertices

C_n = number of binary trees having n internal vertices

(or $n+1$ leaves = external vertices)

$B = \langle L, r, R \rangle$

or L = left subtree, r = root, R = right subtree

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$1, 1, 2, 5, 14, 42, 132, \ldots$

$C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6$

$$C_6 = C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0$$

$$132 \qquad 1\times 42 + 1\times 14 + 2\times 5 + 5\times 2 + 14\times 1 + 42\times 1$$

$\{a_n\}_{n\geq 1}$ $\qquad$ $\{a_1, a_2, \ldots, a_n, \ldots\}$

Some sequences ...

$$\{a_1, a_2, \ldots, a_n, \ldots\}$$

$1, 2, 3, 5, 8, 13, 21, \ldots$ 34

$21 = 13 + 8$

Fibonacci numbers

$F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$

$F_1 = 1$, $F_2 = 2$

Fibonacci

1, 2, 6, 24, 120, ... 720

$$a_n = n\, a_{n-1} \qquad (n \geqslant 1)$$
$$a_1 = 1$$

$$a_n = n!$$
$$= 1 \times 2 \times 3 \times \cdots \times n$$

permutations

1, 2, 3, 5, 7, 11, 13, 17, . . .

19

prime numbers
sequence

experimental combinatorics

guess a formula

$C_n = ?$

$$C_n = \frac{(\text{product})}{(\text{product})}$$

n	C_n	prime numbes decomposition
1	1	1
2	2	2
3	5	5
4	14	2×7
5	42	$2 \times 3 \times 7$
6	132	$2^2 \times 3 \times 11$
7	429	$3 \times 11 \times 13$
8	1430	$2 \times 5 \times 11 \times 13$
9	4862	$2 \times 11 \times 13 \times 17$
10	16796	$2^2 \times 13 \times 17 \times 19$
...	-------	— — — — — —

$$C_n = \frac{(2n)!}{D_n}$$

$$C_{20} = 2^2 \times 3 \times 5 \times 11 \times 13 \times 23 \times 29 \times 31 \times 37$$

$$\{D_n\}_{n\geqslant 1} = \{2, 12, 144, 2880, \ldots ?$$

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1,2,12,144,2880

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Sort: relevance | references | number | modified | created Format: long | short | data

A010790 $\boxed{a(n) = n!*(n+1)!.}$ +30 37

1, 2, 12, 144, 2880, 86400, 3628800, 203212800, 14631321600, 1316818944000, 144850083840000, 19120211066880000, 2982752926433280000, 542861032610856960000, 114000816848279961600000, 27360196043587190784000000, 7441973323855715893248000000 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Let M_n be the symmetrical n X n matrix M_n(i,j)=1/min(i,j); then for n>=0 det(M_n)=(-1)^(n-1)/a(n-1). - Benoit Cloitre, Apr 27 2002

If n women and n men are to be seated around a circular table, with no two of the same sex seated next to each other, the number of possible arrangements is a(n-1). - Ross La Haye, Jan 06 2009

a(n-1) is also the number of (directed) Hamiltonian cycles in the complete bipartite graph K_{n,n}. - Eric W. Weisstein, Jul 15 2011

a(n) is also number of ways to place k nonattacking semi-bishops on an n X n board, sum over all k>=0 (for definition see A187235). - Vaclav Kotesovec, Dec 06 2011

a(n) is number of permutations of {1,2,3,...,2n} such that no odd numbers are adjacent. - Ran Pan, May 23 2015

a(n) is number of permutations of {1,2,3,...,2n+1} such that no odd numbers are adjacent. - Ran Pan, May 23 2015

REFERENCES J. H. Conway and R. K. Guy, The Book of Numbers, Copernicus Press, NY, 1996, pp. 63-65.

$$\{D_n\}_{n\geq 1} = \{2, 12, 144, 2880, \ldots\}$$

OEIS

The on-line encyclopedia of integer sequences

$$D_n = n!\,(n+1)!$$

$$C_n = \frac{(2n)!}{D_n}$$

$$C_n = \frac{(2n)!}{n!\,(n+1)!}$$

$$C_4 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \quad 1 \times 2 \times 3 \times 4 \times 5}$$

$$= 14$$

Catalan
numbers

(1838)

Note sur une Équation aux différences finies ;

Par E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

C_n (2)

Admettant donc la conc[...] [...]ux formules, je vais chercher à en déduire quelqu[...]

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\cdots\frac{4n-6}{n}P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\ 4.\ 5\dots n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3)$$
$$= \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1}C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned}\frac{1}{n+1}C_{2n,n} &= \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1}\\ &+ \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \dots + \frac{1}{n}C_{2n-2,n-1}.\end{aligned}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

$$1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924\dots;$$

qu'on les divise respectivement par

on obtiendr[...]

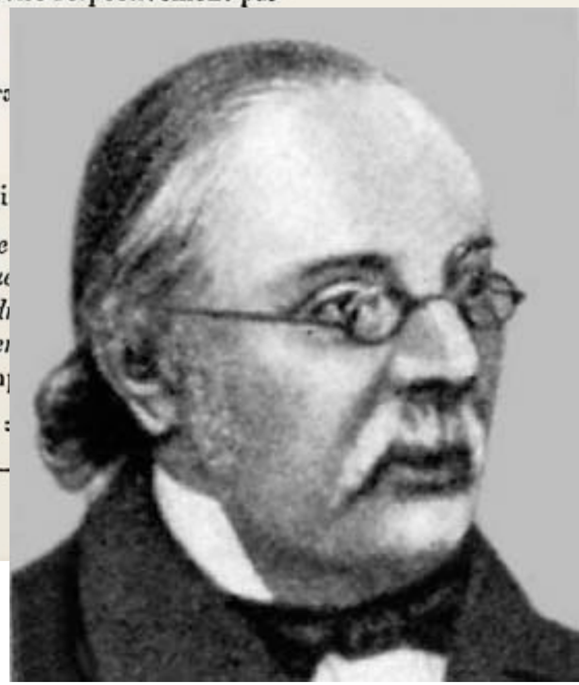
(A)

lesquels joui[...]

Un terme [...] nme des produits qu[...] ème, et dans un ordr[...] n multipliant les te[...]

Par exem[...]

132 [...]



Eugène Catalan (1814-1894)

The arithmetical triangle

binomial coefficient

$$C_n = \frac{(2n)!}{n!\,(n+1)!}$$

$$= \frac{1}{(n+1)}\,\frac{(2n)!}{n!\,n!}$$

$$= \frac{1}{(n+1)}\binom{2n}{n}$$

binomial coefficient

$$\binom{2n}{n} = \frac{(2n)(2n-1)\cdots(n+1)}{n\,(n-1)\cdots 2\times 1}$$

$$\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!}$$

binomial coefficient

$$= \frac{n!}{p!\,(n-p)!}$$

number of subsets having **p** elements of a set having **n** elements

n choose **p**

${}^{n}C_{p}$

C_{n}^{p}

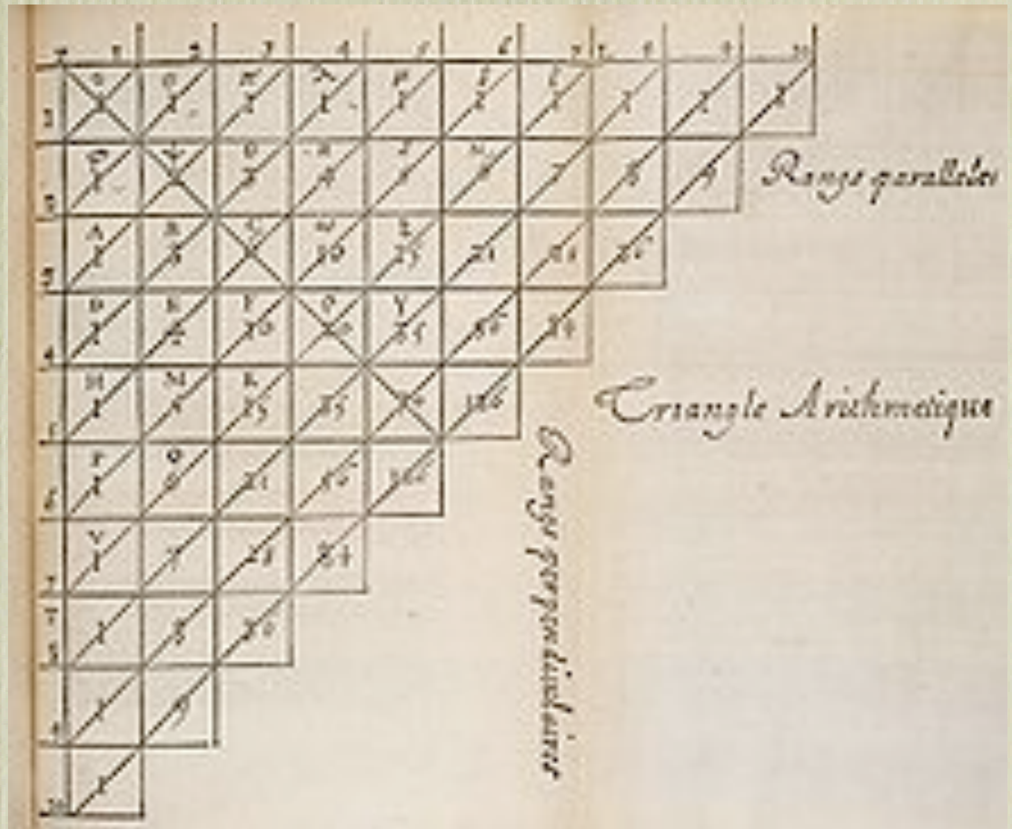
$$\sum_{0 \leq p \leq n} \binom{n}{p} = 2^n$$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

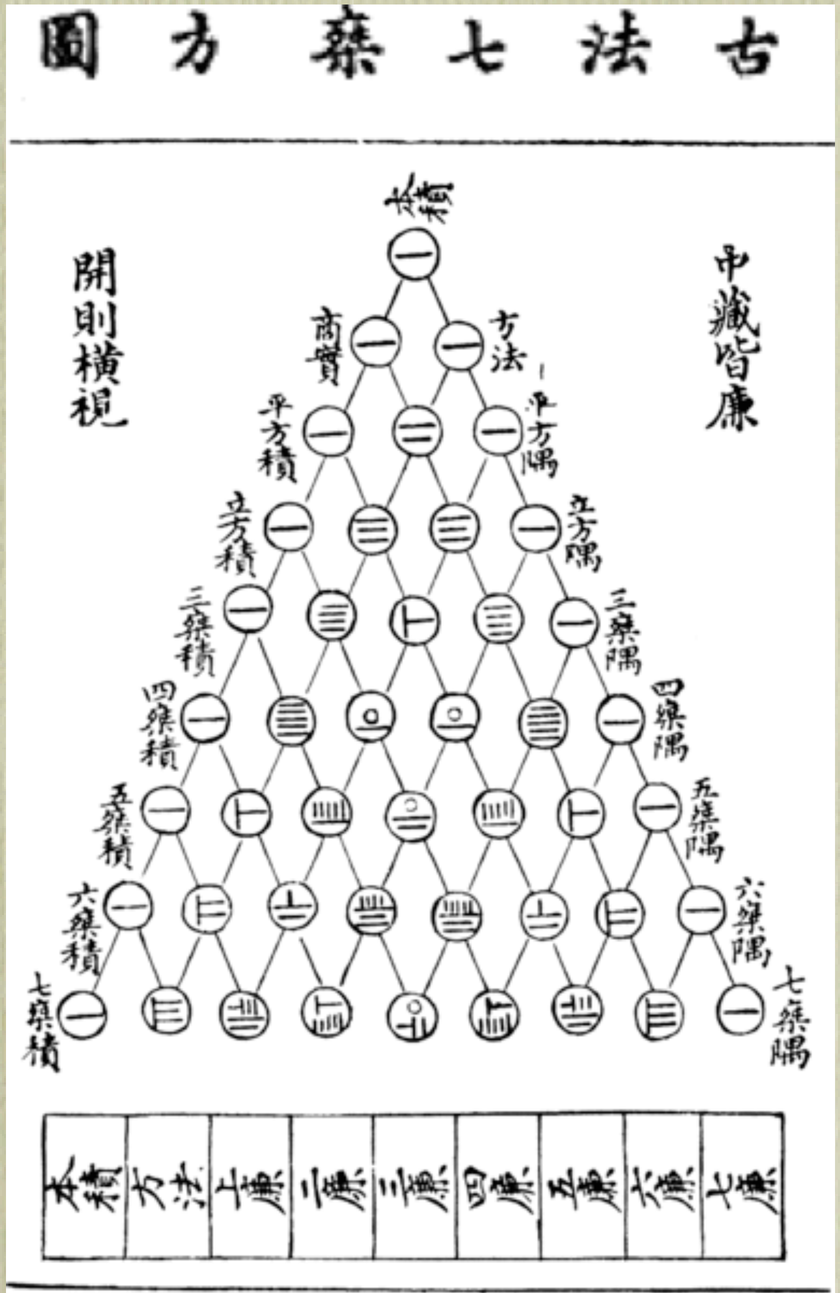
addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

The arithmetical triangle



Pascal triangle
binomial coefficients



Yang Hui triangle
(11th, 12th century)

in Persia
Omar Khayyam
(1048-1131)

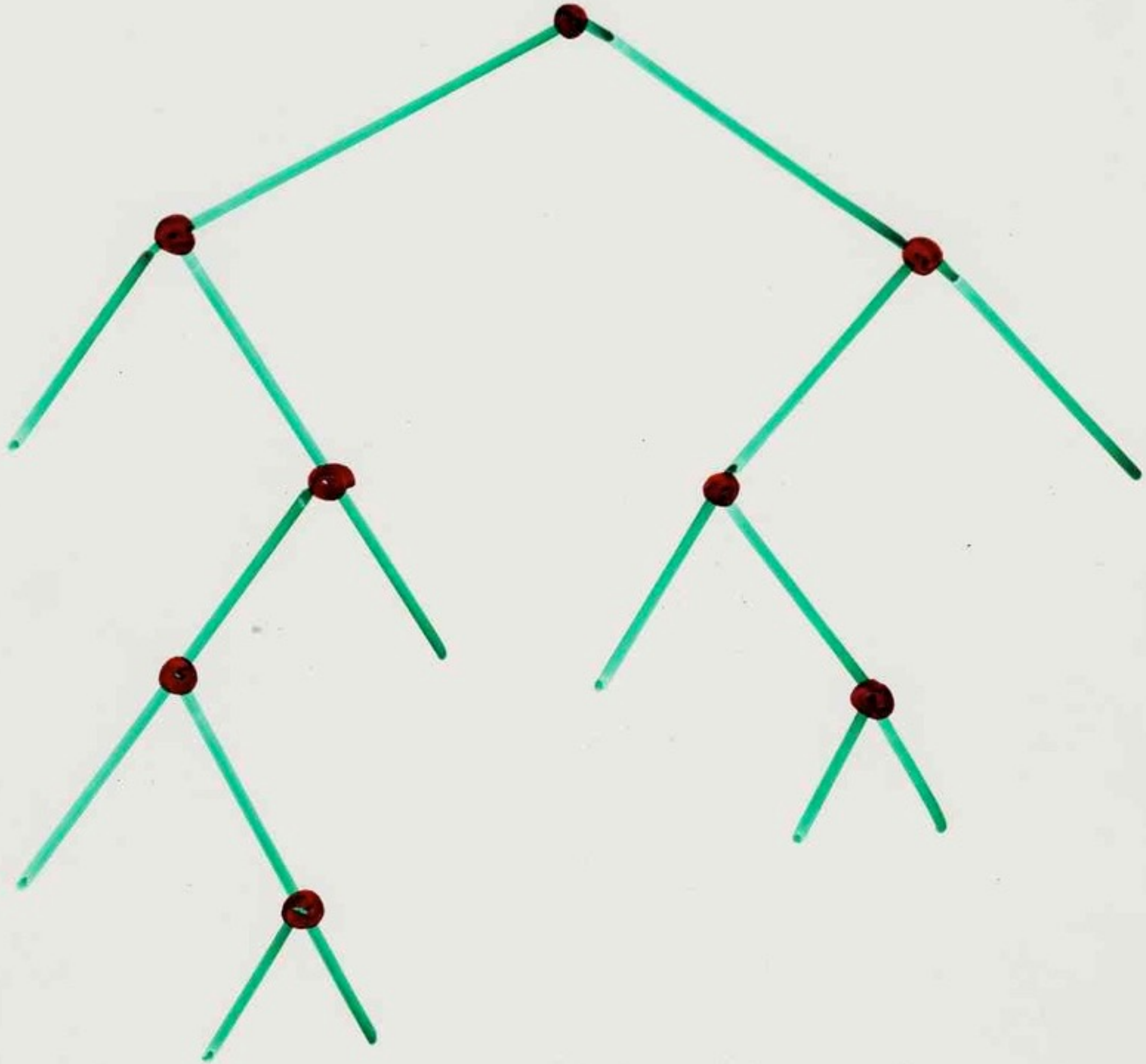
in India
Chandas Shastra by Pingala
2nd century BC

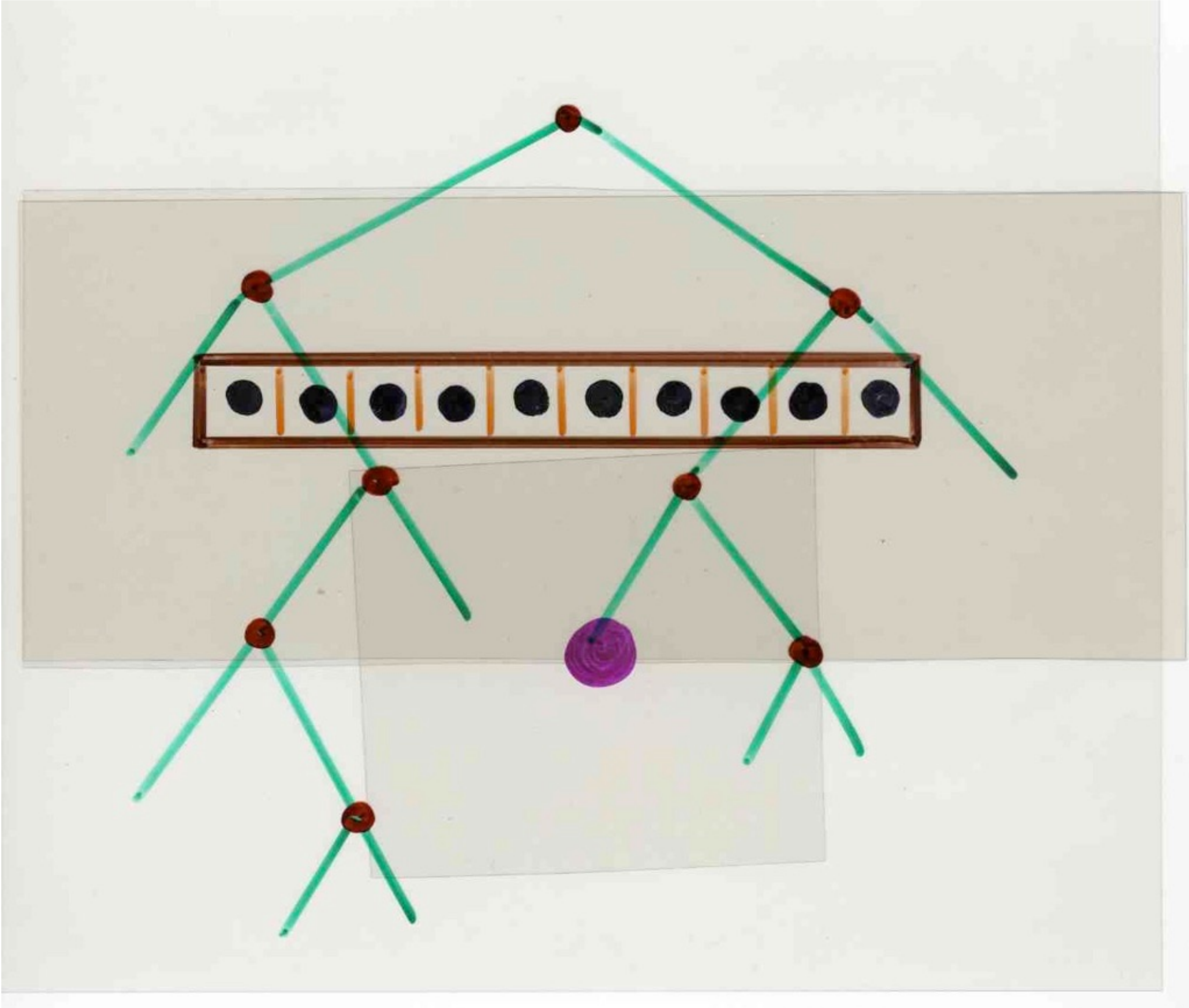
bijective combinatorics

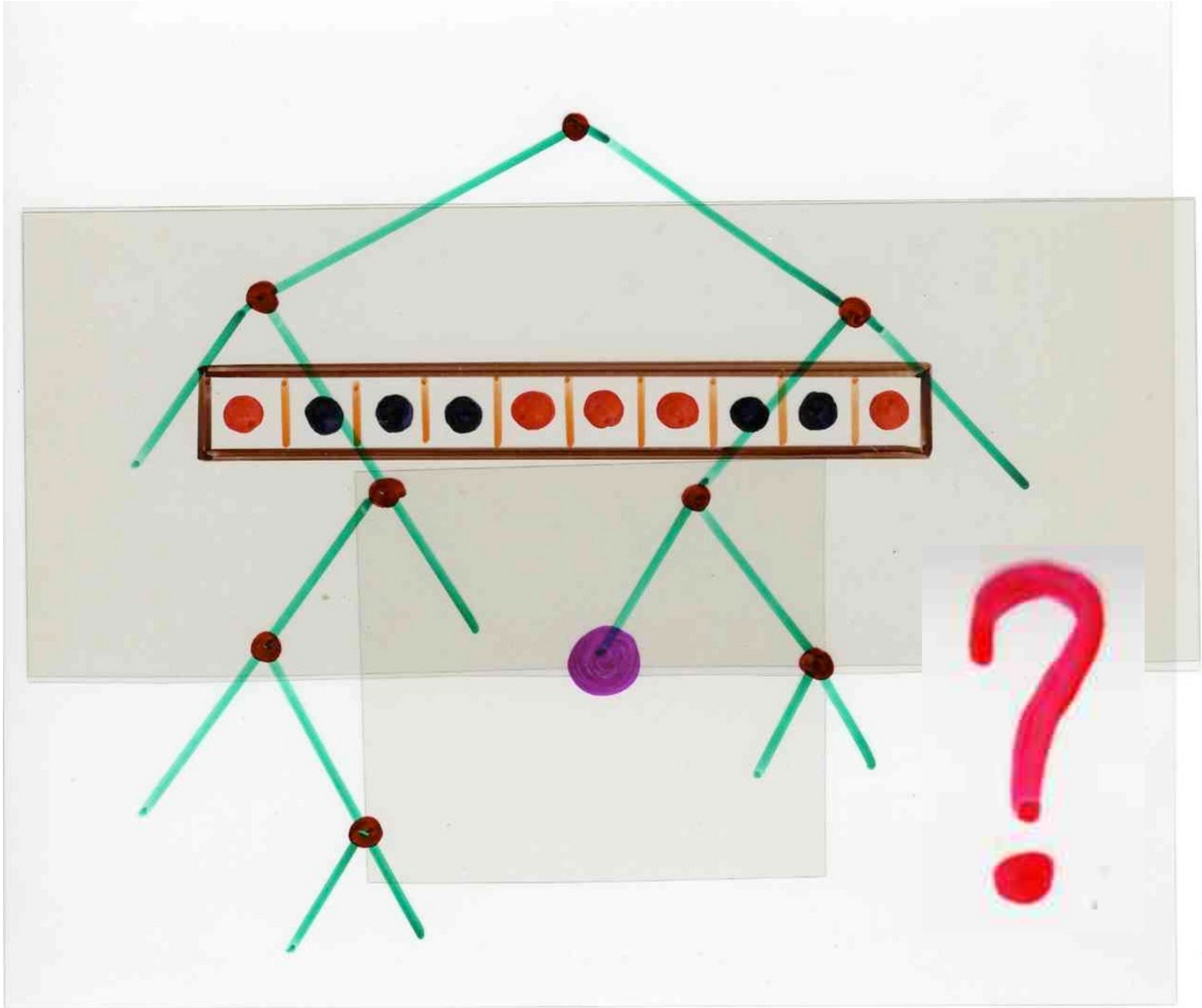
Catalan numbers

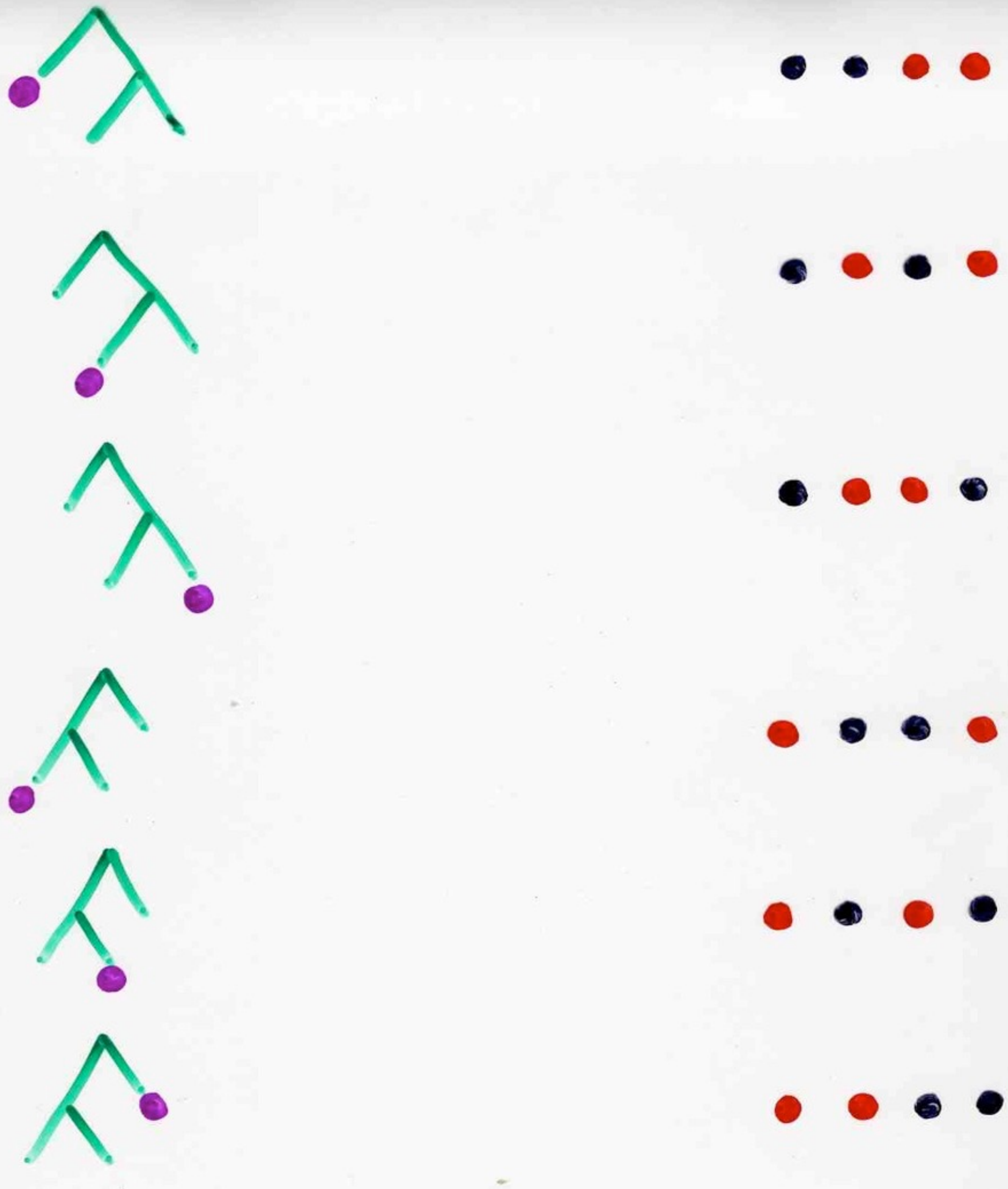
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

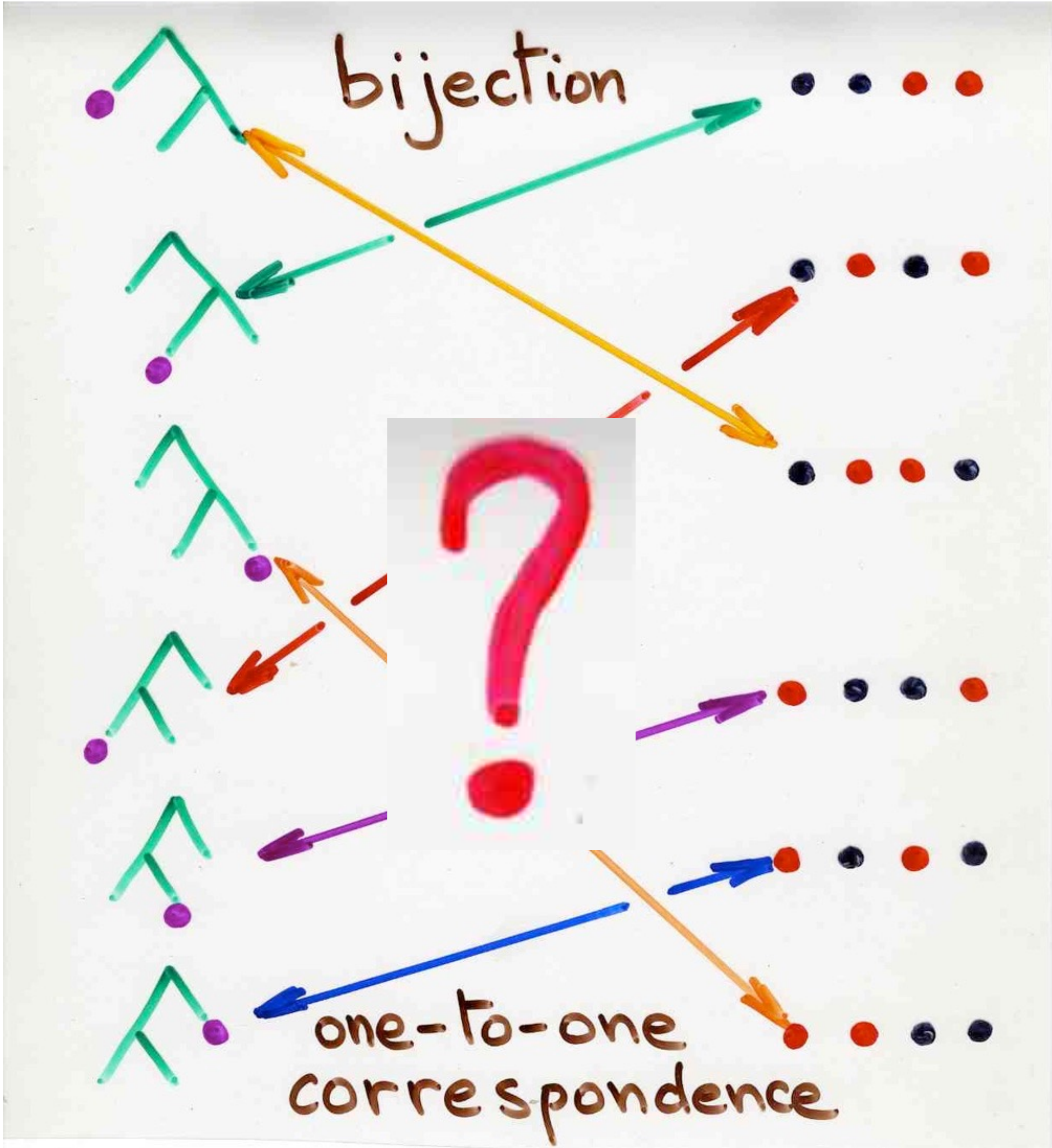
$$(n+1)\,C_n = \binom{2n}{n}$$









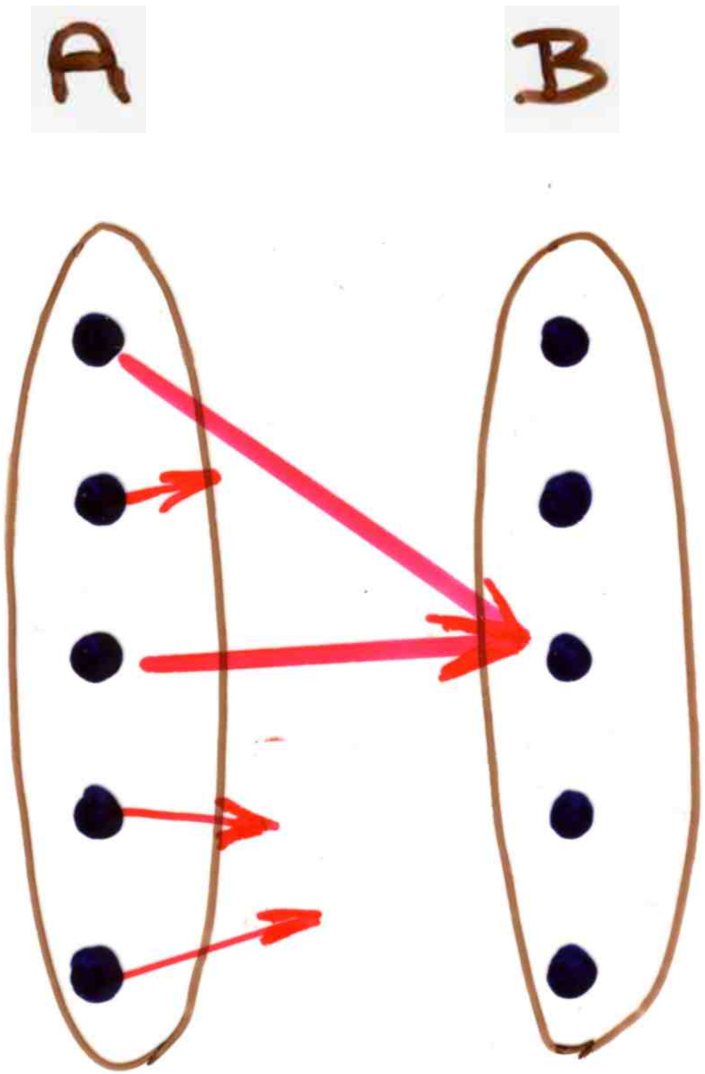

bijection
one-to-one
correspondence

injections

surjections

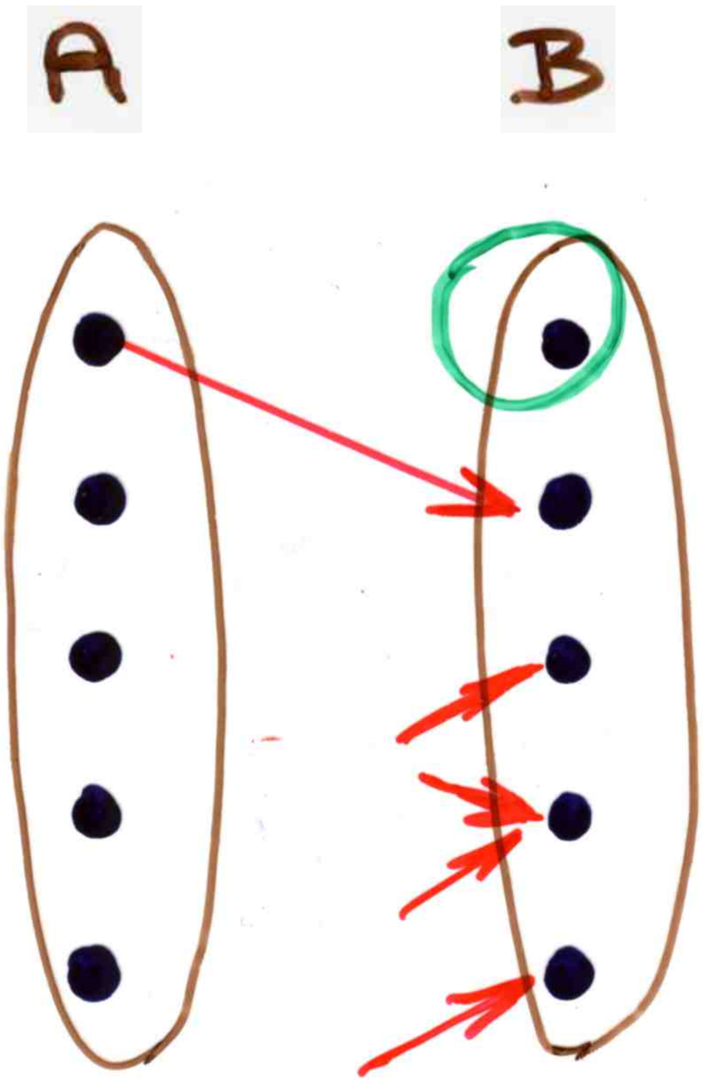
bijections

....



f injective

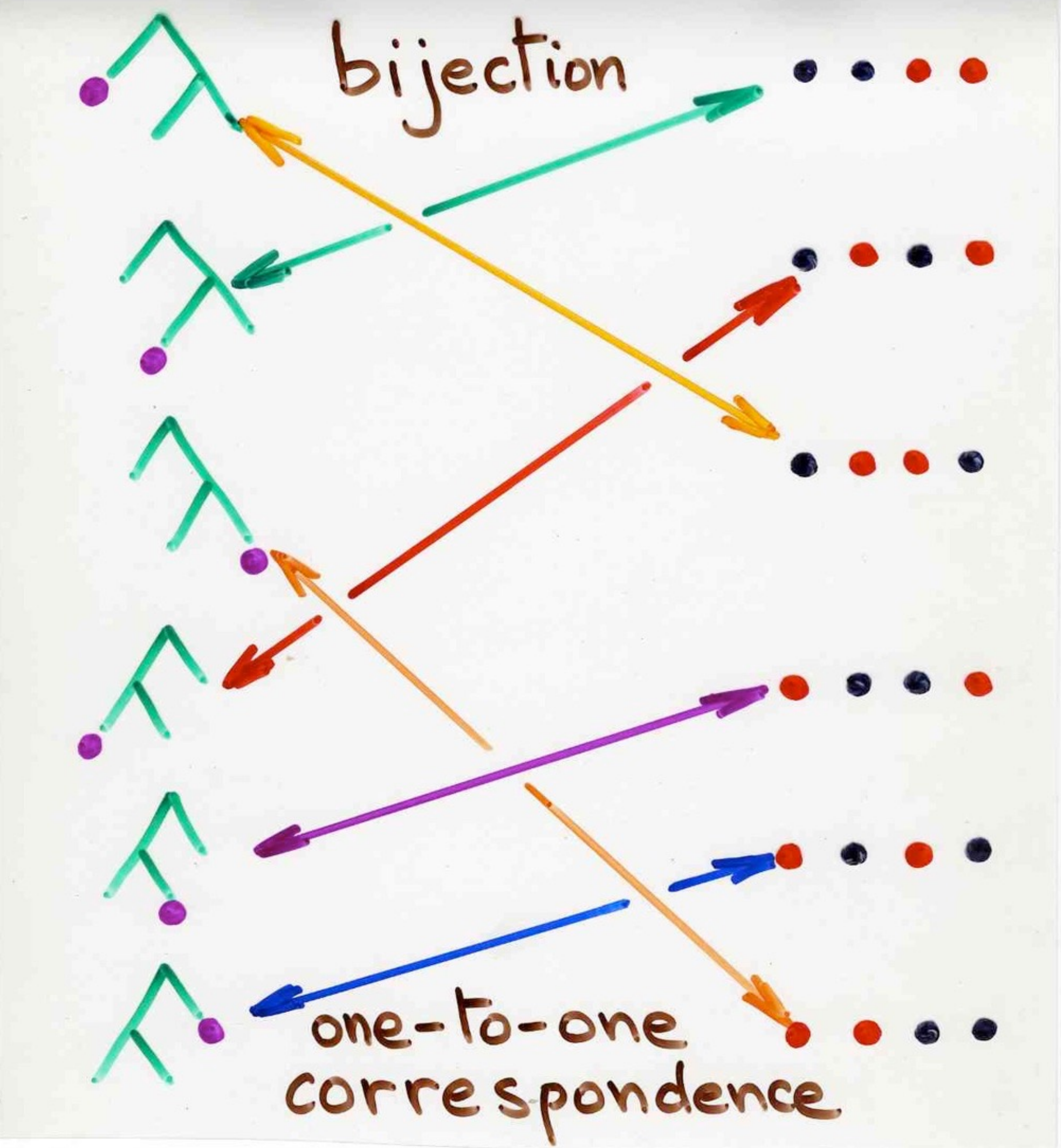
$f: A \longrightarrow B$

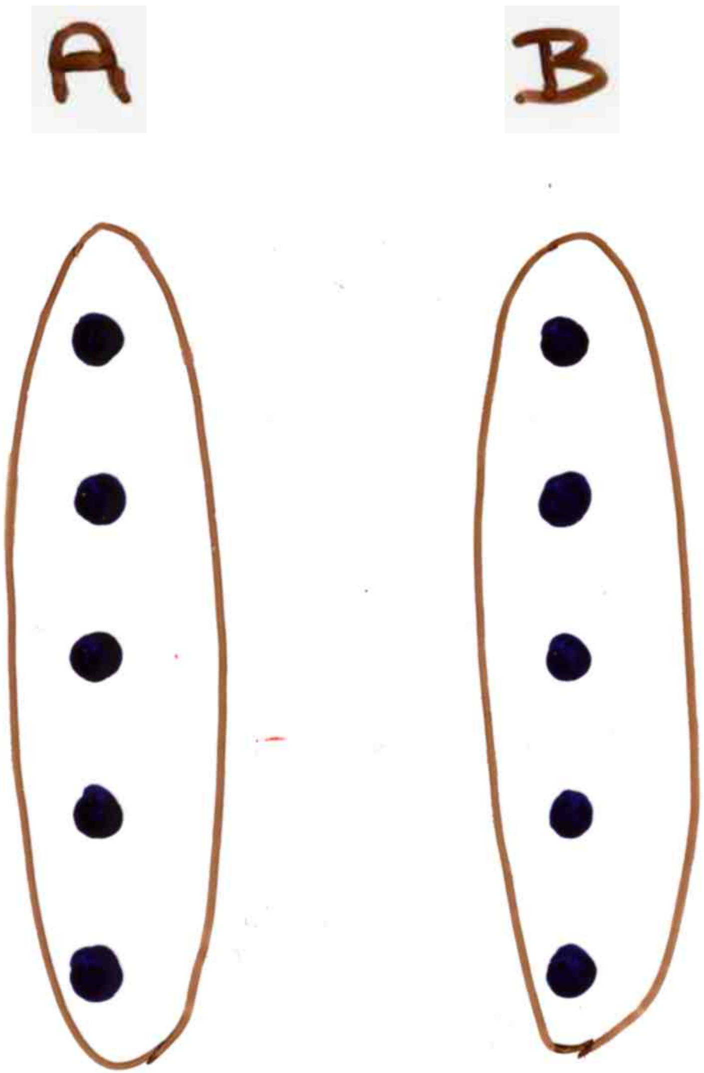


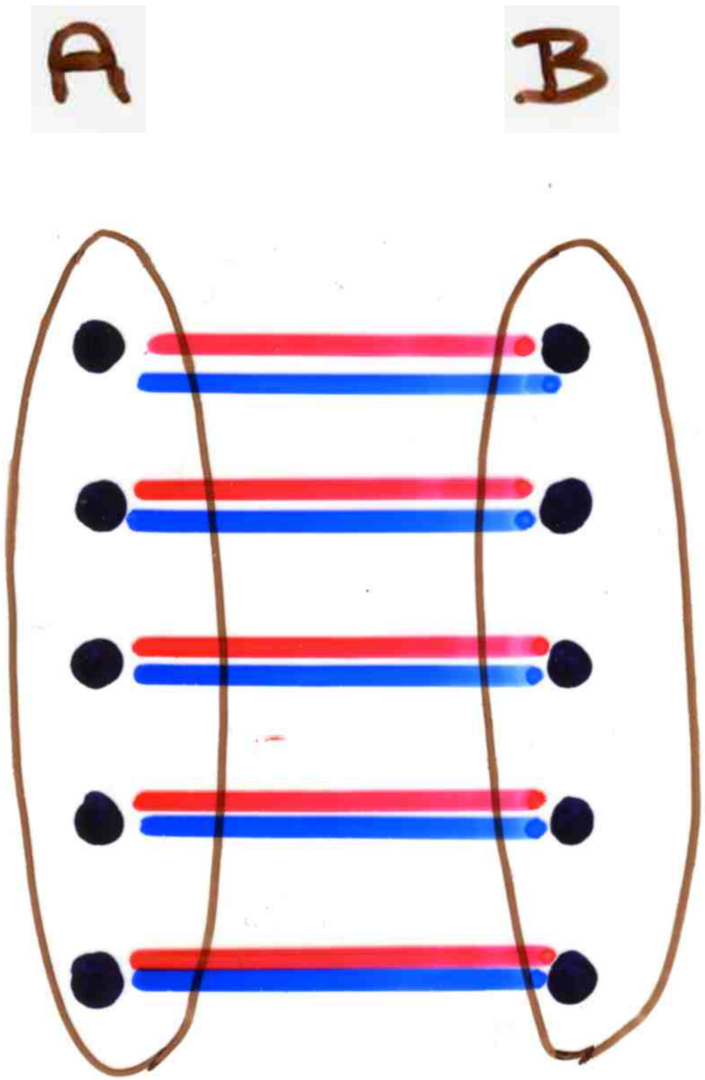
f

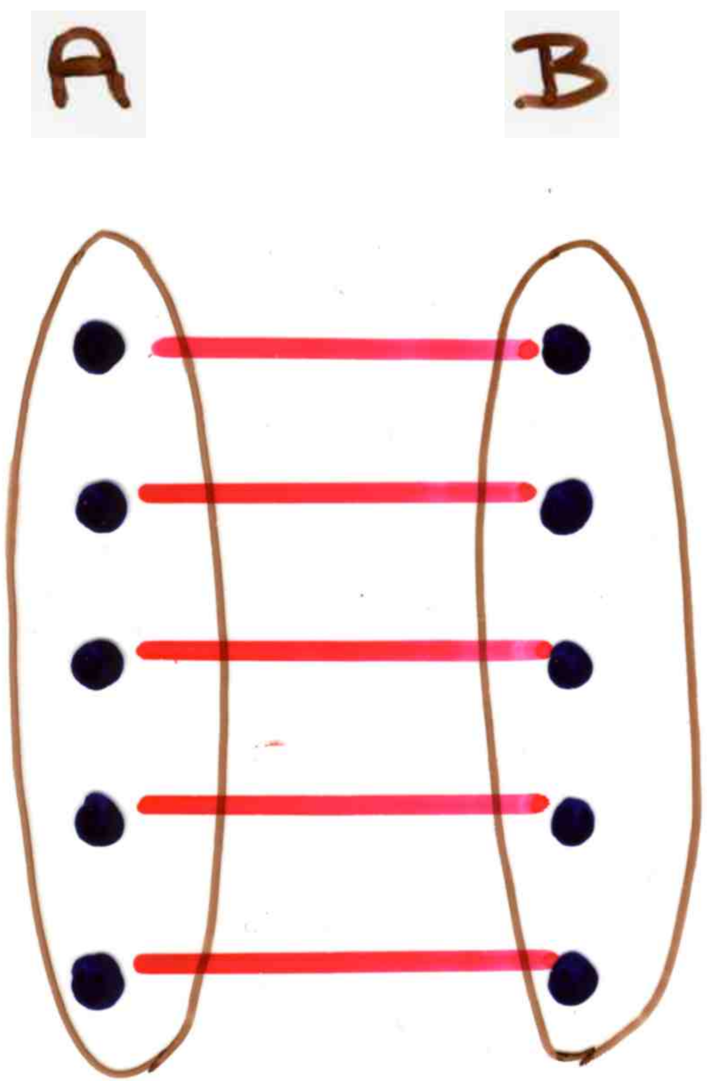
surjective

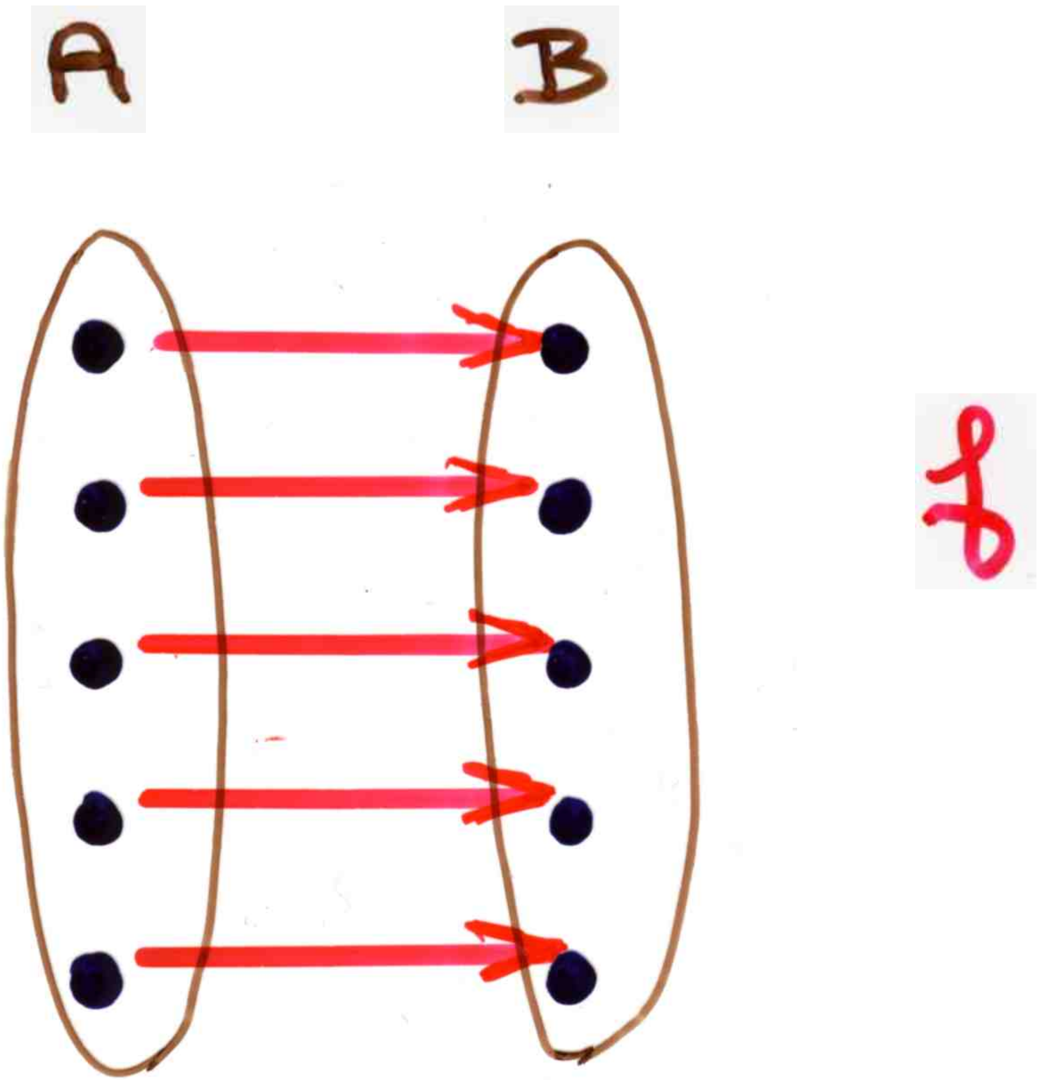
$f : A \longrightarrow B$


bijection
one-to-one
correspondence

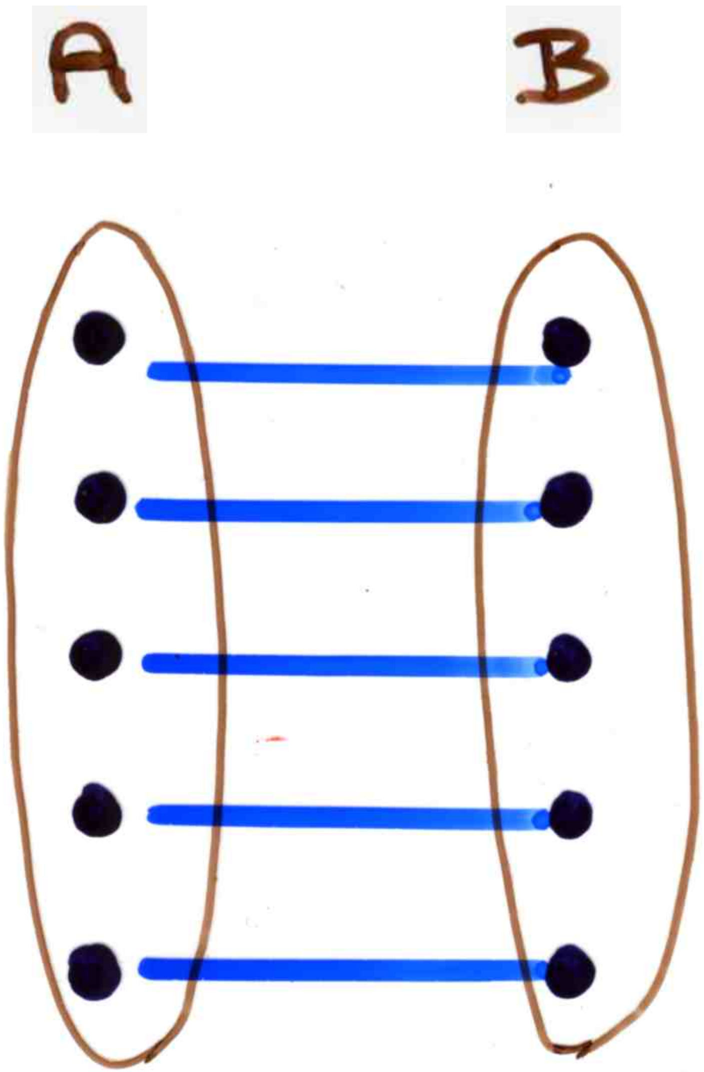

A
B

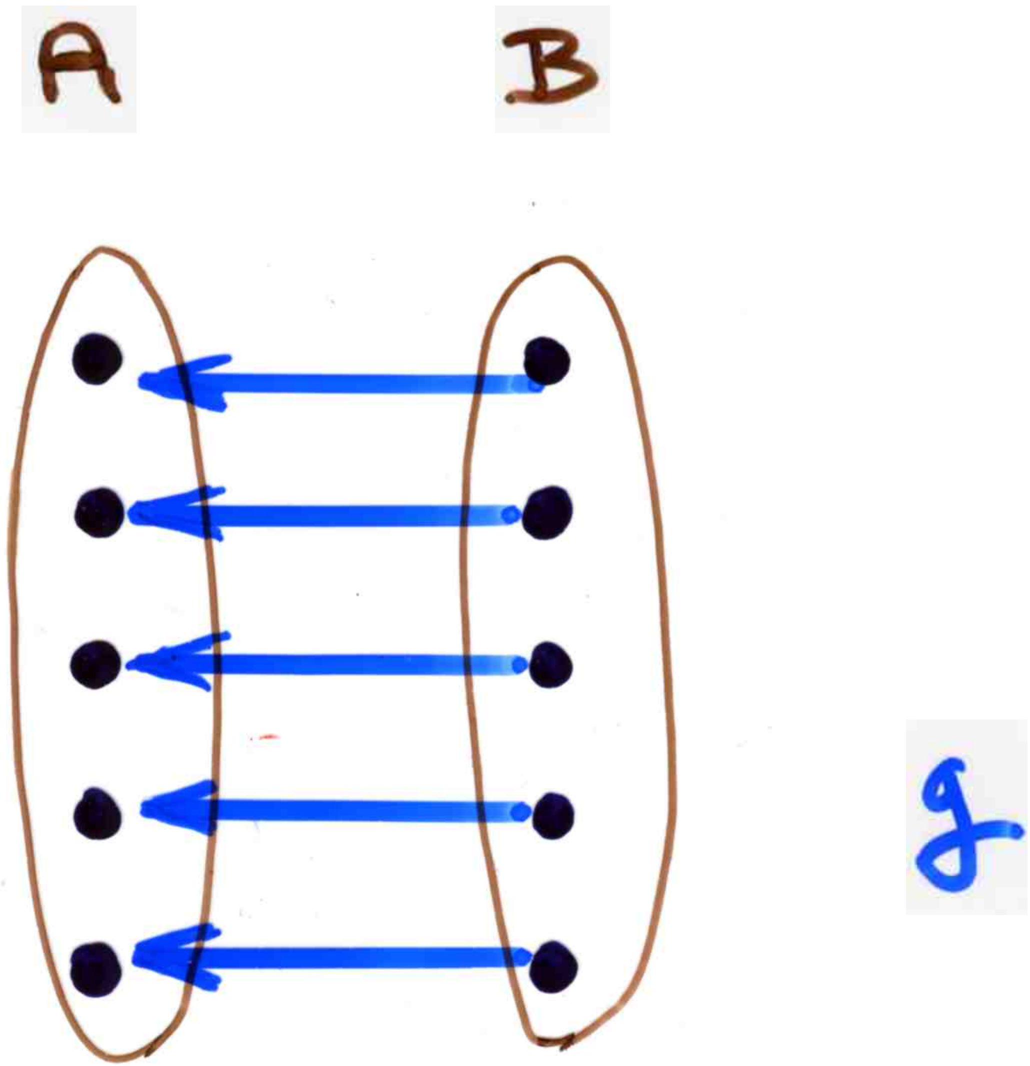

A
B


A
B

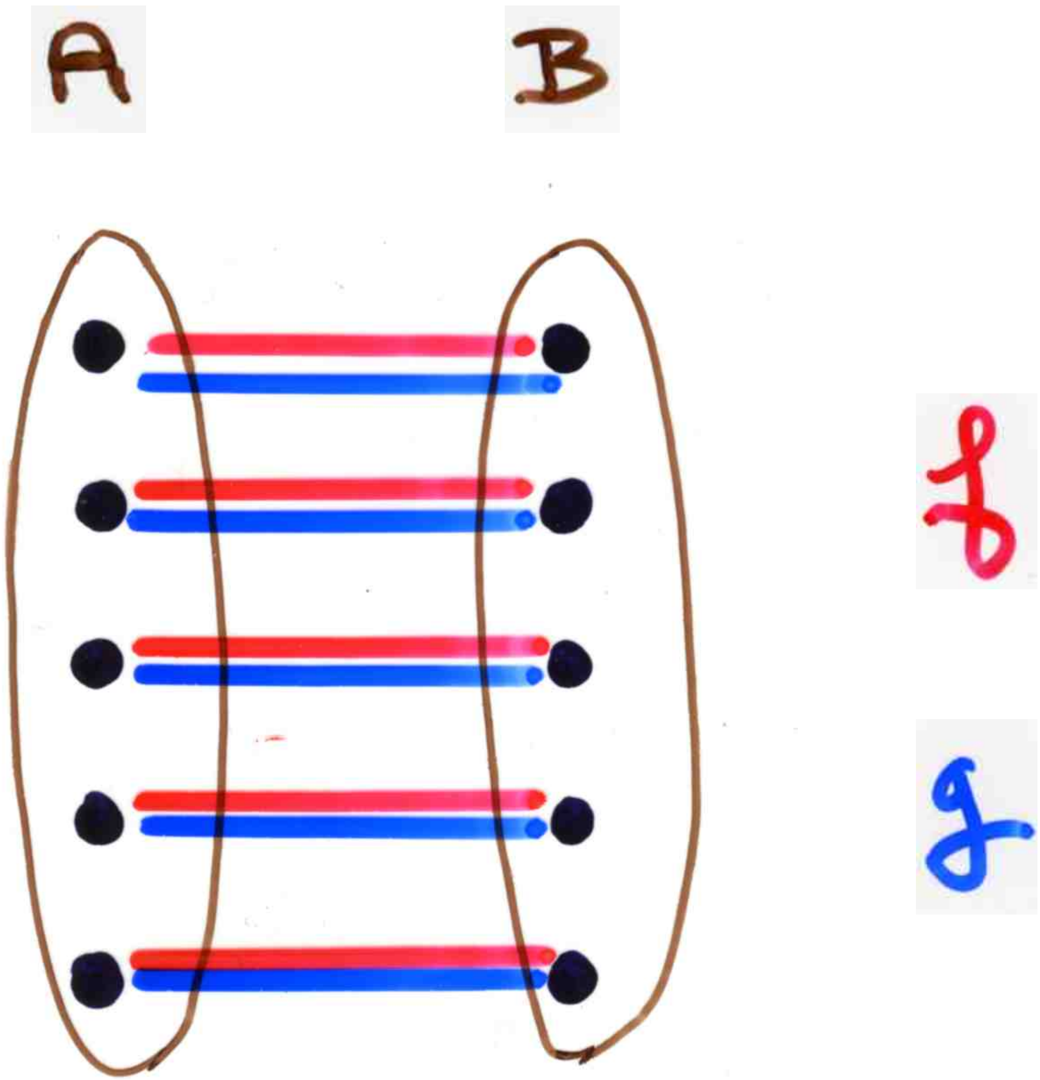


$f : A \to B$


A
B



$g : B \longrightarrow A$



$$f \circ g = Id_A$$

$$g \circ f = Id_B$$

$$f = g^{-1}$$

permutations

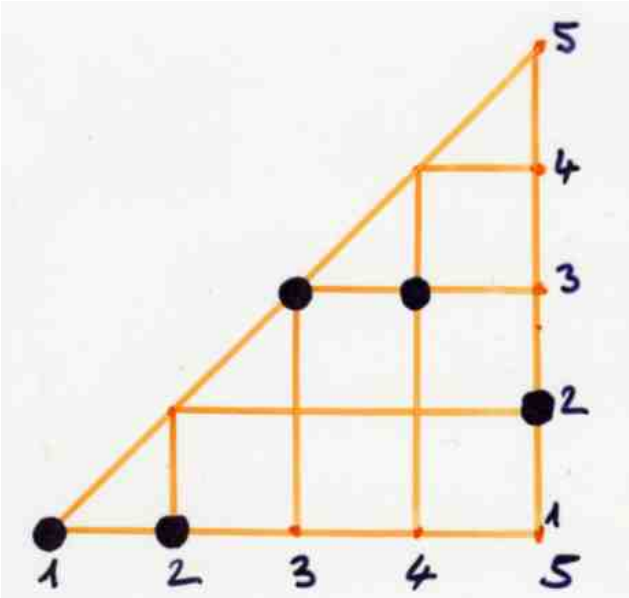
$\sigma = 3\ 4\ 1\ 5\ 2$

permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

permutations

$$\left| \mathfrak{S}_n \right| = n!$$



$f : [1,n] \longrightarrow [1,n]$

$$f : \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 2 \end{array}$$

$f(i) \leq i$ $\quad (1 \leq i \leq n)$

sub-exedante function

$$\left| \mathcal{F}_n \right| = n!$$

1

1

1 2

1 2

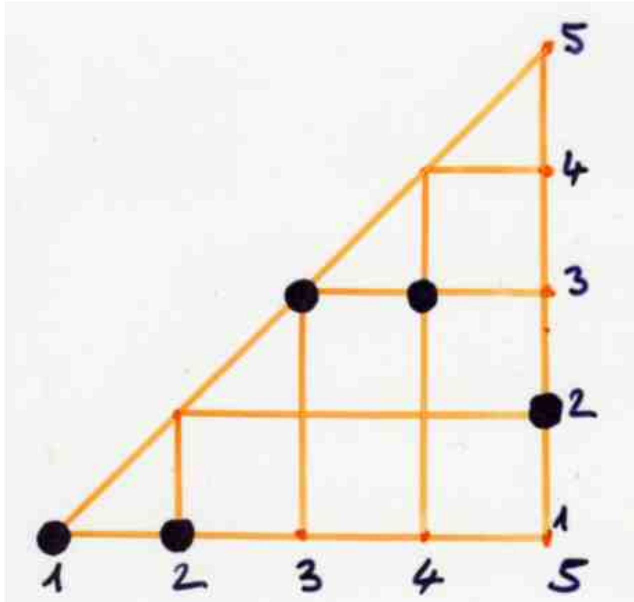
3 1 2

3 1 2

3 4 1 2

3 4 1 2

$\sigma = 3\ 4\ 1\ 5\ 2$



	1	2	3	4	5
f :	1	1	3	3	2

exercise 2

describe the reverse bijection

$\sigma \longrightarrow f$

$$\mathfrak{S}_n \longrightarrow \mathcal{F}_n$$

permutations

sub-exedante function

binomial coefficient

$$\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!}$$

$$\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!}$$

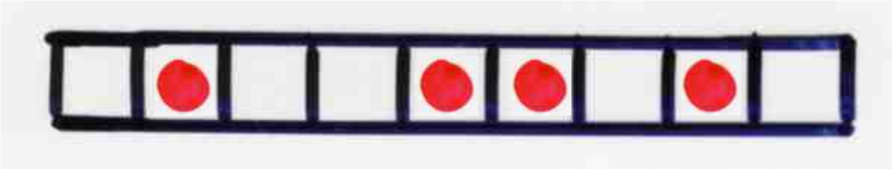
binomial coefficient

$$= \frac{n!}{p!\,(n-p)!}$$

number of subsets having **p** elements of a set having **n** elements

$$\binom{n}{p}\,p!$$

$$n(n-1)\cdots(n-p+1)$$



E set with n elements

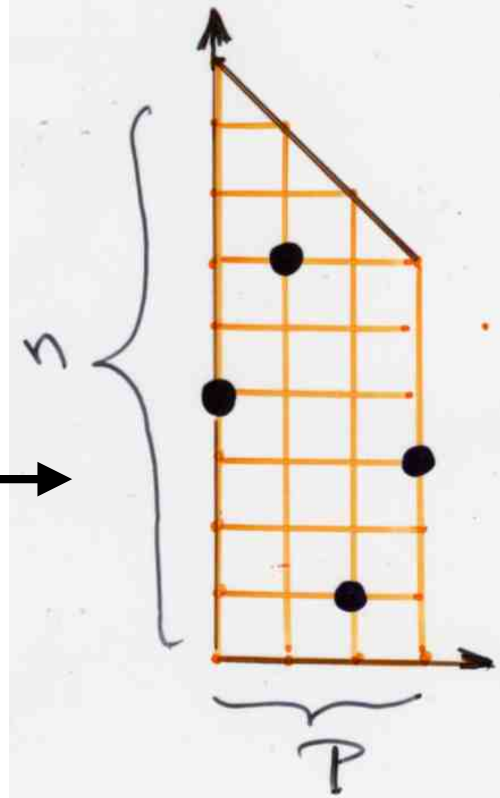
3 1 4 2

permutation σ

$F \subseteq E$ subset with p elements)

E set with n elements

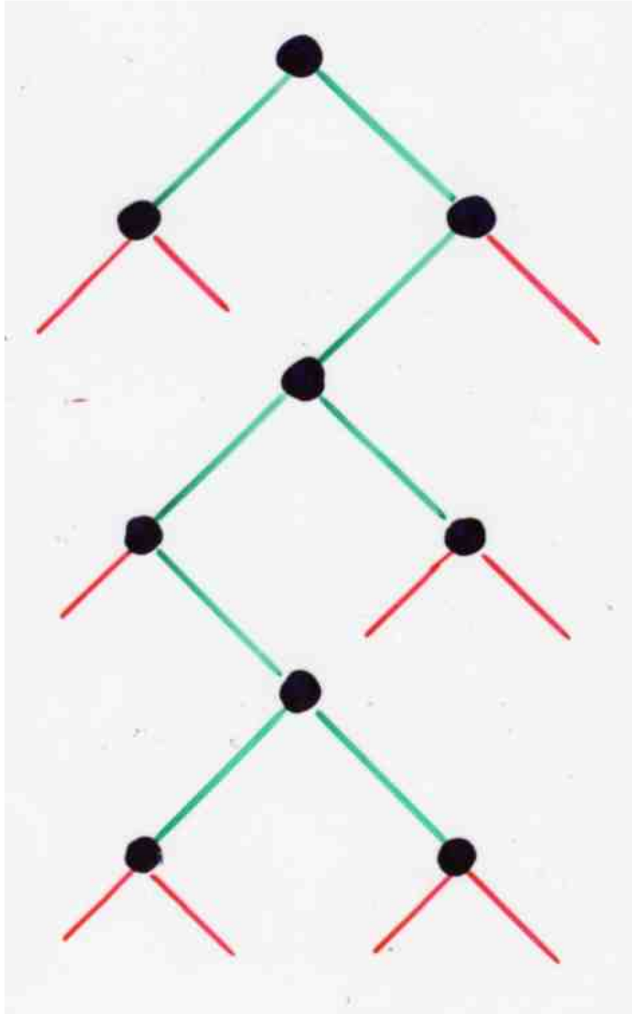
$\left(\begin{array}{c}\text{permutation } \sigma \\ \text{on } p \text{ elements}\end{array}, \begin{array}{c} F \subseteq E \\ \text{subset with } p \text{ elements}\end{array}\right) \longleftrightarrow$

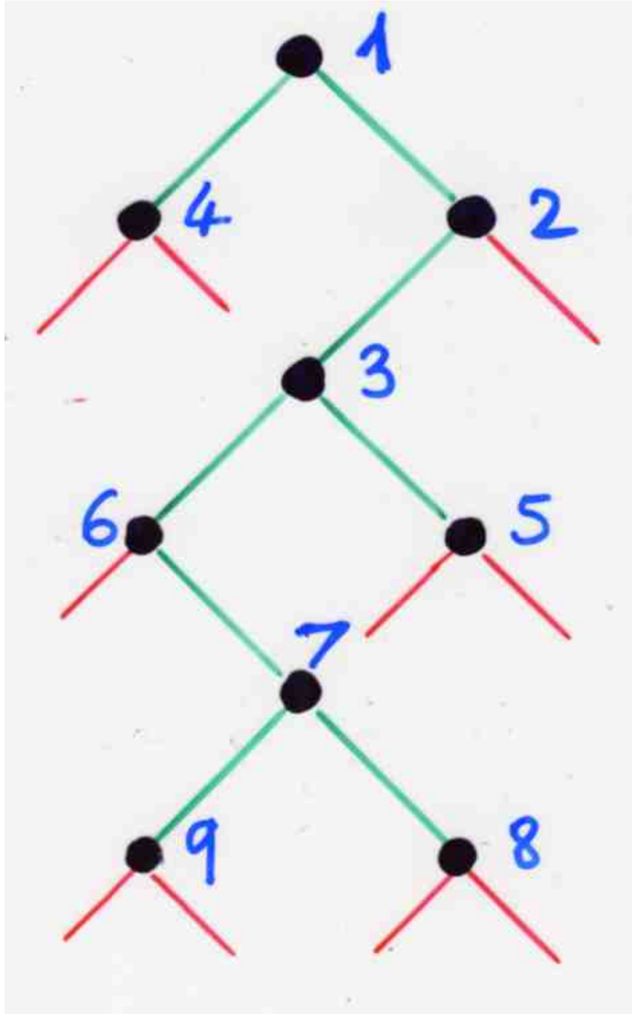


$$\binom{n}{p}\, p!$$

$$n(n-1)\cdots(n-p+1)$$

increasing binary trees




1
4
2
3
6
5
7
9
8

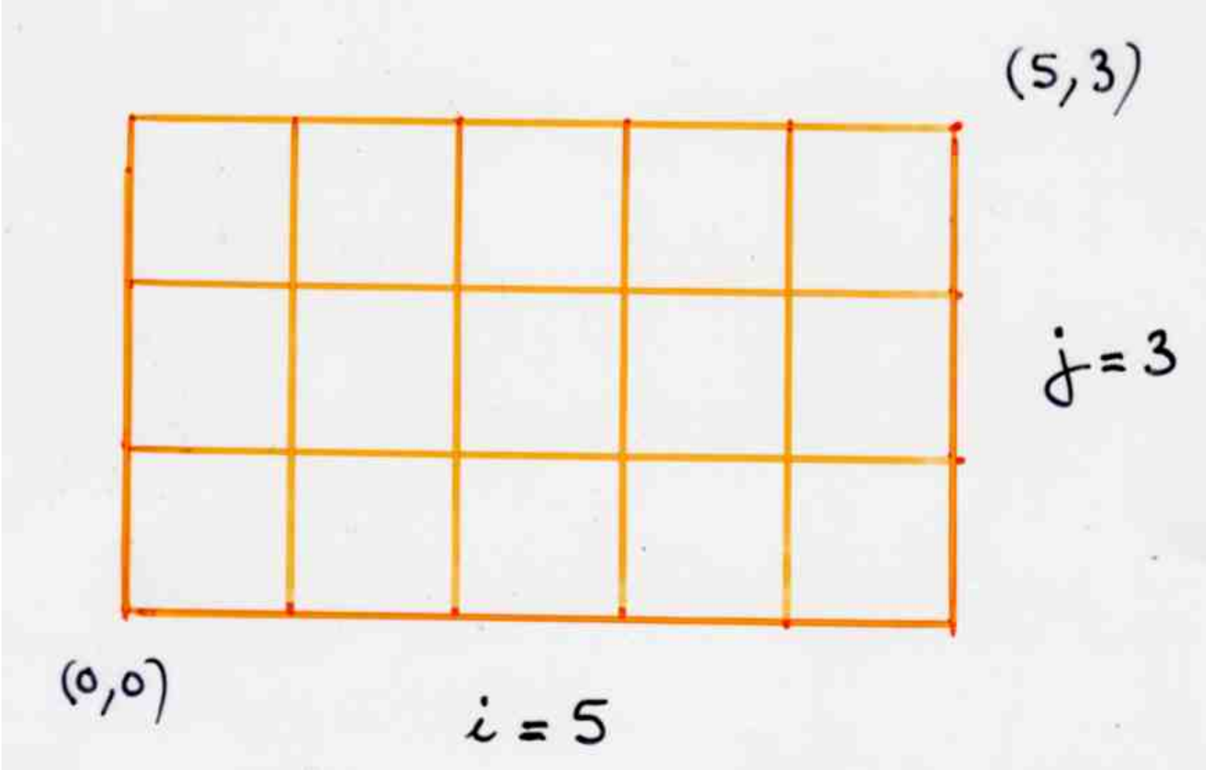
exercise 3

The number of increasing binary trees is $n!$

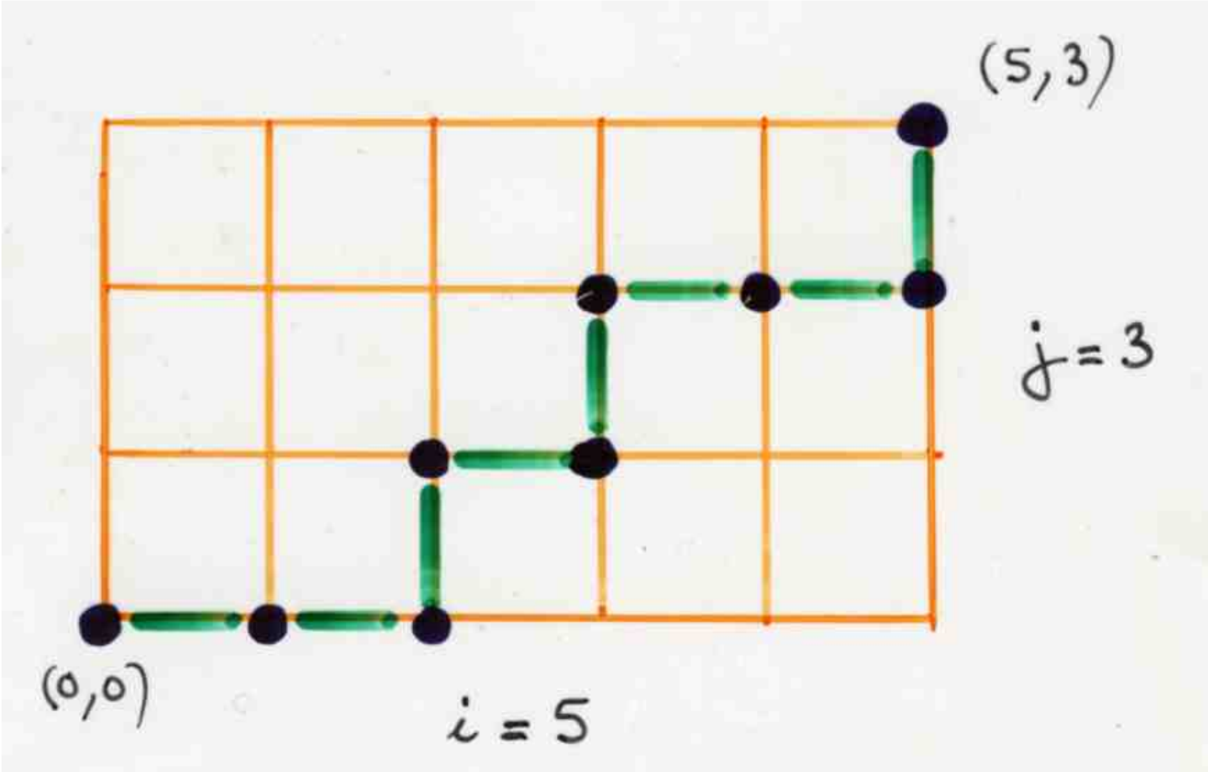
binomials coefficients

with paths

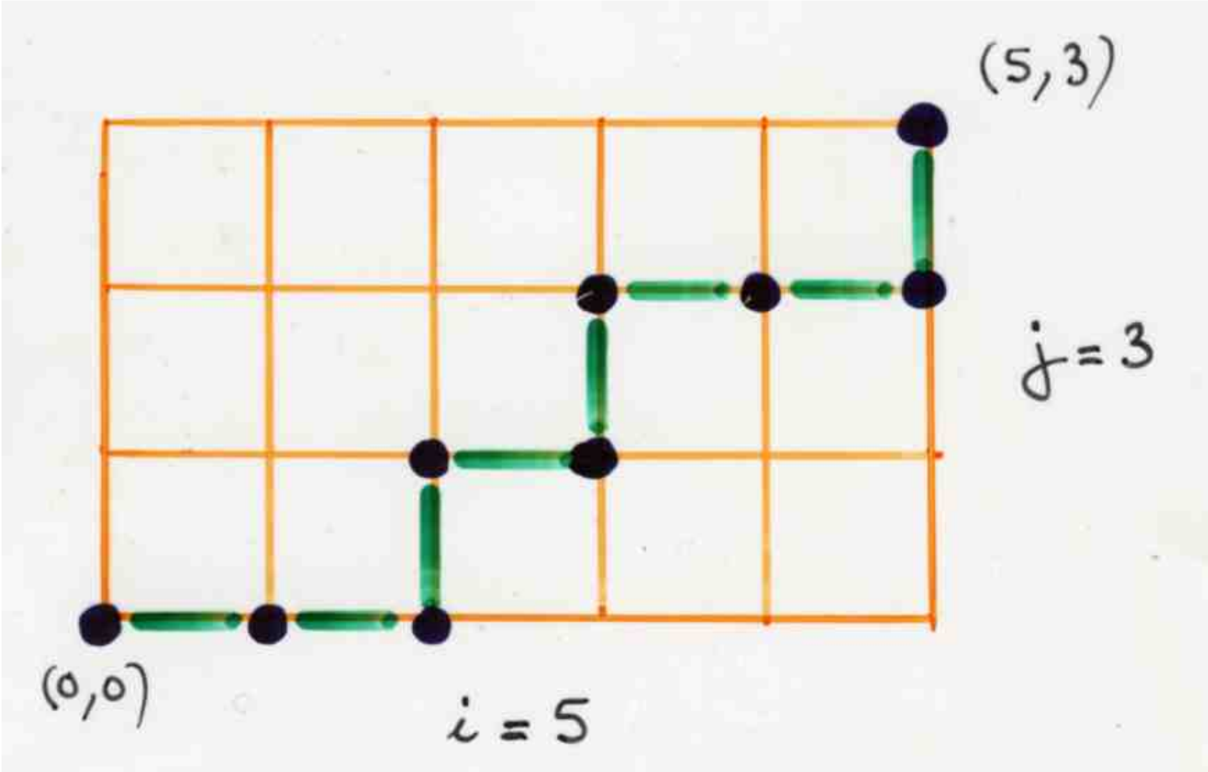
bijection with paths

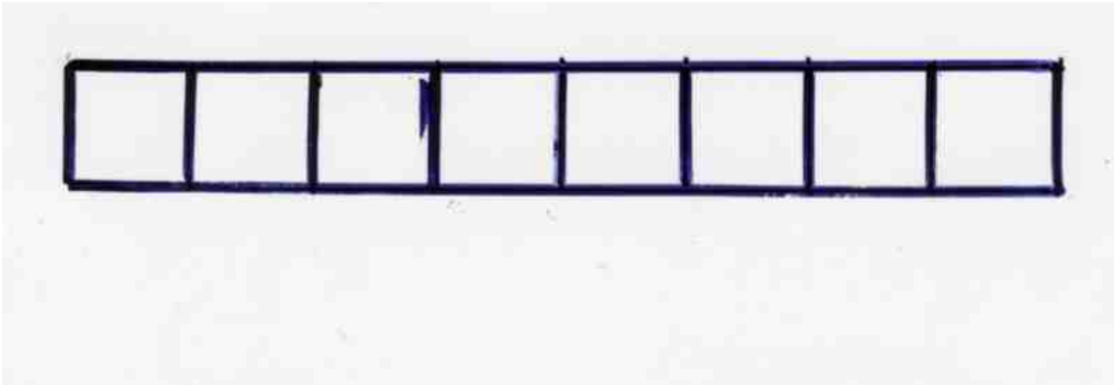


bijection with paths

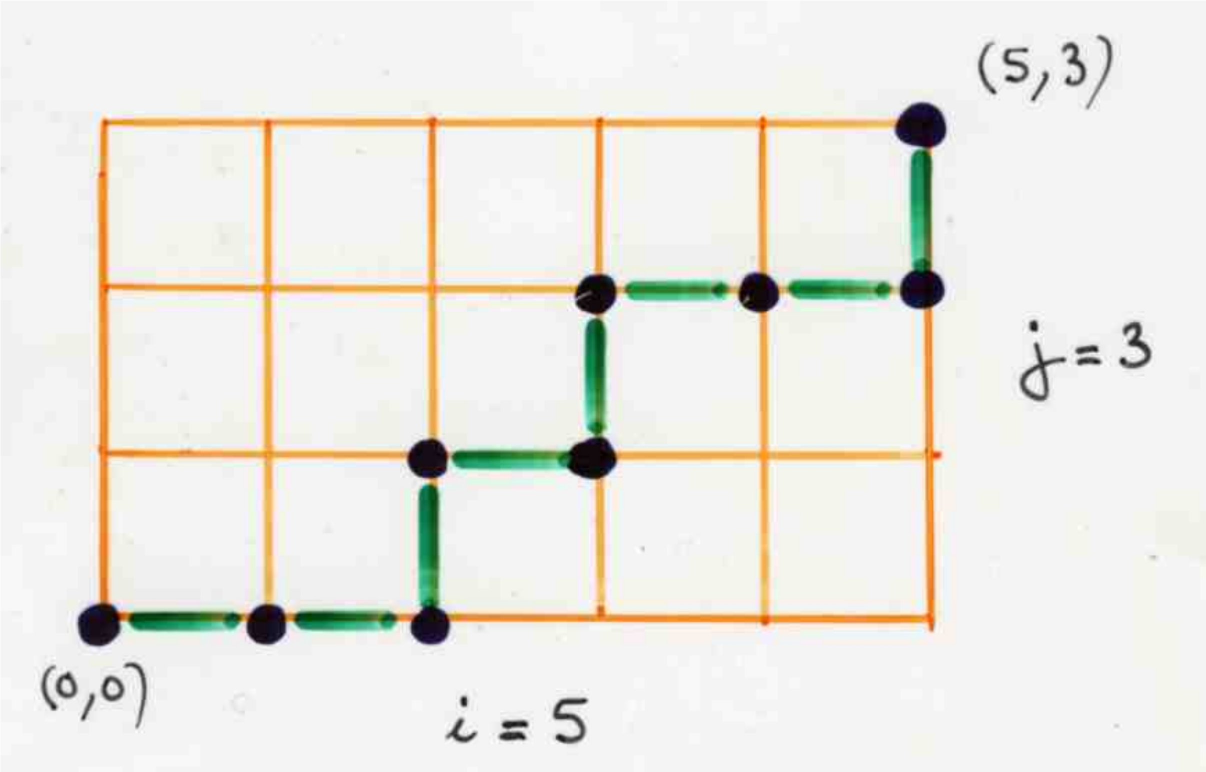


bijection with paths

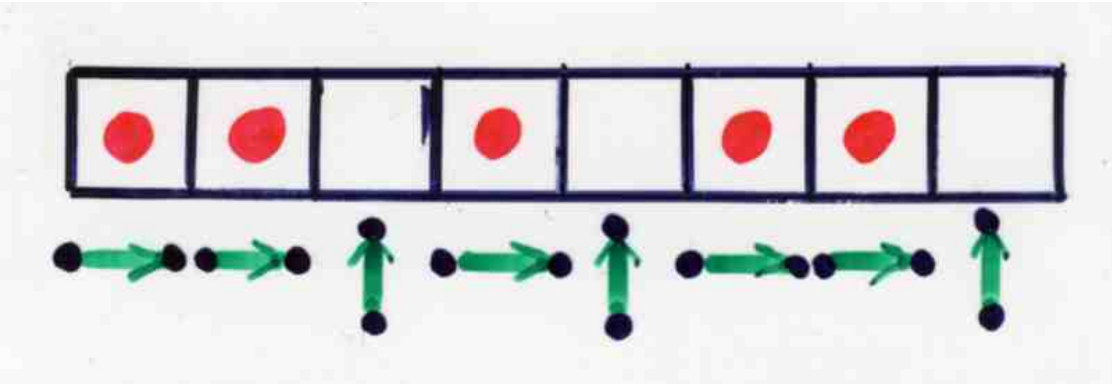




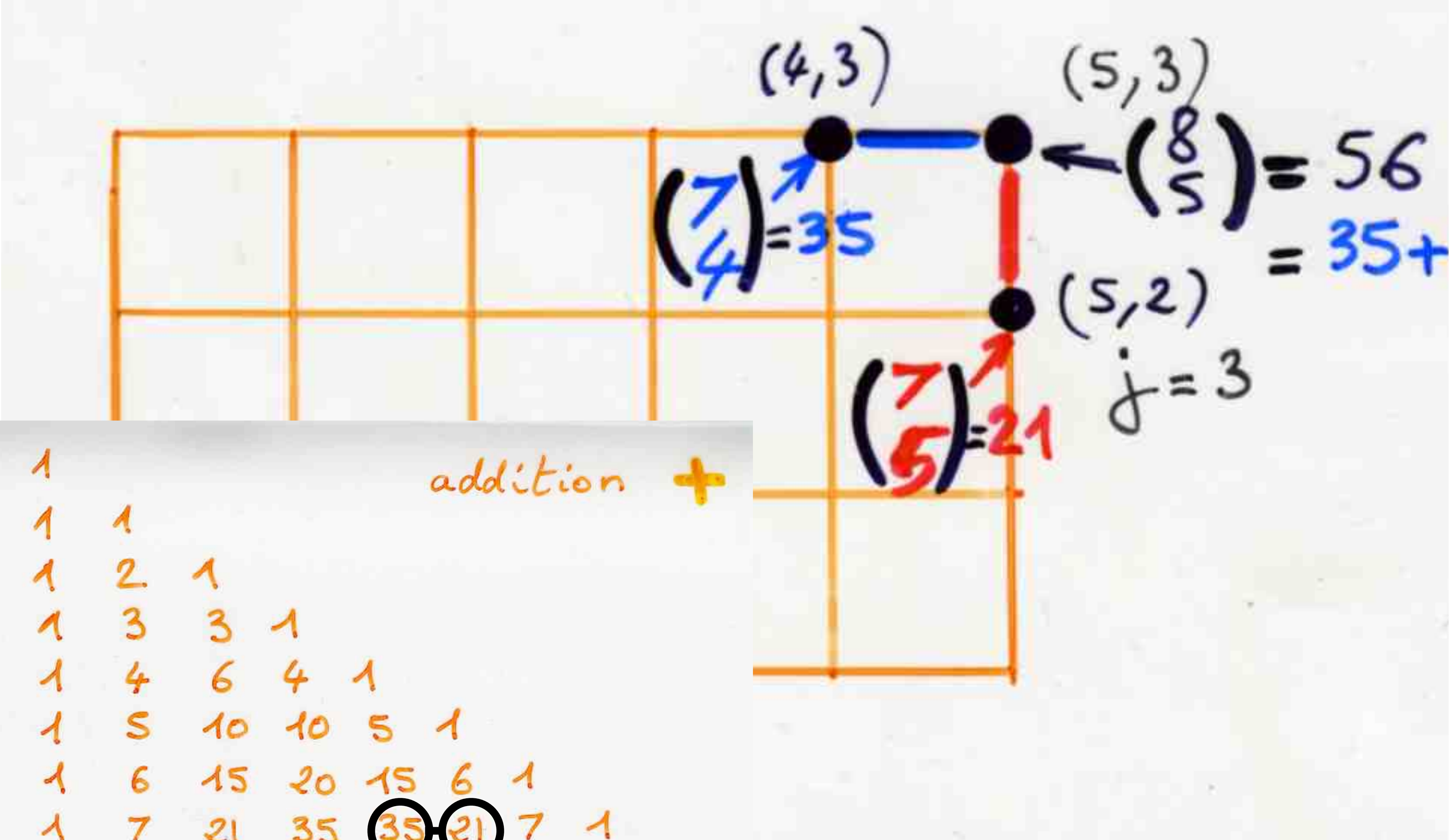
bijection with paths



$\binom{i+j}{i}$

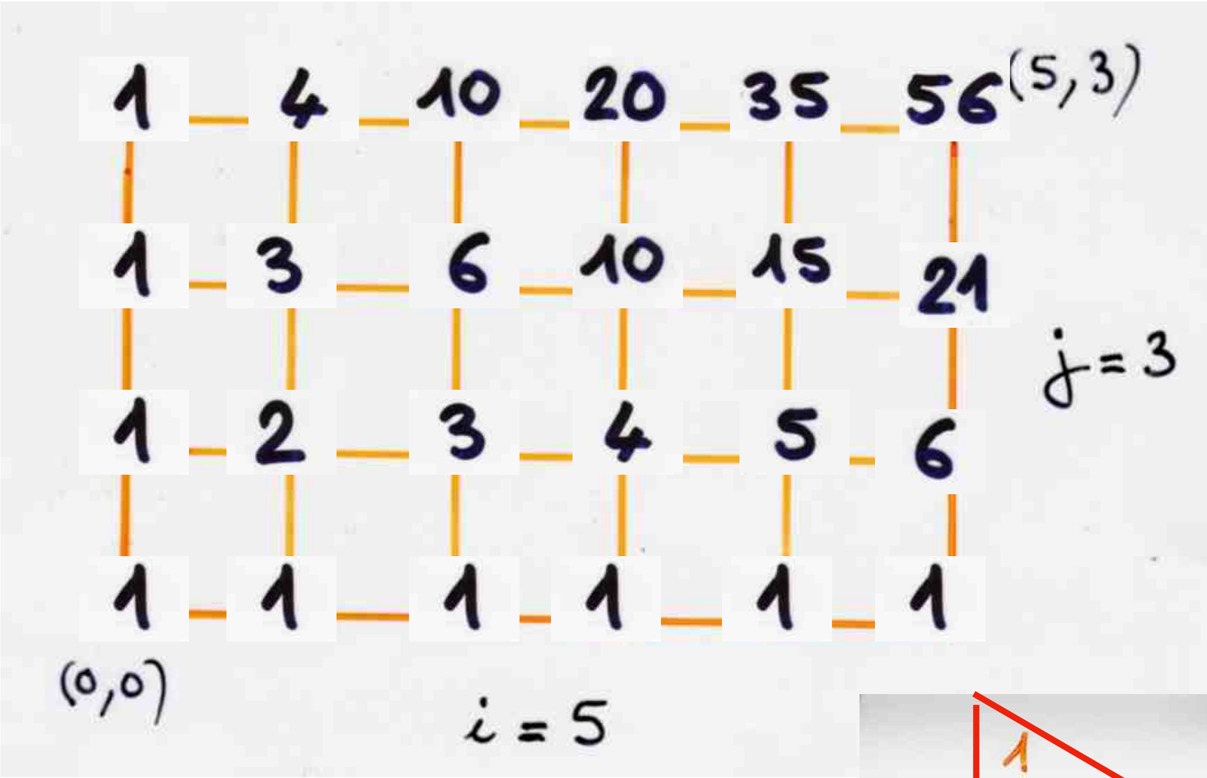


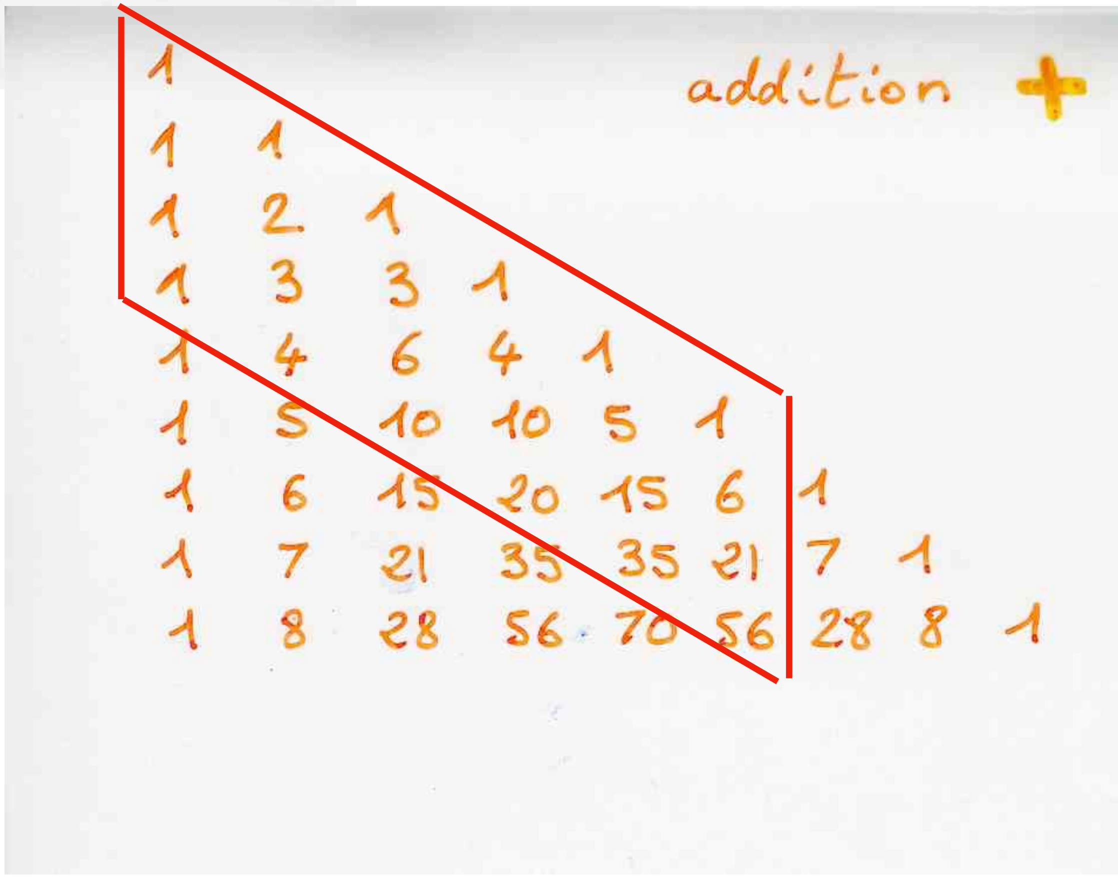
$\binom{8}{5} = 56$



addition +

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1


1 4 10 20 35 56
(5,3)
1 3 6 10 15 21
j=3
1 2 3 4 5 6
1 1 1 1 1 1
(0,0)
i=5


addition +
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

A bijective proof

with binomials coefficients

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$$\sum_{j-1 \leq \ell \leq i+j-1} \binom{\ell}{j-1} = \binom{i+j}{j}$$

exercise 4

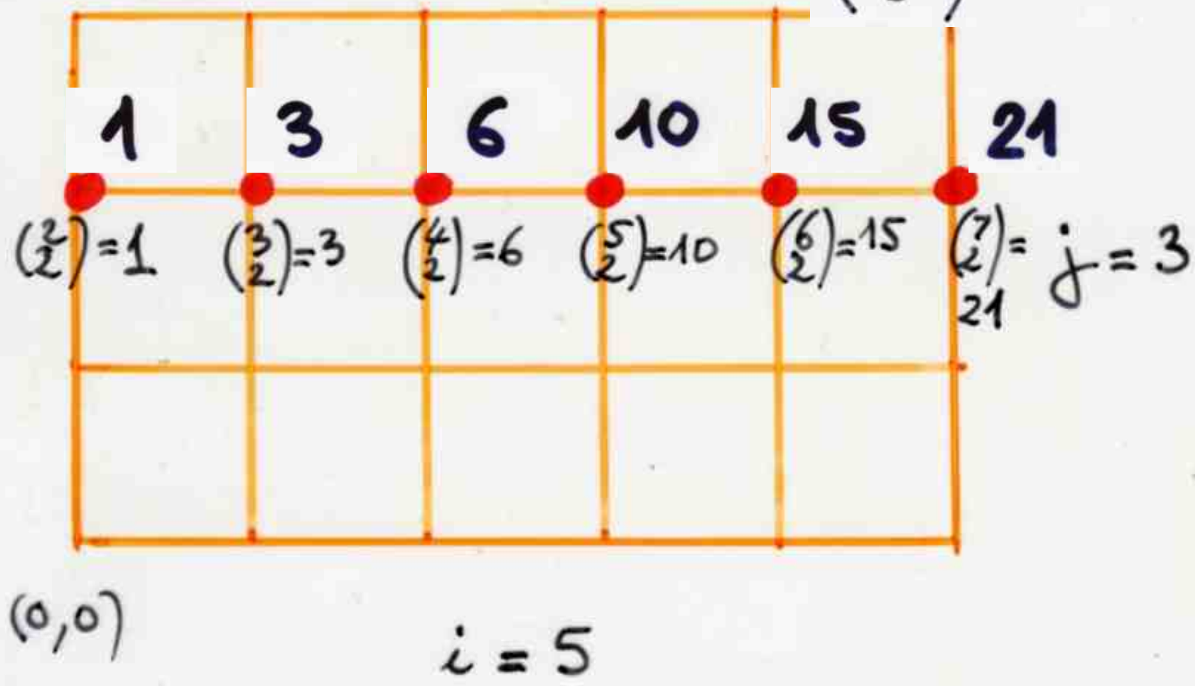
find a bijective proof for the identity

$$\sum_{j-1 \leqslant \ell \leqslant i+j-1} \binom{\ell}{j-1} = \binom{i+j}{j}$$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	7…				

$$\sum_{j-1 \leqslant \ell \leqslant i+j-1} \binom{\ell}{j-1} = \binom{i+j}{j}$$

$$\binom{8}{5} = 56$$



another bijective proof

with binomials coefficients

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8

1
1
2
3
5
8
13
21

Fibonacci numbers

$$\sum_{0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

exercise 5

a) Prove that the number of matchings of the segment $[1, n]$ is the Fibonacci number F_n

exercise 5

$a_{n,k}$ = number of matchings of $\{1,2,\dots,n\}$ with k dimers

b) Give a bijective proof for

$$a_{n,k} = \binom{n-k}{k}$$

thus

$$\sum_{0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

	1								
1	1	1							
1	1	2	1						
2	1	3	3	1					
3	1	4	6	4	1				
5	1	5	10	10	5	1			
8	1	6	15	20	15	6	1		
13	1	7	21	35	35	21	7	1	
21	1	8	28	56	70	56	28	8	1

$$\sum_{0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

Pingala (2nd century B

Pingala

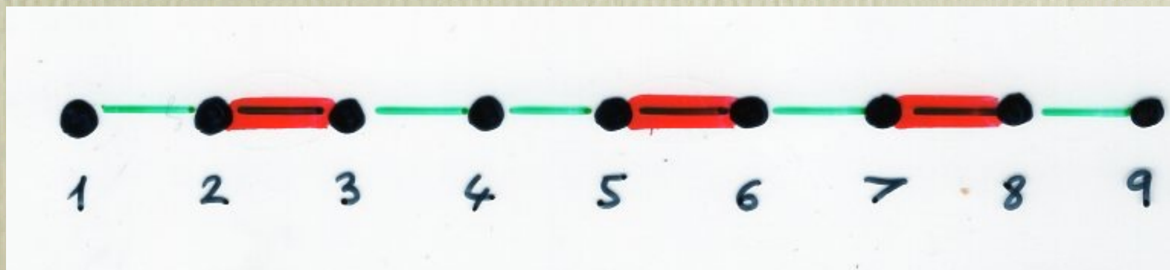
Laghu (short syllabe)

Guru (long syllabe)

two classes of meters in Sanskrit

- Akṣarachandaḥ
  Chandaḥ number of syllables
  later 4 feet (pāda)
- number of mātrās (time measure)
  short syllabe : one mātrās
  long syllabe : two mātrās

relation with Fibonacci numbers ?



at the primary school ....

additions

subtractions

divisions

$$\begin{array}{r} 17 \\ +\ \ 4 \\ \hline 21 \end{array} \qquad \begin{array}{r} 82 \\ +\ \ 9 \\ \hline 91 \end{array} \qquad \begin{array}{r} 13 \\ +\ 21 \\ \hline 34 \end{array}$$

$+$

$$\begin{array}{r} 16 \\ -\ \ 4 \\ \hline 12 \end{array} \qquad \begin{array}{r} 23 \\ -\ 15 \\ \hline 08 \end{array} \qquad \begin{array}{r} 17 \\ -\ \ 8 \\ \hline 09 \end{array}$$

$-$

$15 : 5 = 3$

$19 : 2 = 9,5$

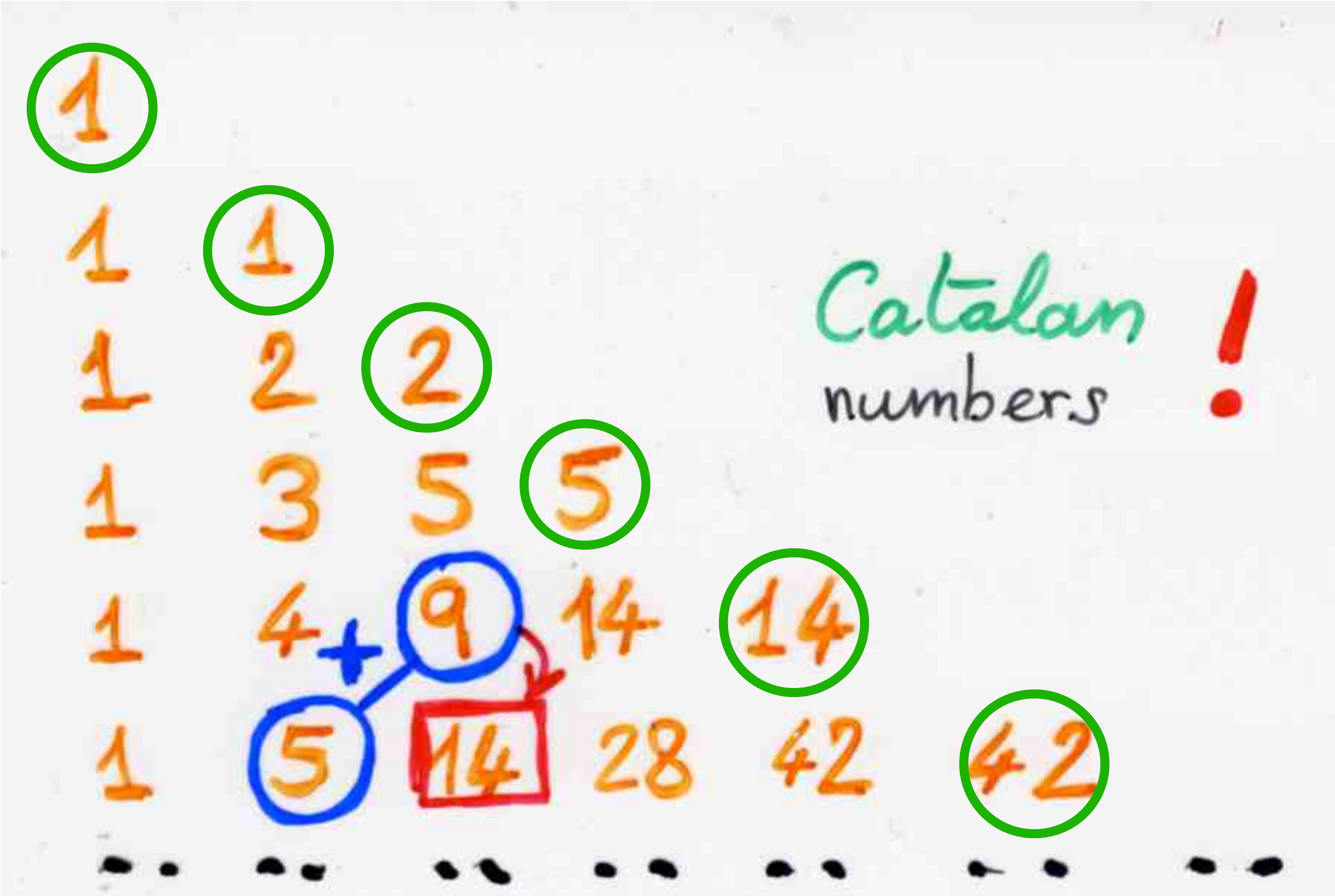
$21 : 7 = 3$

$:$

addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

1						
1	1					
1	2	2				
1	3	5	5			
1	4	9	14	14		
1	5	14	28	42	42	
..	..	..	..	..	..	..


1
1 1
1 2 2
1 3 5 5
1 4 + 9 14 14
1 5 14 28 42 42
Catalan
numbers
!

addition +

−

Catalan numbers !

1

1 1

1 2 − 1 = 1

1 3 3 1

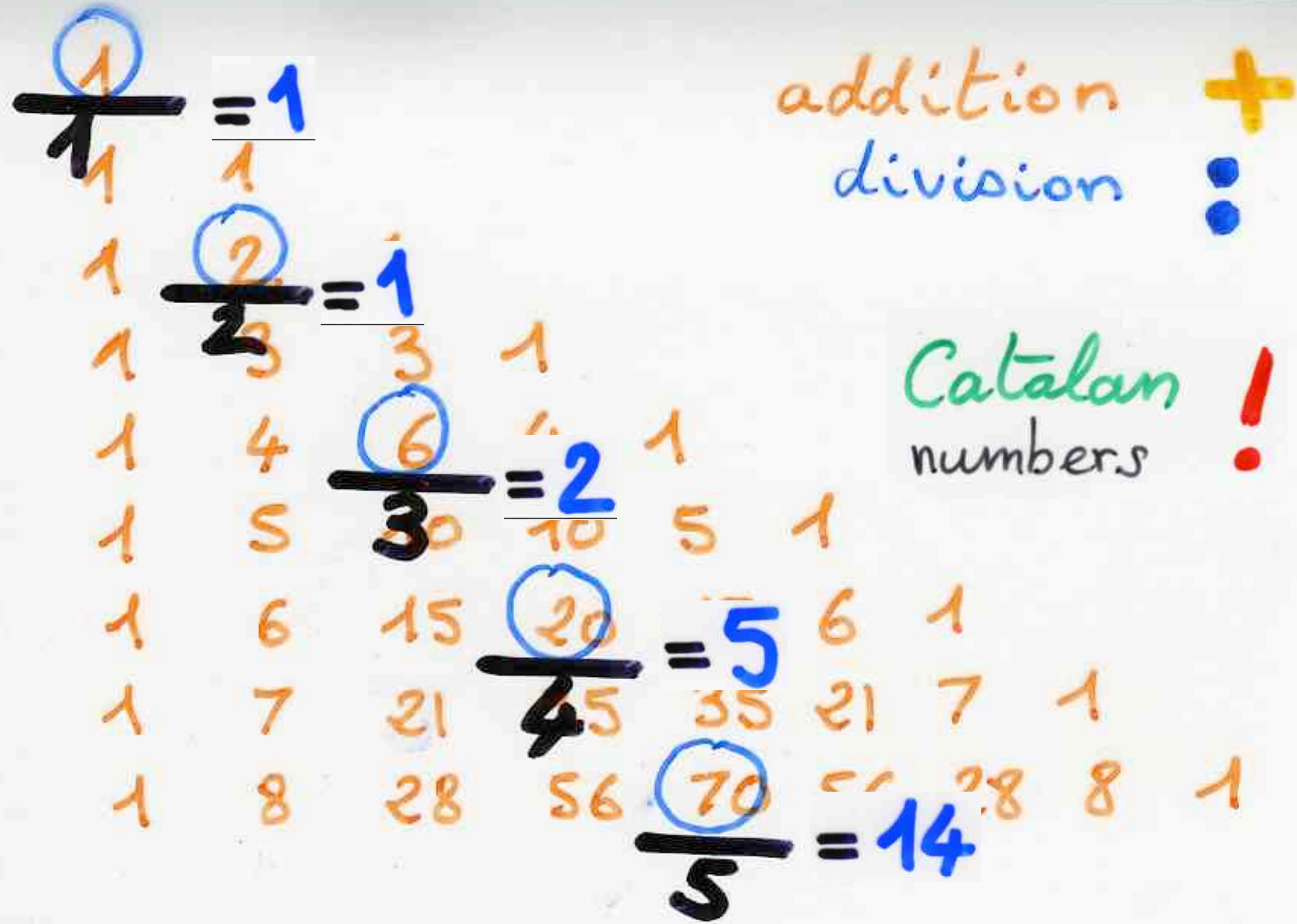
1 4 6 − 4 = 2

1 5 10 10 5 1

1 6 15 20 − 15 = 5 1

1 7 21 35 35 21 7 1

1 8 28 56 70 − 56 = 14 1


addition +
division :
Catalan numbers !
1/1 = 1
2/2 = 1
6/3 = 2
20/4 = 5
70/5 = 14
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

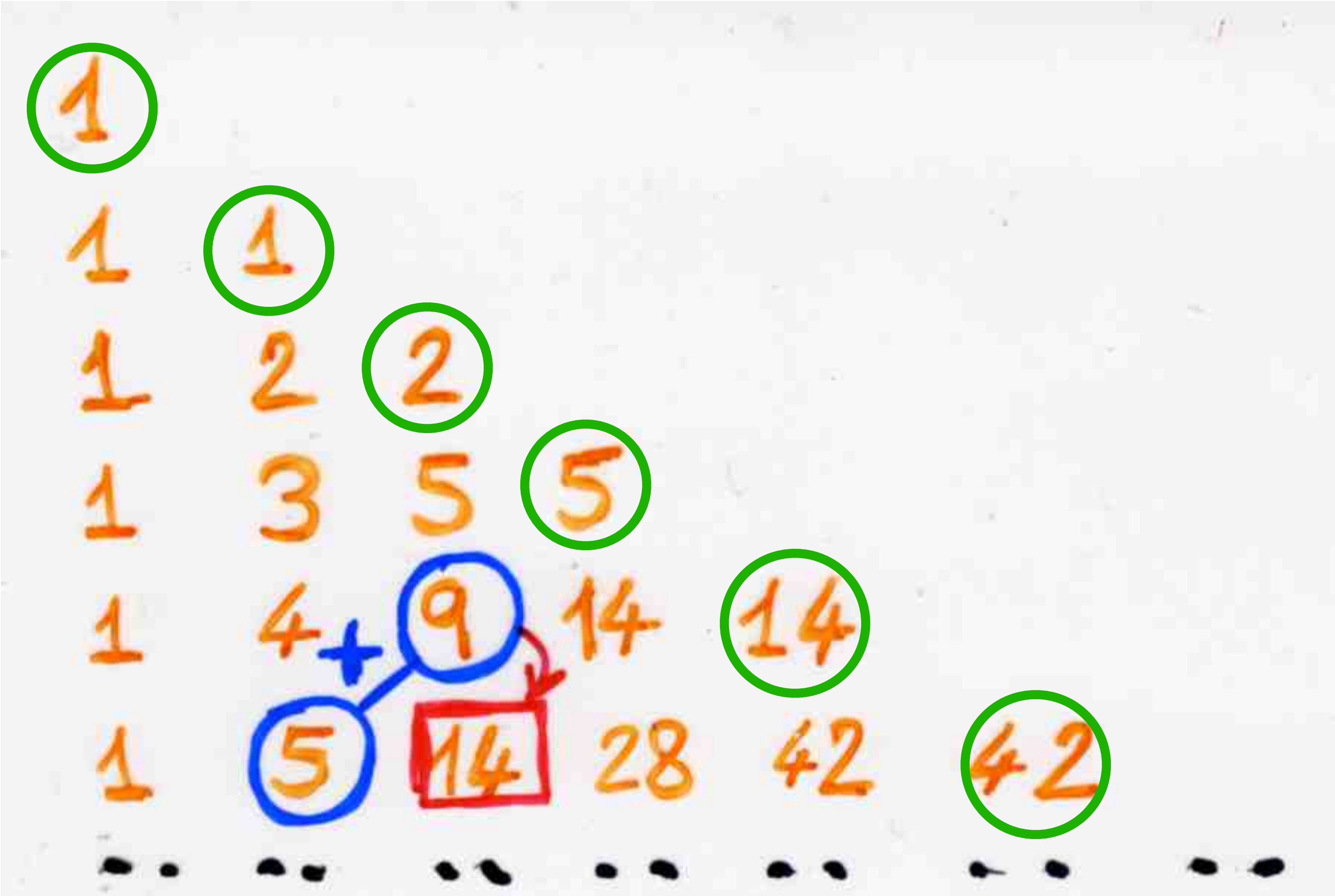
exercise 6

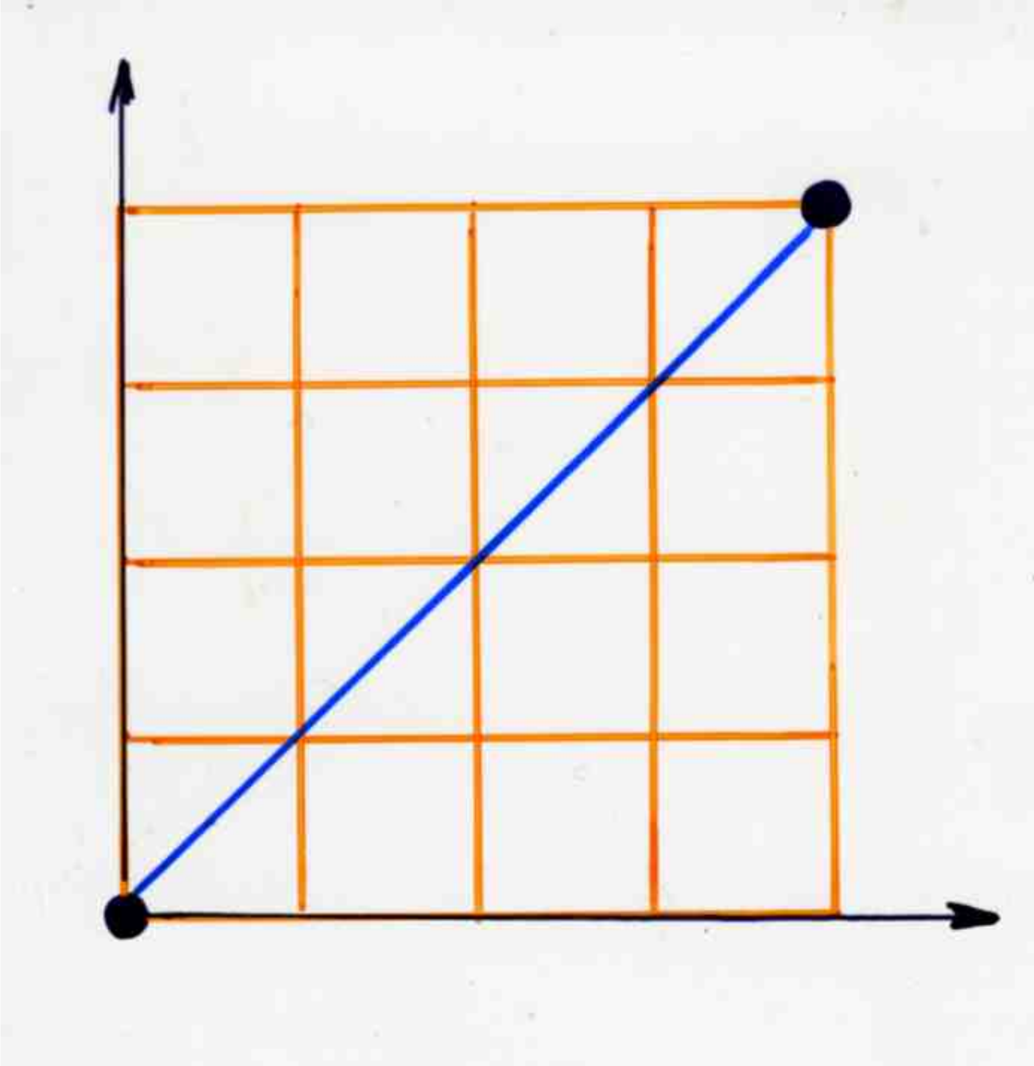
Prove (by calculus) that

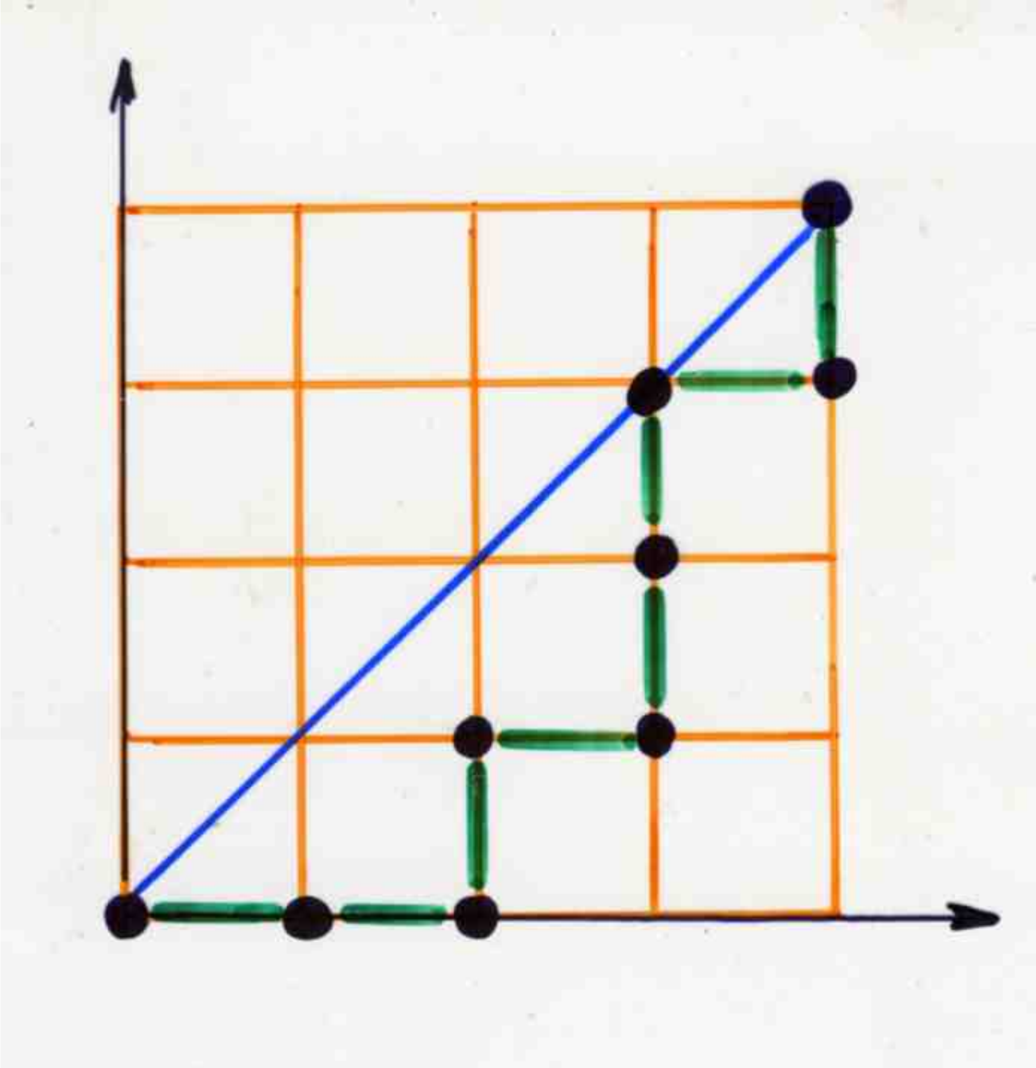
$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{(n+1)}\binom{2n}{n}$$

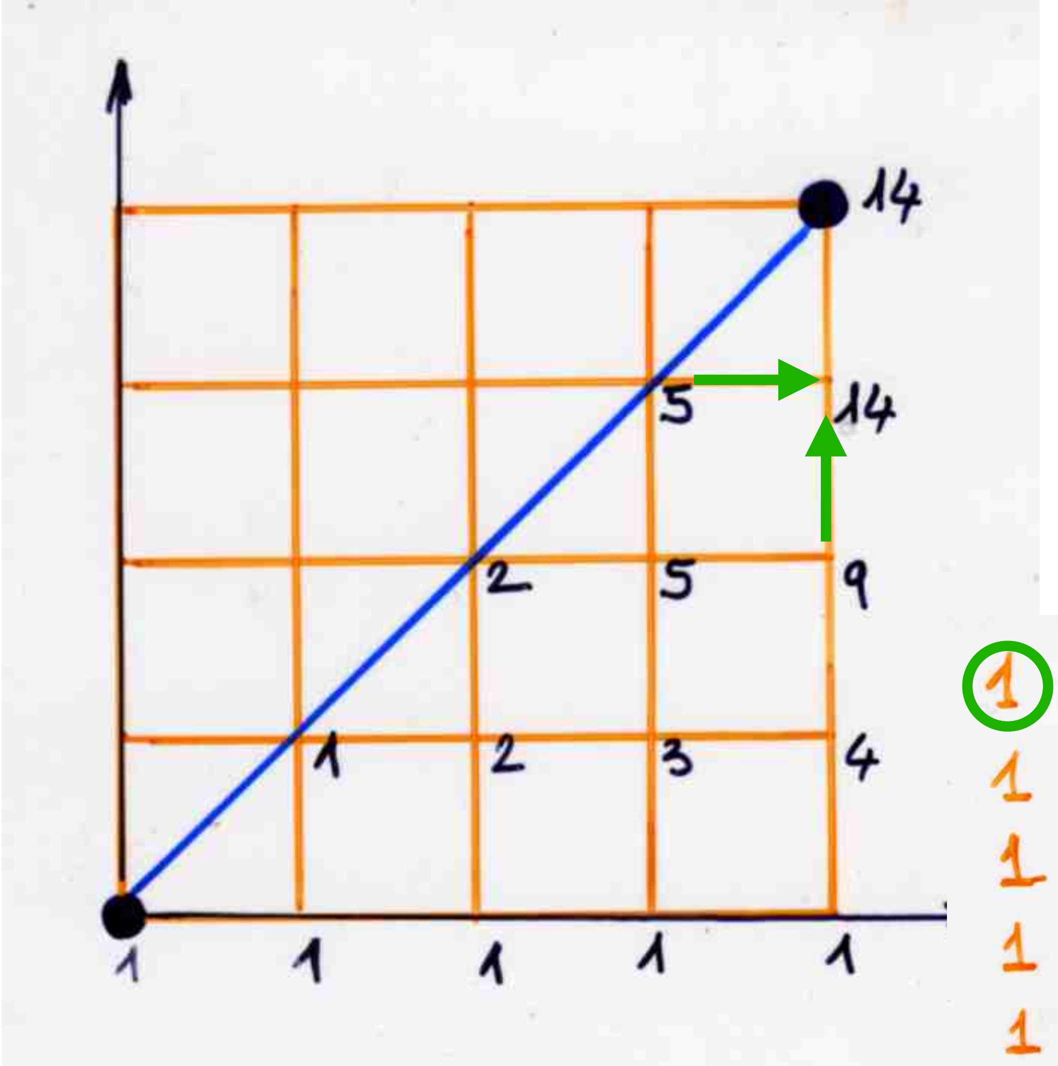
at the primary school ....

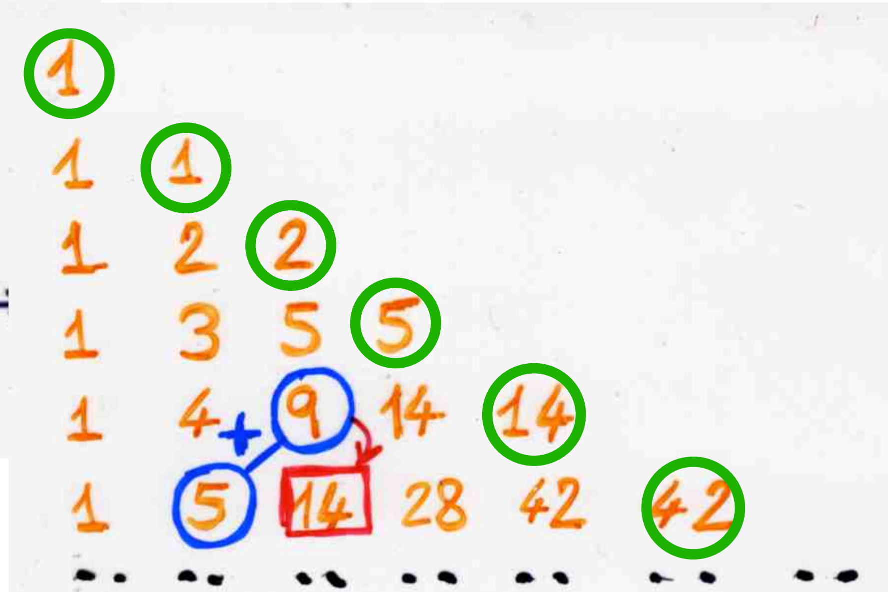
additions


1
1 1
1 2 2
1 3 5 5
1 4 + 9 14 14
1 5 14 28 42 42





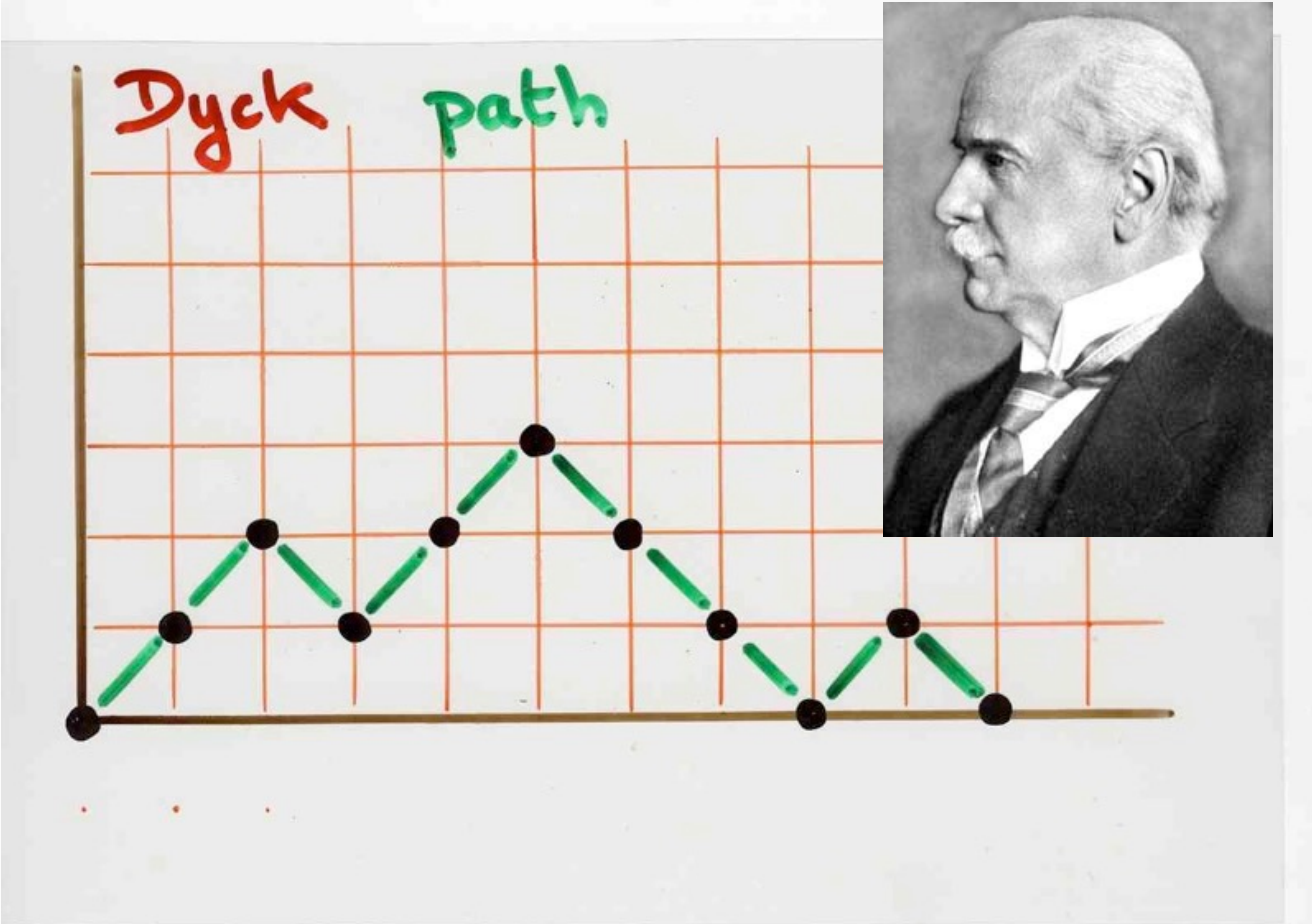

14
5
14
2
5
9
1
2
3
4
1
1
1
1
1

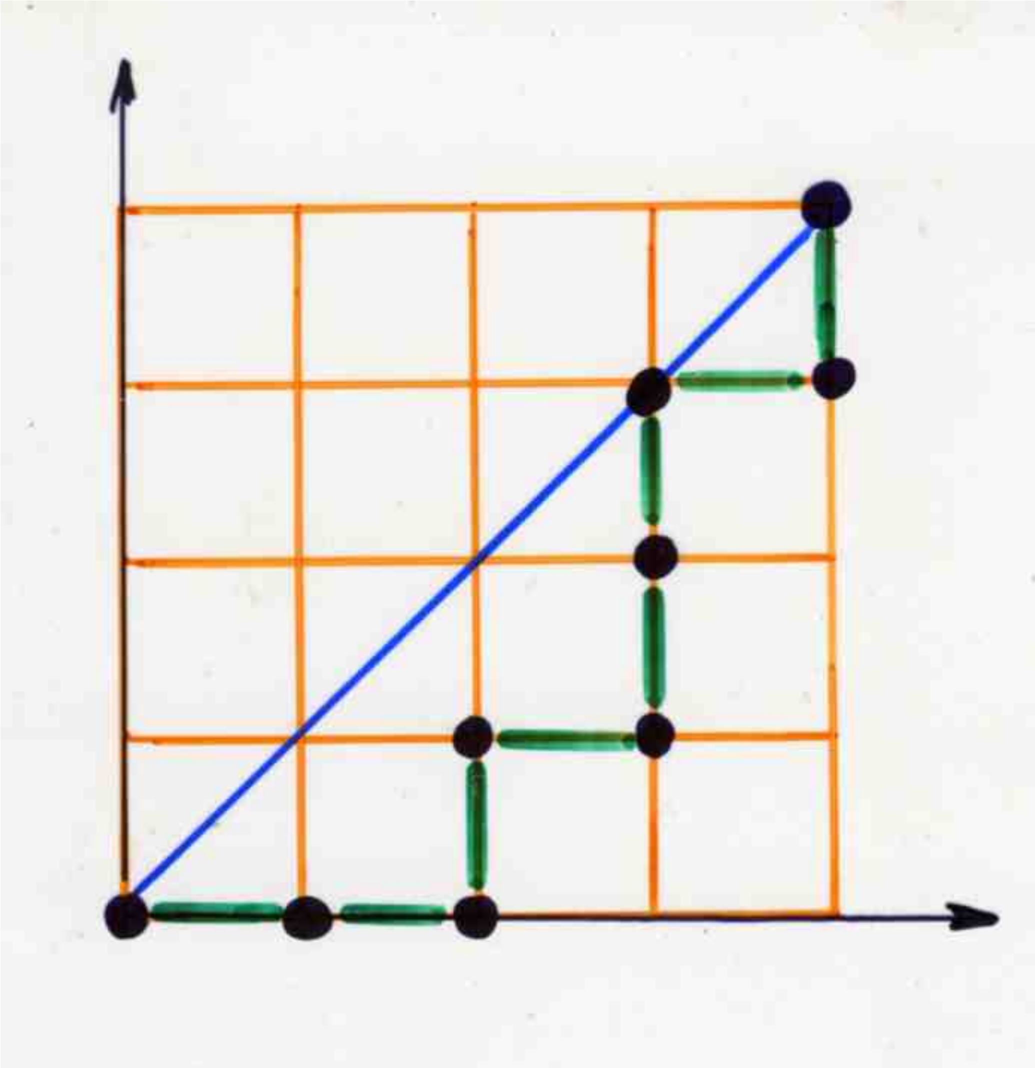

1
1 1
1 2 2
1 3 5 5
1 4 + 9 14 14
1 5 14 28 42 42

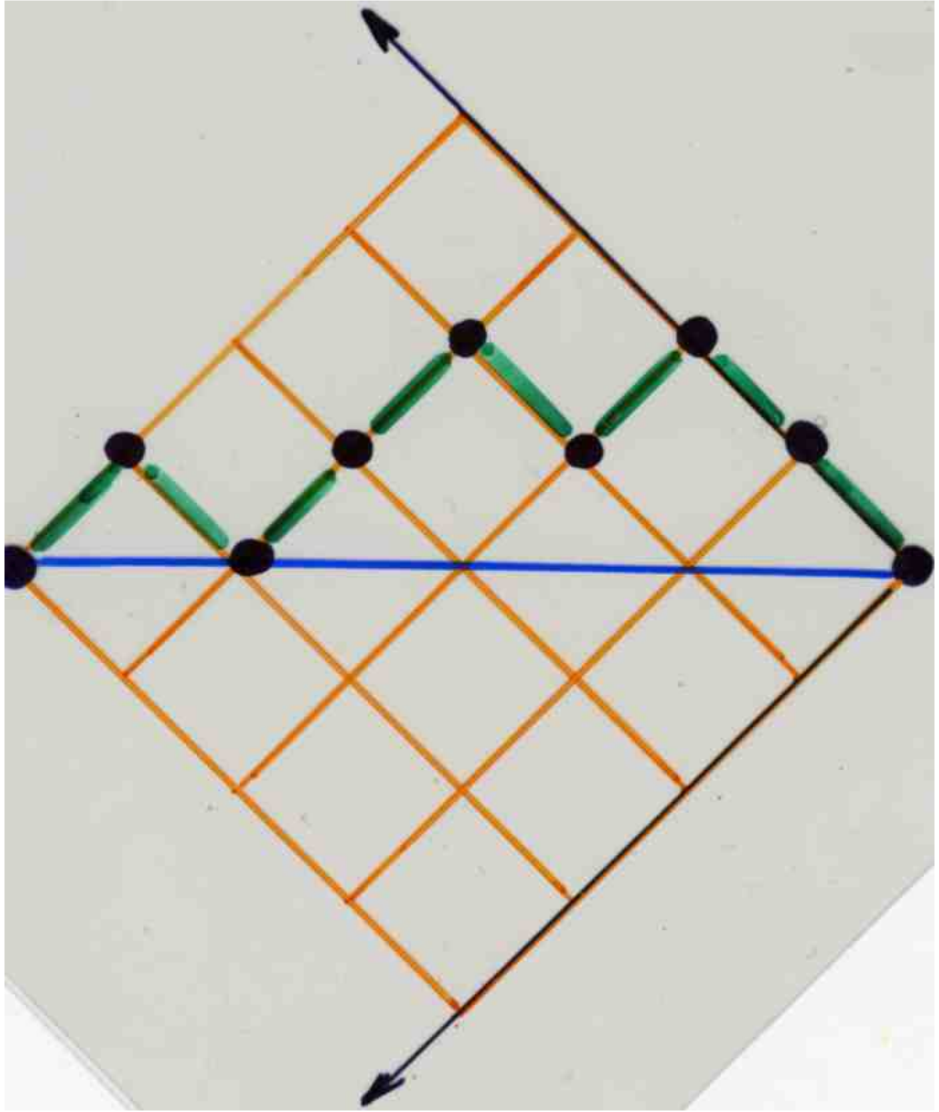
binary trees

and

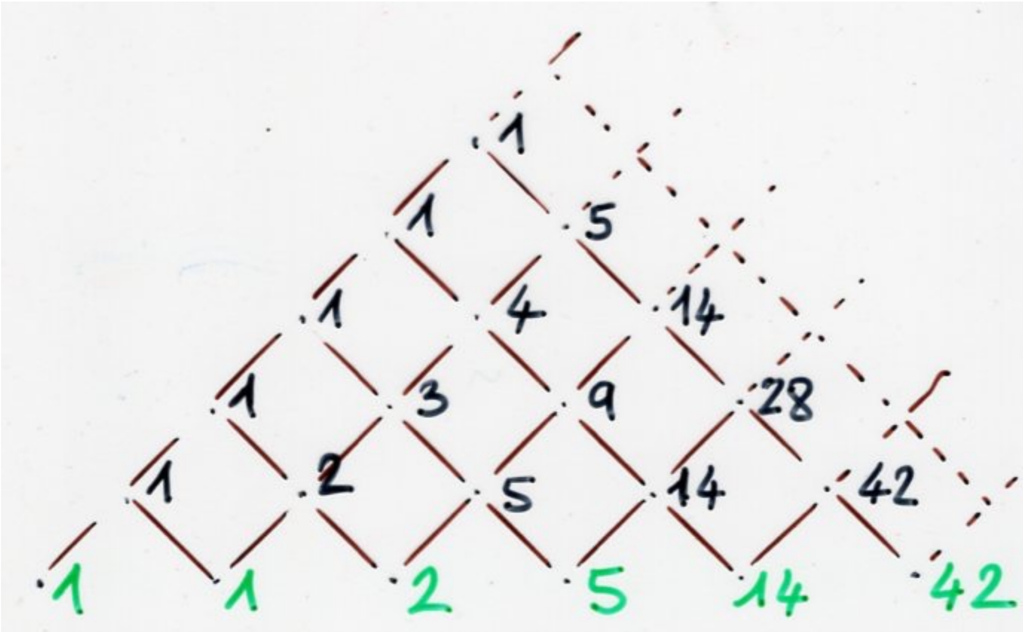
Dyck paths

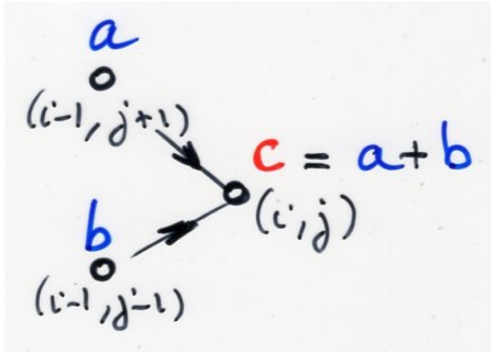

Dyck path



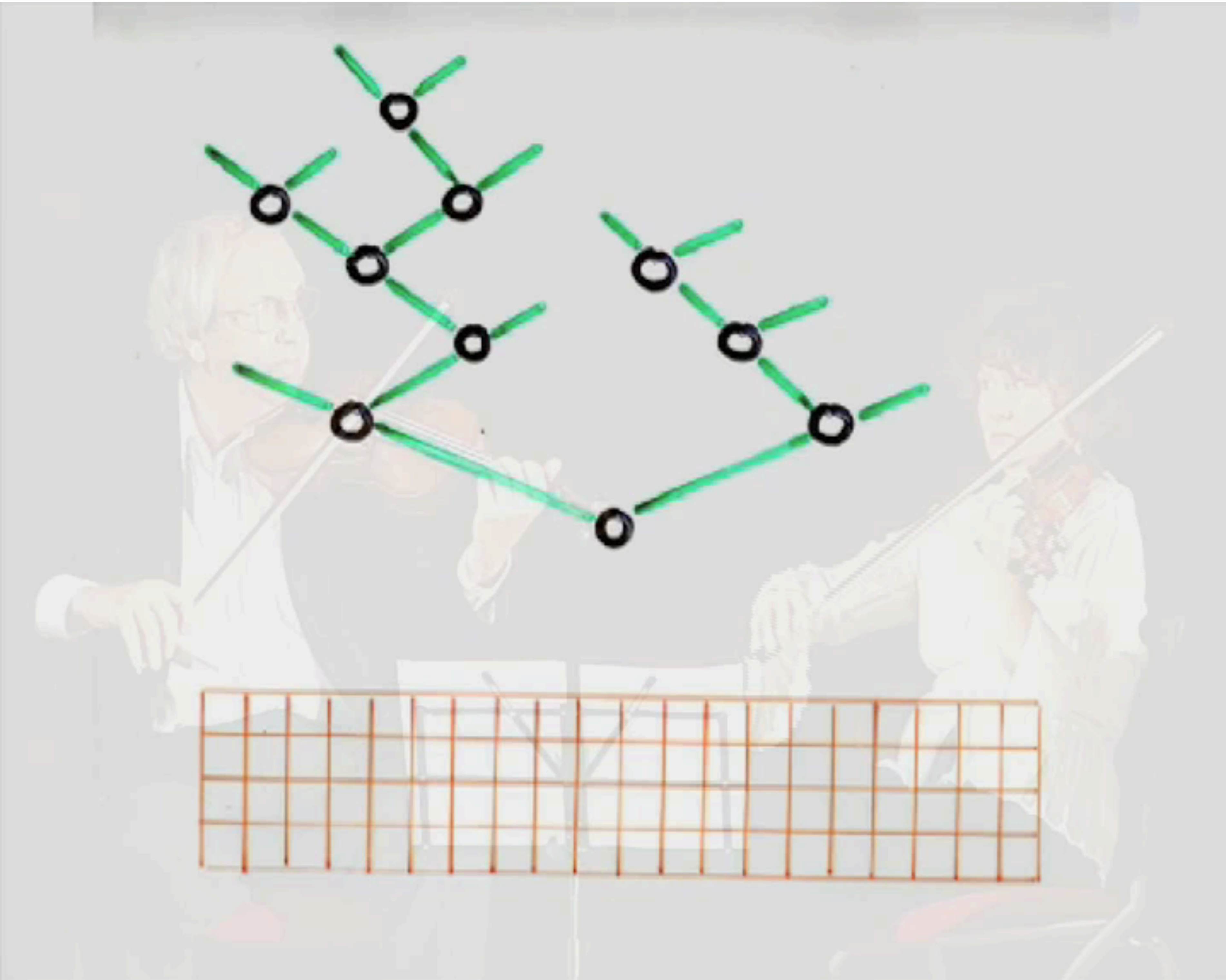


a "Catalan triangle"





1						
1	1					
1	2	2				
1	3	5	5			
1	4 +	9	14	14		
1	5	$\boxed{14}$	28	42	42	
..	..	..	..	..	..	..

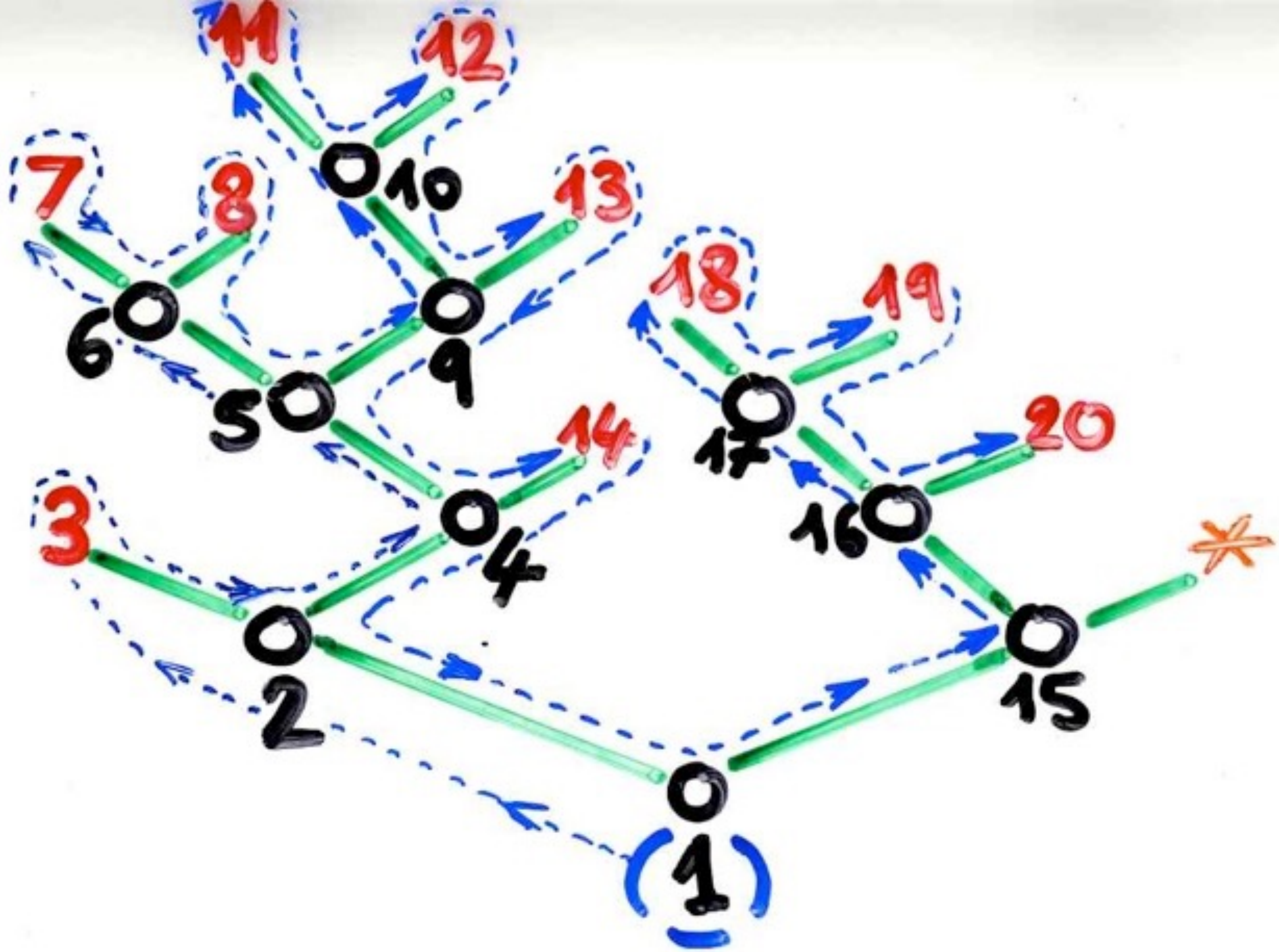


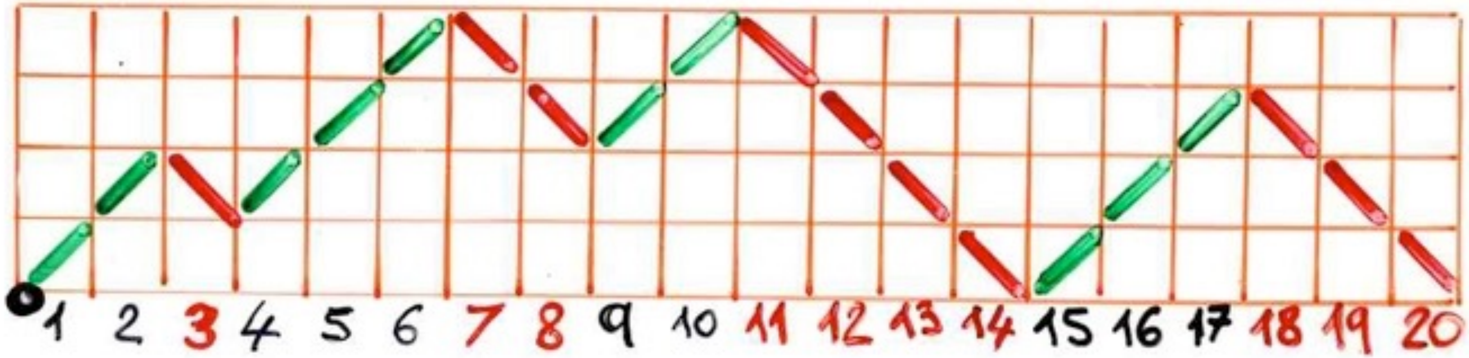
violins:

Mariette Freudentheil

Gérard H.E. Duchamp

Association
Cont'Science


11
12
10
7
8
13
18
19
6
9
5
17
20
14
3
16
4
2
15
1


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

exercise 7

Describe the reciprocal bijection

Dyck path $\longrightarrow$ binary tree

length $2n$

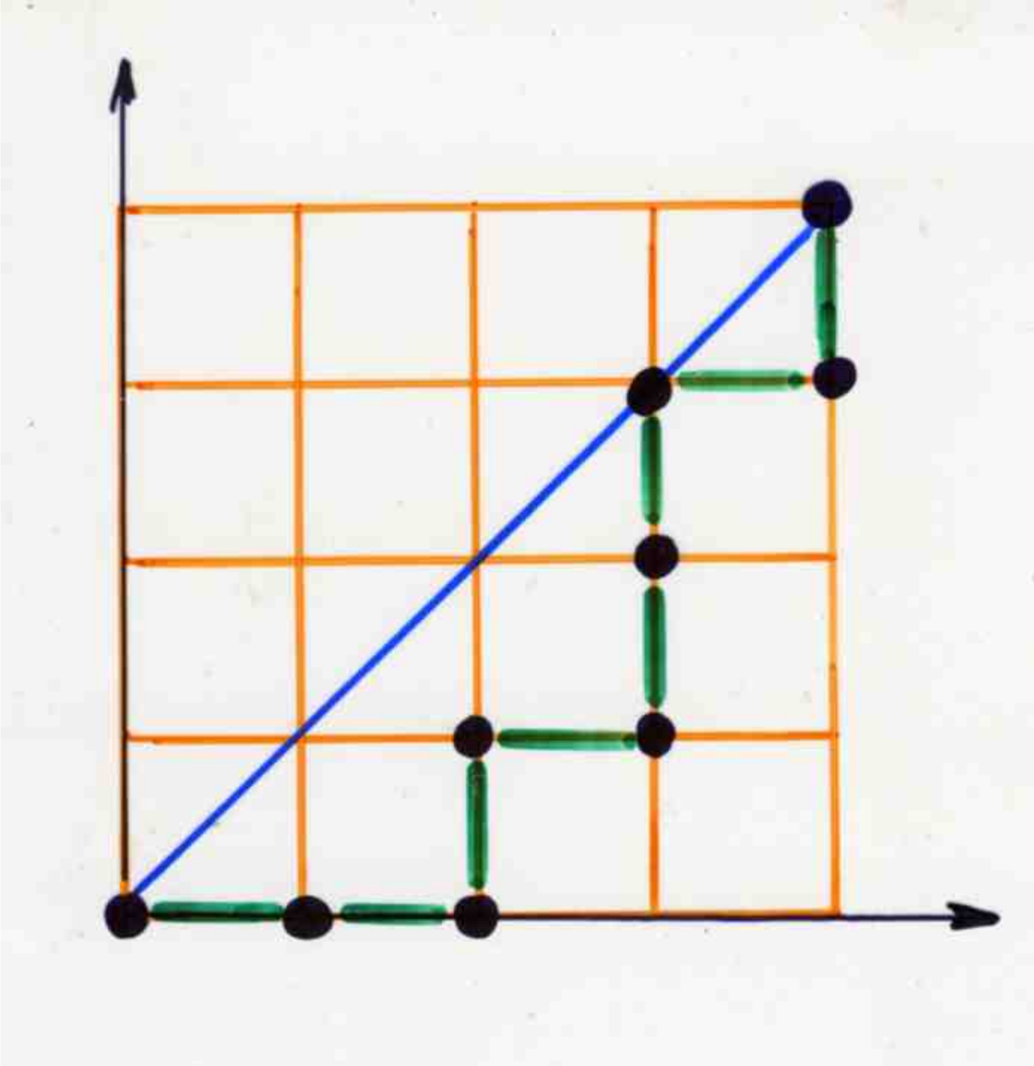
n internal vertices
$(n+1)$ external vertices

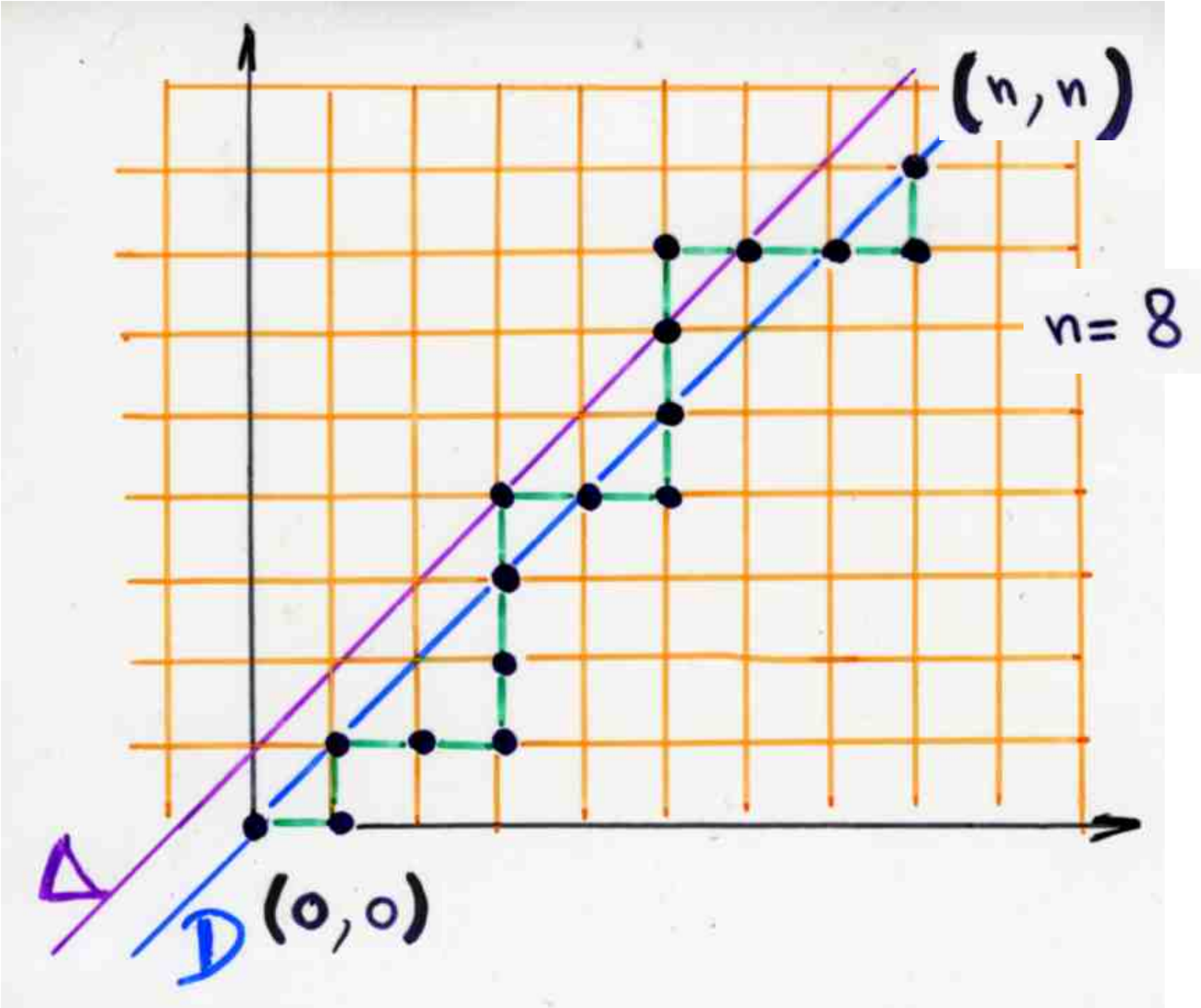
C_n

at the primary school ....

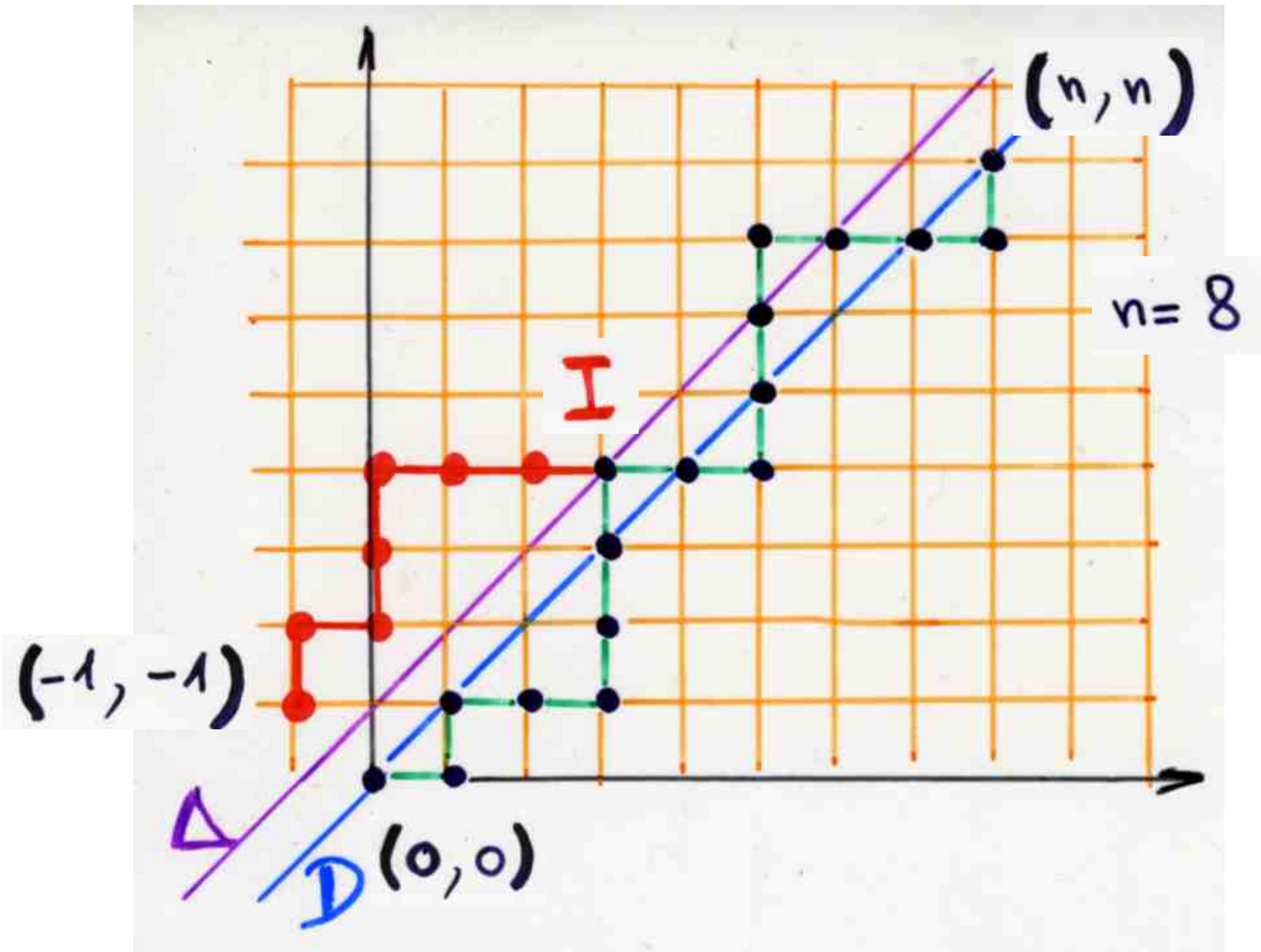
subtractions

The reflexion principle





$$\binom{2n}{n}$$



$$\binom{2n}{n} \qquad \binom{2n}{n-1}$$

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

1 addition +

1 1 −

1 2 − 1 = 1

1 3 3 1

1 4 6 − 4 = 2

1 5 10 10 5 1

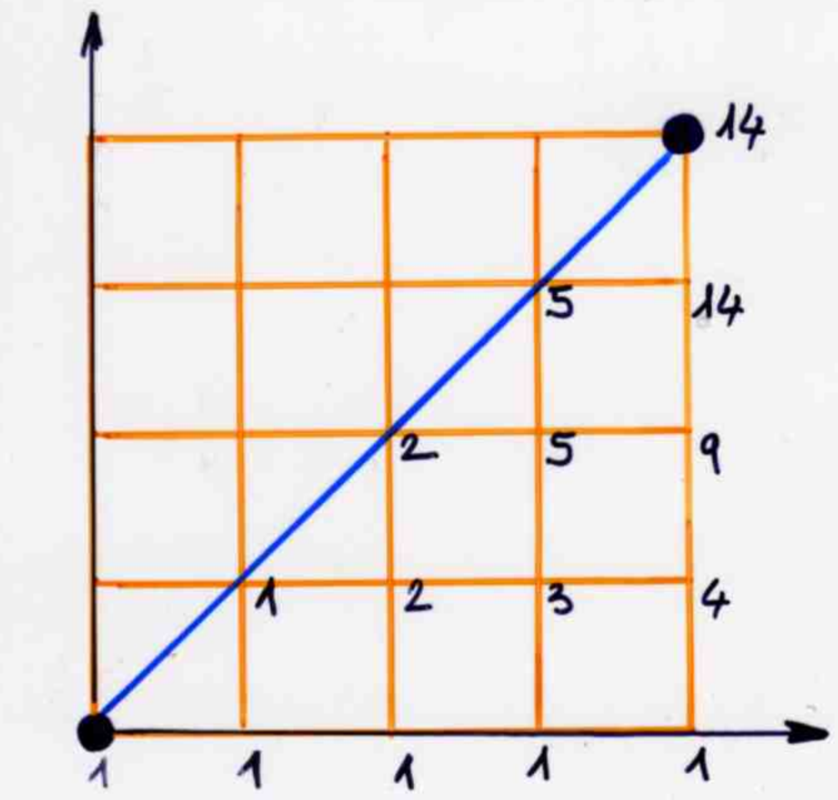
1 6 15 20 − 15 = 5 1

1 7 21 35 35 21 7 1

1 8 28 56 70 − 56 = 14 1

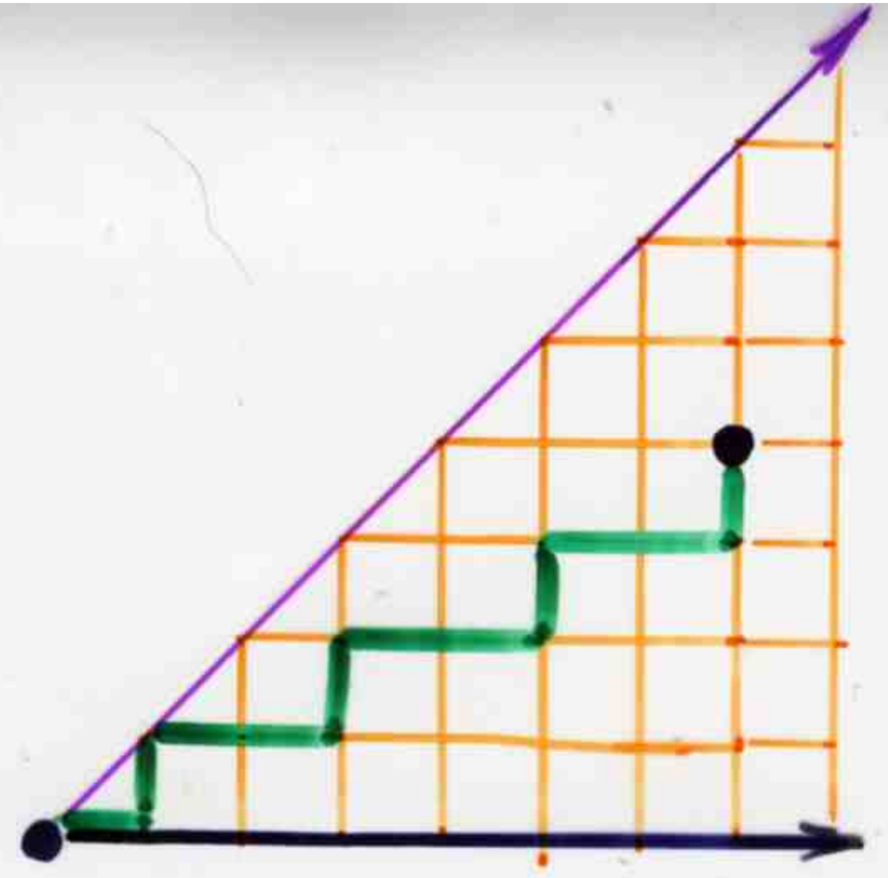
$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

a "Catalan triangle"



"ballot problem"

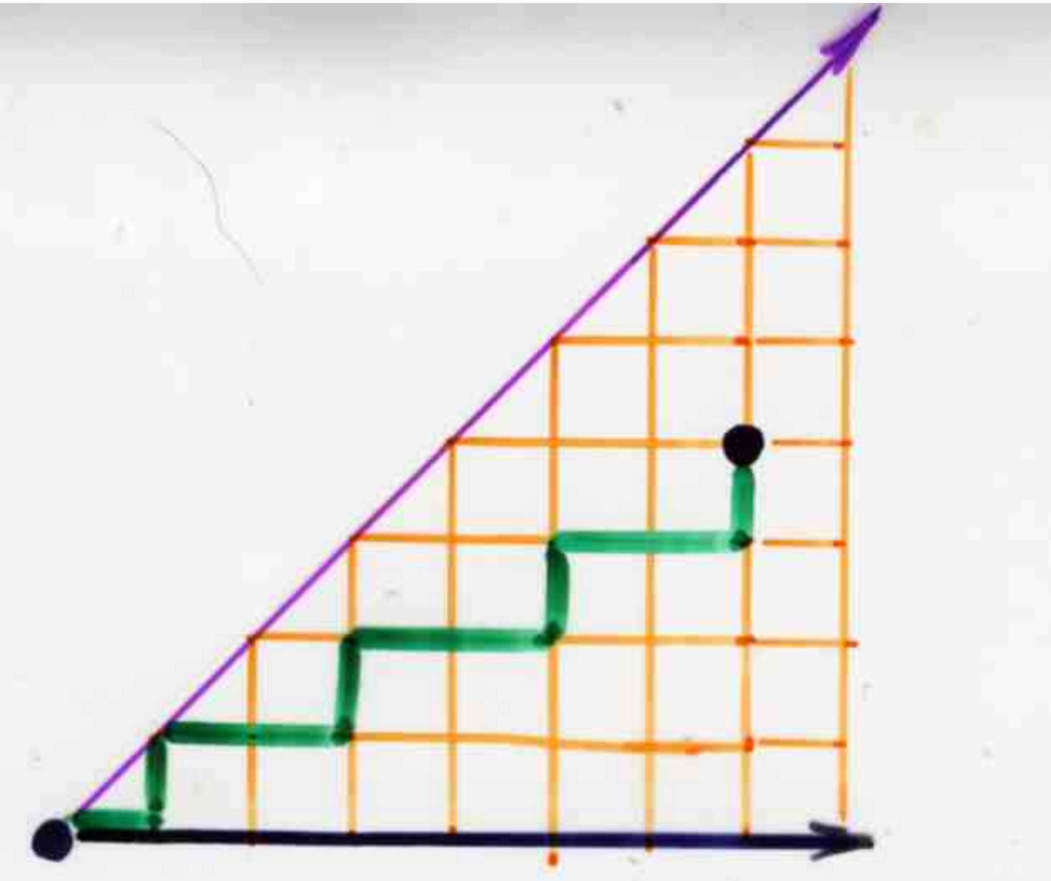
J. Bertrand
D. André
(1887)



"ballot numbers"

exercise 8

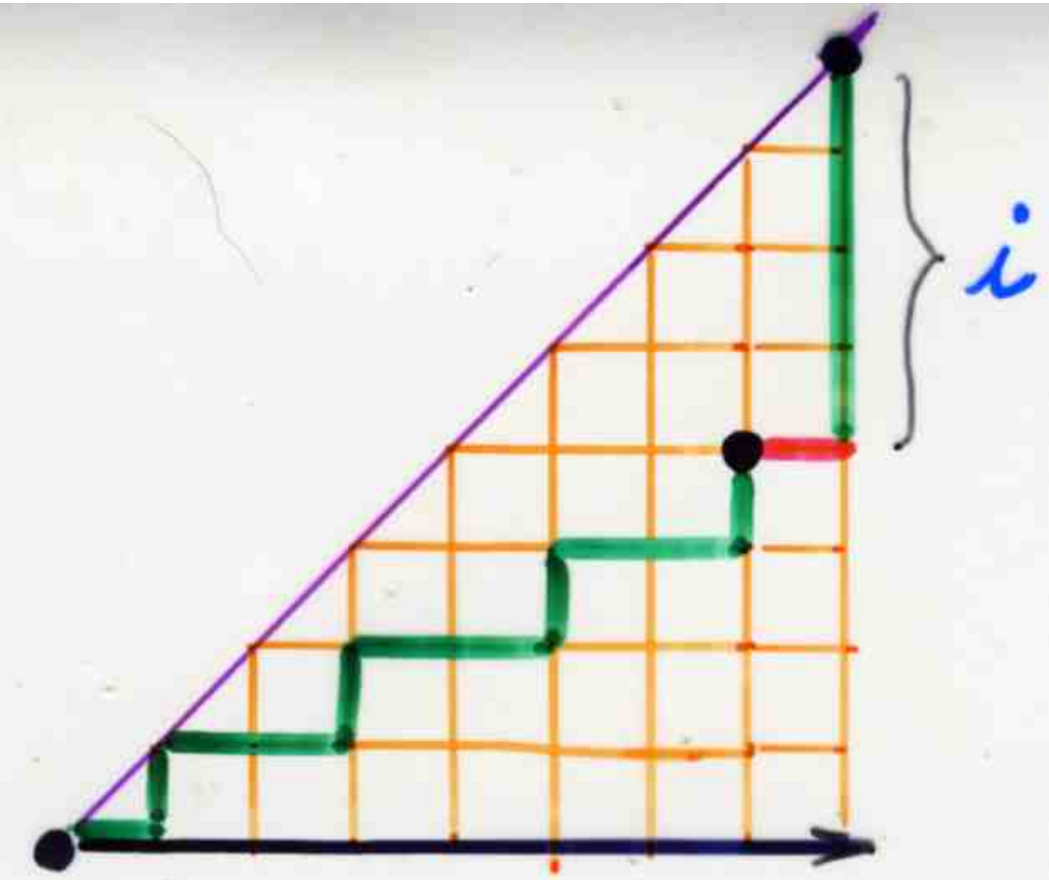
$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$



"ballot numbers"

exercise 8

$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$



$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

DU CALCUL

DES

DÉRIVATIONS;

PAR L. F. A. ARBOGAST,

De l'Institut national de France, Professeur de Mathématiques à Strasbourg.

A STRASBOURG,

DE L'IMPRIMERIE DE LEVRAULT, FRÈRES.

AN VIII (1800).

enfin

$$A_{m,n} = \qquad \ldots\ldots(3)$$

$$m\{A_{0,n}.\gamma'^m + A_{0,n+1}.m\gamma'^{m-1}\beta + A_{0,n+2}.\frac{m.m-1}{1.\ \ 2}\gamma'^{m-2}\beta^2$$
$$+ \text{etc.} + A_{0,n+m-1}.m\gamma'\beta^{m-1} + A_{0,n+m}.\beta^m\},$$

périeur ou inférieur ayant lieu suivant que m est pair ou impair. s'accorde avec celui que l'on peut déduire d'une solution différente exemple, donnée par LAPLACE dans les mémoires de Paris, année XVII, page 267.

ait n négatif dans la formule (1) ci-dessus, on trouve, en rejetant et celles de leurs parties où les indices de D sont négatifs et ceux de ou positifs > 0, que cette formule se réduit à la suivante :

$$A_{m,-n} = \qquad \ldots\ldots(4)$$

$${}_{0}\mathfrak{D}^m.(\alpha^n.\beta^{l-n-1}) - A_{0,1}\mathfrak{D}^m.(\alpha^{n+1}.\beta^{l-n-2}) + A_{0,2}\mathfrak{D}^m.(\alpha^{n+2}.\beta^{l-n-3}) - \text{etc.}\}$$

à cause que $\alpha = 0$ et que sa seule dérivée D est β, devient

$$A_{m,-n} = \qquad \ldots\ldots(5)$$

$$.\beta^n\mathfrak{D}^{m-n}.\beta^{l-n-1} - A_{0,1}.\beta^{n+1}\mathfrak{D}^{m-n-1}.\beta^{l-n-2} + A_{0,2}.\beta^{n+2}\mathfrak{D}^{m-n-2}.\beta^{l-n-3} - \text{etc.}\}.$$

uit que $A_{m,-n}$ n'est zéro qu'autant que m est $< n$. Ainsi la série e s'étend, sous forme de triangle, dans la quatrième région.

EXEMPLE VI.

tant donné le commencement de la table suivante, où chaque terme de la somme de celui qui le précède dans la même ligne horizontale lui qui le suit d'un rang dans la ligne horizontale immédiatement re, avec la condition que chacun des termes de la première ligne ale soit égal à l'unité : on demande le terme général de cette table :

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

uation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m-1$ au m, et elle devient

at the primary school ....

divisions

addition +

division :

$$\frac{1}{1} = 1$$

1 1

1 2 1

$$\frac{2}{2} = 1$$

1 3 3 1

1 4 6 4 1

$$\frac{6}{3} = 2$$

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\frac{20}{4} = 5$$

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

$$\frac{70}{5} = 14$$

Catalan number

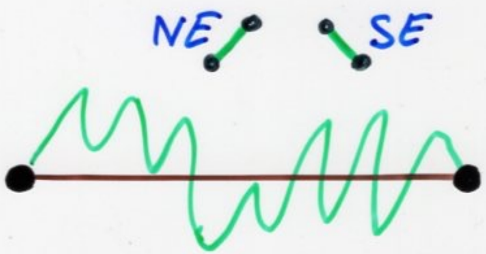
$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

$$(n+1)\,C_n = \binom{2n}{n}$$

$$(n+1)C_n = \binom{2n}{n}$$

$C_n + nC_n$ $\longleftrightarrow$?

Dyck path



bilateral Dyck path

exercise 9

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

$$\Longleftrightarrow$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

bijective proof for

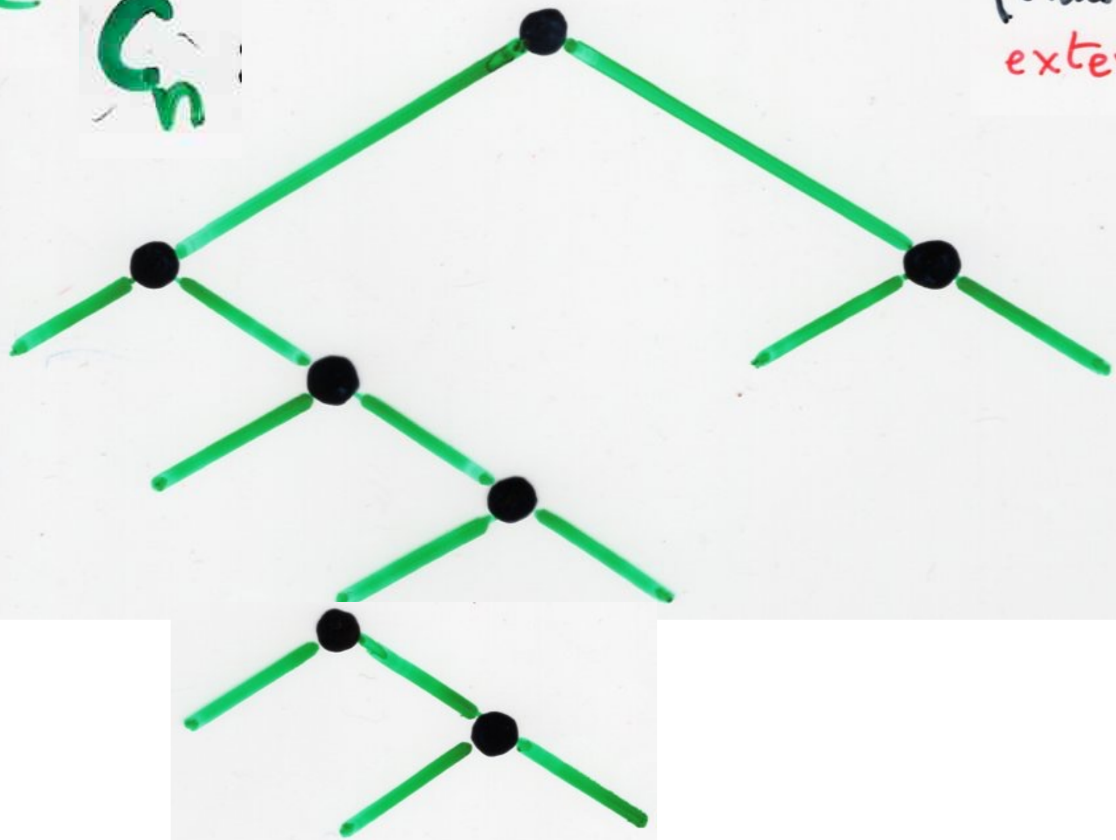
the multiplicative recurrence
of Catalan numbers

$$2(2n+1)C_n = (n+2)C_{n+1}$$

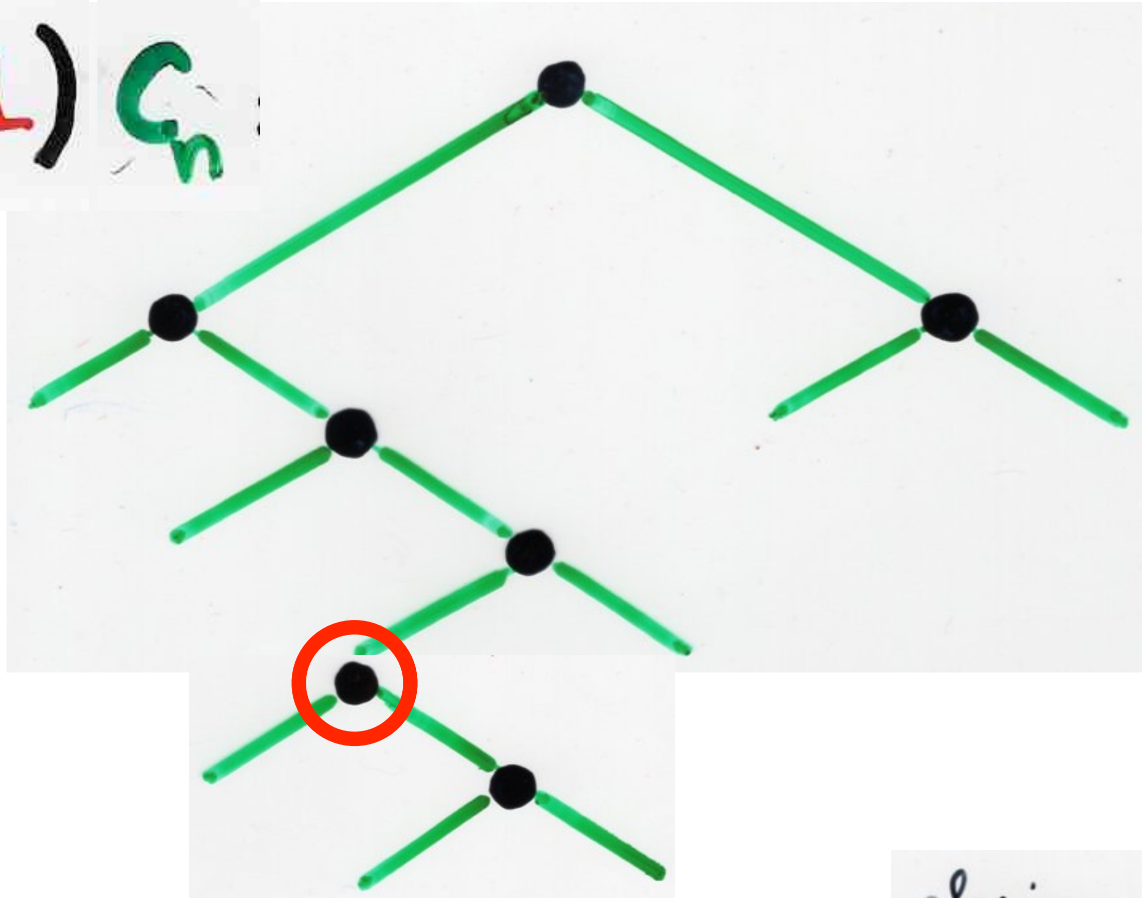
choice of a
complete
binary tree

C_n :

with n
internal vertices
(thus $(n+1)$
external vertices)

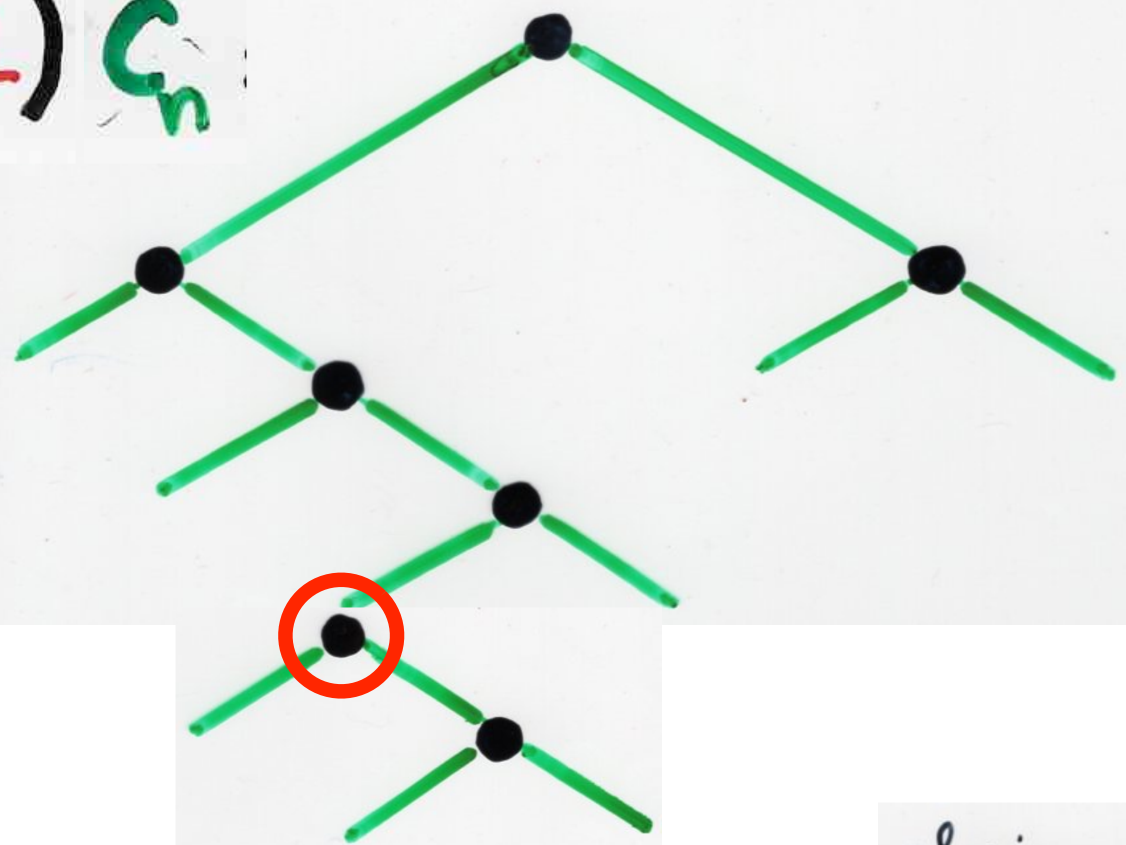


$(2n+1)\, C_n$:



choice of a vertex (internal or external)

$(2n+1)\,C_n$:



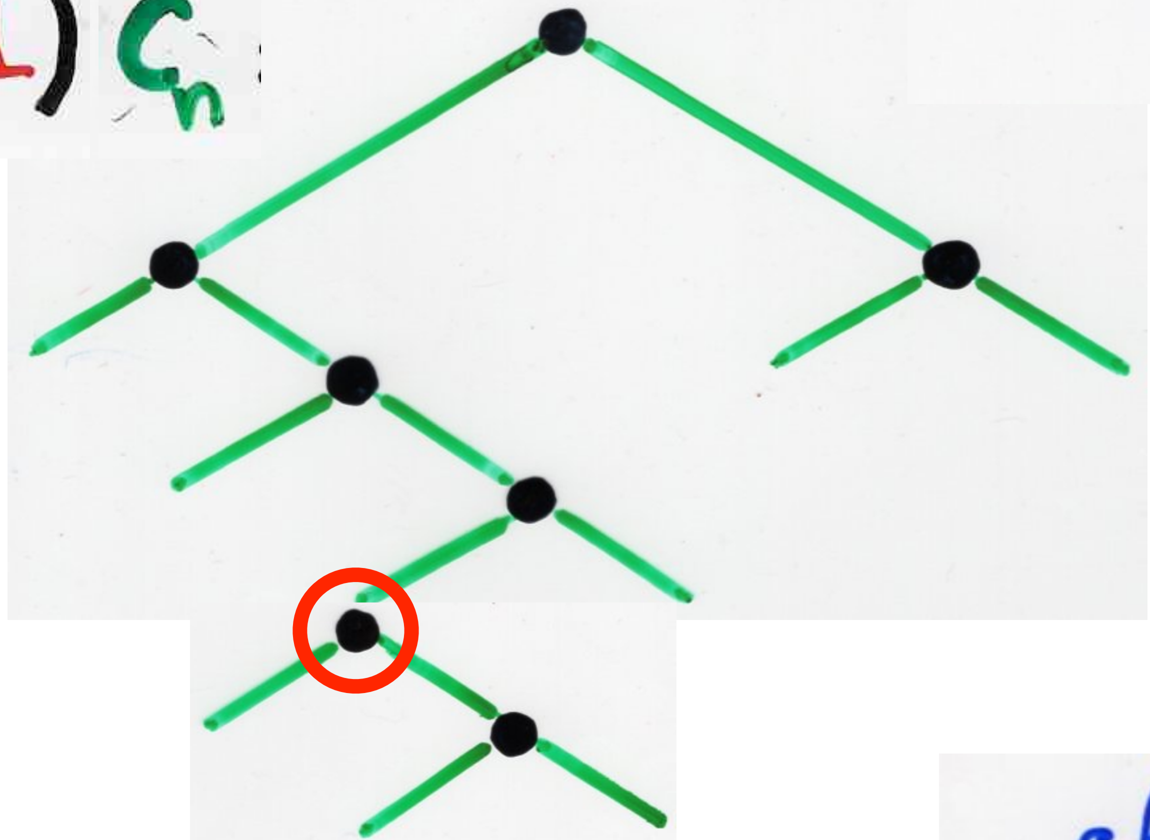
sub-tree

choice of a vertex (internal or external)

$2\,(2n+1)\,C_n$:

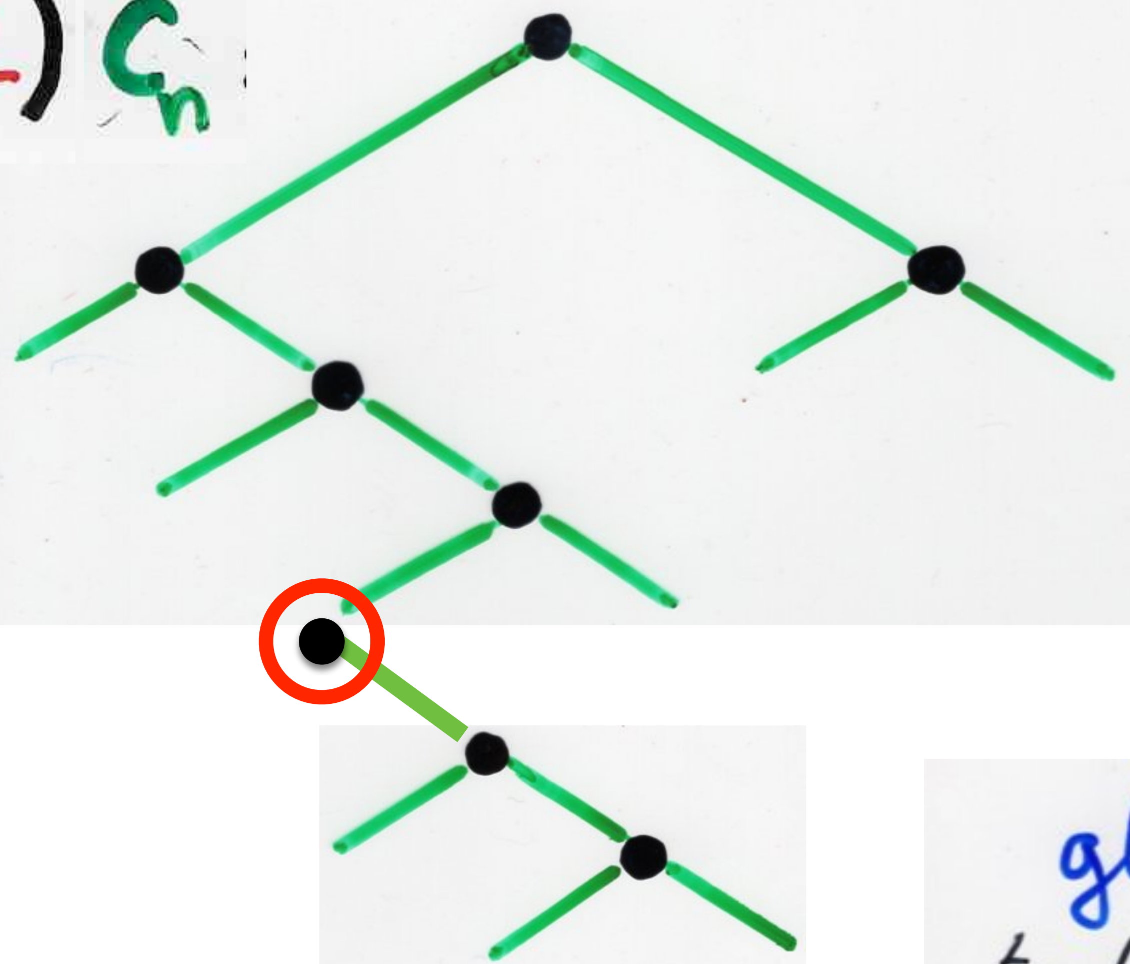
choice left

right



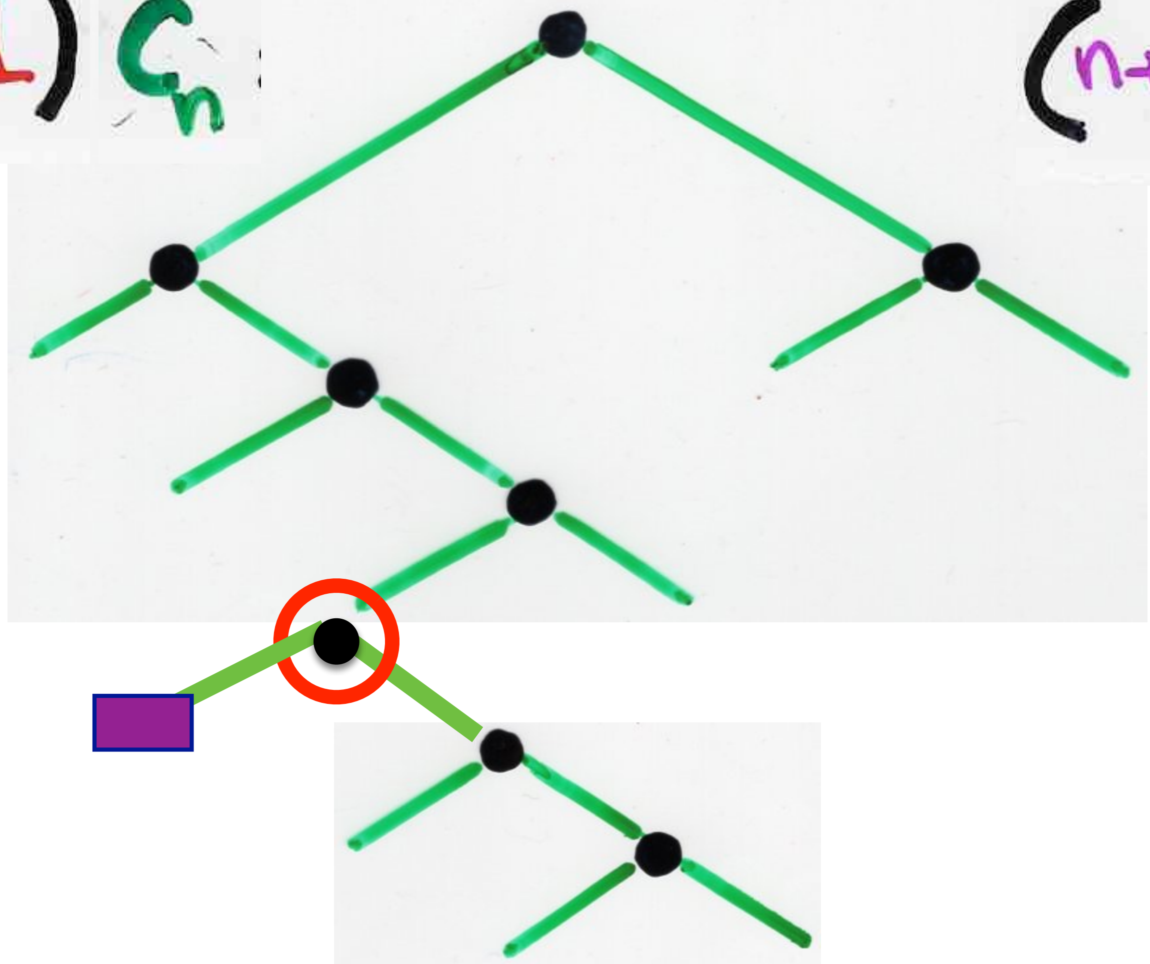
gliding
to the right

$2\,(2n+1)\,C_n$:



gliding to the right

$$2\,(2n+1)\,C_n \qquad (n+2)\,C_{n+1}$$



add a leaf
(= external vertex)

reciprocal

bijection

$$2(2n+1)C_n = (n+2)C_{n+1}$$

exercise 10

Describe the reciprocal bijection

$$\left(\underset{\text{vertex}}{\text{external}} \;,\; \underset{(n+1)\text{ vertices}}{\underset{\text{binary tree}}{\mathcal{B}}} \right) \longrightarrow$$

$$\left(\{\ell, r\} \;,\; \underset{\left(\substack{\text{internal}\\ \text{or}\\ \text{external}}\right)}{\text{vertex}} \;,\; \underset{n\text{ vertices}}{\underset{\text{binary tree}}{\mathcal{B}}} \right)$$


random binary tree

random

increasing

binary tree

binary search tree

decision tree

2 :	4000	6000									
3 :	2000	3000	5000								
4 :	1000	2000	3000	4000							
5 :	500	1000	2000	3000	3500						
6 :	250	500	1000	2000	3000	32[illegible]					
7 :	125	250	500	1000	2000	3000	3125				
8 :	63	125	250	500	1000	2000	3000	[illegible]			
9 :	31	63	125	250	500	1000	20[illegible]	[illegible]	[illegible]		
10:	15	31	63	125	250	500	10[illegible]	[illegible]	[illegible]	[illegible]016	
11 :	7	15	31	63	125	250	500	1000	2000	3000	3024