Une bijection insolite pour les arbres binaires

A strange bijection for binary trees

(2)

GT LaBRI, Bordeaux
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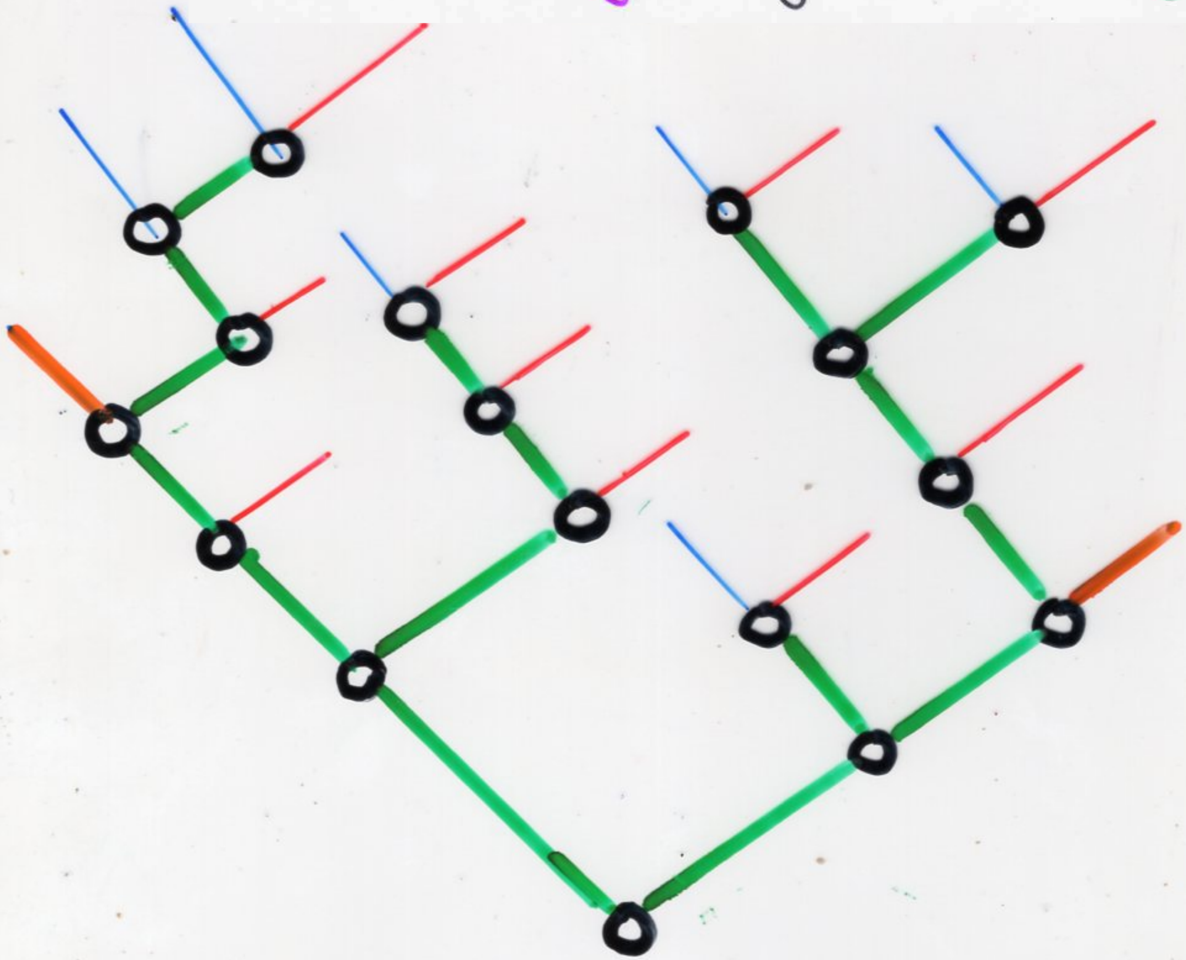
From the previous talk

GT LaBRI,   9 December 2019

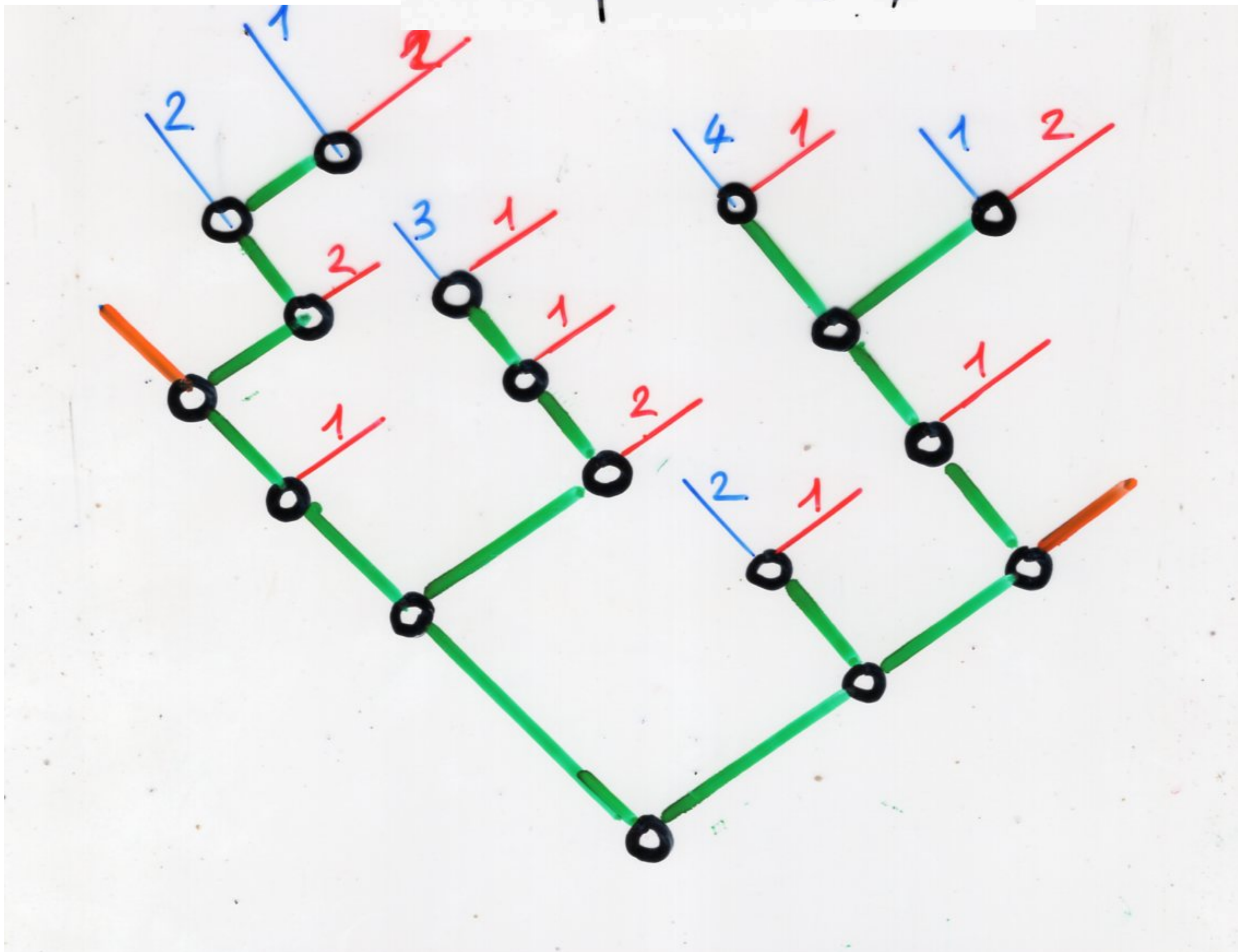
The Tamil bijection

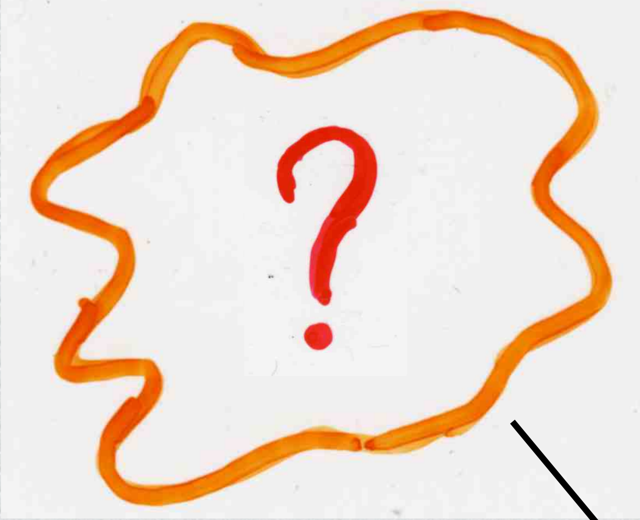
w canopy of a binary tree



2 1 2 2 1 3 1 1 2 2 1 4 1 1 2 1

the pair (w, h)





$\mathcal{B} \longrightarrow (w, h)$

binary tree

The Tamil bijection

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

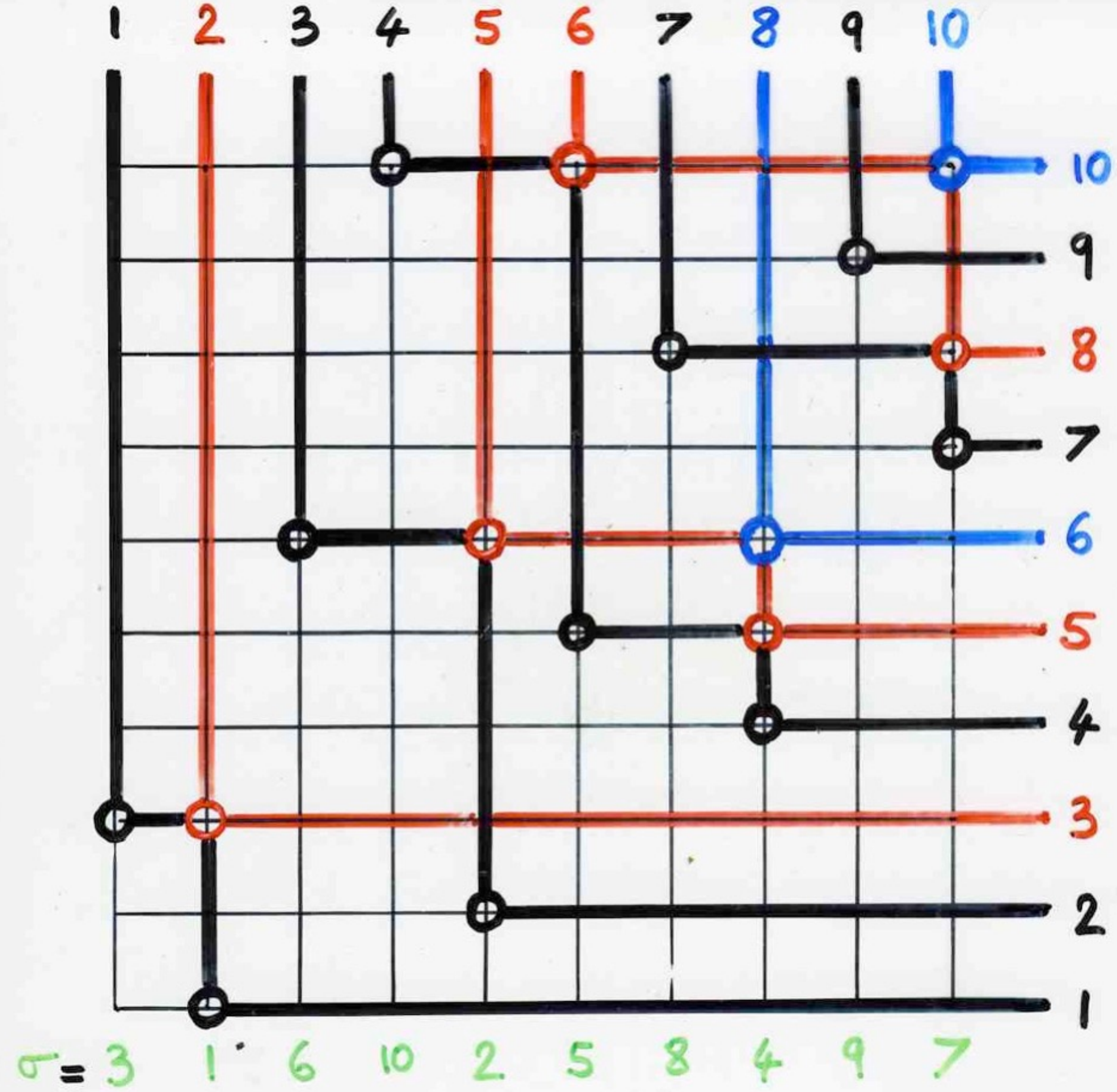
Schensted's insertions

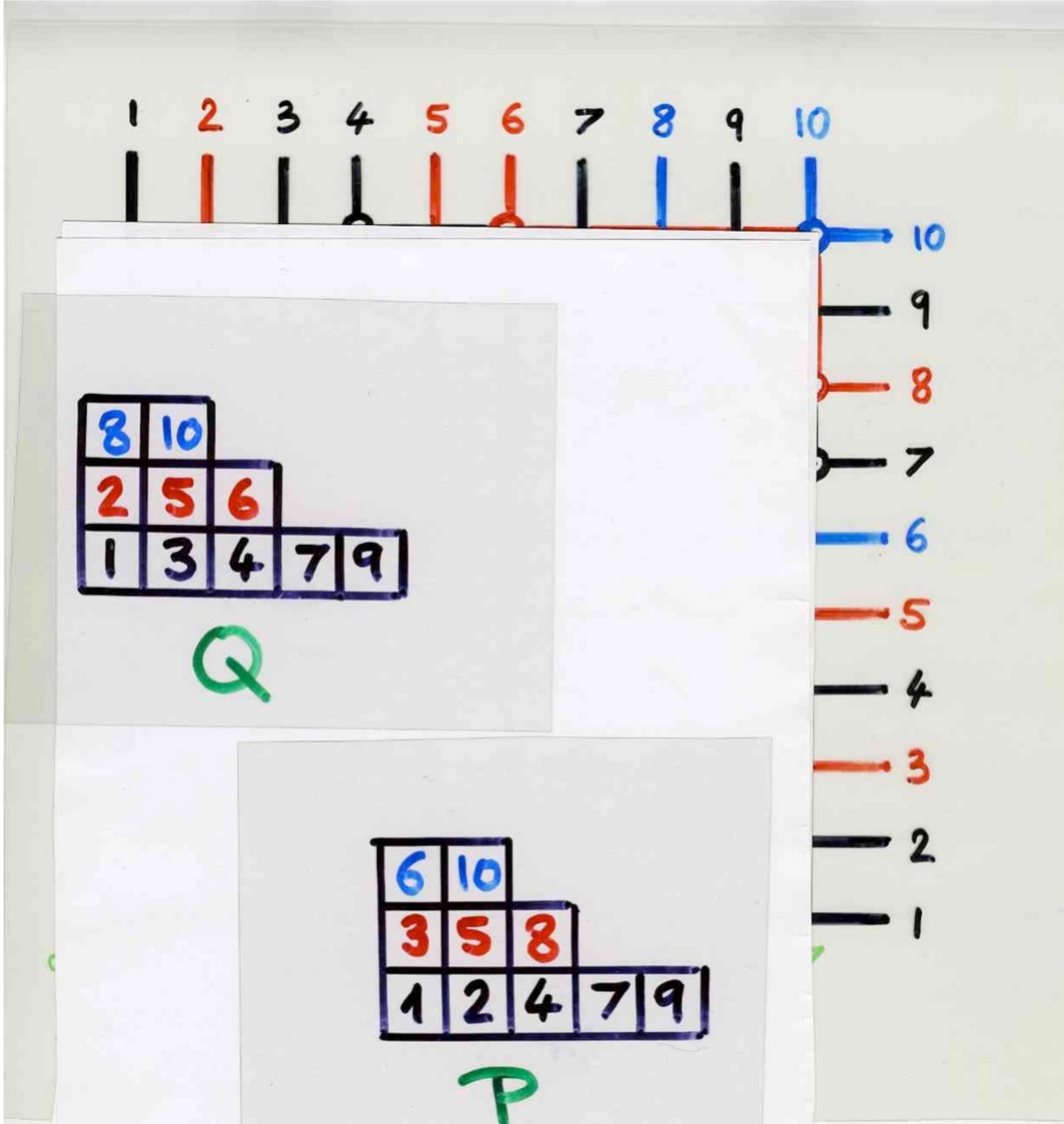
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

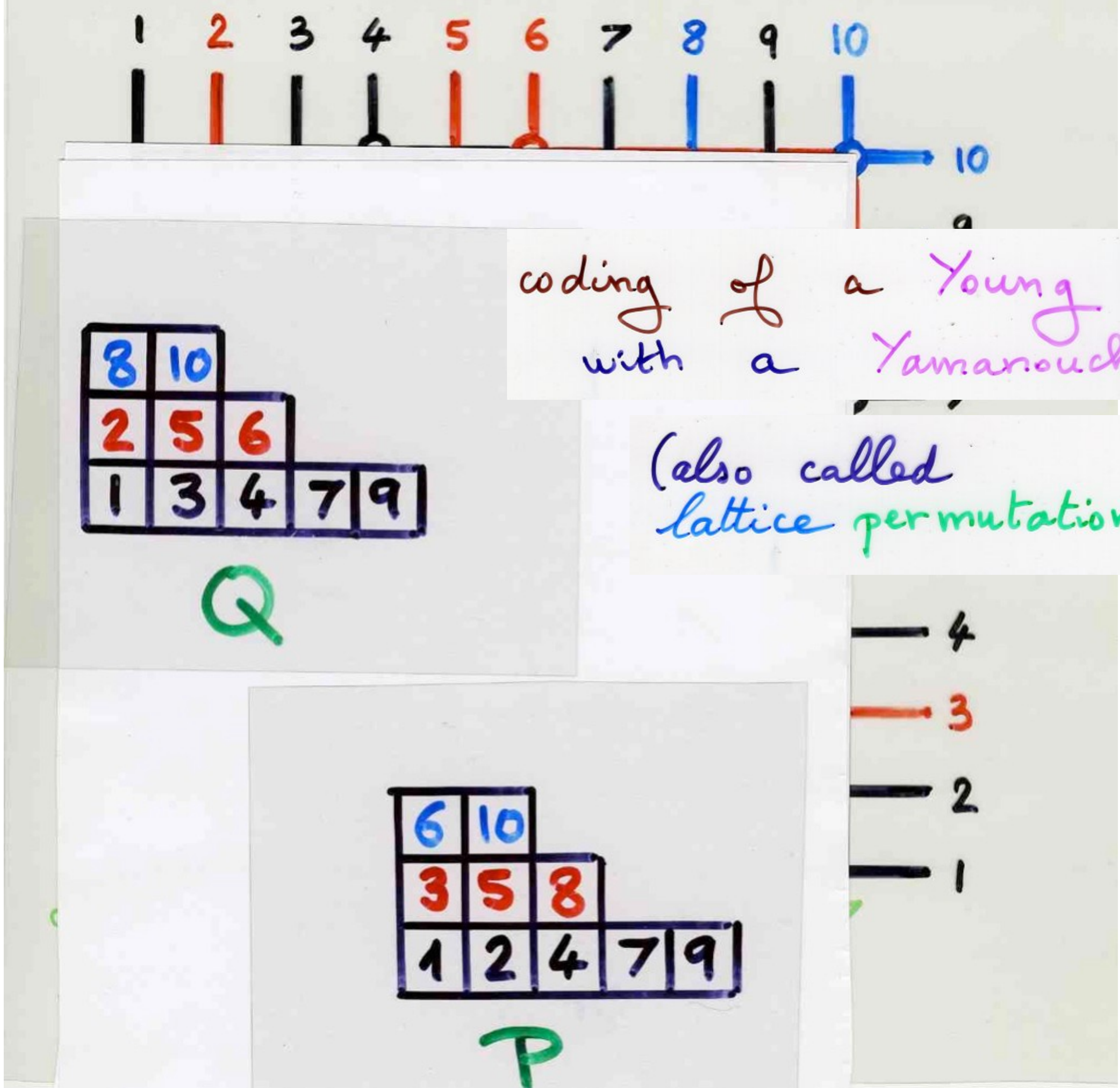
geometric version
with
"light" and "shadow"





1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P

= 1 2 1 1 2 2 1 3 1 3



coding of a Young tableau with a Yamanouchi word

(also called lattice permutation)

$= 1\ 2\ 1\ 1\ 2\ 2\ 1\ 3\ 1\ 3$

Definition Yamanouchi word w

$w \in \{1, 2, \dots\}^*$

free monoid generated by the alphabet $1, 2, \dots,$

such that:

for every factorization $w = uv$

$$|u|_1 \geq |u|_2 \geq \cdots \geq |u|_i \geq \cdots$$

number of occurrences of the letter i in u

$$UD = DU + Id$$

commutations

<u>Lemma</u> Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w)\, D^i U^j$$

normal ordering in physics

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i}\, D^i U^i$$

$$c_{n,0} = n!$$

permutations

quadratic algebra Q

generators

$\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

for every $i \in I$, $j \in J$

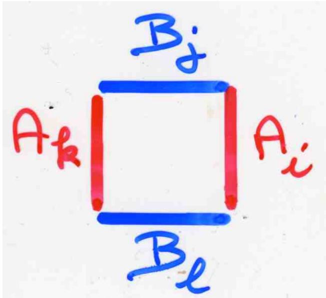
$\mathcal{A} \cap \mathcal{B} = \emptyset$

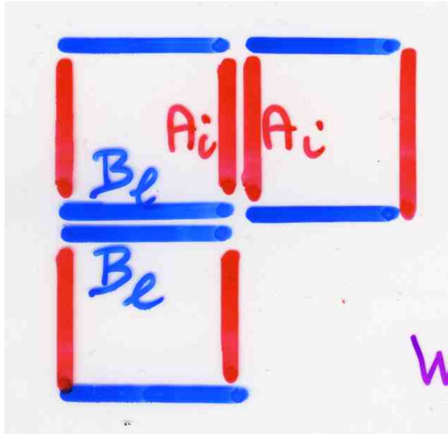
commutations

$$B_j A_i = \sum_{k\,l} c_{ij}^{kl} A_k B_l$$

R set of rewriting rules $B_j A_i \to c_{ij}^{kl} A_k B_l$

Wang tile





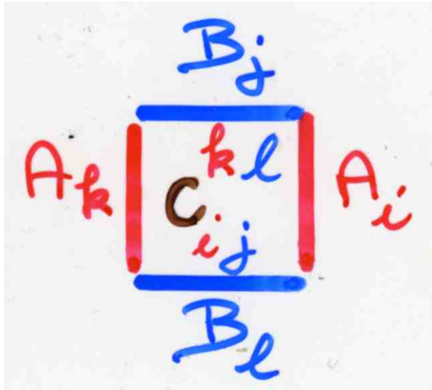
Wang tiling

<u>Definition</u> complete Q-tableau

Wang tiling of the Ferrers diagram F

$$B_j A_i \longrightarrow c_{i\,j}^{k\,l}\, A_k B_l$$

weight of a Wang tile



Definition

weight of a complete Q-tableau T

$$\mathrm{wgt}(T) = \prod_{\substack{\text{cells}\\ \text{of } F}} c_{i\,j}^{k\,l} \in \mathbb{K}[X]$$

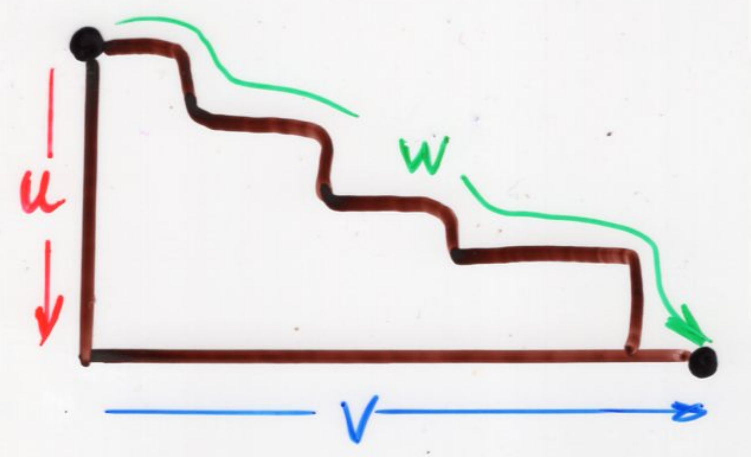
Lemma In **Q** every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

Proposition For any words
$w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{T} \mathrm{wgt}(T)$$

T complete Q-tableau
$uwb(T) = w$
$lwb(T) = uv$



Q-tableaux

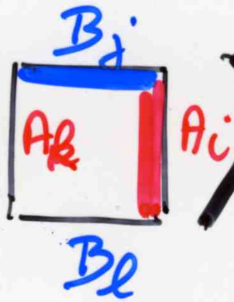
L set of "labels"

$\varphi : R \longrightarrow L$

set of rewriting rules

$B_j A_i \longrightarrow c_{ij}^{kl} A_k B_l$

$$\varphi\left(B_j A_i \longrightarrow c_{ij}^{kl} A_k B_l\right)$$

or for short $\varphi\Big($  $\Big) \in L$

$$\varphi : R \longrightarrow L$$

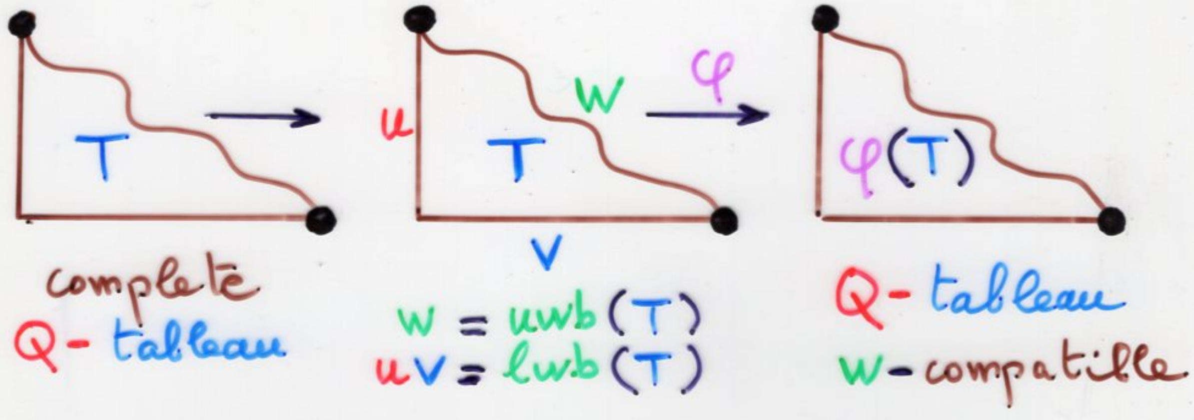
$$(*) \qquad B_j A_i = \cdots + c_{ij}^{kl} A_k B_l + \cdots\cdots$$

$$\downarrow \varphi \qquad \Downarrow \varphi \qquad \downarrow \varphi$$

all distinct

Definition Q-tableau

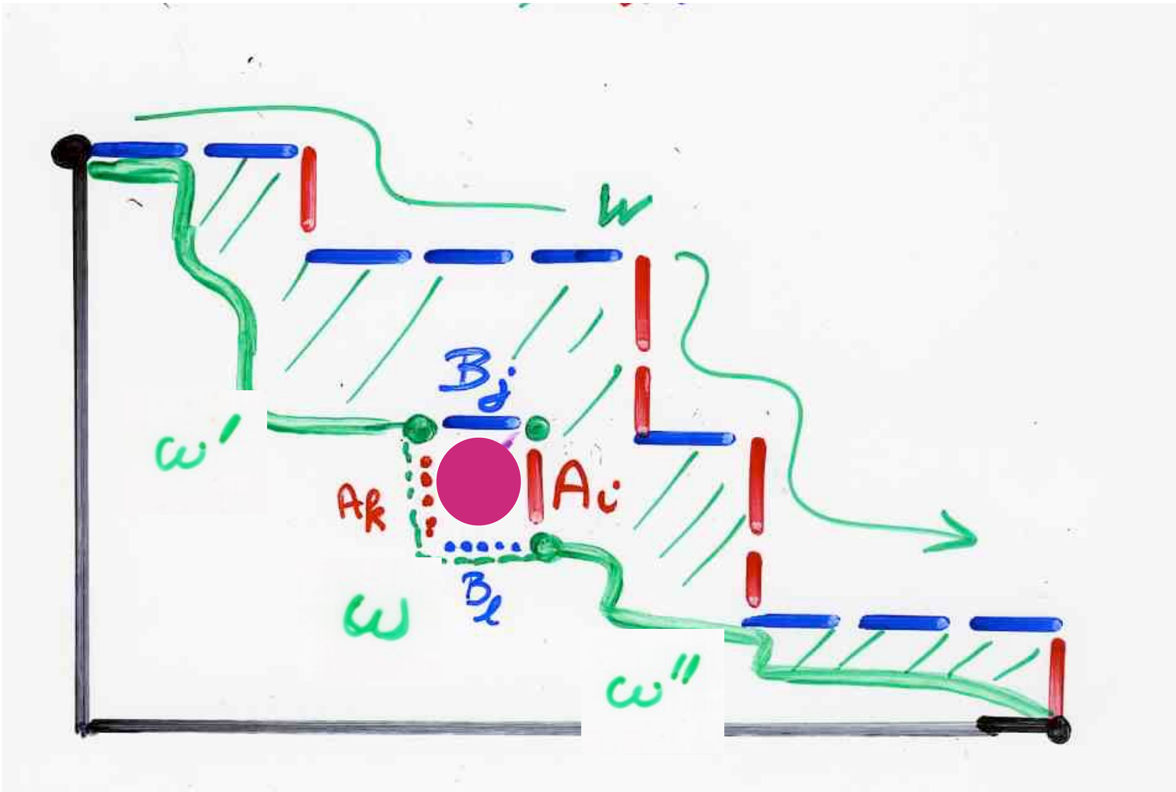
is the "image" by φ satisfying $(*)$
of a complete Q-tableau



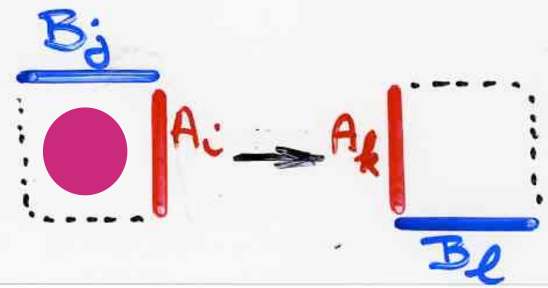
Proposition for $w \in (\alpha \cup \beta)^*$ fixed

$\left\{ \begin{array}{l} \text{set of Q-tableaux} \\ \text{w-compatible} \end{array} \right\} \overset{\varphi}{\longleftrightarrow} \left\{ \begin{array}{l} \text{set of complete} \\ \text{Q-tableaux } T \\ \text{with } uwb(T) = w \end{array} \right\}$

are in bijection by φ



"planar" rewriting

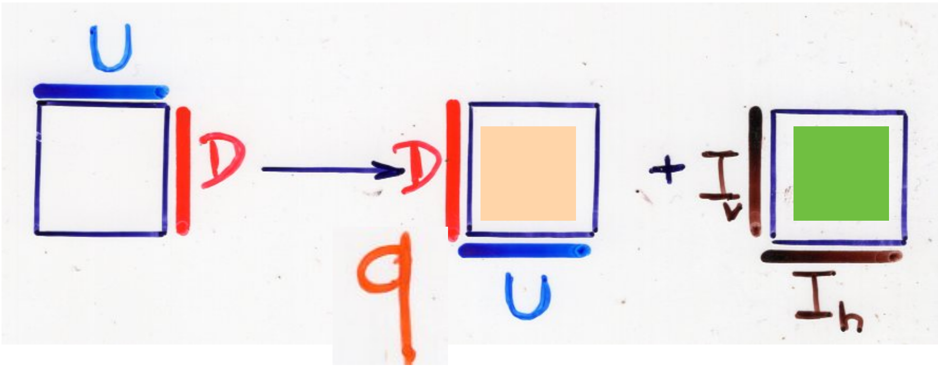


$\varphi : R \to L$

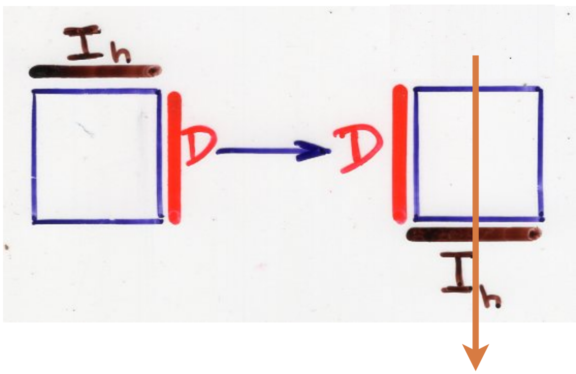
L set of "labels"

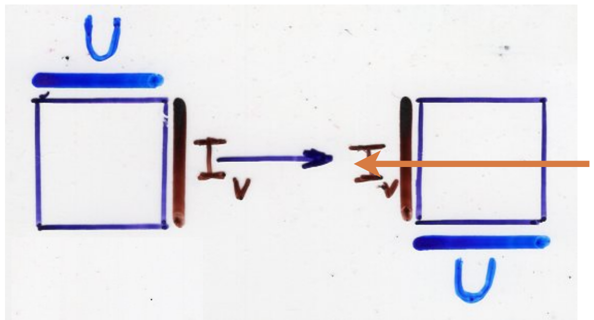
$$UD = qDU + I$$

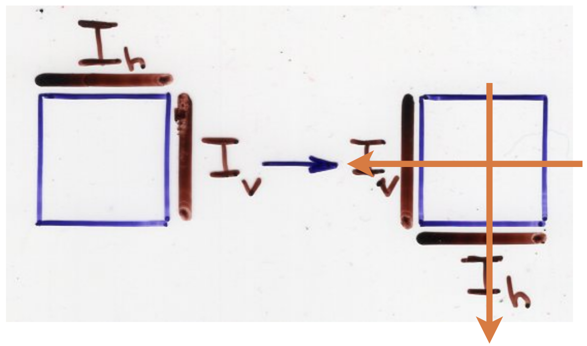
$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$



"complete"
Q-tableau

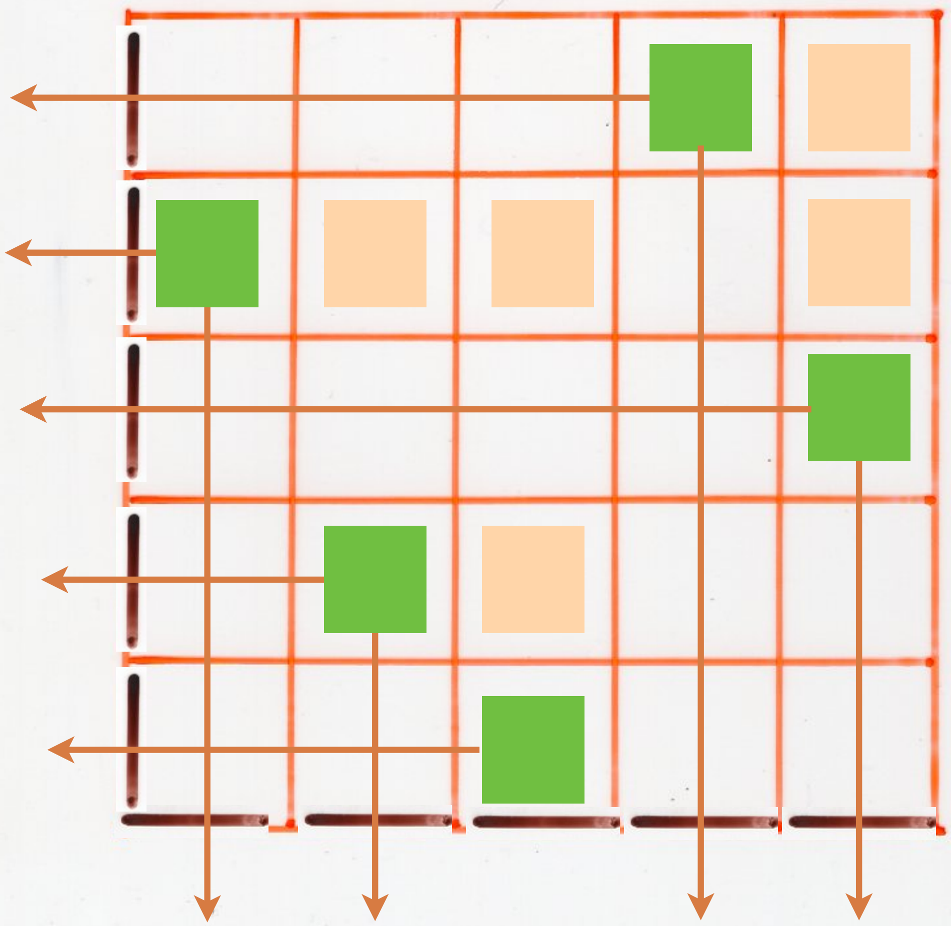






permutations

"complete" Q-tableau

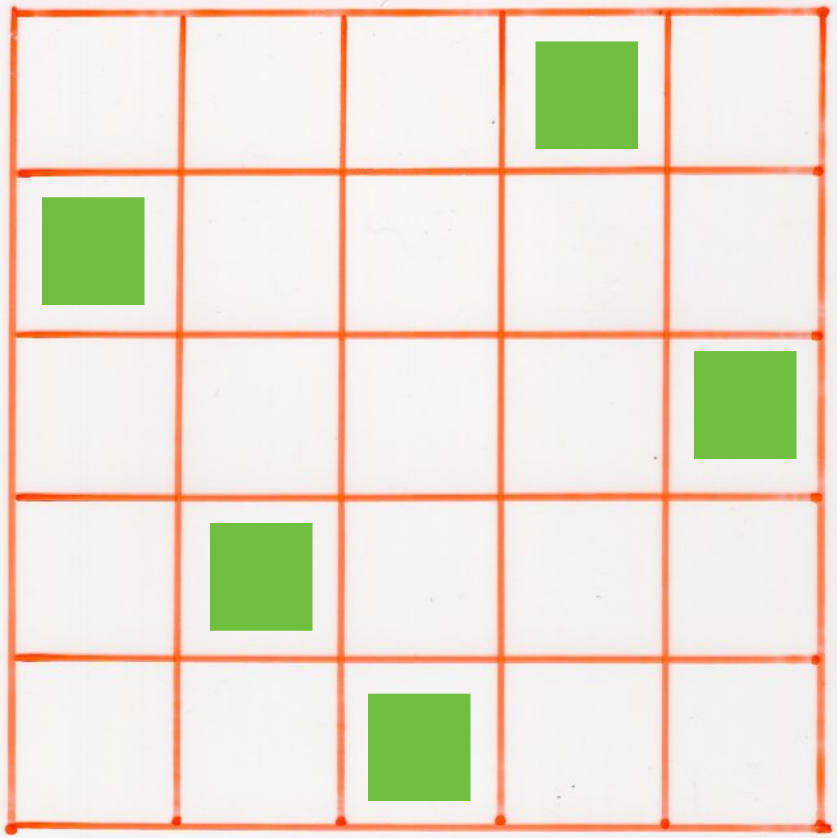


examples

$L = \{\square, \blacksquare\}$

permutations

Q-tableaux

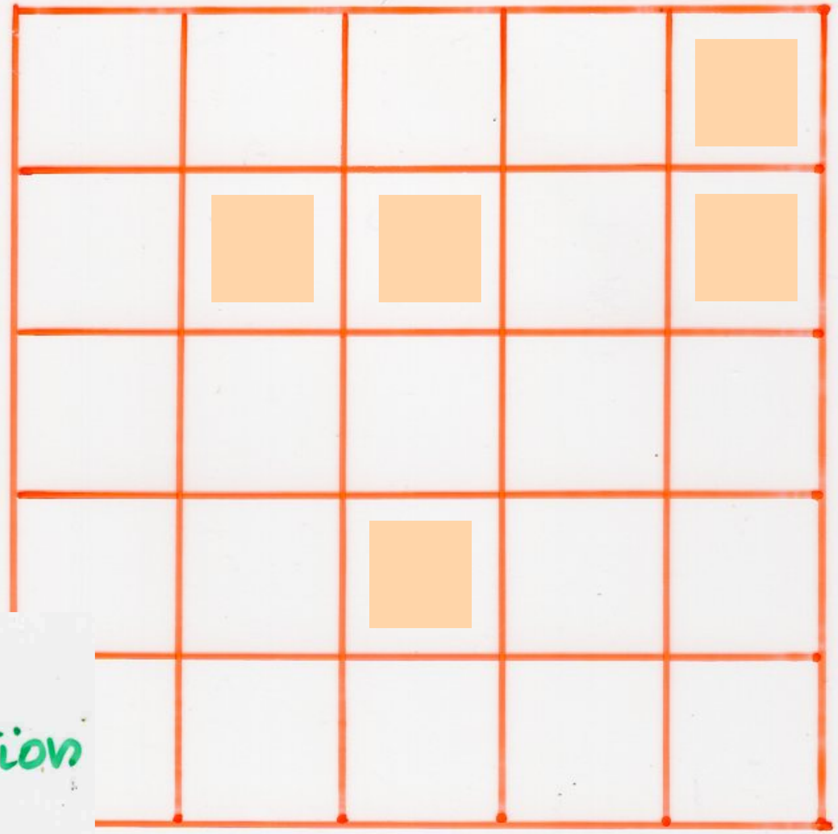


examples

$L = \{\square, \blacksquare\}$

permutations

Q-tableaux

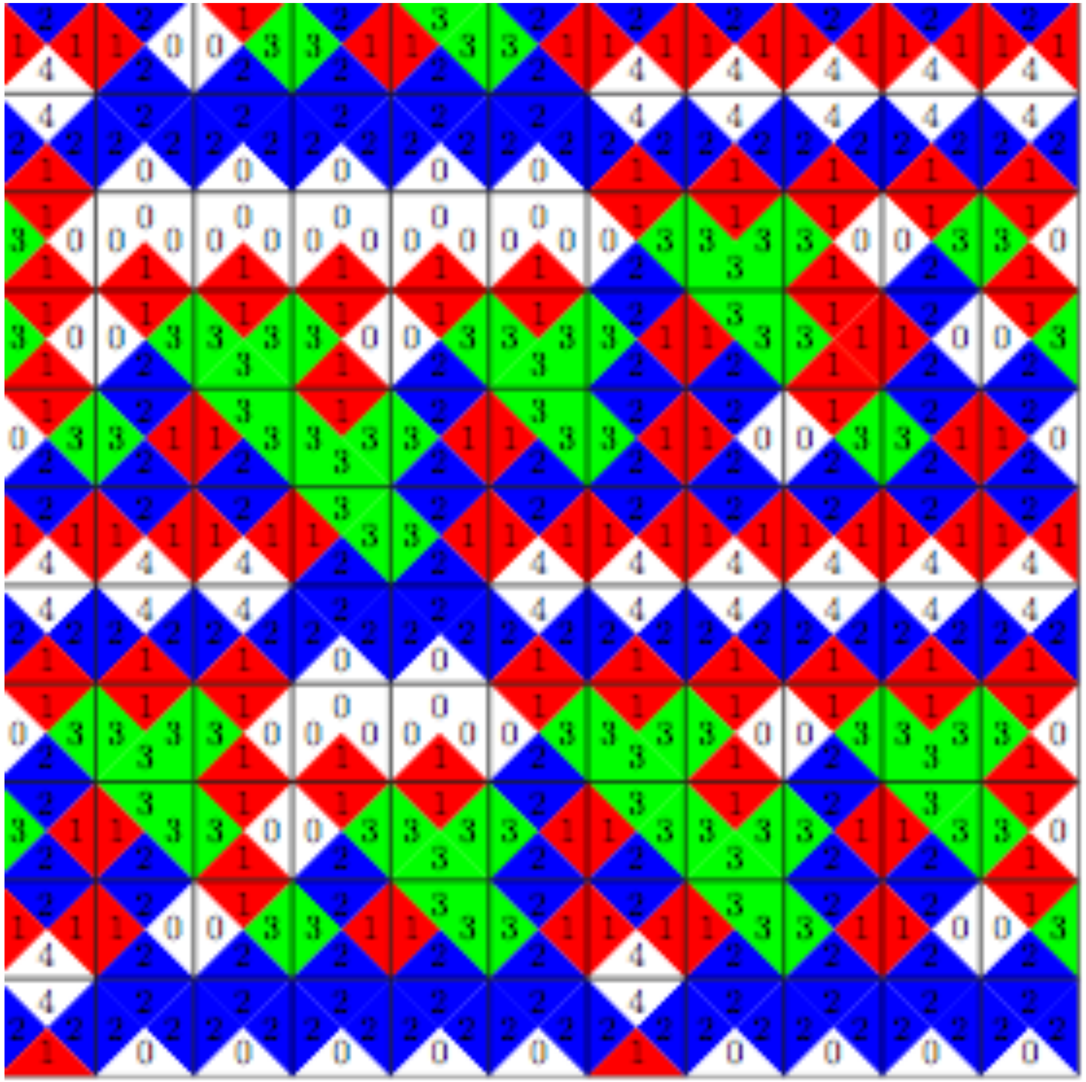


Rothe diagram of a permutation (1800)

Wang tilings

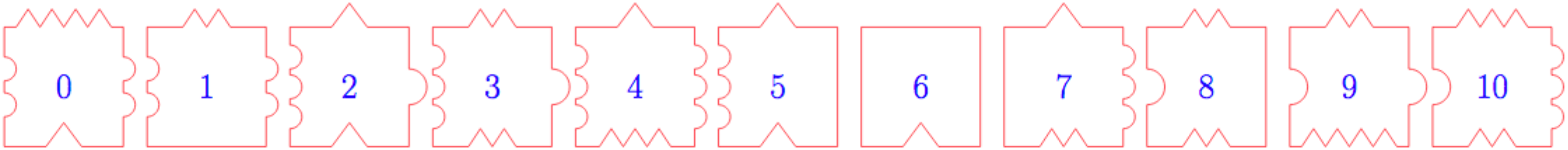
Enumeration ?

Jeandel - Rao (2015)
aperiodic Wang tiling
11 tiles



S. Labbé

JR's 11 tiles ... as closed topological disks using base-1 rep. of $\mathbb{N}$:



152244 valid 7×7 solutions but only 483 $(+\varepsilon)$ extend to $\mathbb{Z} \times \mathbb{Z}$.

(Solution found by Pauline Hubert and Antoine Abram, Sep 2018) quadratic algebra Q

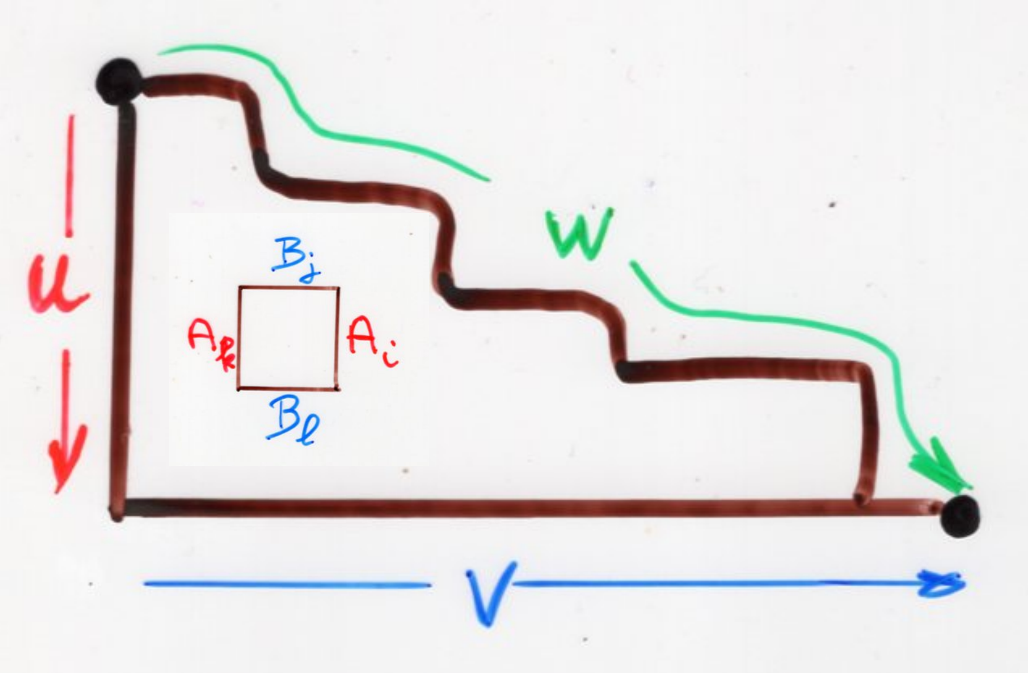
$$B_j A_i = \sum_{k\,l} c_{ij}^{kl} A_k B_l$$

$$c(u, v; w) = \sum_{T} \mathrm{wgt}(T)$$

complete Q-tableau

$\mathrm{uwb}(T) = w$

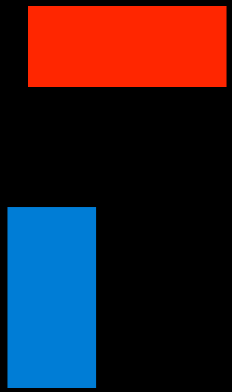
$\mathrm{lwb}(T) = uv$

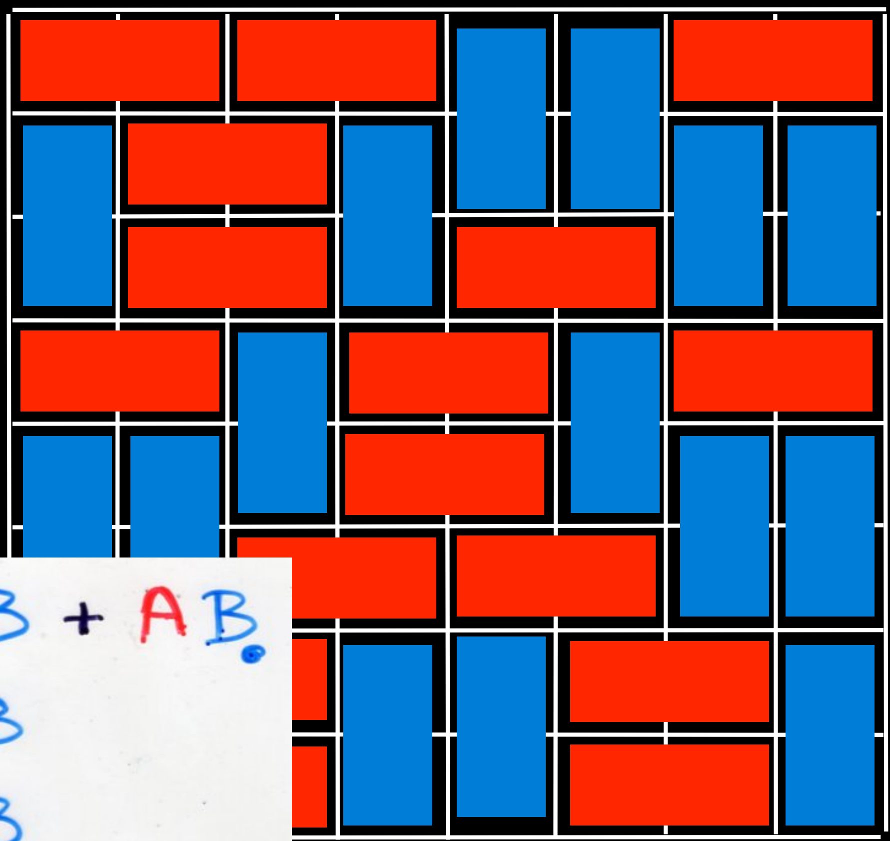


many examples

in general $F(w)$ is a rectangle

pavages d'un échiquier avec des dominos

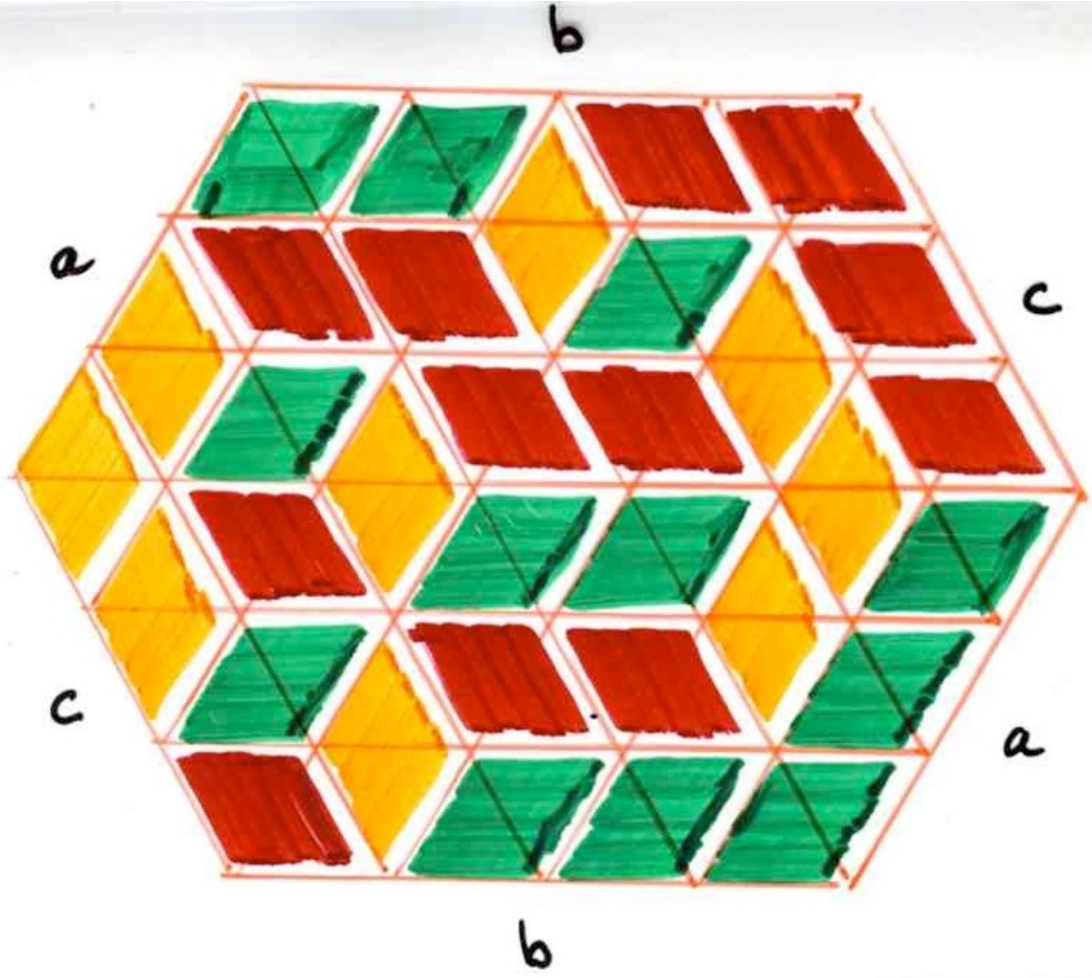




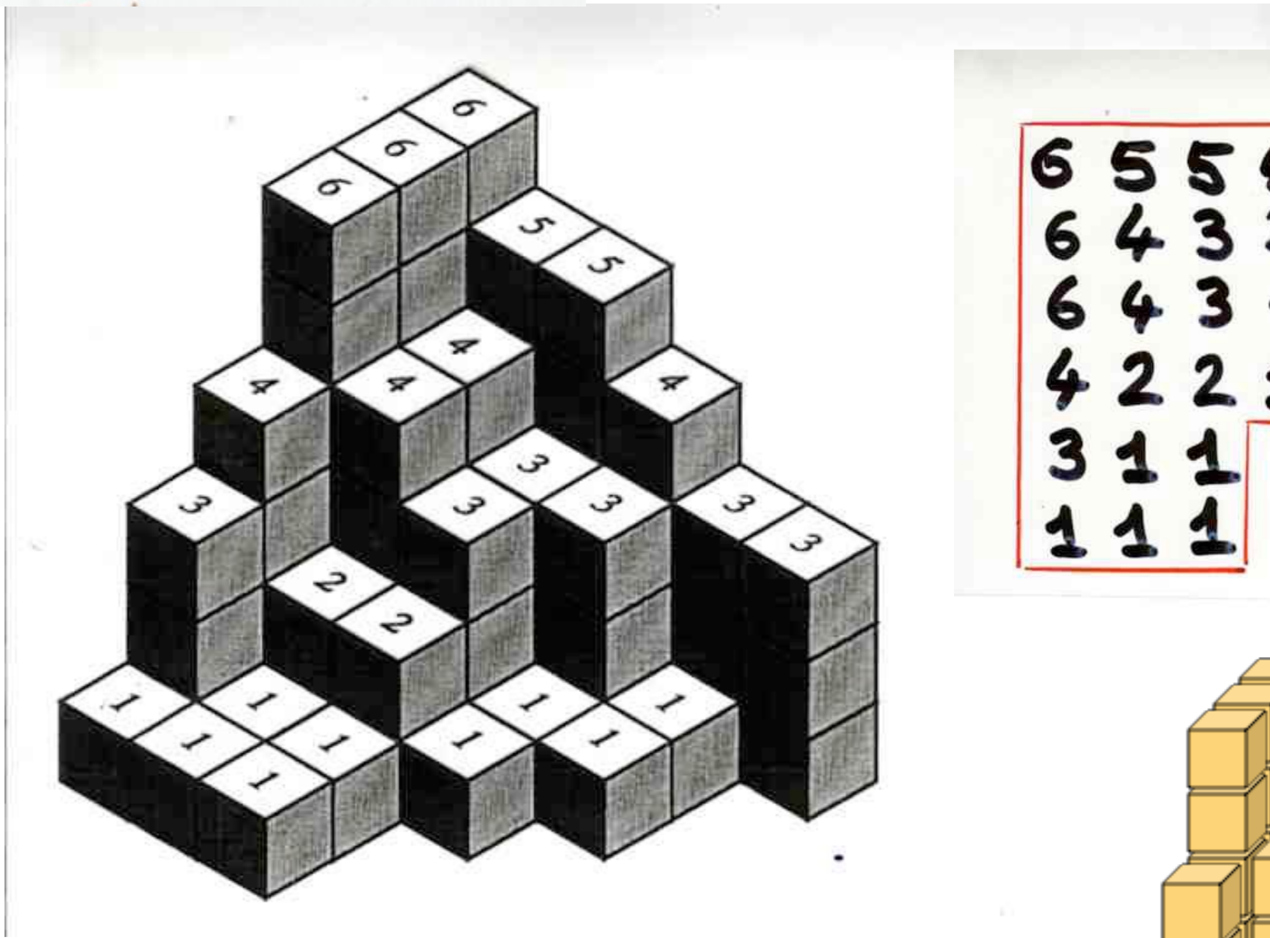
$$\begin{cases} B\,A = A_{\bullet}B + A\,B_{\bullet} \\ B_{\bullet}A = A\,B \\ B\,A_{\bullet} = A\,B \\ B_{\bullet}A_{\bullet} = 0 \end{cases}$$

Q-tableaux

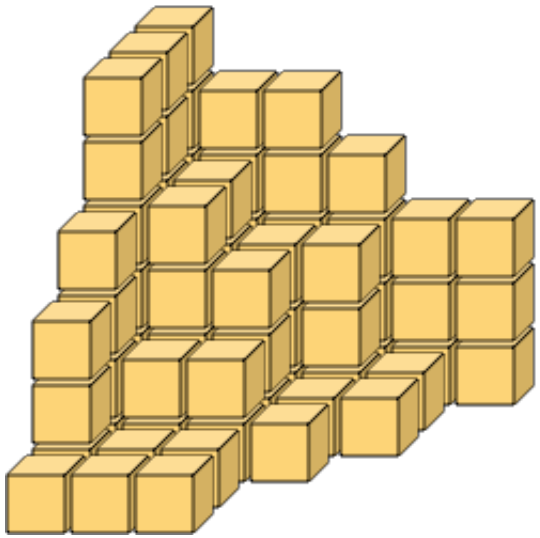
Q-tableaux



Q-tableaux



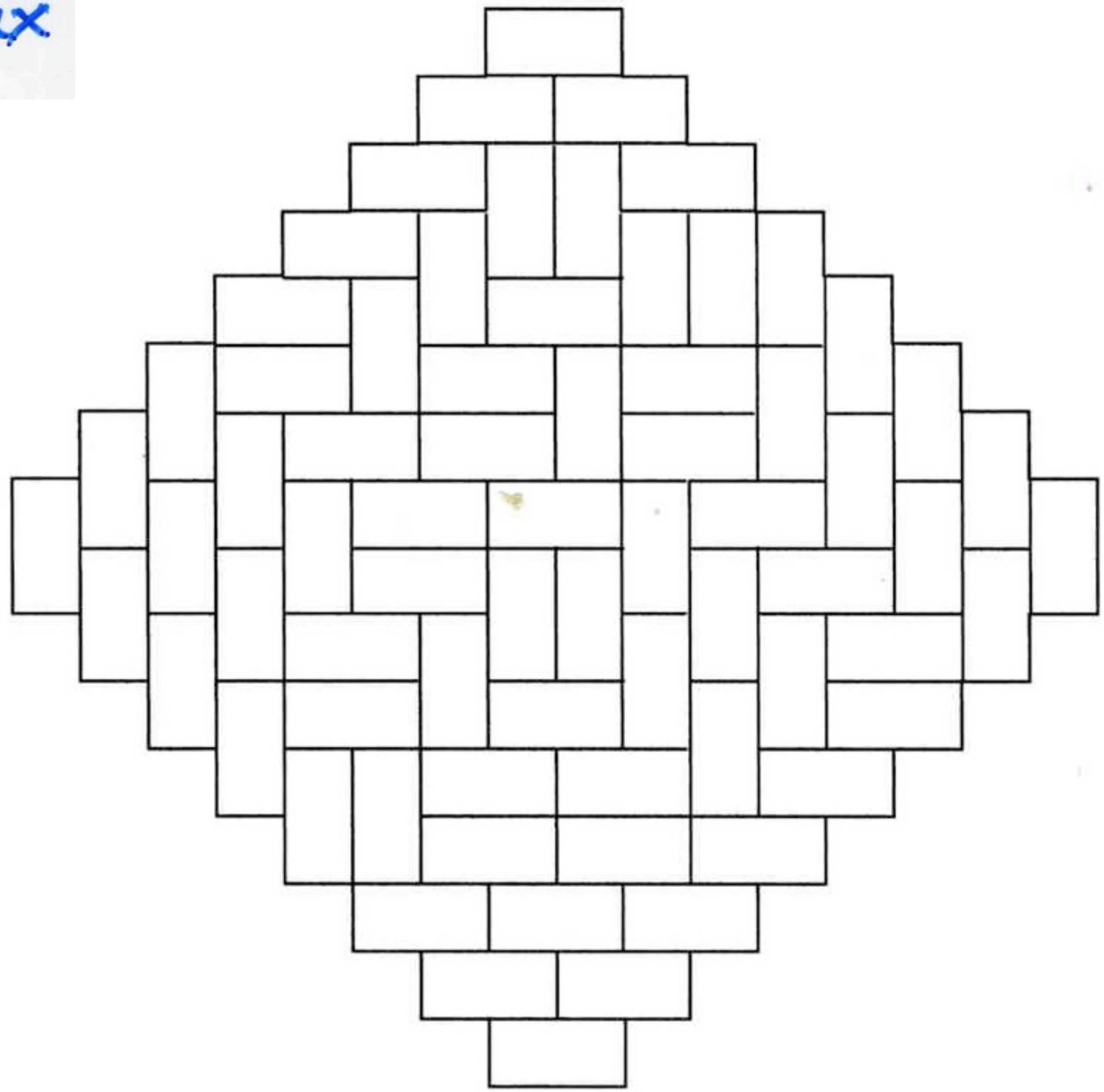
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



plane partitions

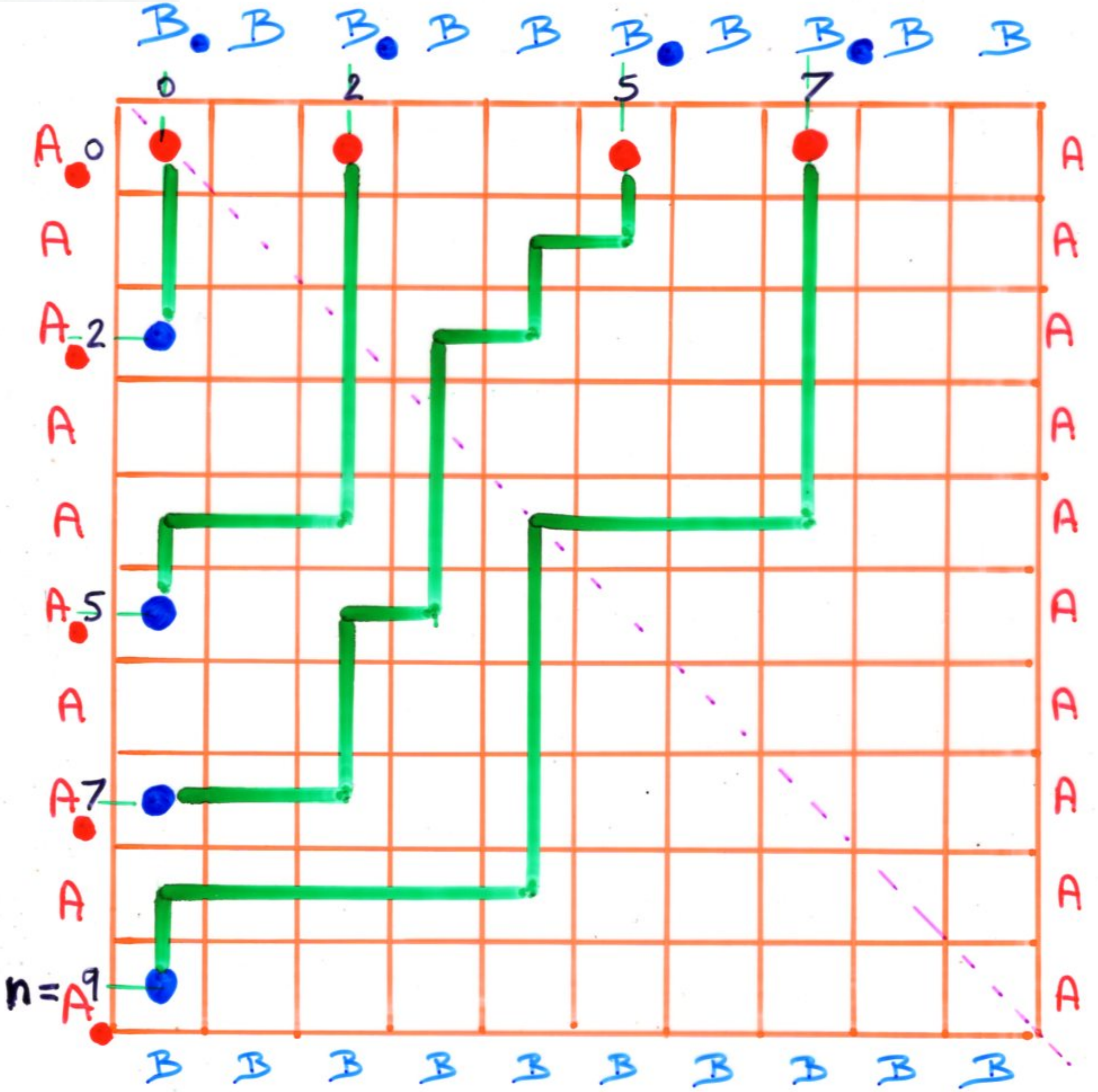
Q-tableaux

Aztec tiling

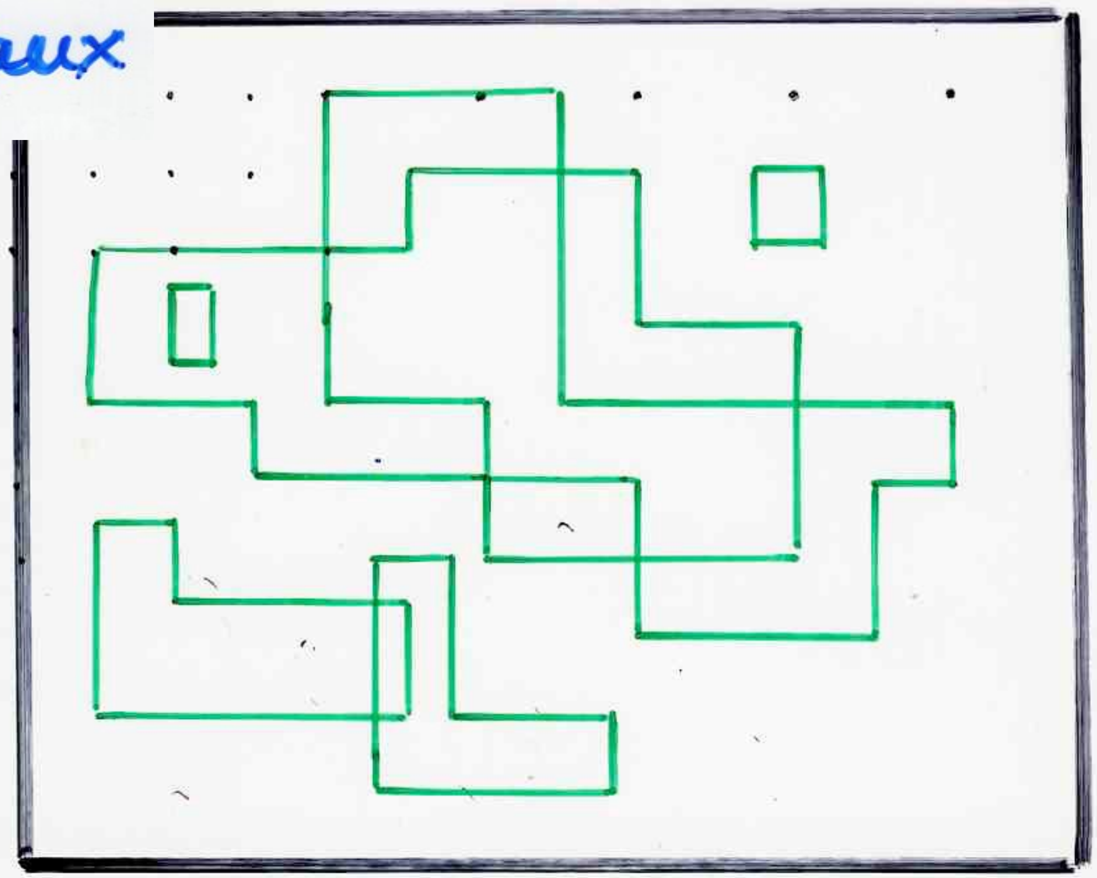


Q-tableaux

non-intersecting configuration of paths



Q-tableaux



"closed" graph

Ising model

Ising model

$$vu = B^m A^n$$
$$uv = A^n B^m$$

Q-tableaux

Def- **ASM** alternating sign matrix

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

(i) entries: 0, 1, −1

(ii) sum of entries in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right.$ = 1

(iii) non-zero entries alternate in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right.$

Reverse Q-tableaux

example with the Weyl-Heisenberg algebra

$$UD = DU + Id$$

reverse Q-tableau

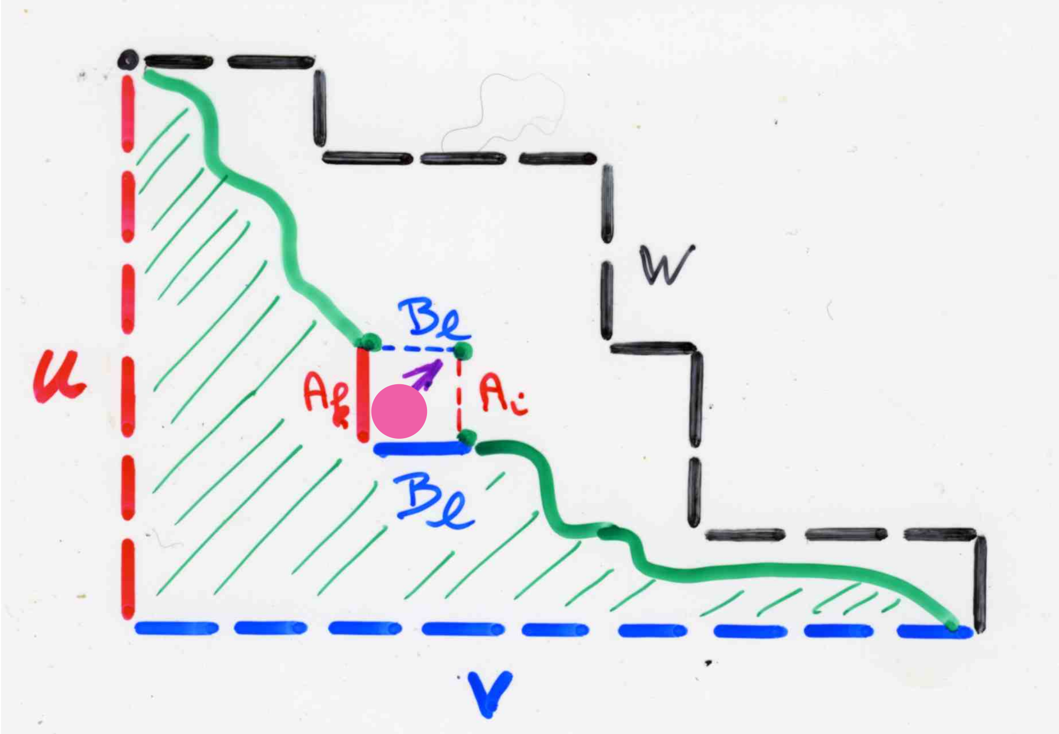
$$\varphi : R \longrightarrow L$$

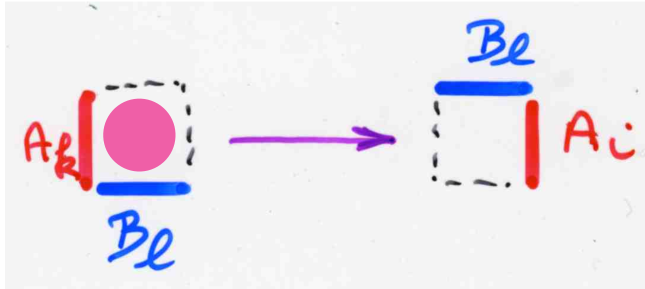
$$\begin{aligned} &= \cdots + \cdots \\ B_j A_i &= \cdots + A_k B_l + \cdots \\ B_* A_* &= \cdots A_k B_l + \cdots \\ B_* A_* &= \cdots + \cdots + A_k B_l + \cdots \\ &= \cdots \end{aligned}$$

(†)

$$\varphi\left(B_* A_* \longrightarrow A_k B_l\right)$$

all distinct





L set of "labels"

$\varphi : R \to L$ (+)

bijection

complete Q-tableau T $\longleftrightarrow$ reverse Q-tableau

$lwb(T) = uv$ "uv-compatible"

Weil-Heisenberg algebra

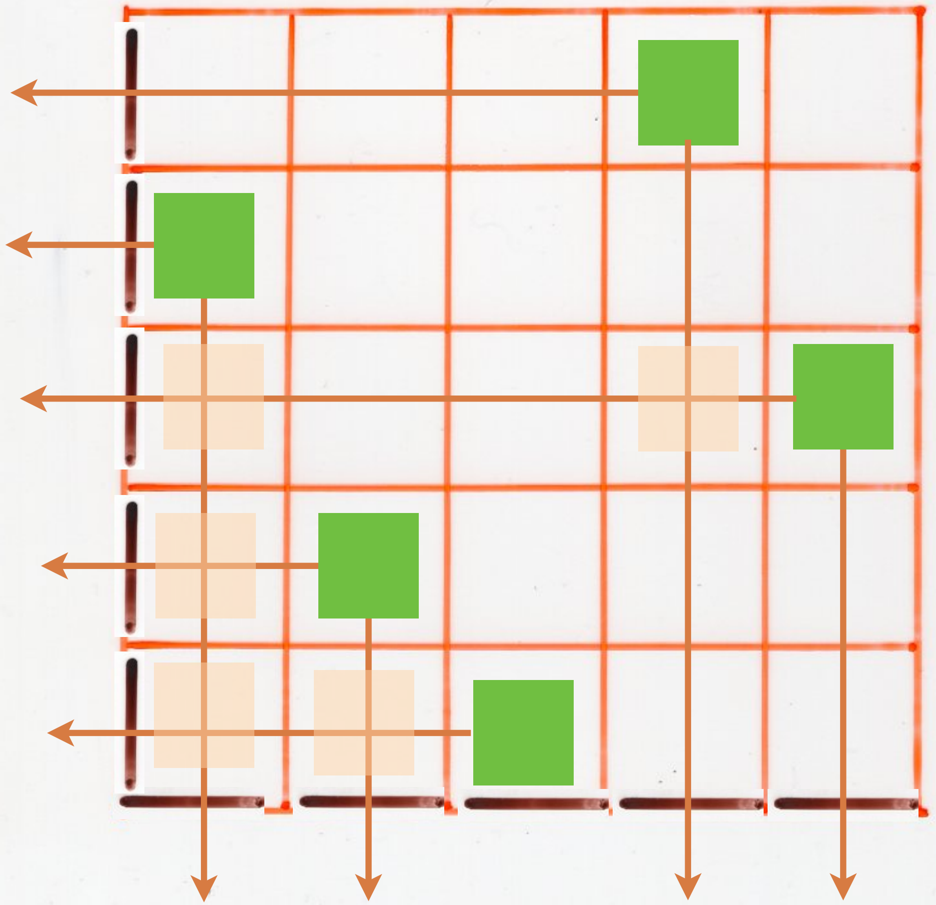
$$Q \begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$

$$UD \longrightarrow tYX$$

$$XY \longrightarrow q_2 YX$$

$$Q\begin{cases} UD = q_1\, DU + t\, \boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\, \boxed{YX} \end{cases}$$

$$UD \longrightarrow t\,YX$$
$$XY \longrightarrow q_2\,YX$$



$$Q\begin{cases} UD = q_1\, DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$

$$UD \longrightarrow tYX$$

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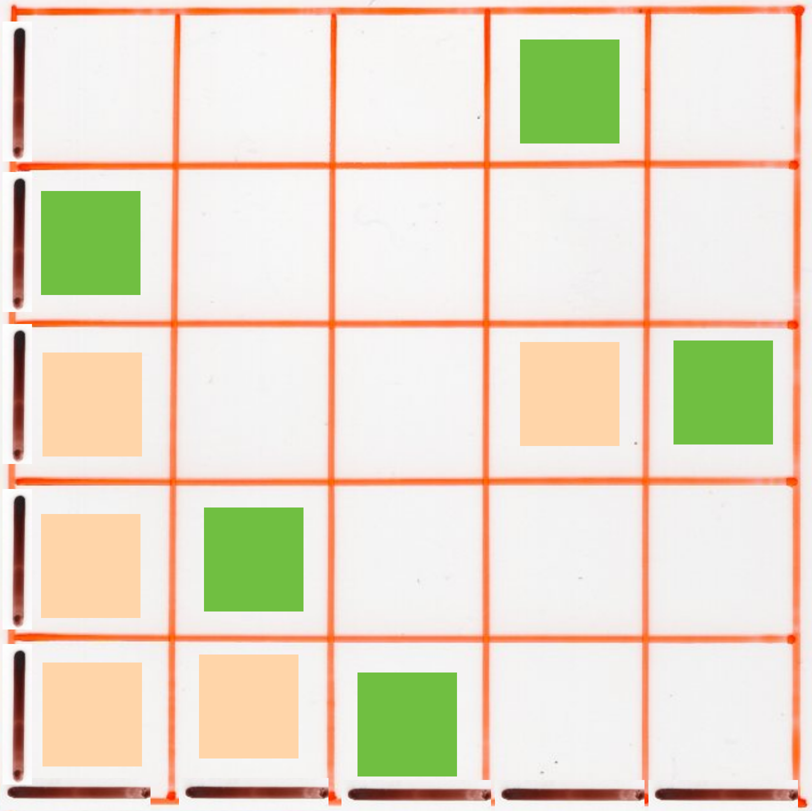
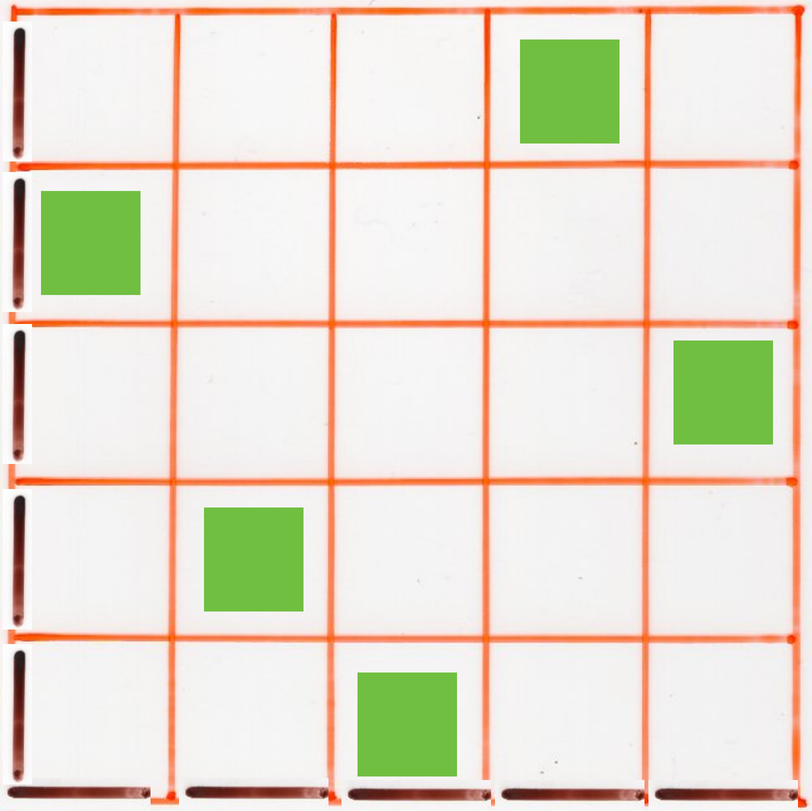


reverse Q-tableau

$$\mathbf{Q}\begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$

$$UD \longrightarrow t\,YX$$
$$XY \longrightarrow q_2\,YX$$

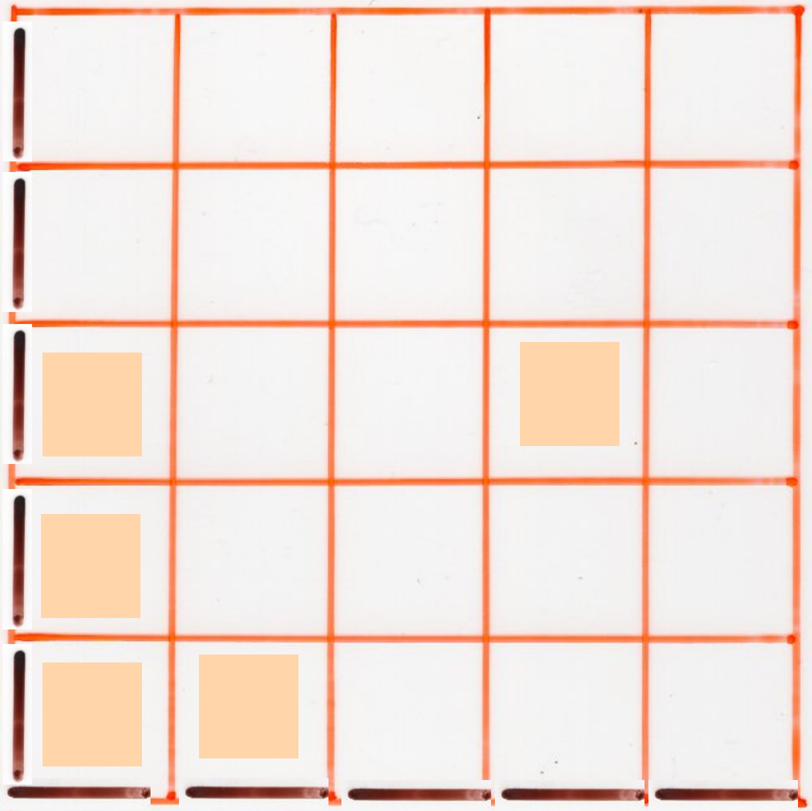


reverse Q-tableau

$$Q\begin{cases} UD = q_1\, DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$

$$UD \longrightarrow tYX$$

$$XY \longrightarrow q_2\, YX$$



reverse Q-tableau

reverse quadratic algebra

reverse quadratic algebra

$$Q: \left\{ B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \right.$$

$\forall\, i \in I,\ \forall\, j \in J$

$$Q^{\dagger} \left\{ A_k B_l = \sum_{i,j} c_{ij}^{kl} B_j A_i \right.$$

$\forall\, k \in I,\ l \in J$

(possibly 0)

Q quadratic algebra

Q^{+} reverse quadratic algebra

reverse Q- tableaux

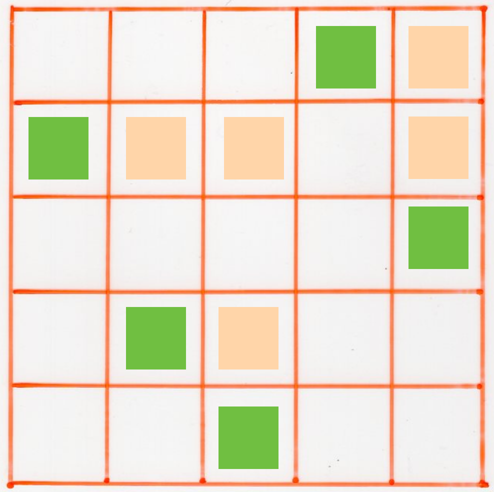
= Q^{+}- tableaux

complete Q- tableau

= complete Q^{+}- tableau

Weyl-Heisenberg algebra

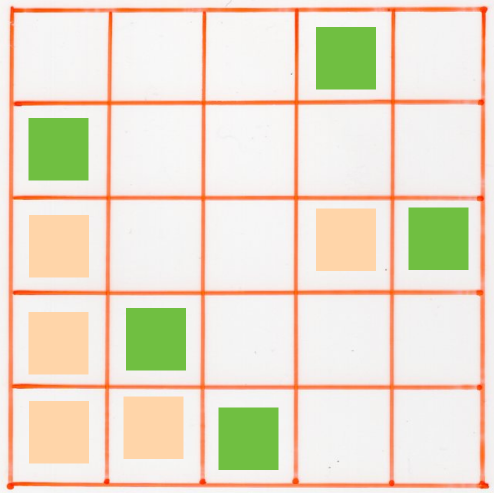
Q-tableaux



$$Q \begin{cases} UD = q_1 DU + tYX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$

self-reciprocal quadratic algebra

$$Q^{\dagger} \begin{cases} YX = q_2 XY + tUD \\ YU = UY \\ DX = XD \\ DU = q_1 UD \end{cases}$$



reverse Q-tableau

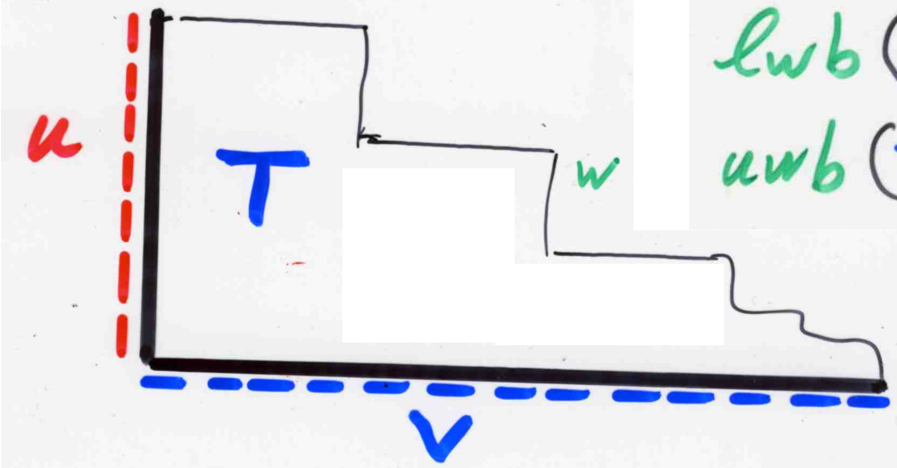
Duplication of equations

in a quadratic algebra

Suppose

$$\begin{cases} B_* A_* = \cdots + \cdots \\ B_j A_i = \cdots + A_k B_l + \cdots \\ \quad \vdots \qquad \vdots \; = \cdots \\ B_* A_* = \underbrace{\cdots\cdots\cdots}_{\text{terms all distinct}} \end{cases}$$

$$\varphi : R \to L = \{ \square \} \qquad (\dagger)$$



$lwb(T) = uv$

$uwb(T) = w$

$u \in \mathcal{A}^*$

$v \in \mathcal{B}^*$

complete Q-tableau

$=$ complete Q^+-tableau

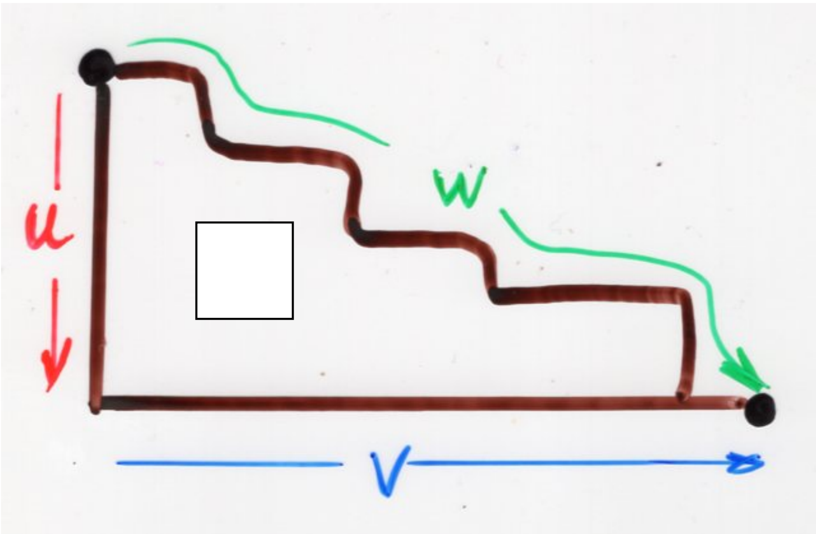
($=$ reverse complete Q-tableau)

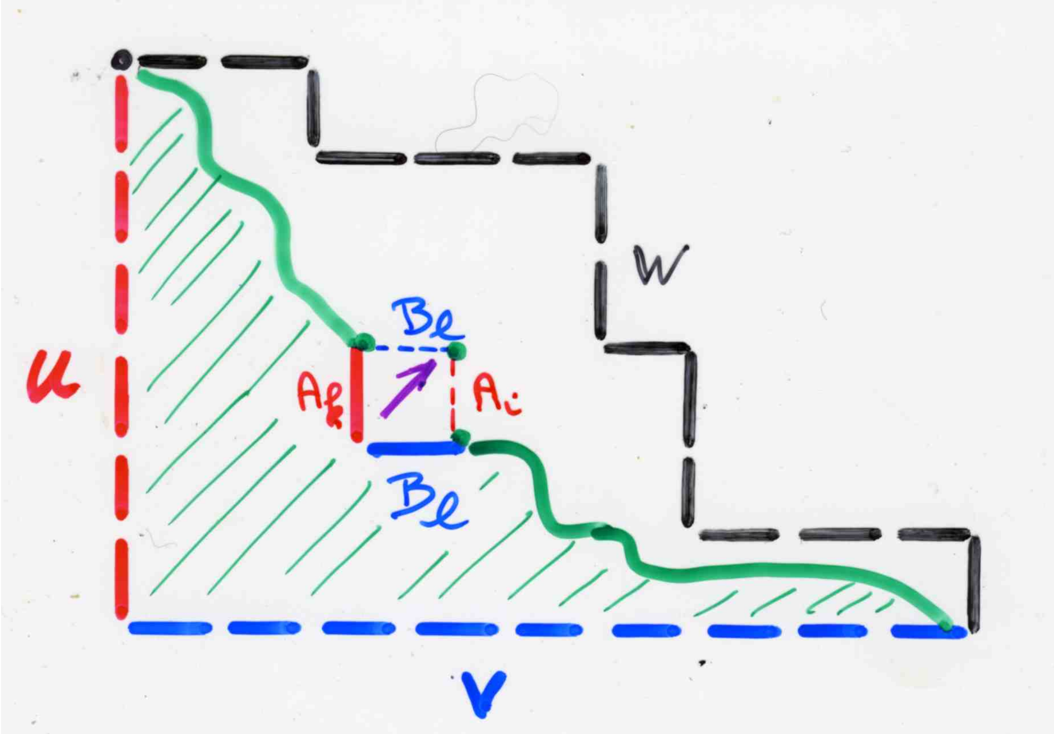
$\updownarrow$

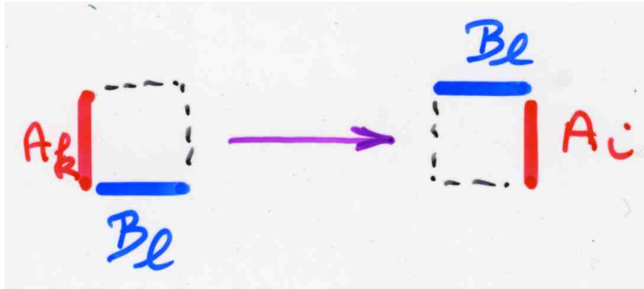
reverse Q-tableau

$\updownarrow$

(u, v) coding of the complete Q-tableau T







(u, v) coding of the complete Q-tableau T

$$(\dagger) \qquad \left\{ \begin{array}{l} B_* A_* = \; \underbrace{\text{-- - + - - - - -}}_{} \\ B_j A_i = \dots + A_k B_l + \dots \\ \;\vdots \quad \vdots \; = \text{-- - - - - - -} \\ B_* A_* = \; \underbrace{\dots\dots\dots}_{\text{terms all distinct}} \end{array} \right.$$

Duplication of equations

in quadratic algebra

Example: $UD = DU + Id$

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$\begin{cases} UD = DU + YX \\ UY = YU \\ XU = UX \\ XY = YX \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$X_2 Y_2 = Y_3 X_3$$

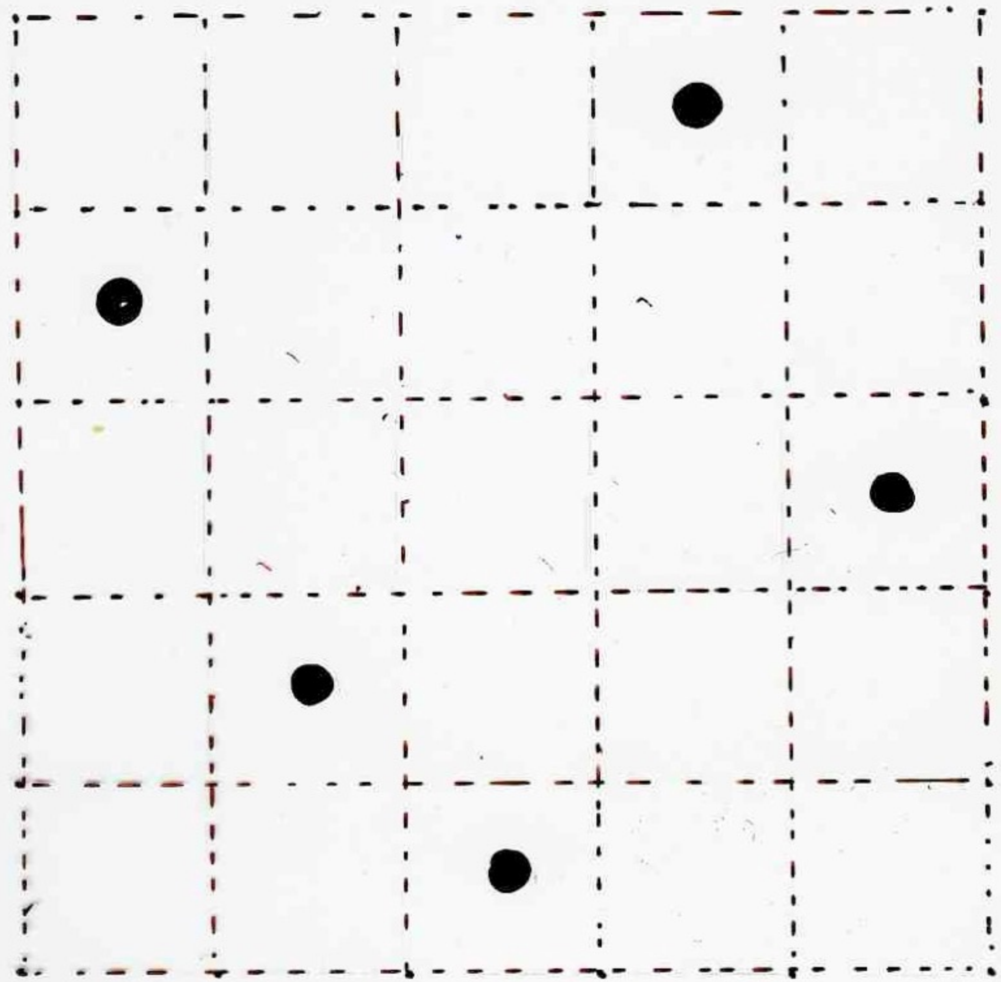
$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

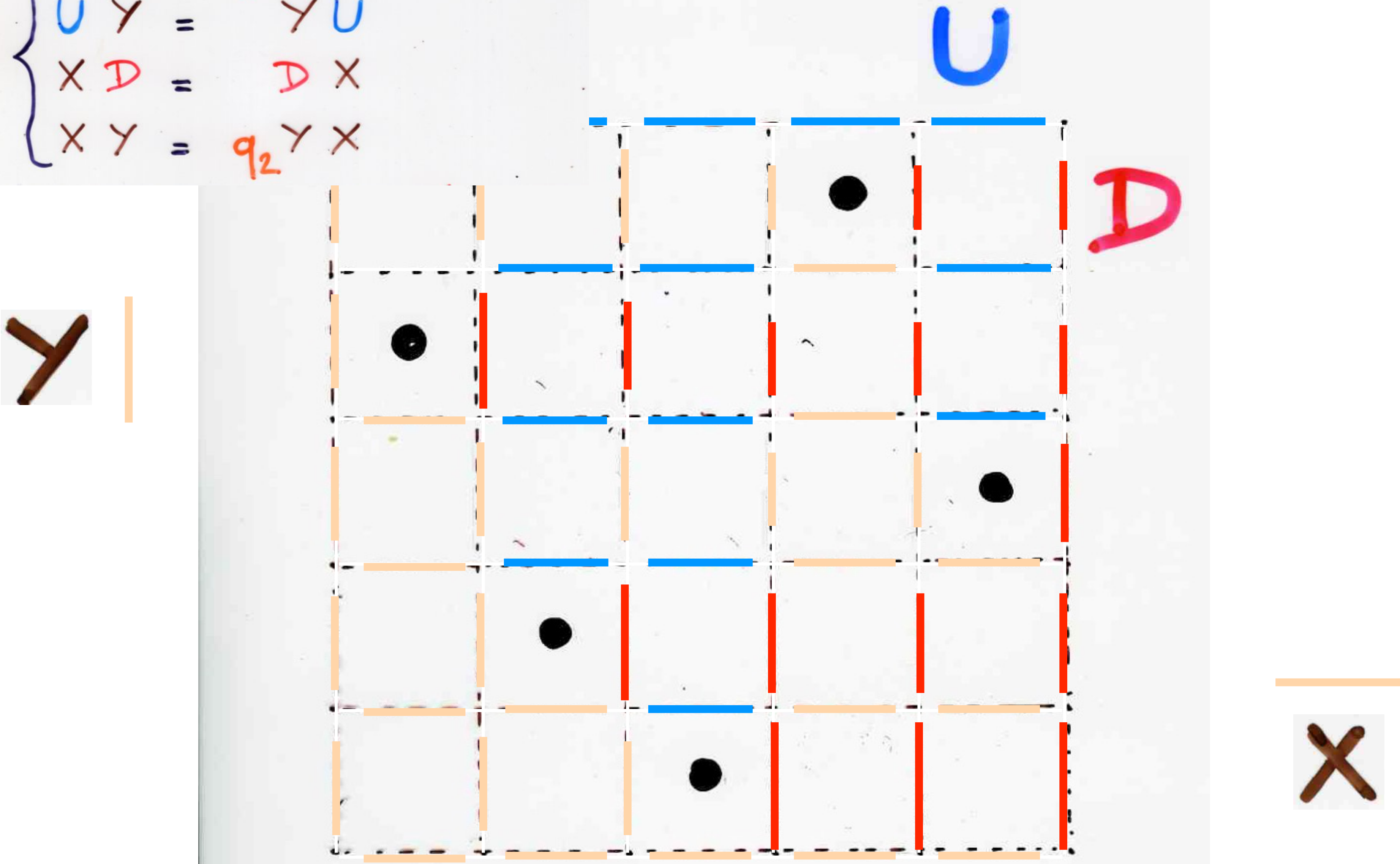
$$UD = DU + Y_1 X_1$$
$$X_1 Y_1 = Y_2 X_2$$
$$X_2 Y_2 = Y_3 X_3$$
$$\cdots\cdots$$
$$X_i Y_i = Y_{i+1} X_{i+1}$$
$$\cdots\cdots$$

$$U Y_i = Y_i U$$
$$X_j D = D X_j$$

$$X_j Y_i = Y_i X_j$$
for $i \neq j$

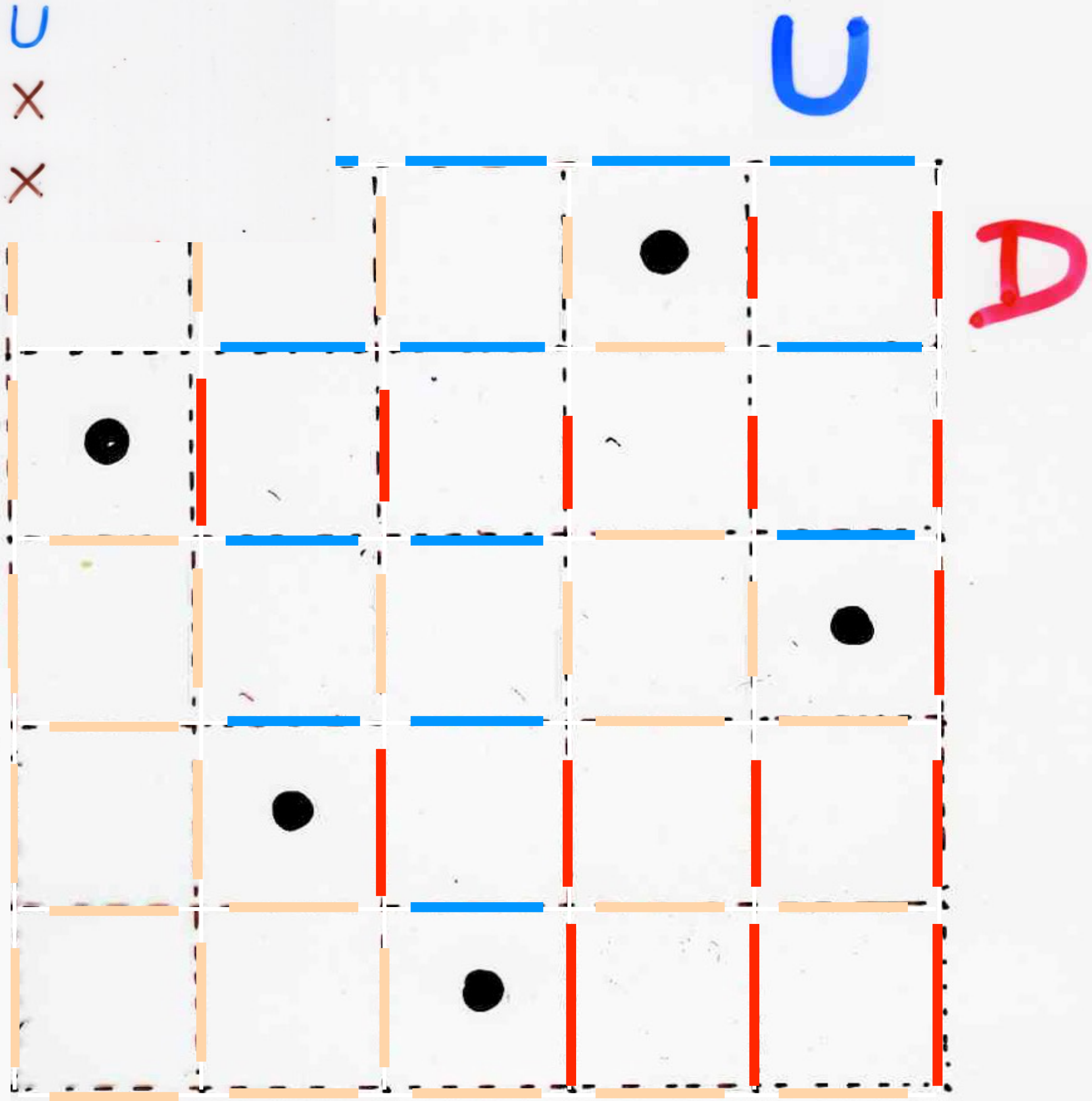


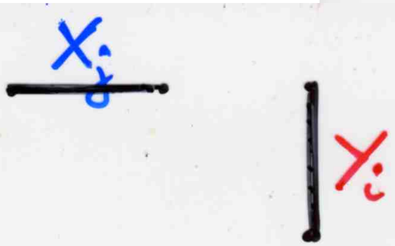
$$Q\begin{cases} UD = q_1 DU + t\,YX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$



$$\mathbf{Q}\begin{cases} UD = q_1 DU + t\,YX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$

$$UD = DU + Y_1 X_1$$



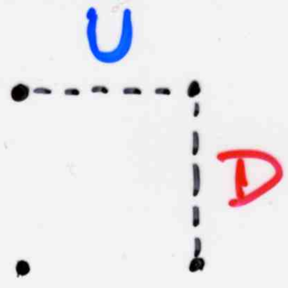


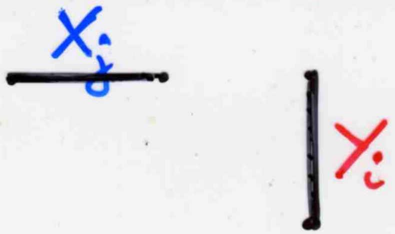
$$UY_i = Y_i U$$
$$X_j D = D X_j$$

$$X_j Y_i = Y_i X_j$$
for $i \neq j$

$$X_i Y_i = Y_{i+1} X_{i+1}$$

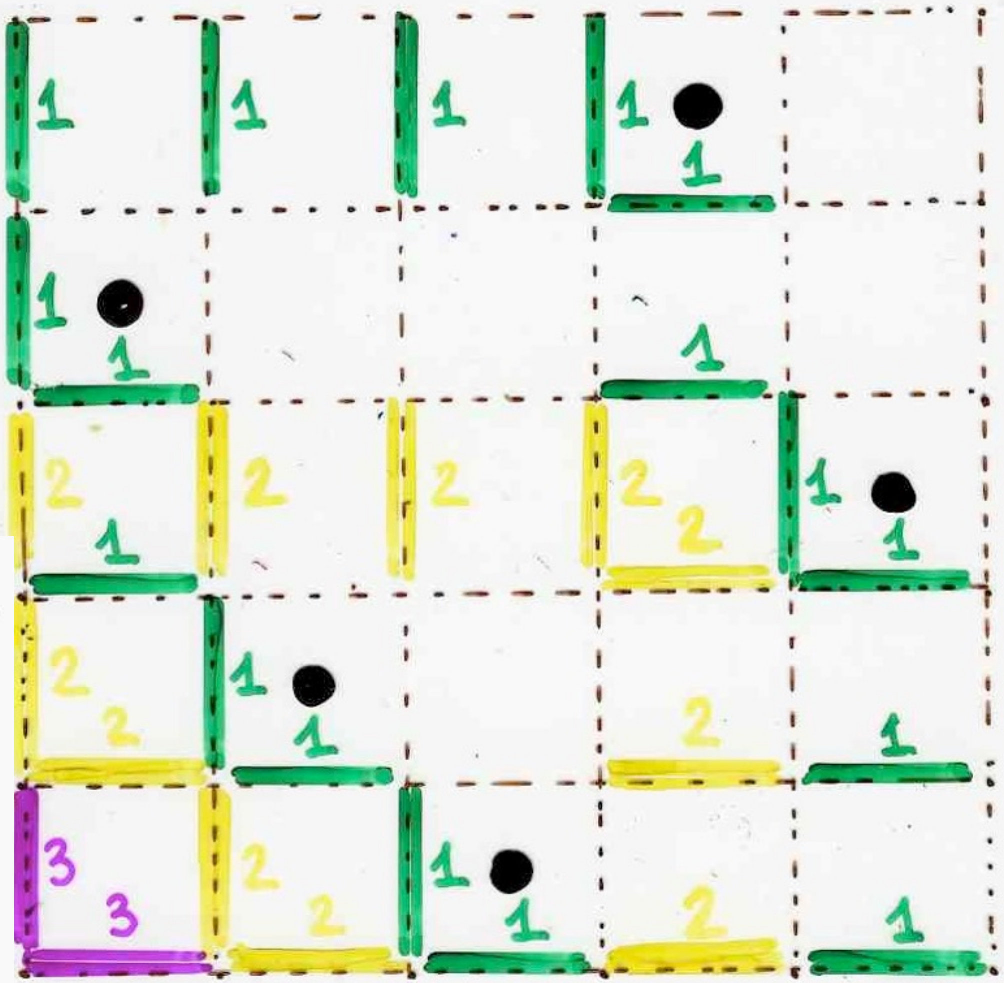
$UD = DU + Y_1 X_1$





$UY_i = Y_iU$

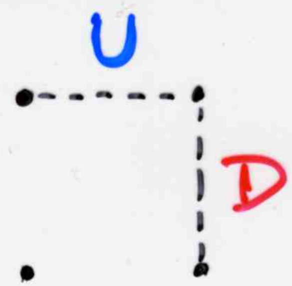
$X_jD = DX_j$

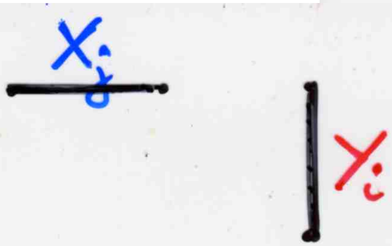


$X_jY_i = Y_iX_j$ for $i \neq j$

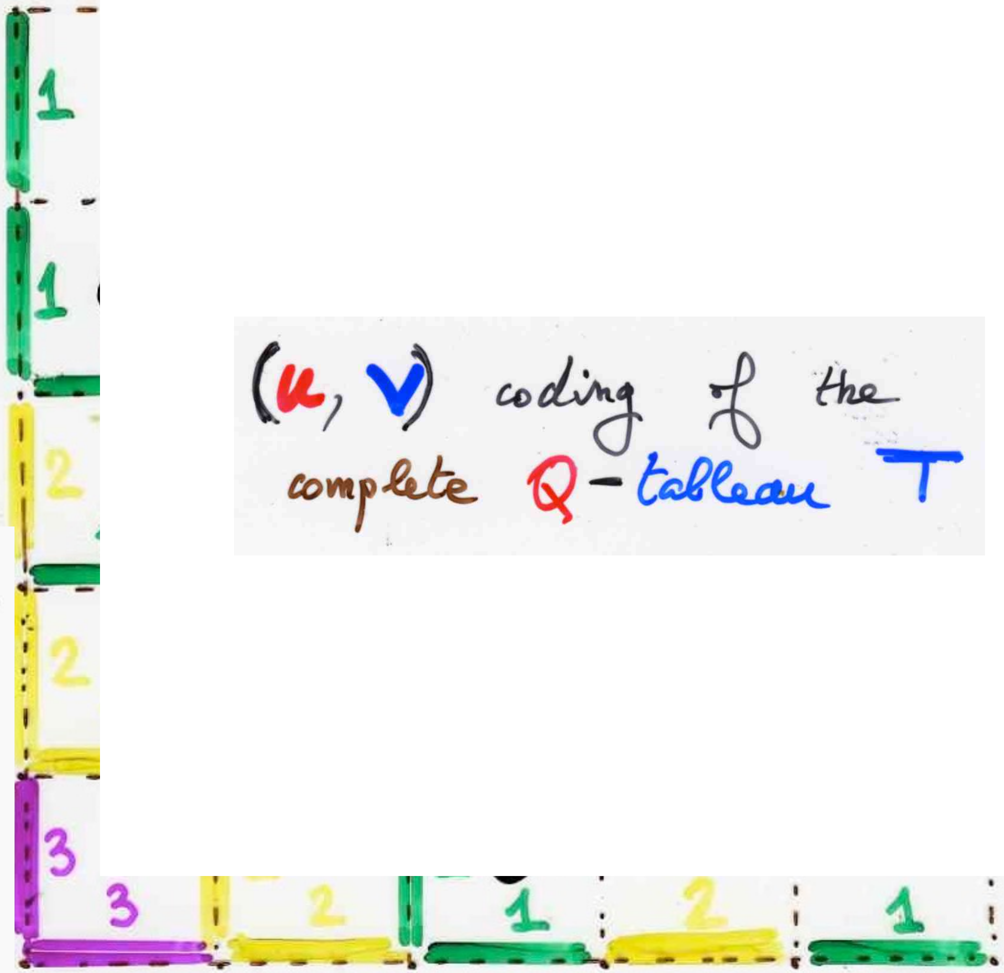
$X_iY_i = Y_{i+1}X_{i+1}$

$$UD = DU + Y_1 X_1$$





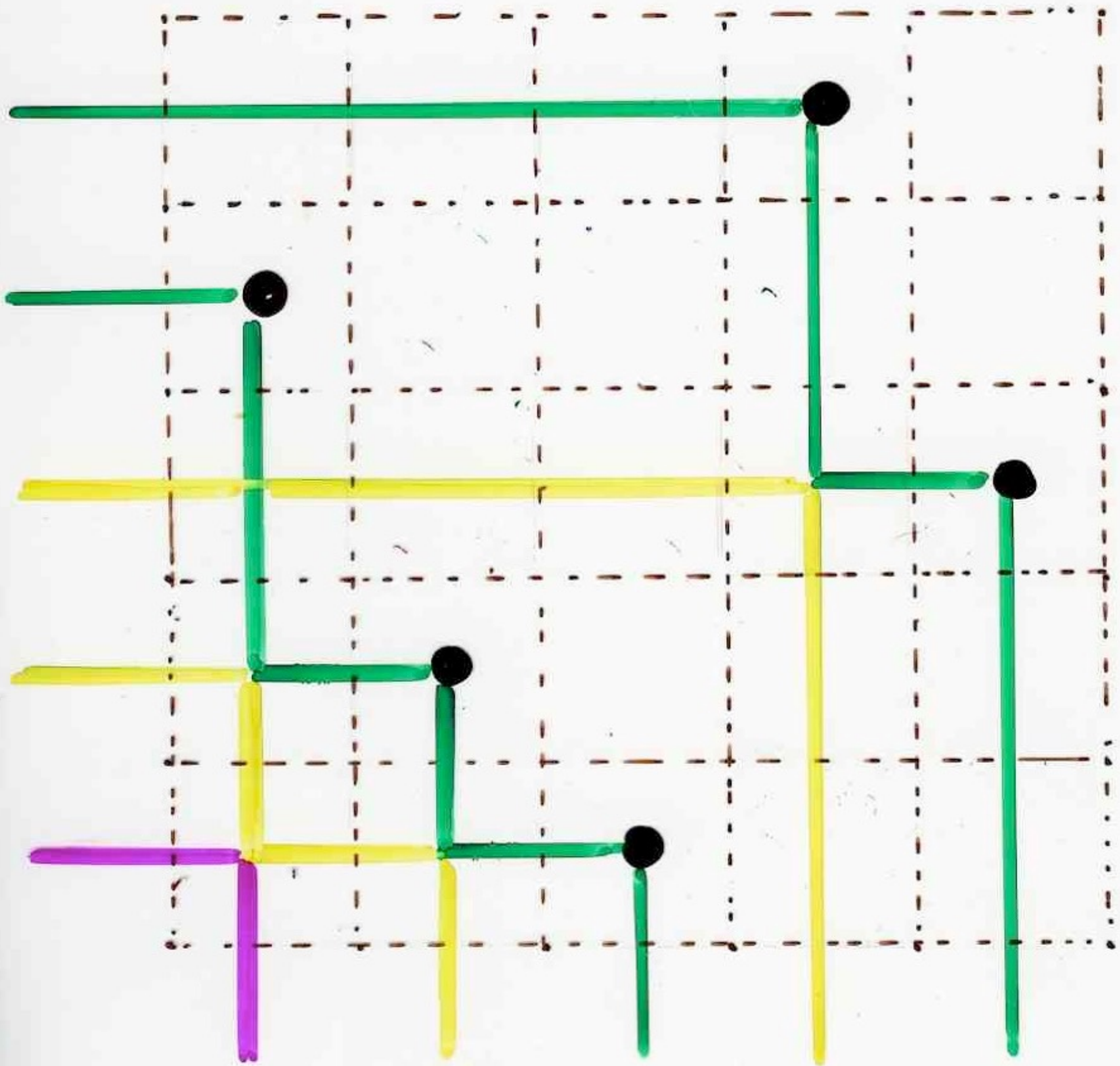
$$U Y_i = Y_i U$$
$$X_j D = D X_j$$



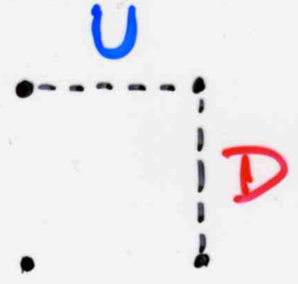
(u, v) coding of the complete Q-tableau T

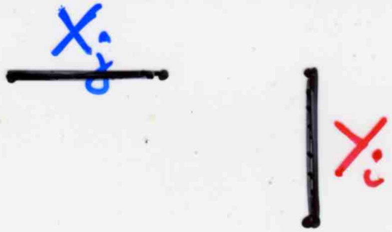
$$X_j Y_i = Y_i X_j$$
for $i \neq j$

$$X_i Y_i = Y_{i+1} X_{i+1}$$

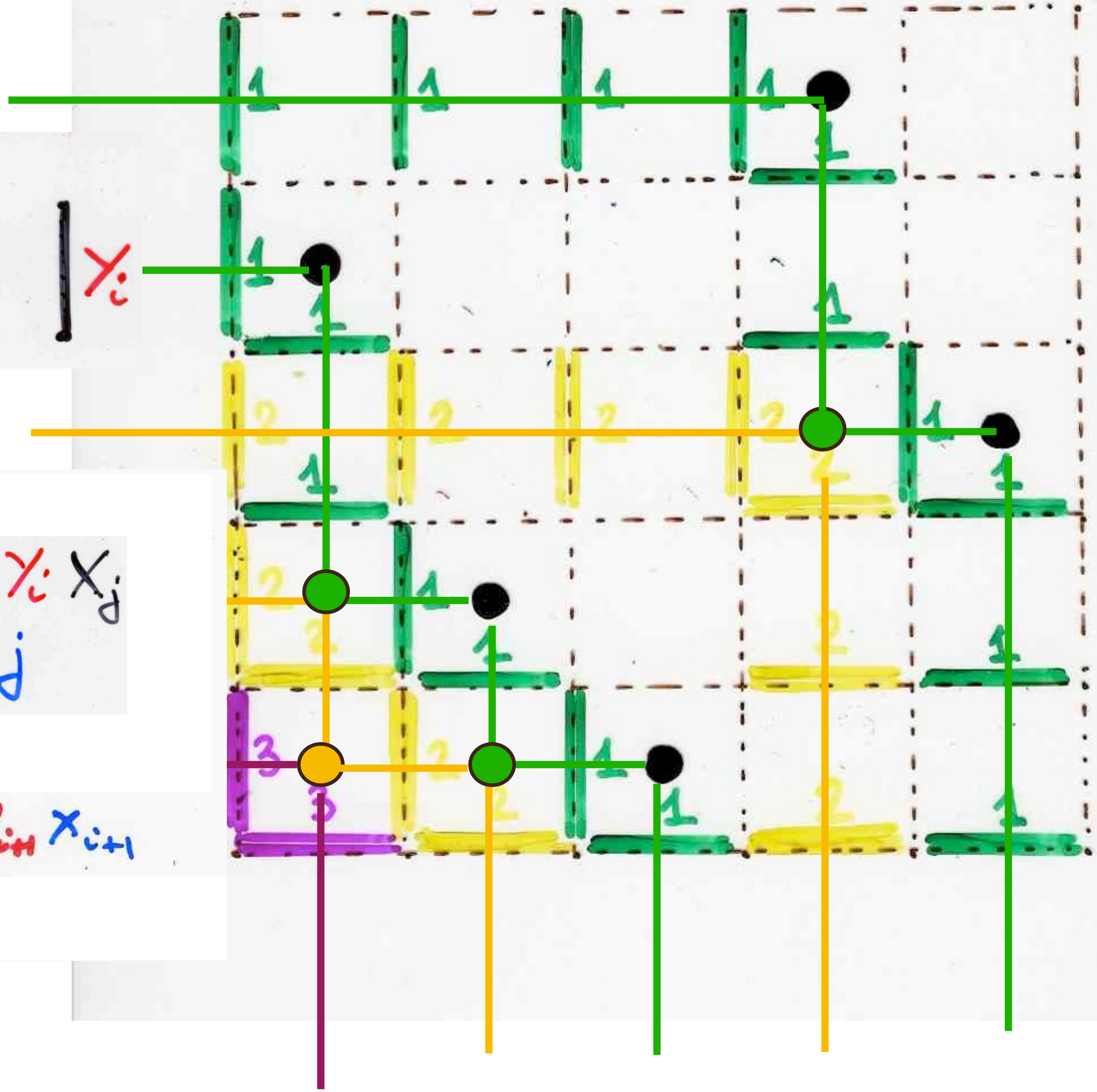


$$UD = DU + Y_1 X_1$$



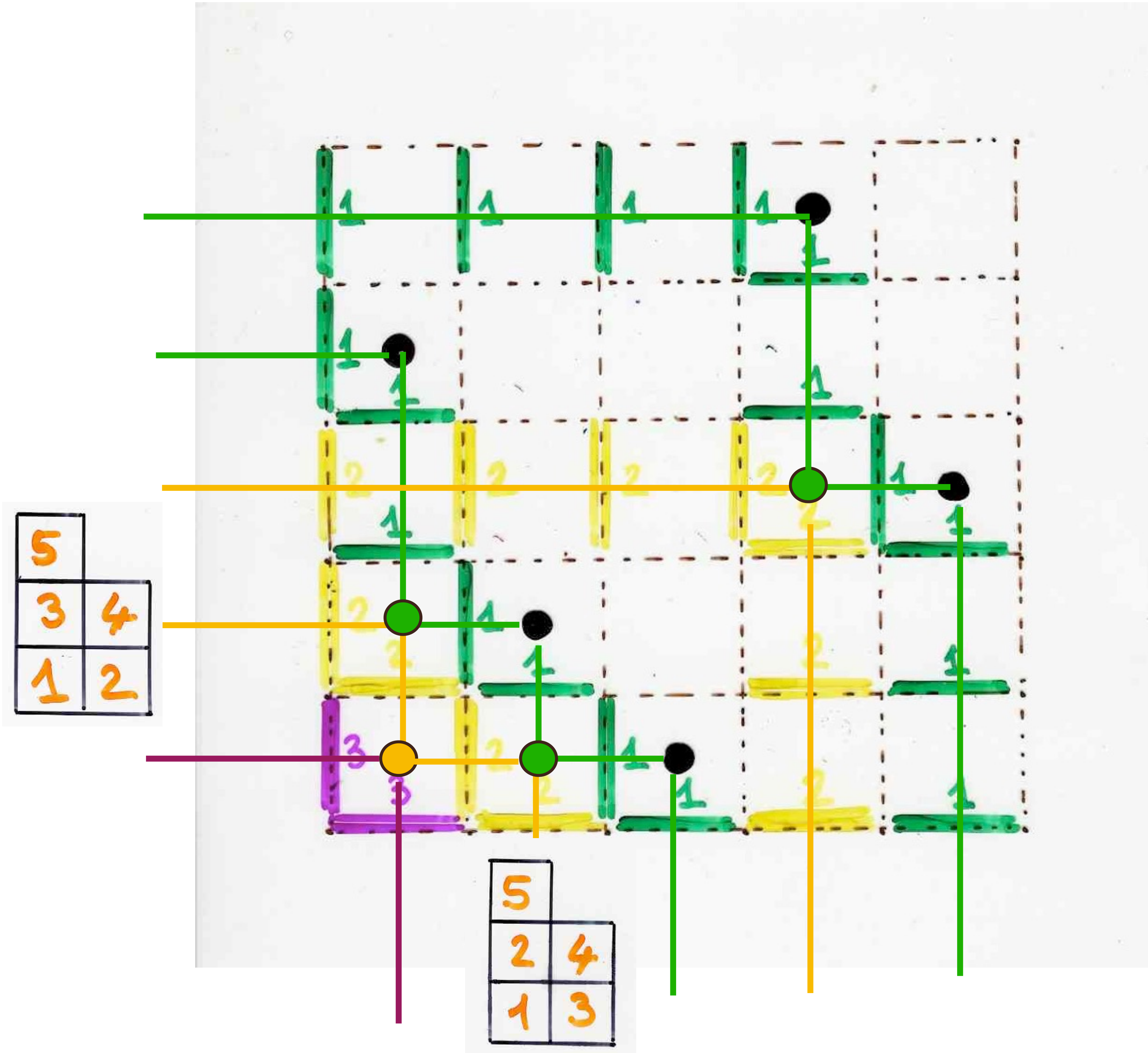


$$U Y_i = Y_i U$$
$$X_j D = D X_j$$



$$X_j Y_i = Y_i X_j$$
for $i \neq j$

$$X_i Y_i = Y_{i+1} X_{i+1}$$

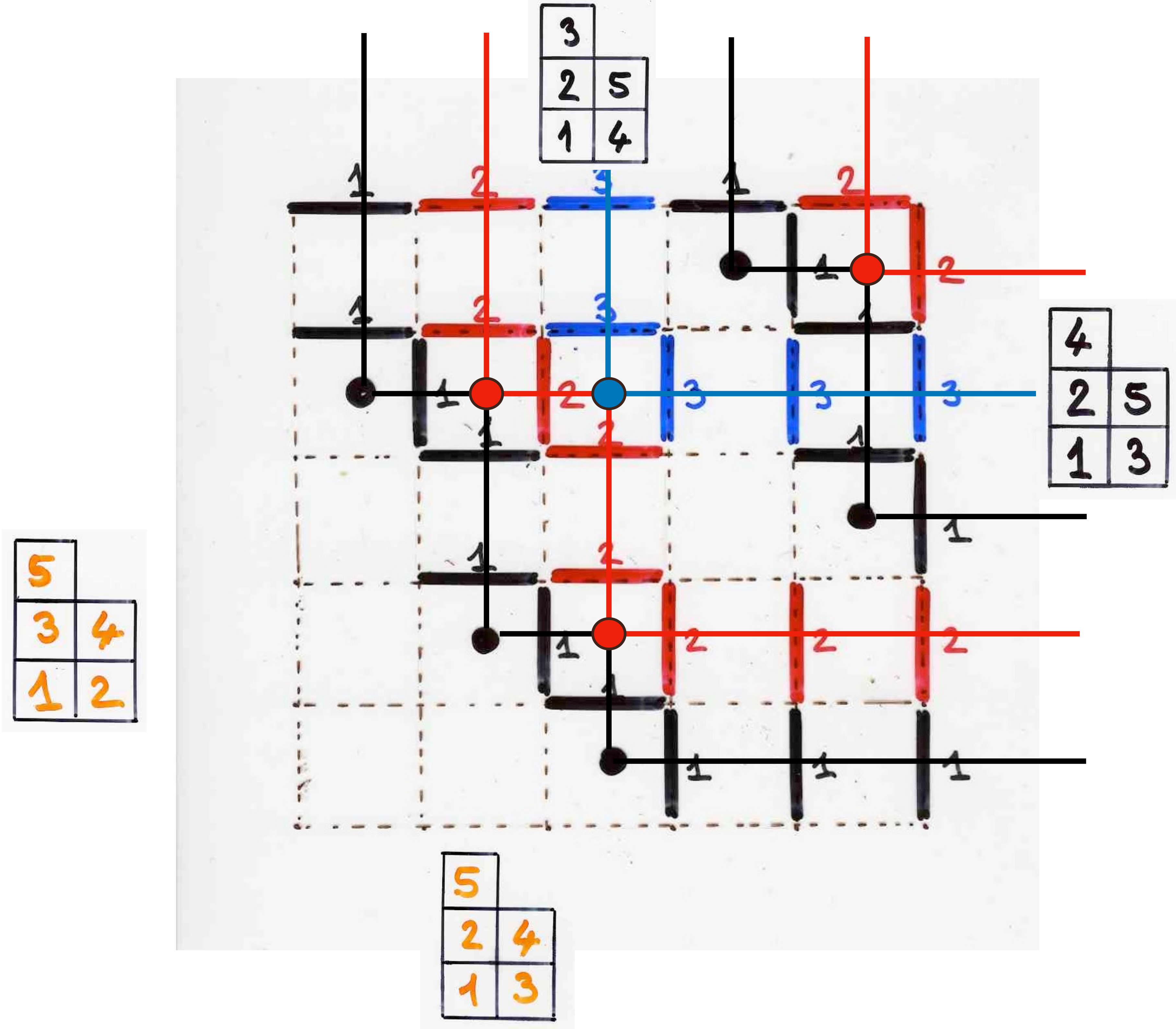


5
3 4
1 2
5
2 4
1 3

Weil-Heisenberg algebra

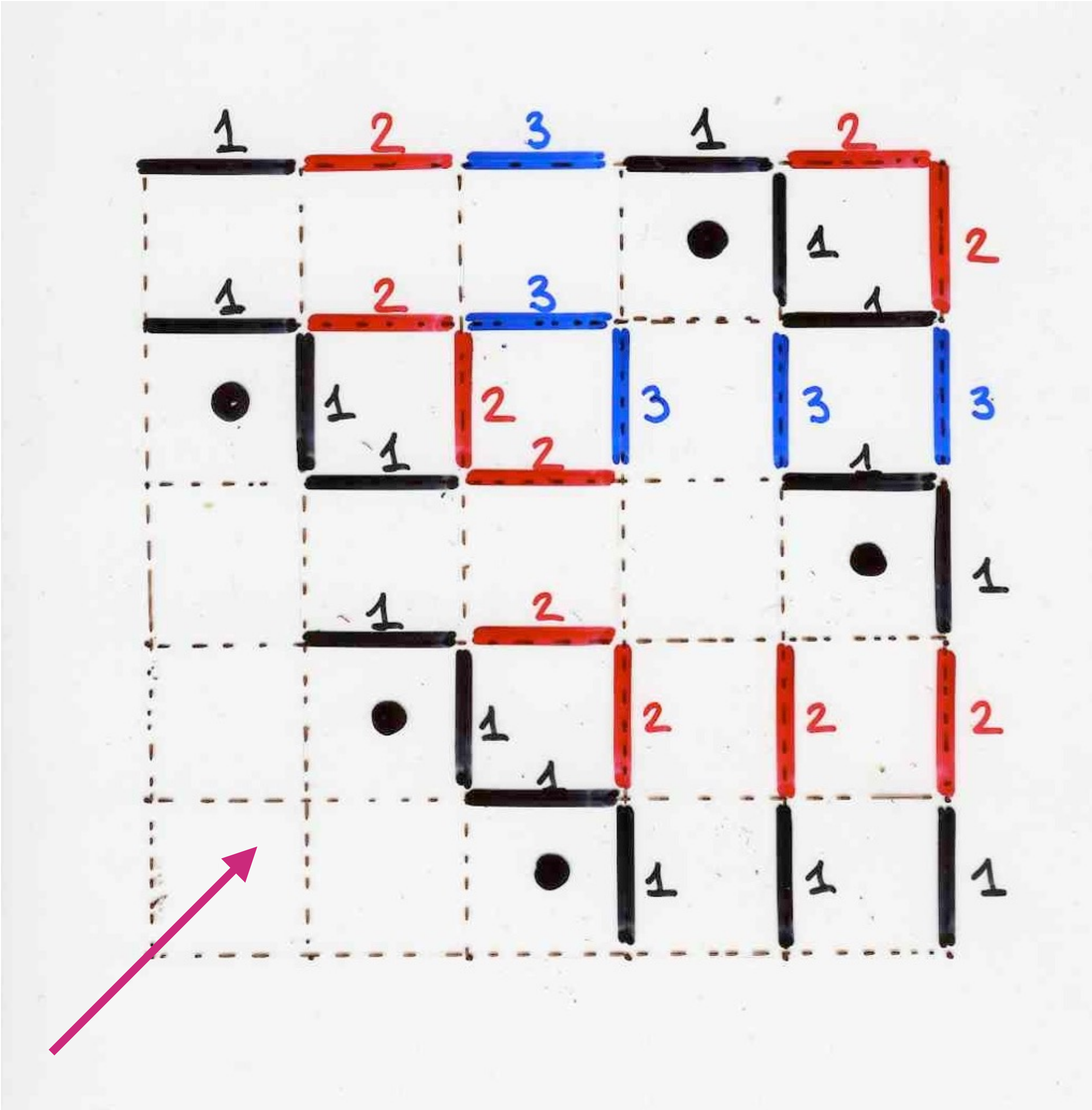
$$
Q \begin{cases} UD = q_1 DU + tYX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}
$$

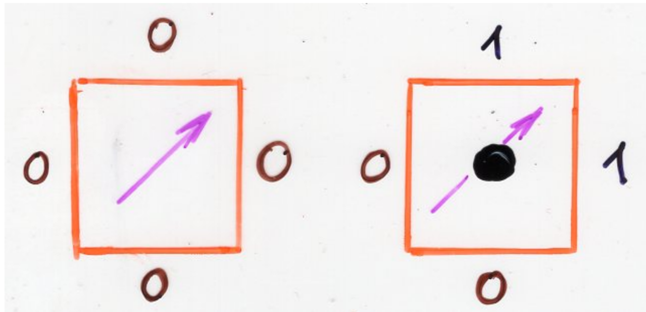
$$
Q^{+} \begin{cases} YX = q_2 XY + tUD \\ YU = UY \\ DX = XD \\ DU = q_1 UD \end{cases}
$$

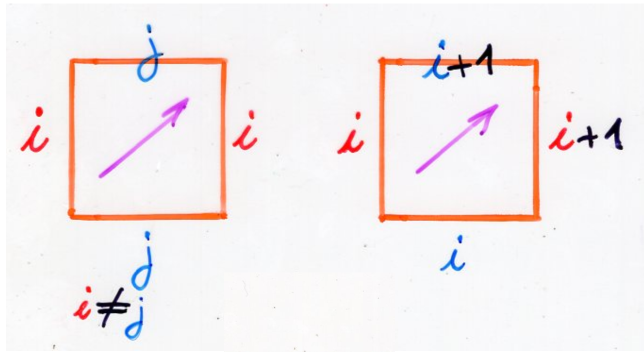


edge local rules

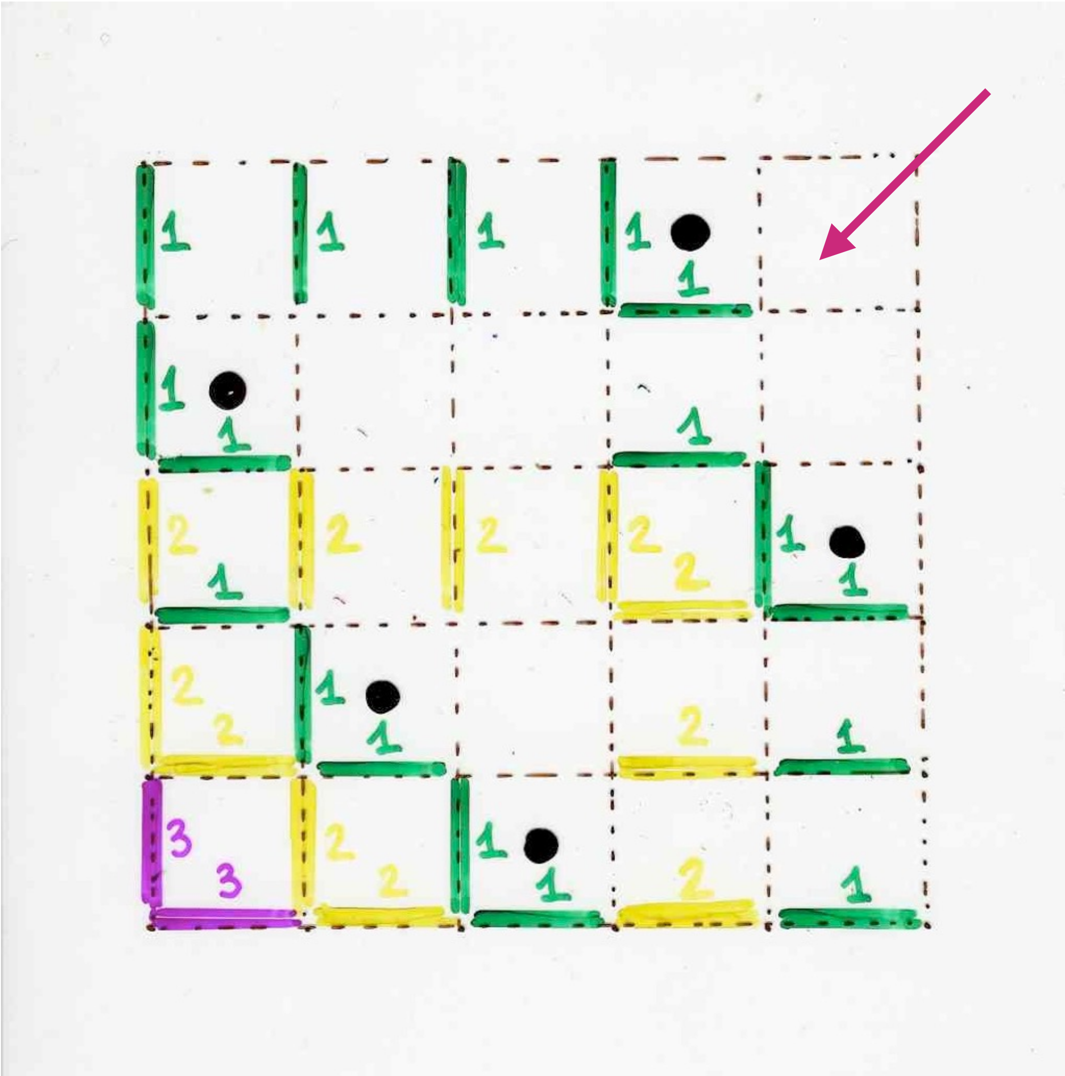
"local rules" on the edges

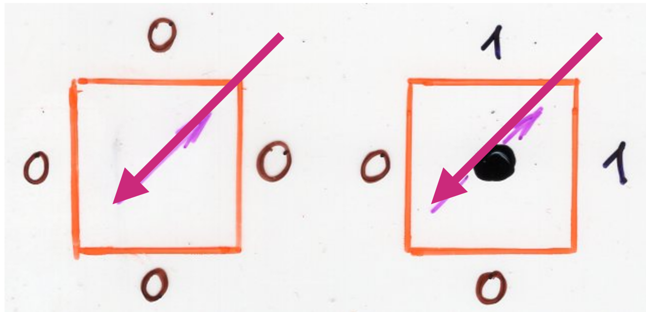


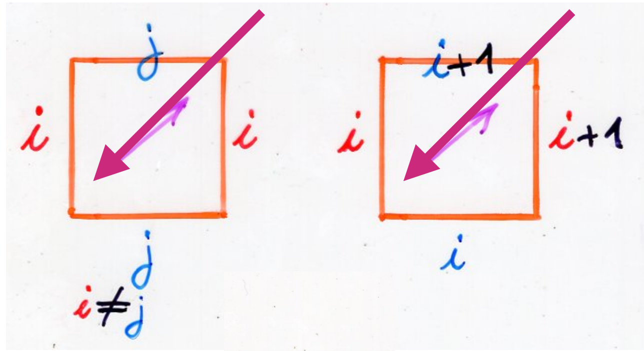




"local rules"
on the edges







The Robinson-Schensted correspondence

G. de B. Robinson, 1938

- Schensted insertions algorithm C. Schensted, 1961
- Geometric version X.V. 1976

edge local rules

« Demultiplication of equations » in the algebra $UD = DU + Id$

- "local" rules on the vertices. or "growth diagrams"

S. Fomin, 1986, 1994

Combinatorial representation
of the quadratic algebra. $UD = DU + Id$

another « demultiplication »
of the algebra $UD=DU+Id$

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

another "duplication" of the commutation relations of the algebra Q

$$UD = DU + Y_0 X$$

$$XY_0 = Y_1 X$$

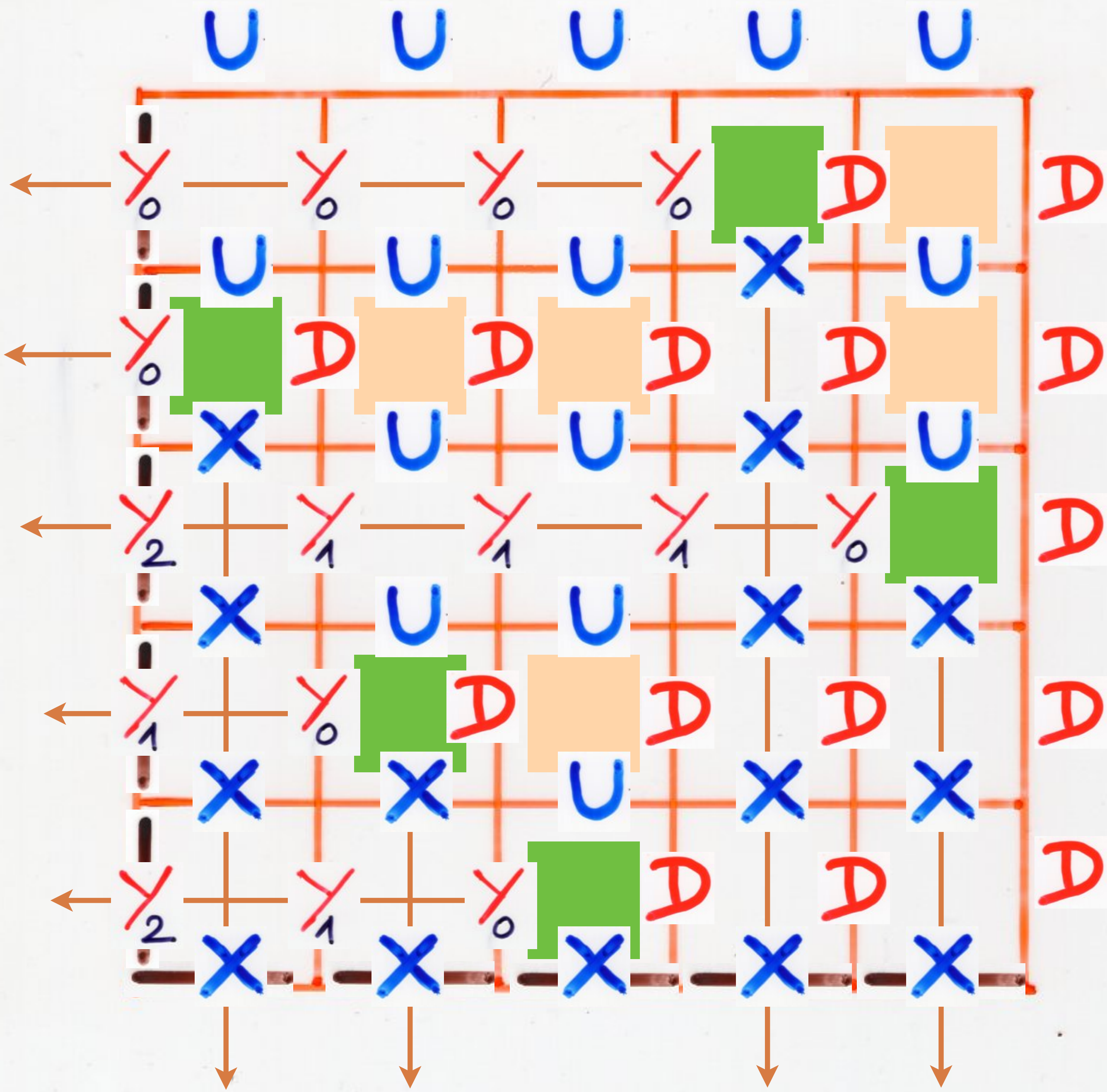
$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

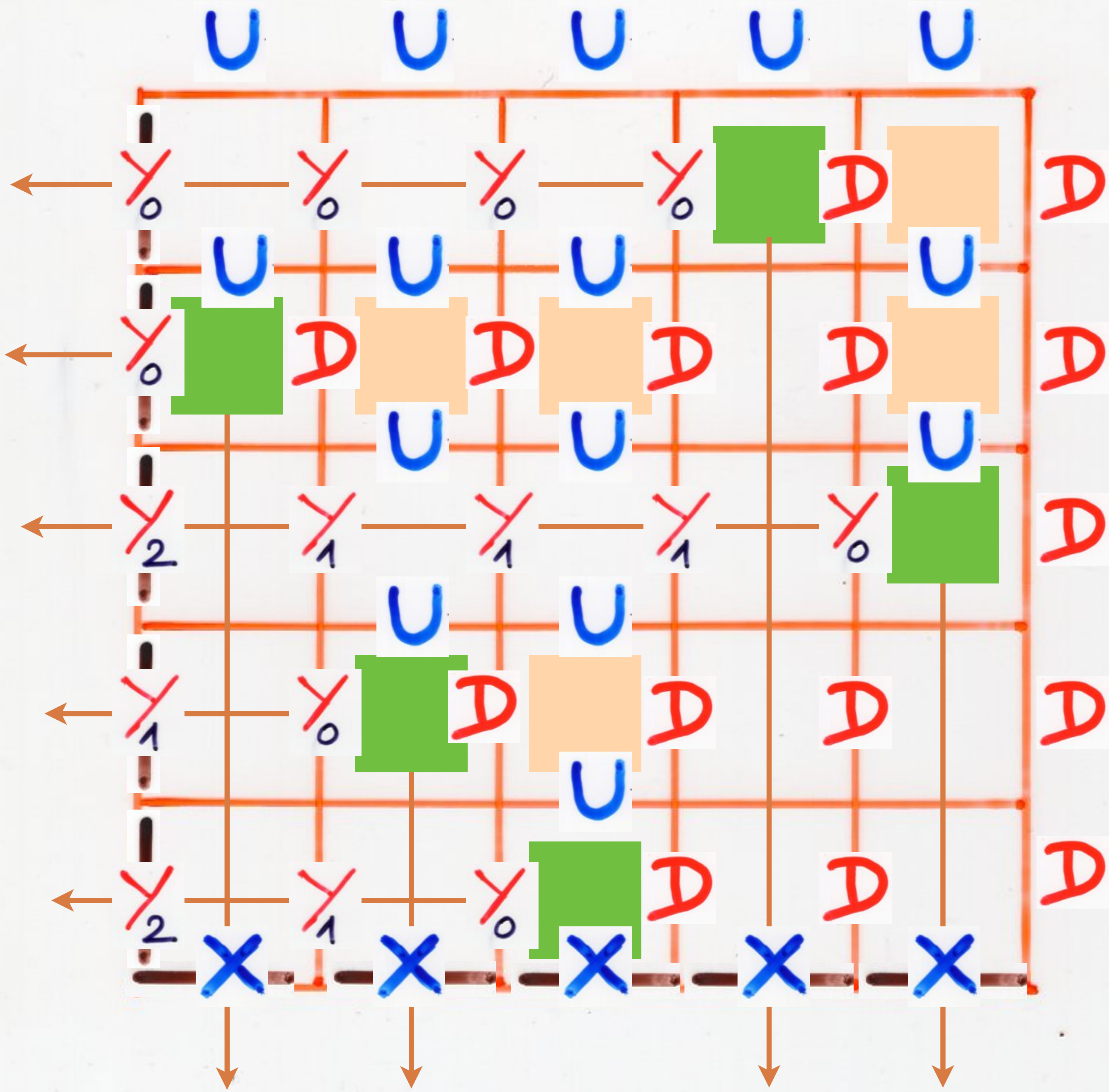
another "duplication" of the commutation relations of the algebra Q

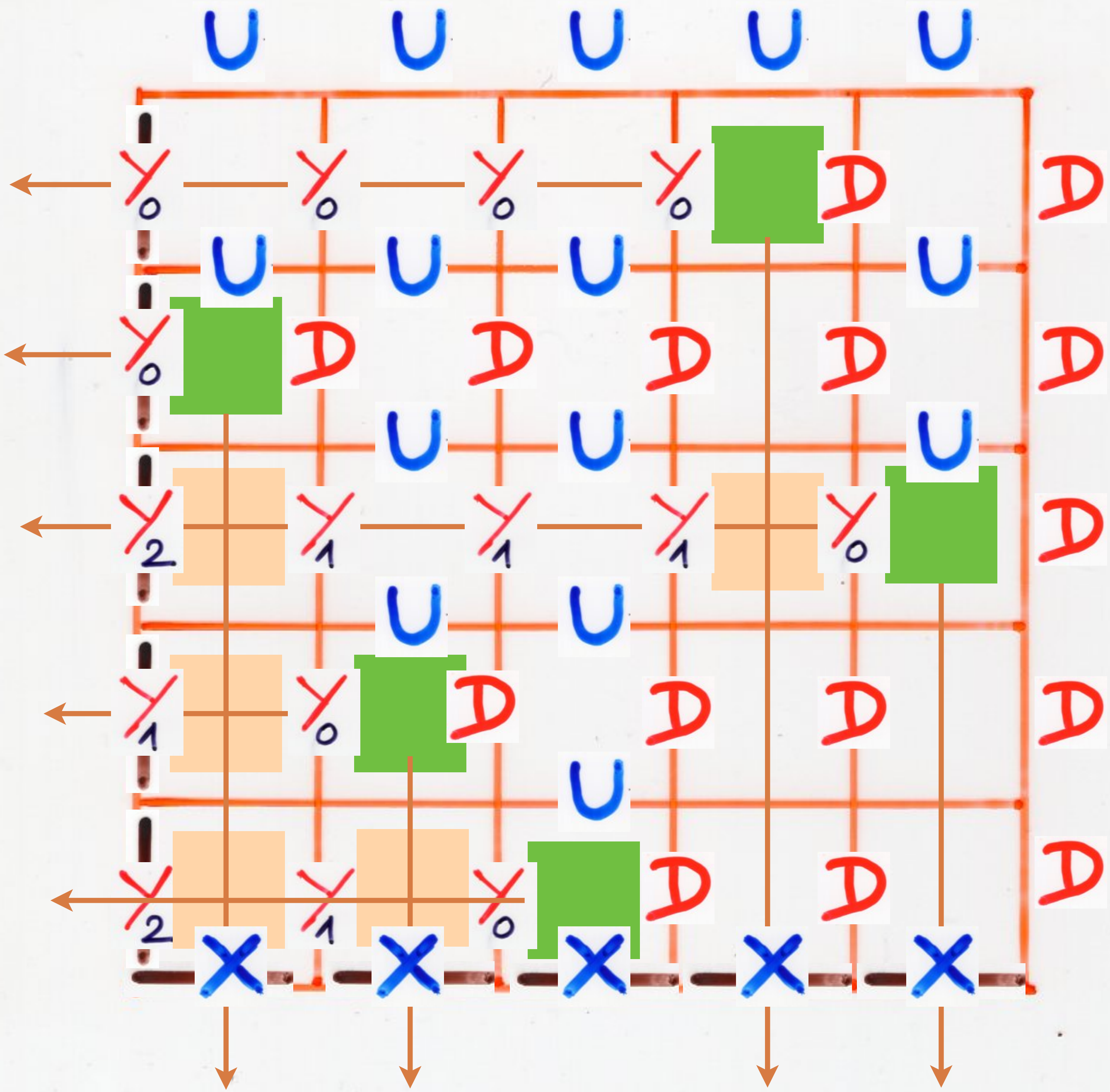
$$UD = DU + Y_0 X$$

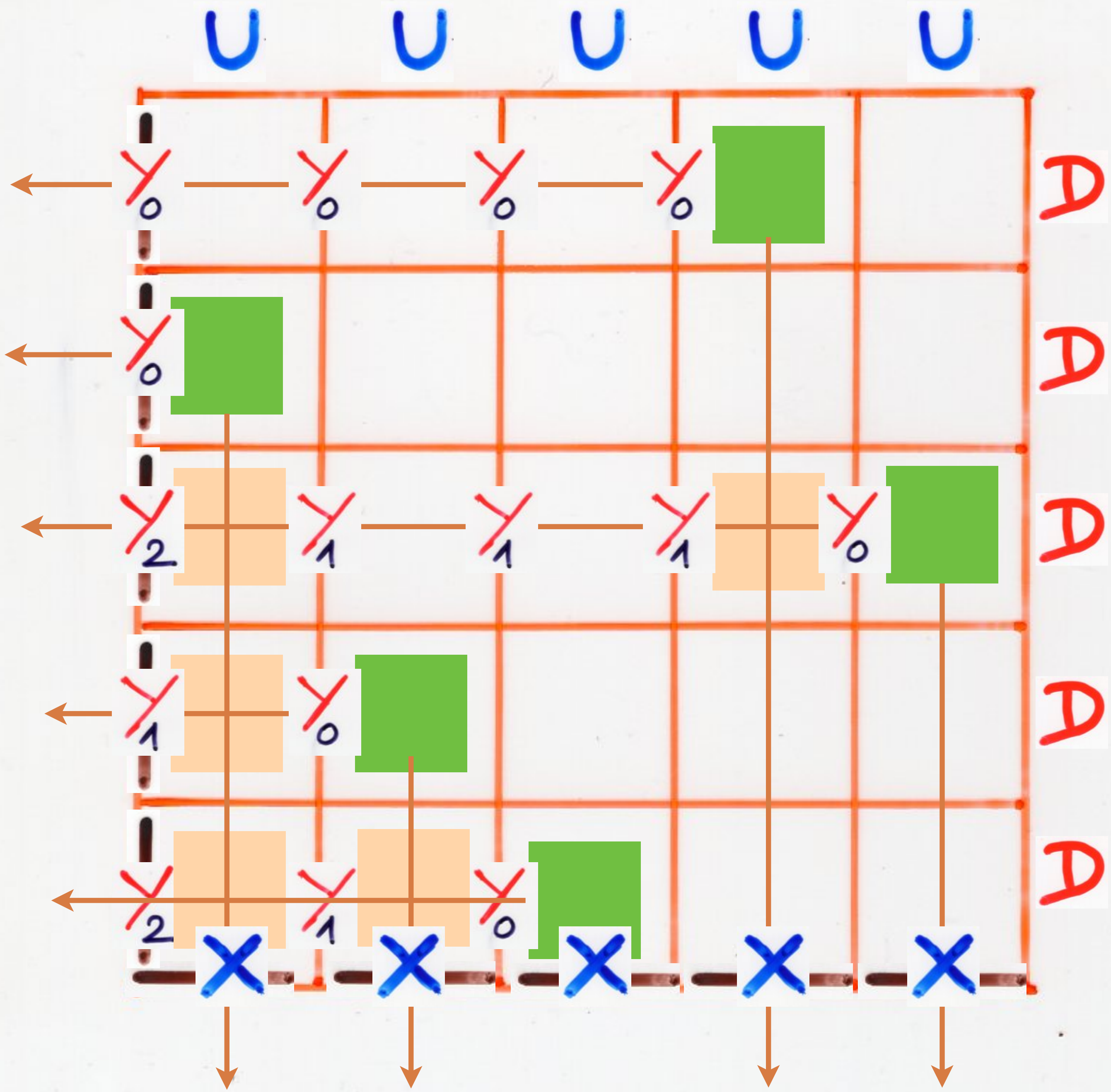
$$\begin{cases} X Y_0 = Y_1 X \\ X Y_1 = Y_2 X \\ X Y_2 = Y_3 X \\ \cdots\cdots \\ X Y_i \quad Y_{i+1} X \\ \cdots\cdots \end{cases}$$

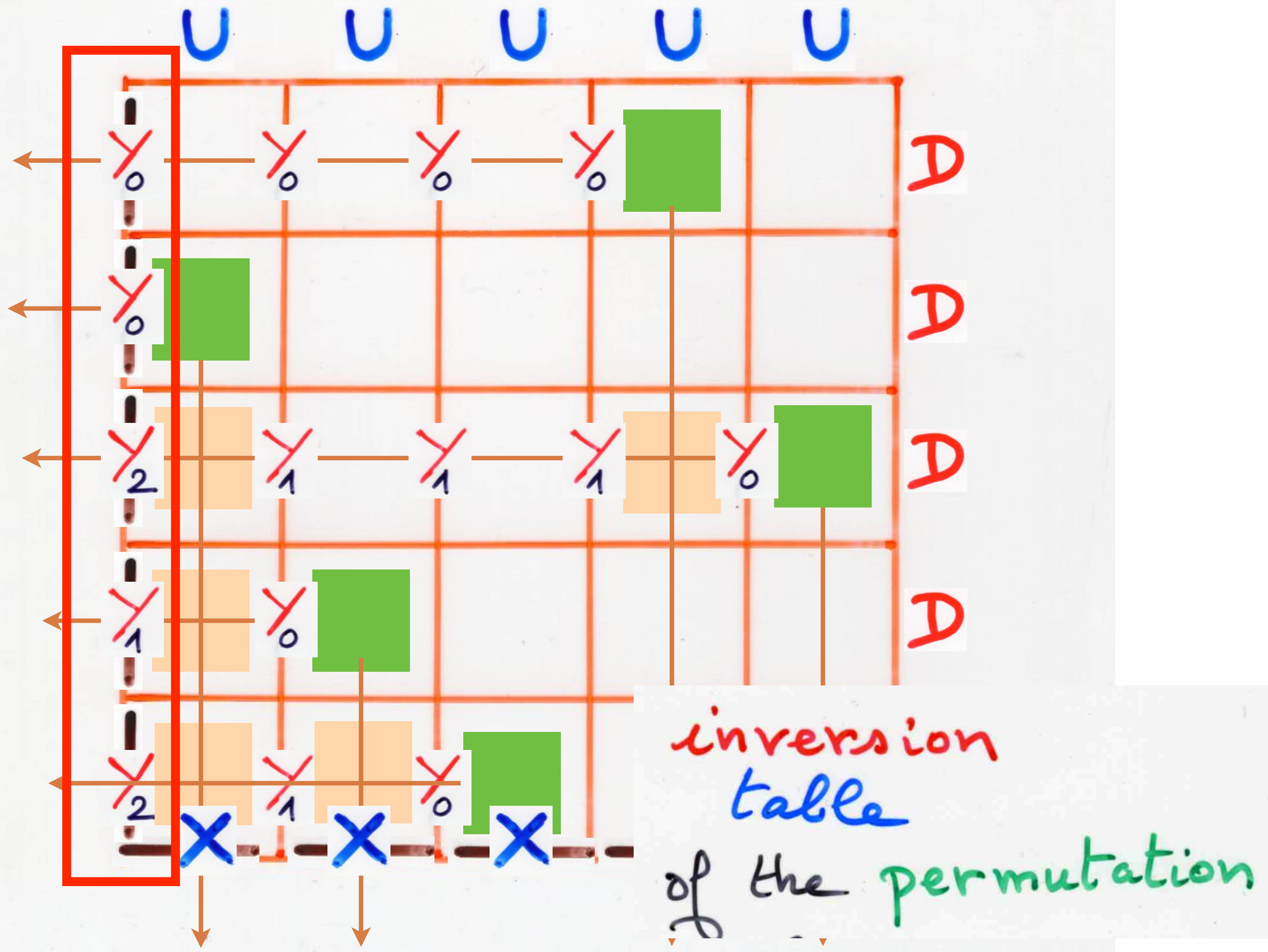
$$XU = UX$$
$$UY_i = Y_i U$$




U
D
X
Y_0
Y_1
Y_2



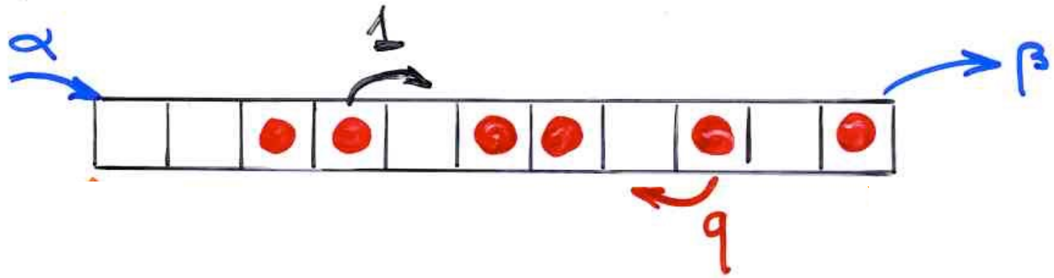



U U U U U
D
D
D
D
y0 y0 y0 y0
y0
y2 y1 y1 y1 y0
y1 y0
y2 y1 y0
X X X
inversion table of the permutation

The PASEP

toy model in the physics of
dynamical systems far from equilibrium

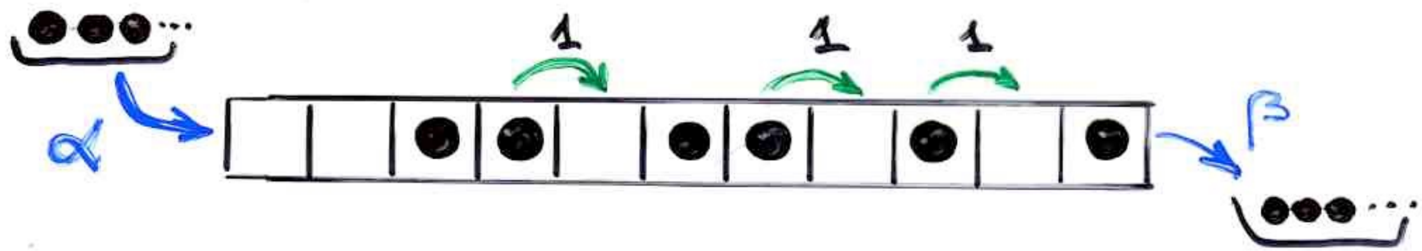
ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

TASEP

"totally asymmetric exclusion process"



The PASEP algebra

$$DE = qED + E + D$$

The PASEP algebra

$$DE = qED + E + D$$

analog of the normal ordering

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

tableau

unique

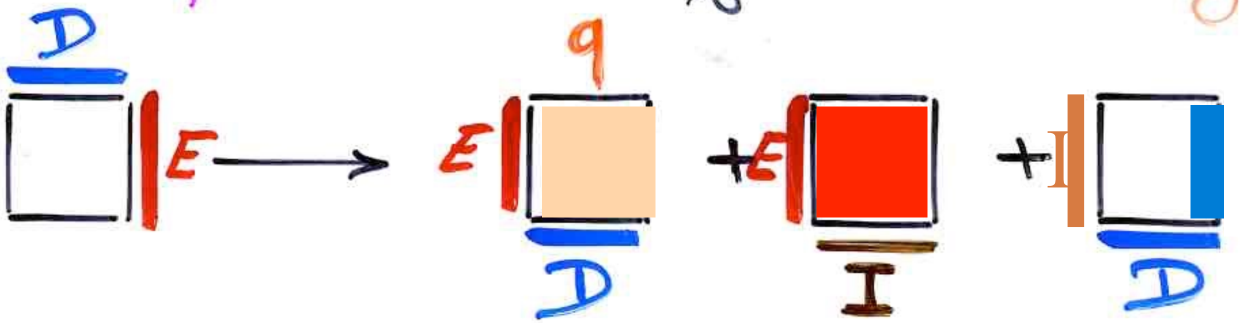
Tableaux for the PASEP algebra

The PASEP algebra

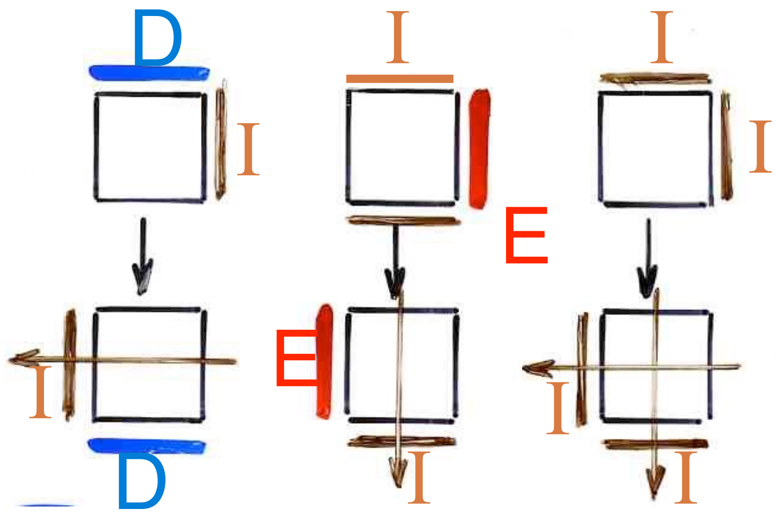
$$DE = qED + E + D$$

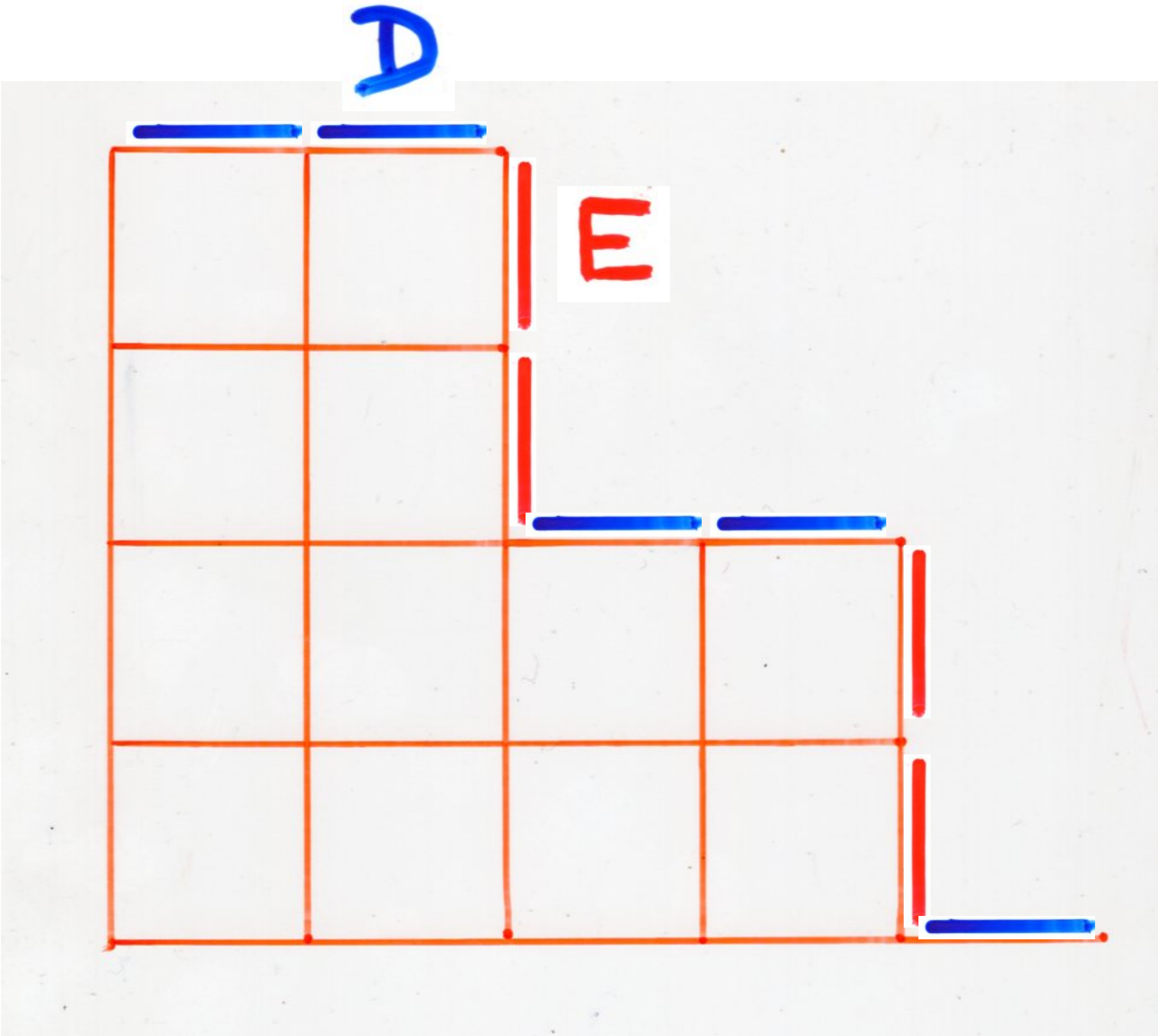
$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$

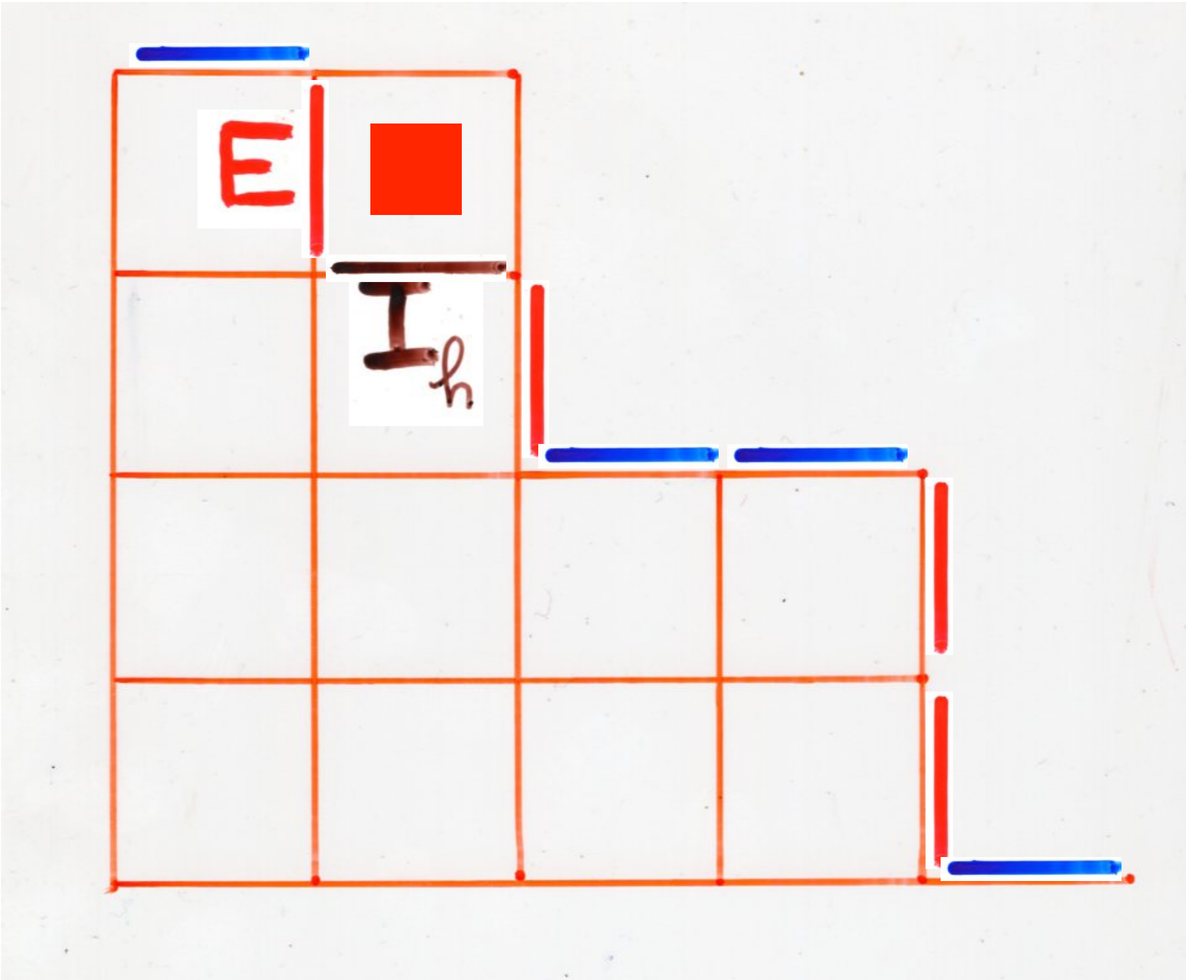
"planarization" of the rewriting rules



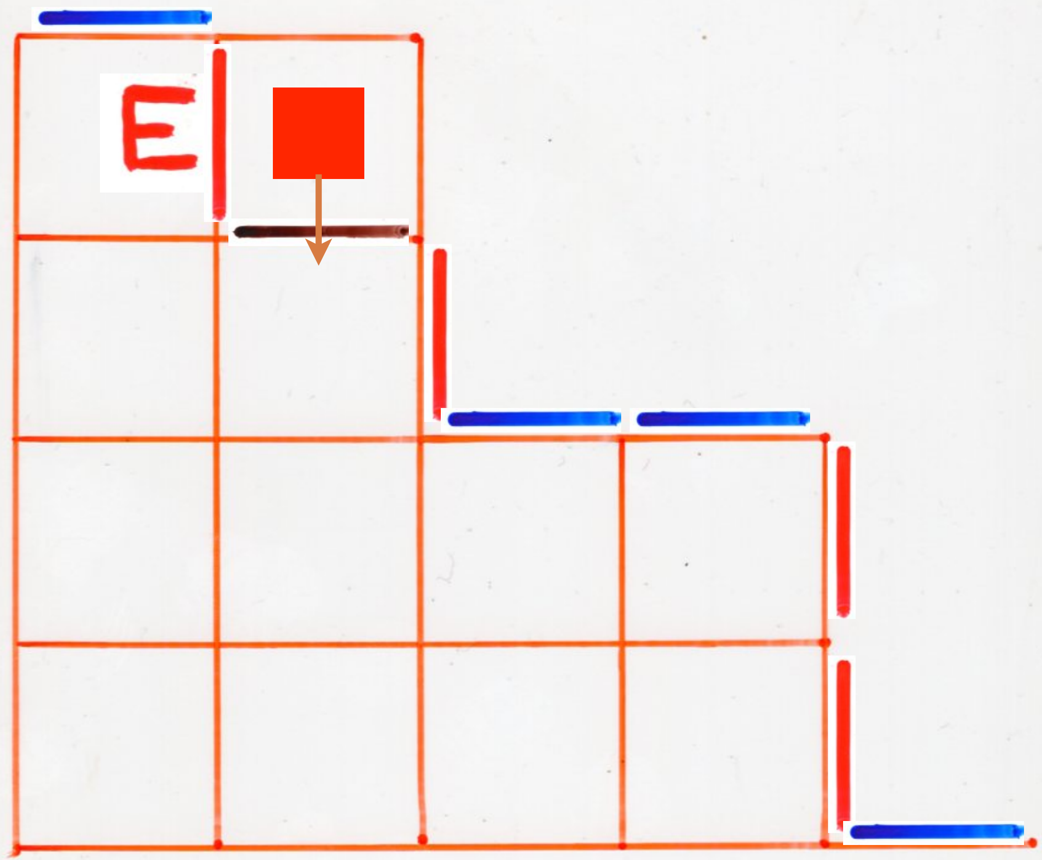
I identity

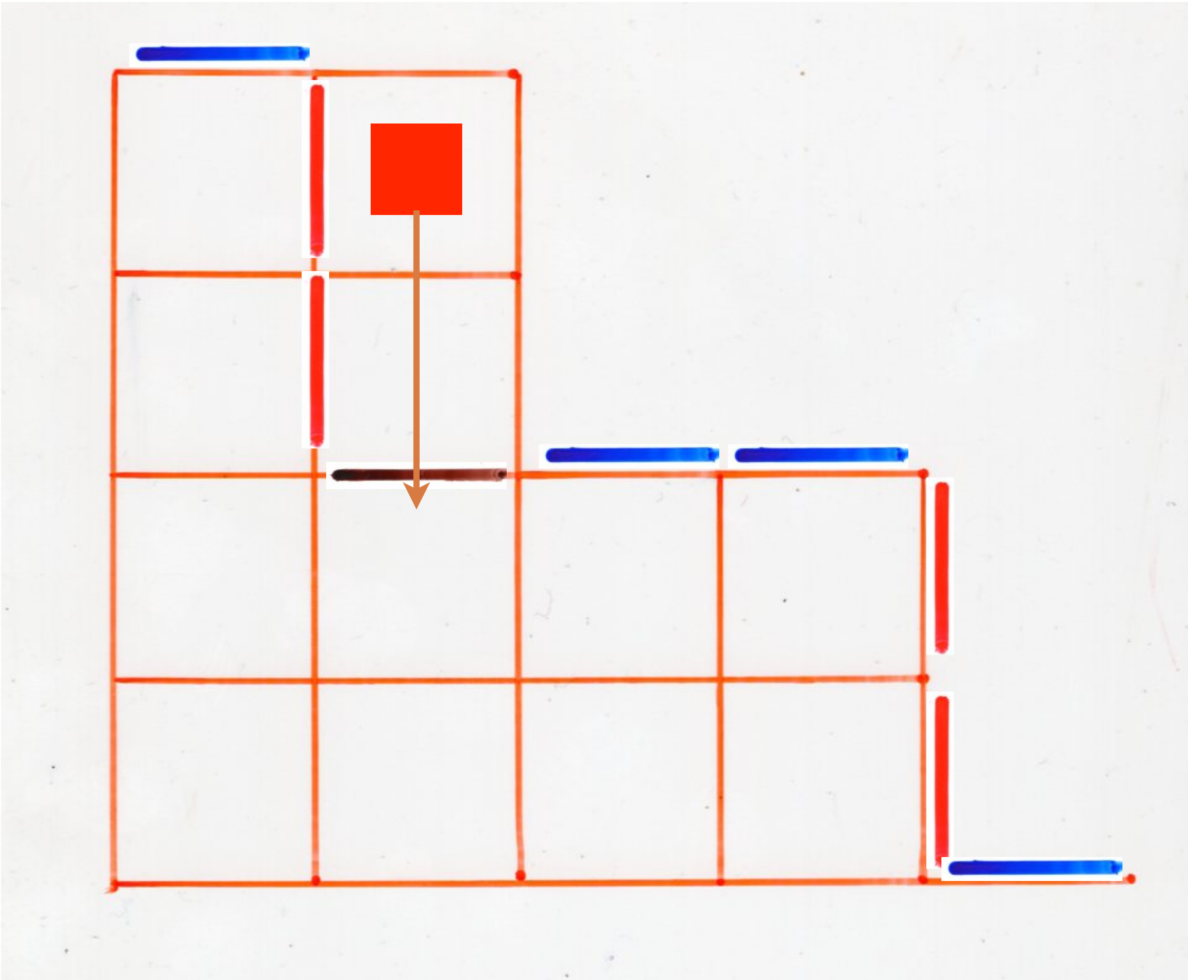


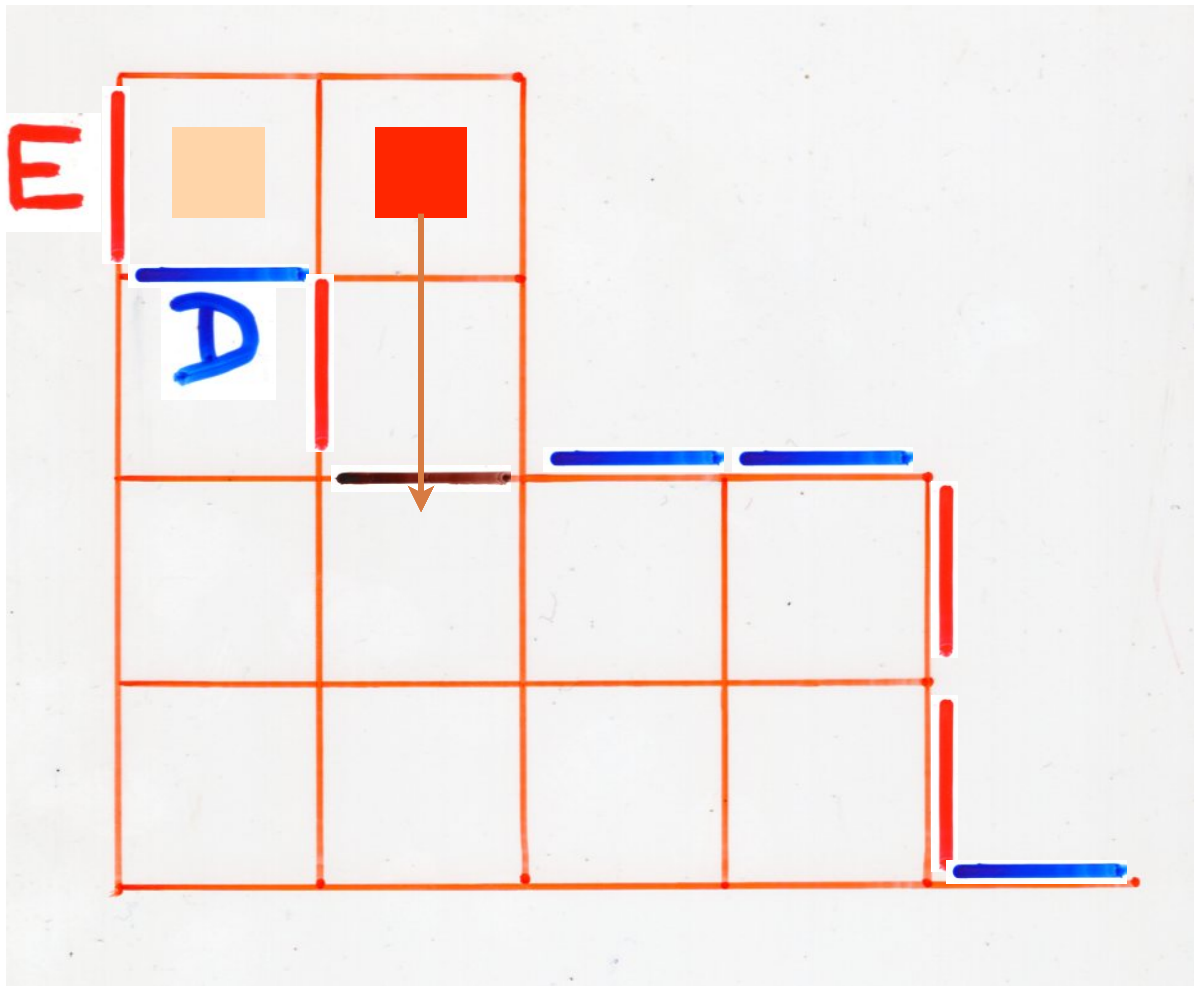

D
E

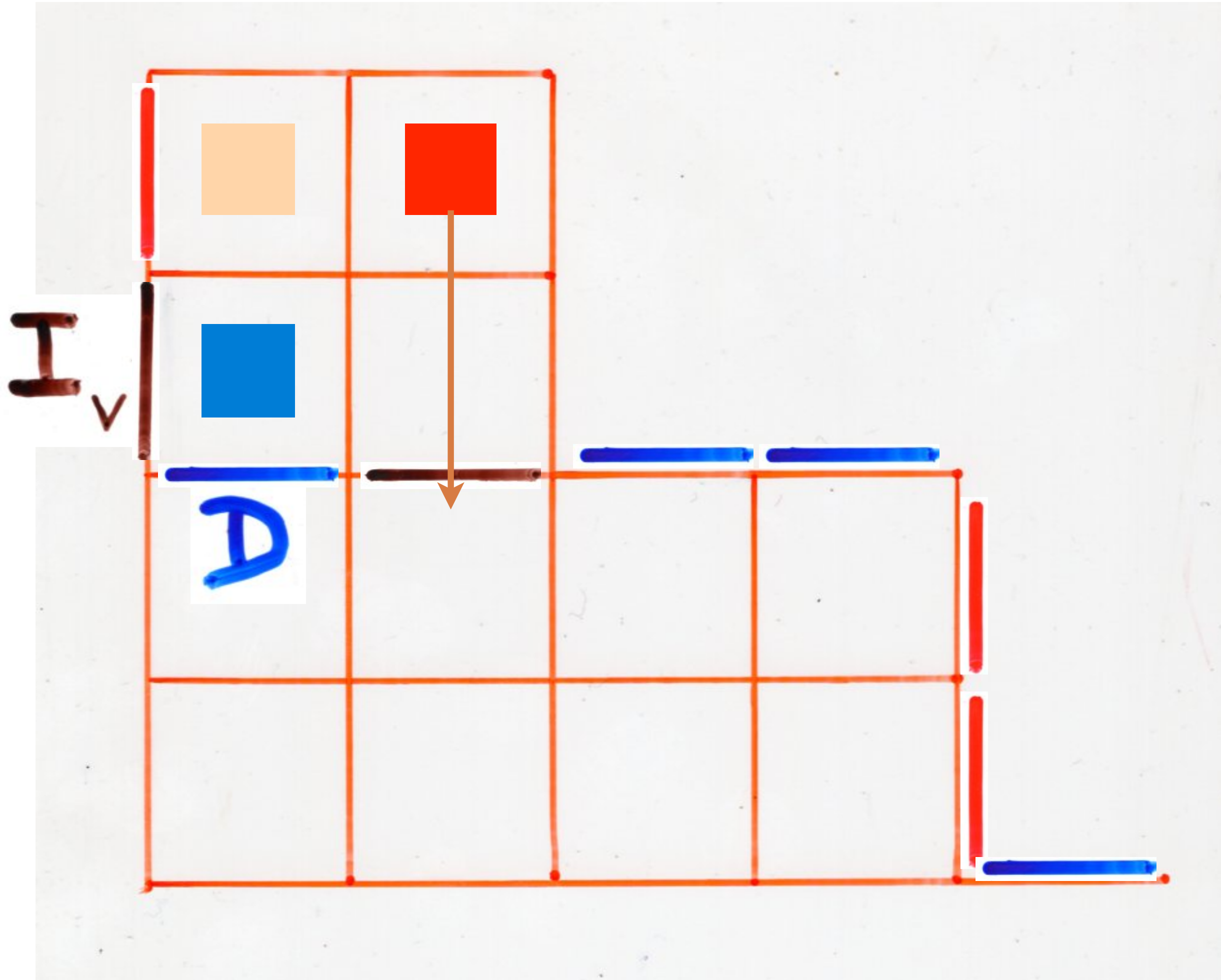


E
I_h

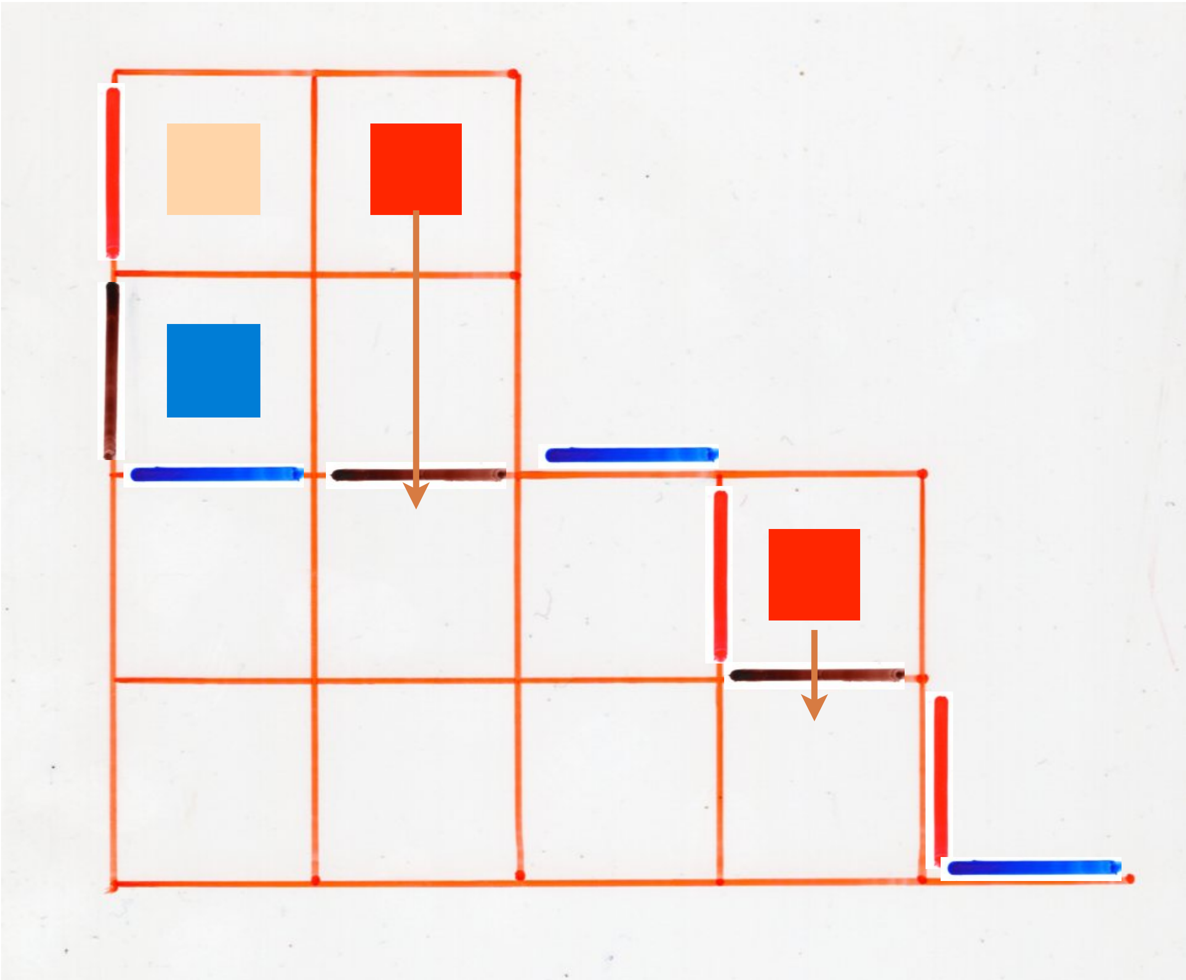

E

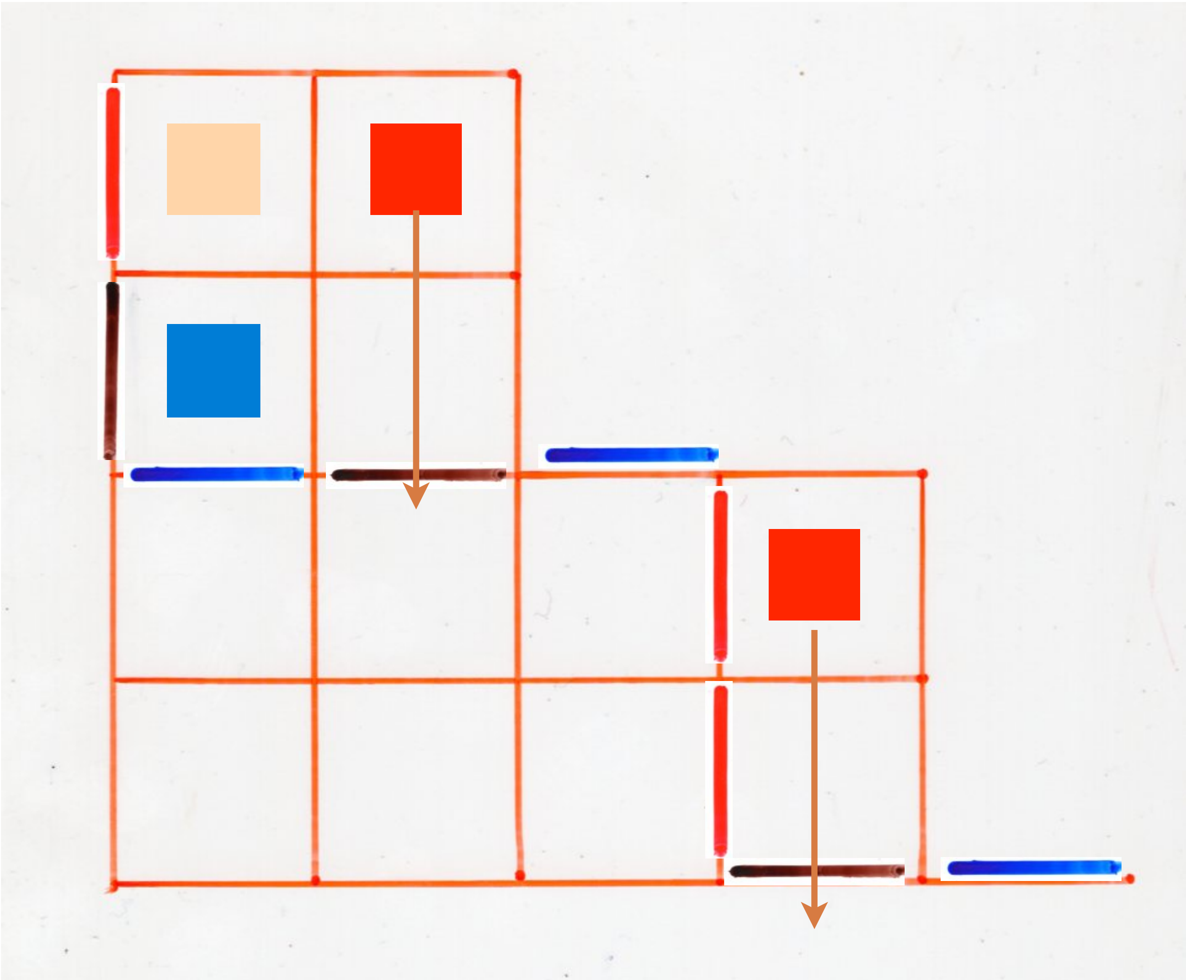


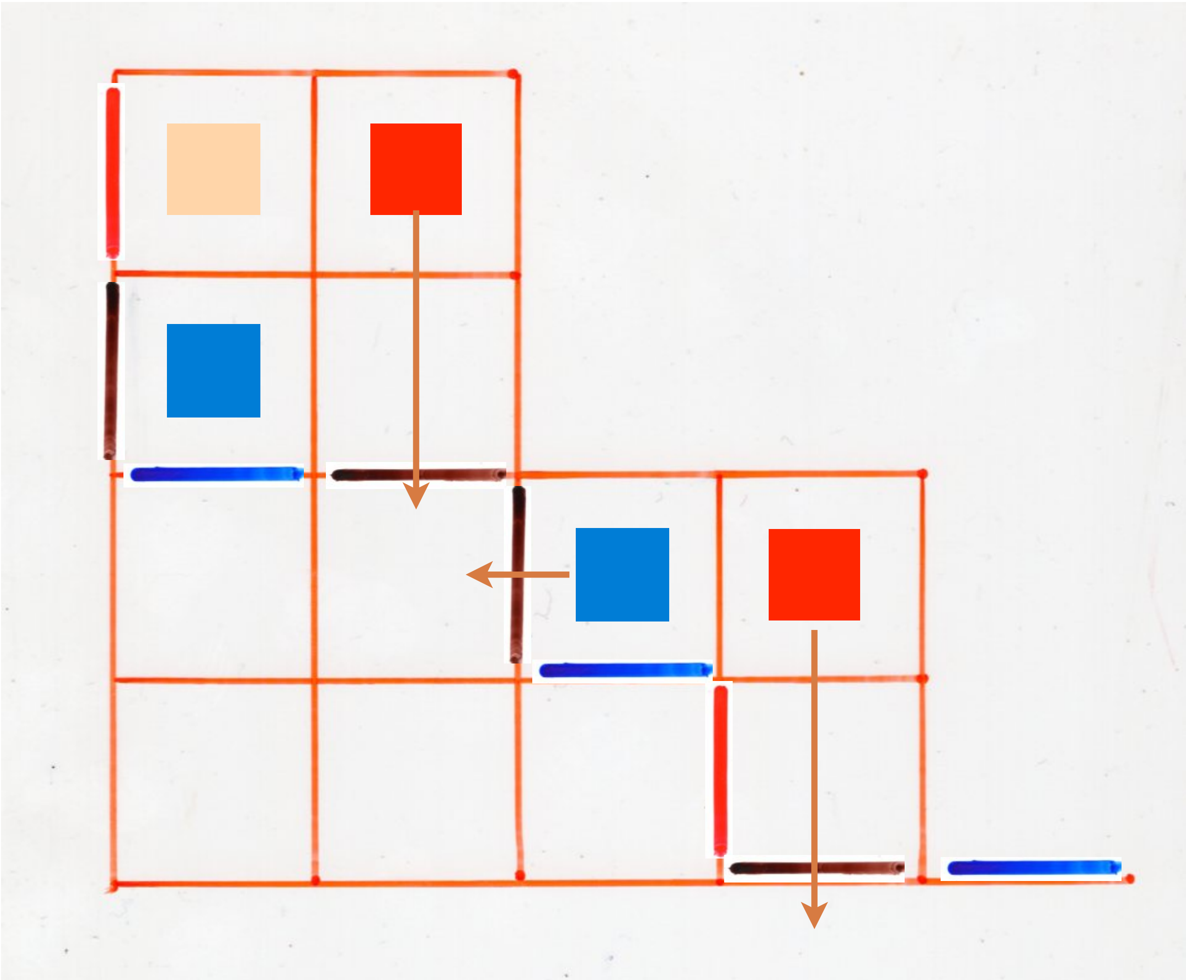

E
D

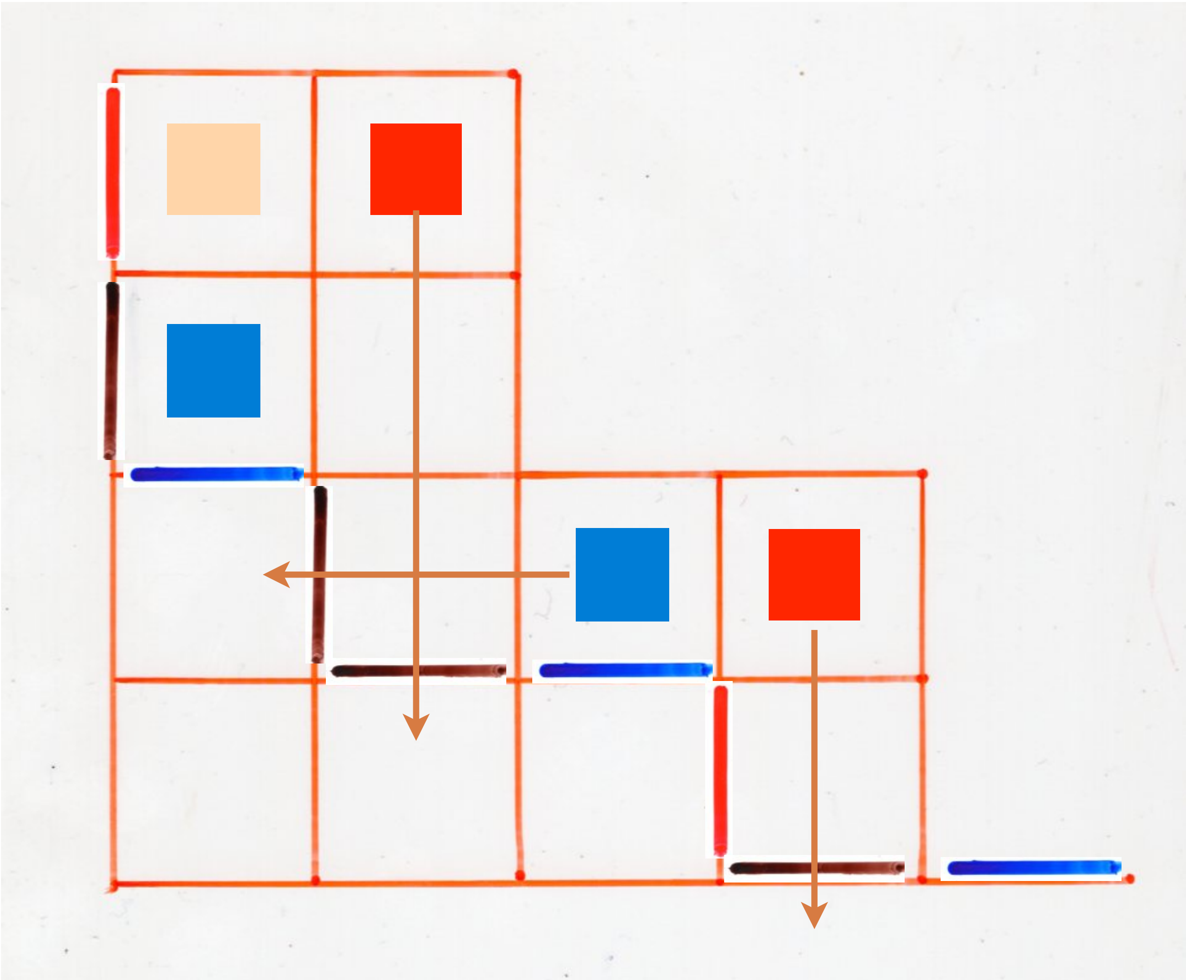


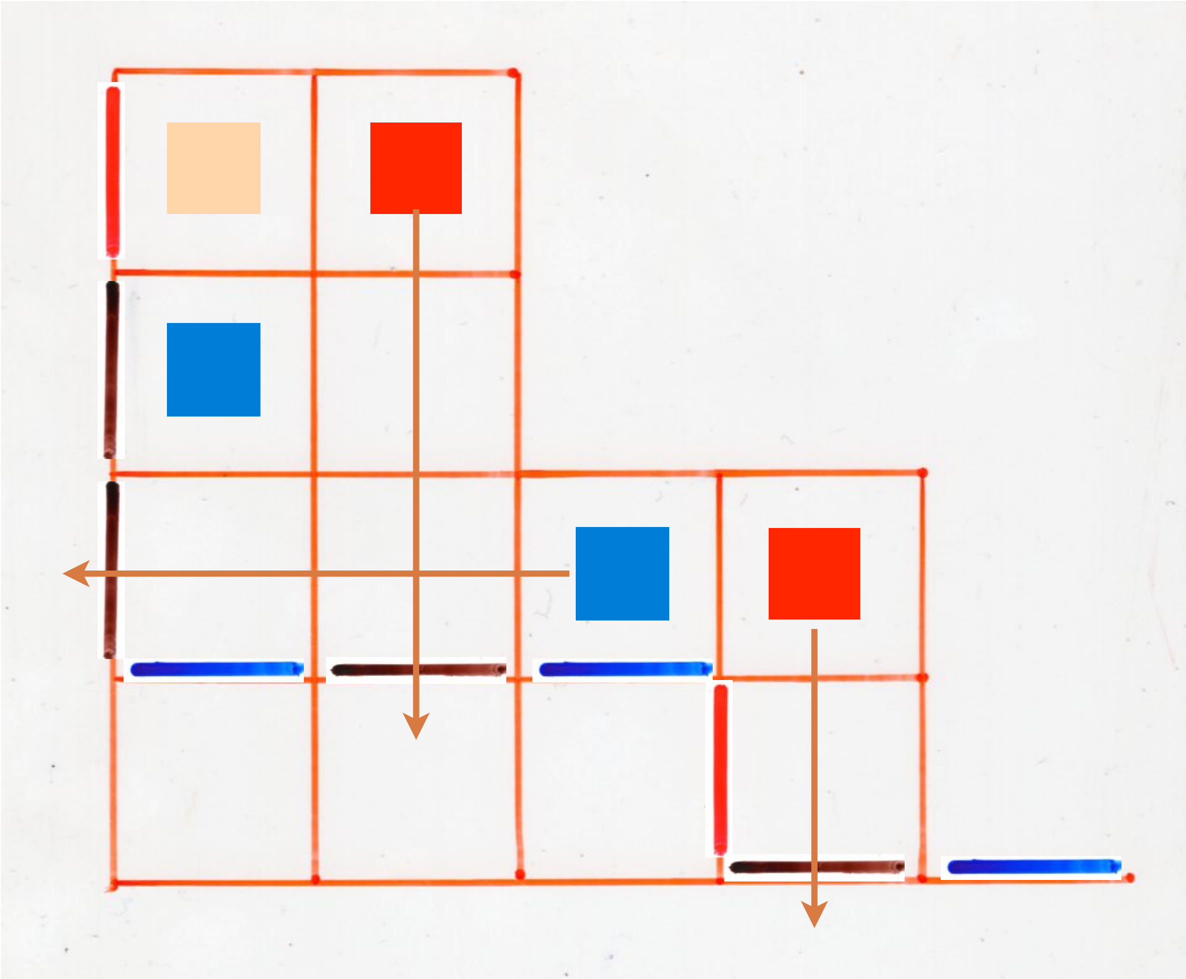
I
V
D

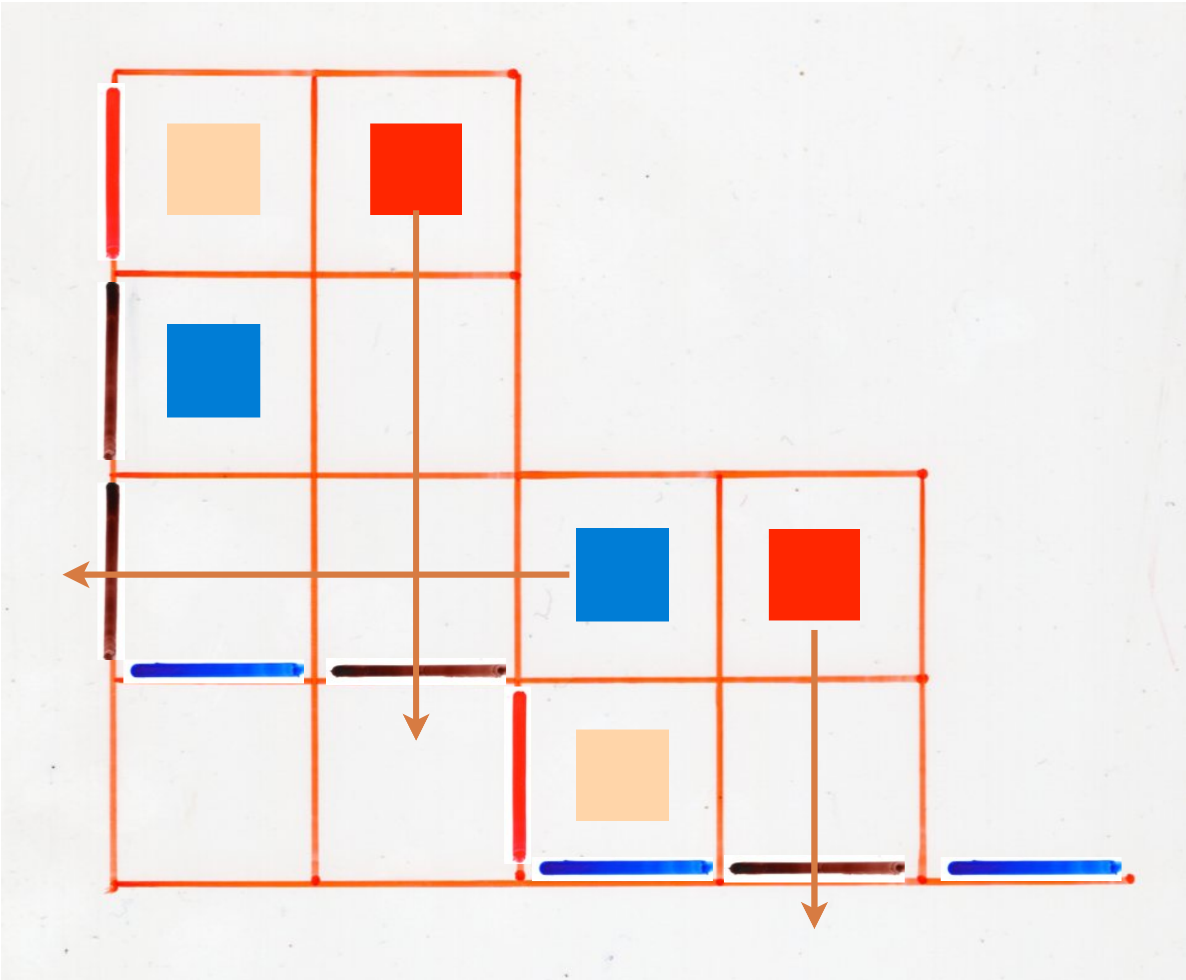


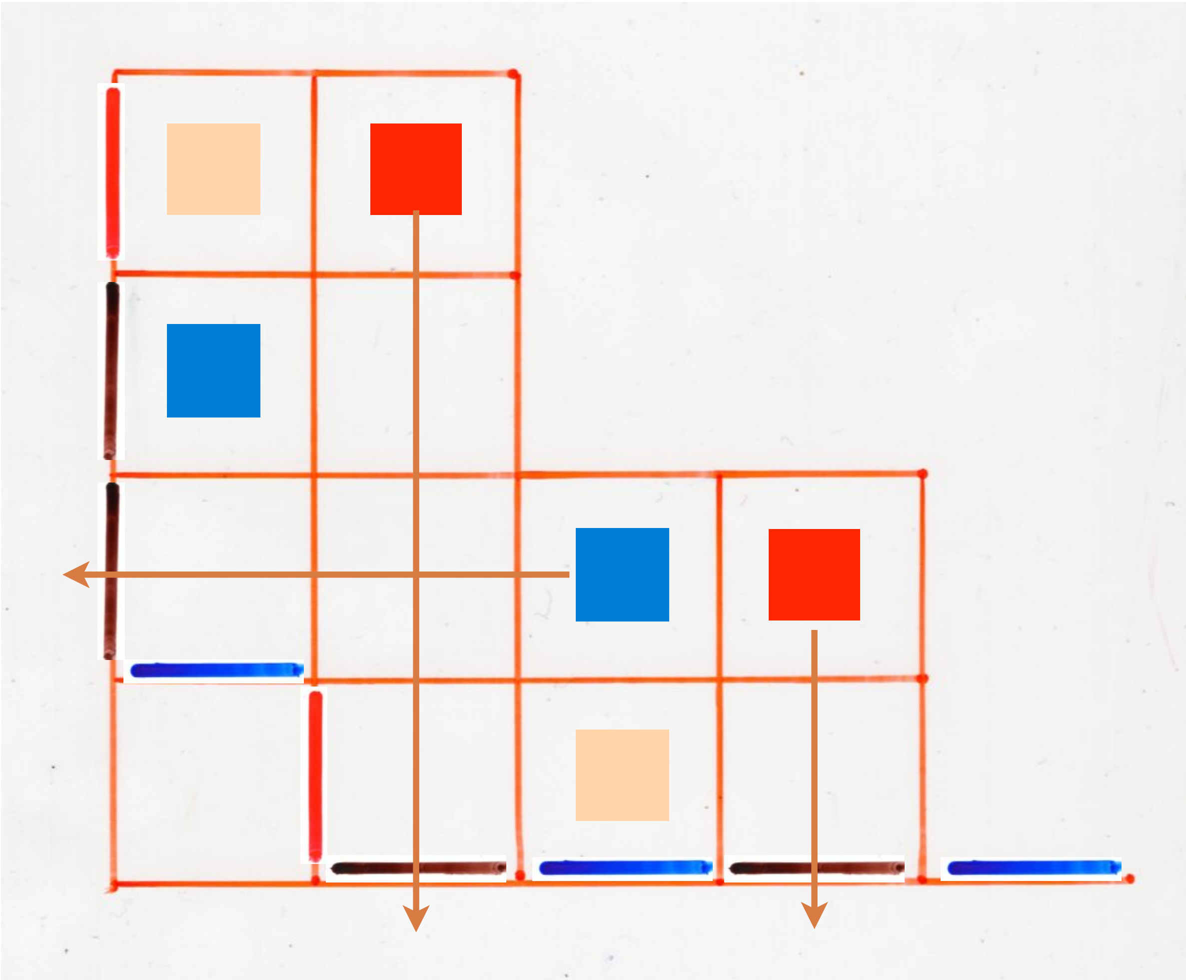


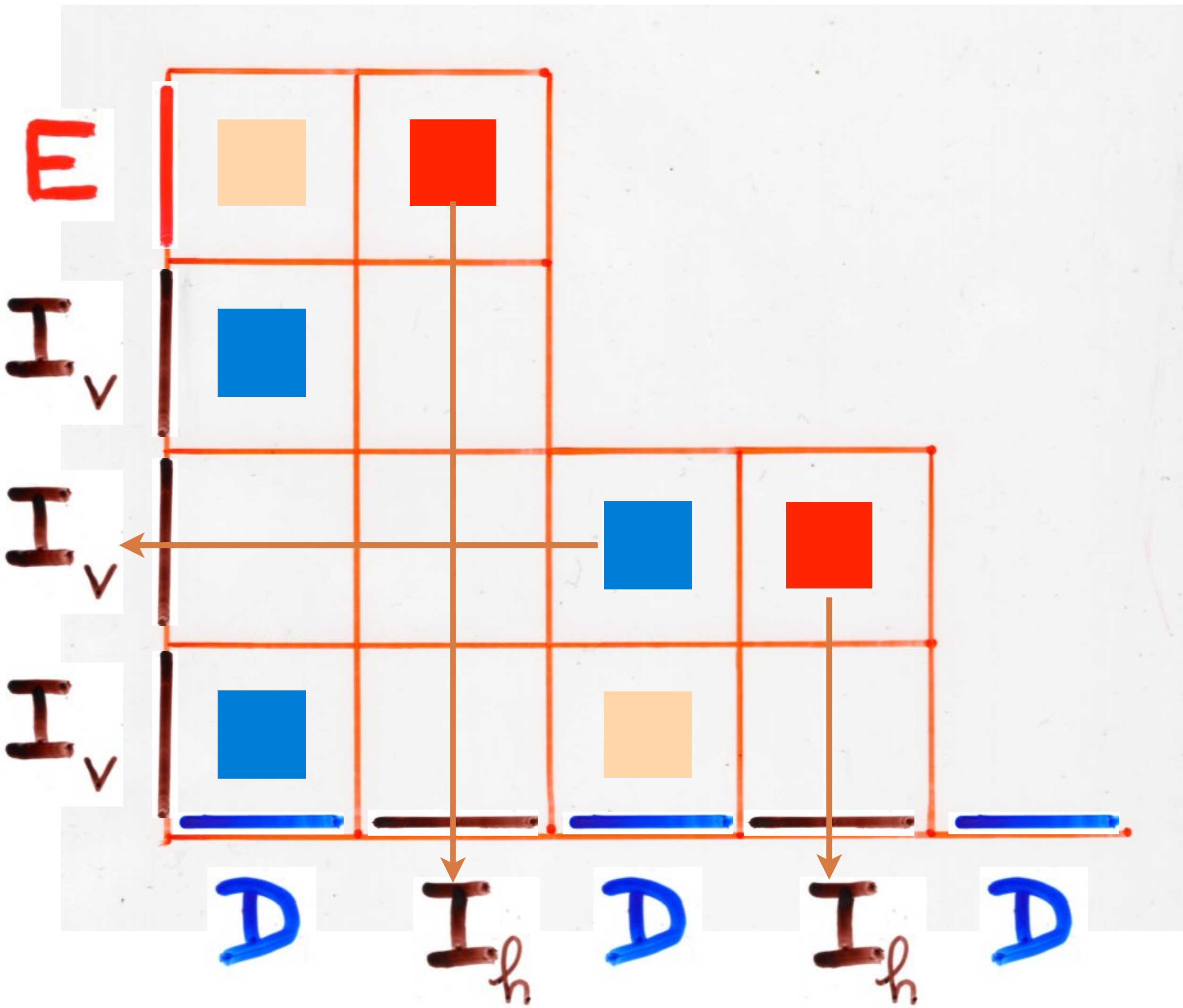


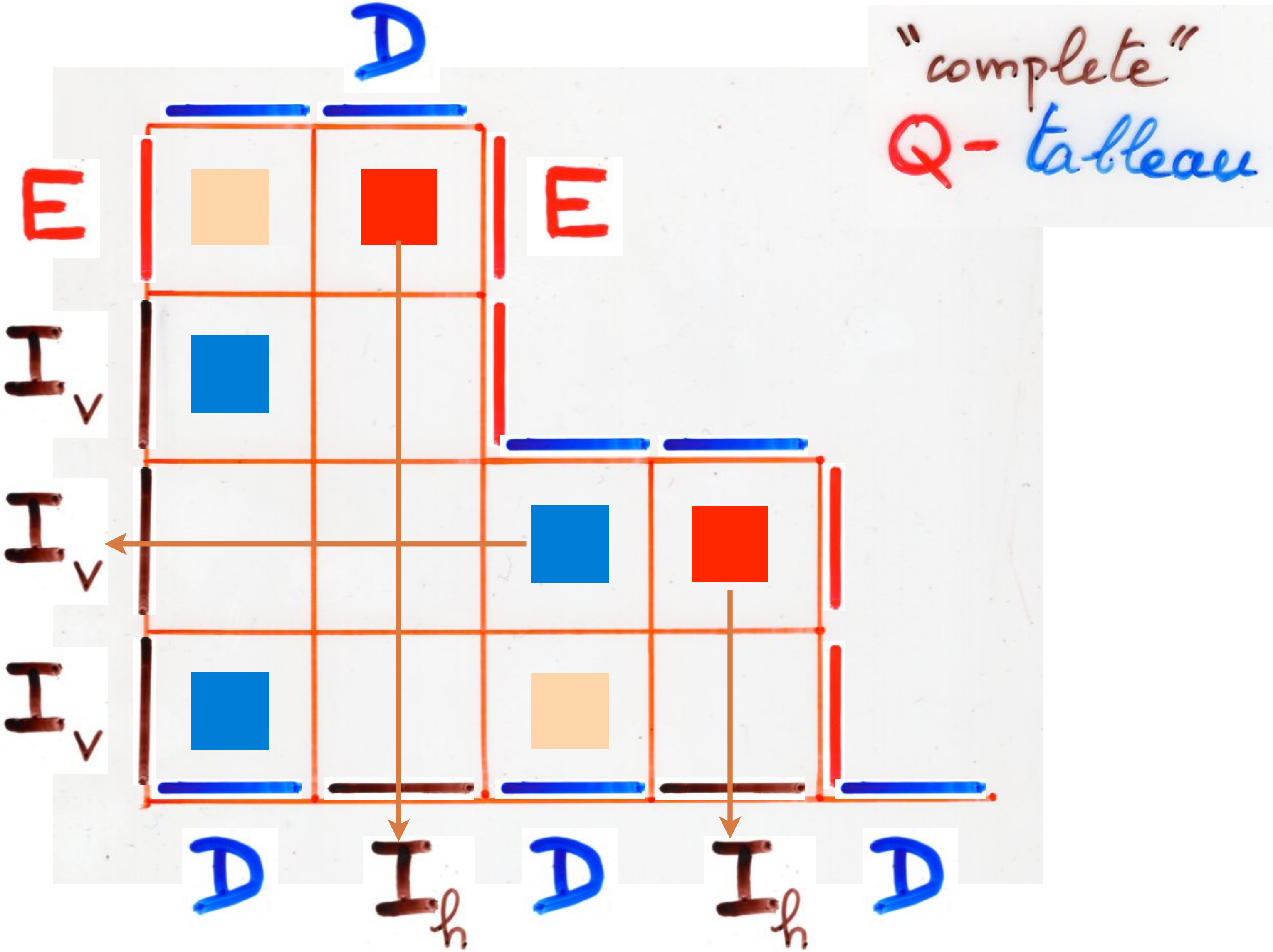









E
I_v
I_v
I_v
D
I_h
D
I_h
D


"complete"
Q- tableau
D
E
E
I_v
I_v
I_v
D
I_h
D
I_h
D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

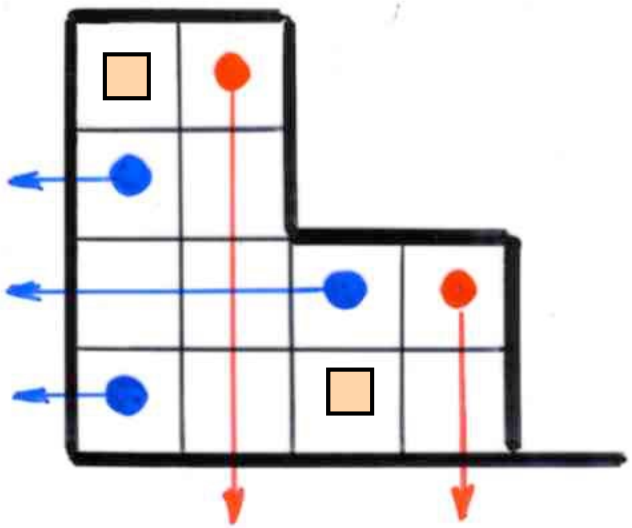
"complete"
Q- tableau

unique

$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without



The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

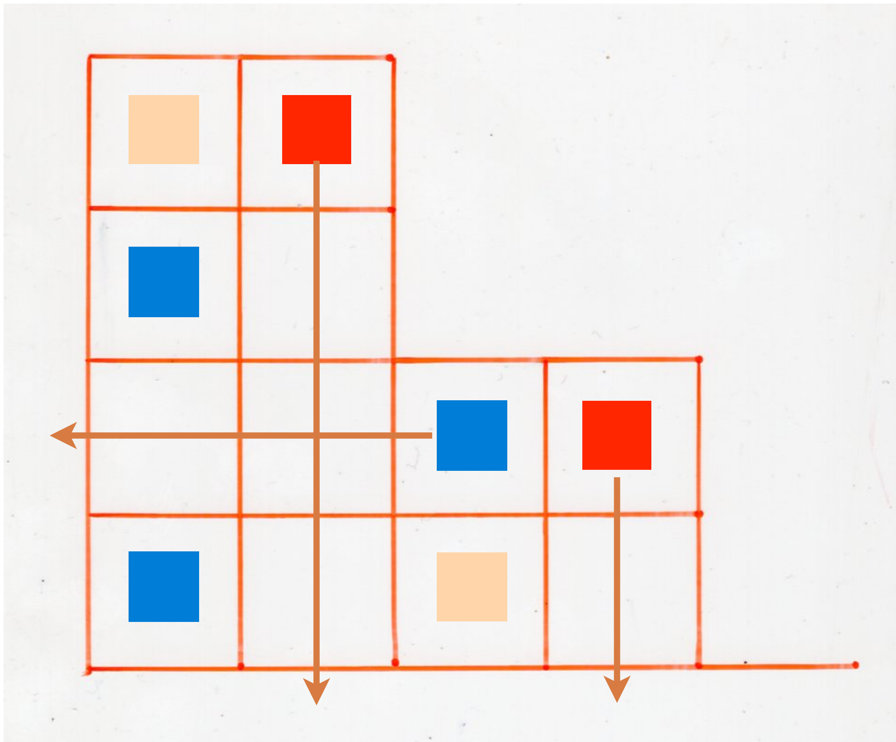
word

tableau

unique

complete Q-tableau

$$DE = qED + E I_h + I_v D$$
$$D I_v = I_v D$$
$$I_h E = E I_h$$
$$I_h I_v = I_v I_h$$



The PASEP algebra

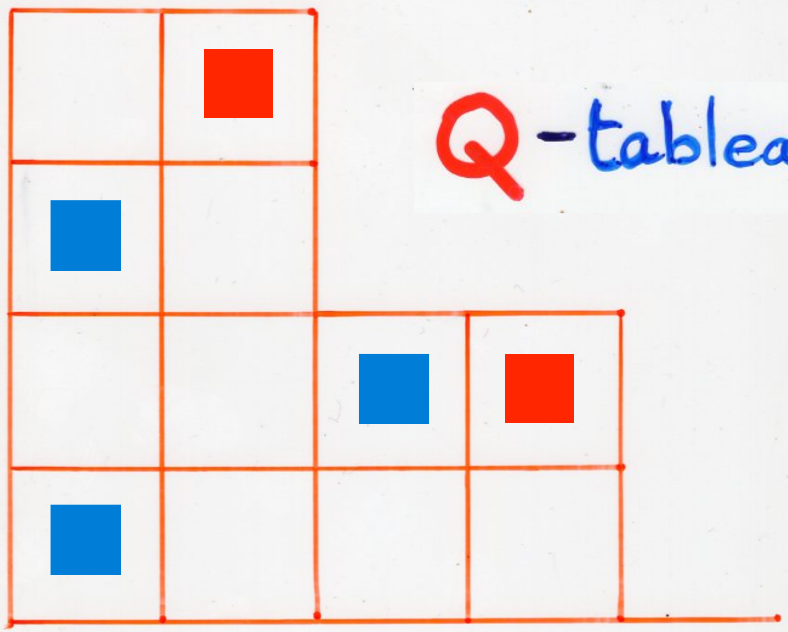
$$DE = qED + E + D$$

$$DE = \boxed{}\,ED + E I_h + I_v D$$
$$D I_v = \boxed{}\, I_v D$$
$$I_h E = \boxed{}\, E I_h$$
$$I_h I_v = \boxed{}\, I_v I_h$$

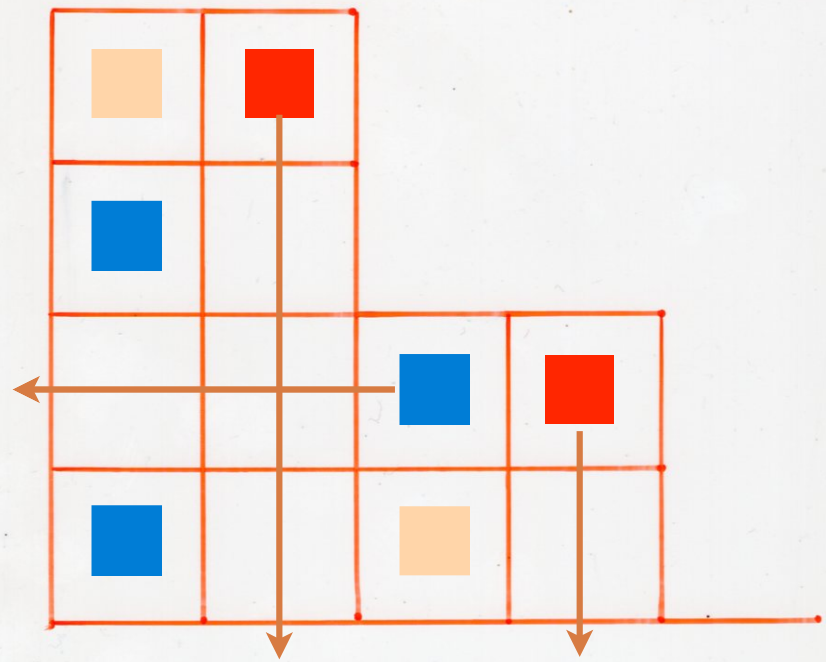
$$DE = qED + E I_h + I_v D$$
$$D I_v = I_v D$$
$$I_h E = E I_h$$
$$I_h I_v = I_v I_h$$

alternative tableau

complete Q-tableau



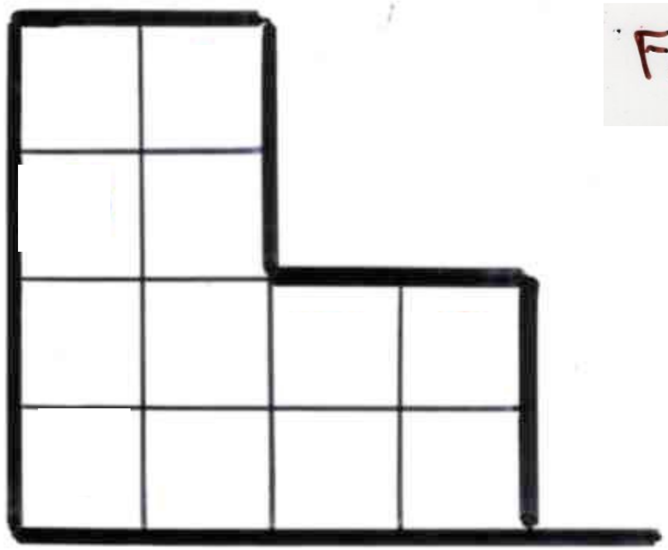
Q-tableaux



alternative tableaux

alternative tableau

Definition



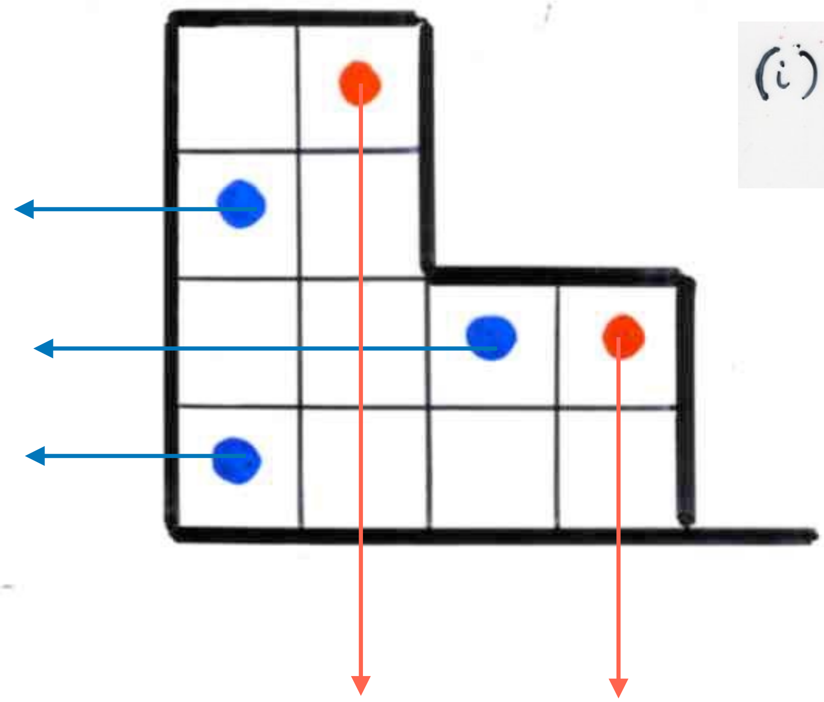
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

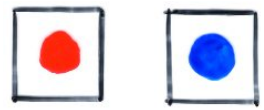
$\mathbf{n} = (\text{number of rows}) + (\text{number of columns})$

alternative tableau

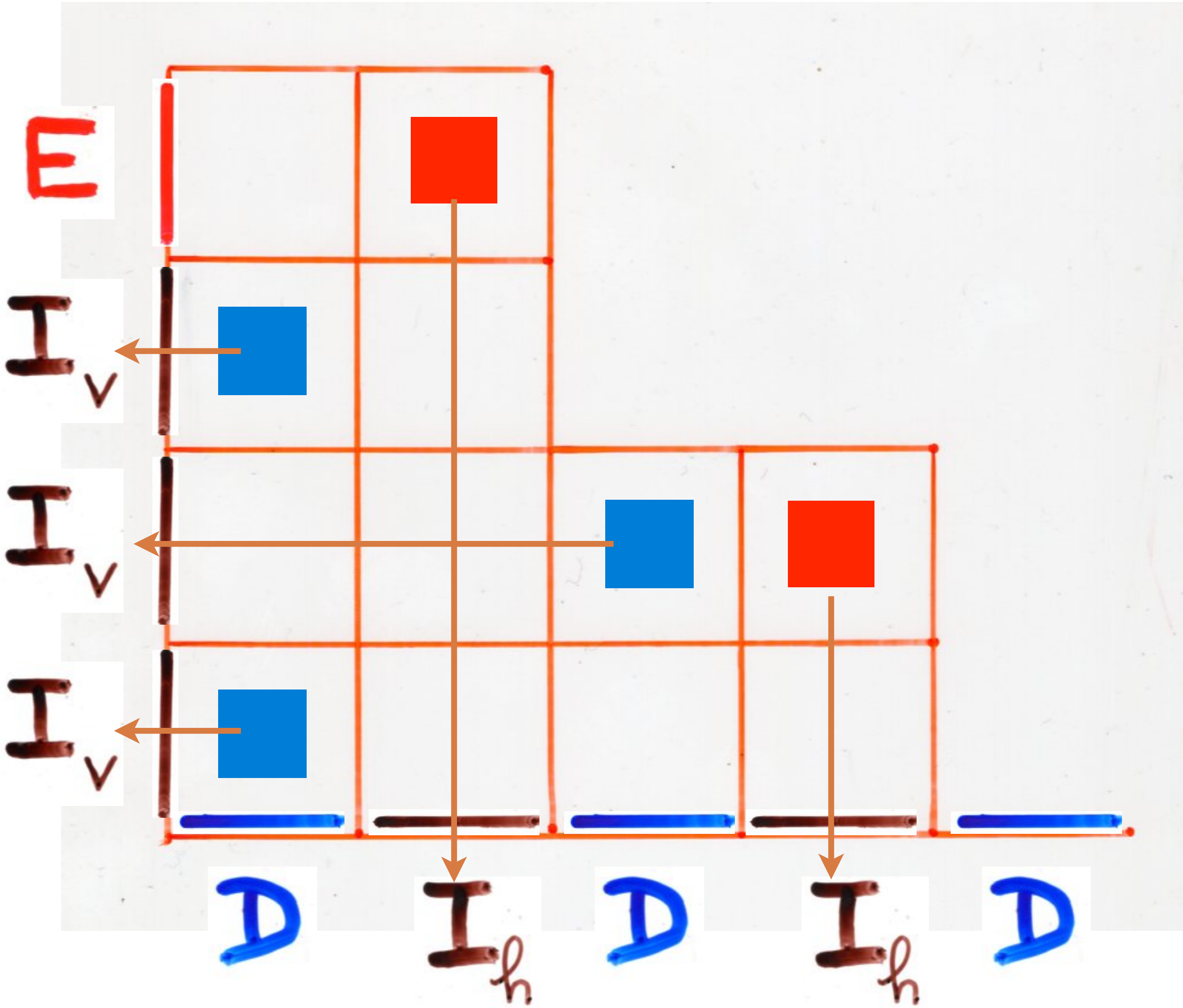


Definition

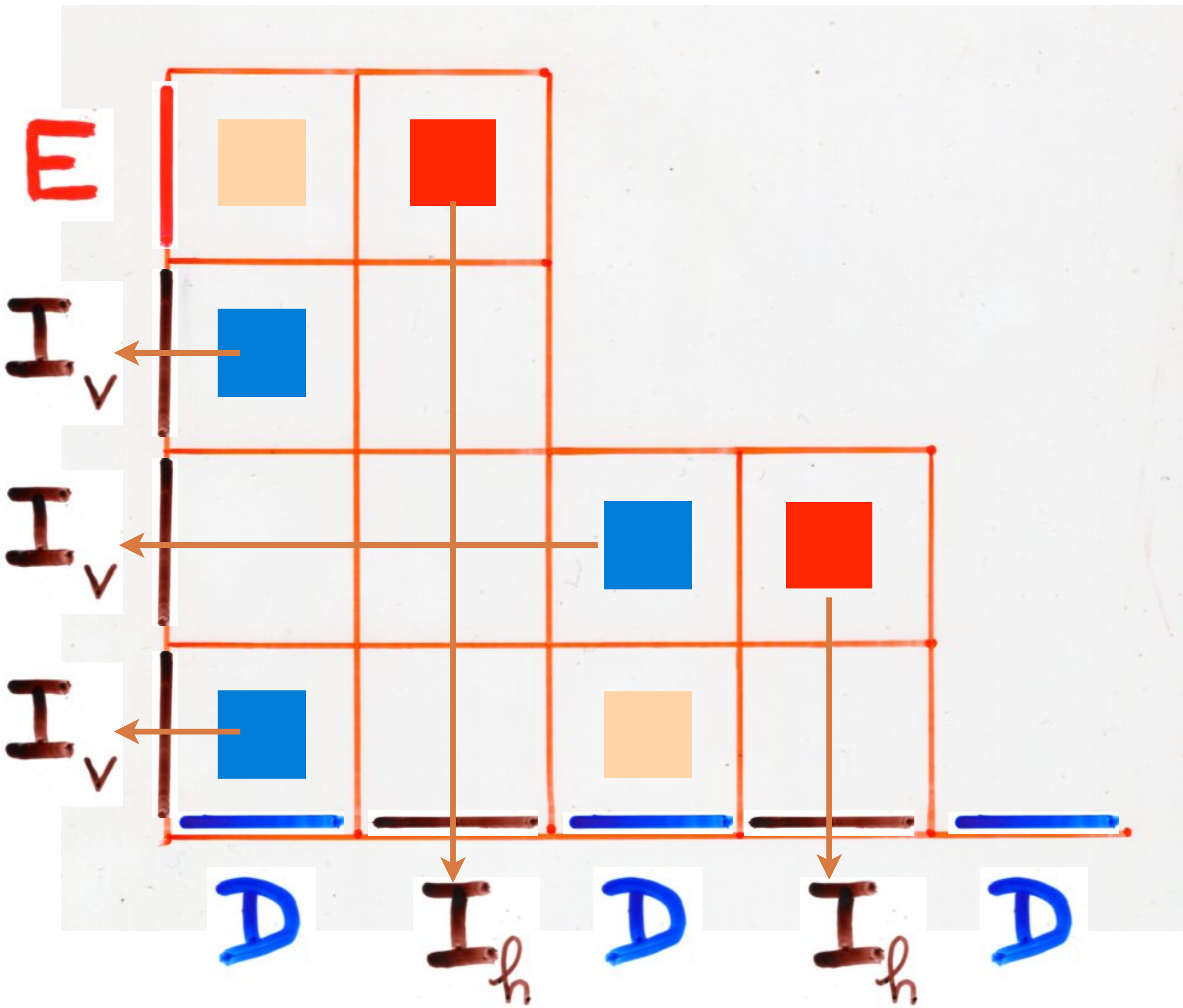
(i) some cells are coloured red or blue



(ii)
- no coloured cell at the left of a blue cell
- no coloured cell below a red cell



E
I_v
I_v
I_v
D
I_h
D
I_h
D



E
I_v
I_v
I_v
D
I_h
D
I_h
D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

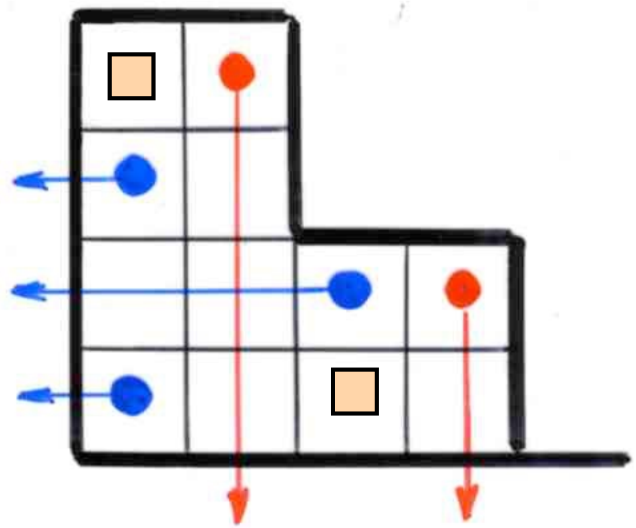
alternative tableau

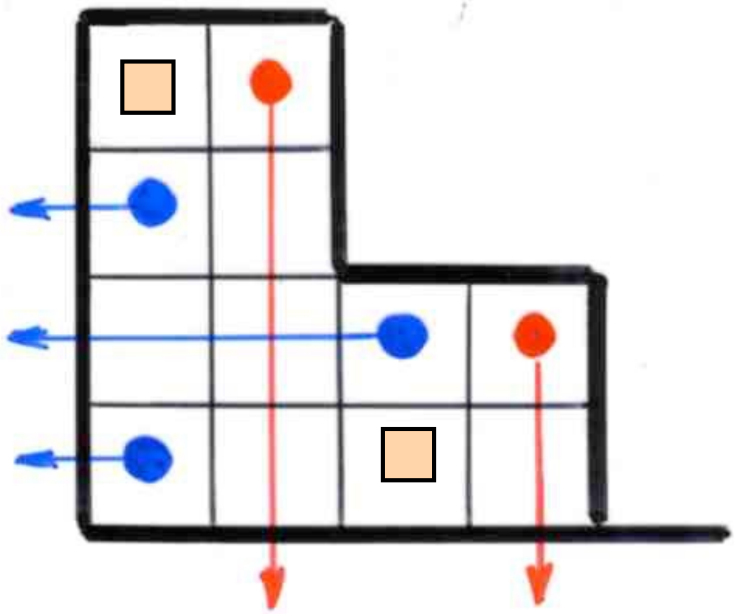
unique

$k(T)$ = nb of cells $\square$

$i(T)$ = nb of rows without $\bullet$

$j(T)$ = nb of columns without $\bullet$

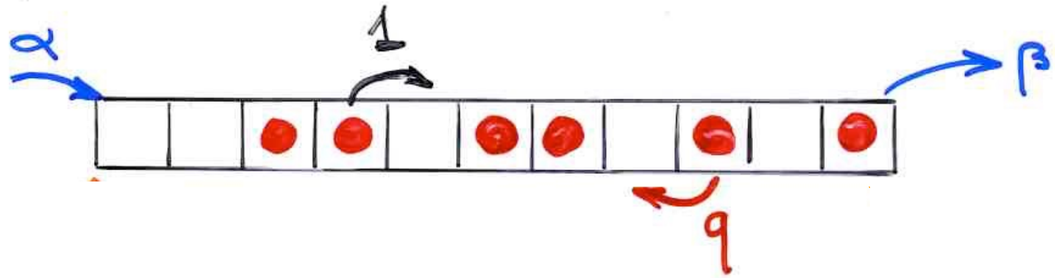




Prop. The number of alternative tableaux of size n is $(n+1)!$

computation of the "stationary probabilities"

PASEP with 3 parameters q, α, β

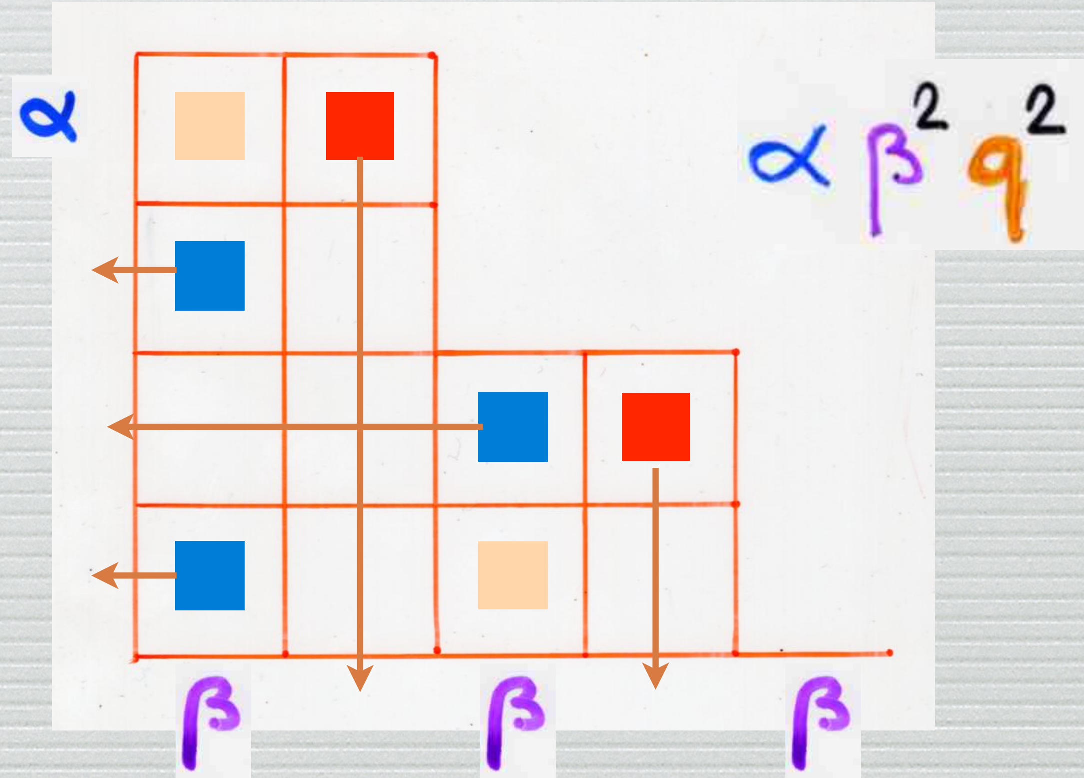


PASEP

Partition function

Z_n

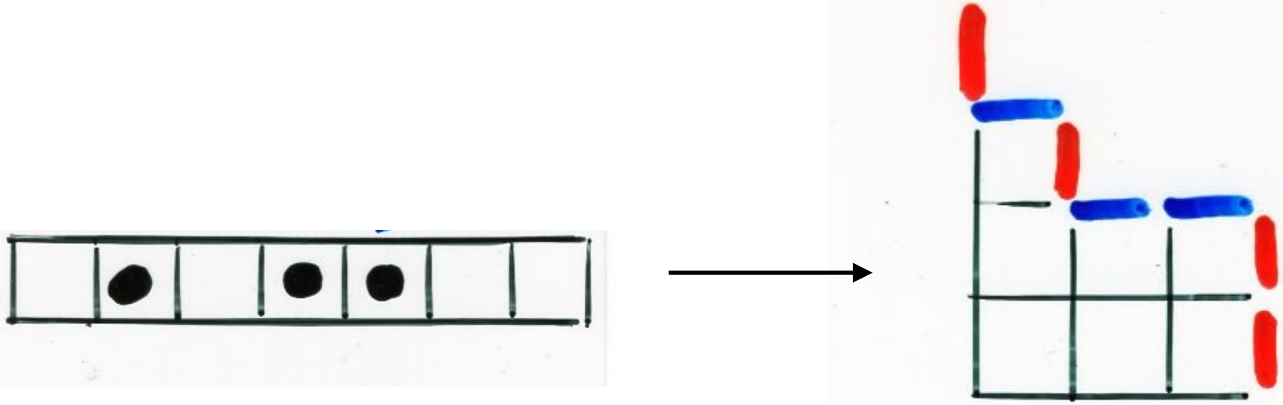
Sum of the weight of all tableaux of size n



q $k(T)$ = nb of cells

α $i(T)$ = nb of rows without

β $j(T)$ = nb of columns without



Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\mathrm{proba}_{\tau}(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

alternative tableaux profile τ

alternative tableau X.V. (2008)

permutation tableau

S. Corteel, L. Williams (2007, 2008, 2009) seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

colum vector V

row vector W

$$\begin{cases} DE = qED + E + D \\ \langle W | (\alpha E - \gamma D) = \langle W | \\ (\beta D - \delta E) | V \rangle = | V \rangle \end{cases}$$

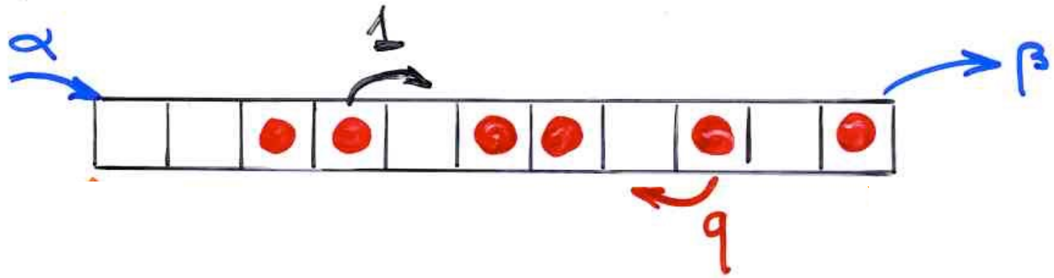
$q = 0$

The TASEP

$q = 0$

The PASEP algebra $\quad DE = \qquad E + D$

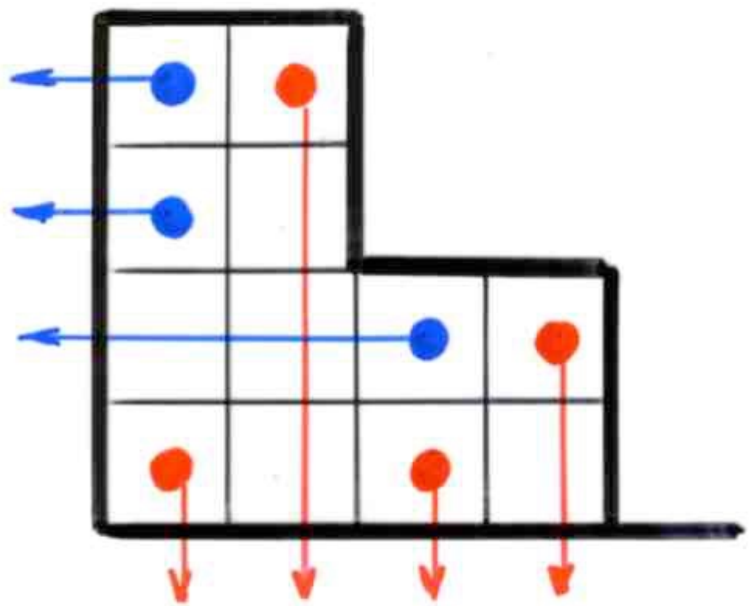
TASEP



Definition Catalan alternative tableau

alternative tableau T without cells □

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

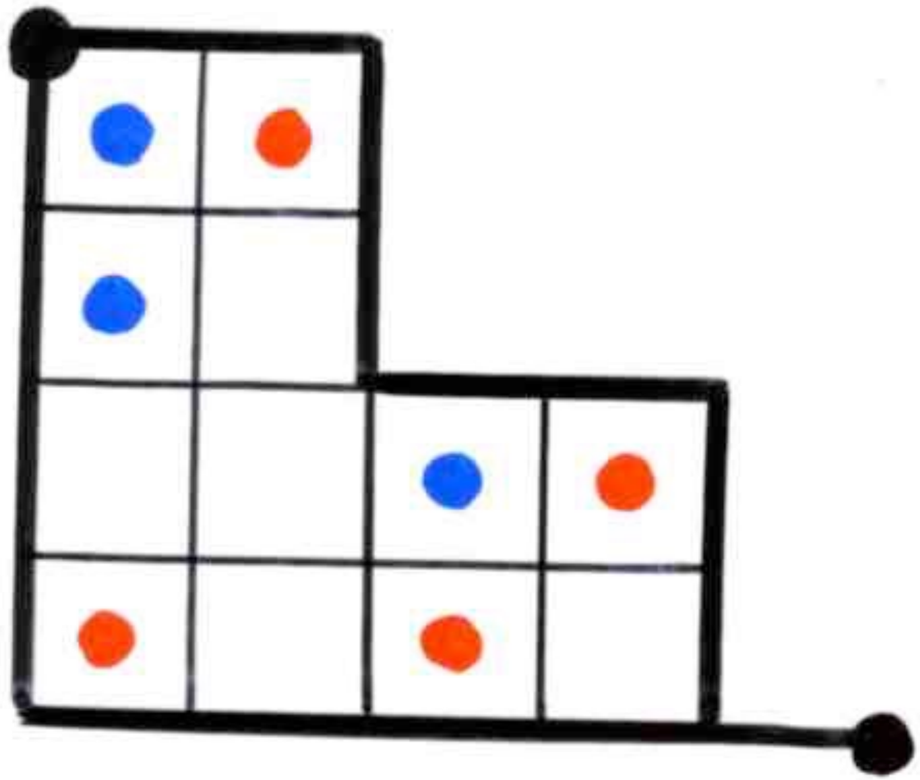
Catalan numbers

bijection

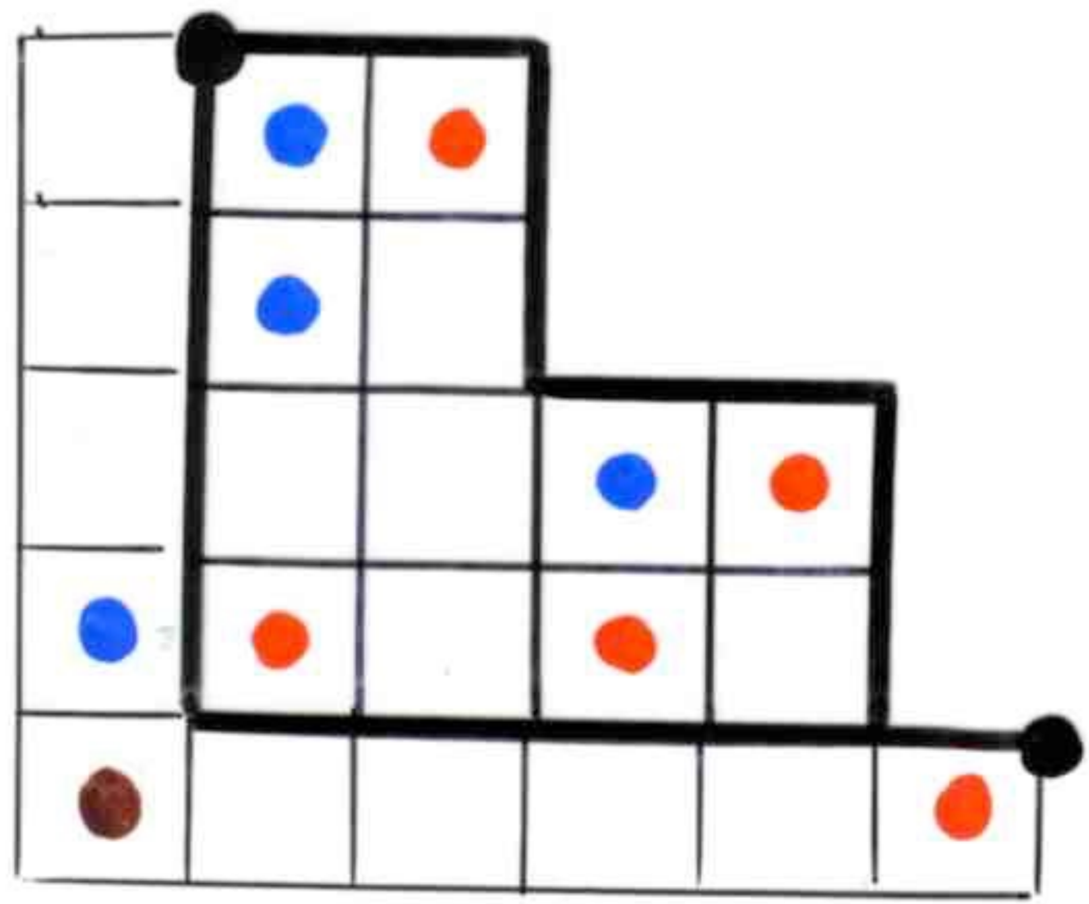
Catalan alternative tableaux

binary trees

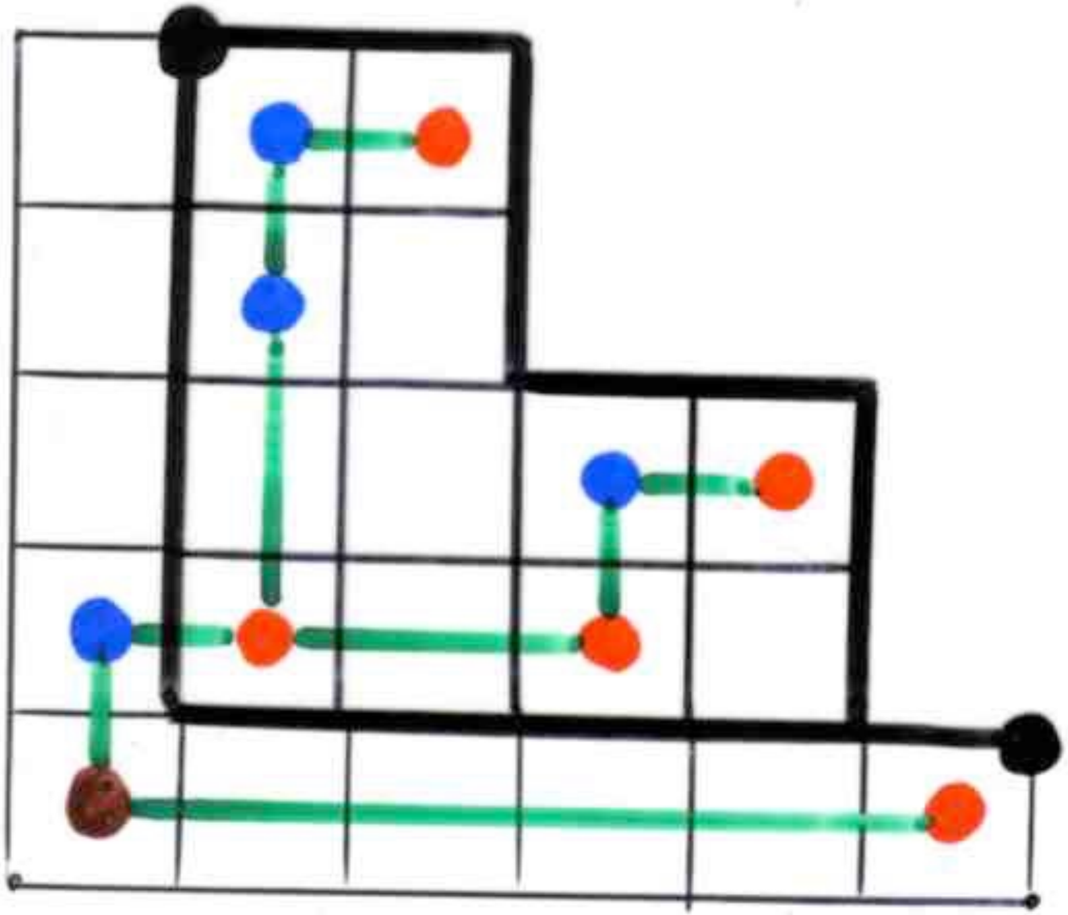
A Catalan alternative tableau



the extended Catalan alternative tableau

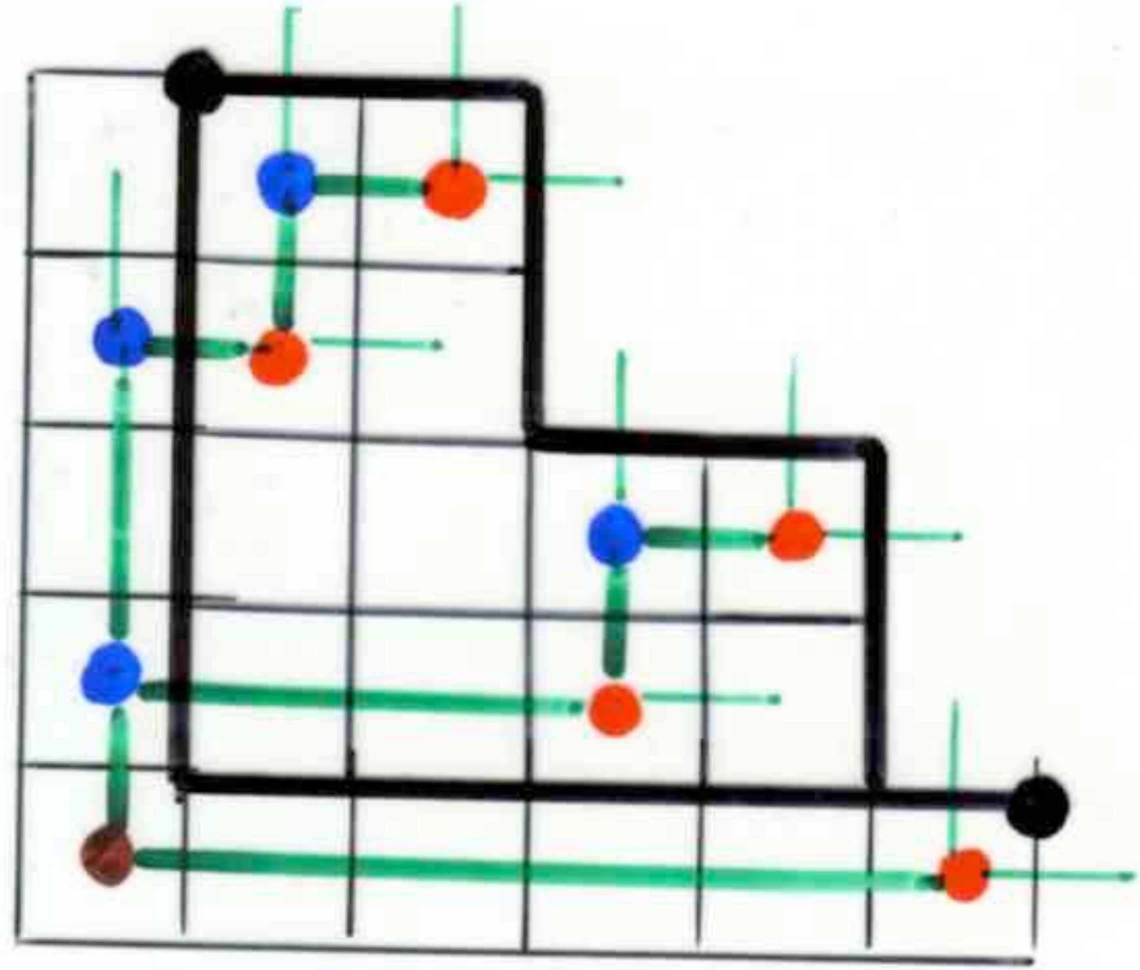


for each blue point add a vertical (green) edge below the point
for each red point add an horizontal (green) edge at the left of he point



one get a binary tree

the associated extended binary tree
(also called complete)



Reverse Q-tableaux

for the PASEP algebra

PASEP algebra

$$
Q \left\{
\begin{aligned}
DE &= q\,ED + EX + YD \\
XE &= EX \\
DY &= YD \\
XY &= YX
\end{aligned}
\right.
$$

$$
Q \left\{
\begin{aligned}
DE &= \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE &= \square\, EX \\
DY &= \square\, YD \\
XY &= \square\, YX
\end{aligned}
\right.
$$

alternative tableaux

PASEP algebra

$$Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

reverse Q - tableau

PASEP algebra

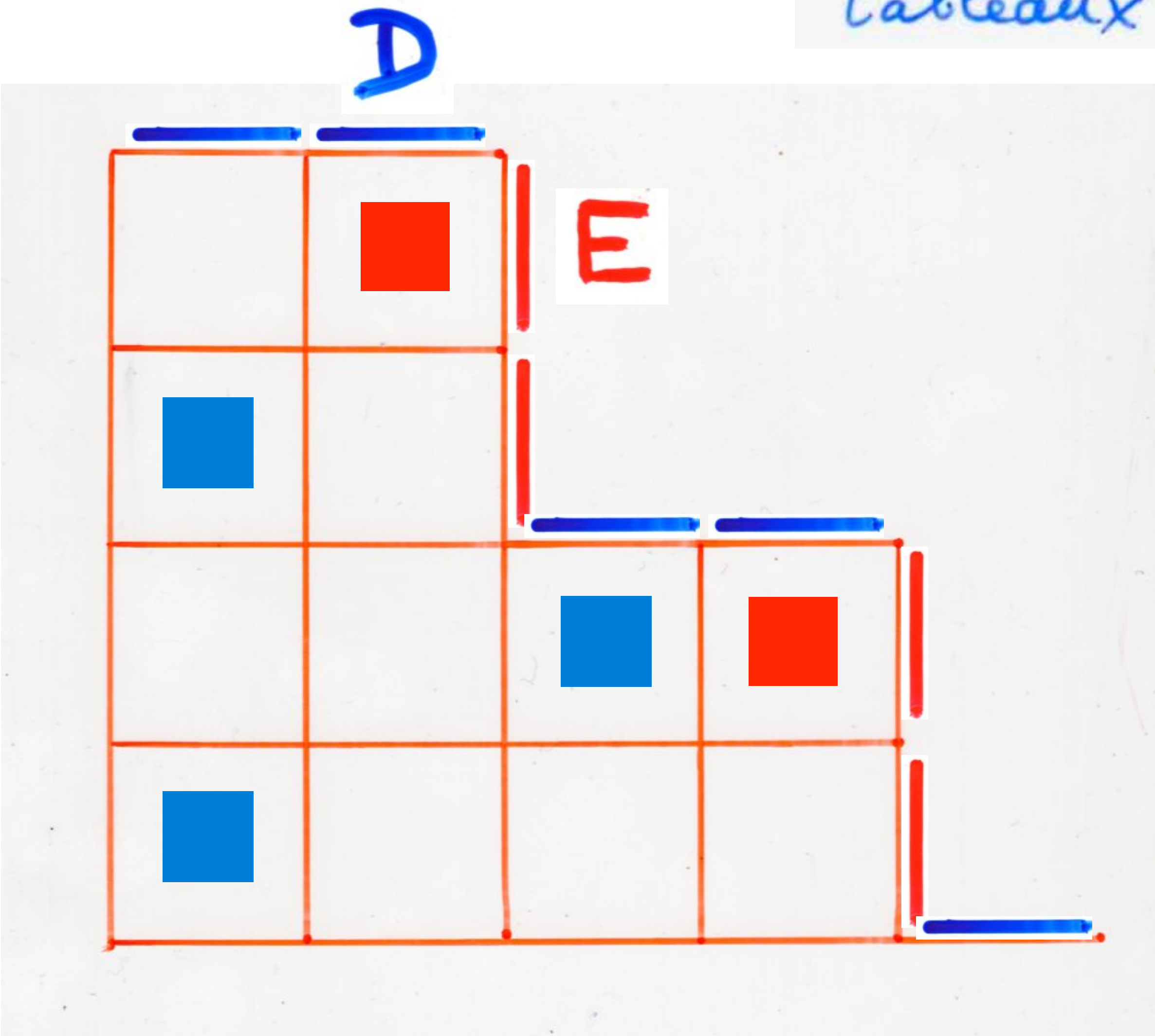
$$
Q\left\{
\begin{aligned}
DE &= \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE &= \square\, EX \\
DY &= \square\, YD \\
XY &= \square\, YX
\end{aligned}
\right.
$$

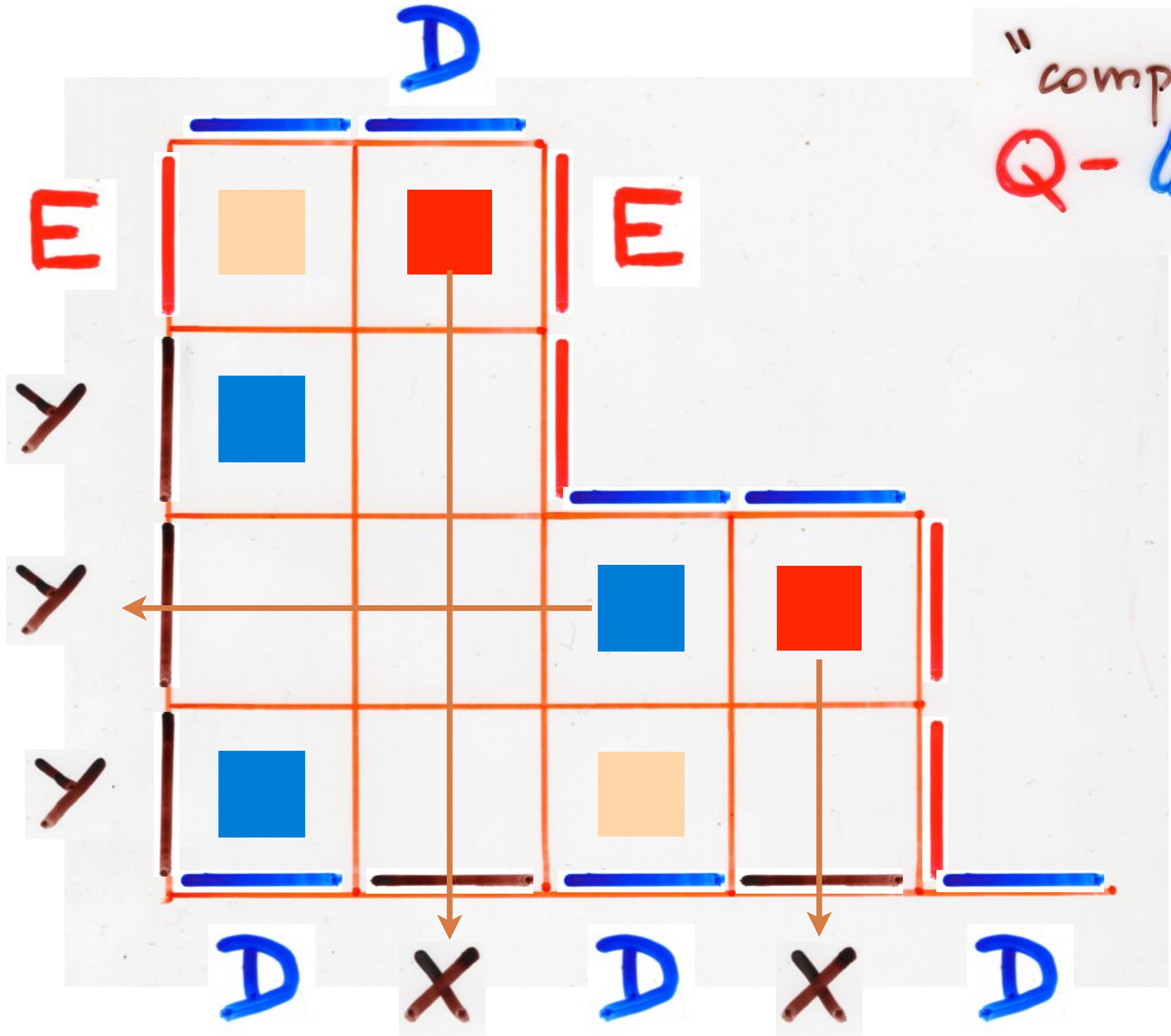
reverse Q-tableau

$$
Q^{+}\left\{
\begin{aligned}
ED &= \square\, DE \\
EX &= \square\, XE + \blacksquare\, DE \\
YD &= \square\, DY \quad \blacksquare\, DE \\
YX &= \square\, XY
\end{aligned}
\right.
$$

Q-tableaux

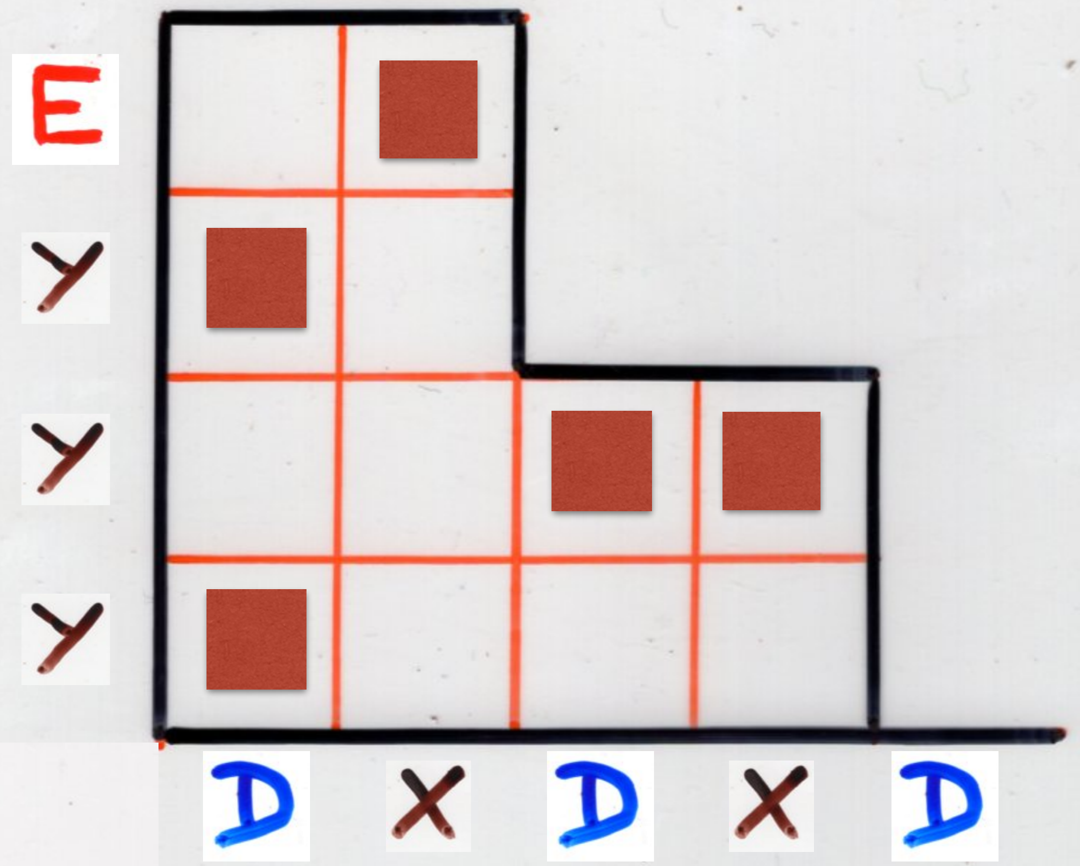
alternative tableaux





"complete"
Q- tableau

reverse Q - tableau

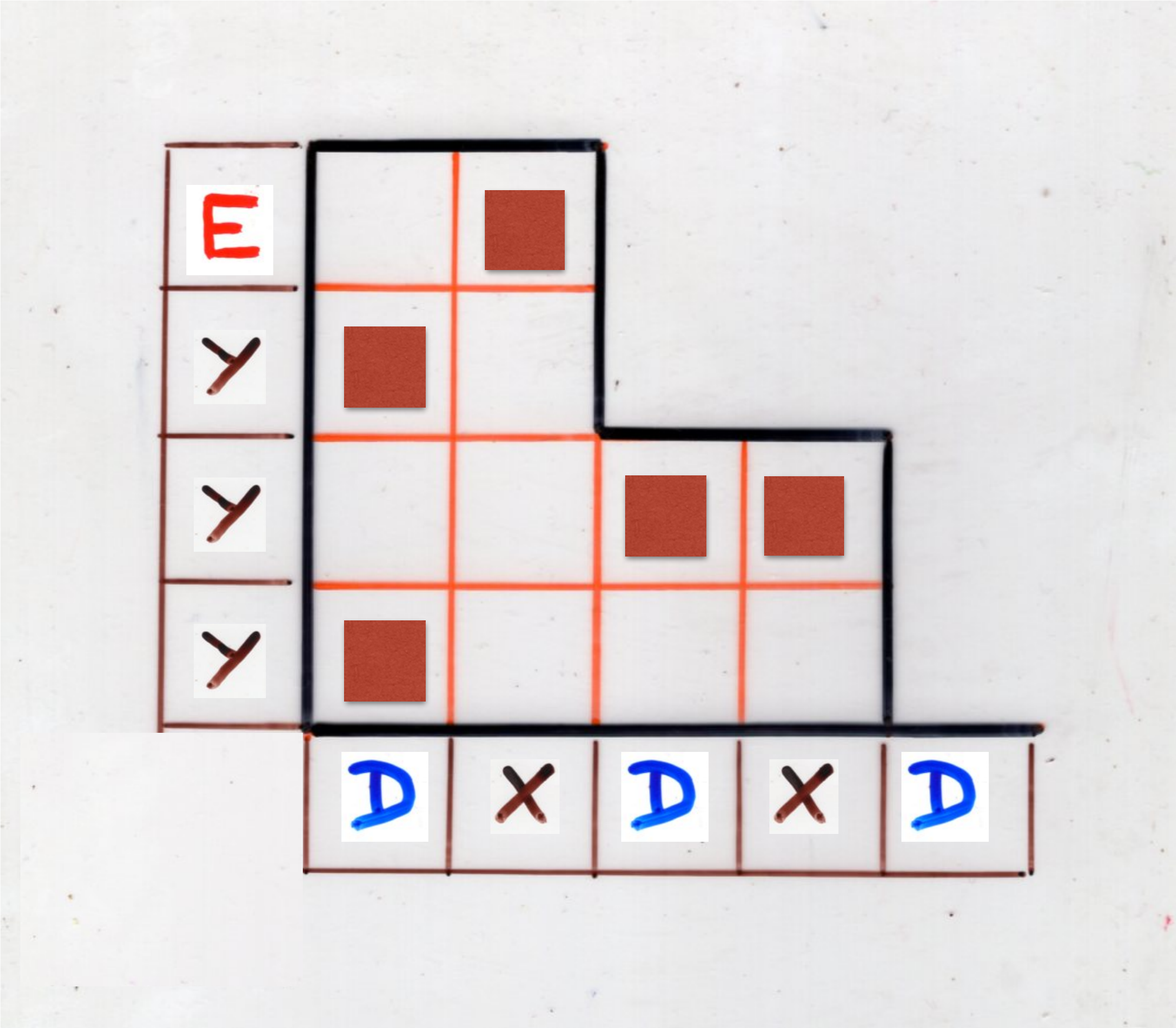


Coding of the reverse Q-tableau

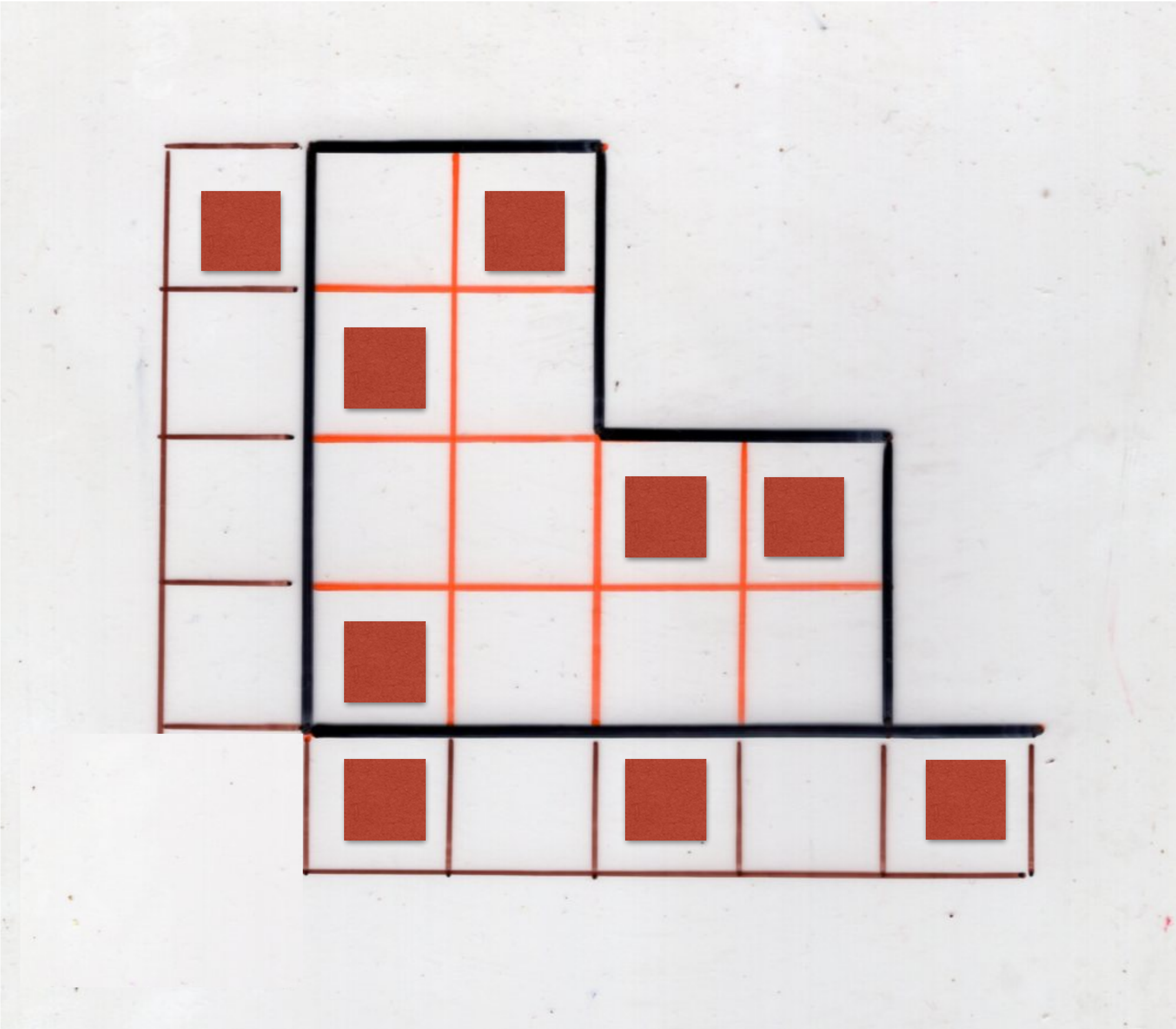
with

a tree-like tableau

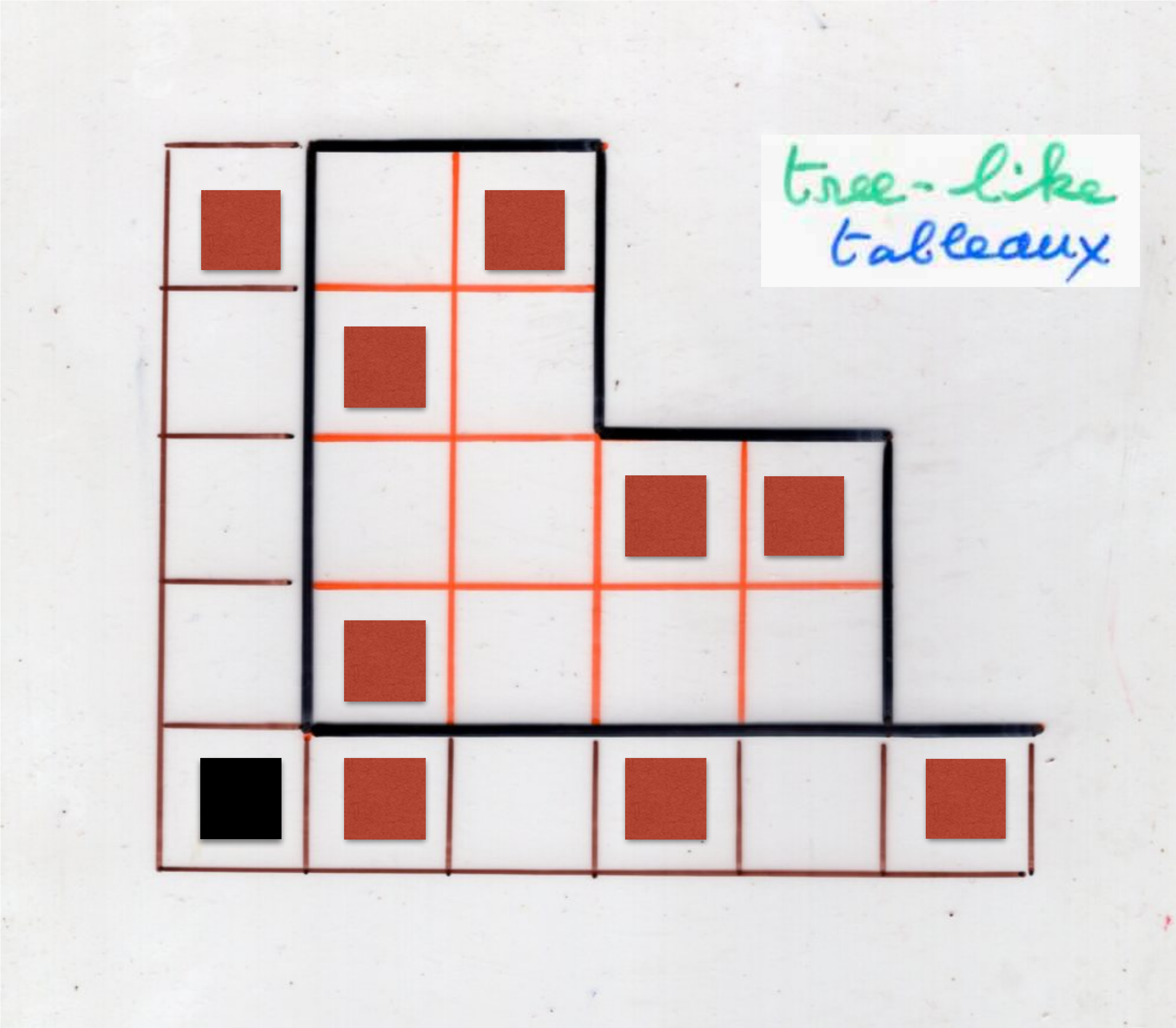
reverse Q - tableau



reverse Q - tableau



reverse Q - tableau



Definition Tree-like tableaux

Aval, Boussicault, Nadeau (2011)

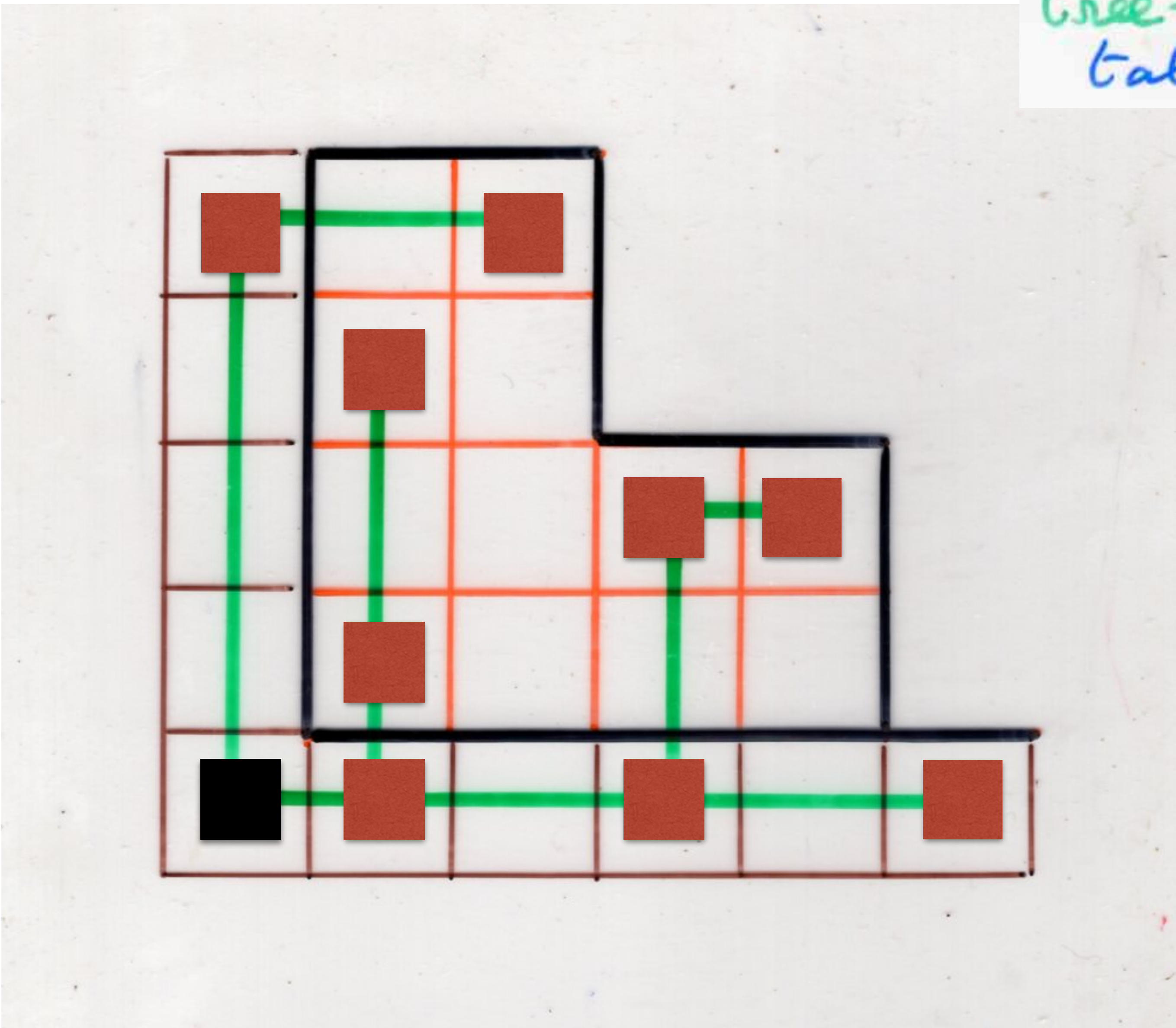
Ferrers diagram F with cell

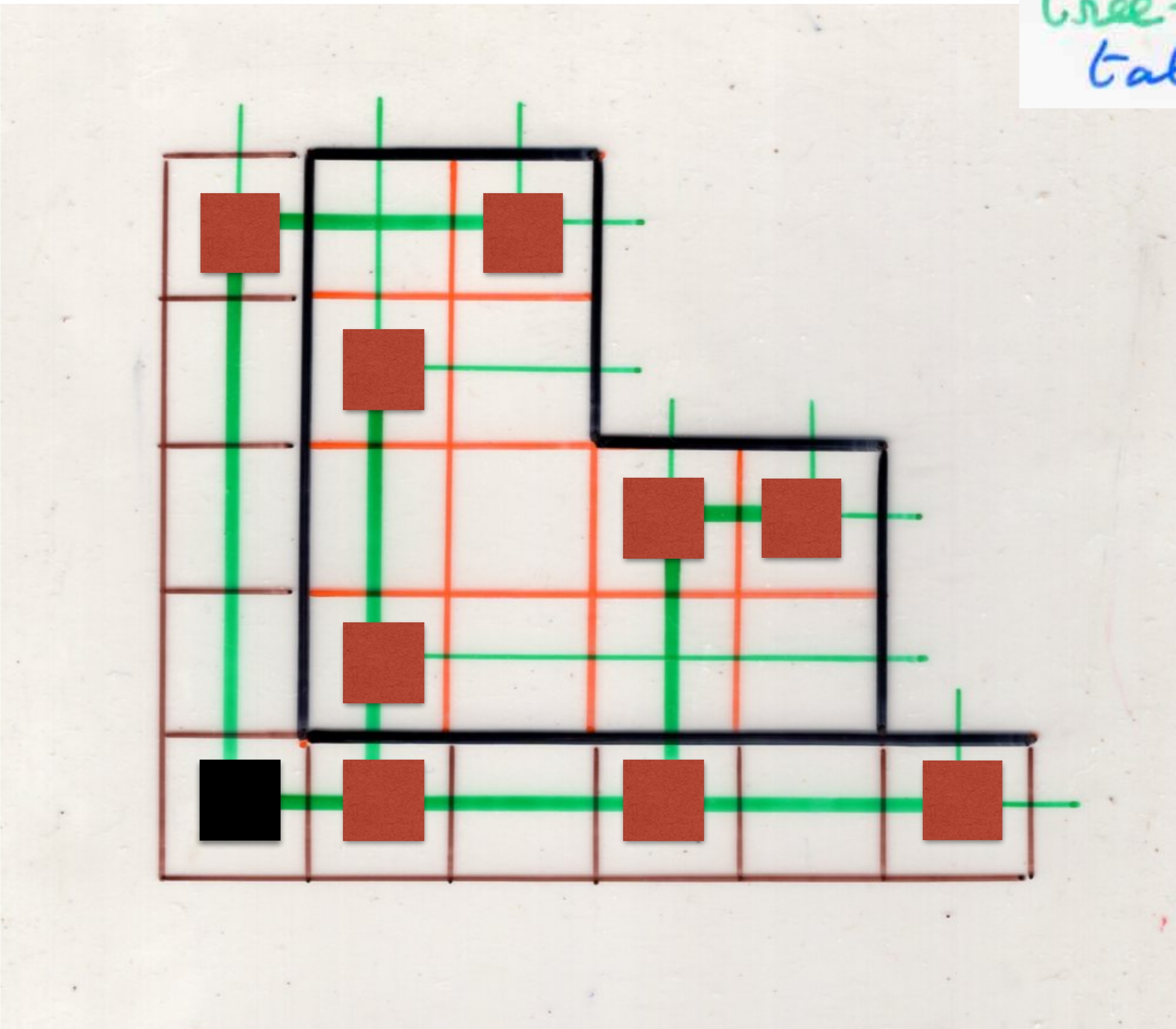
empty cell pointed cell

(i) the botom left cell is pointed (called the root cell)

(ii) for every non-root pointed cell c, there exist a pointed cell below c in the same column, or a pointed cell to its left in the same row, but not both

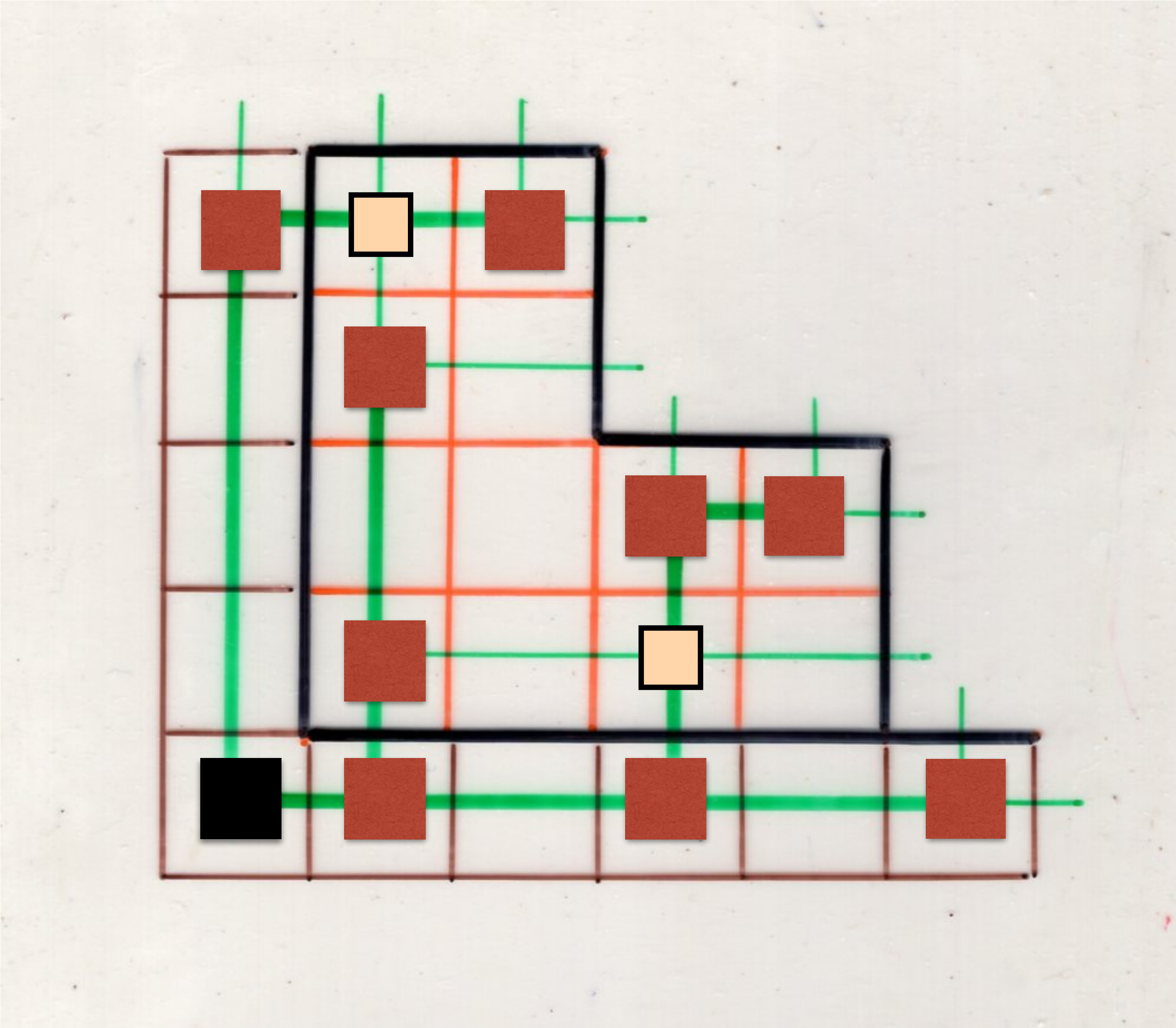
(iii) every column and every row passesses at least one pointed cell tree-like tableaux

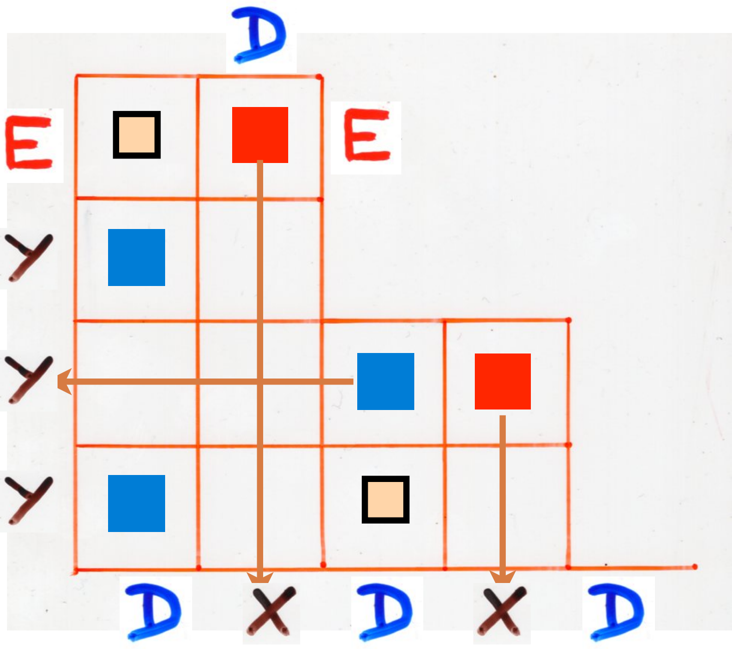


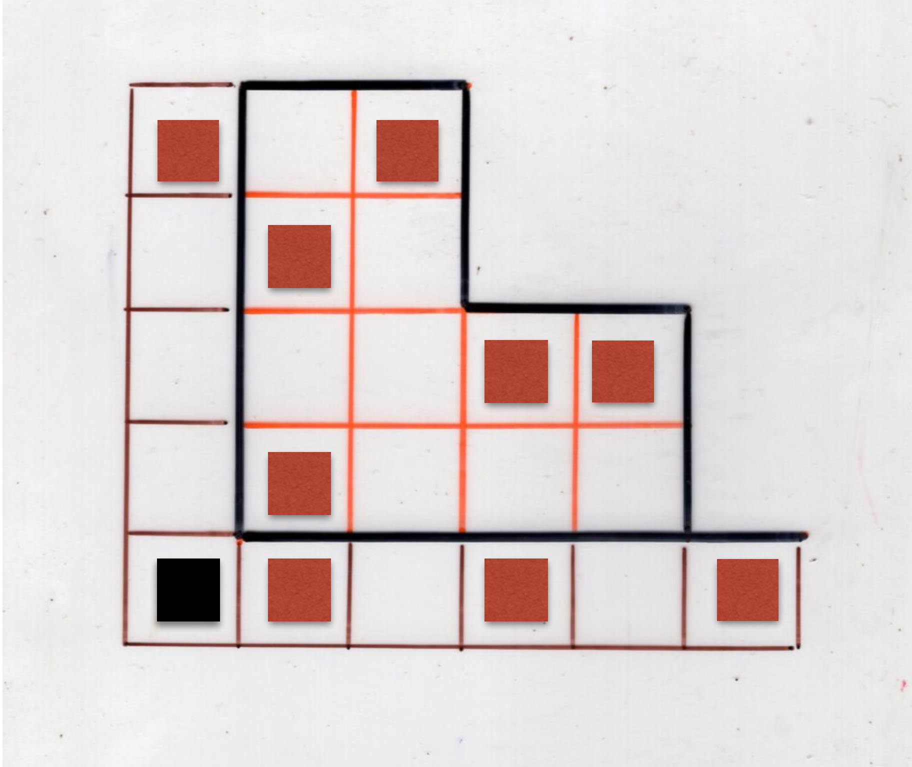


tree-like
tableaux

tree-like tableaux

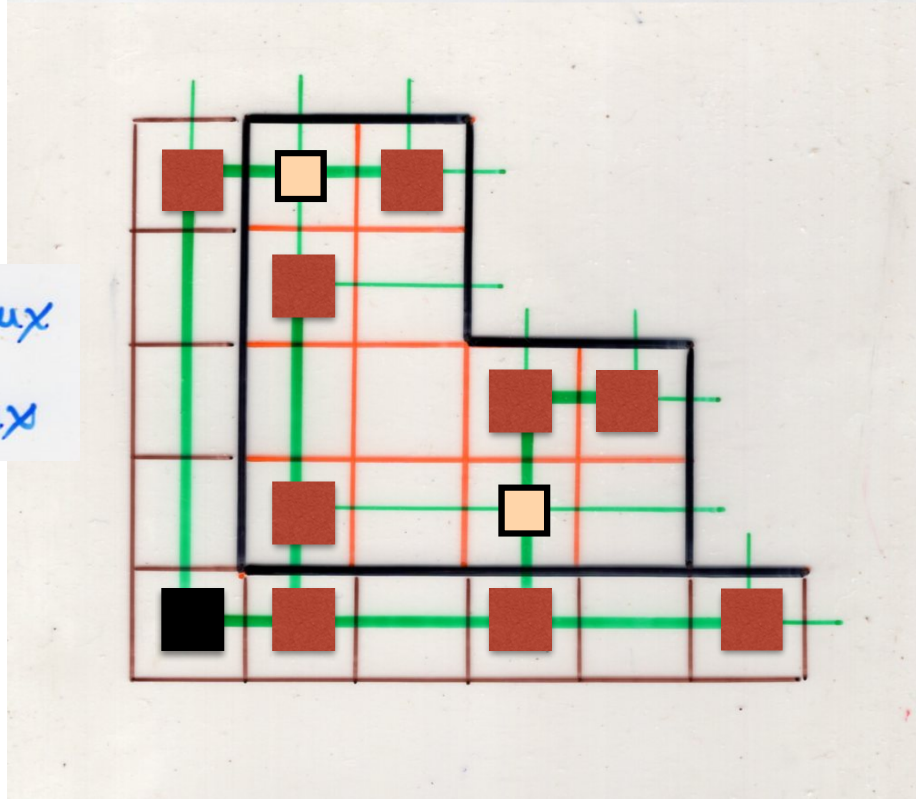




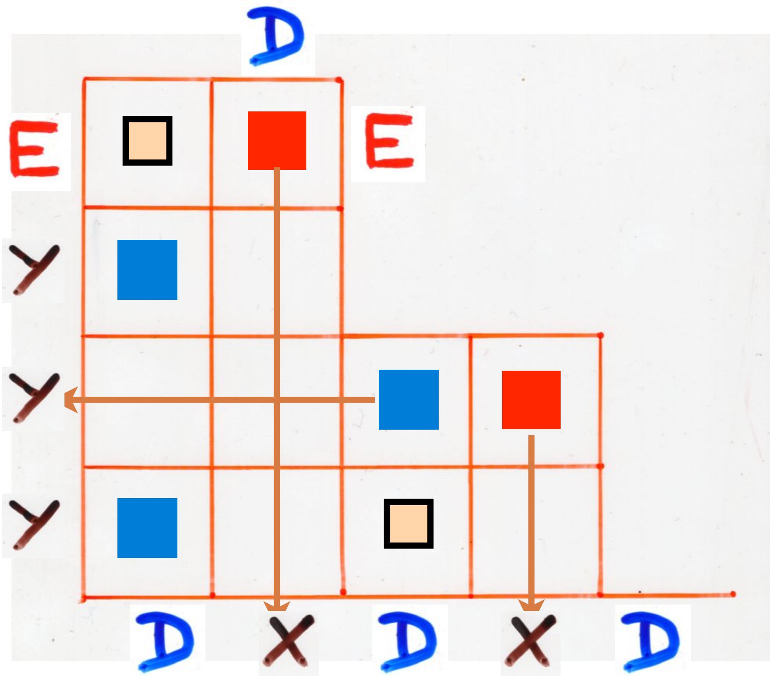


bijection alternative tableaux

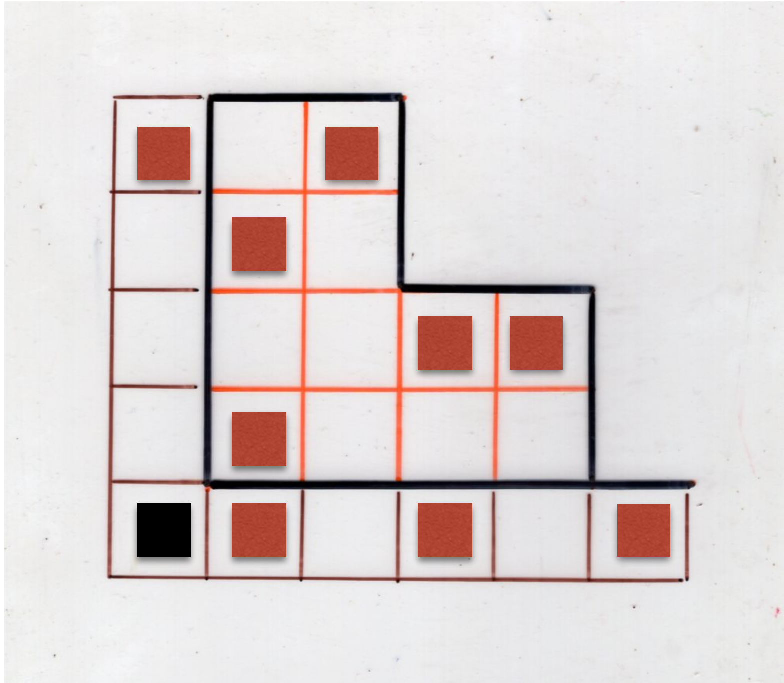
tree-like tableaux



alternative tableaux

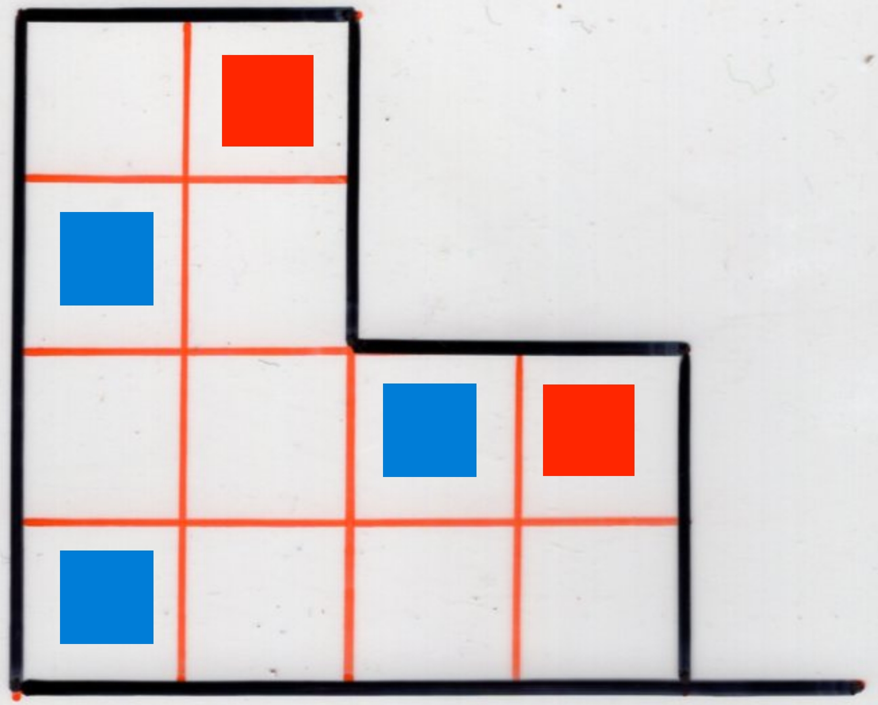


tree-like tableaux

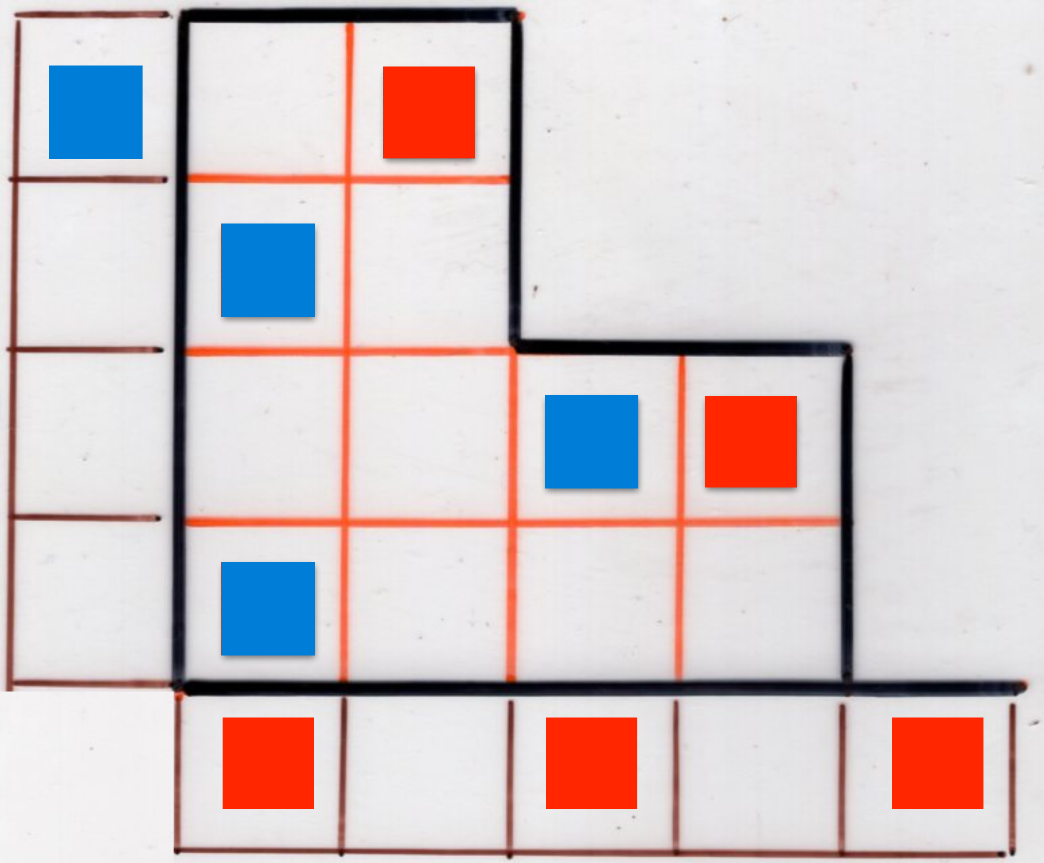


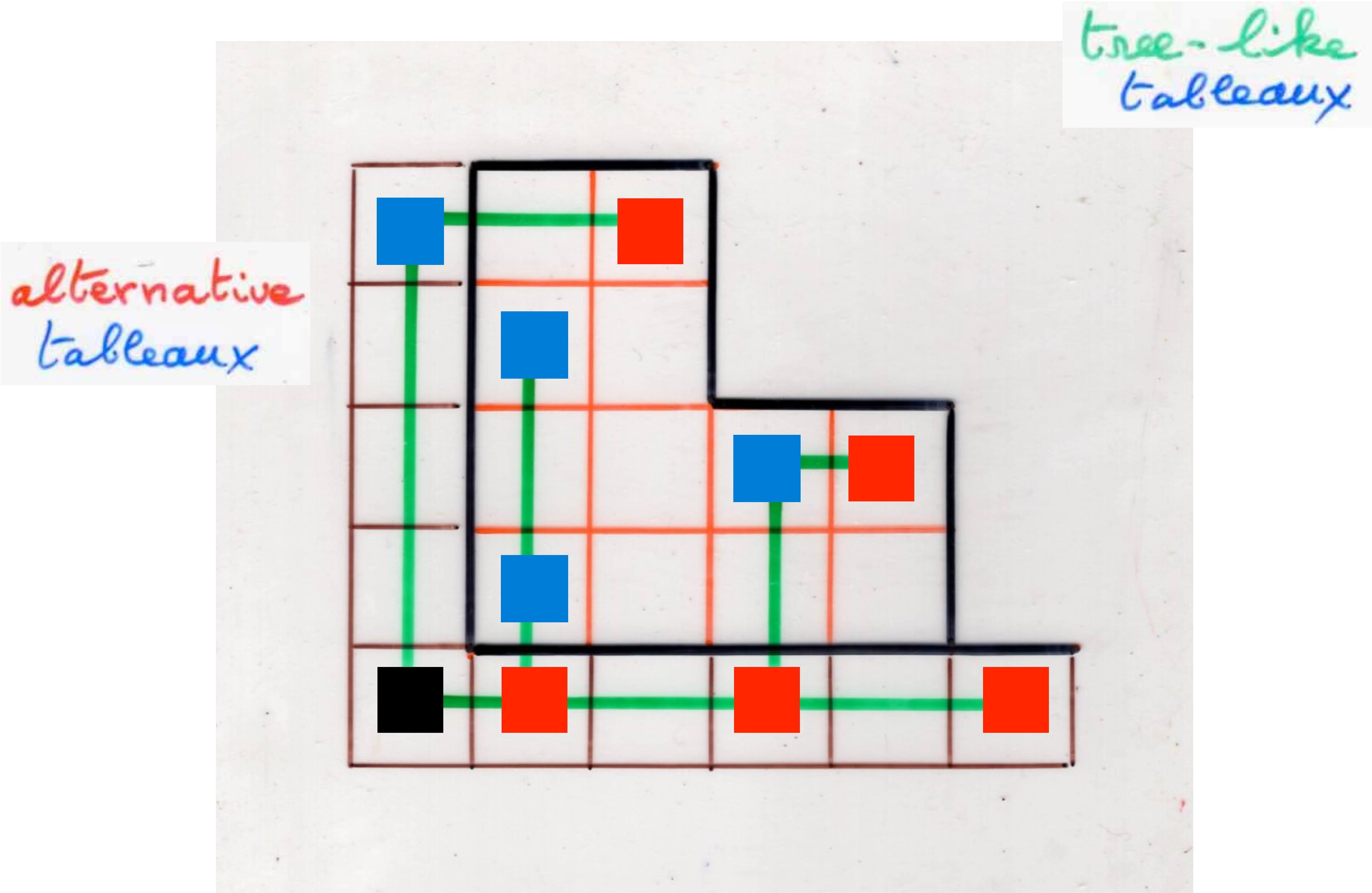
tree-like tableaux are "the" reverse Q-tableaux of alternative tableaux

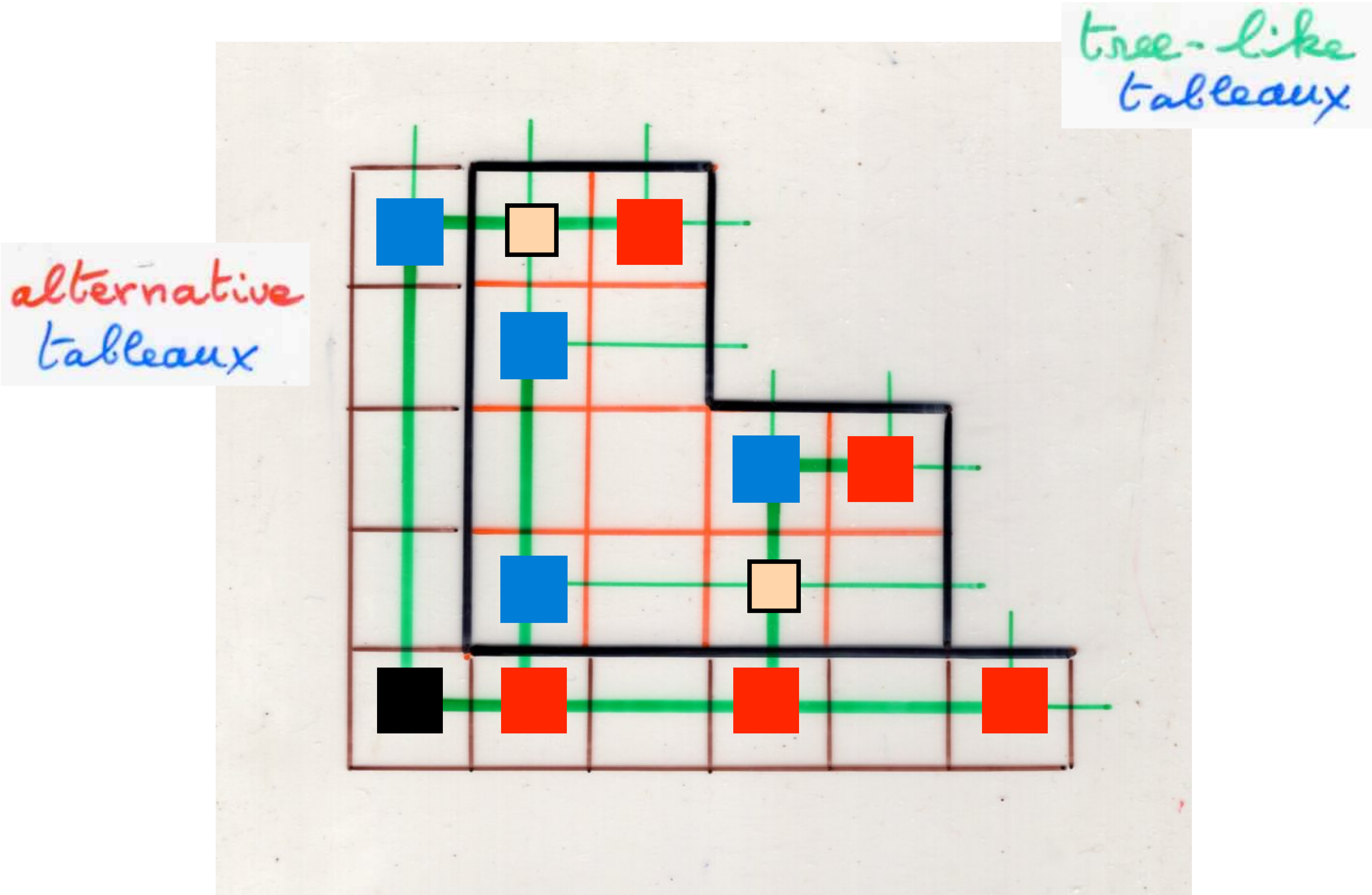
alternative
tableaux



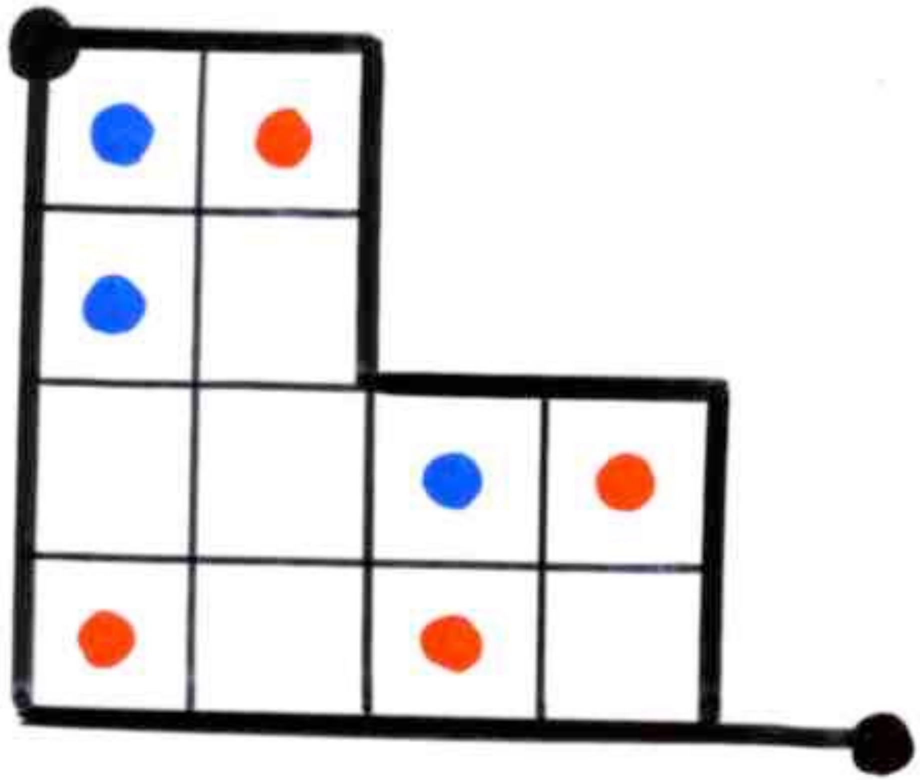
alternative
tableaux



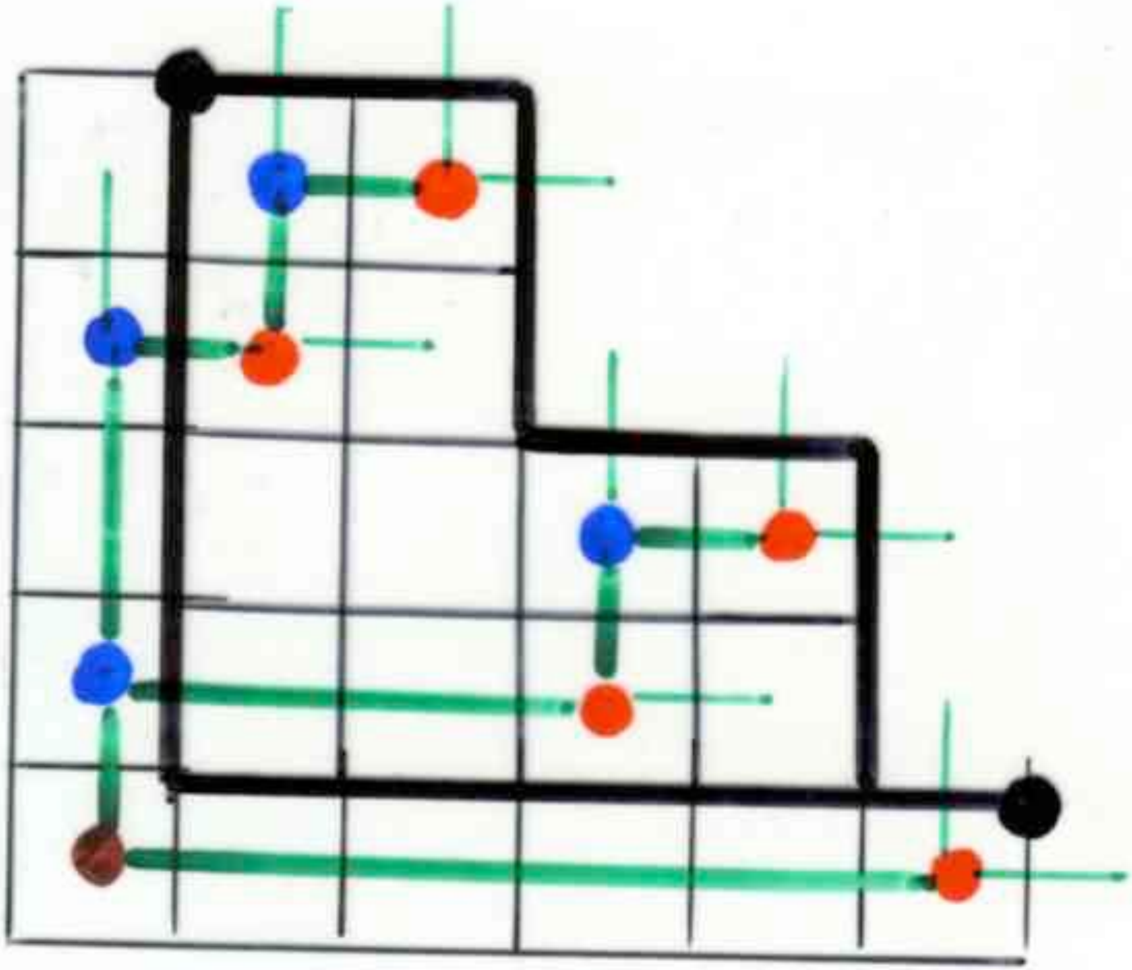

tree-like
tableaux
alternative
tableaux


tree-like
tableaux
alternative
tableaux

A Catalan alternative tableau



the associated extended binary tree
(also called complete)



The Tamil bijection

demultiplication of equations
in the
reverse PASEP quadratic algebra

PASEP algebra

$$
Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}
$$

reverse PASEP algebra

$$
Q^{+} \begin{cases} ED = qDE \\ EX = XE + DE \\ YD = DY \qquad DE \\ YX = XY \end{cases}
$$

$$\begin{cases} EX = XE + DE \\ YD = DY + DE \\ YX = XY \\ ED = 0 \end{cases}$$

$q = 0$

The Tamil bijection

$$B \longrightarrow (w, h)$$

reverse PASEP algebra

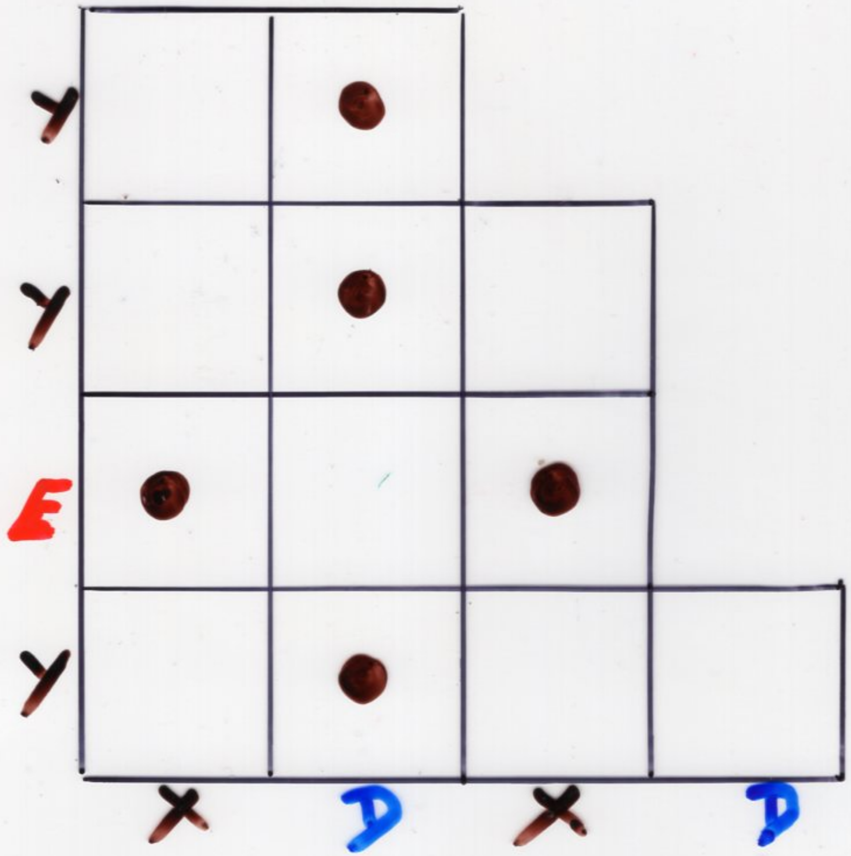
$$Q^{+} \begin{cases} ED = qDE \\ EX = XE + DE \\ YD = DY \quad DE \\ YX = XY \end{cases}$$

reverse PASEP algebra

$$
Q^{+}\begin{cases} ED = q\,DE \\ EX = XE + DE \\ YD = DY \quad DE \\ YX = XY \end{cases}
$$

$$
\begin{cases} E_i D_j = q\,\boxed{D_{j+1} E_{i+1}} \\ E_i X = X E_i + \boxed{D_1 E_{i+1}} \\ Y D_j = D_j Y + \boxed{D_{j+1} E_1} \\ YX = XY \end{cases}
$$

$i, j \geq 1$

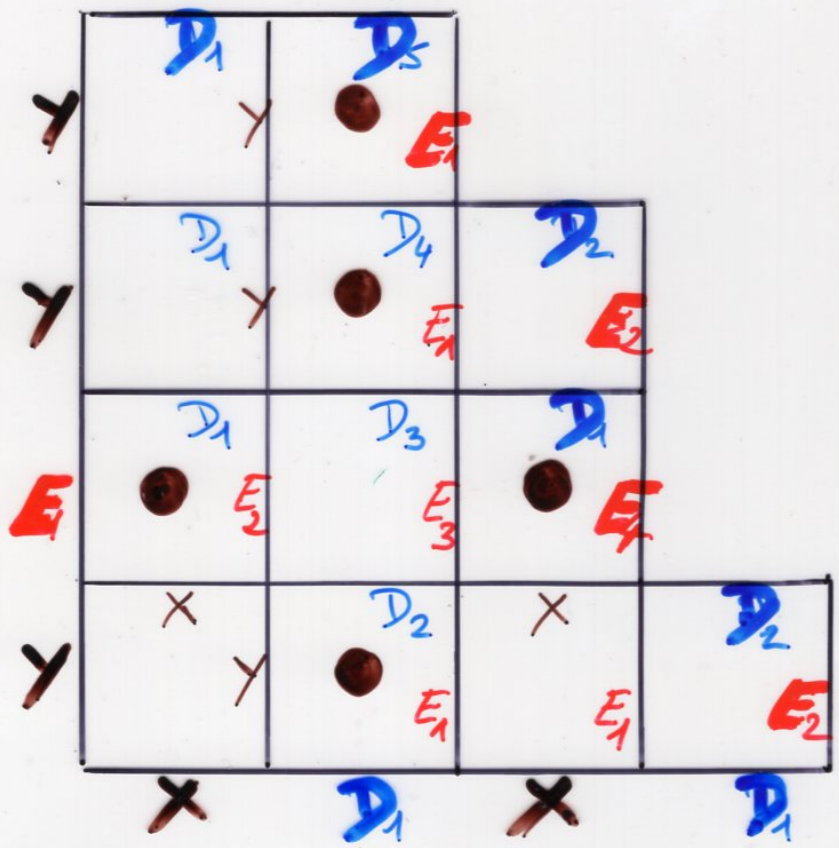

Y
Y
E
Y
X
D
X
D

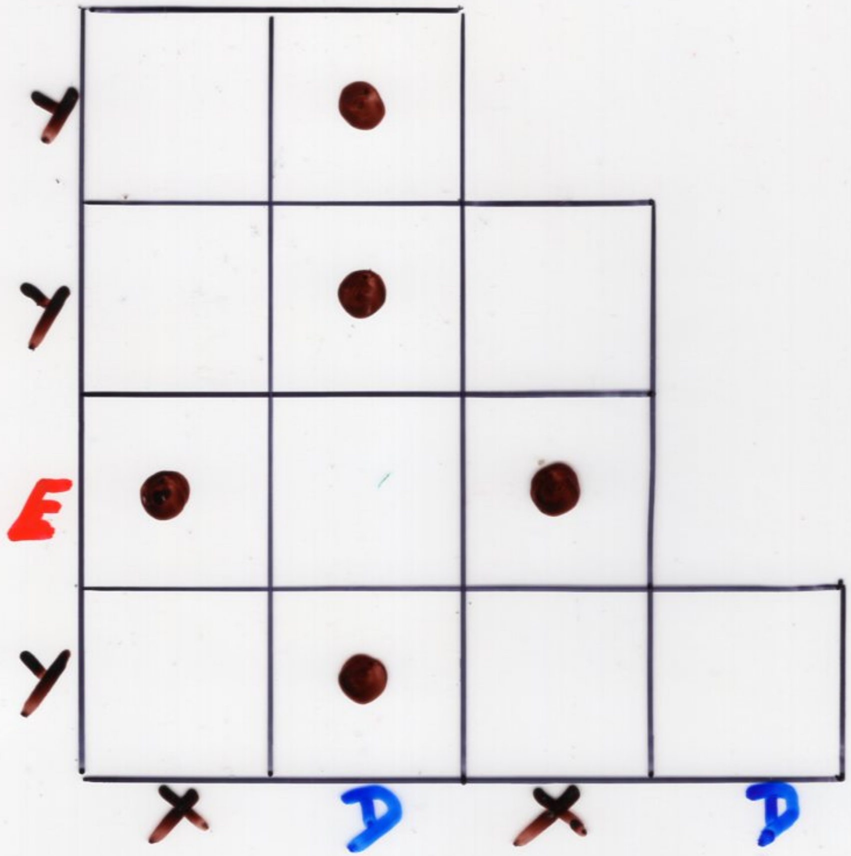
The Tamil bijection

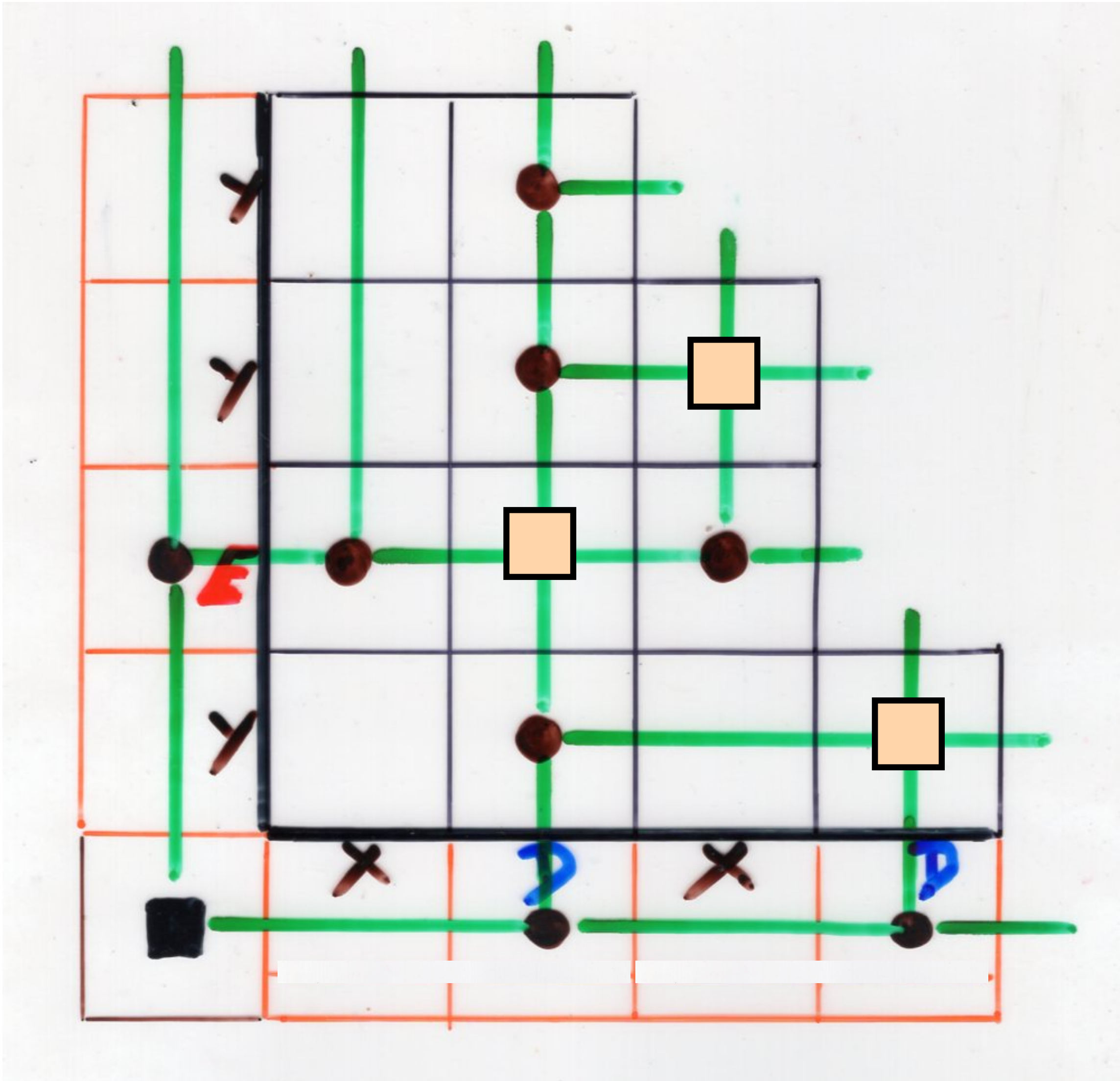
alternative or tree-like tableaux (size n)

some words $w \in \{D_i, E_j \,;\, i,j \geq 1\}^*$, $|w| = n$

$$w = D_1\, D_5\, E_1\, D_2\, E_2\, E_4\, D_2\, E_2$$

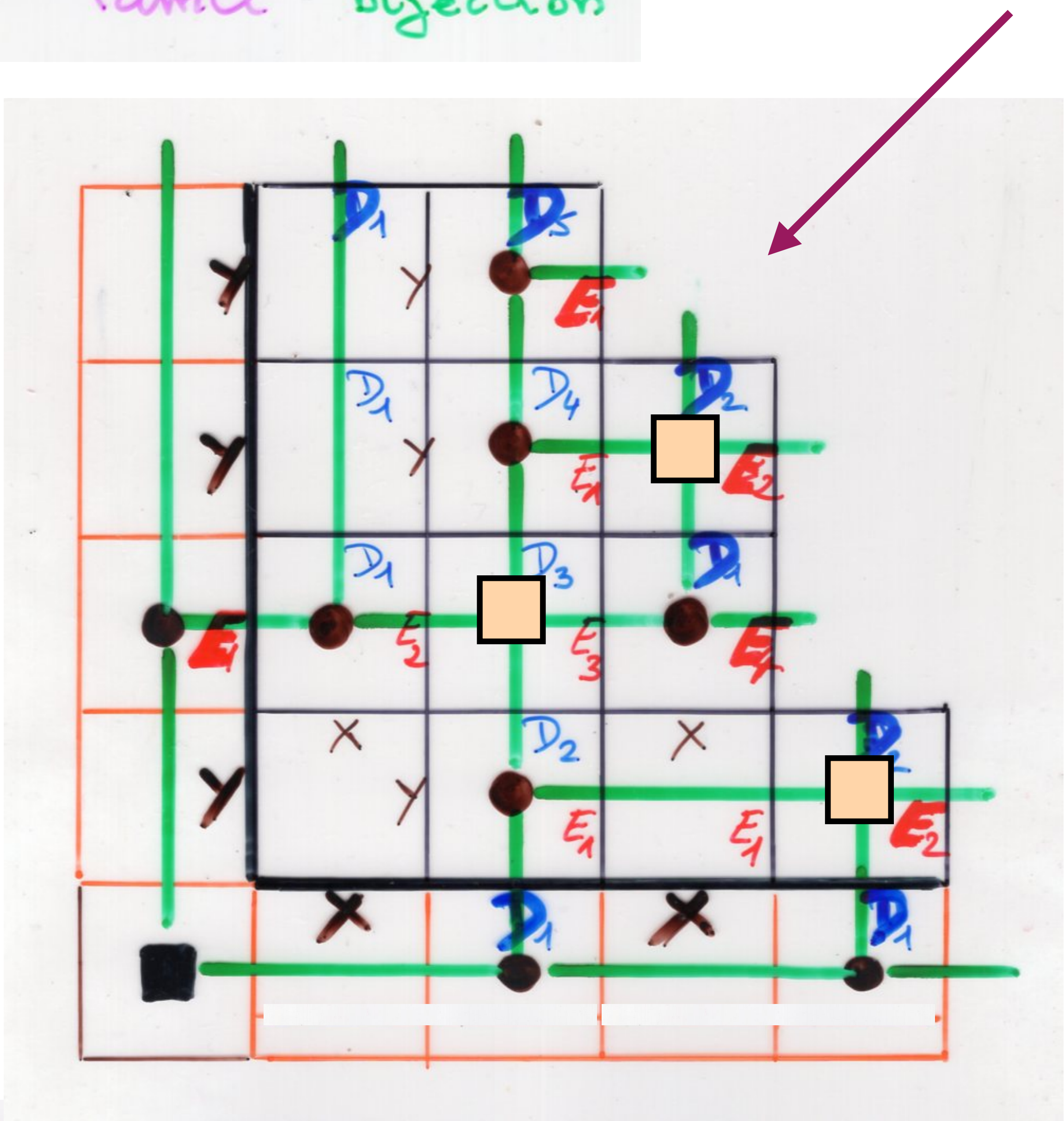



Y
Y
E
Y
X
D
X
D

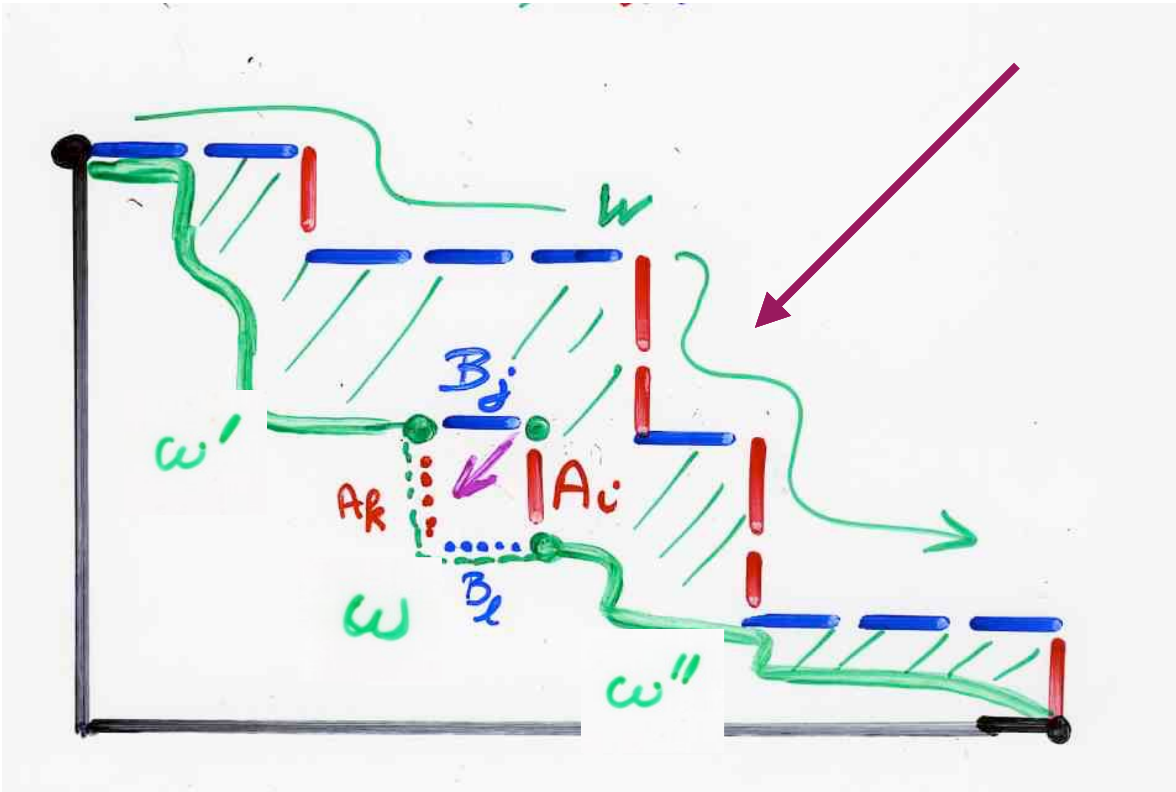


Y
Y
Y
X
X

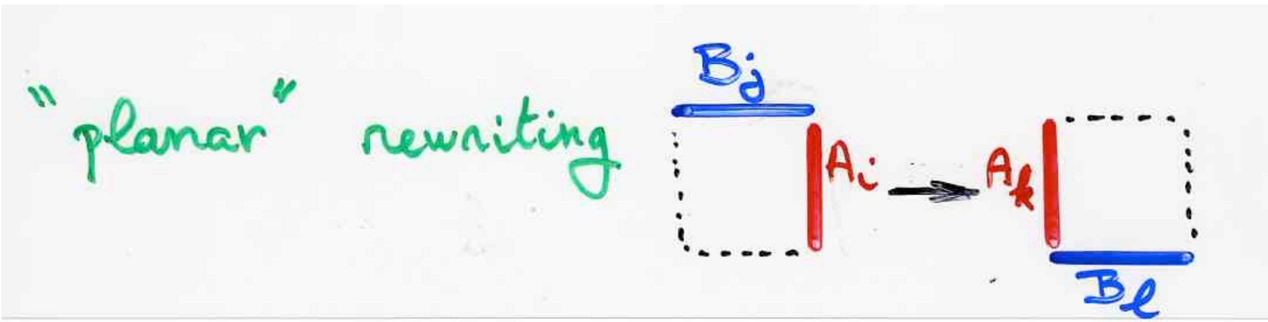
The Tamil bijection



$w = D_1 D_5 E_1 D_2 E_2 E_4 D_2 E_2$

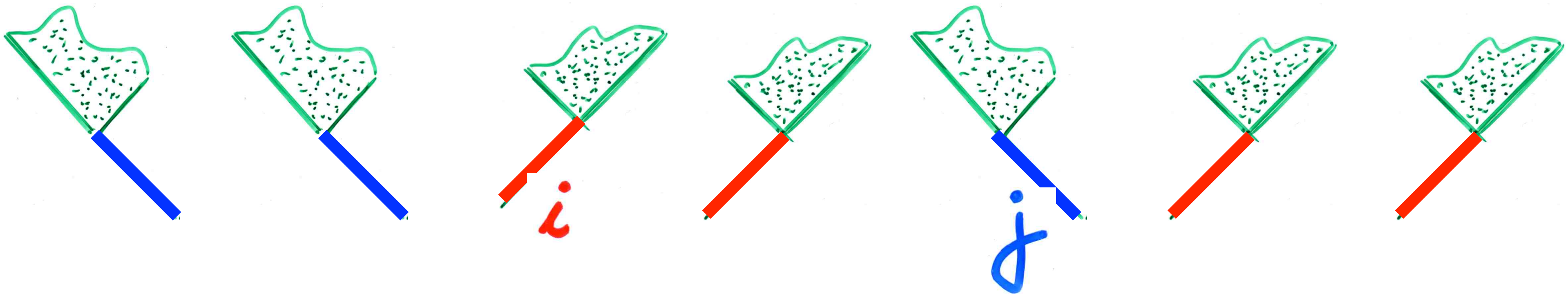


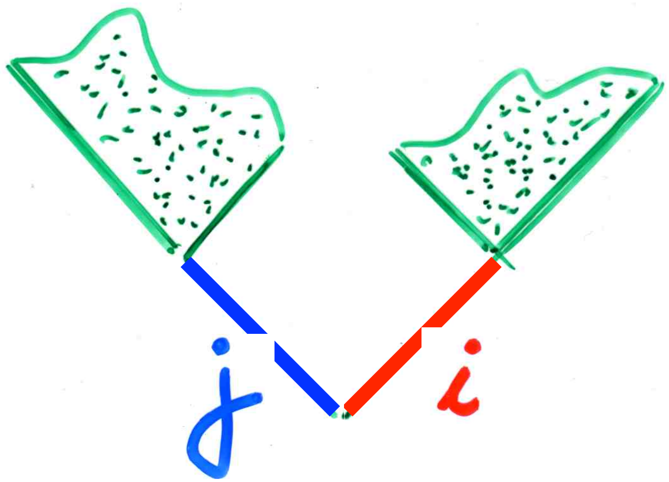
w
B_j
ω'
A_k
A_i
ω
B_ℓ
ω''



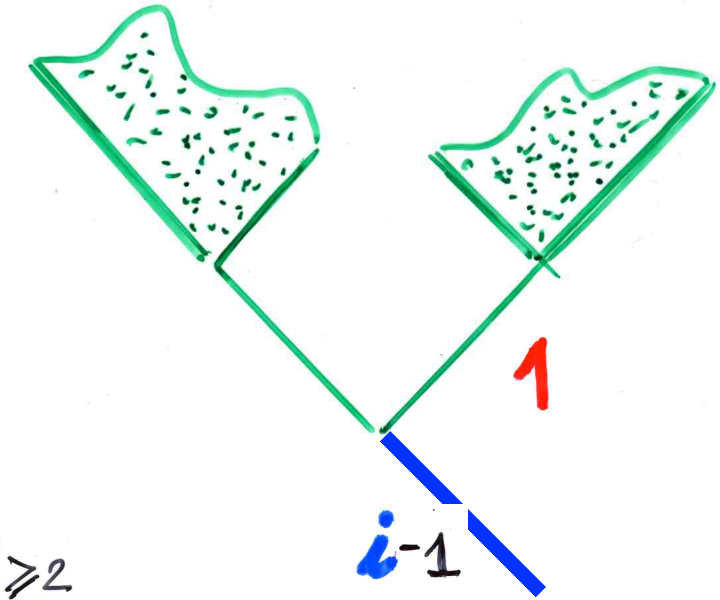
"planar" rewriting
B_j
A_i
A_k
B_ℓ

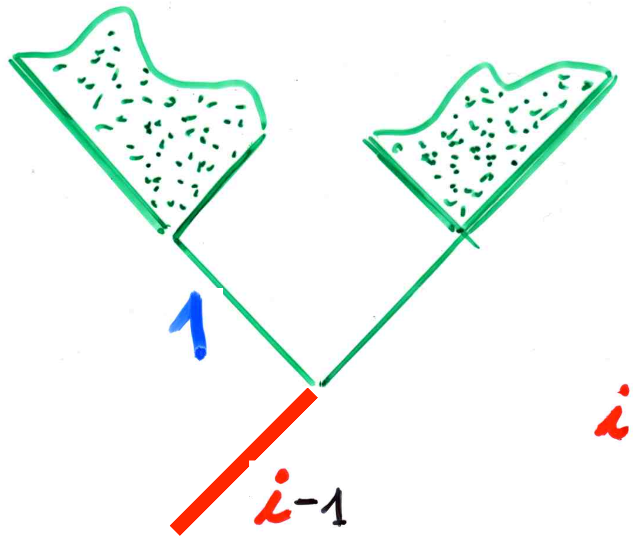

2 1 2 2 2 1 2 2 1


i
j

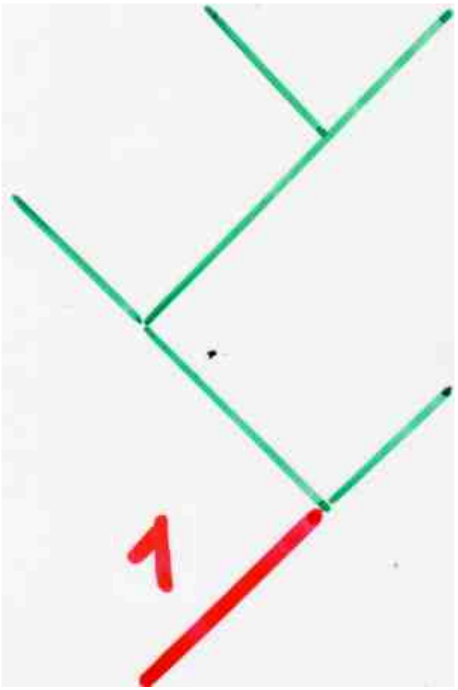

j
i

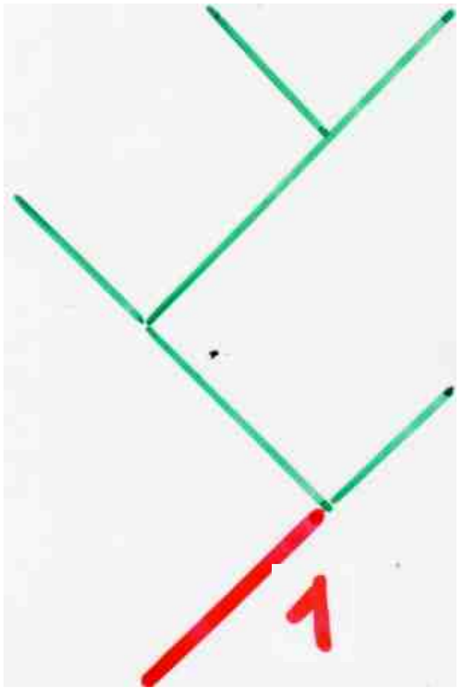
$i \geqslant 2$

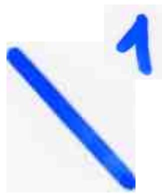


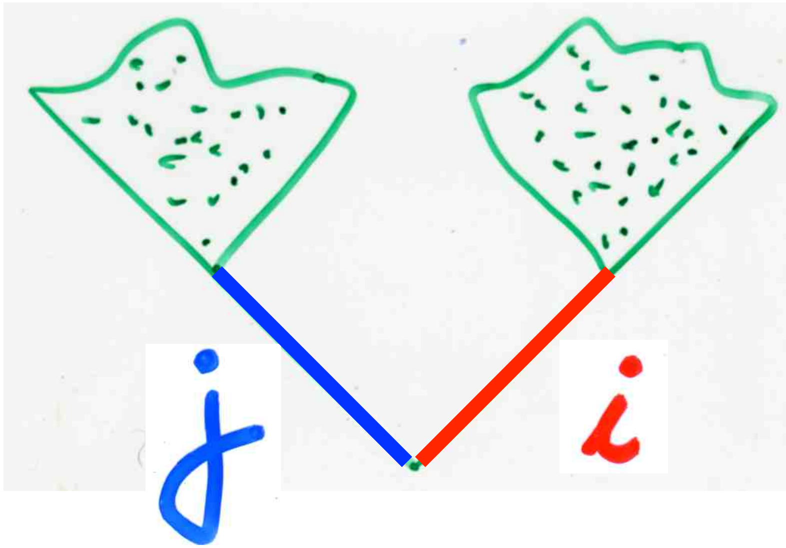


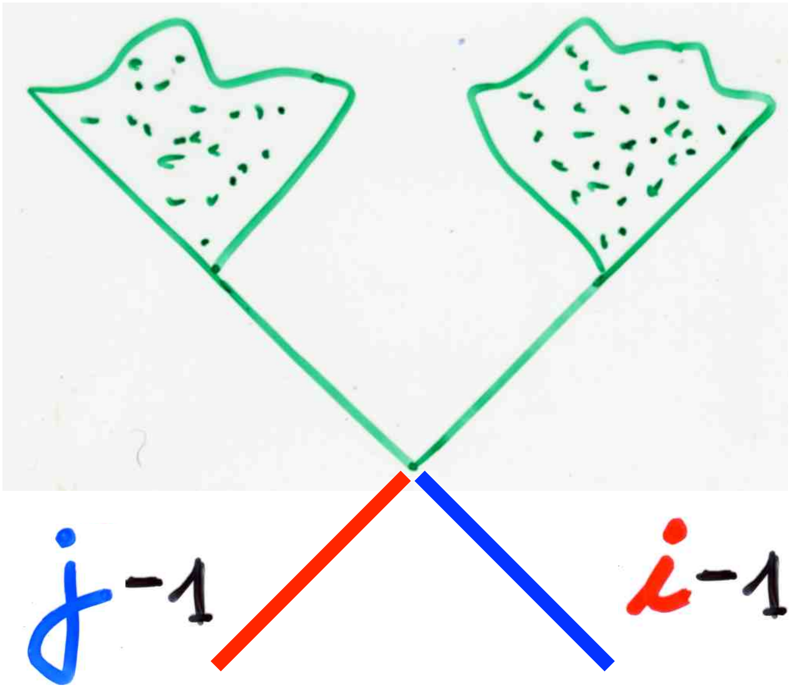
$i \geqslant 2$


1

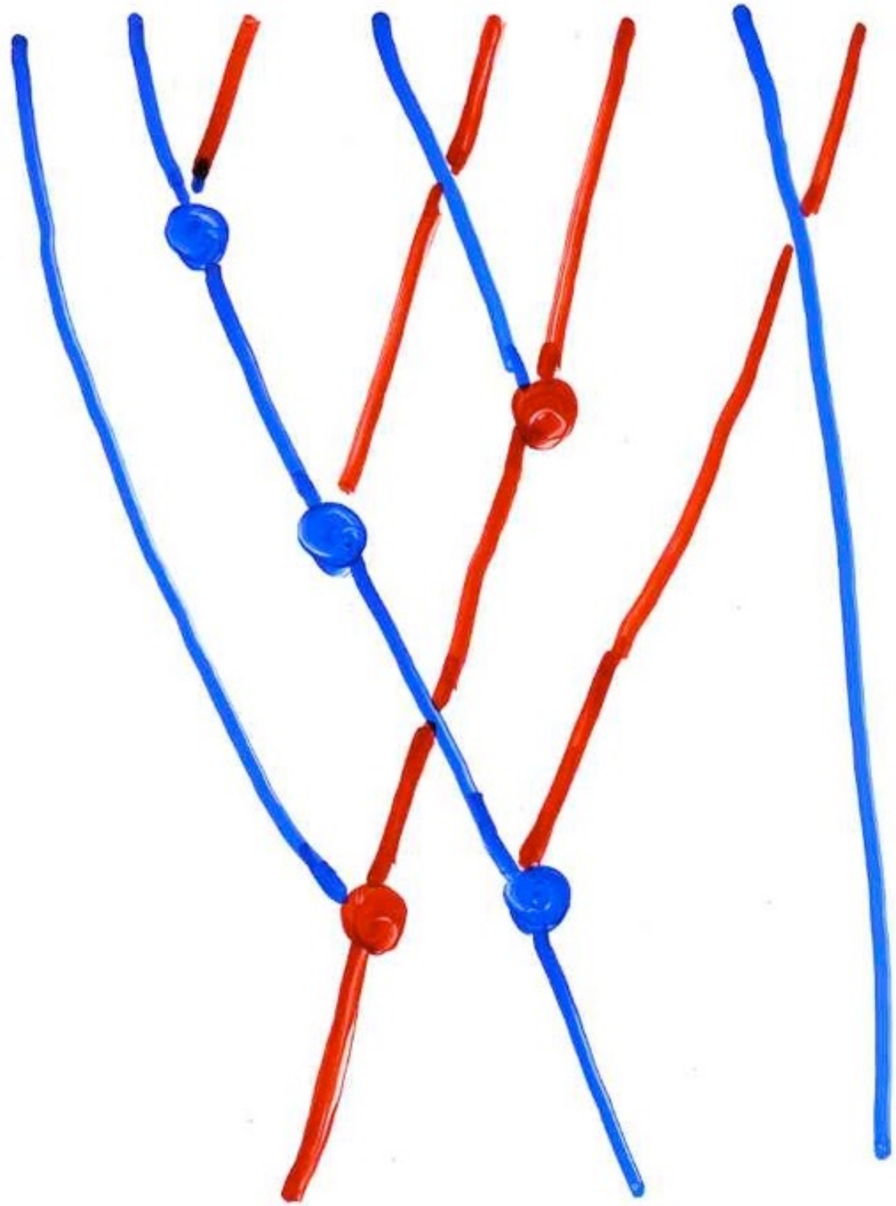

1

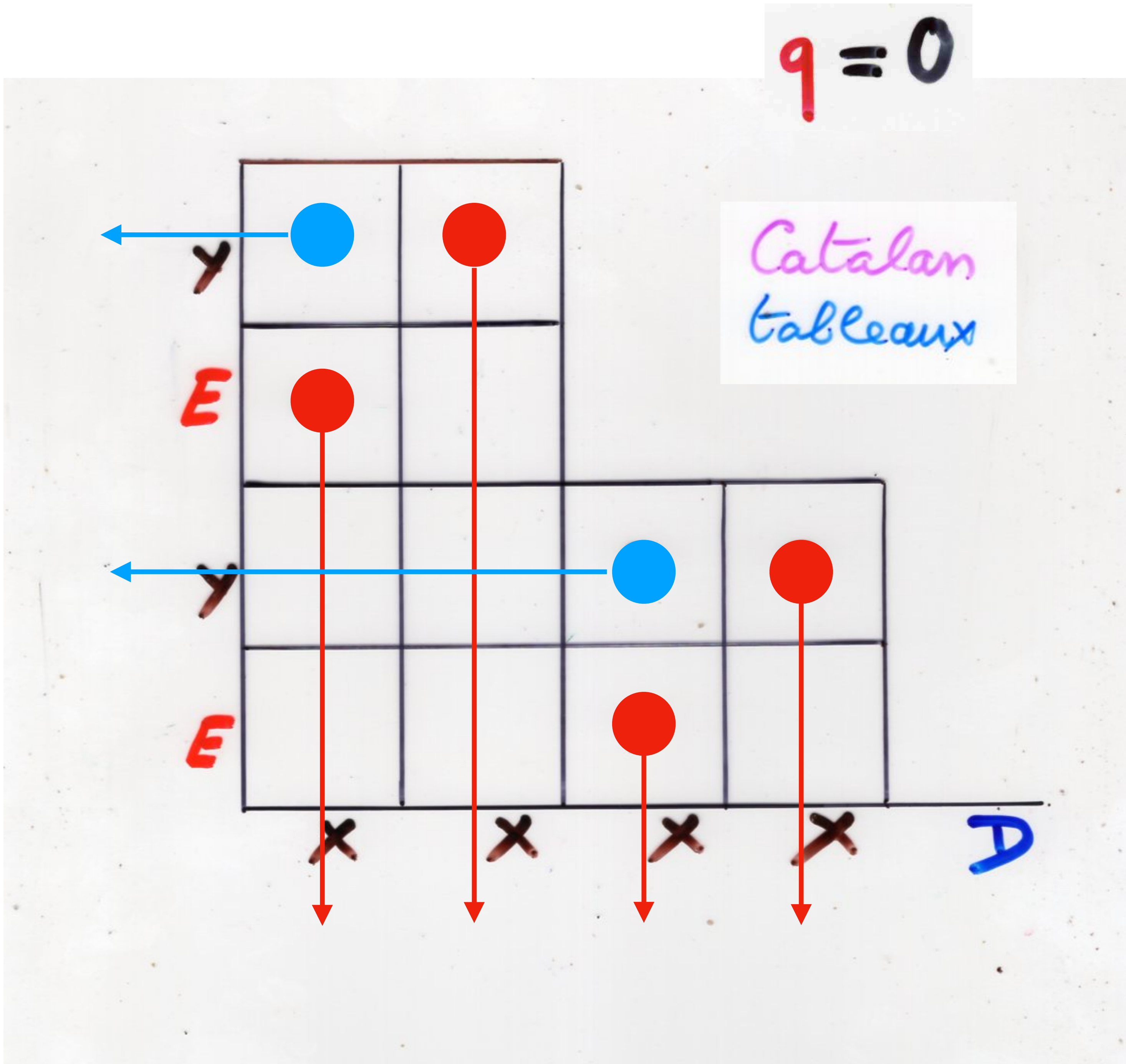

1


j
i

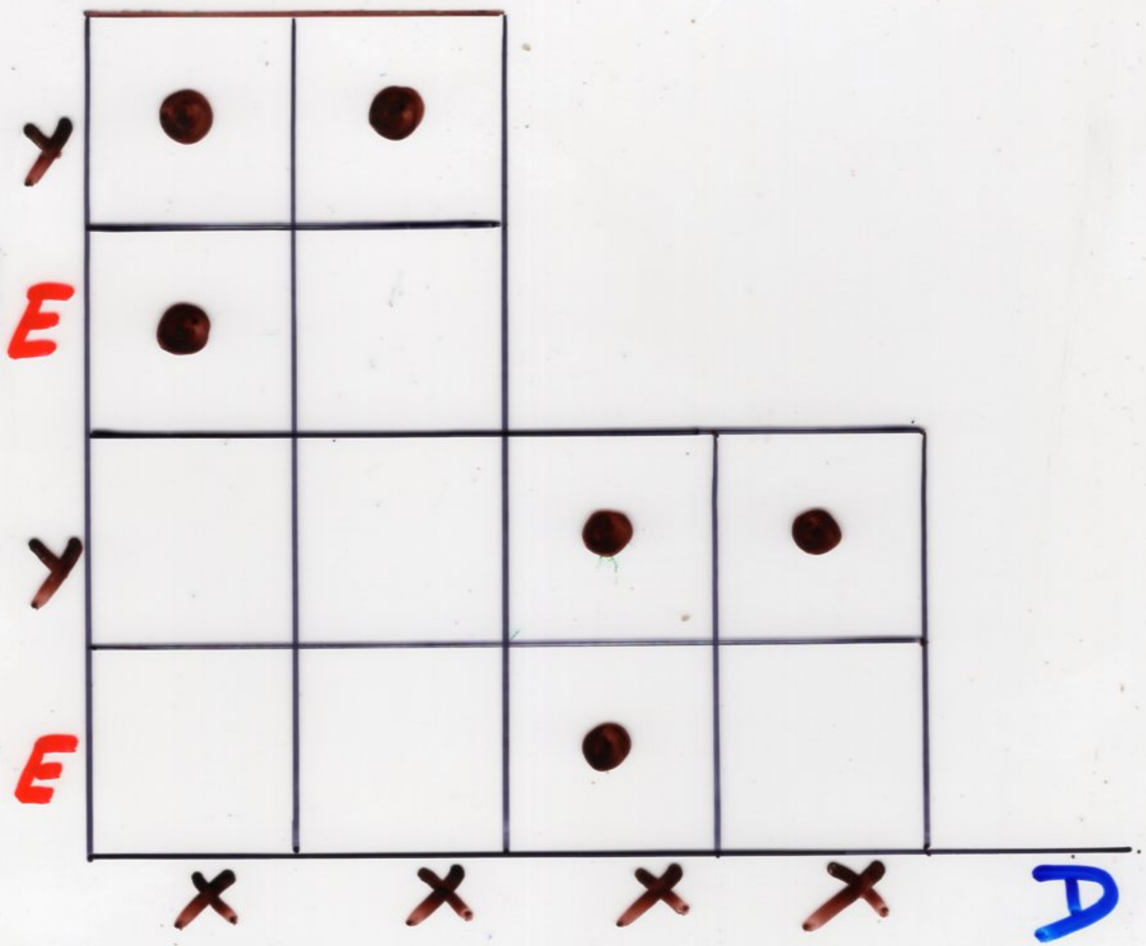

j^{-1}
i^{-1}

$(n+1)!$

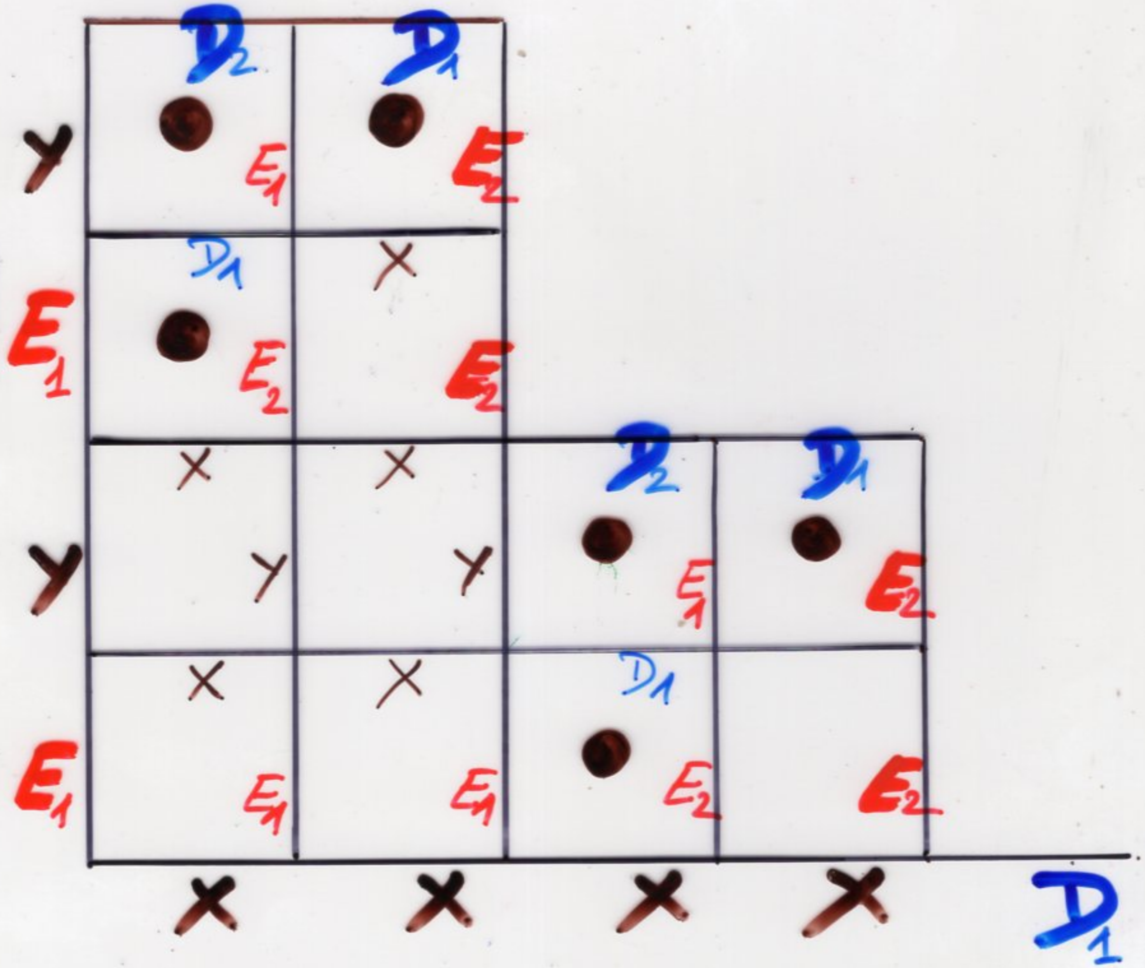




q = 0
Catalan
tableaux
y
E
y
E
x
x
x
x
D

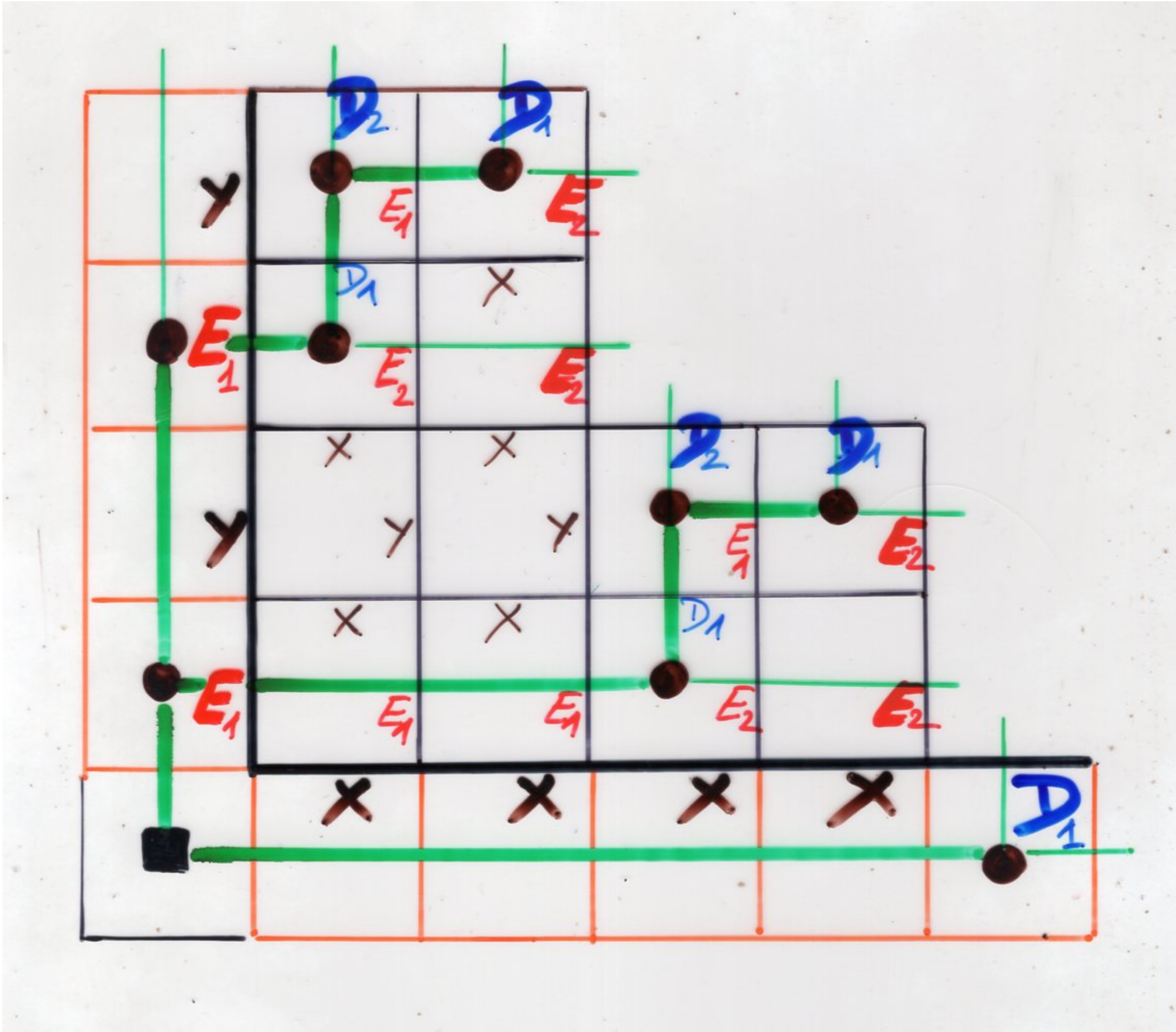


y
E
y
E
x
x
x
x
A



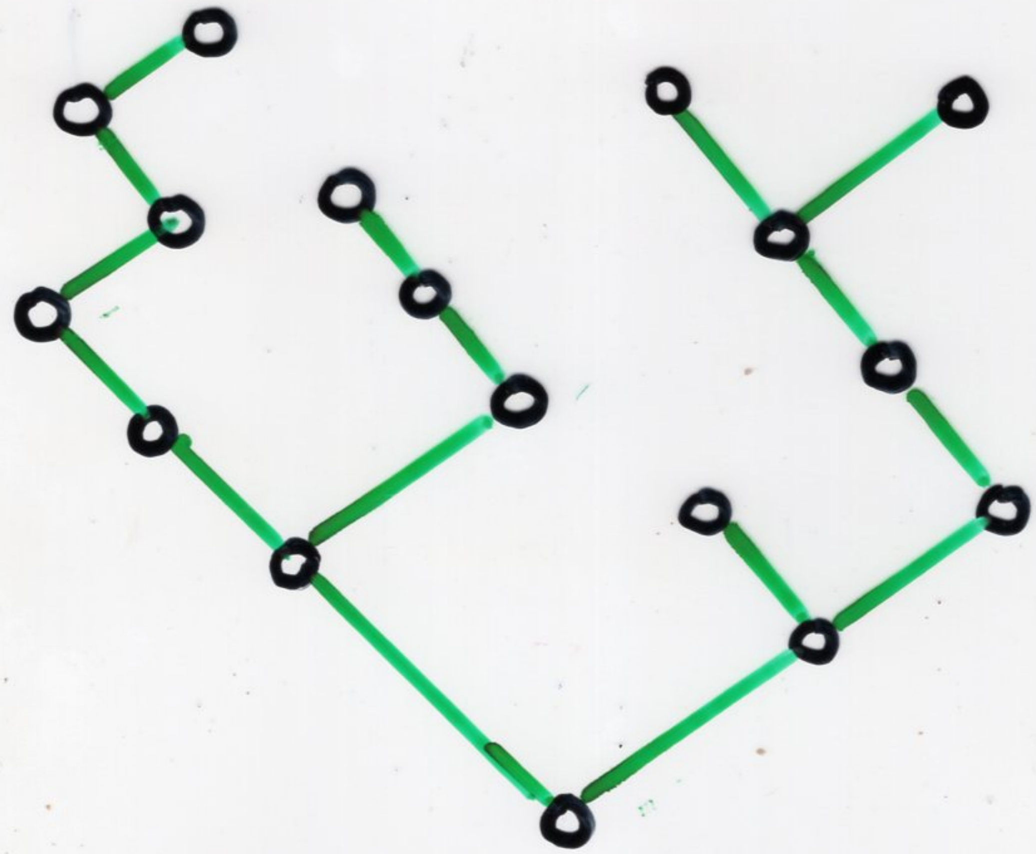
D2
D1
Y
E1
E2
D1
X
E1
E2
E2
X
X
D2
D1
Y
Y
Y
E1
E2
X
X
D1
E1
E1
E1
E2
E2
X
X
X
X
D1

$$w = D_2\, D_1\, E_2\, E_2\, D_2\, D_1\, E_2\, E_2\, D_1$$



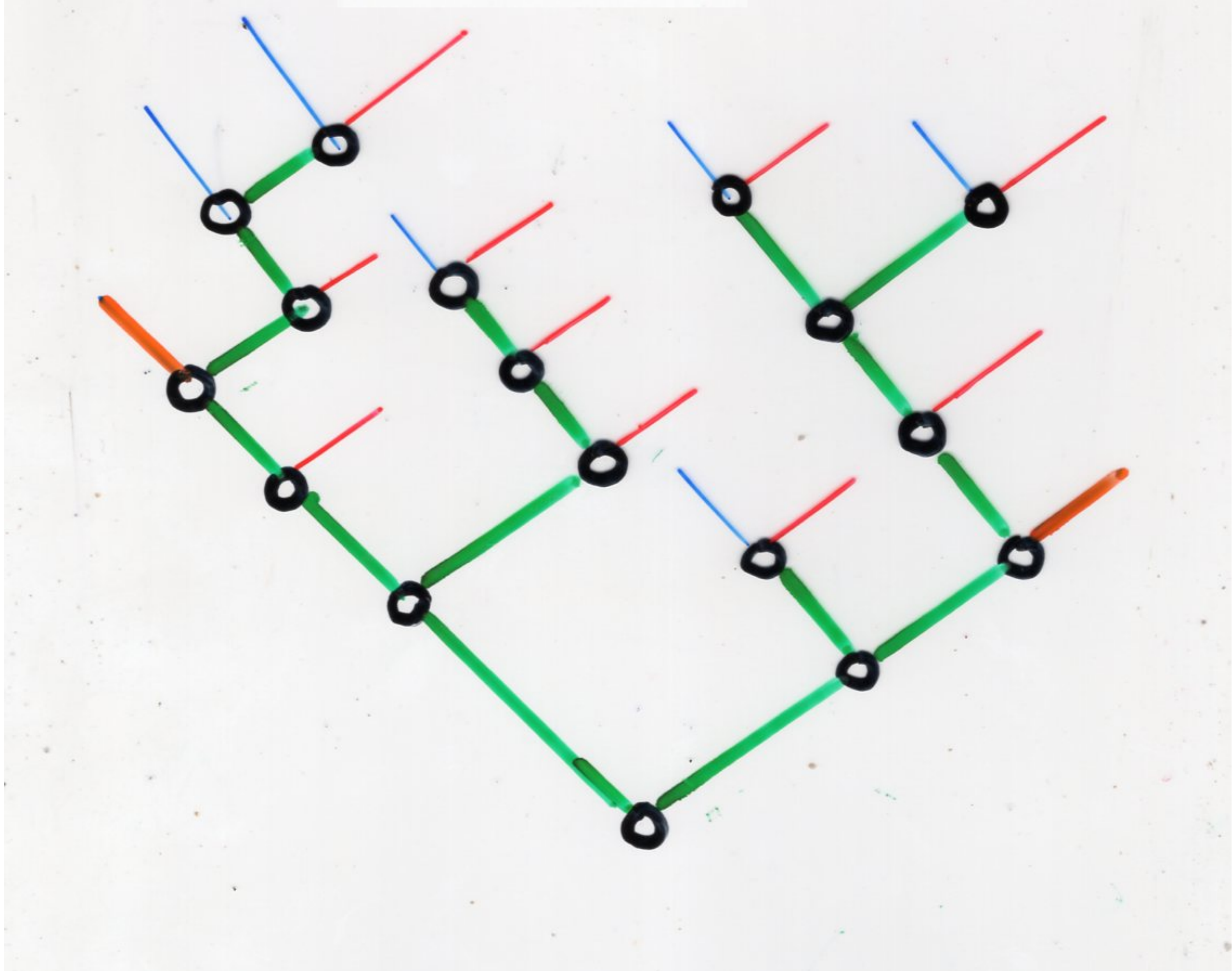
The Tamil bijection

binary trees

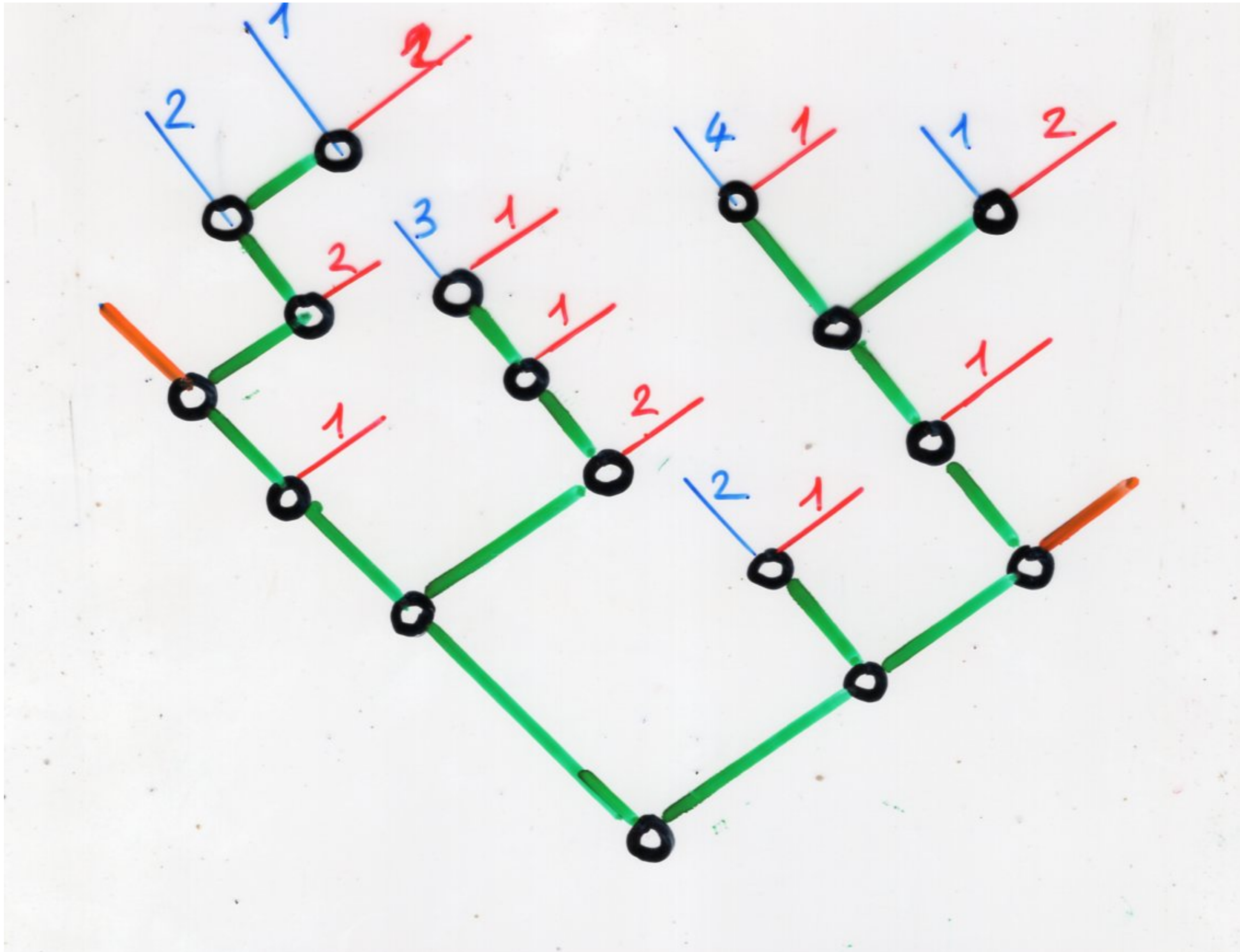


The Tamil bijection

binary trees



$$w = 2\ 1\ 2\ 2\ 1\ 3\ 1\ 1\ 2\ 2\ 1\ 4\ 1\ 1\ 2\ 1$$



Bonnes

fêtes

à

tous !