

Une bijection insolite pour les arbres binaires

(1)

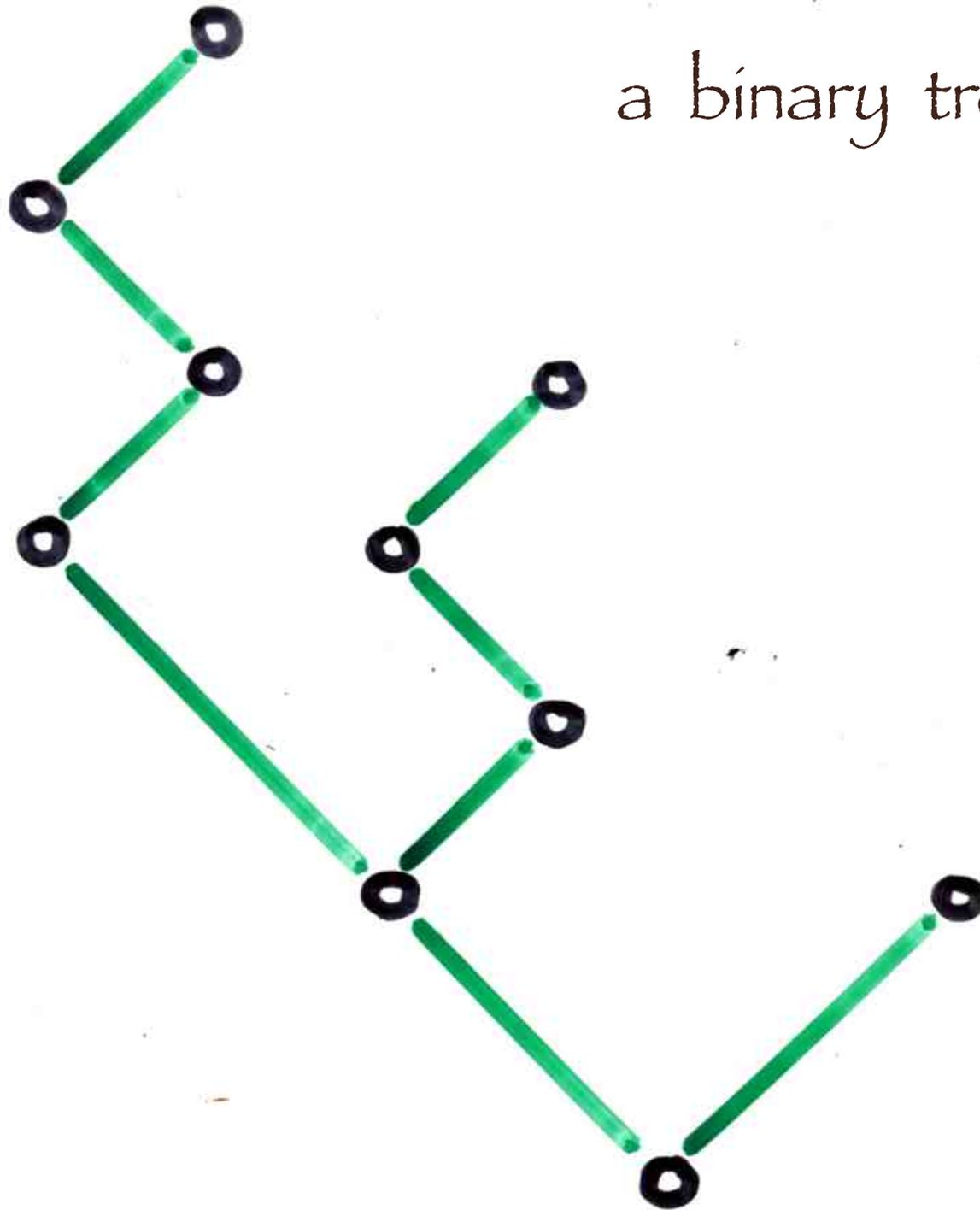
GT LaBRI, Bordeaux
9 Décembre 2019

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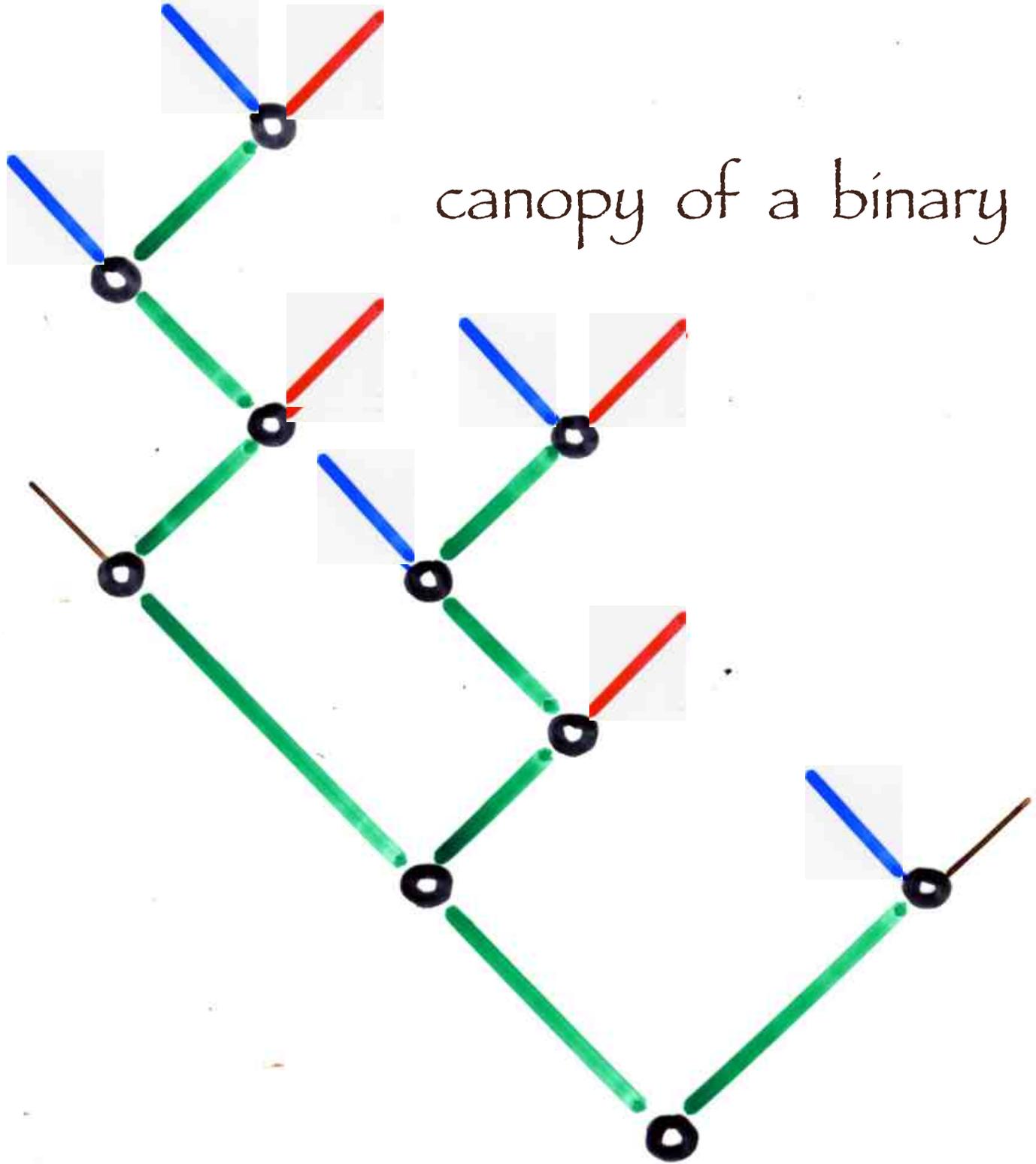
A strange bijection for binary trees

(Part I)

a binary tree



canopy of a binary tree





w
canopy of a binary tree

permutations

increasing binary trees

canopy

up-down sequence

See BJC I, Ch 4a, 74-94

algebraic structures
Hopf algebra

descent
algebra

Loday-Ronco
algebra

Reutenauer
Malvenuto
algebra

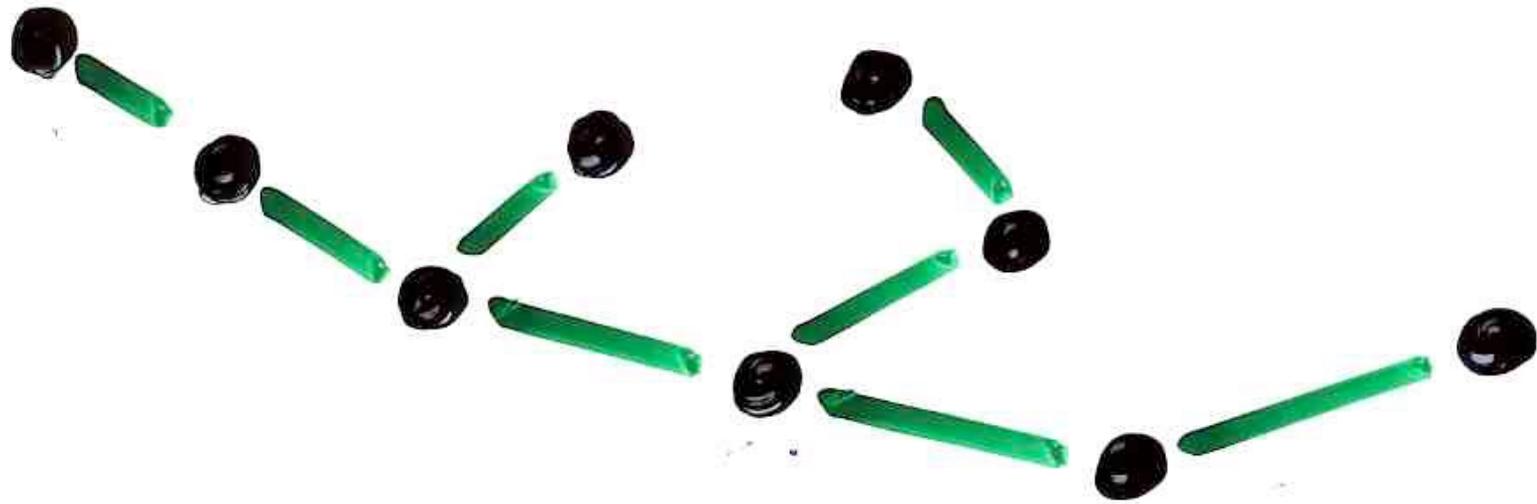
dim

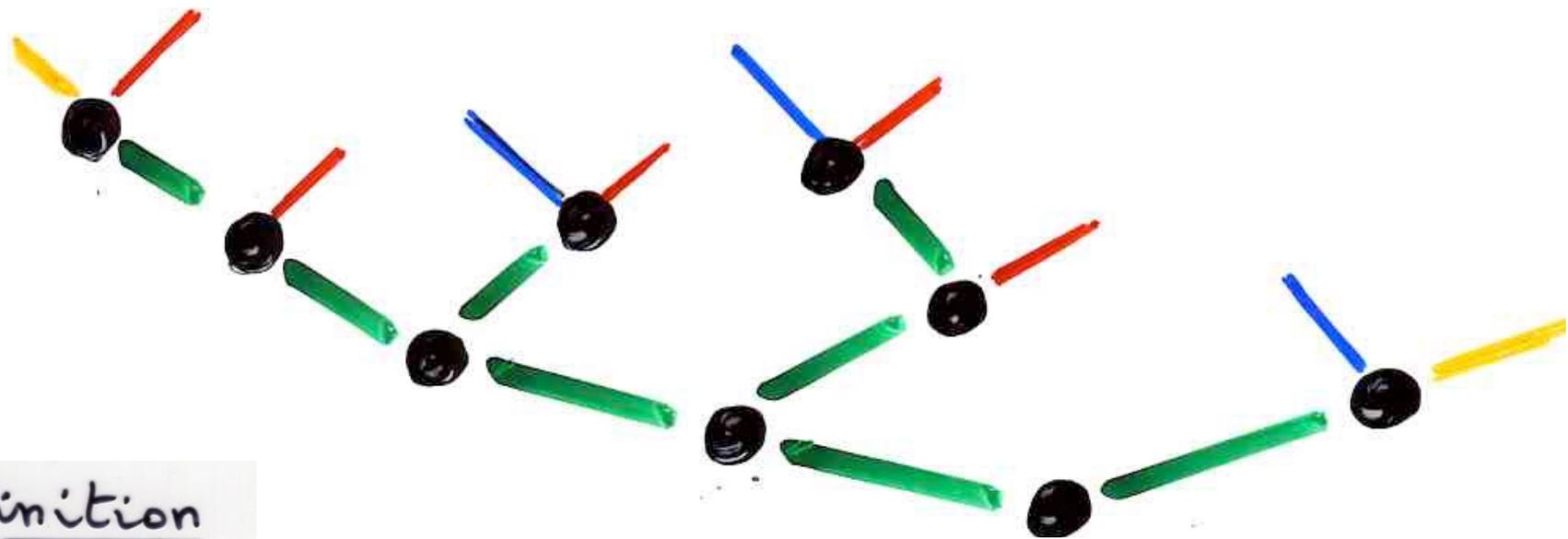
$$2^{n-1}$$

$$C_n$$

$$n!$$

Catalan





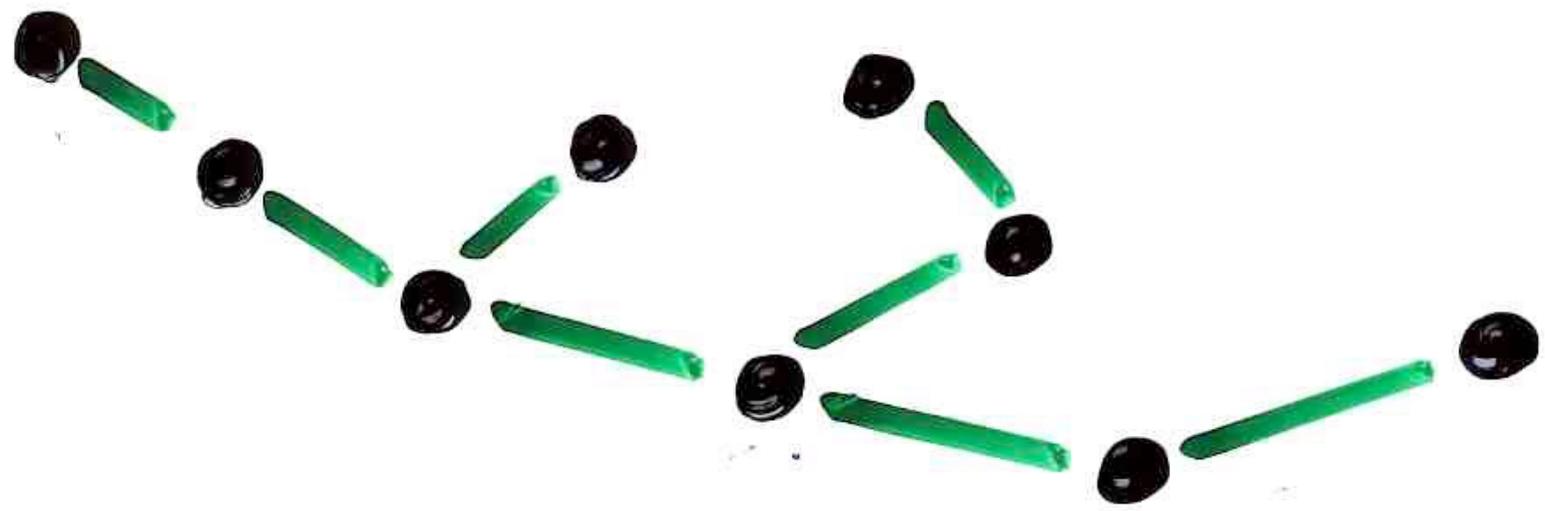
Definition

canopy of a binary tree

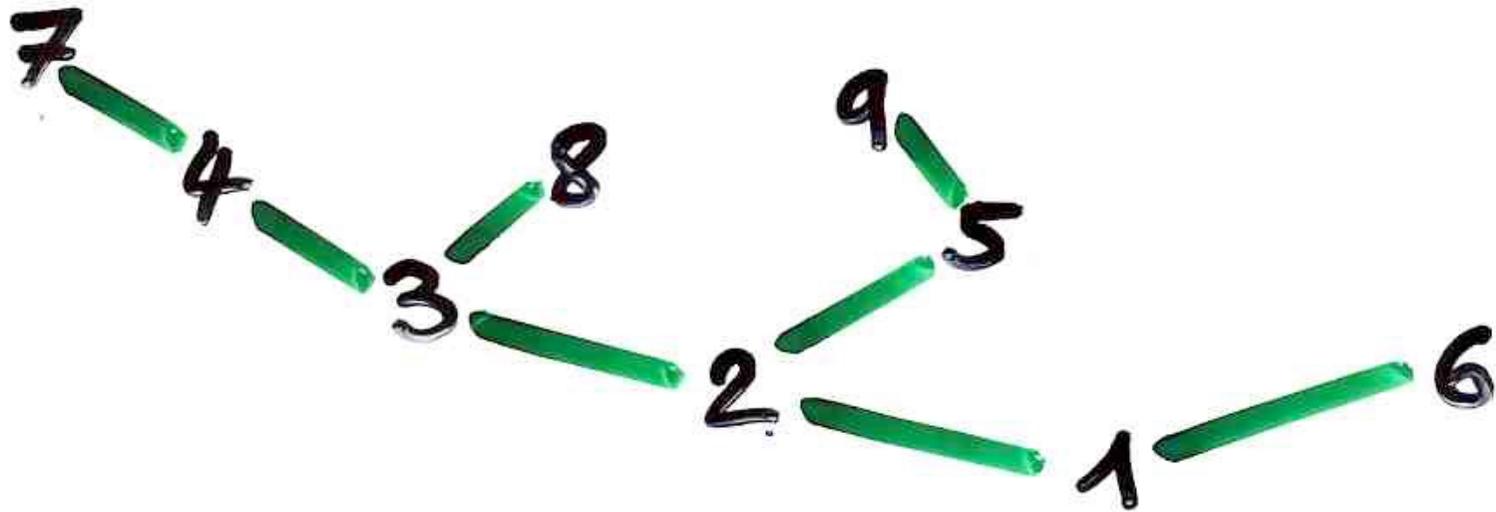
$$C(B) = \begin{array}{ccccccc} / & / & \backslash & / & \backslash & / & / & \backslash \end{array}$$

Loday, Ronco (1998)
(2012)

Increasing binary tree



Increasing binary tree



Bijection

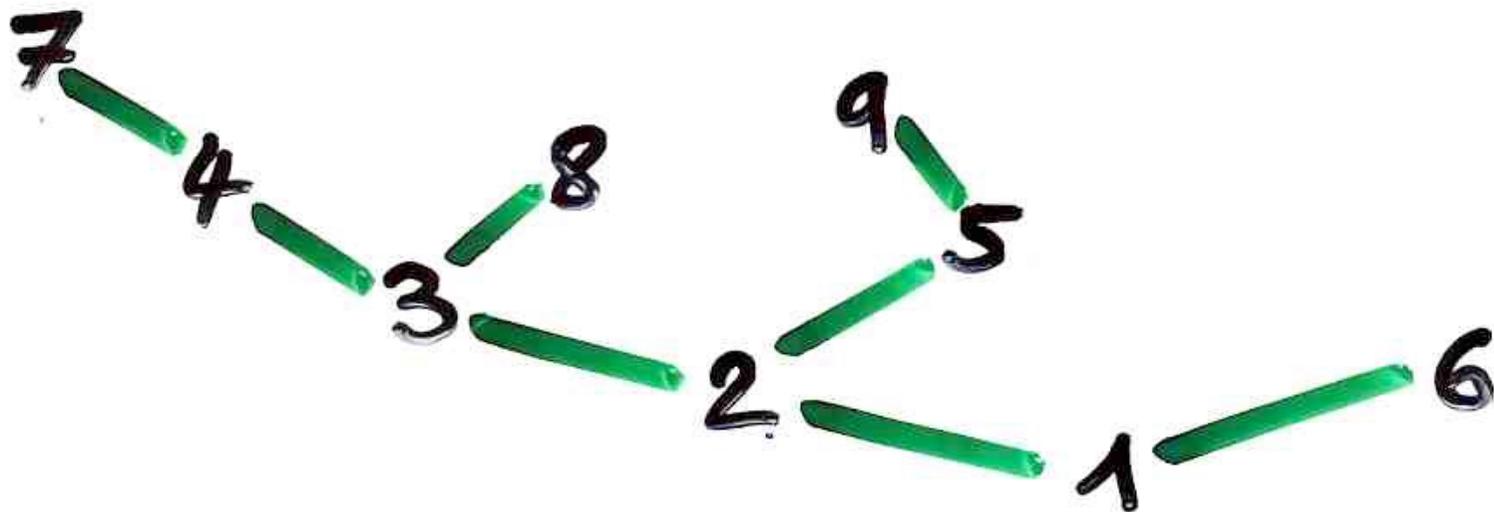
increasing
binary
tree



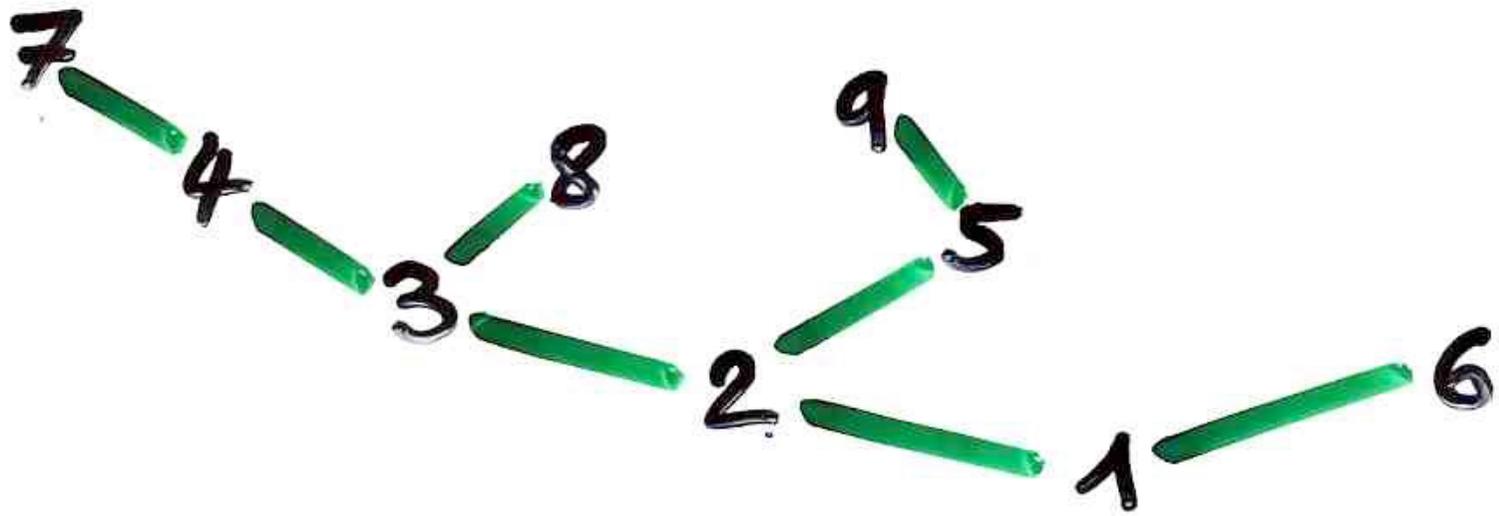
permutation

T

σ



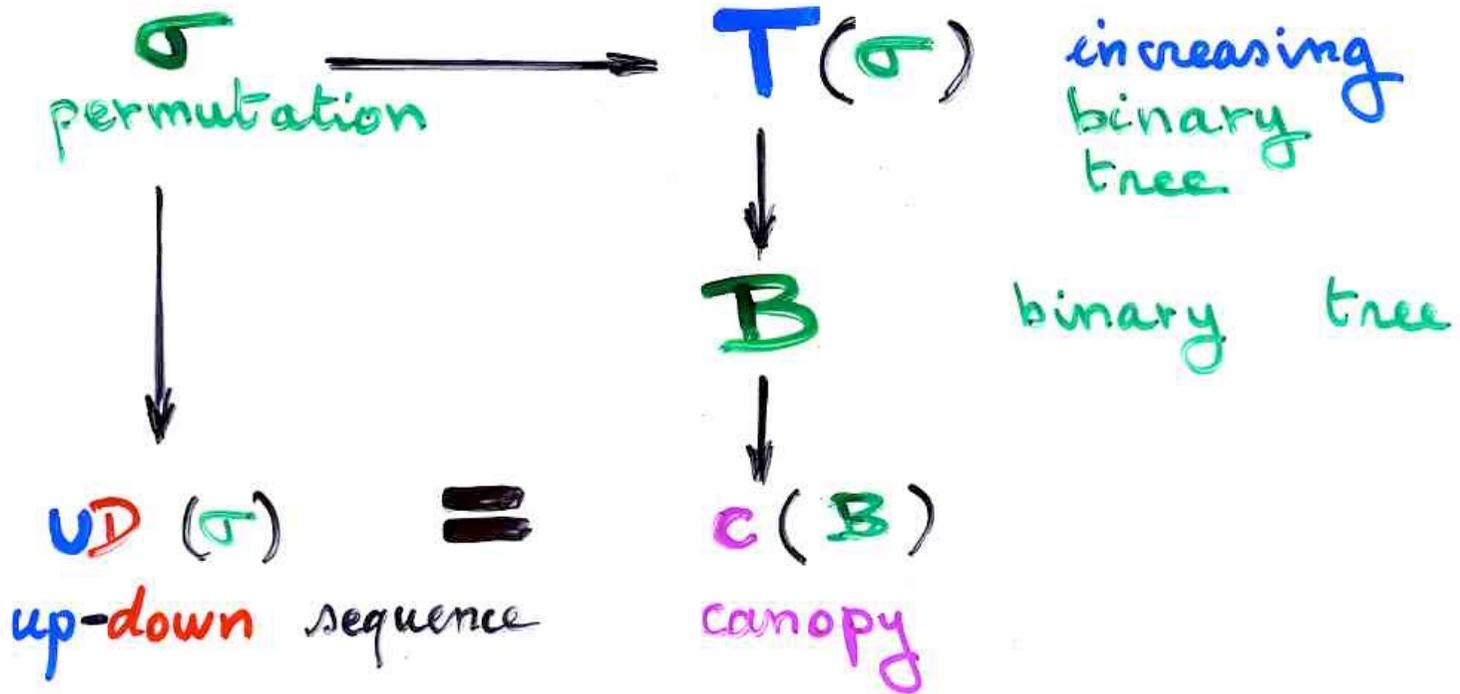
$$\sigma = 7 \ 4 \ 3 \ 8 \ 2 \ 9 \ 5 \ 1 \ 6$$

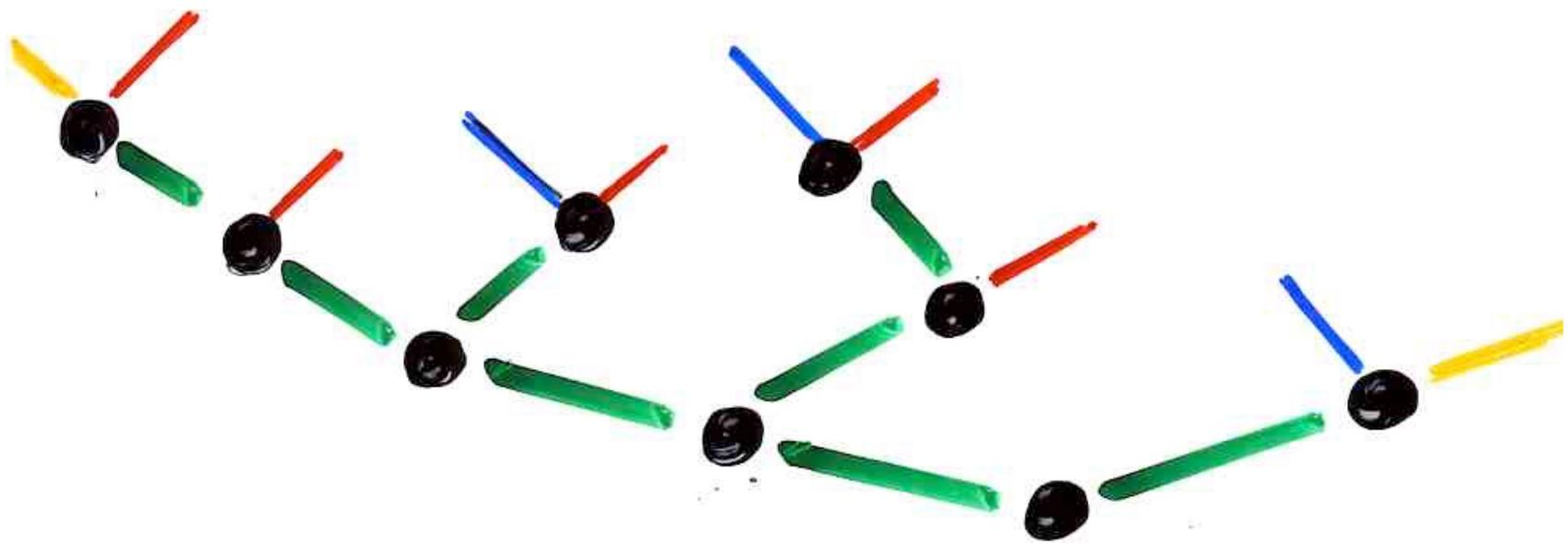


$$\sigma = 7 \setminus 4 \setminus 3 / 8 \setminus 2 / 9 \setminus 5 \setminus 1 / 6 \dots$$

up-down
sequence

- - + - + - - +

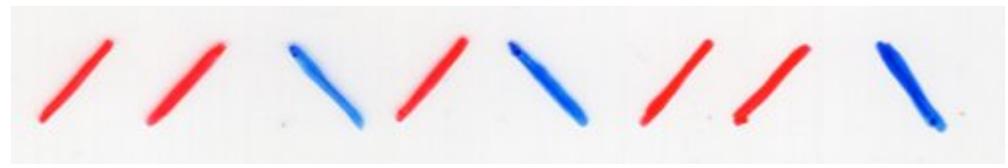




$\sigma = 7 \backslash 4 \backslash 3 / 8 \backslash 2 / 9 \backslash 5 / 1 / 6 \dots$

up-down
sequence

- - + - + - - +



algebraic structures
Hopf algebra

descent
algebra

Loday-Ronco
algebra

Reutenauer
Malvenuto
algebra

dim

$$2^{n-1}$$

$$C_n$$

$$n!$$

Catalan

combinatorial structures

hypercube

Boolean lattice
inclusion

dim 2^{n-1}

associahedron

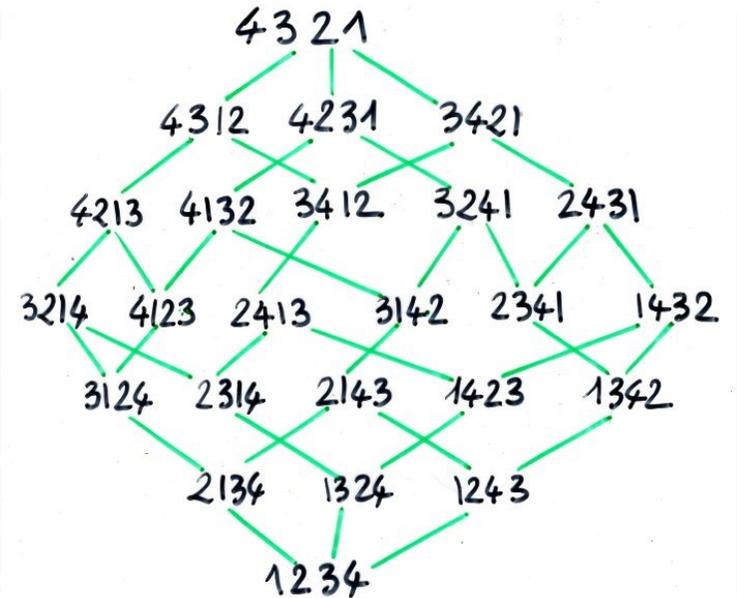
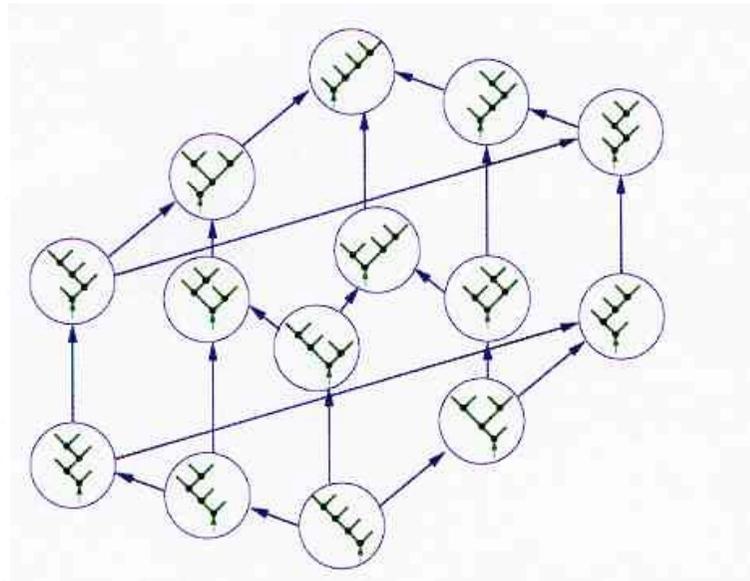
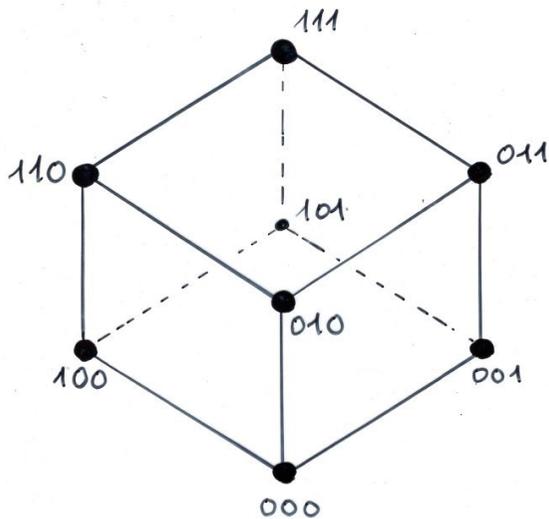
Tamari order

C_n
Catalan

permutahedron

weak Bruhat order

$n!$



Enumeration

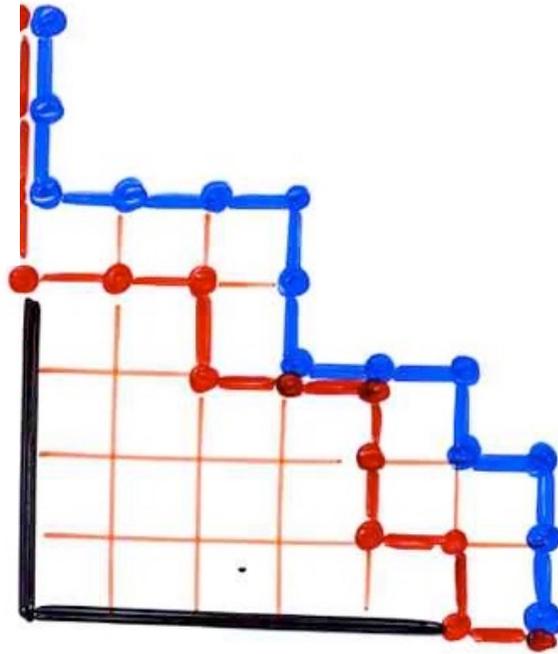
How many binary trees
With a given canopy w ?

Kreweras's determinant
Narayana (1955)

$$\det \left(\begin{matrix} \lambda_i + 1 \\ j - i + 1 \end{matrix} \right)_{1 \leq i, j \leq k}$$

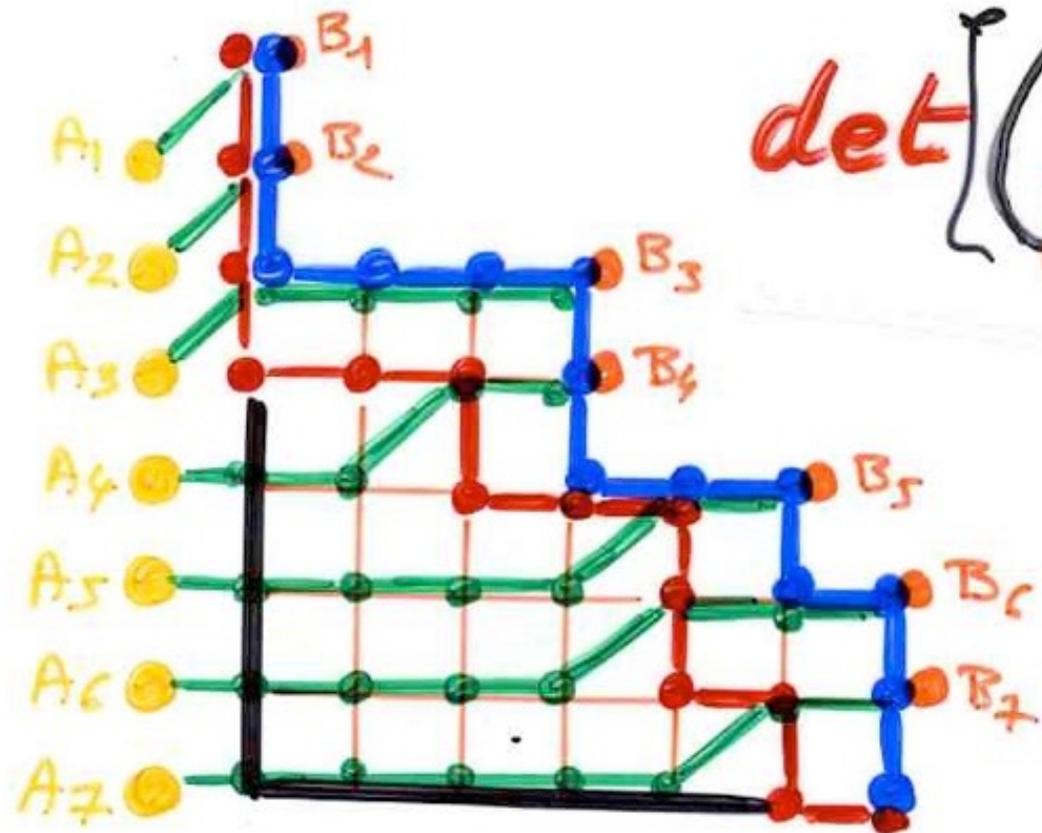
$k =$ nb of 0's in λ

$\lambda_i =$ nb of 1's to the left of the i th zero



$$\lambda = (0, 0, 3, 3, 5, 6, 6)$$

LGV Lemma



$$\det \left[\begin{matrix} \lambda_i + 1 \\ j - i + 1 \end{matrix} \right]_{1 \leq i, j \leq k}$$

$$\lambda = (0, 0, 3, 3, 5, 6, 6)$$

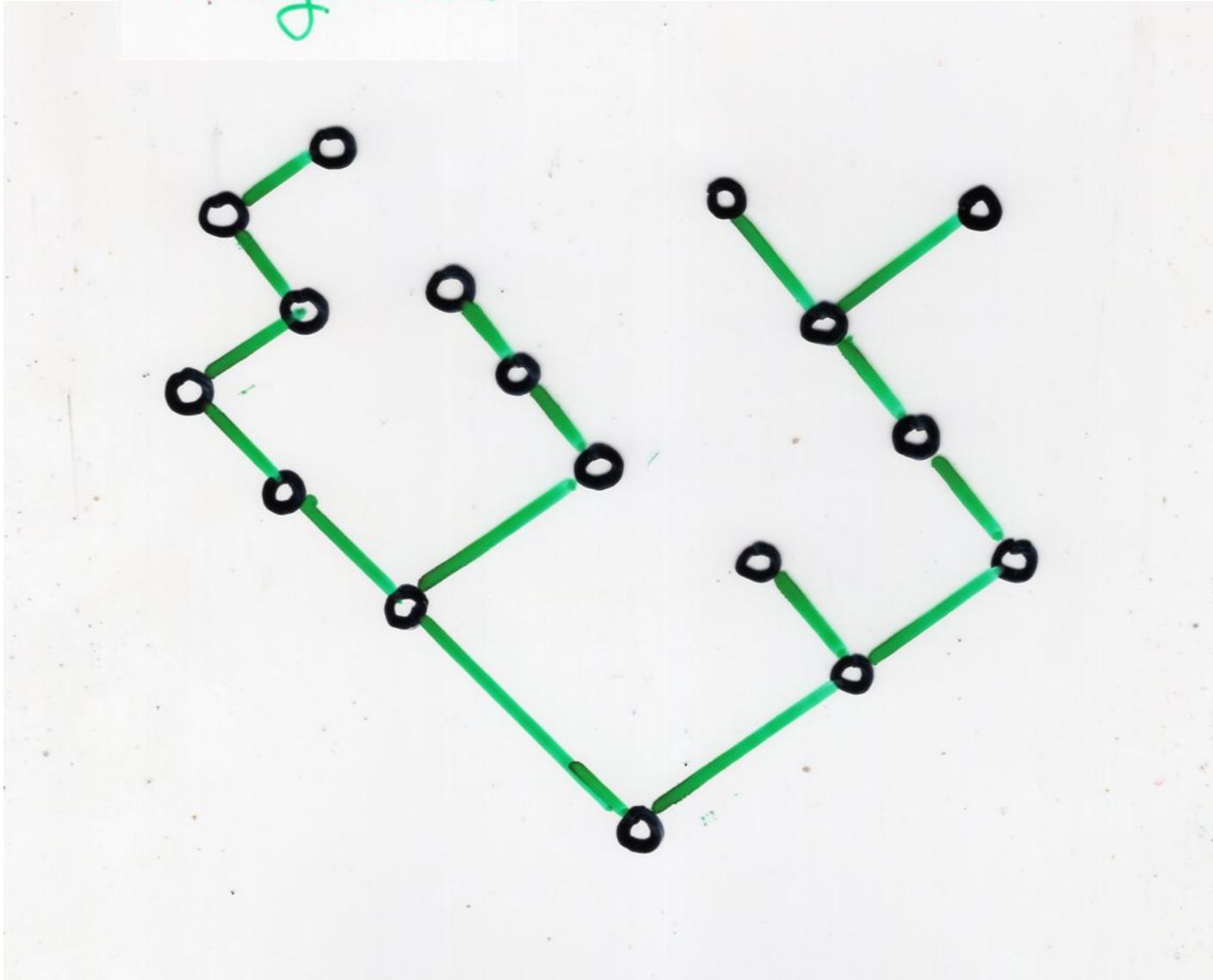
$$(\lambda_1, \dots, \lambda_k)$$

The Tamil bijection

$B \longrightarrow (w, h)$

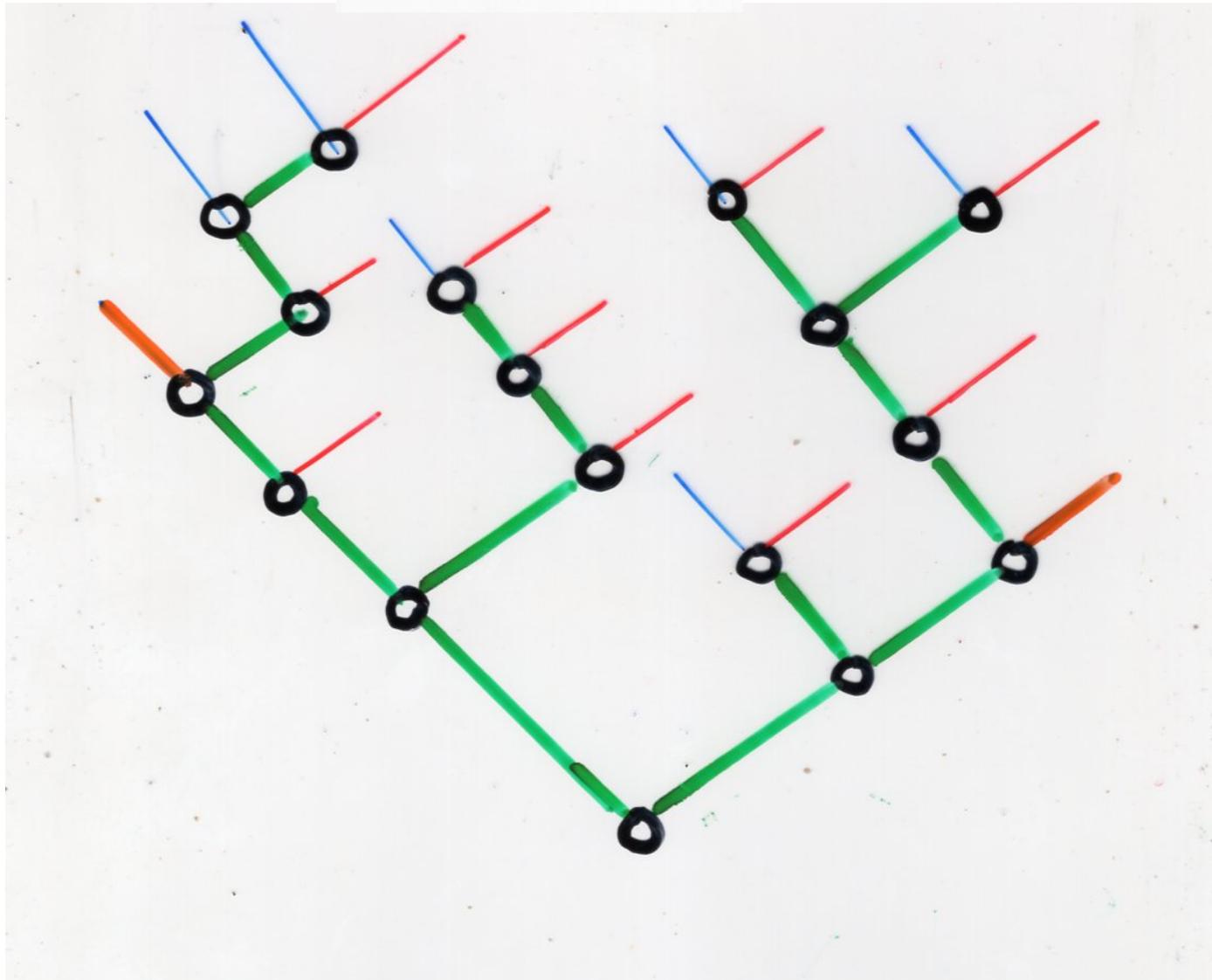
The Tamil bijection

binary trees



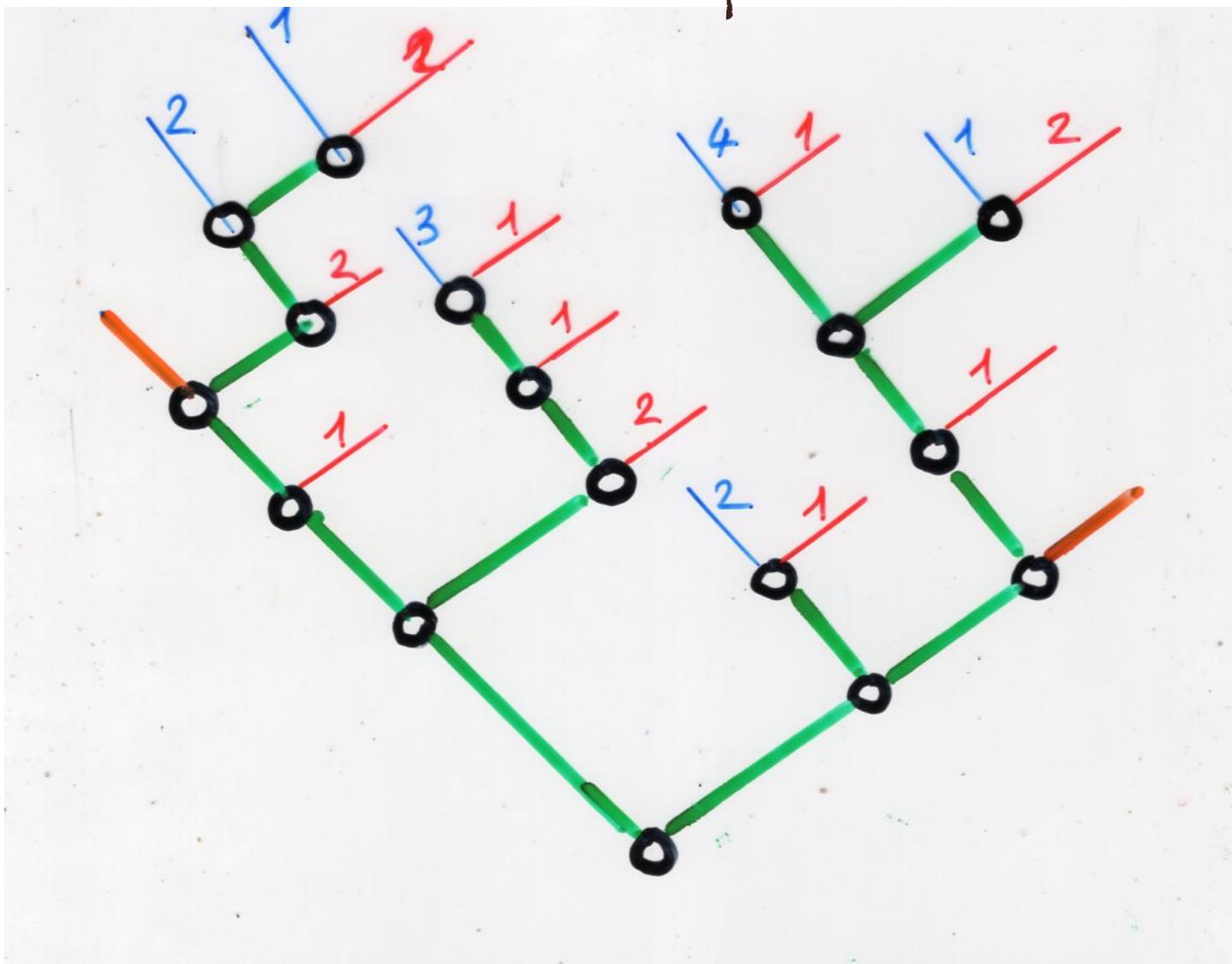
The Tamil bijection

binary trees

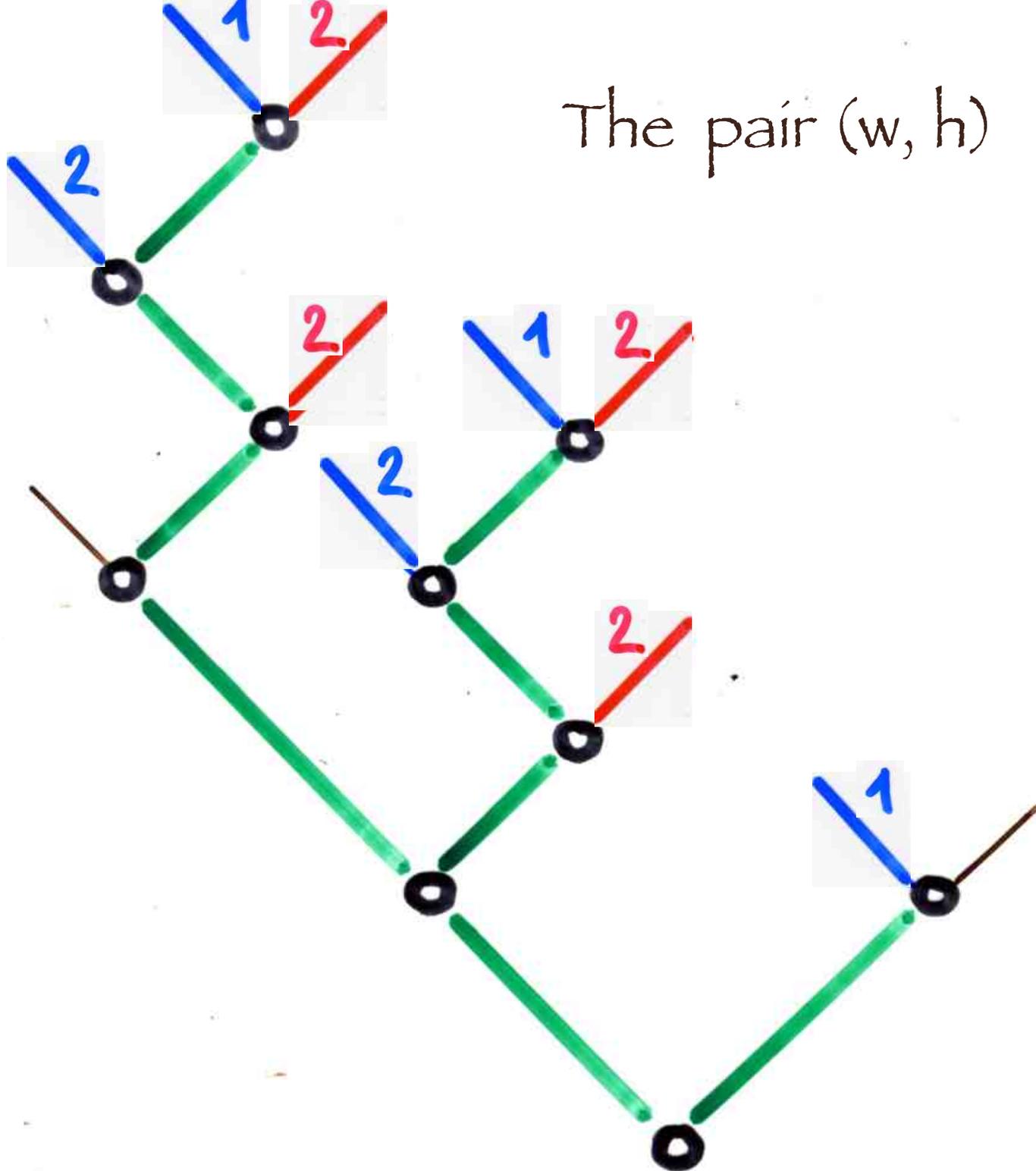


2 1 2 2 1 3 1 1 2 2 1 4 1 1 2 1

The pair (w, h)



The pair (w, h)

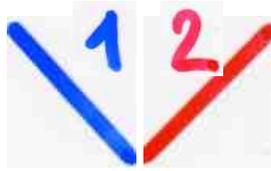


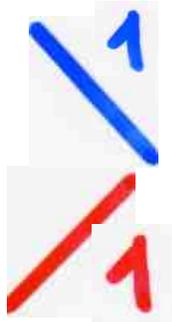


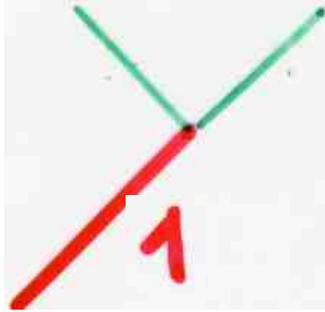
The pair (w, h)

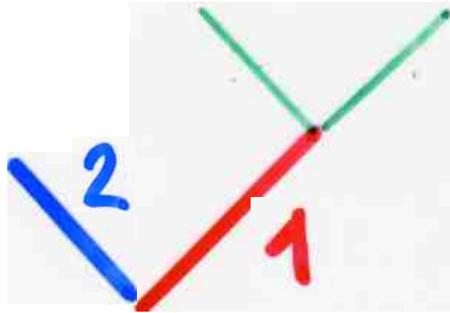
From the pair (w, h) to the binary tree B

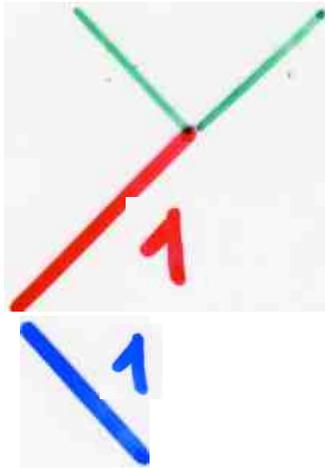


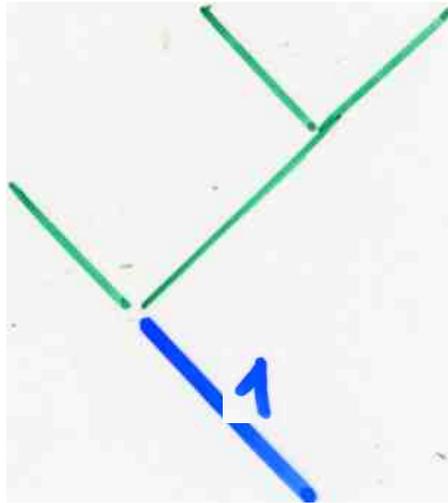


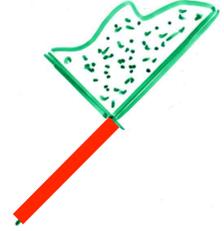
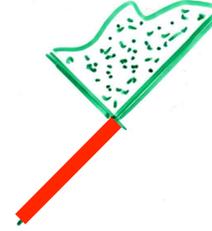
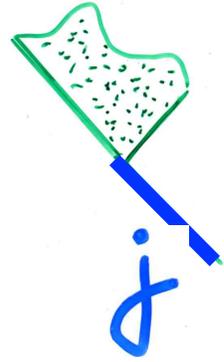
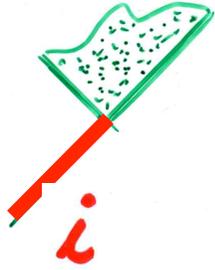
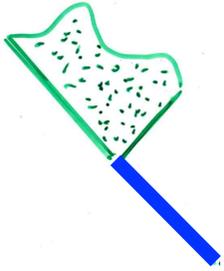
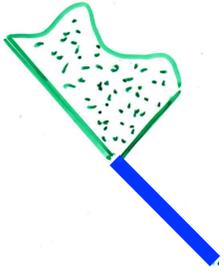


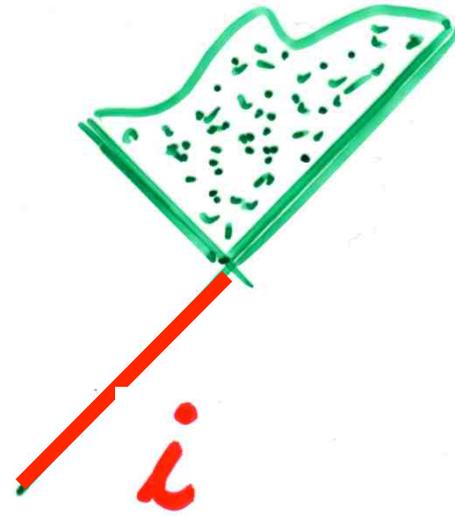
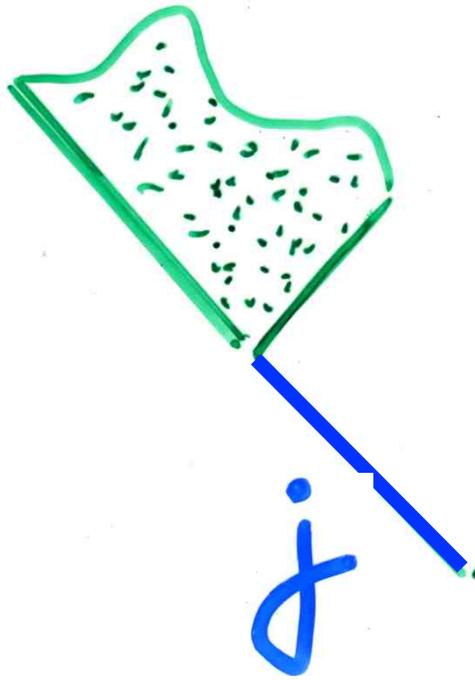


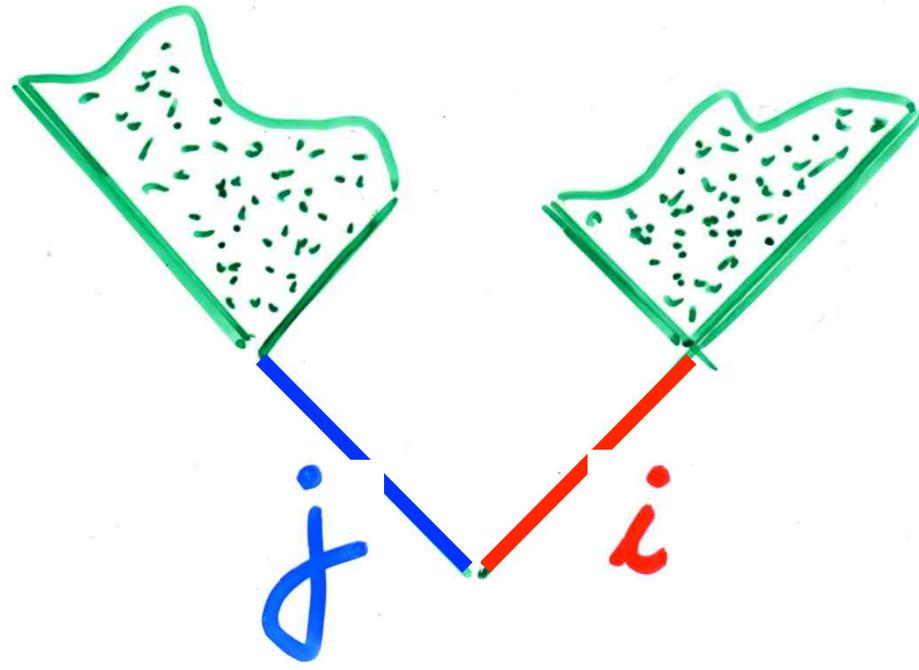


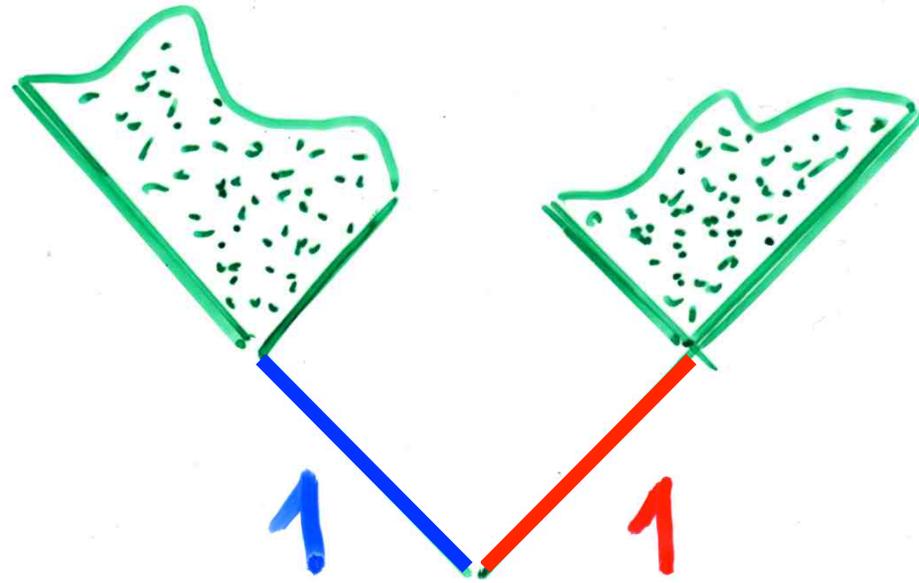


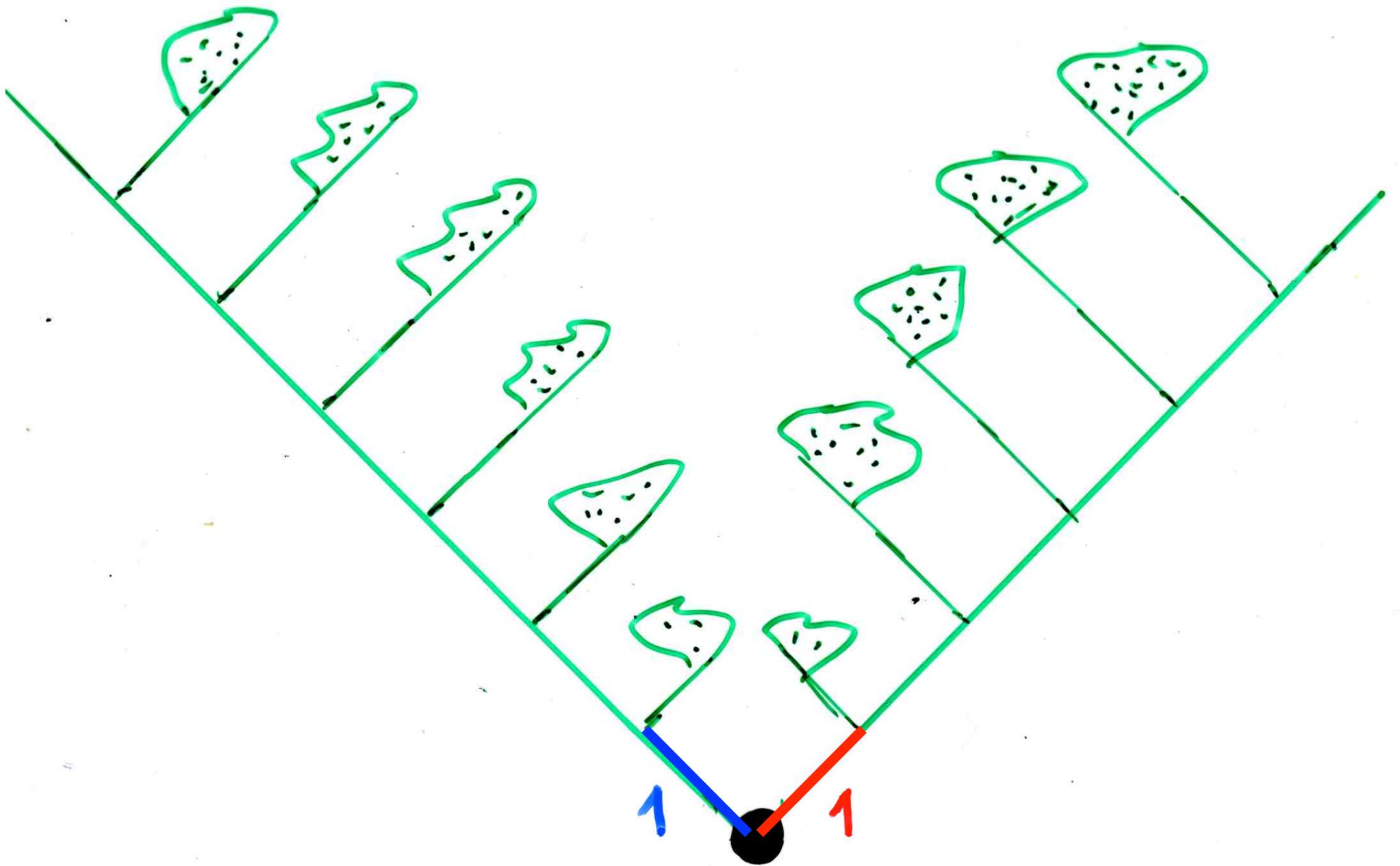


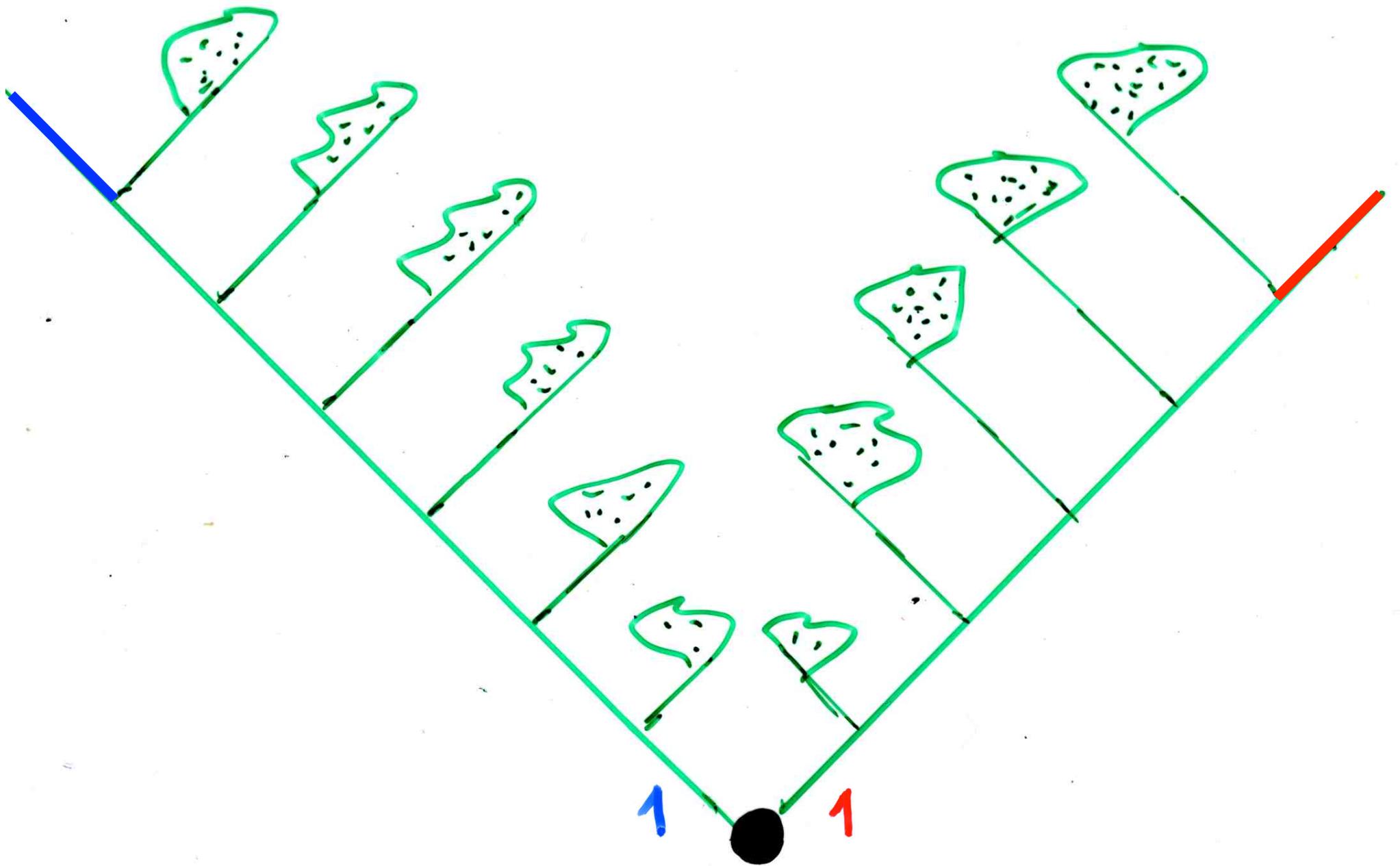


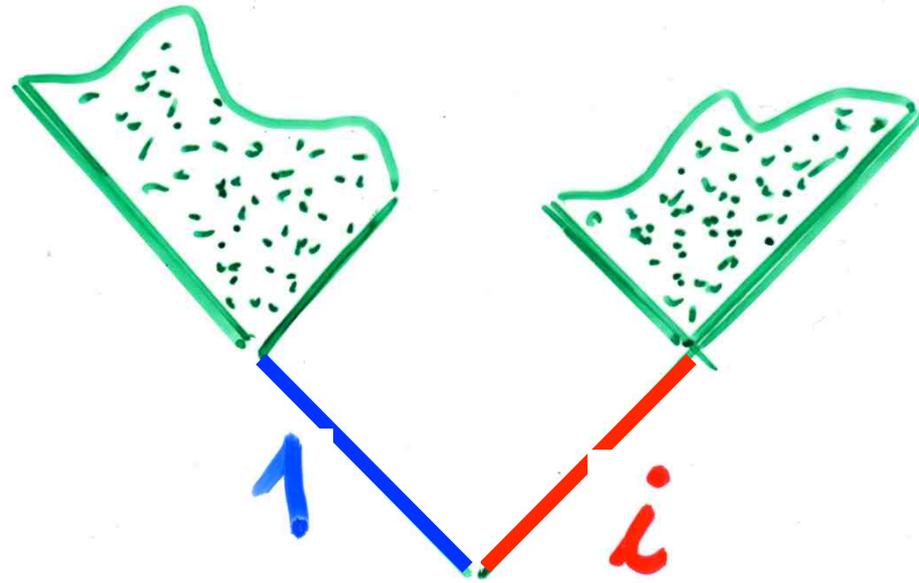




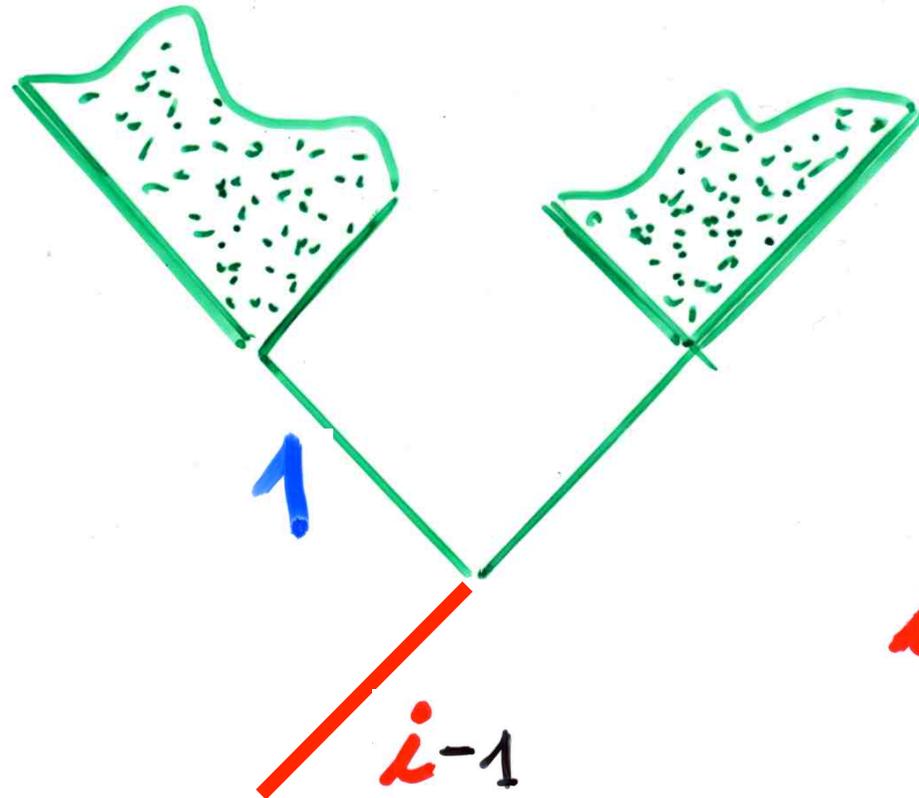






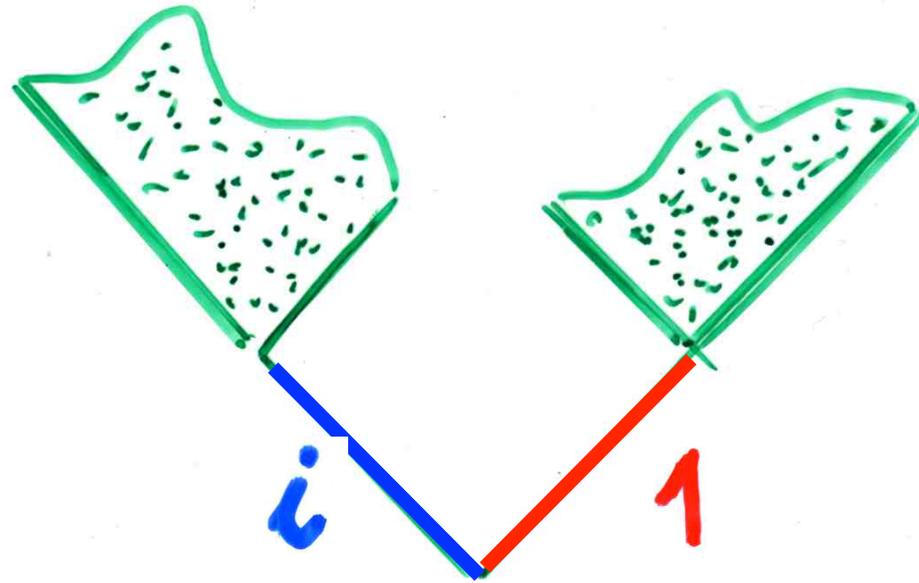


$i \geq 2$

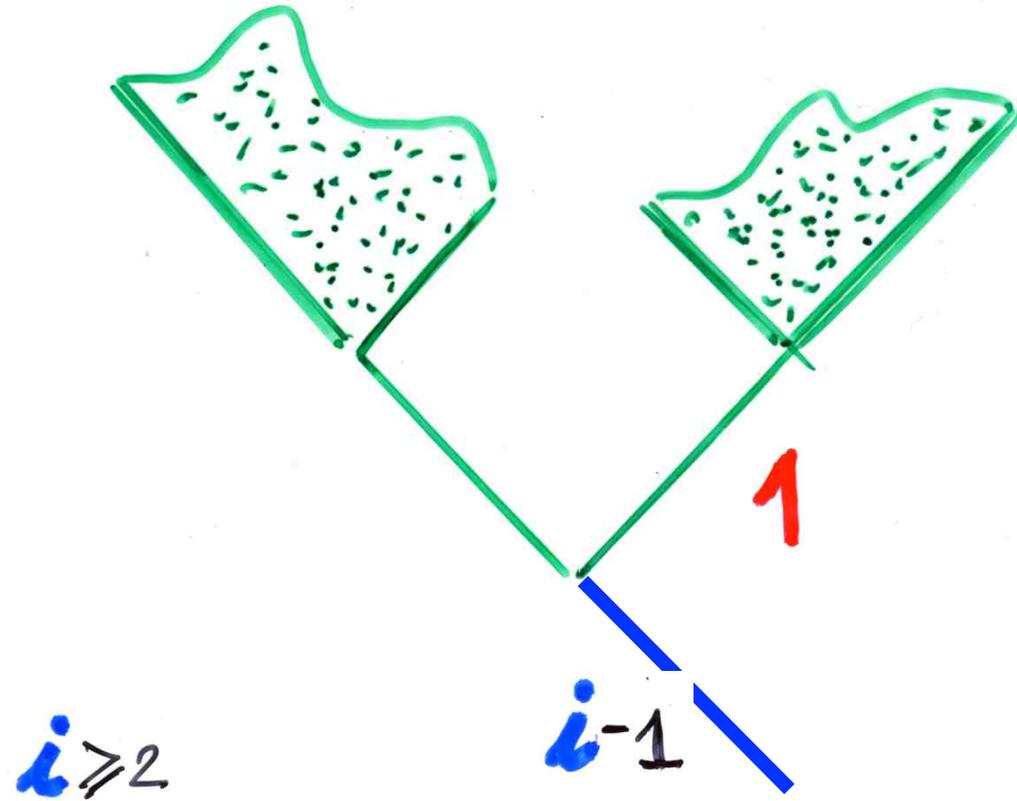


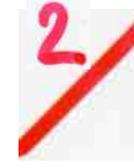
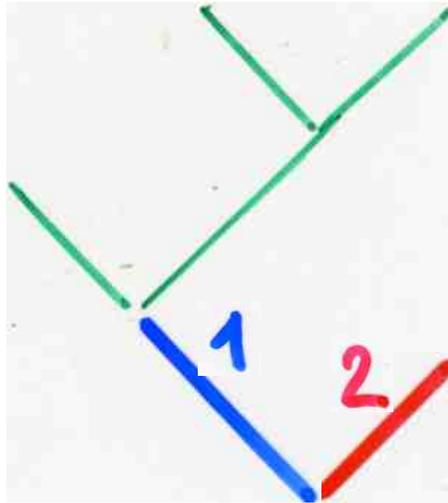
$i-1$

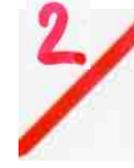
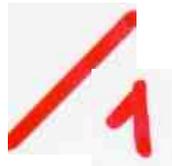
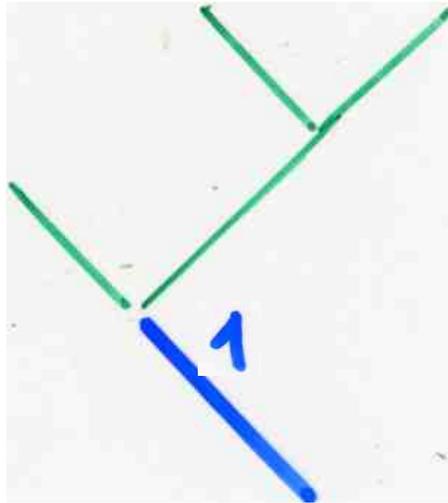
$i \geq 2$

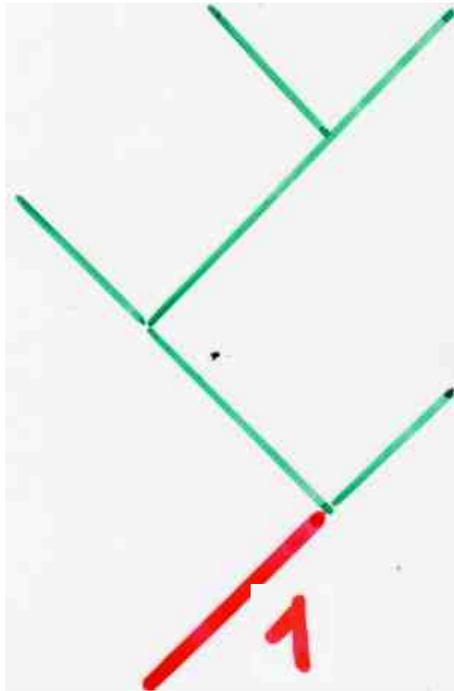


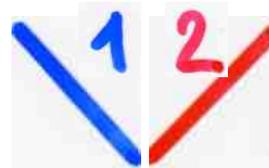
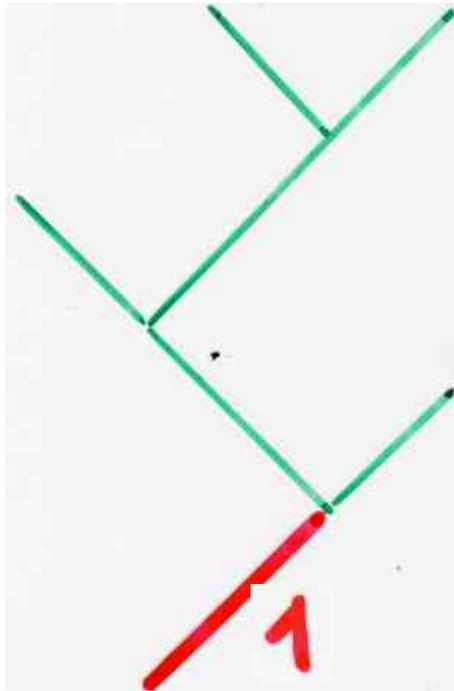
$i \geq 2$

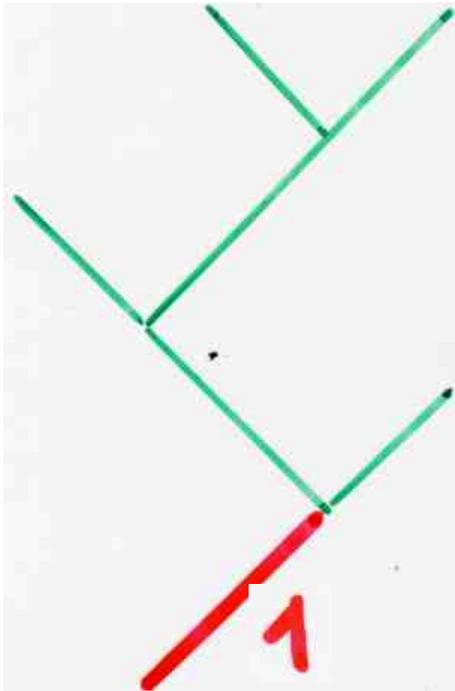


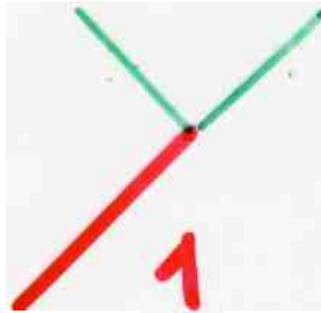
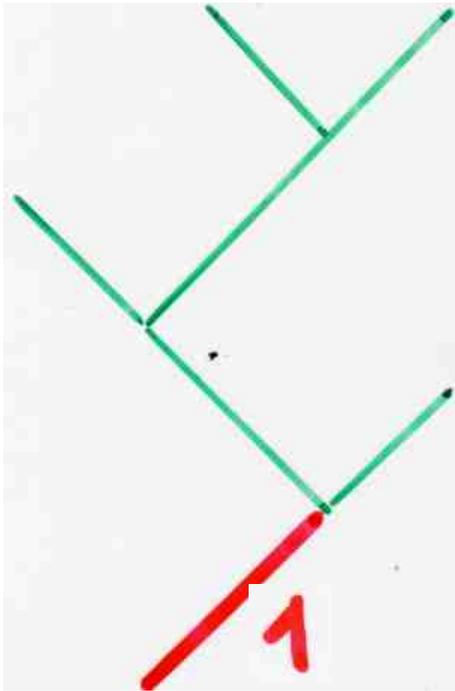


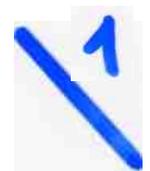
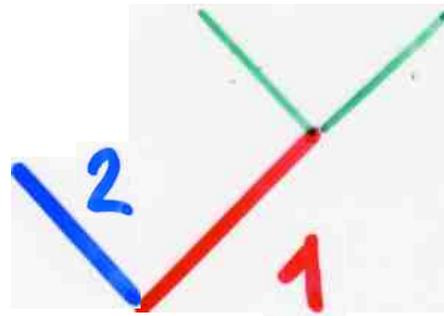
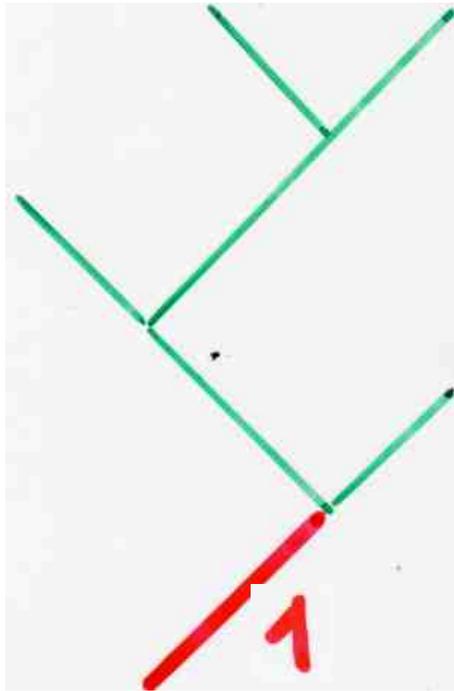


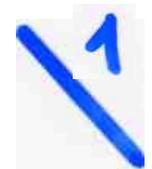
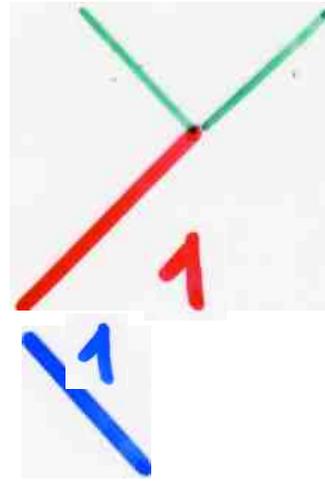
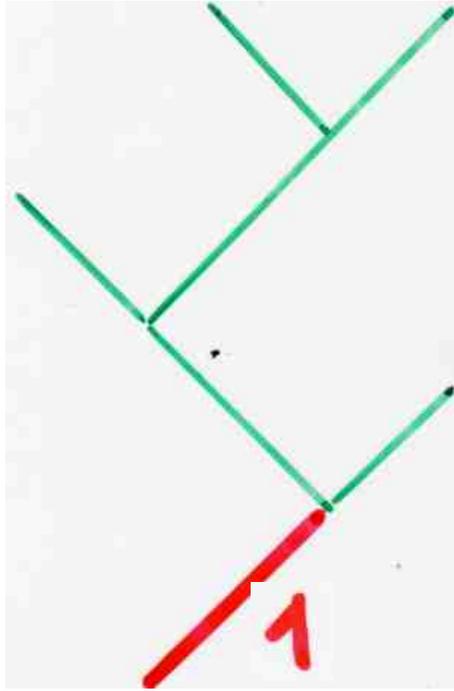


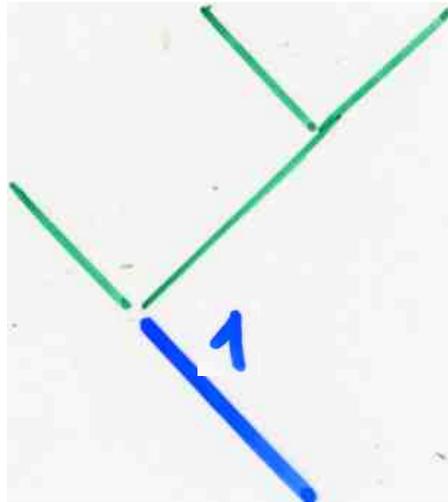
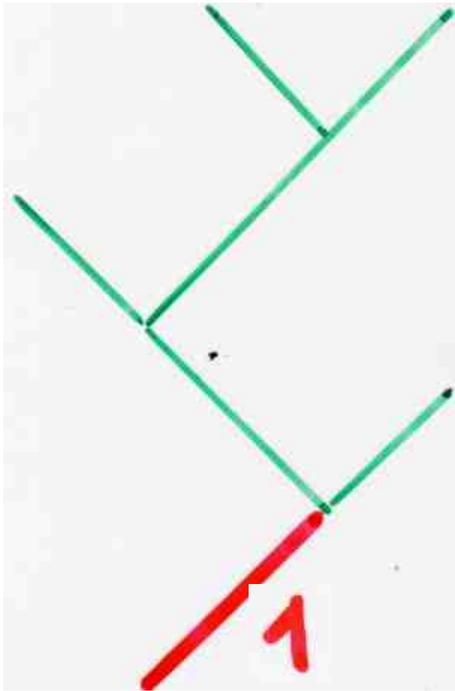


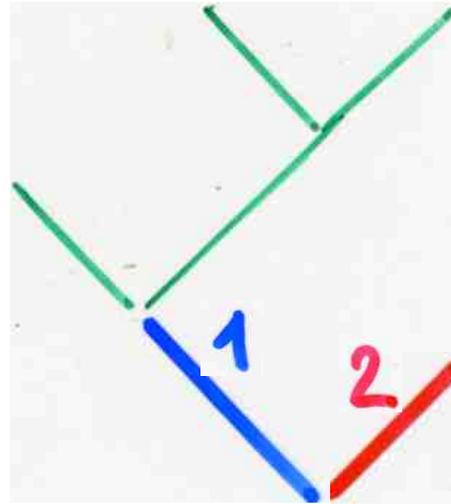
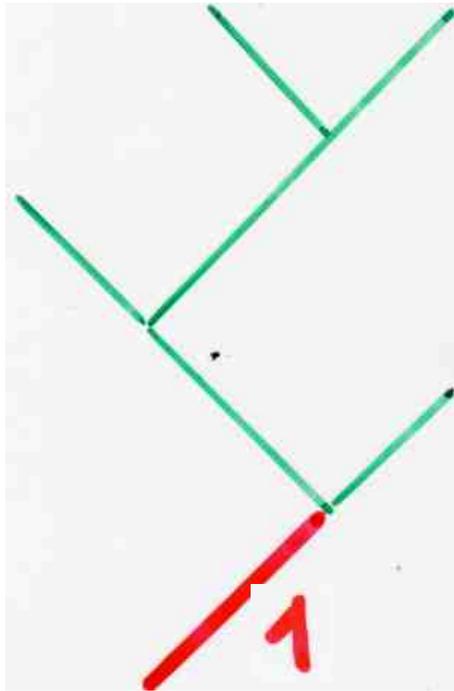


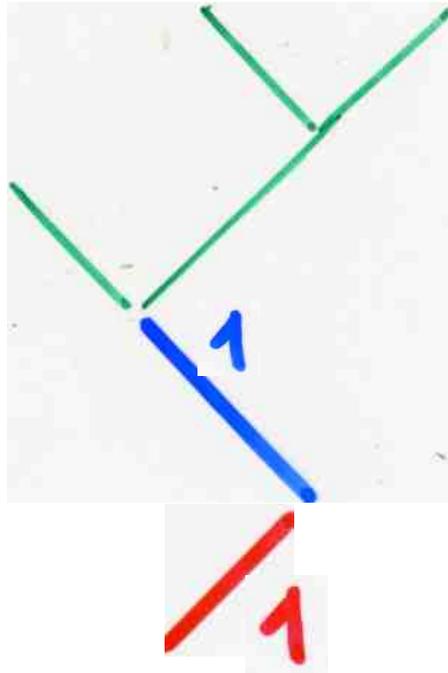
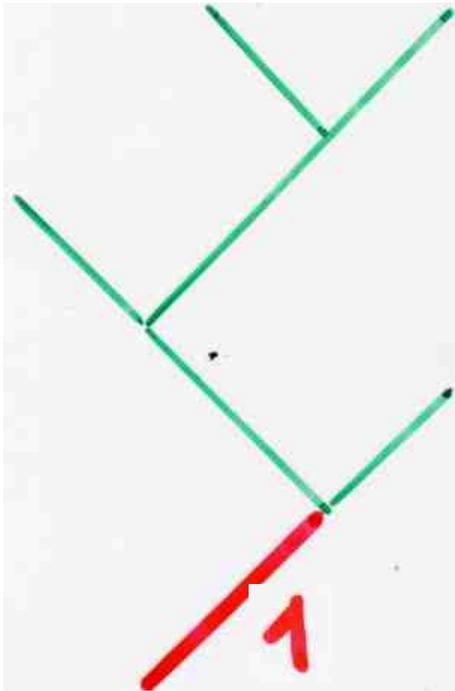


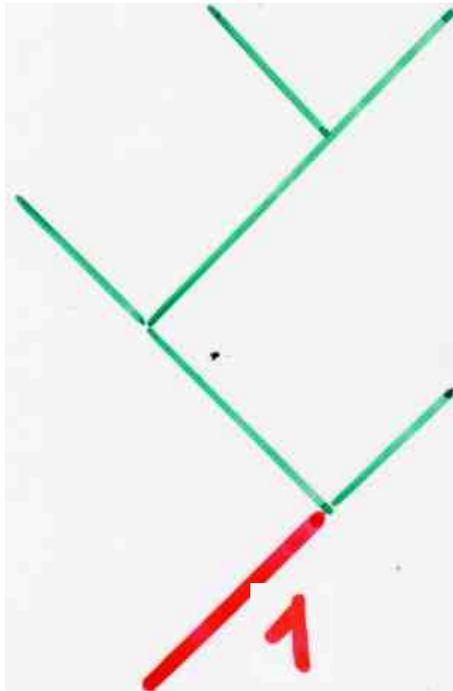
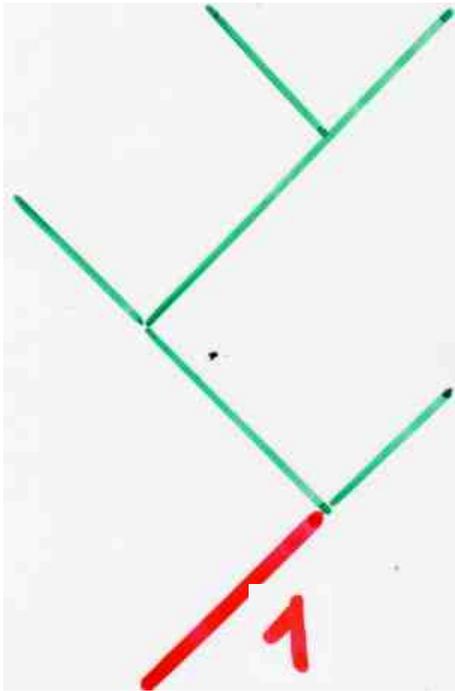


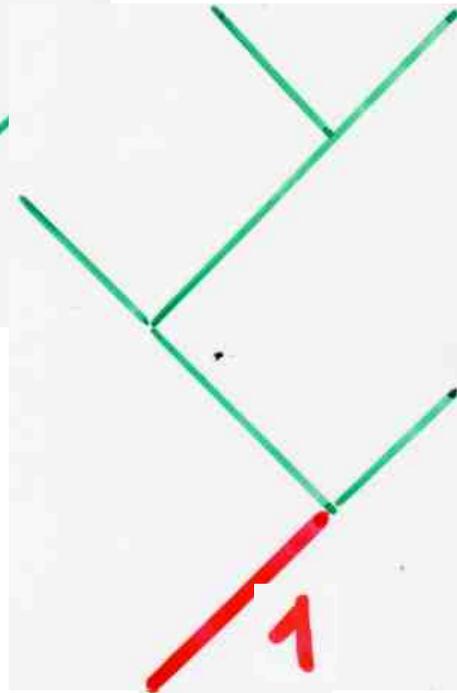
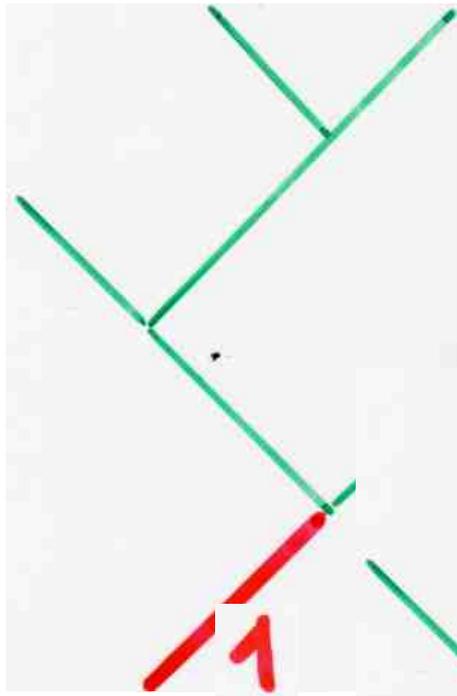


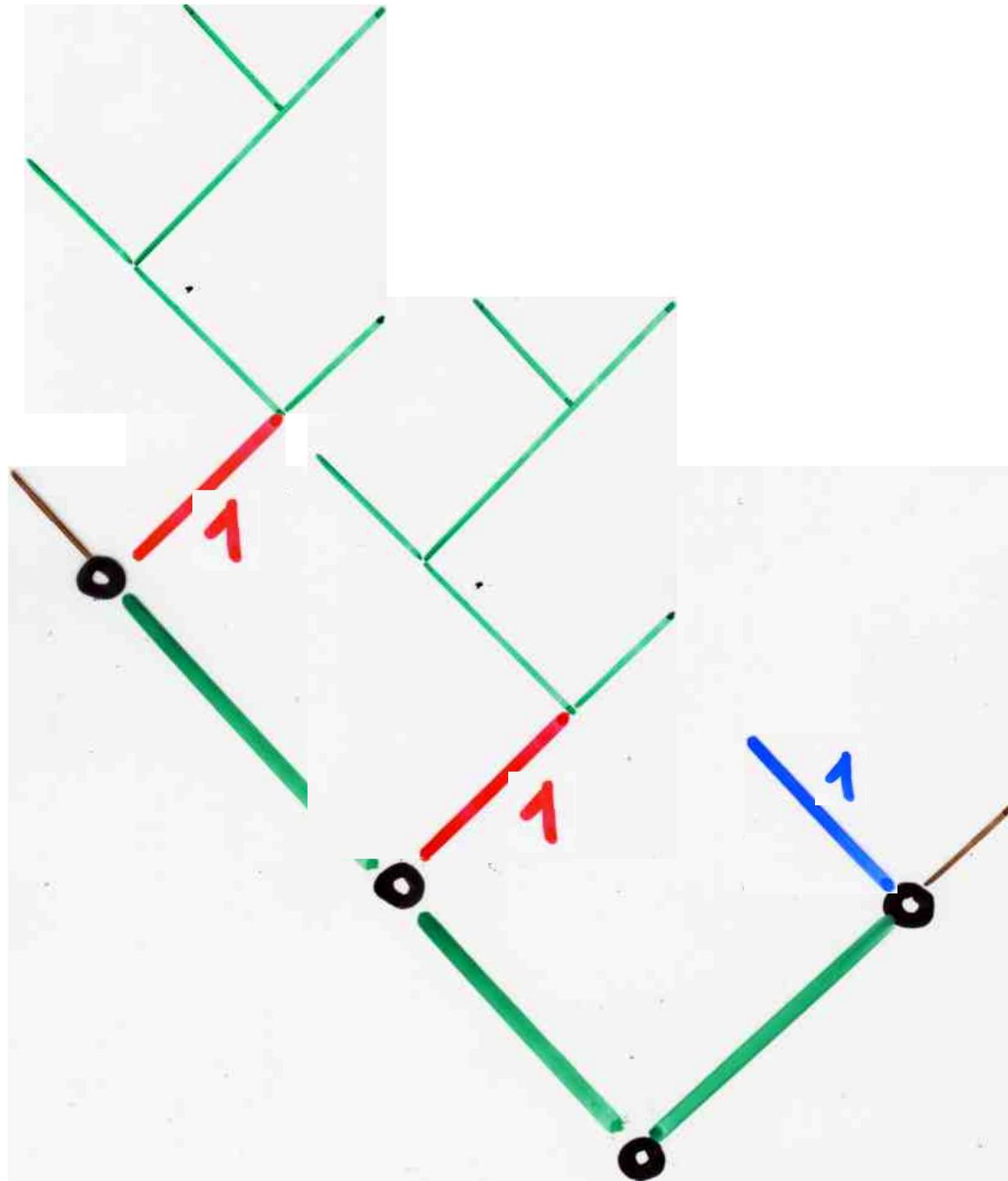


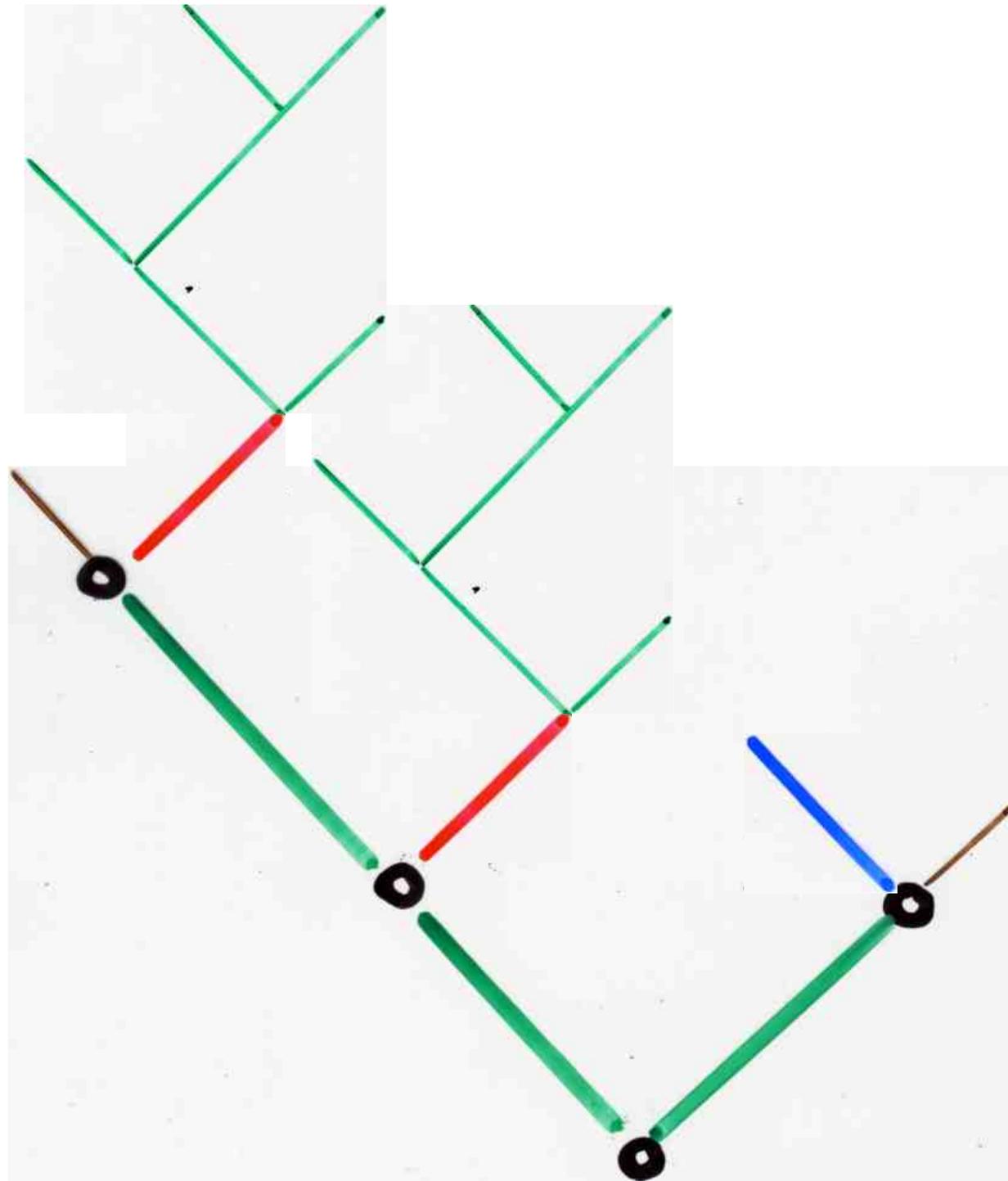


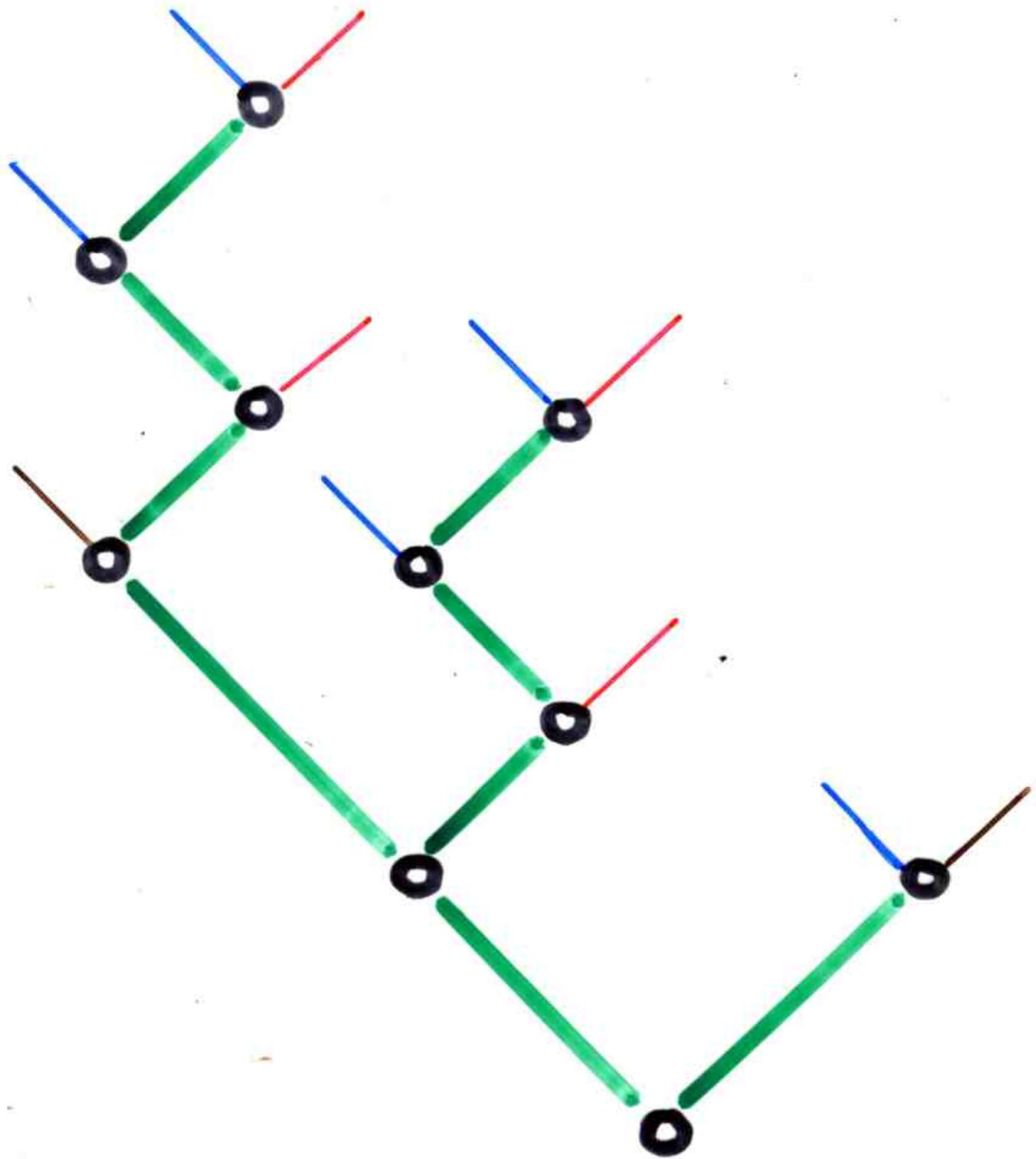


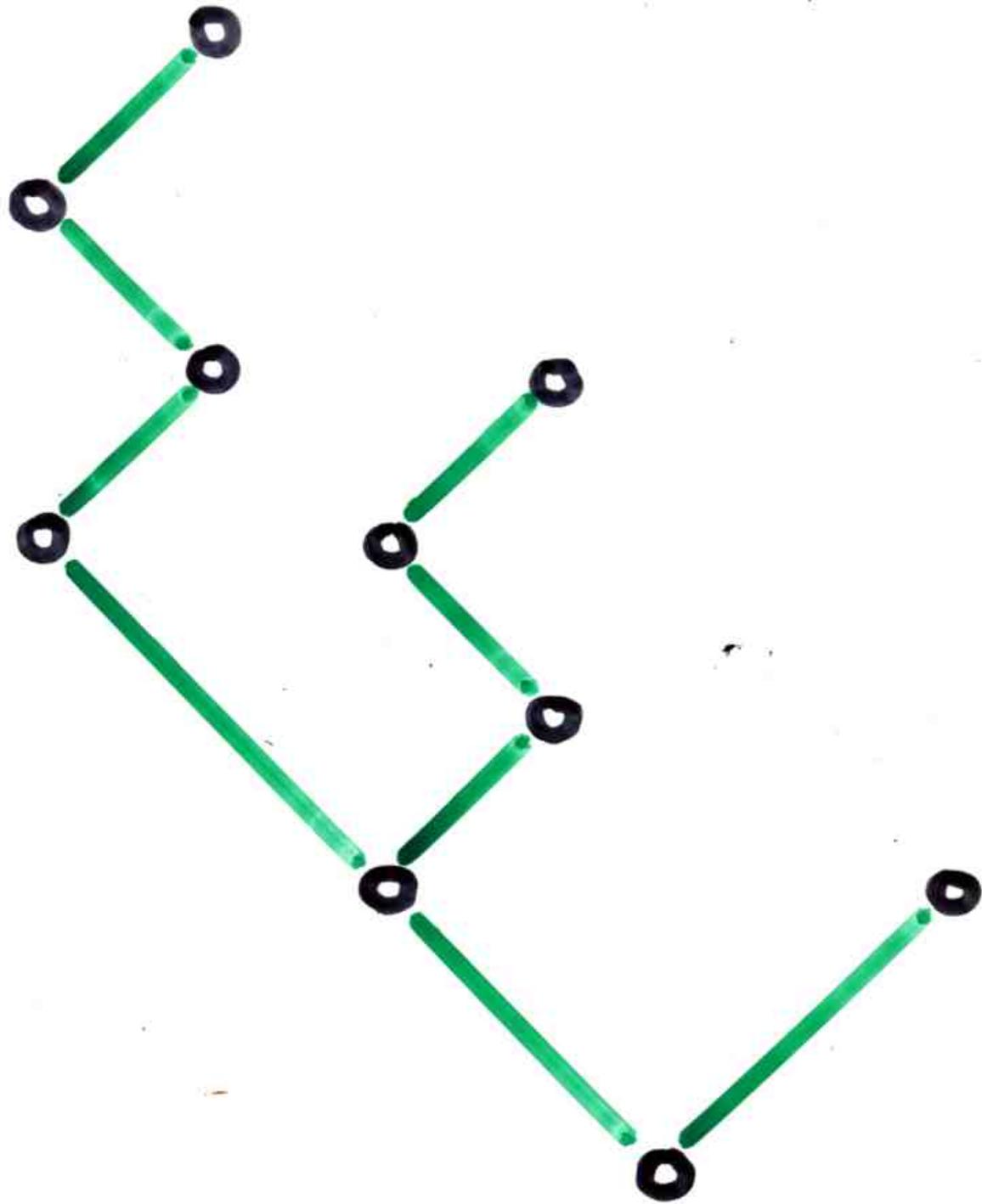


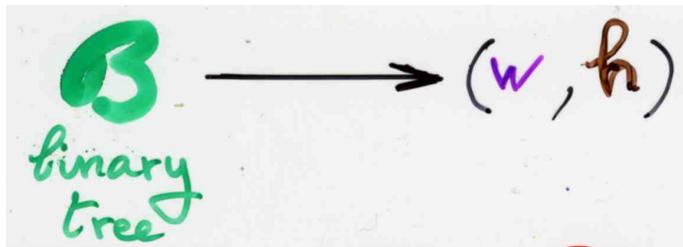












$$w = c(\beta)$$

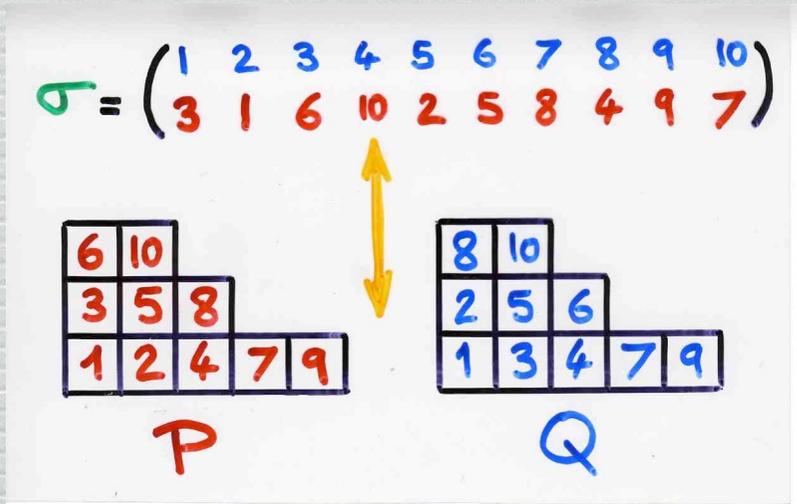
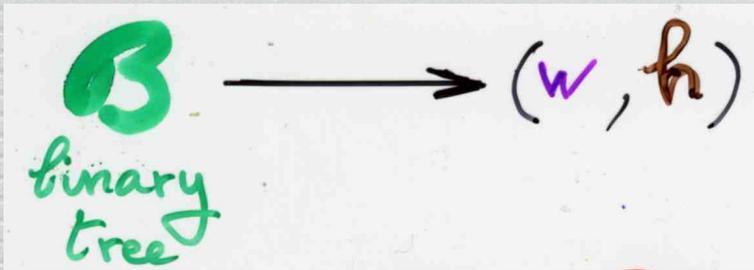
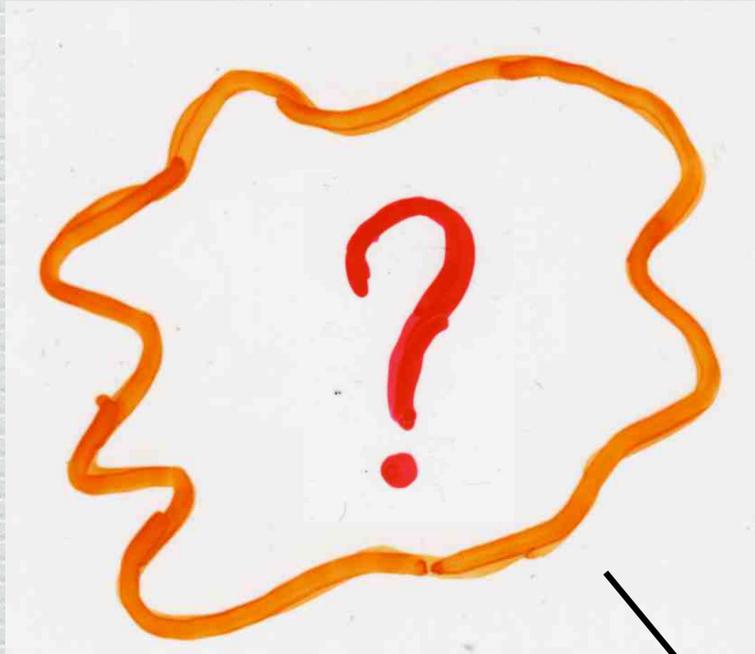
canopy

h ?

$$A = B$$

$$\sum_{n \geq 0} a_n t^n = \sum_{n \geq 0} b_n t^n$$

$$f(t) = g(t)$$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

6	10			
3	5	8		
1	2	4	7	9

P



8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape

The Robinson-Schensted correspondence

G. de B. Robinson, 1938

- Schensted insertions algorithm
- Geometric version

C. Schensted, 1961

X.V. 1976

RS with Schensted's insertions

G =

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q

recording
tableau

P

insertion
tableau

$P =$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q

1					

recording
tableau

P

3					

insertion
tableau

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1					

3					
1					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1	3				

3					
1	6				

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1	3	4			

3					
1	6	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1	3	4			

3					
1	6	10			2

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5				
1	3	4			

3	6				
1	2	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5				
1	3	4			

3	6				
1	2	10			5

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5				
1	3	4			

3	6			10	
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		4

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10		5	
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10		5	
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

			6		
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			8
1	2	4	7	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			8
1	2	4	7	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						10
3	5	8				
1	2	4	7	9		

$\rho =$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

$Q =$

=

8	10				
2	5	6			
1	3	4	7	9	

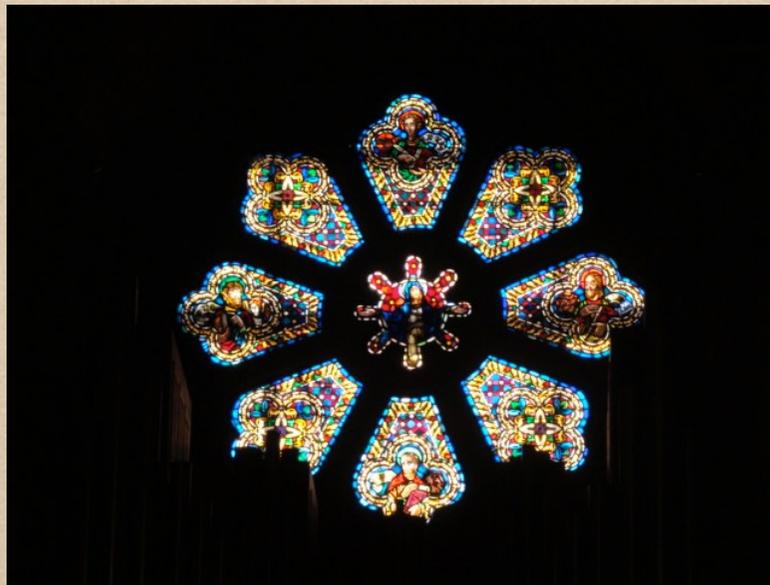
$P =$

=

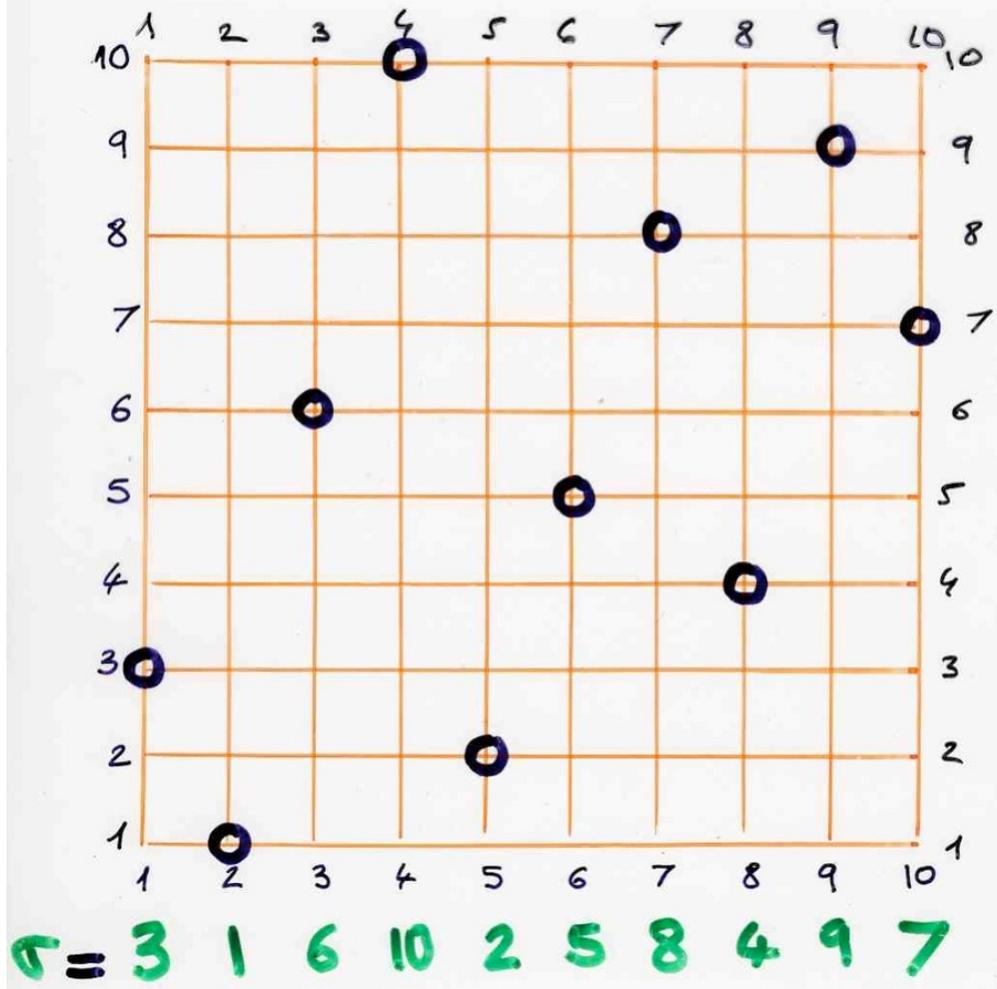
6	10				
3	5	8			
1	2	4	7	9	

end of the
RS algorithm

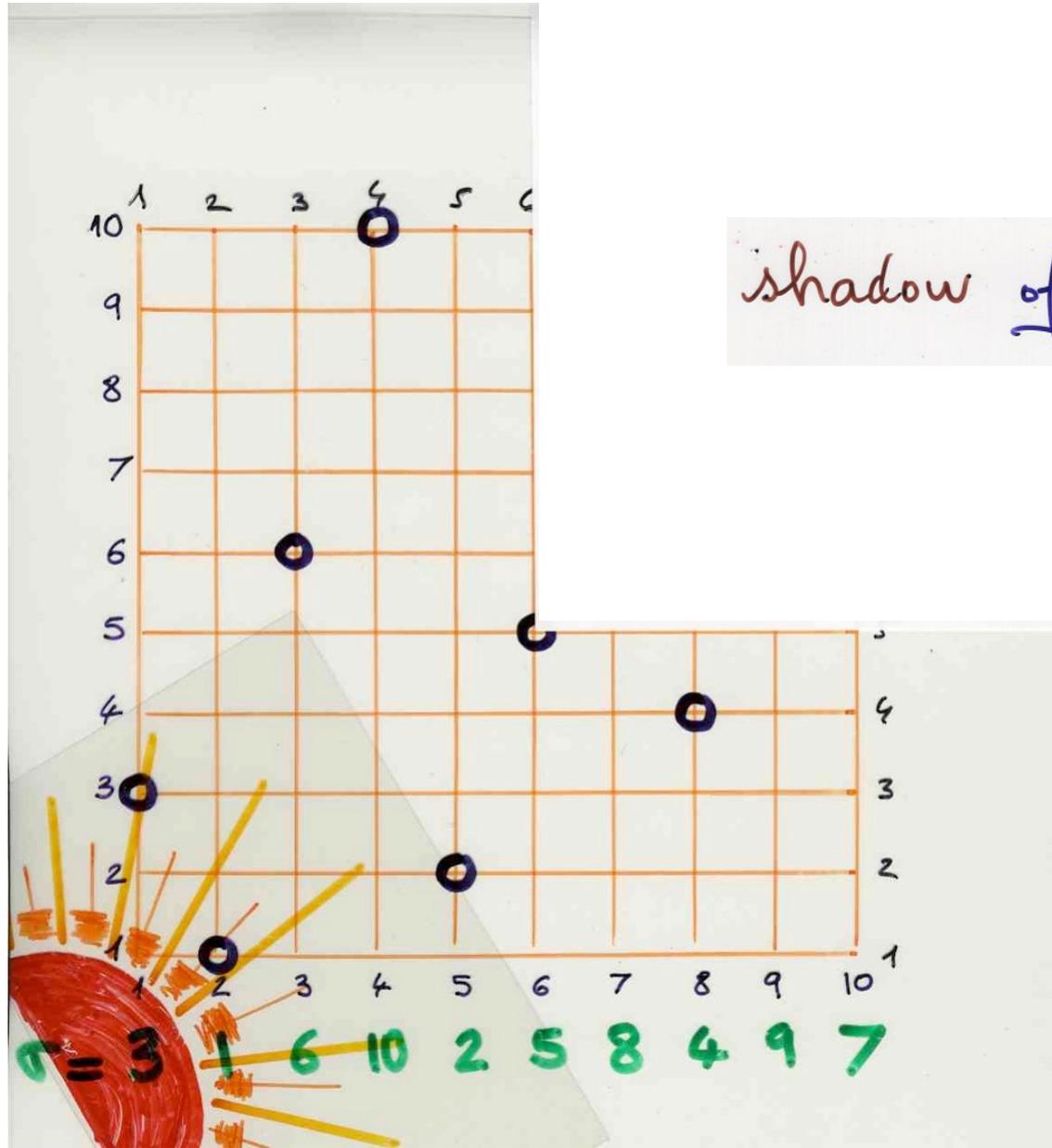
A geometric version of RS
with "light" and "shadow lines"



$$\{(i, \sigma(i))\}_{i=1, \dots, n} \subseteq [1, n] \times [1, n]$$

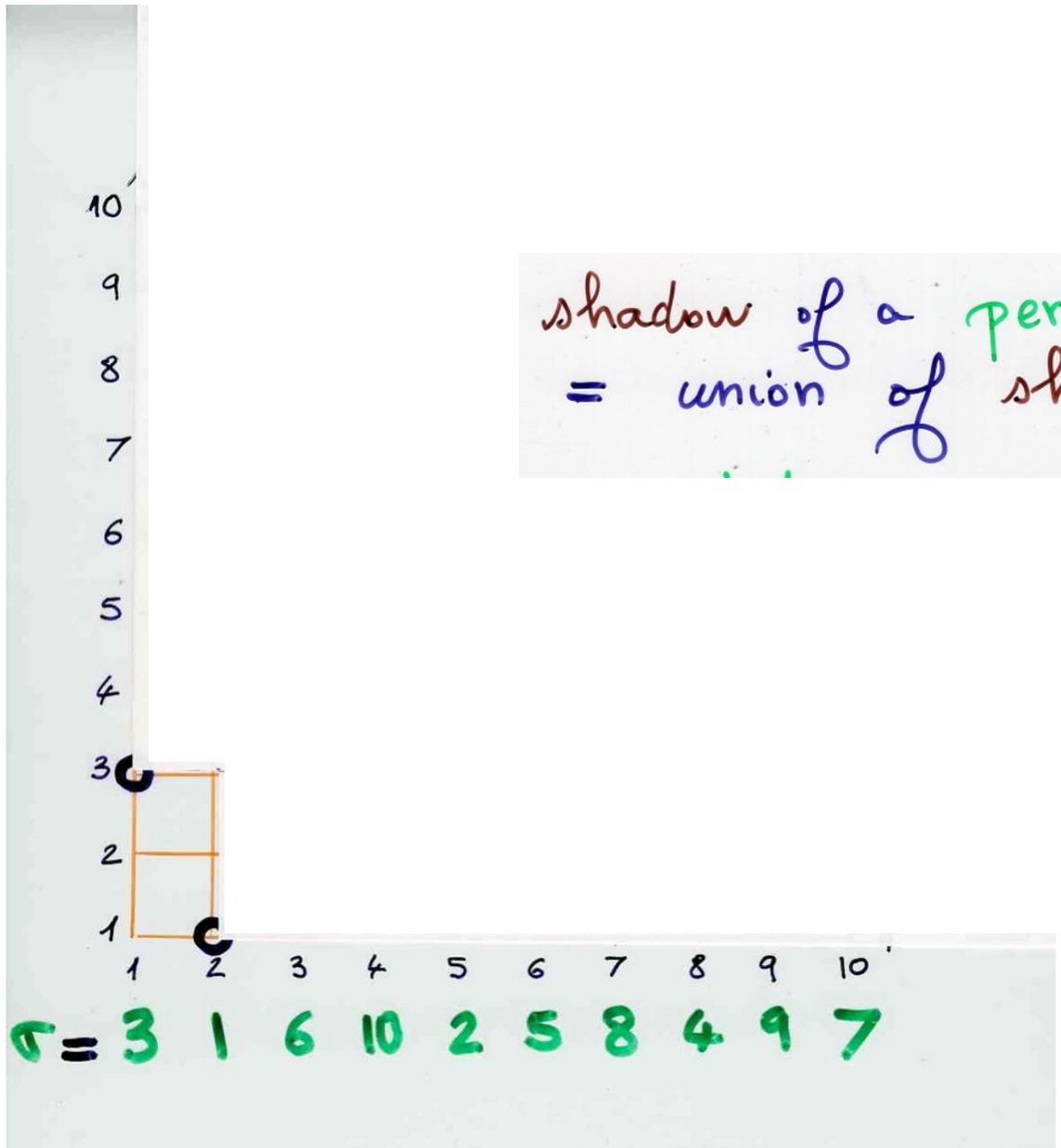


graph of a permutation σ

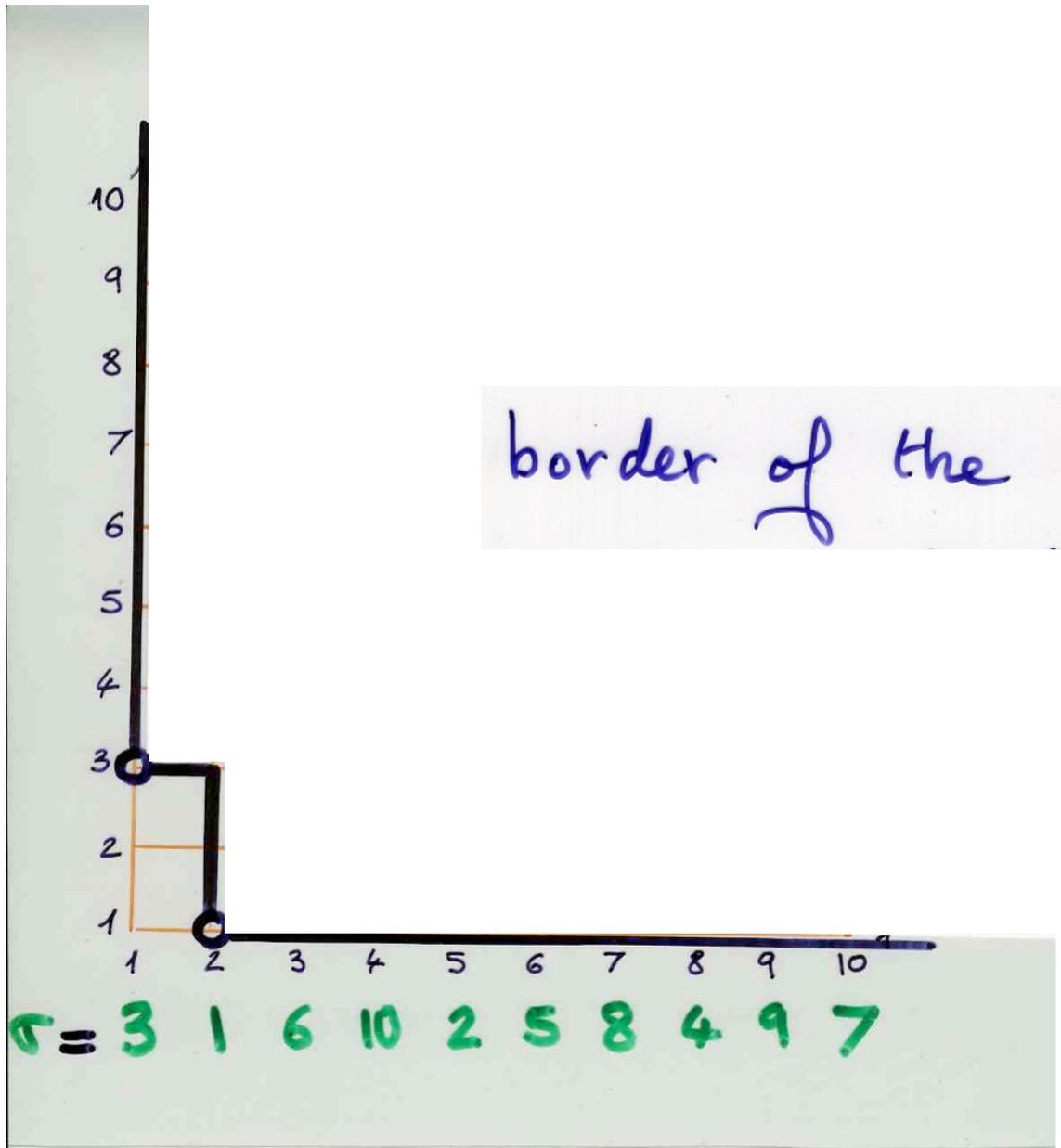


shadow of a point ●

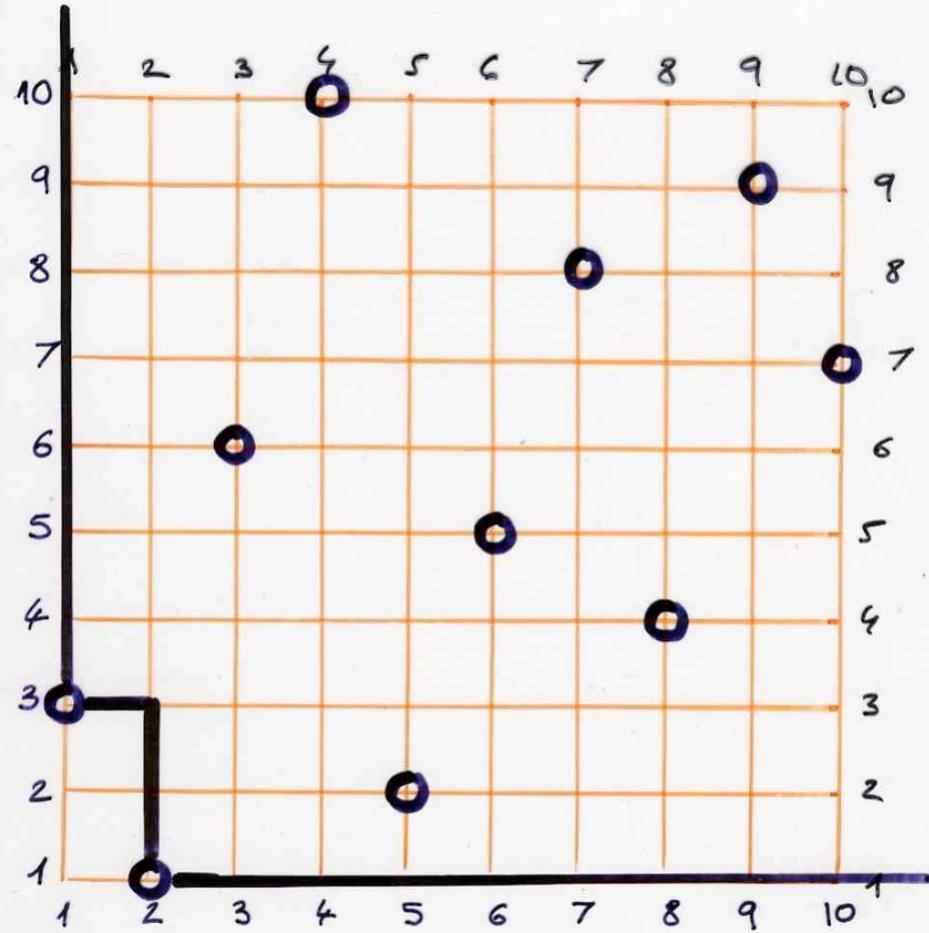
shadow of a permutation
= union of shadows



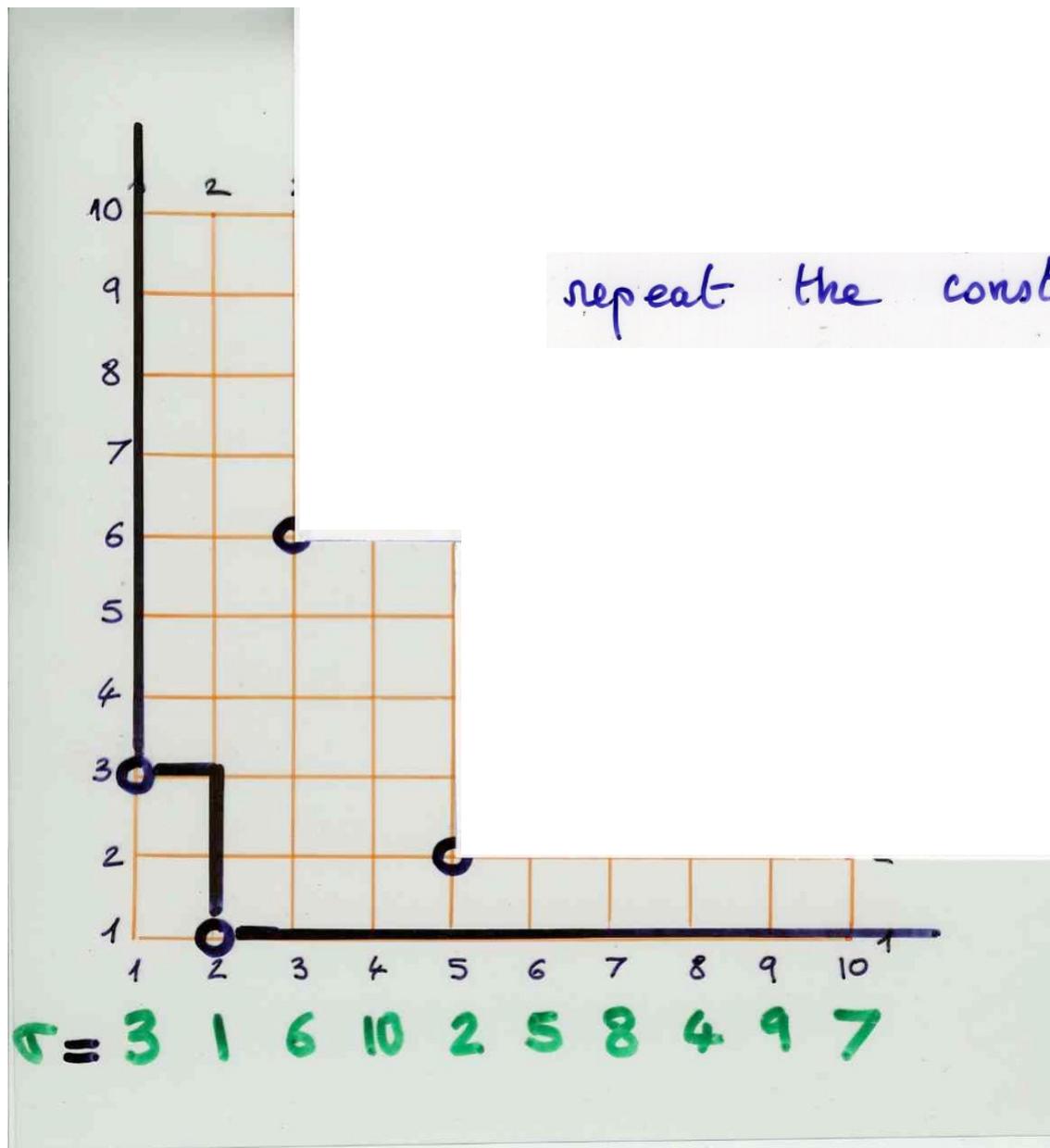
border of the shadow

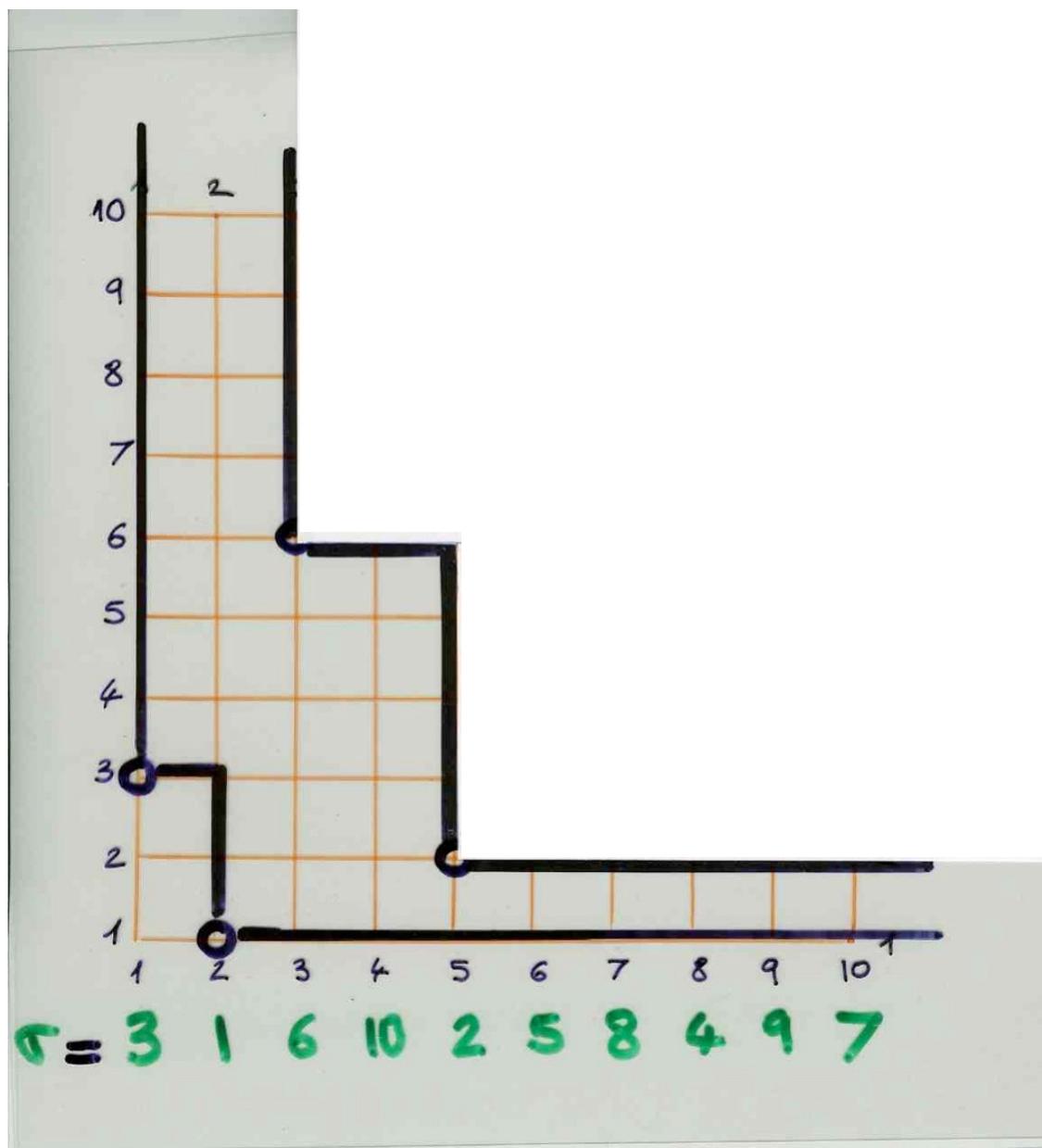


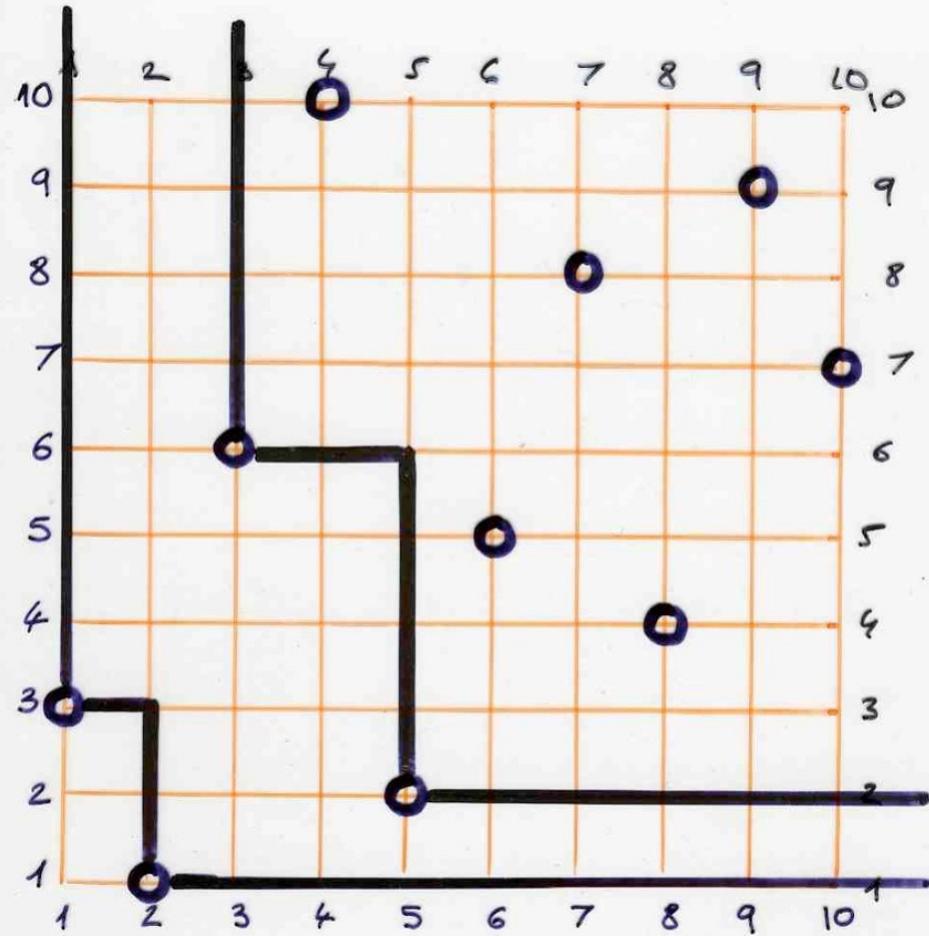
$\sigma = 3, 1, 6, 10, 2, 5, 8, 4, 9, 7$



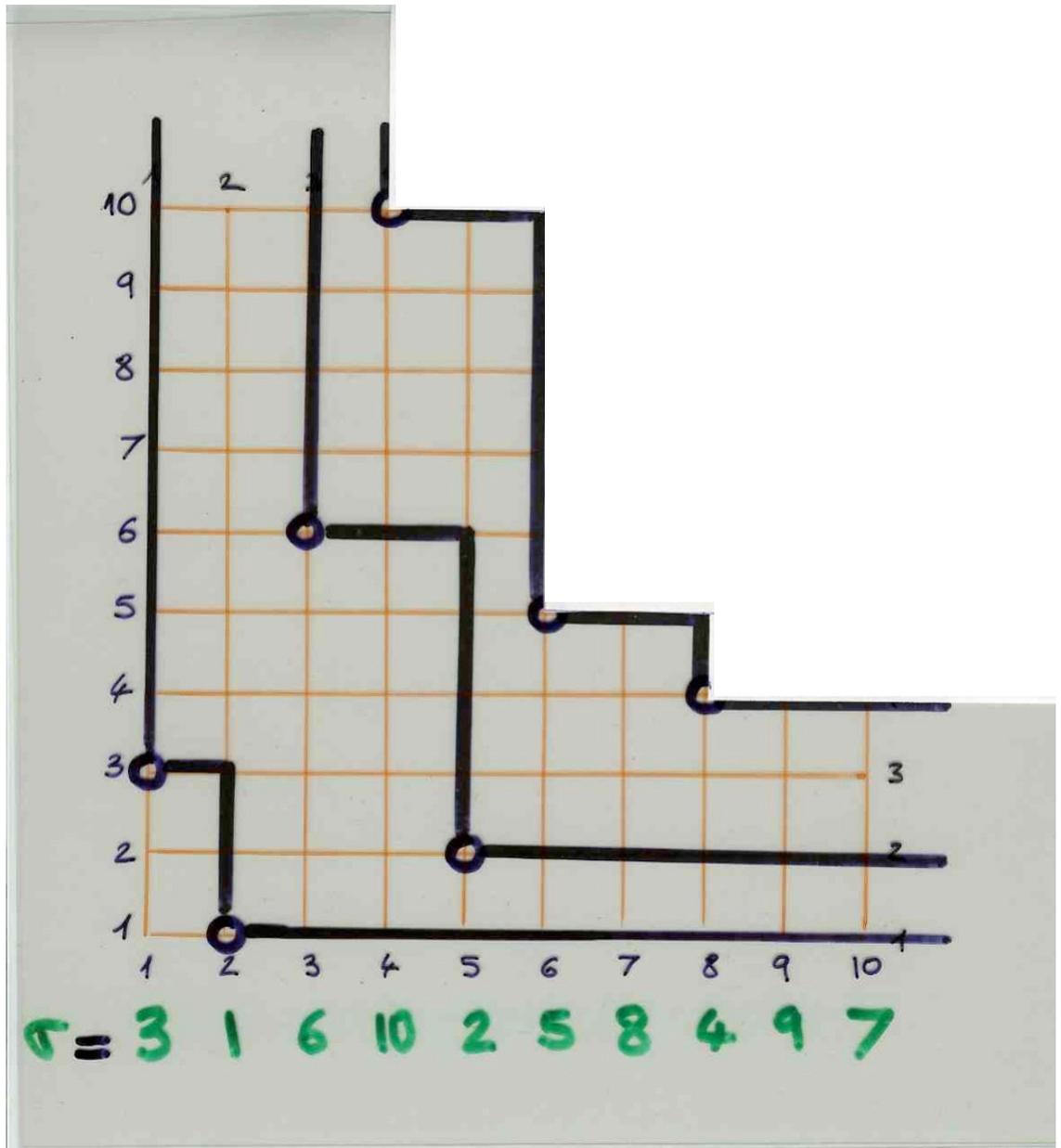
$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



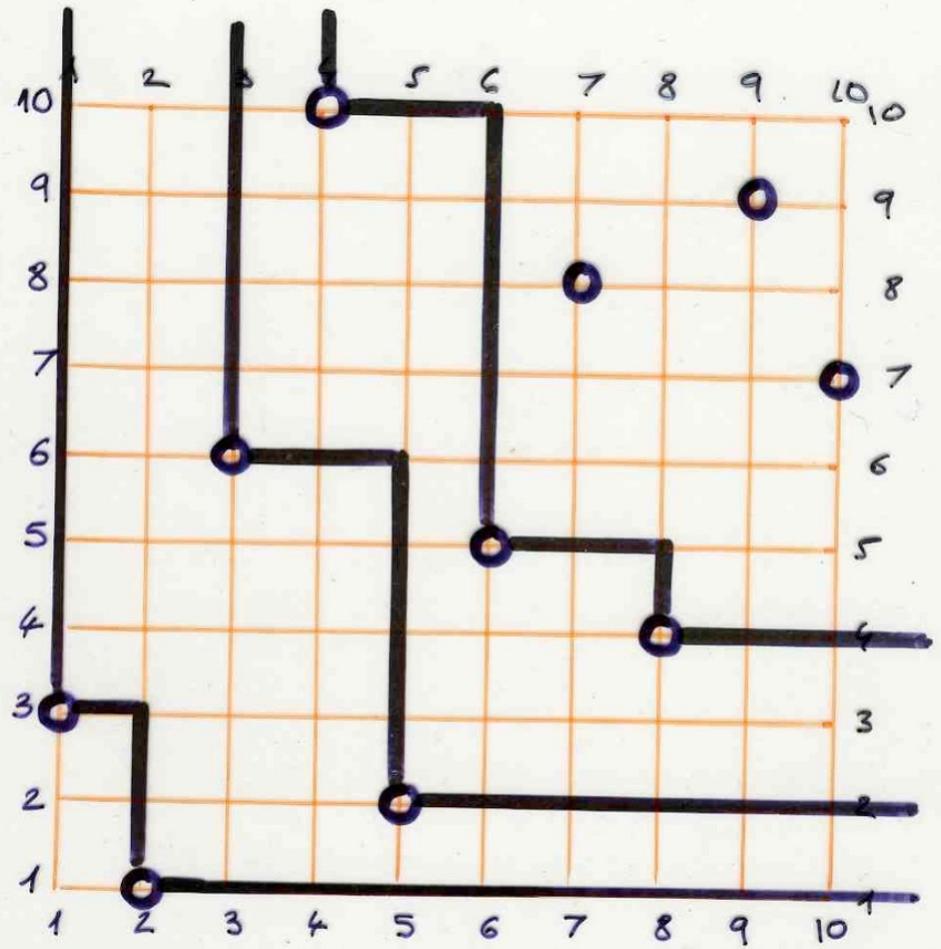




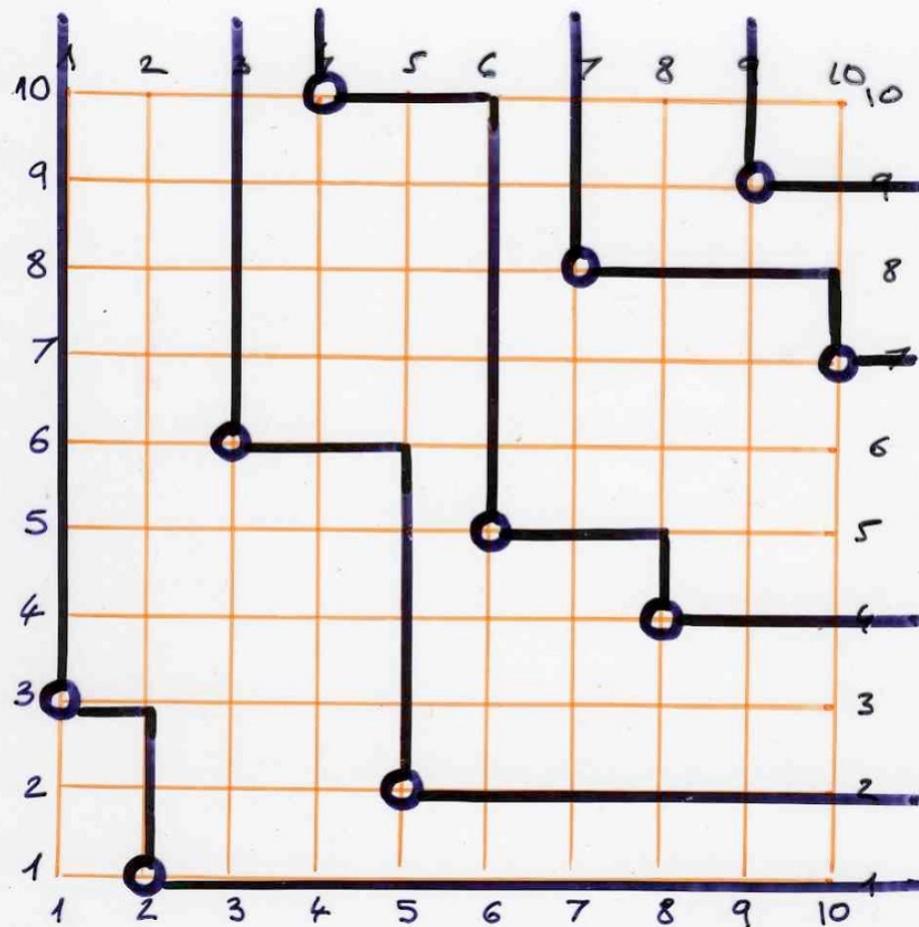
$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



$$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

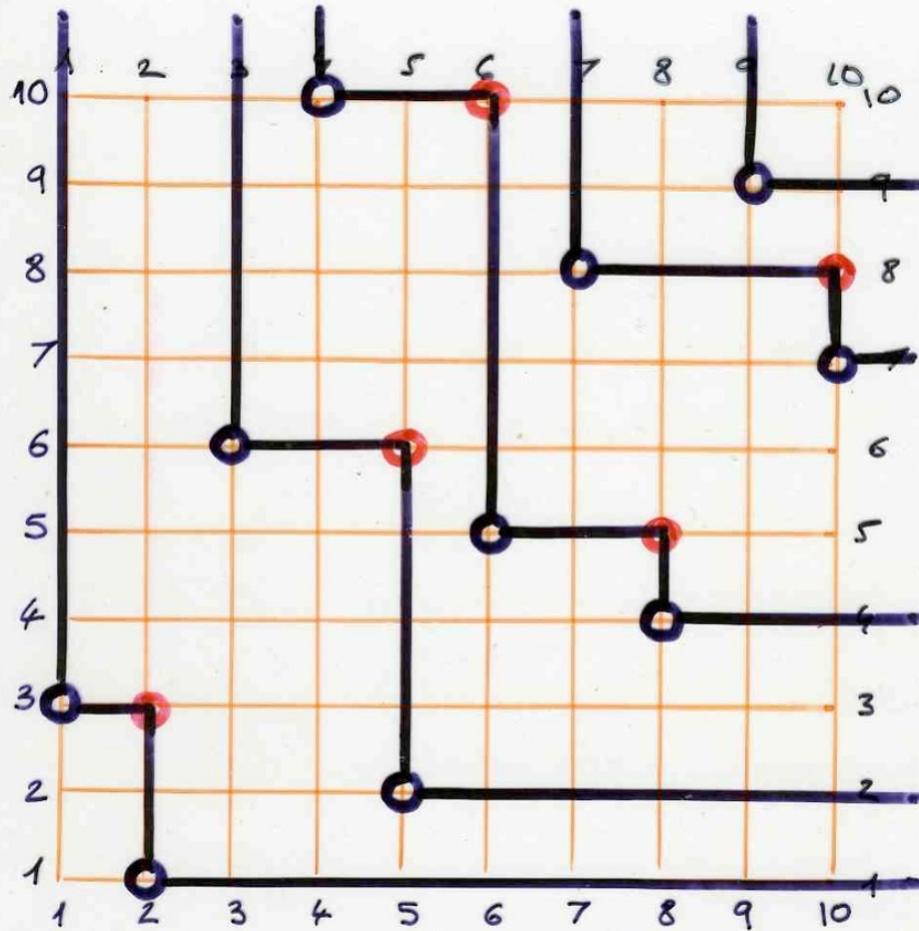


$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



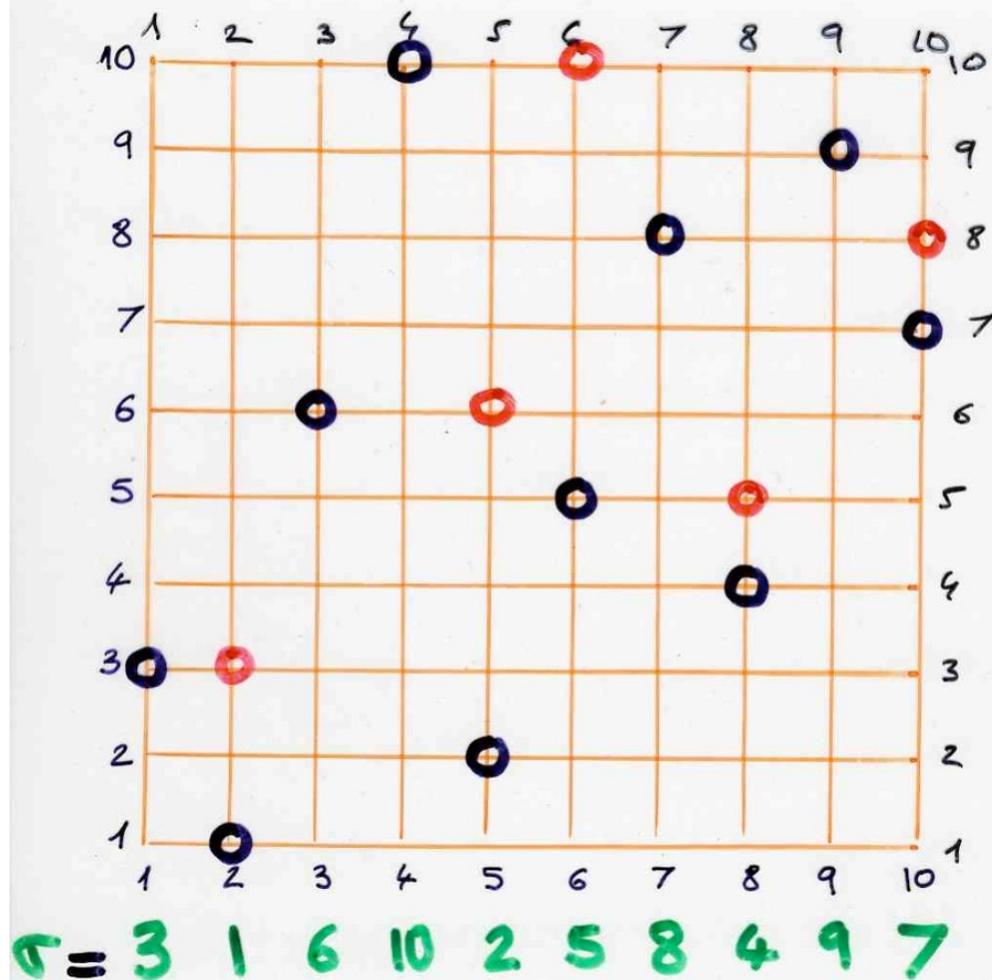
$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

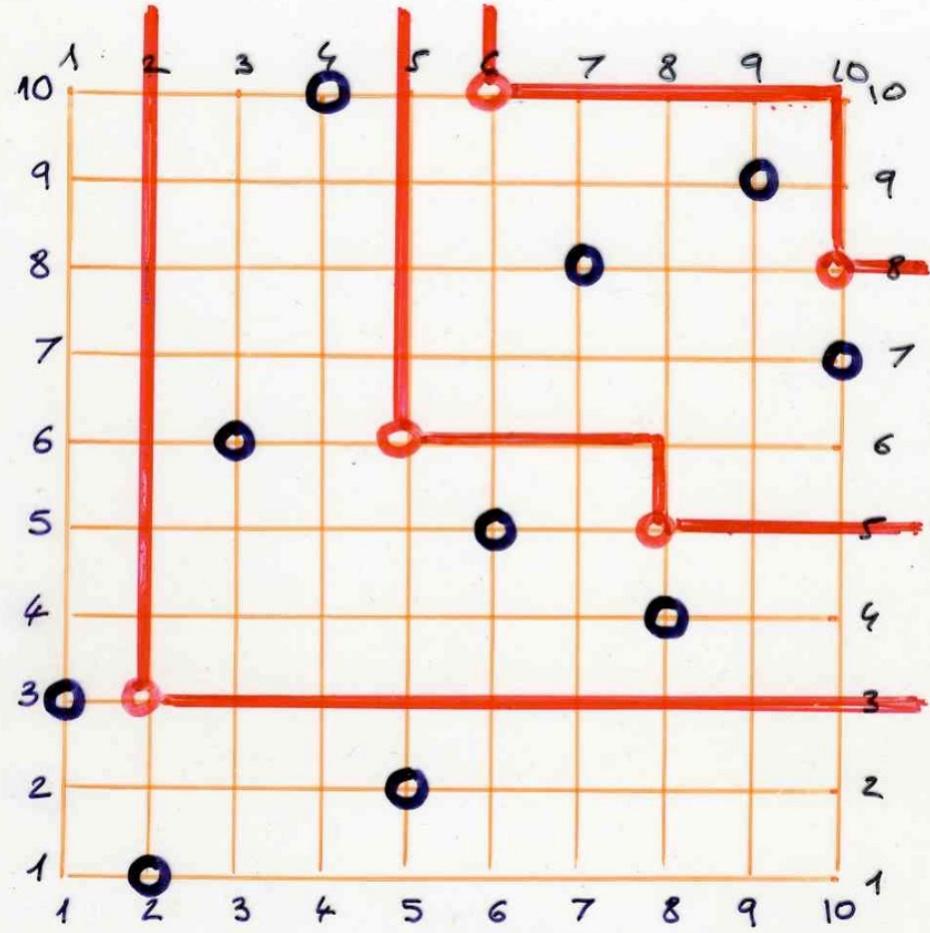
red points ●



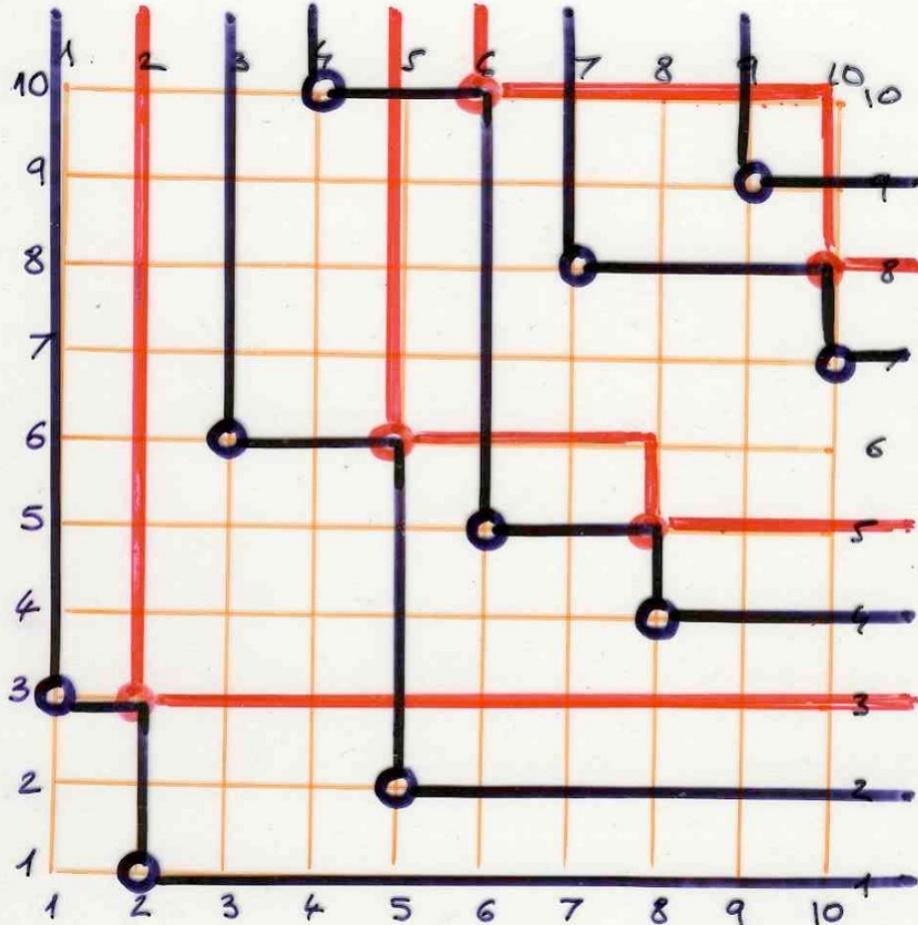
$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

repeat with the red points
 the construction of successive shadows



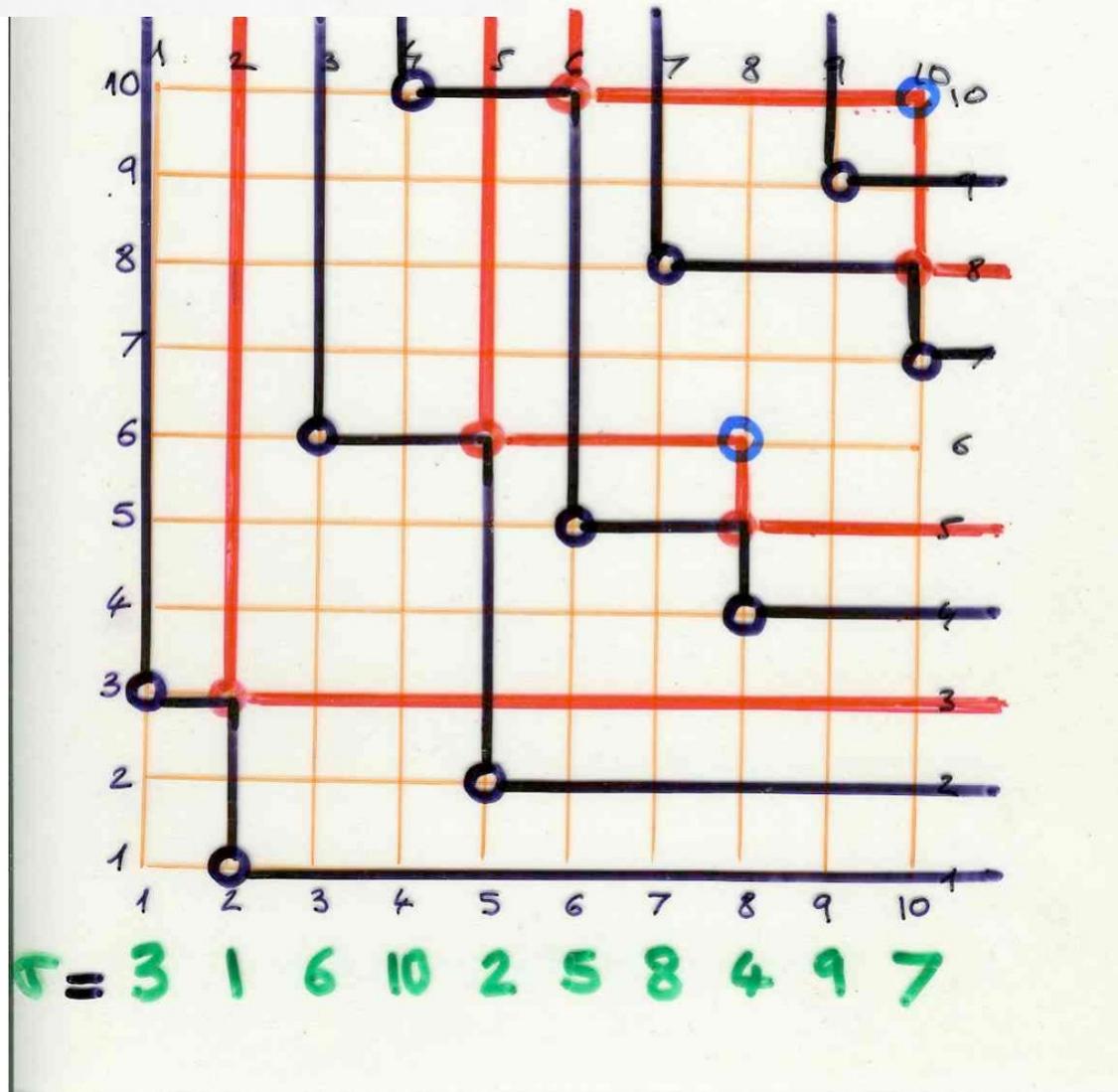


$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

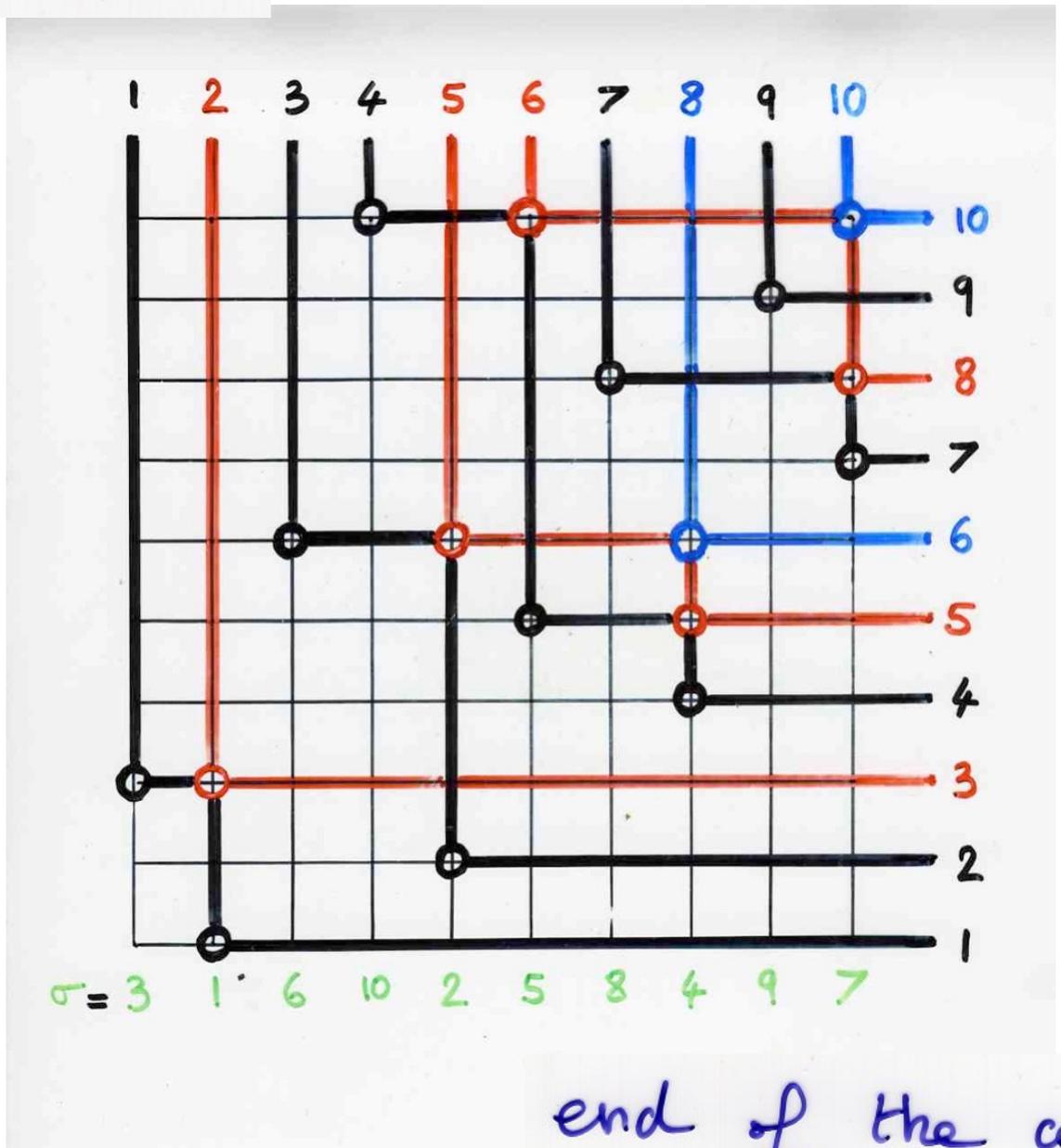


$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

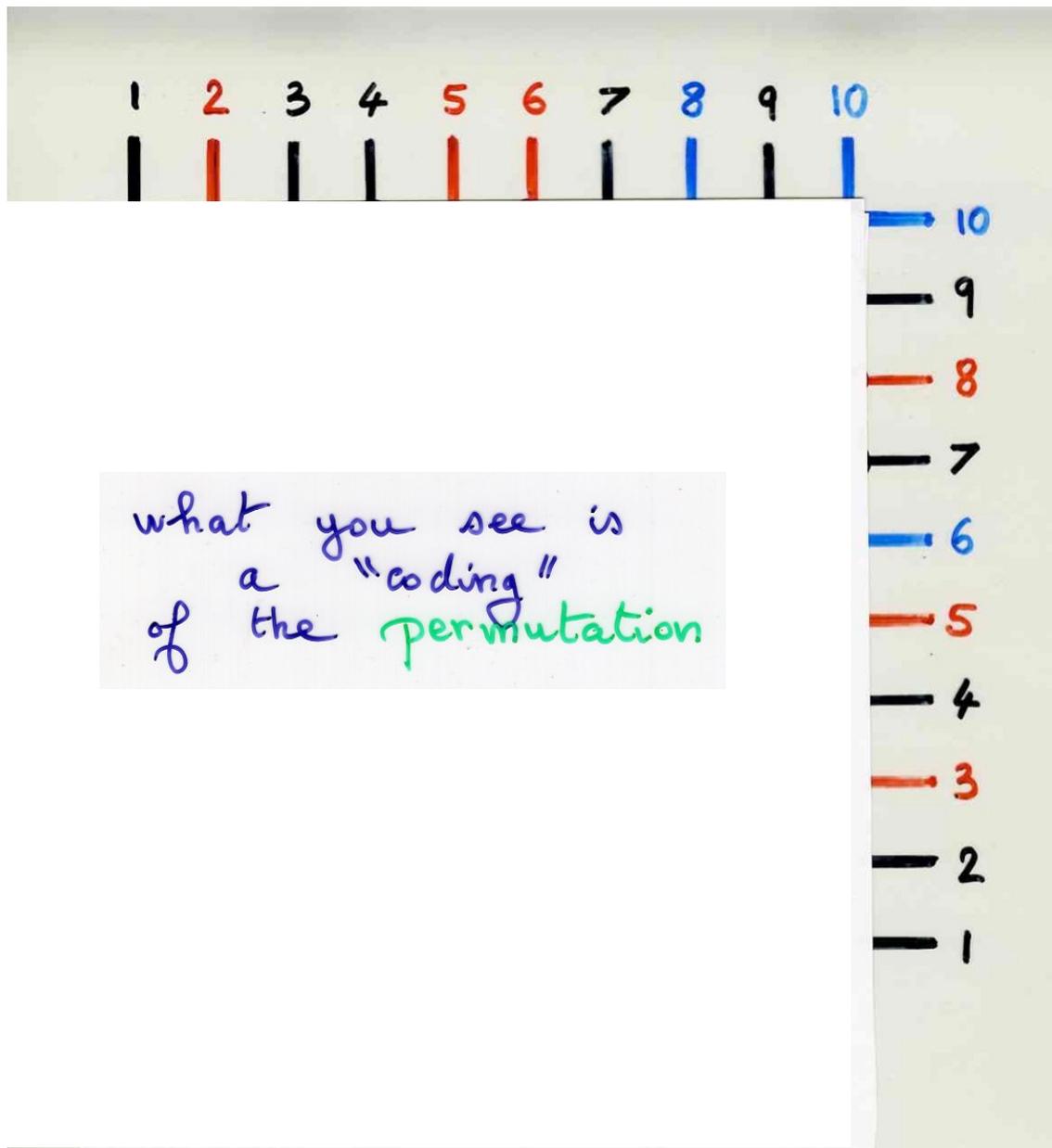
blue points ●

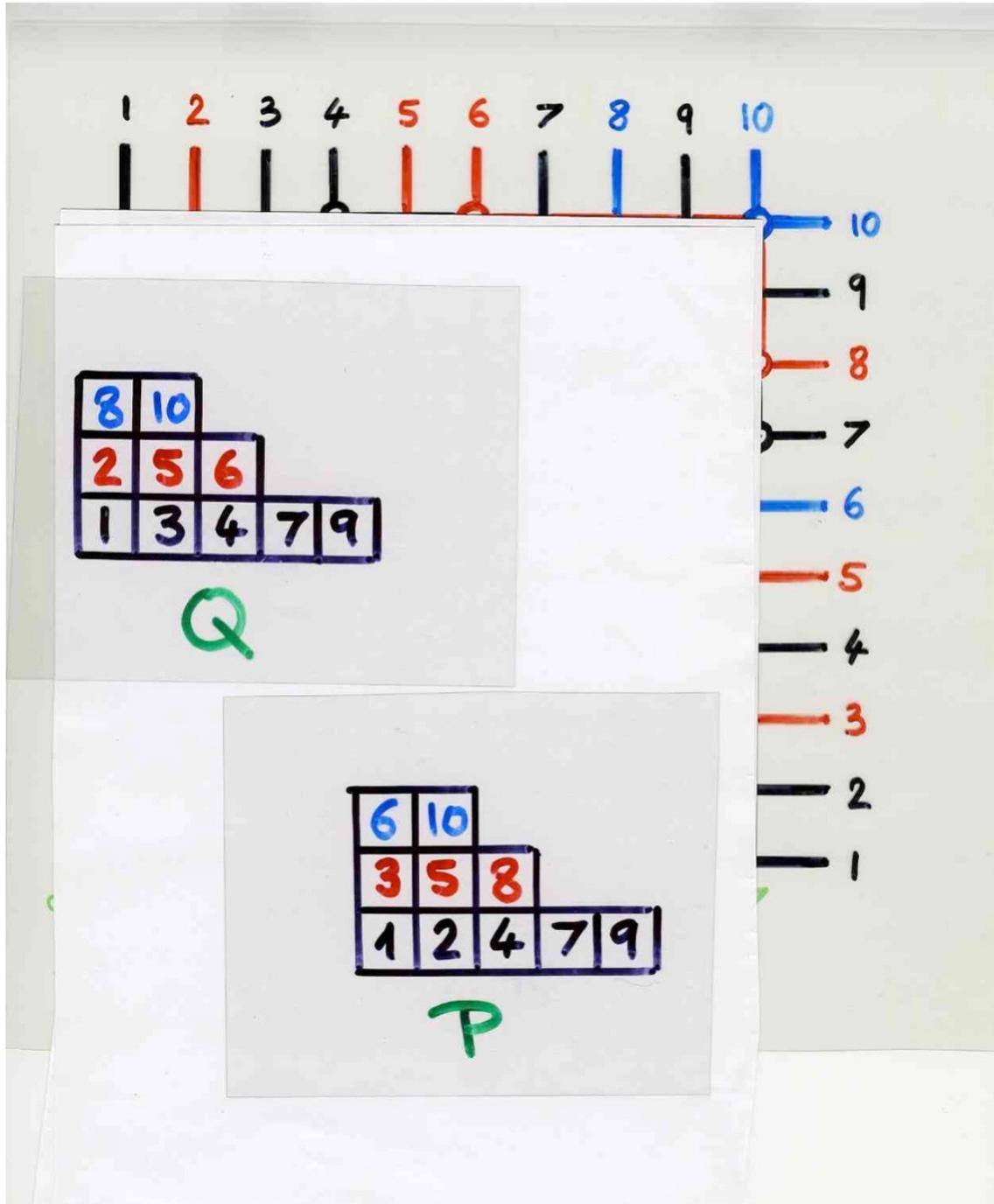


no green points ●

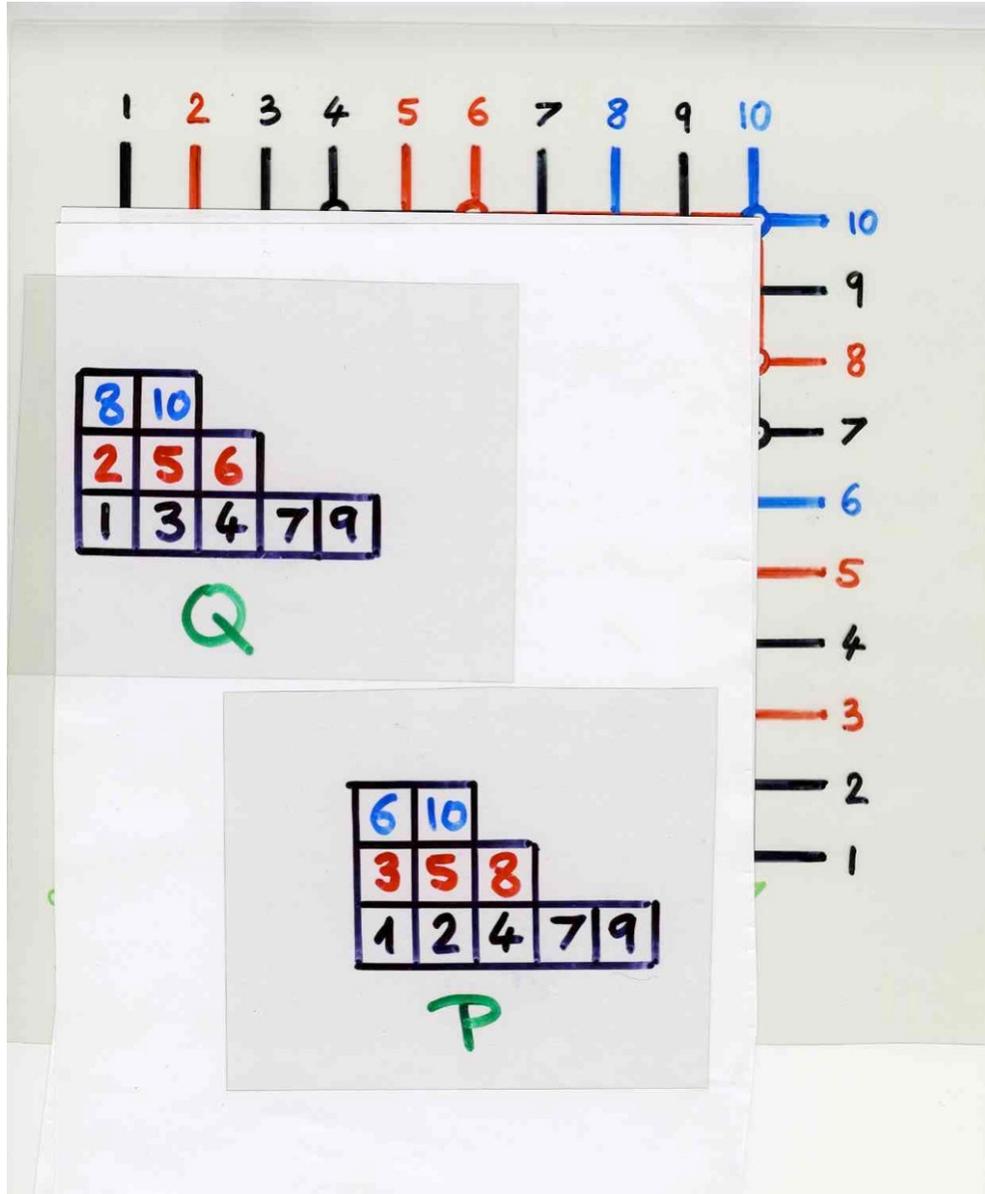


end of the construction





geometric version
with
"light" and "shadow"

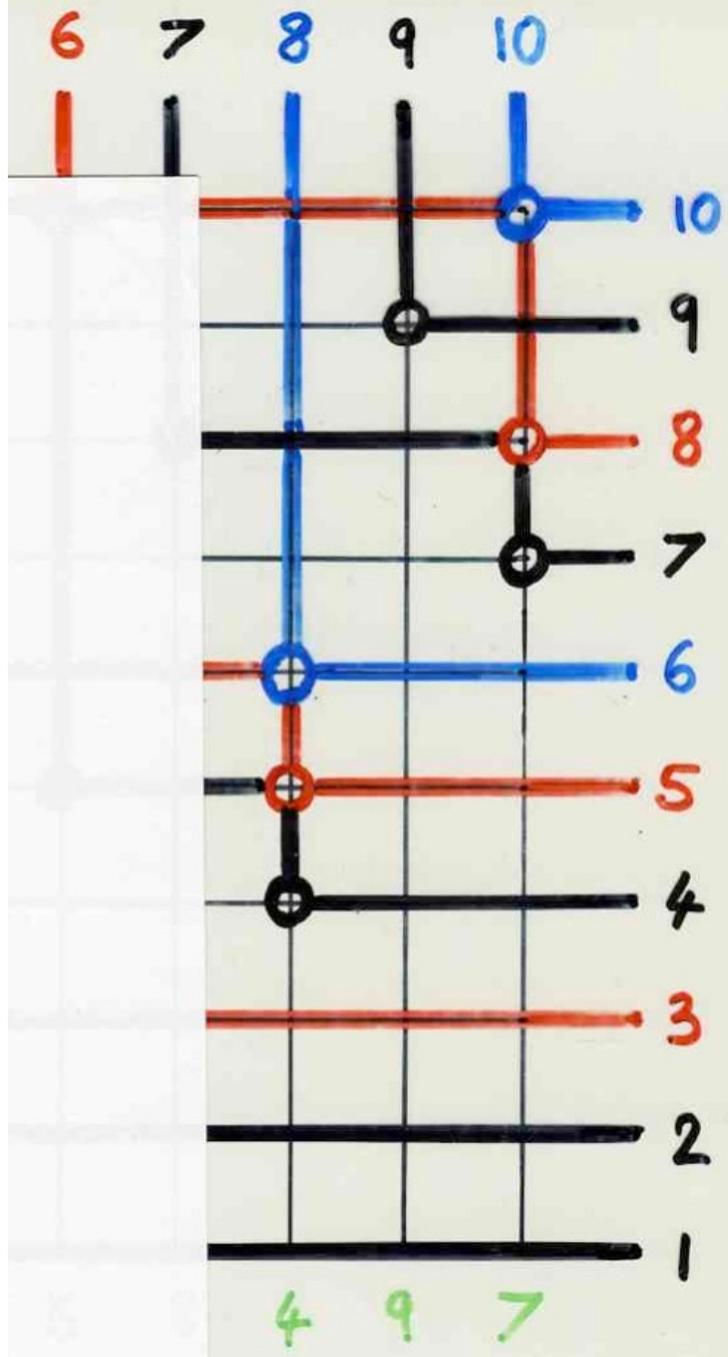


Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

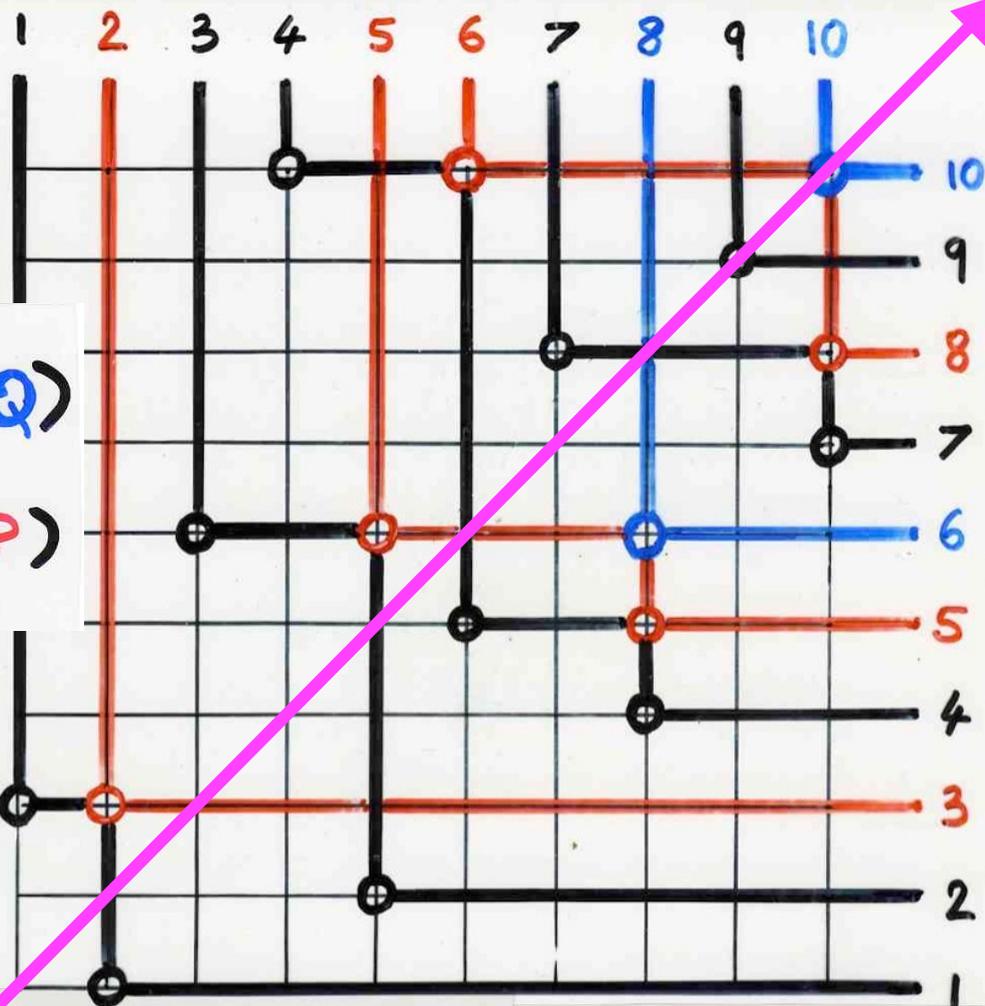
6	10				
3	5	8			
1	2	4	7	9	



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		4



$g \leftrightarrow (P, Q)$
 $g^{-1} \leftrightarrow (Q, P)$

$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence

G. de B. Robinson, 1938

- Schensted insertions algorithm

C. Schensted, 1961

- Geometric version

X.V. 1976

- Growth diagrams

S. Fomin, 1986, 1994

- Edge local rules

- Combinatorial Representation of a quadratic algebra

$$UD = DU + Id$$

the quadratic algebra

$$UD = DU + Id$$

$$UD = DU + Id$$

commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering
in physics

$$UD = DU + Id$$

commutations

$$UD \rightarrow DU$$

$$UD \rightarrow Id$$

rewriting rules

UUDD

$$UUDD = UDUD + UD$$

$$= DUUD + 2UD$$

$$= (DUUD + DU) + 2(DU + Id)$$

$$= (DDUU + 2DU) + 2(DU + Id)$$

$$= DDUU + 4DU + 2Id$$

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

$$UD = DU + Id$$

commutations

$$UD \rightarrow DU$$

$$UD \rightarrow Id$$

rewriting rules

homogenization
of the system
of commutations
relations

$$\begin{cases} UD = DU + I_v I_h \\ UI_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \rightarrow DU \\ UI_v \rightarrow I_v U \\ I_h D \rightarrow D I_h \\ I_h I_v \rightarrow I_v I_h \end{cases}$$

$$UD \rightarrow I_v I_h$$

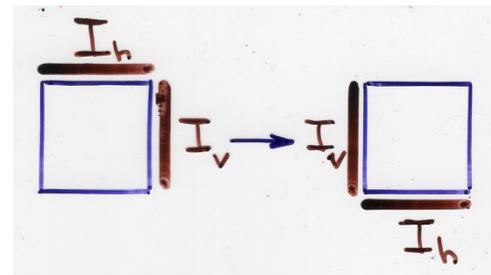
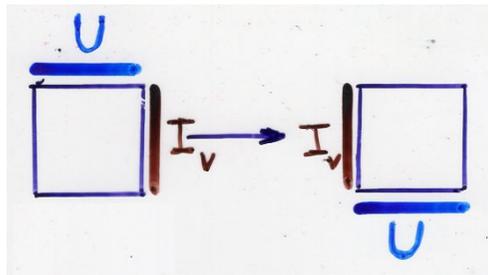
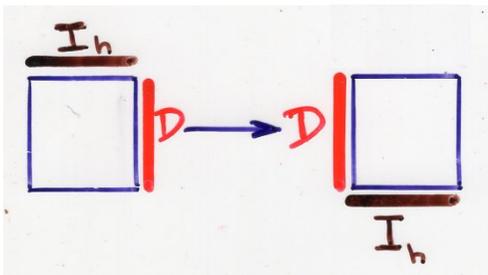
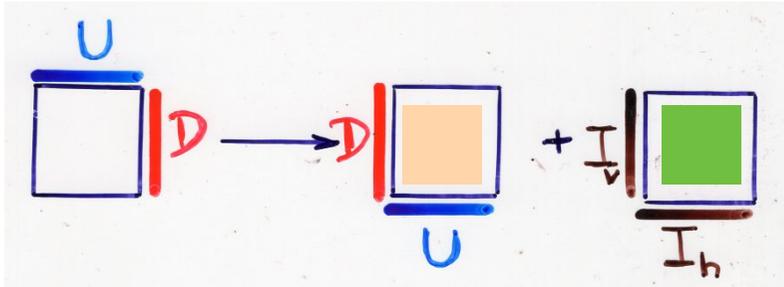
rewriting rules

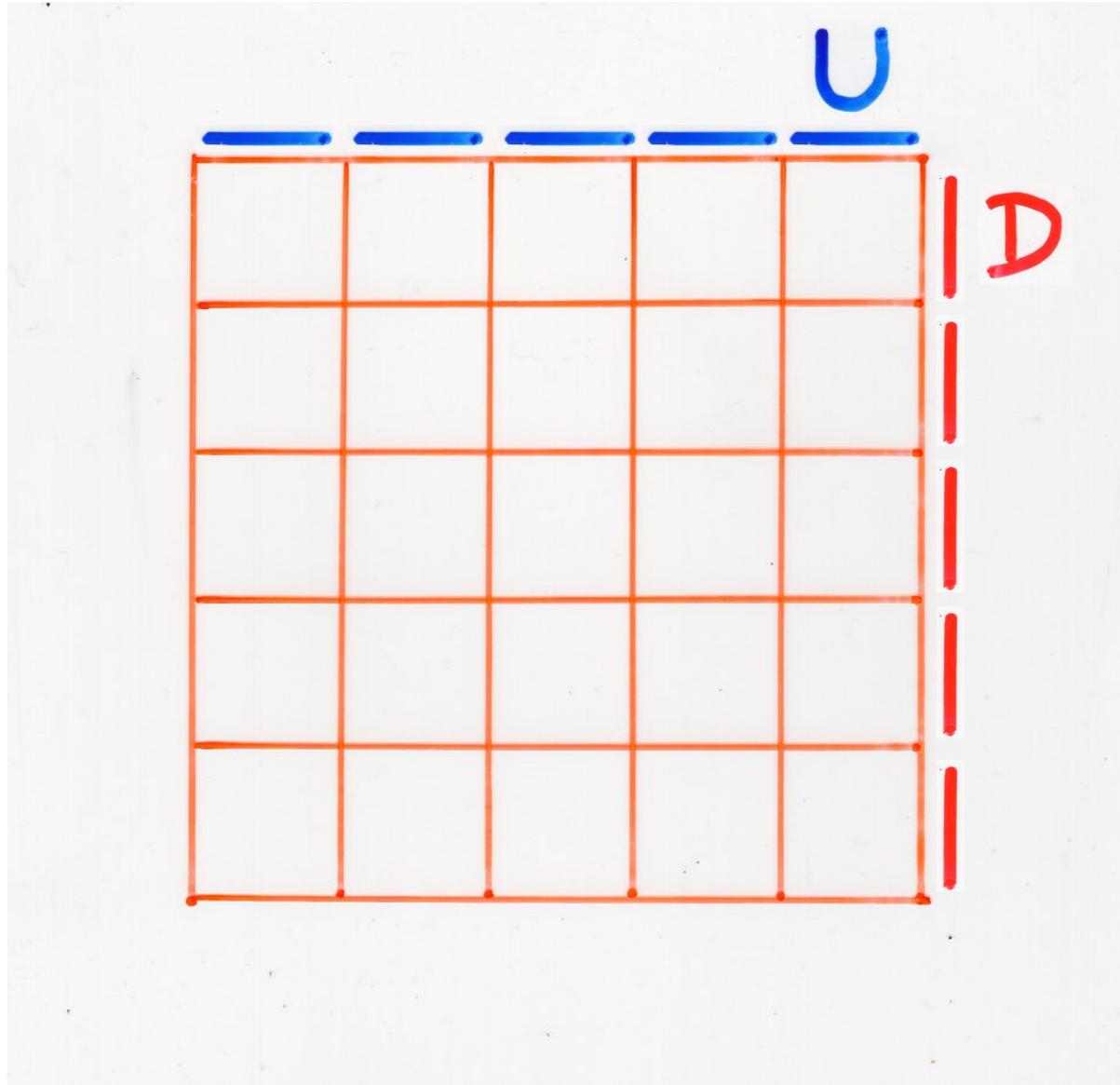
Planarization of the rewriting rules

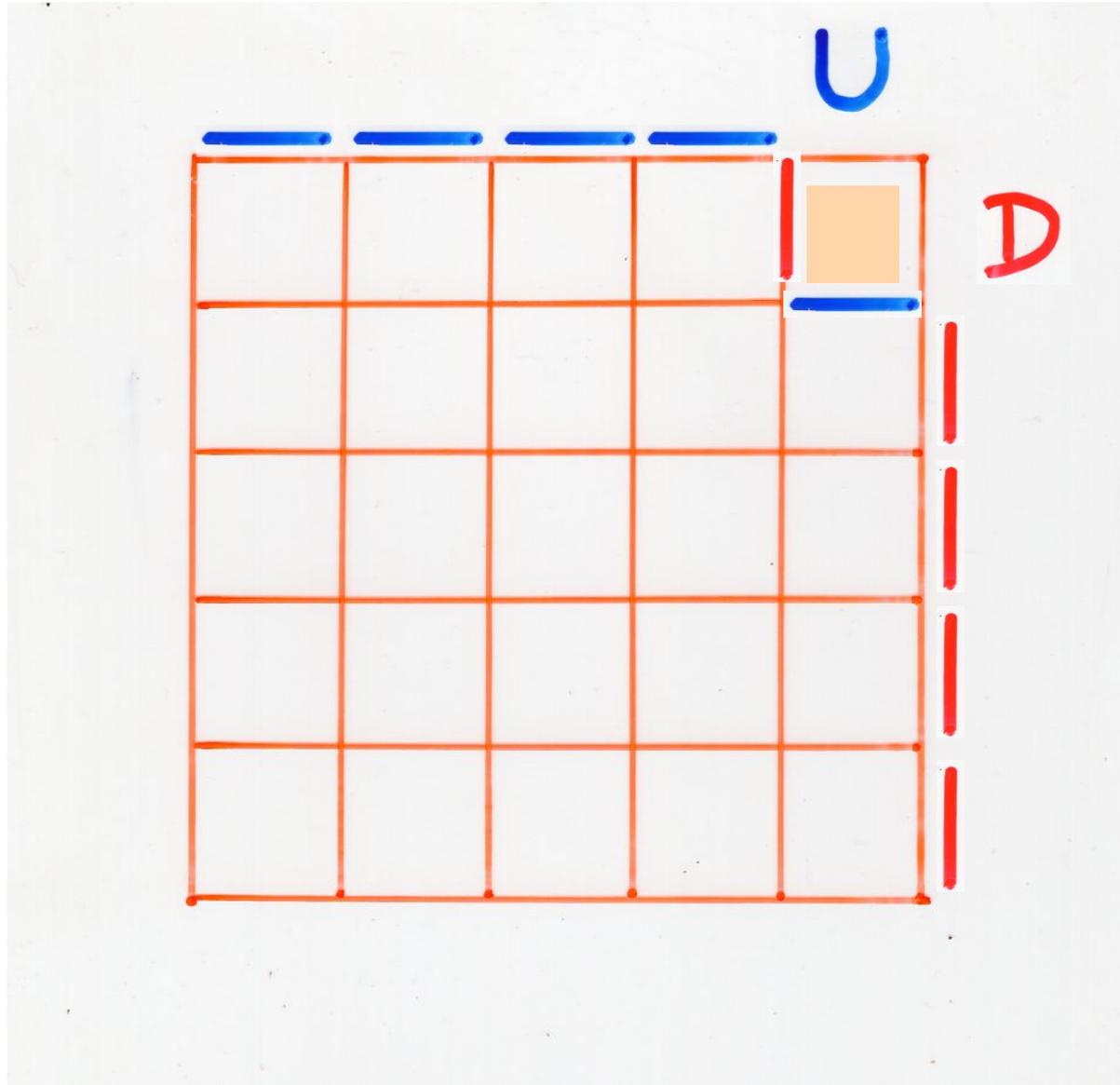
$$\left\{ \begin{array}{l}
 UD \rightarrow DU \\
 UI_v \rightarrow I_v U \\
 I_h D \rightarrow D I_h \\
 I_h I_v \rightarrow I_v I_h
 \end{array} \right. \quad UD \rightarrow I_v I_h$$

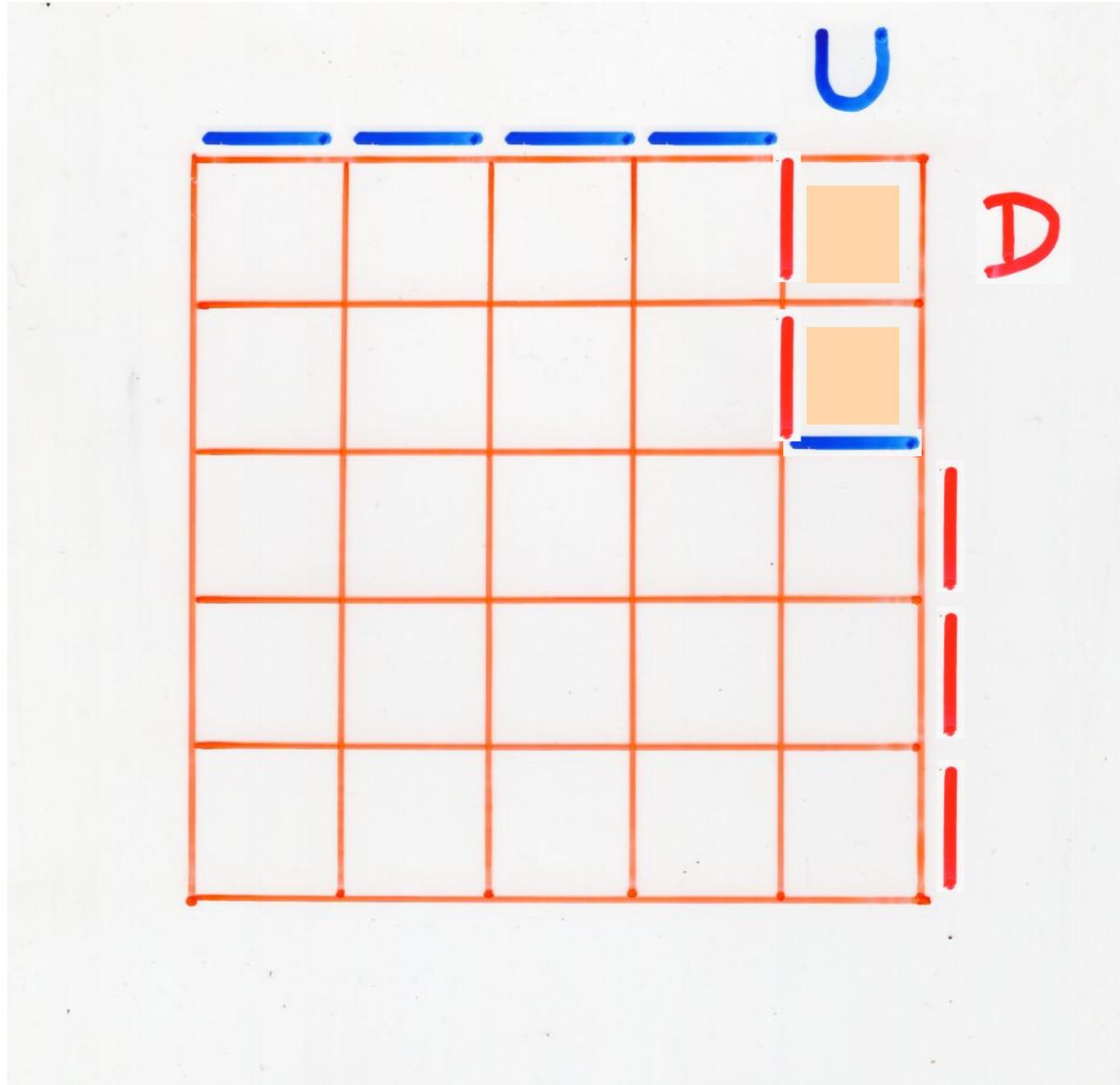
rewriting rules

planarization of the rewriting rules

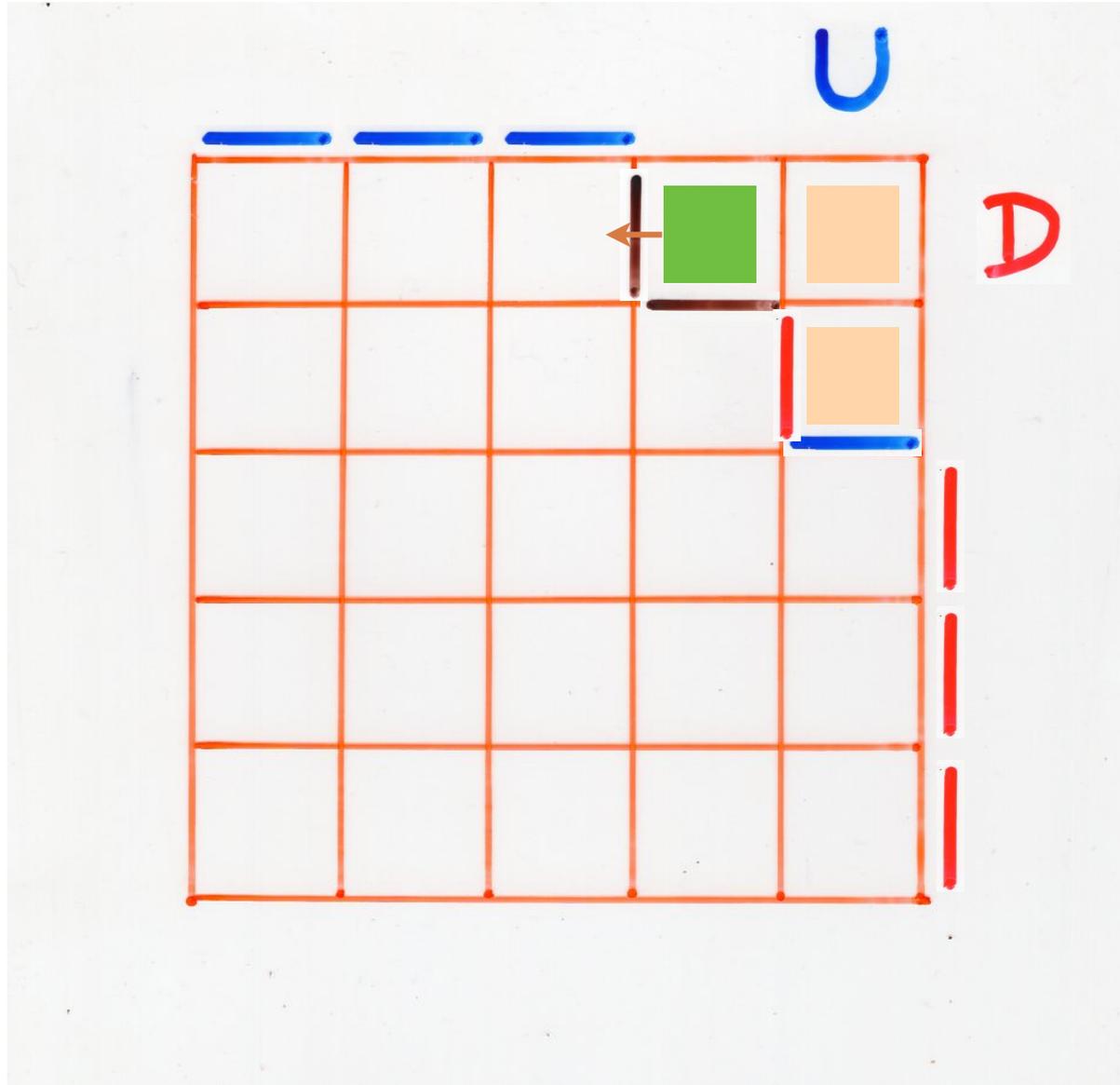




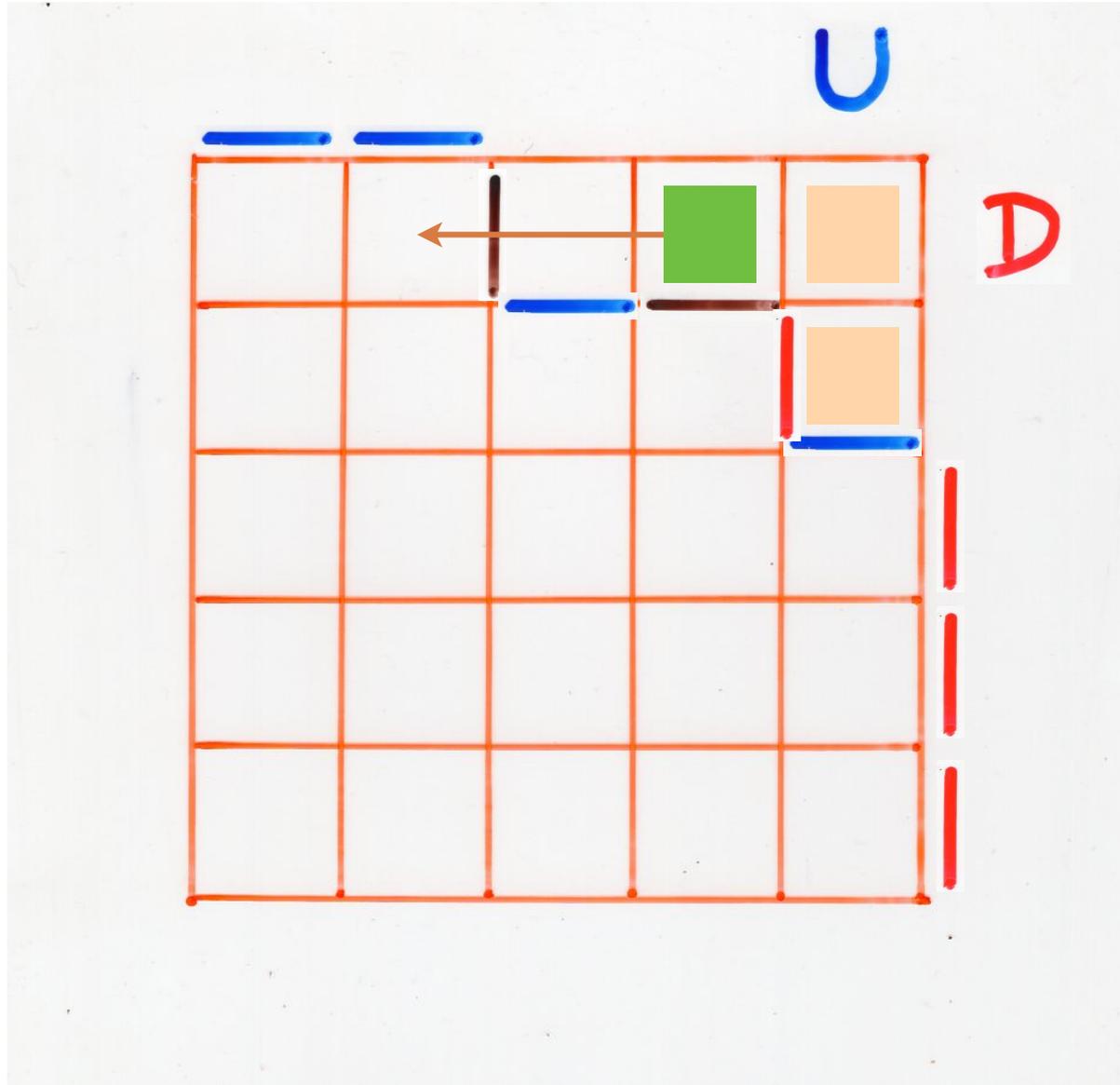


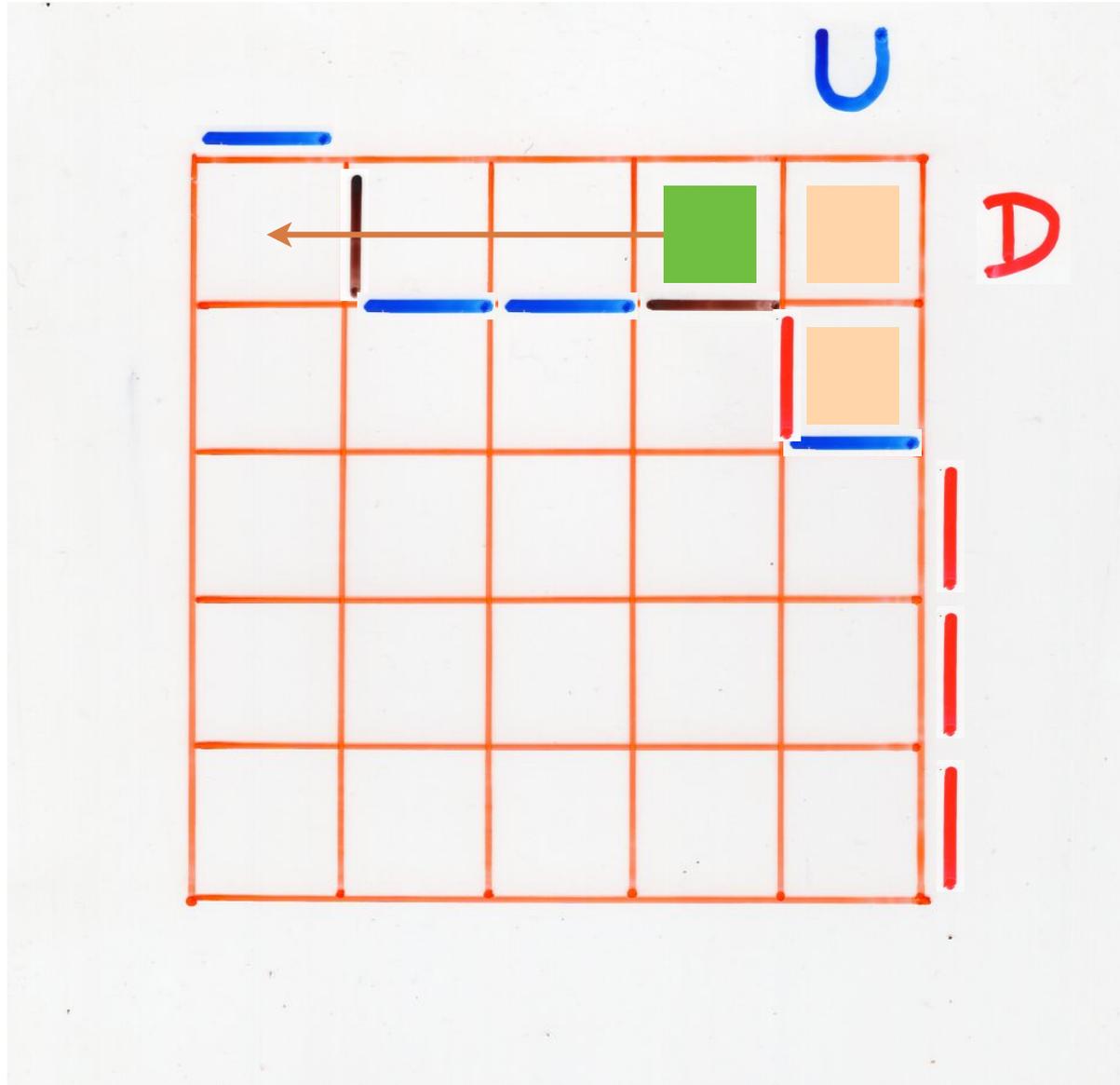


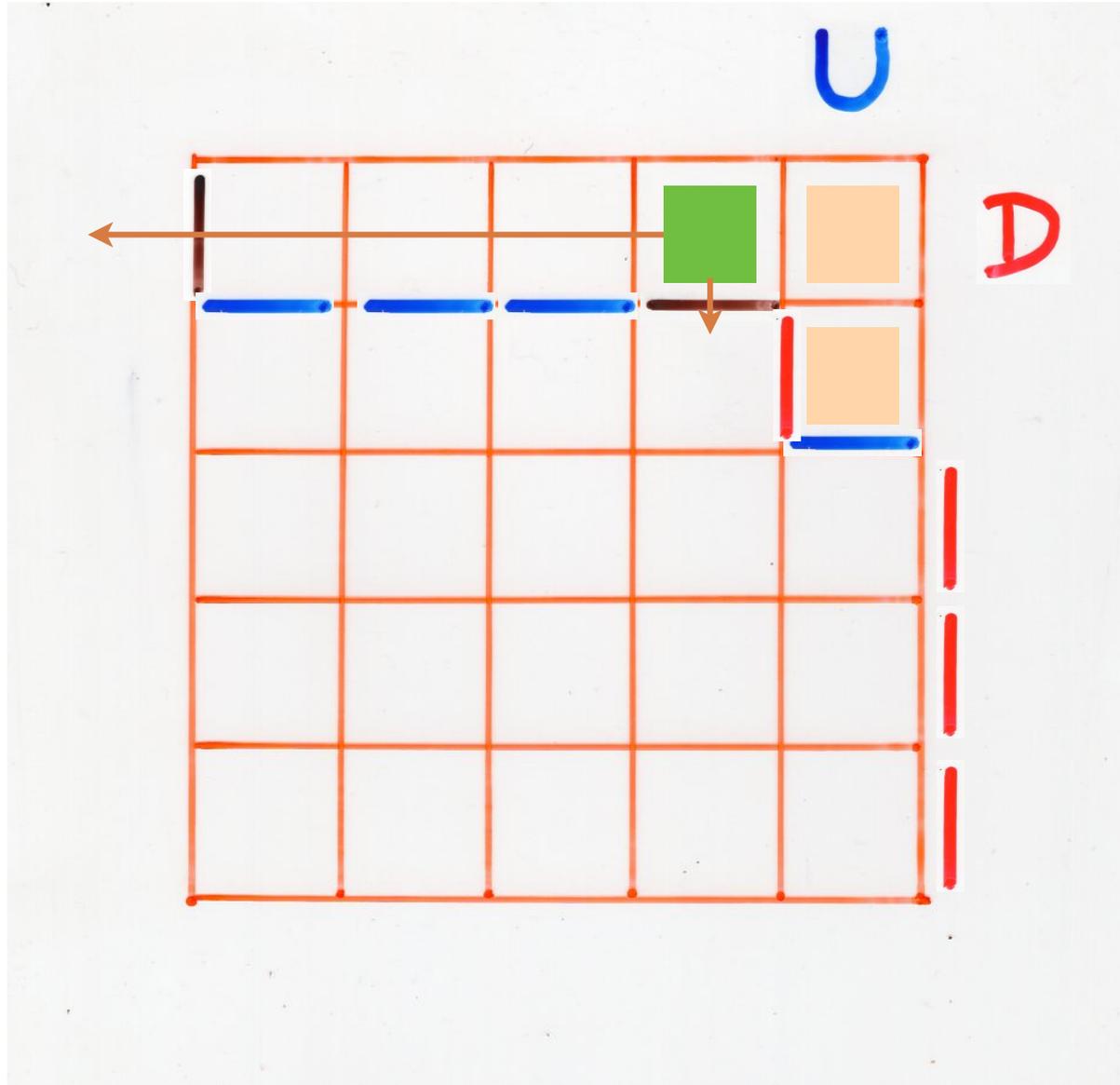
I_h

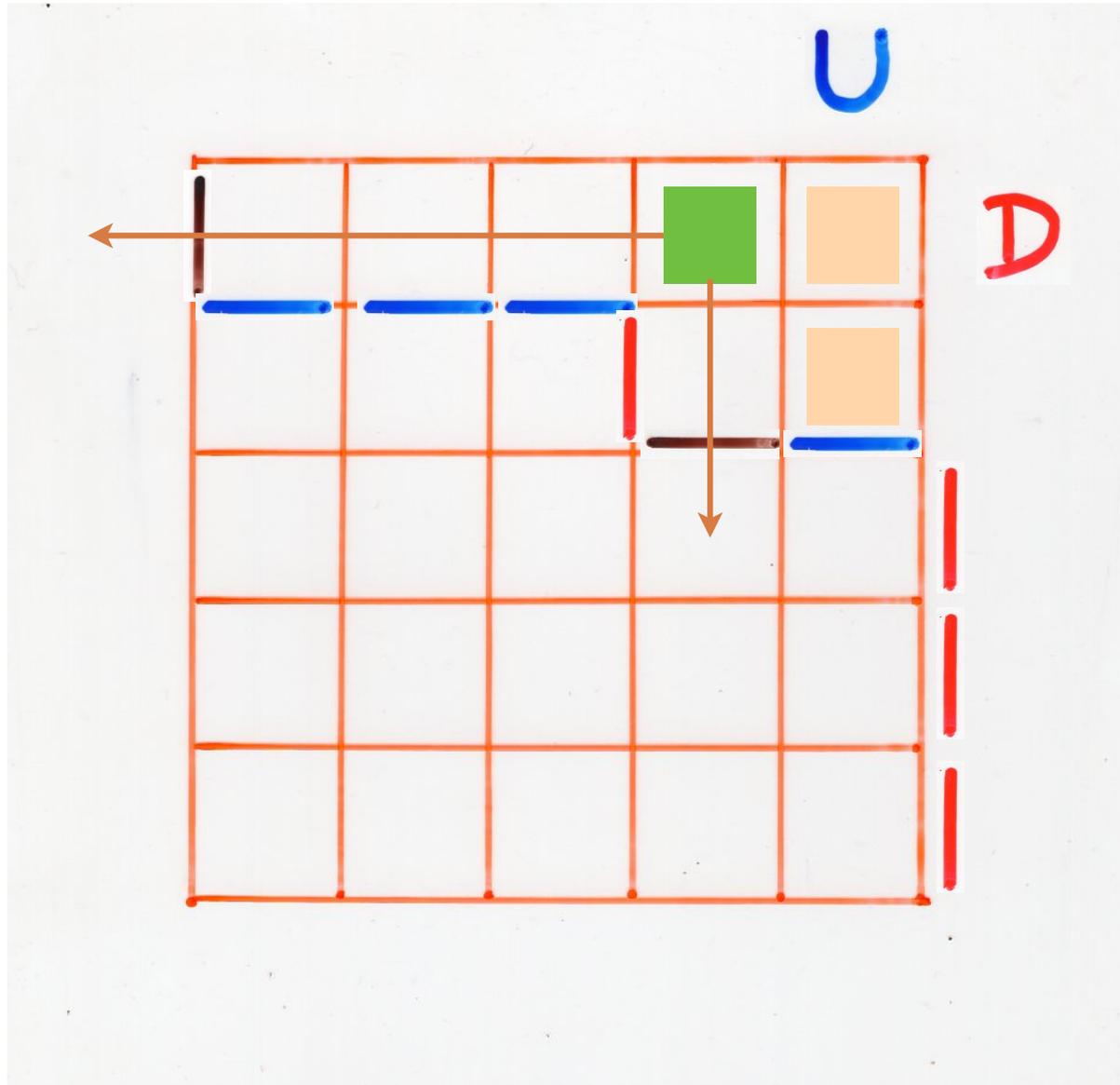


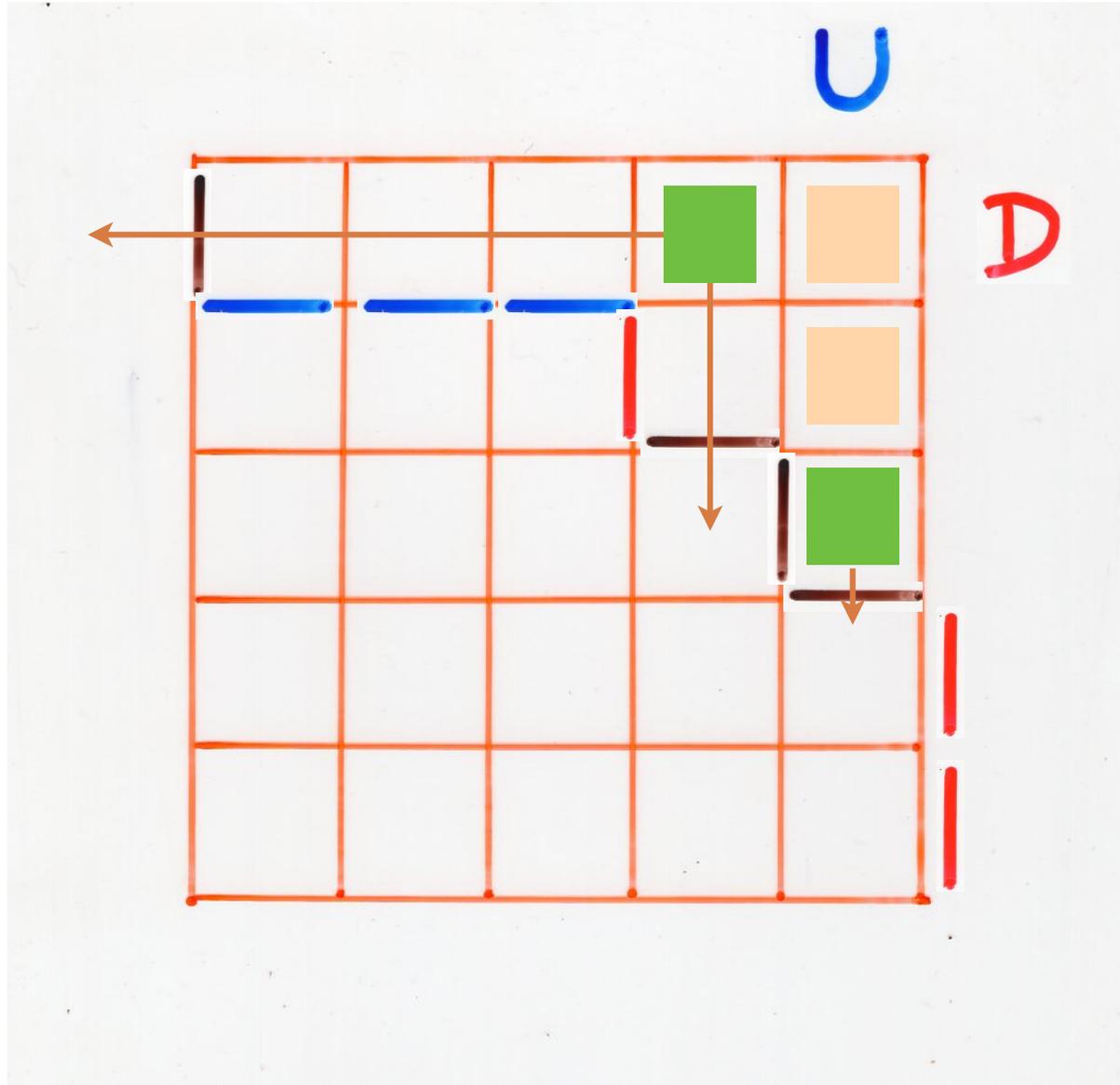
I_v

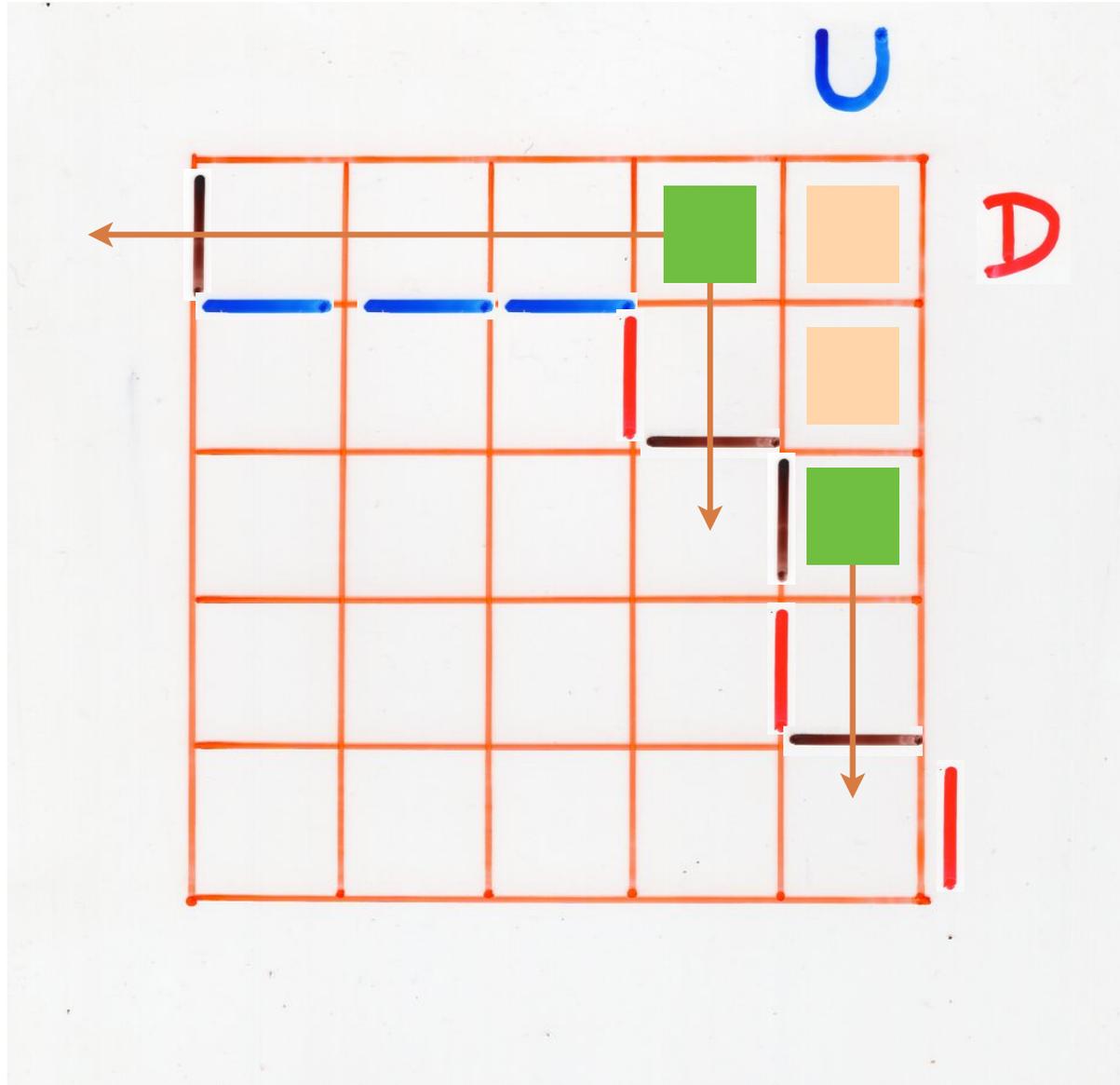


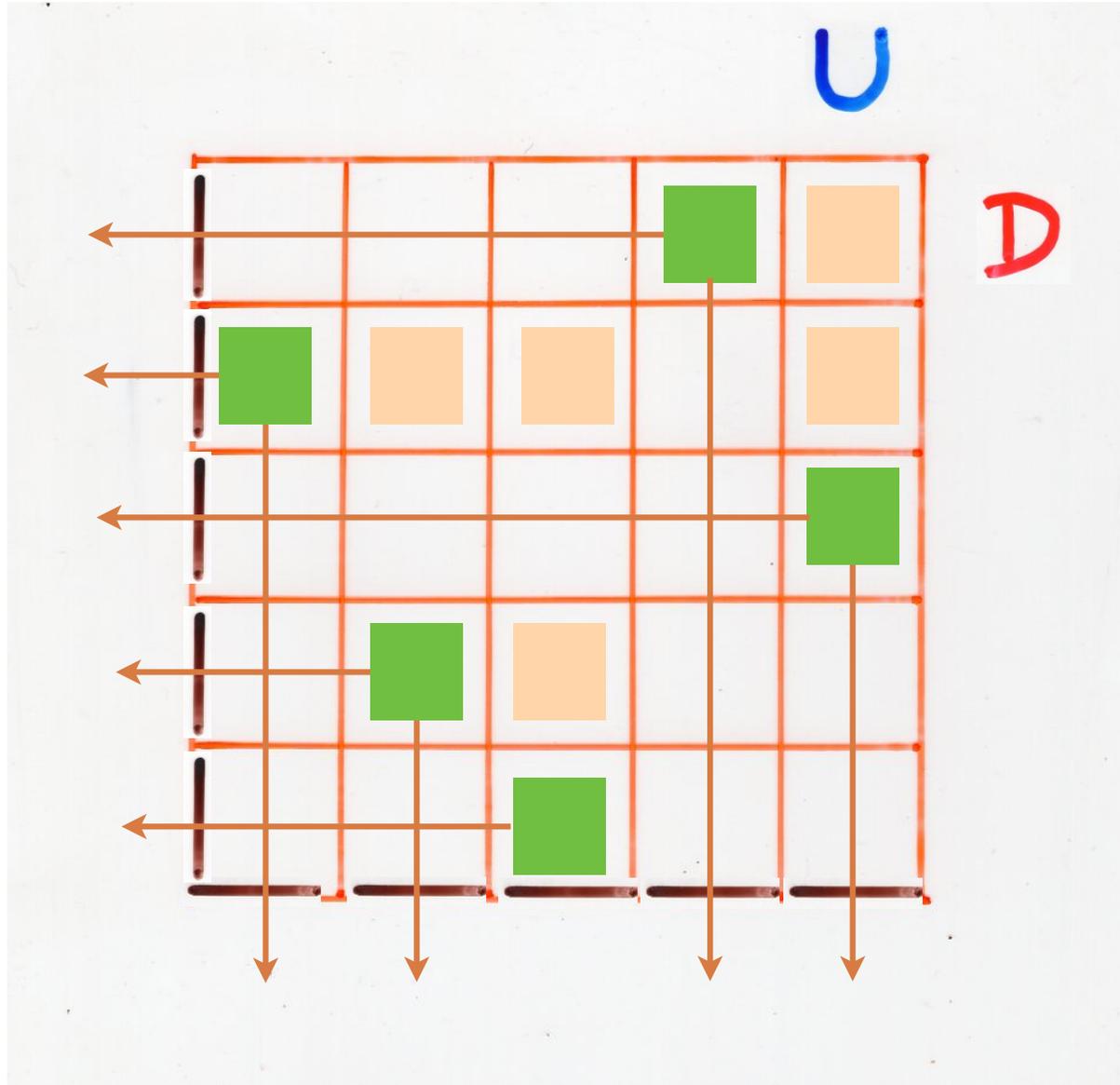












$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

quadratic algebra Q

Q -tableaux

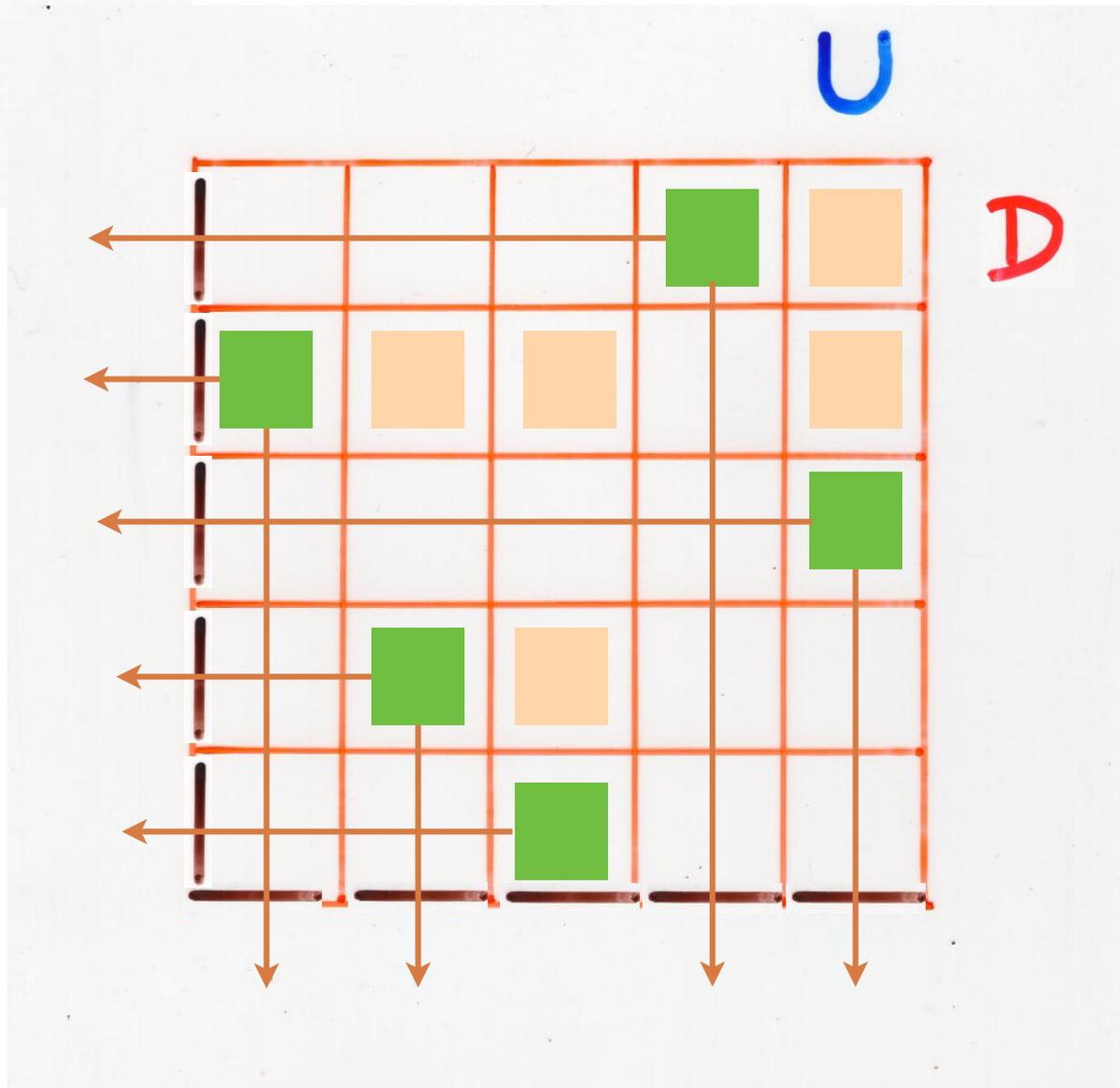
$$UD = DU + Id$$

permutations

complete Q -tableau

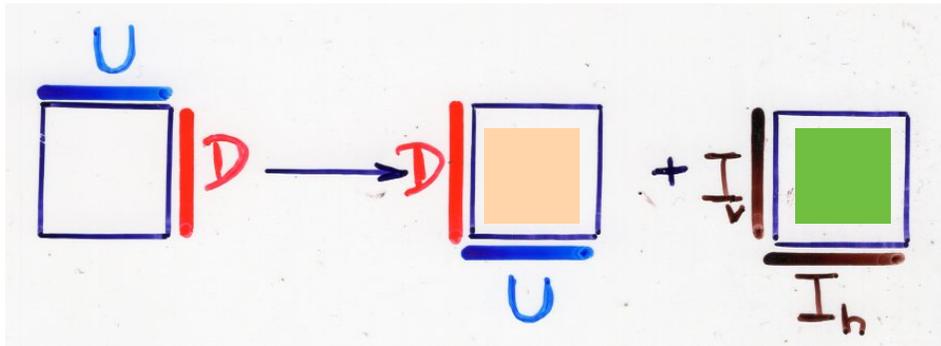
$$\begin{cases} U \mathcal{D} = \mathcal{D} U + I_v I_h \\ U I_v = I_v U \\ I_h \mathcal{D} = \mathcal{D} I_h \\ I_h I_v = I_v I_h \end{cases}$$

complete Q -tableau

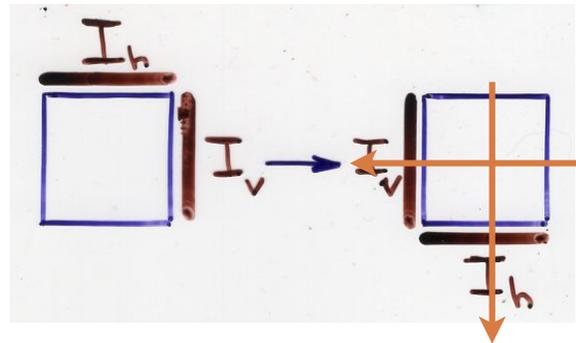
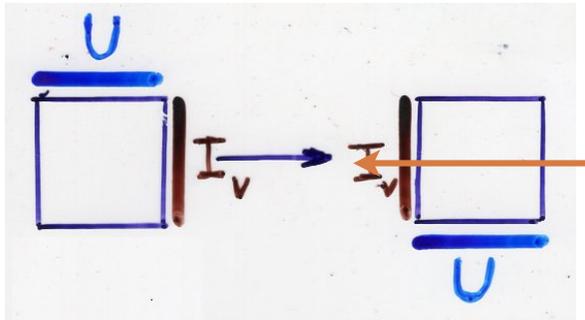
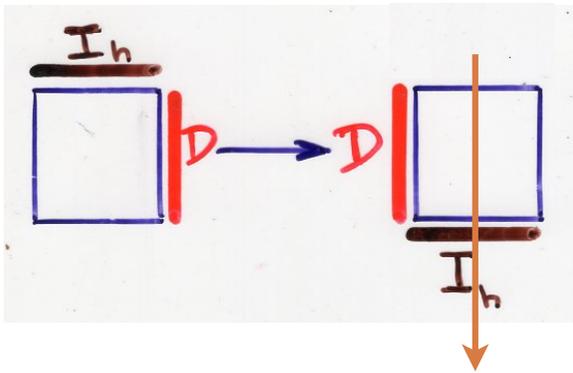


$$\begin{cases} U \mathcal{D} = q \mathcal{D} U + I_v I_h \\ U I_v = I_v U \\ I_h \mathcal{D} = \mathcal{D} I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$U \mathcal{D} = q \mathcal{D} U + I$$

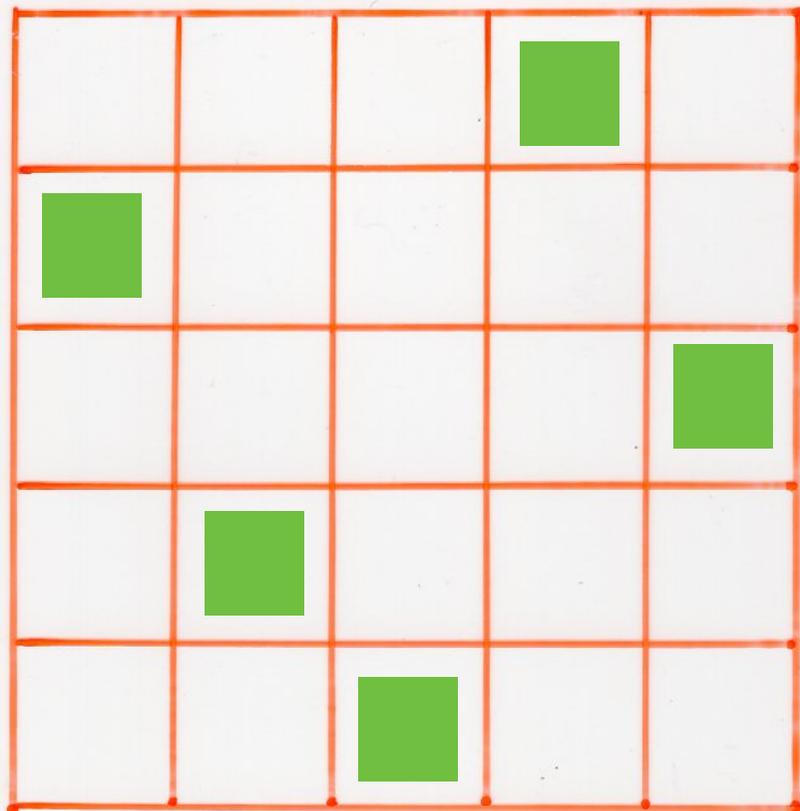


"complete"
Q-tableau



quadratic algebra \mathcal{Q}

$$UD = qDU + I$$



permutation
as a \mathcal{Q} -tableau

Extension

Q quadratic algebra

complete Q-tableaux

Q-tableaux

quadratic algebra

Q

generators

$$\mathcal{B} = \{B_j\}_{j \in J}$$
$$\mathcal{A} = \{A_i\}_{i \in I}$$

for every $i \in I$
 $j \in J$

$$\mathcal{A} \cap \mathcal{B} = \emptyset$$

commutations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l$$

Lemma In Q every word $w \in (A \cup B)^*$ can be written in a unique way

$$w = \sum_{\substack{u \in A^* \\ v \in B^*}} c(u, v; w) uv$$

The monomials

$$\{uv, u \in A^*, v \in B^*\}$$

form a basis of the algebra

$$Q = \mathbb{C} \langle A+B \rangle / \mathcal{J}$$

non-commutative polynomials
with variables $A+B$

$$(A \cup B)$$

\mathcal{J} ideal generated by
the commutations relations

This polynomial can be obtained by successive rewriting rules from w

$$B_j A_i \rightarrow \sum c_{ij}^{kl} A_k B_l$$

until there is no more such occurrence

Lemma This polynomial is independent of the order of rewritings

quadratic
algebra \mathbb{Q}

complete \mathbb{Q} -tableau

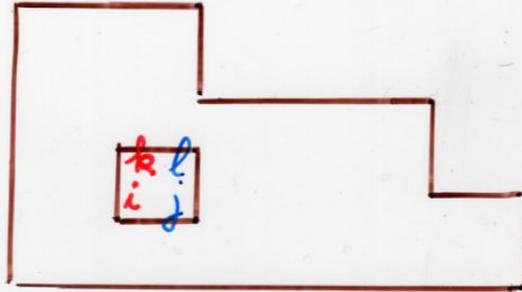
R set of rewriting rules

$$B_j A_i \rightarrow c_{ij}^{kl} A_k B_l$$

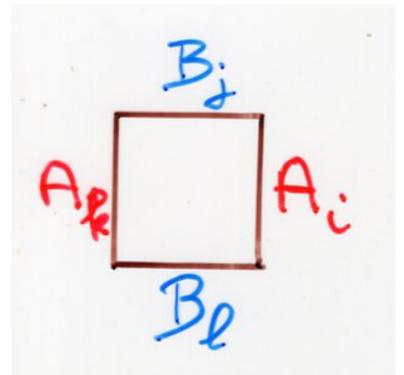
Definition

complete Q -tableau

Ferrers diagram F
where each cell is
labeled by the set
 R of rewriting rules
with "compatibility" condition



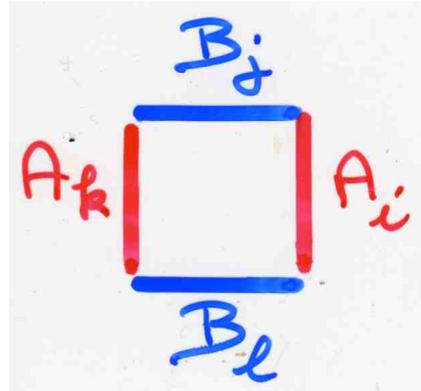
or



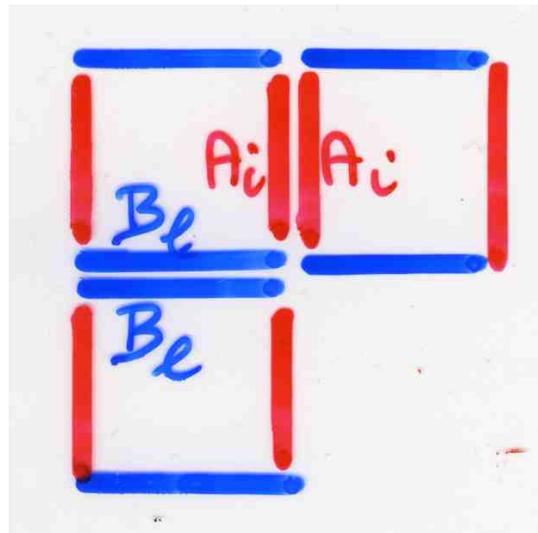
R set of rewriting rules

$$B_j A_i \rightarrow c_{ij}^{kl} A_k B_l$$

Wang tile



Wang tiling



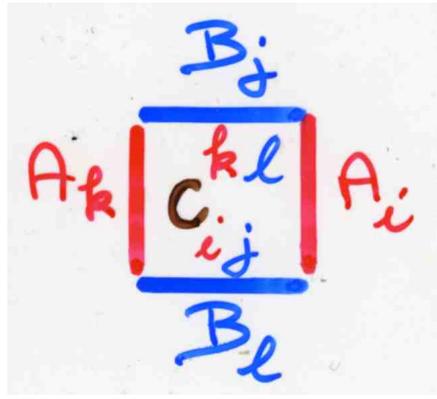
Definition

complete Q -tableau

Wang tiling

of the Ferrers
diagram F

weight of
a Wang tile



$$B_j A_i \rightarrow c_{ij}^{kl} A_k B_l$$

Definition

weight of a complete Q -tableau T

$$\text{wgt}(T) = \prod_{\substack{\text{cells} \\ \text{of } F}} c_{ij}^{kl} \in \mathbb{K}[X]$$

Lemma In Q every word $w \in (d \cup \beta)^*$ can be written in a unique way

$$w = \sum_{\substack{u \in d^* \\ v \in \beta^*}} c(u, v; w) uv$$

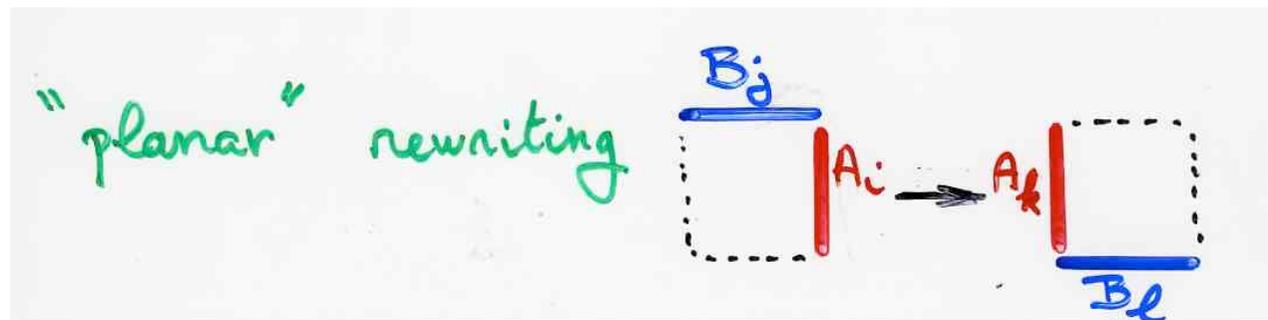
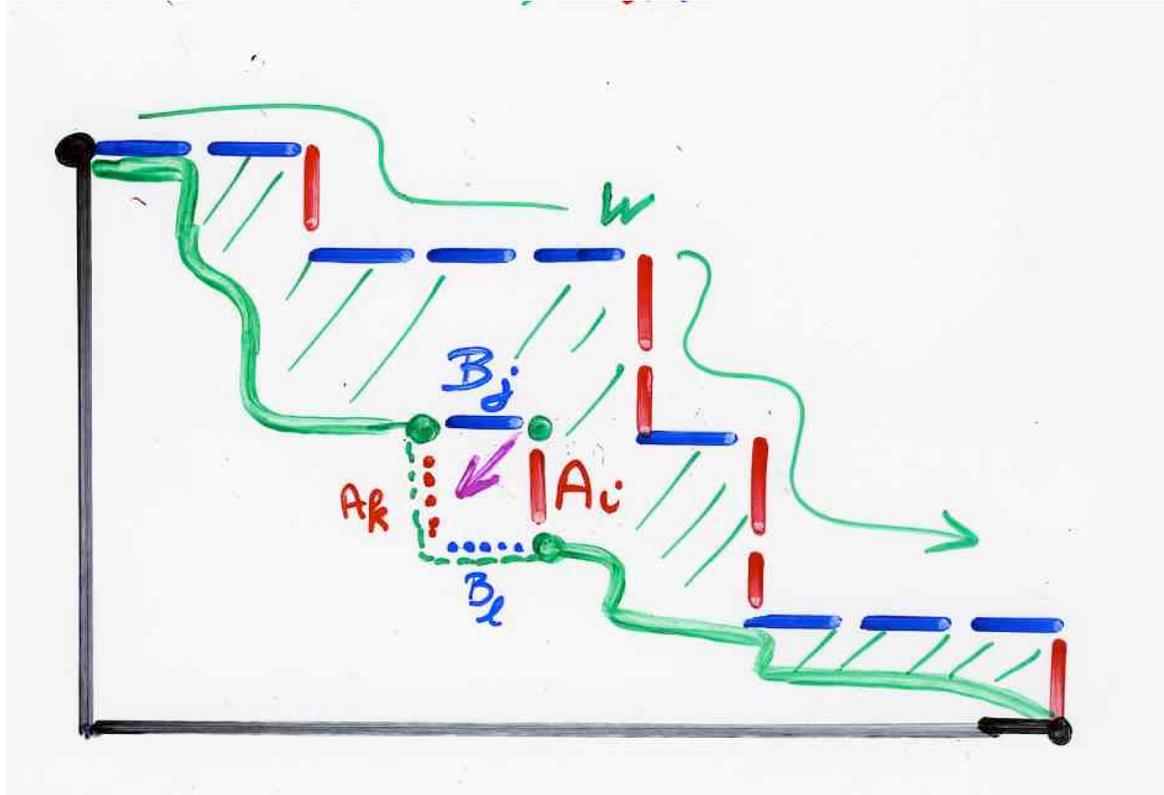
Proposition For any words $w \in (d \cup \beta)^*$, $u \in d^*$, $v \in \beta^*$

$$c(u, v; w) = \sum_{\mathbf{T}} \text{wgt}(\mathbf{T})$$

complete Q -tableau

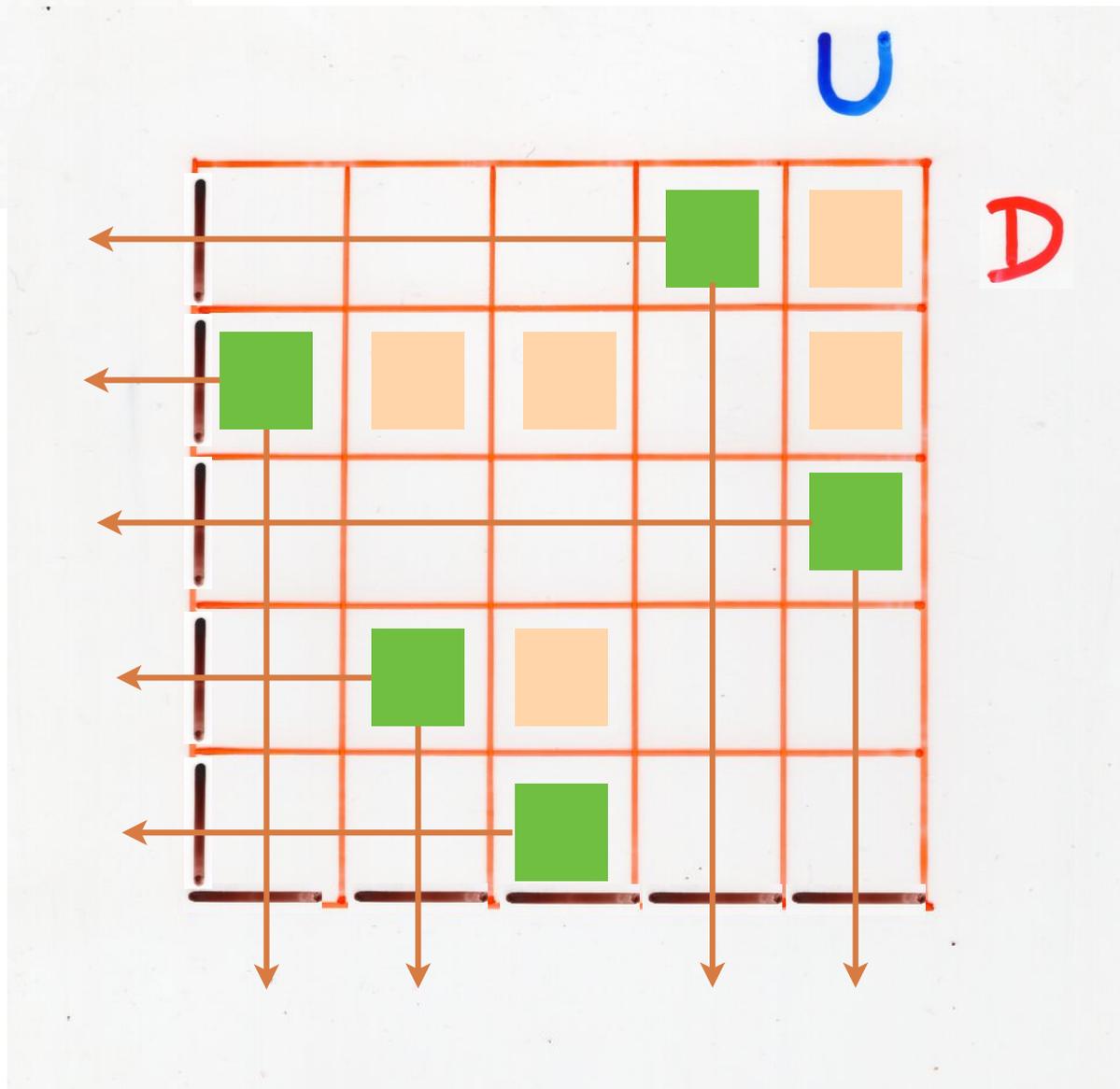
$$uwb(\mathbf{T}) = w$$
$$lwb(\mathbf{T}) = uv$$

complete skew Q -tableau
between w and w



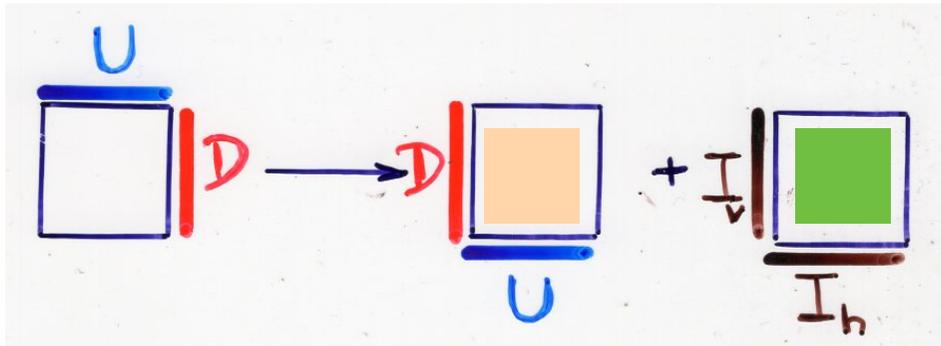
$$\begin{cases} U \mathcal{D} = \mathcal{D} U + I_v I_h \\ U I_v = I_v U \\ I_h \mathcal{D} = \mathcal{D} I_h \\ I_h I_v = I_v I_h \end{cases}$$

complete Q -tableau

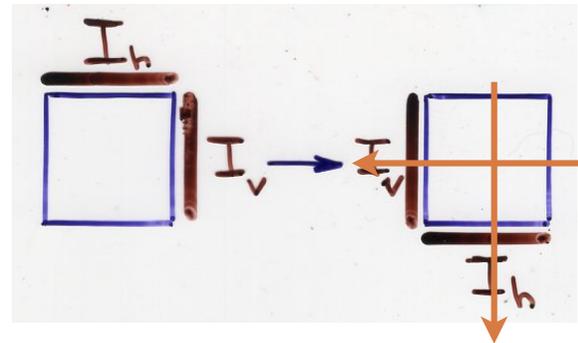
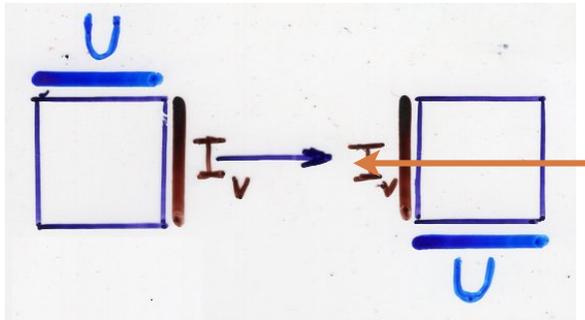
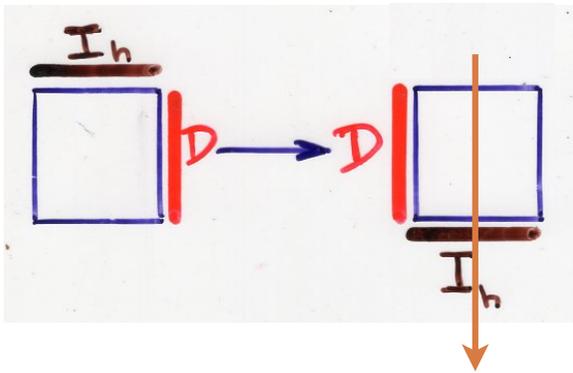


$$\begin{cases} U \mathcal{D} = q \mathcal{D} U + I_v I_h \\ U I_v = I_v U \\ I_h \mathcal{D} = \mathcal{D} I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$U \mathcal{D} = q \mathcal{D} U + I$$



"complete"
Q-tableau



$$UD = qDU + I$$

$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{aligned} w &= U^n D^n \\ uv &= I_v^n I_h^n \end{aligned}$$

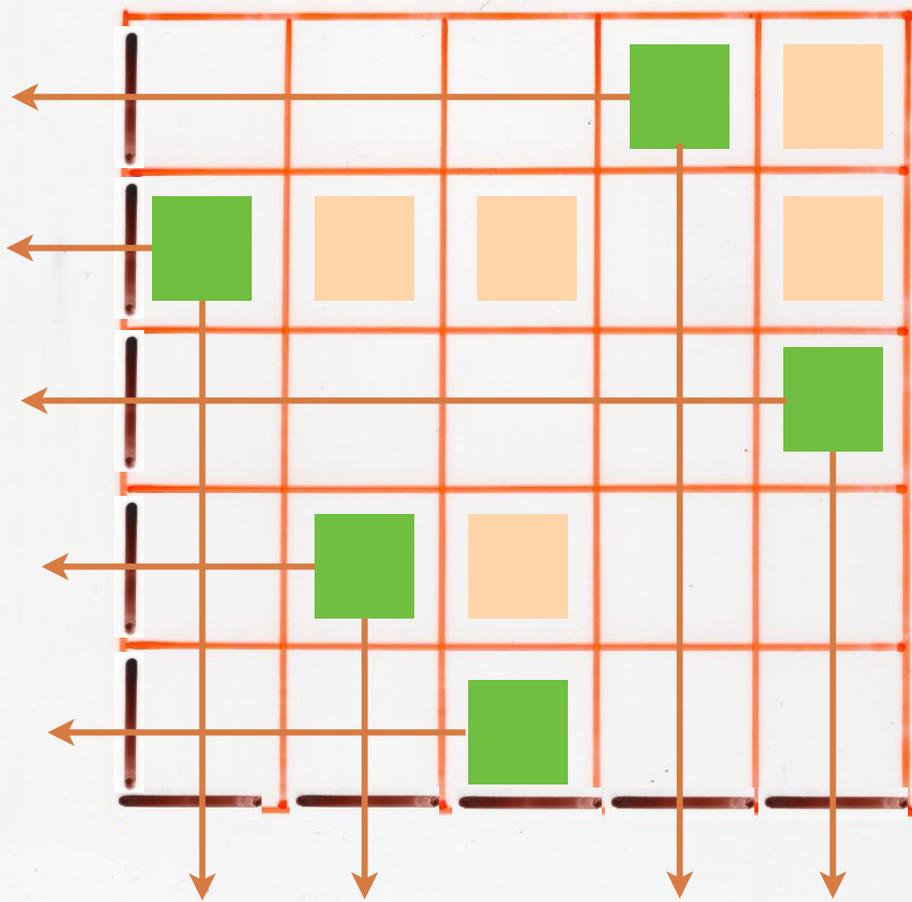
$$c(u, v; w) = n!$$

complete Q -tableau

$$\begin{aligned} \text{uwb}(T) &= U^n D^n \\ \text{lwb}(T) &= I_v^n I_h^n \end{aligned}$$

\longleftrightarrow Permutations
 G_n

$$\begin{cases}
 U \mathcal{D} = q \mathcal{D} U + I_v I_h \\
 U I_v = I_v U \\
 I_h \mathcal{D} = \mathcal{D} I \\
 I_h I_v = I_v I
 \end{cases}$$



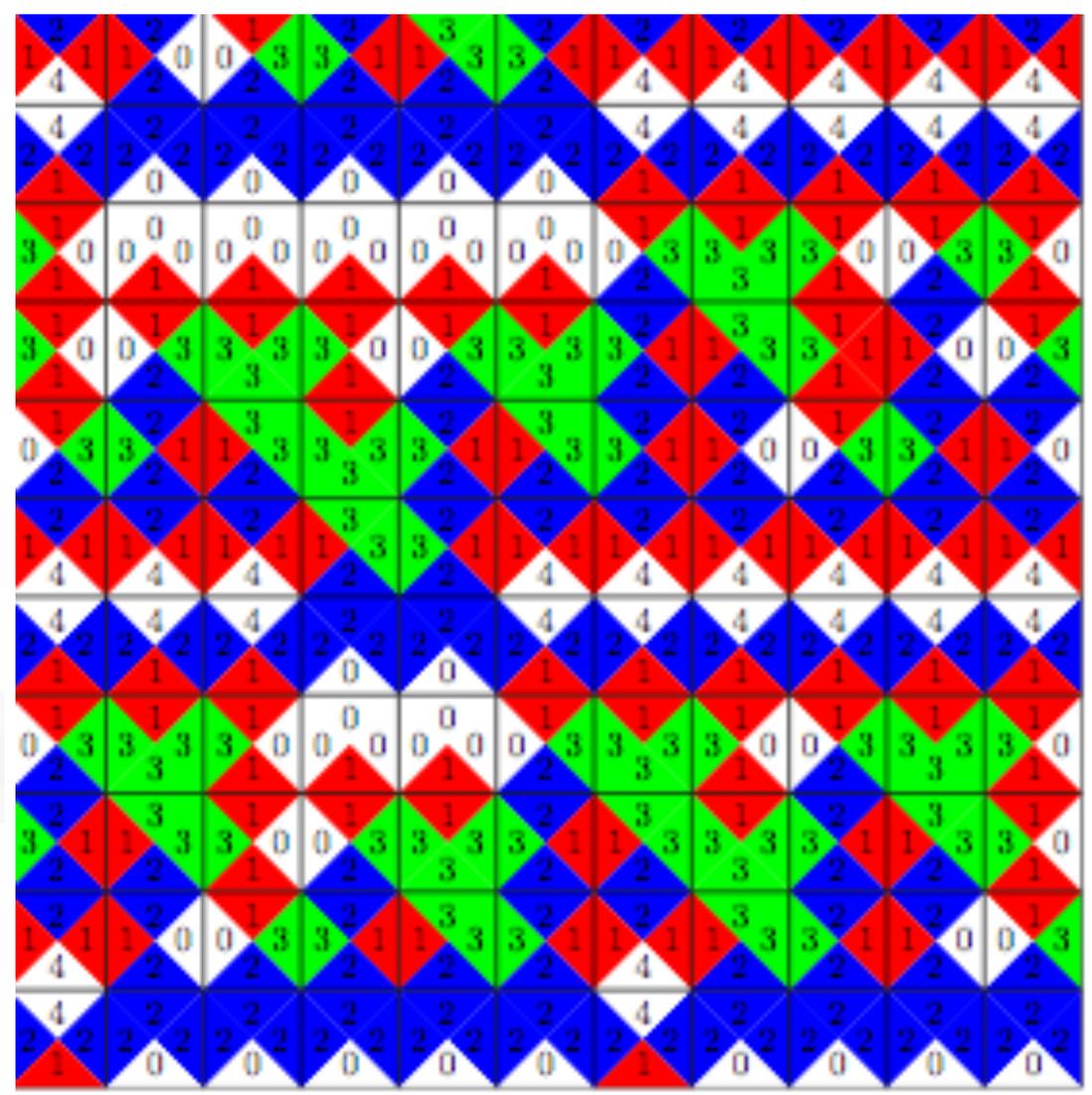
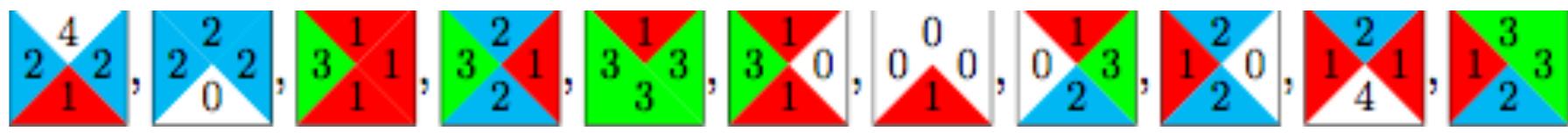
number of
inversions
of a permutation σ

"complete"
Q-tableau

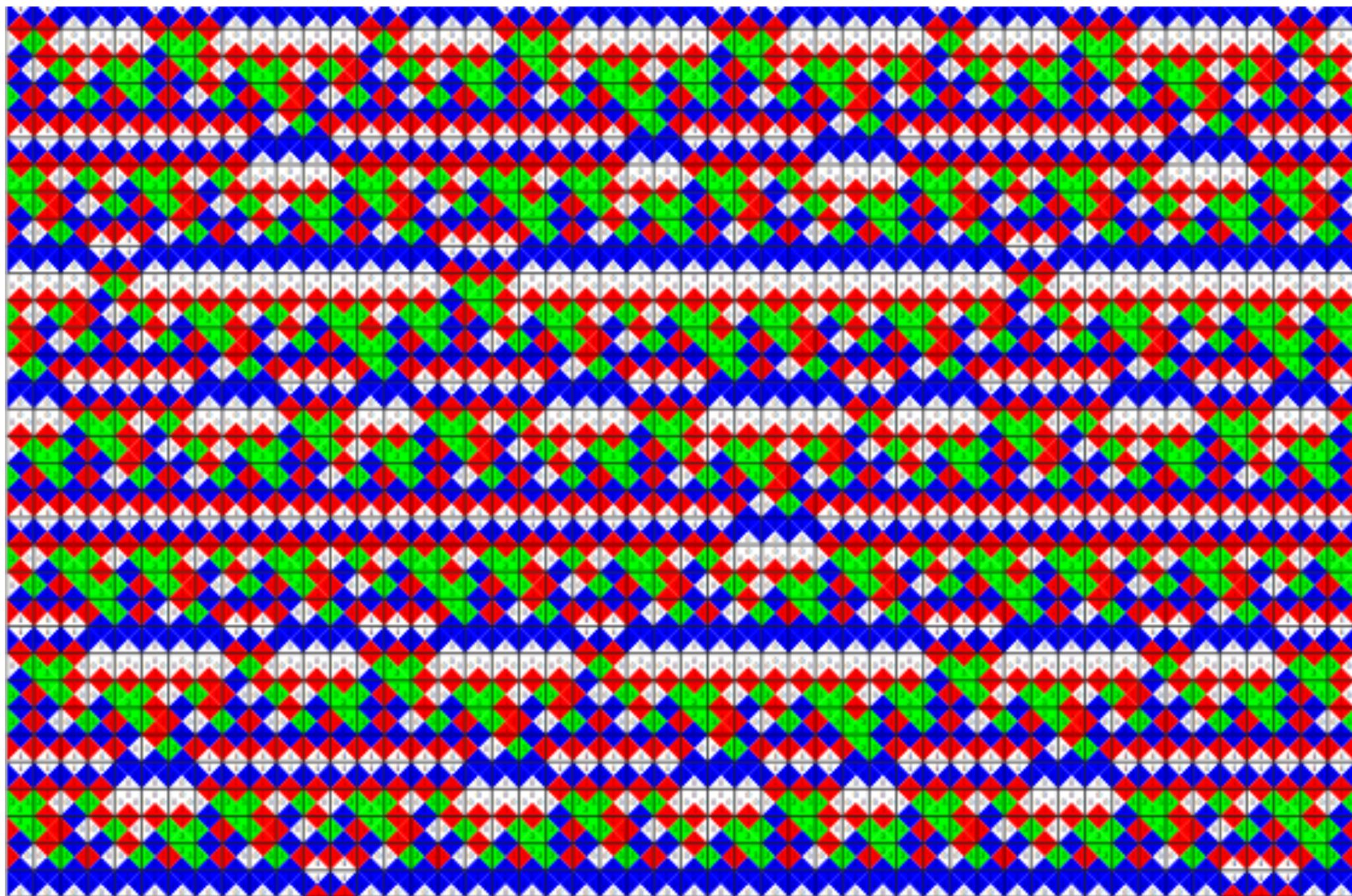
Wang tiling

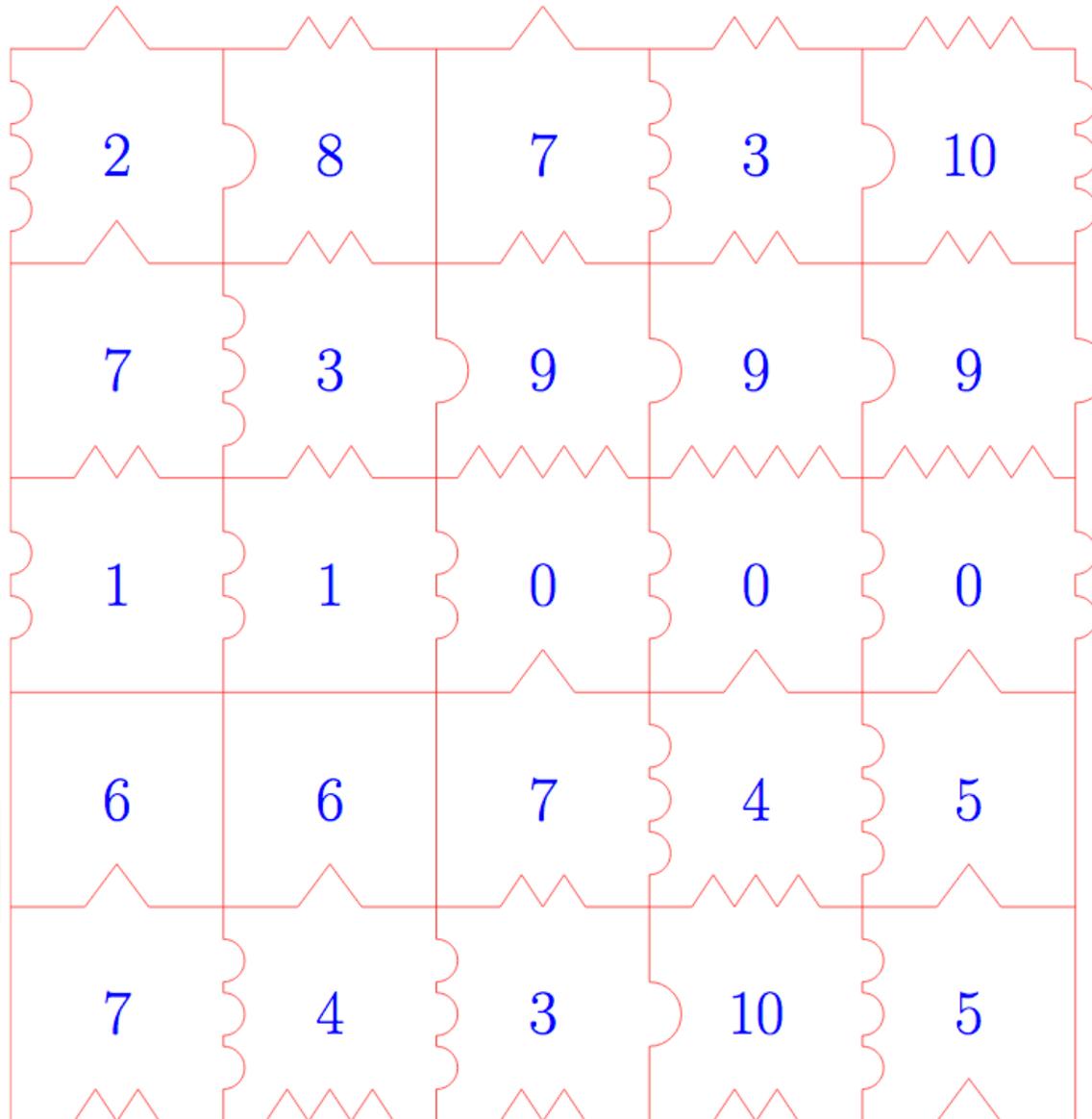
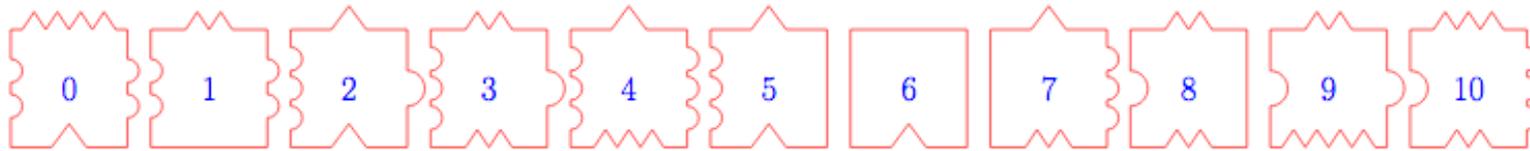
- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

Jeandel - Rao (2015)
aperiodic Wang tiling
11 tiles



S. Labbé





S. Labbe

FIGURE 9. A 5×5 pattern with Jeandel-Rao tiles ready for laser cut (red means cut and blue means engrave). Tiles should have 3cm size when printed in A4 format.

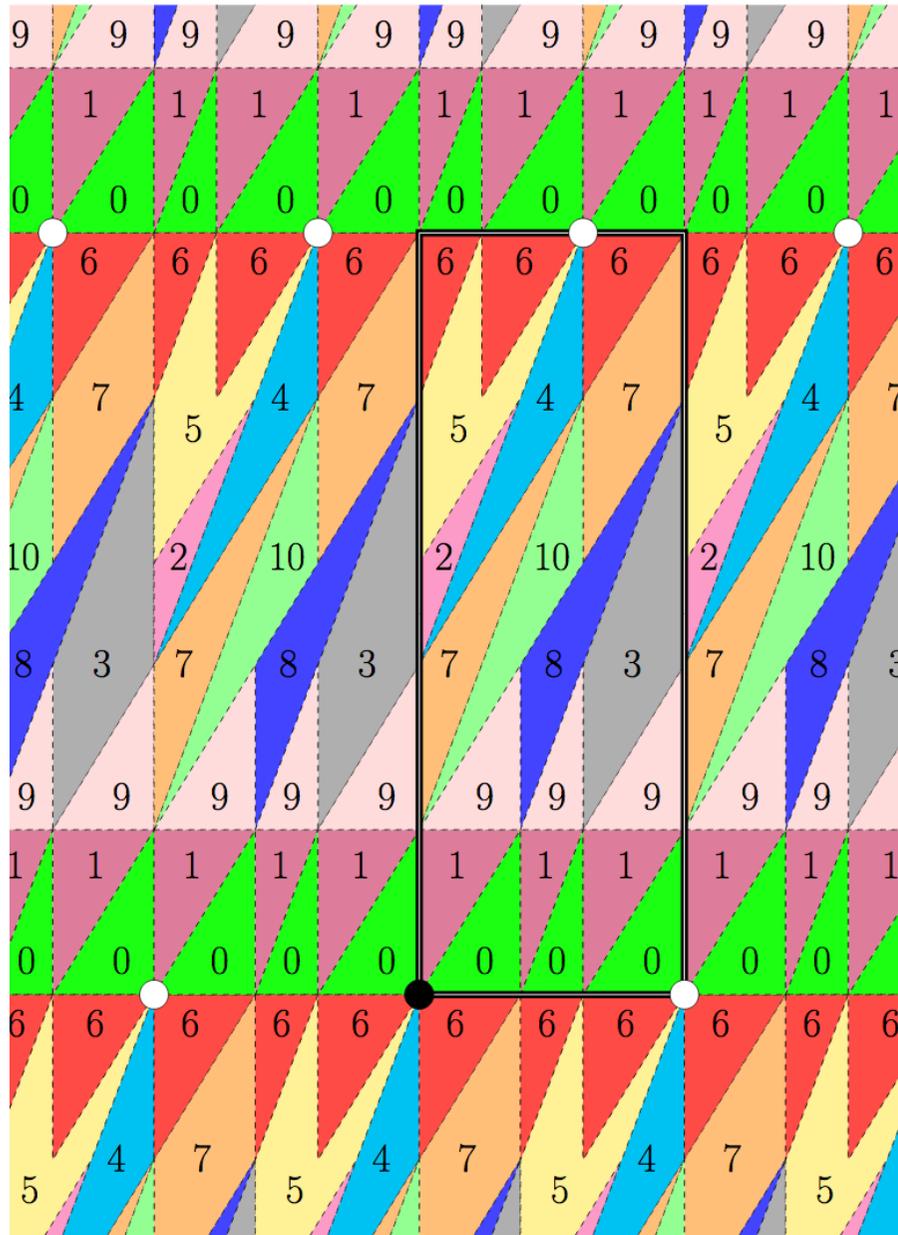


FIGURE 10. The Universal solver for Jeandel-Rao tilings. Any pattern in the minimal subshift of Jeandel-Rao tilings is the coding of the orbit of some starting point by the action of horizontal and vertical translations by 1 unit (3cm when printed in A4 format).

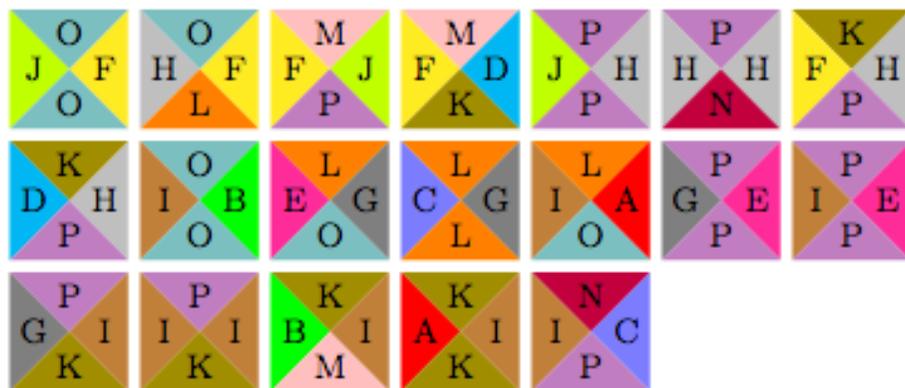
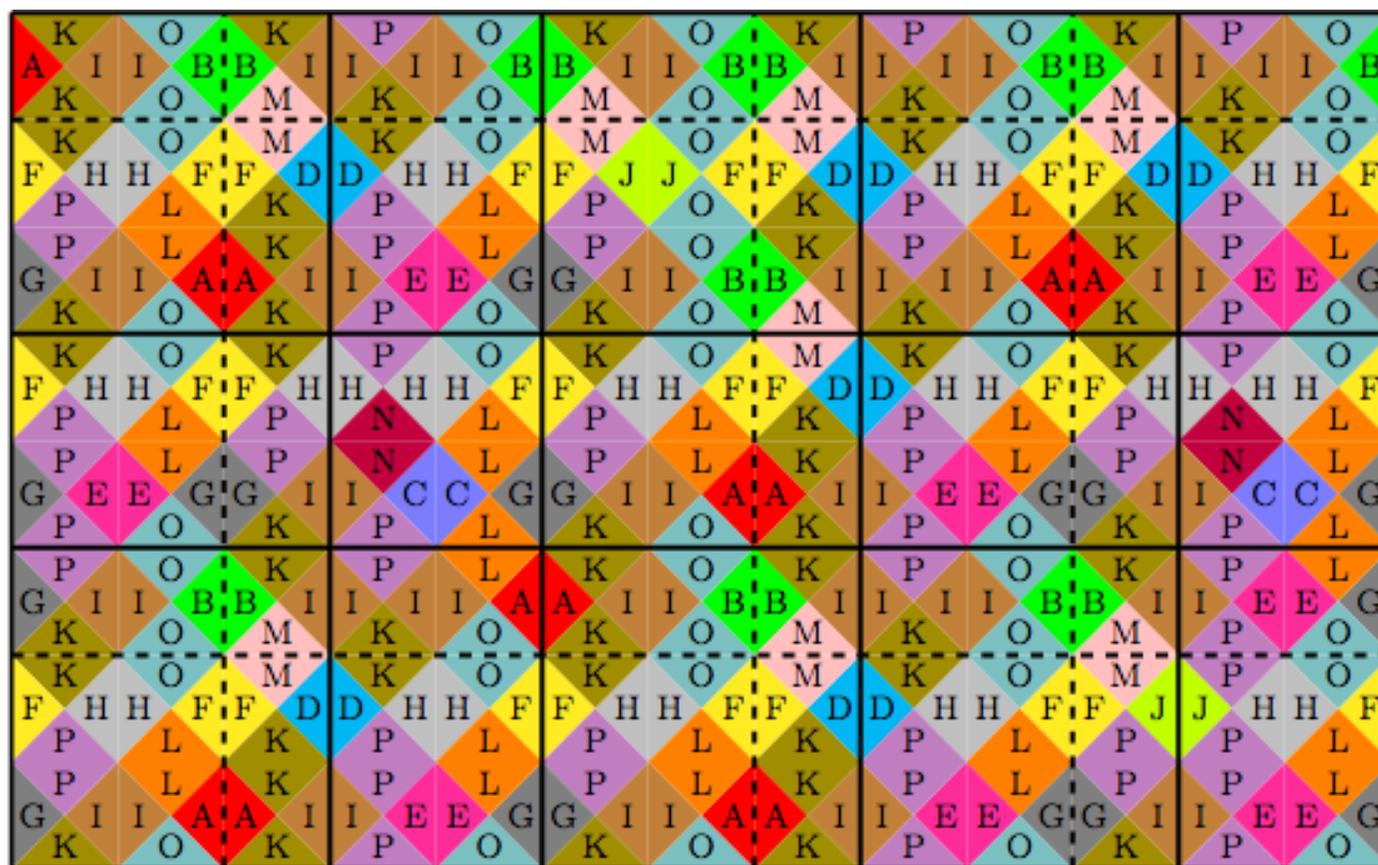


FIGURE 1. The set \mathcal{U} of 19 Wang tiles.



S. Labbé

Q-tableau:
definition

quadratic
algebra \mathbb{Q}

\mathbb{Q} -tableaux

complete \mathbb{Q} -tableau

L set of "labels"

$$\varphi : R \rightarrow L$$

set of rewriting rules

$$B_j A_i \rightarrow C_{ij}^{kl} A_k B_l$$

$$\varphi (B_j A_i \rightarrow C_{ij}^{kl} A_k B_l)$$

or for short

$$\varphi \left(\begin{array}{|c|c|} \hline B_j & A_i \\ \hline A_k & B_l \\ \hline \end{array} \right) \in L$$

$$\varphi : R \rightarrow L$$

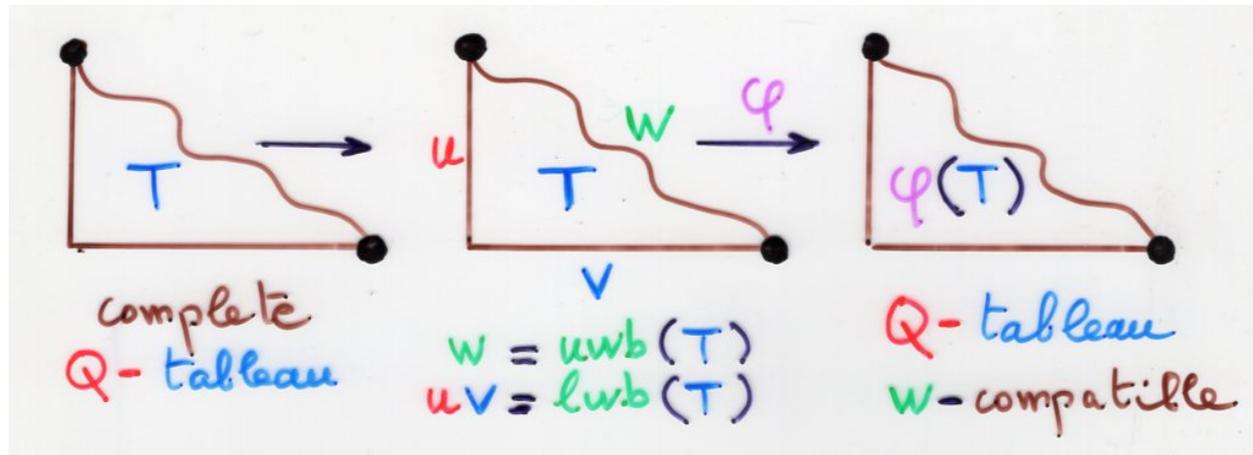
(*) $B_j A_i = \dots + c_{ij}^{kl} A_k B_l + \dots$

$\downarrow \varphi$ $\downarrow \varphi$ $\downarrow \varphi$

all distinct

Definition Q -tableau

is the "image" by φ satisfying (*) of a complete Q -tableau



Proposition for $w \in (d \cup \beta)^*$ fixed

$\left\{ \begin{array}{l} \text{set of } Q\text{-tableaux} \\ w\text{-compatible} \end{array} \right\} \xleftrightarrow{\varphi} \left\{ \begin{array}{l} \text{set of complete } Q\text{-tableaux } T \\ \text{with } uwb(T) = w \end{array} \right\}$

are in bijection by φ

$$c(u, v; w) = \sum_{\mathcal{T}} \text{wgt}(\mathcal{T})$$

complete Q-tableau

$$uwb(\mathcal{T}) = w$$

$$lwb(\mathcal{T}) = uv$$

complete Q-tableau

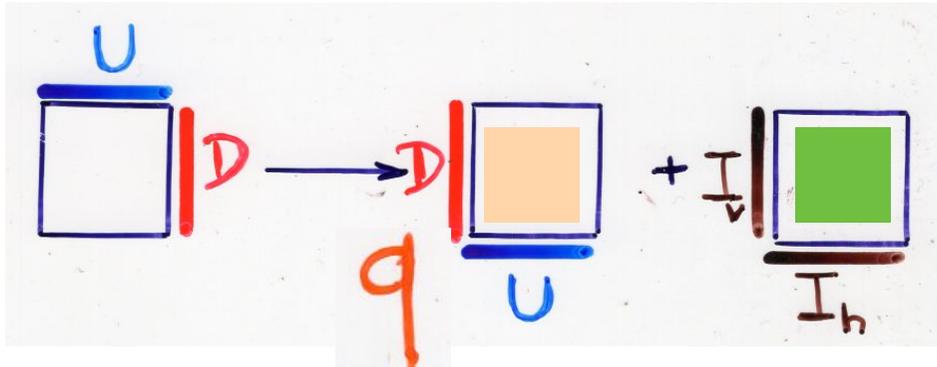
Q-tableaux

Q-tableaux: example

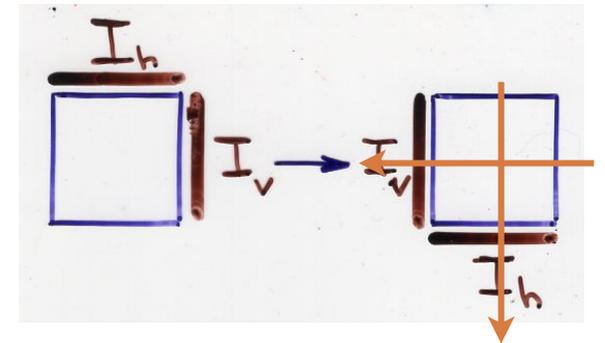
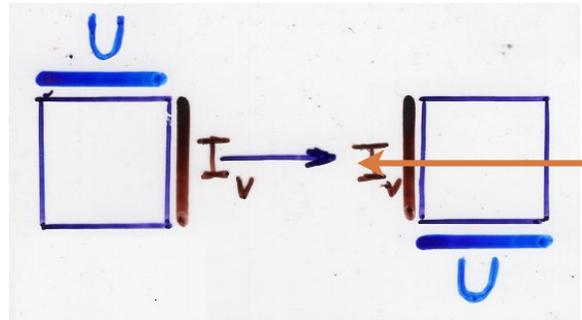
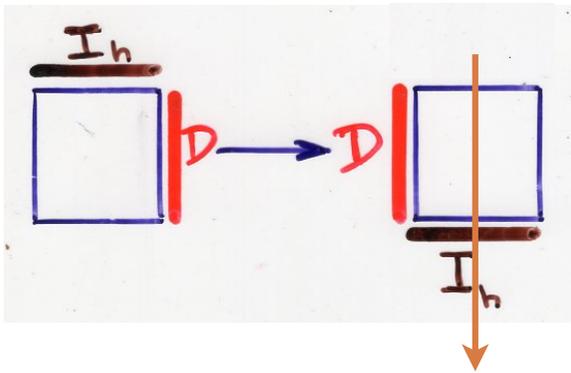
$$UD = DU + Id$$

$$\begin{cases} UD = qDU + I_v I_h \\ UI_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$UD = qDU + I$$



"complete"
Q-tableau



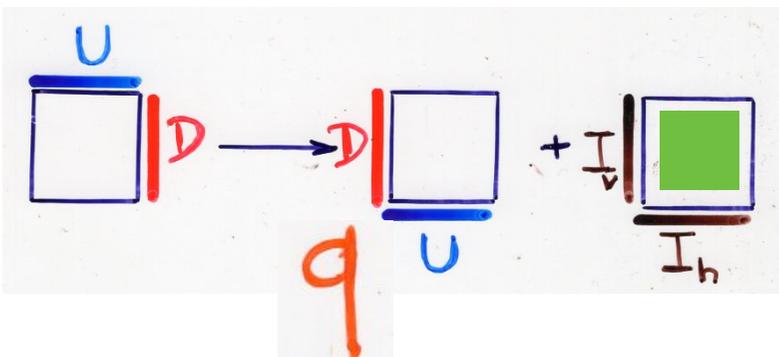
$$\left\{ \begin{array}{l} U D = q D U + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$

Q-tableaux

$$\varphi: R \longrightarrow L$$

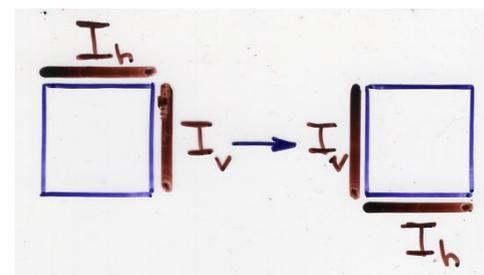
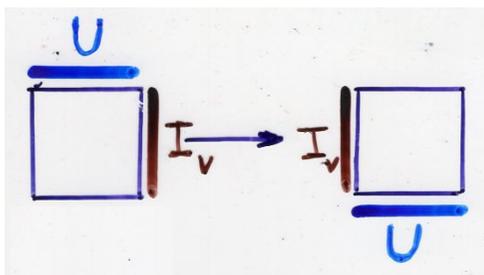
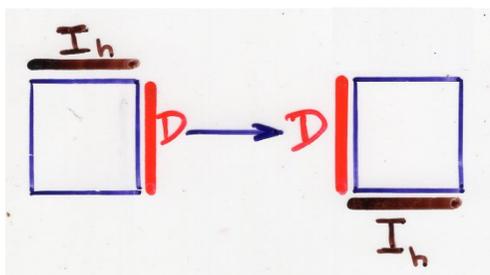
map

L a set of "labels"
(for the cell of $[n] \times [n]$)

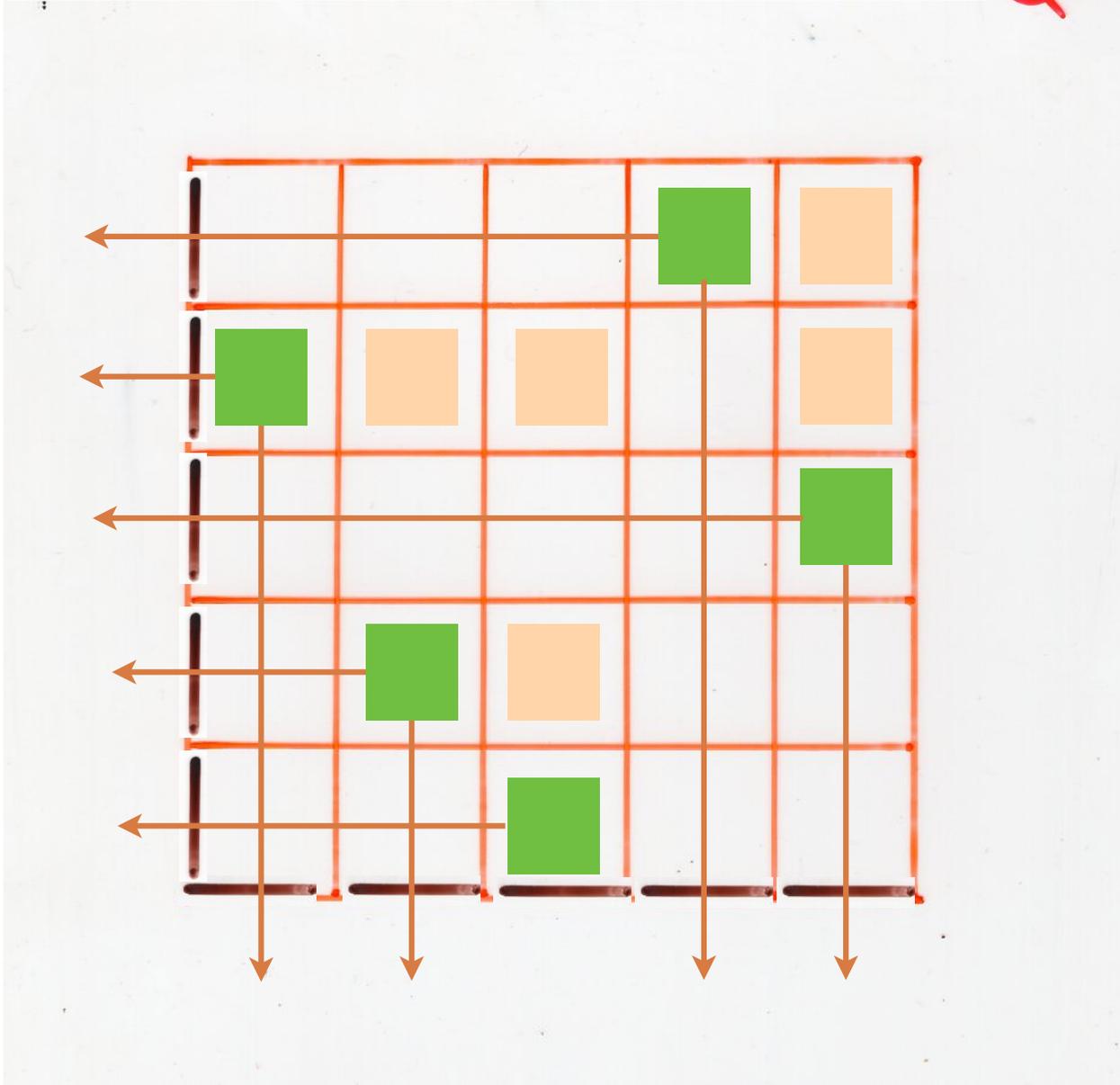


examples

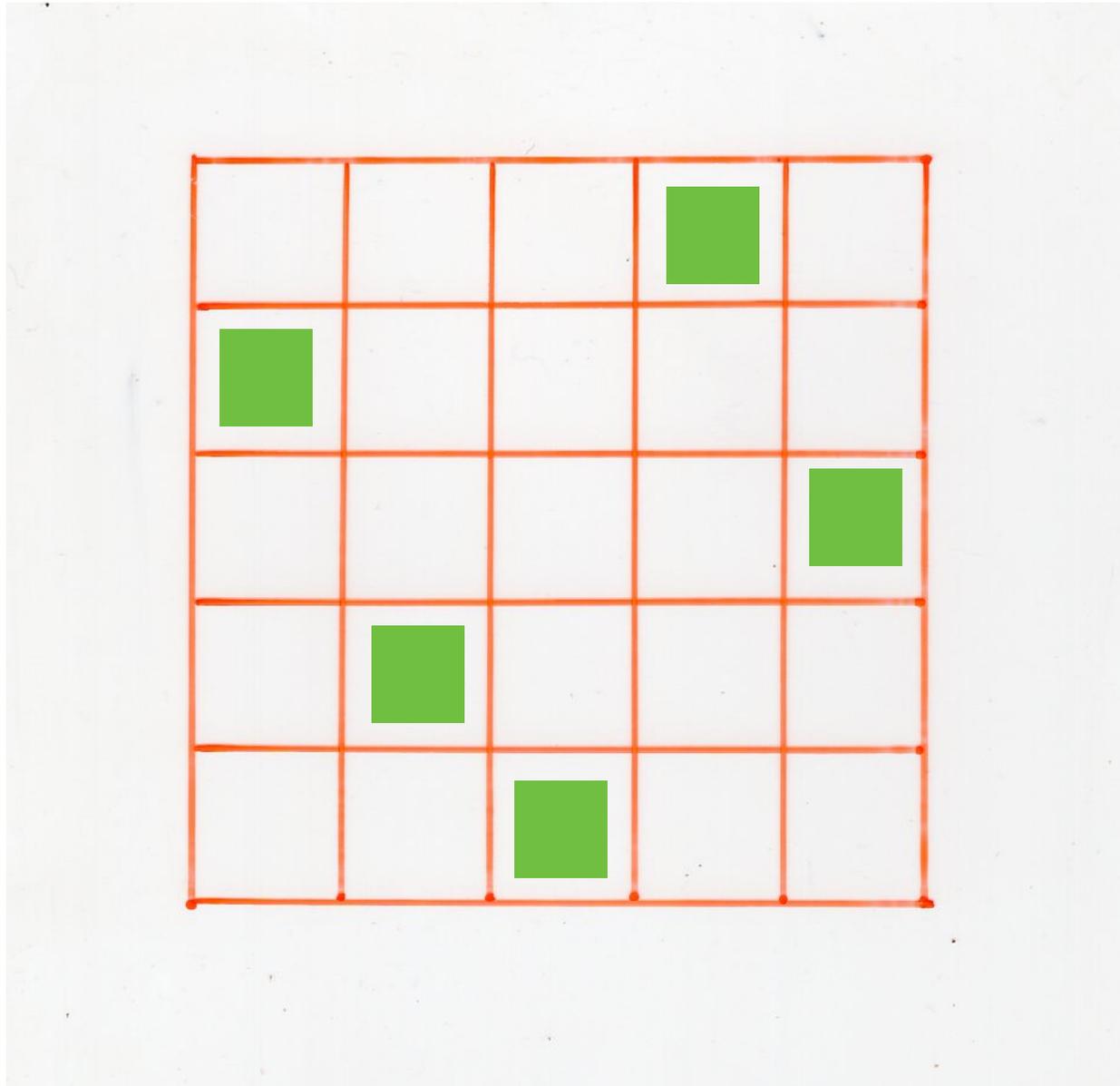
$$L = \{ \square, \text{green square} \}$$



"complete"
Q-tableau



Q-tableaux



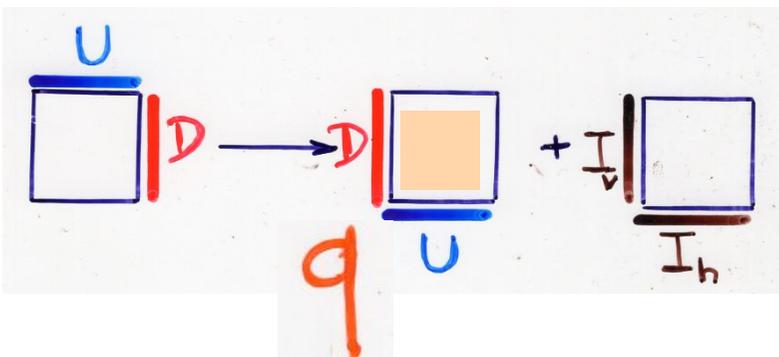
$$\begin{cases} U \mathcal{D} = q \mathcal{D} U + I_v I_h \\ U I_v = I_v U \\ I_h \mathcal{D} = \mathcal{D} I_h \\ I_h I_v = I_v I_h \end{cases}$$

Q-tableaux

$$\varphi: R \longrightarrow L$$

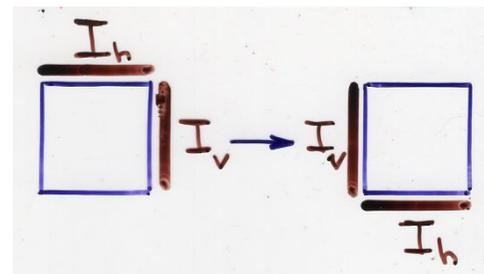
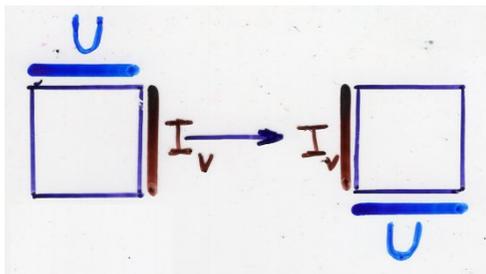
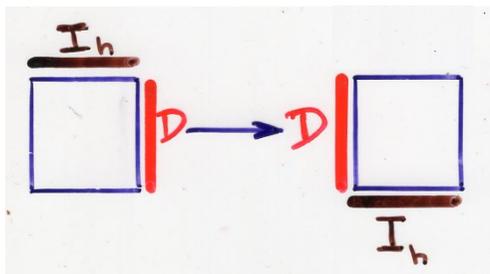
map

L a set of "labels"
(for the cell of $[n] \times [n]$)

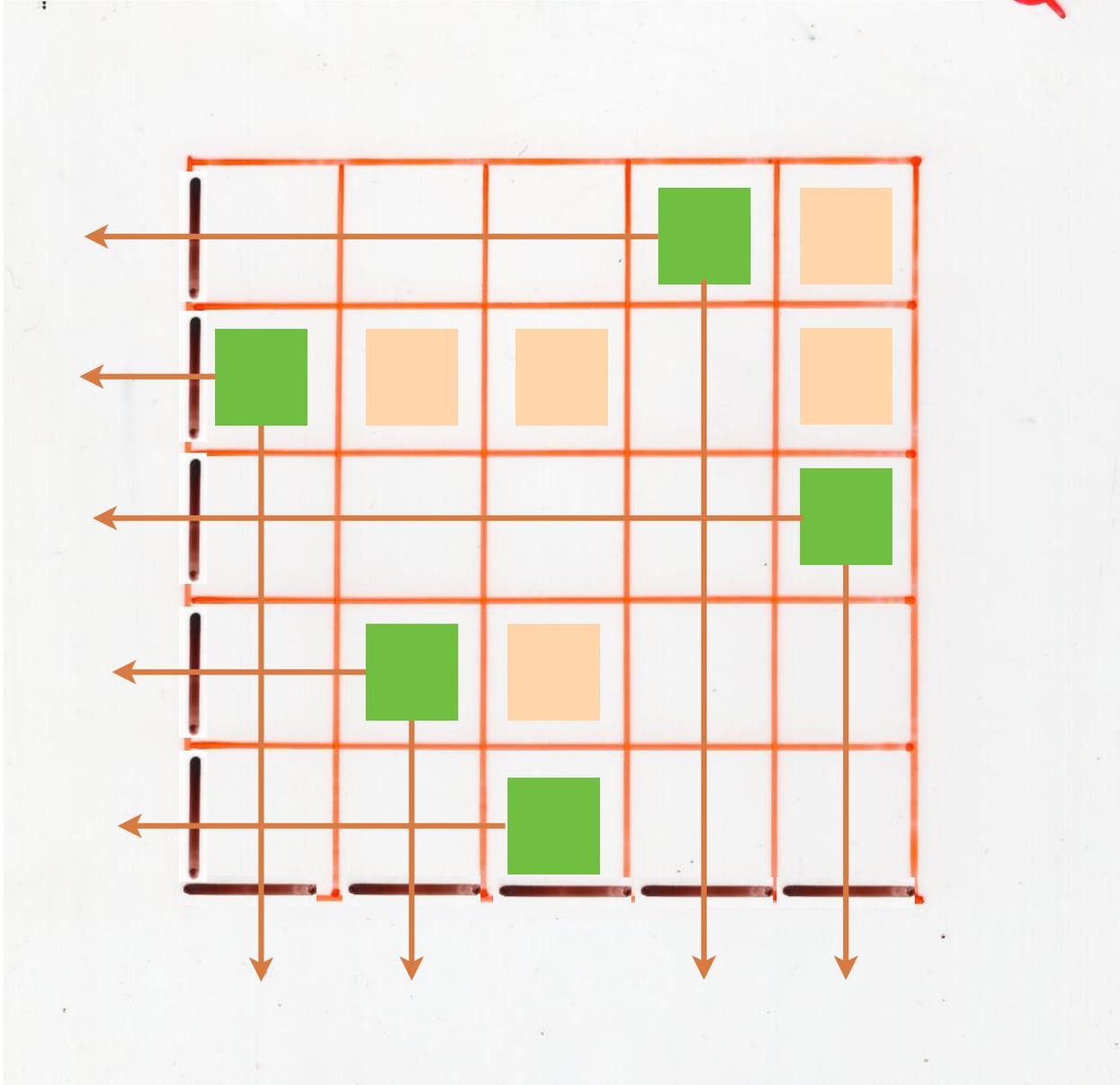


examples

$$L = \{ \square, \square \}$$

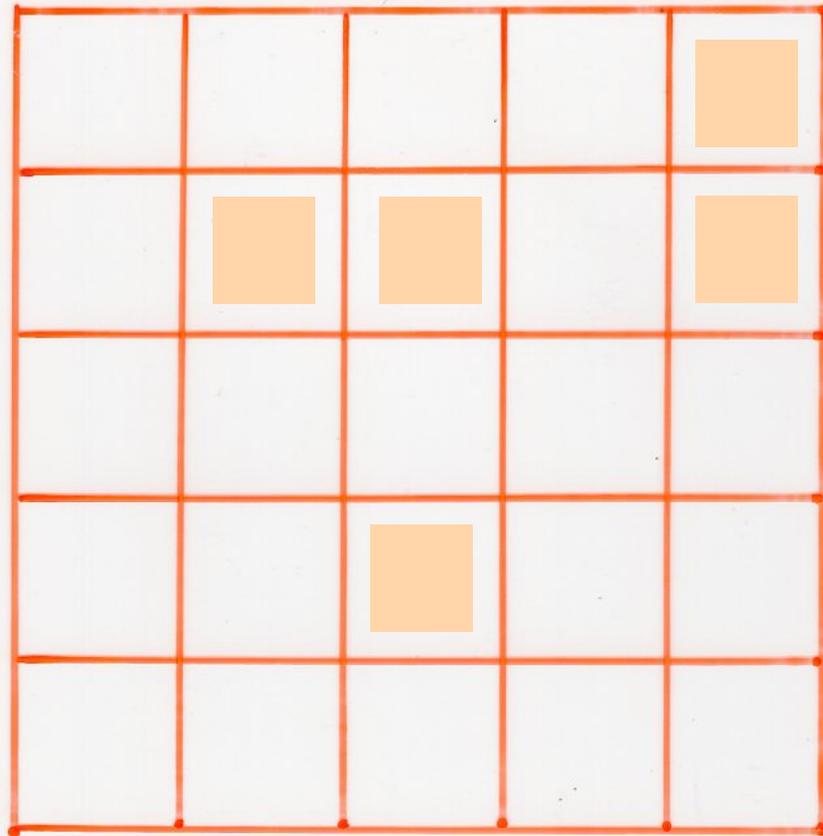


"complete"
Q-tableau



Rothe diagram
of a permutation
(1800)

Q-tableaux



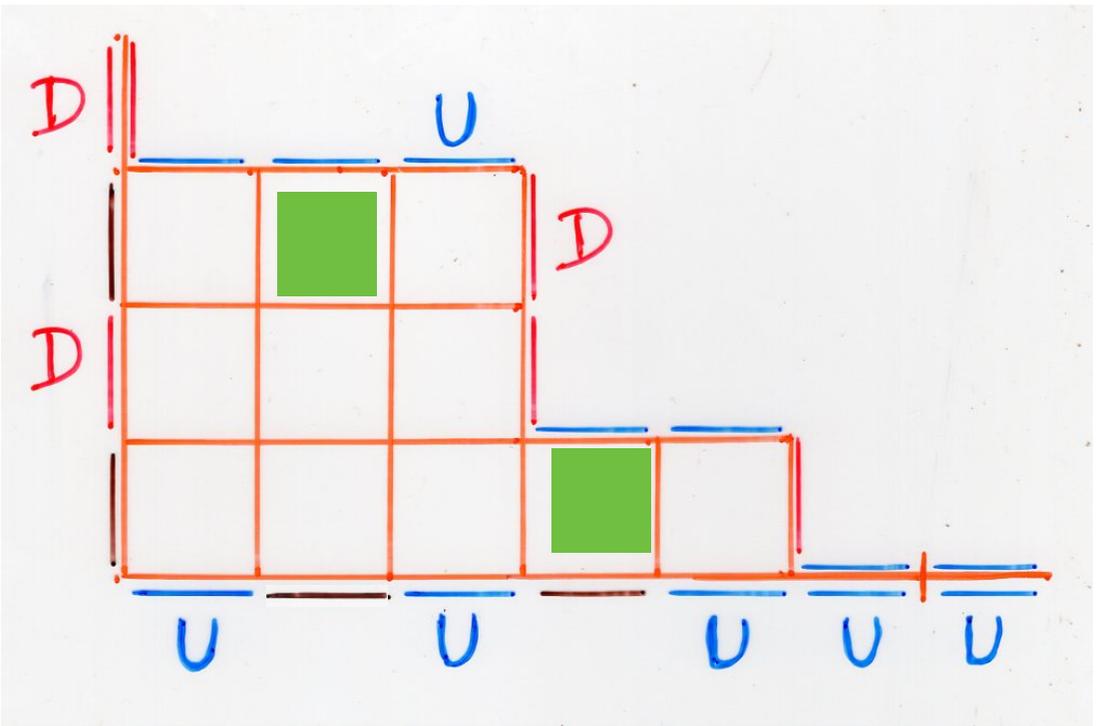
Rooks
placement

$$w = D U^3 D^2 U^2 D U^2$$

$$w \rightarrow F = F(w)$$

F Ferrers diagram

Rooks placement



A strange bijection for binary trees

Part II: → GT 16 De 2019