Lattice paths and heaps

Marches aléatoires, combinatoire et interactions
Lattice paths, combinatorics and interactions
CIRM, Luminy, 25 June 2021

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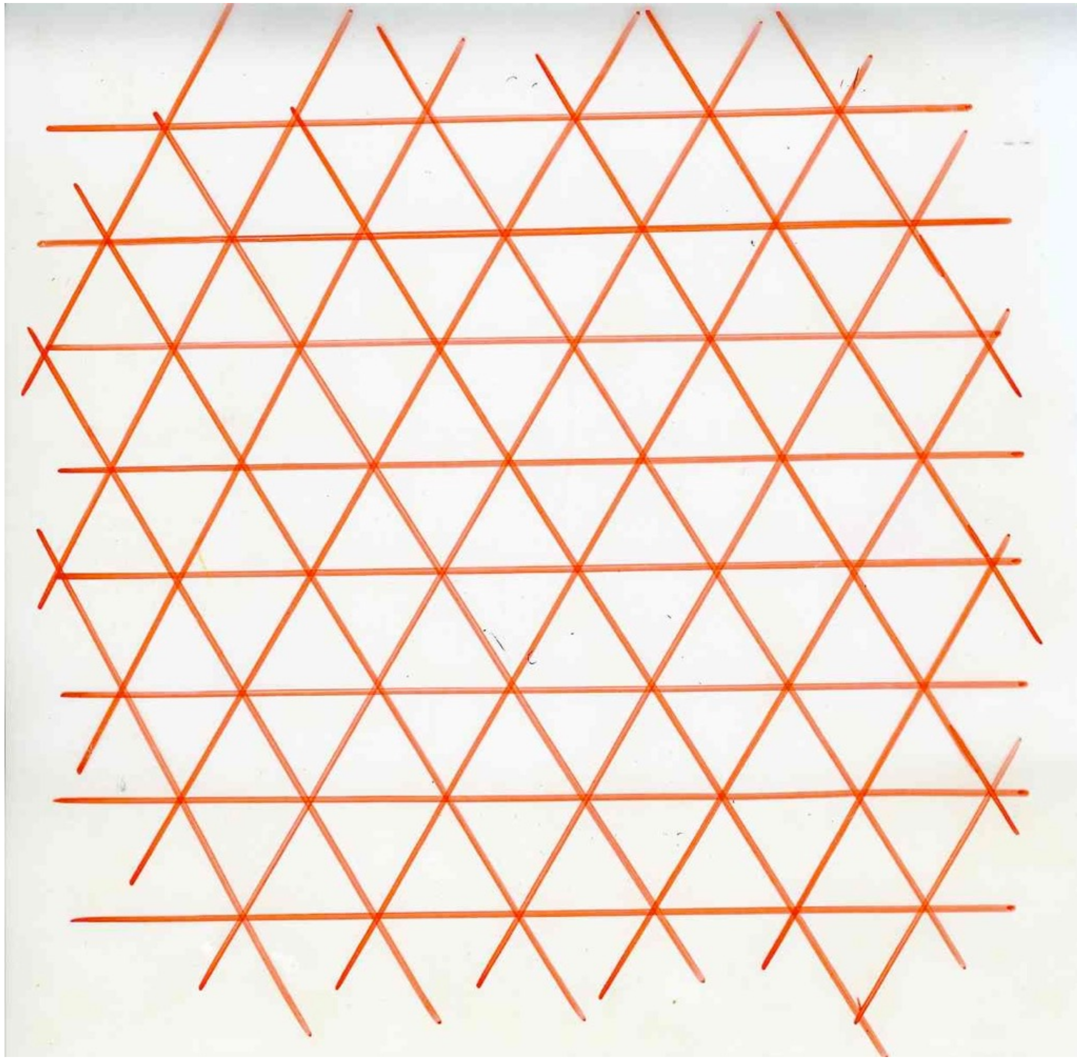
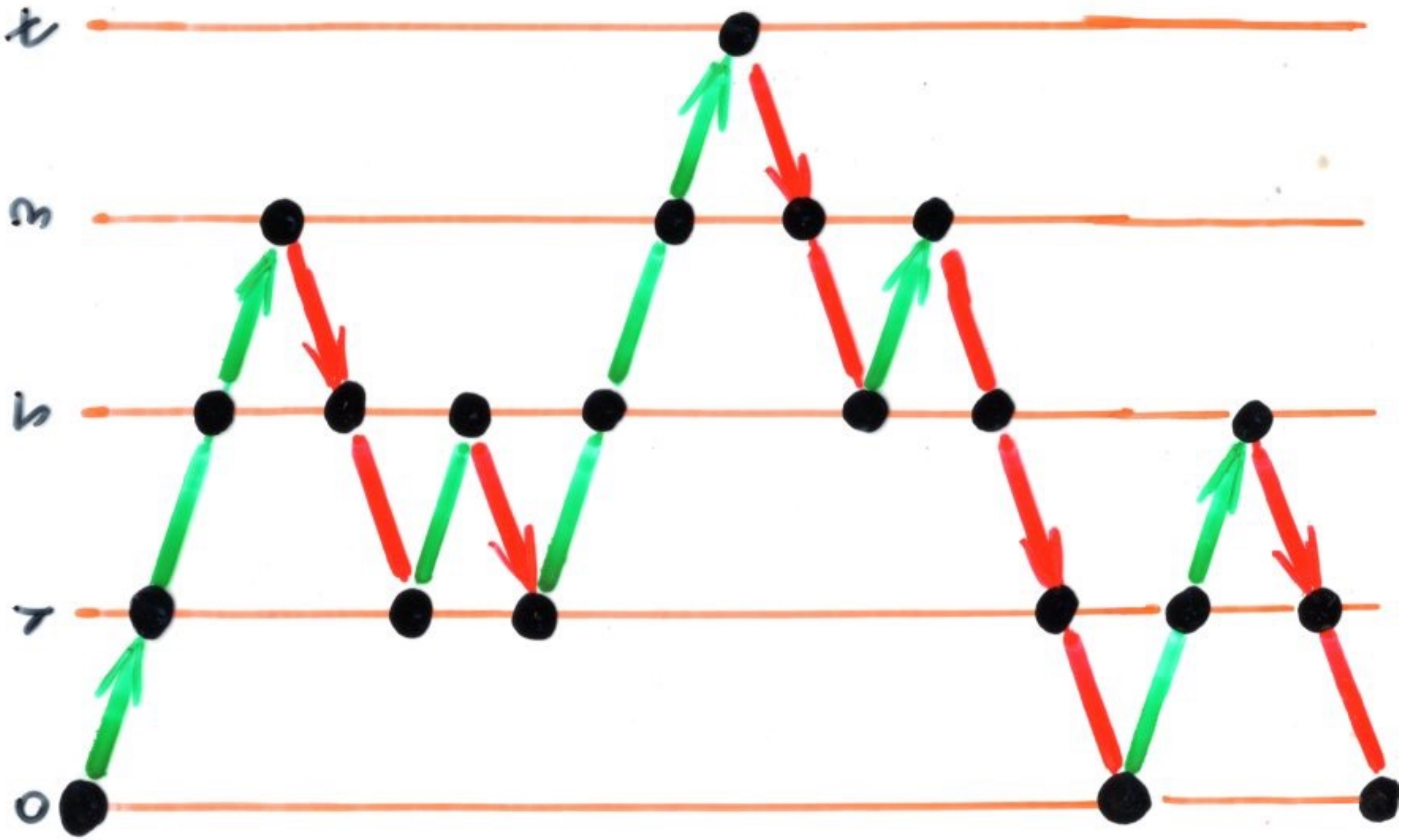
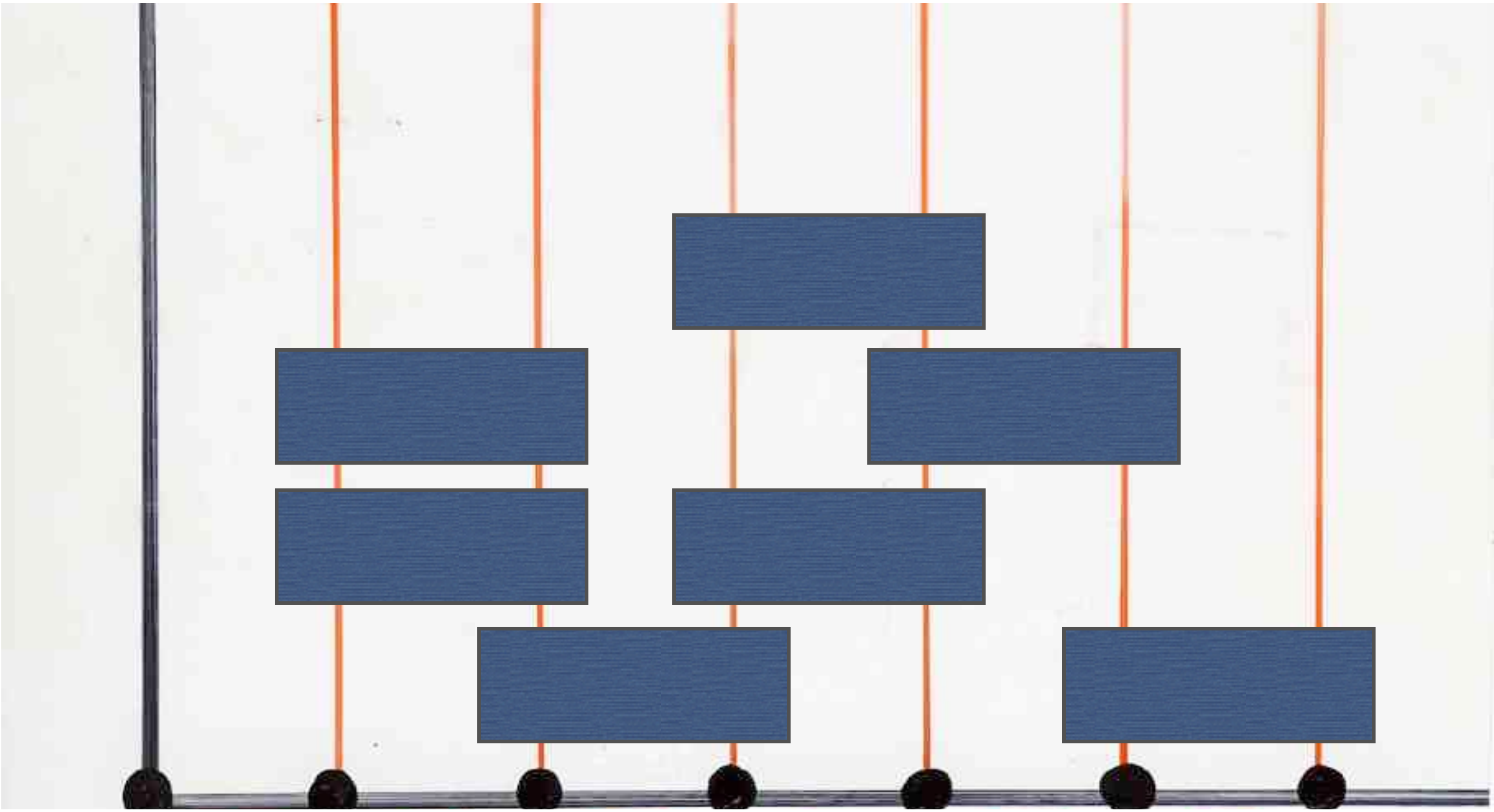
lattice





Dyck path

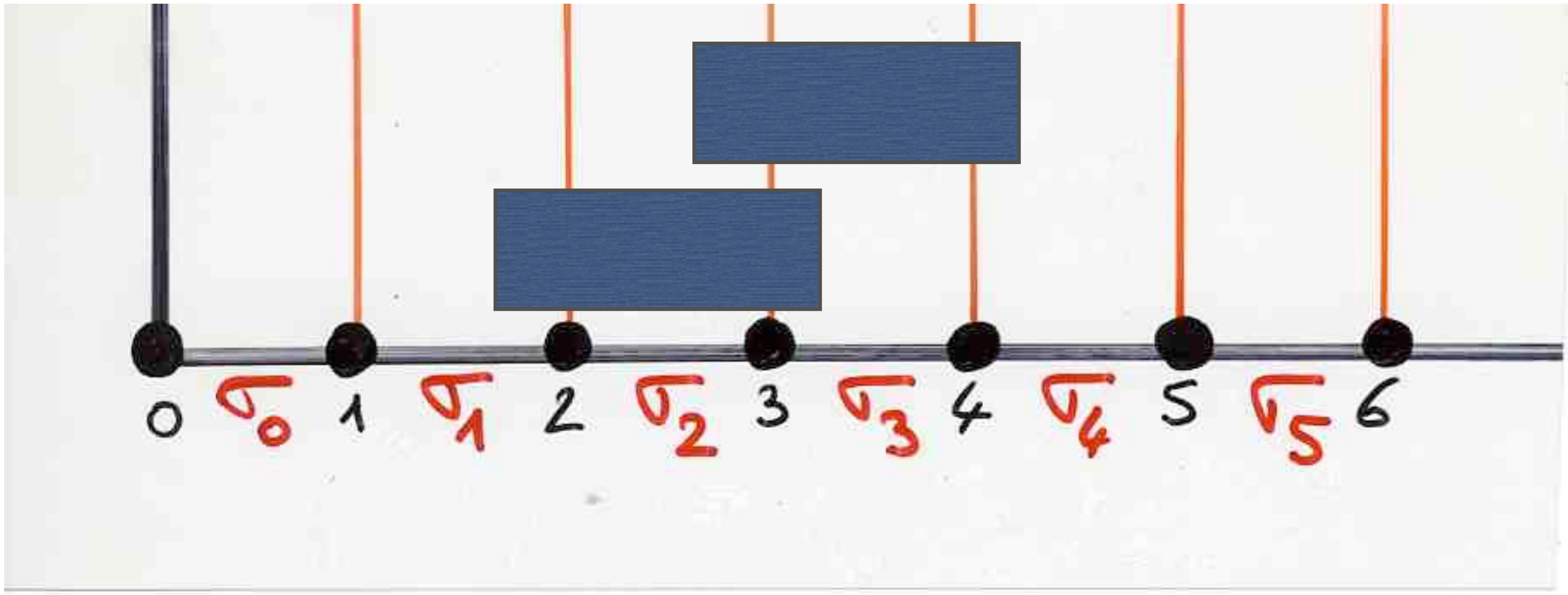


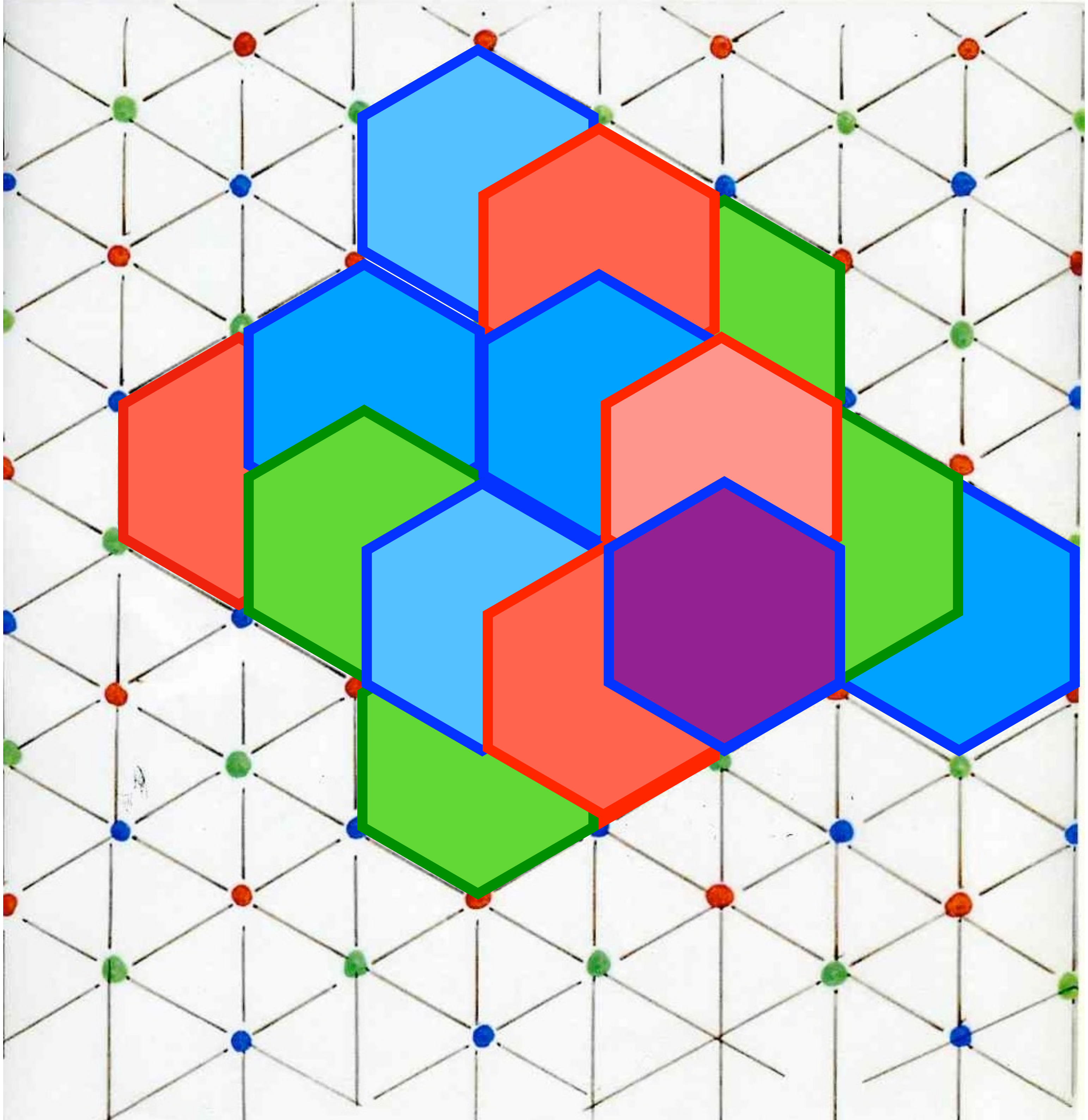
heap of dimers

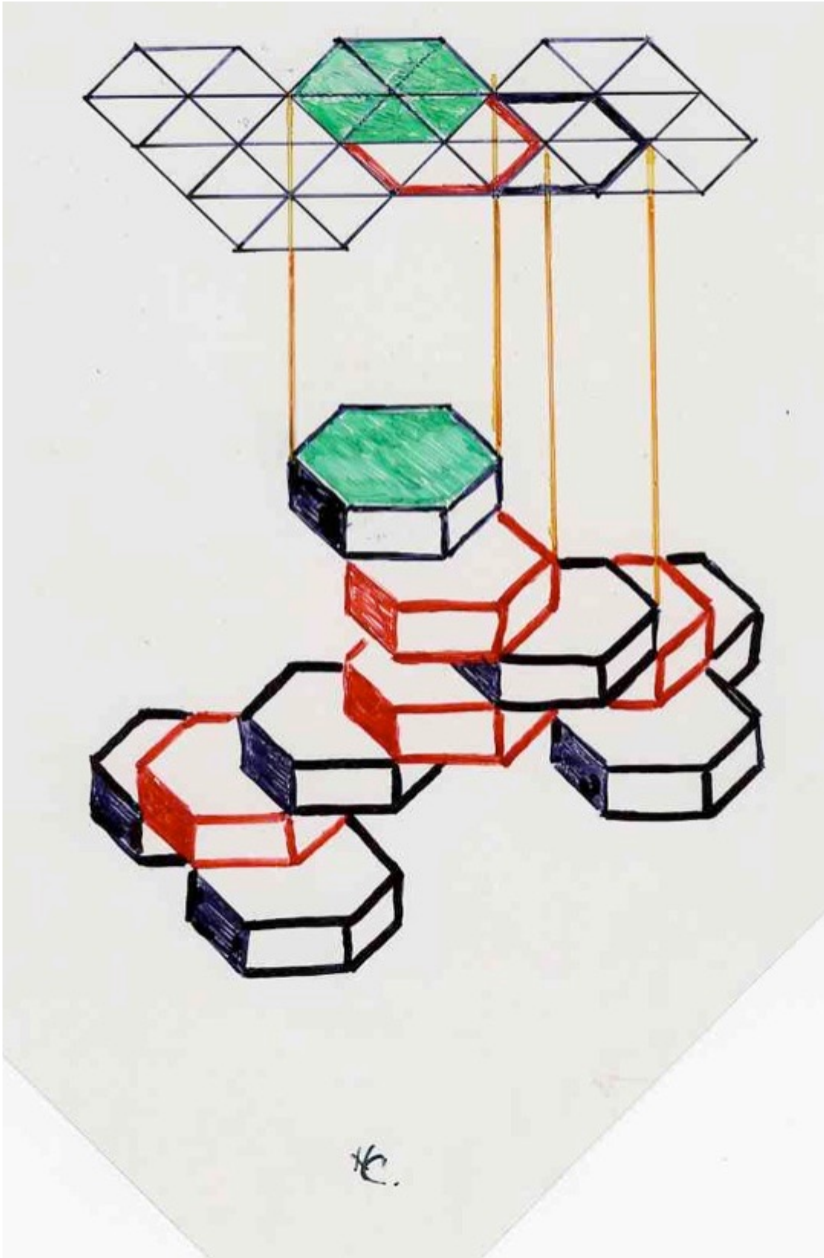
ex: heaps of dimers on $\mathbb{N}$

$\mathcal{P} = \{ [i, i+1] = \sigma_i \ , i \geq 0 \}$ set of basic pieces

$\mathcal{C}$ dependency relation $\sigma_i \cap \sigma_j \neq \emptyset$







arXiv

- T. Helmuth, A. Shapira Aug. 2020

  - *Loop-erased random walk as a spin system observable*,

- A.M. Garsia, G. Ganzberger Sept. 2020

  Fibonacci polynomials

- M.V. Tamm, N. Pospelov, S. Nechaev Oct. 2020

  Growth rate of 3D heaps of pieces

- P.-L. Giscard Nov 2020

  Counting walks by their last erased self-avoiding polygons using sieves,

arXiv

- E. Bagno, R. Biagioli, F. Jouhet, Y. Roichman Dec 2020

Block number, descents and Schur positivity of fully commutative elements in B_n

- J. Cigler, C. Krattenthaler Dec 2020

Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity

(70 pp)

- L. Fredes, J.-F. Marckert Feb 2021

Aldous-Broder theorem: extension to the non reversible case and new combinatorial proof,

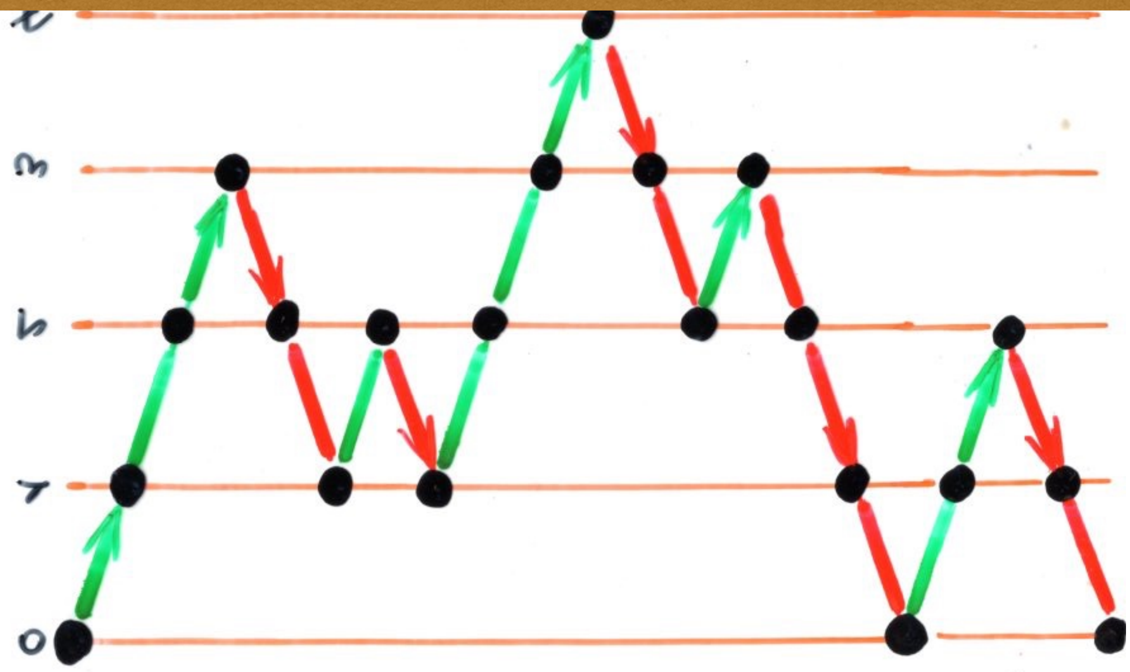
arXiv

• J. Cigler, C. Krattenthaler Dec 2020

Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity

(70 pp)

Reciprocity



Dyck path

$$C^{(k)}_{2n}$$

$$\sum_{n \geq 0} C^{(k)}_{2n} \, t^{2n}$$

Rational

$(a_n)_{n\geq 0}$

Rational

$(a_n)_{n<0}$

combinatorial meaning ?

combinatorial "reciprocity law"

E. Ehrhart (1959, 1967, 68, 1973)

$$f(t) = \sum_{n\geq 0} a_n t^n$$

R. Stanley (1974)

$$-f(1/t) = \sum_{n\geq 1} a_{-n} t^n$$

book Beck - Sanyal (2018)

MathOverflow

Johann Cigler

26 Sept 2020
28 Sept

$$C^{(2k+1)}_{-2n} = \det\left(C^{(2k+1)}_{2n+2i+2j+2} \right)_{0 \leq i,j \leq k-1}$$

alternating sequence

$$a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{2n-1}$$

S. Hopkins

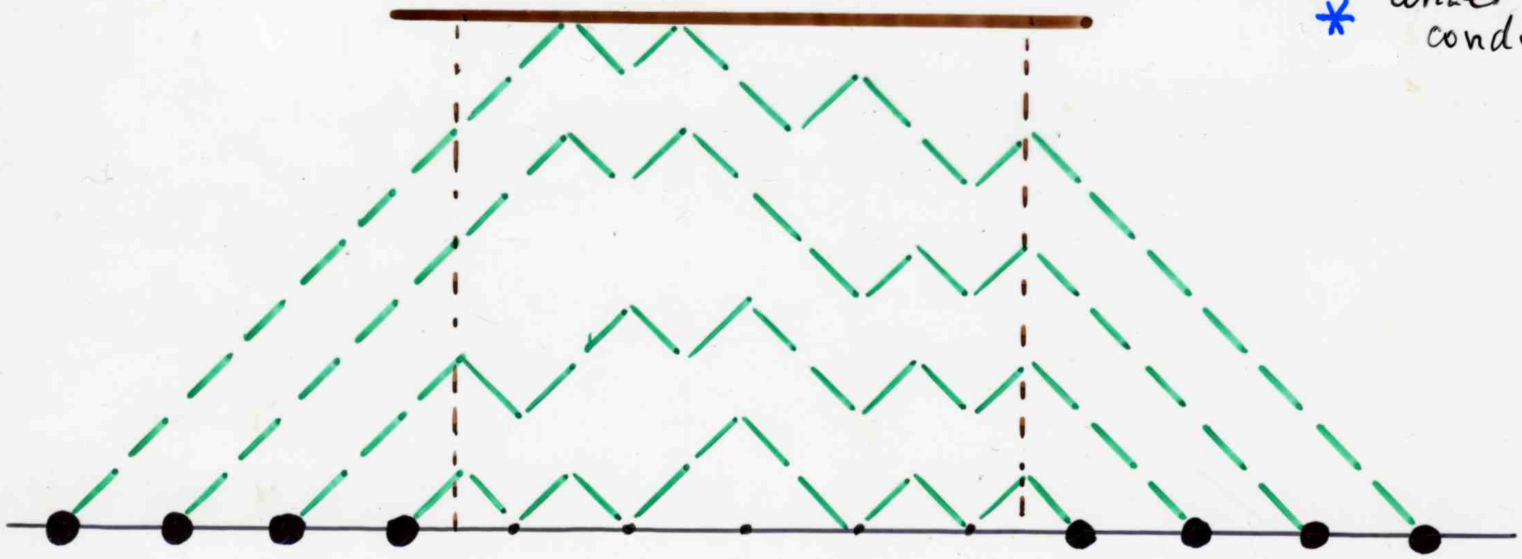
R. Stanley

30 Sept 2020

Hankel determinants

"LGV Lemma"

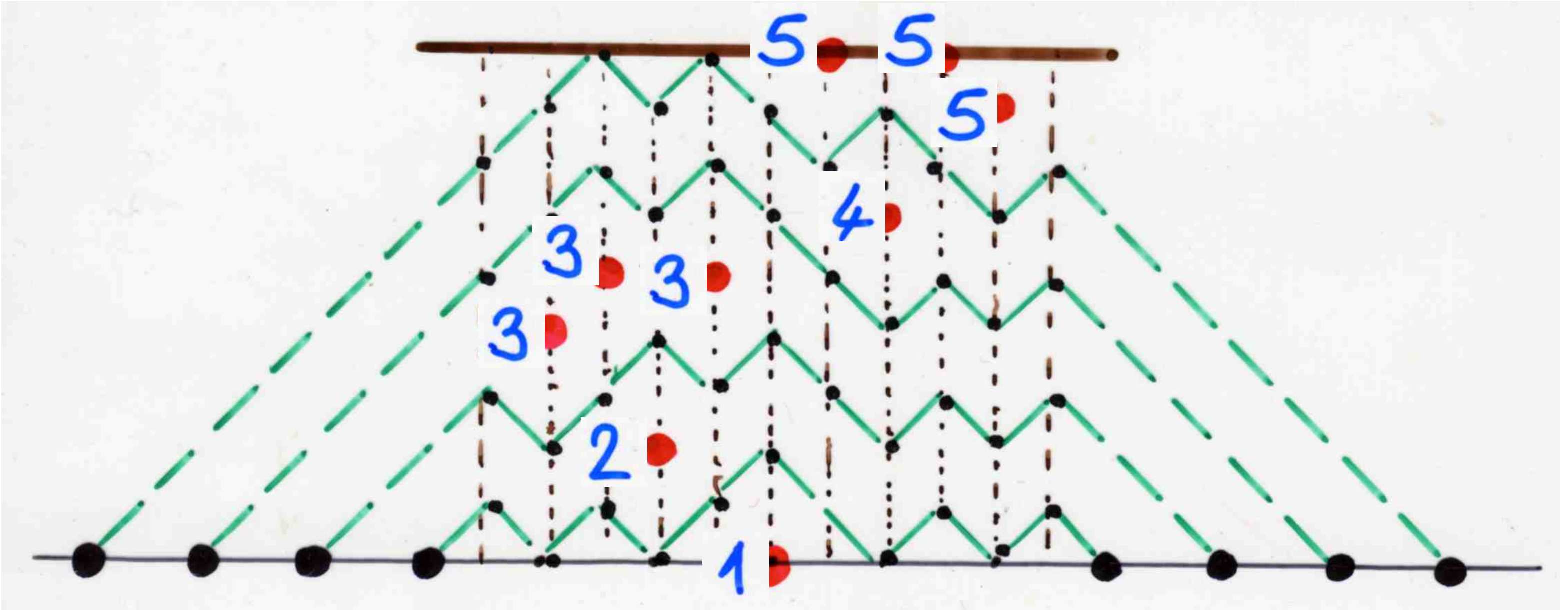
determinant $\overset{*}{=}$ non-intersecting paths



* under certain conditions

Hankel determinant

$$\det\left(C^{(2k+1)}_{2n+2i+2j+2}\right)_{0\le i,j\le k-1}$$



$$3 \leq 3 \geq 2 \leq 3 \geq 1 \leq 5 \geq 4 \leq 5 \geq 5$$

alternating sequence

$$1 \leq a_i \leq k \qquad a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots\cdots \geq a_{2n-1}$$

$$? = \left| \mathcal{A}_{2n}^{(k)} \right|$$

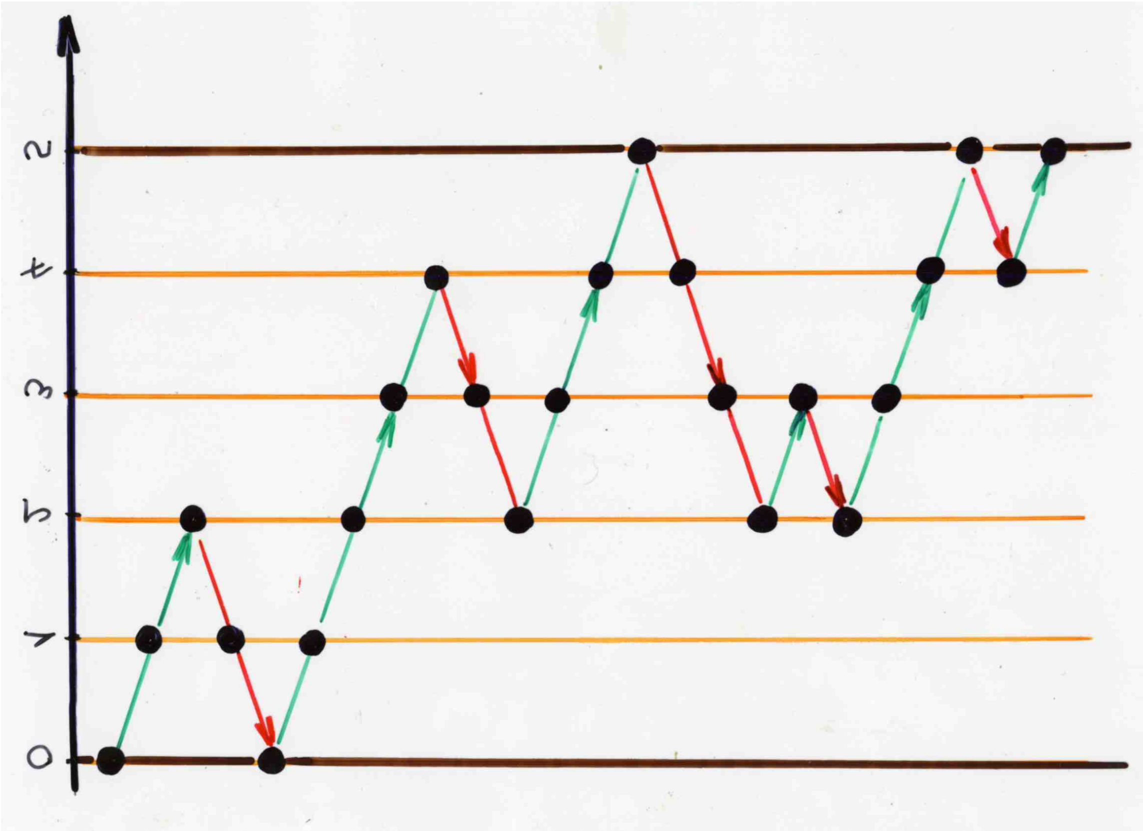
$1 \leqslant a_i \leqslant k$ $\qquad$ $a_1 \leqslant a_2 \geqslant a_3 \leqslant a_4 \geqslant \cdots \leqslant a_{2n}$

$$C_{-2n}^{(2k-1)} = \left| \mathcal{A}_{2n-1}^{(k)} \right|$$

alternating sequence

$1 \leqslant a_i \leqslant k$ $\qquad$ $a_1 \leqslant a_2 \geqslant a_3 \leqslant a_4 \geqslant \cdots\cdots \geqslant a_{2n-1}$

$$(-1)^{k+1}\,\mathbf{D}^{(2k-1)}_{-2n-2k} = \left|\mathcal{A}^{(k)}_{2n}\right|$$



$$\mathbf{D}^{(k)}_{2n} = \mathbf{C}^{(k)}_{2n+k}(0 \to k)$$

About the « LGV Lemma »

See the video-book « ABjC »: *The Art of Bijective Combinatorics,*

Part I, *An introduction to enumerative, algebraic and bijective combinatorics*

IMSc, Chennai, 2016, Chapter 5a, pp 3-28

www.viennot.org/abjc1-ch5.html

About Hankel determinants

See the video-book « ABjC », Part IV, *Combinatorial theory of orthogonal polynomials and continued fractions*

IMSc, Chennai, 2019, Chapter 4a, pp 39-56, pp 61-70

www.viennot.org/abjc4-ch4.html

slide added after the talk

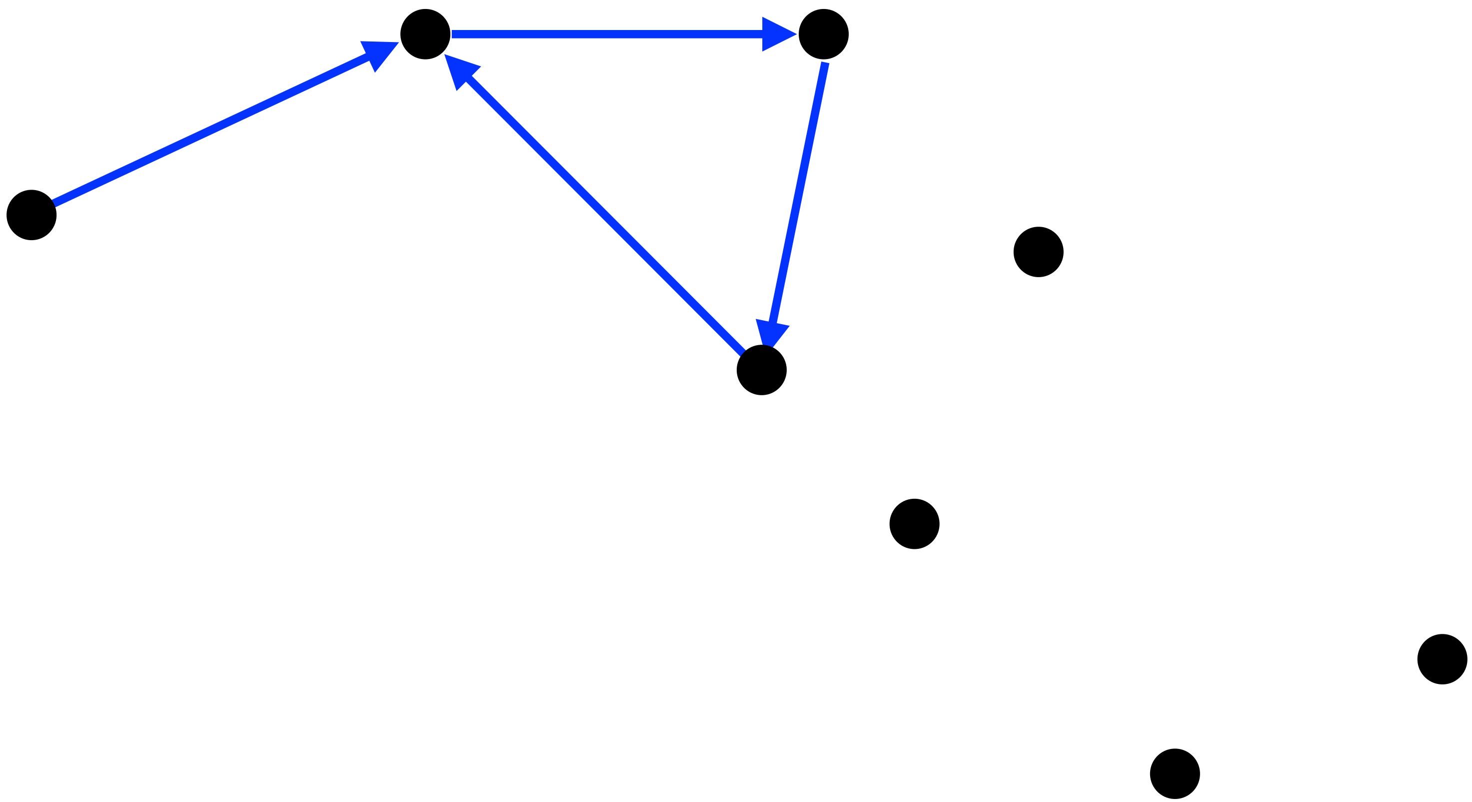
generating function

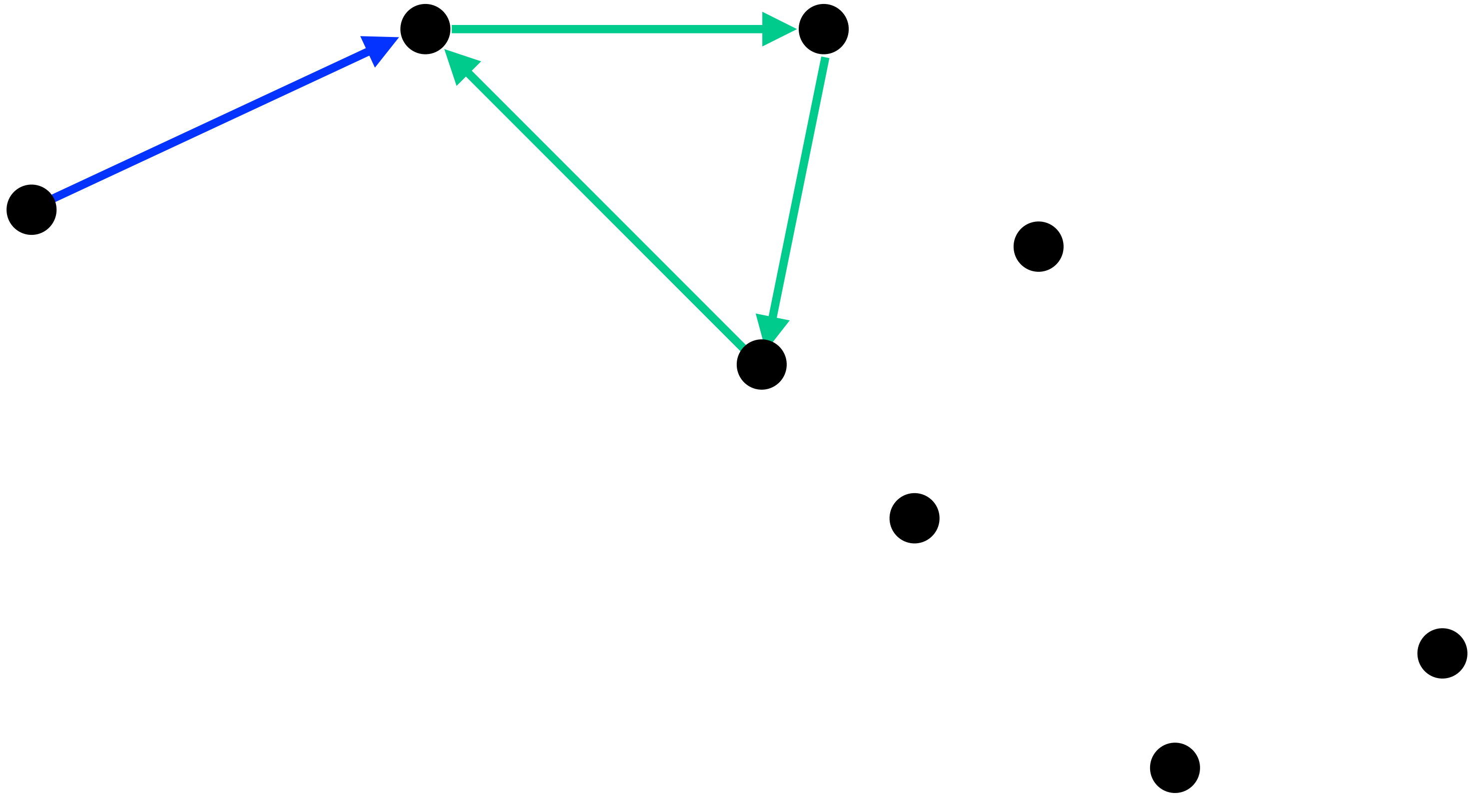
for bounded Dyck paths

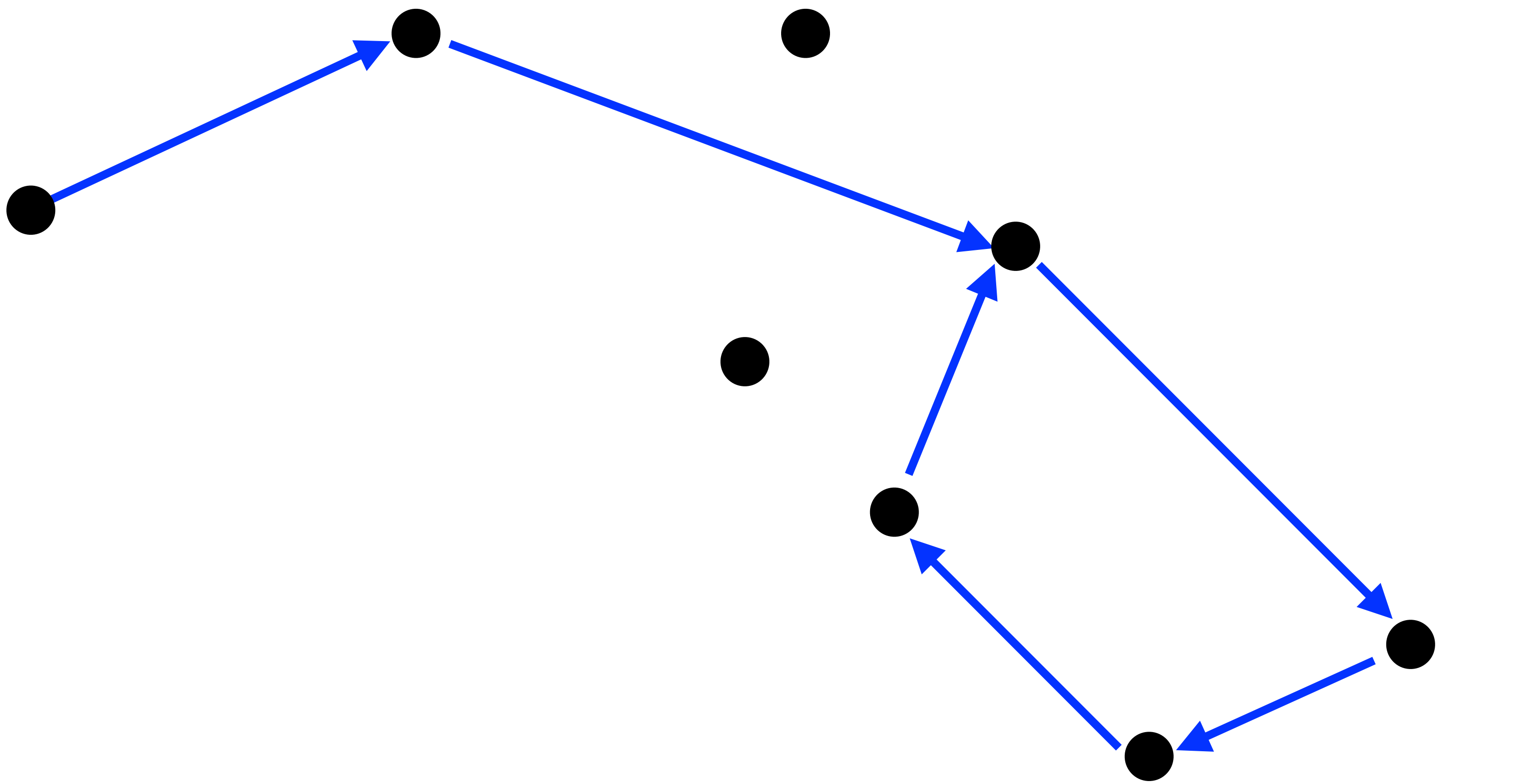
first basic lemma

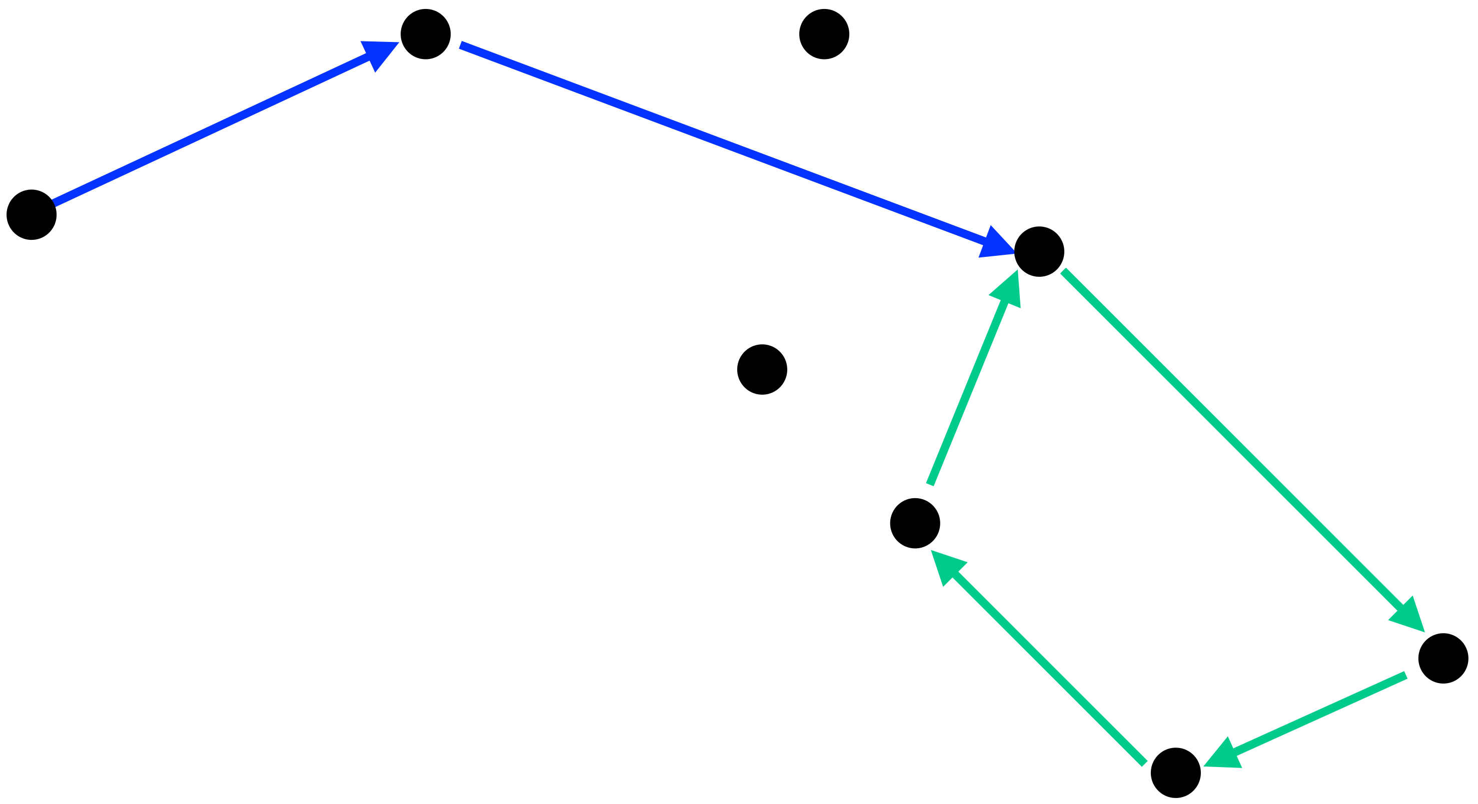
the bijection paths — heaps

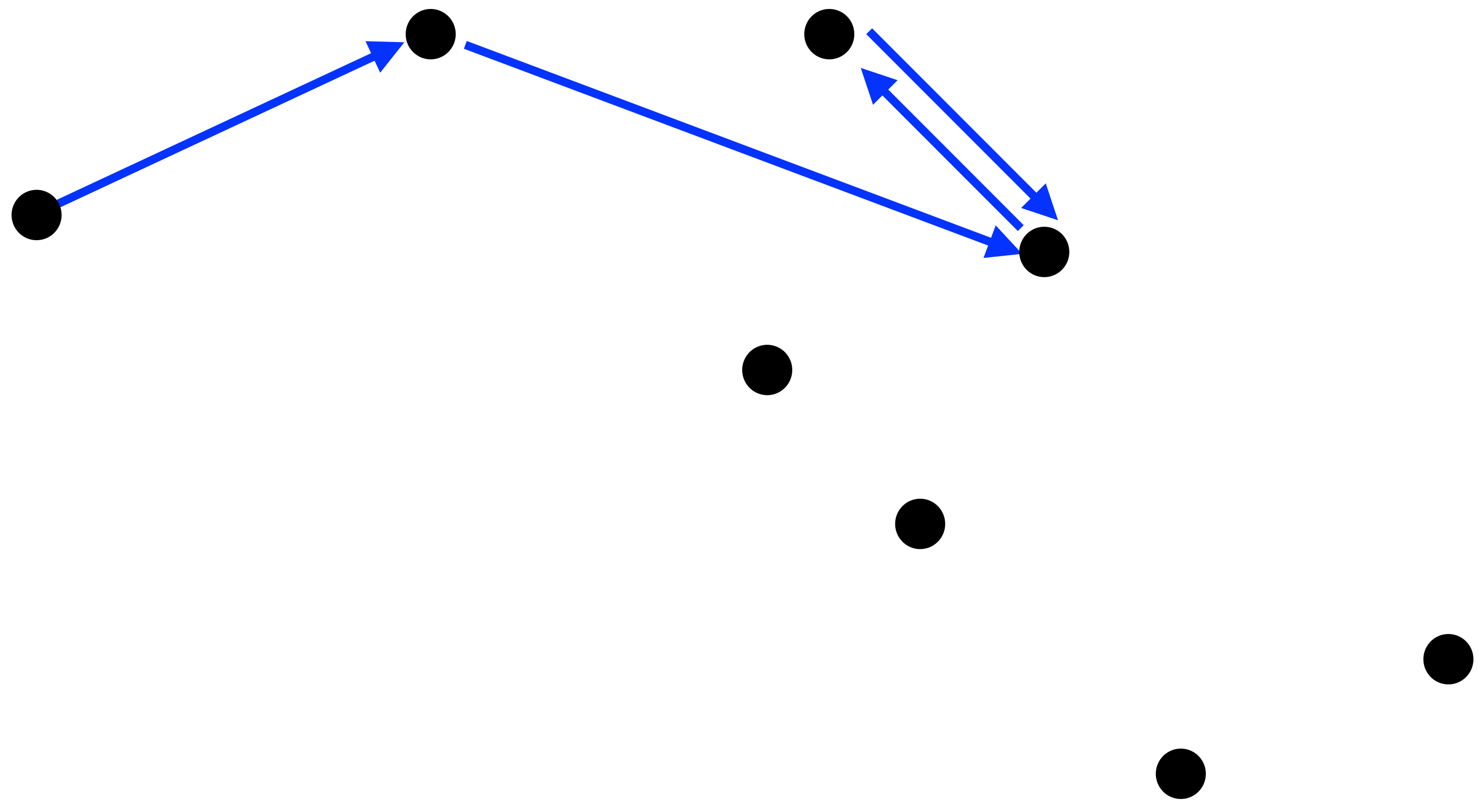
LERW
Loop erased random walk

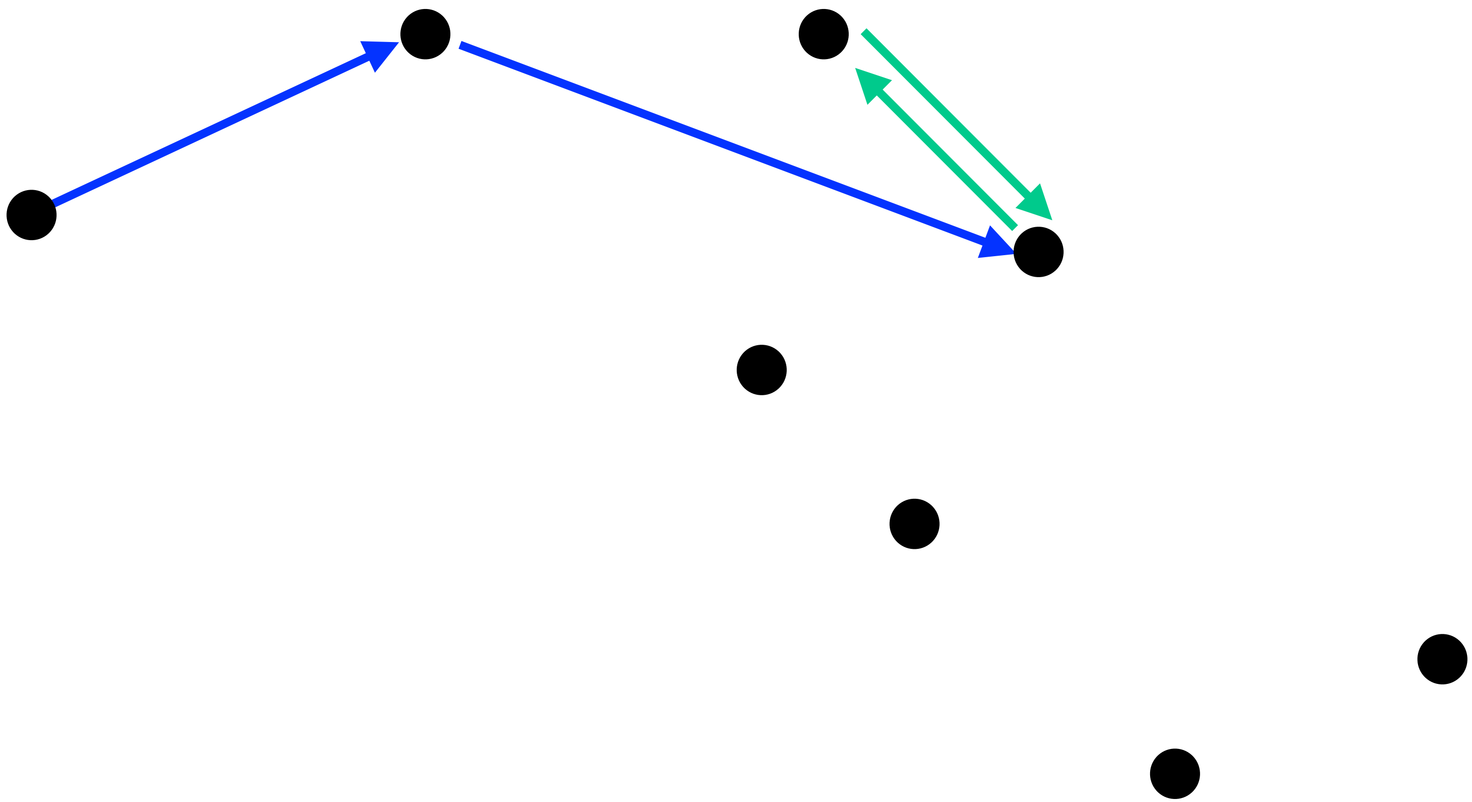


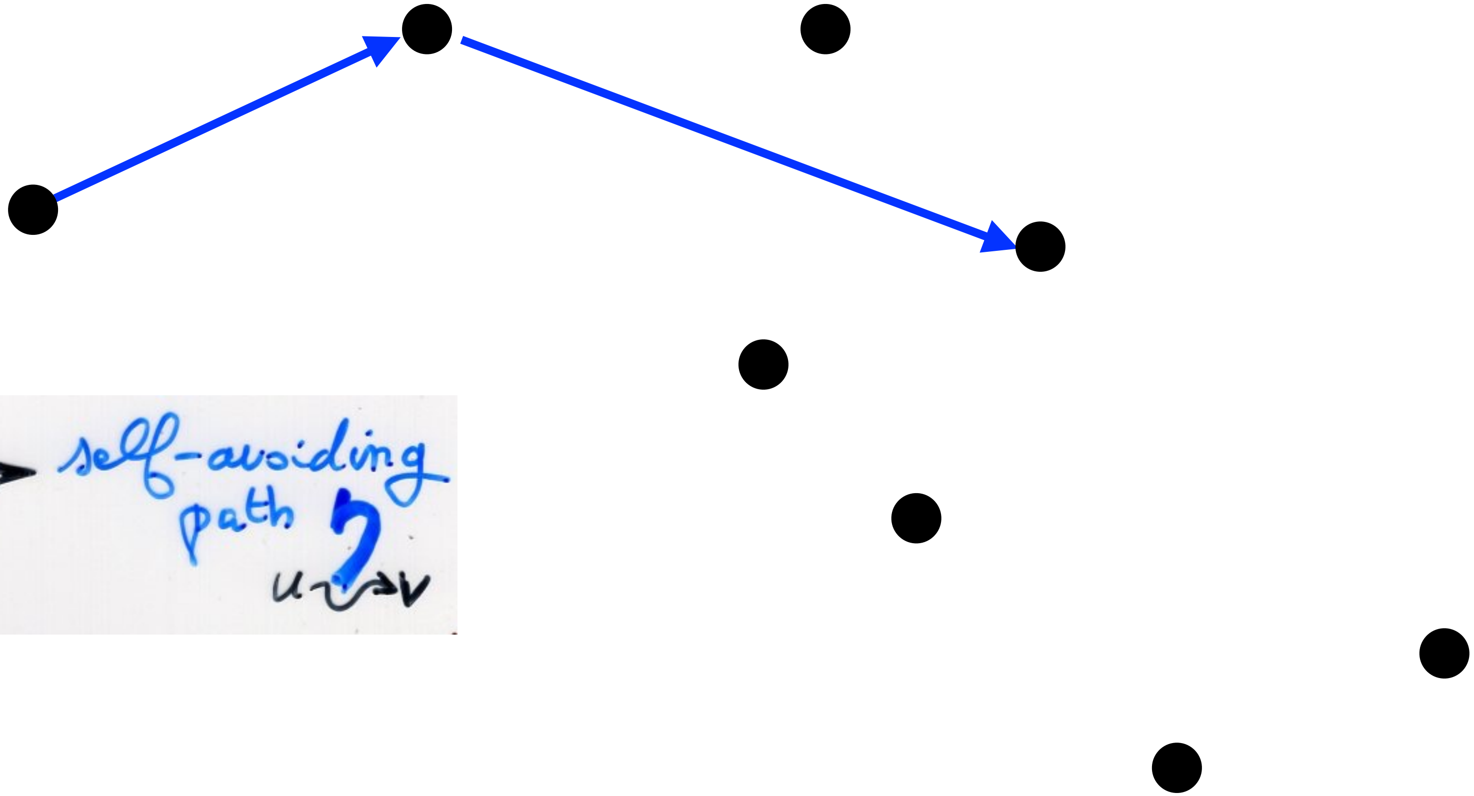












path ω $\longrightarrow$ self-avoiding path η

$u \rightsquigarrow v$ $\quad$ $u \rightsquigarrow v$

LERW

Loop-erased random walk

Lawler (1980)

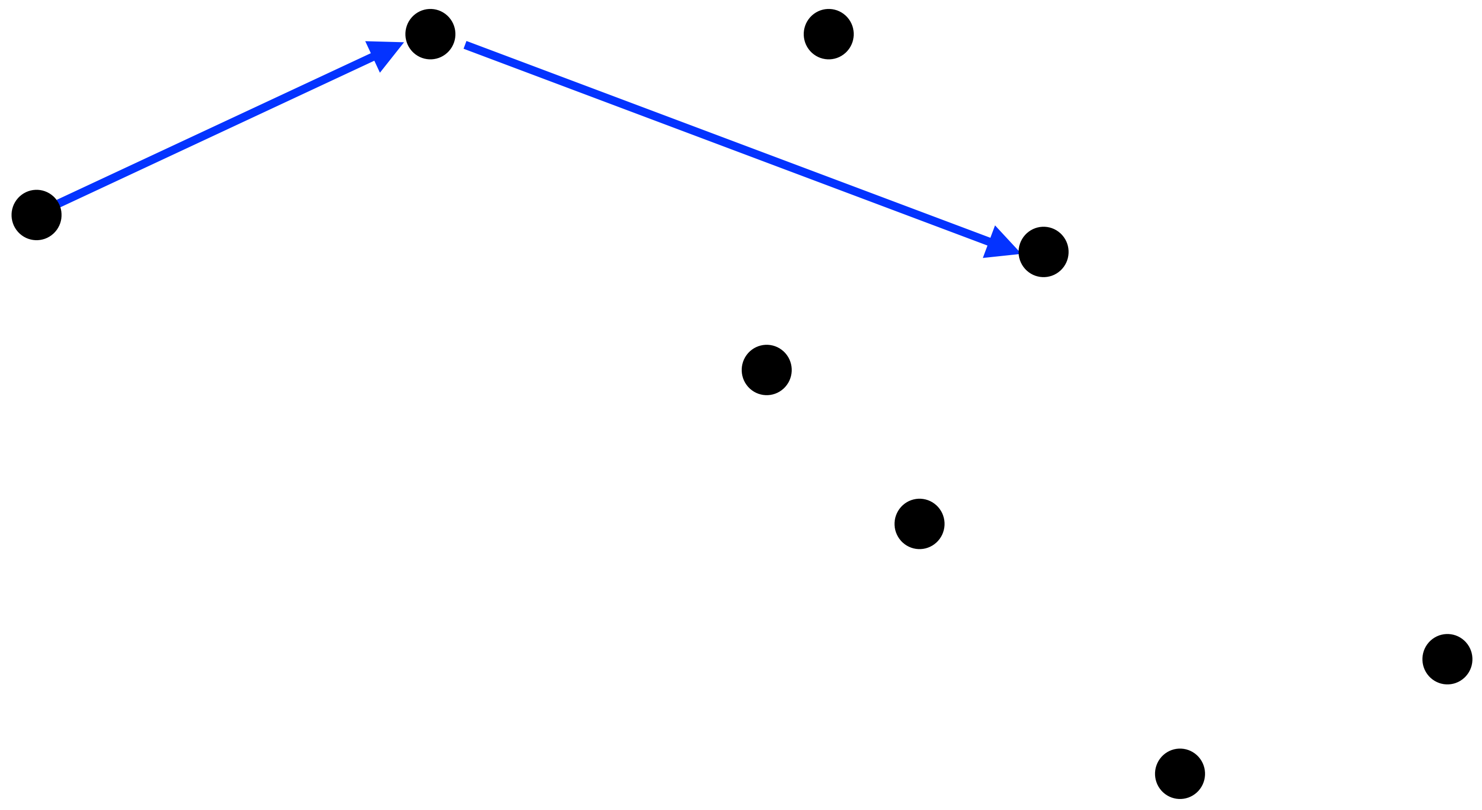
ω random path on X

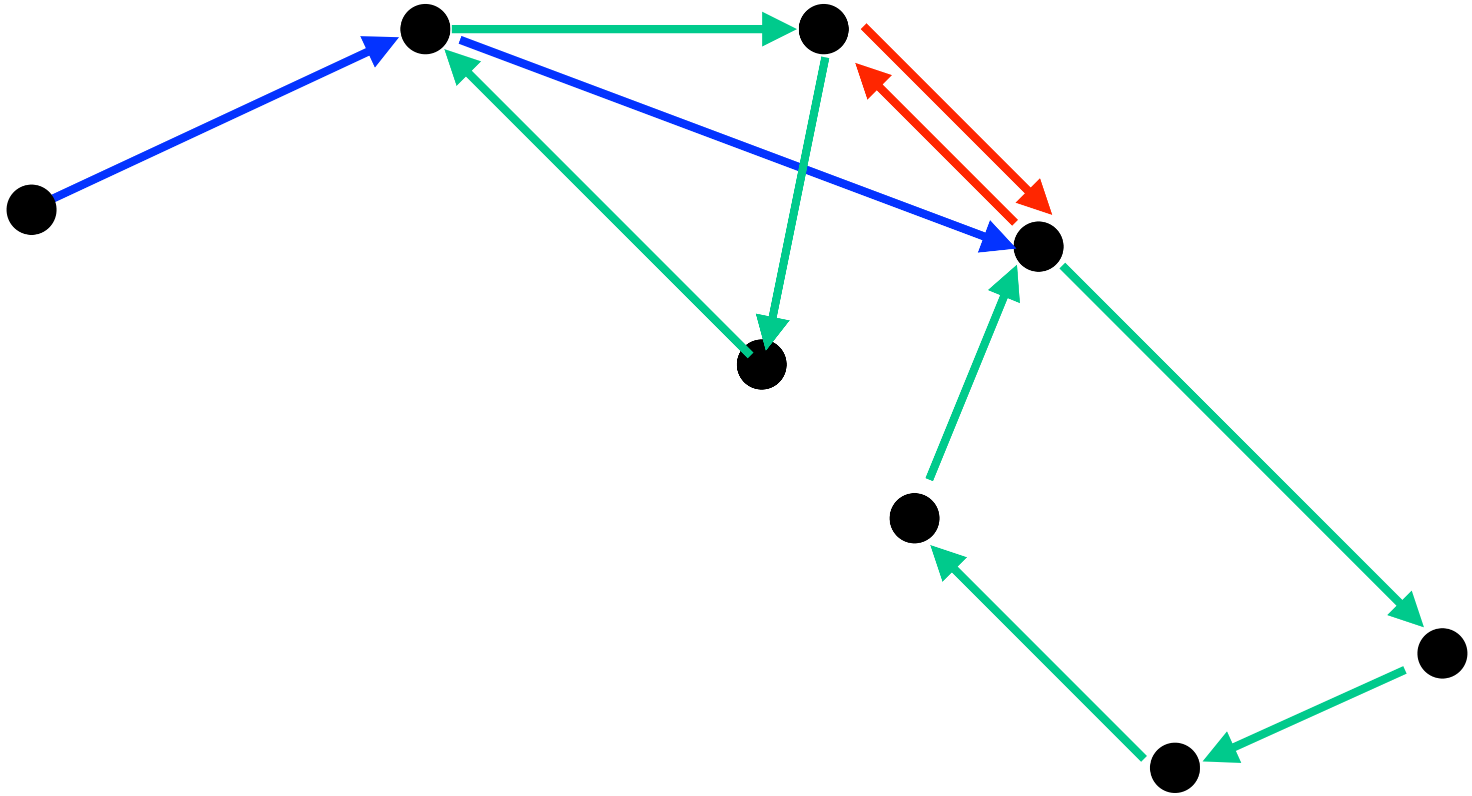
path ω $u \rightsquigarrow v$ $\longrightarrow$ self-avoiding path γ $u \rightsquigarrow v$

LERW $\longrightarrow$ SLE_2

Schramm-Loewner evolution

the bijection paths — heaps





$$\omega \longrightarrow (\eta\ ;\ (\dot\gamma_1, \ldots, \dot\gamma_{r_n}))$$

self-avoiding path $u \rightsquigarrow v$

sequence of pointed cycles

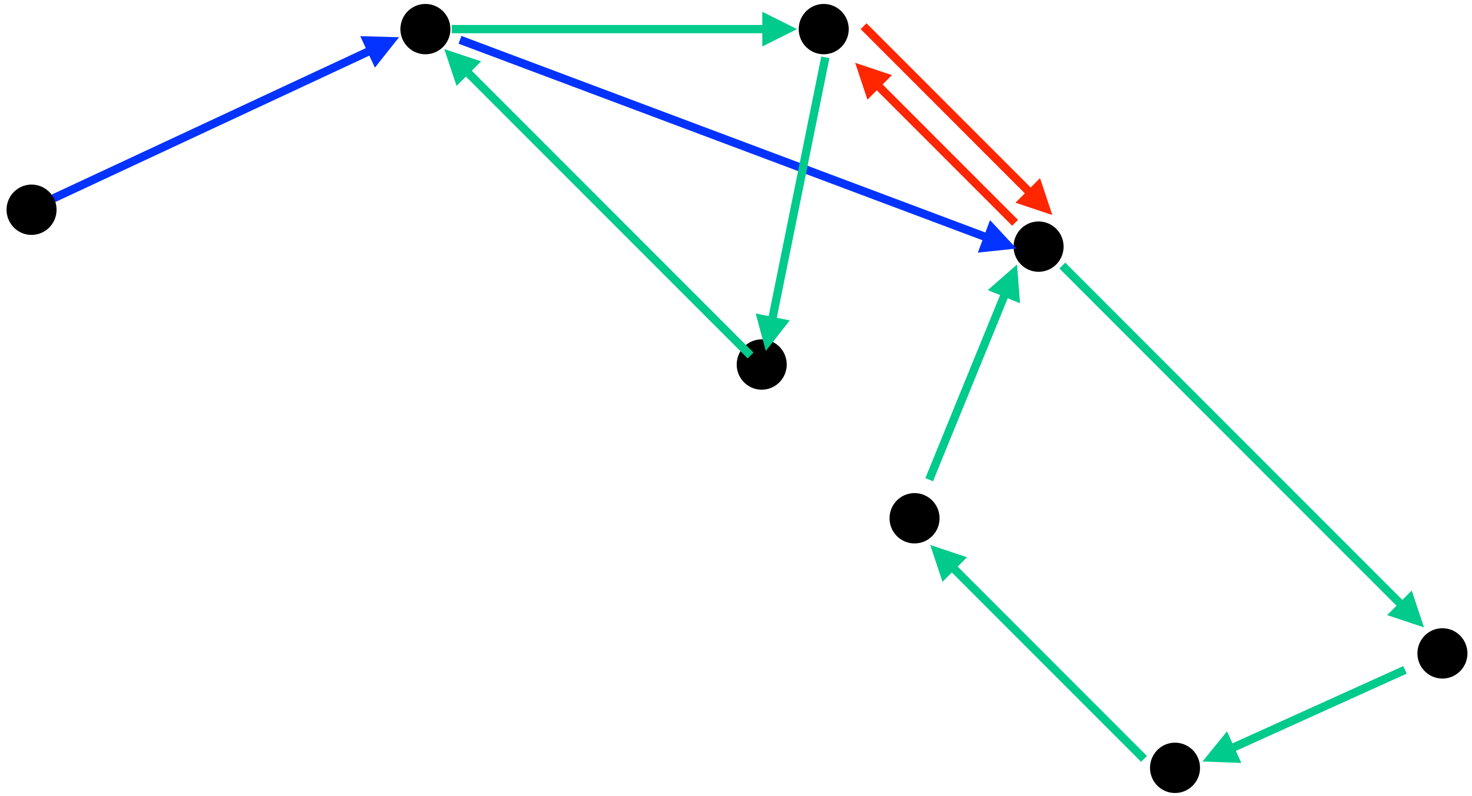
from the pair $(\eta\ ;(\dot\gamma_1, \ldots ; \dot\gamma_{r_n}))$ we can reconstruct the path ω

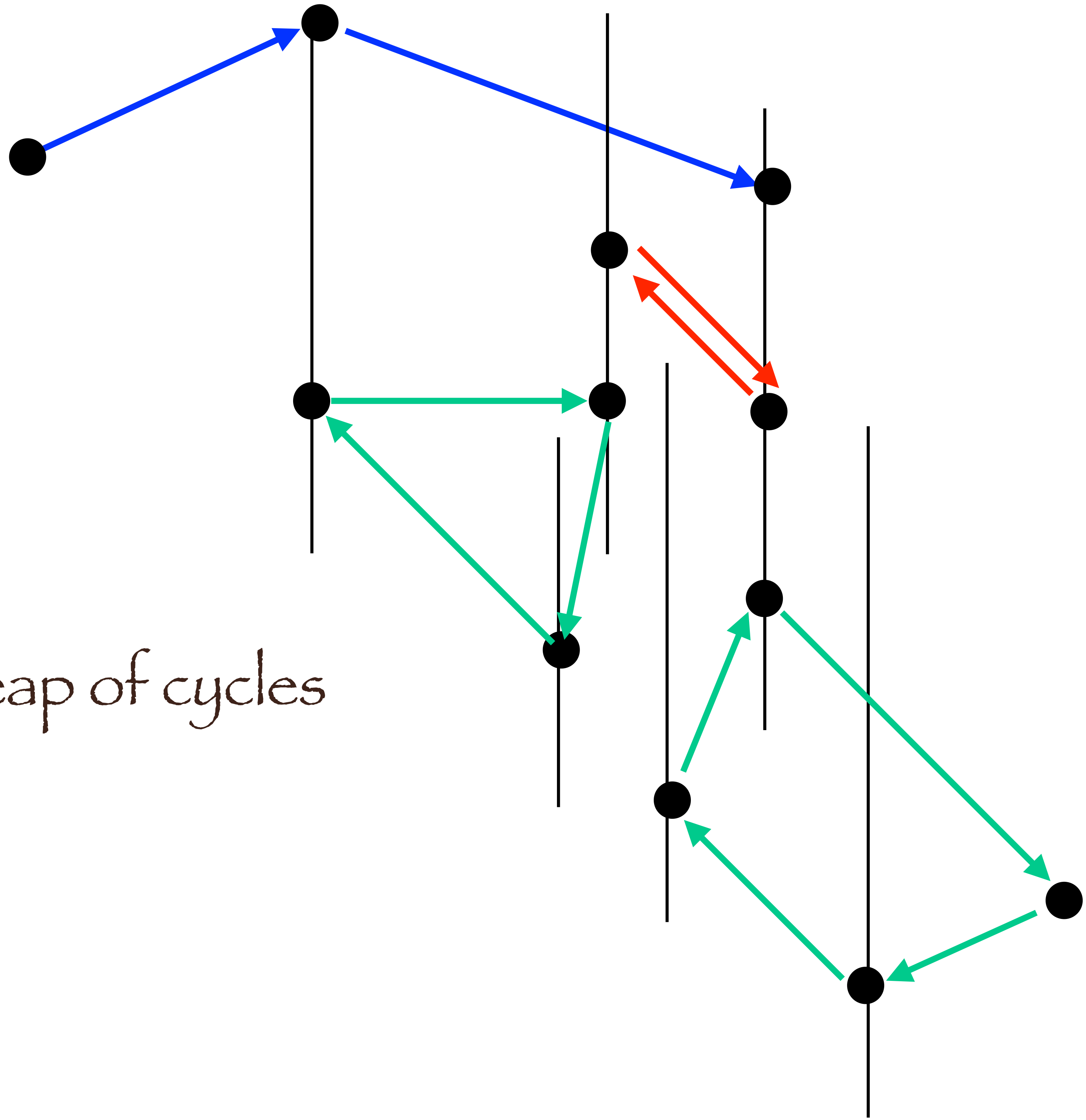
$$(\dot\gamma_1, \ldots, \dot\gamma_{r_n}) \longrightarrow E = \gamma_1 \odot \cdots \odot \gamma_{r_n}$$

$$\omega \longrightarrow (\eta\ , E)$$

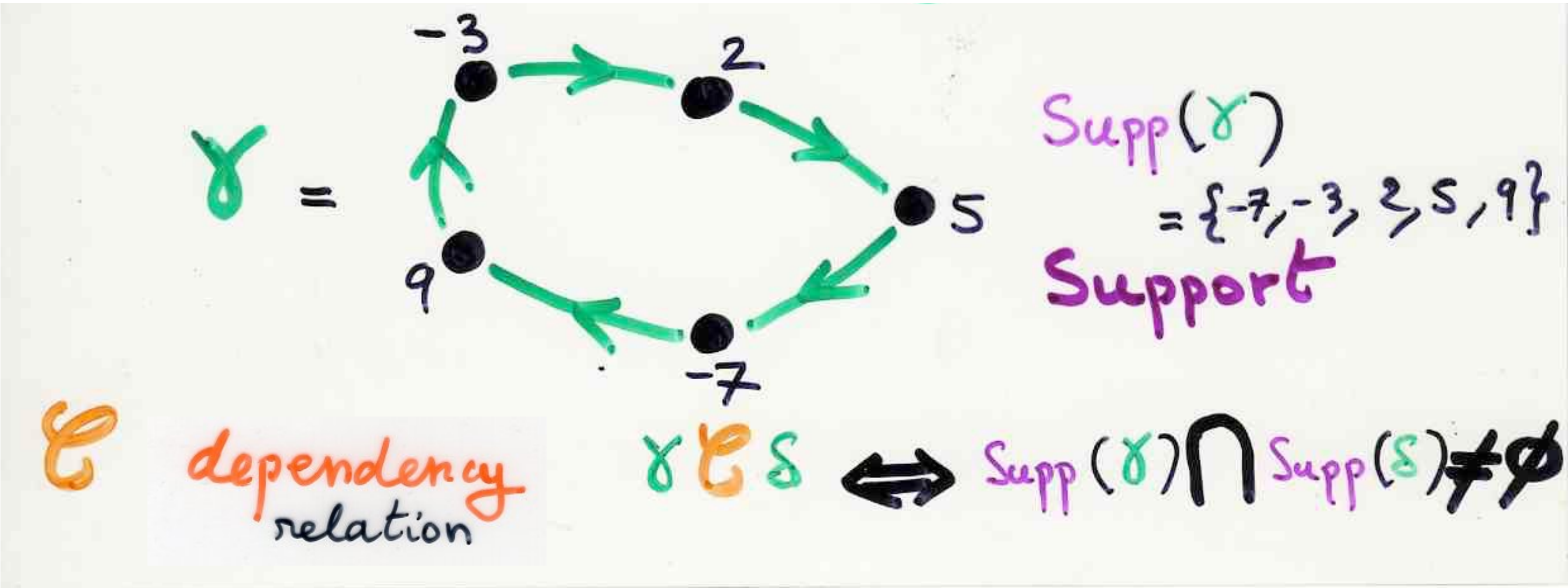
heaps of cycles on X

monoid





Heap of cycles


γ =
-3
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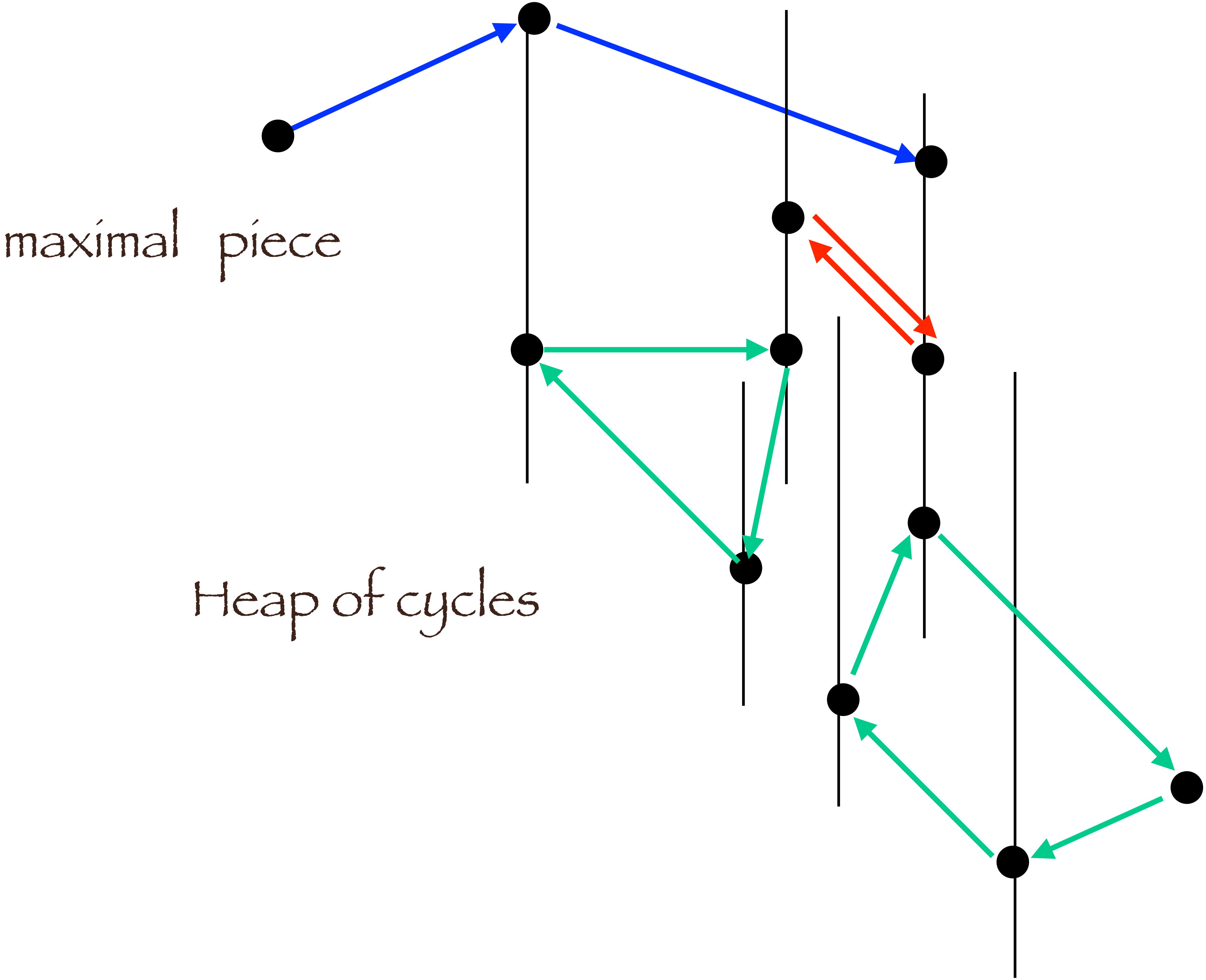
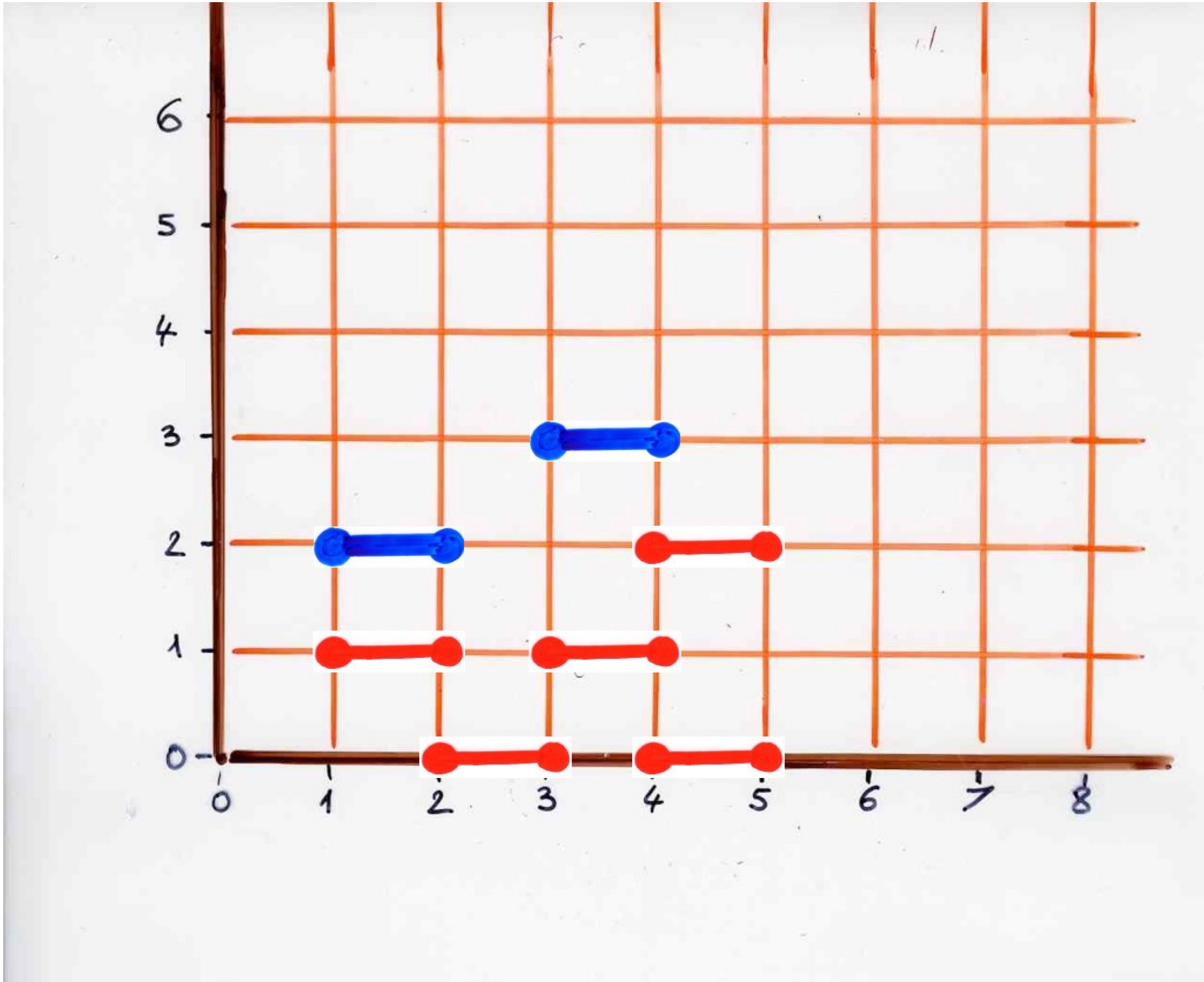
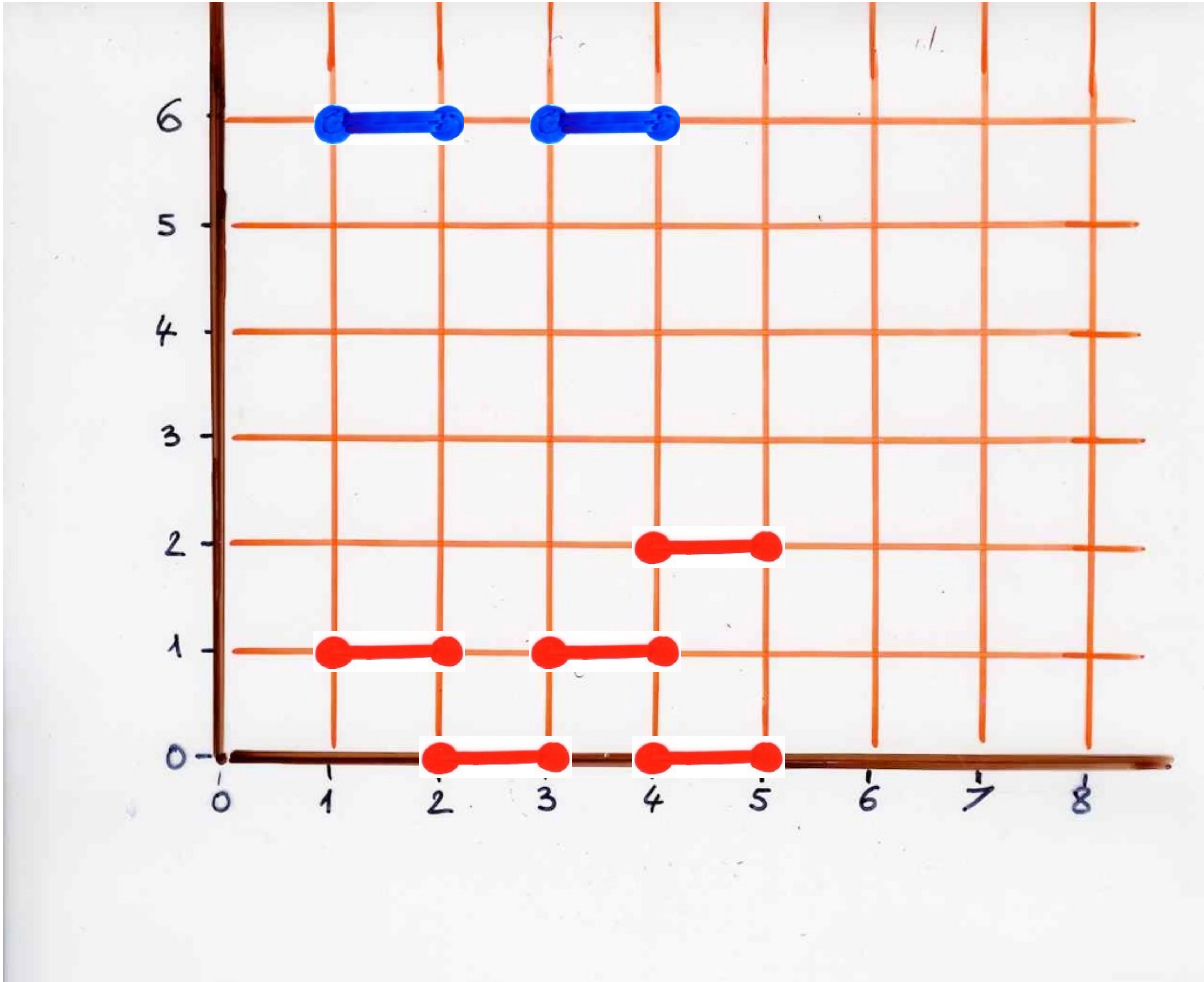
5
-7
9
Supp(γ)
= {-7, -3, 2, 5, 9}
Support
C dependency relation
γ C δ ⟺ Supp(γ) ∩ Supp(δ) ≠ ∅


maximal piece
Heap of cycles

maximal pieces



maximal pieces



Bijection $u, v \in X$

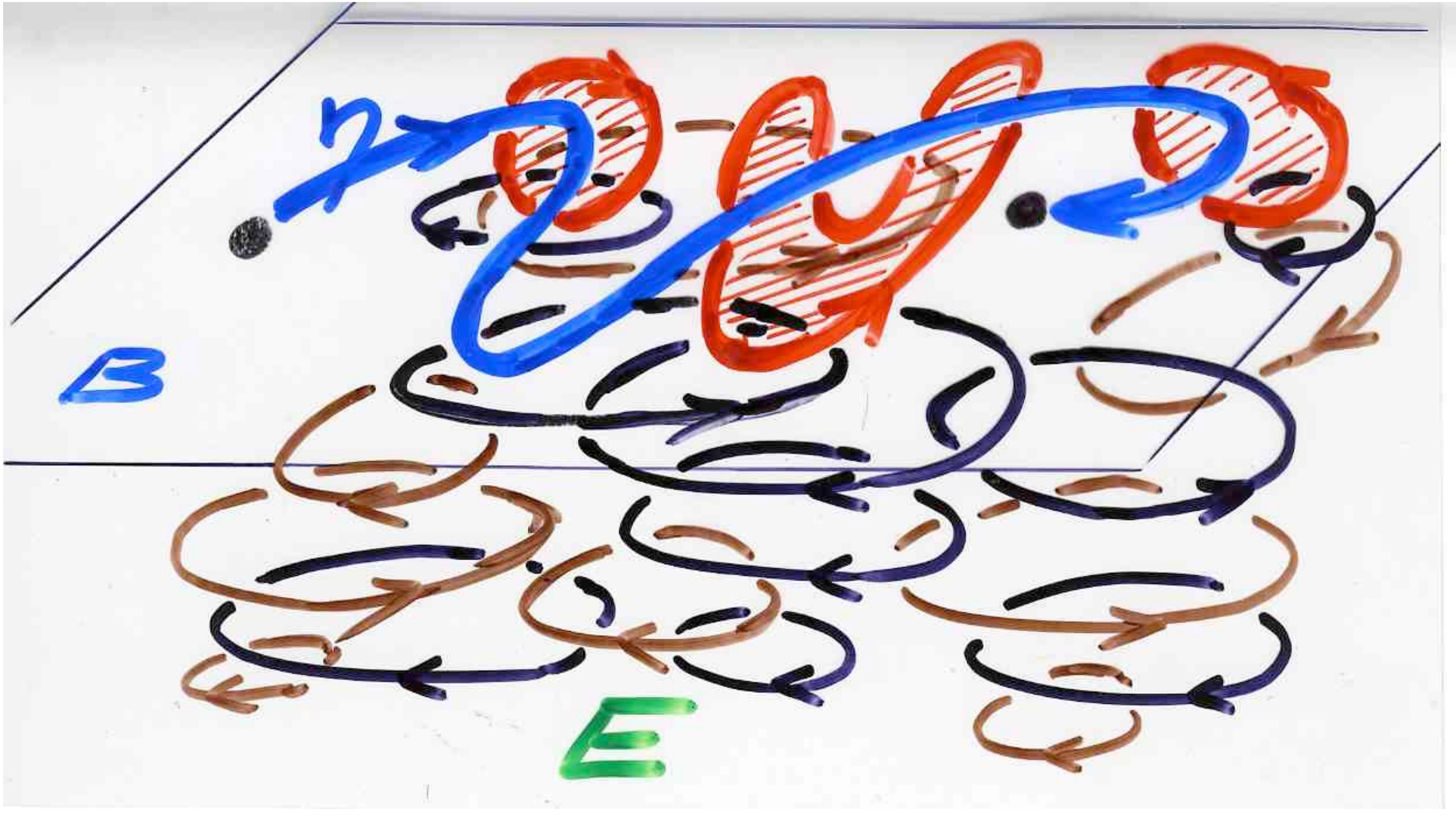
path ω on X going from u to v $\overset{\chi}{\longleftrightarrow}$ (η, E)

- η self-avoiding path going from u to v

- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η

($\underset{\text{cycle}}{\alpha}$ and $\underset{\text{path}}{\eta}$ has a common vertex

The bijection χ



Bijection $u, v \in X$

path ω on X going from u to v $\xleftrightarrow{\chi}$ (η, E)

Weight on the edges of the path

for any $s, t \in X$ the numbers of occurrences of the edge (s,t) in ω and in (η, E) are the same.

$$\Rightarrow \quad v(\omega) = v(\eta)\,v(E)$$

- T. Helmuth, A. Shapira Aug. 2020

  \- *Loop-erased random walk as a spin system observable*,

- P.-L. Giscard Nov 2020

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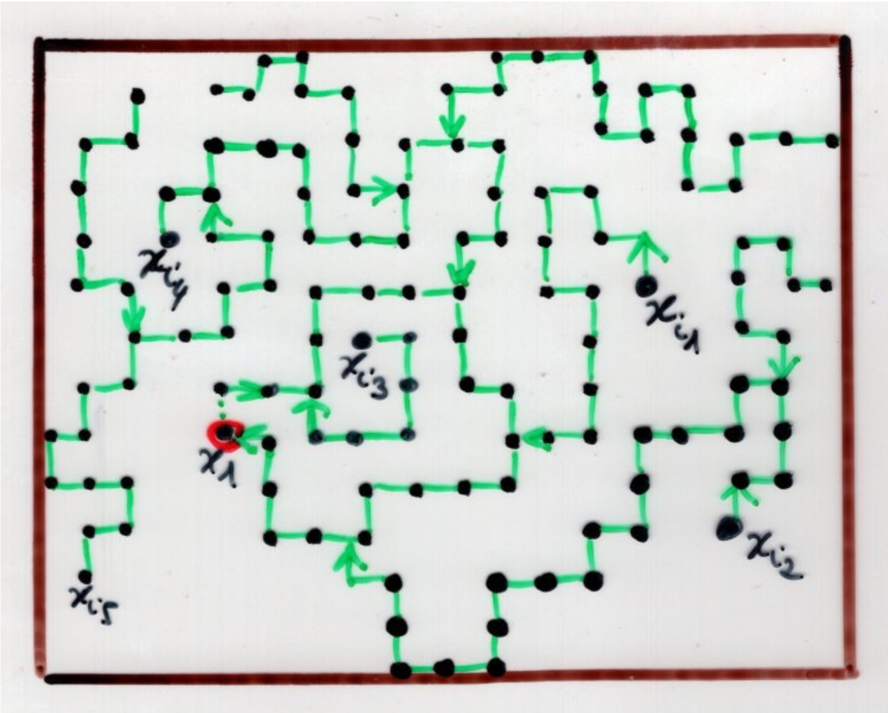


spanning tree

Wilson's algorithm

More details in the video-book « ABjC », Part II, *Commutations and heaps of pieces with interactions in physics, mathematics and computer science*

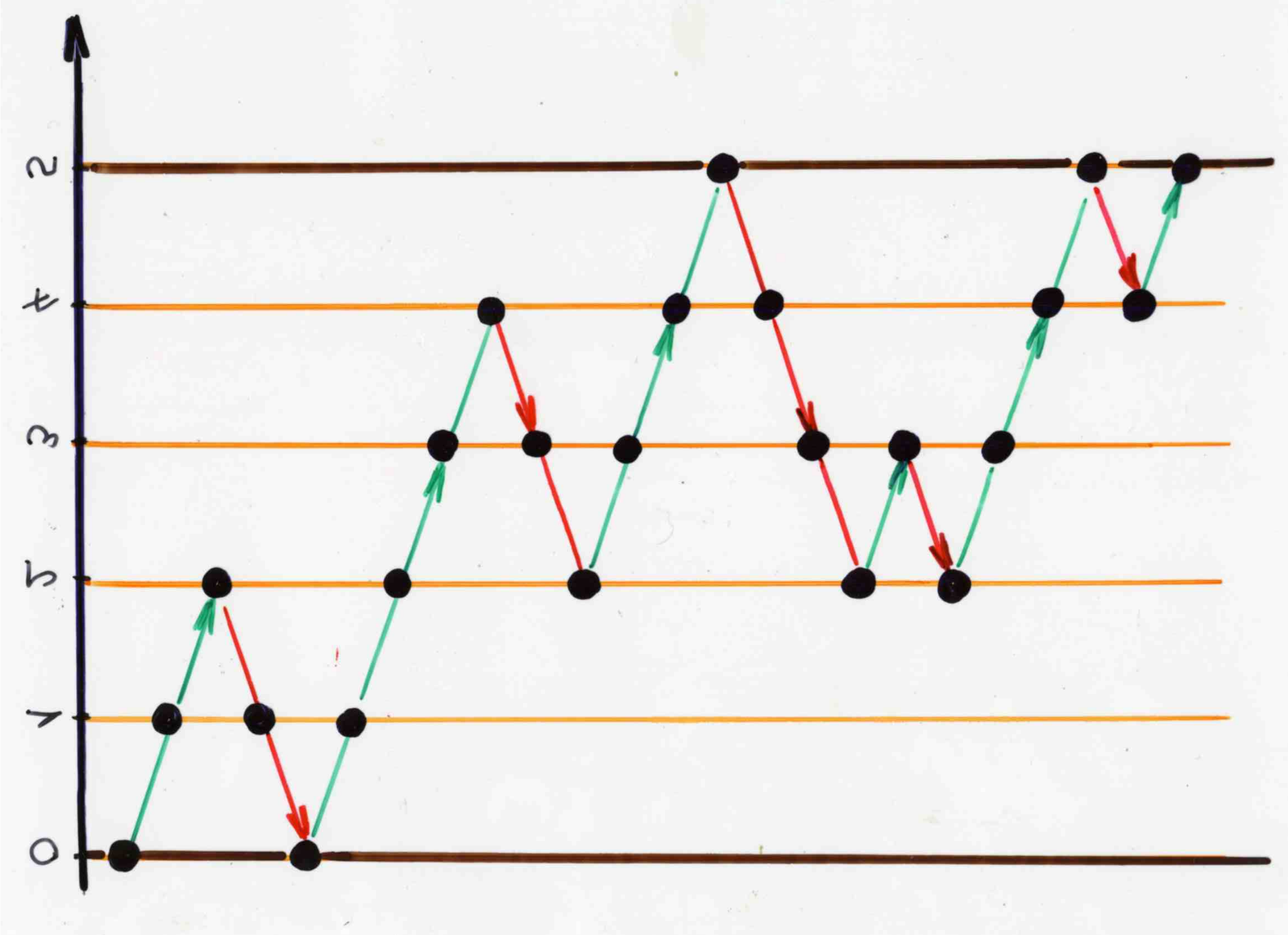
IMSc, Chennai, 2017 www.viennot.org/abjc2.html

Chapter 3b, www.viennot.org/abjc2-ch3.html

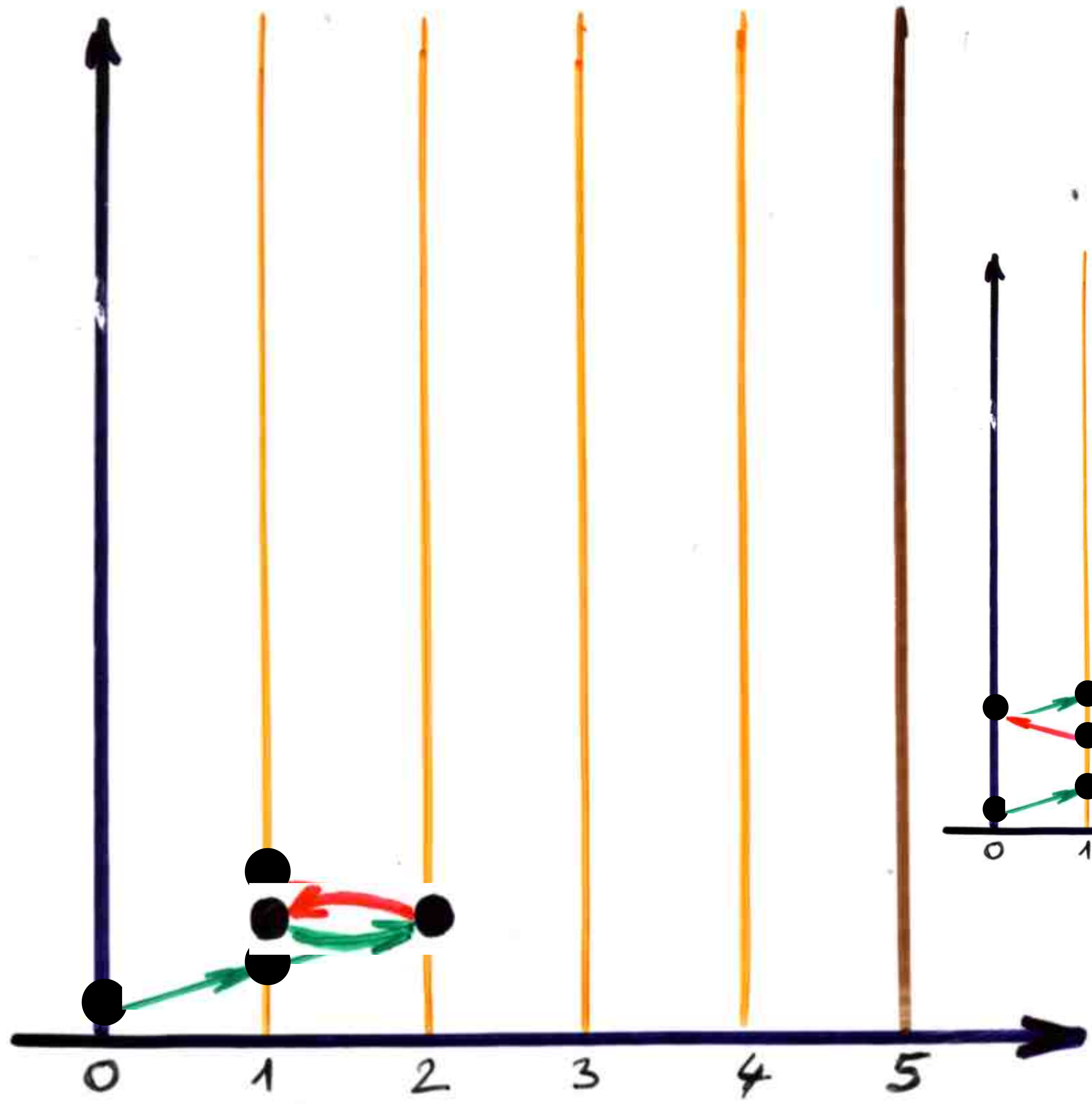
- Description of the bijection paths — heaps, pp 26-39
- LERW, pp 66-72
- Spanning trees, pp 73-79
- Wilson algorithm, pp 80-91 and Chapter 5b, pp 66-79

slide added after the talk

Examples with Dyck paths



$$D_{2n}^{(k)} = C_{2n+k}^{(k)}(0 \to k)$$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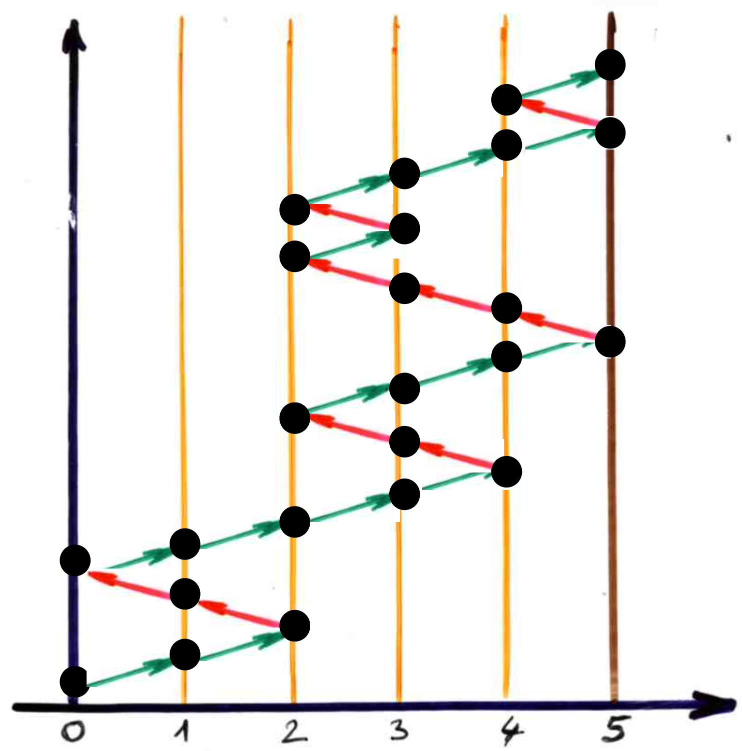
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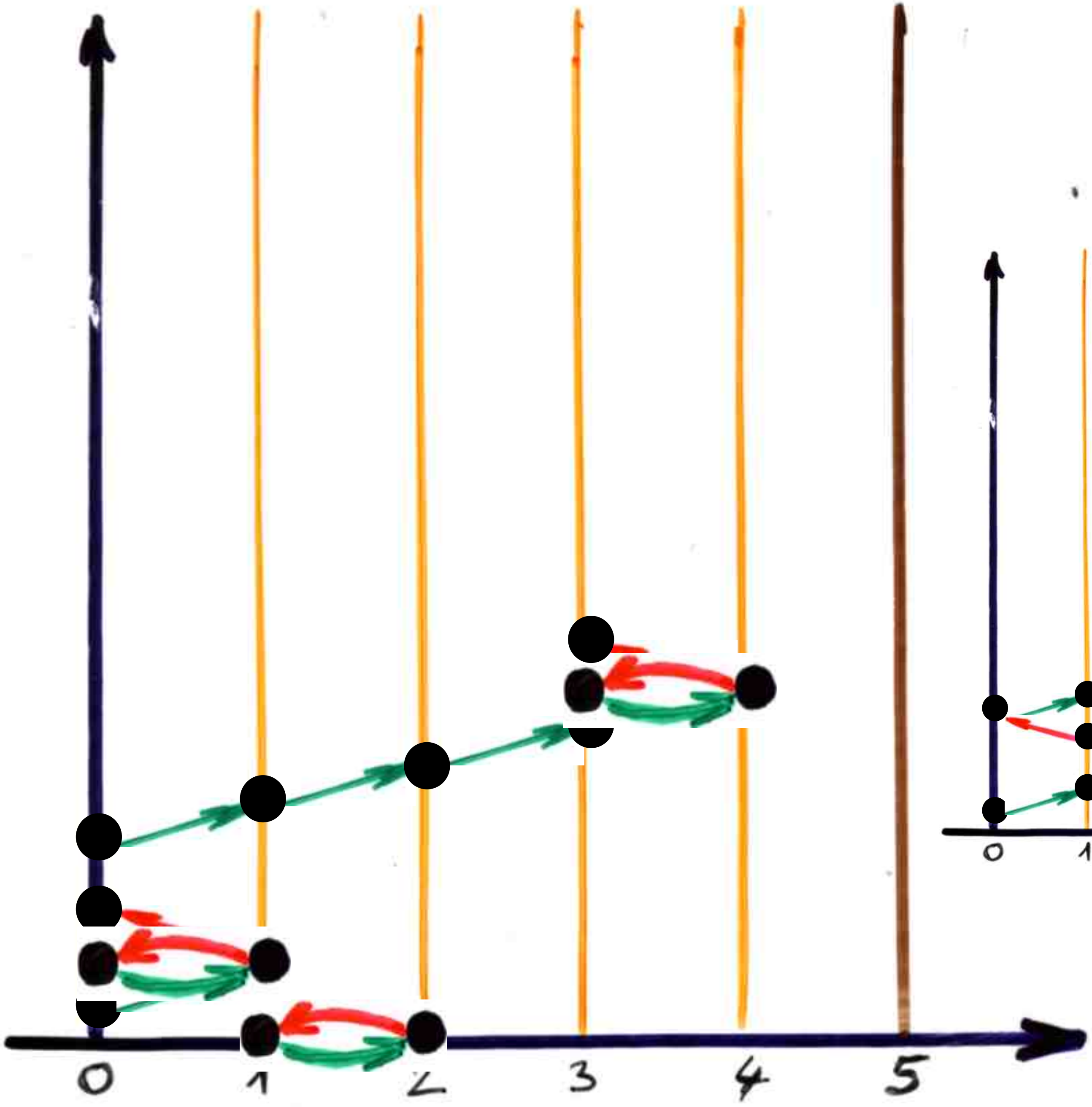
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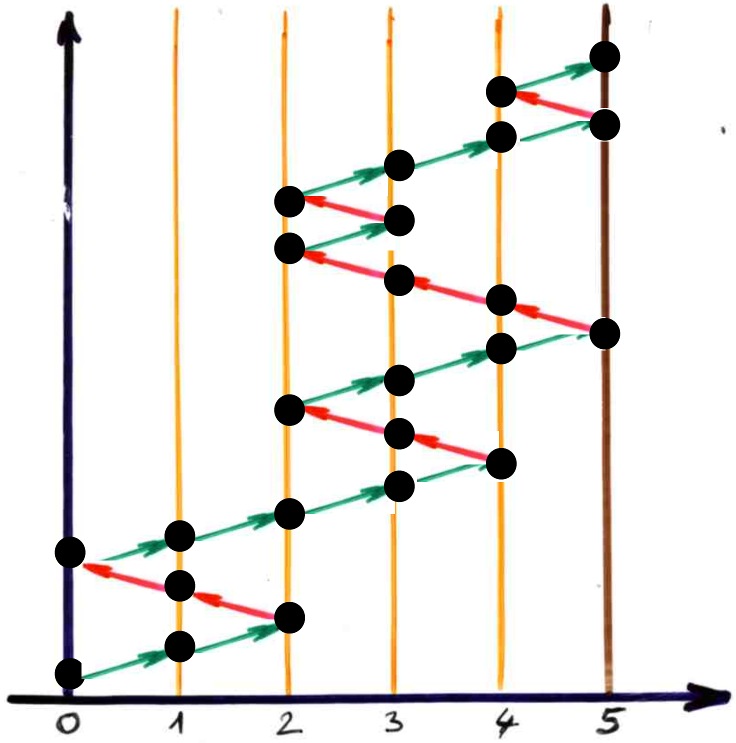
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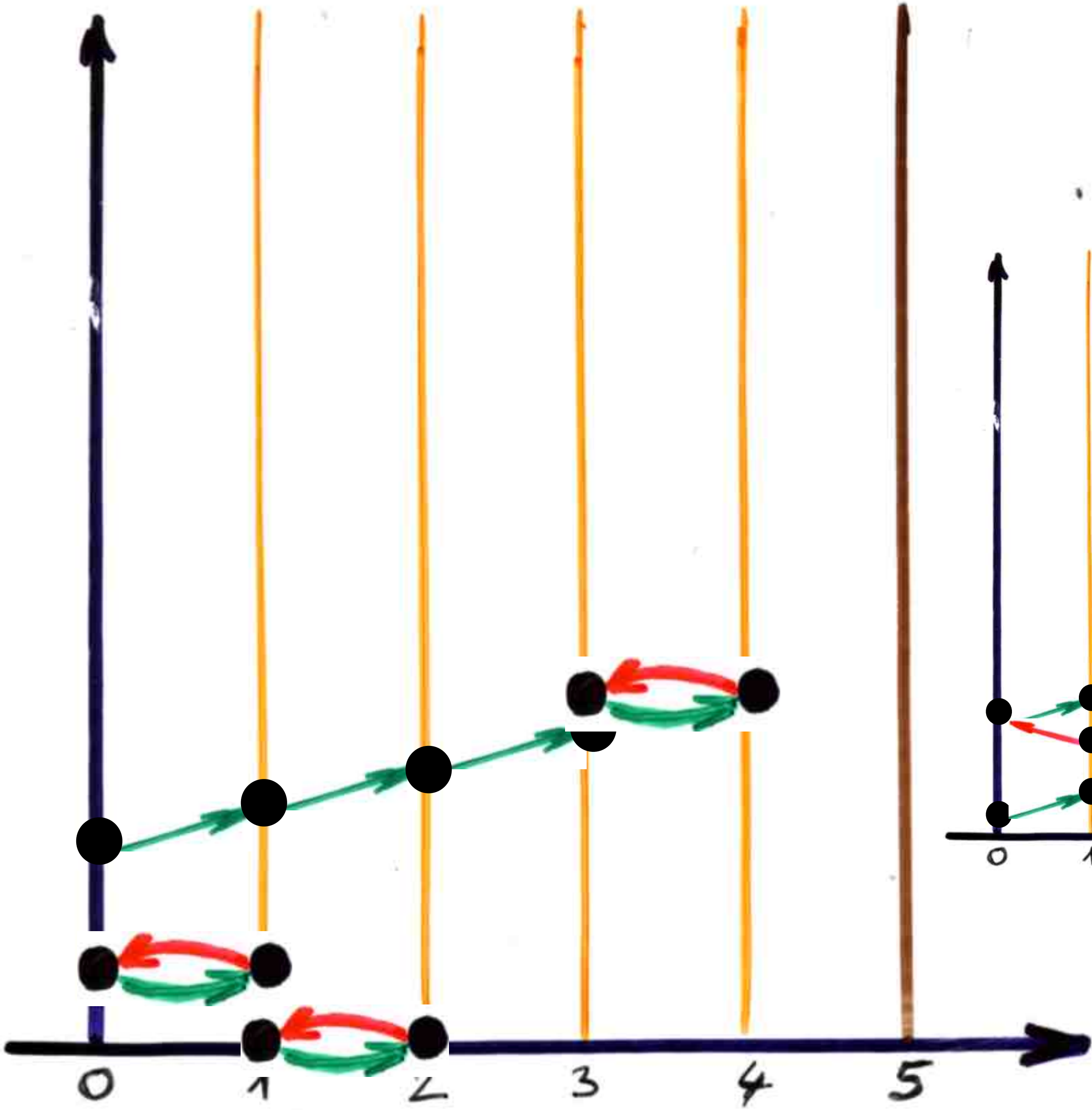

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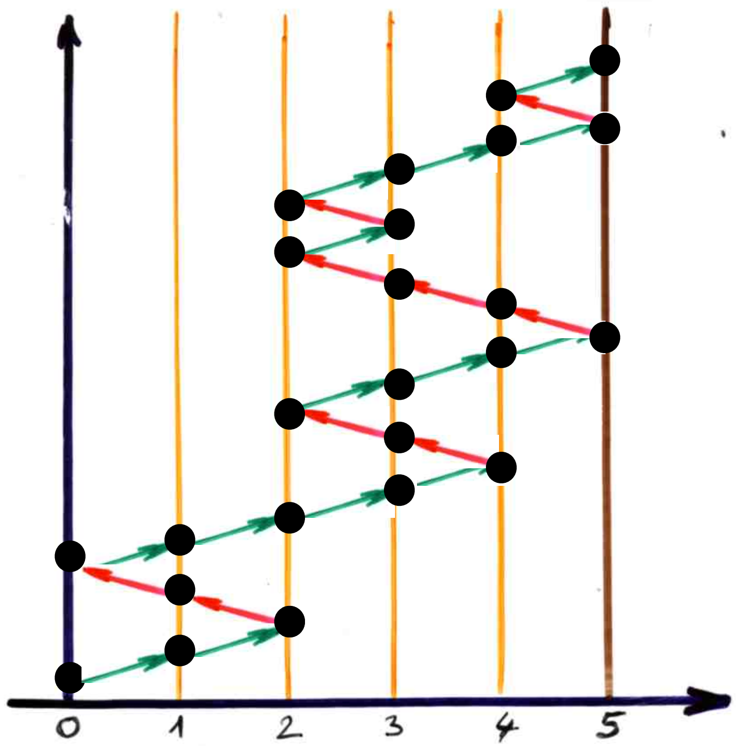

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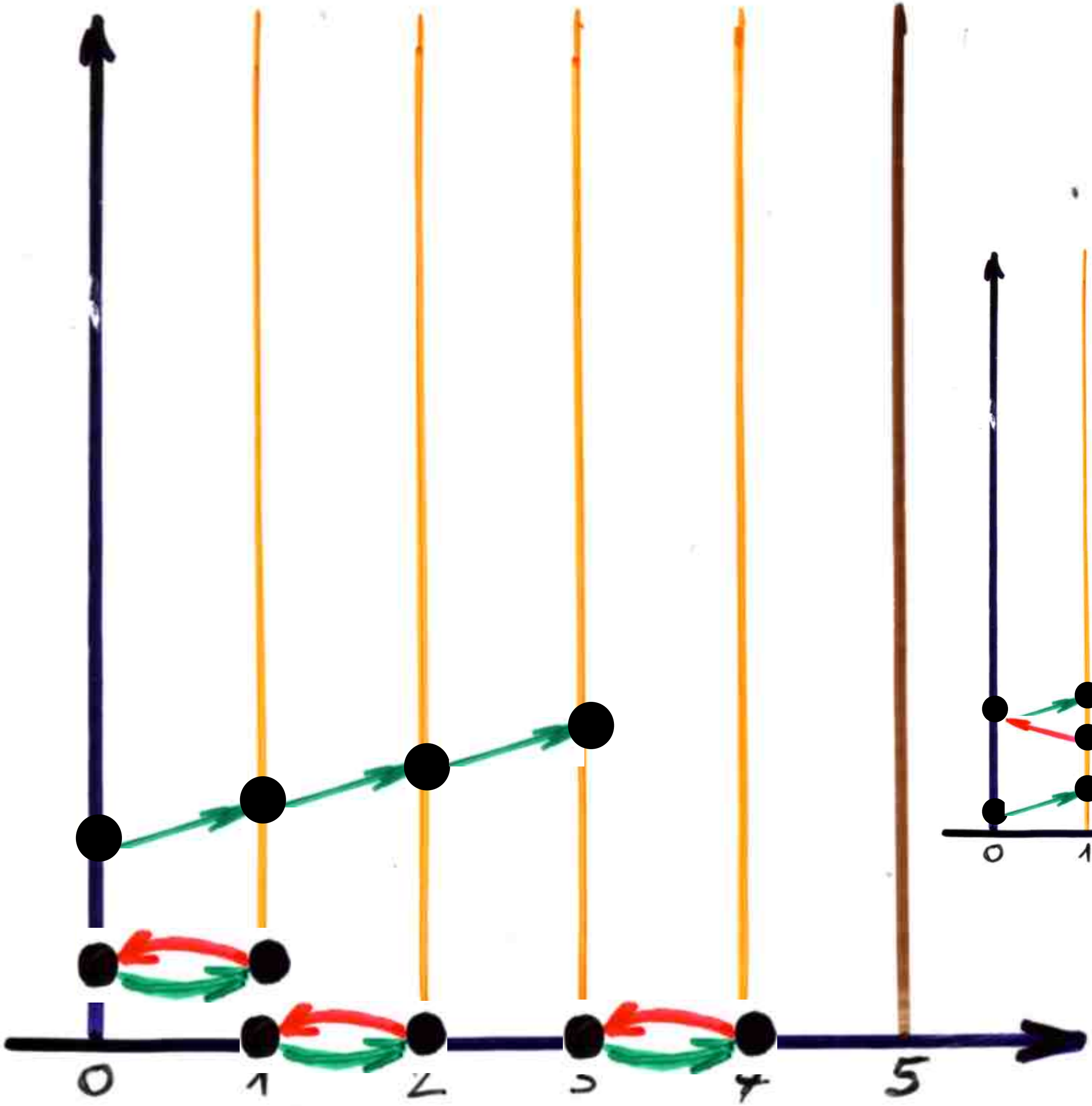
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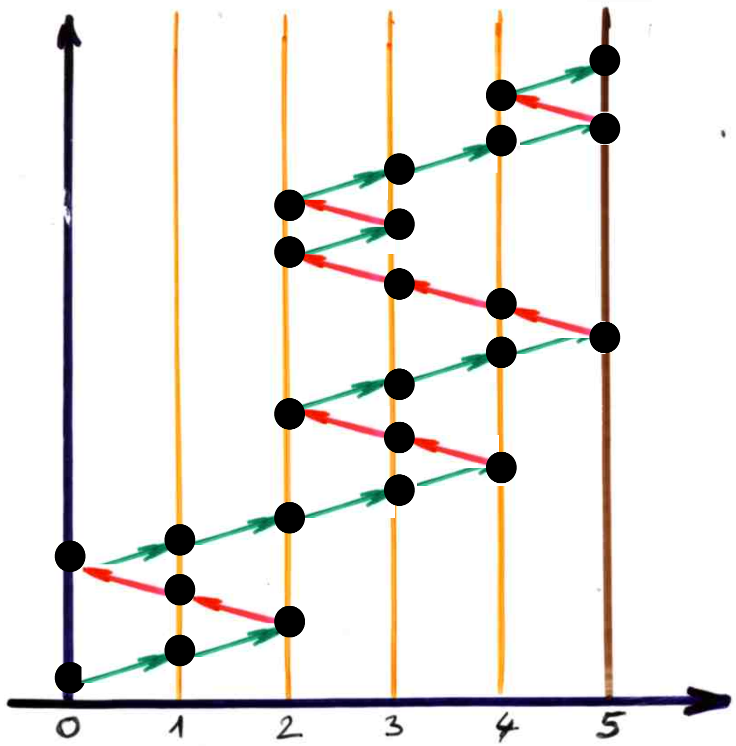
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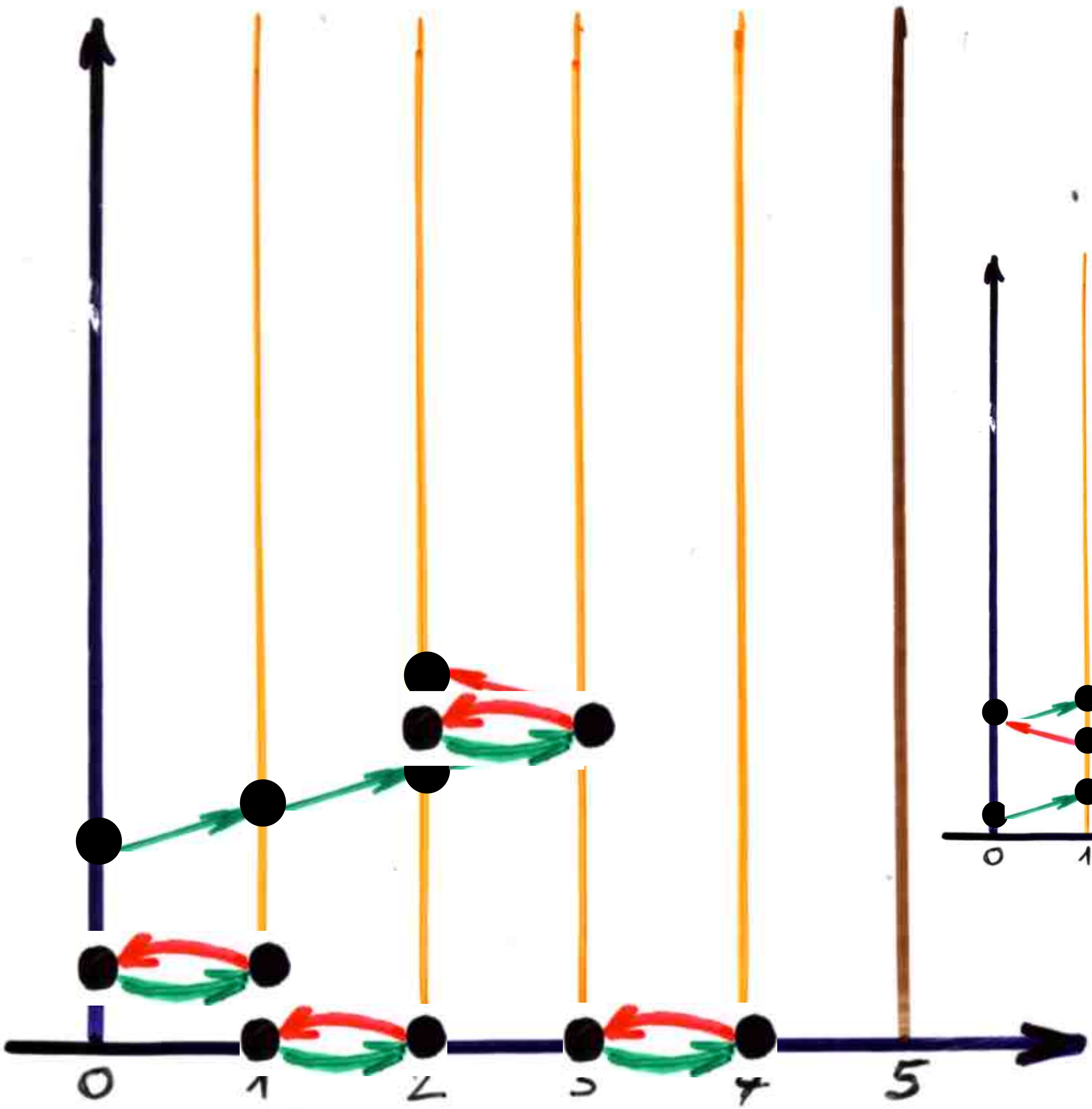
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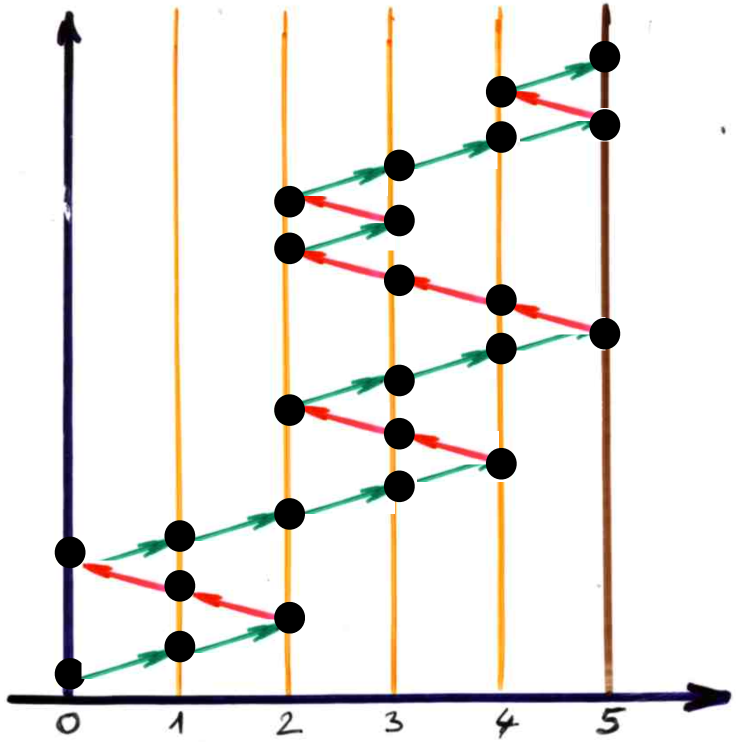
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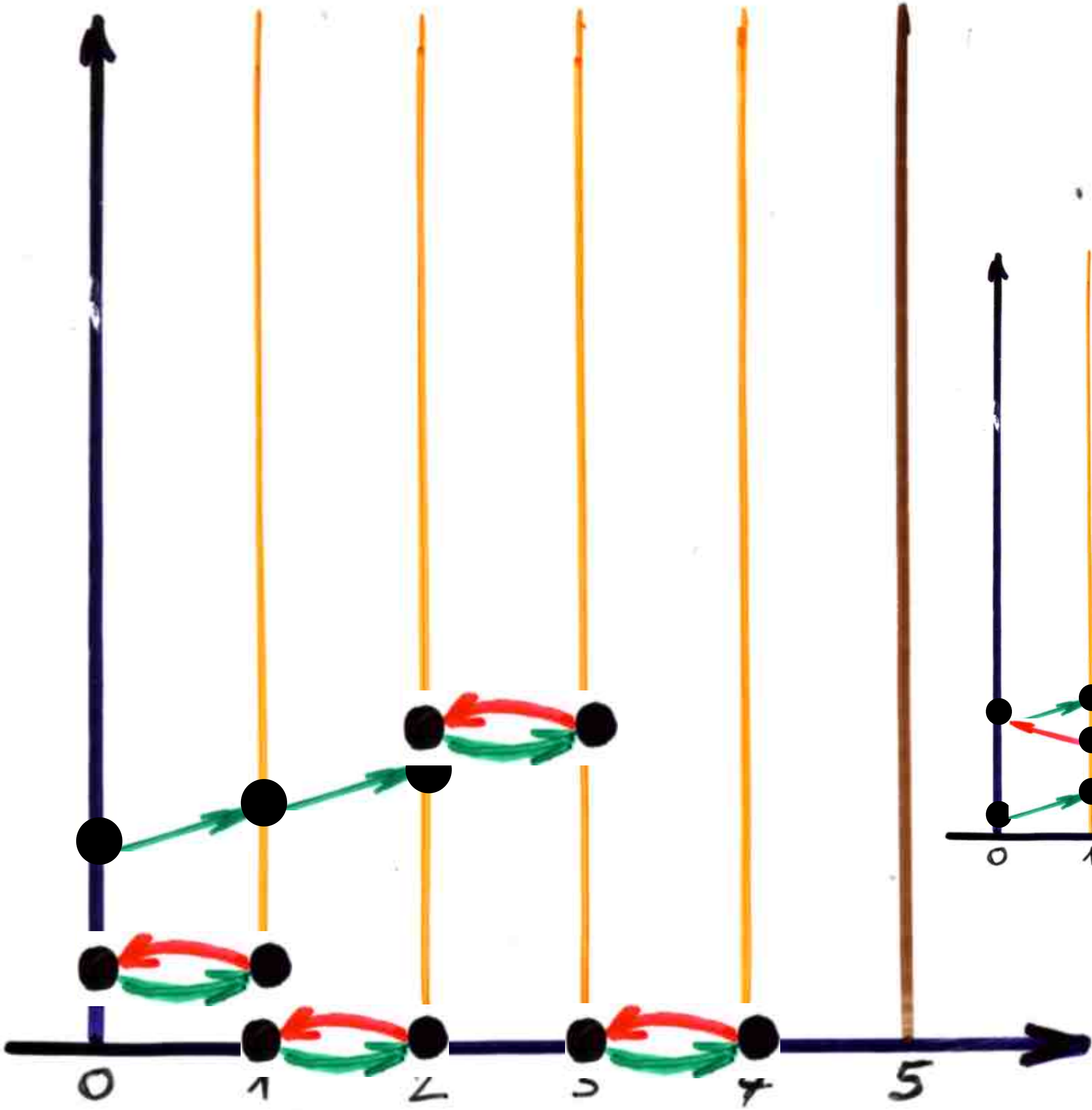
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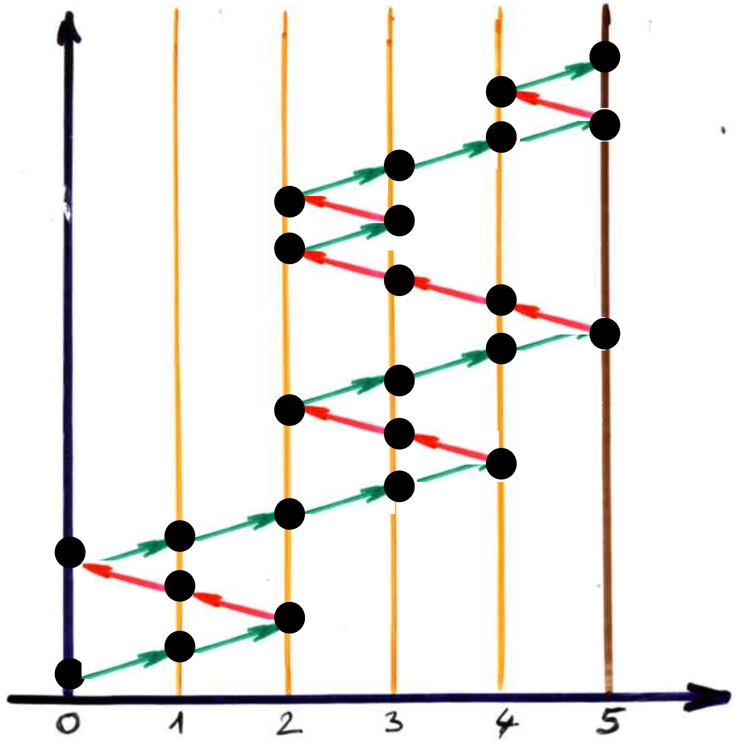
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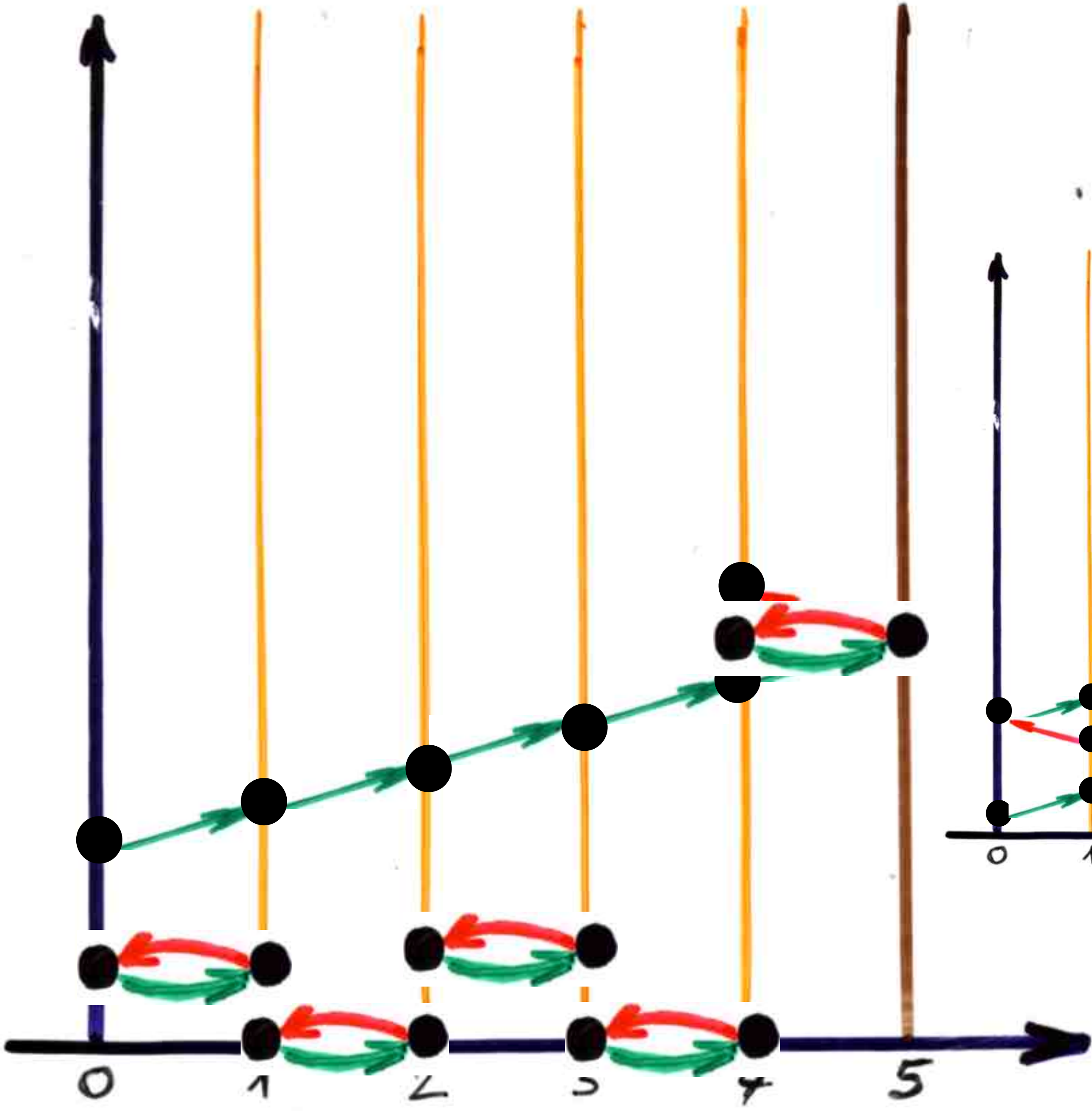
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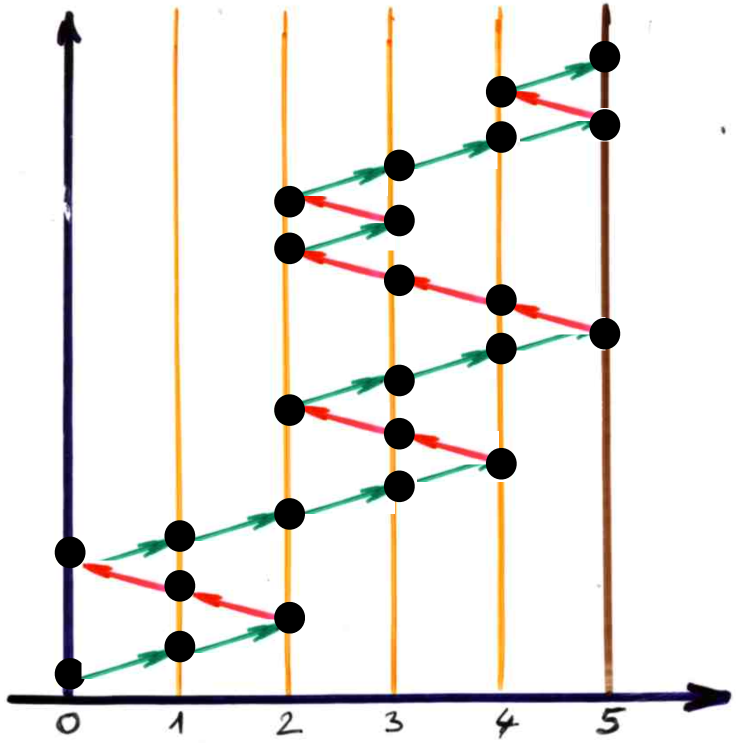
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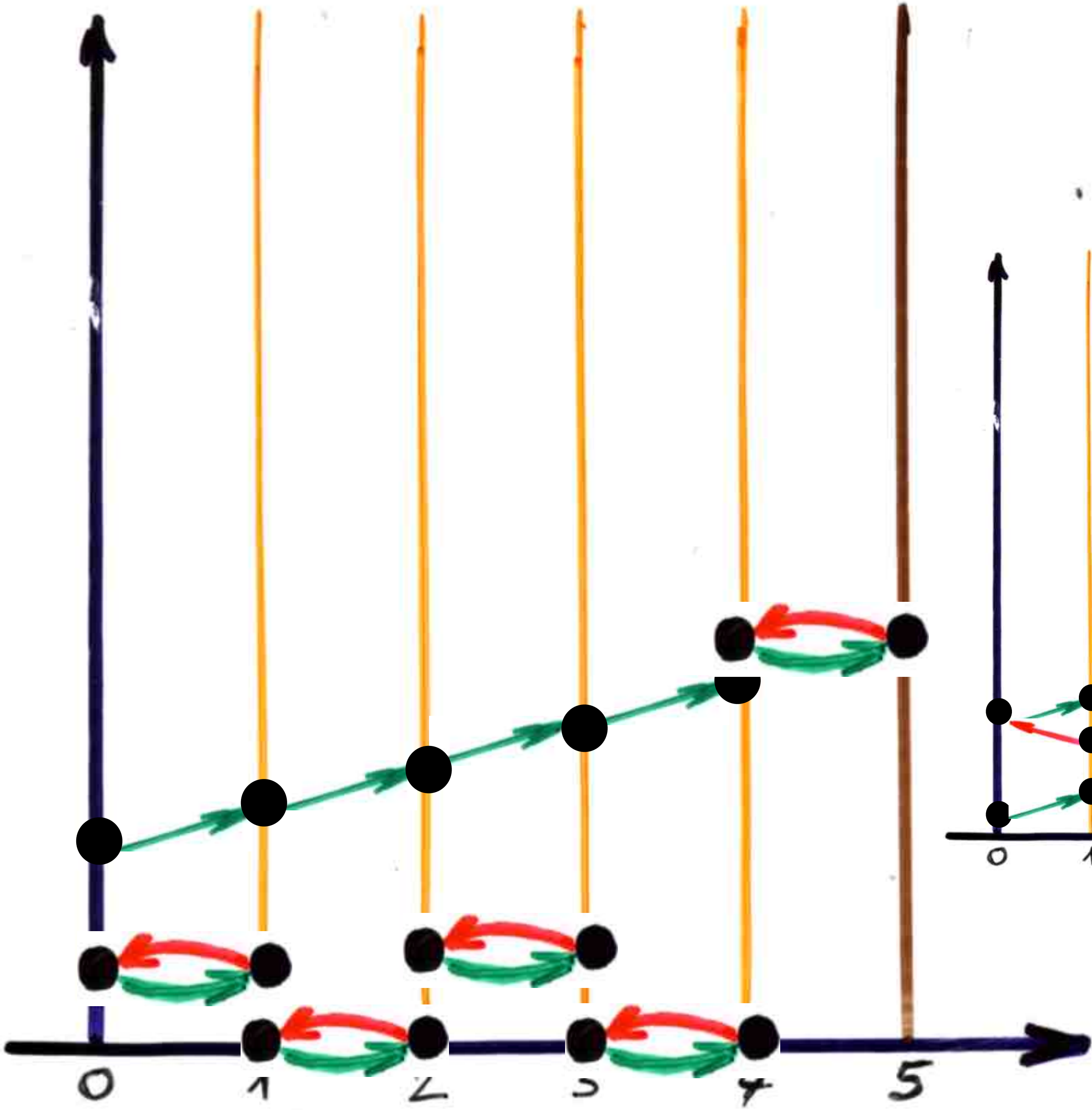
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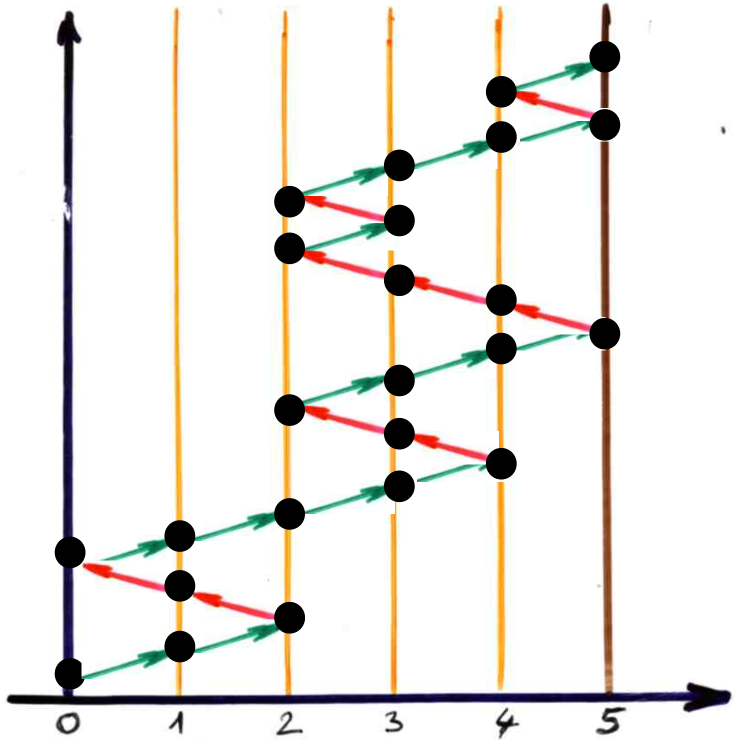
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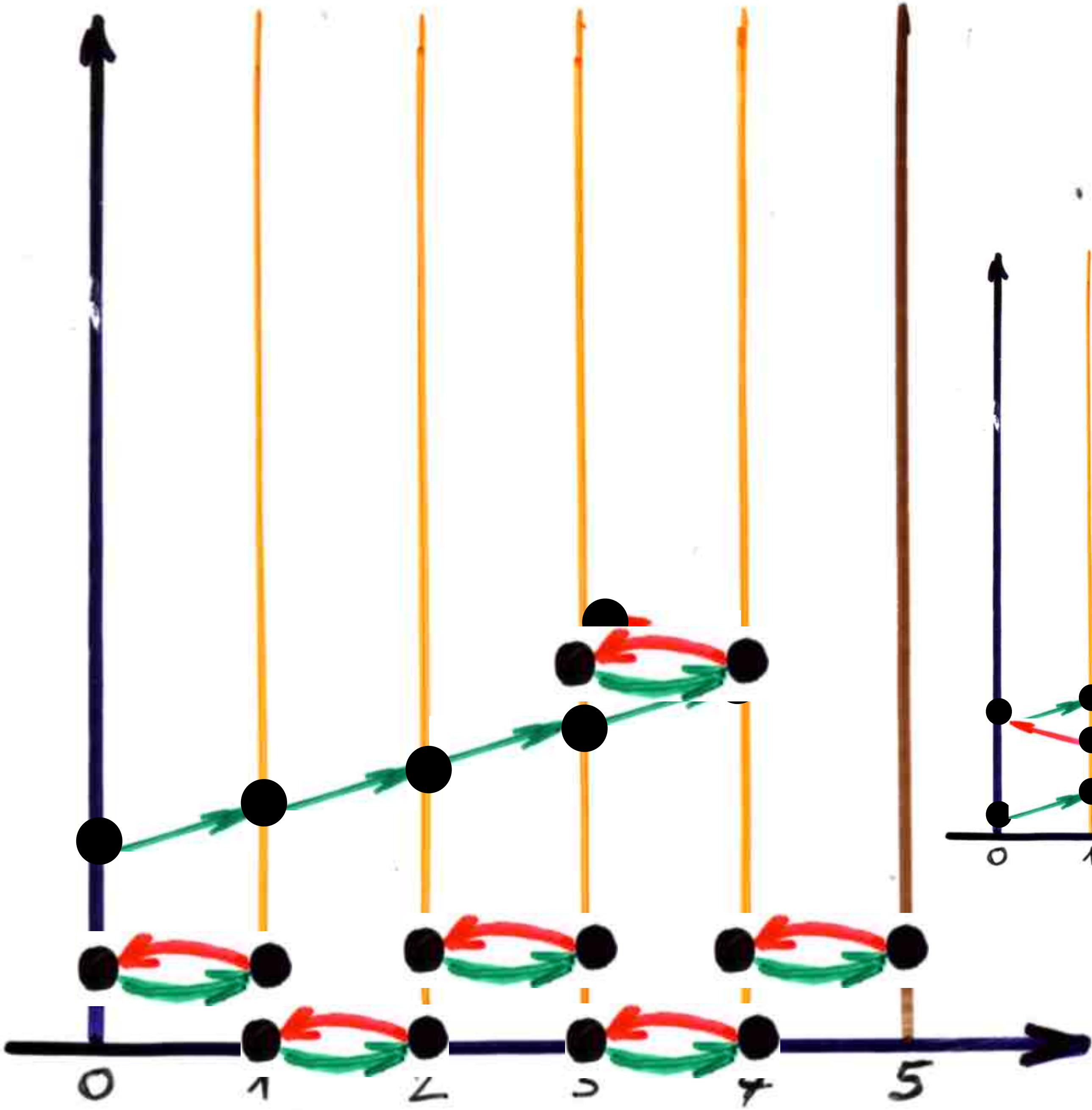
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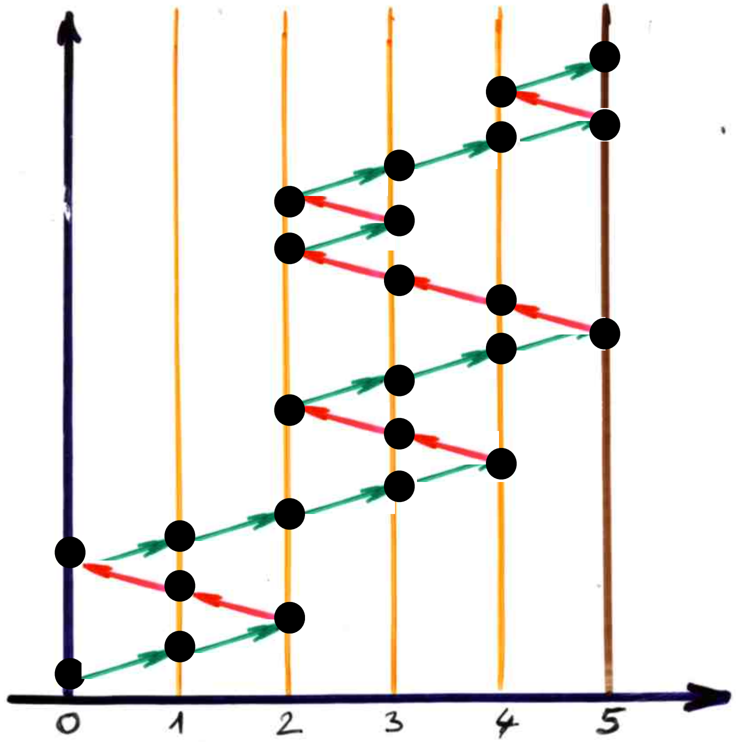
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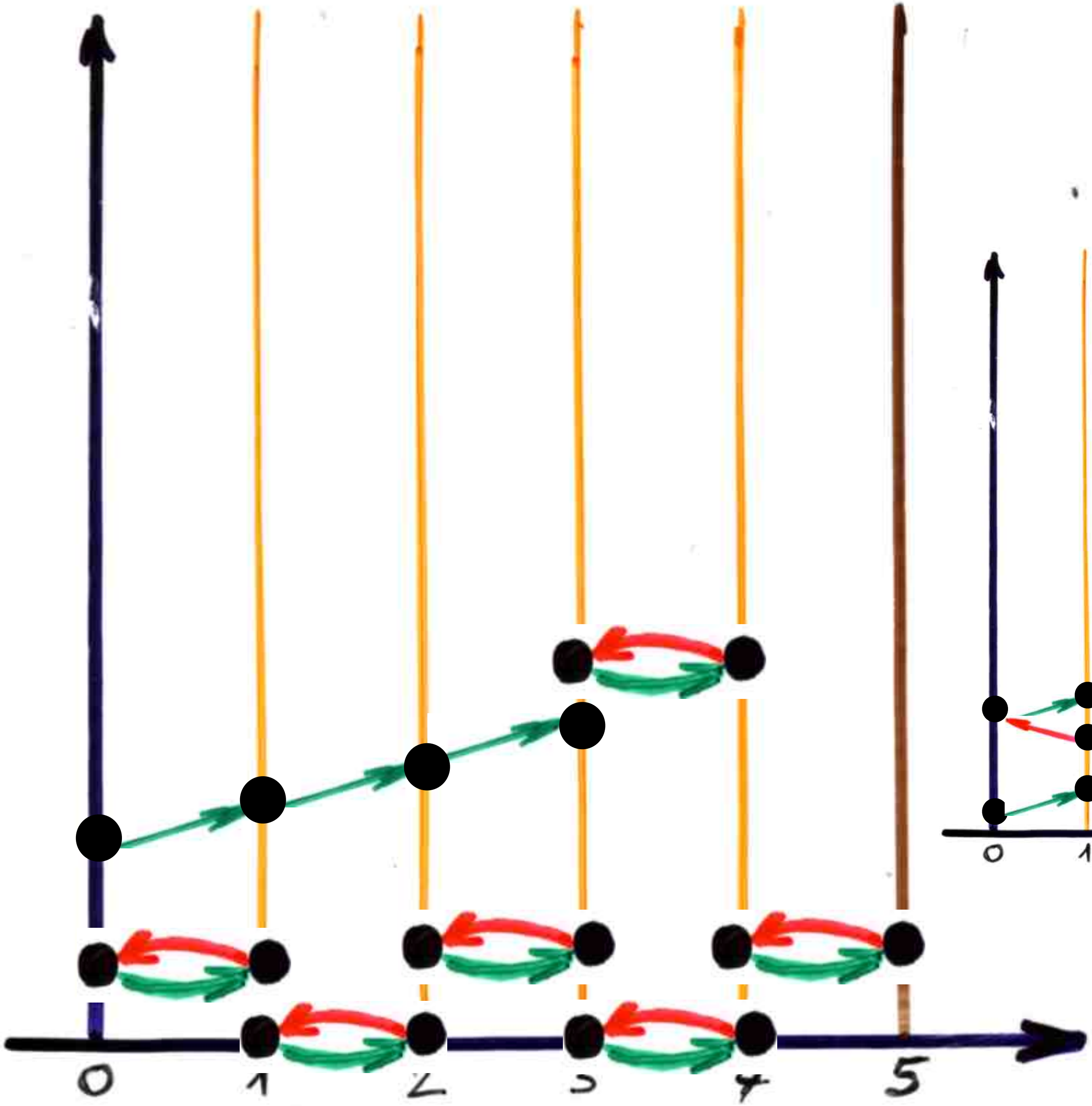
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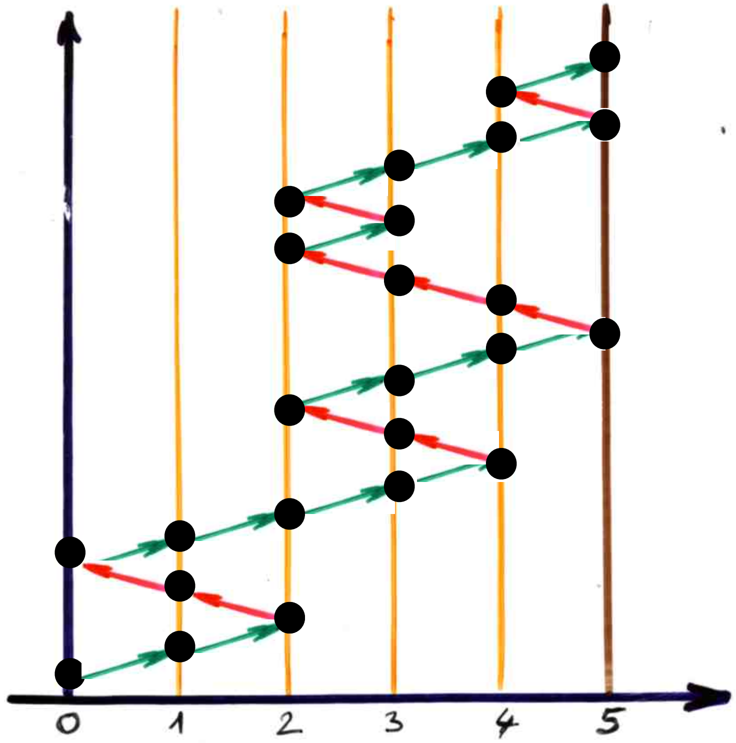
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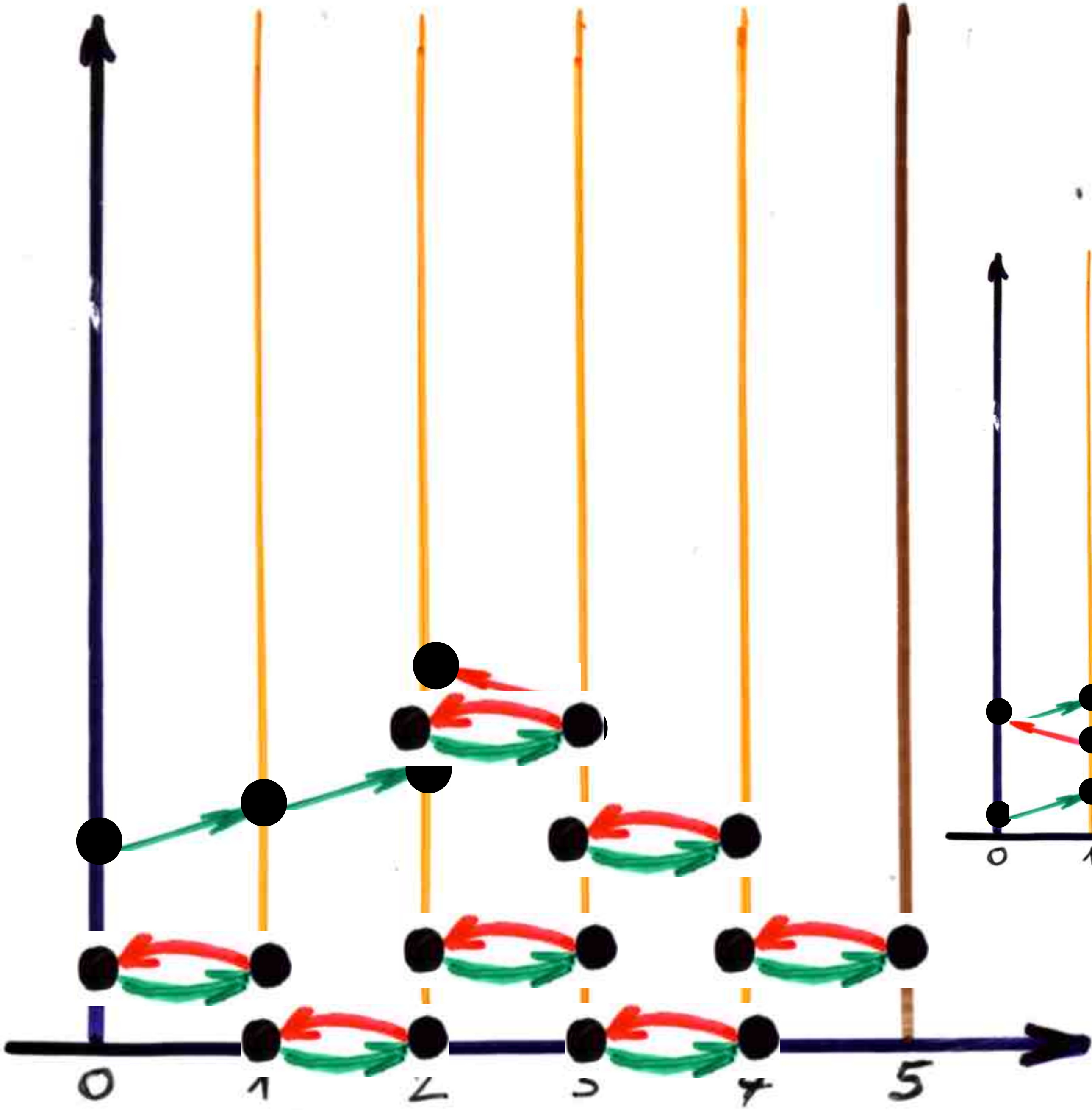
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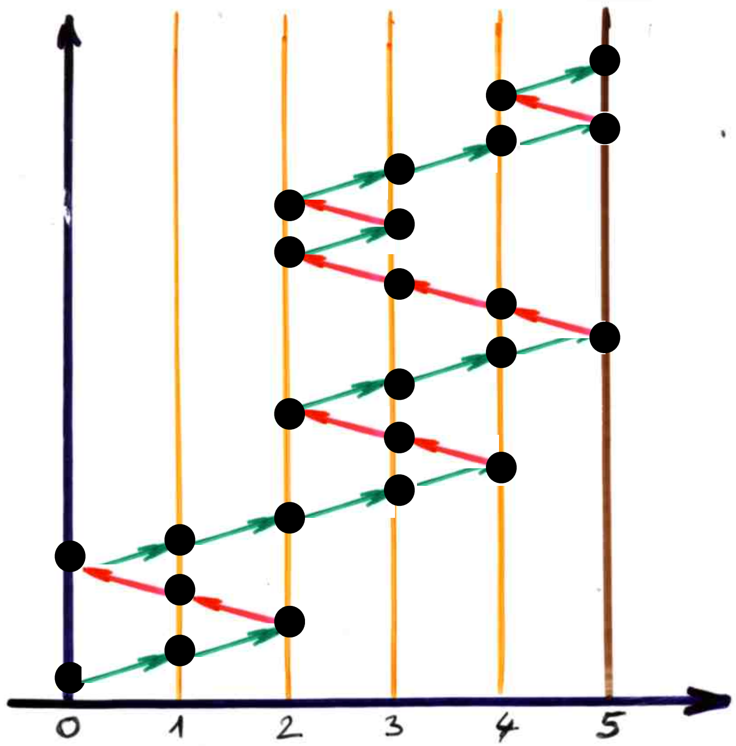
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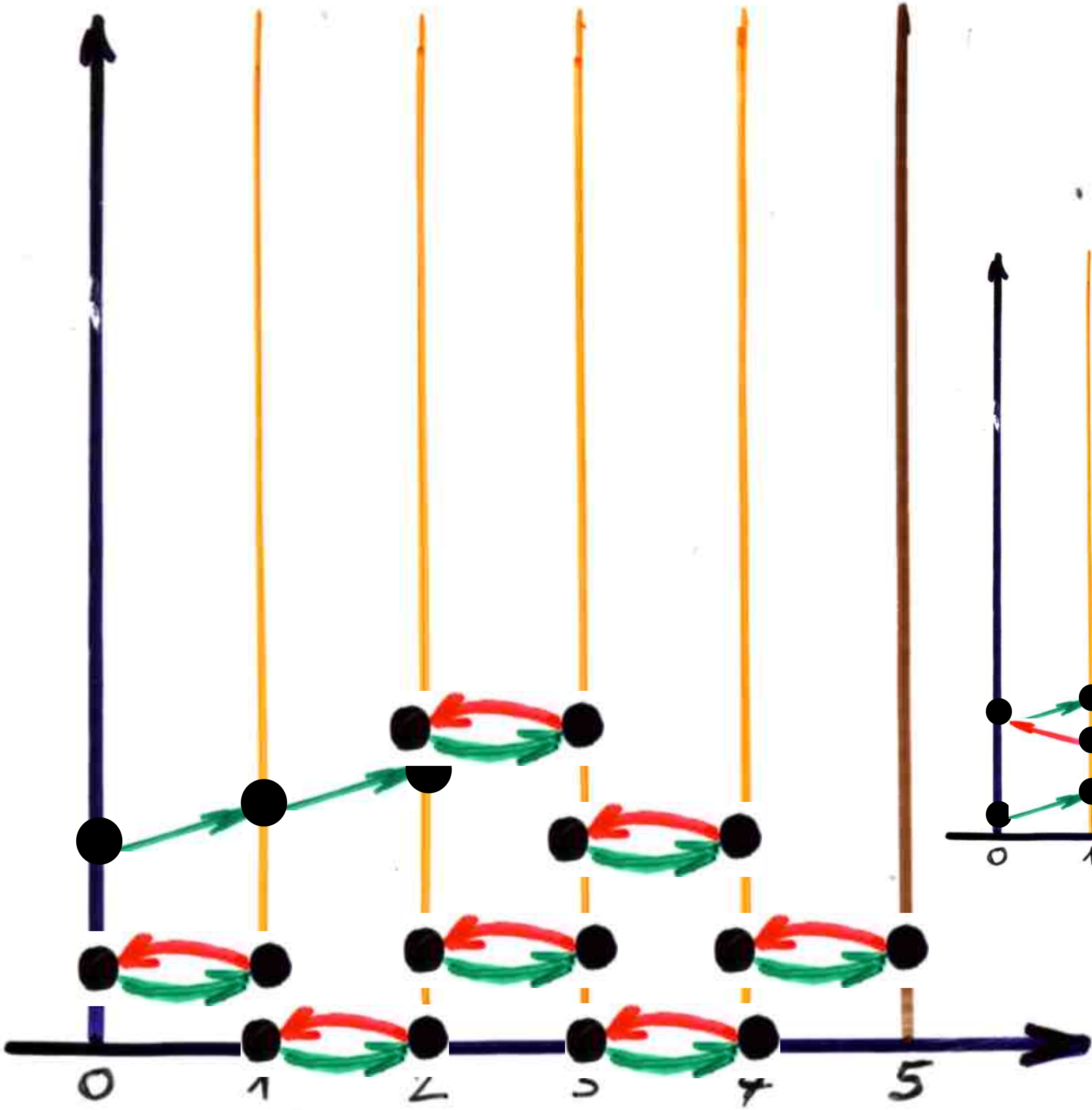
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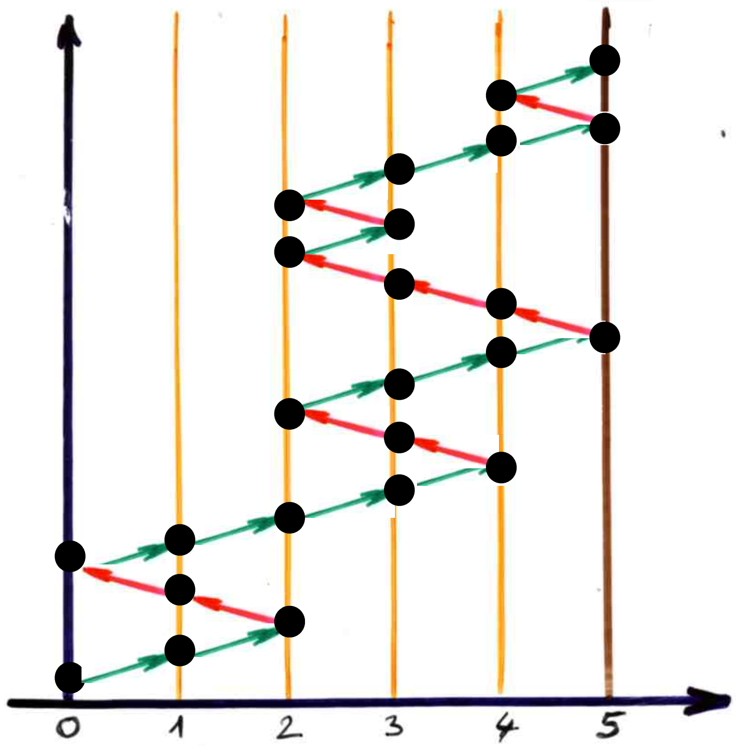
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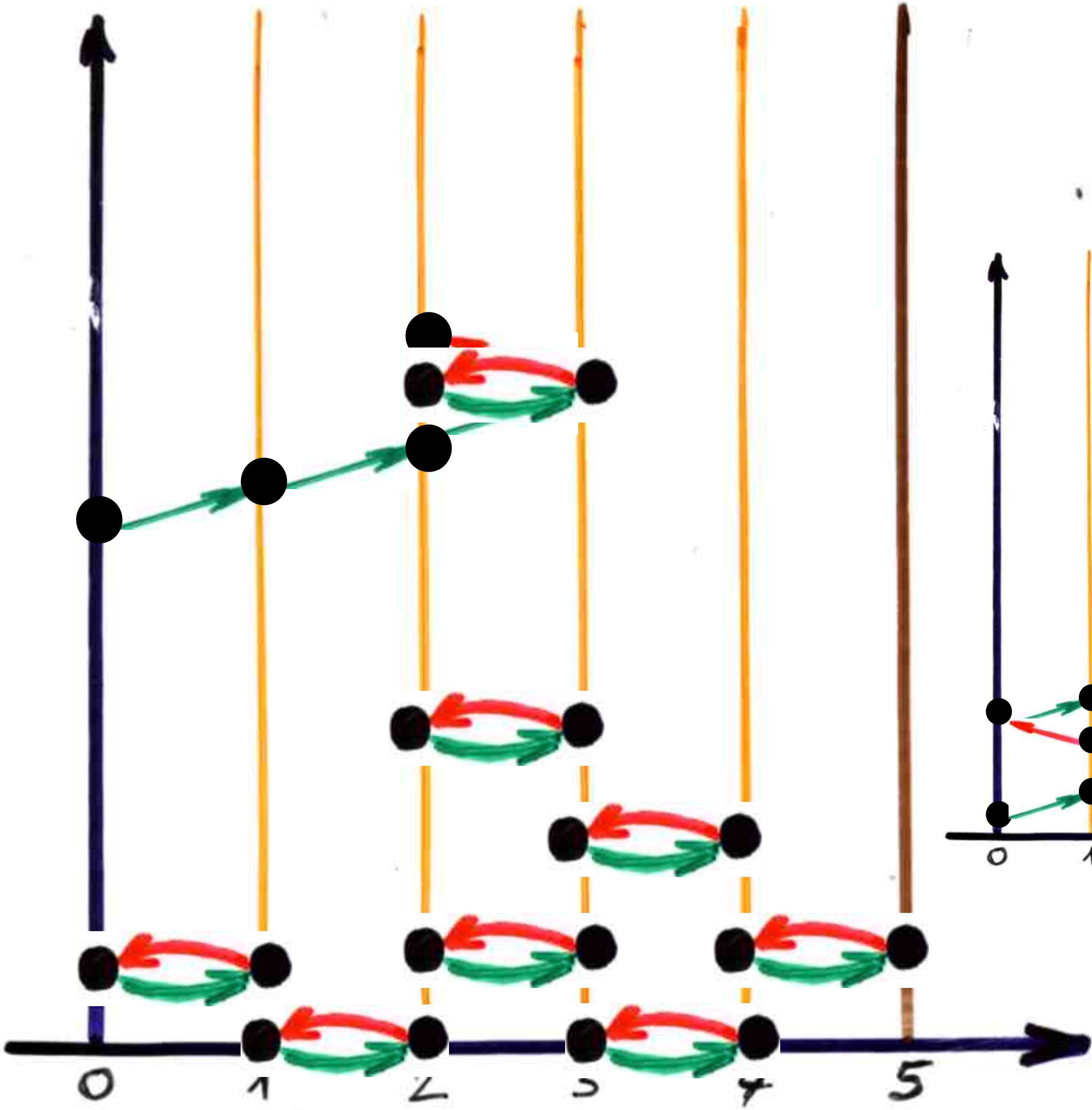
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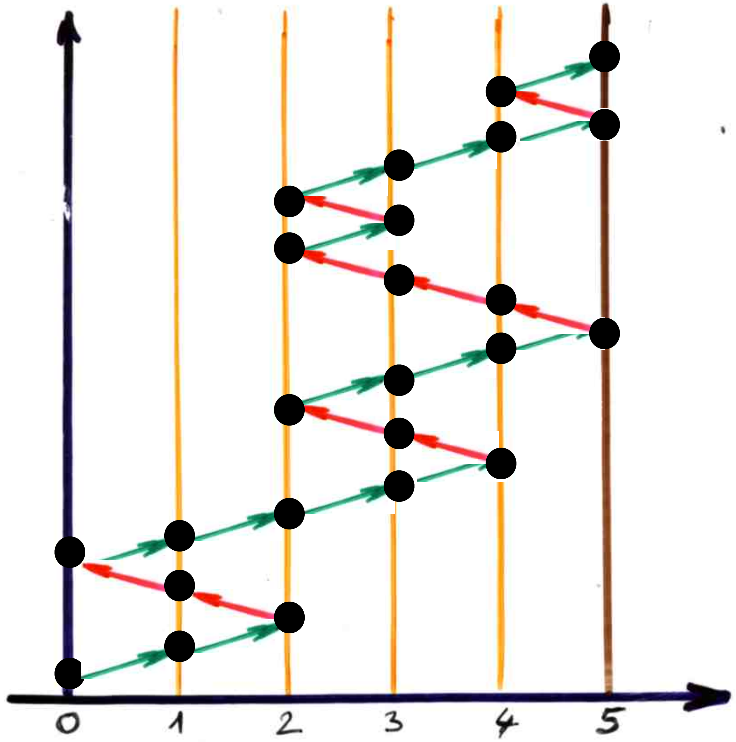
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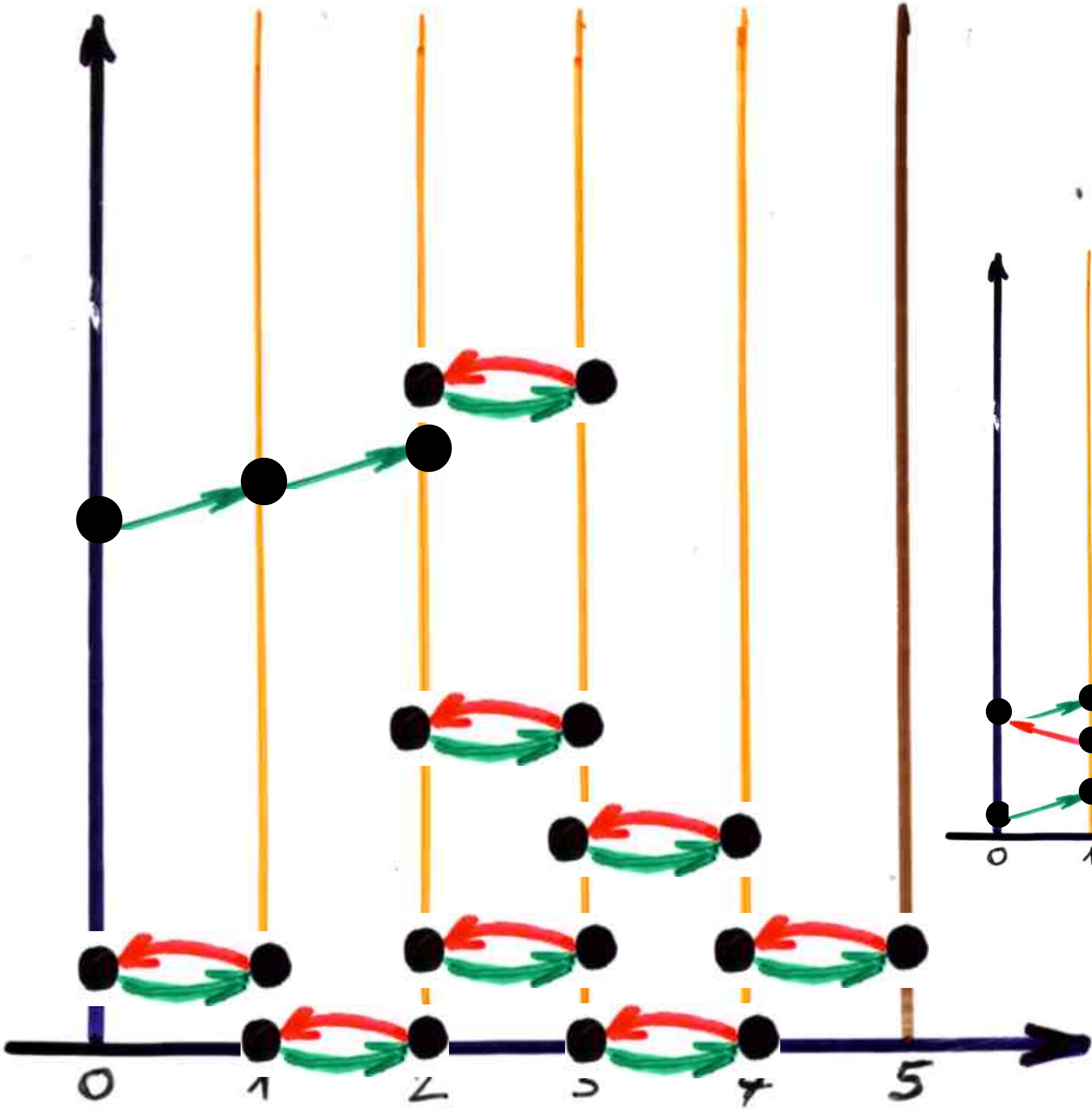
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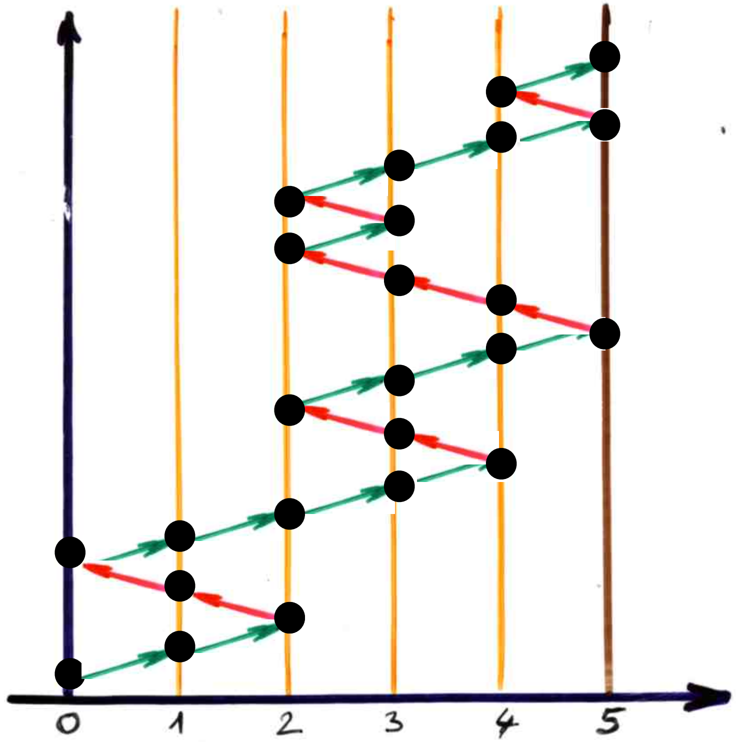
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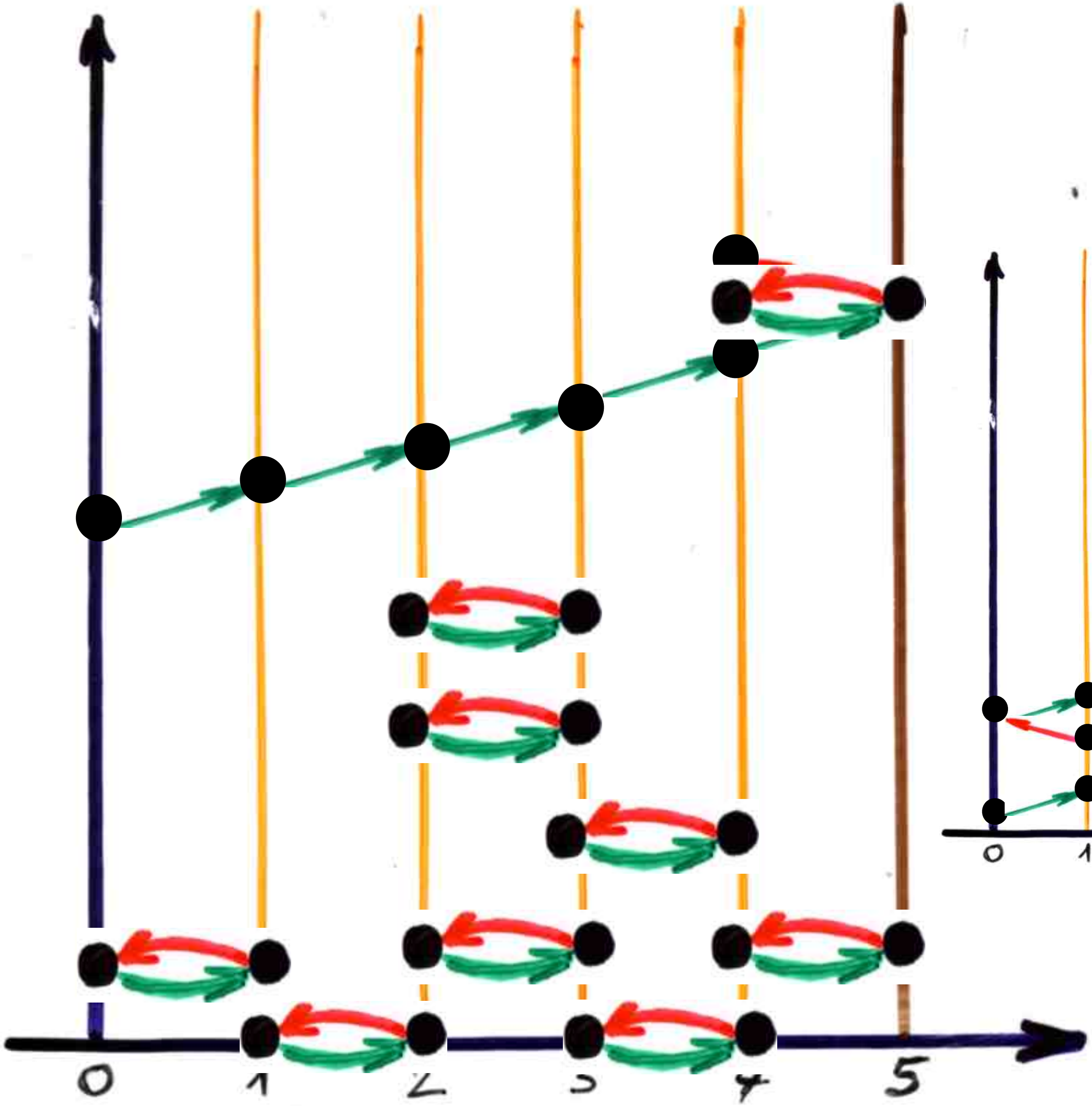
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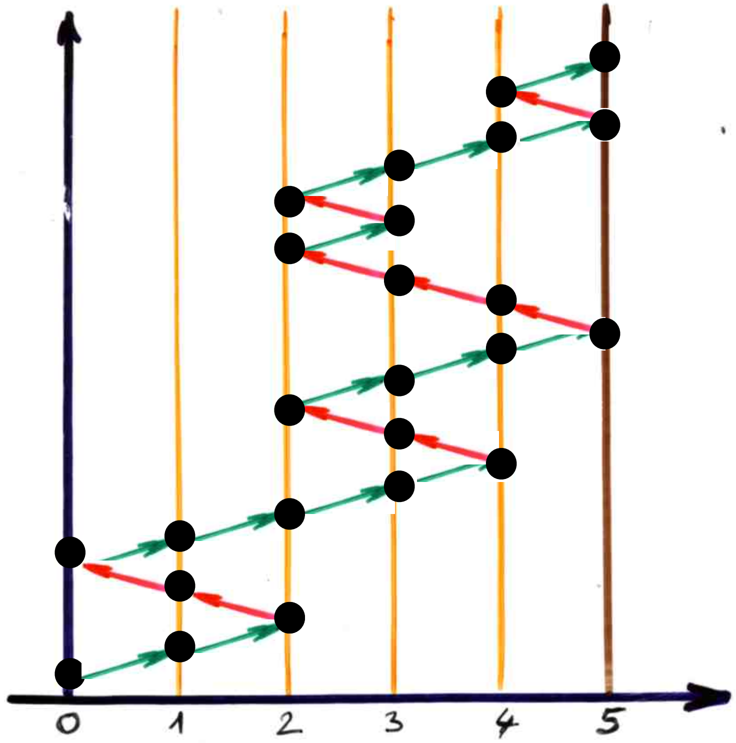
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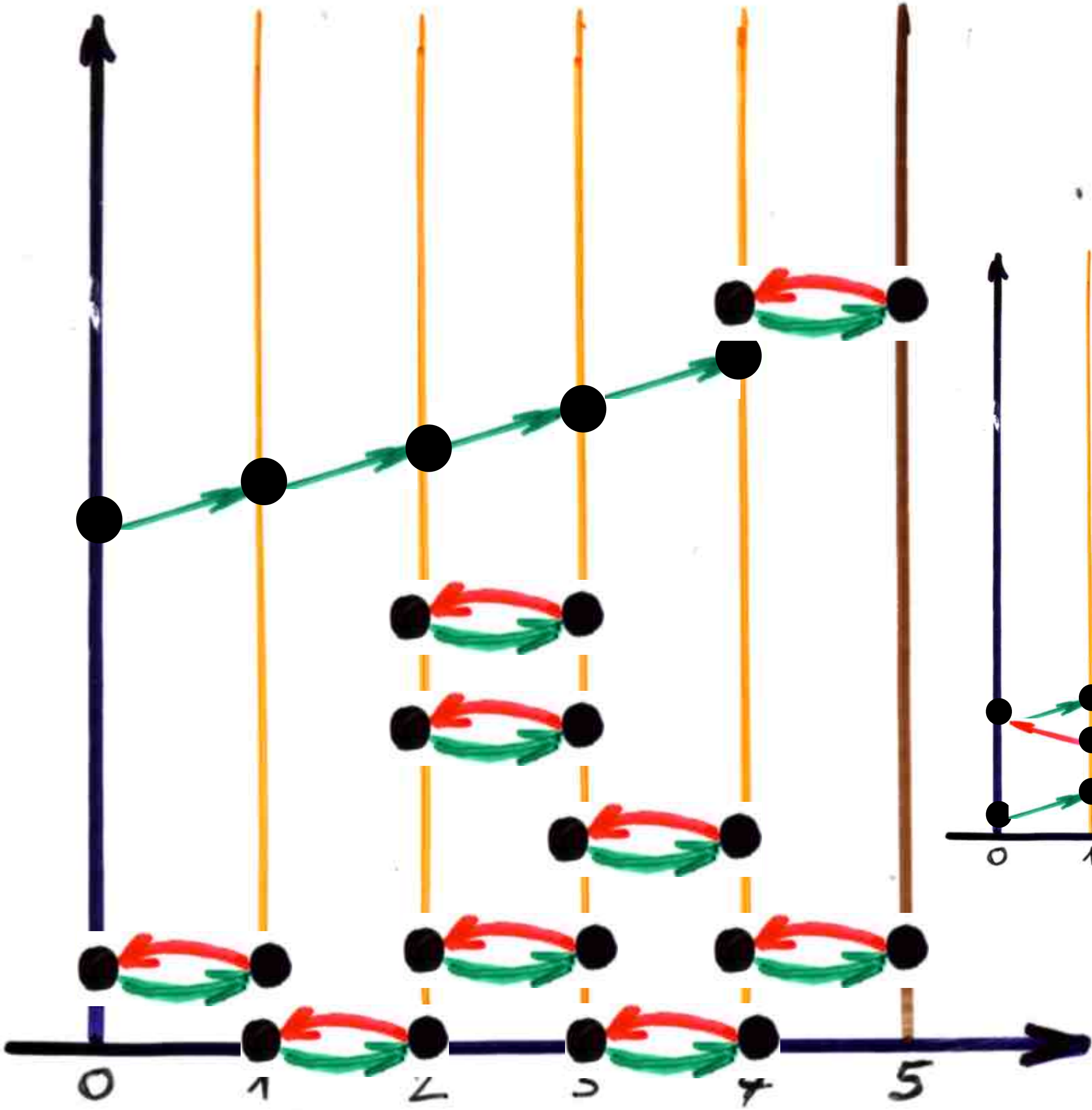
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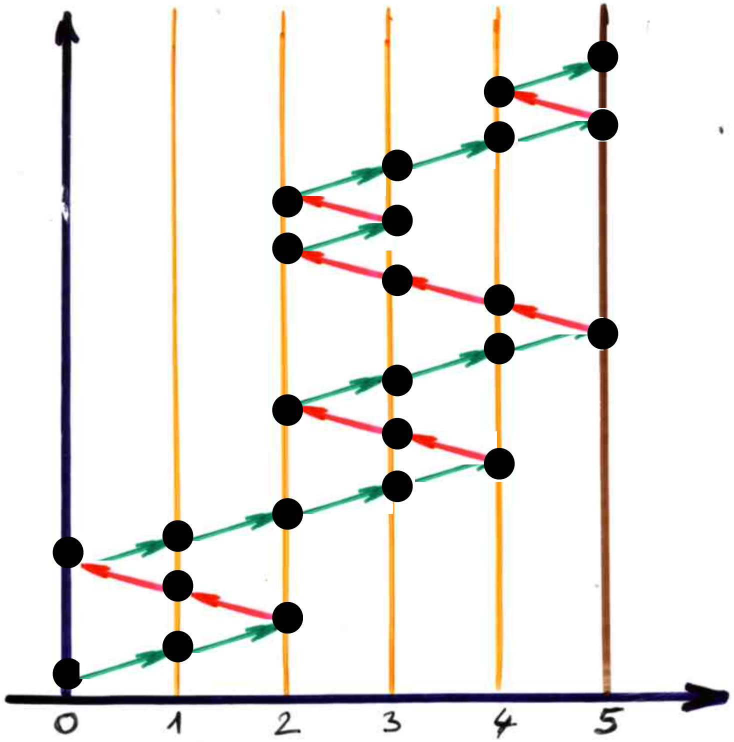
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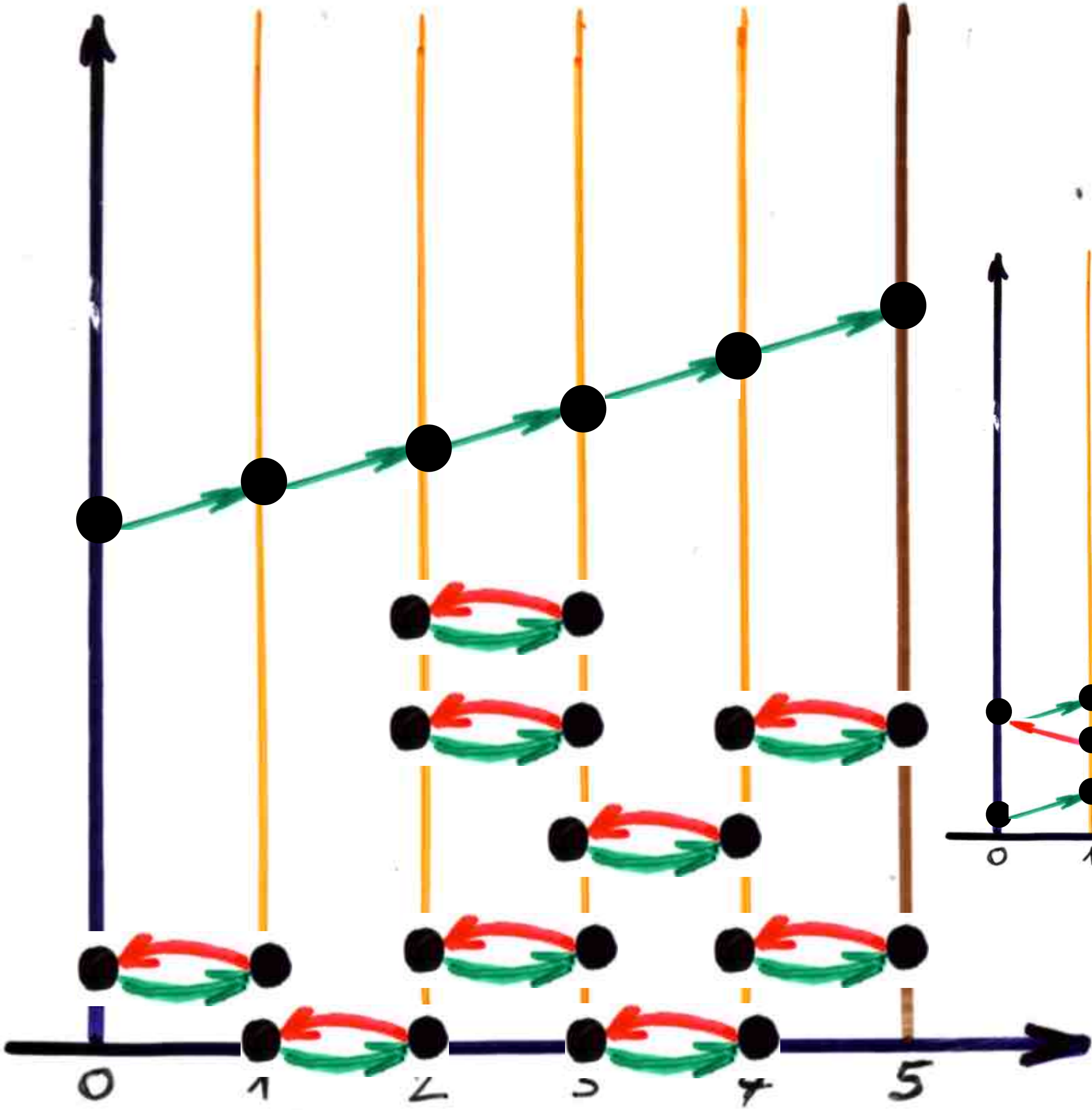
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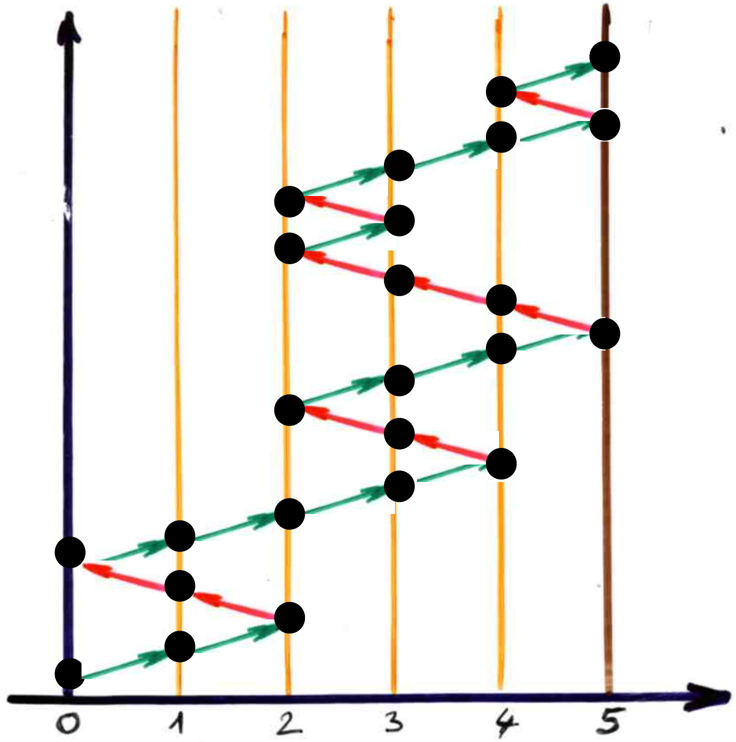
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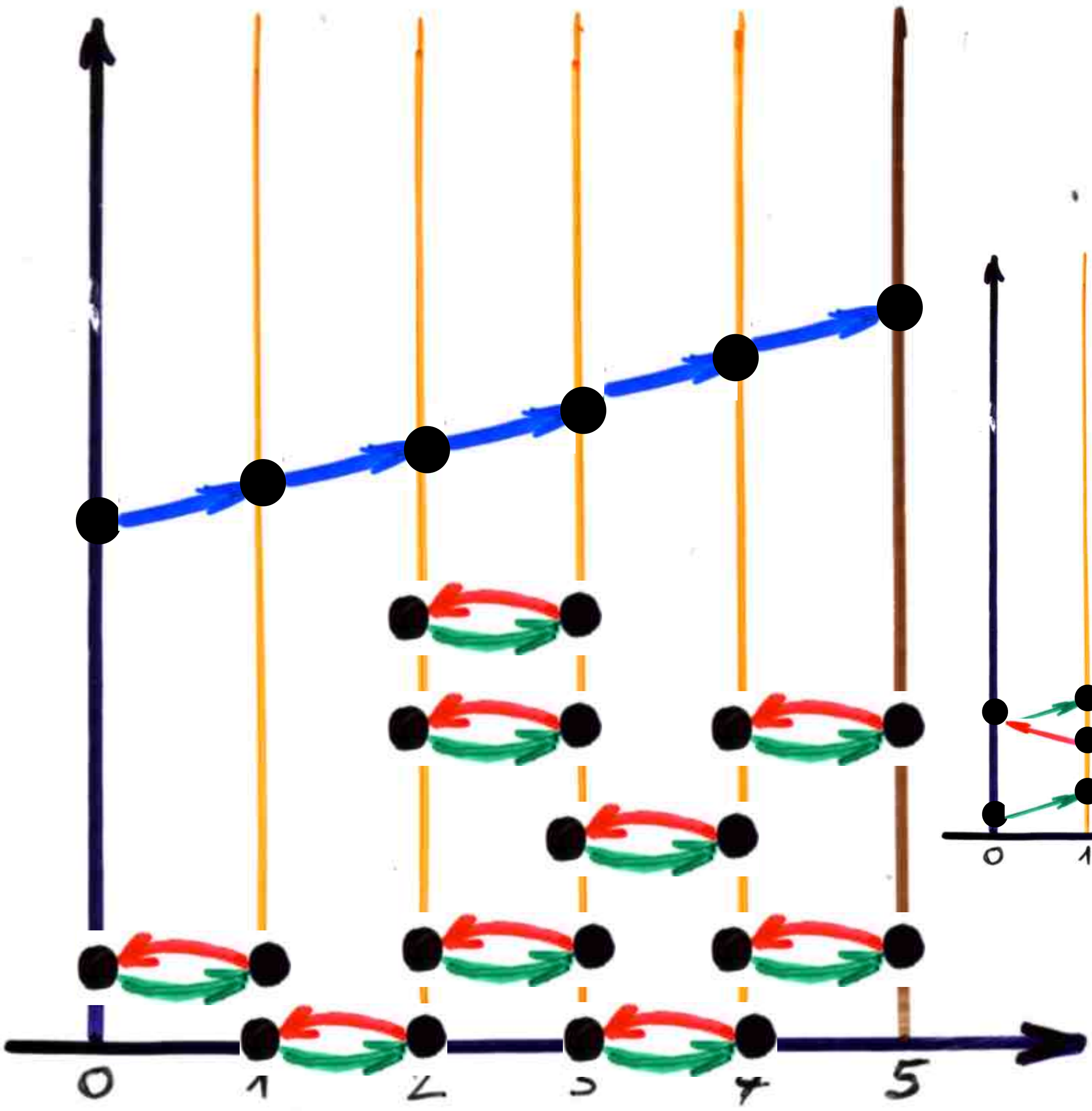
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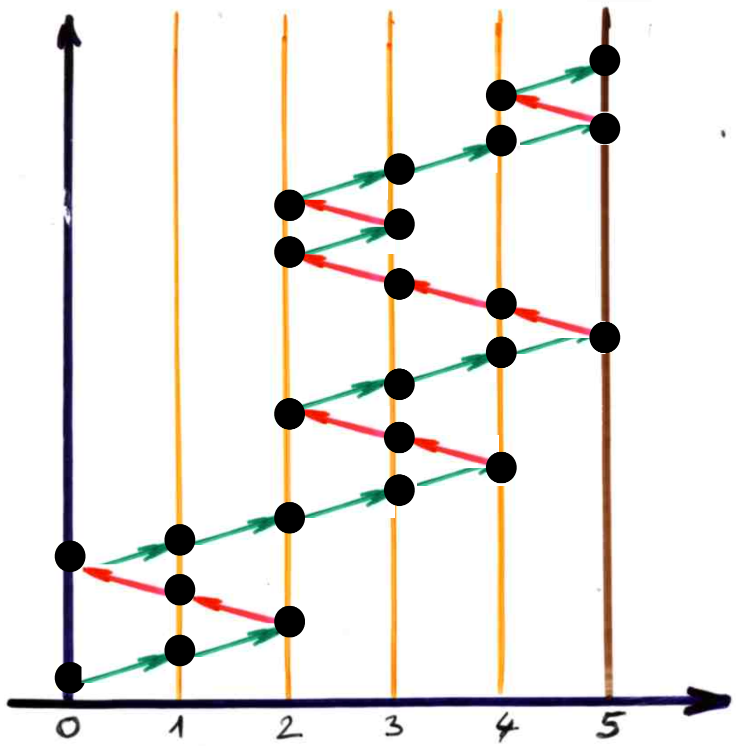
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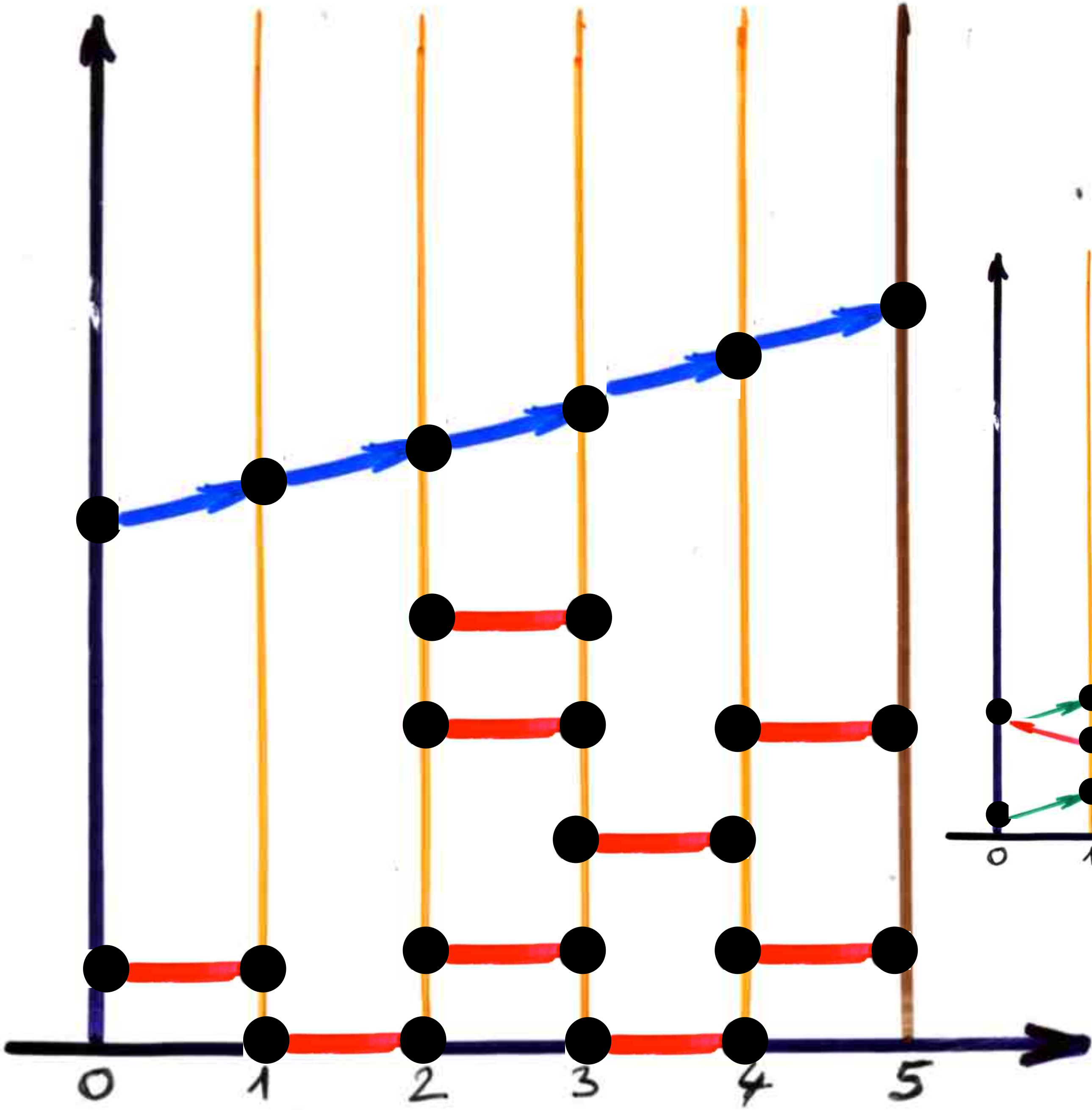
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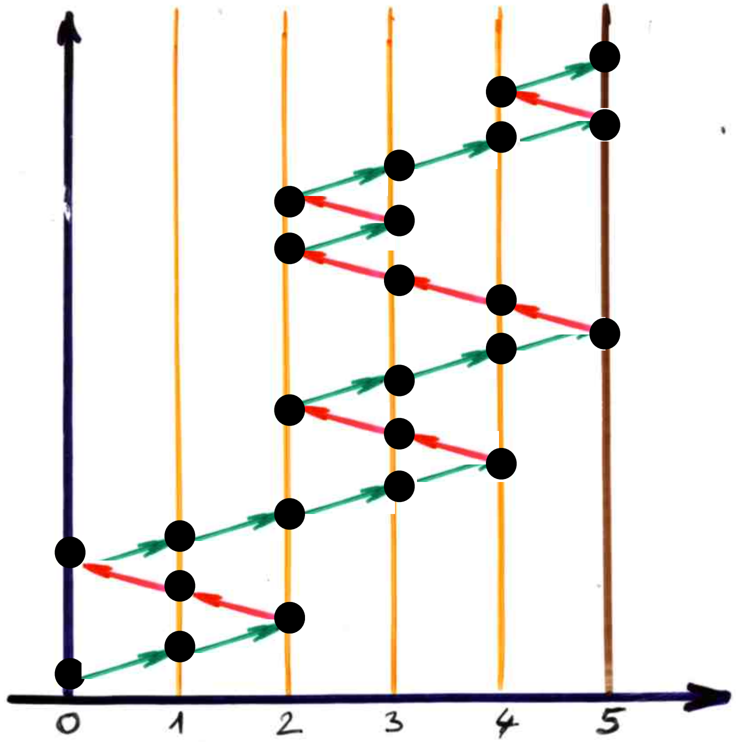
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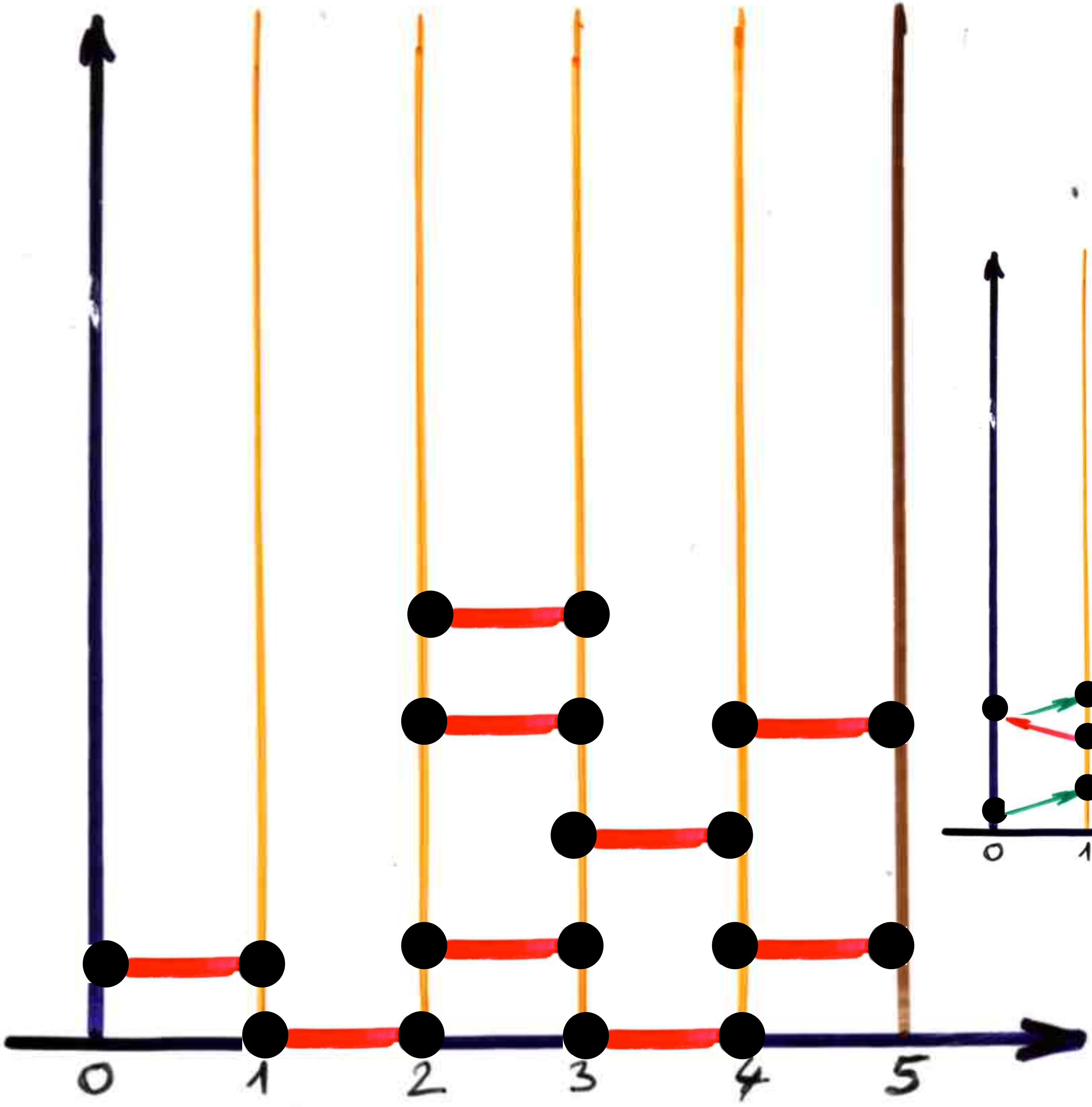
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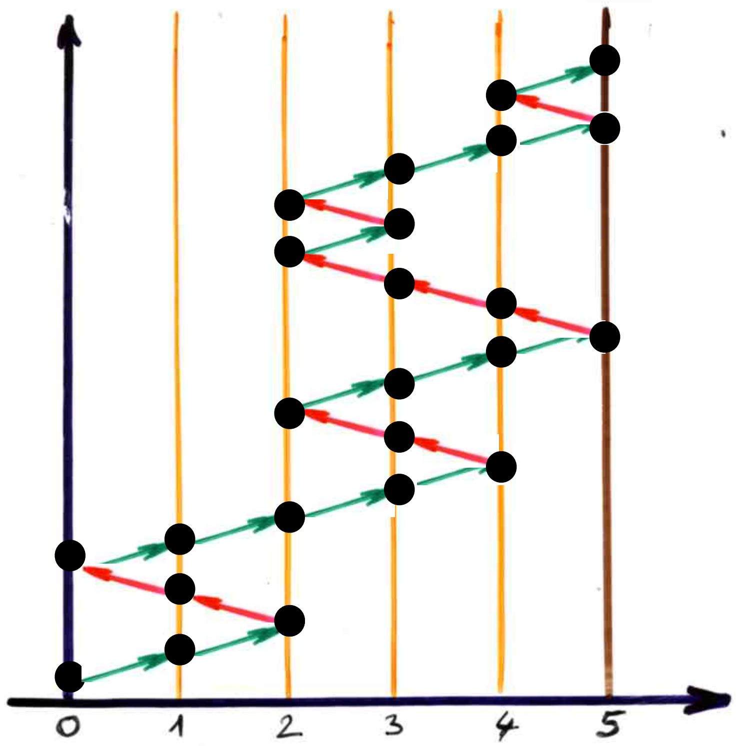
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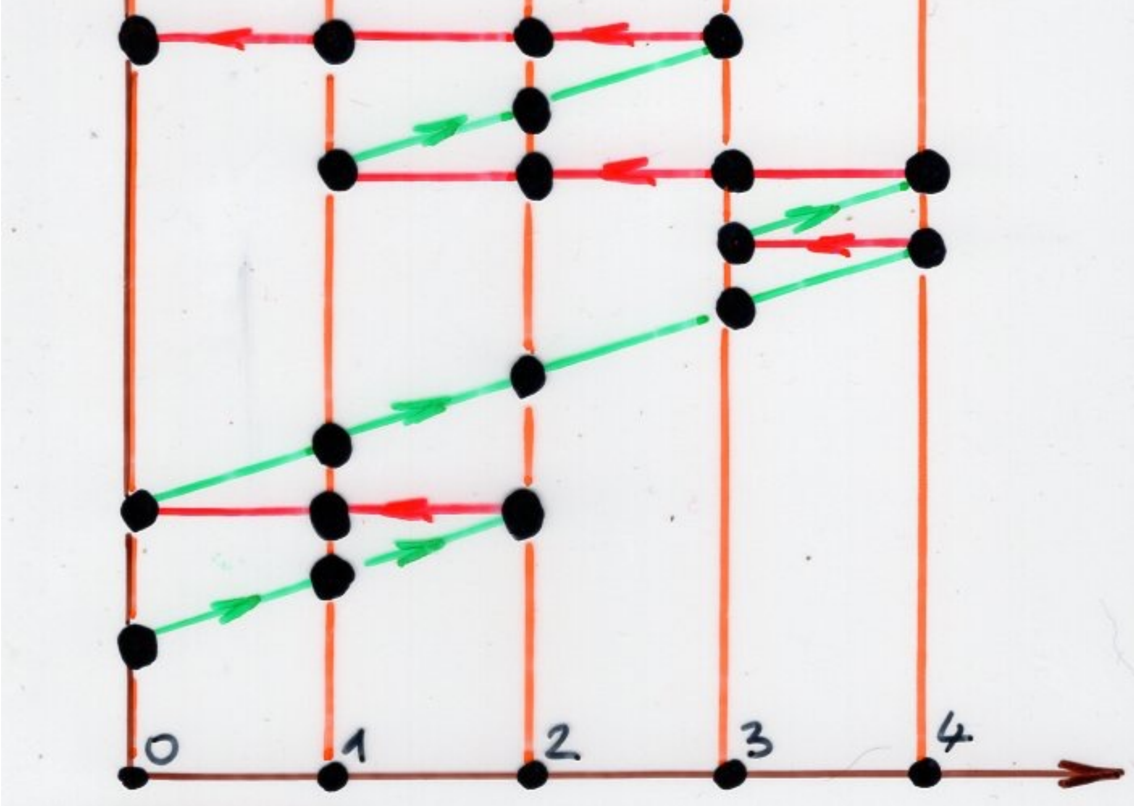
Dyck paths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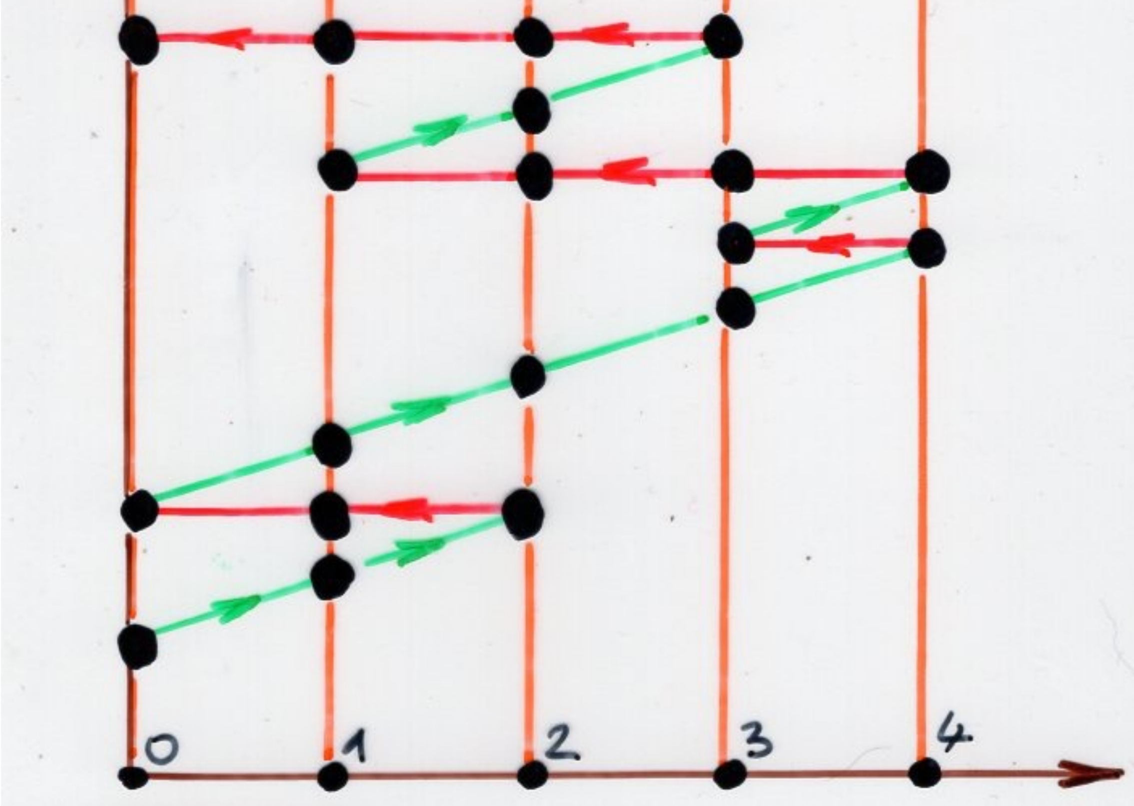
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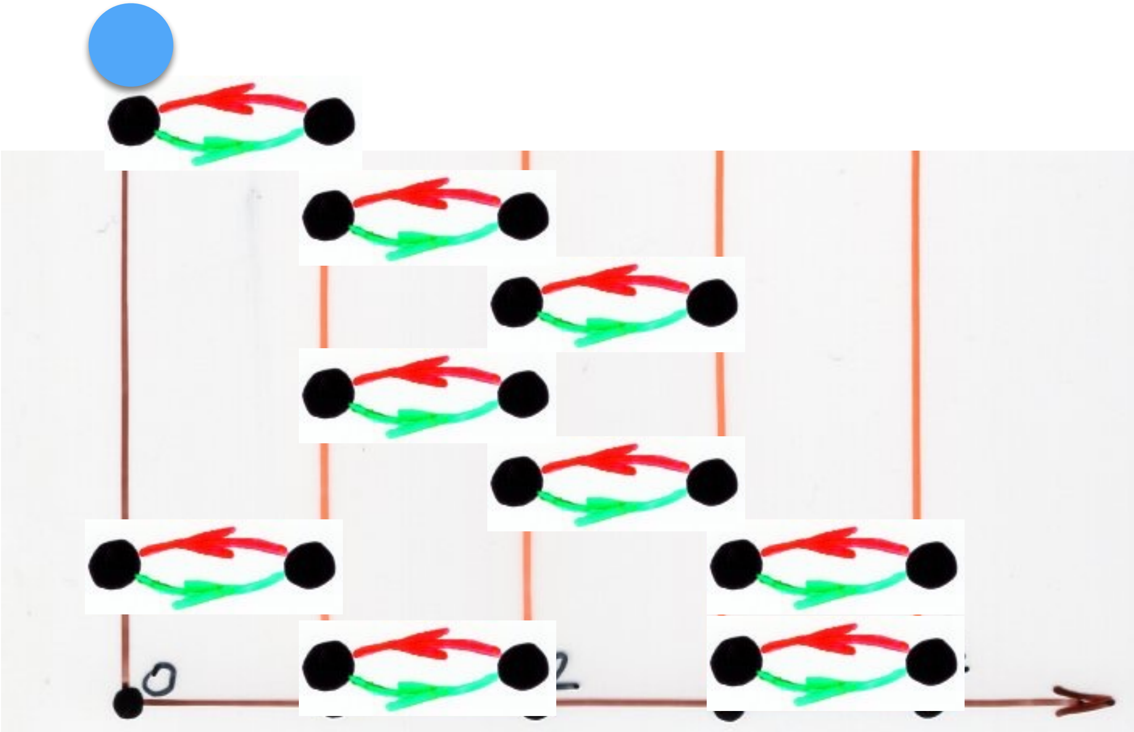
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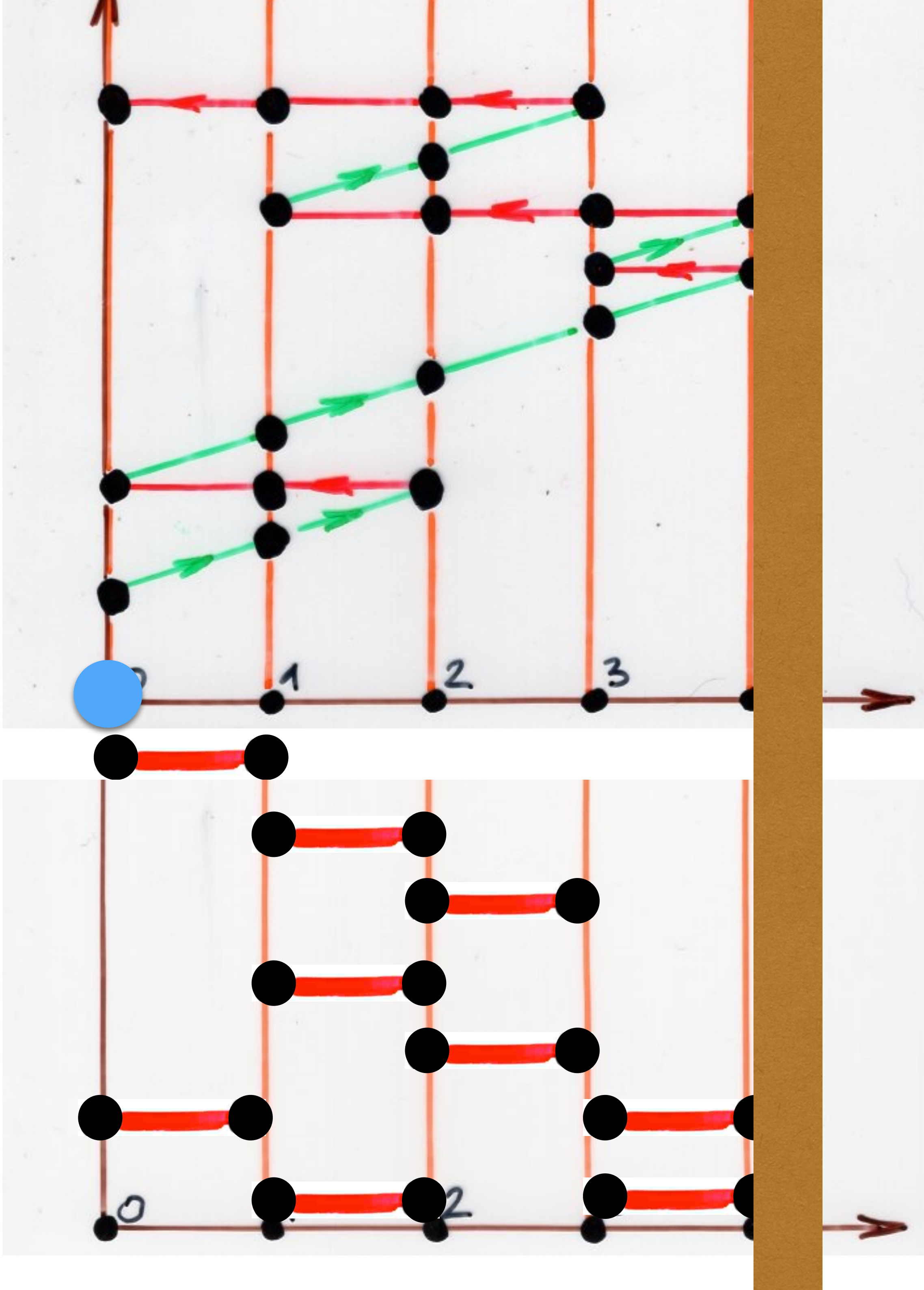
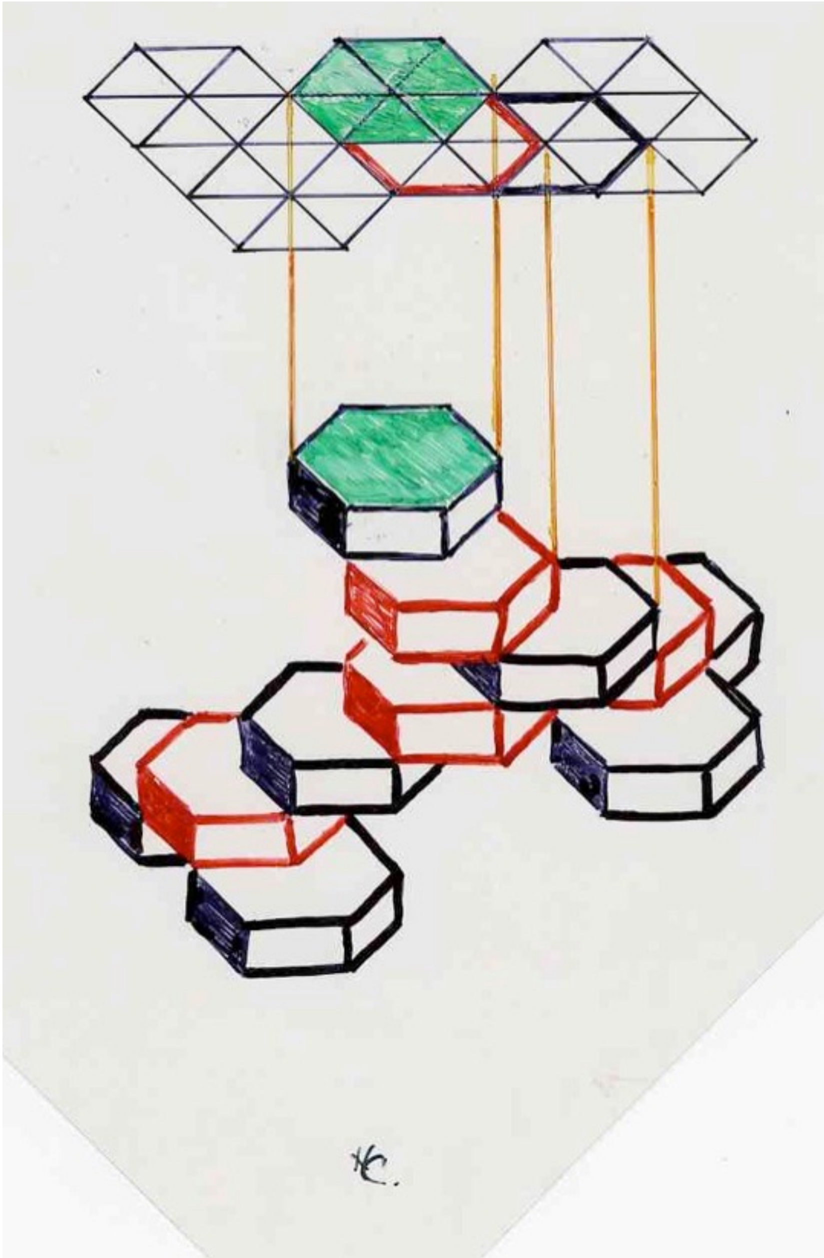
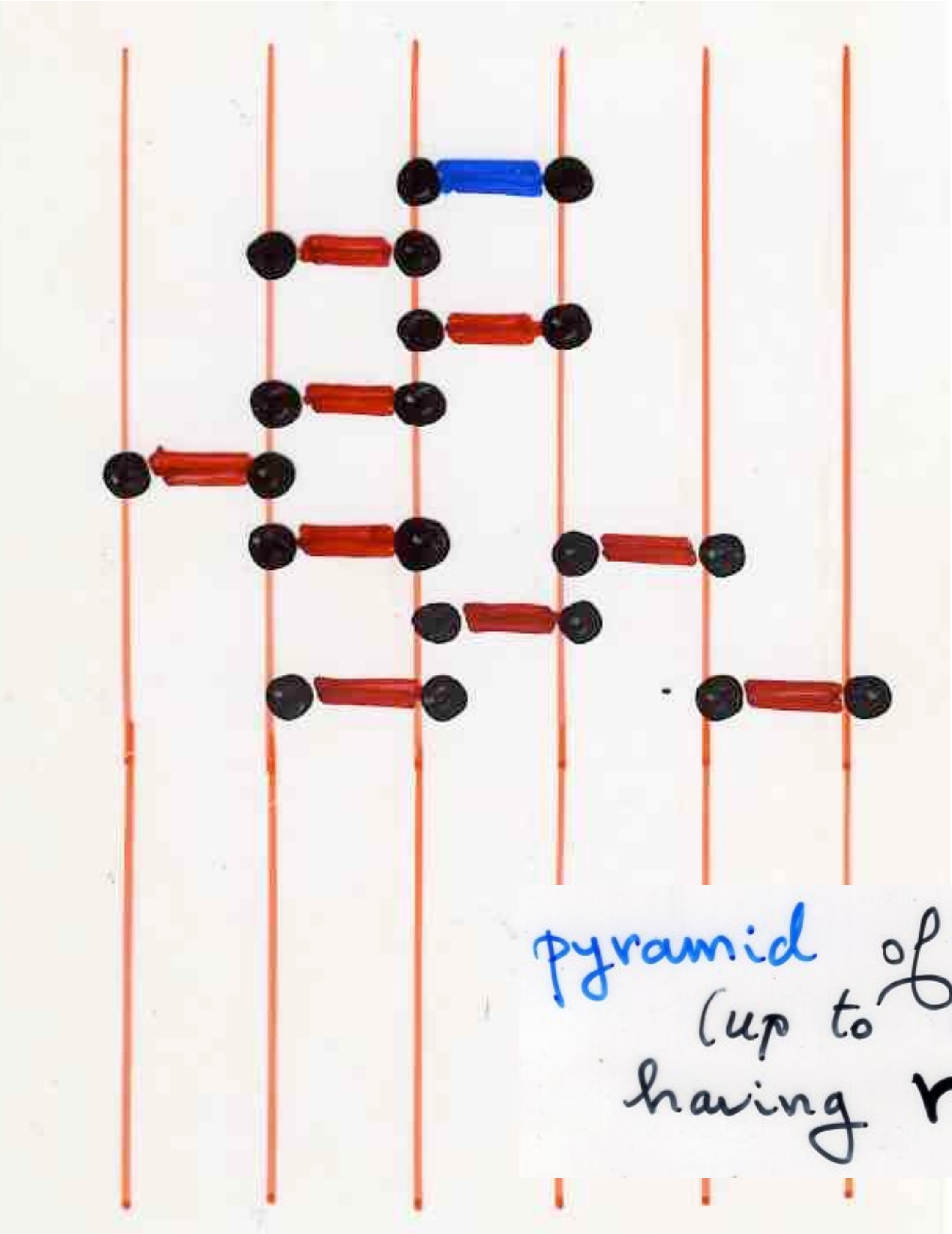


$C_{2n}^{(k)}$

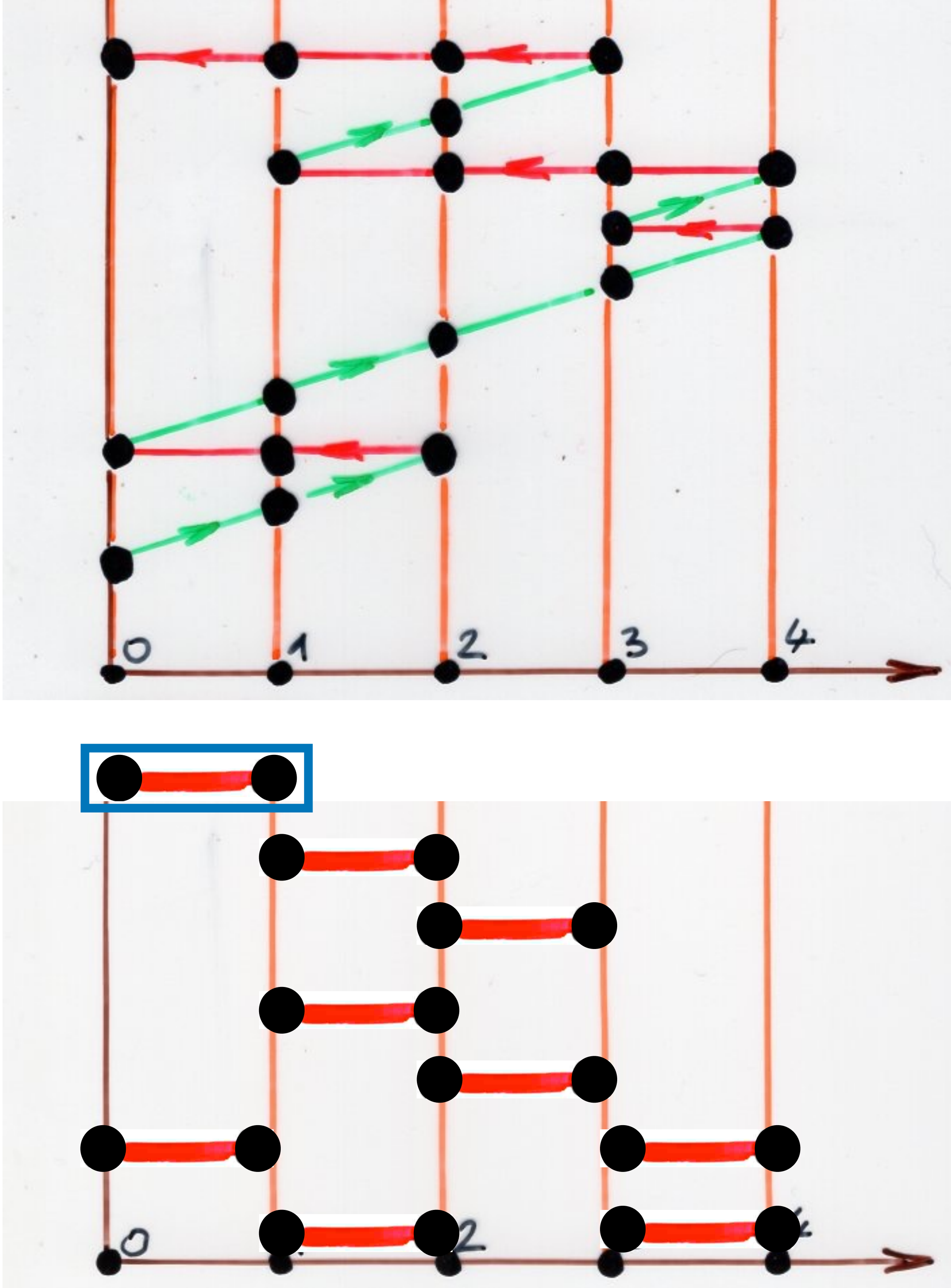
pyramid:
only one
maximal piece





pyramid:
only one
maximal piece

pyramid of dimers on $\mathbb{Z}$
(up to translation)
having n dimers



semi-pyramid:
maximal piece
= (0,1)

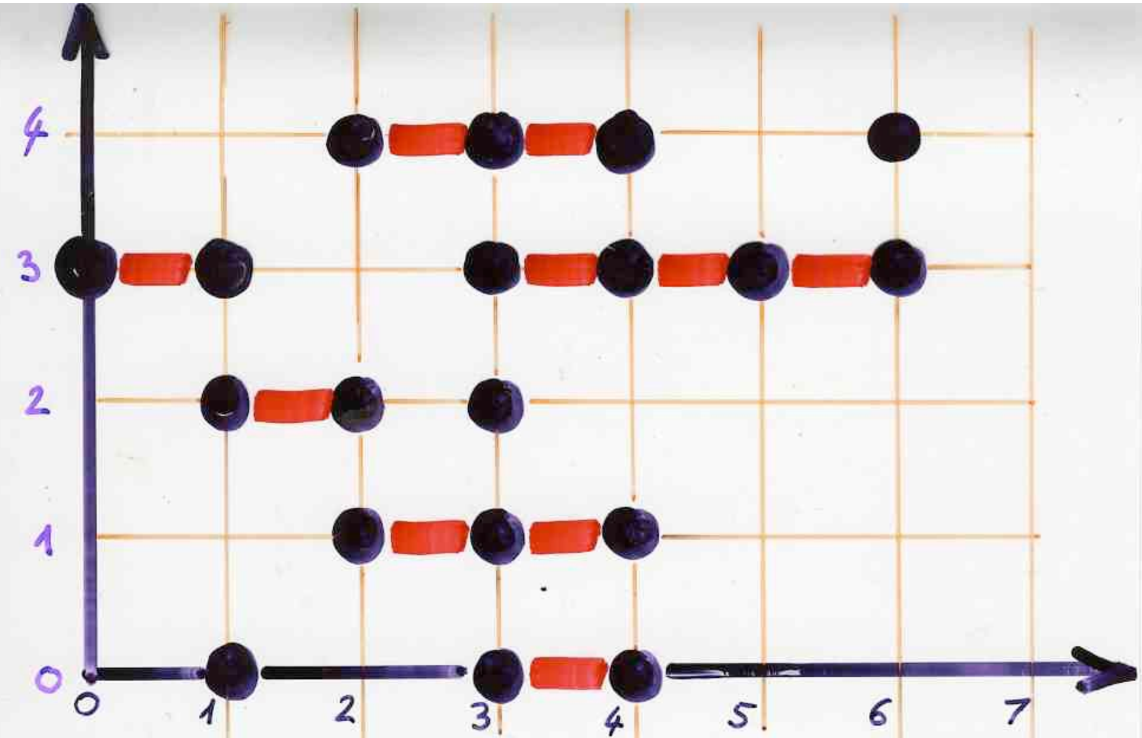
Bijection alternating sequences
heaps of segments

(even case)

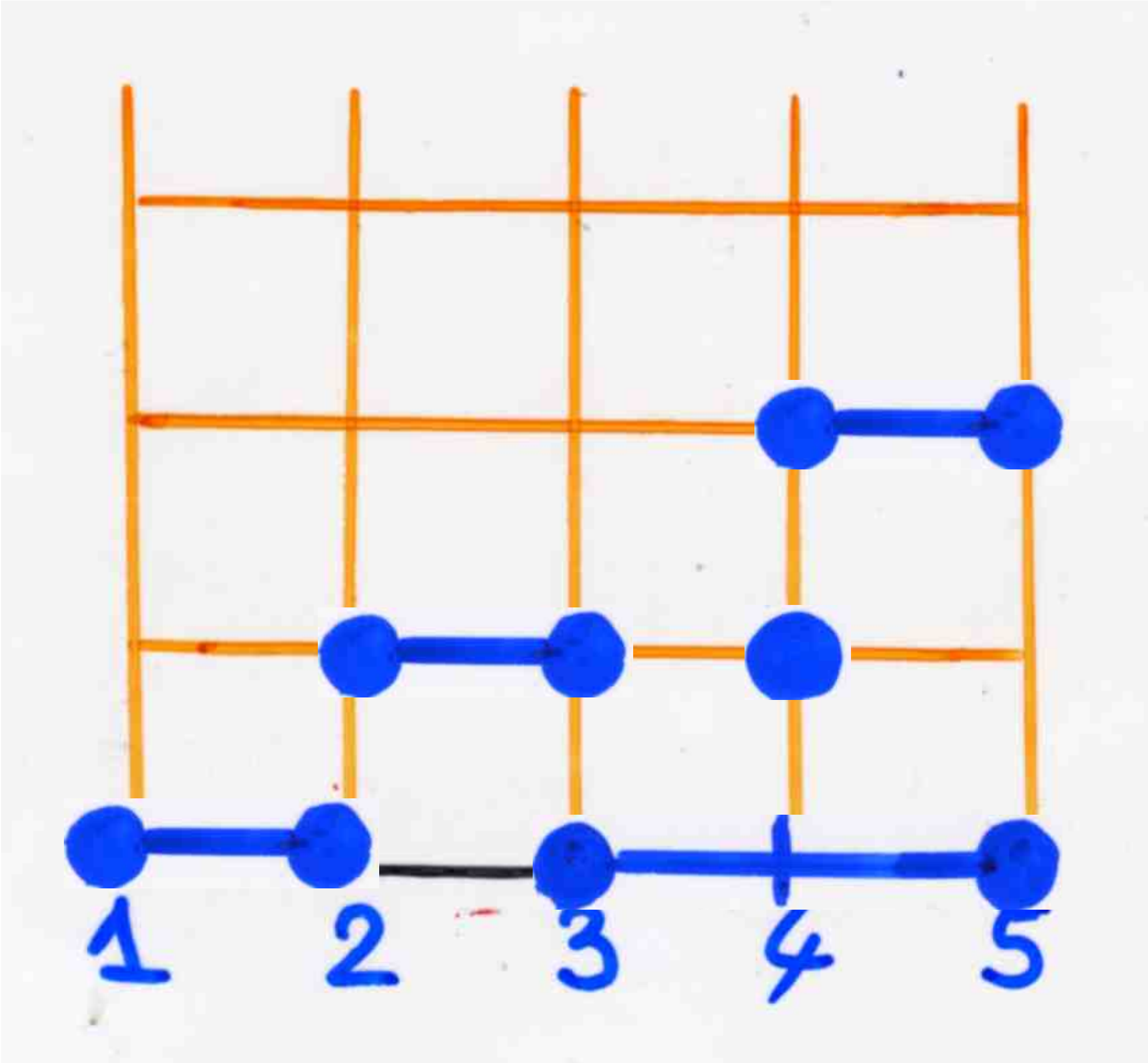
<u>**ex**</u>: heap of segments over $\mathbb{N}$

$$P = \{\, [a,b] = \{a, a+1, \dots, b\},\ 0 \le a \le b \,\}$$

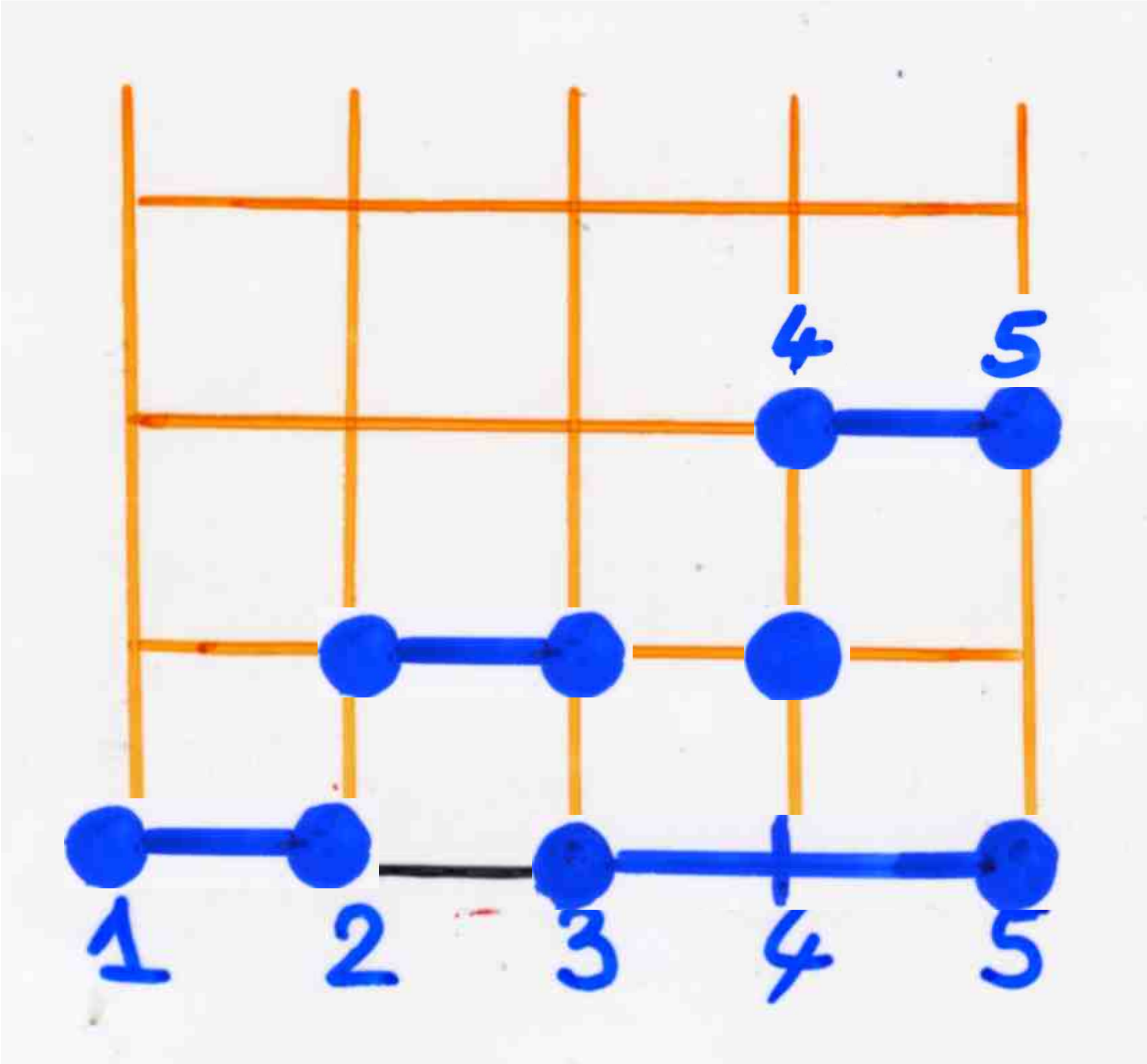
$$\mathcal{C} \qquad [a,b]\ \mathcal{C}\ [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$



heap of segments

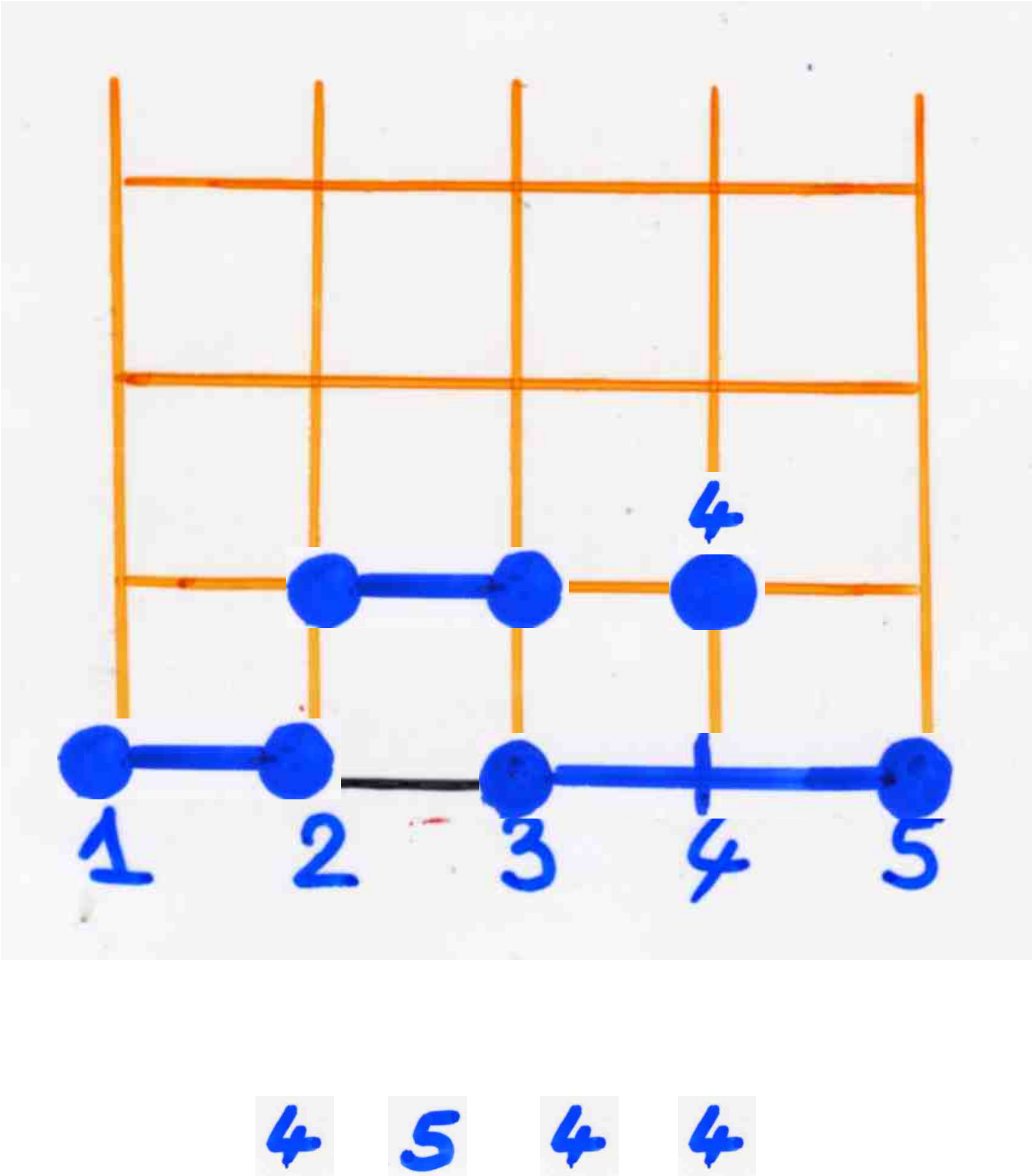


right most
maximal piece

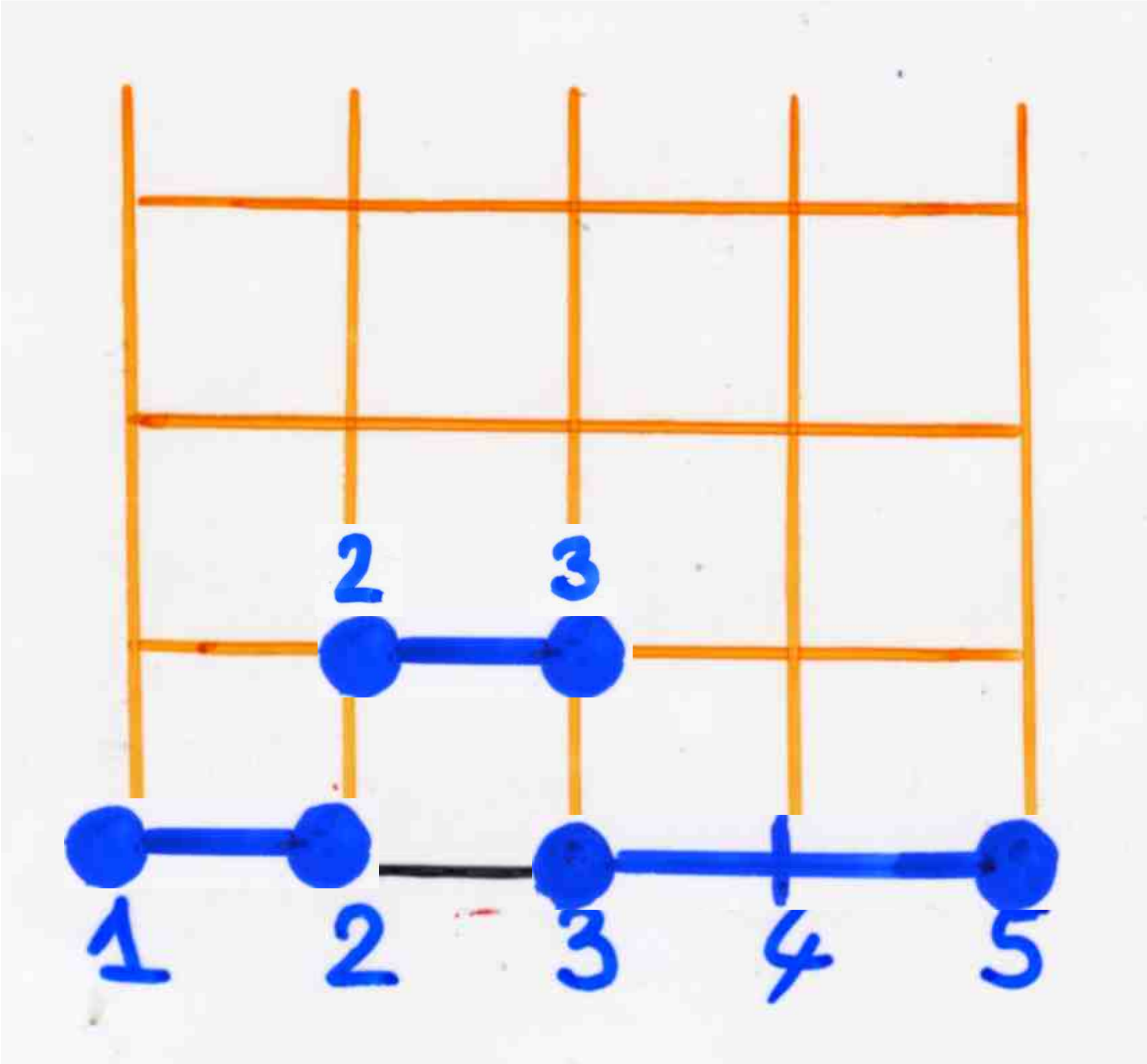


right most
maximal piece

4 5

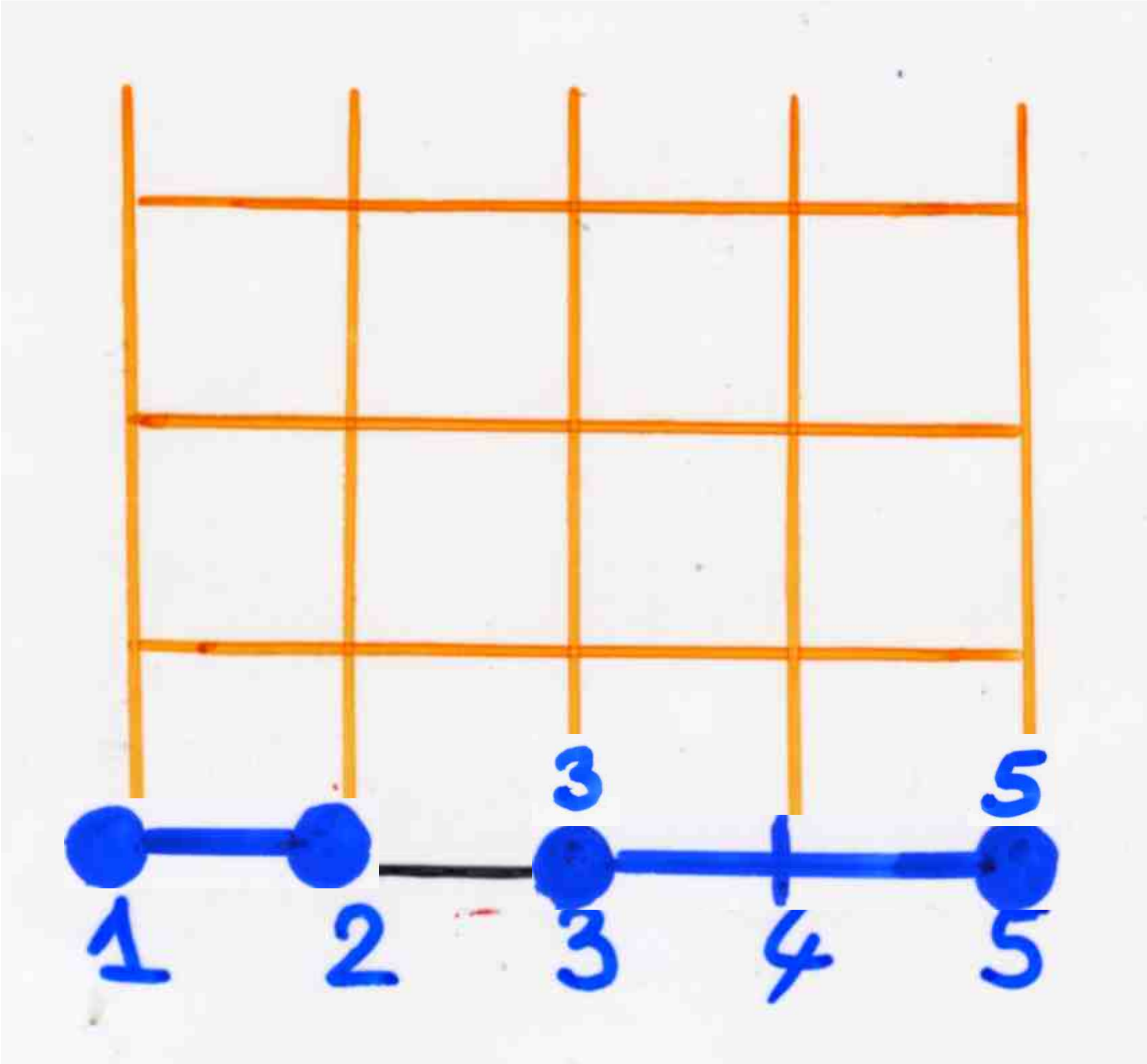


right most
maximal piece



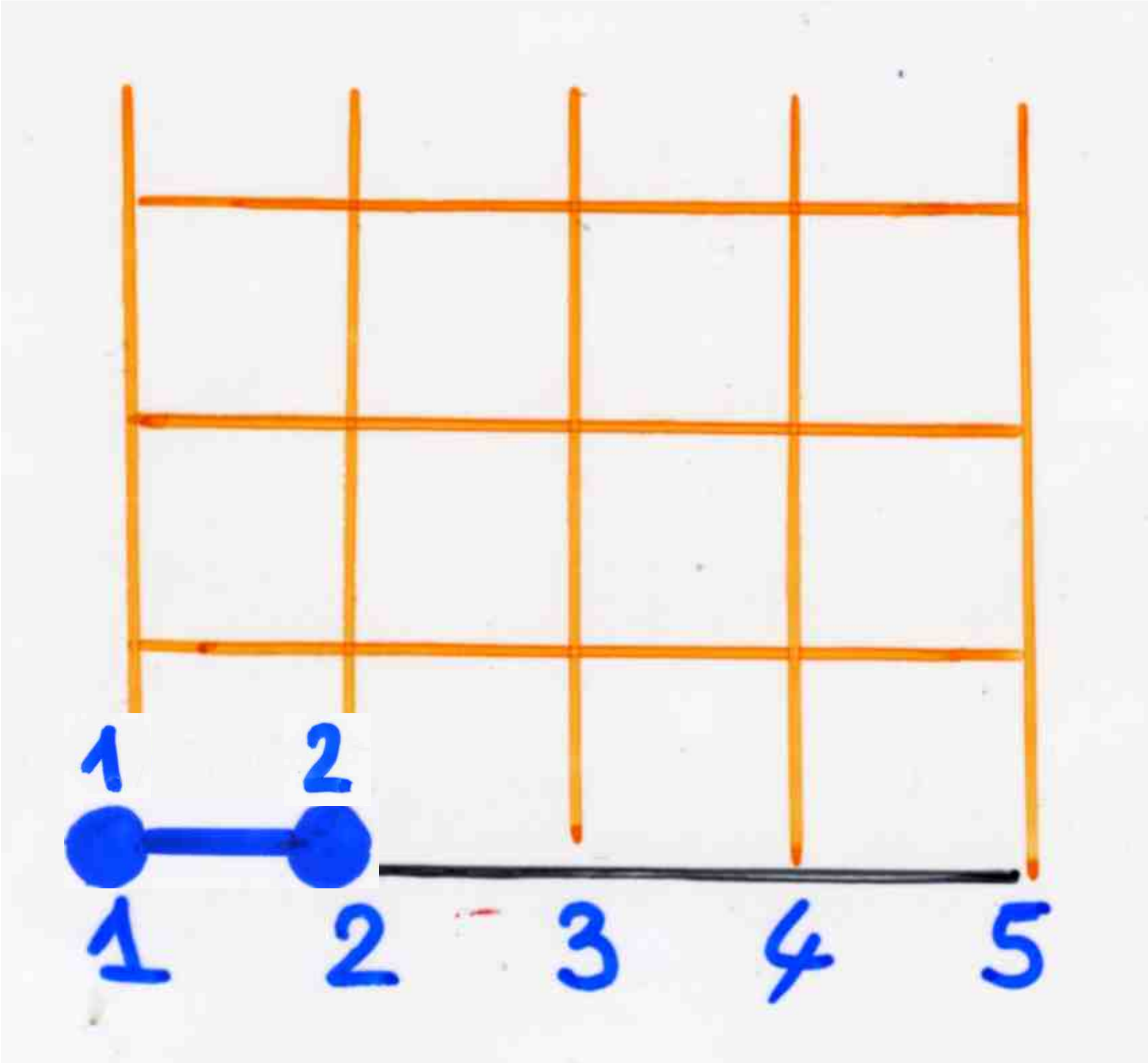
right most maximal piece

4 5 4 4 2 3



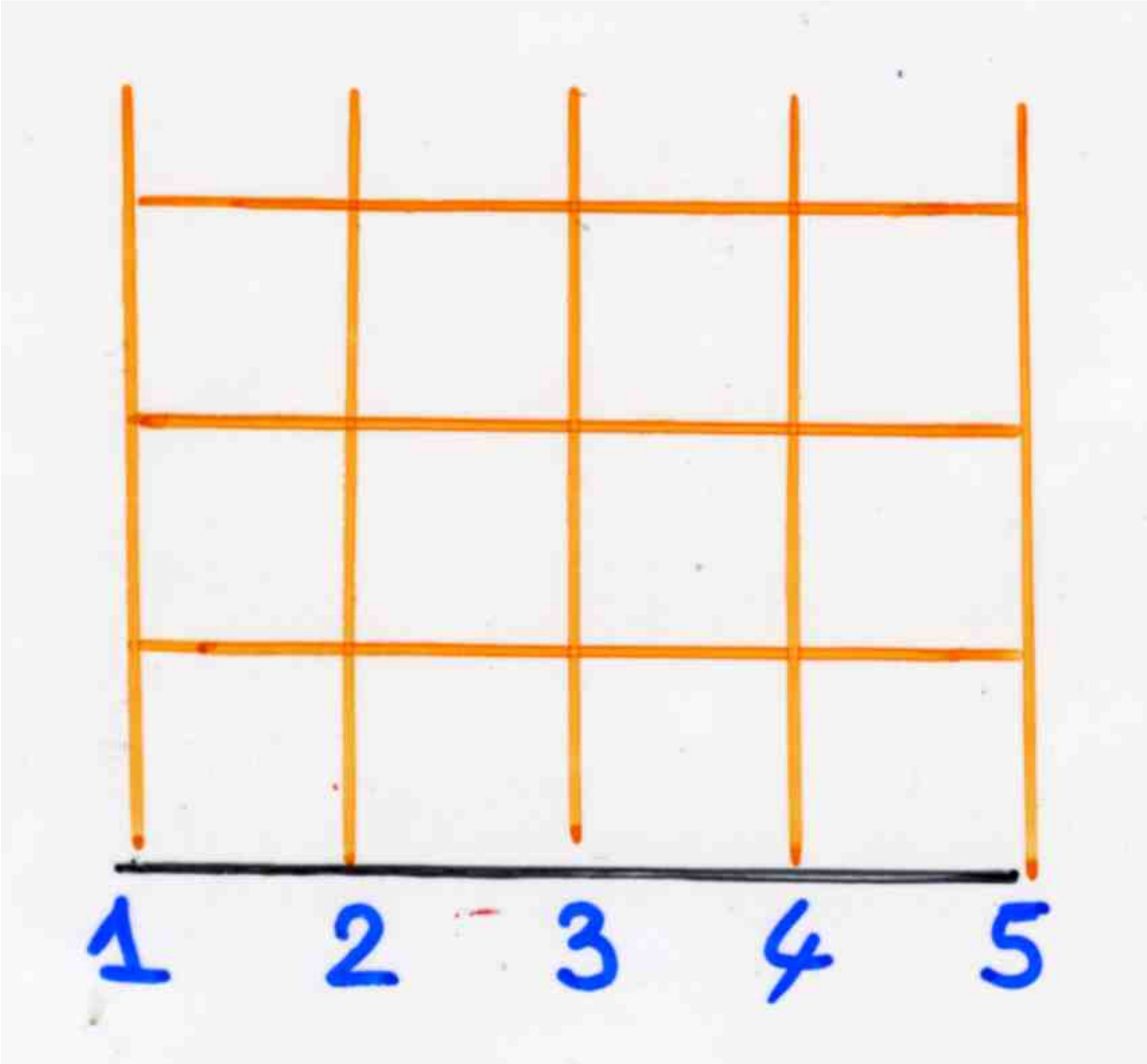
right most
maximal piece

4 5 4 4 2 3 3 5



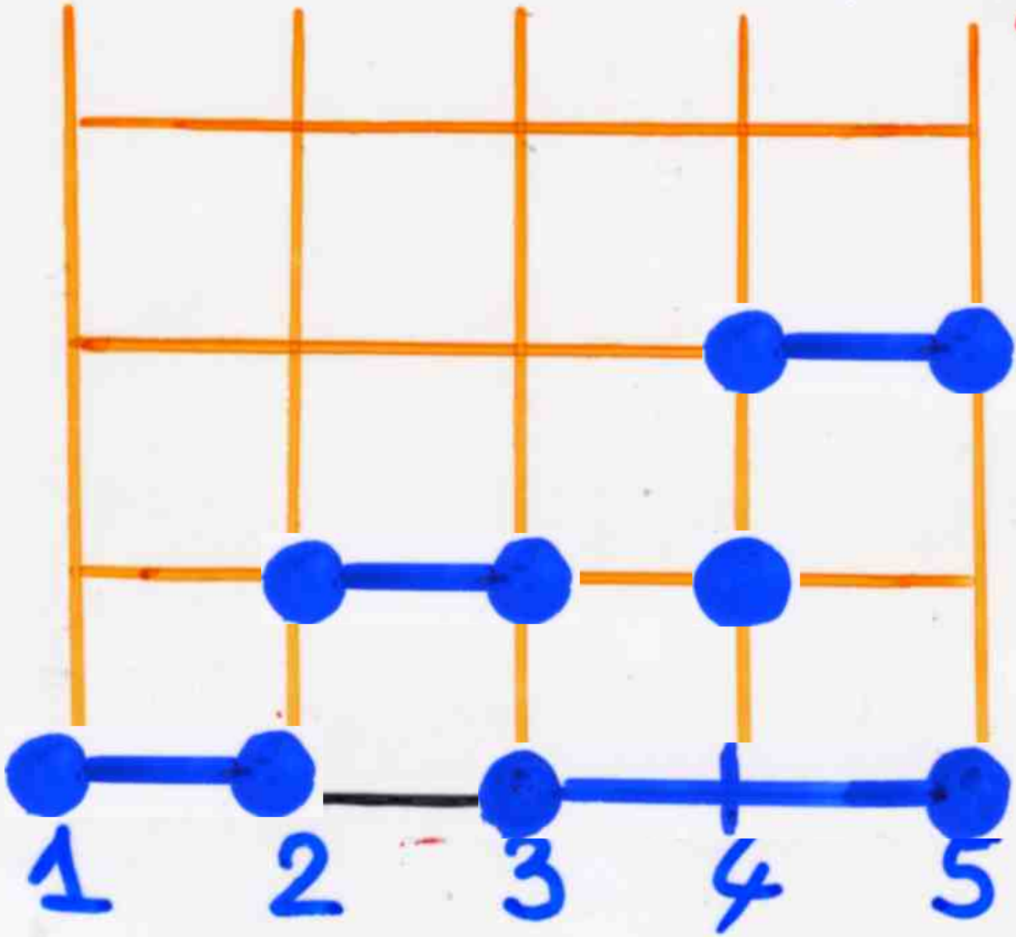
right most
maximal piece

4 5 4 4 2 3 3 5 1 2



right most
maximal piece

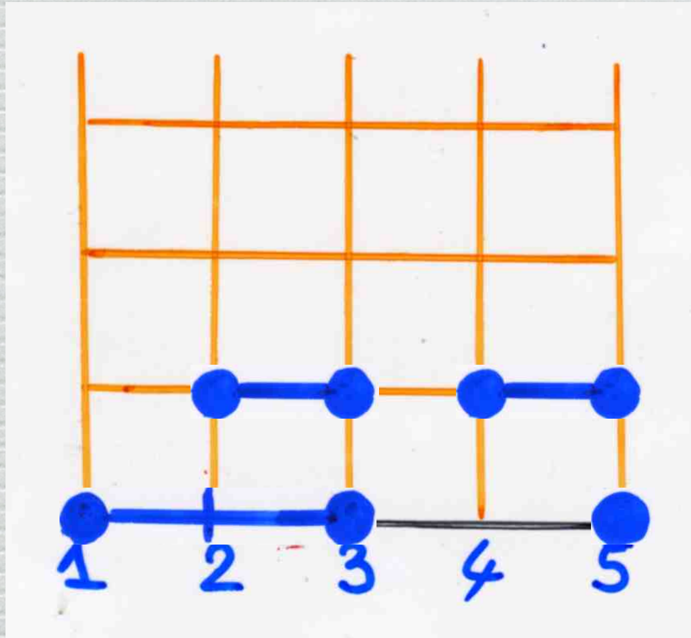
$$4 \leq 5 \geq 4 \leq 4 \geq 2 \leq 3 \geq 3 \leq 5 \geq 1 \leq 2$$



alternating sequence

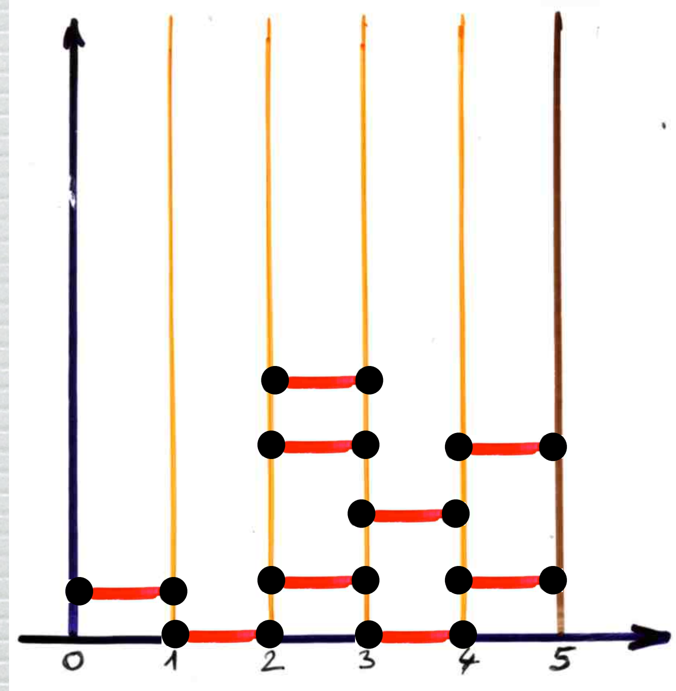
$1 \leq a_i \leq k$

$4 \leq 5 \geq 4 \leq 4 \geq 2 \leq 3 \geq 3 \leq 5 \geq 1 \leq 2$



reciprocity

$\longleftrightarrow$



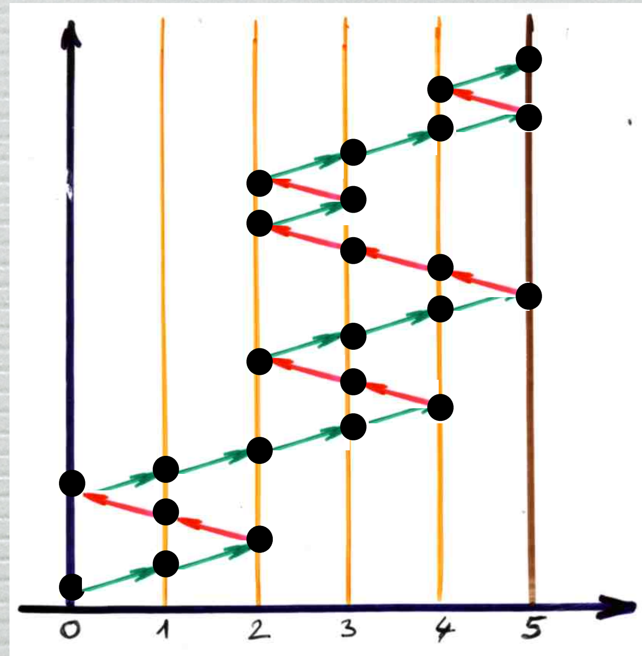
alternating sequence

duality

$a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{2n}$

(even case)

$D_{2n}^{(k)} = C_{2n+k}^{(k)}(0 \to k)$

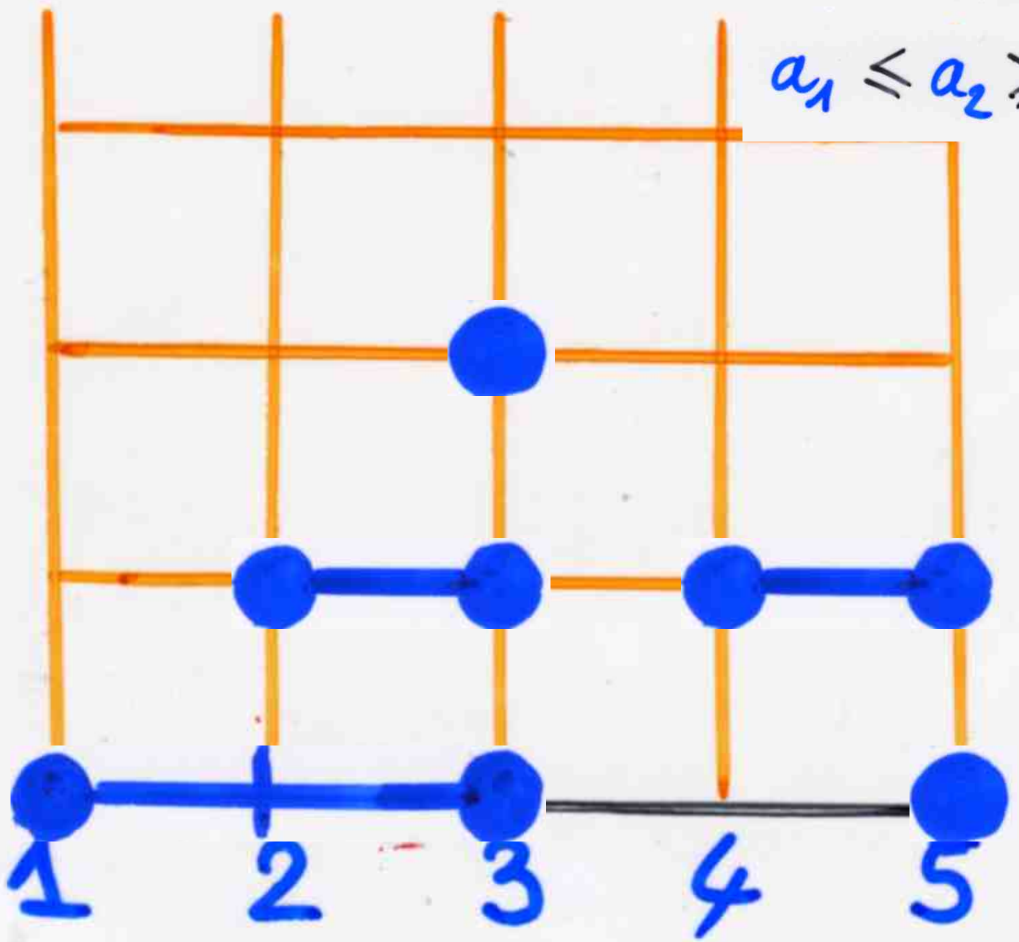


Bijection alternating sequences
heaps of segments

(odd case)

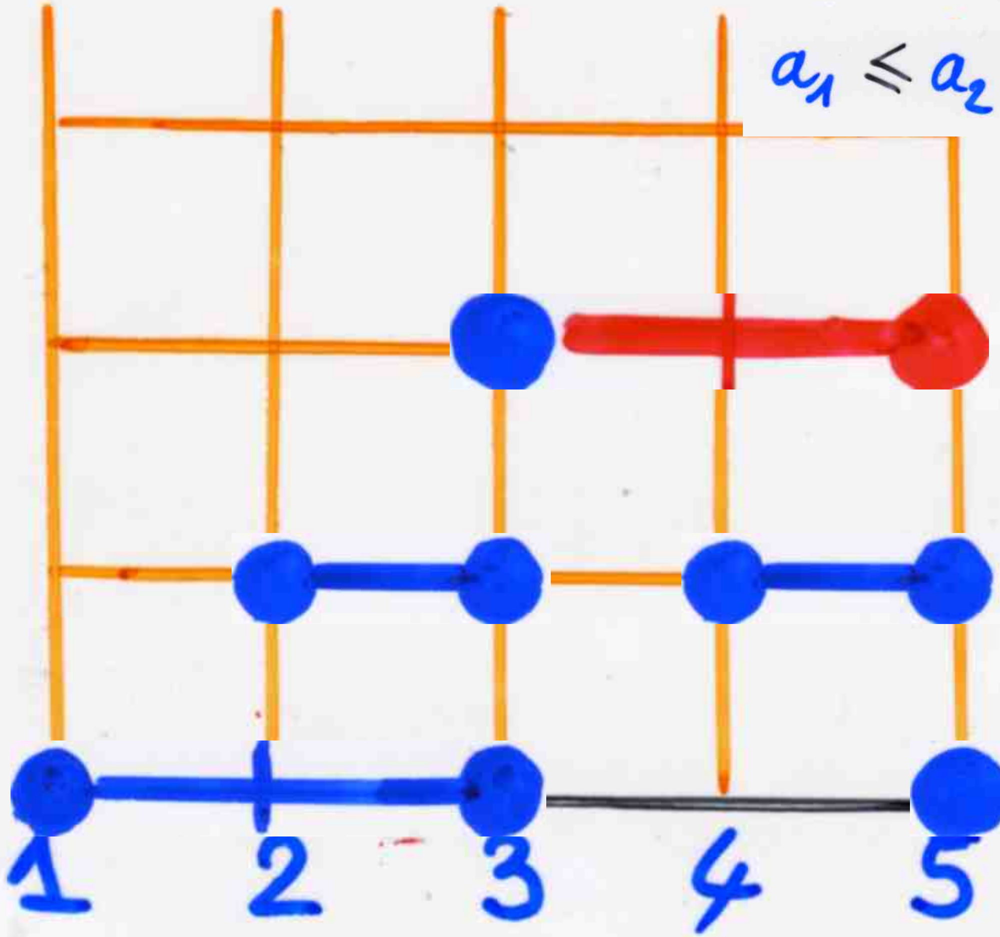
alternating sequence

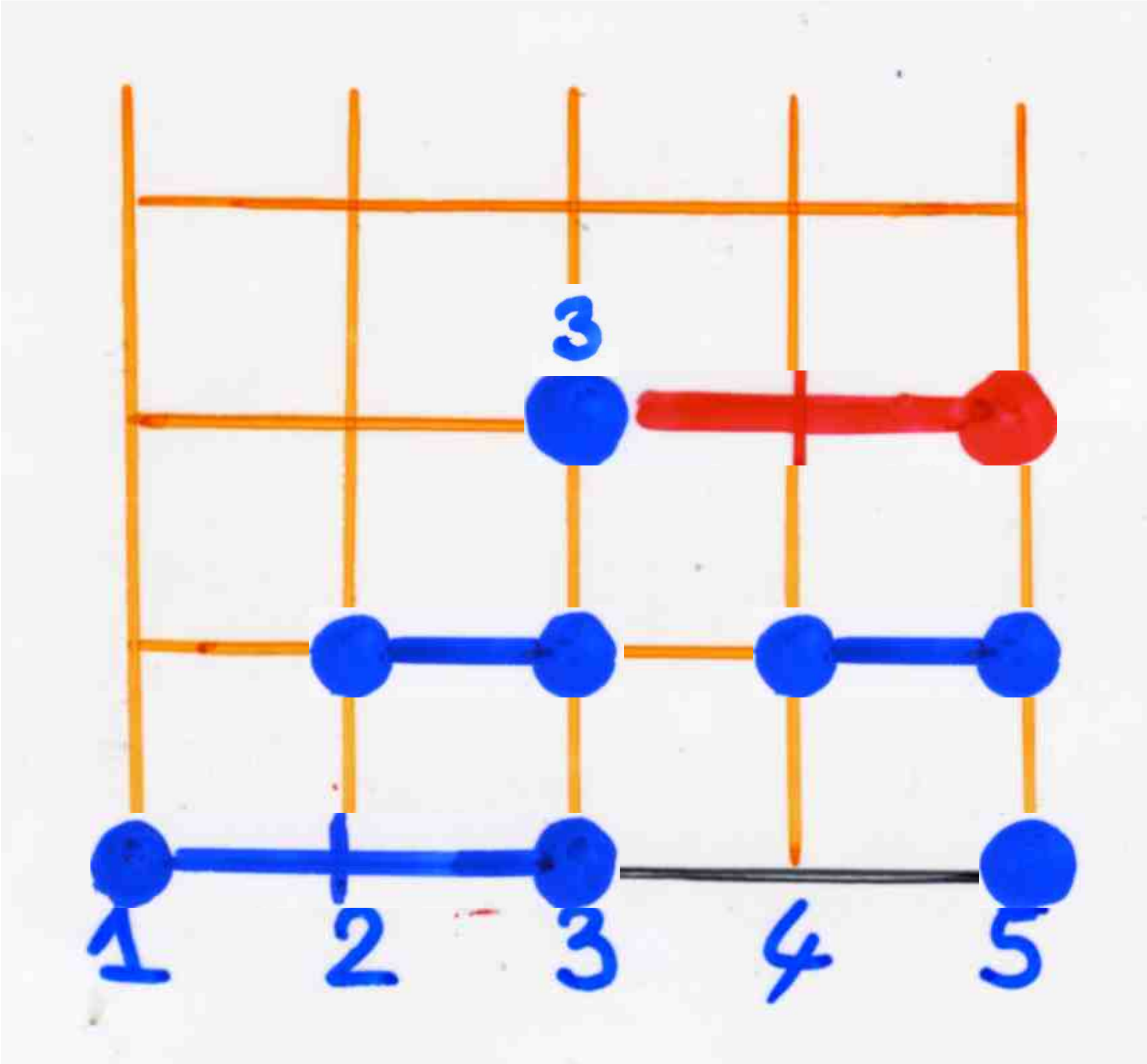
$$a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{2n-1}$$



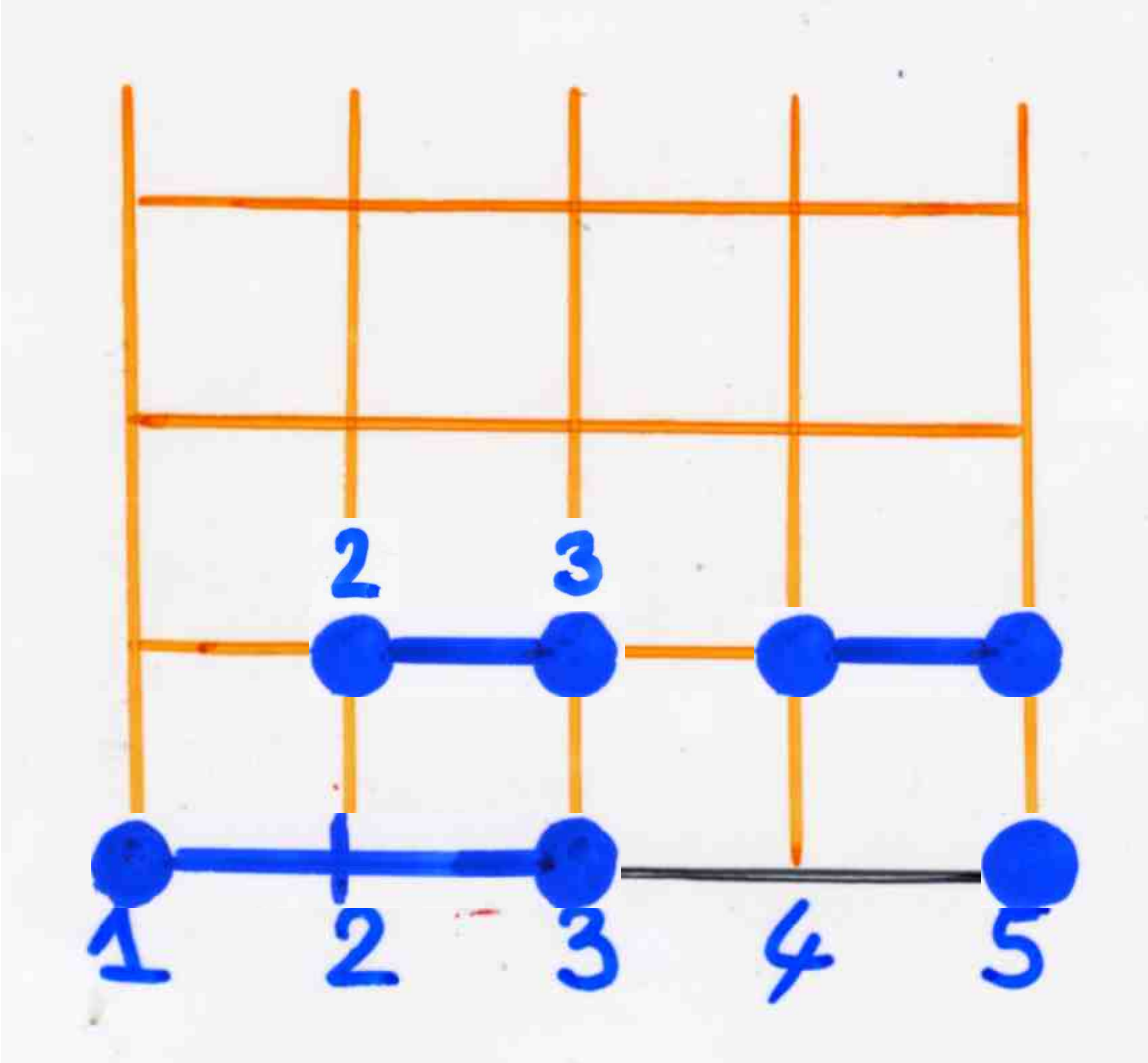
alternating sequence

$a_1 \leqslant a_2 \geqslant a_3 \leqslant a_4 \geqslant \cdots\cdots \geqslant a_{2n-1}$

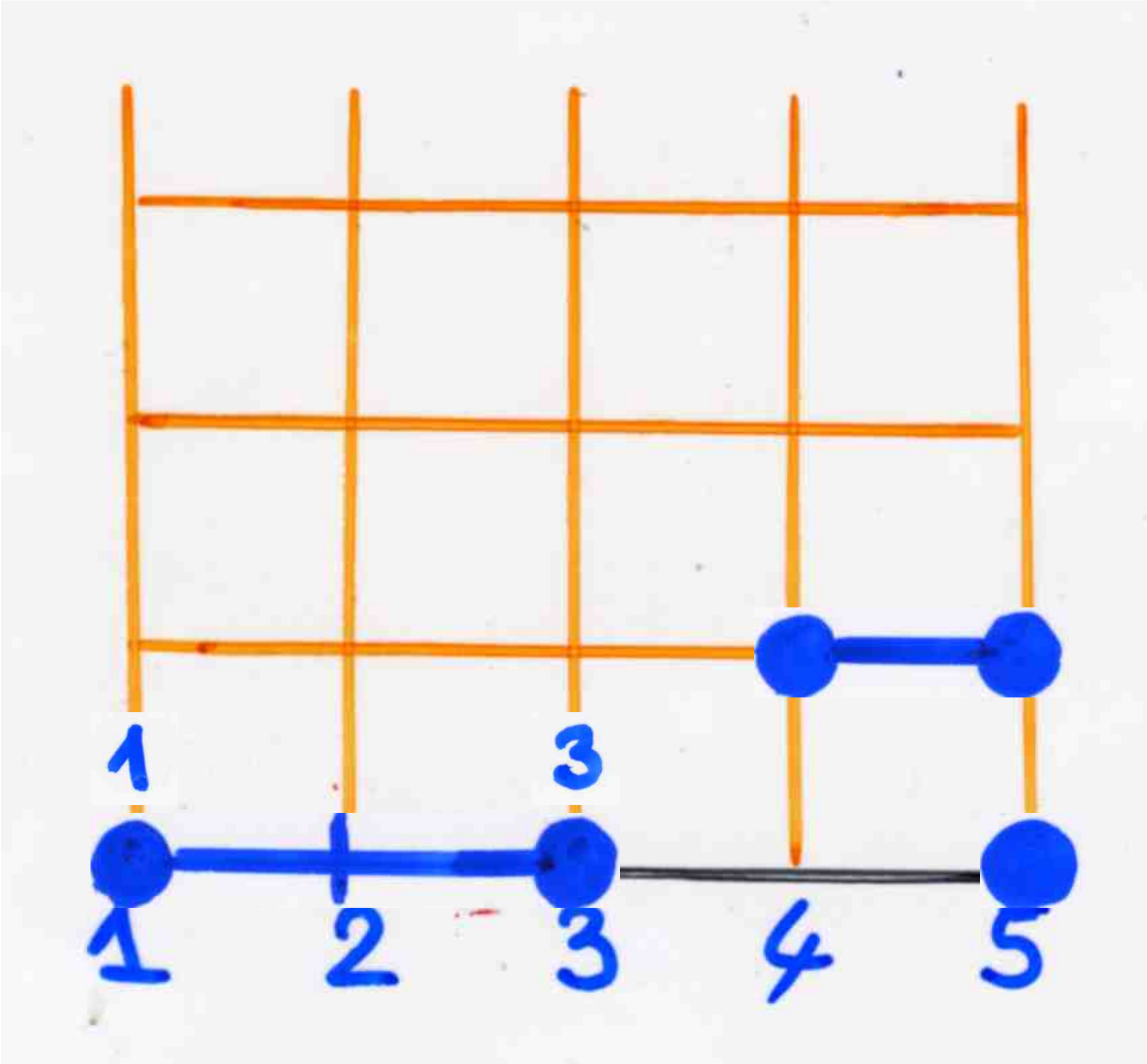




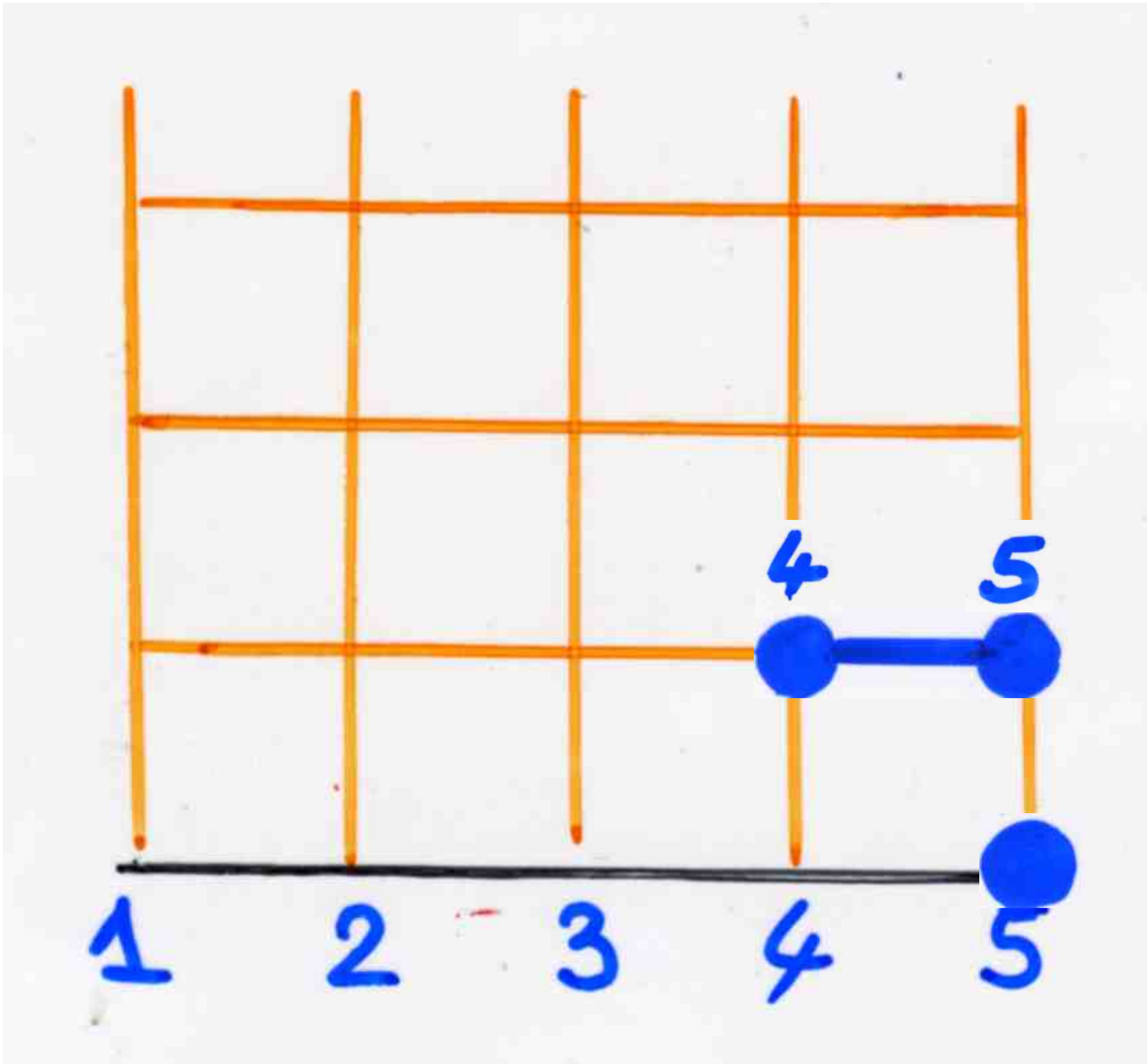
Left most
maximal piece



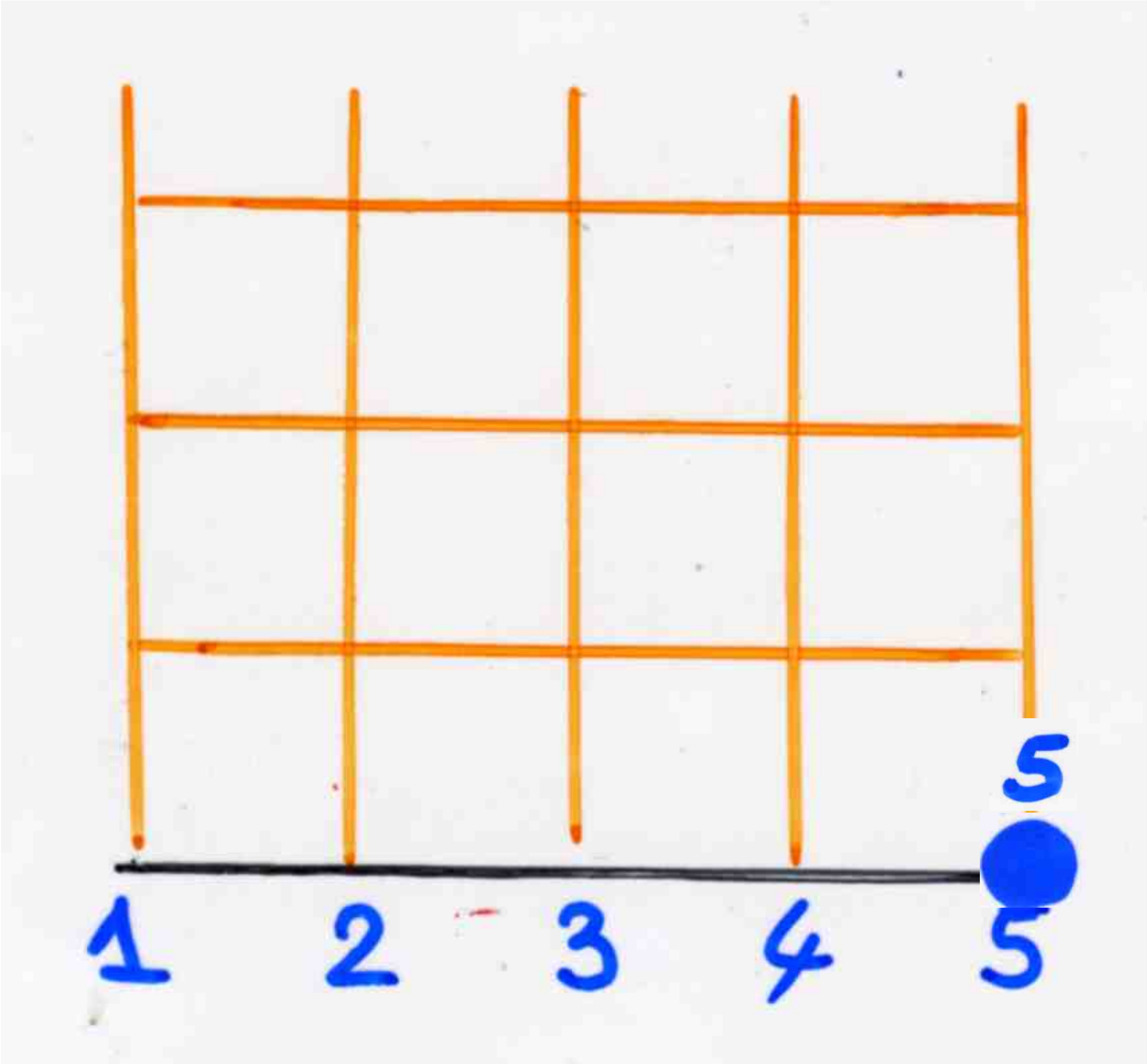
3 3 2



3 3 2 3 1



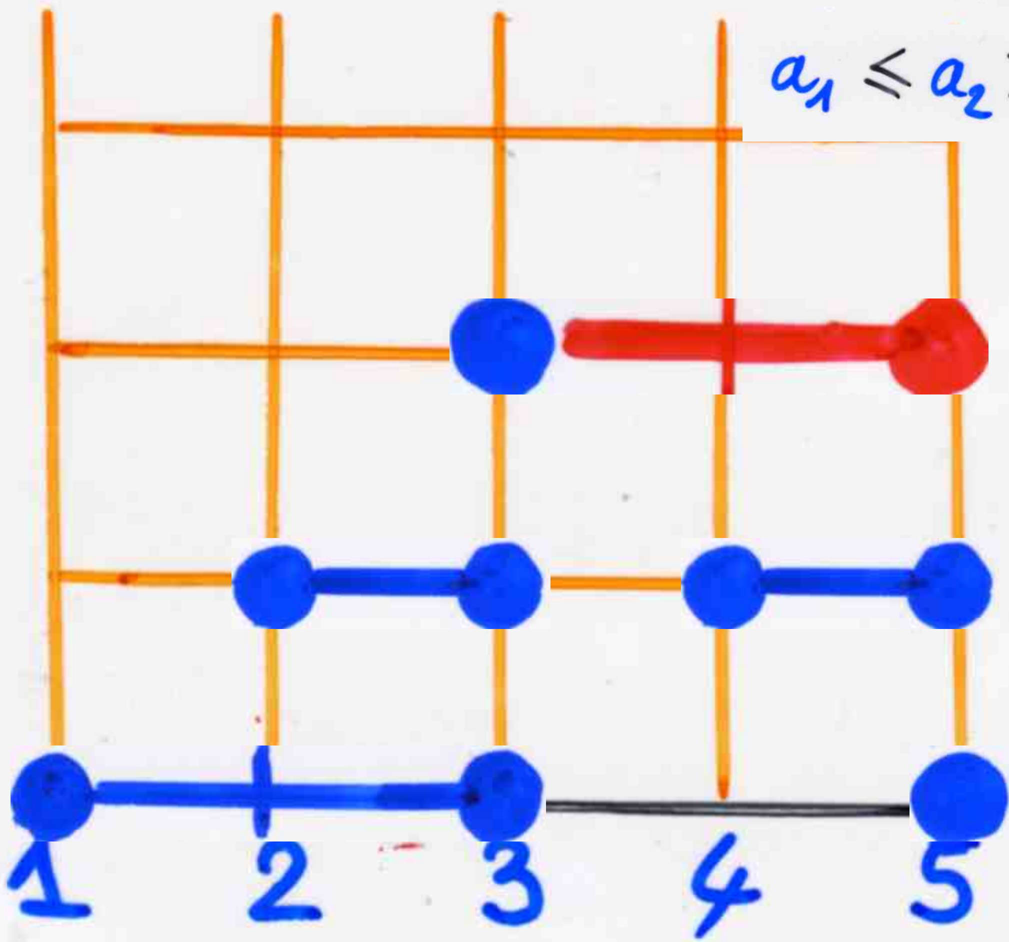
3 3 2 3 1 5 4



3 3 2 3 1 5 4 5 5

alternating sequence

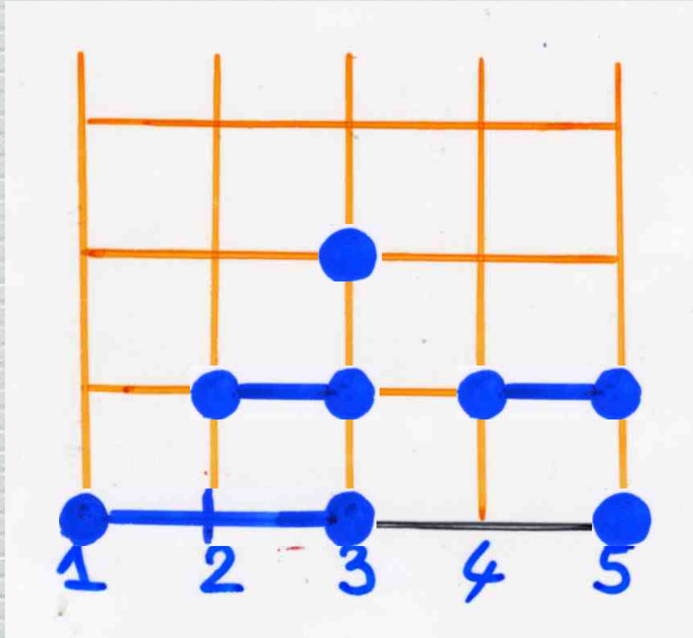
$$a_1 \leqslant a_2 \geqslant a_3 \leqslant a_4 \geqslant \cdots\cdots \geqslant a_{2n-1}$$

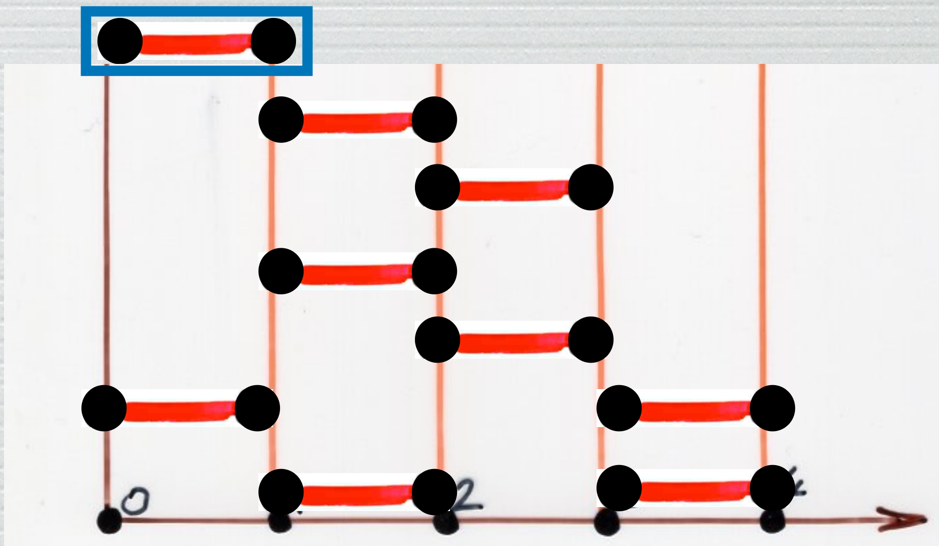


$$1 \leqslant a_i \leqslant k$$

$$3 \leqslant 3 \geqslant 2 \leqslant 3 \geqslant 1 \leqslant 5 \geqslant 4 \leqslant 5 \geqslant 5$$

reciprocity



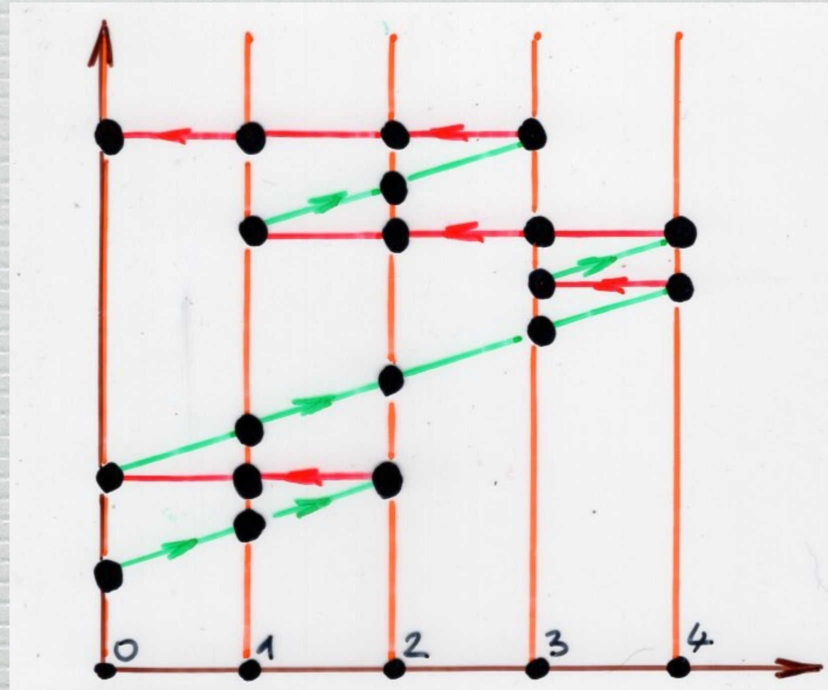


$a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots\cdots \geq a_{2n-1}$

$C_{2n}^{(k)}$

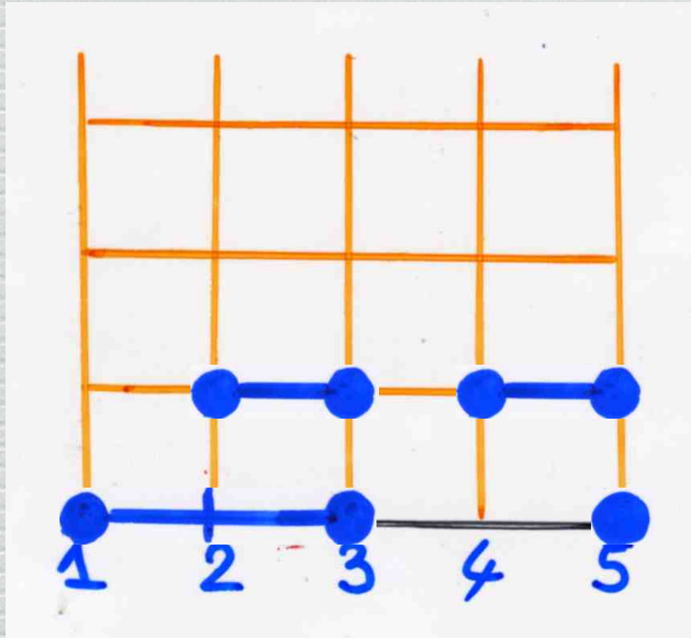
duality

(odd case)



Taking the left most or
rigth most maximal piece. ....

Use it later ...

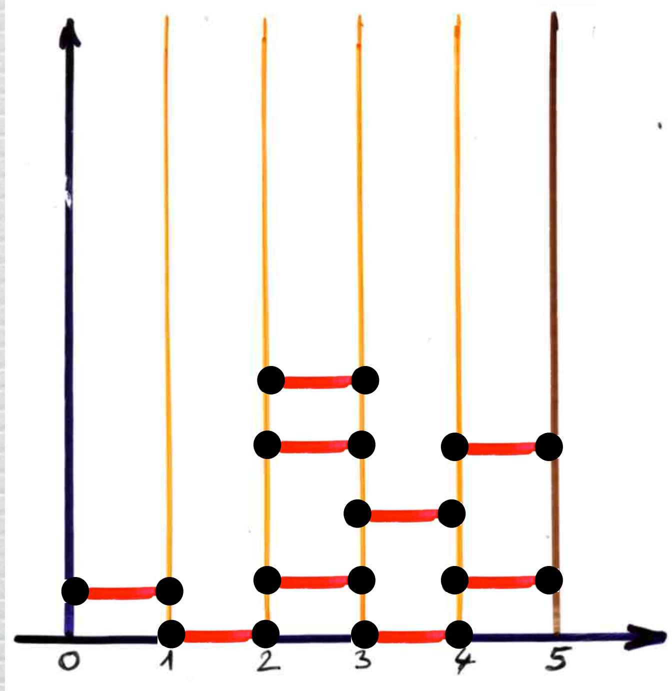


reciprocity

$\longleftrightarrow$

duality

(even case)



$$-f(1/t) = \sum_{n\geq 1} a_{-n} t^n$$

$$f(t) = \sum_{n\geq 0} a_n t^n$$

Second basic lemma on heaps: the inversion lemma

1/D

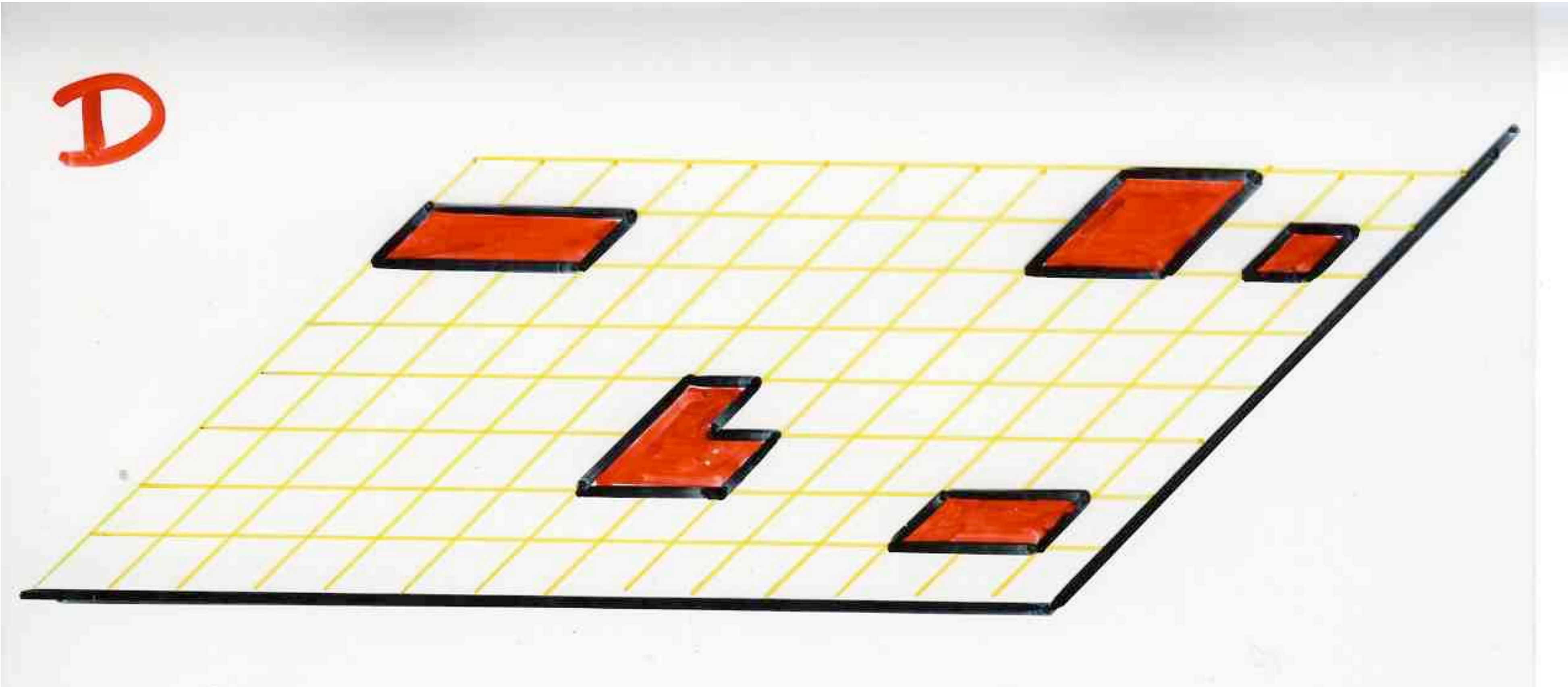
the inversion lemma

$$(\text{Heaps}) = \frac{1}{(\text{Trivial heaps})}$$

trivial
heap

F

all pieces (α, i)
at level 0



weight
valuation

$v(E)$

- $v : P \longrightarrow \mathbb{K}[x, y, \ldots]$

basic
piece

- $v(\alpha, i) = v(\alpha)$

piece

- $v(E) = \prod_{(\alpha,i) \in E} v(\alpha\ i)$

heap

the inversion lemma

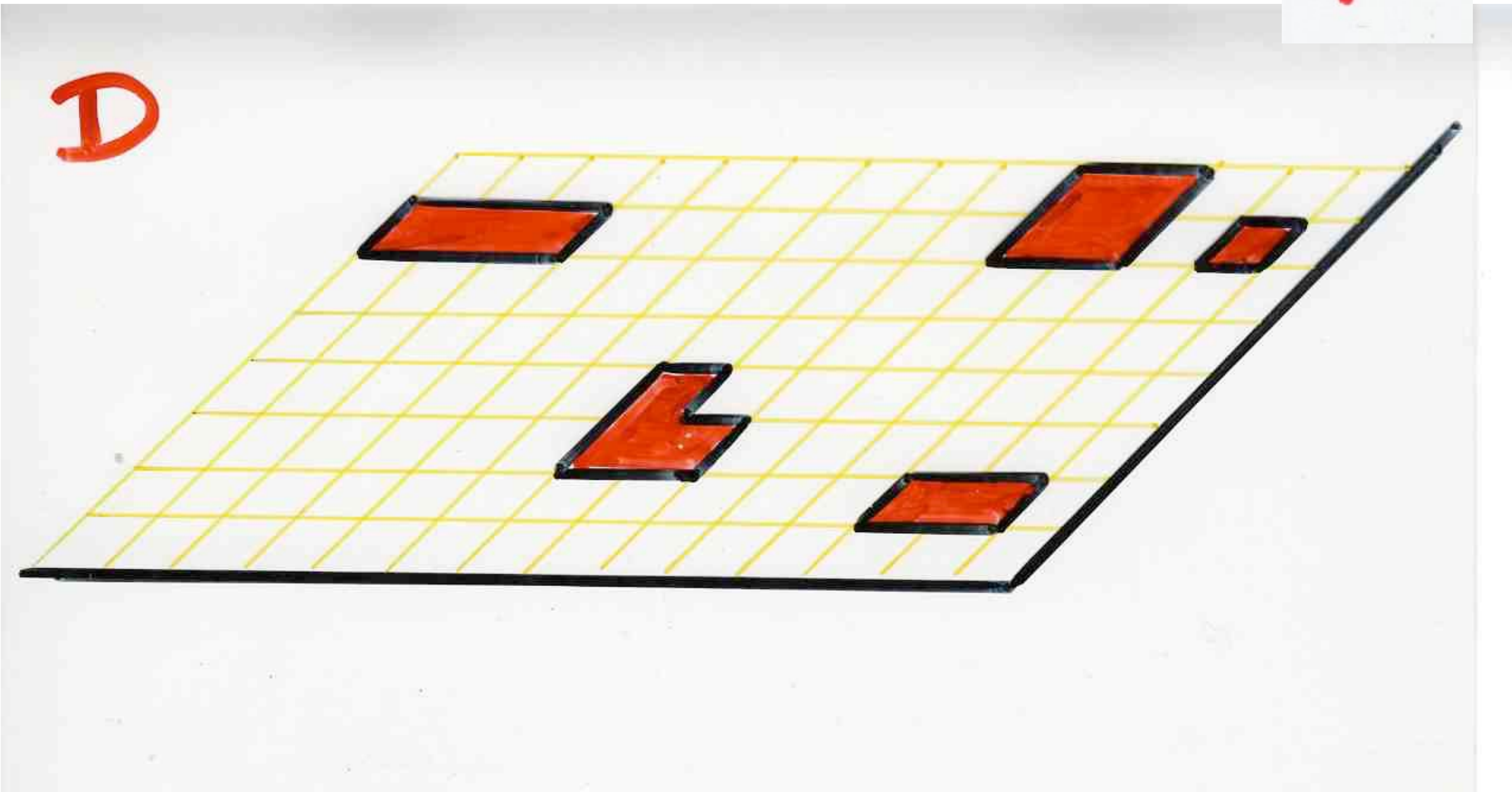
$$\left(\sum_{\substack{E\\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F\\ \text{trivial}\\ \text{heaps}}} (-1)^{|F|} v(F)\right)}$$

the inversion lemma

$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right)}$$

D


F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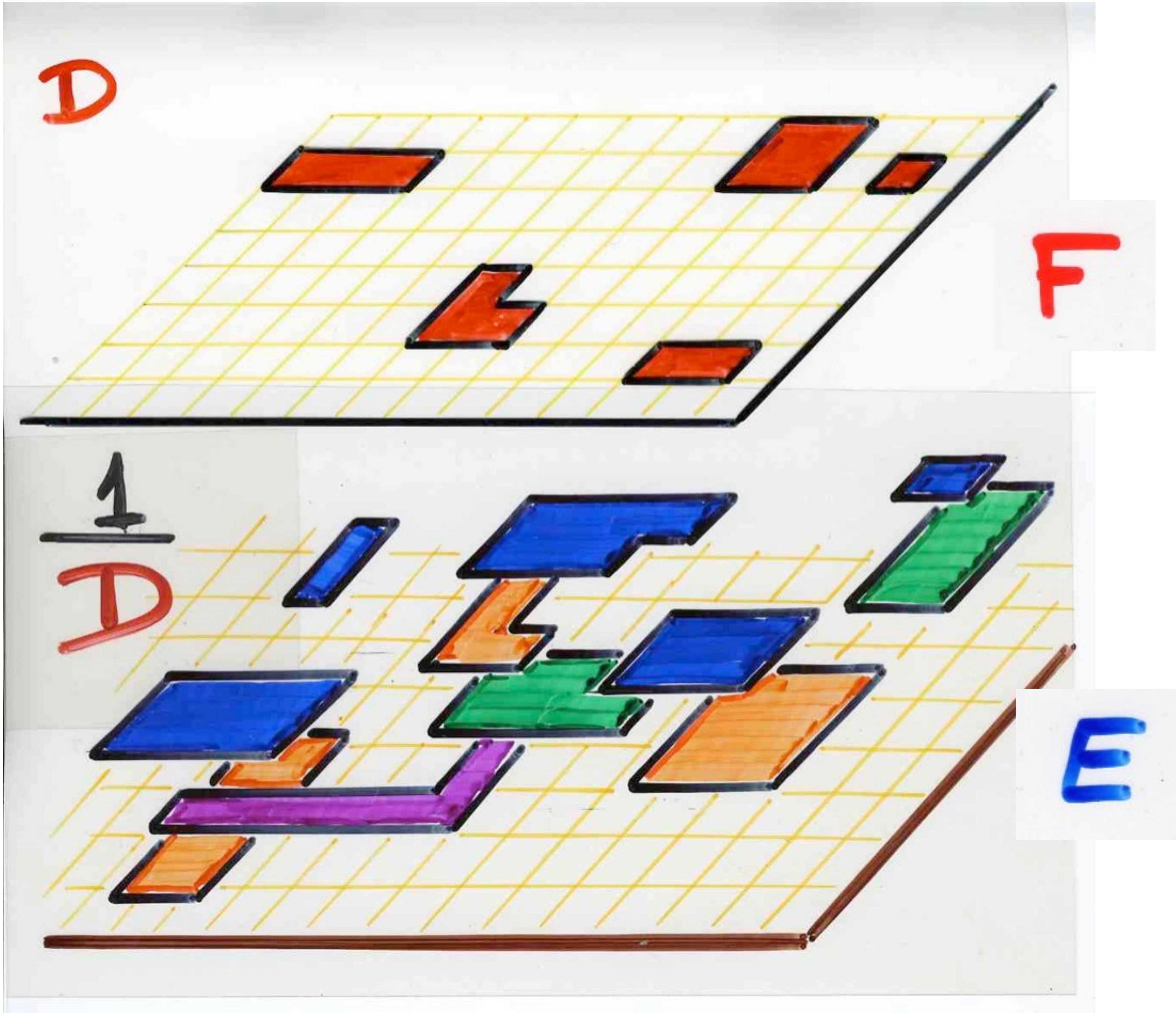
D
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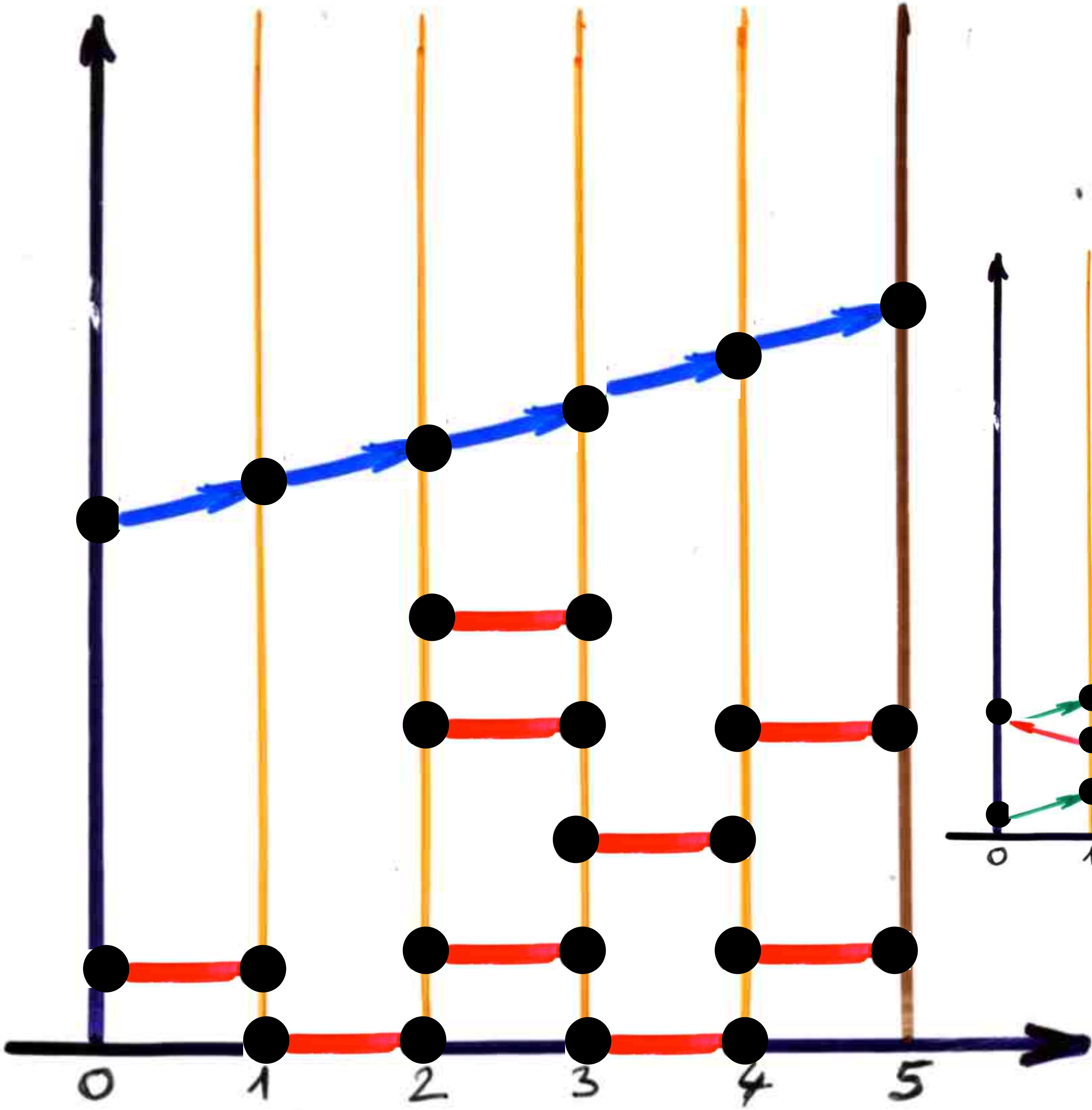
examples:

heaps of dimers
on a segment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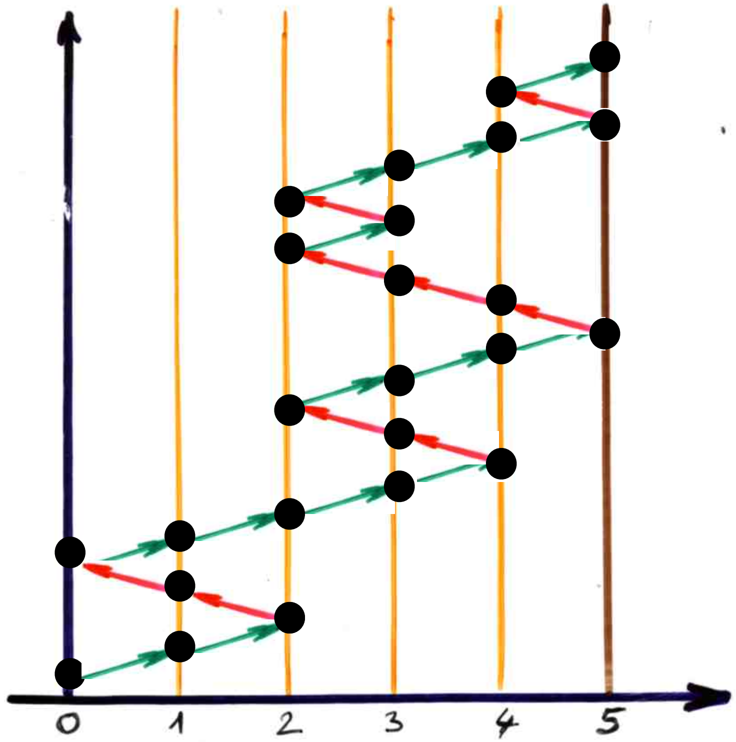
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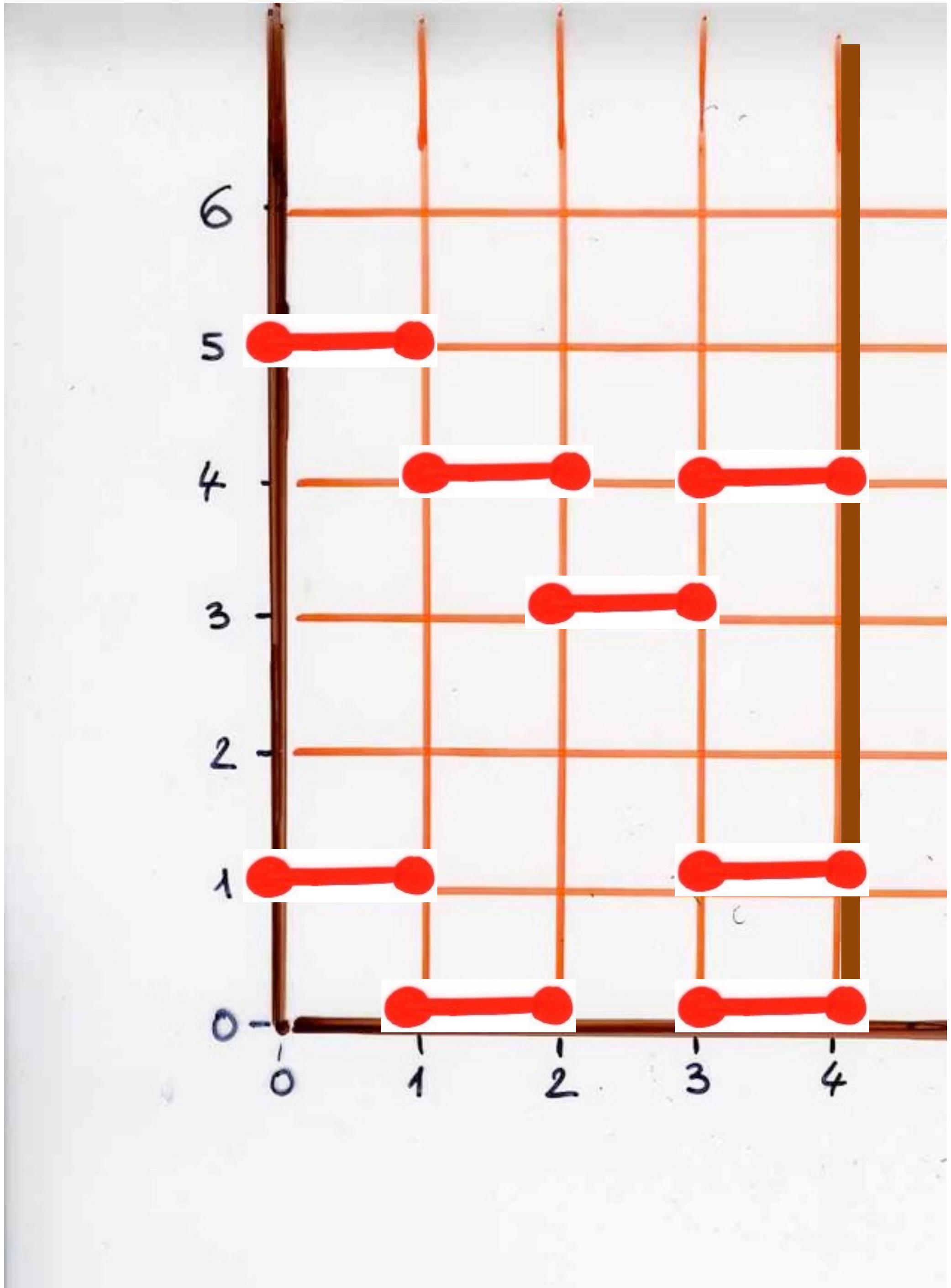
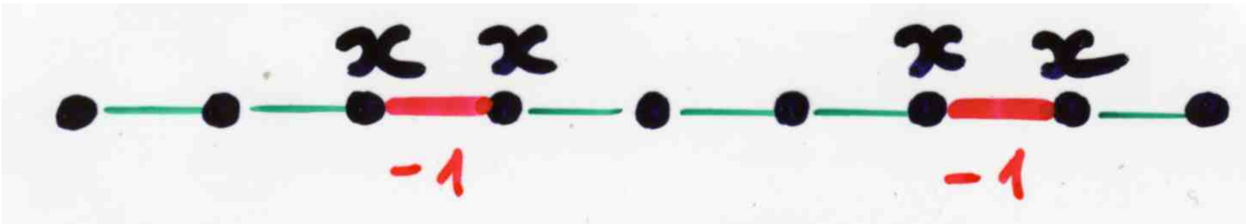


generating function
of heaps of dimers
on the segment $[0, k]$
(enumerated by the number of dimers)

$$\frac{1}{F_{k+1}(t)}$$

$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k$$

$$= \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0,n-1]}} (-x)^{|\alpha|}$$



$$F_n(x^2)$$

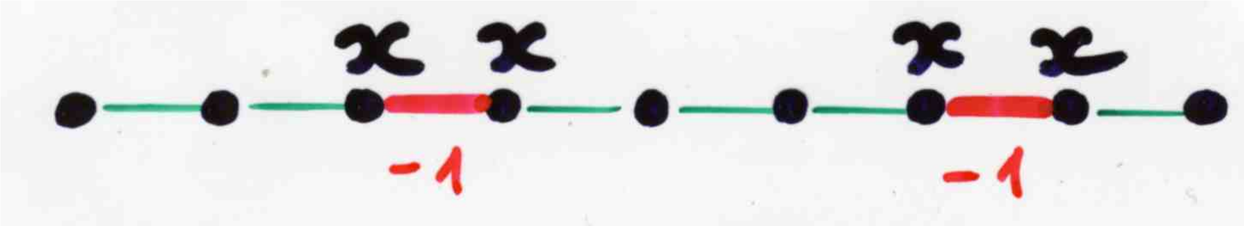
Reciprocity

reciprocity

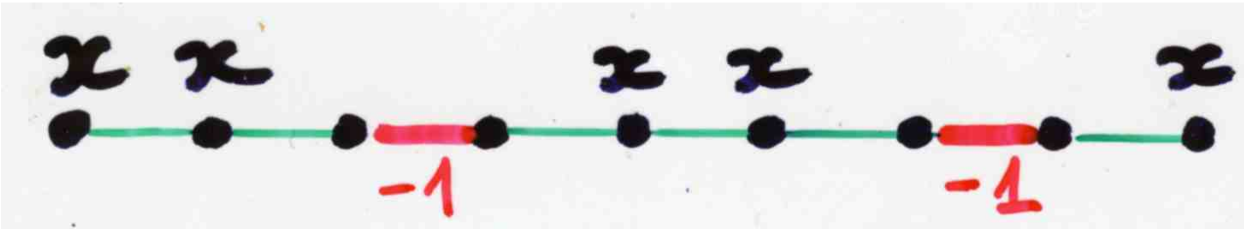
$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

$$\frac{1}{F_n(x^2)}$$

$$-f(1/t) = \sum_{n \geqslant 1} a_{-n} t^n$$



$F_n(x^2)$

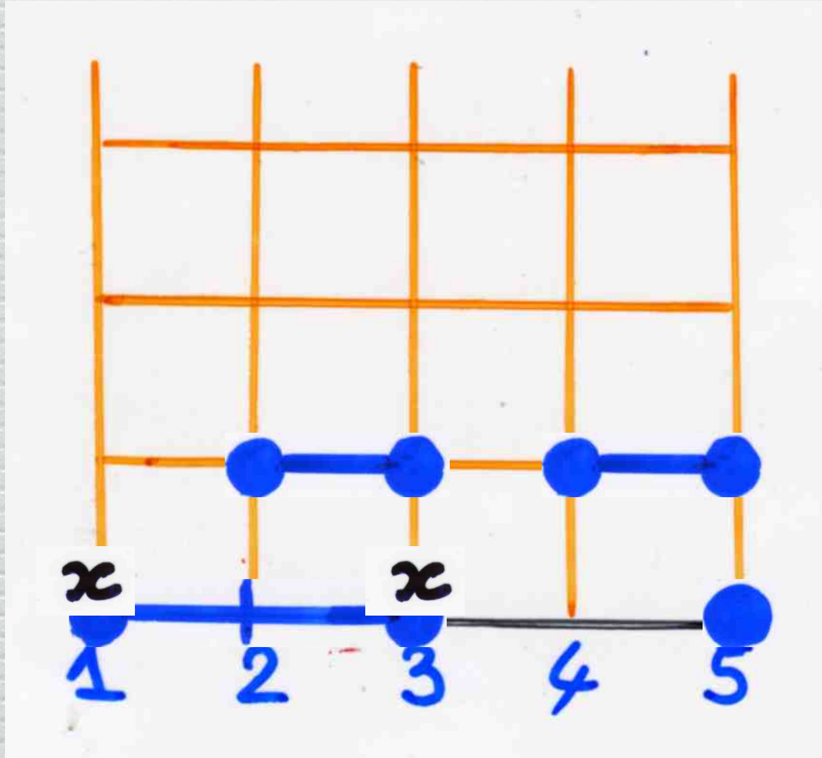


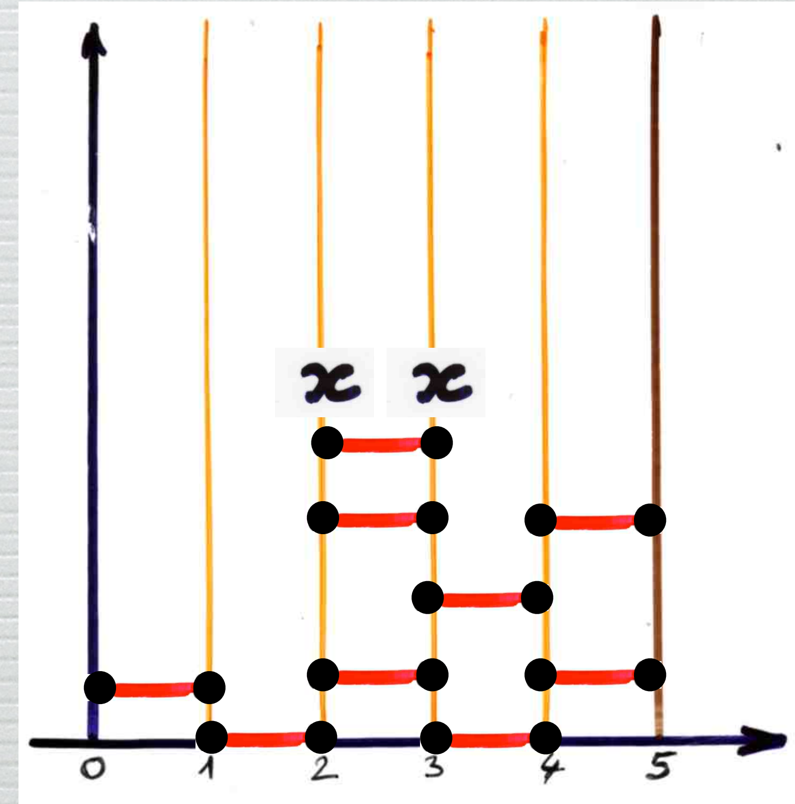
$S_n(x)$

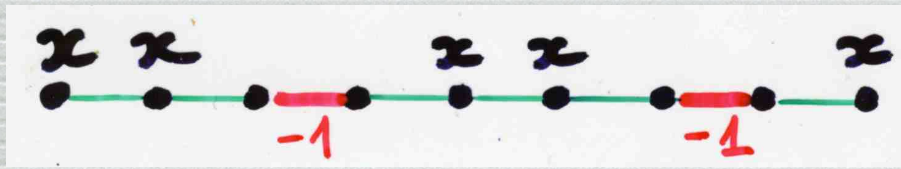
$$S^*_n(x) = x^n S_n(1/x) = \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0,..,n-1]}} (-x^2)^{|\alpha|} = F_n(x^2)$$

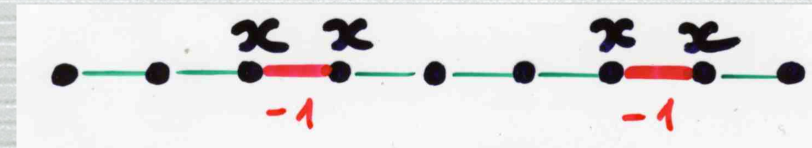
reciprocal polynomial

reciprocity (even case)





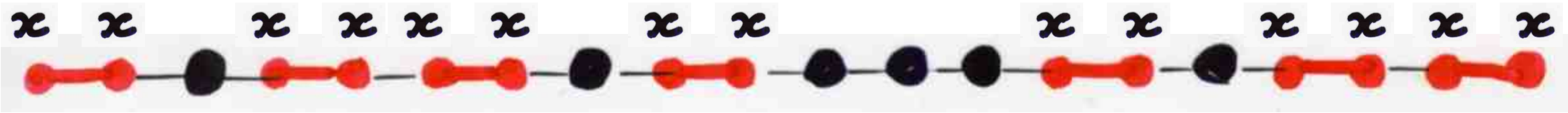


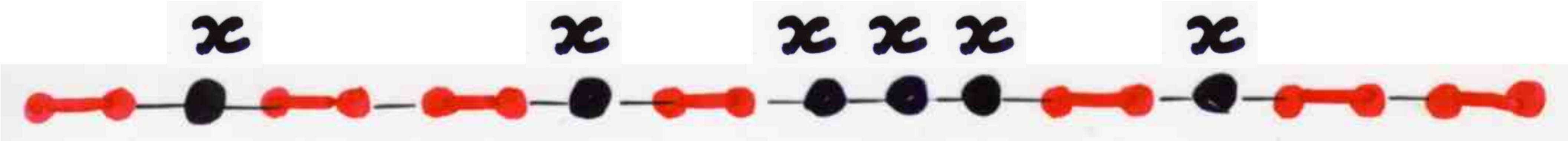


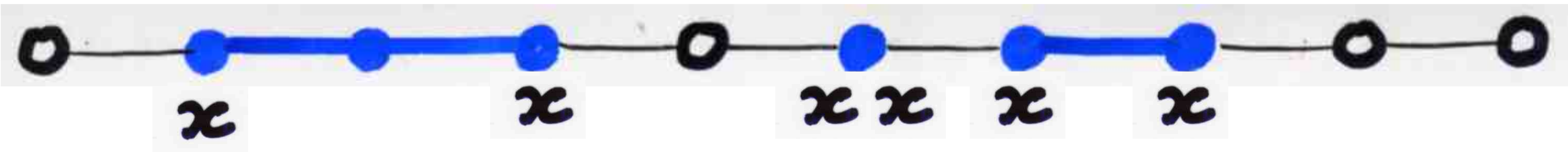
$S_n(x)$

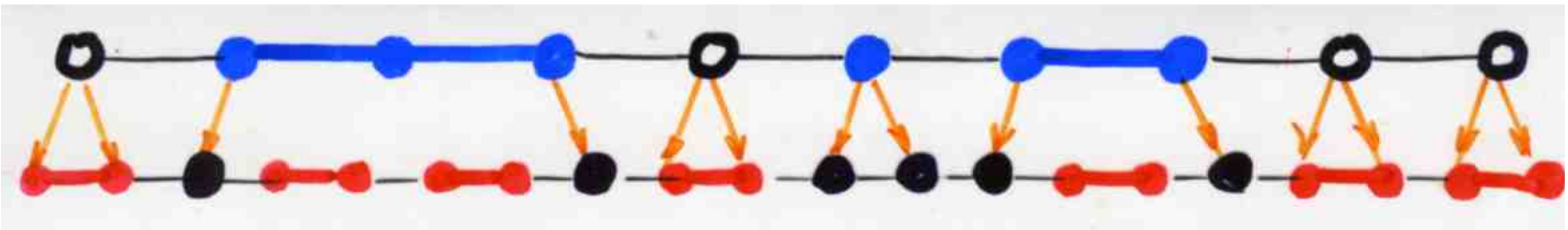
$F_n(x^2)$

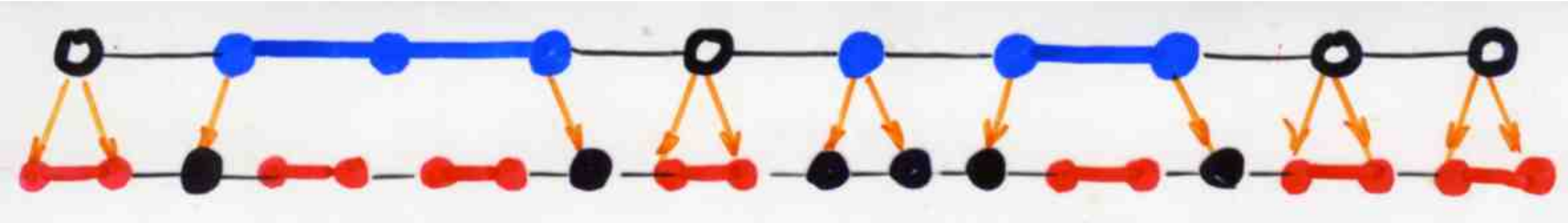
$S^*_n(x) = x^n S_n(1/x)$


x x x x x x x x x x x x x x x


x x x x x x

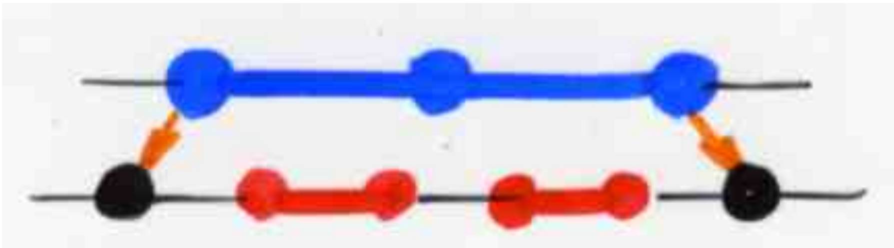

x x x x x x



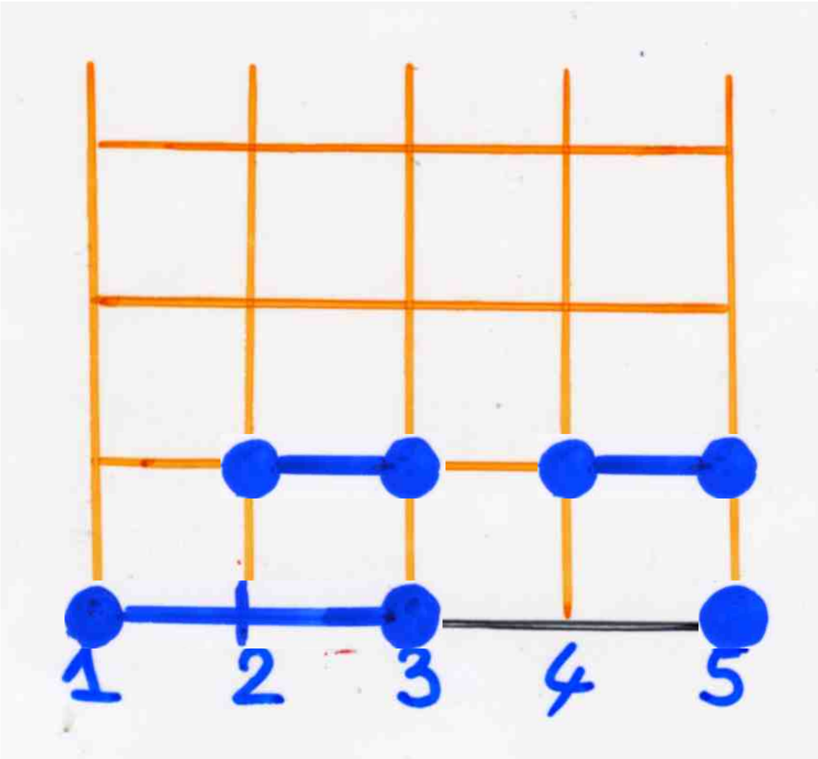


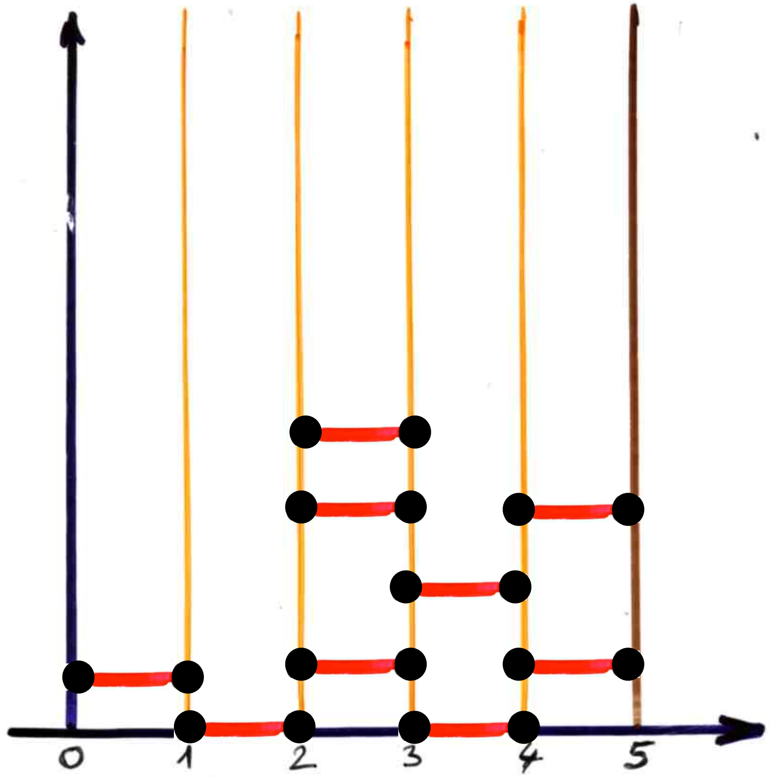
$k = 10$

$2k = 20$



reciprocity





$$\left| \mathcal{A}_{2n}^{(k)} \right| = (-1)^{k+1} \mathcal{D}_{-2n-2k}^{(2k-1)}$$

Fibonacci
and
Tchebychev polynomials

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind

$$U_n(x) = S_n(2x)$$

$$S_n^*(x) = x^n S_n(1/x) = F_n(x^2)$$

About Tchebychev and Fibonacci polynomials

More details in the video-book « ABjC », Part I,

An introduction to enumerative, algebraic and bijective combinatorics

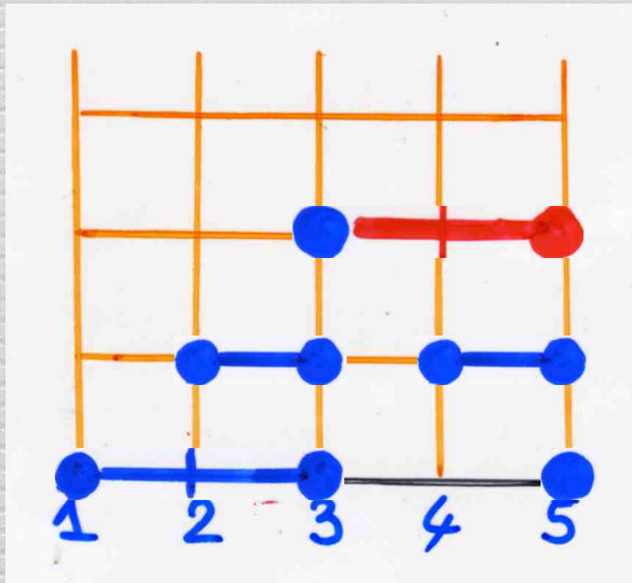
IMSc, Chennai, 2016, Chapter 1c, pp 30-49

www.viennot.org/abjc1-ch1.html

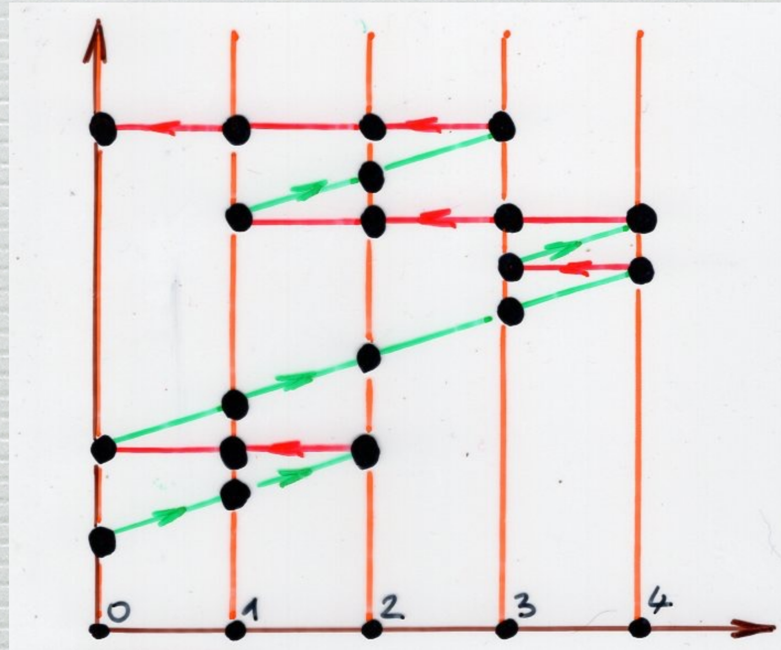
slide added after the talk

reciprocity (even case)

$$\left|\mathcal{A}^{(k)}_{2n-1}\right| = C^{(2k-1)}_{-2n}$$



$\longleftrightarrow$



reciprocity (odd case)

extension of the inversion lemma

N/D

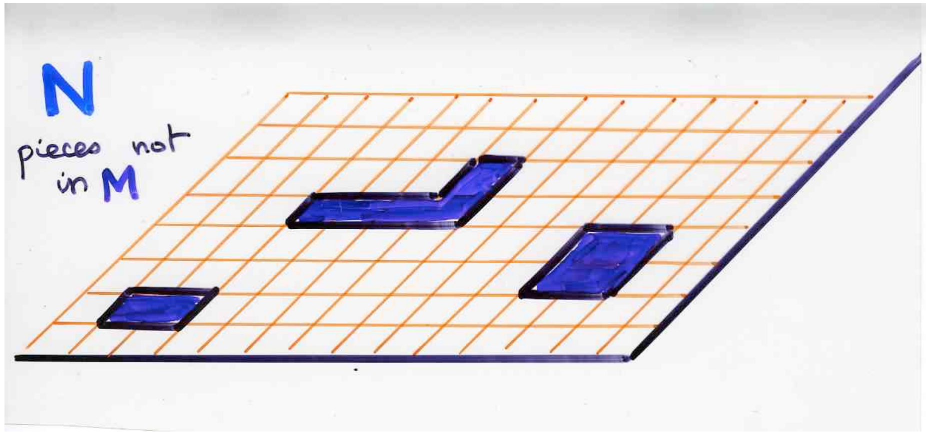
extension of the inversion lemma

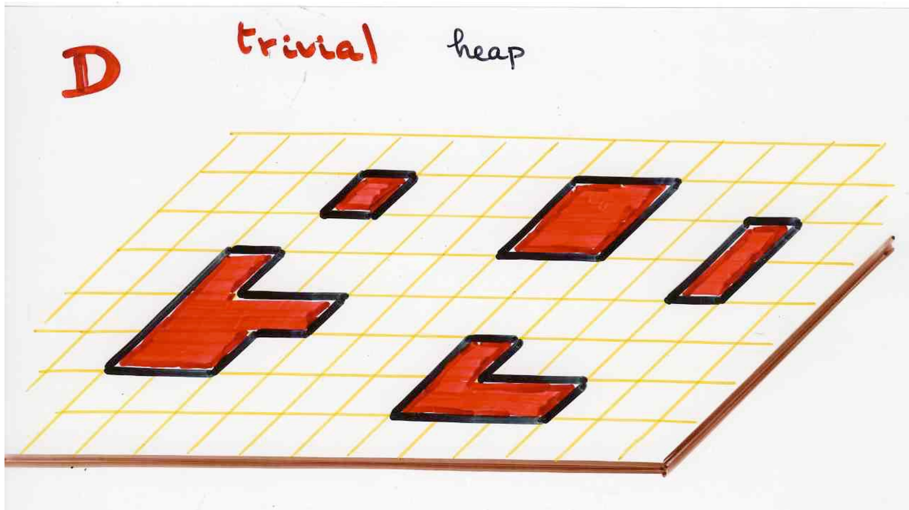
$M \subseteq P$

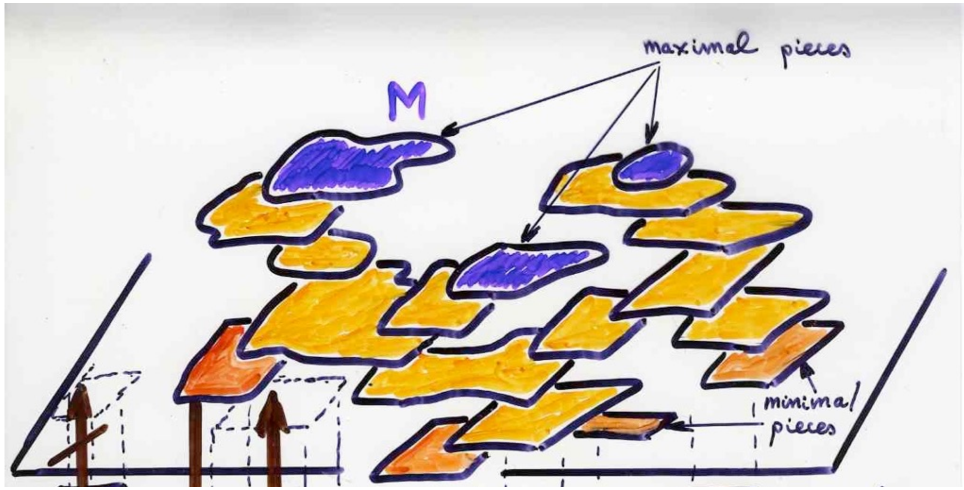
$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

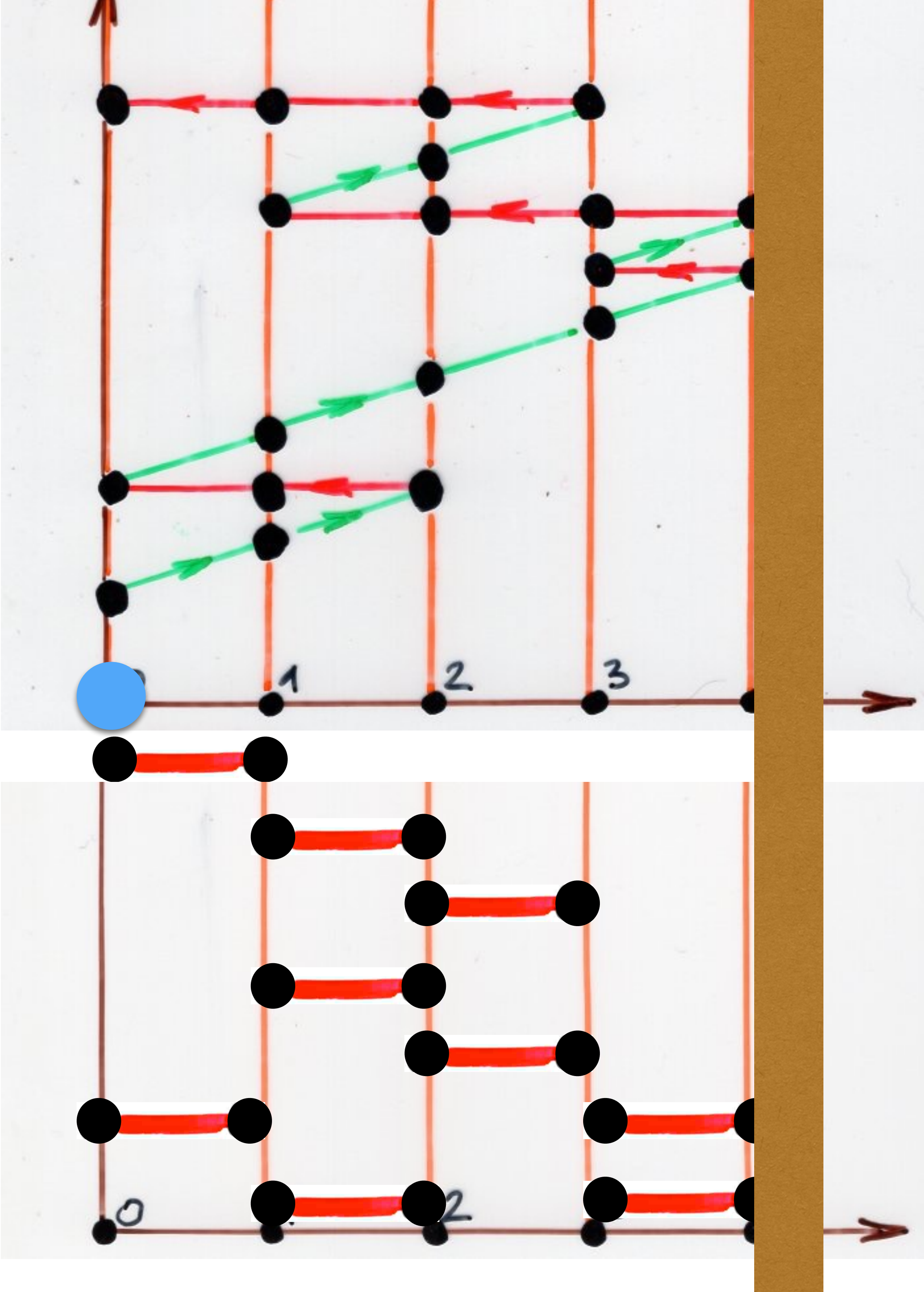
$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$

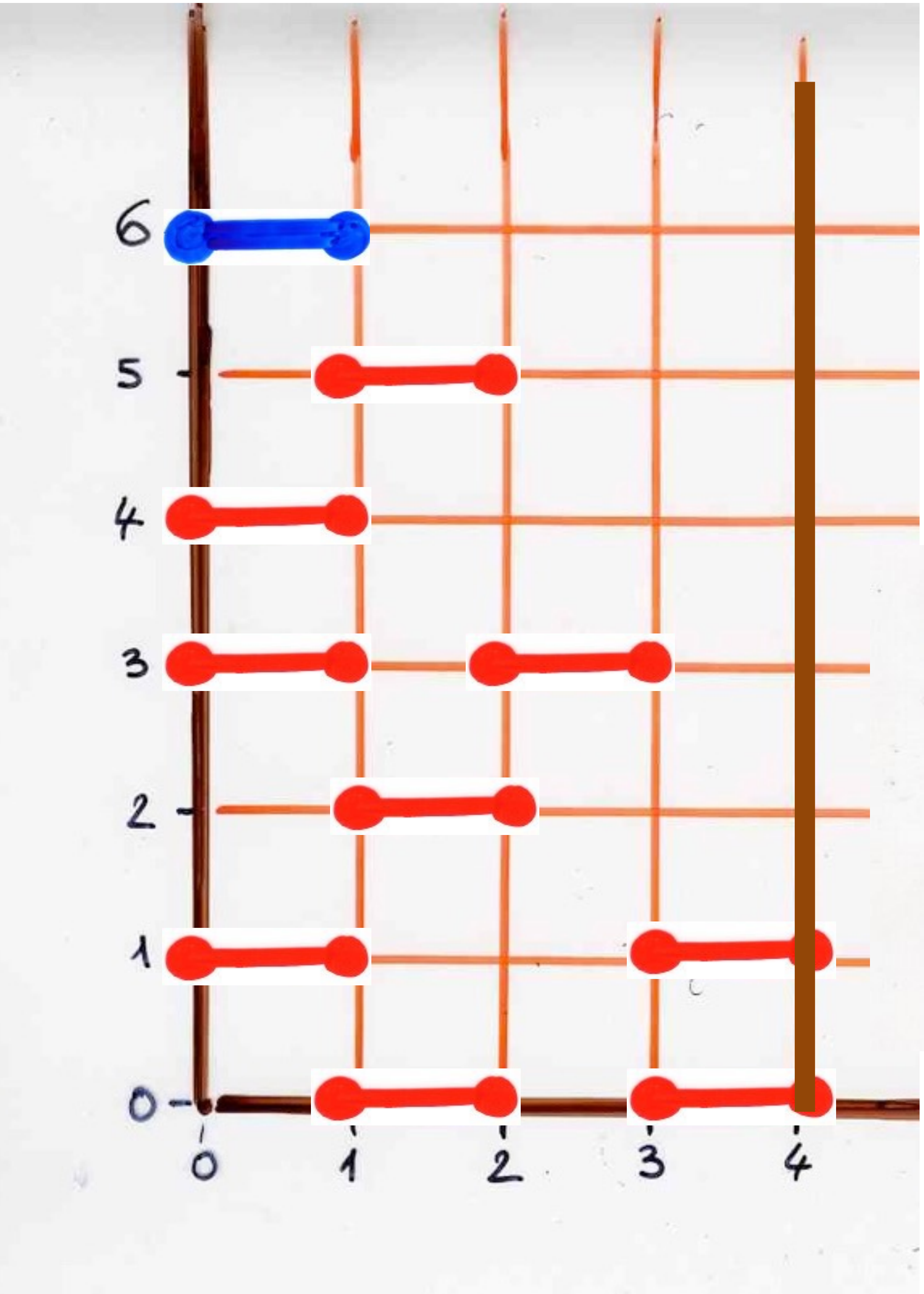

N
pieces not in M


D
trivial heap


maximal pieces
M
minimal pieces



$$C_{2n}^{(k)}$$

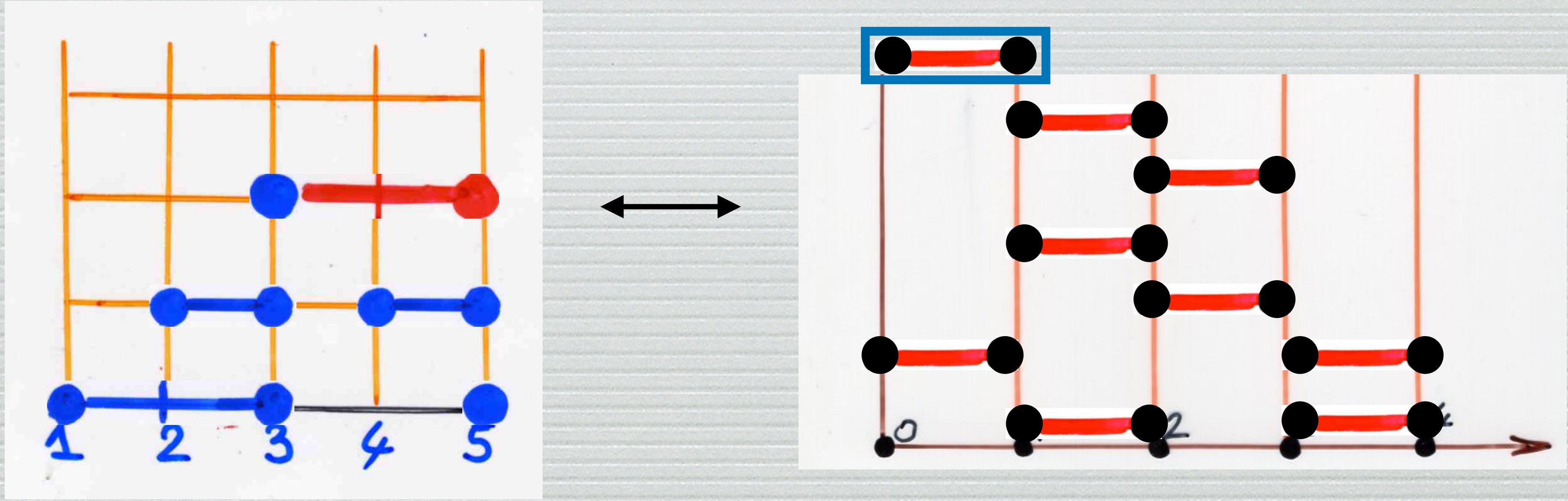


$$\frac{F_k(t)}{F_{k+1}(t)}$$

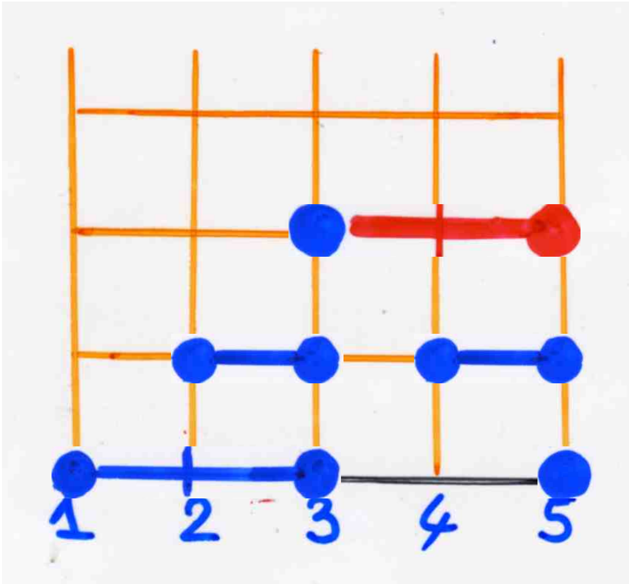
generating function
of semi-pyramids of dimers
on the segment $[0, k]$
(enumerated by the
number of dimers)

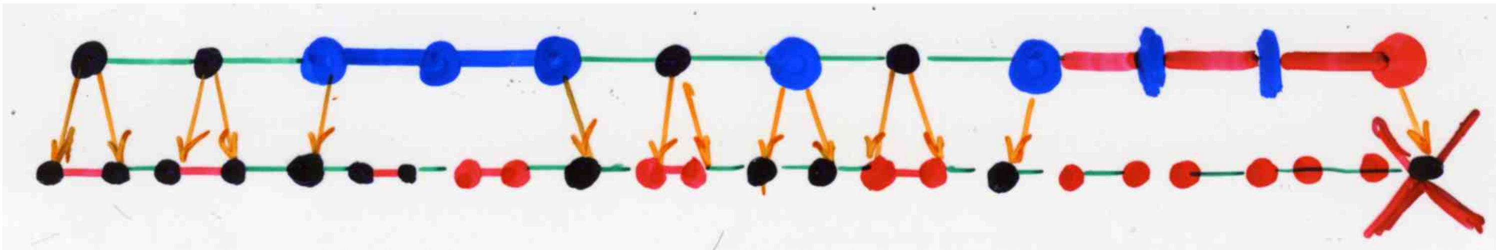
reciprocity odd case

$$\left|\mathcal{A}^{(k)}_{2n-1}\right| = C^{(2k-1)}_{-2n}$$

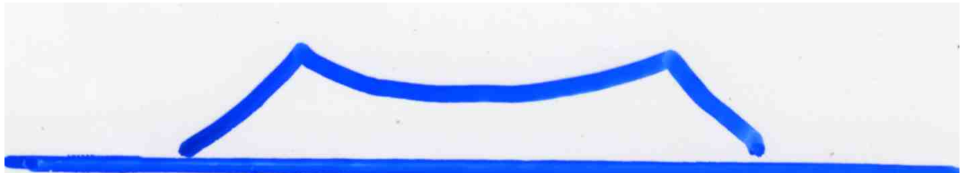


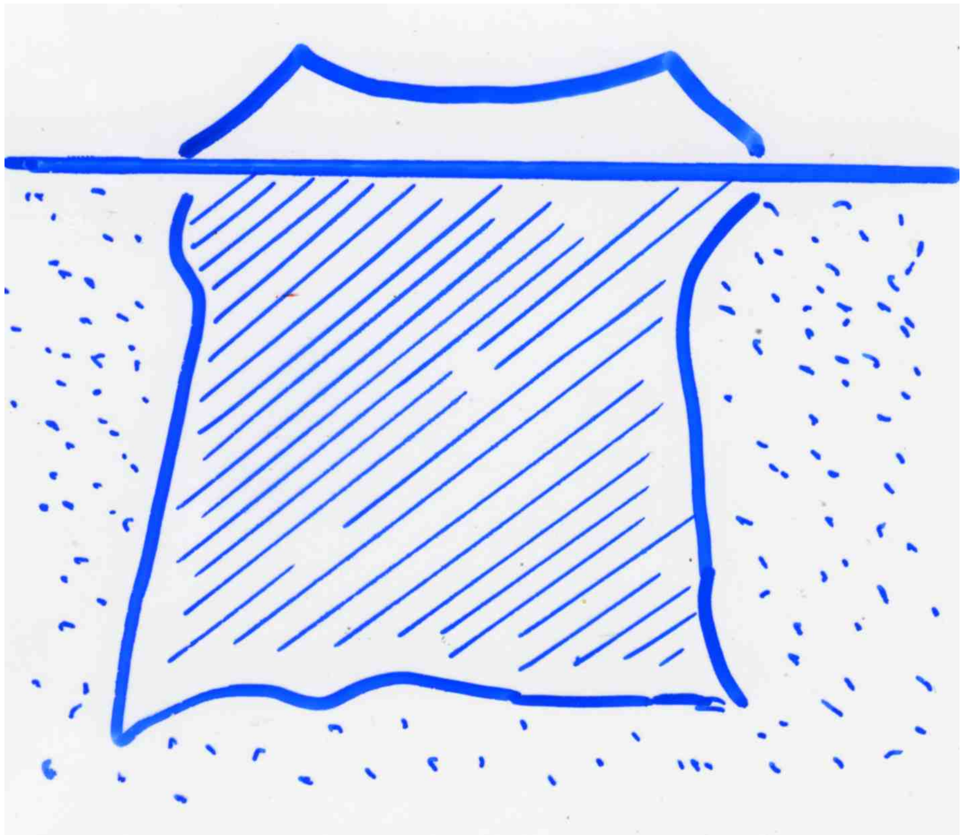
$$\frac{F_k(t)}{F_{k+1}(t)}$$

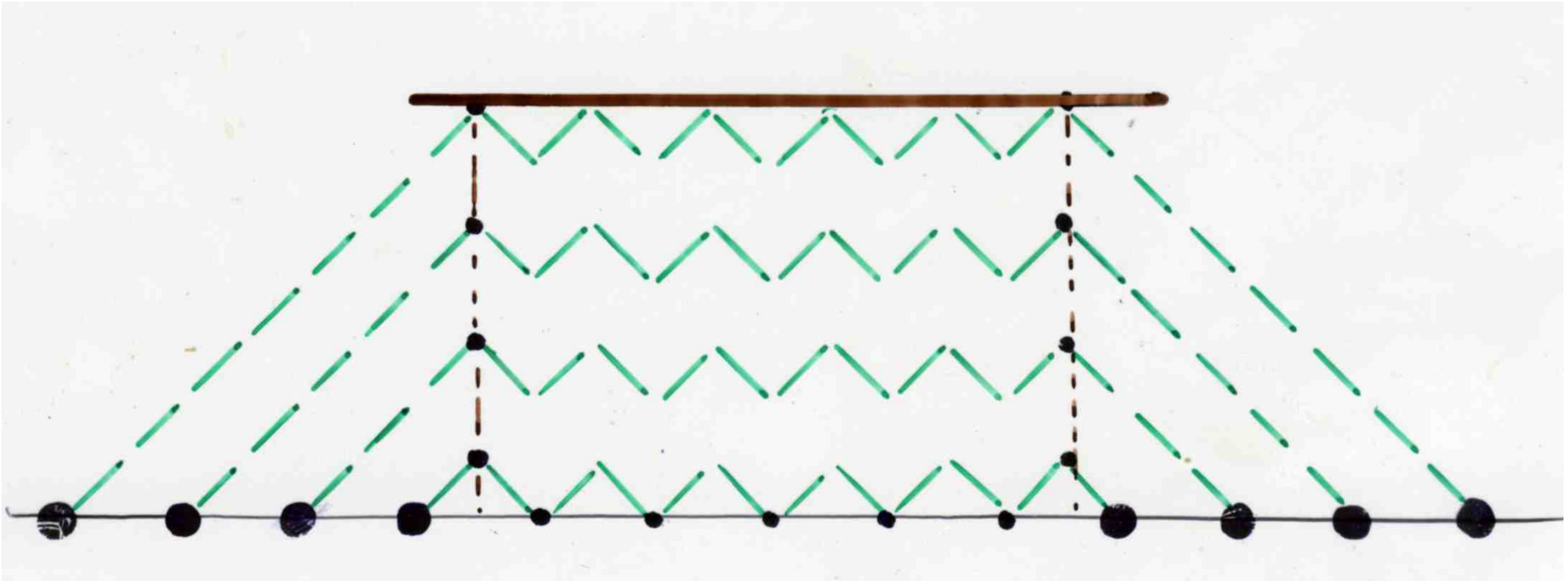

1
2
3
4
5

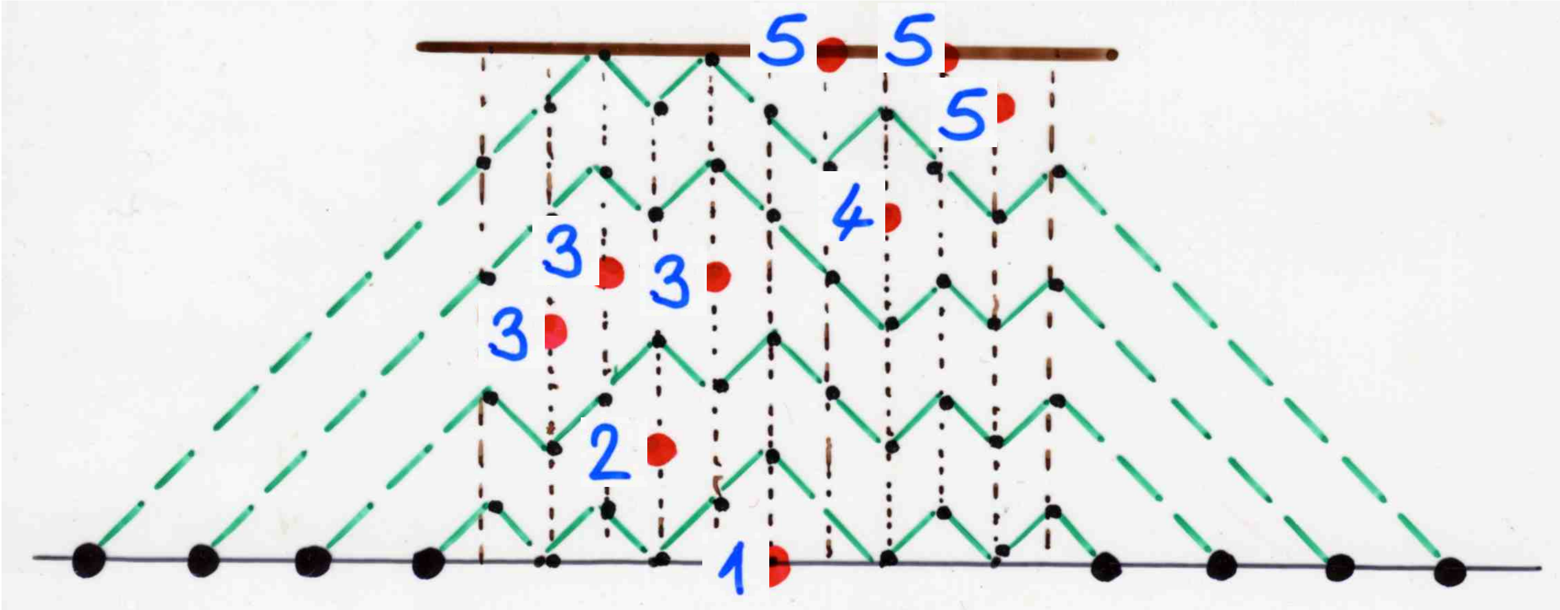


Tip of the iceberg …

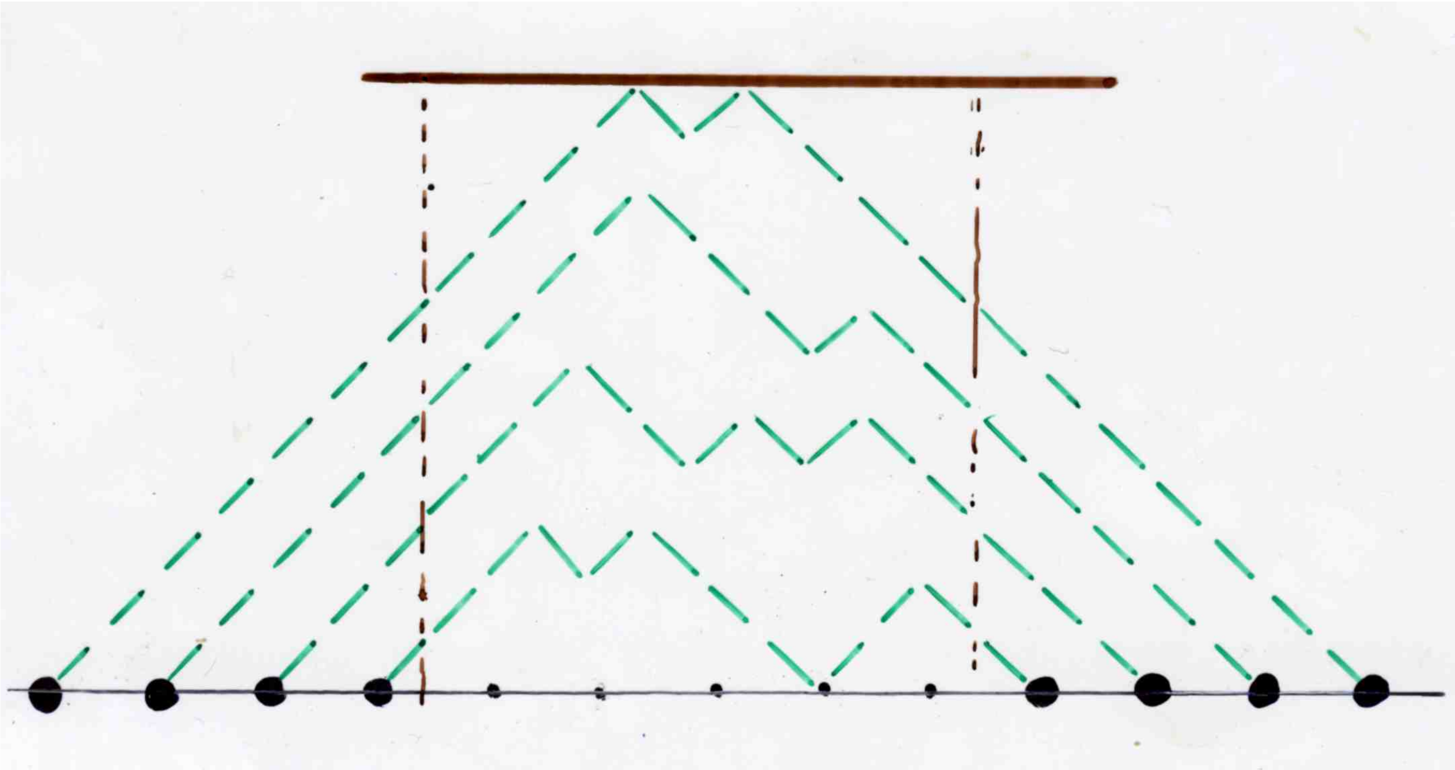


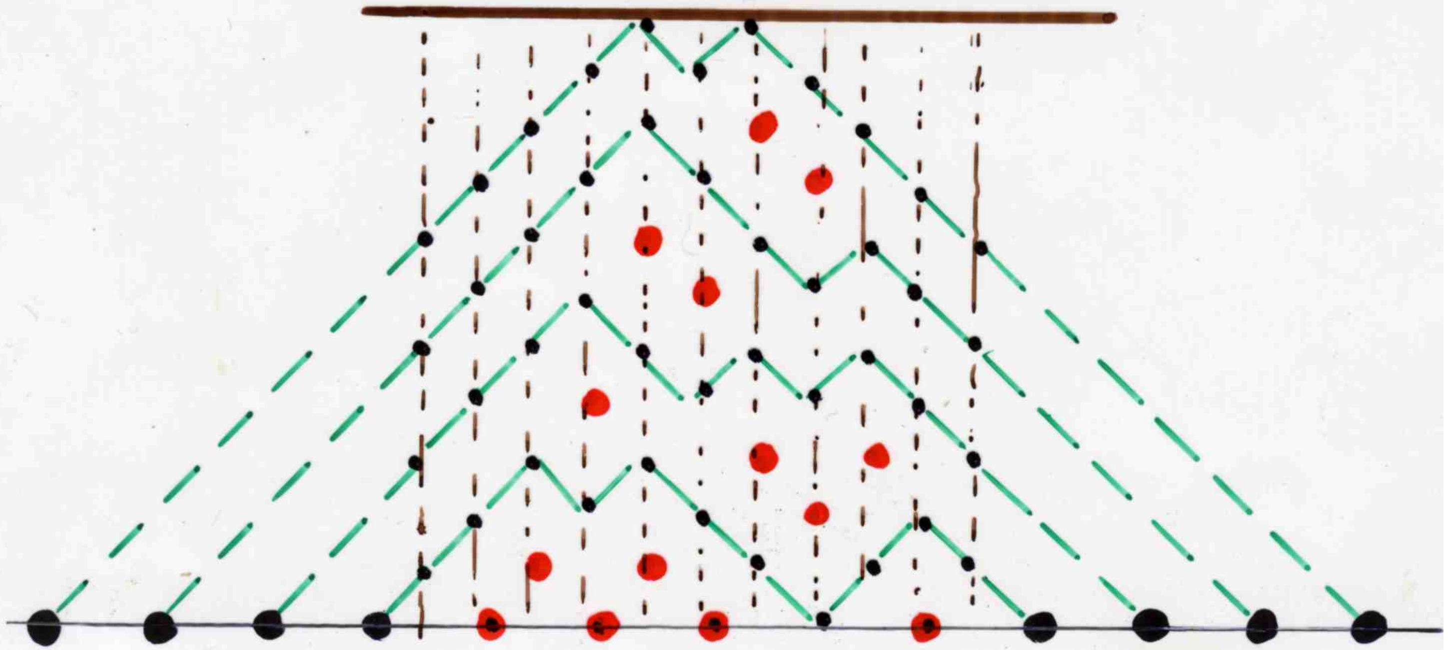


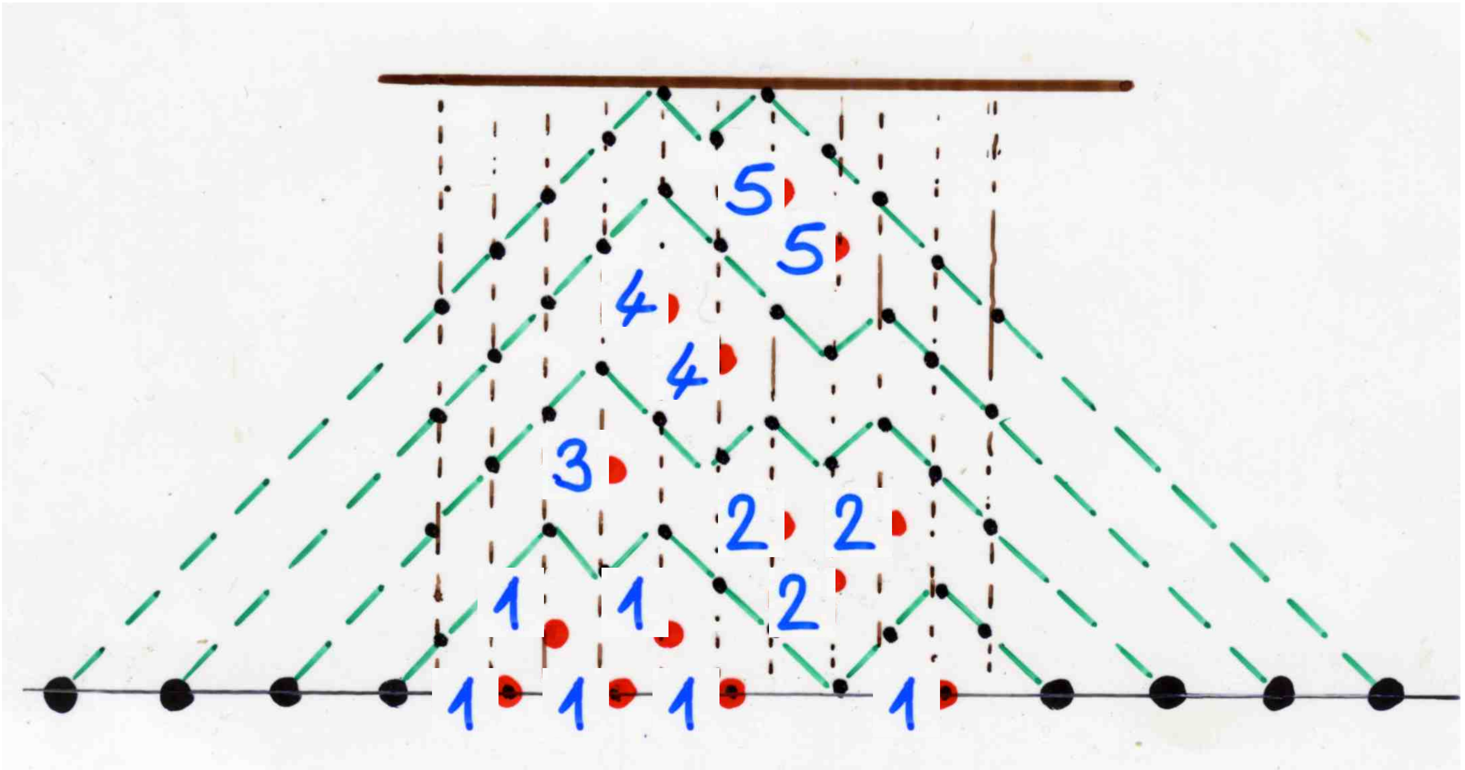




5
5
5
4
3
3
3
2
1







3 4 4 5 5

1 1 1 1 1 2 2 2 1

Orthogonal polynomials
continued fractions

A.M. Garsia, Ö. Eğecioğlu (2020)
Lecture in algebraic Combinatorics

M.E.H. Ismail

talk this meeting
Monday 31

More details in the video-book « ABjC », Part IV, *A combinatorial theory of orthogonal polynomials and continued fractions*

IMSc, Chennai, 2019 www.viennot.org/abjc4.html

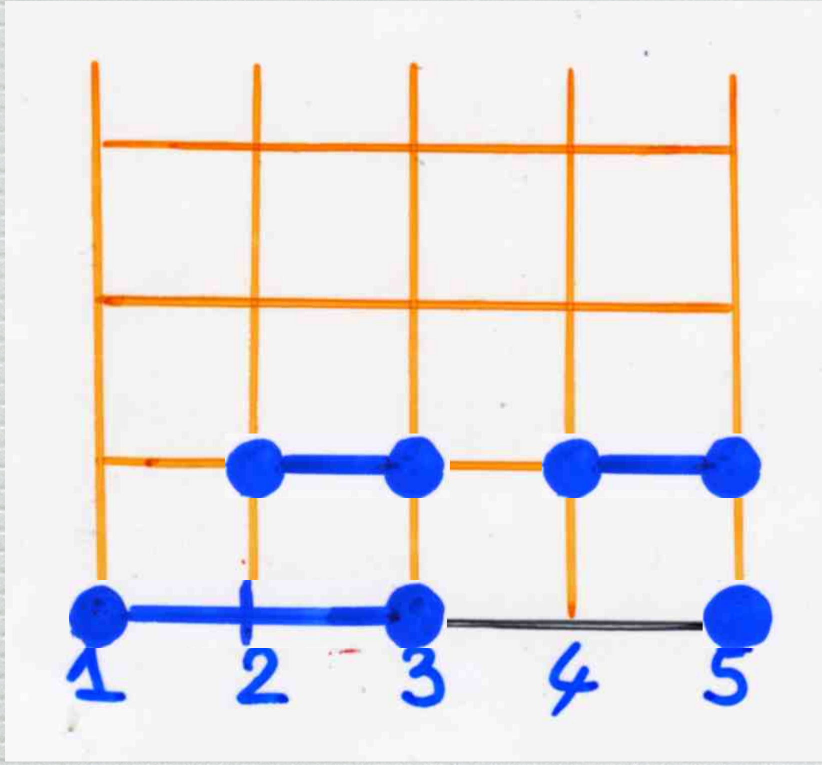
Interpretation of continued fractions and moments of orthogonal polynomials with semi-pyramids of dimers and monomers

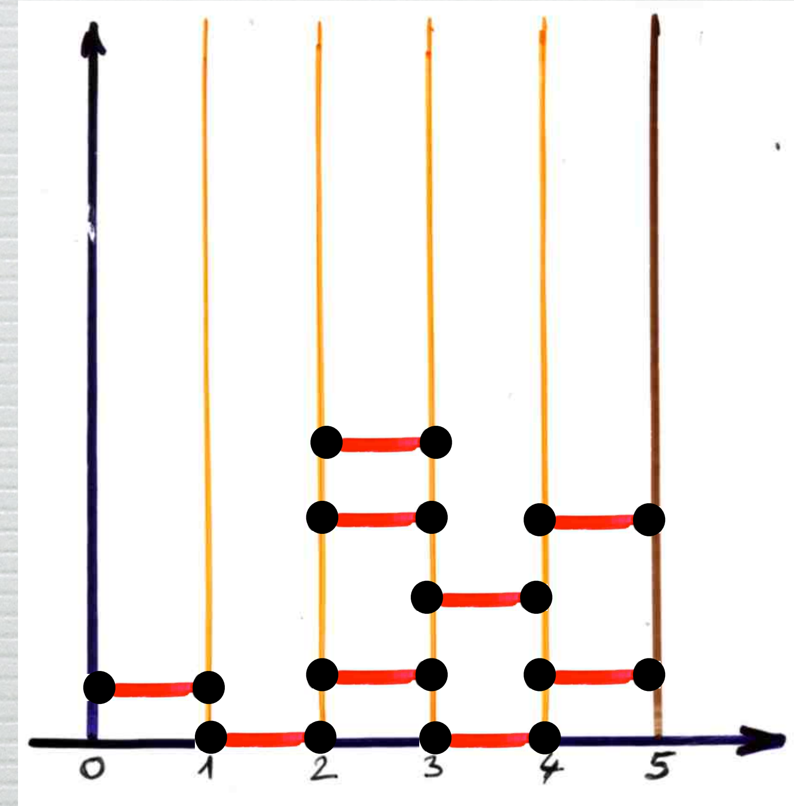
Chapter 3b, pp 137-147

www.viennot.org/abjc4-ch3.html

slide added after the talk

The duality





In the context of
fully commutative elements
in Coxeter groups

before that:

taking the rightmost maximal picece

remind something related to ....

Ramanujan

Andrews theorem about the «reciprocal» of Ramanujan continued fraction

quasi-partitions of n

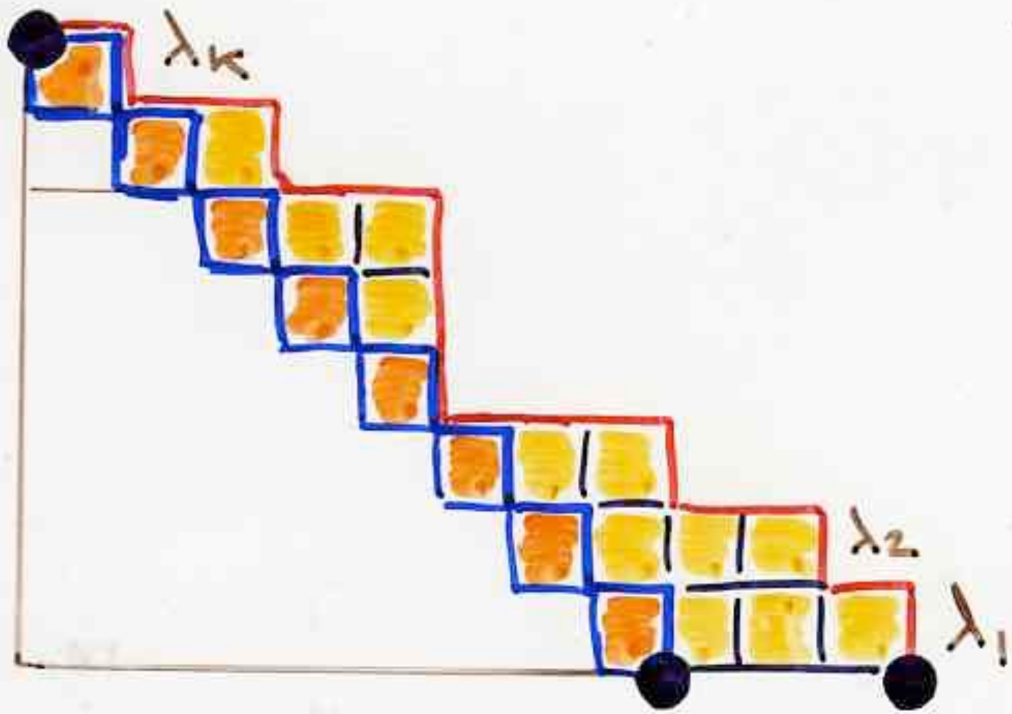
G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$1 + \lambda_i \geq \lambda_{i+1}$$

$$i = 1, \ldots, k-1$$

$$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$$

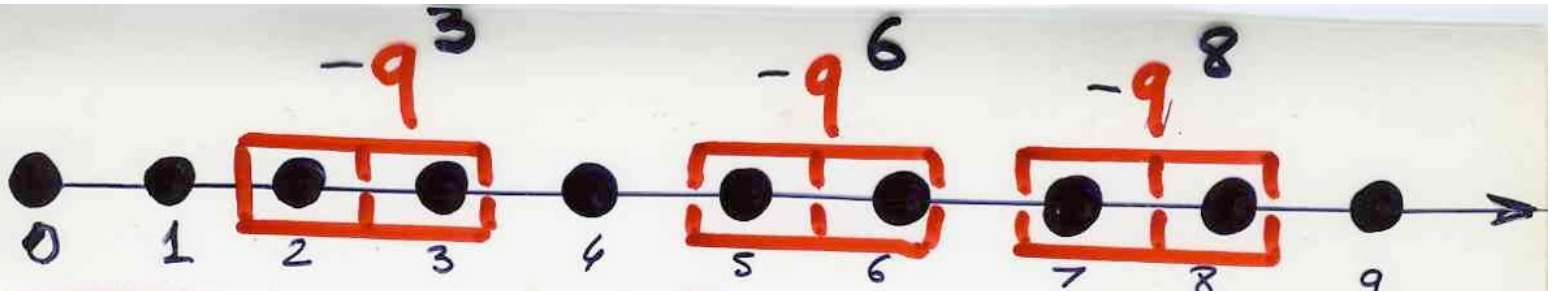


$$\frac{1}{R_I} = \sum_{\substack{\lambda \\ \text{quasi-} \\ \text{partitions}}} (-1)^{\ell(\lambda)} q^{|\lambda|}$$

G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

Rogers-Ramanujan 1st identity

$$\mathbf{D} = \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$



$$\mathbf{D} = \sum_{E} (-1)^{|E|} v(E)$$

E trivial heaps of dimers on $\mathbb{N}$

$$v([k-1, k]) = -q^k$$

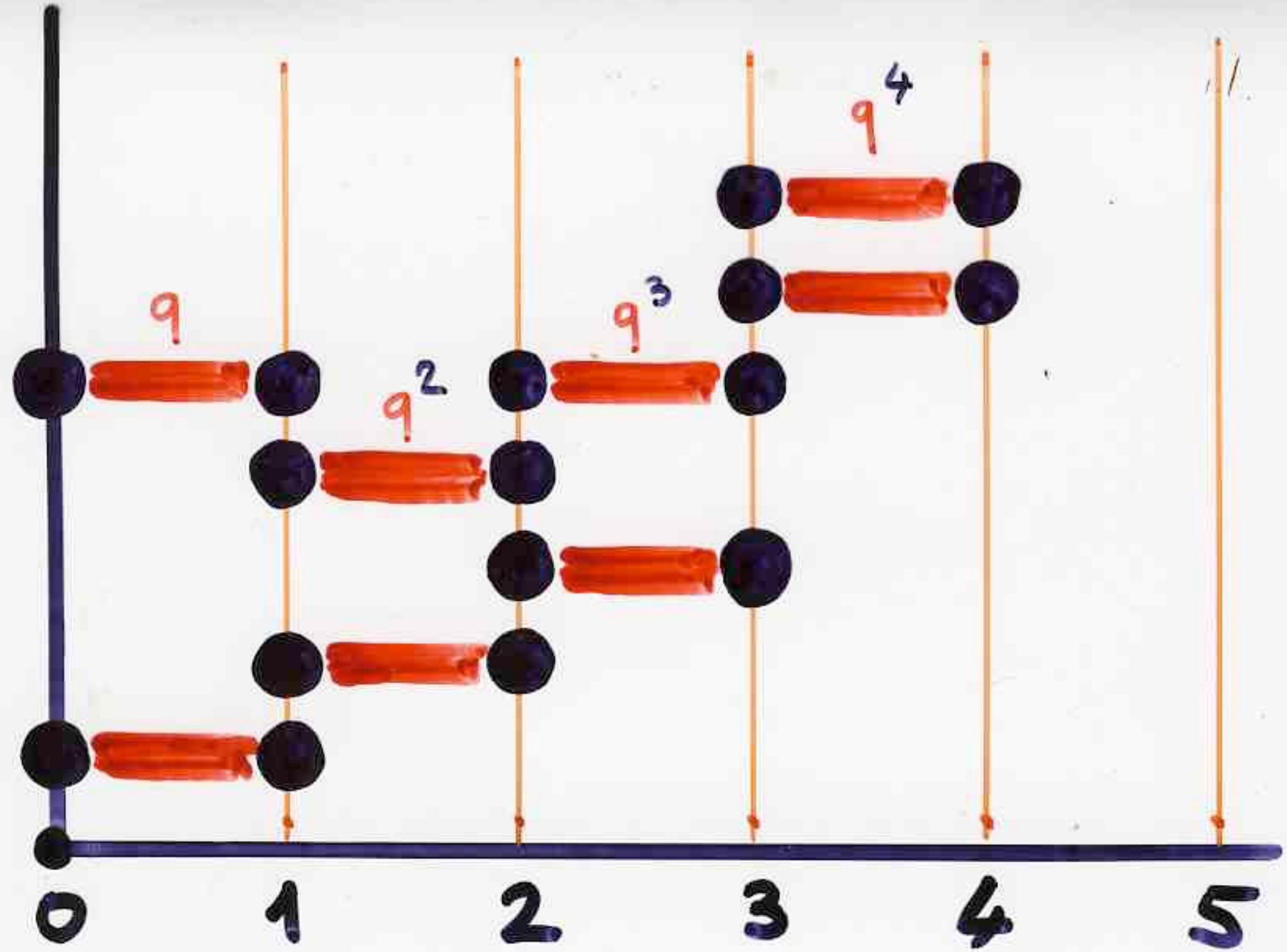
weighted heap $v(E)$

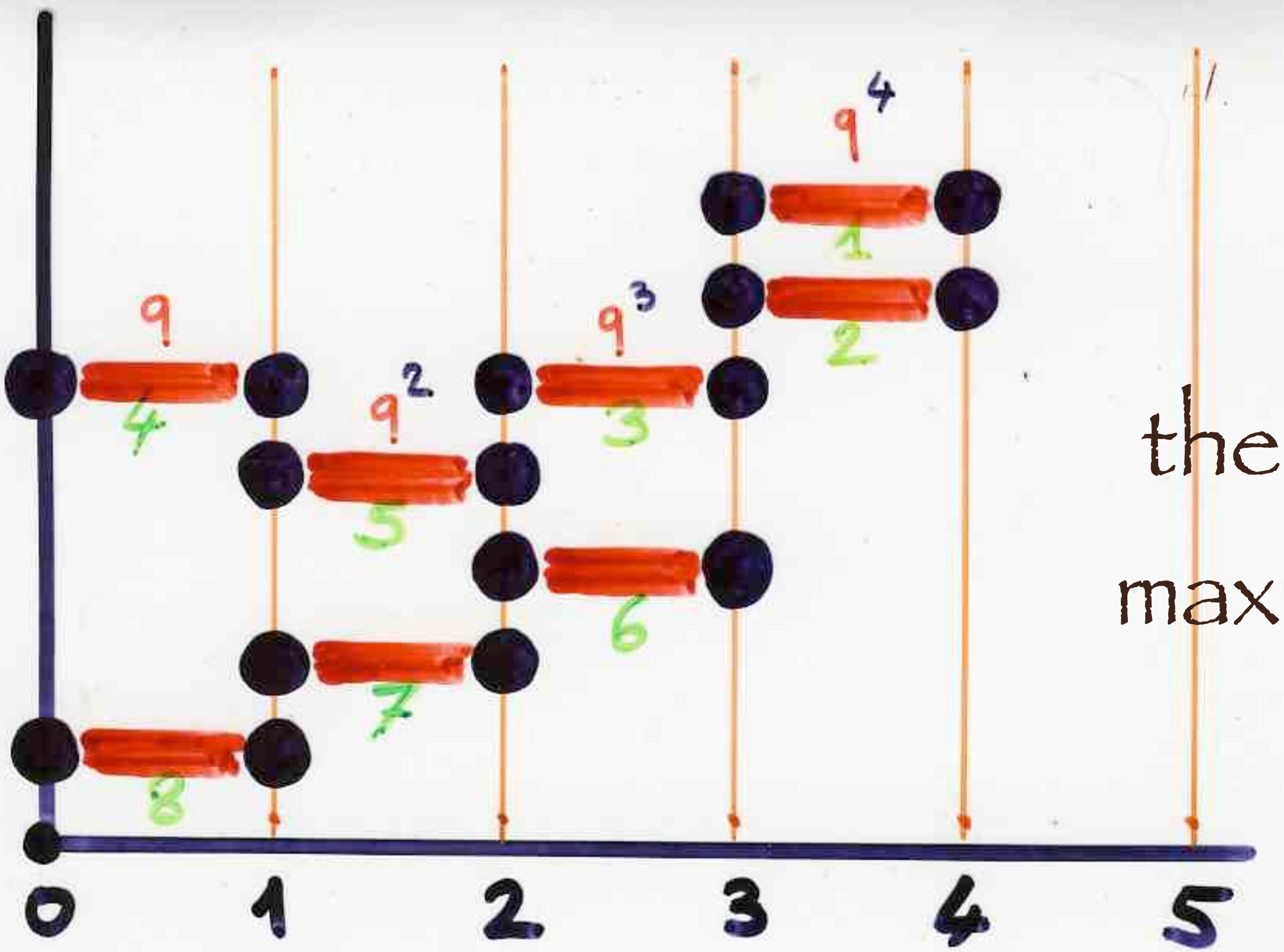
$$v(E) = \prod_{\alpha \in E} v(\alpha)$$

$v(\alpha) = v(\pi(\alpha))$ π "projection"

$$v([i-1, i]) = -q^{i}$$

$$\frac{1}{D} = \sum_{\substack{E \\ \text{heaps} \\ \text{of} \\ \text{dimers}}} v(E)$$


q
q^2
q^3
q^4
0
1
2
3
4
5

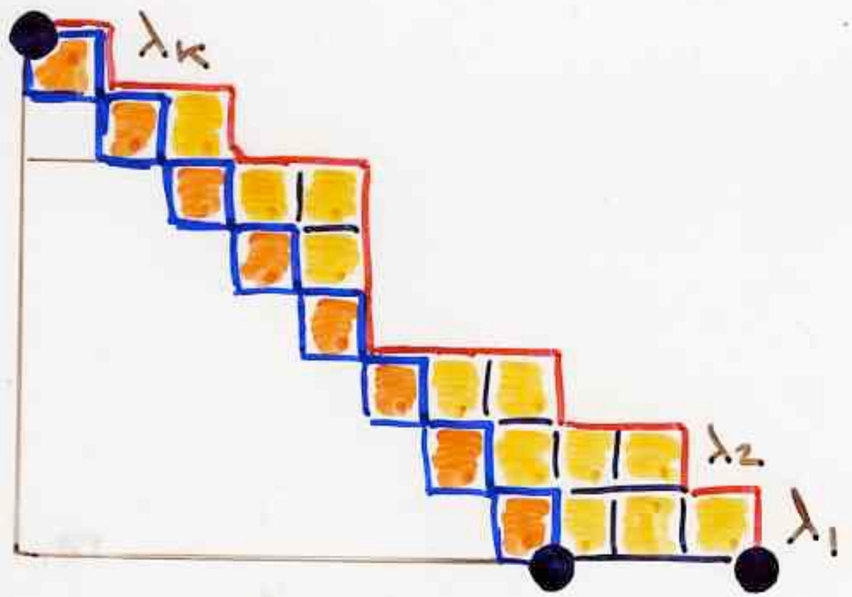


taking
the rightmost
maximal picece

$$H \longrightarrow \lambda = (\underset{1}{4}, \underset{2}{4}, \underset{3}{3}, \underset{4}{1}, \underset{5}{2}, \underset{6}{3}, \underset{7}{2}, \underset{8}{1})$$

quasi-partition

$$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$$



More details in the video-book « ABjC »

« Proofs without words: the exemple of the Ramanujan continued fraction »
colloquium IMSc, Chennai, February 21, 2019

slides and video available at Part II of ABjC,

« Some lectures related to the course »

www.viennot.org/abjc2-lectures.html

or at Part IV of ABjC, Chapter 3,

www.viennot.org/abjc4-ch3.html

paper:

X.V., bijections for the Rogers-Ramanujan reciprocal, J. Indian Math. Soc., 52 (1987) 171-183

slide added after the talk

Some history …

ballot problem

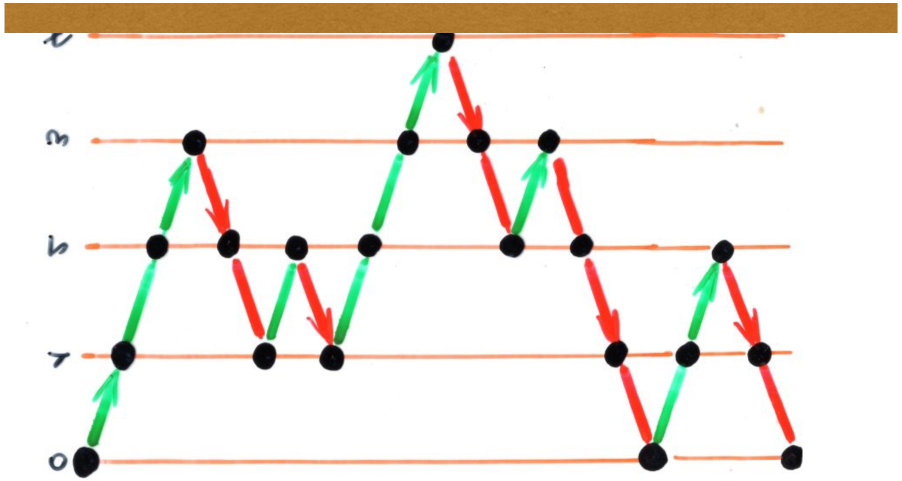
I. Pak "Catalan page"

H. Delannoy (1886)

J. Bertrand (1887)

D. André (1887)

W.A. Whitworth (1879)



$$\sum_{n\geq 0} C_{2n}^{(k)} t^{2n}$$

Rational

N.G. de Bruijn, D.E. Knuth, S.O. Rice (1972)

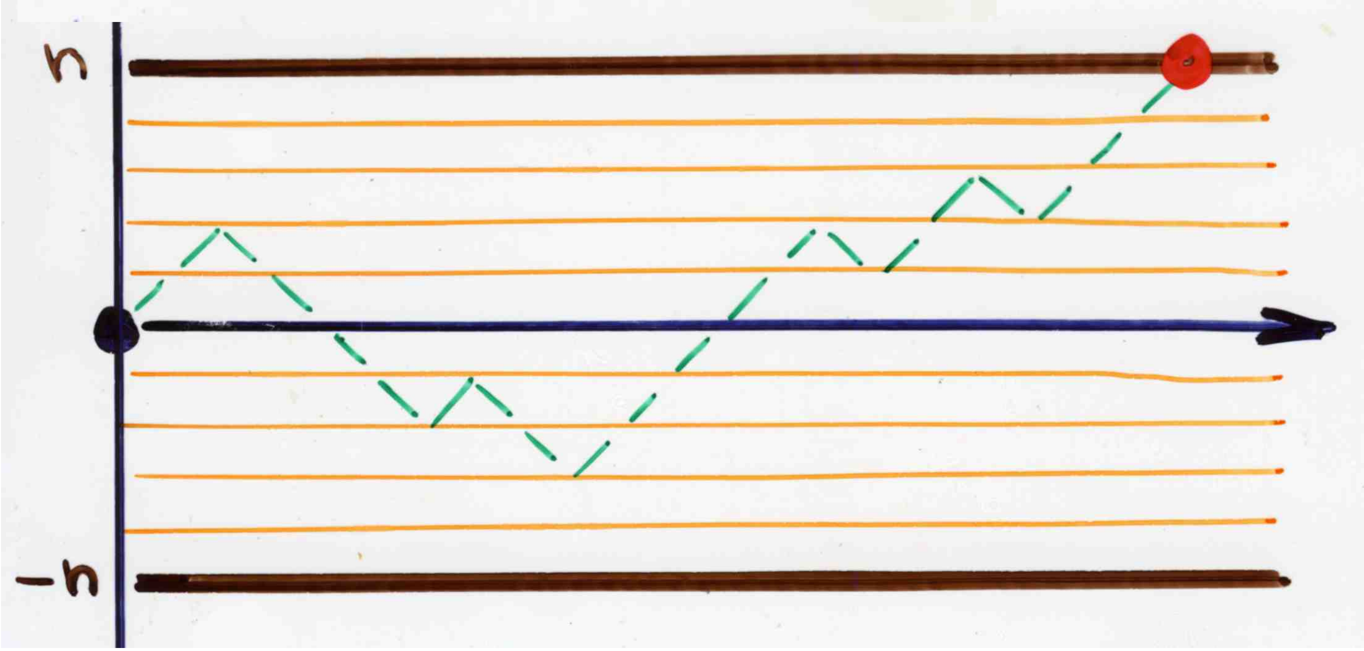
H. Delannoy
(1888)

H. Delannoy (1888)

"Sur la durée du jeu"

" méthode de l'échiquier "

E. Rouché (1888)



$$\frac{(-1)^{m-n}}{n}\sum_{1\le k\le n}(-1)^{k-1}\sin\left(\frac{(2k-1)\pi}{2n}\right)\cos^{m-1}\left(\frac{(2k-1)\pi}{2n}\right) =$$

E. Rouché (1888)

H. Delannoy (1888)

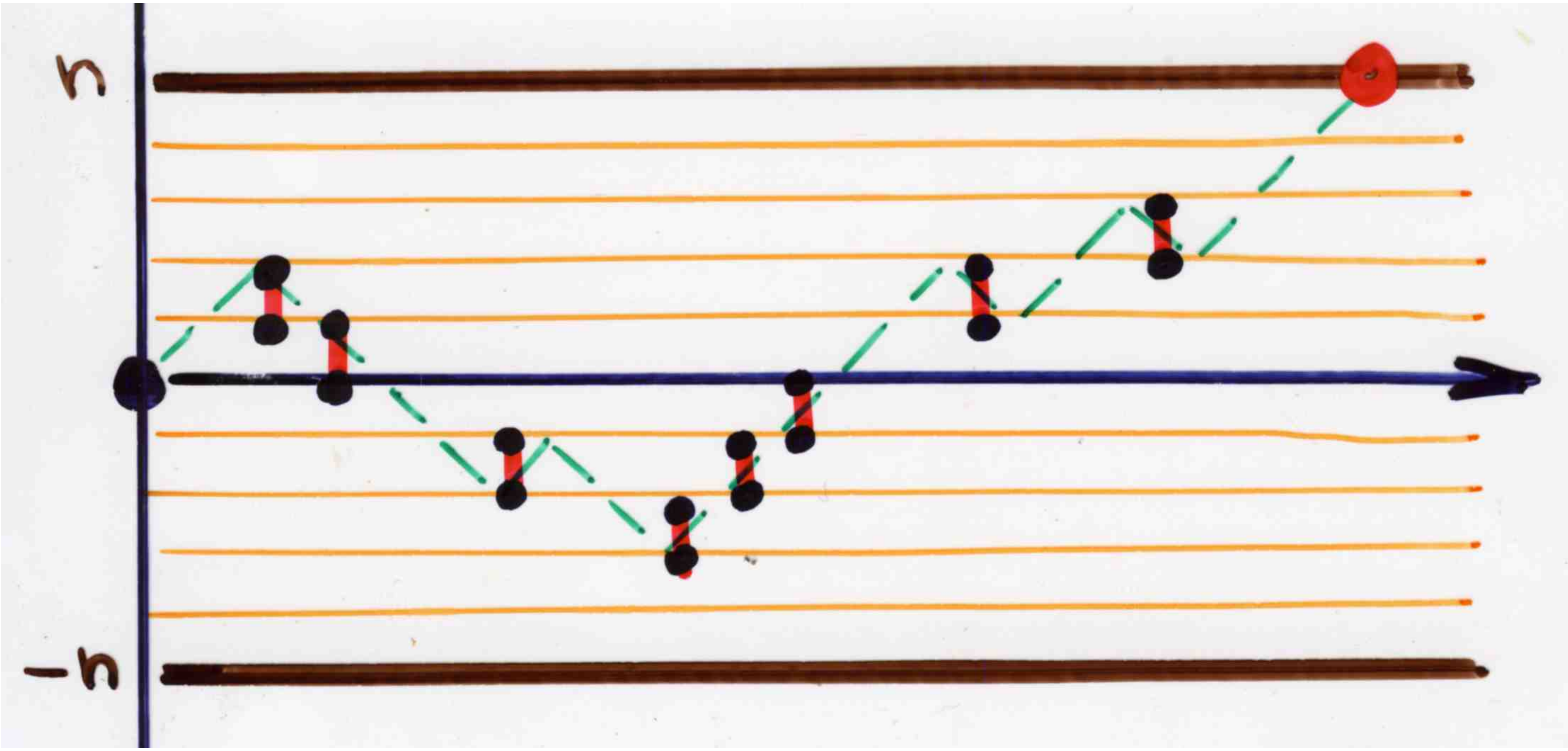
$$\frac{n}{2^{m-1}}\sum_{0\le k\le q/n}(-1)^{k}\,\frac{2k+1}{\frac{m+n}{2}+kn}\binom{m-1}{q-kn}$$

C. Banderier, S. Schwer
5th lattice paths ... Athens (2002)

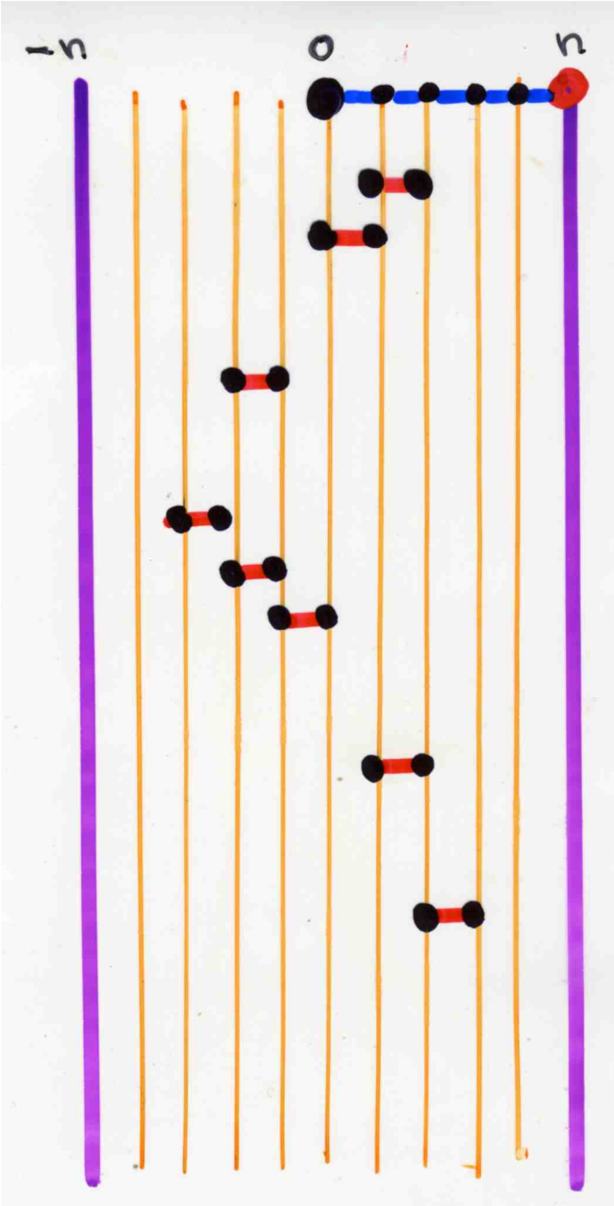
colloque Guéret (2015)

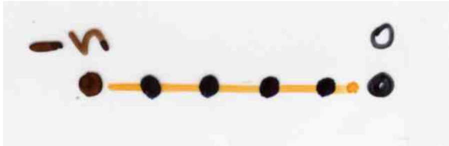
C. Goldstein

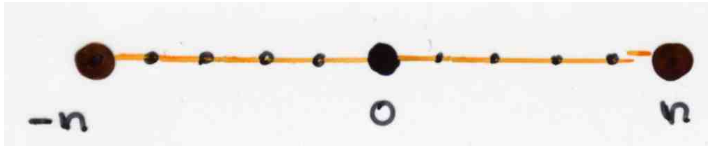
"Catalan et les Effa-Rouchés"

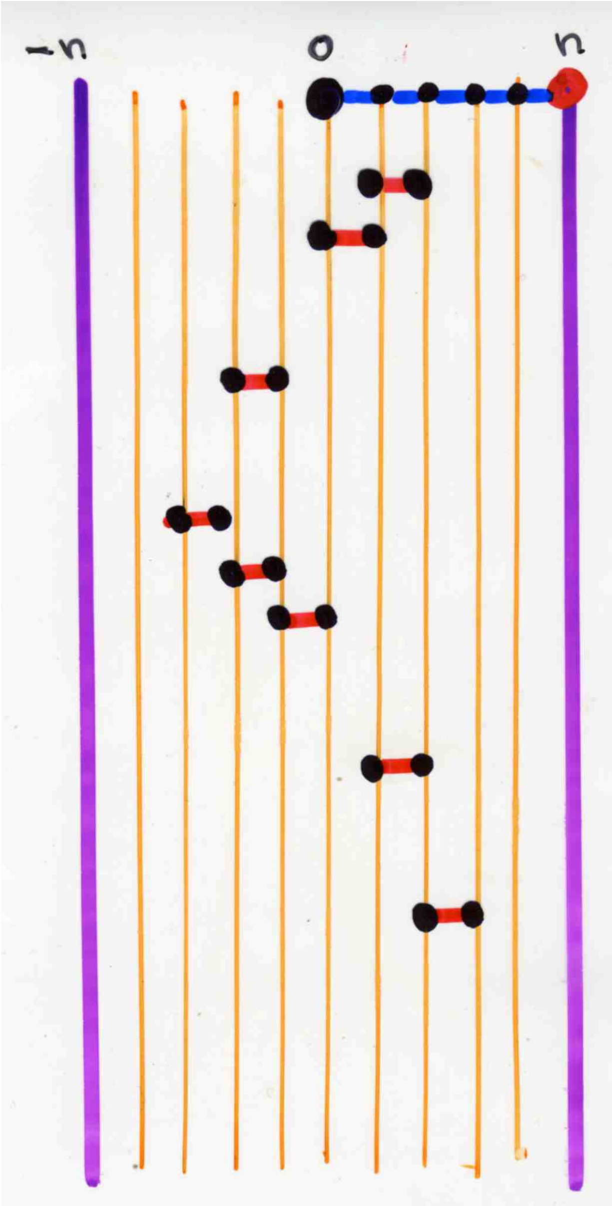


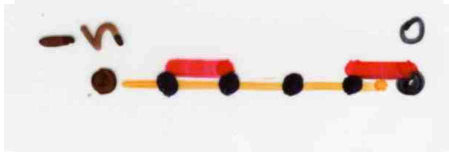
n
-n

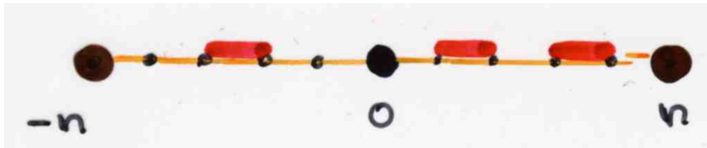

-n
0
n


-n
0


-n
0
n

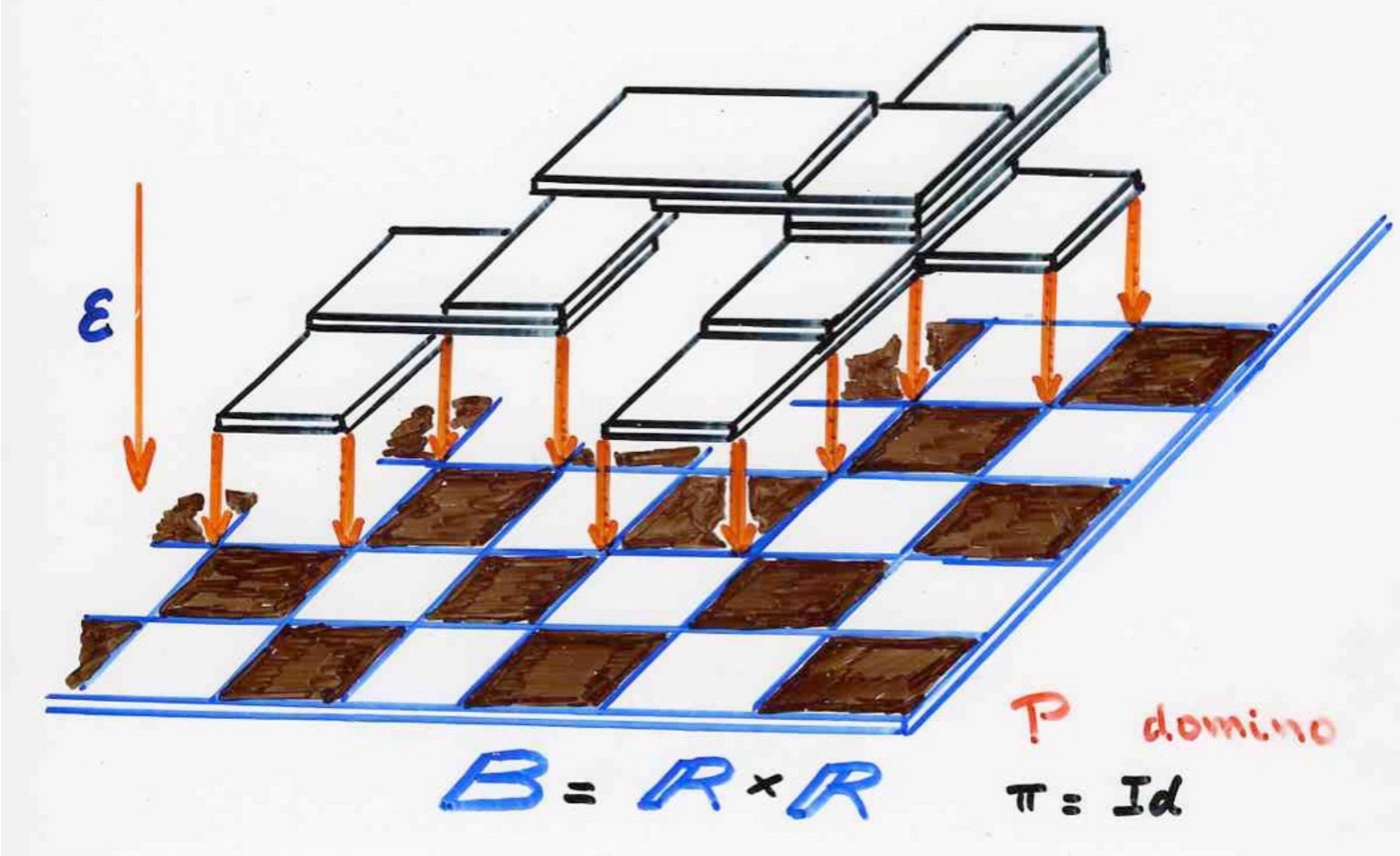

-n
0
n


-n
0


-n
0
n

H. Delannoy

" méthode de l'échiquier "



heap of dimers

Donc on a enfin
$$A_{m,n} = \quad \ldots\ldots(3)$$
$$\pm \beta'^{-m}\{A_{0,n}.\gamma'^{m} + A_{0,n+1}.m\gamma'^{m-1}\beta + A_{0,n-2}.\frac{m.m-1}{1.\ 2}\gamma'^{m-2}\beta^{2}$$
$$+ \text{etc.} + A_{0,n+m-1}.m\gamma'\beta^{m-1} + A_{0,n+m}.\beta^{m}\},$$
le signe supérieur ou inférieur ayant lieu suivant que m est pair ou impair. Ce résultat s'accorde avec celui que l'on peut déduire d'une solution différente du même exemple, donnée par LAPLACE dans les mémoires de Paris, année 1779, n.° XVII, page 267.

Si l'on fait n négatif dans la formule (1) ci-dessus, on trouve, en rejetant les termes et celles de leurs parties où les indices de D sont négatifs et ceux de D' négatifs ou positifs $>$ 0, que cette formule se réduit à la suivante :
$$A_{m,-n} = \quad \ldots\ldots(4)$$
$$\pm \beta'\{A_{0,0}\underset{\beta}{D}^{m}.(\alpha^{n}.\beta'^{-n-1}) - A_{0,1}\underset{\beta}{D}^{m}.(\alpha^{n+1}.\beta'^{-n-2}) + A_{0,2}\underset{\beta}{D}^{m}.(\alpha^{n+2}.\beta'^{-n-3}) - \text{etc.}\}$$
laquelle, à cause que $\alpha = 0$ et que sa seule dérivée D est β, devient
$$A_{m,-n} = \quad \ldots\ldots(5)$$
$$\pm \beta'\{A_{0,0}.\beta^{n}\underset{\beta}{D}^{m-n}.\beta'^{-n-1} - A_{0,1}.\beta^{n+1}\underset{\beta}{D}^{m-n-1}.\beta'^{-n-2} + A_{0,2}.\beta^{n+2}\underset{\beta}{D}^{m-n-2}.\beta'^{-n-3} - \text{etc.}\}.$$
D'où il suit que $A_{m,-n}$ n'est zéro qu'autant que m est $< n$. Ainsi la série récurrente s'étend, sous forme de triangle, dans la quatrième région.

EXEMPLE VI.

246. Étant donné le commencement de la table suivante, où chaque terme est formé de la somme de celui qui le précède dans la même ligne horizontale et de celui qui le suit d'un rang dans la ligne horizontale immédiatement supérieure, avec la condition que chacun des termes de la première ligne horizontale soit égal à l'unité : on demande le terme général de cette table :

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m-1$ au lieu de m, et elle devient

L.F.A. Arbogast

$\pm \mathfrak{C}'\{A_{0,0}\mathrm{D}^m.(\alpha^n.\mathfrak{C}^{l-n-1}) - A_{0,1}\mathrm{D}^m.(\alpha \ldots$

laquelle, à cause que $\alpha = 0$ et que sa seule dérivée D est $\mathfrak{C}$, devient

$$A_{m,-n} = \qquad \ldots\ldots\ldots (5)$$

$$\ldots \mathfrak{C}^{n} + {}^{1}\mathrm{D}^{m-n-1}.\mathfrak{C}^{l-n-2} + A_{0,2}.\mathfrak{C}^{n+2}\mathrm{D}^{m-n-2}.\mathfrak{C}^{l-n-3} - \text{etc.}\}.$$

... zéro qu'autant que m est $< n$. Ainsi la série récurrente s'étend, sous forme de triangle, dans la quatrième région.

EXEMPLE VI.

246. Étant donné le commencement de la table suivante, où chaque terme est formé de la somme de celui qui le précède dans la même ligne horizontale et de celui qui le suit d'un rang dans la ligne horizontale immédiatement supérieure, avec la condition que chacun des termes de la première ligne horizontale soit égal à l'unité : on demande le terme général de cette table :

An VIII

(1800)

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m-1$ au lieu de m, et elle devient

ballot numbers

DU CALCUL

DES

DÉRIVATIONS;

PAR L. F. A. ARBOGAST,

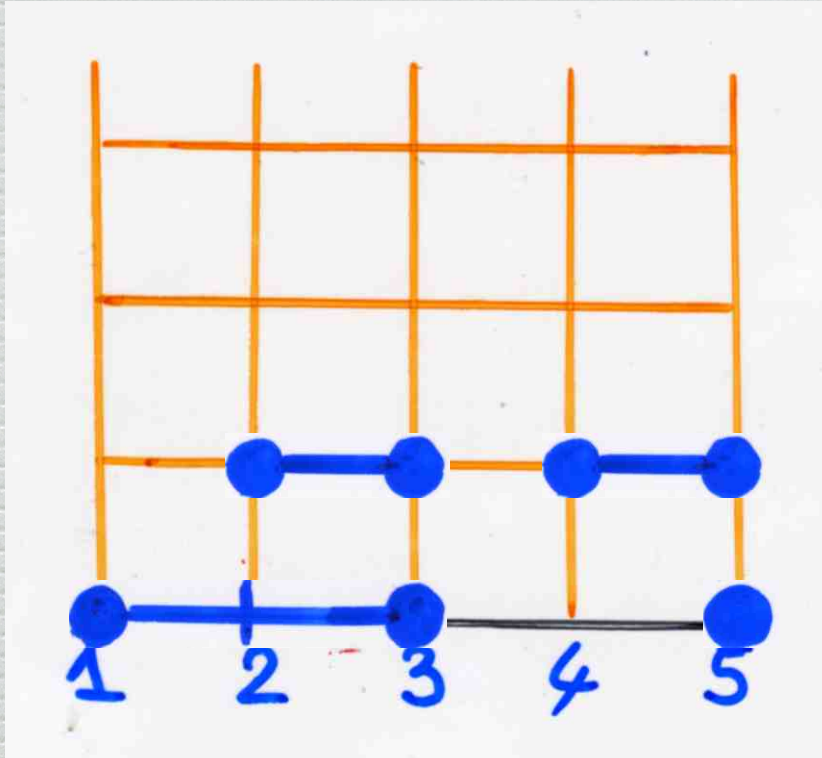
De l'Institut national de France, Professeur de Mathématiques à Strasbourg.

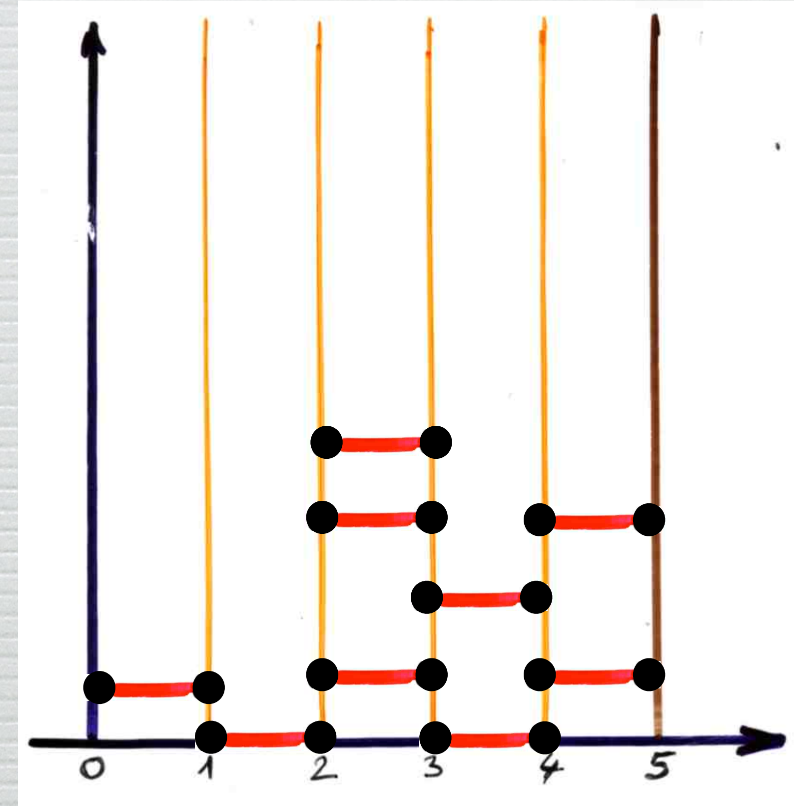
A STRASBOURG,

DE L'IMPRIMERIE DE LEVRAULT, FRÈRES.

AN VIII (1800).

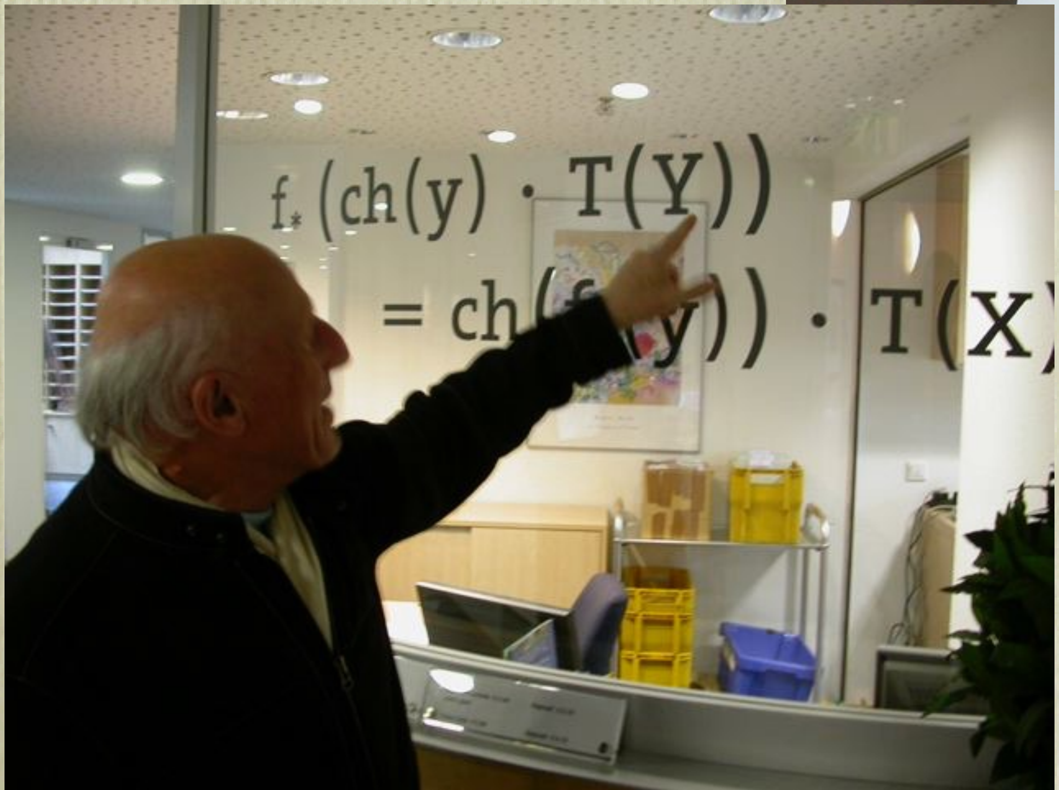
The duality





In the context of
fully commutative elements
in Coxeter groups

Heaps of pieces
and
commutations


$f_*(ch(y) \cdot T(Y))$

D. Foata, (1965)

thesis Doct. Etat
"Etude algébrique de certains problèmes
d'Analyse Combinatoire et du Calcul des Probabilités"

In 1969 P. Cartier and D.Foata introduced the "**commutation monoid**s", that is monoids of words defined up to a certain commutation of letters, in the monograph:

P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics, no. 85, Springer–Verlag, Berlin, New York, 1969.

http://www.mat.univie.ac.at/~slc/

with an appendix
by C. Krattenthaler

"**heaps of pieces**", as a geometric interpretation of these "commutation monoids"

G.X.V., *Heaps of pieces. I. Basic definitions and combinatorial lemmas,* in Combinatoire énumérative (Montréal, Québec,1985), vol. 1234 of Lecture Notes in Math., pp321–350, Springer, Berlin, 1986.

A companion paper:

G.X.V, *Problèmes combinatoires posés par la physique statistique*, Astérisque, SMF, tome 121–122 (1985), **Séminaire Bourbaki**, 36ème année 1983/84, exposé 626, Feb 1984, p225–246

« Video-book » The Art of bijective combinatorics

Part II, Comutations and heaps of pieces with interactions in physics, mathematics and computer science

IMSc, Chennai, 2007

www.viennot.org/abjc2.html

- $a \, C \, b \Leftrightarrow b \, C \, a$
- ~~$a \, C \, a$~~

commutation relation C antireflexive symmetric

$\equiv_C$ congruence of A^* generated by the commutations

$$ab \equiv ba \quad \text{iff} \quad a \, C \, b$$

ex : heaps of dimers on $\mathbb{N}$

$$P = \{ [i, i+1] = \sigma_i \ , \ i \geq 0 \}$$

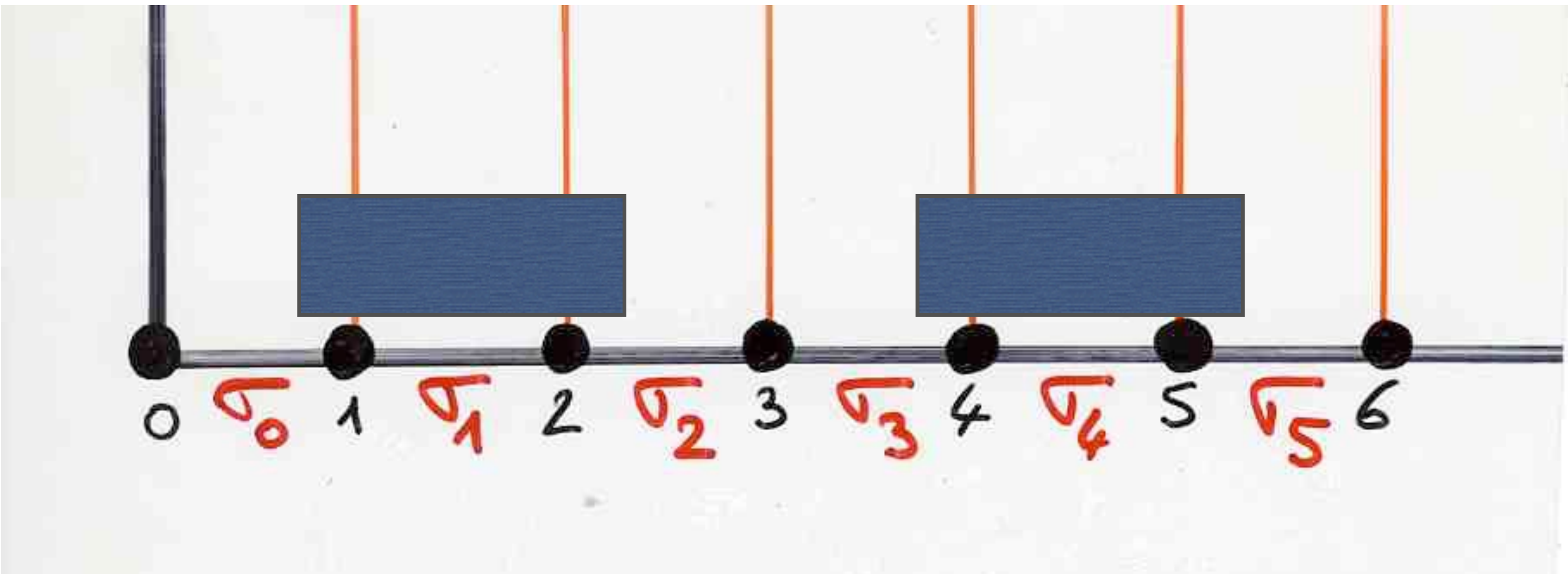
$\mathcal{C}$

C commutations

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i \quad \text{iff } |i-j| \geq 2$$

commutation class

$$\sigma_i \cap \sigma_j = \emptyset$$

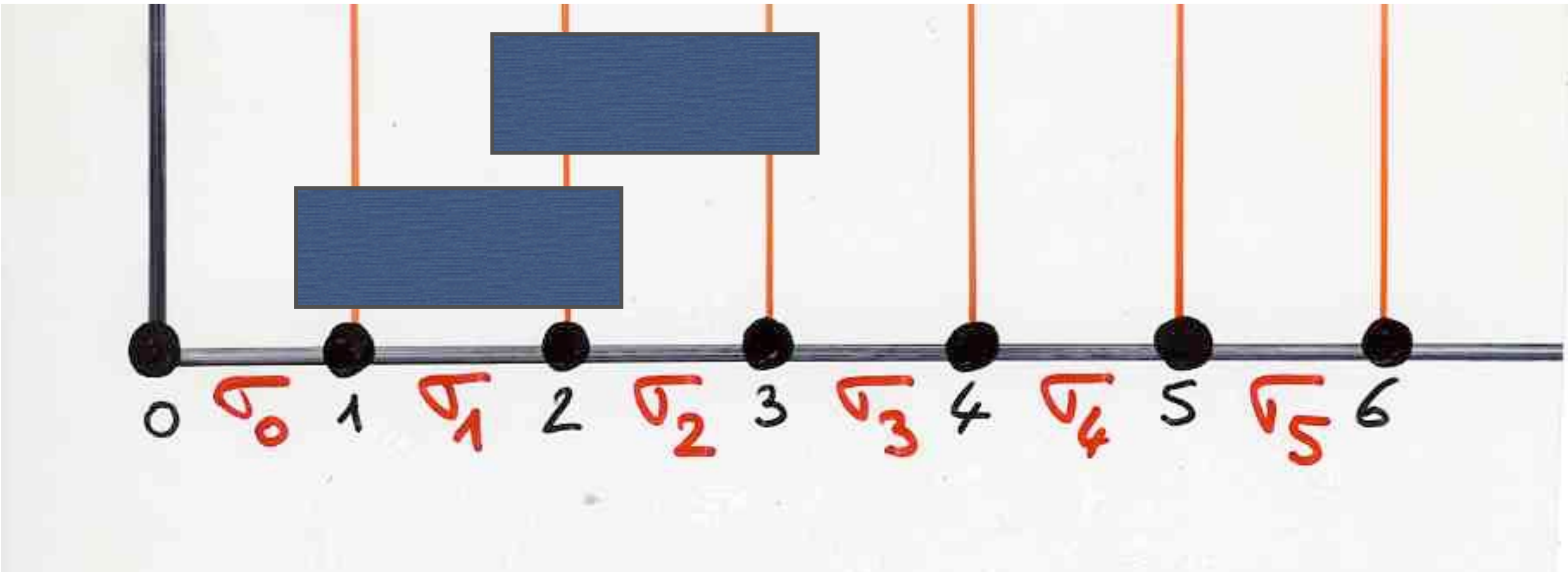


ex : heaps of dimers on $\mathbb{N}$

$P = \{ [i, i+1] = \sigma_i \ , i \geq 0 \}$

$\mathcal{C}$

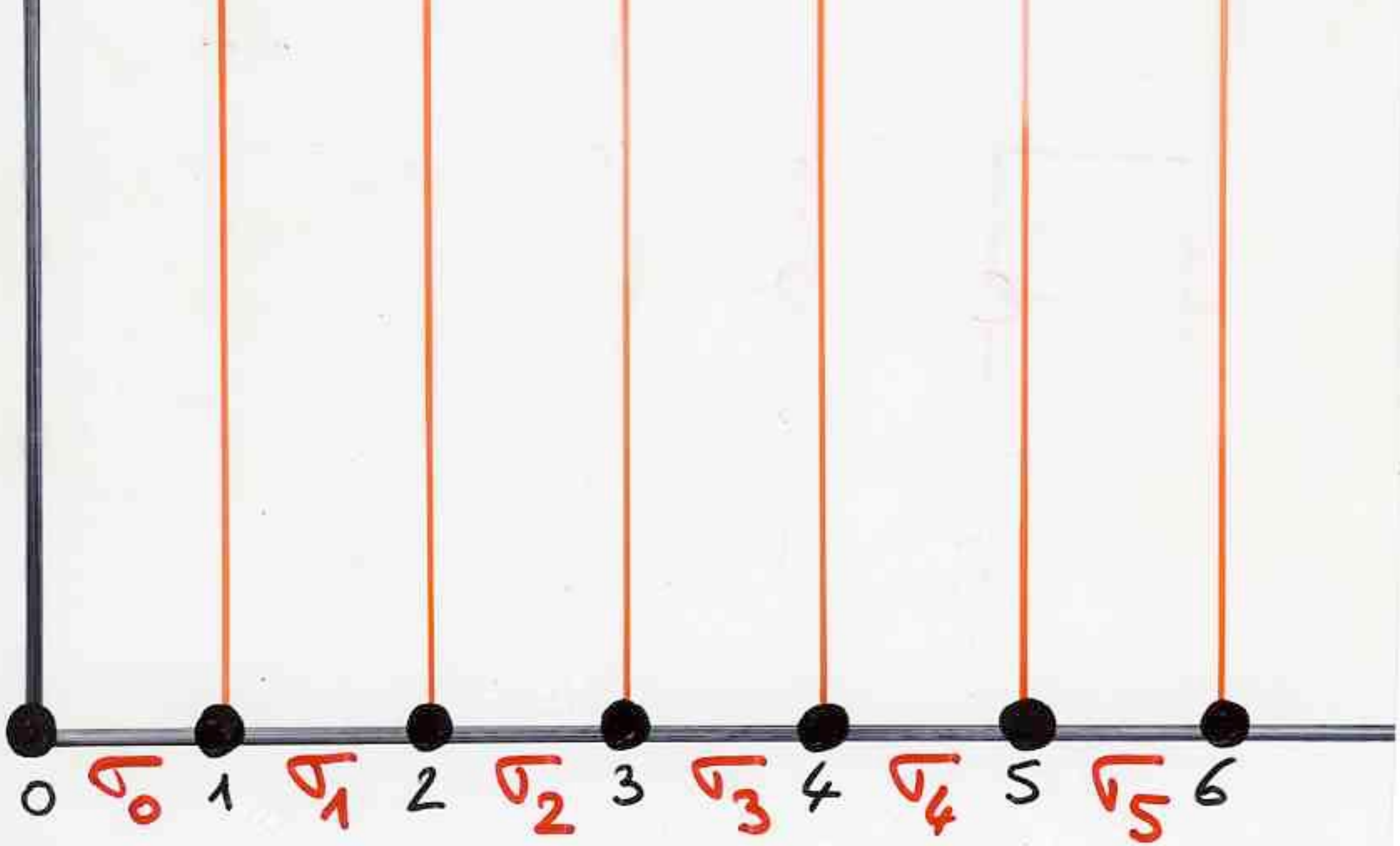
dependency relation $\sigma_i \cap \sigma_j \neq \emptyset$



$$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$$

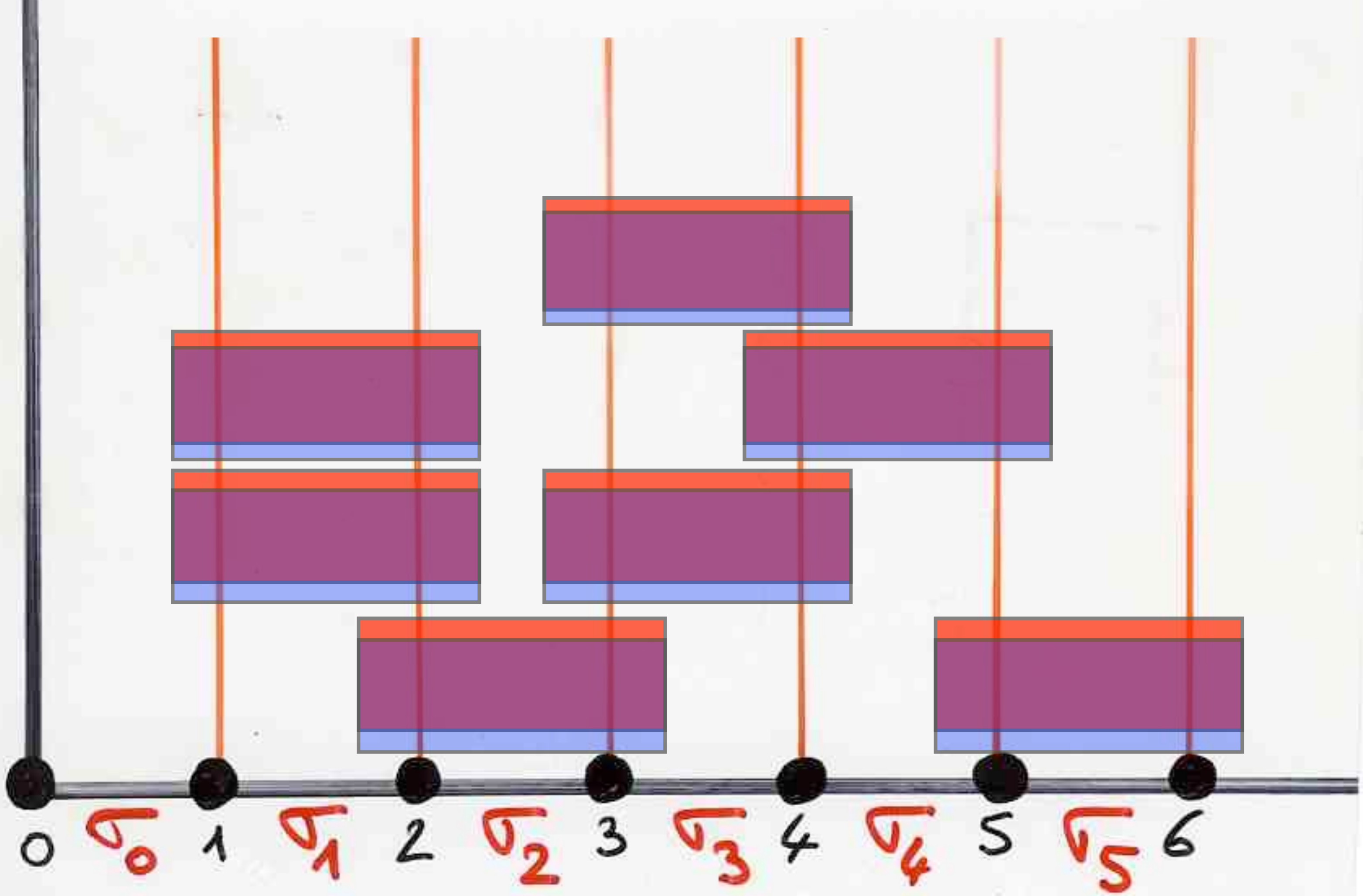
$$w = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$$

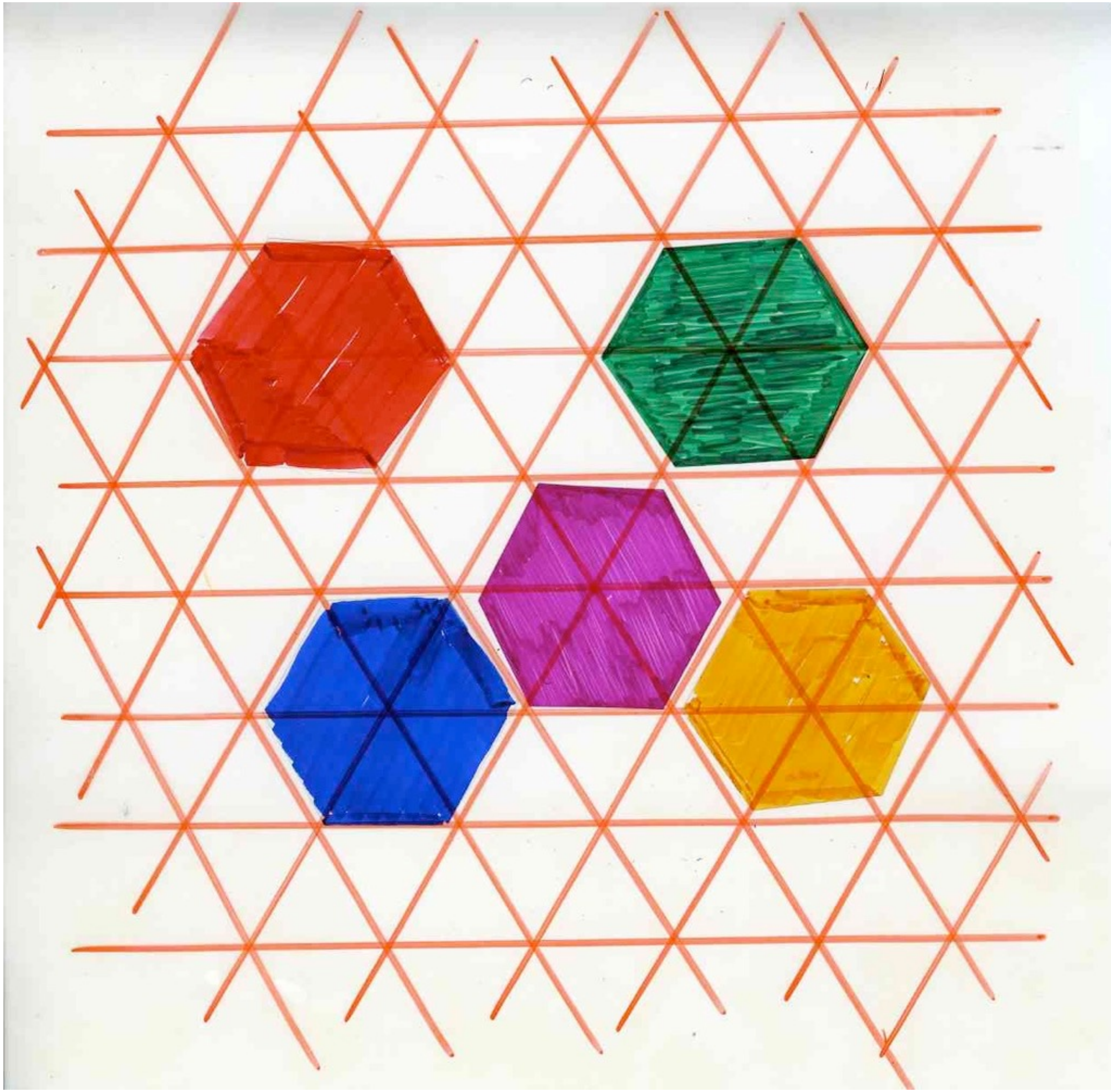
commutation class



$$W = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$$

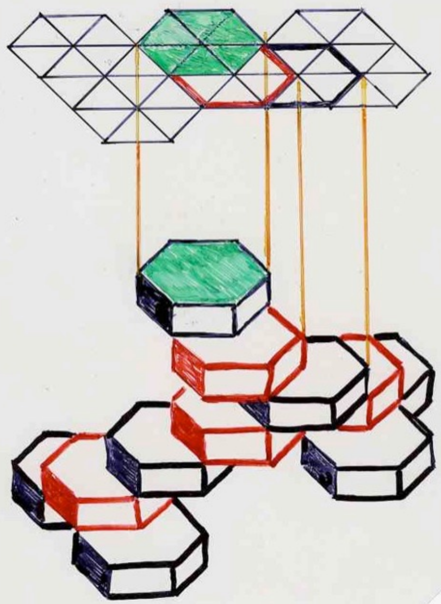
$$W = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$$



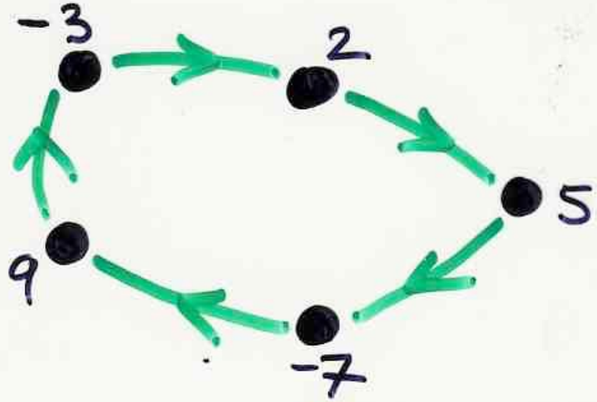


Heaps of "hard hexagons"

$$-p(-t) = y$$



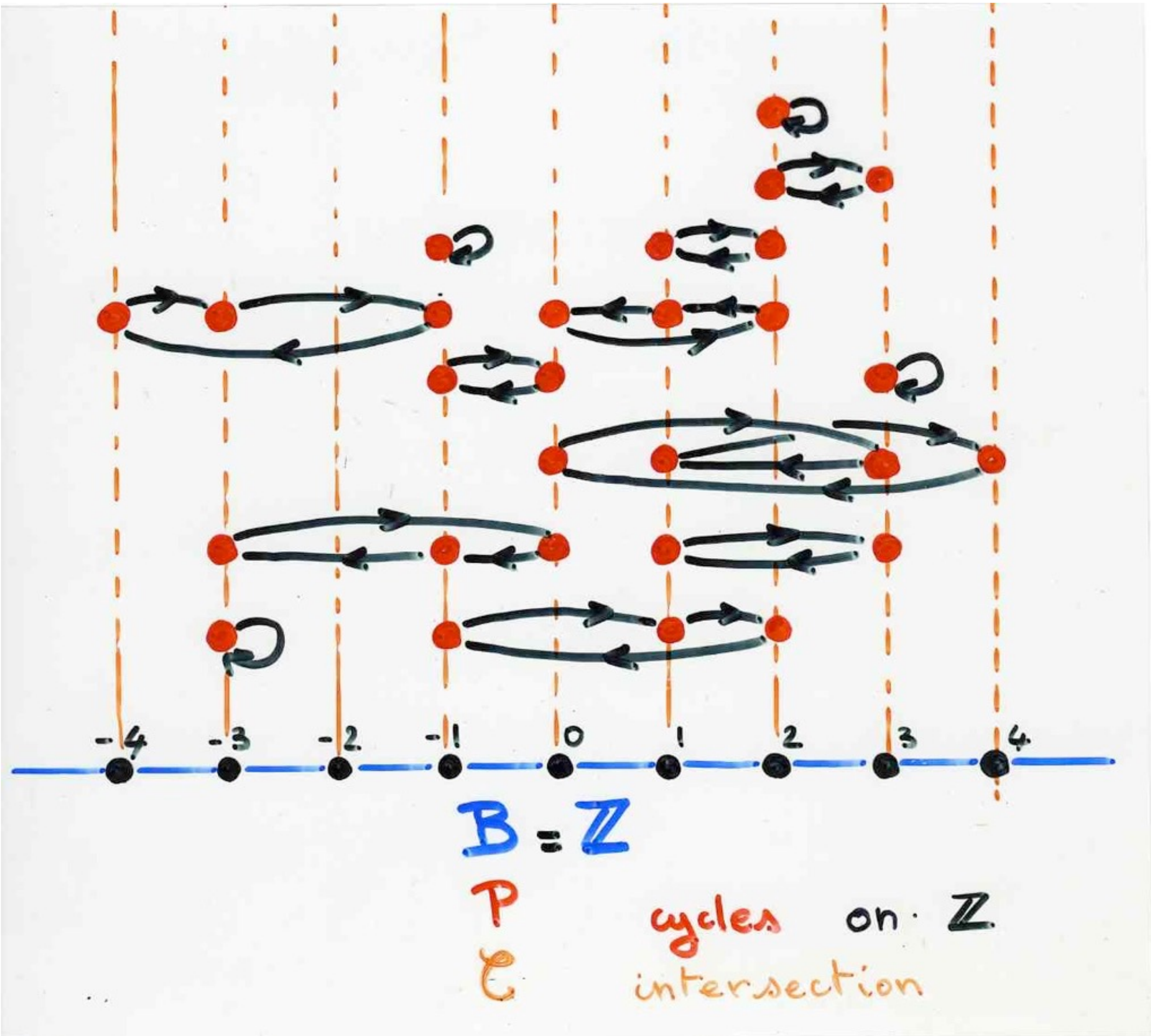
basic pieces $\mathcal{P} = \{$ cycles on $\mathbb{Z} \}$

$\gamma =$ 

$\mathrm{Supp}(\gamma) = \{-7, -3, 2, 5, 9\}$

Support

$\mathcal{C}$ dependency relation $\quad \gamma\, \mathcal{C}\, \delta \Longleftrightarrow \mathrm{Supp}(\gamma) \cap \mathrm{Supp}(\delta) \neq \emptyset$

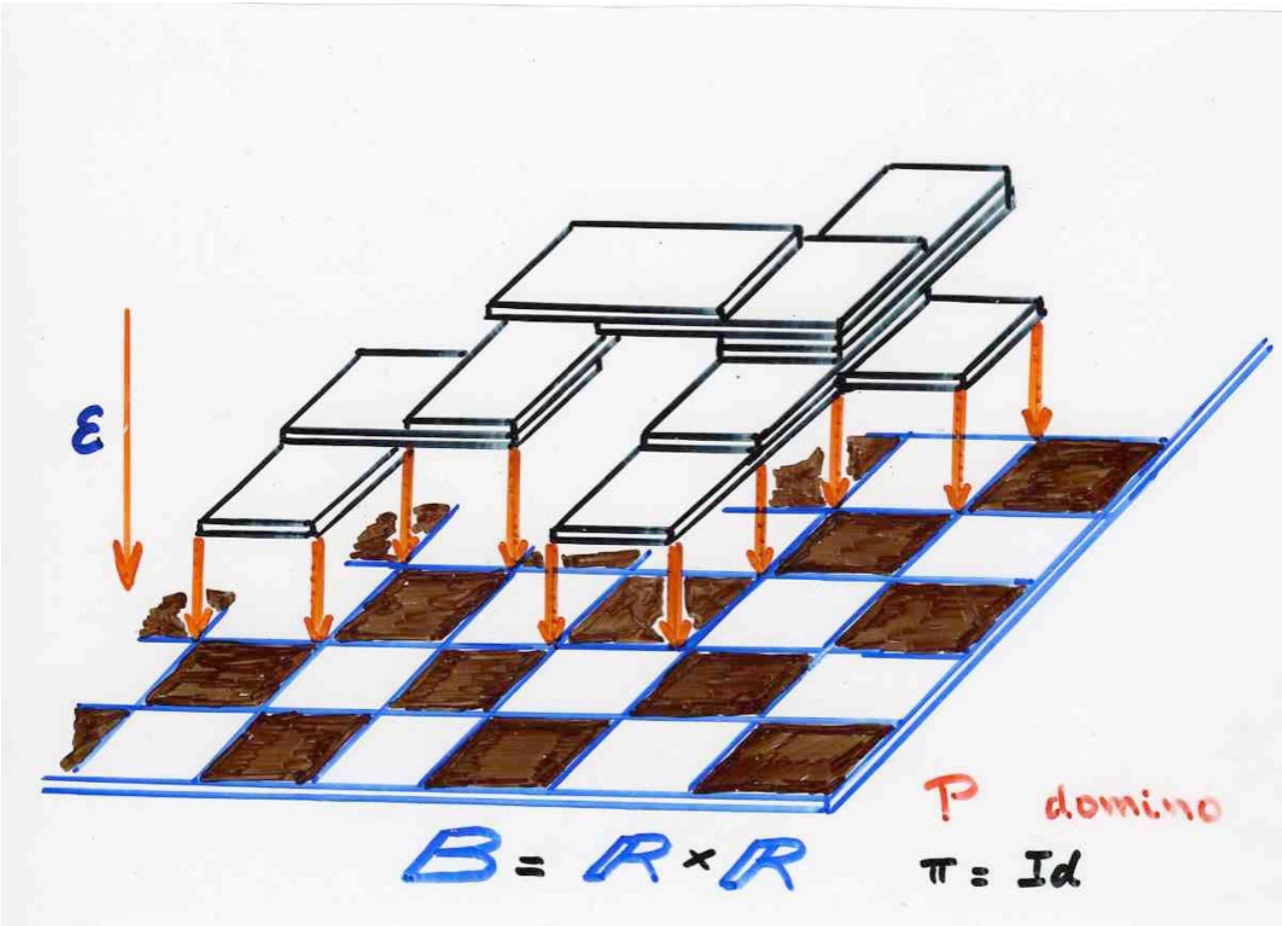


-4 -3 -2 -1 0 1 2 3 4
B = ℤ
P cycles on ℤ
C intersection

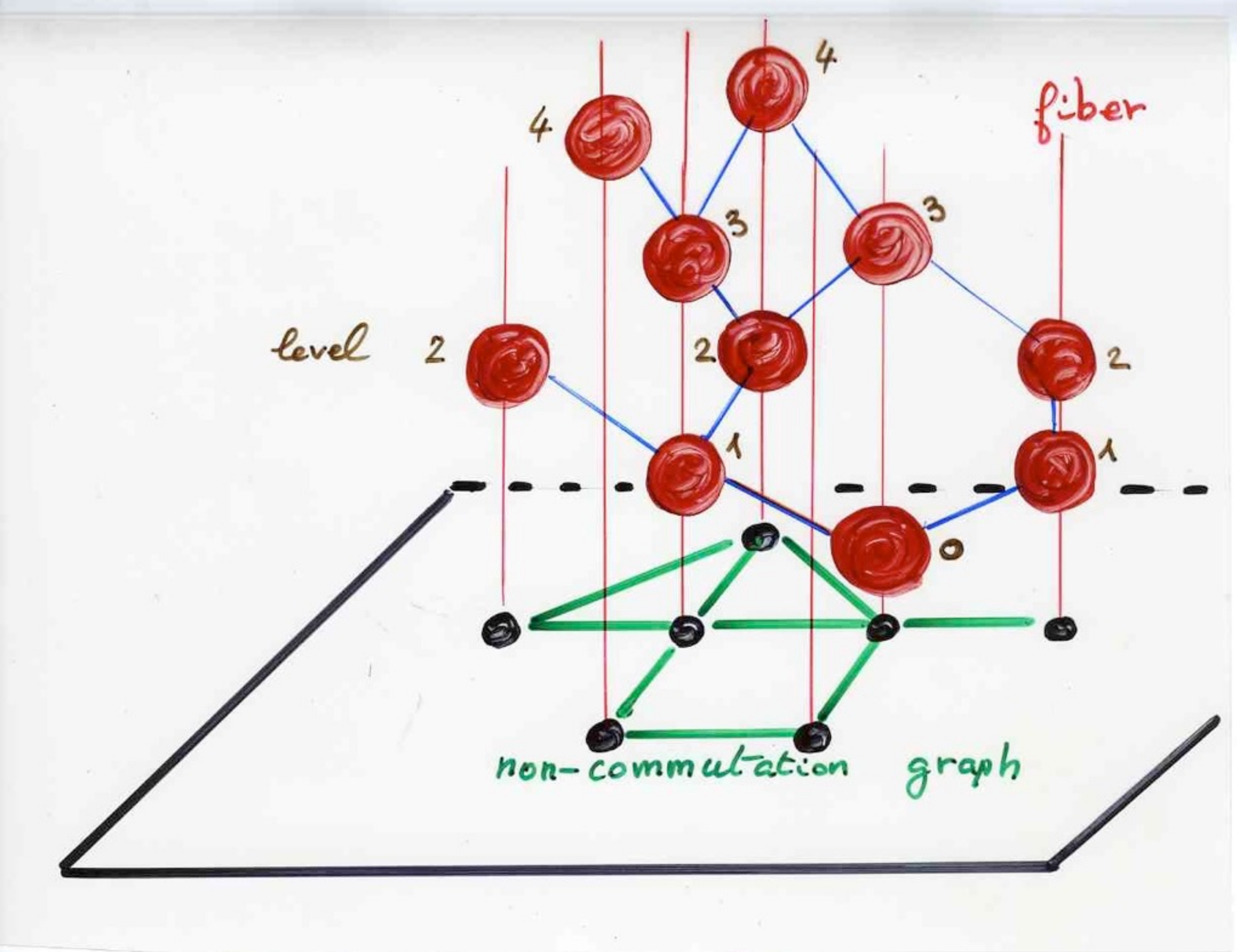
- definition by level

- definition with poset over a graph

Heaps of "hard dimers"
on a chessboard



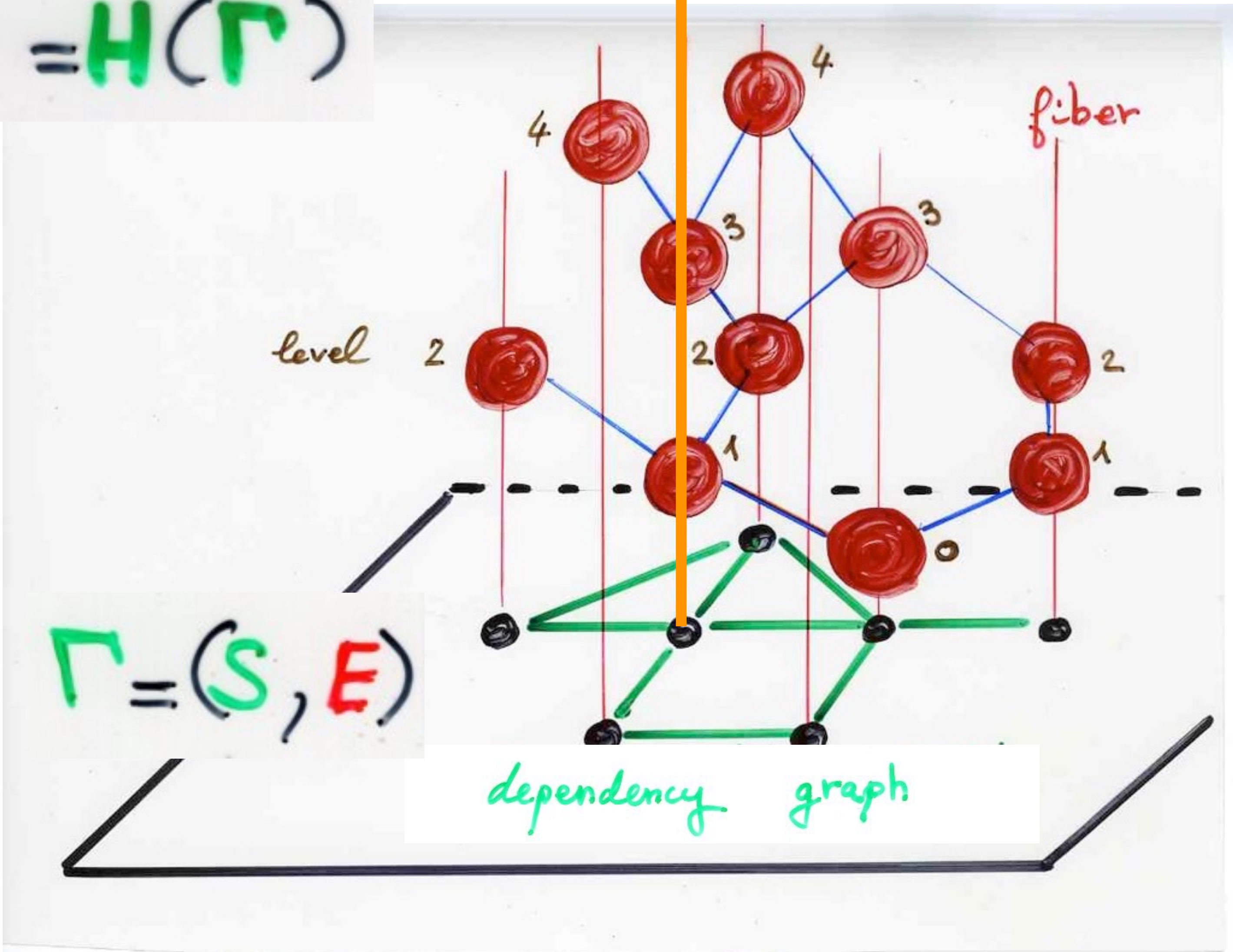
fully commutative elements in Coxeter groups


fiber
4
4
3
3
level 2
2
2
1
1
0
non-commutation graph

Coxeter graph Γ

fiber over $s \in S$

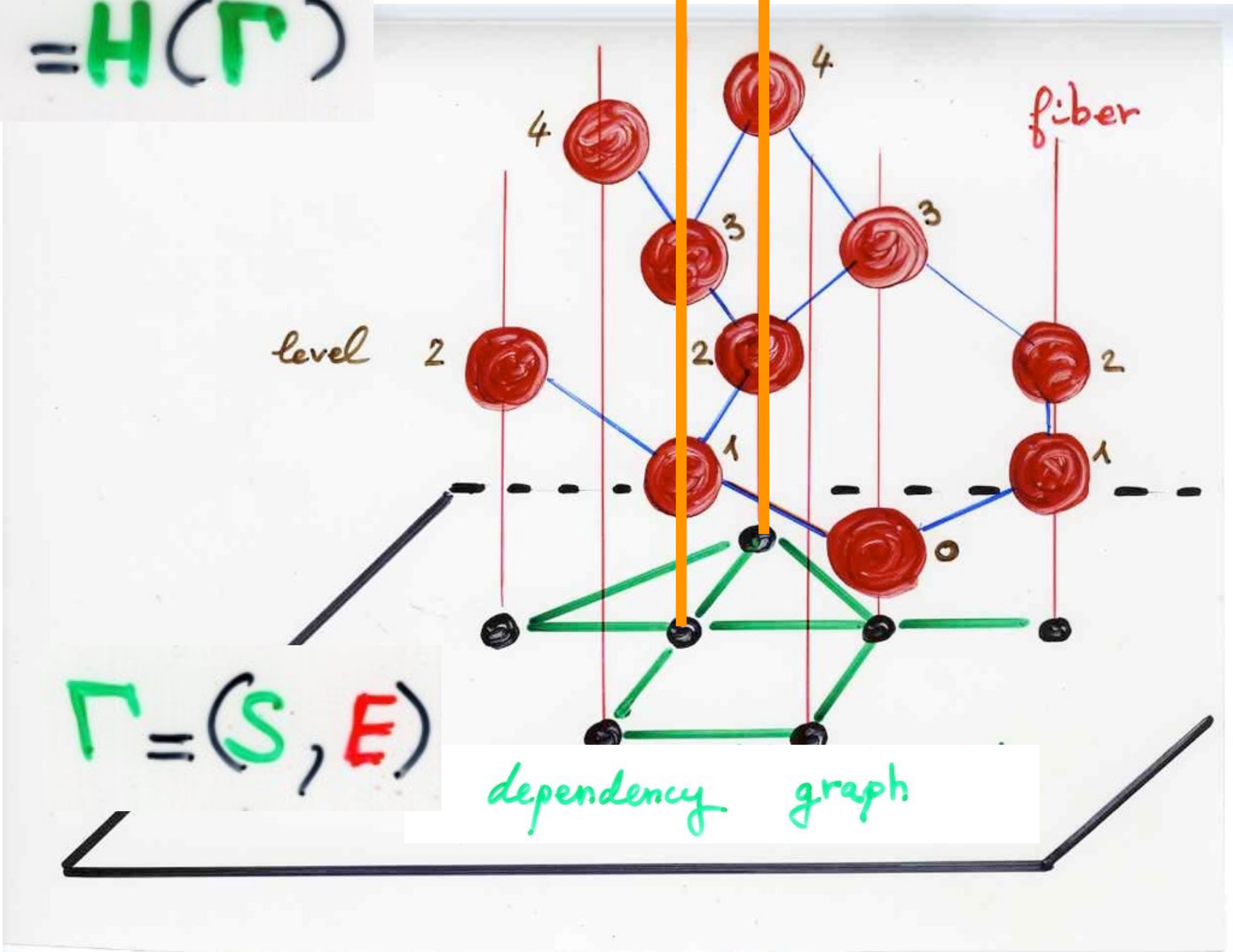
$H(W,S) = H(\Gamma)$



Coxeter graph Γ

fiber over $\{s,t\}$ edge of Γ

$\mathbf{H}(W,S) = \mathbf{H}(\Gamma)$



Symmetric group $\mathfrak{S}_n$

S_n

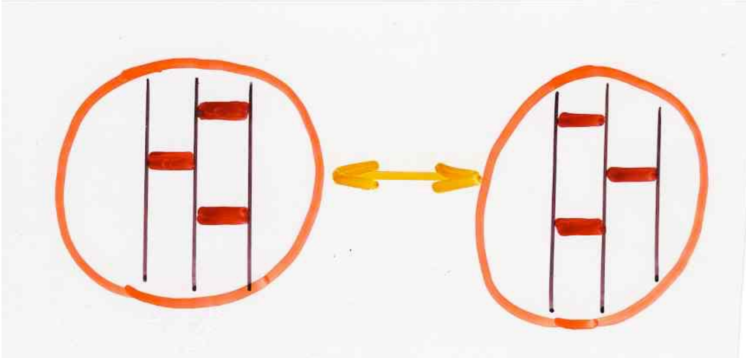
$n!$ permutations

$$\sigma_i = (i, i+1) \qquad i = 1, 2, \ldots, n-1$$

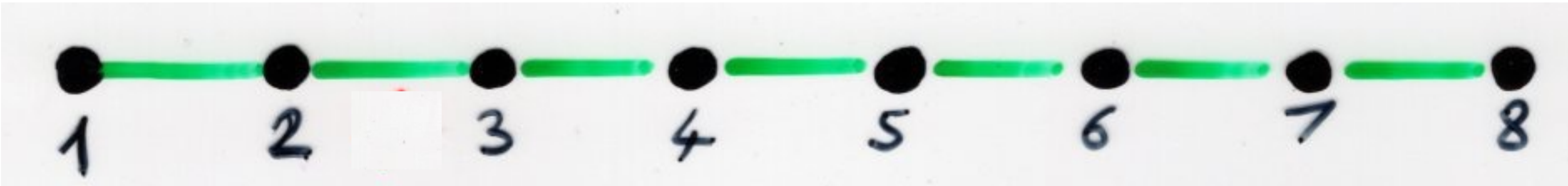
transposition of two consecutive elements

$$\begin{cases} \text{(i)} & \sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| \geq 2 \\ \text{(ii)} & \sigma_i^2 = 1, \\ \text{(iii)} & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Moore-Coxeter
Yang-Baxter



Coxeter graph



__Definition__ An element w of the Coxeter group W is *fully commutative* iff $R(w)$ is reduced to one commutation class.

The corresponding heap $H(w)$ will also be called *fully commutative* (FC)

Prop $\sigma \in S_n$ permutation

- (321) - avoiding

$\Updownarrow$

- only one commutation class

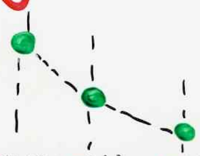
(Billey, Jockusch, Stanley) (1993)

$\Updownarrow$

- no  or 

(counted by $C_n = \frac{1}{n+1}\binom{2n}{n}$
Catalan numbers

(321) - avoiding permutations

no occurence of 

$\sigma(i) > \sigma(j) > \sigma(k)$
$i < j < k$

• E. Bagno, R. Biagioli, F. Jouhet, Y. Roichman
Dec 2020

Block number, descents and Schur positivity of fully commutative elements in B_n

affine Coxeter groups
Biagioli, Jouhet, Nadeau (2014, 2015)
" " " , Bousquet-Mélou
(2016)

Hanusa, Jones (2010)

seminal papers

→ Stembridge (1996, 98)

- classification of Coxeter groups with a finite number of FC elements
- enumeration in each of these cases

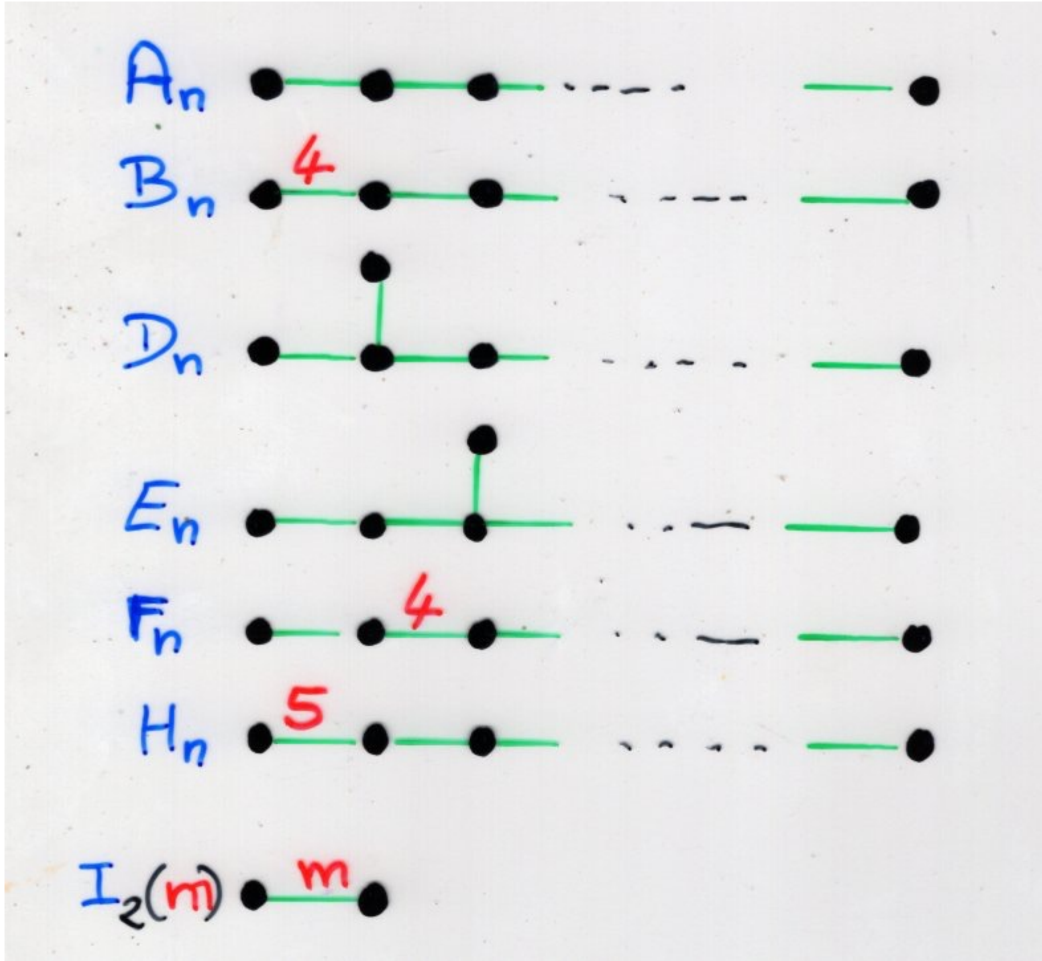
  → always algebraic generating functions

→ Fan (1995) for $m_{s,t} \leq 3$ (simply laced)

→ Graham (1995)

FC elements in any Coxeter group W naturally index ~~a~~ a basis of the generalized Temperley-Lieb algebra of W

finite Coxeter groups



A_n

B_n

D_n

E_6 E_7 E_8

F_4

H_3 H_4

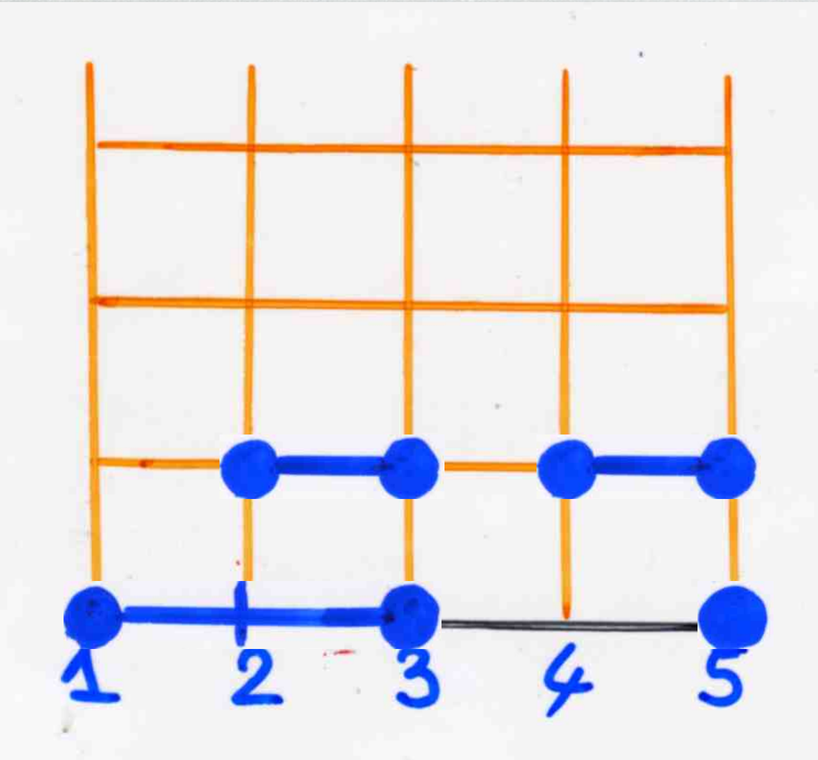
$I_2(m)$

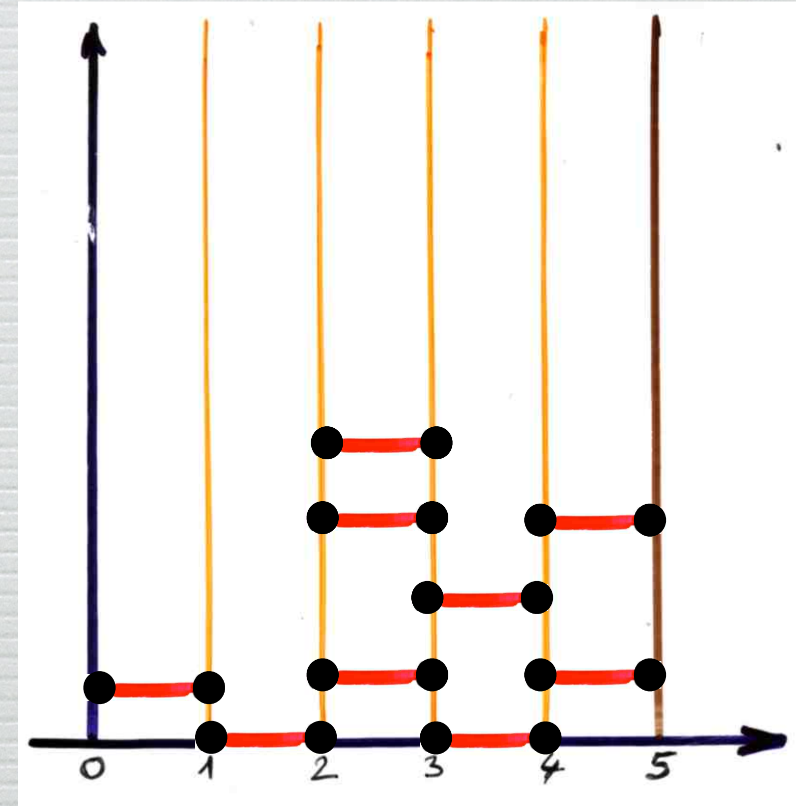
The list of FC-finite Coxeter groups

$I_2(5) = H_2$

$I_2(6) = G_2$

The duality





In the context of

fully commutative elements

in the symmetric group

« Video-book » The Art of bijective combinatorics

Part II, Comutations and heaps of pieces
with interactions
in physics, mathematics and computer science

IMSc, Chennai, 2007

www.viennot.org/abjc2-ch6.html

Chapter 6a, Heaps and Coxeter group

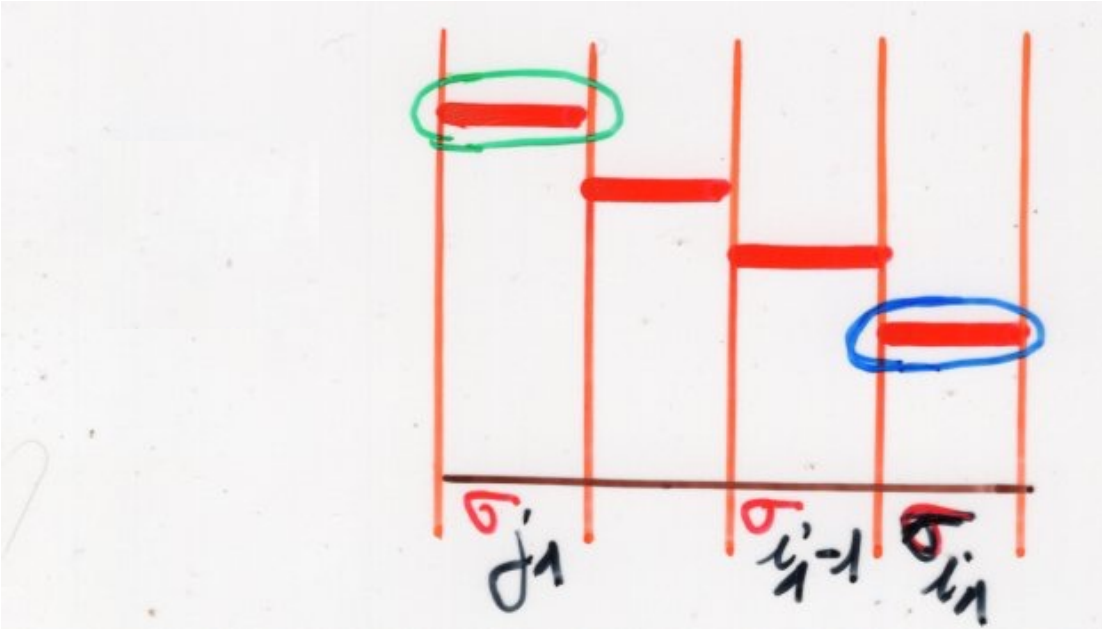
Fully commutative elements (FC) in Coxeter groups, definition slide 31

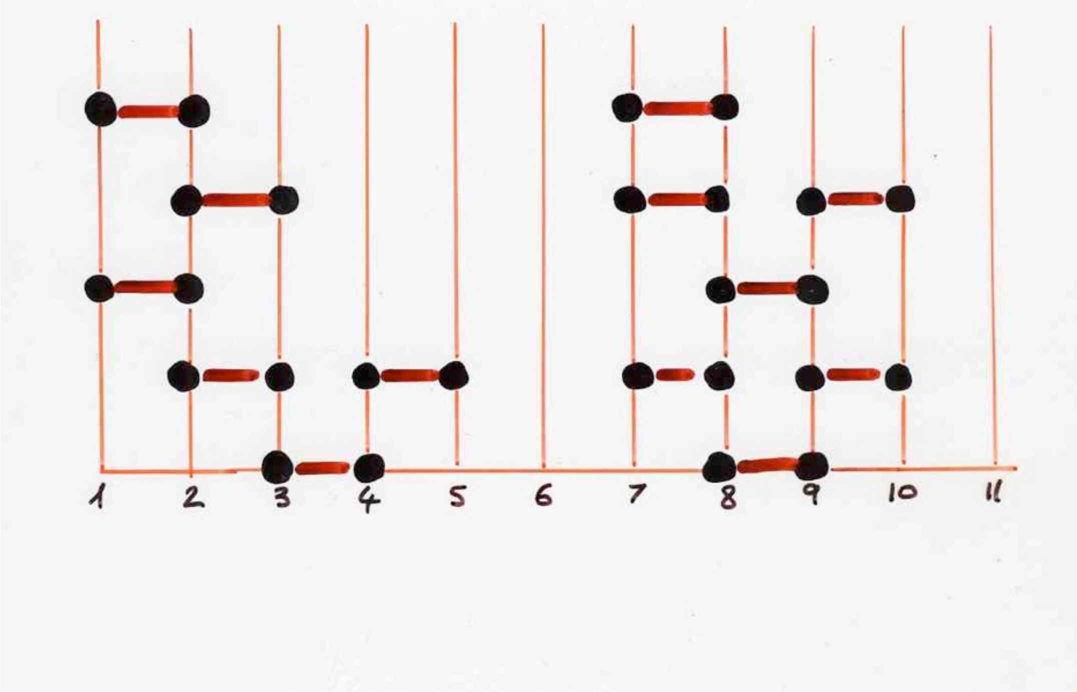
the stairs decomposition of a heap of dimers

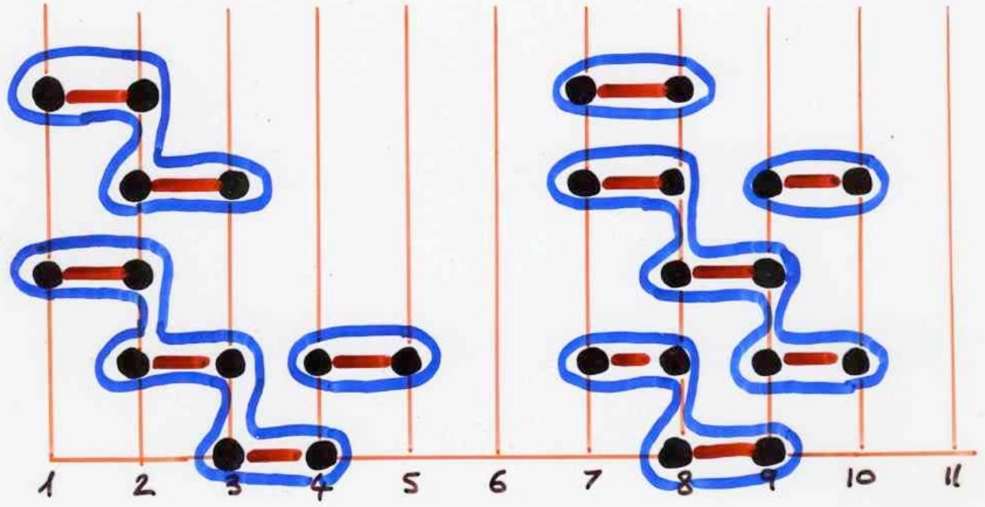
a stair is

a convex chain of dimers

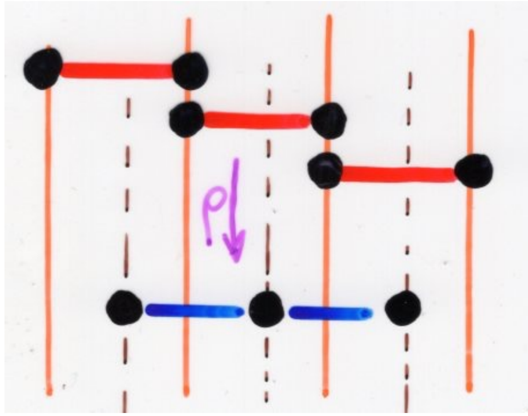
$\sigma_i \prec \sigma_{i-1} \prec \cdots \prec \sigma_k$



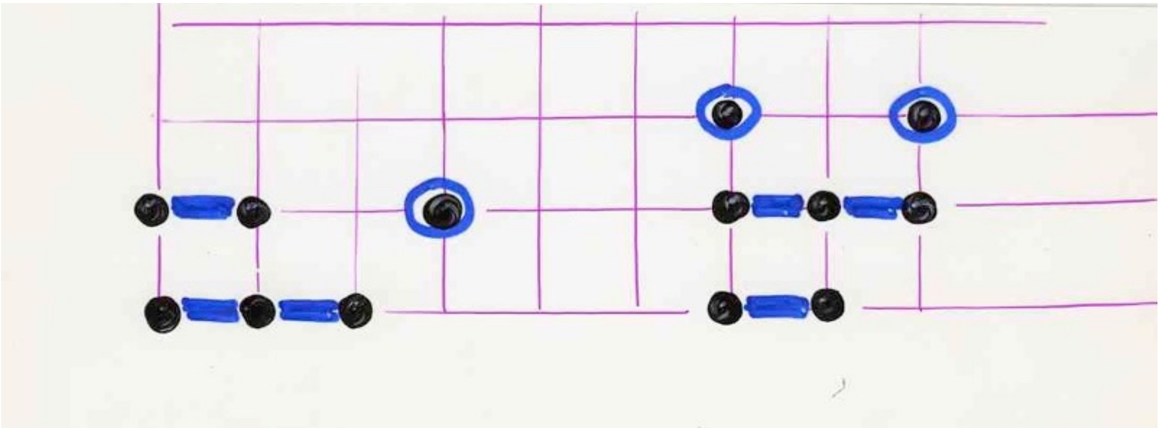

1 2 3 4 5 6 7 8 9 10 11



stairs decomposition



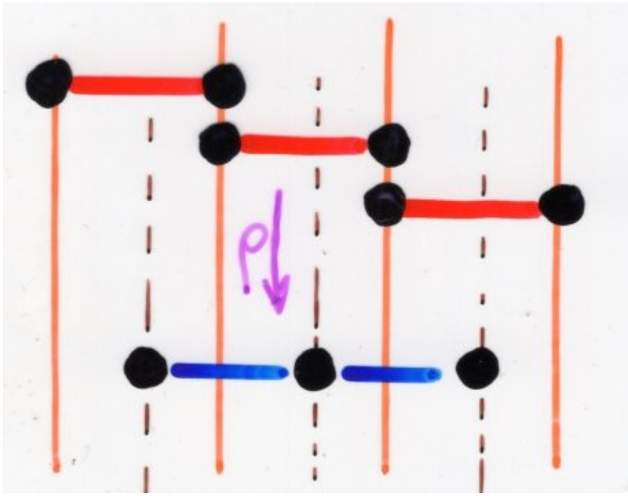
substitution



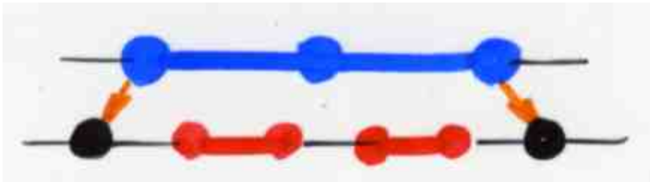
Proposition The stairs decomposition of a heap of dimers on $\mathbb{N}$ gives a bijection ρ

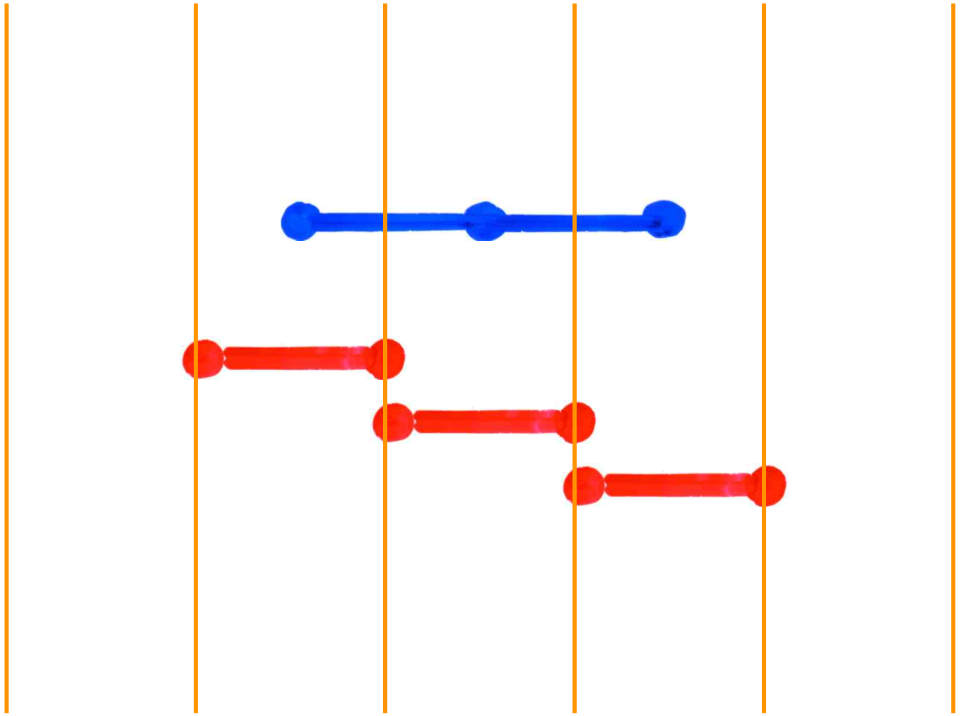
$$\text{heap of dimers on } \mathbb{N} \xrightarrow{\rho} \text{heap of segments on } \mathbb{N}$$

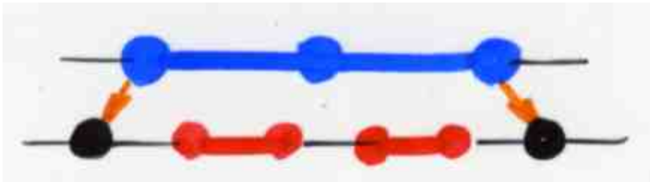
substitution

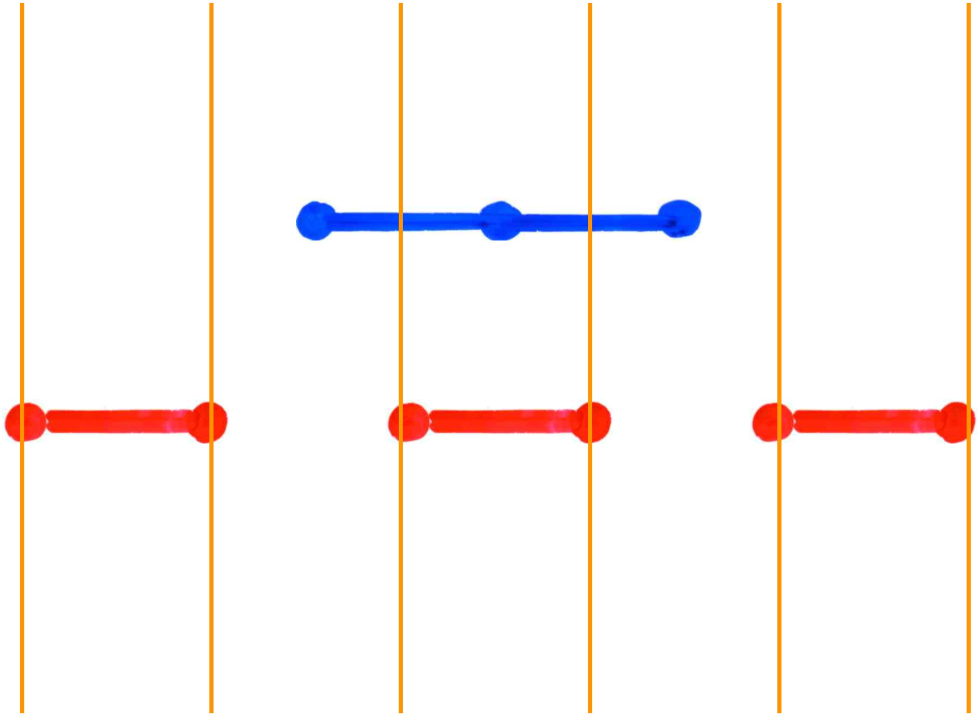


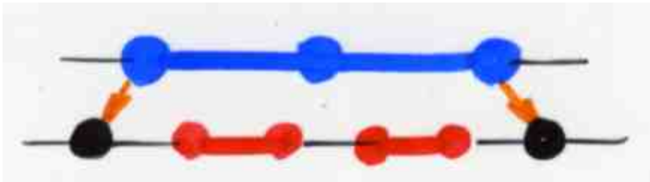
A funny remark .....

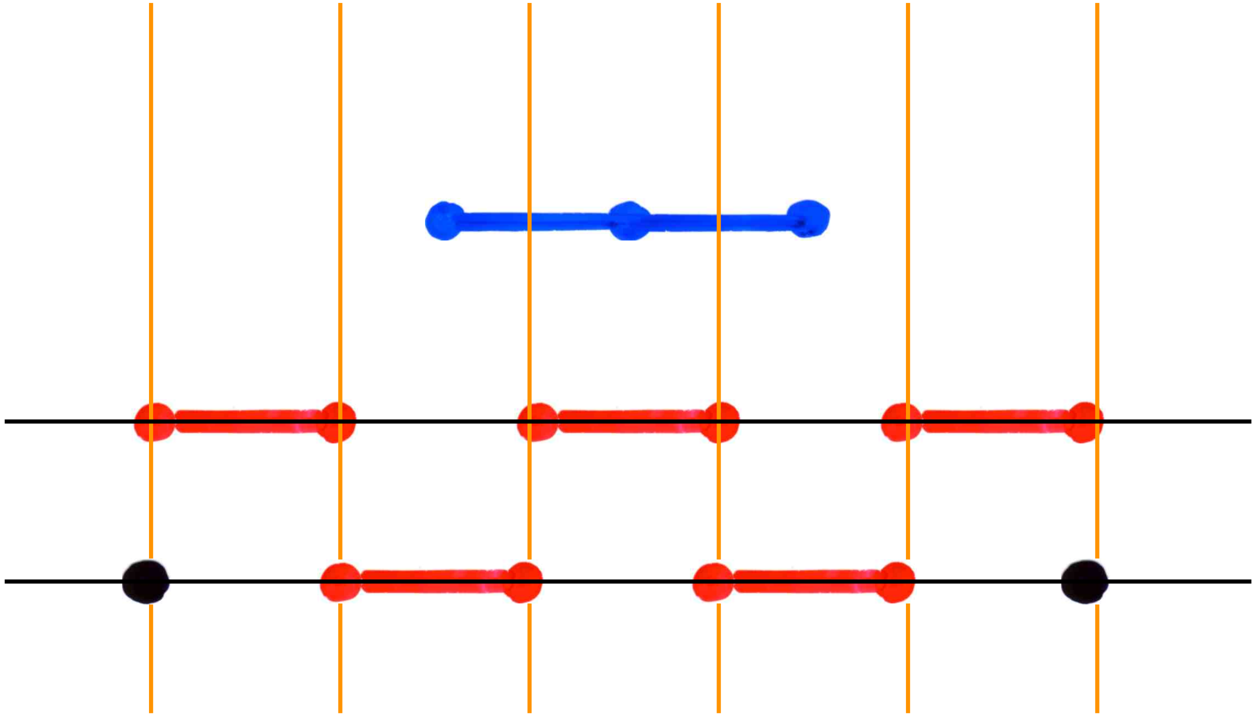


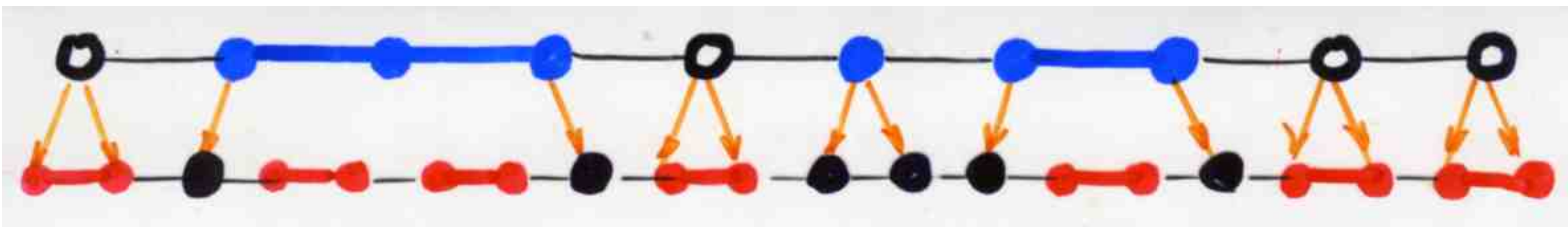


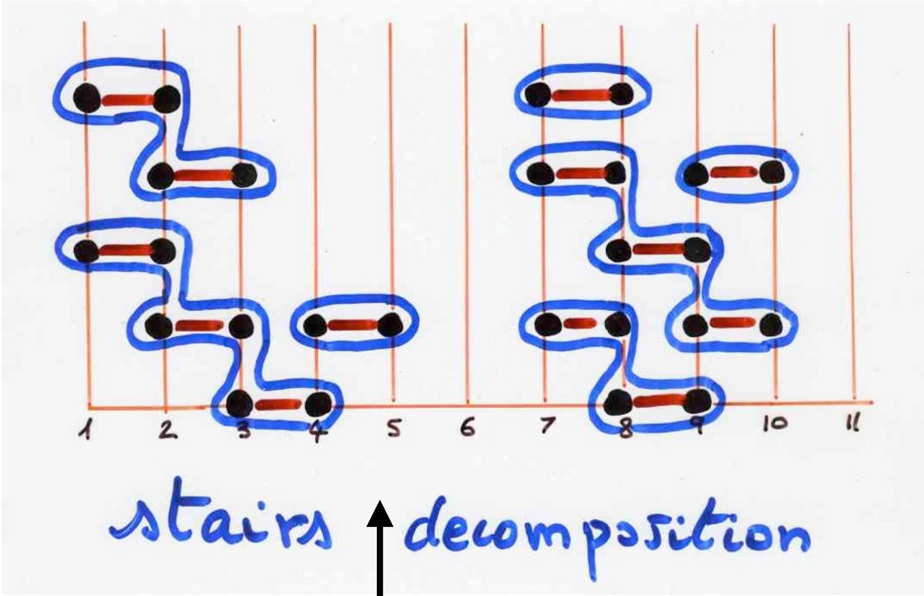


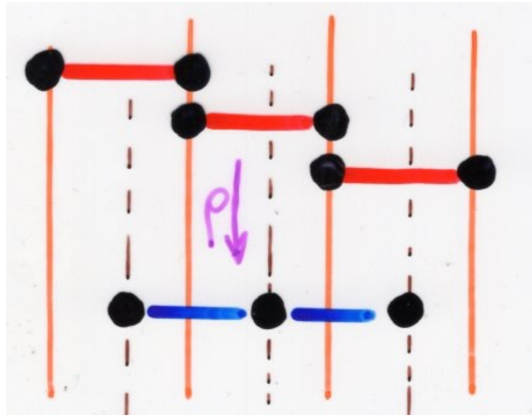


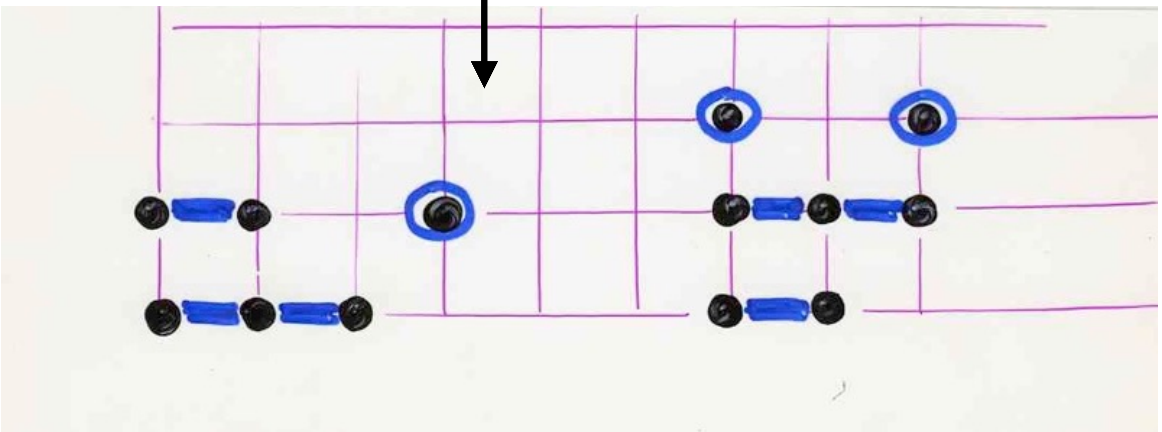


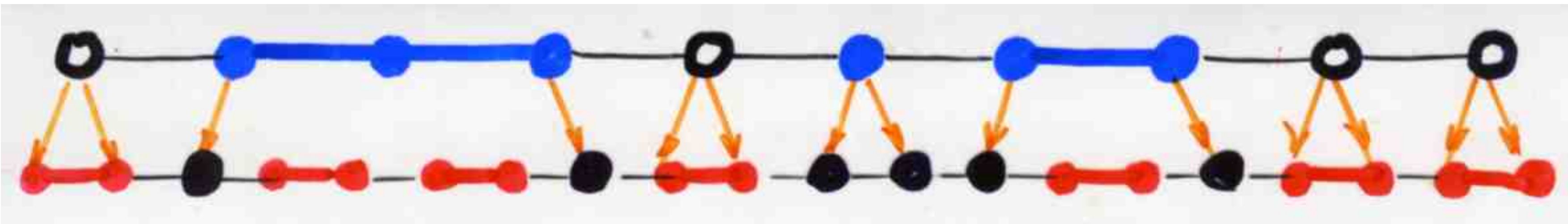





1 2 3 4 5 6 7 8 9 10 11
stairs decomposition


ρ





total order
of the stairs
in a heap
of dimers

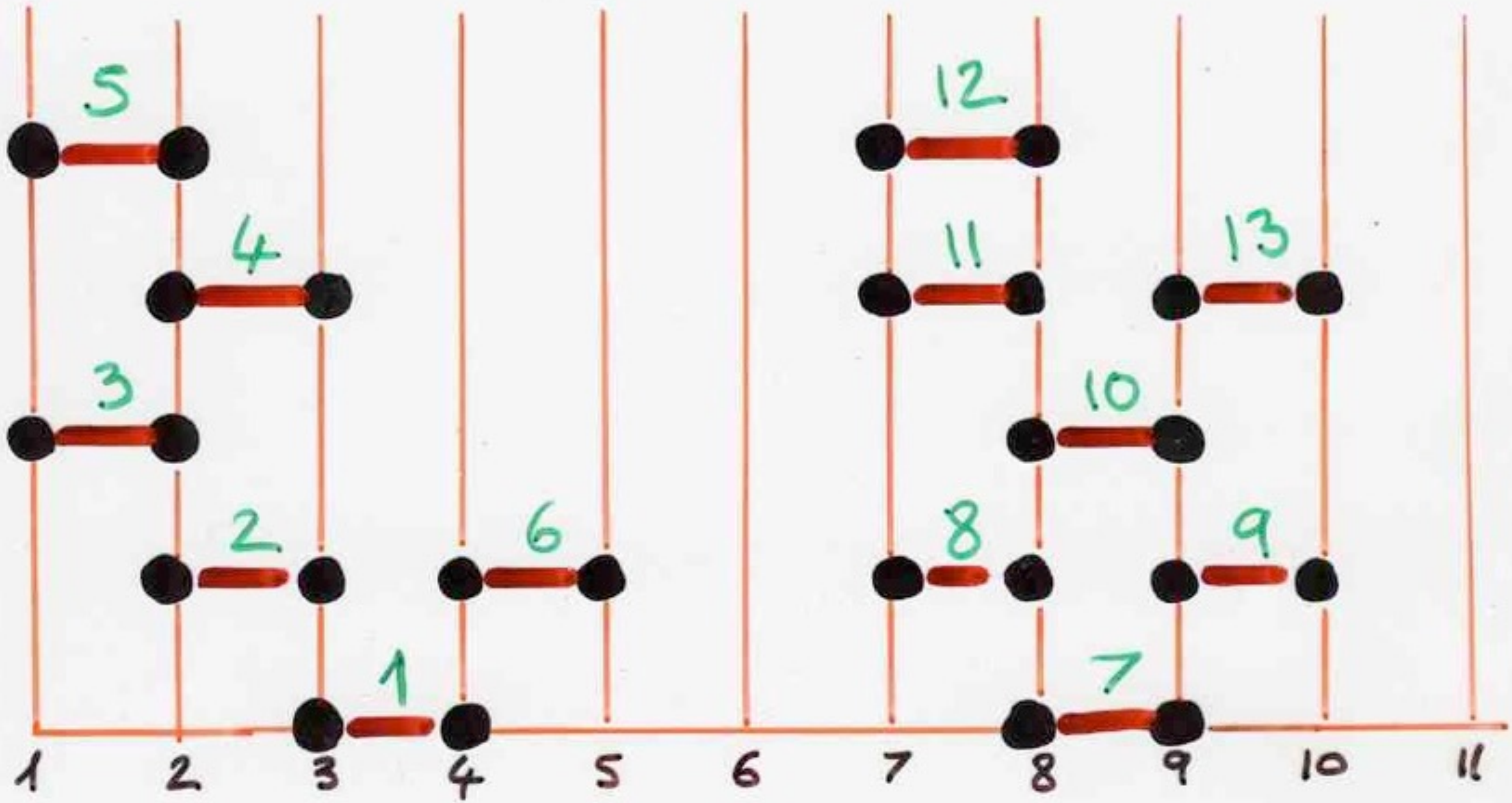
Normal form of a commutation class

- Cartier-Foata normal form

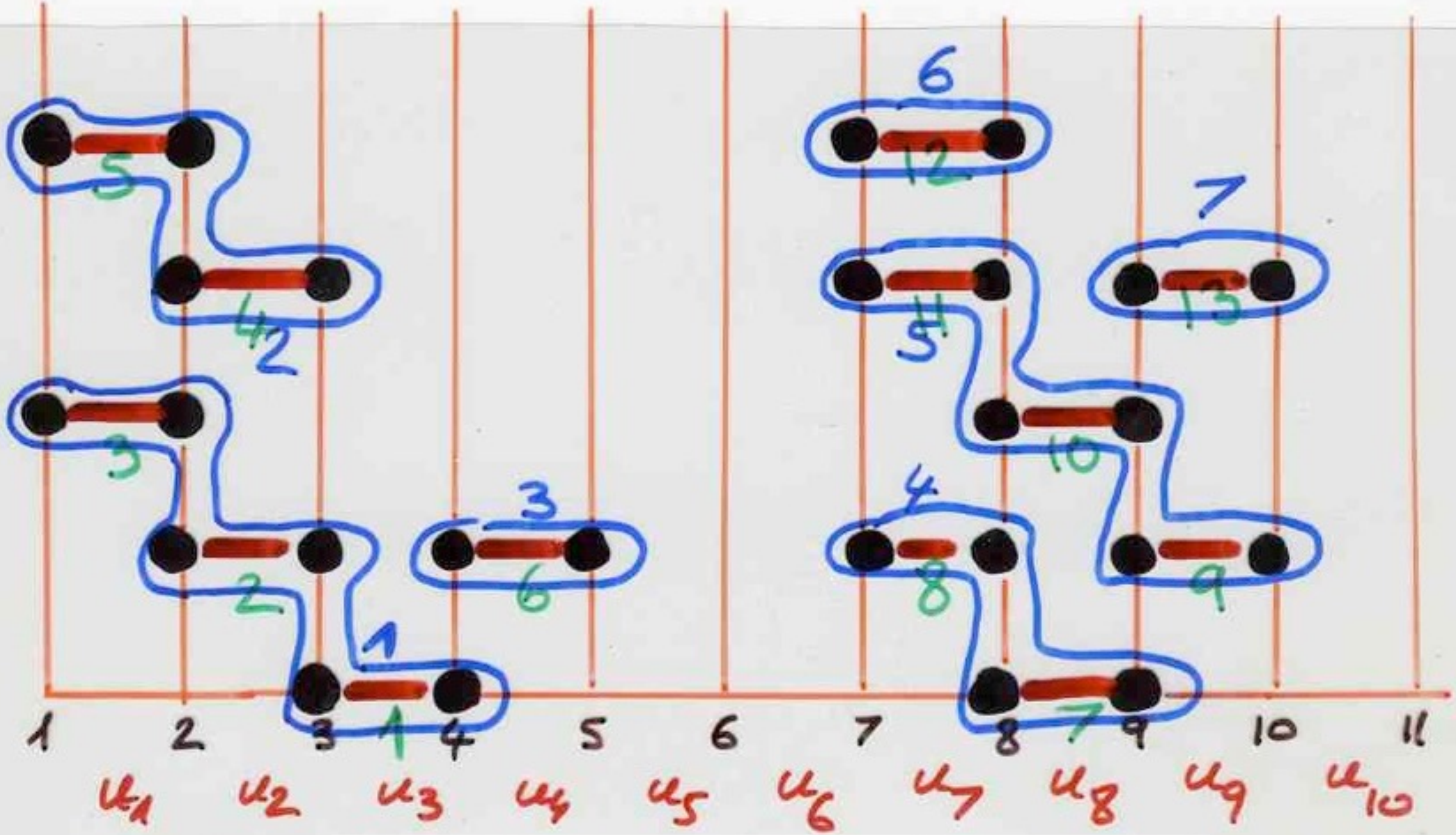
- lexicographic normal form
  [« Knuth normal form »]

Taking the left most or
rigth most maximal piece. ....

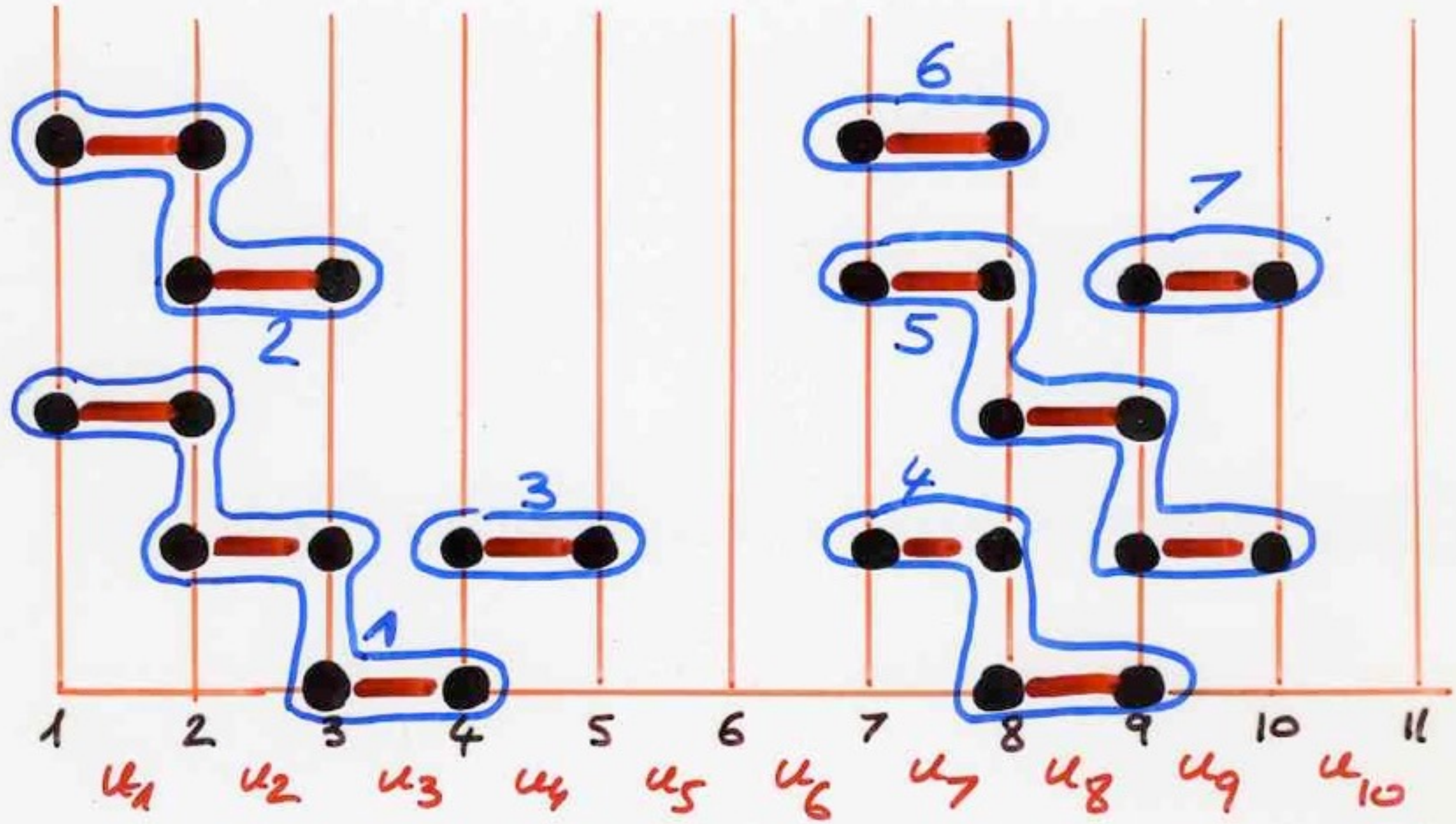
equivalent to the so-called
Lexicographic normal form



lexicographic normal form



lexicographic normal form



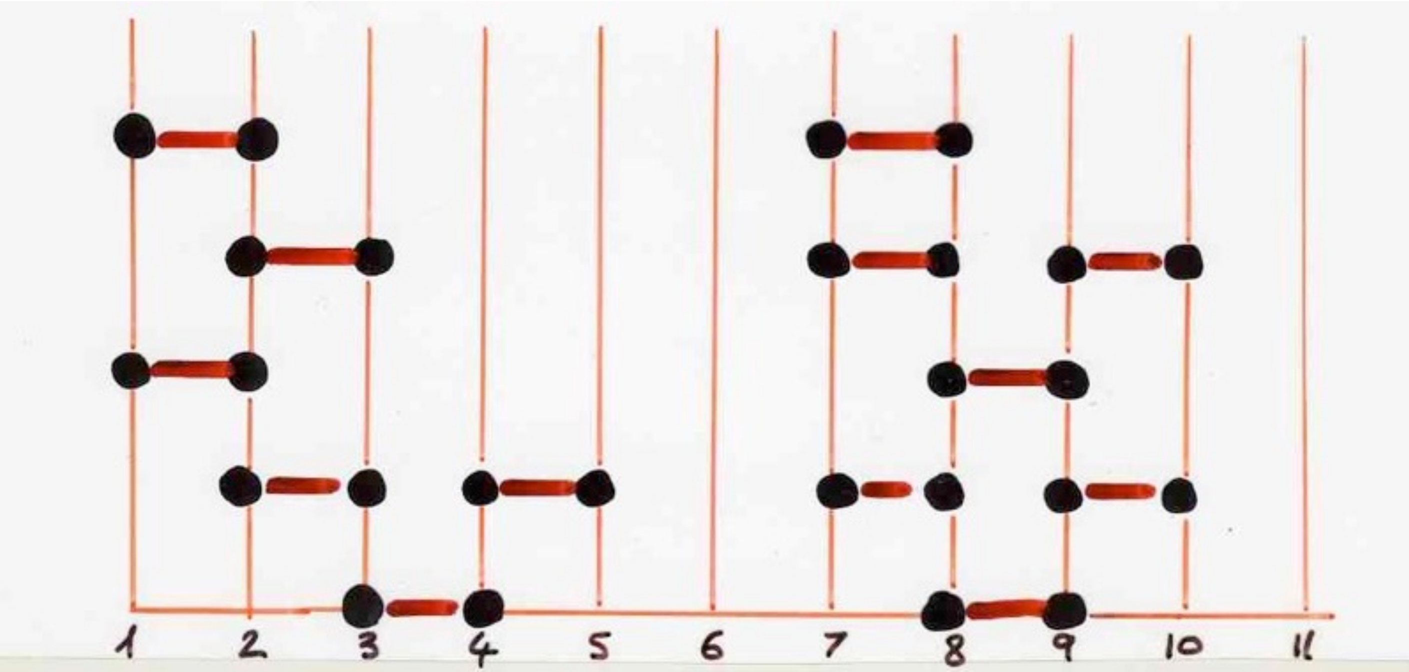
1
2
3
4
5
6
7
1 2 3 4 5 6 7 8 9 10 11
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10}

ordering the segments

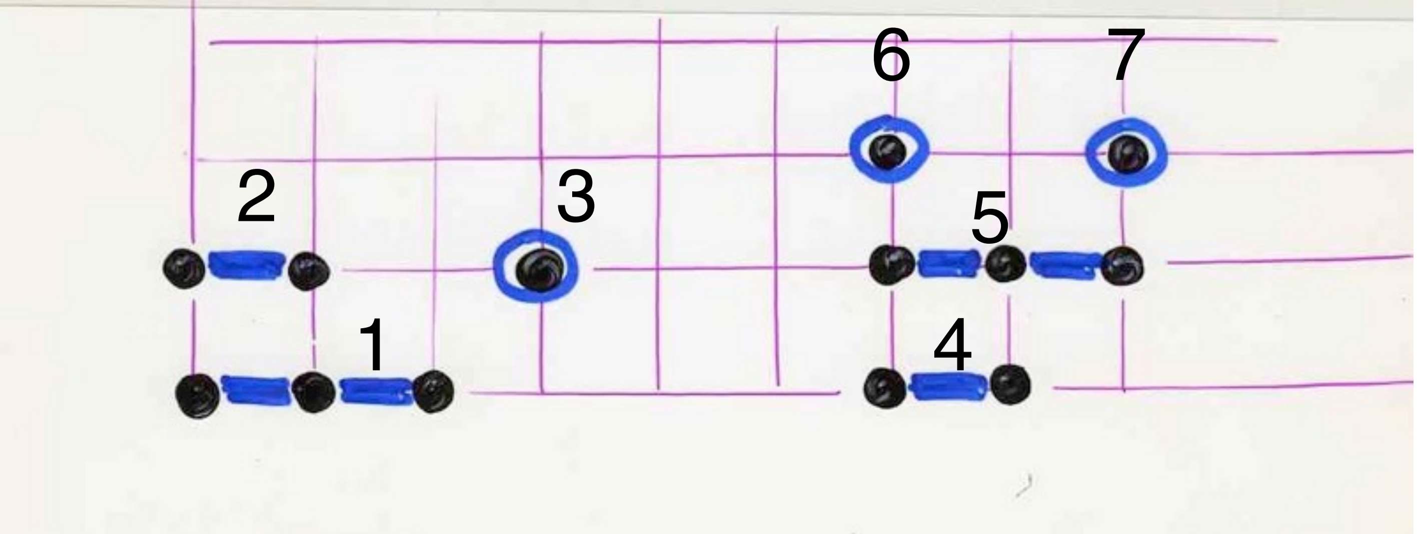
total order of the segments in a heap of segments

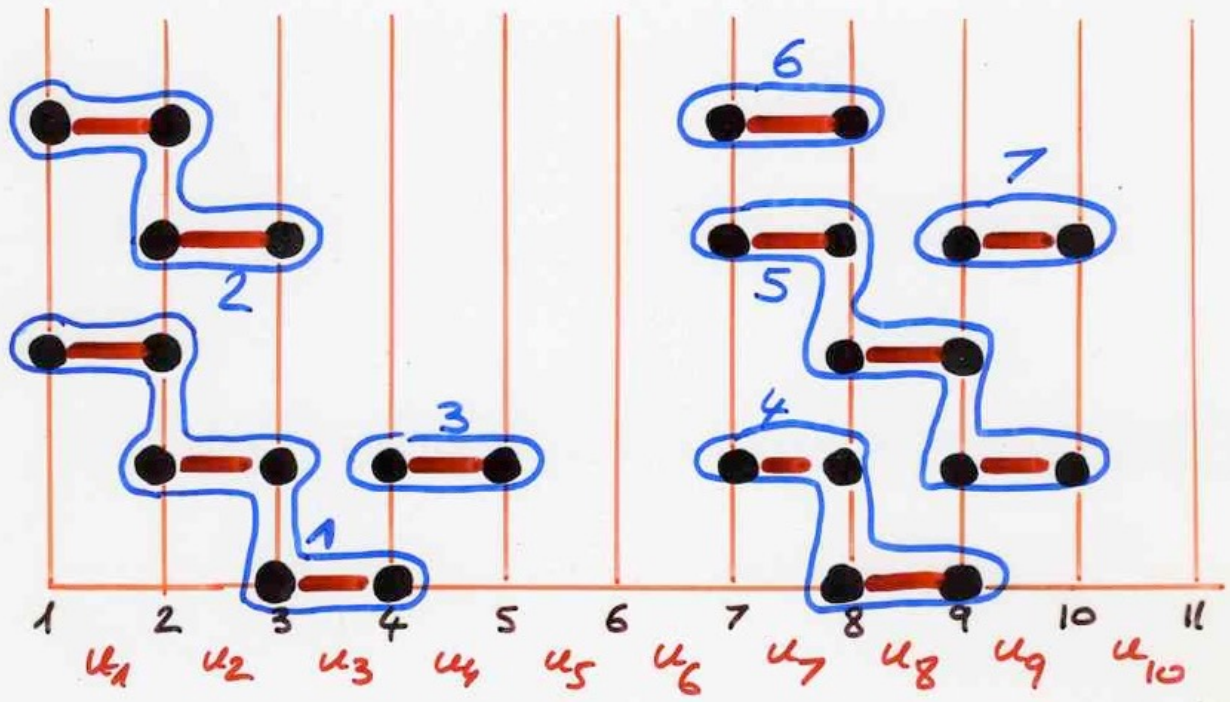
$\longrightarrow$

total order of the stairs in a heap of dimers

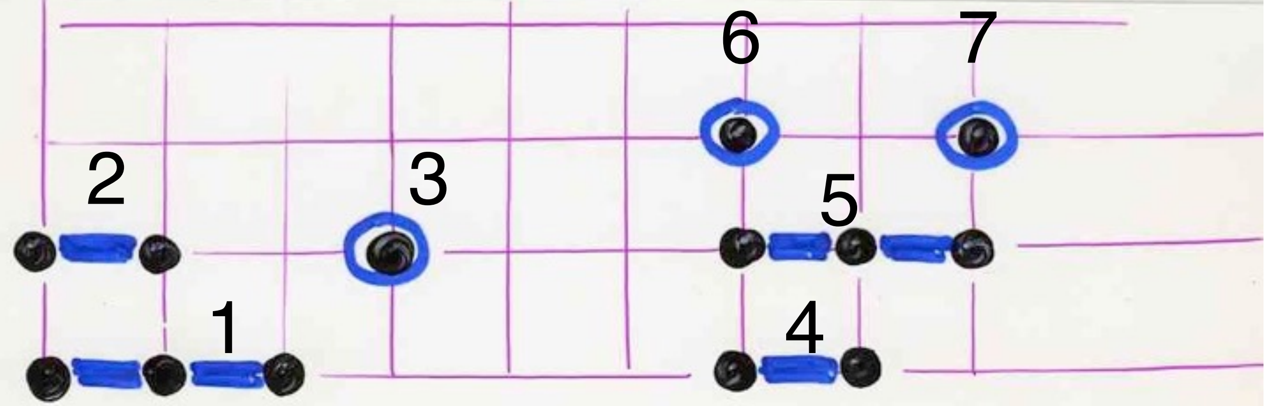


lexicographic normal form





6
7
2
5
3
4
1
1 2 3 4 5 6 7 8 9 10 11
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10}

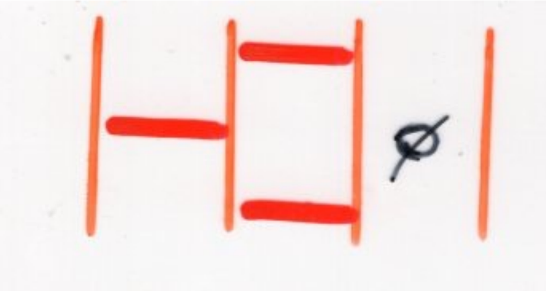


6
7
2
3
5
1
4

the stair lemma

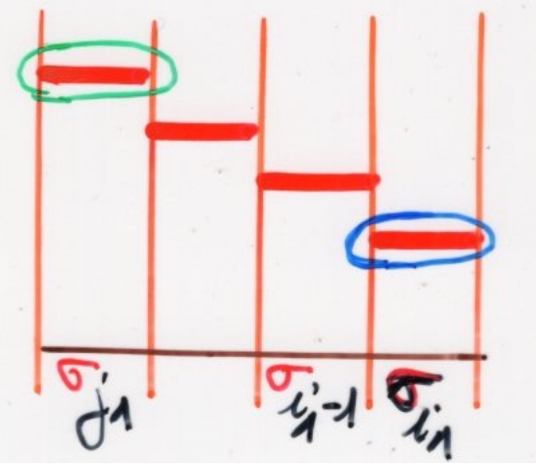
The stair lemma

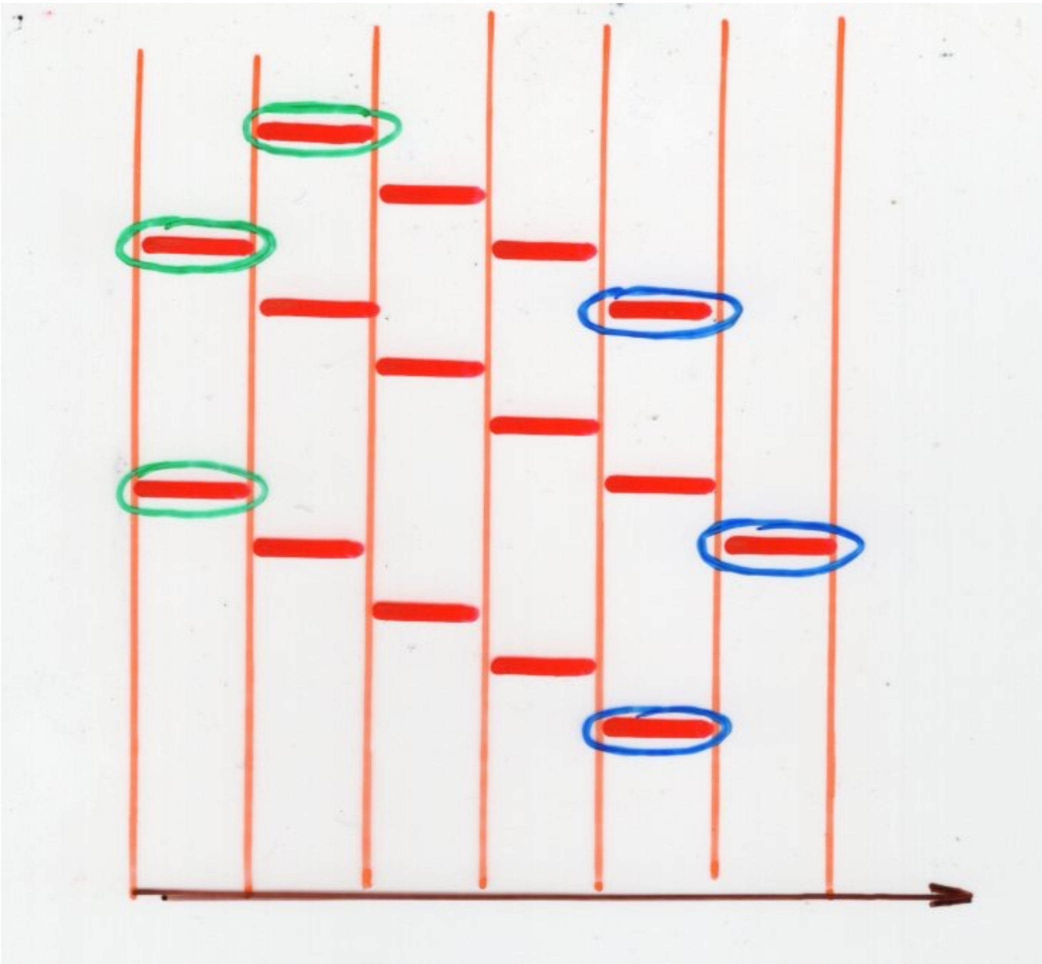
no occurences of

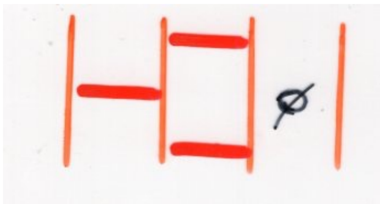


$\Longleftrightarrow$ $\min(S_1) \prec \cdots \prec \min(S_k)$

$\Longleftrightarrow$ $\max(S_1) \prec \cdots \prec \max(S_k)$

$S_i =$ 

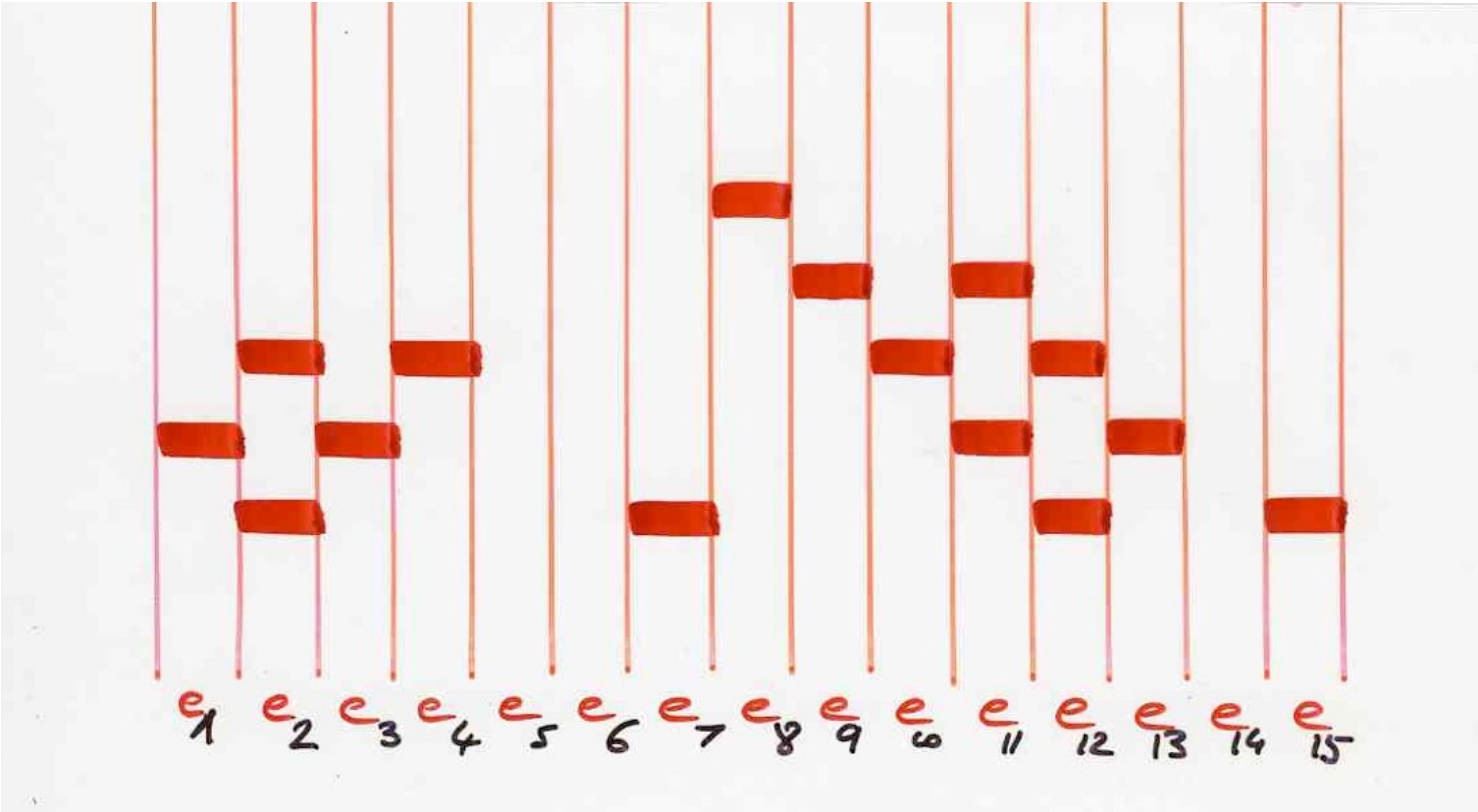




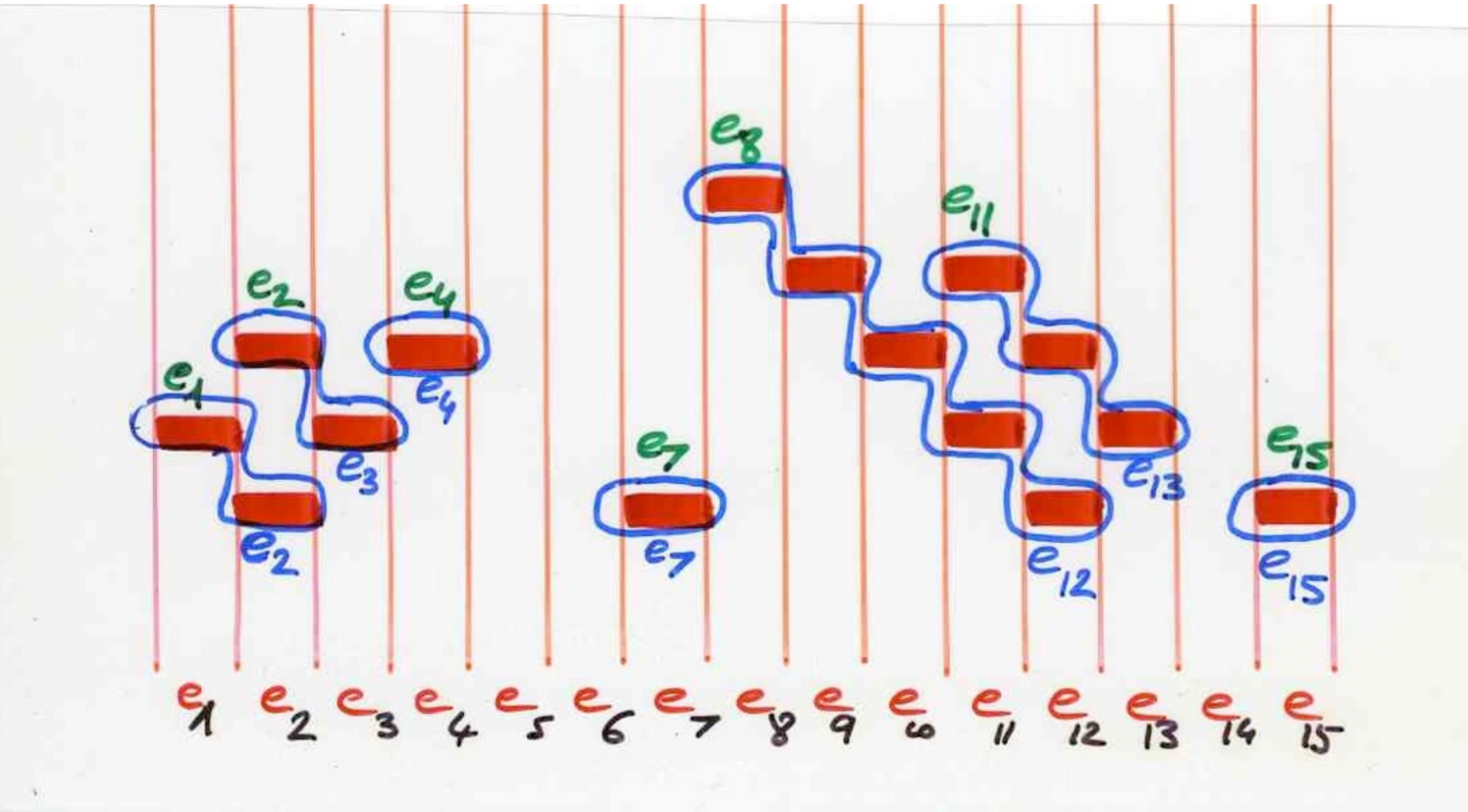
$\Longleftrightarrow$ $\min(S_1) \prec \cdots \prec \min(S_k)$

$\Longleftrightarrow$ $\max(S_1) \prec \cdots \prec \max(S_k)$

fully commutative heaps
(of dimers)



e 1 e 2 e 3 e 4 e 5 e 6 e 7 e 8 e 9 e 10 e 11 e 12 e 13 e 14 e 15

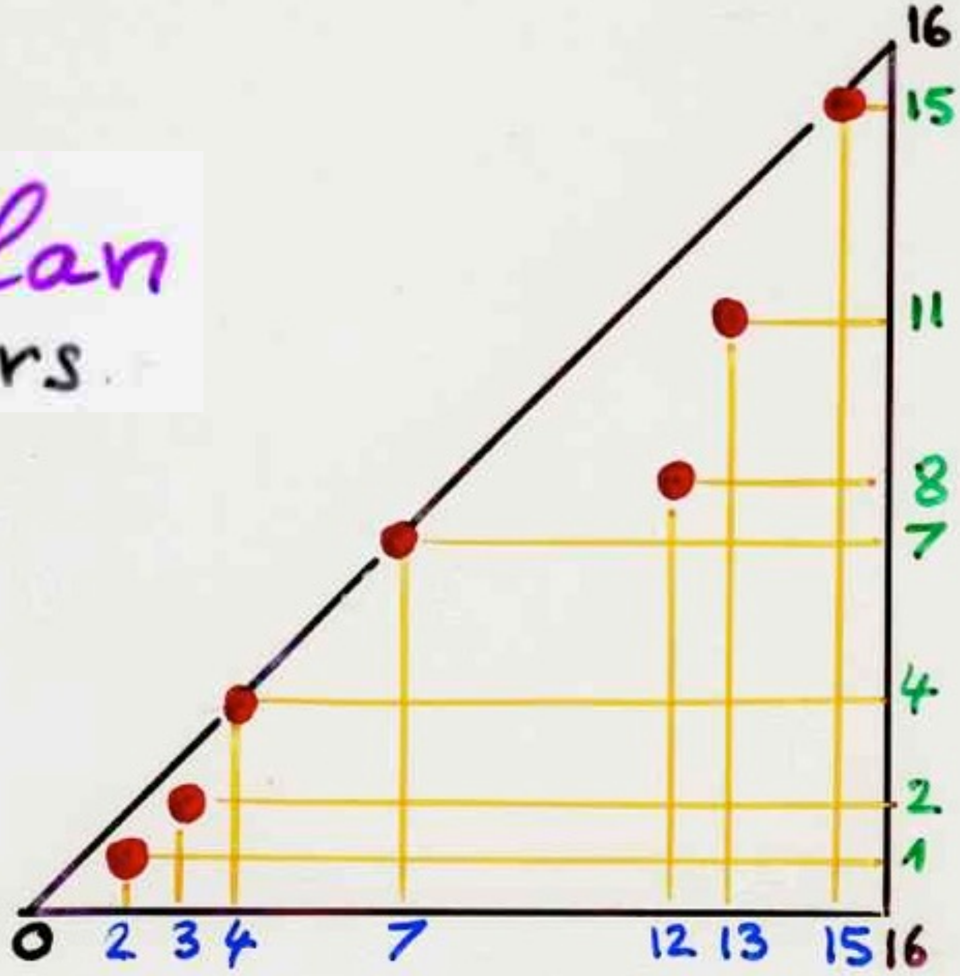


e1
e2
e2
e3
e4
e4
e7
e7
e8
e11
e12
e13
e15
e15
e1 e2 e3 e4 e5 e6 e7 e8 e9 e10 e11 e12 e13 e14 e15

Catalan numbers

$$\begin{array}{ccccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{array}$$

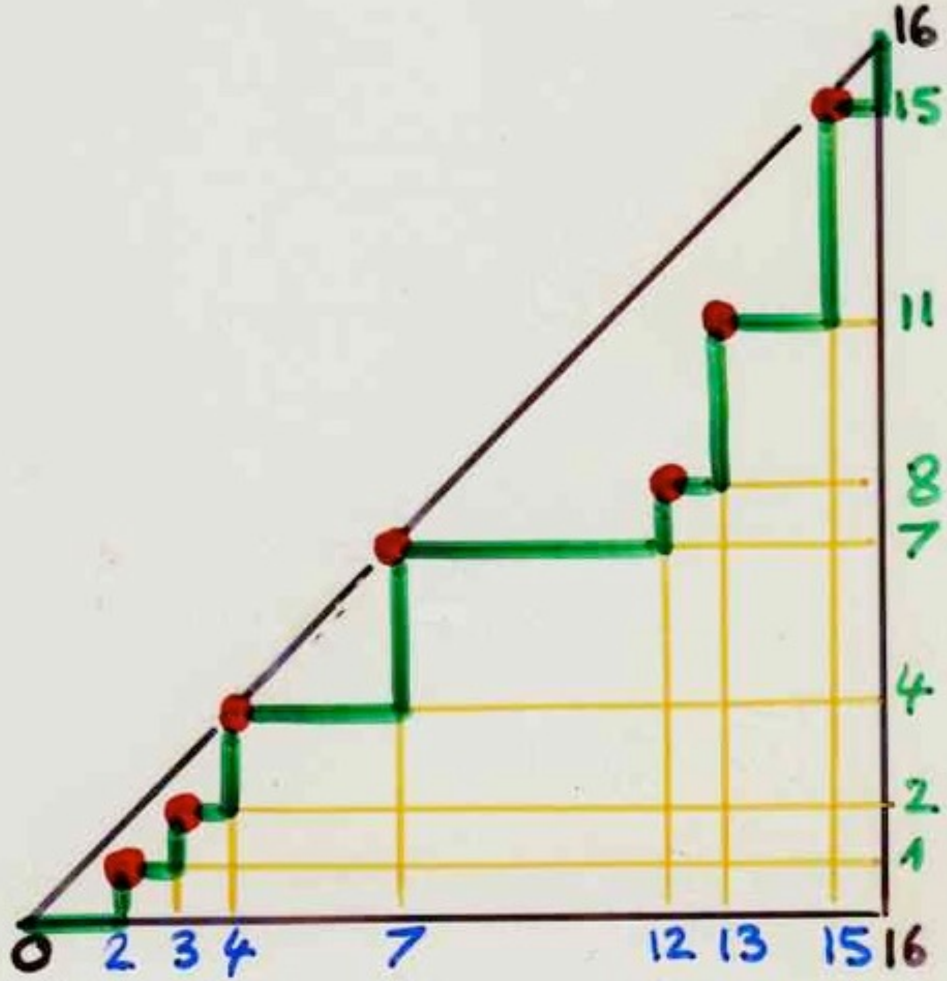
Catalan numbers



$$
\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}
$$

Dyck

path



$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$

More details in the video-book:

« *ABjC* », Part II, *Commutations and heaps of pieces* with interactions in physics, mathematics and computer science

IMSc, Chennai, 2017, Chapter 6, Heaps and Coxeter groups

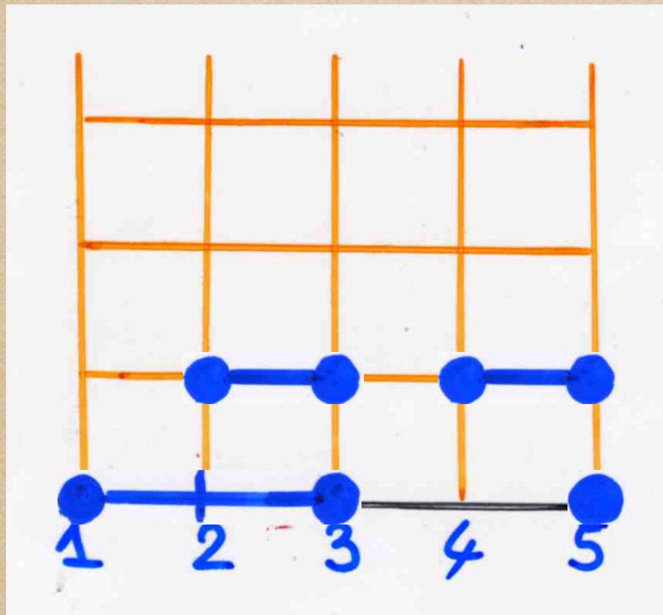
www.viennot.org/abjc2-ch6.html

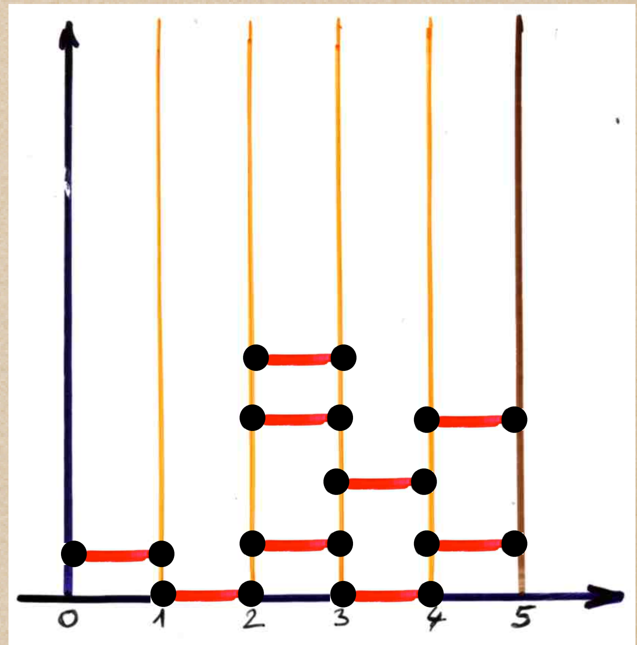
Ch 6a, the heap monoid of a Coxeter group, reduced decomposition,

fully commutative elements of Coxeter group, stair decommposition of a heap of dimers, fully commutative heaps of dimers, relation with parallelogram polyominoes, bijection FC elements — (321) avoiding permutations

slide added after the talk

The duality

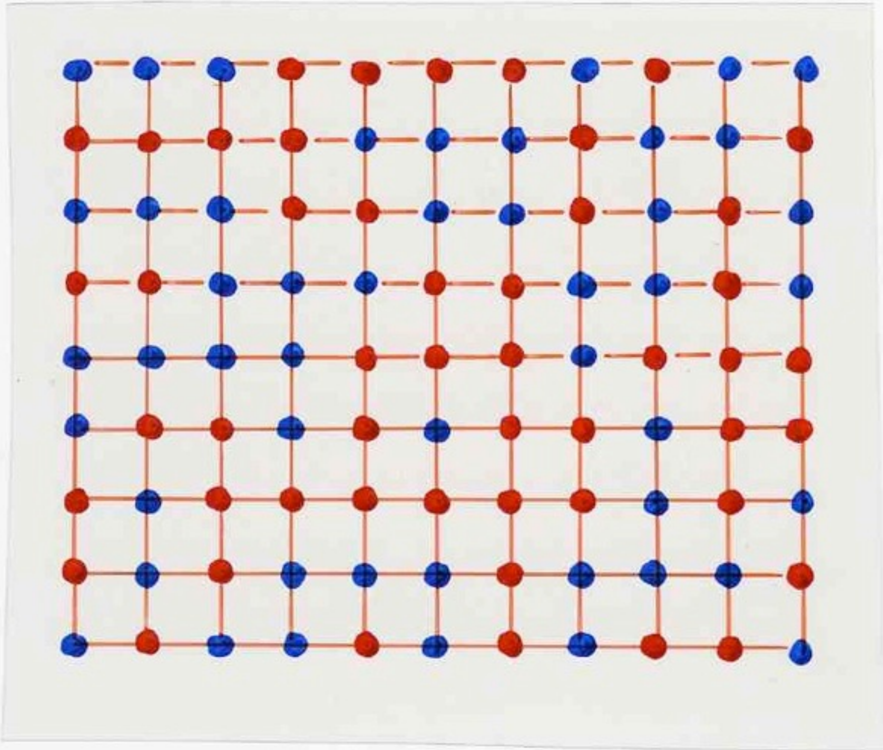




In the context of

The first and second bijection paths-heaps

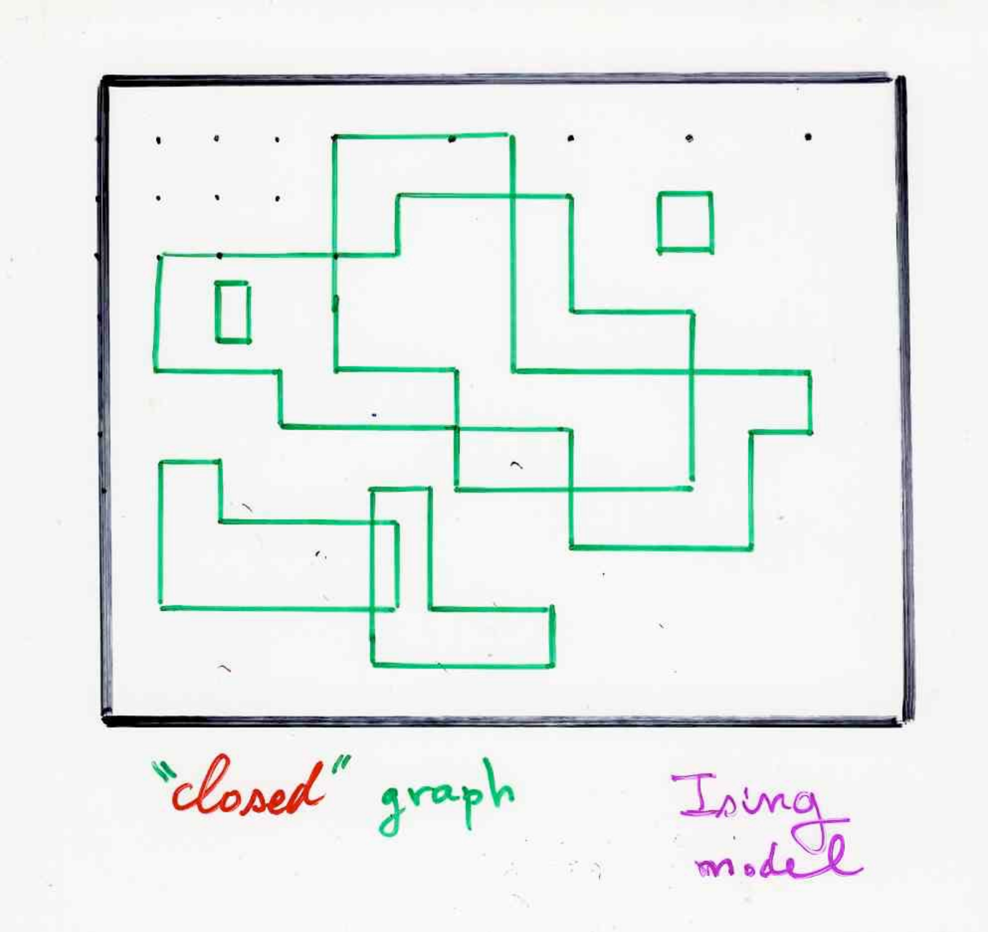
Onsager (1944)



Ising
model

Kastelyn (1961)

Fisher – Temperley (1961)



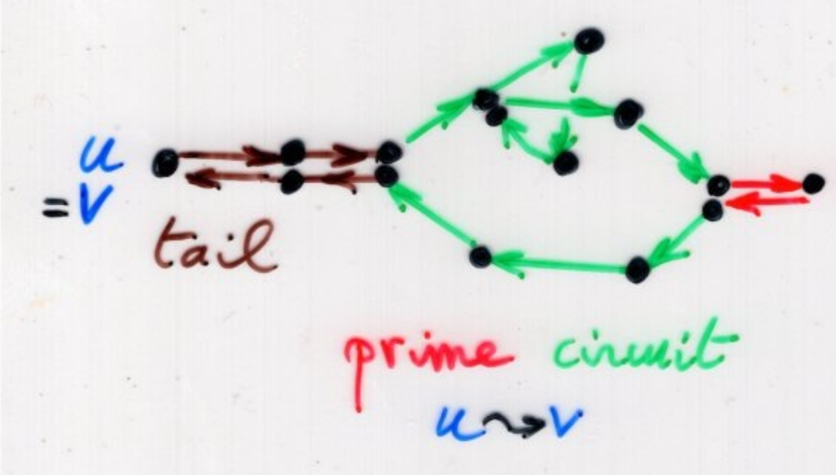
Kac-Ward (1952)

Sherman (1960)

Helmuth (2012)

• T. Helmuth, A. Shapira Aug. 2020

- *Loop-erased random walk as a spin system observable,*



backtracking

- no tail
- no backtracking

second
bijection ψ

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

$\underset{u \rightsquigarrow v}{\eta}$ trail

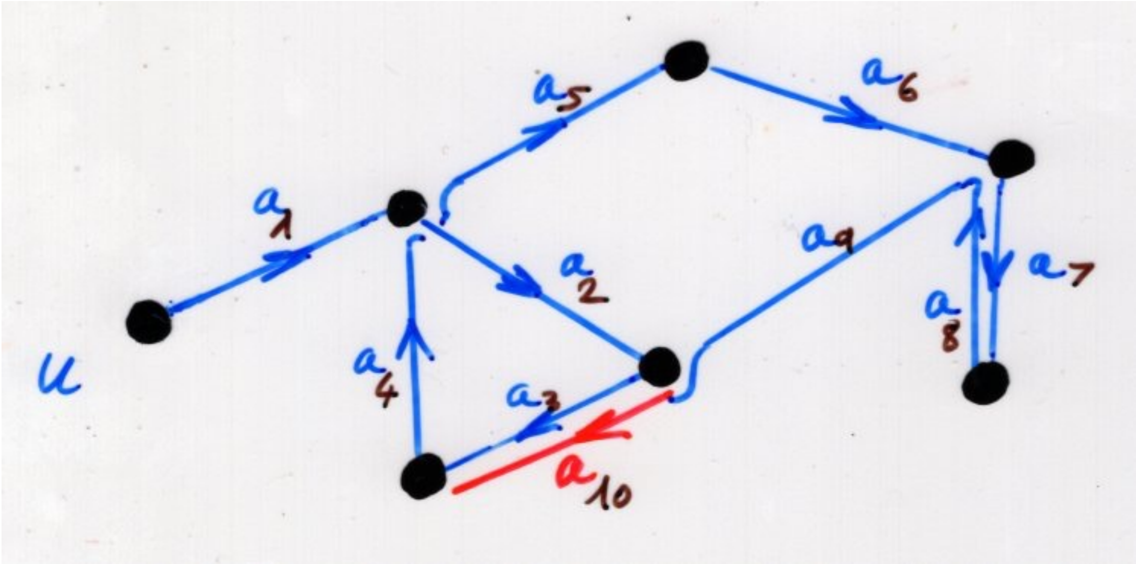
trail = path having all oriented edges distinct

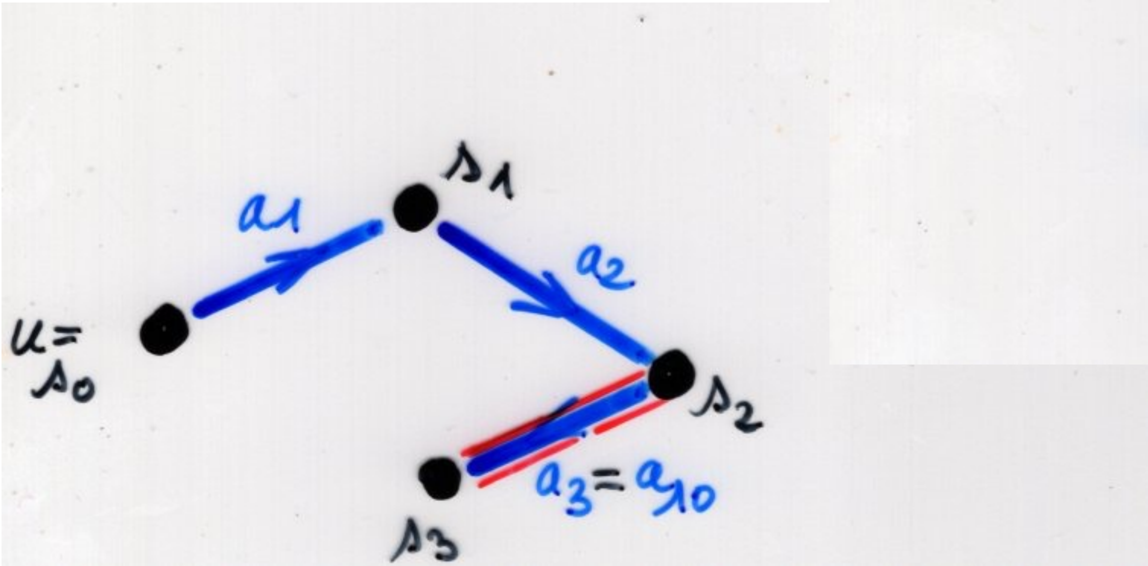
F heap of "oriented loops"

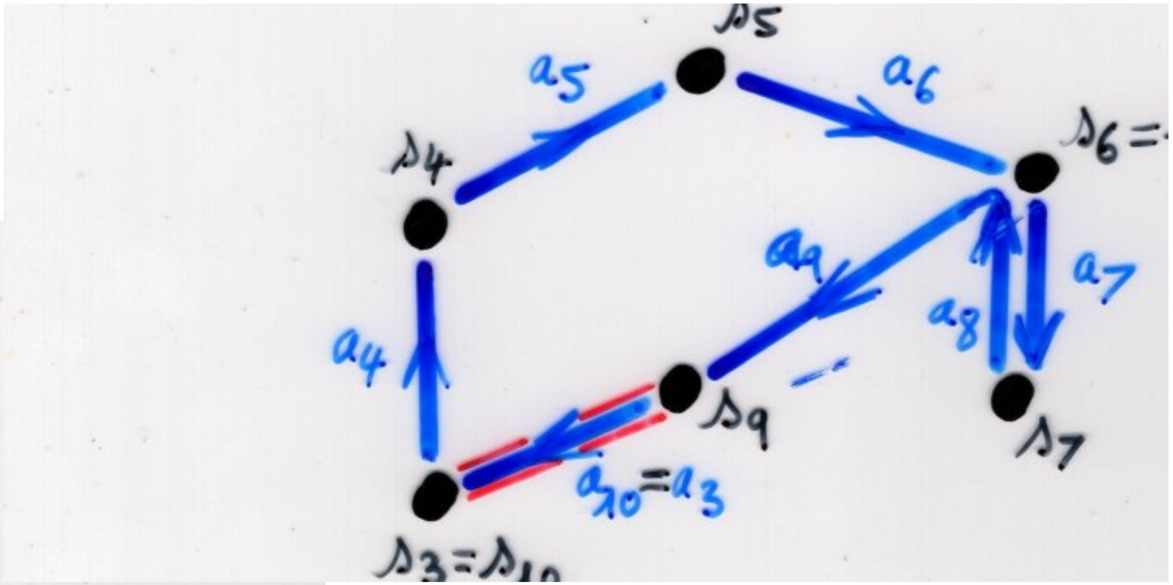
oriented loop

equivalence class of trail

trail $\underset{u \rightsquigarrow u}{\eta}$ up to a circular permutation of its edges


u
a1
a2
a3
a4
a5
a6
a7
a8
a9
a10


u = s0
a1
s1
a2
s2
a3 = a10
s3


s5
a5
a6
s4
s6 =
a9
a7
a8
a4
s9
s7
a10 = a3
s3 = s10

Proposition

$\underset{\text{path}}{\omega} \xrightarrow{\psi} (\eta, F)$

ω (no tail
non backtracking

$\Longleftrightarrow$

$\left\{ \right.$
- η is non backtracking, no tail

and

- each oriented loops of F is non backtracking

More details in the video-book

« ABjC », Part II, *Commutations and heaps of pieces with interactions in physics, mathematics and computer science*
IMSc, Chennai, 2017, Chapter 5, Heaps and algebraic graph theory
www.viennot.org/abjc2-ch5.html

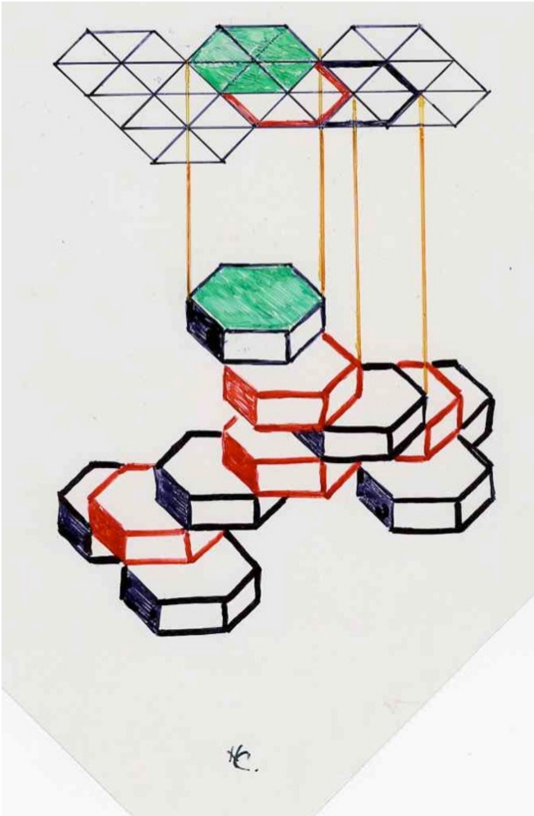
Ch 5b, the second bijection paths — heaps of oriented loops, pp 21-31

T. Helmut, « Ising model observables and non-backtracking walks",
J. Math. Phys. **55**(8), 1–28, 2014. arXiv: 1209.3996v3 [math.CO].

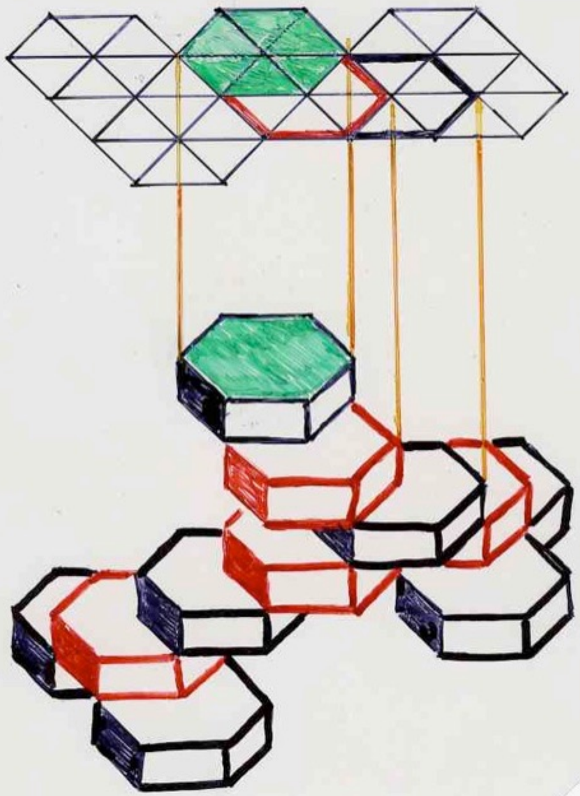
slide added after the talk

The logarithmic Lemma

$$t \frac{d}{dt} \log \left(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{|E|} \right) = \sum_{\substack{P \\ \text{pyramid}}} v(P)\, t^{|P|}$$



$$-\rho(-t) = y$$



generating function
for the density
of Baxter hard hexagons
gas model

algebraic equation
degree 12

pyramid of hexagons

hand made slide: H. Crapo

Zeta function of a graph

$$\zeta(s) = \prod_{p} \left(\frac{1}{1-p^{-s}} \right)$$

p prime number

Euler identity

$$Z_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

some "prime" over the graph G

Ihara-Selberg zeta function of a graph $Z_G(t)$

Ihara-Selberg zeta function of a graph

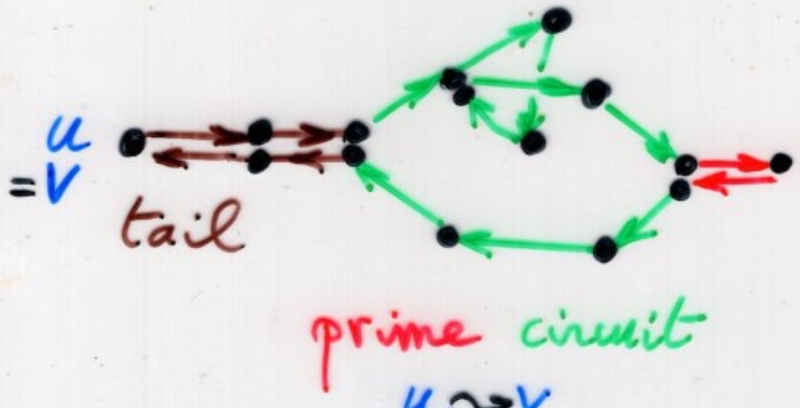
Ihara (1966)

$$(i) \quad \zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class

prime circuit

no backtracking



backtracking

(- no tail
- no backtracking

Ihara-Selberg zeta function of a graph

$$\text{(i)} \quad \zeta_G(t) = \prod_{[C]} \frac{1}{\left(1 - t^{|C|}\right)}$$

$$\text{(ii)} \quad \zeta_G(t) = \frac{1}{\det(1 - Ht)}$$

$$\text{(iii)} \quad \zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I - tA + t^2(D - I)\right)}$$

$$t \frac{d}{dt} \log \zeta_G(t)$$

Bass formula

More details in the video-book:

« *ABjC* », Part II, *Commutations and heaps of pieces*

with interactions in physics, mathematics and computer science

IMSc, Chennai, 2017, Chapter 5, Heaps and algebraic graph theory

Ch 5b, zeta function of a graph, pp 7-20

www.viennot.org/abjc2-ch5.html

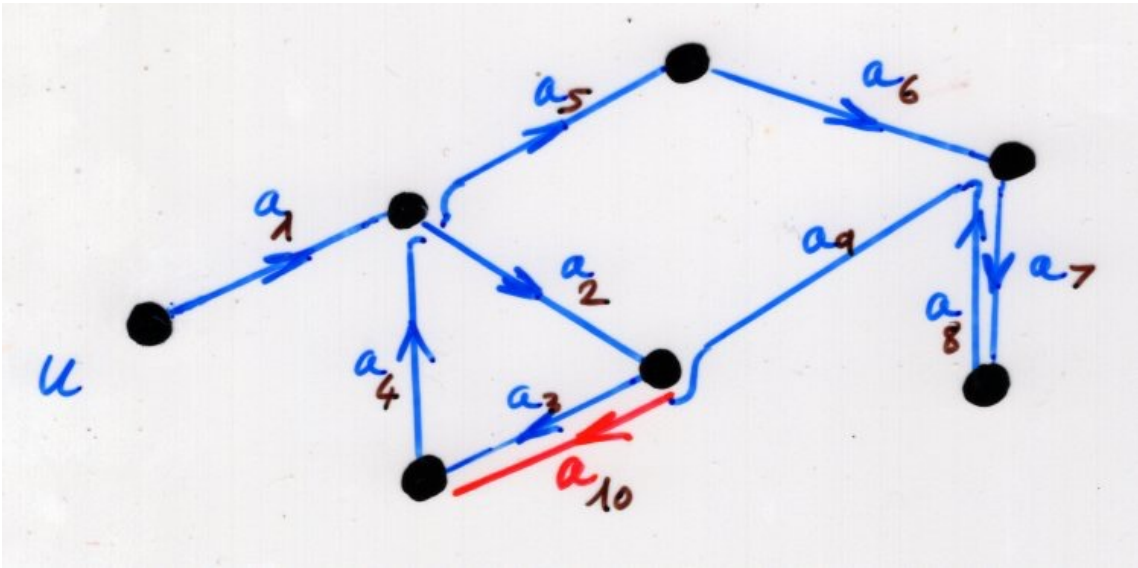
slide added after the talk

bijections

Dyck paths

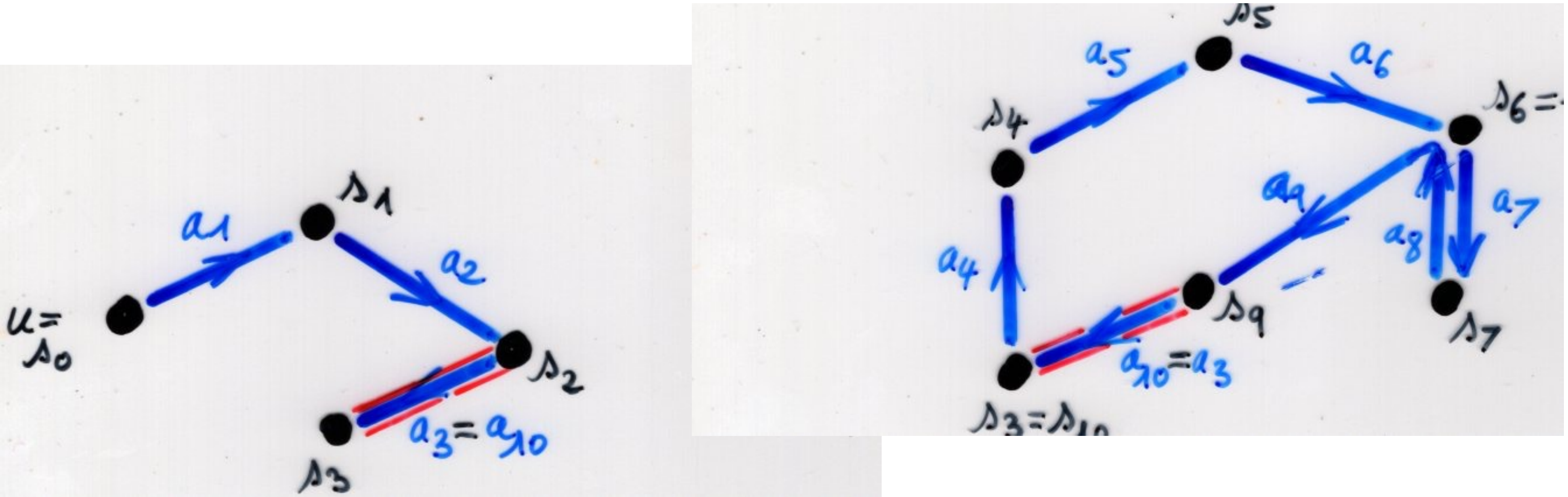
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

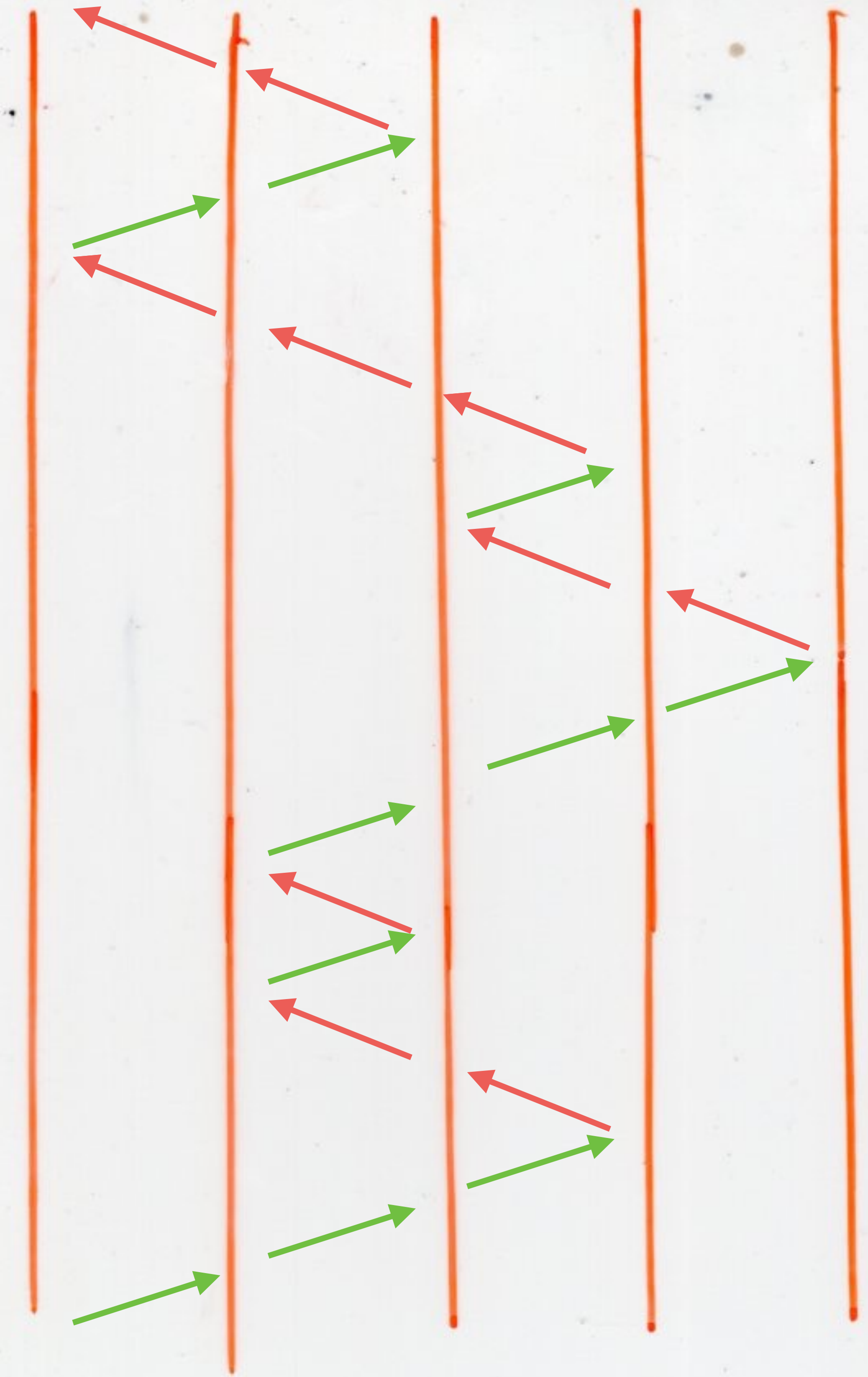
heaps of oriented loops

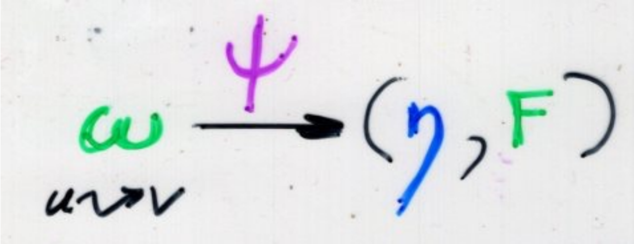


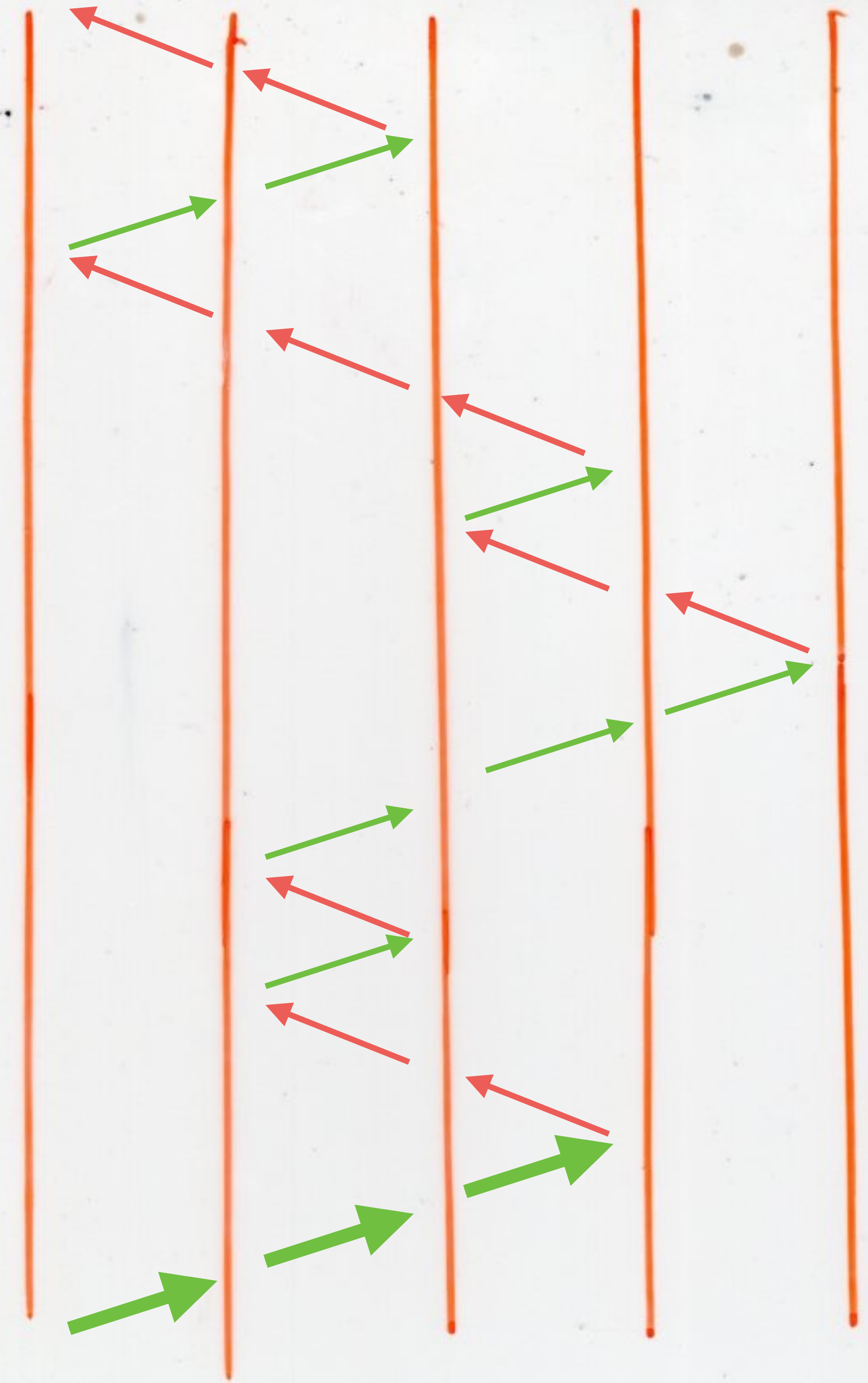
$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\Psi} (\eta, F)$

Ch 5b, p 21-29

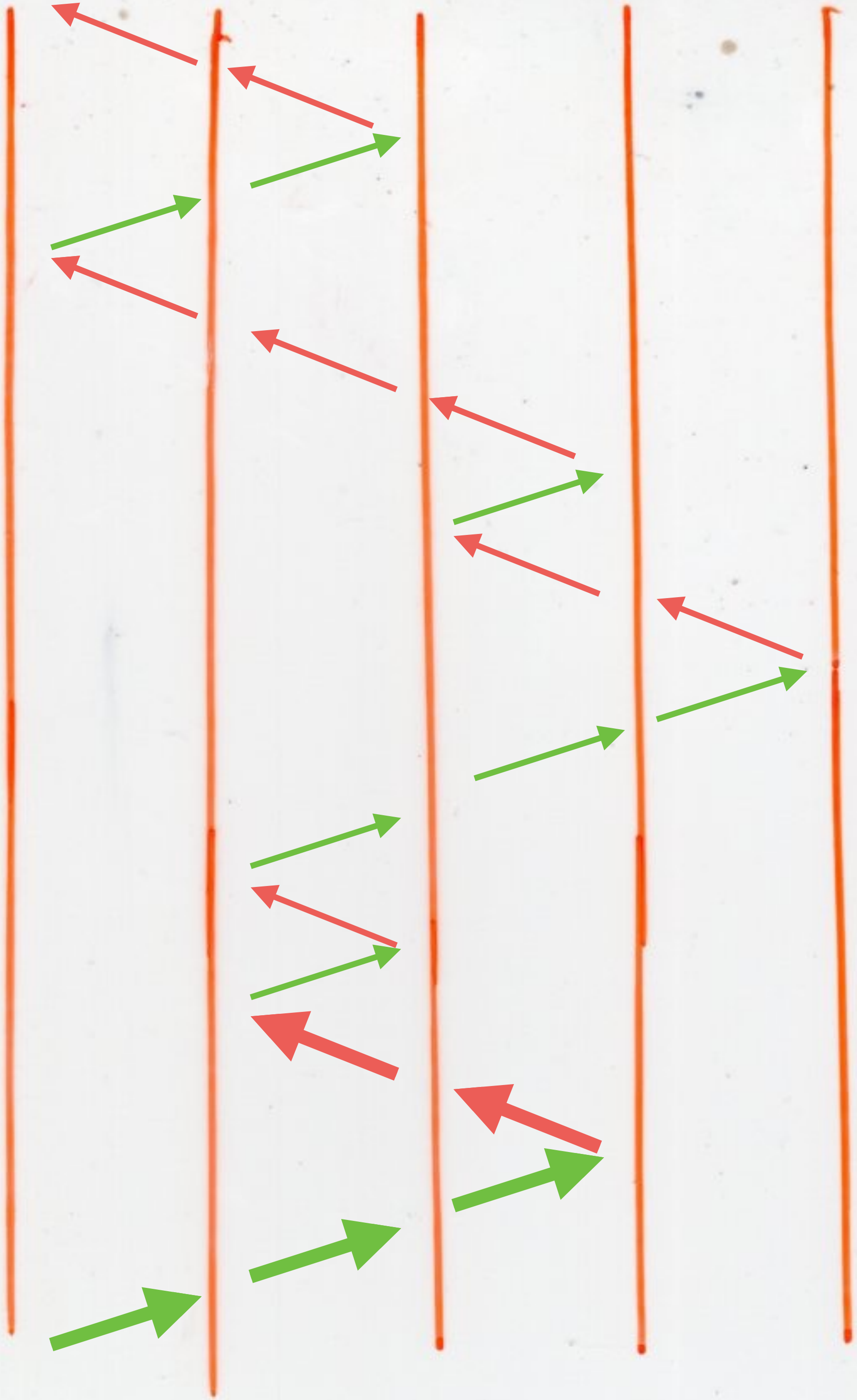




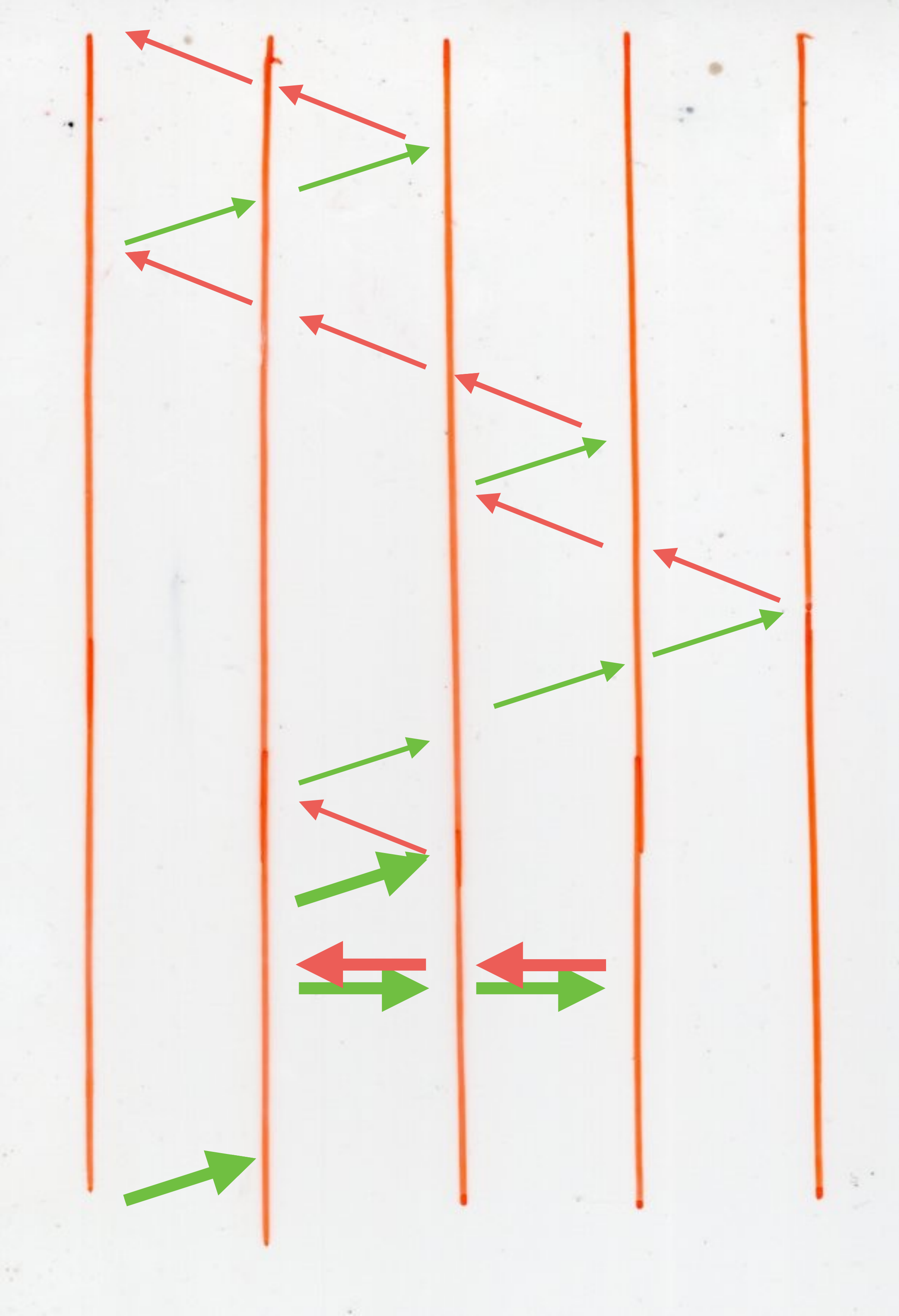

$\omega \xrightarrow{\psi} (\eta, F)$
$u \rightsquigarrow v$



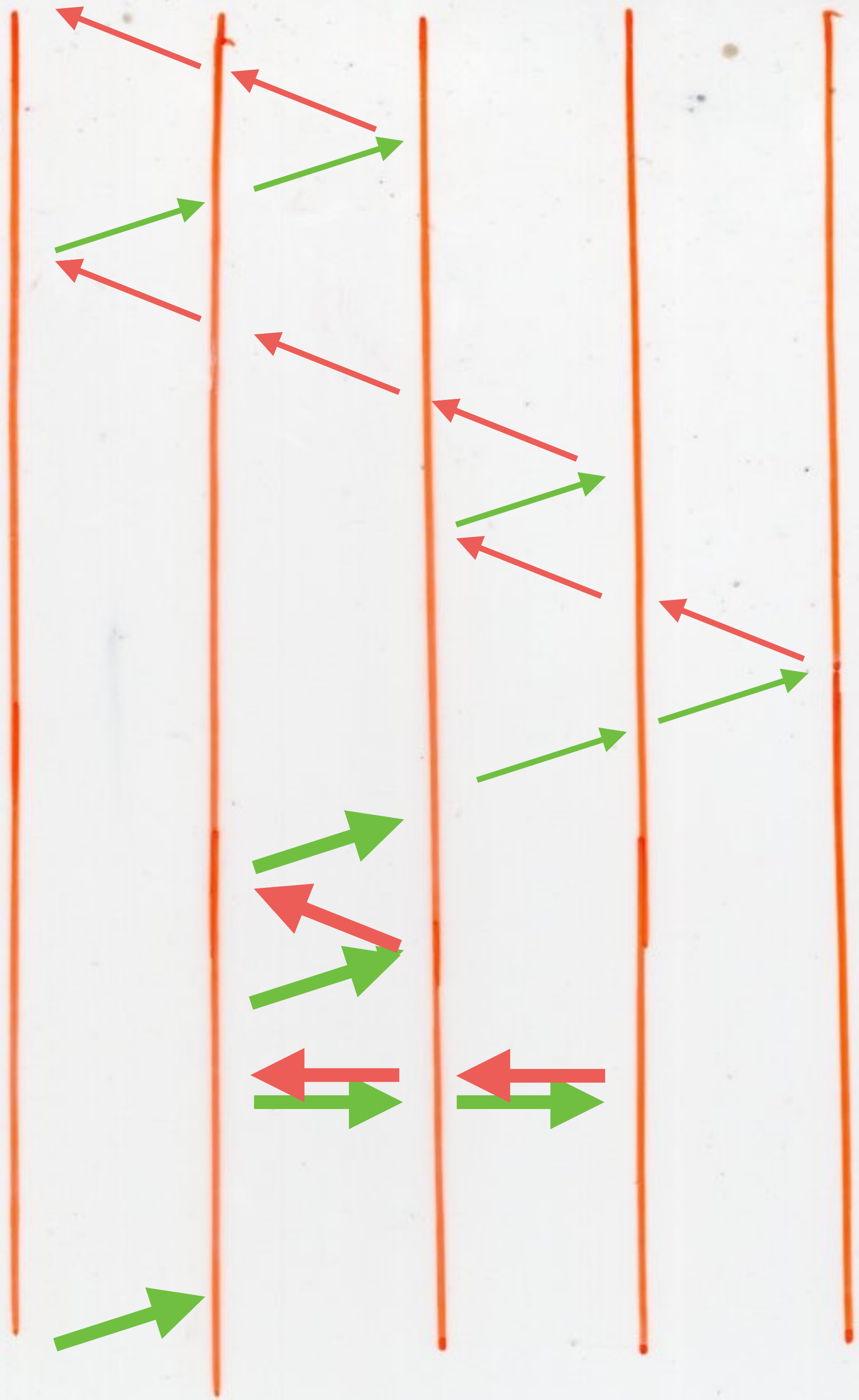
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



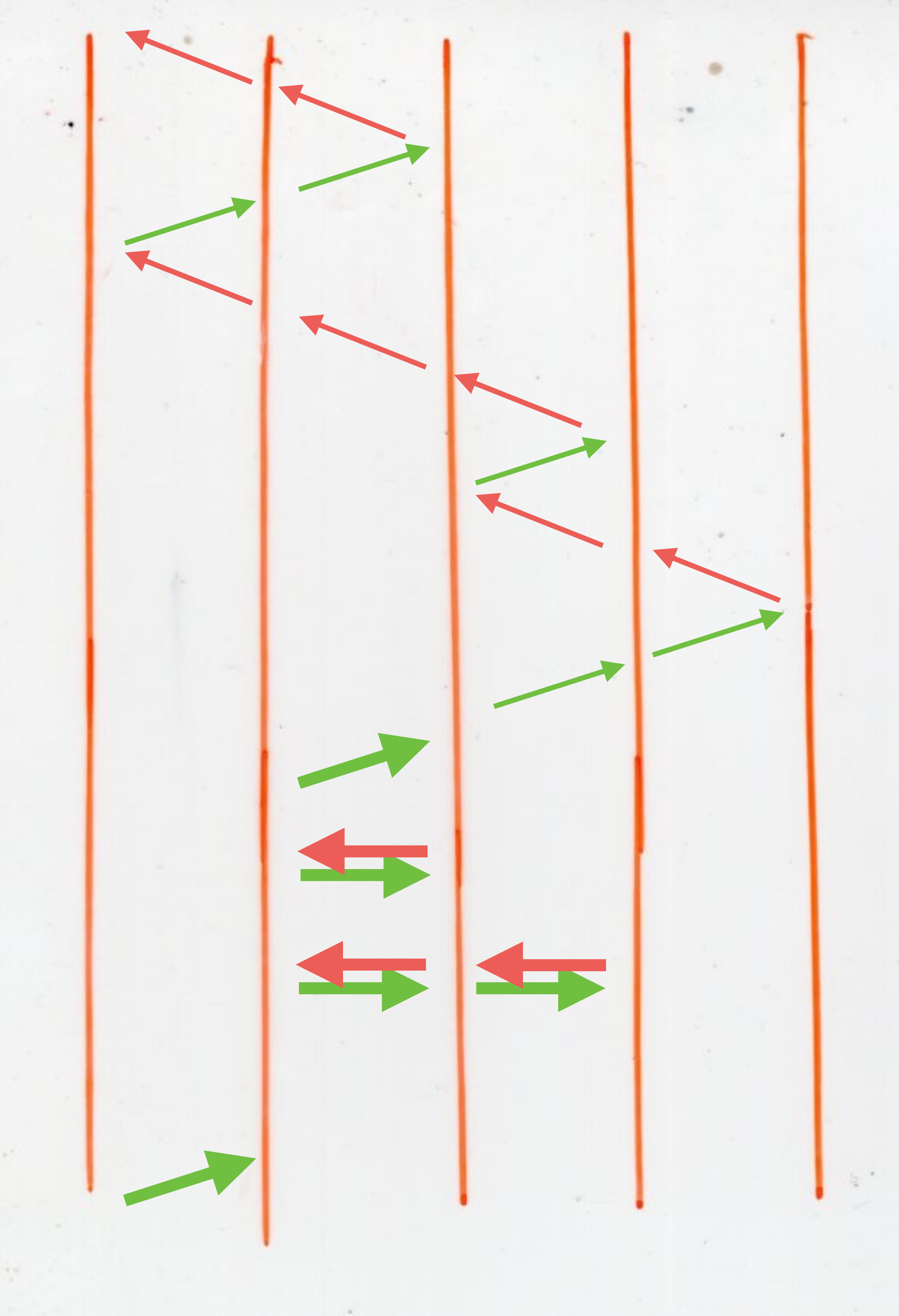
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



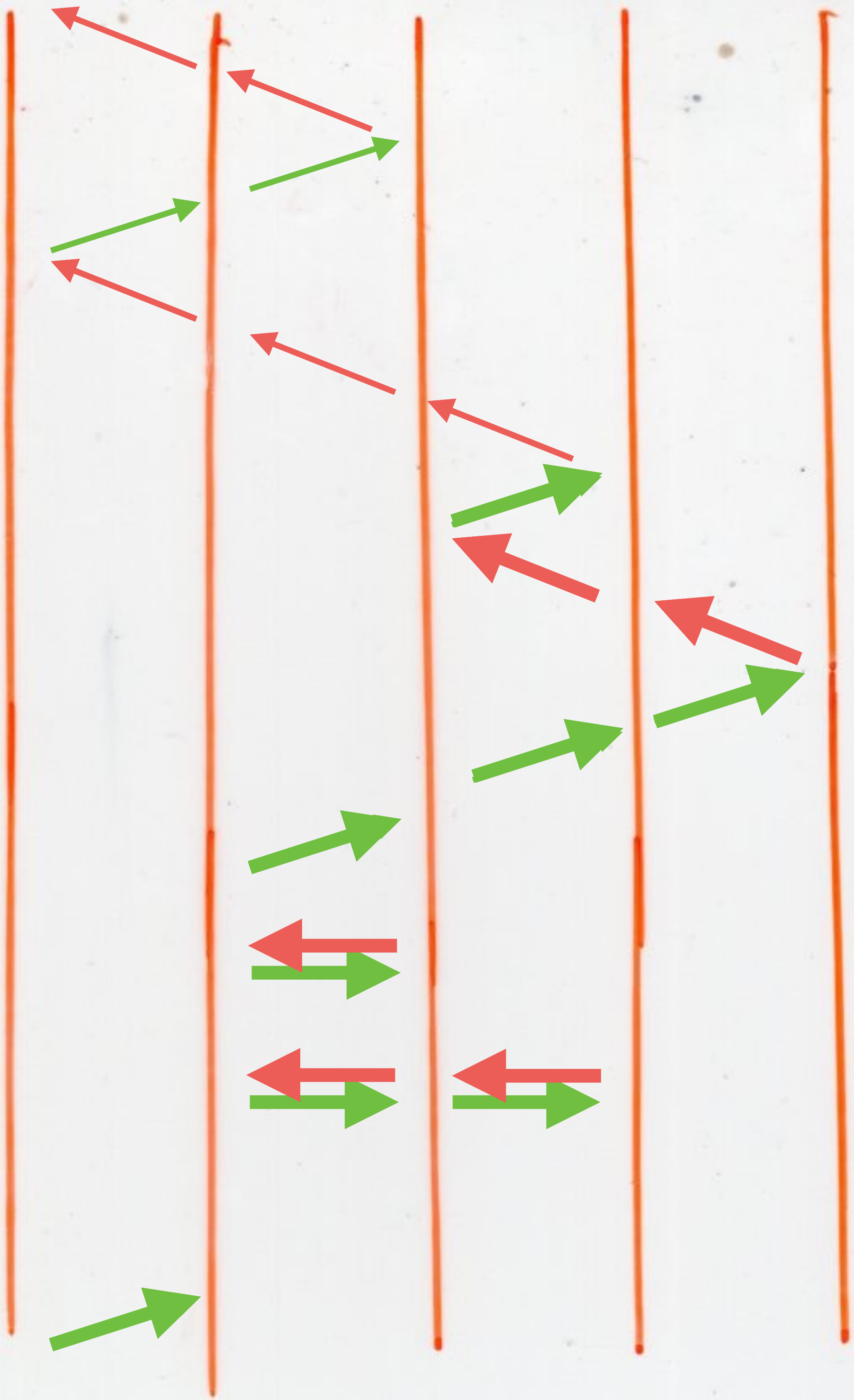
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



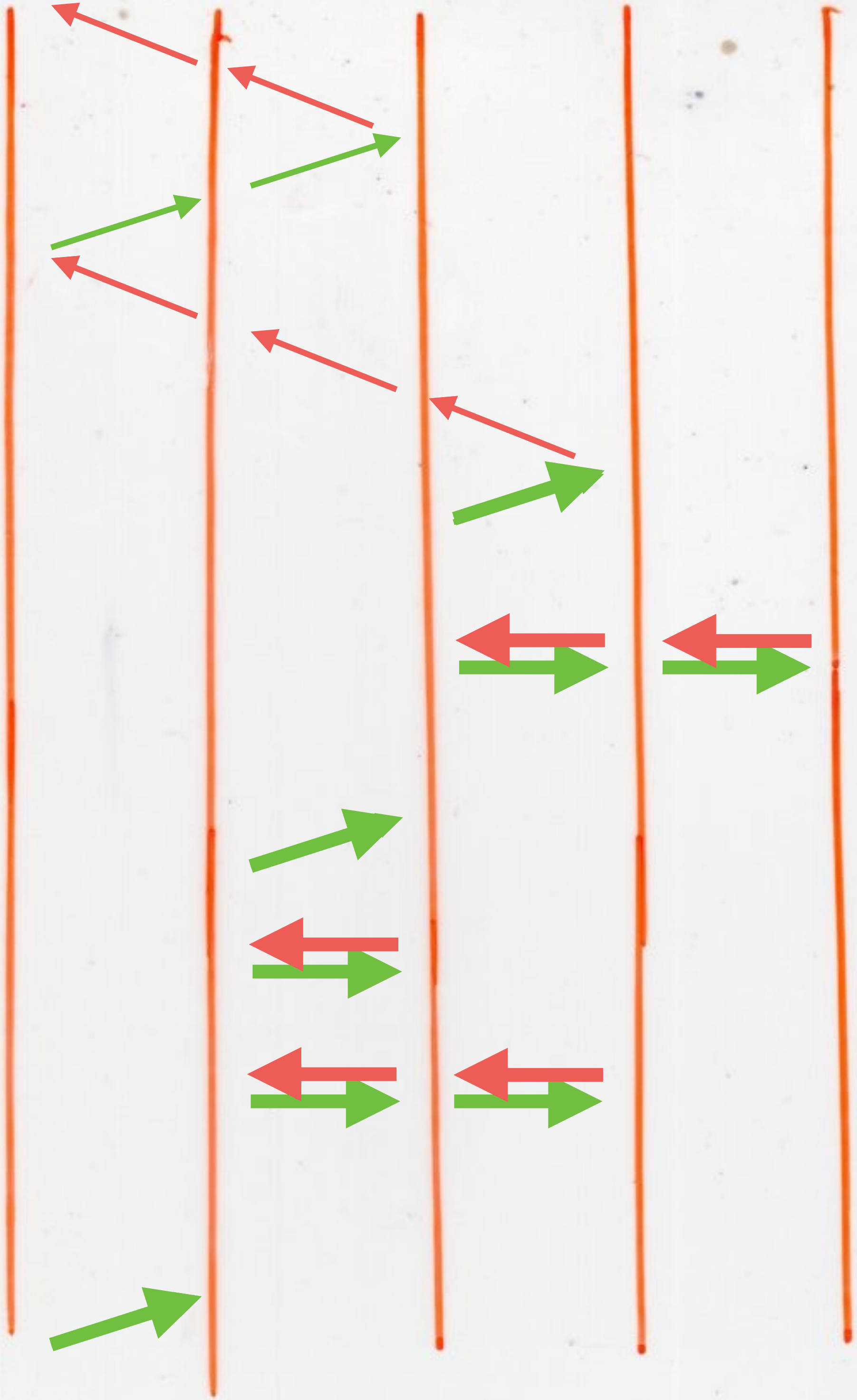
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



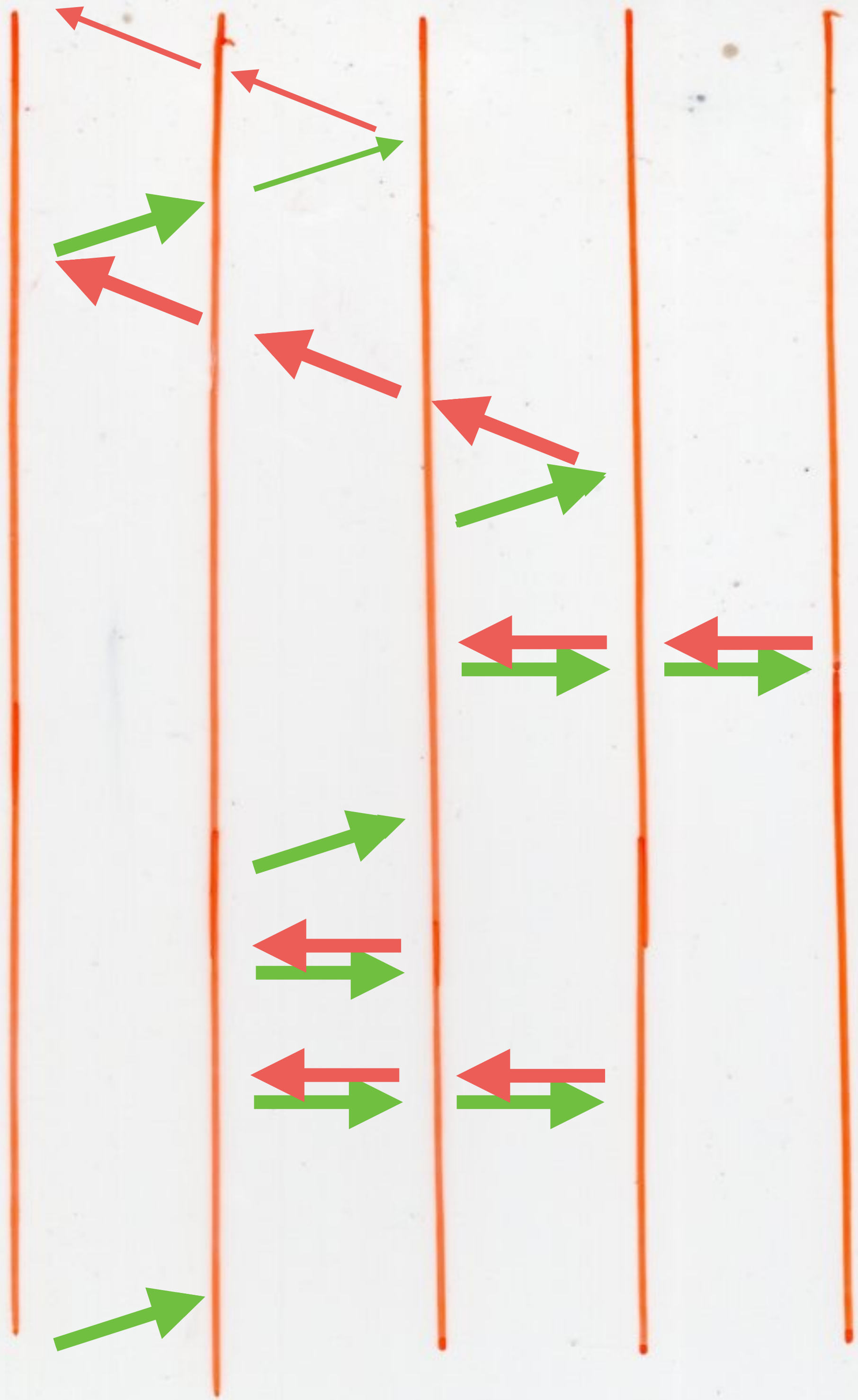
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



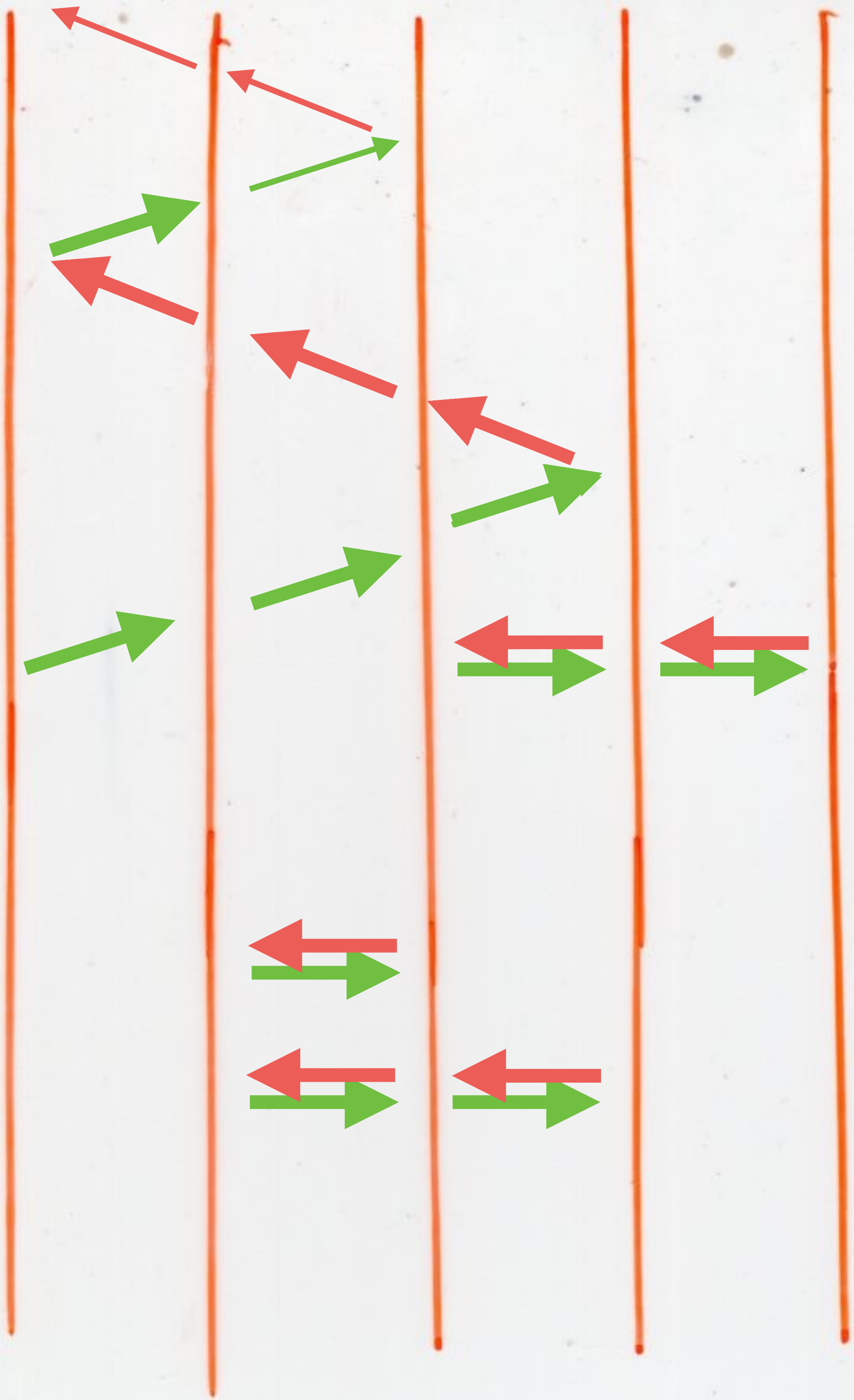
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



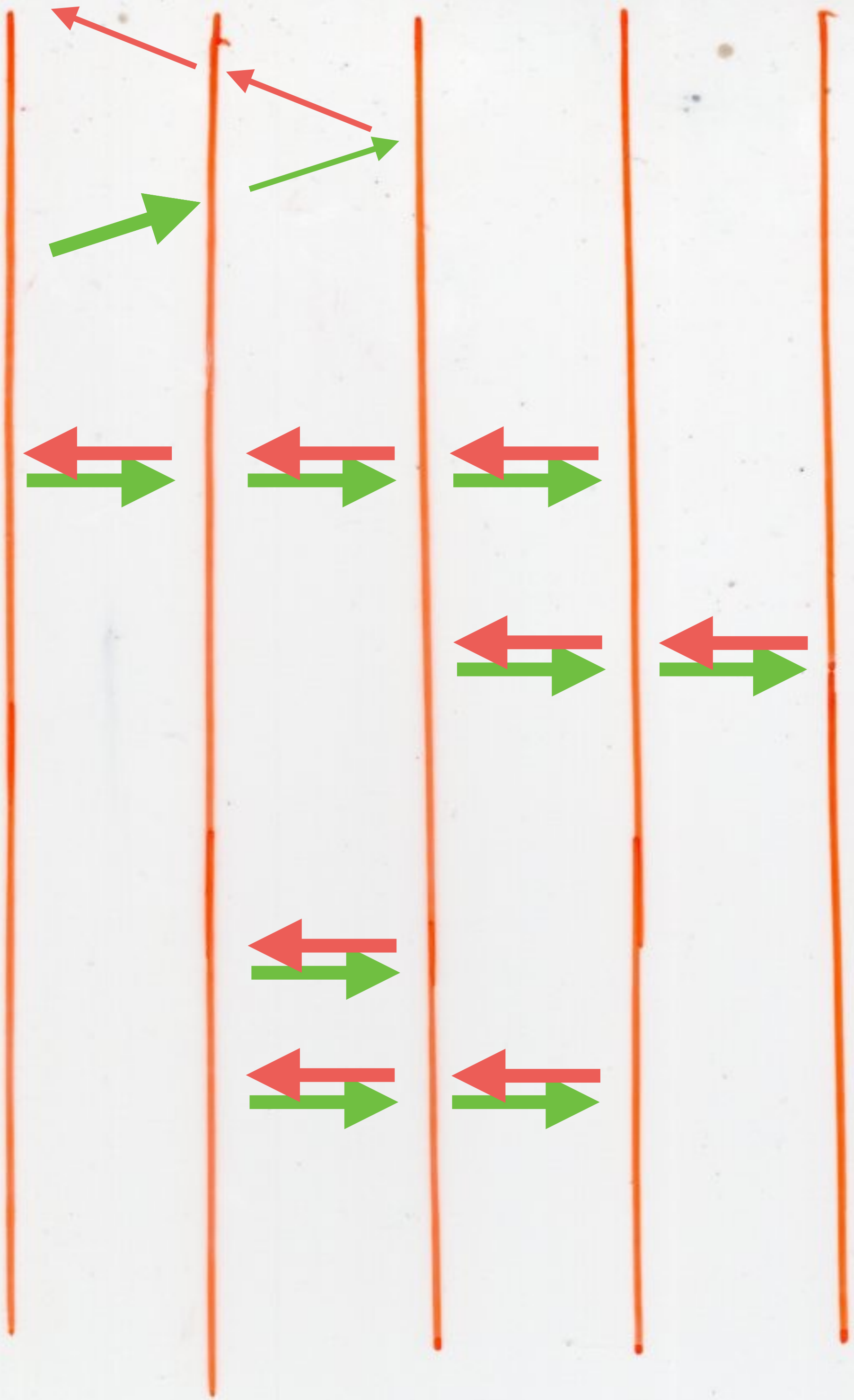
$$\underset{u\rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



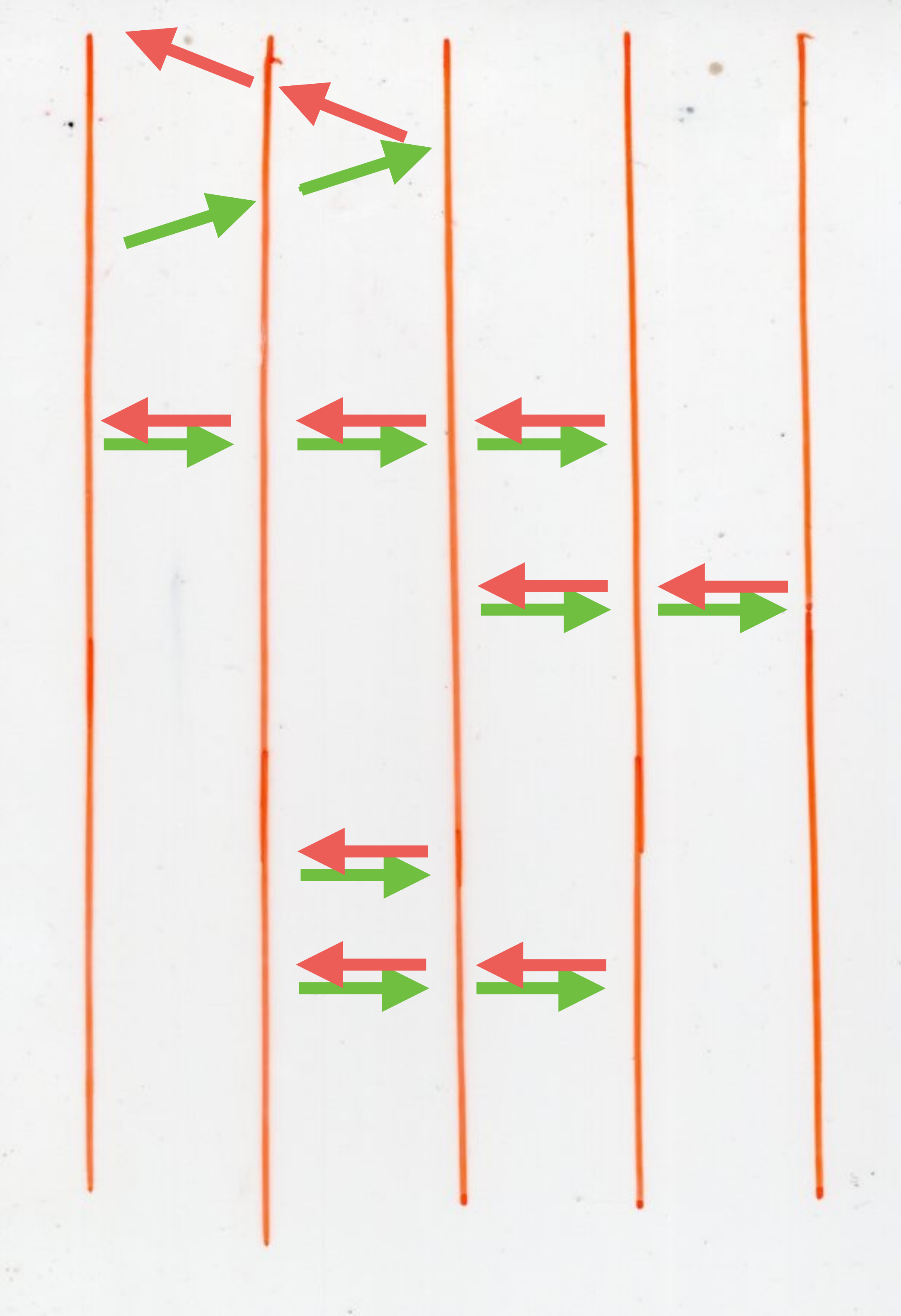
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



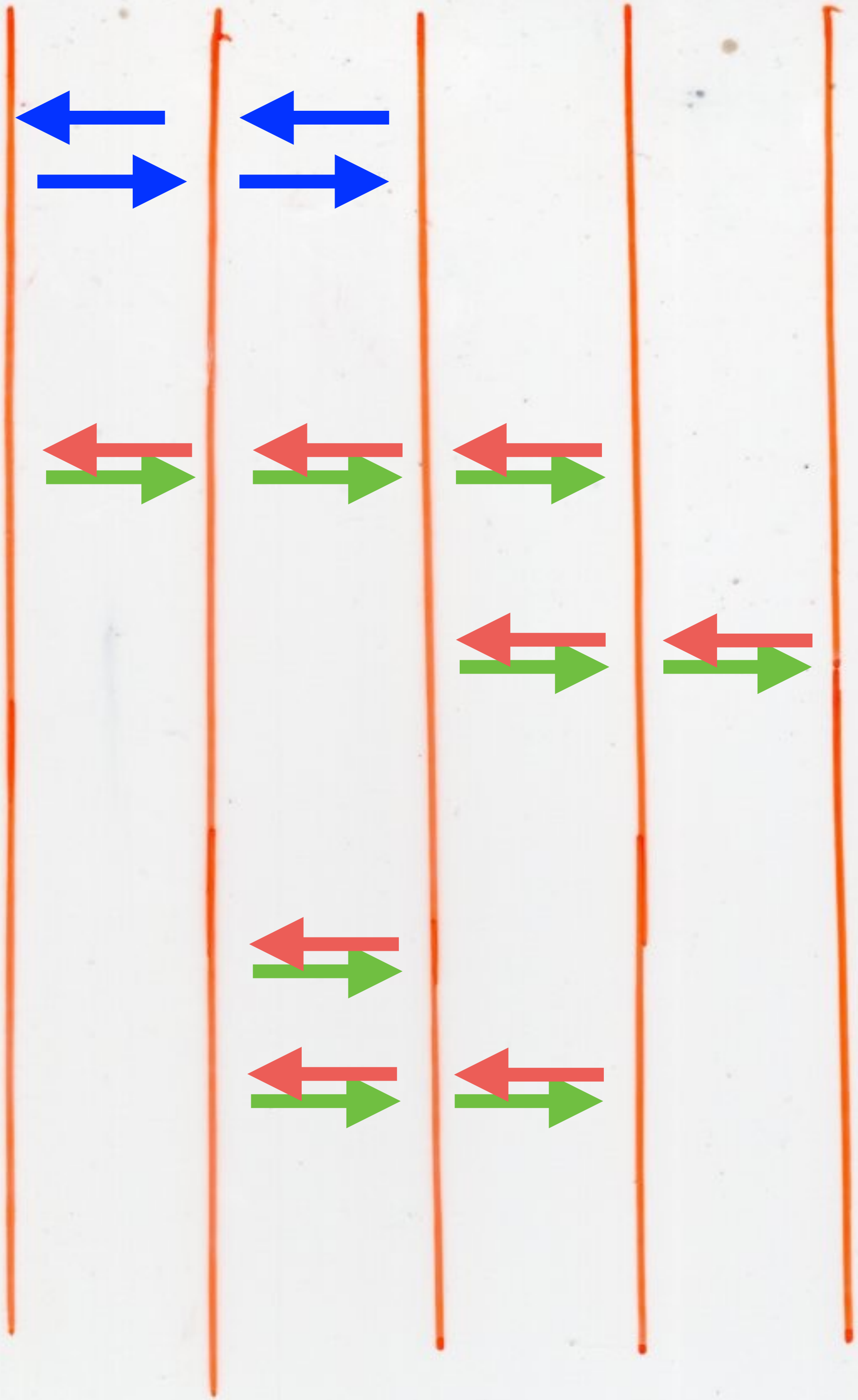
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



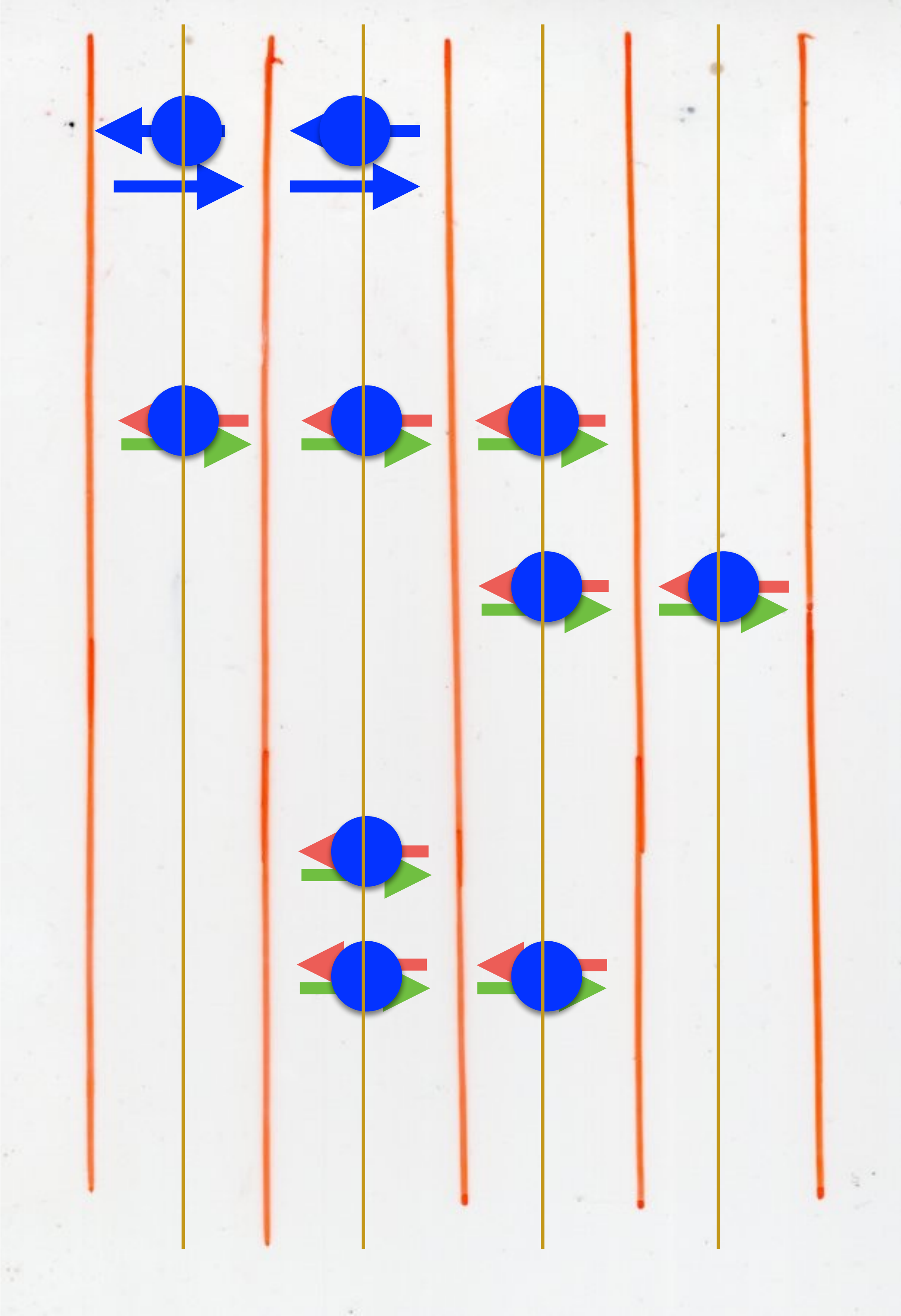
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



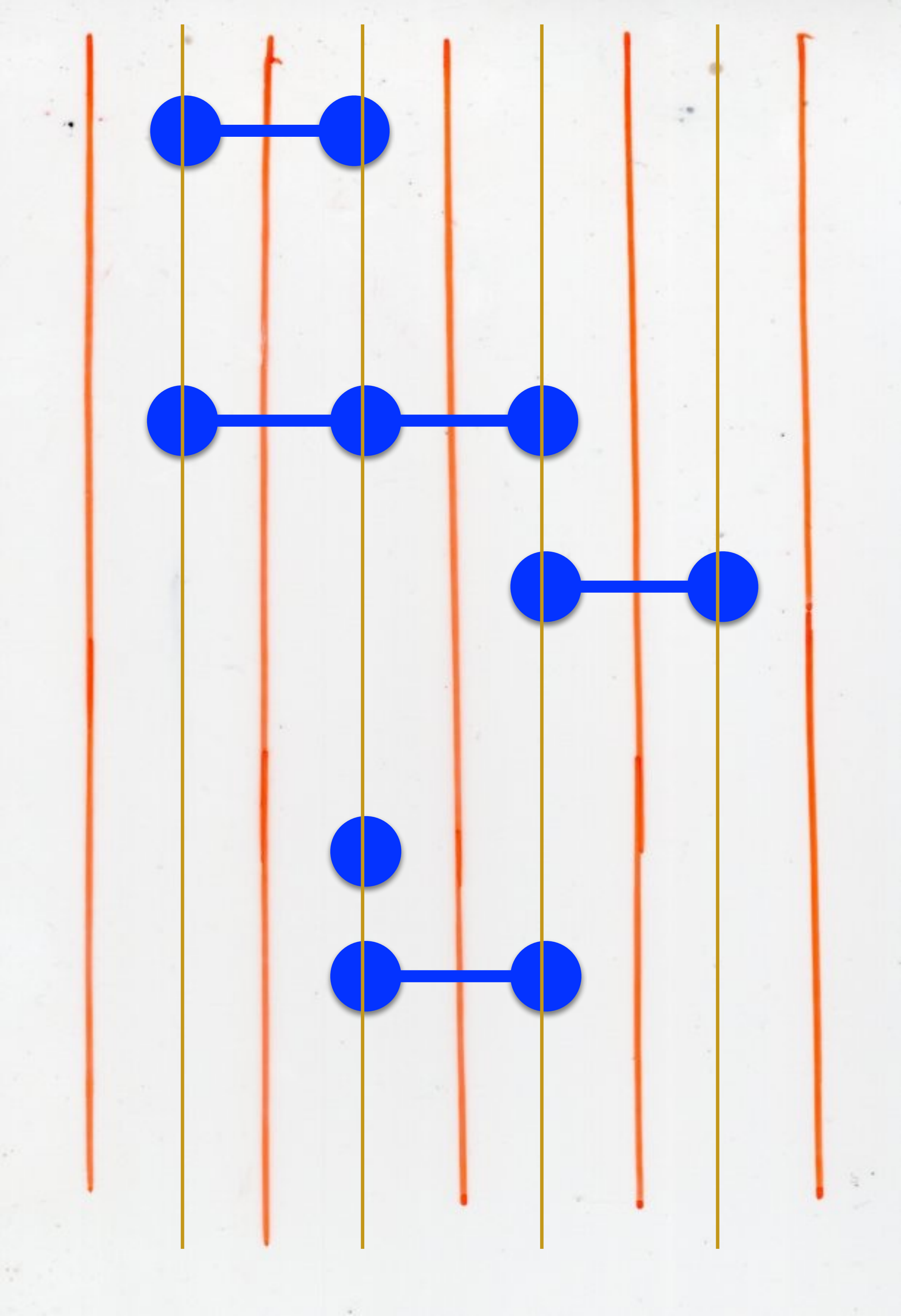
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

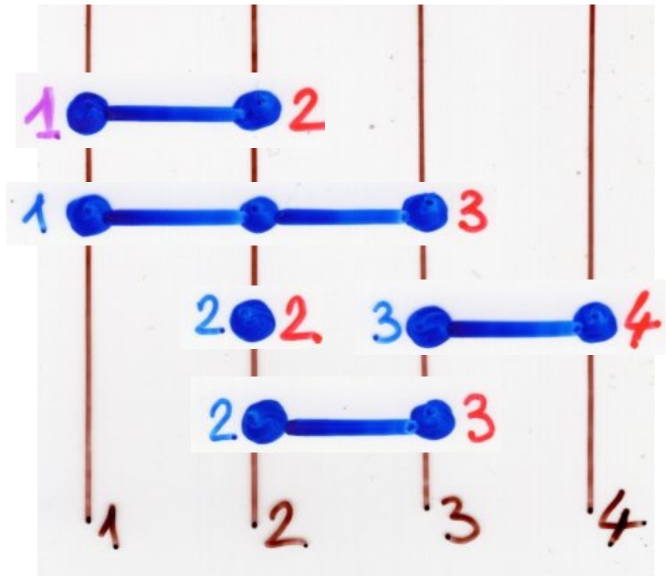


$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

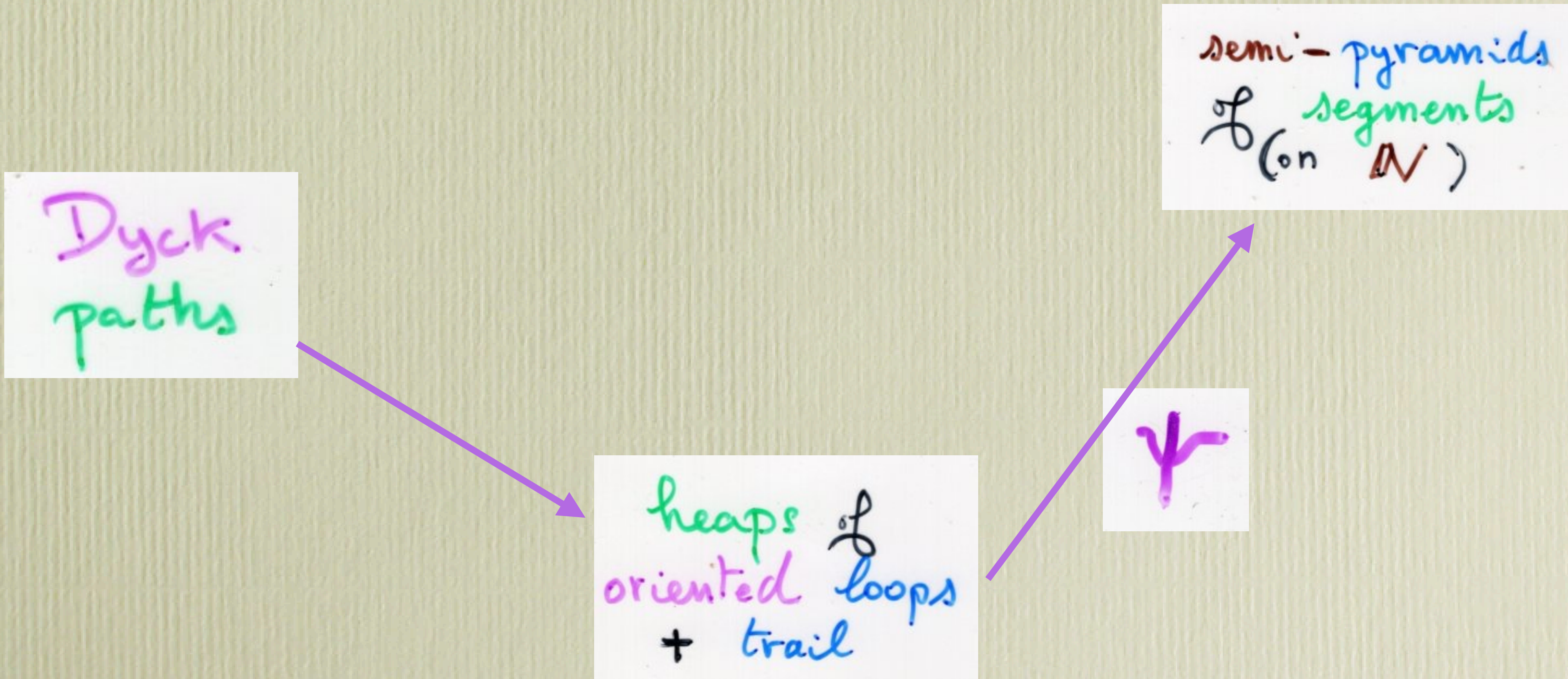


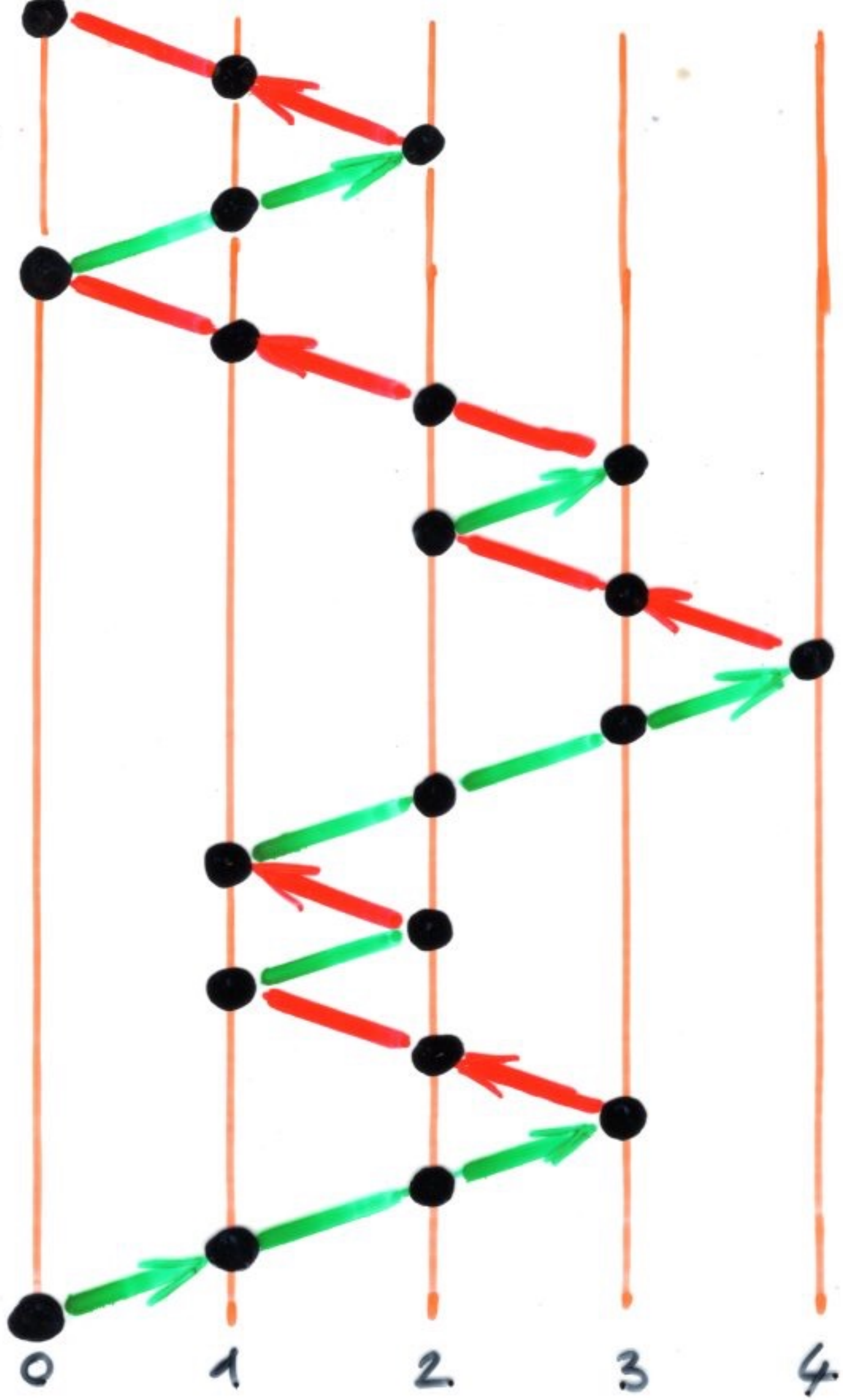
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



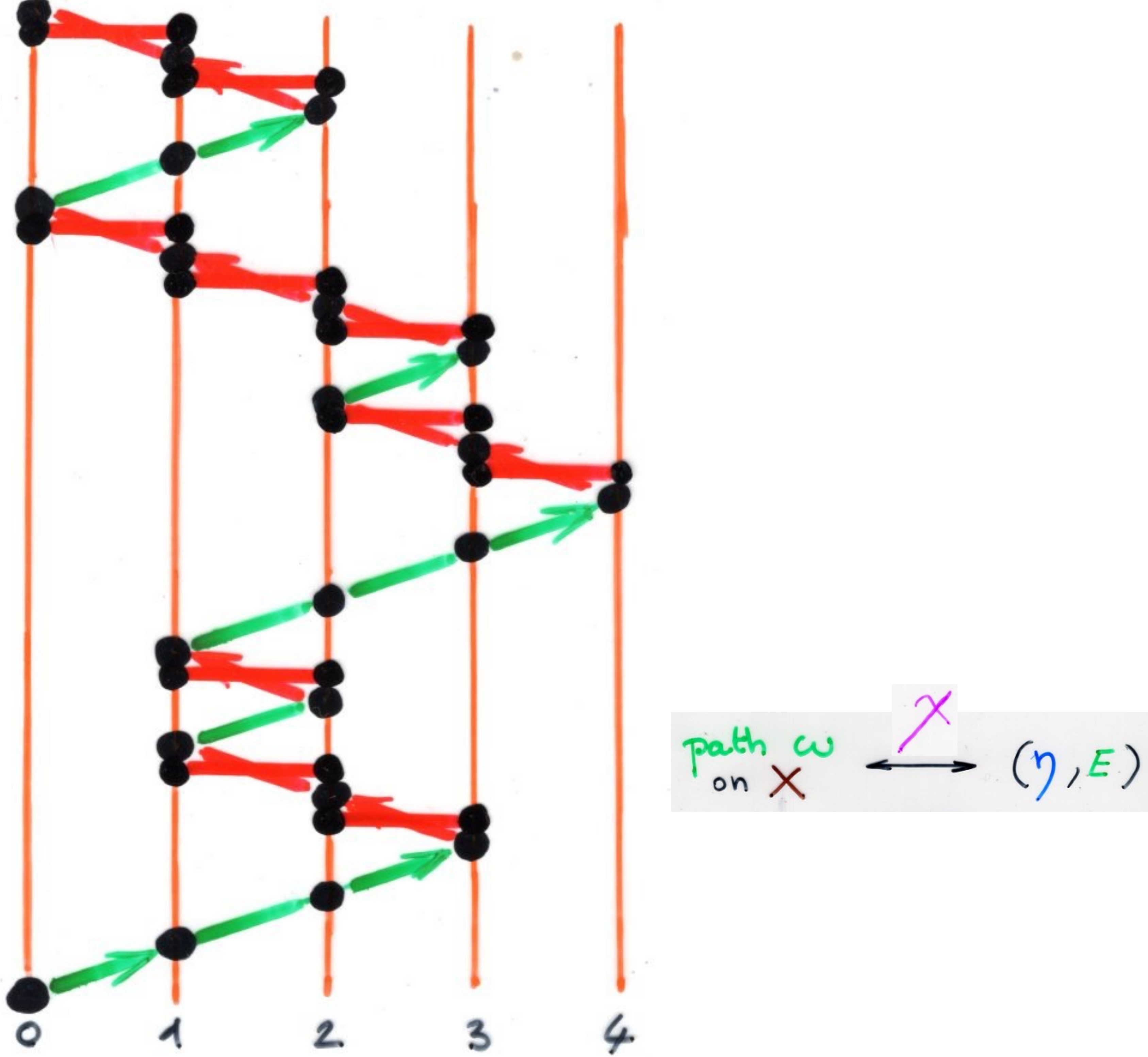

1 2
1 3
2 2 3 4
2 3
1 2 3 4

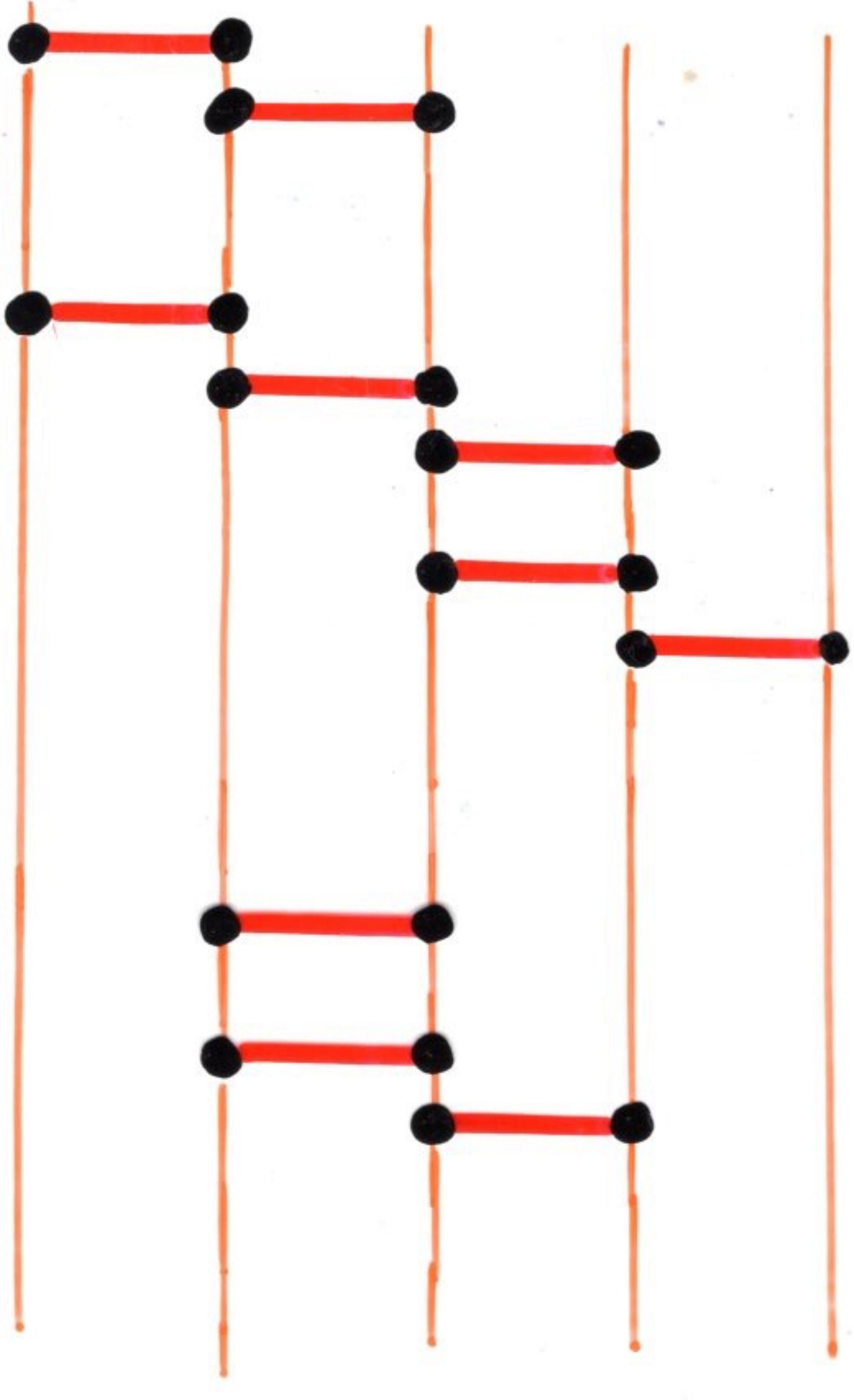
A festival of bijections ....



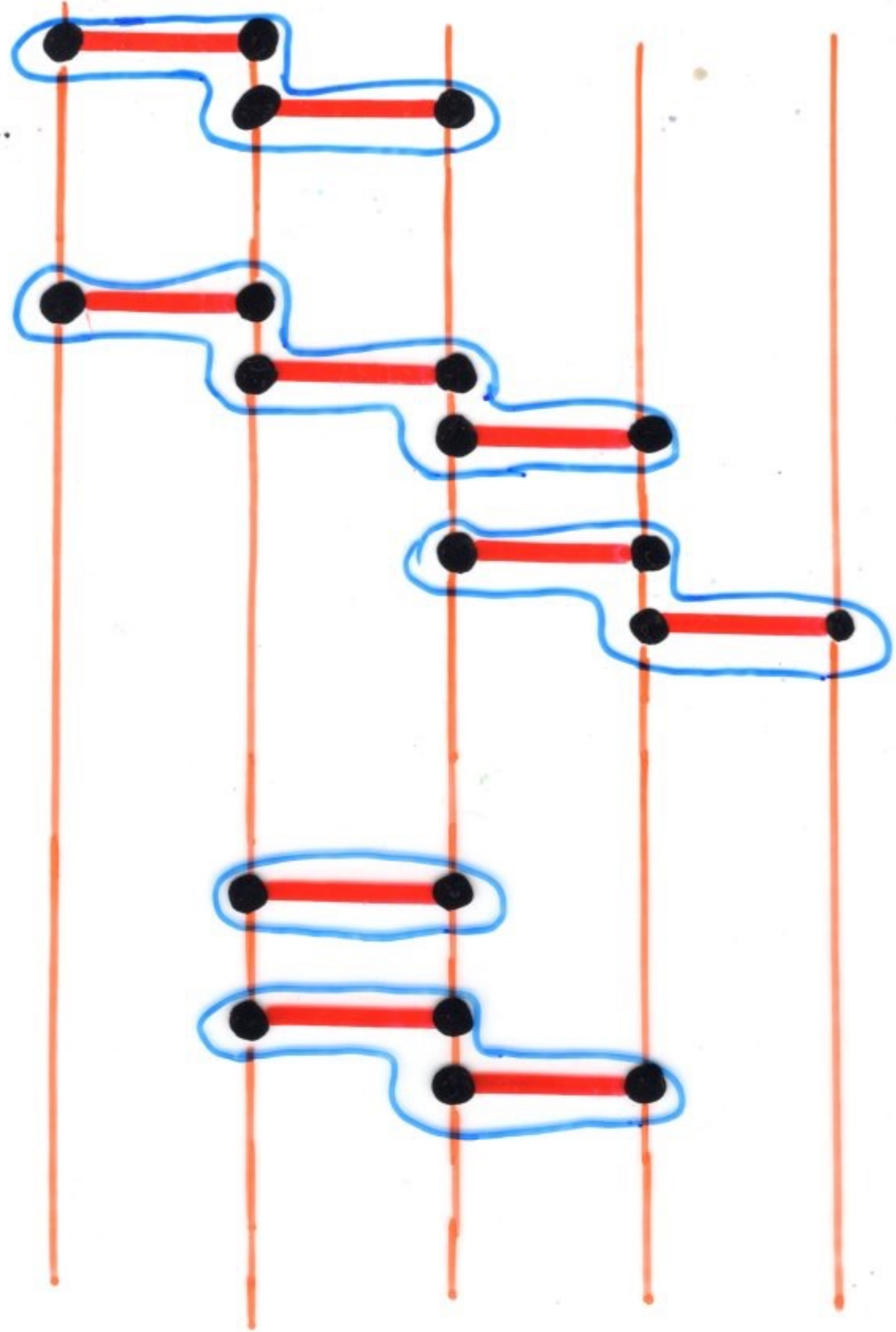


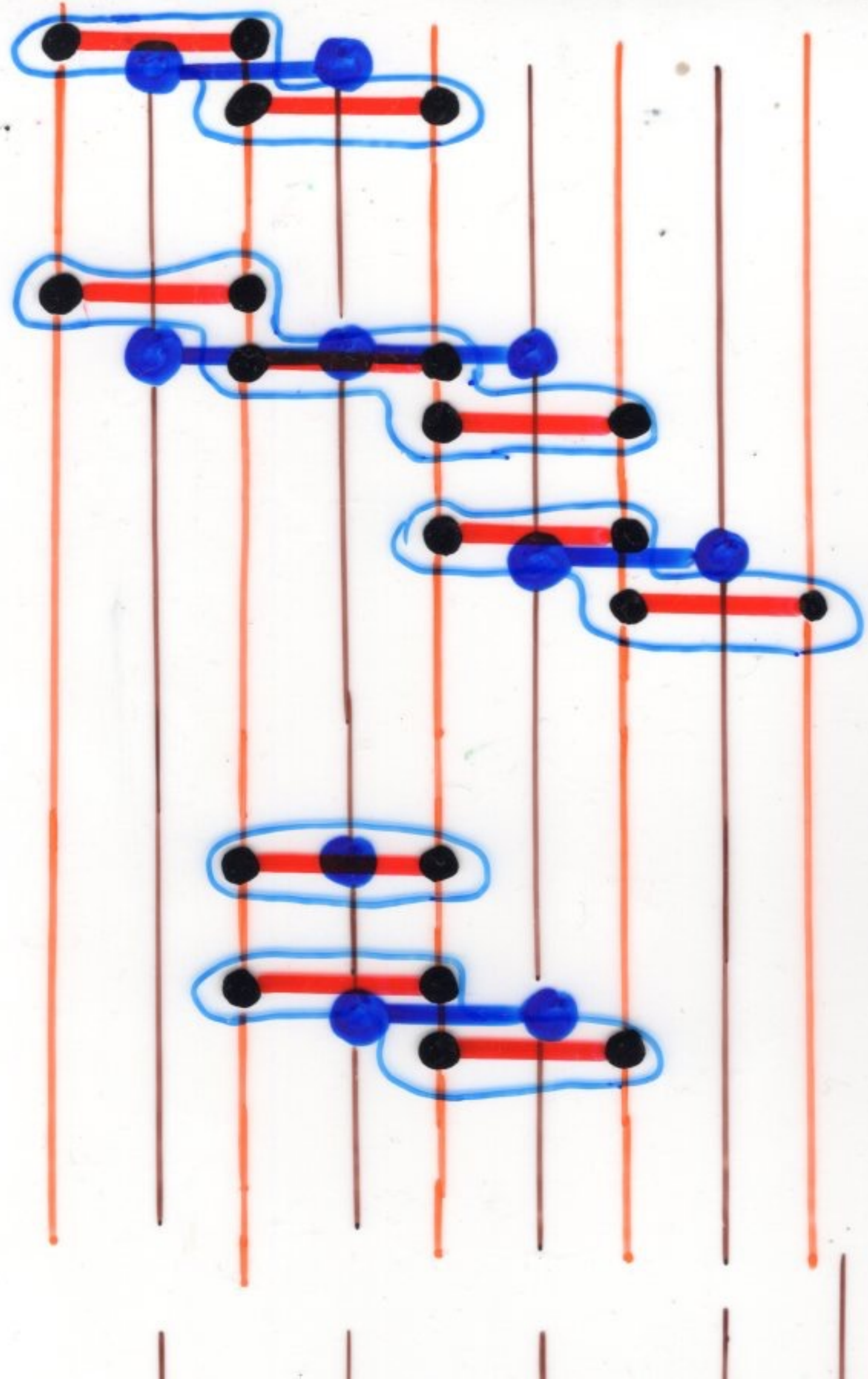
path ω on X $\xleftrightarrow{\chi}$ (η, E)

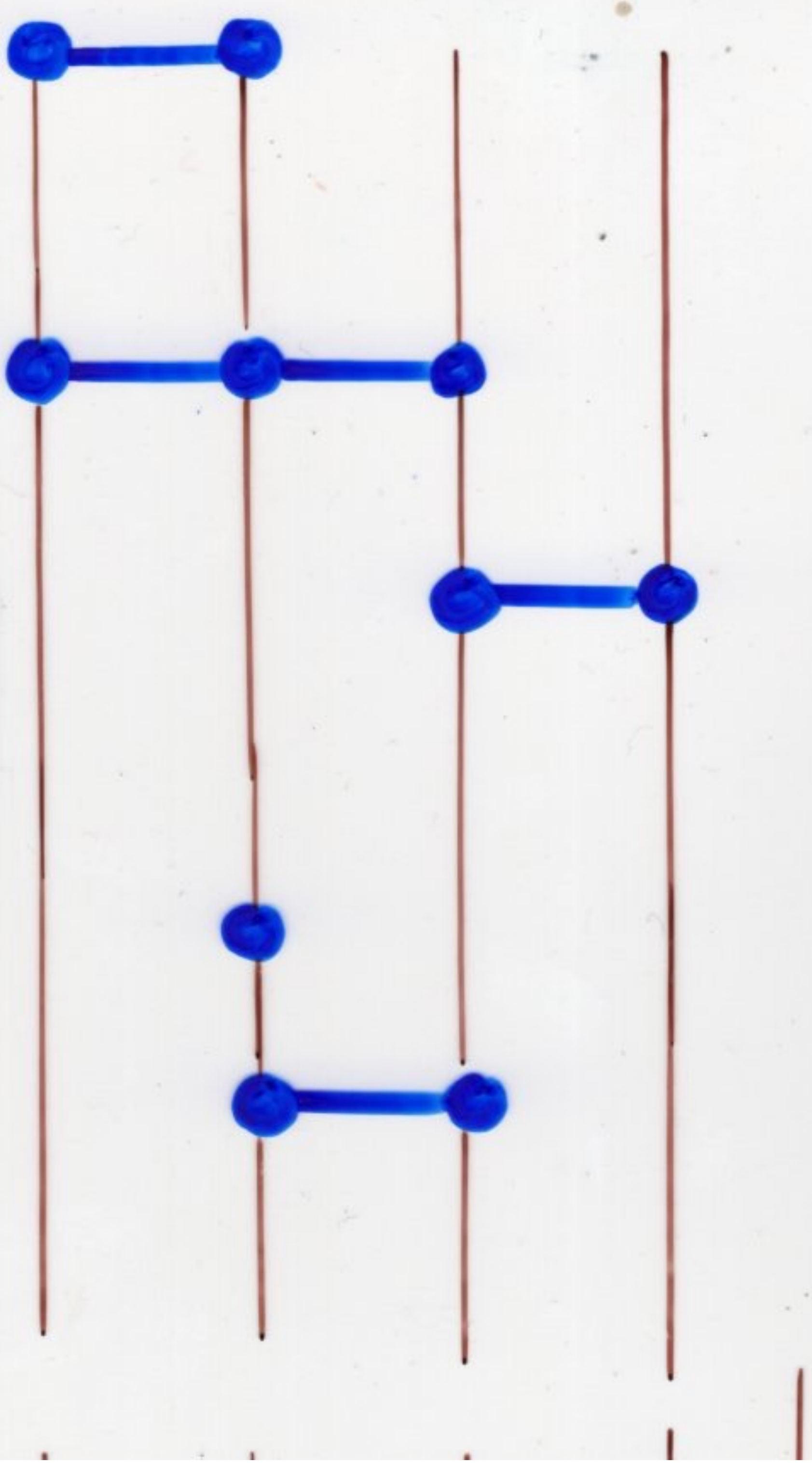

0
1
2
3
4
path ω
on X
χ
(η, E)

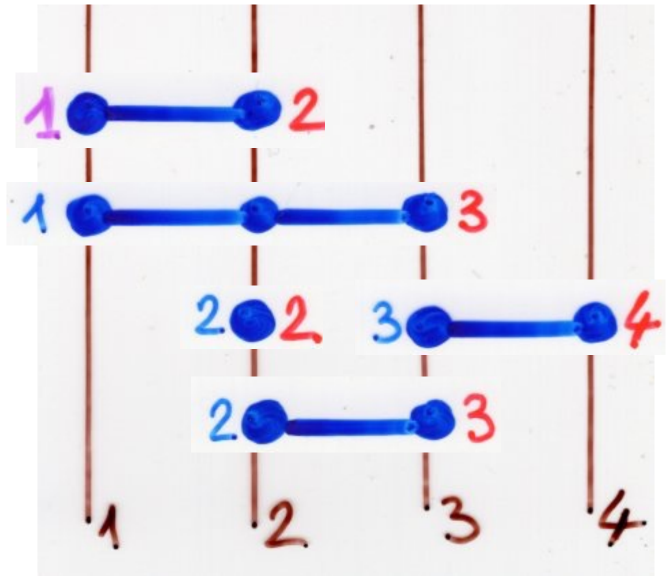


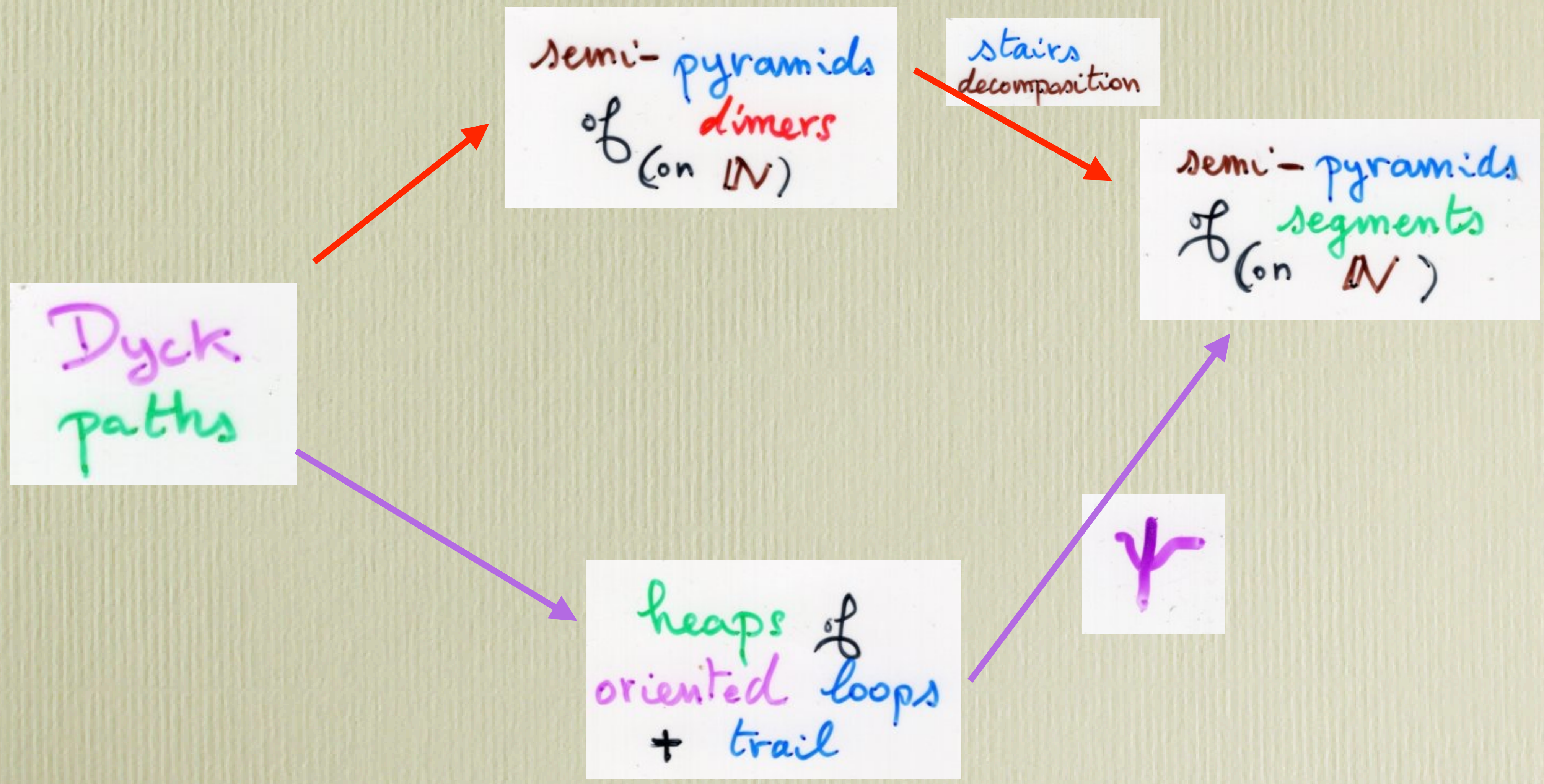
path ω on X $\overset{\chi}{\longleftrightarrow}$ (η, E)



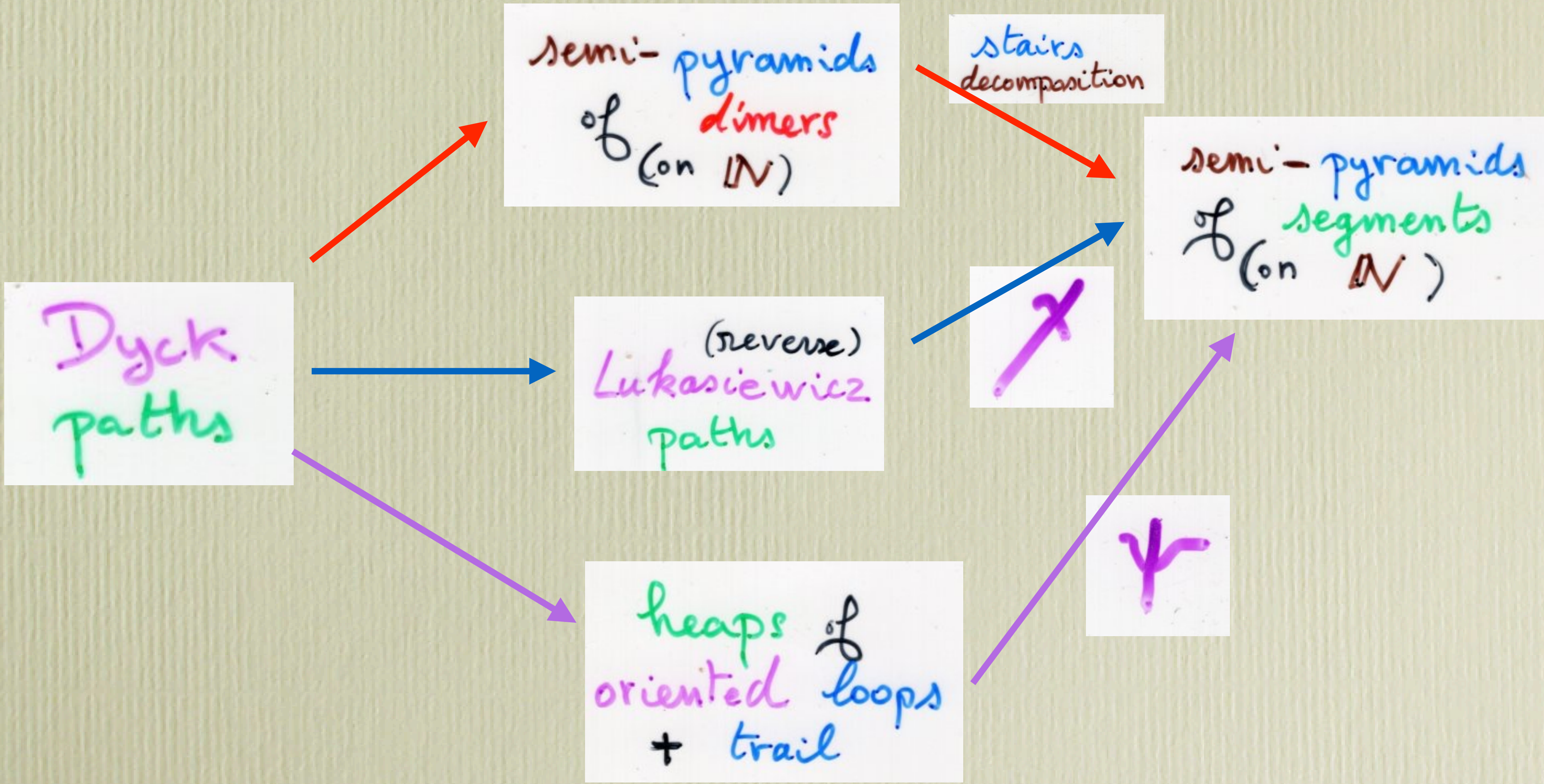




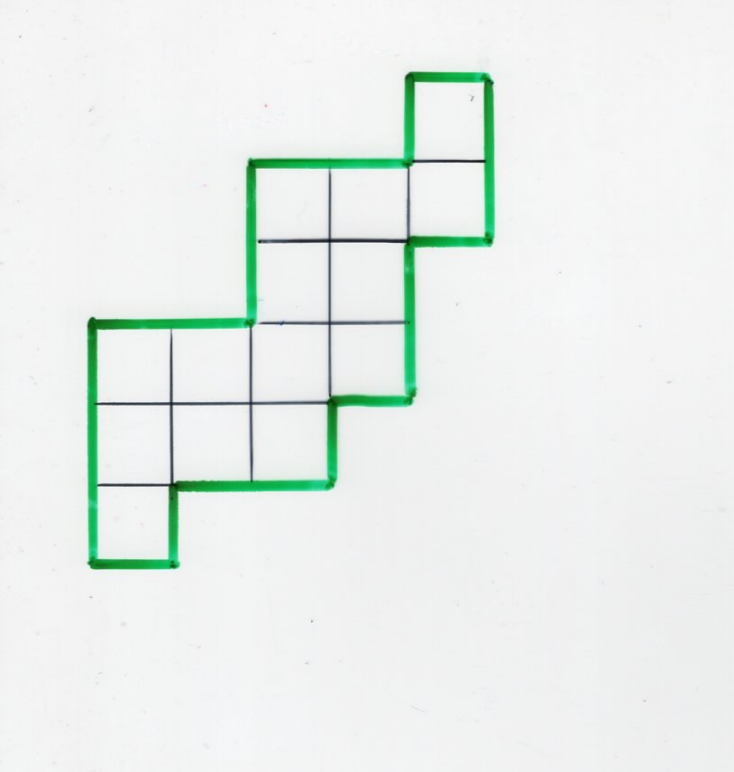

1 2
1 3
2 2
3 4
2 3
1 2 3 4



Dyck paths
semi-pyramids of dimers (on ℕ)
stairs decomposition
semi-pyramids of segments (on ℕ)
heaps of oriented loops + trail
ψ



Dyck paths
semi-pyramids of dimers (on ℕ)
(reverse) Lukasiewicz paths
heaps of oriented loops + trail
stairs decomposition
χ
ψ
semi-pyramids of segments (on ℕ)



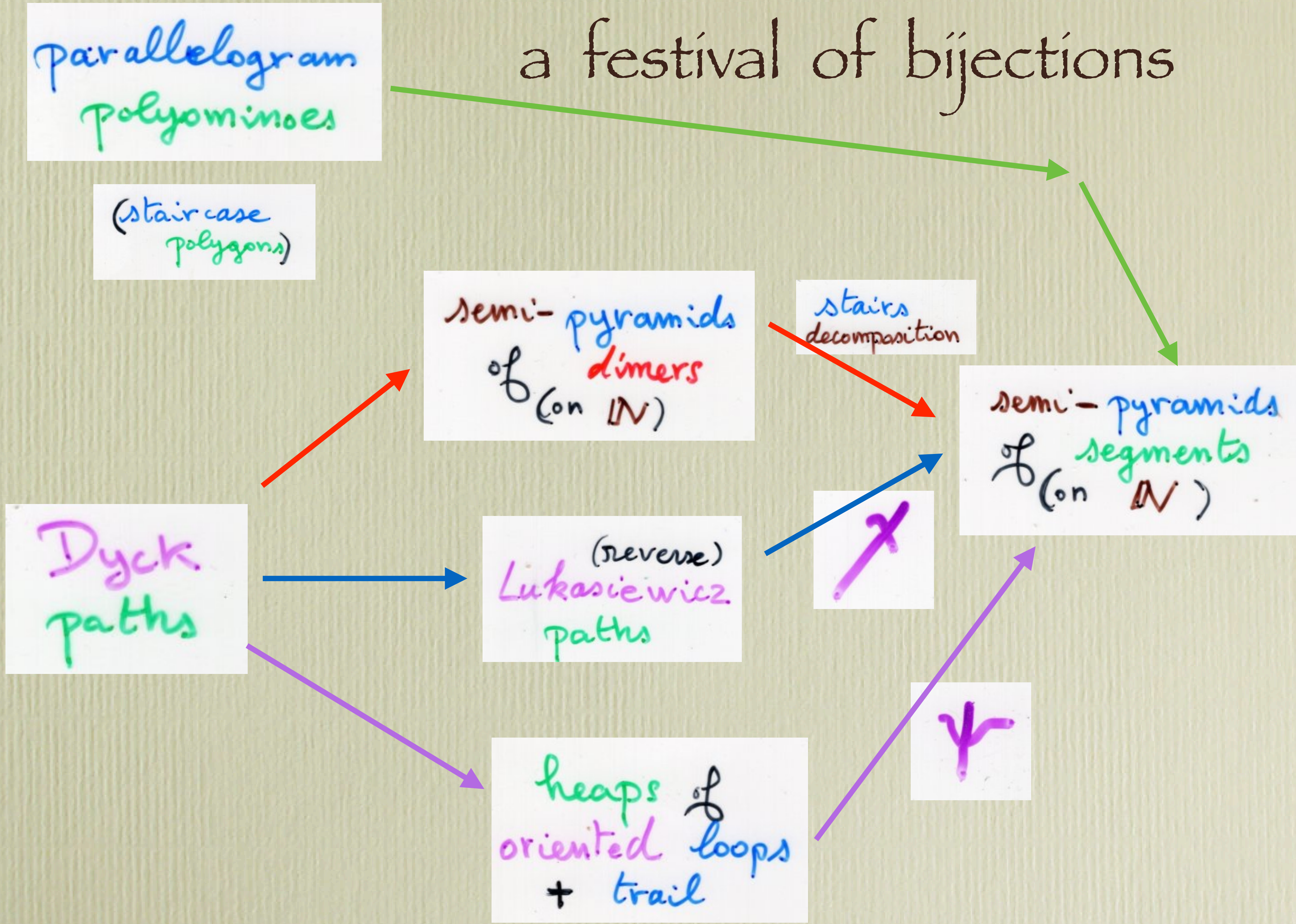
parallelogram polyominoes

staircase polygons

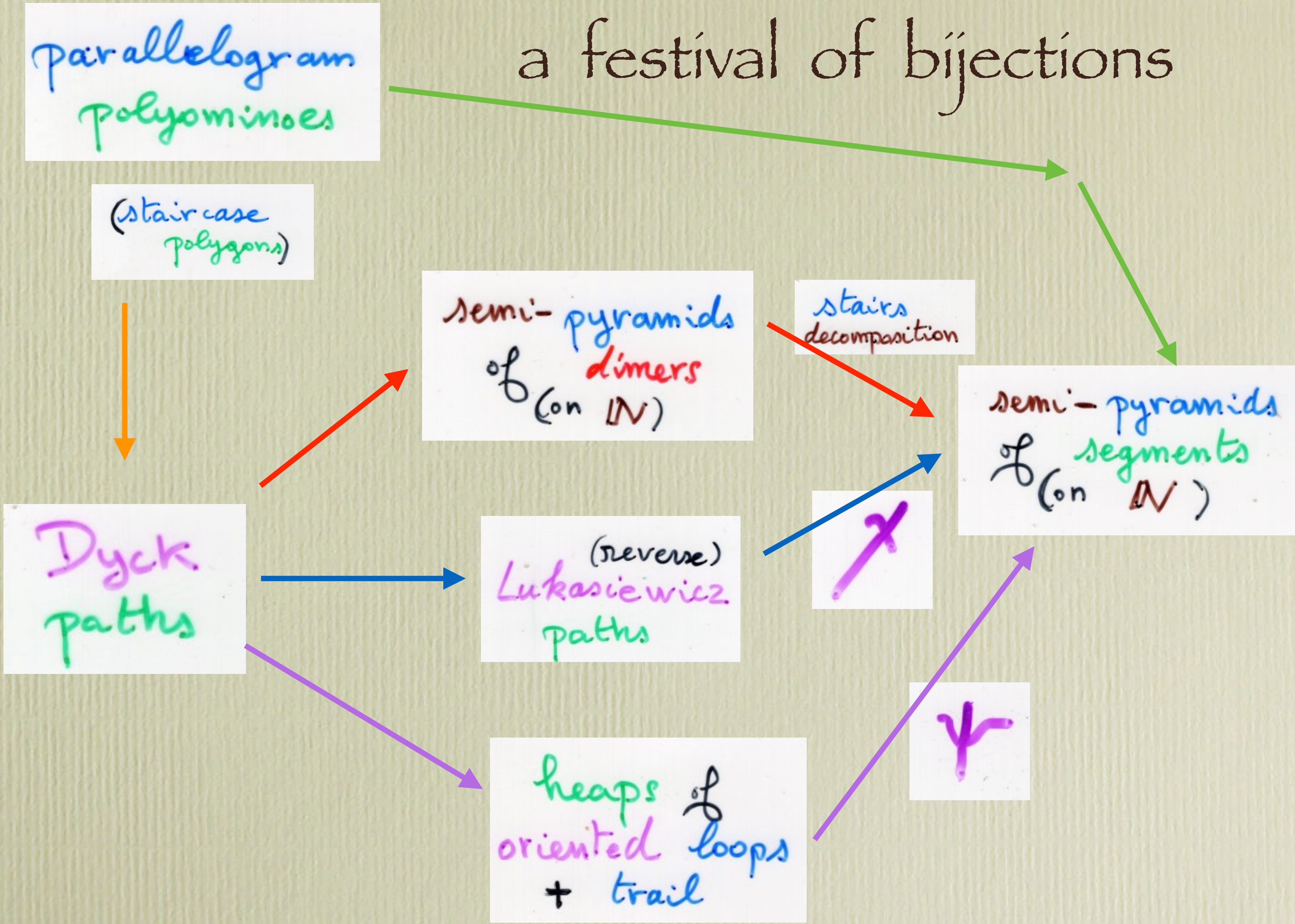
M. Bousquet-Mélou, X.V. (1992)

q-Bessel functions

a festival of bijections



a festival of bijections



« ABjC »

« Video-book » The Art of bijective combinatorics

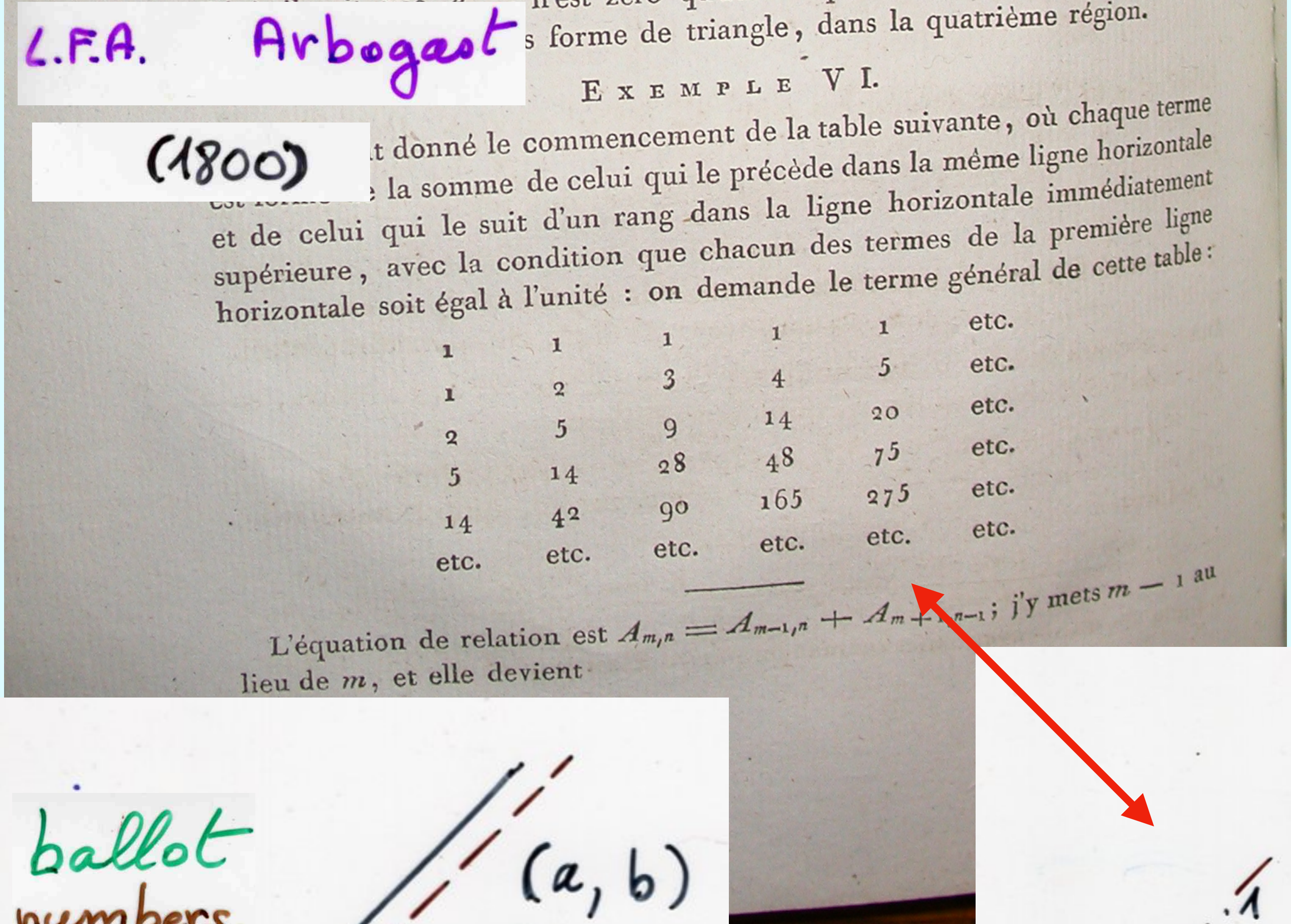
Part II, Comutations and heaps of pieces
with interactions
in physics, mathematics and computer science

IMSc, Chennai, 2007

www.viennot.org/abjc2.html

Thank you !

Merci infiniment !



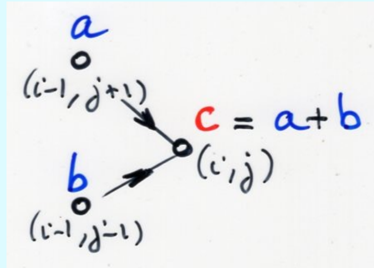
s forme de triangle, dans la quatrième région.

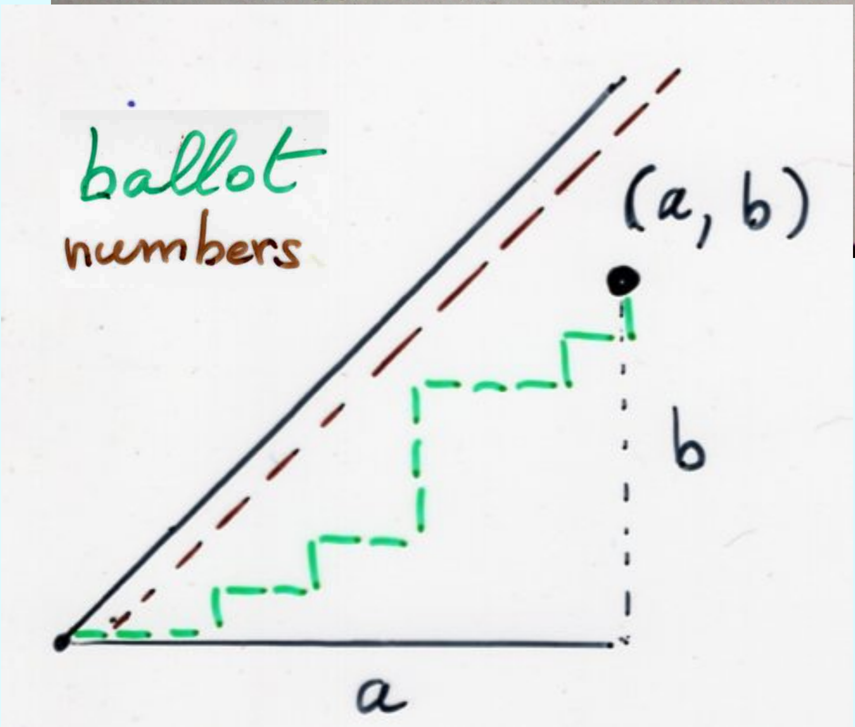
EXEMPLE VI.

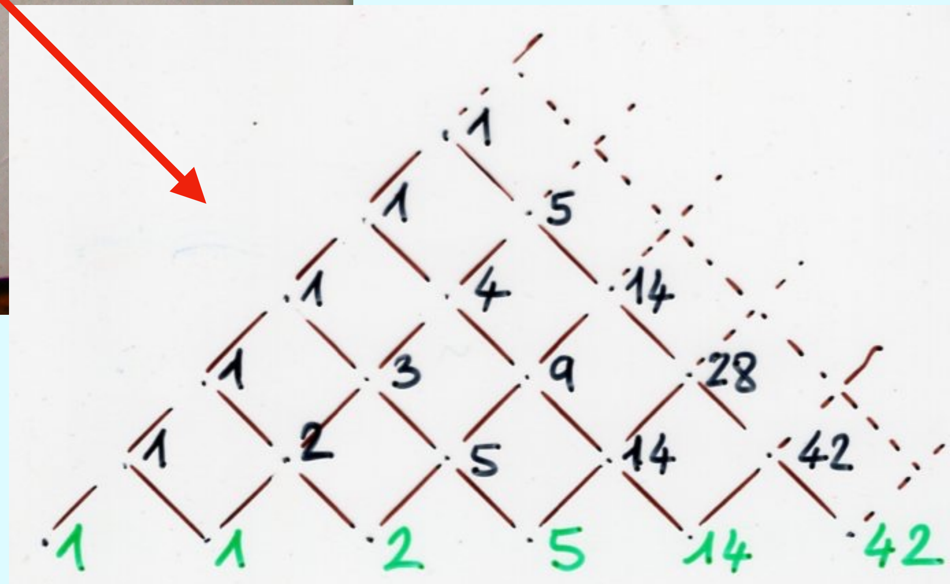
t donné le commencement de la table suivante, où chaque terme la somme de celui qui le précède dans la même ligne horizontale et de celui qui le suit d'un rang dans la ligne horizontale immédiatement supérieure, avec la condition que chacun des termes de la première ligne horizontale soit égal à l'unité : on demande le terme général de cette table:

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m-1$ au lieu de m, et elle devient





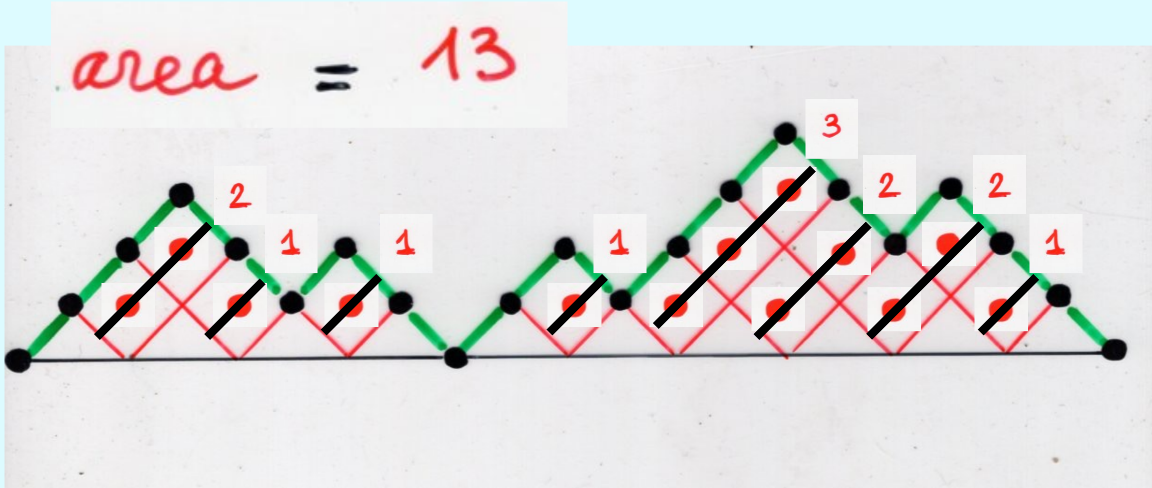


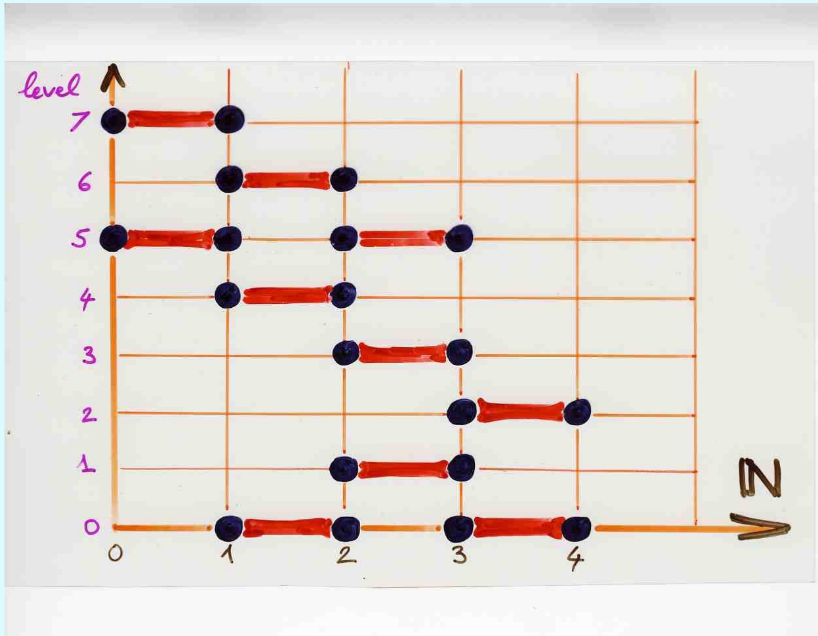
In response to the question of Doron Zeilberger …

« Arbogast triangle » and lattice paths ?

In response to the question of Philippe Di Francesco

Do usual statistics on paths lift to heaps (eg. area under the path ?)





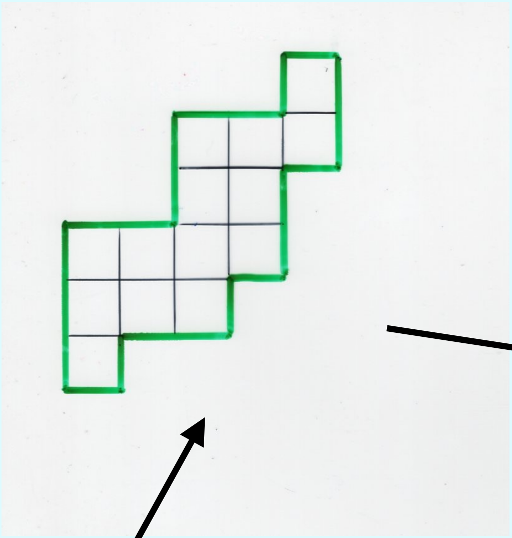
ω Dyck path $\longrightarrow$ P semi-pyramid of dimers on $\mathbb{N}$

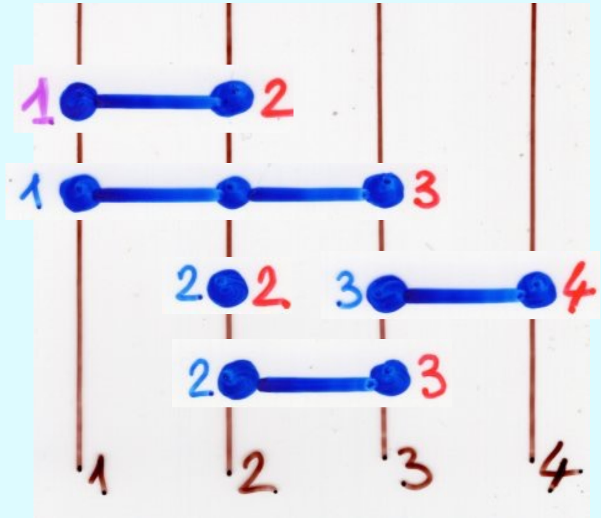
$$v([k-1, k]) = q^{k-1} t$$

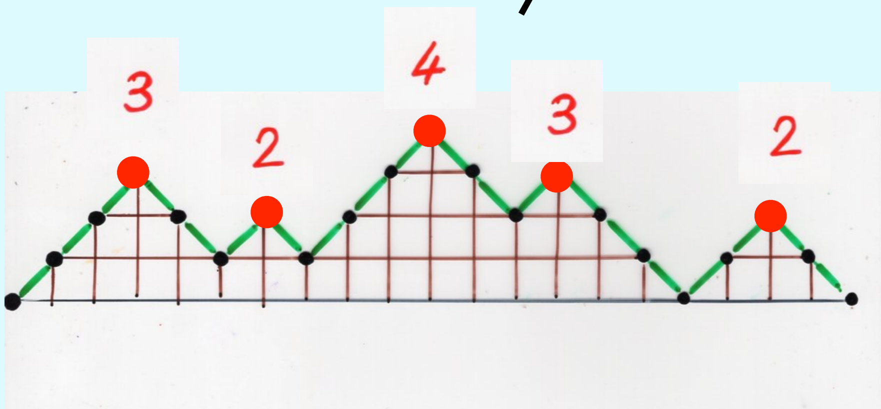
$$v(P) = q^{\mathrm{area}(\omega)} t^{|\omega|/2}$$

In response to the question of Philippe Di Francesco

Do usual statistics on paths lift to heaps (eg. area under the path ?)







another example:

sum of the height of the peaks

of a Dyck path

see Chapter 7c, ABjC, Part II

www.viennot.org/abjc2-ch7.html

In addition to the link given by Cyril Banderier

About the « LGV Lemma »

See the video-book « ABjC »

The Art of Bijective Combinatorics, Part I,

An introduction to enumerative, algebraic and bijective combinatorics

IMSc, Chennai, 2016, Chapter 5a, pp 3-28

www.viennot.org/abjc1-ch5.html

« LGV Lemma »

In addition to the link given by Cyril Banderier

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

4Lindström used the term "pairwise node disjoint paths". The term "non-intersecting," which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

5By a curious coincidence, Lindström's result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the "Lindstrom–Gessel–Viennot theorem." It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called "Slater determinant" in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an "ancestor" of the Lindström–Gessel–Viennot determinant.

6There exist however also several interesting applications of the general form of the Lindström–Gessel–Viennot theorem in the literature, see [10, 16, 51].

In response to the question of Philippe Biane

Reciprocity with Rieman zeta function ?

Ch 5b, zeta function of a graph, pp 7-20

www.viennot.org/abjc2-ch5.html

In my answer I mentioned P.-L. Giscard relating number theory and heaps.

See for example: P.-L. Giscard and P. Rochet Algebraic Combinatorics on Trace Monoids: Extending Number Theory to Walks on Graphs, *SIAM J. Discrete Math.*, 2017, 31(2), 1428–1453.

Remarks on some question of vocabulary

Trace monoids were introduced by Mazurkiewicz in computer science as model for concurrency. These monoids are the same as the Cartier-Foata monoids. Unaware of the heaps interpretation of commutation monoids, the authors introduced the term « hikes » for an equivalence class of cycles, which is equivalent to « heaps of cycles », themselves in bijection with the so-called « rearrangements » in Cartier-Foata monography.

In response to the question of Joones Turunen

In the question of P. Di Francesco, I was talking about Lorentzian quantum gravity in 2 dimension, where the theory of heaps can plays a very useful role. There is a bijection between semi-pyramids of dimers (enumerated by Catalan numbers) and certain Lorentzian triangulations. (P. Di Francesco, E. Guitter, C. Kristjansen, X.V., following some work of J. Ambjorn and R. Loll).

See the video-book « ABjC », part II, Chapter 7c, Lorentzian triangulations in 2D quantum gravity, the curvature parameter of the 2D space-time, connected heaps of dimers. www.viennot.org/abjc2-ch7.html (pp 47-84).

Taking in account the parameter « curvature » of the space-time, curiously, appears the stair decomposition of heaps of dimers introduced in the talk (pp 85-104)

General Lorentzian triangulations are in bijection with connected heaps of dimers (or multidirected animals). The generating function is not D-finite, formula given by M. Bousquet-Mélou and A. Rechnitzer. A bijective proof is given by X.V. with the introdcution of the « Nordic decomposition » of heap of dimers. (pp 105-127)

In response to the second question of Bishal Deb

There are extensions of the LGV Lemma to paths in a graph making cycles.

See the works of P. Lalonde, S. Fomin, Talaska, Carrozza, Krajewski, Tanasa, .…

The interpretation of P.Lalonde with heaps of cycles is the most elegant. A bijective proof can be given, simplifying Lalonde's proof, using the interpretation of S. Fomin which uses LERW.

Talaska proof is a consequence of Lalonde interpretation and the second basic Lemma of heaps theory N/D.

Carrozza, Krajewski, Tanasa proof make use of Grassmann algebra and integral related fo physics.

See the video-book « ABjC » *The Art of Bijective Combinatorics*, Part II, *Commutations and heaps of pieces, IMSc, Chennai, 2017,*

Chapter 4c, Jacobi dual identity, extension of LGV Lemma with heaps, relation with Fomin's theorem on LERW. www.viennot.org/abjc2-ch4.html

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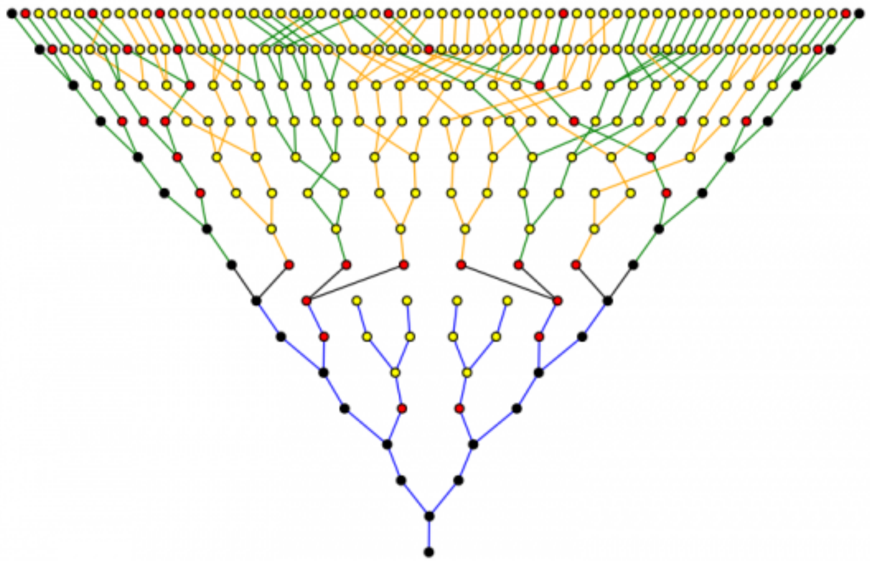
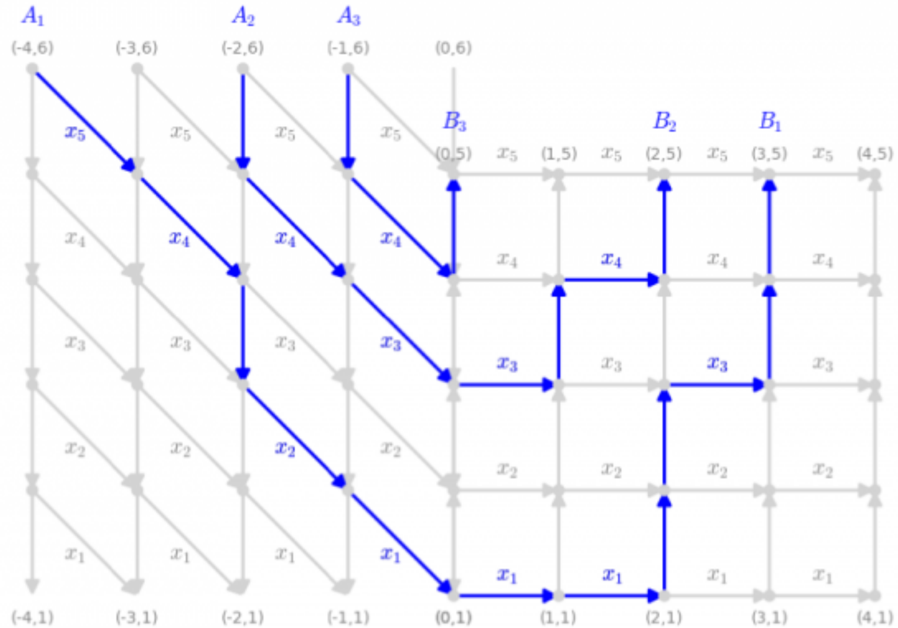
where the video-book « ABjC » was created (2016-2019)





Macdonald tree in Young'graph

(A. Ayyer, A. Prasad, S. Spallone, 2016)

Bijective proof of Giambelli identity with lattice paths and LGV Lemma

(J. Stembridge, 1990)

Thank you !