Growth diagrams and edge local rules

RSK revisited

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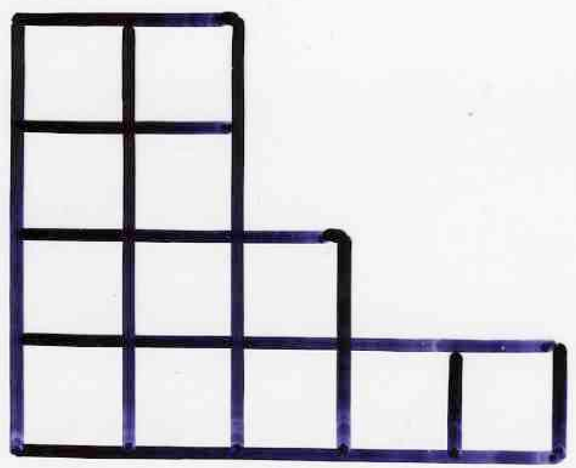
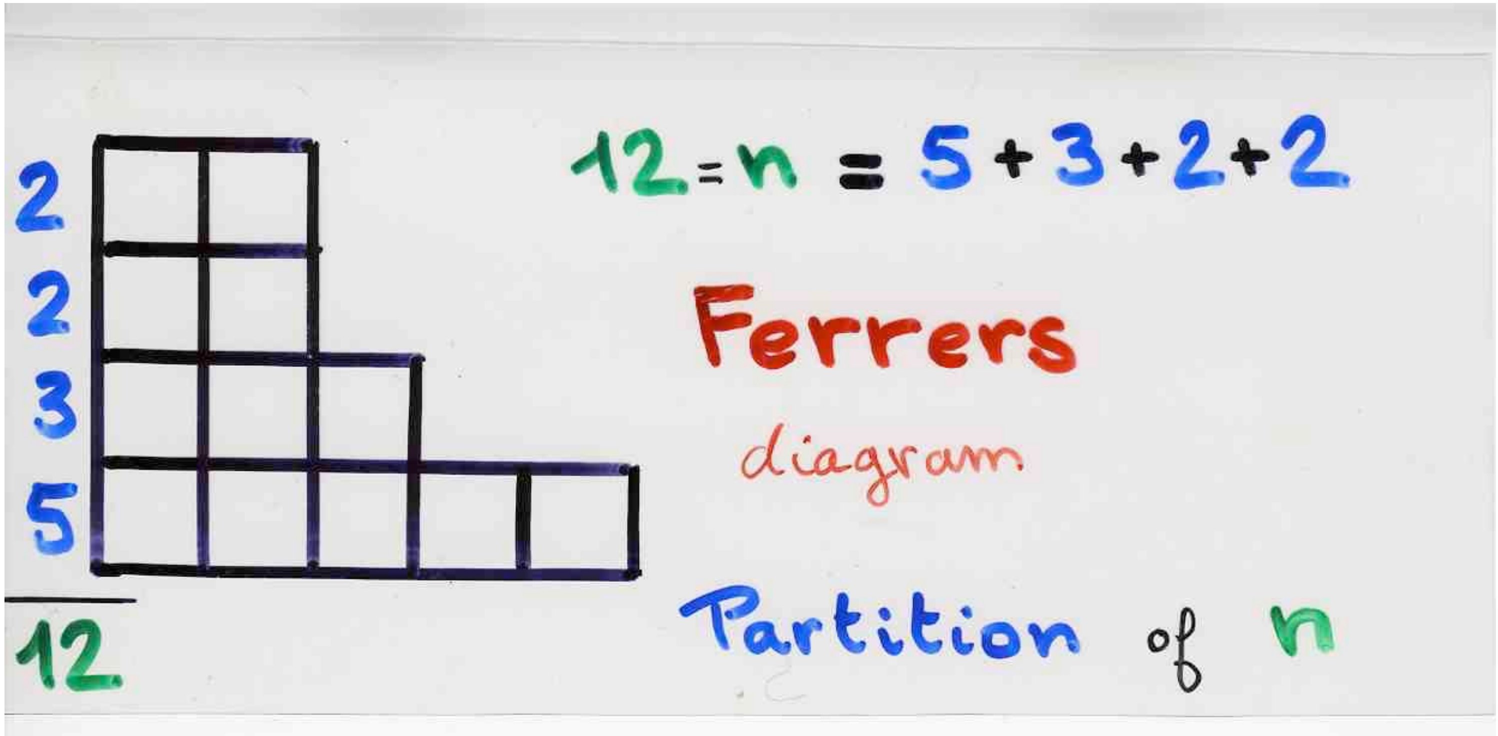
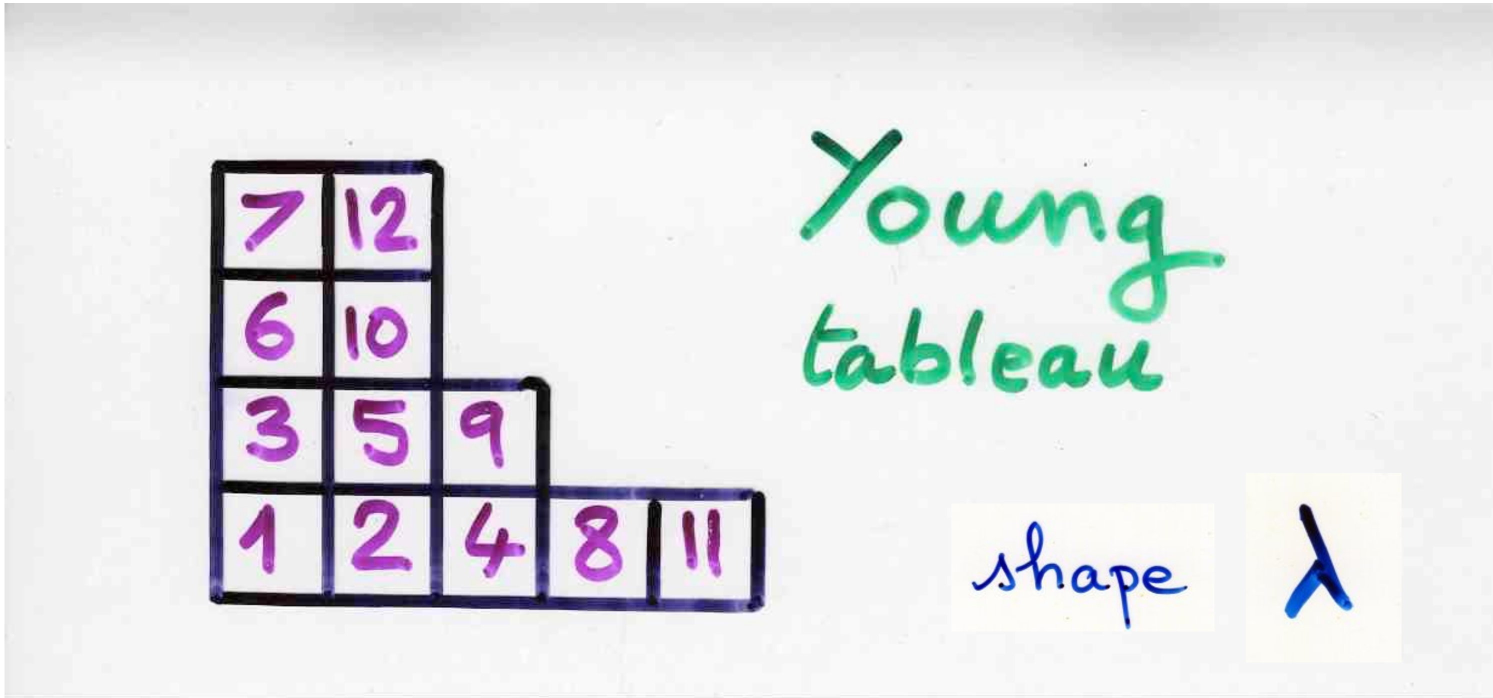
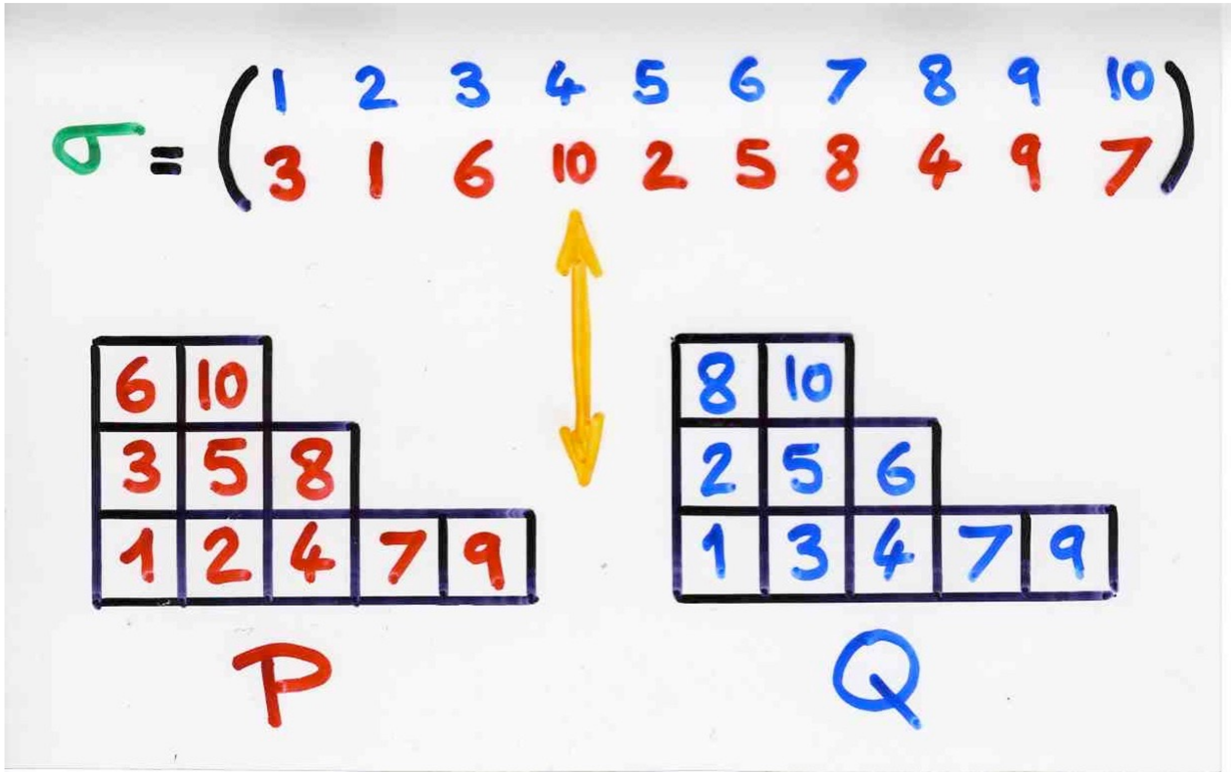
RS
The Robinson-Schensted correspondence




12 = n = 5 + 3 + 2 + 2
Ferrers
diagram
Partition of n
2
2
3
5
12


7 12
6 10
3 5 9
1 2 4 8 11
Young
tableau
shape
λ



The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape

f_λ = number of Young tableaux with shape λ

$$n! = \sum_{\lambda} (f_\lambda)^2$$

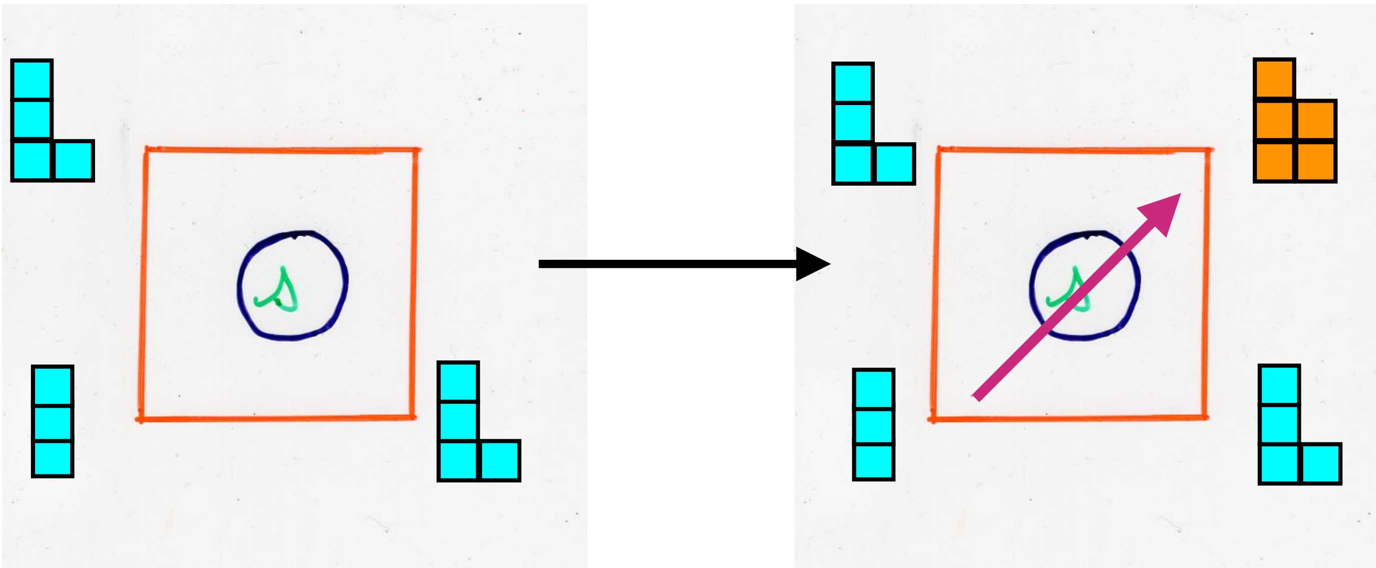
λ partition of n

"local" algorithm on a grid or "growth diagrams"

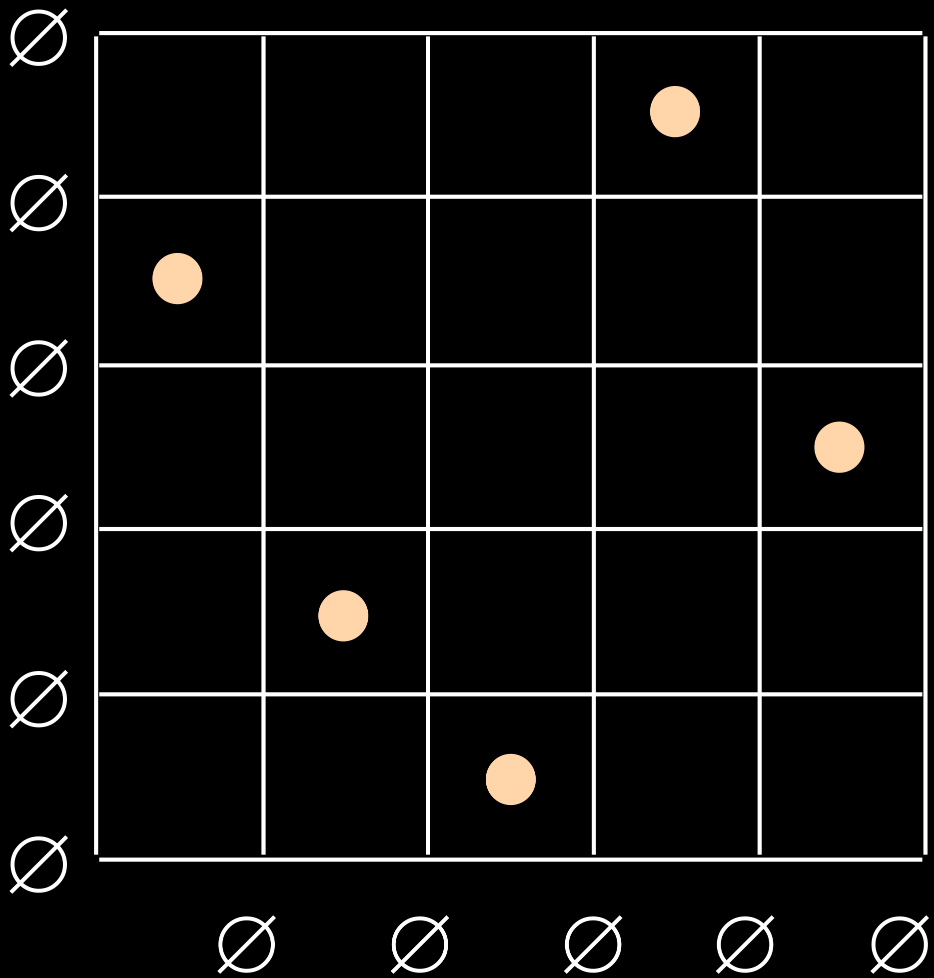
S. Fomin, 1986, 1994

C.Krattenthaler

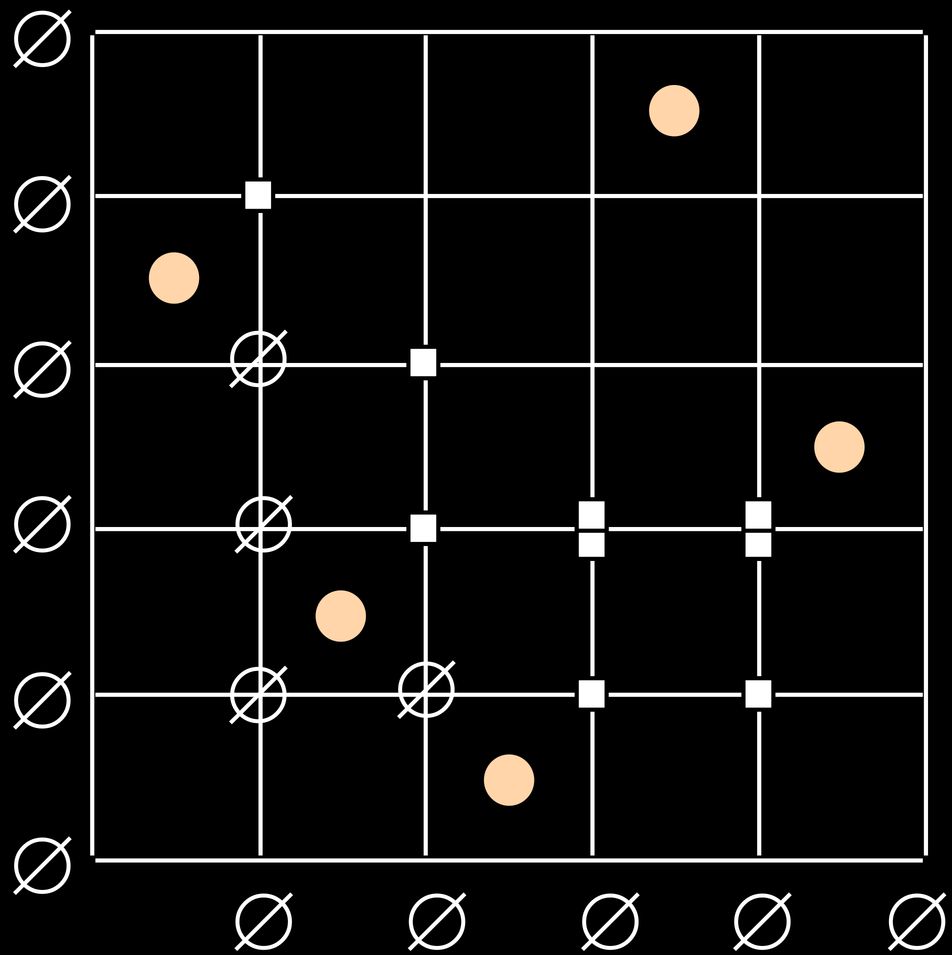
Fomin's "local rules"
"growth diagrams"



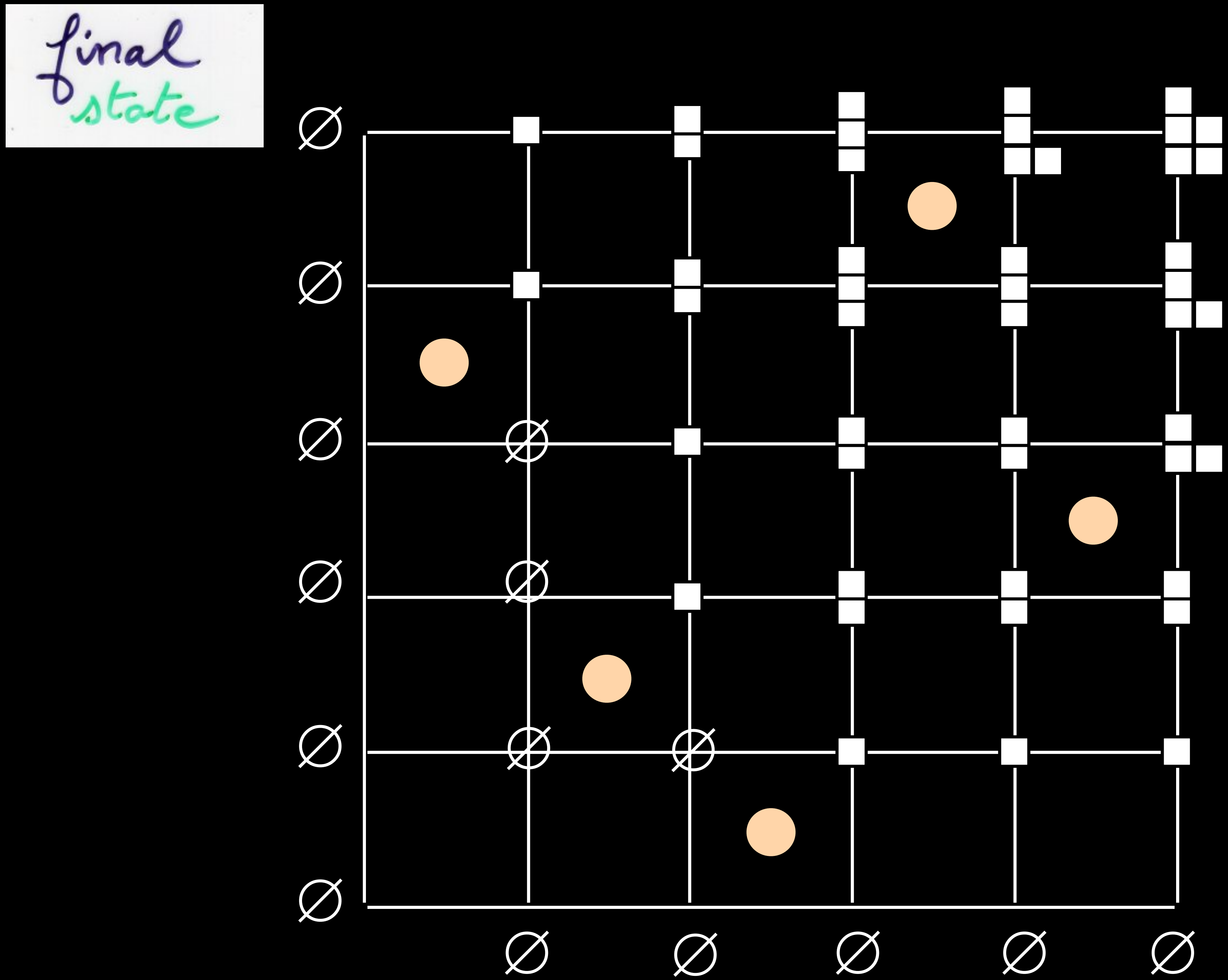
initial *state*



during the *labeling* process

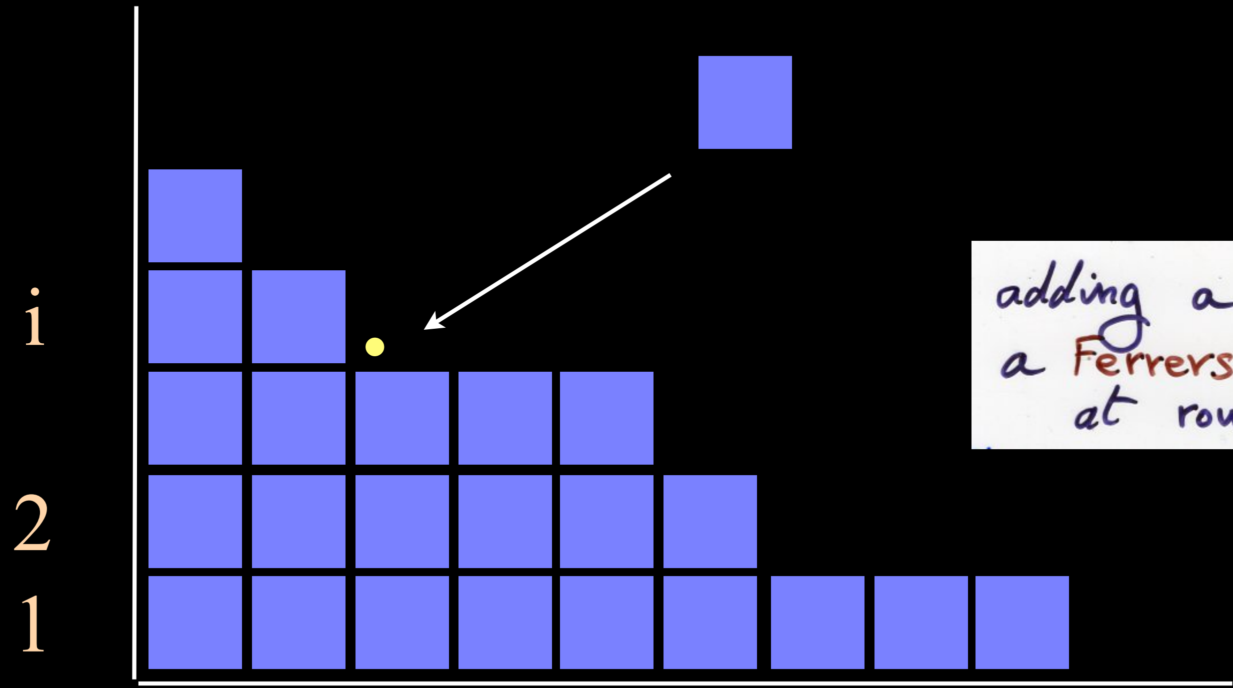


$\sigma = 4, 2, 1, 5, 3$


final
state

notations

operator U_i

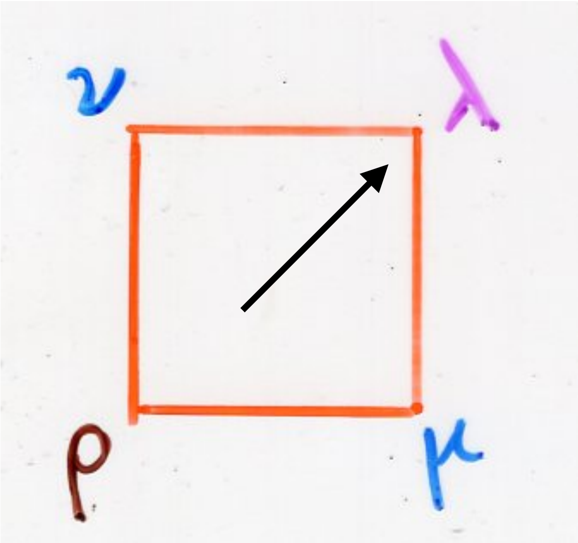


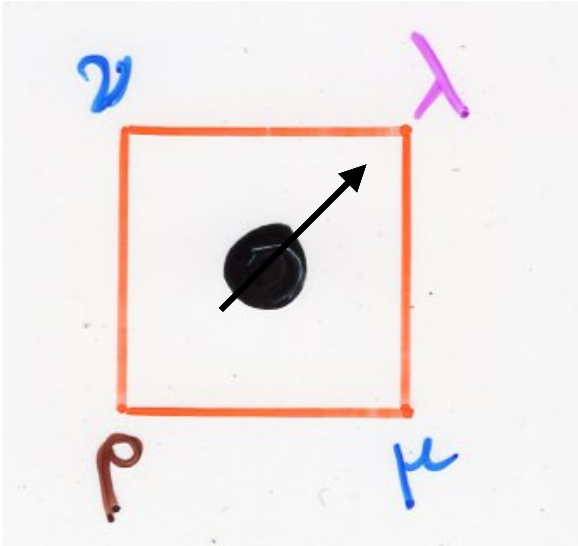
adding a cell in a Ferrers diagram ρ at row i

$$U_i(\rho) = \rho + (i)$$

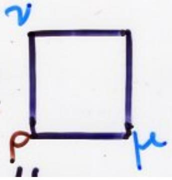
"growth diagrams"

"local rules"





"local rules"

(i) $\rho = \mu = \nu$ and 

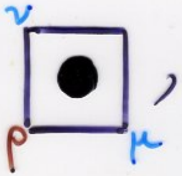
then $\lambda = \rho$

(ii) $\rho = \mu \neq \nu$, then $\lambda = \nu$

(iii) $\rho = \nu \neq \mu$, then $\lambda = \mu$

(iv) ρ, μ, ν pairwise $\neq$, then $\lambda = \mu \cup \nu$

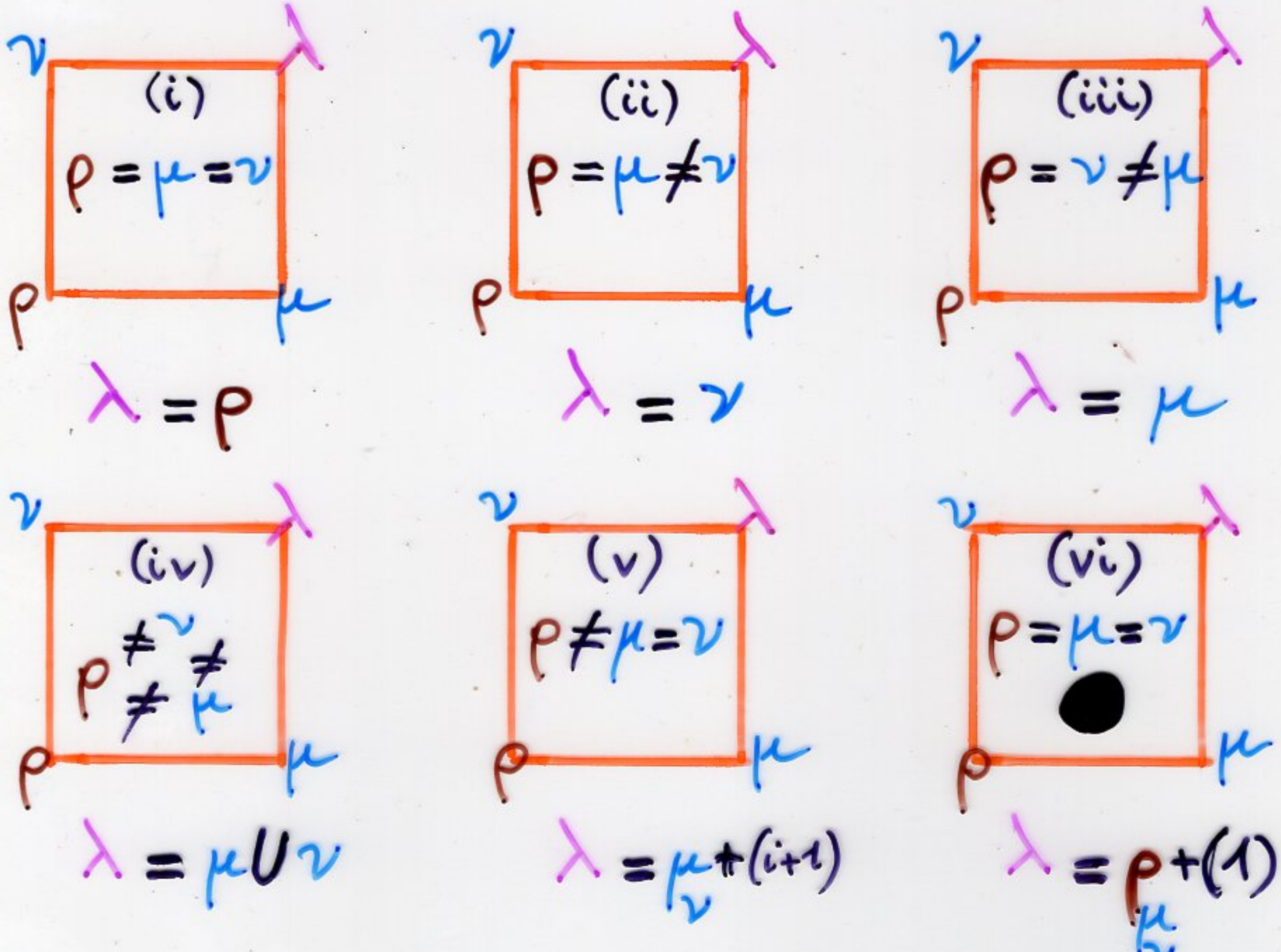
(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$
given that $\mu = \nu$ and ρ differ in the i-th row
[in fact $\mu = \nu = \rho + (i)$]

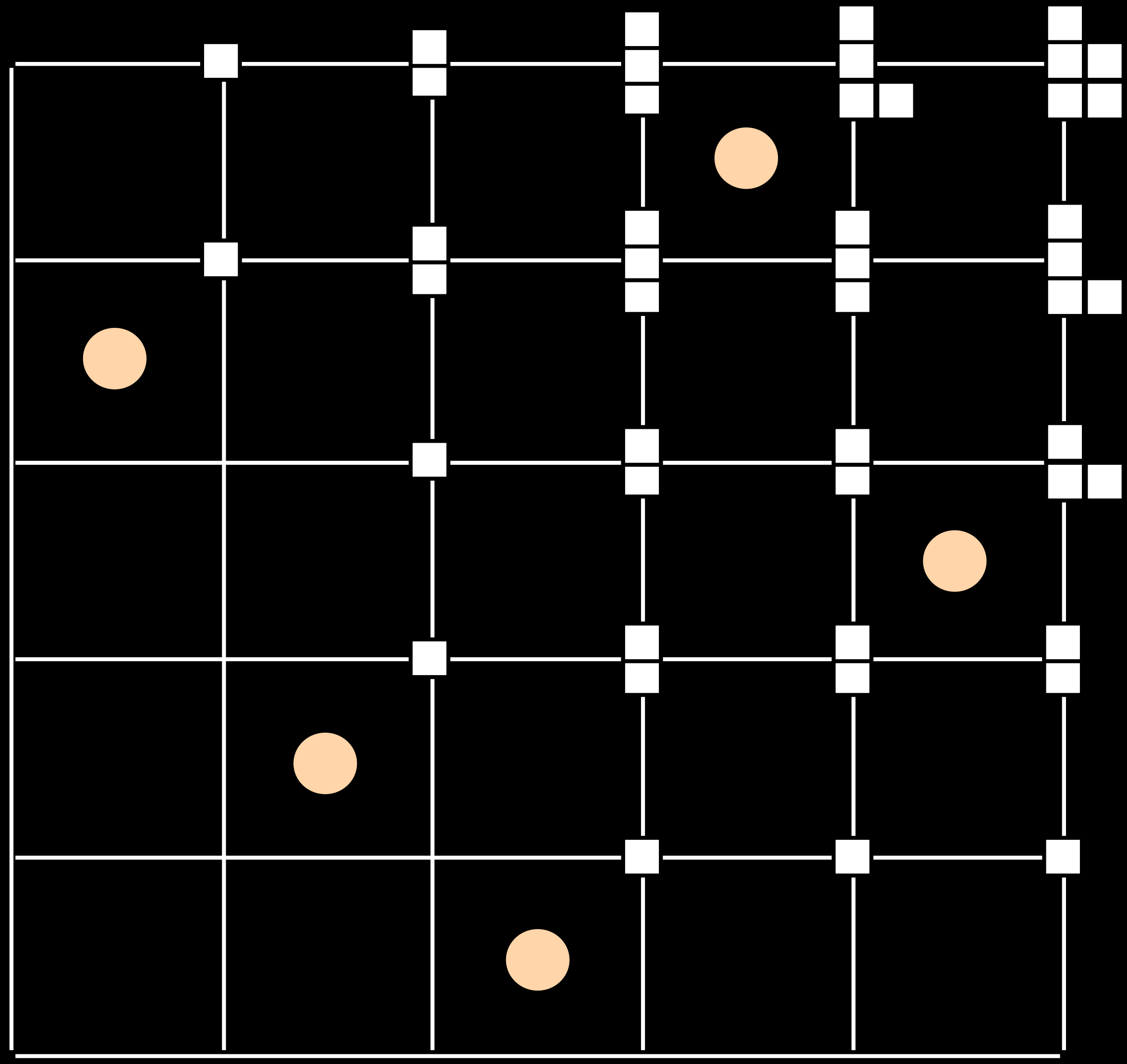
(vi) $\rho = \mu = \nu$ and 

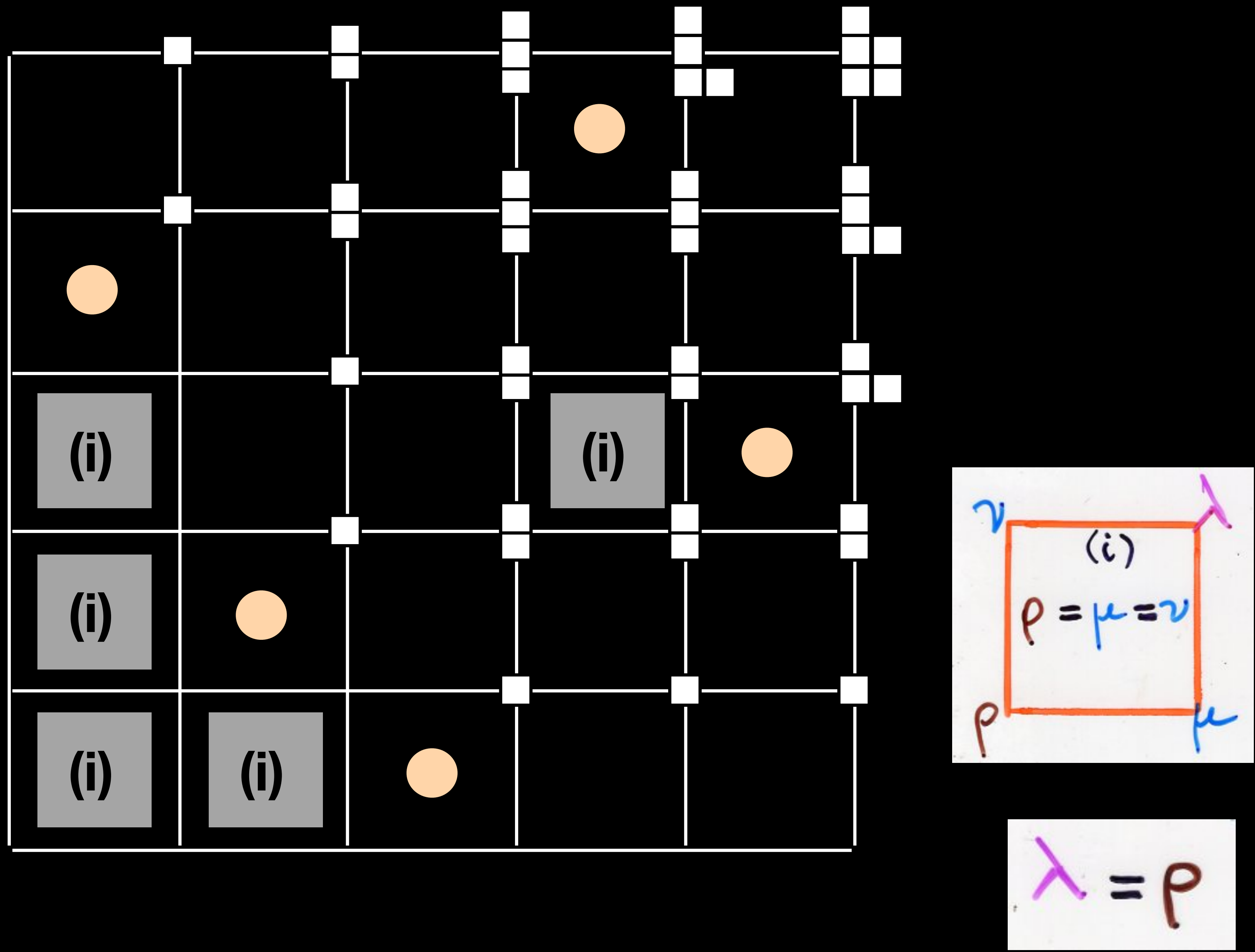
, then $\lambda = \mu + (1)$

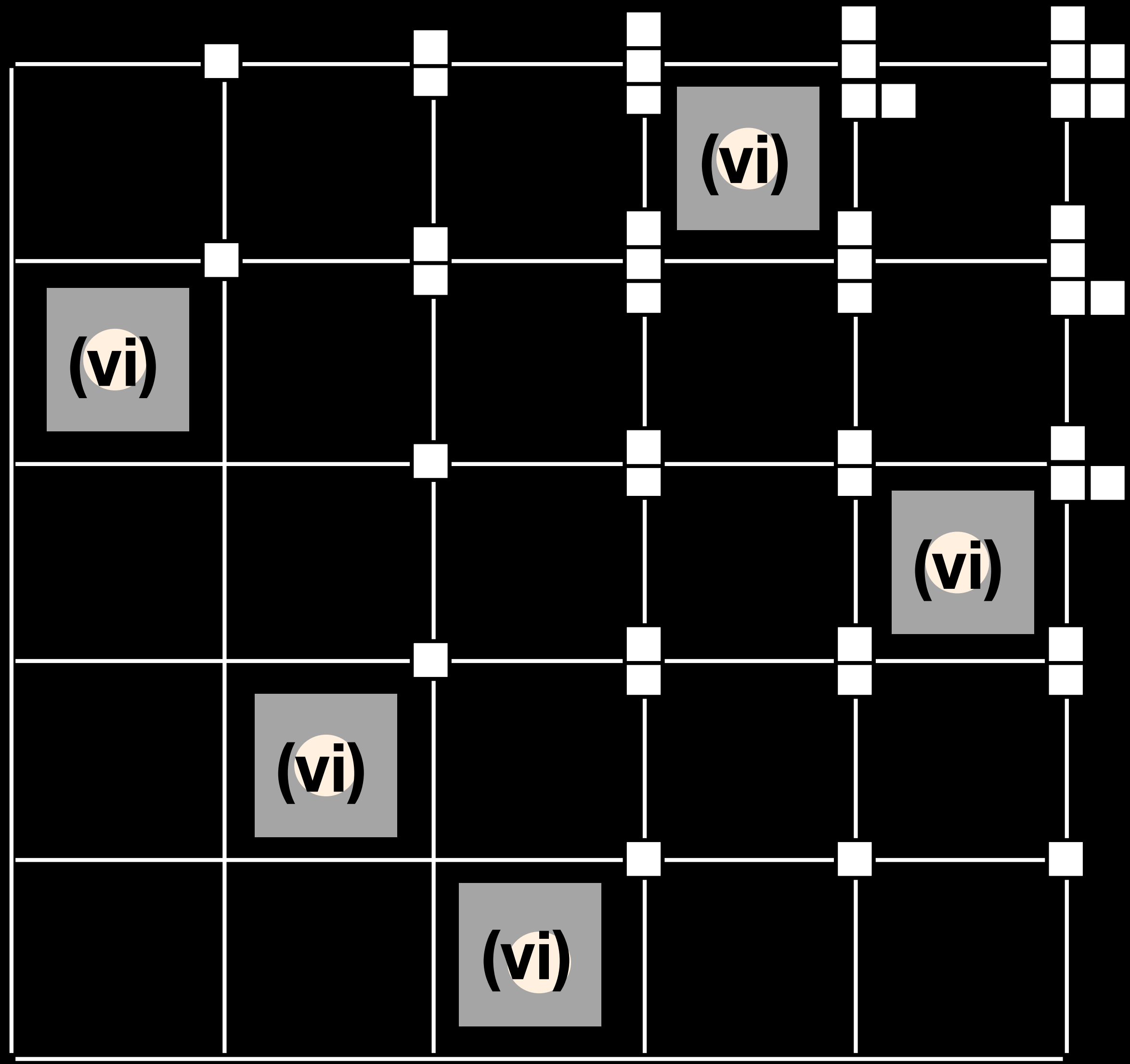
C.Krattentheler, (2006).

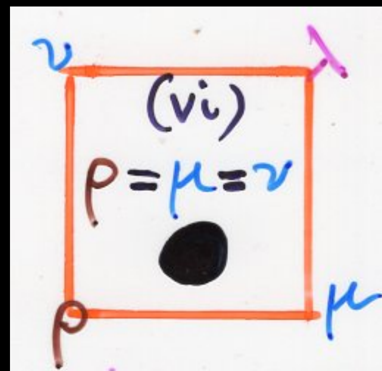
GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

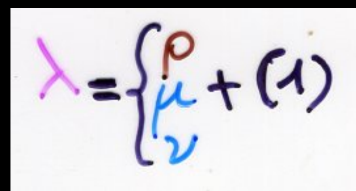

(i)
ρ = μ = ν
λ = ρ
(ii)
ρ = μ ≠ ν
λ = ν
(iii)
ρ = ν ≠ μ
λ = μ
(iv)
λ = μ ∪ ν
(v)
ρ ≠ μ = ν
λ = μ ν +(i+1)
(vi)
ρ = μ = ν
λ = ρ μ ν +(1)

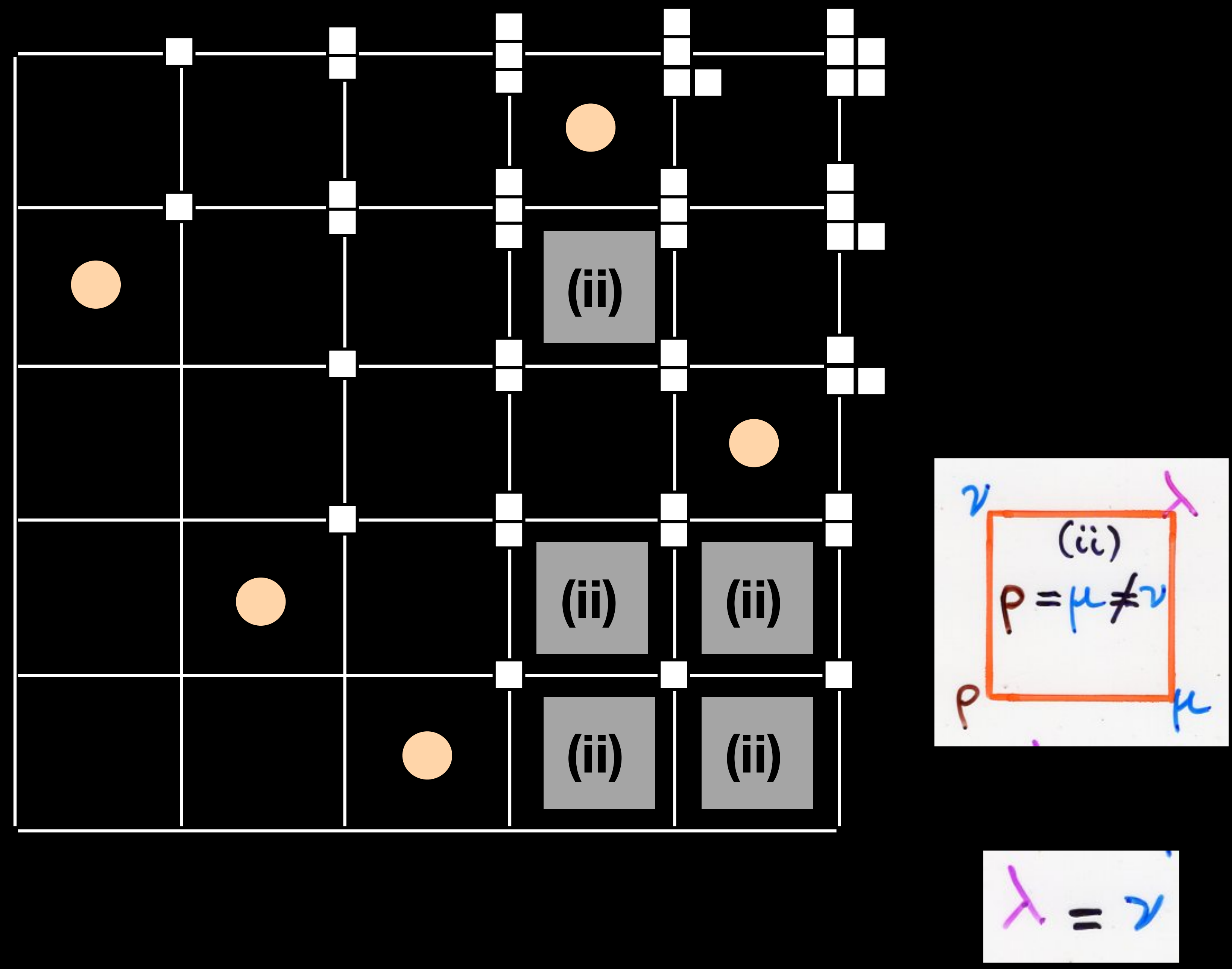



(i)
(i)
(i)
(i)
(i)
(i)
ν
λ
ρ
μ
ρ = μ = ν
λ = ρ

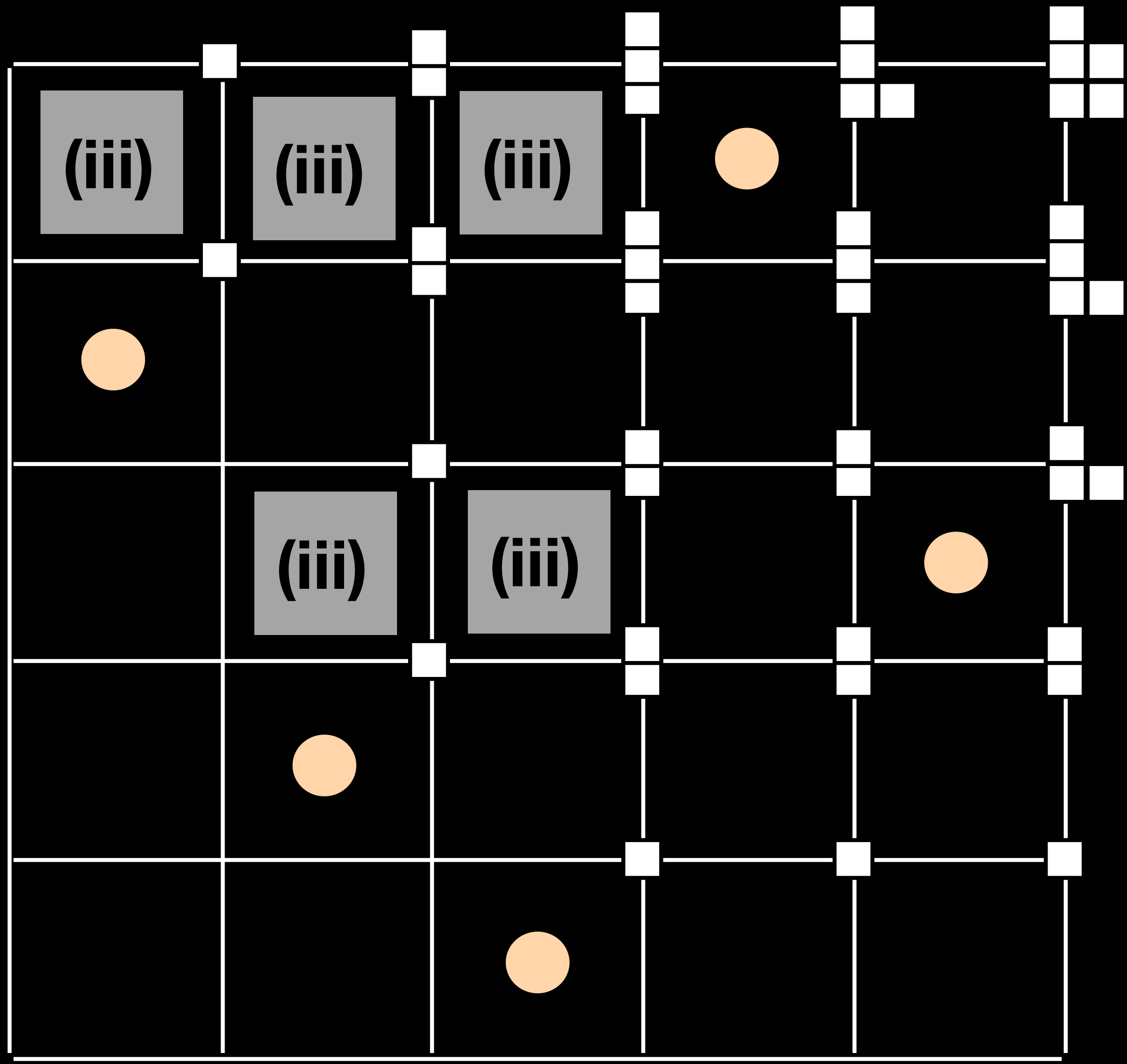

(vi)
(vi)
(vi)
(vi)
(vi)
(vi)

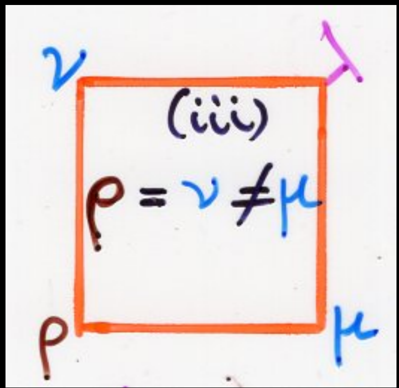

ν
λ
(vi)
ρ = μ = ν
ρ
μ

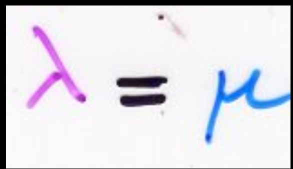

λ = { ρ μ ν + (1)

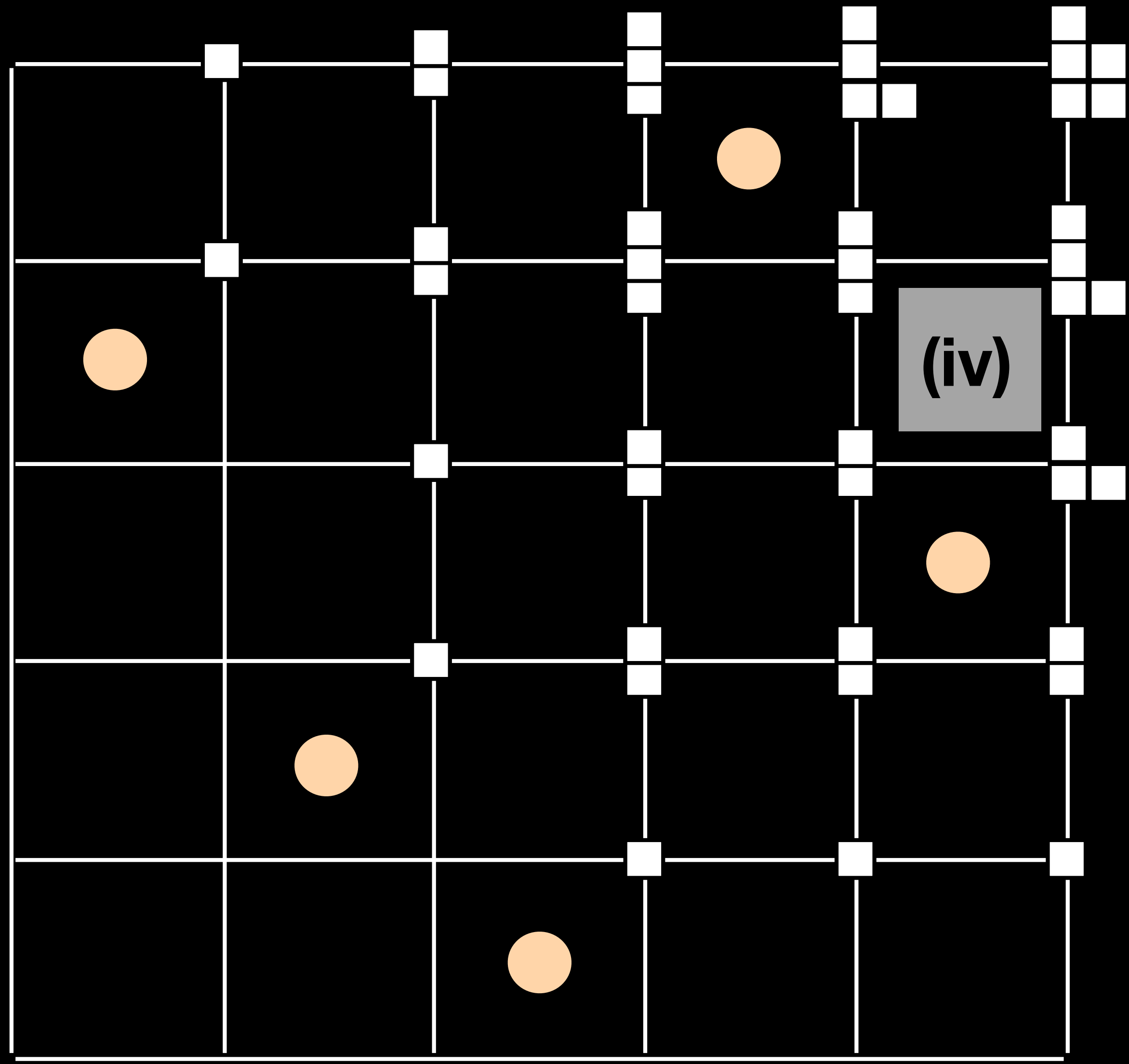


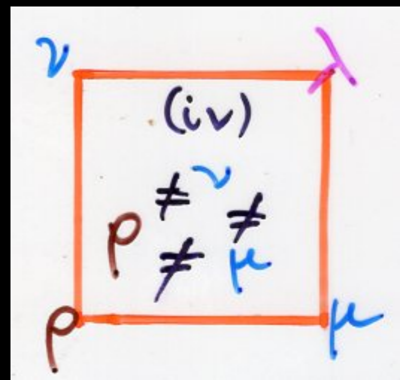
(ii)
(ii)
(ii)
(ii)
(ii)
(ii)
ρ = μ ≠ ν
ν
λ
ρ
μ
λ = ν

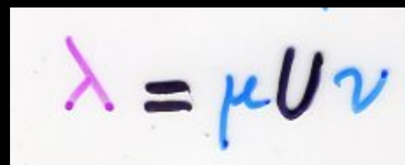

(iii)
(iii)
(iii)
(iii)
(iii)

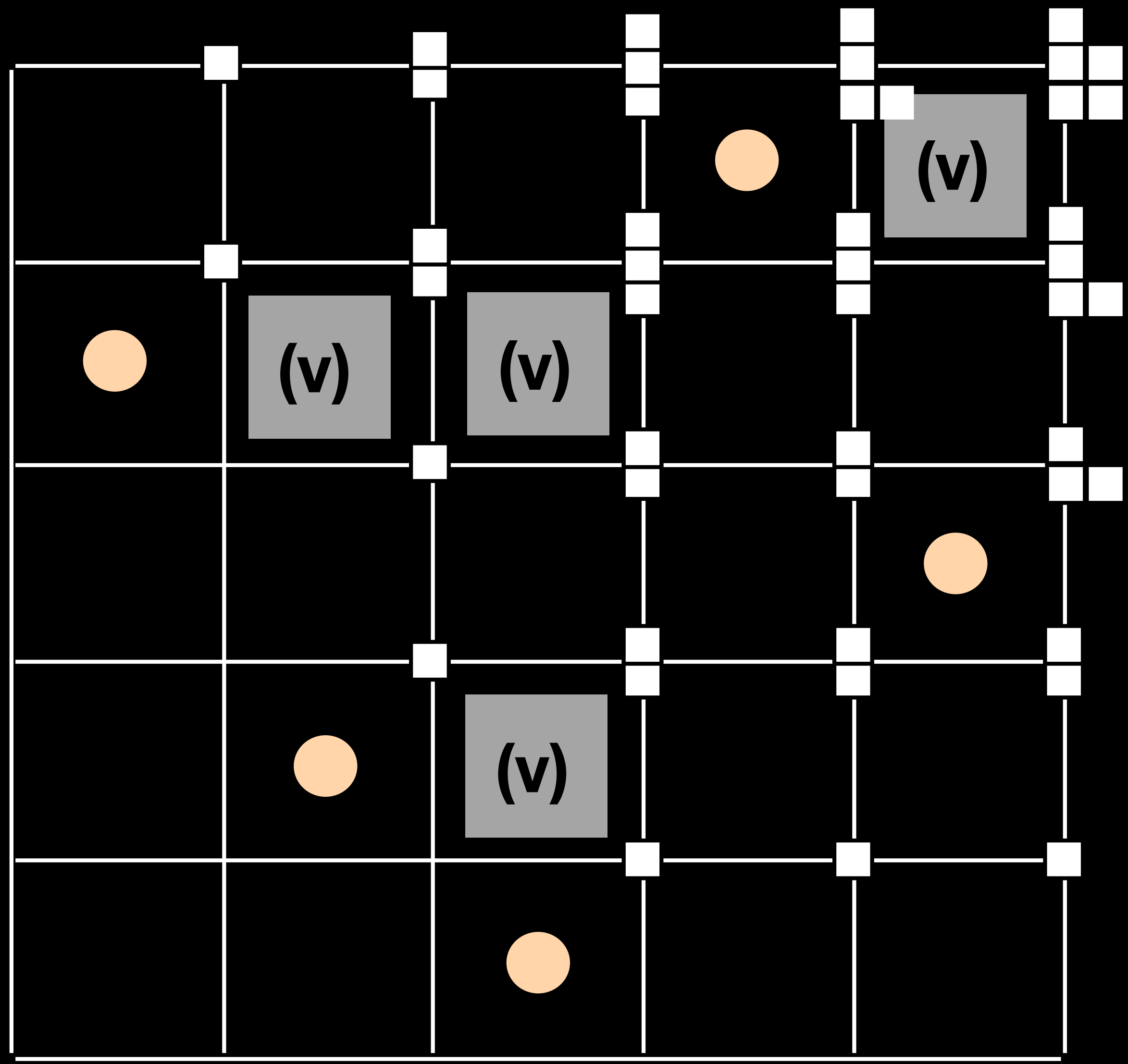

ν
λ
(iii)
ρ = ν ≠ μ
ρ
μ

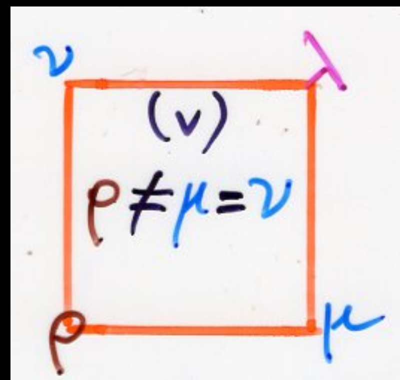

λ = μ


(iv)

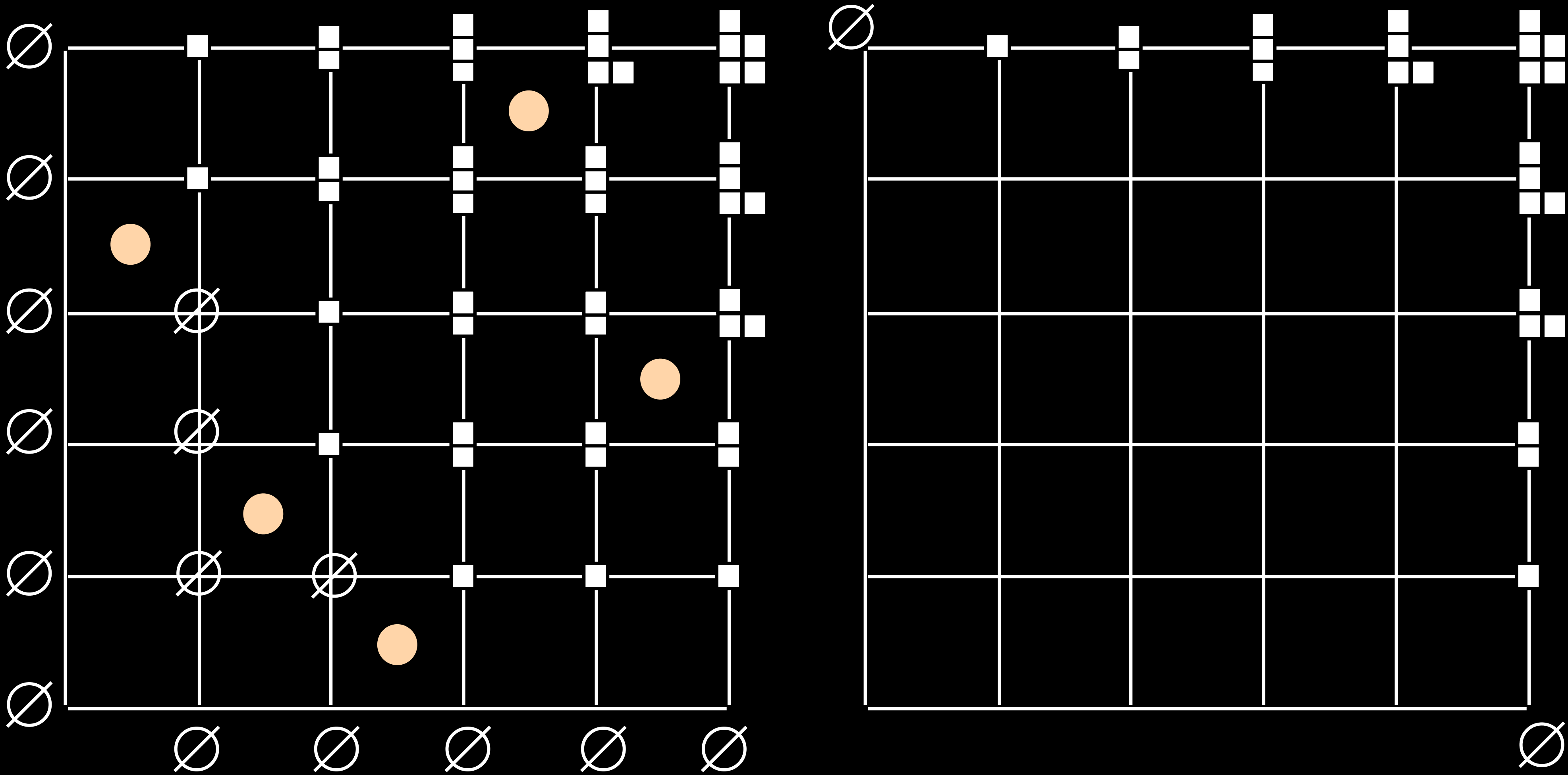

ν
λ
(iv)
ρ ≠ ν ≠ μ ≠
ρ
μ

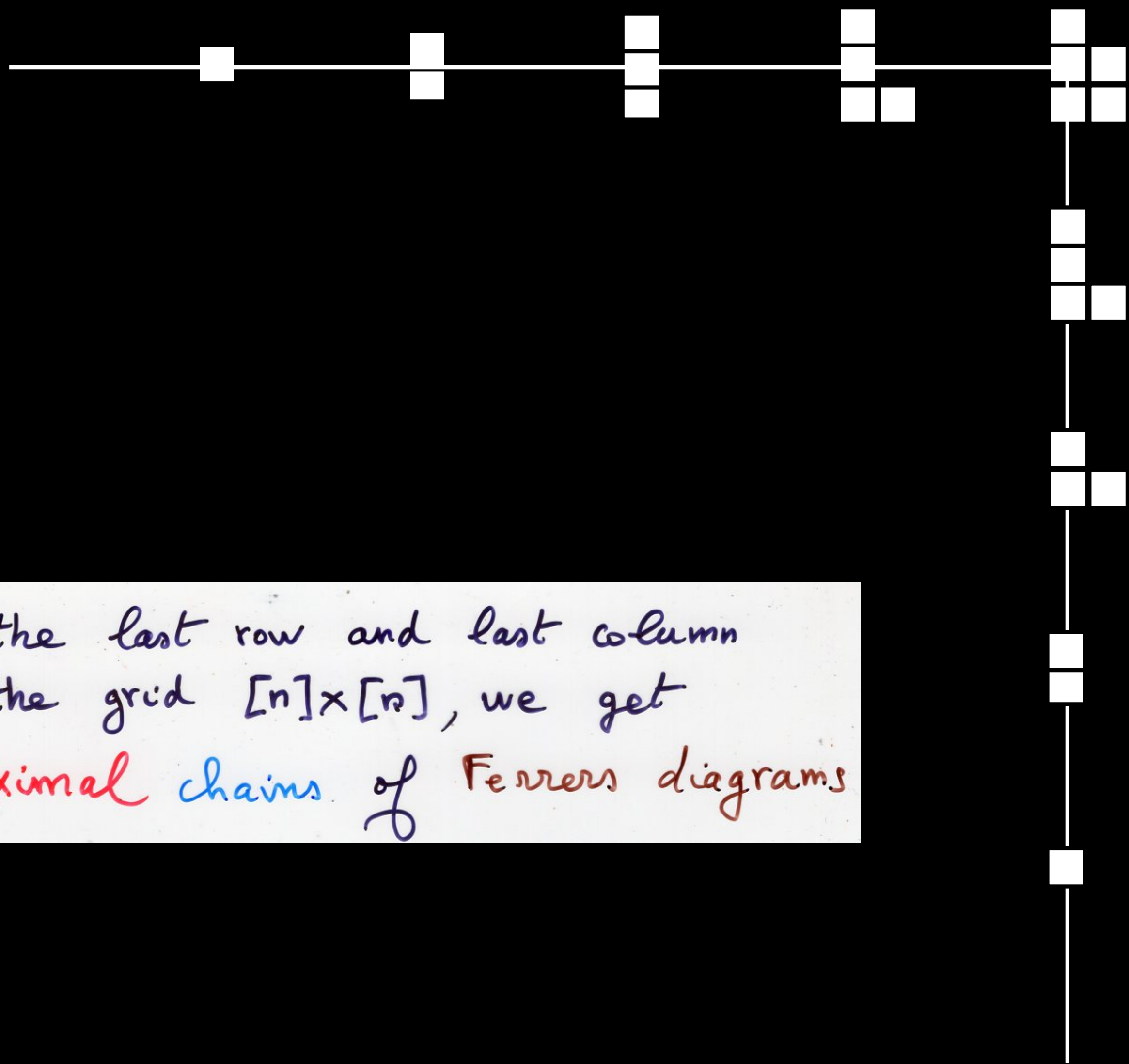

λ = μ ∪ ν



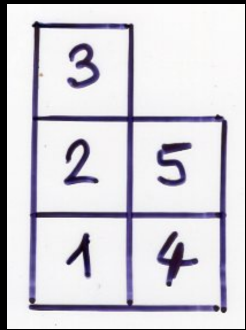


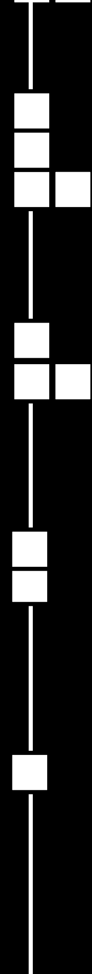
$$\lambda = \left\{ \begin{matrix} \mu \\ \nu \end{matrix} \right. + (i+1)$$



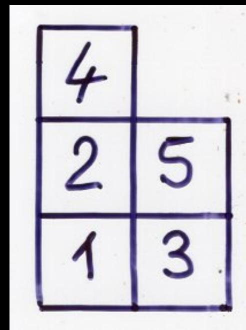


- in the last row and last column of the grid $[n]\times[n]$, we get maximal chains of Ferrers diagrams

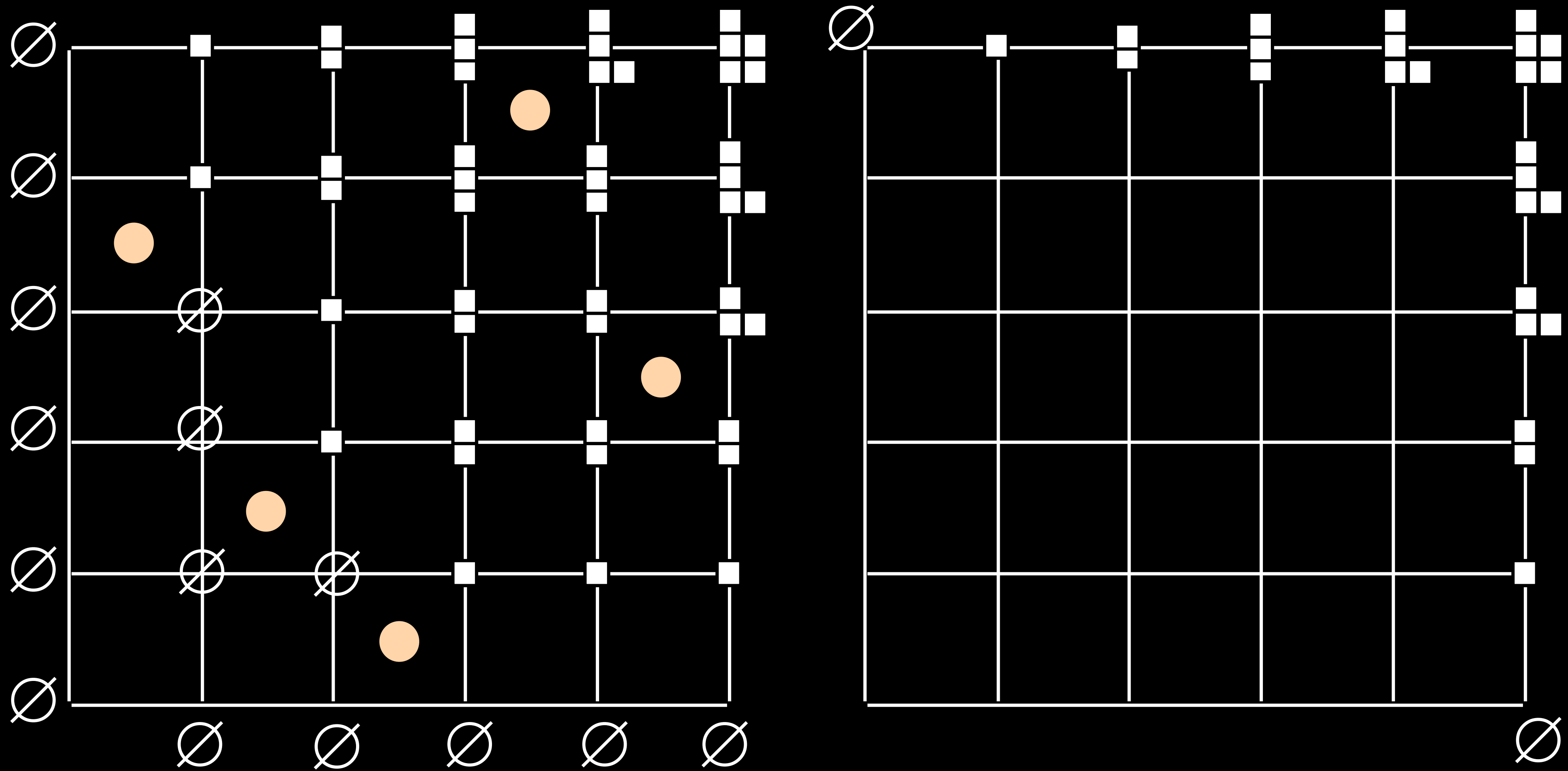




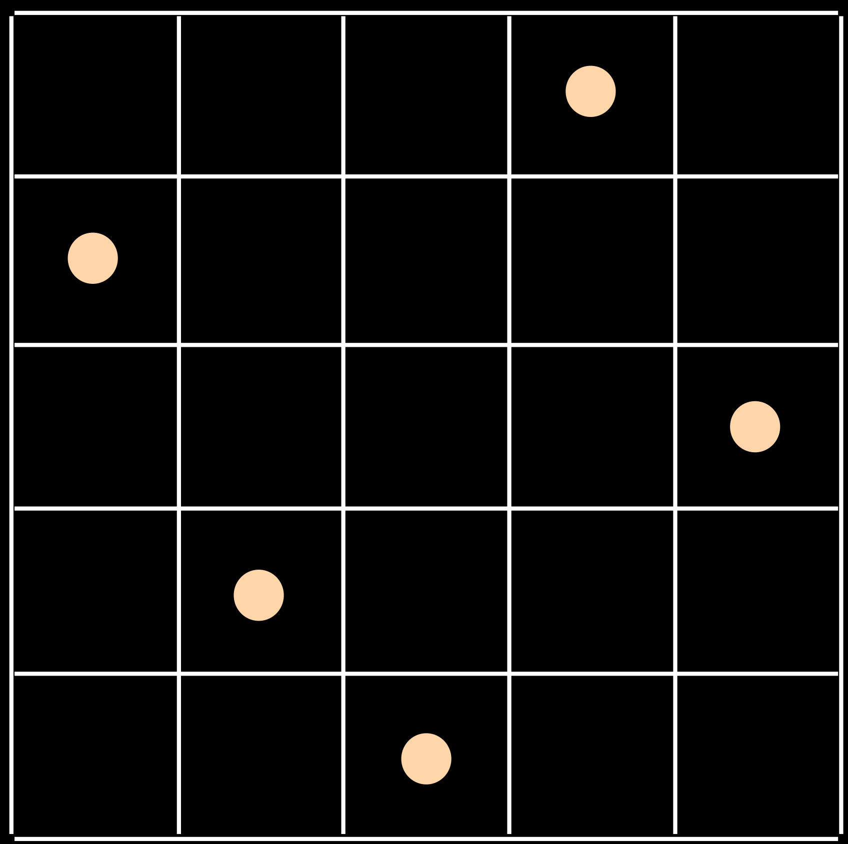
- in the last row and last column of the grid $[n]\times[n]$, we get maximal chains of Ferrers diagrams

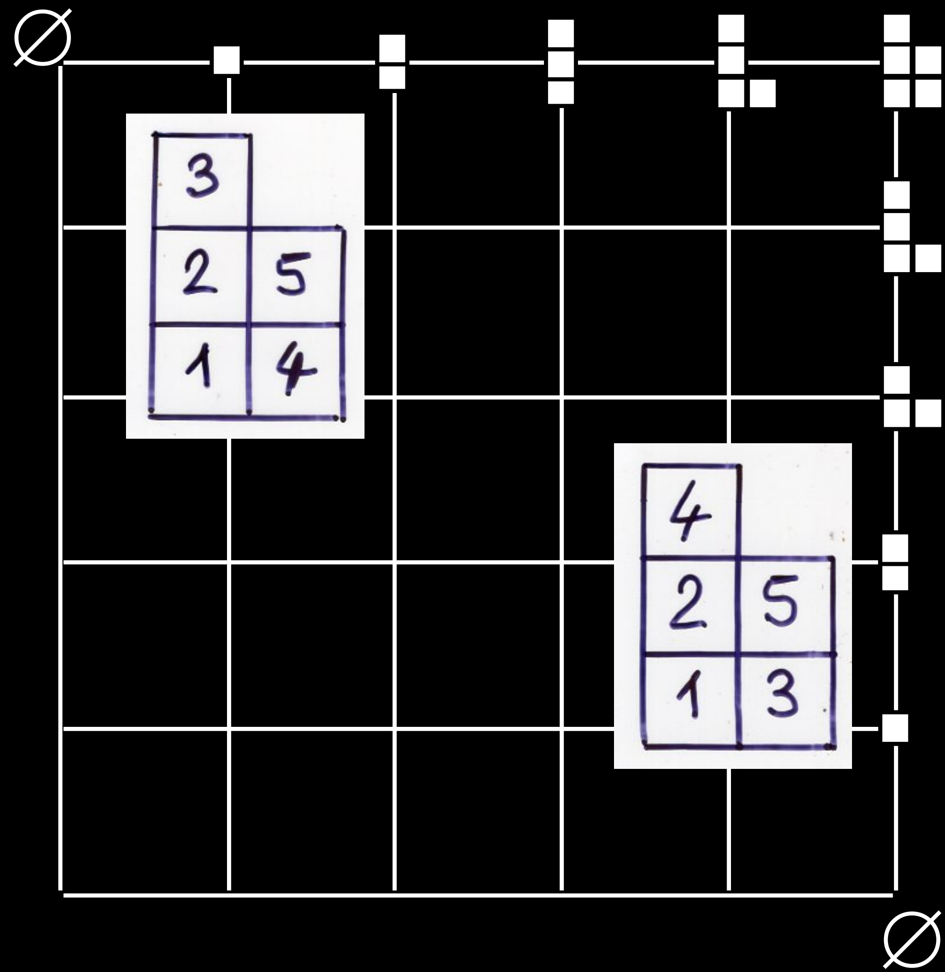


- these maximal chains encode a pair (P, Q) of Young tableaux having the same shape



- the algorithm can be reversed : from the pair (P, Q) , get back the permutation

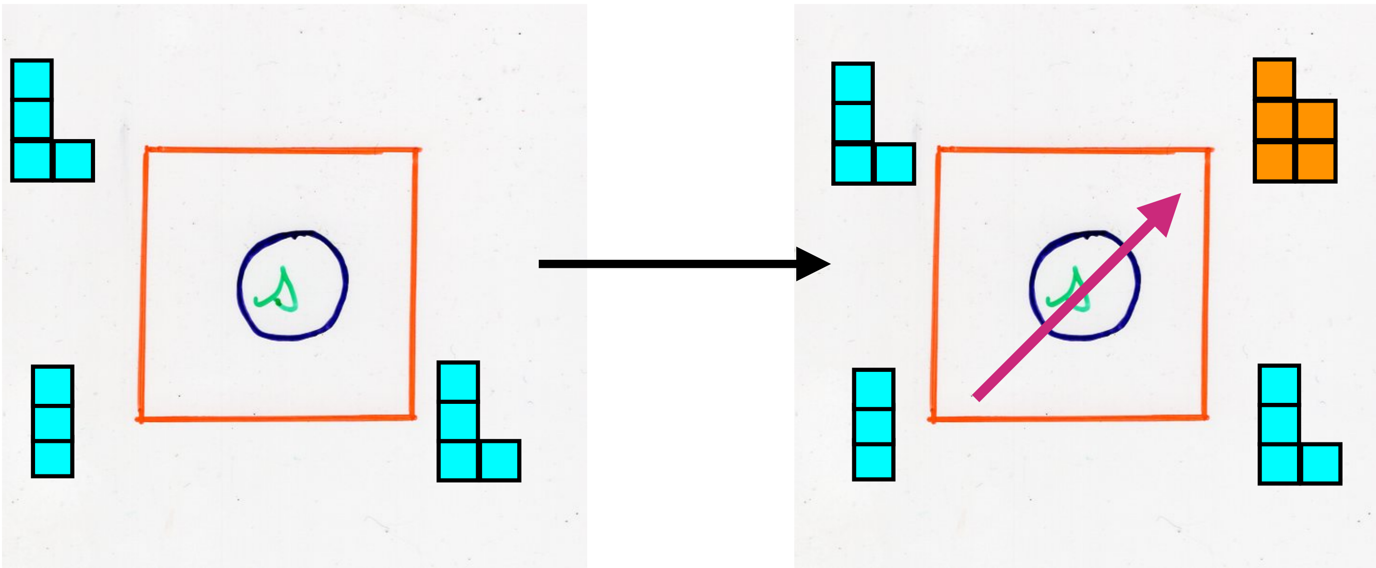




- this bijection is the same as the Robinson-Schensted correspondence

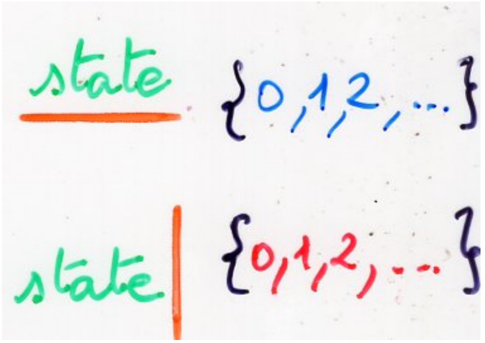
edge local rules

Fomin's "local rules"
"growth diagrams"

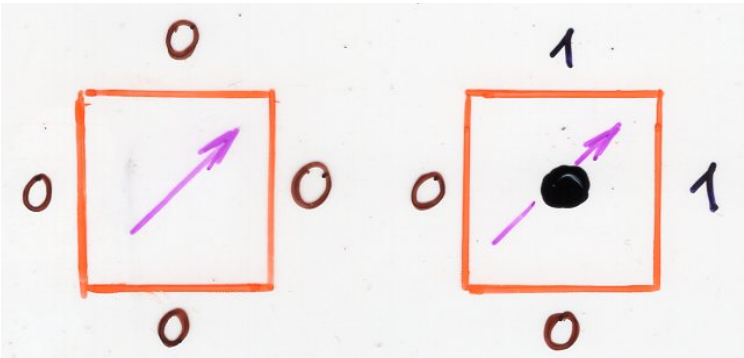


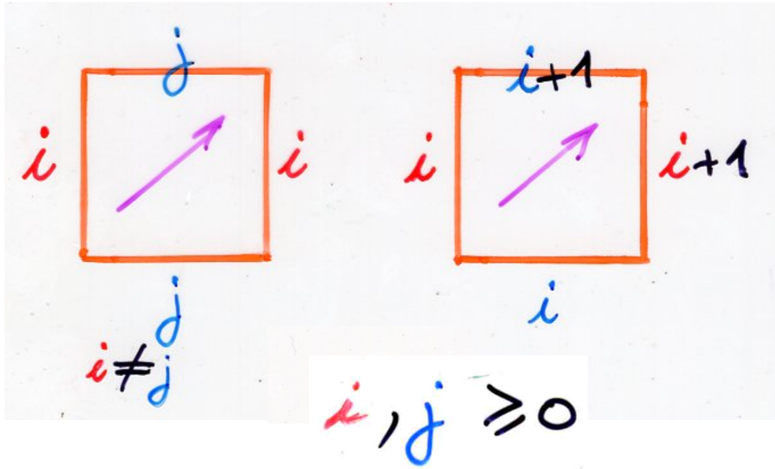
"local rules"
on the vertices

"local rules" on the edges

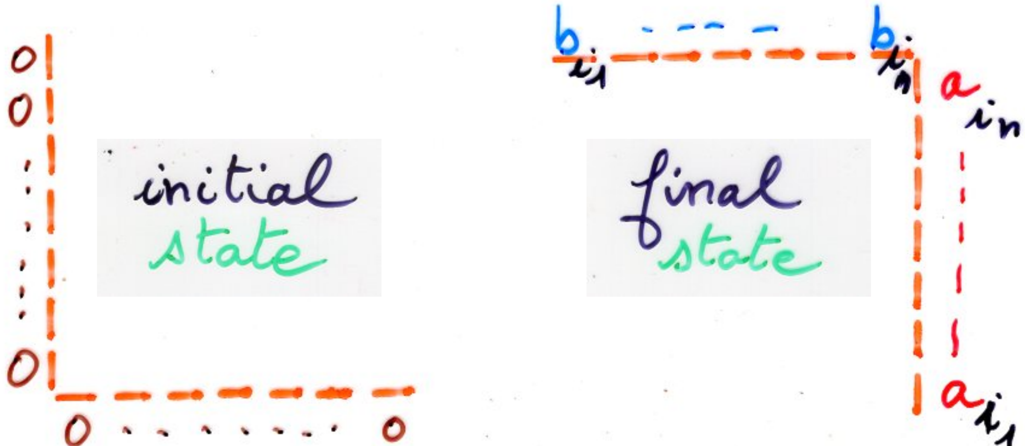


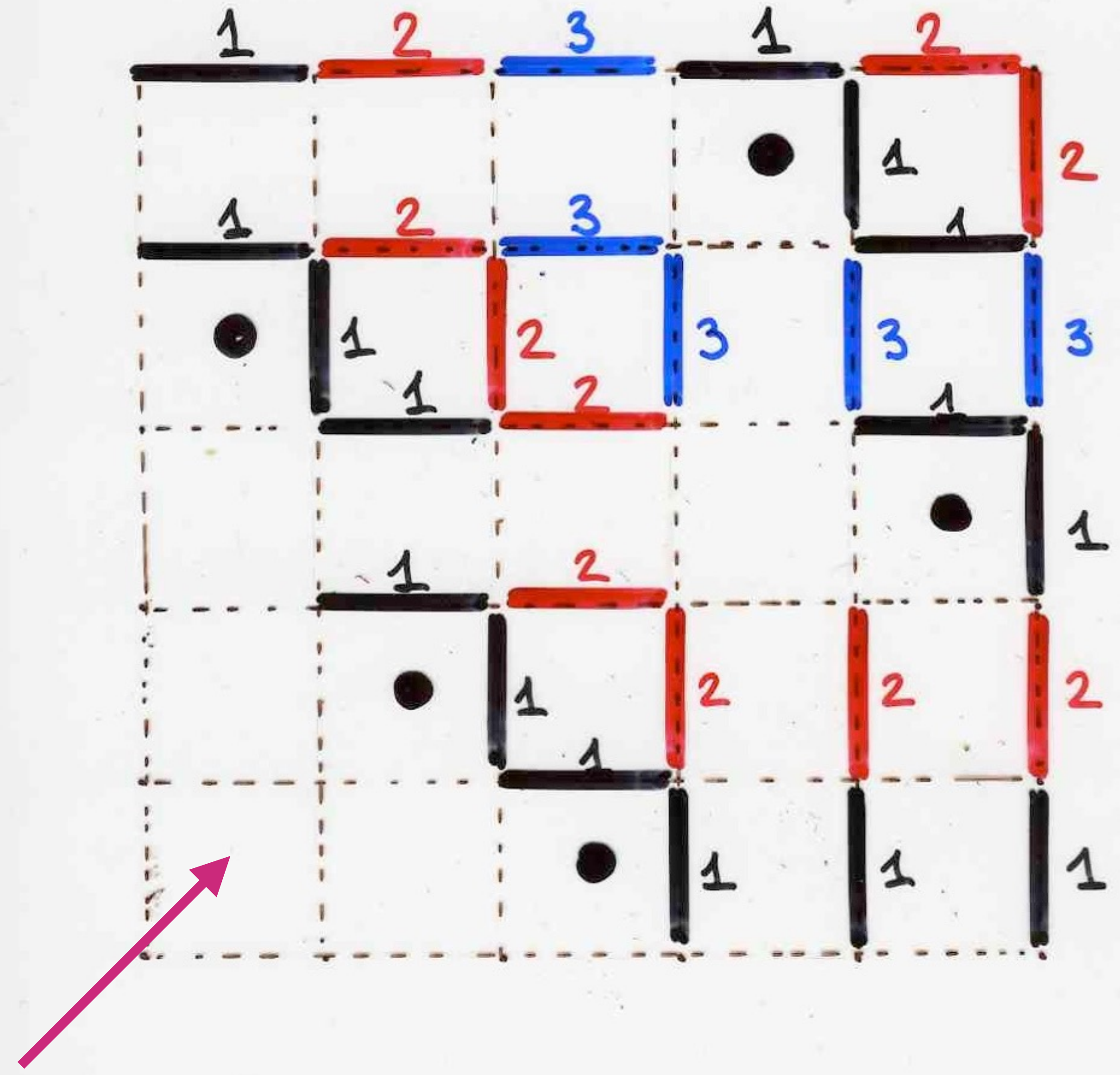
set of labels $L = \{\square, \boxdot\}$





"planar rewriting"




1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1

Definition Yamanouchi word w

$w \in \{1, 2, \dots\}^*$

free monoid generated by the alphabet $1, 2, \dots,$

such that:

for every factorization $w = uv$

$$|u|_1 \geq |u|_2 \geq \dots \geq |u|_i \geq \dots$$

↑ number of occurrences of the letter i in u

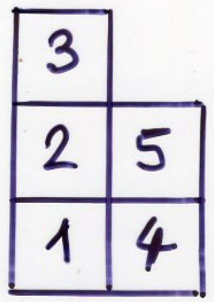
coding of a Young tableau
with a Yamanouchi word

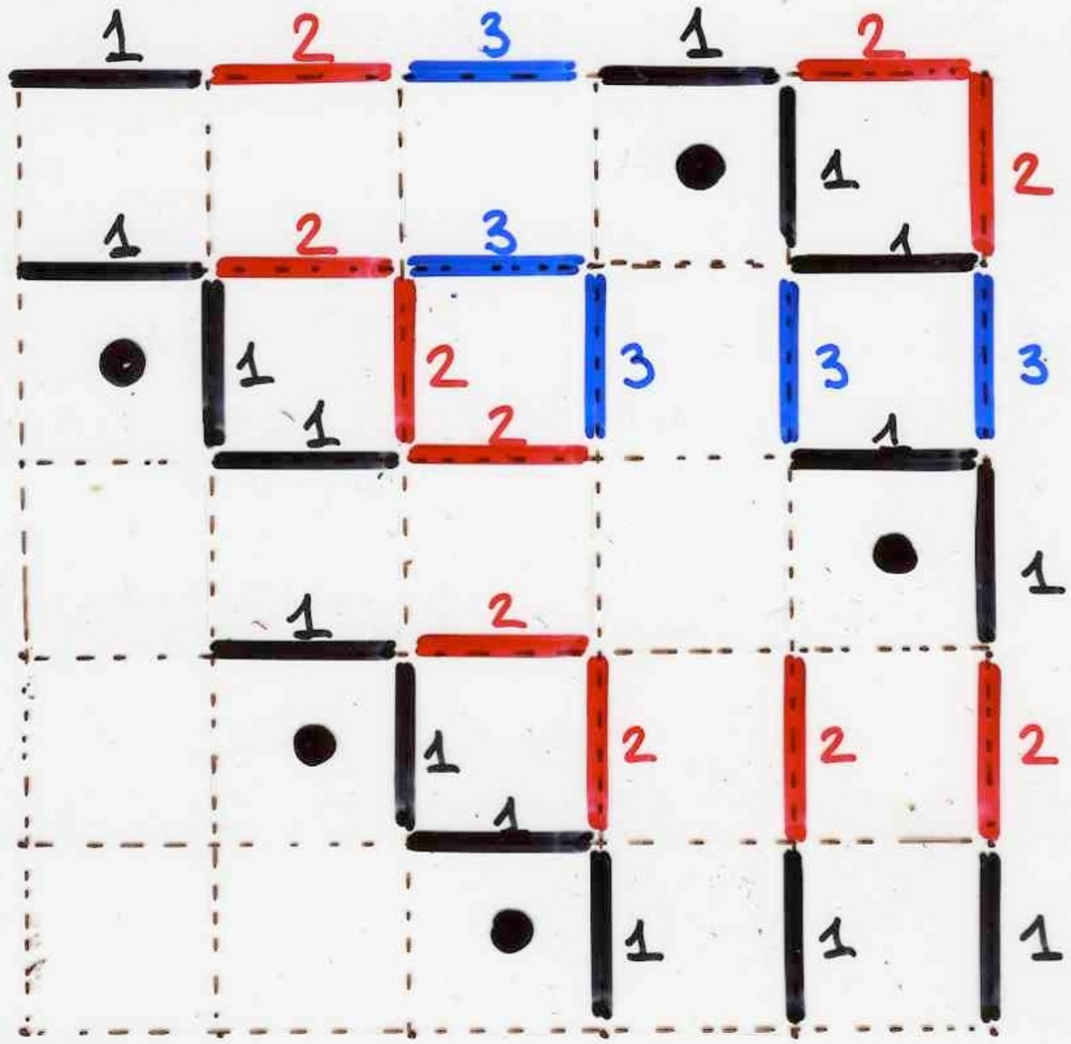
(also called
lattice permutation)

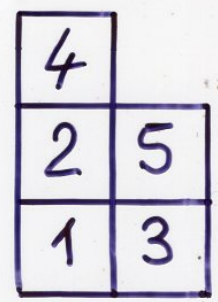
w =	1	2	1	1	2	2	1	3	1	3
	1	2	3	4	5	6	7	8	9	10

Q =

8	10			
2	5	6		
1	3	4	7	9

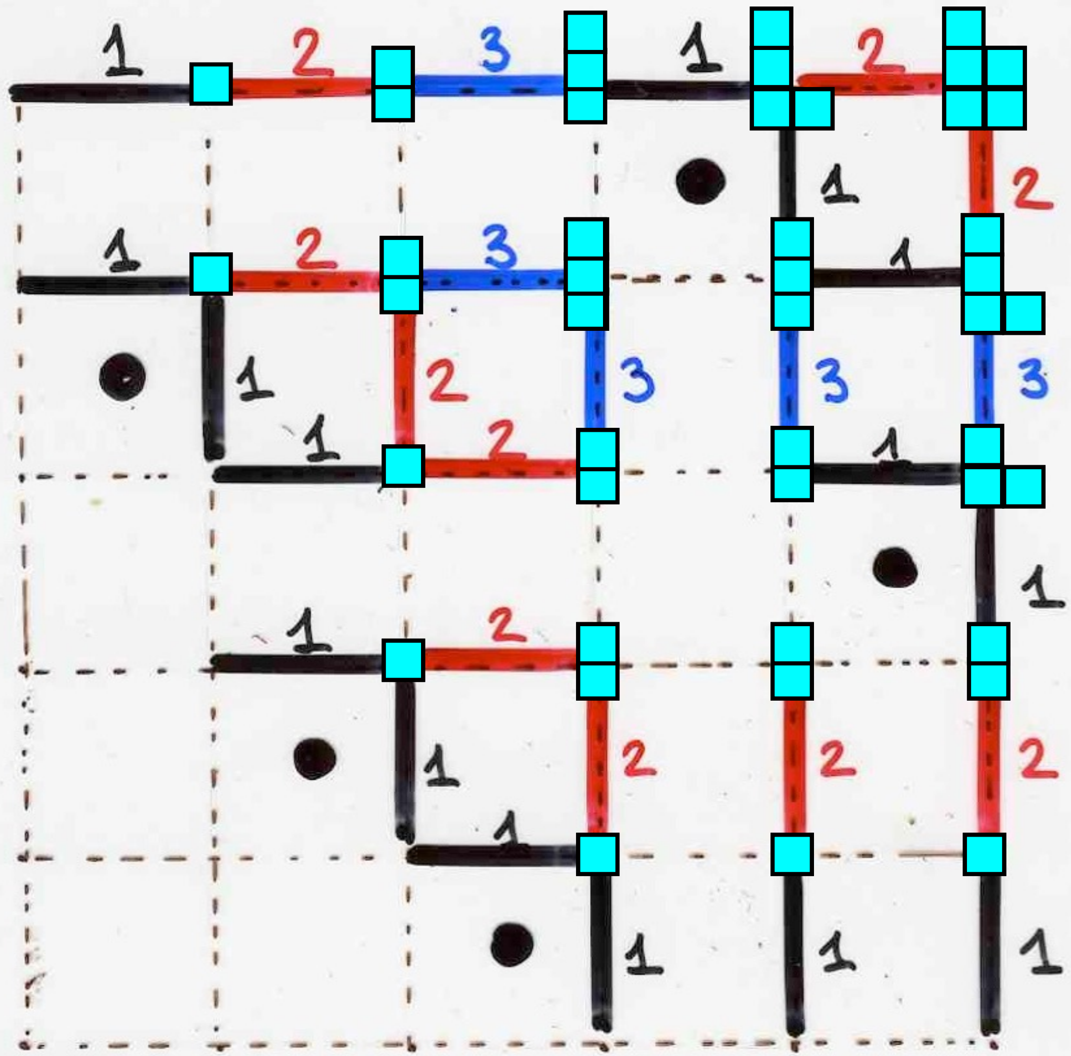

3
2 5
1 4



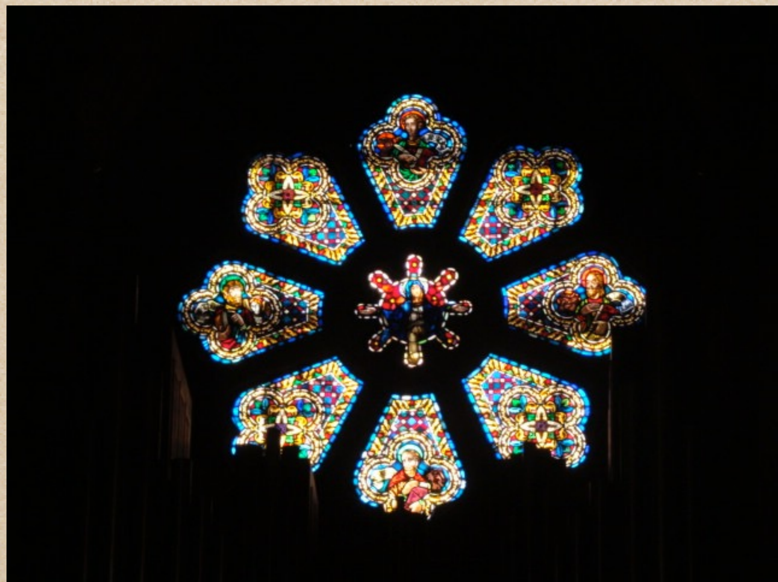

4
2 5
1 3

Proposition

The two processes « growth diagrams » and « edge local rules » are equivalent

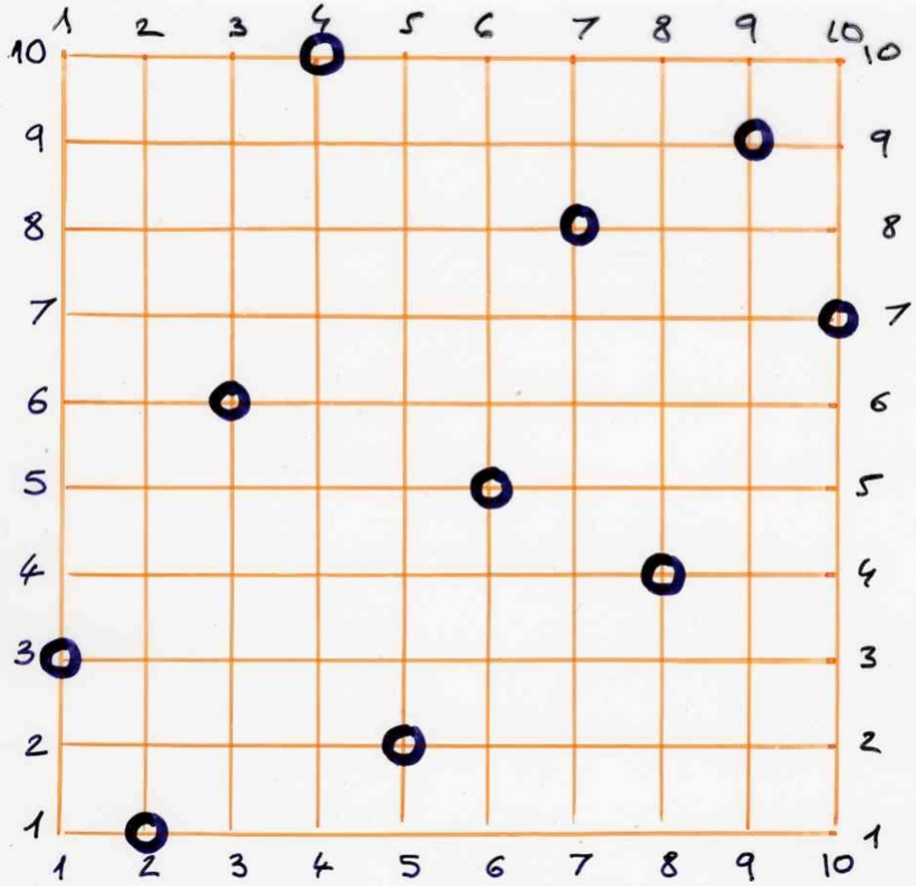

1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1

A geometric version of RSK with "light" and "shadow lines"



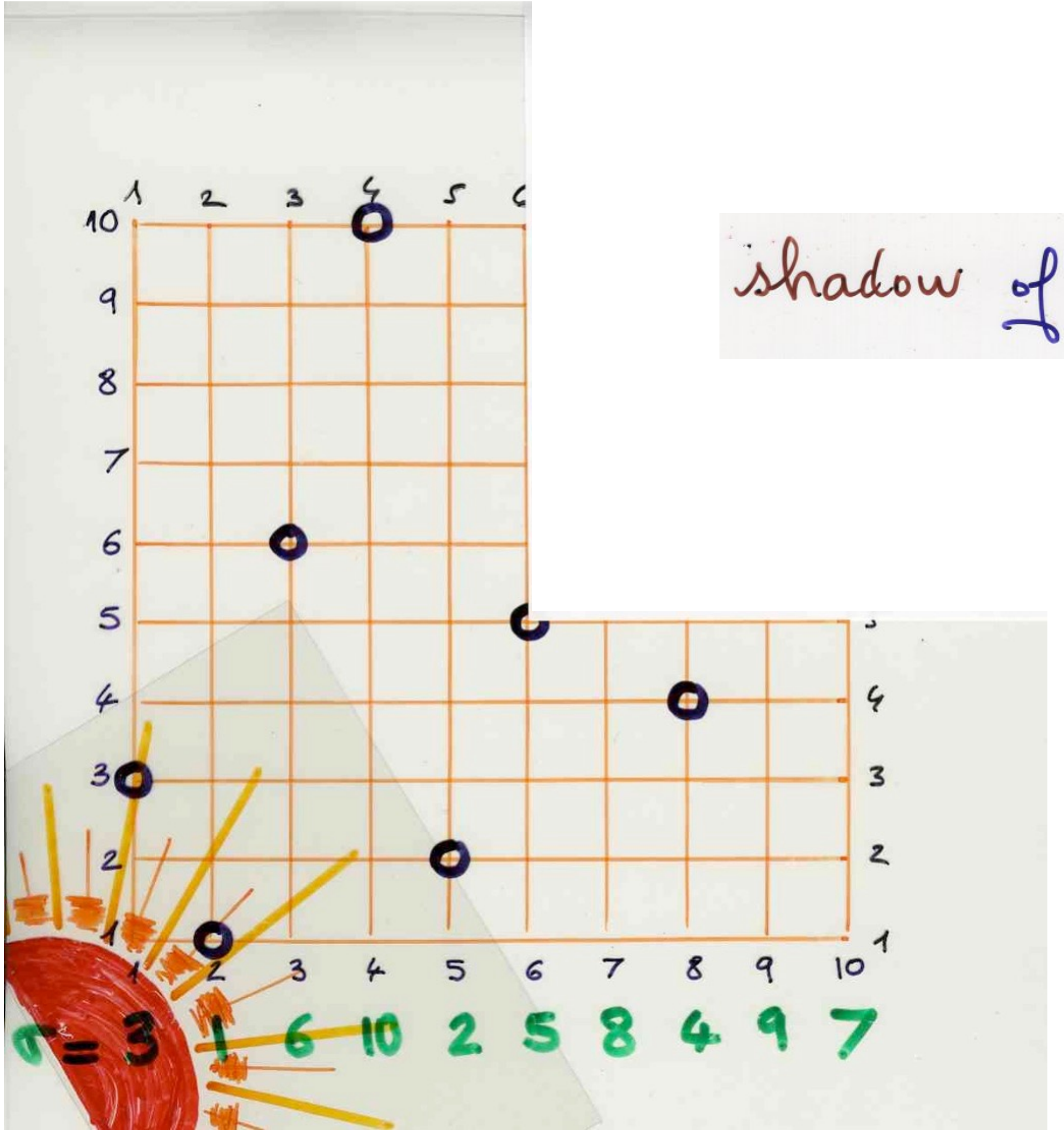
X.V. 1976

$$\{(i,\sigma(i))\}_{i=1,\dots,n} \subseteq [1,n]\times[1,n]$$



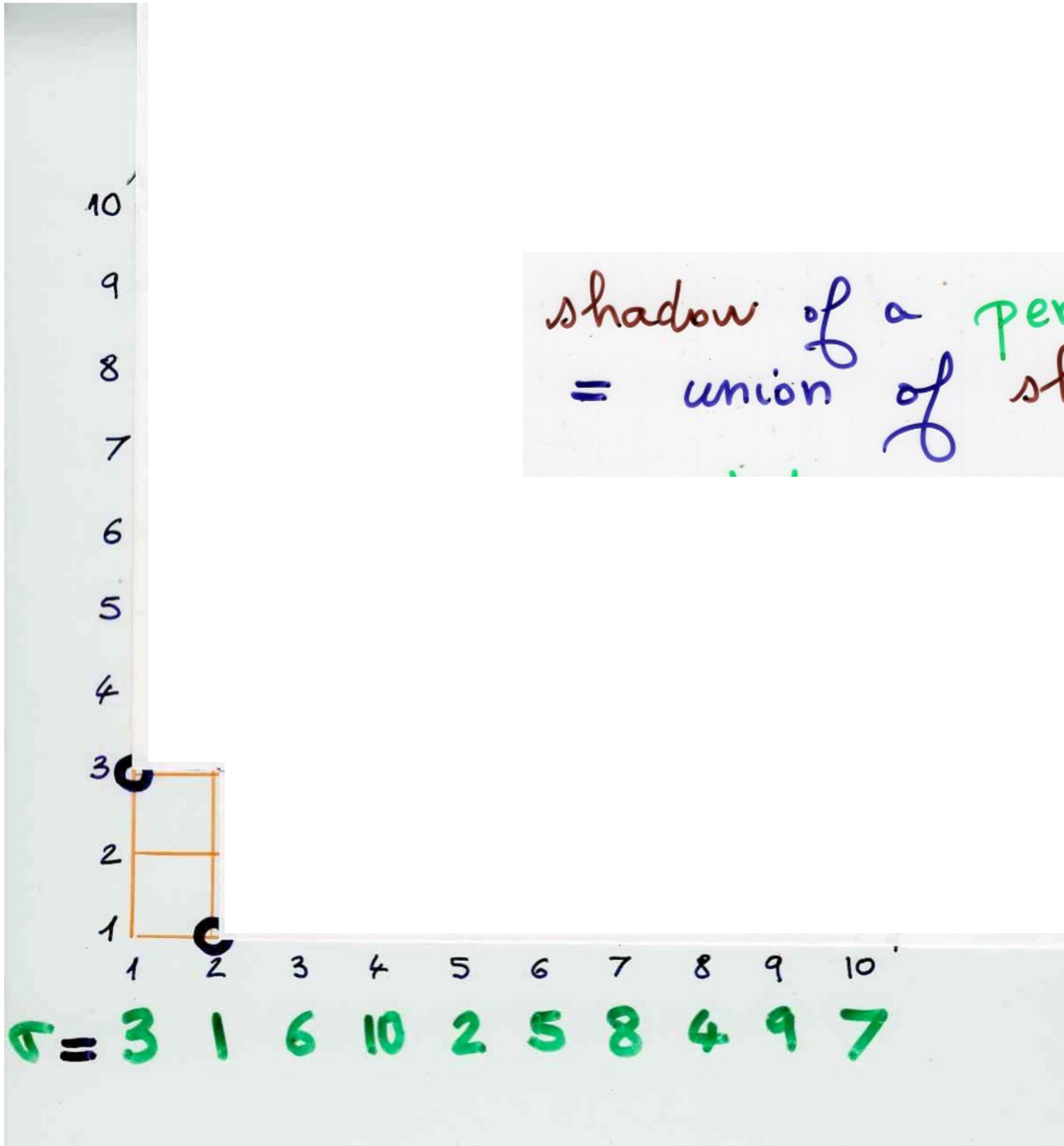
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

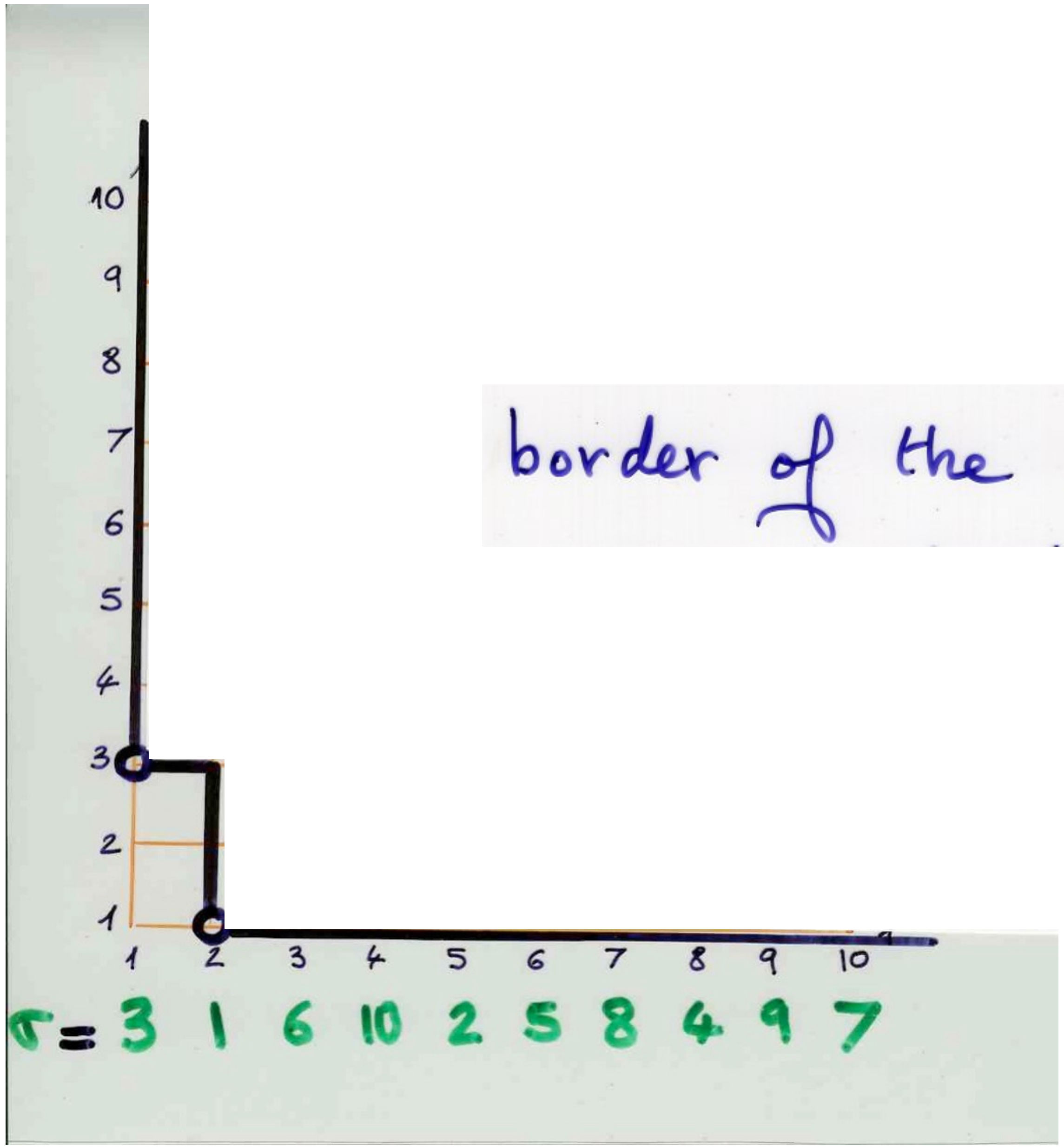
graph of a permutation σ



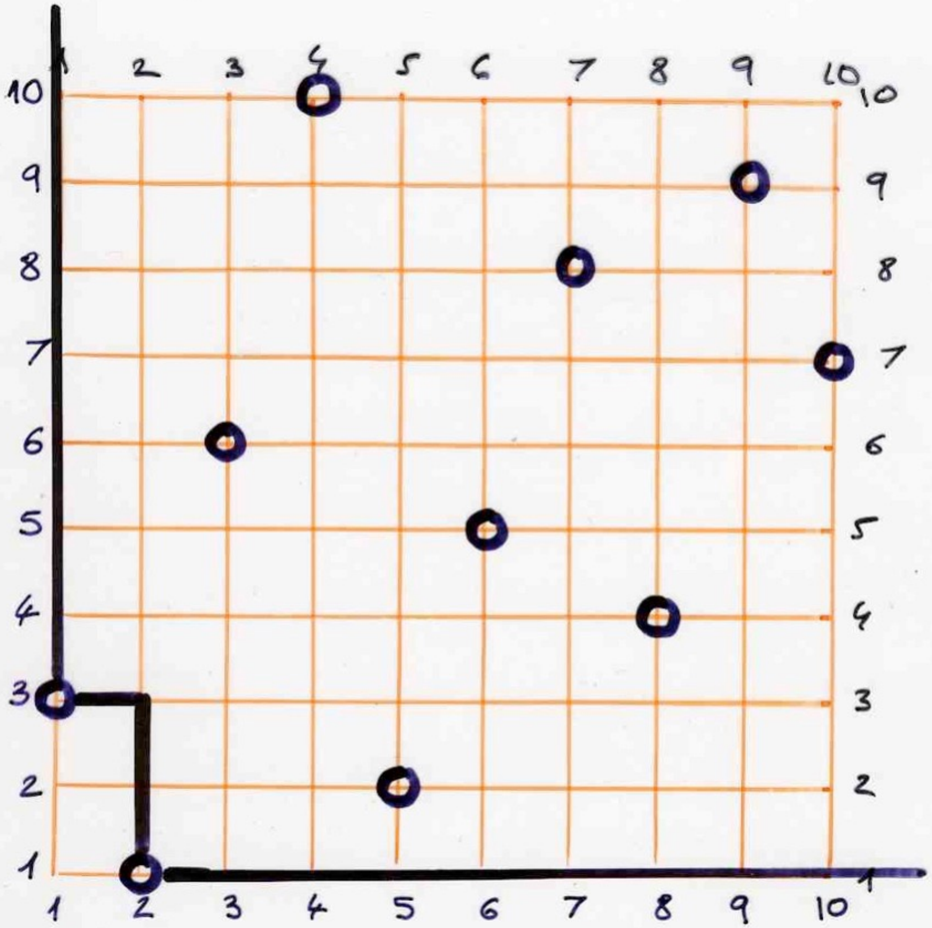
shadow of a point ●

shadow of a permutation
= union of shadows

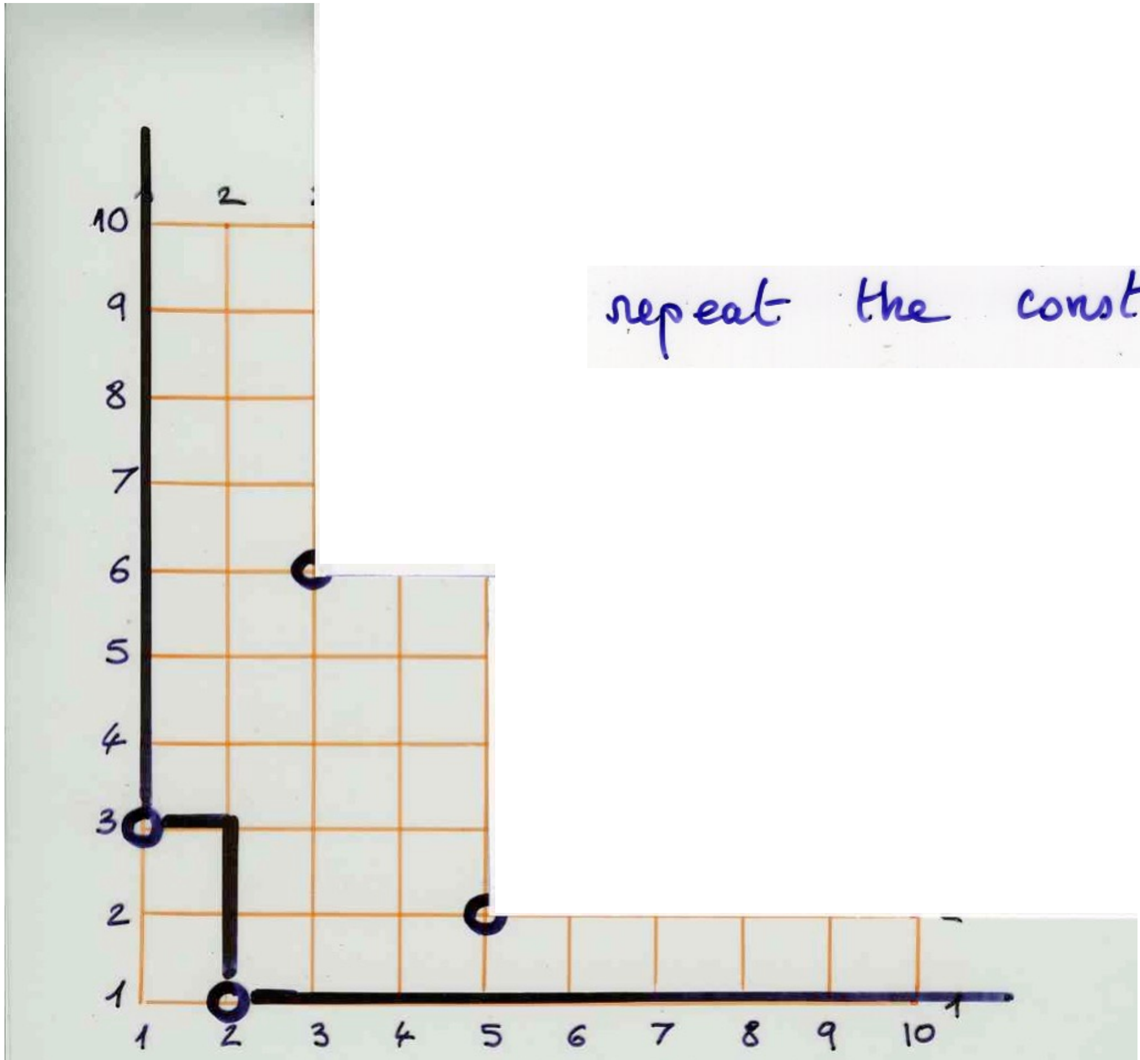




border of the shadow

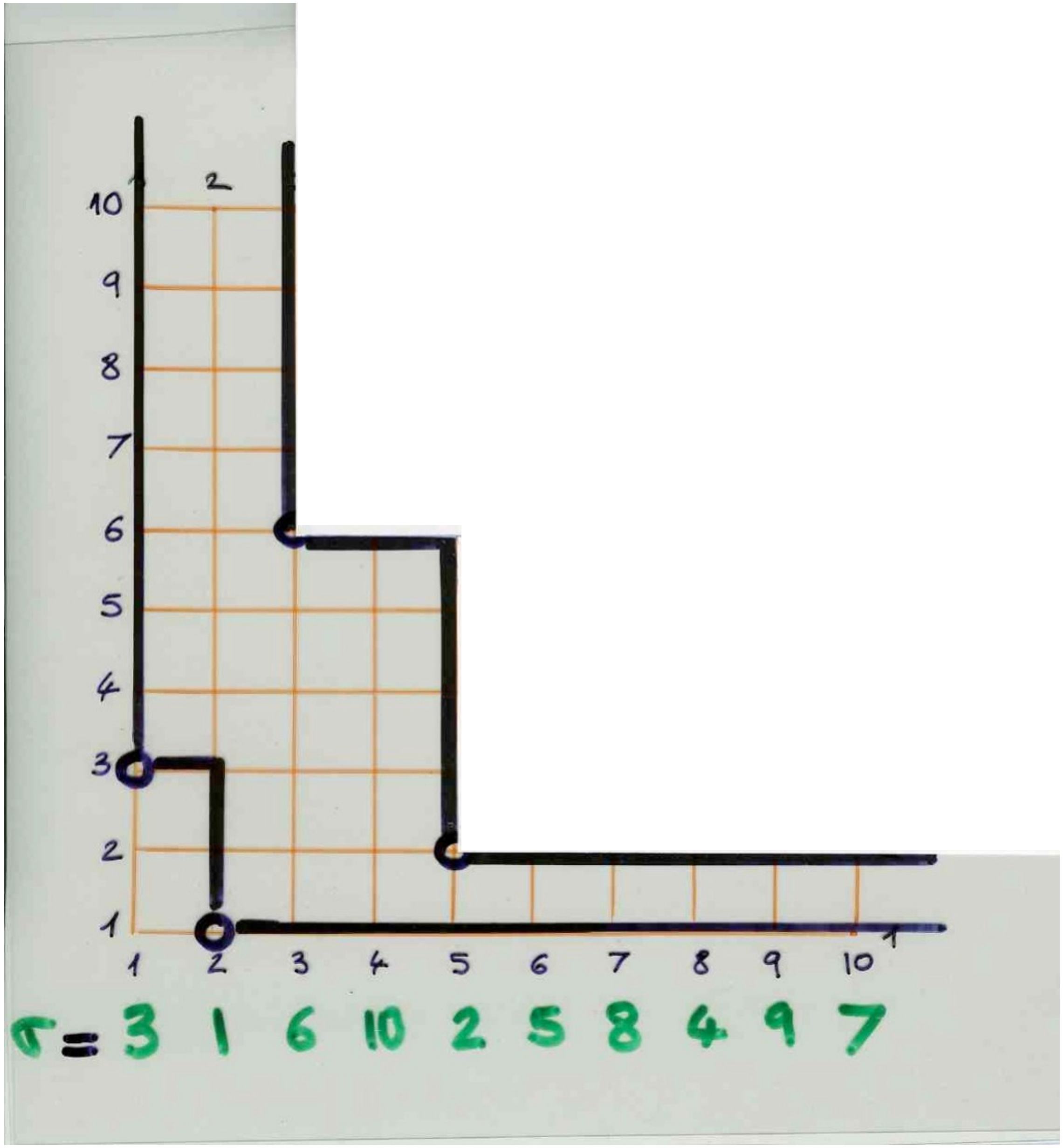


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

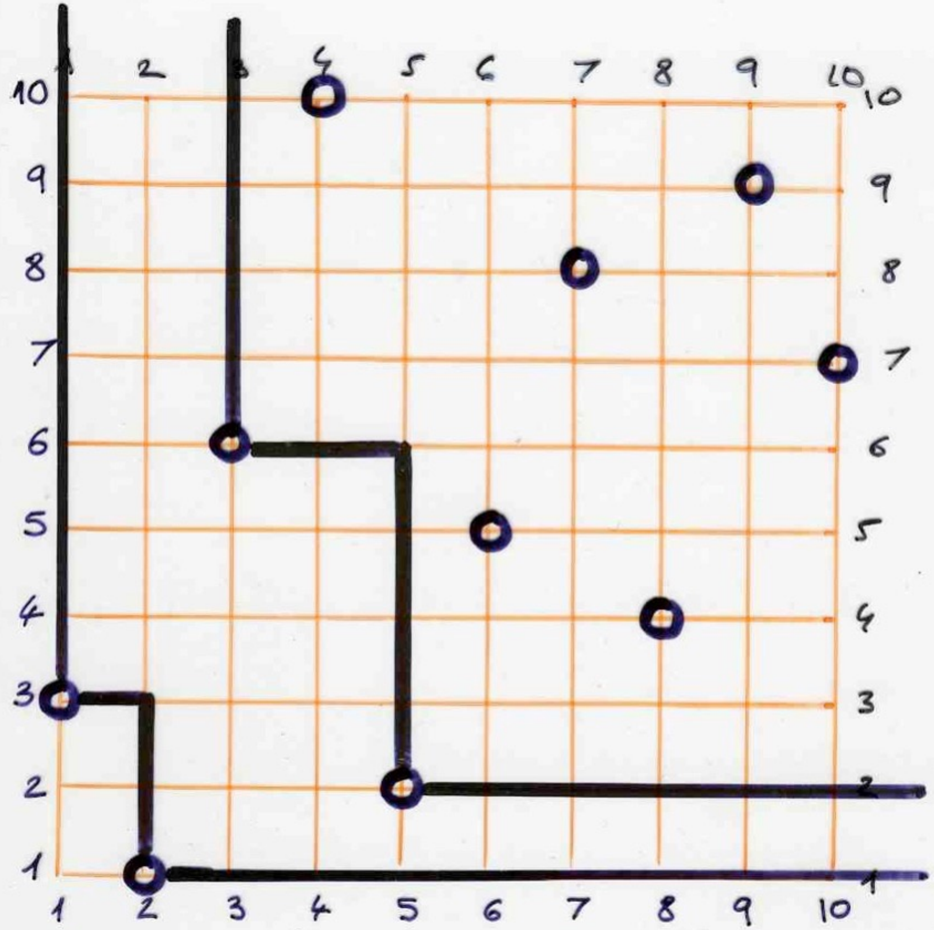


repeat the construction ...

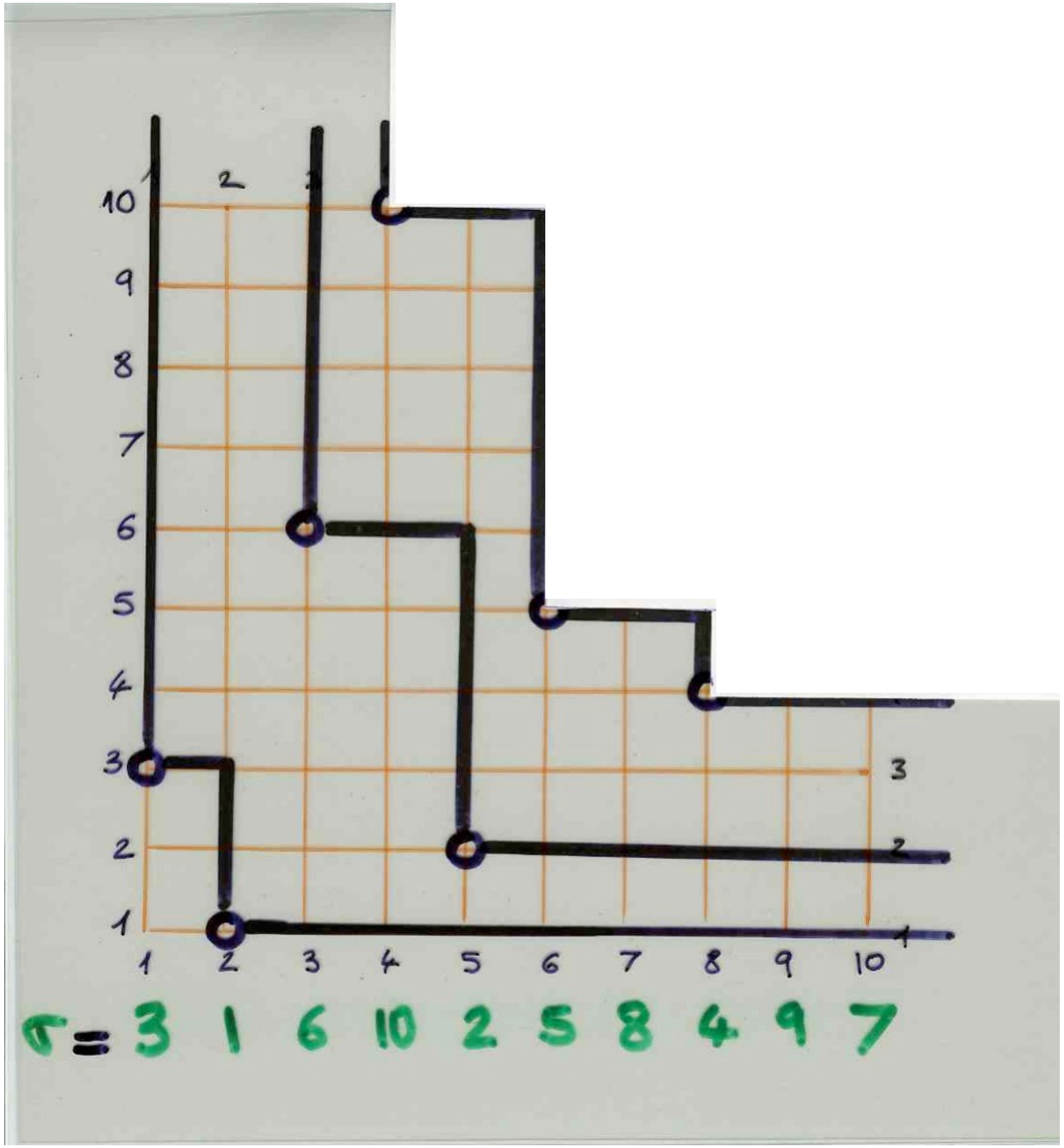
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



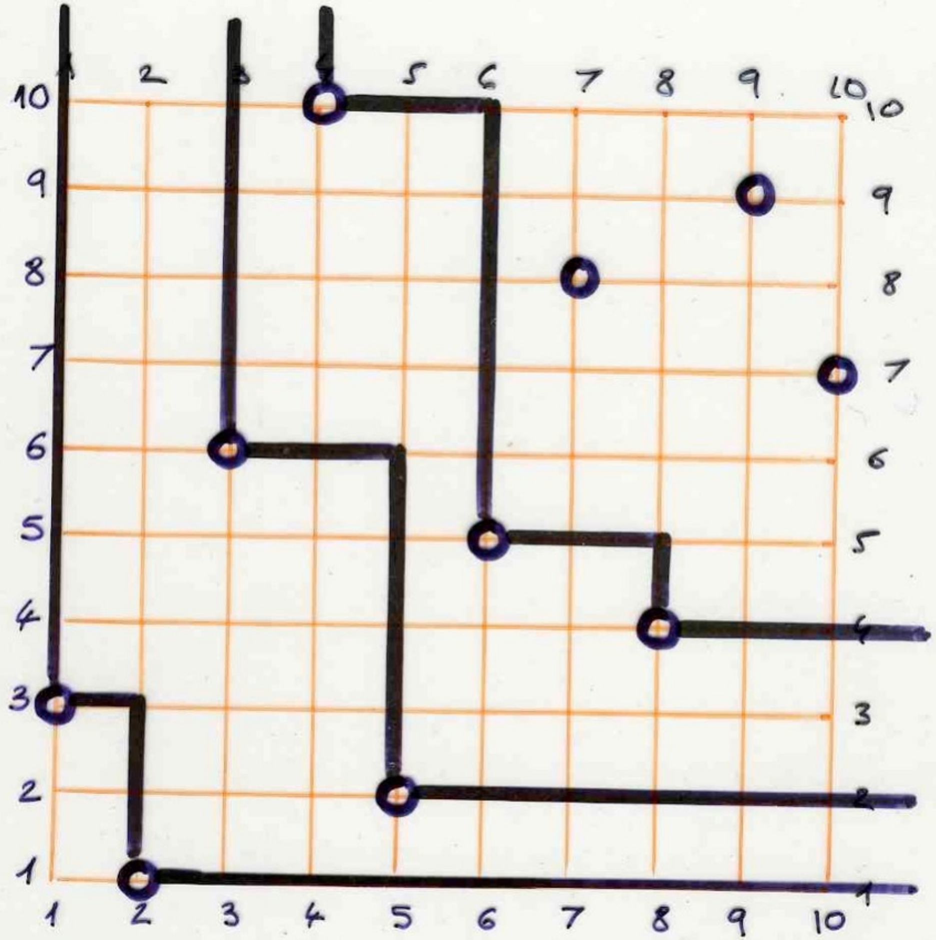
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7



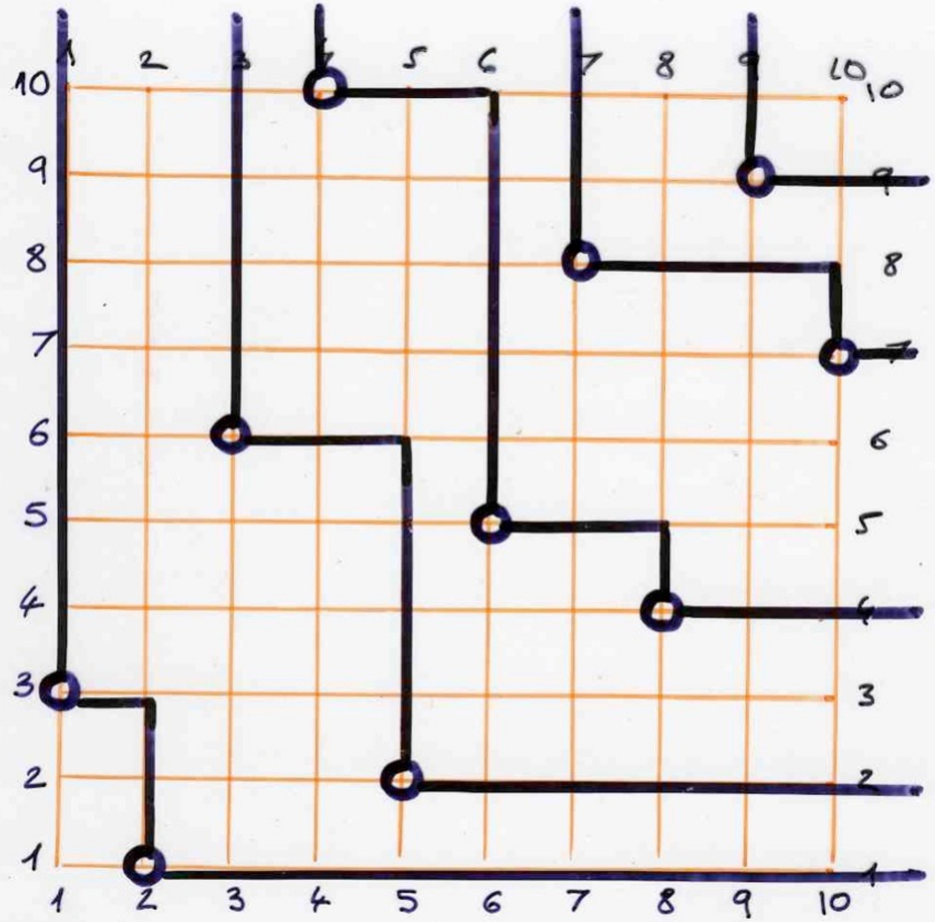
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7

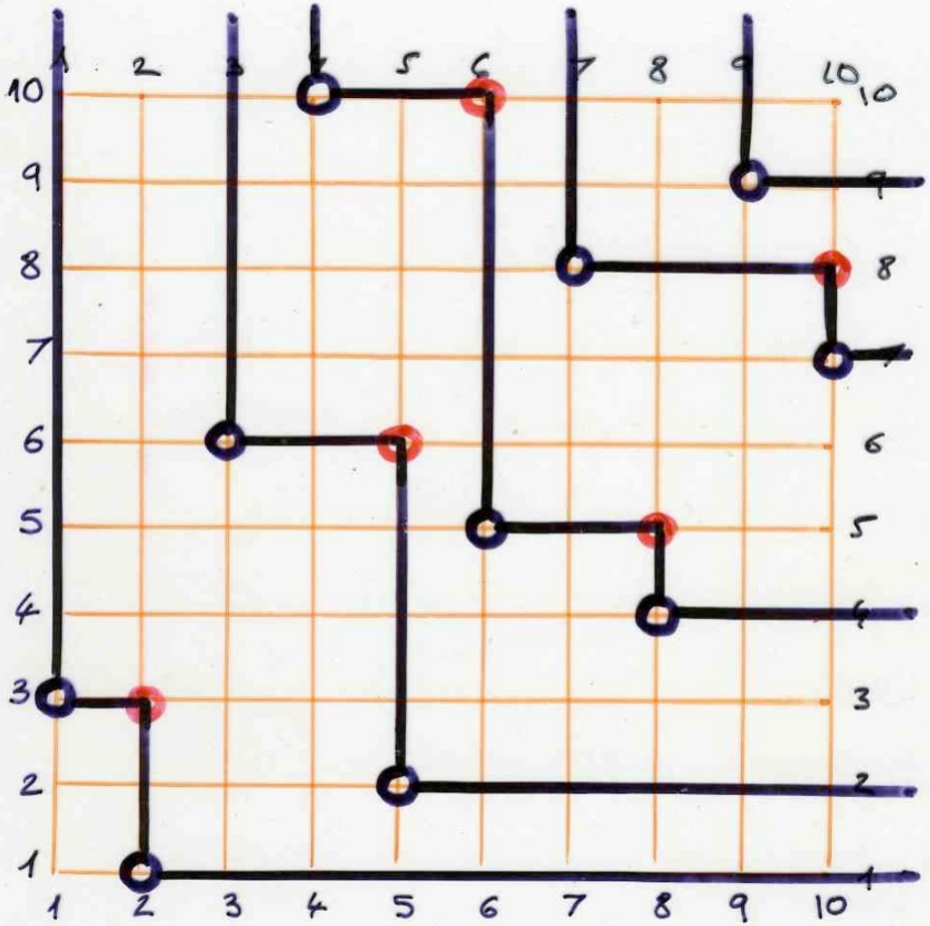


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



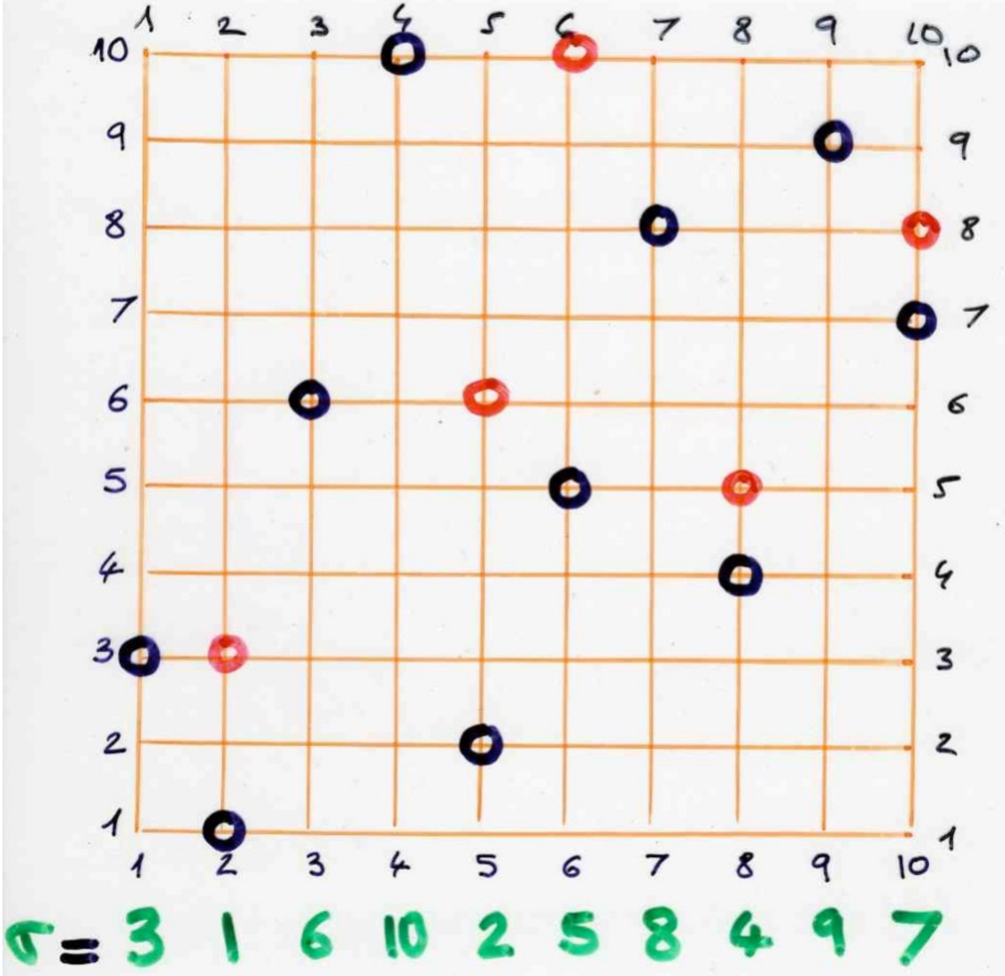
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

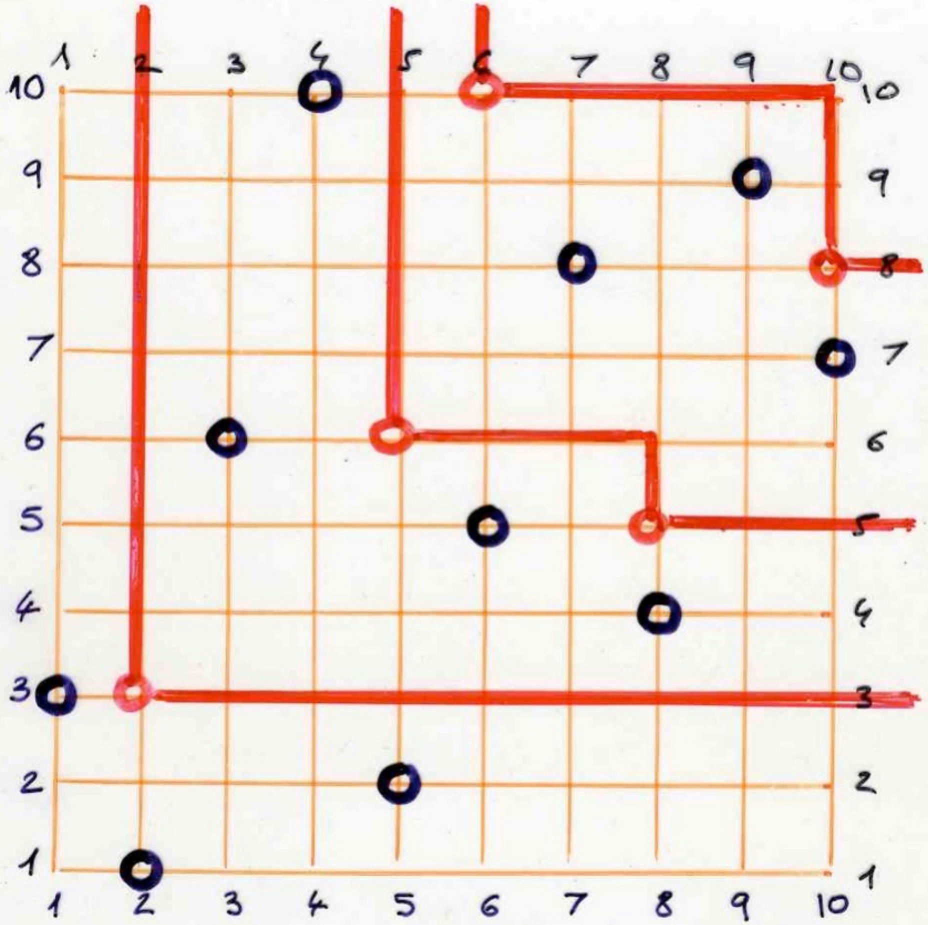
red points



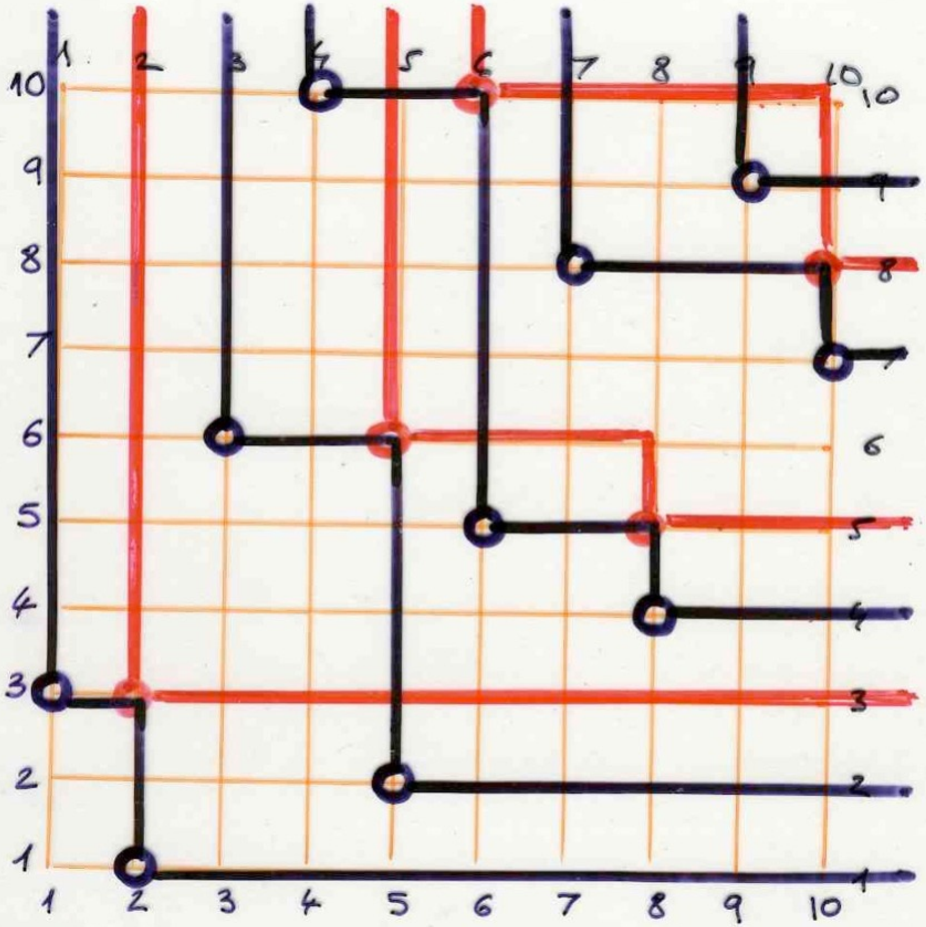
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

repeat with the red points
the construction of successives shadows



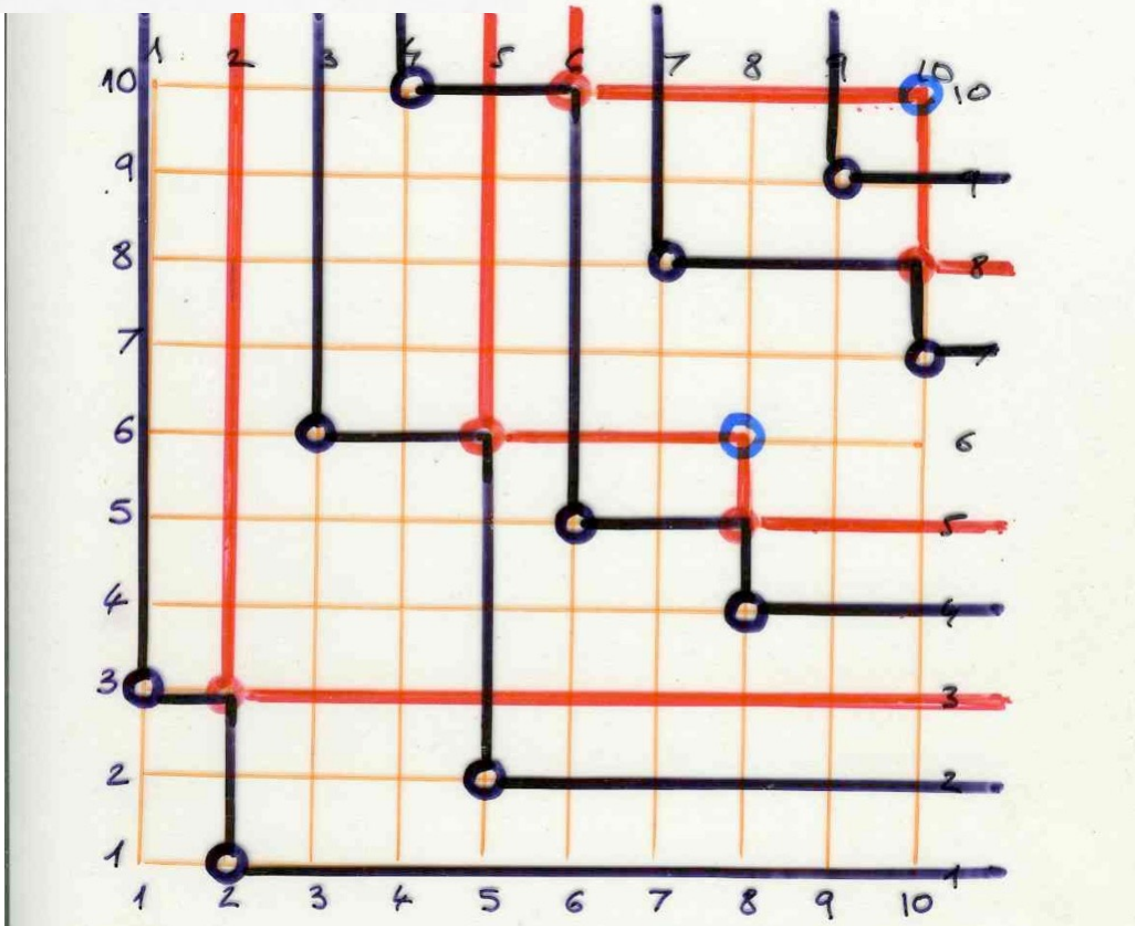


σ = 3 1 6 10 2 5 8 4 9 7



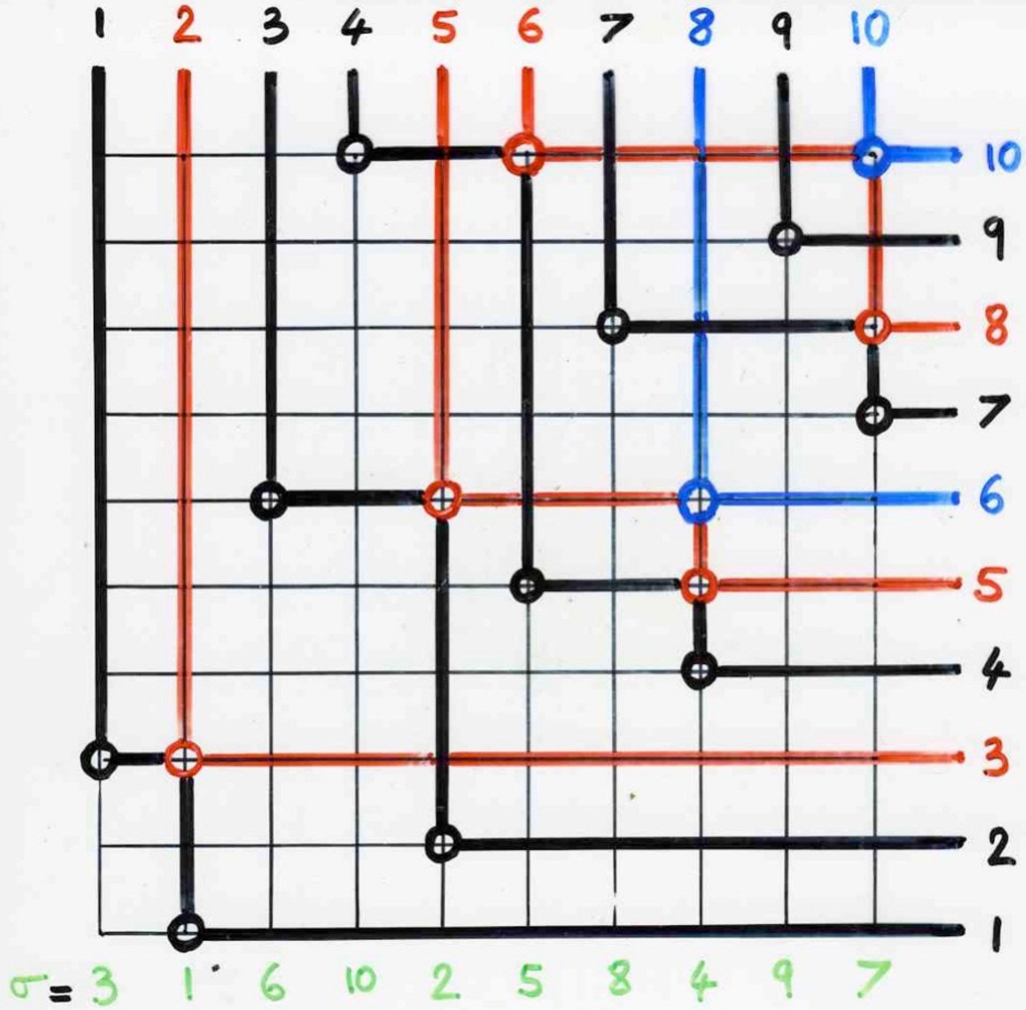
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

blue points ●

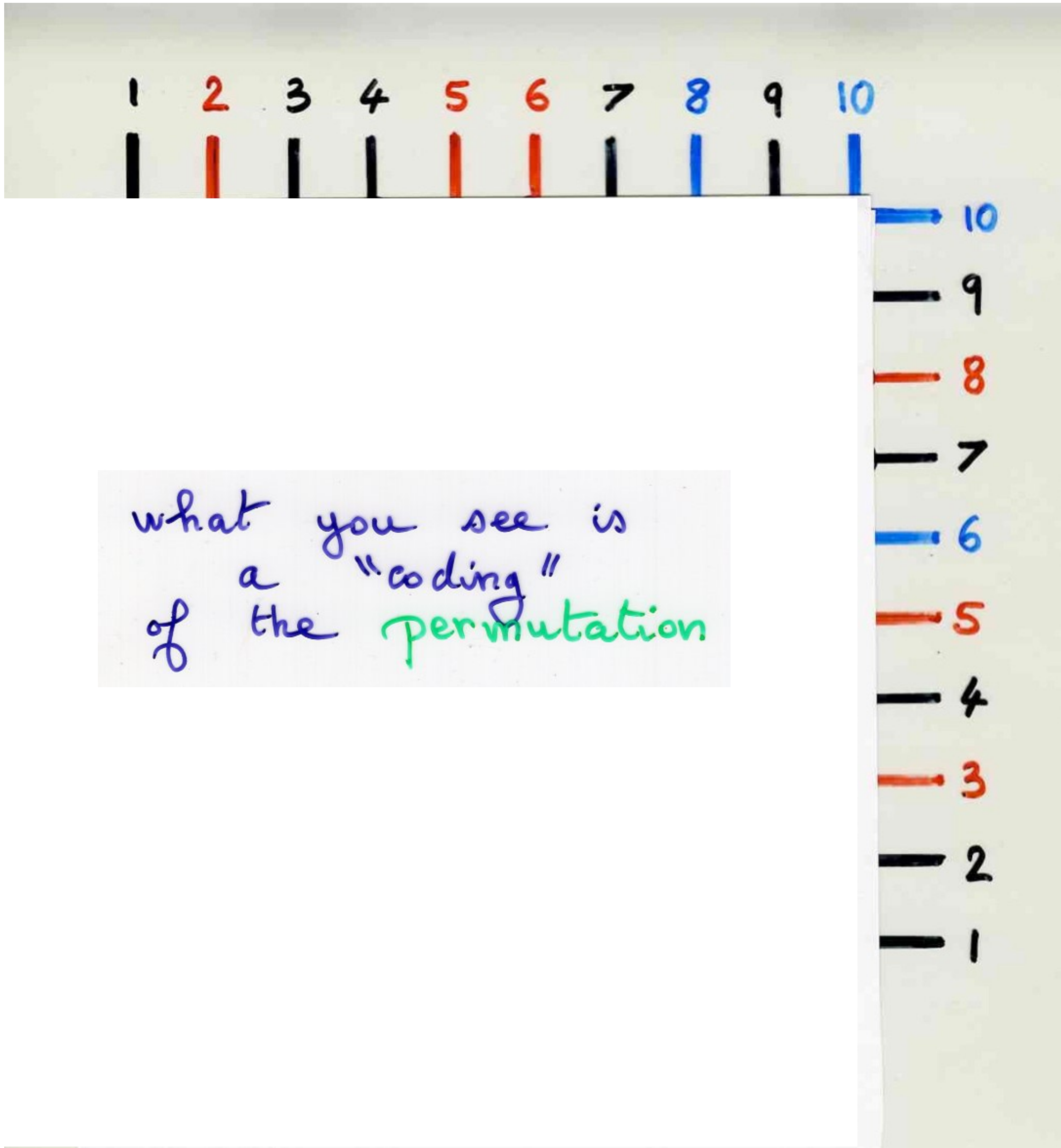


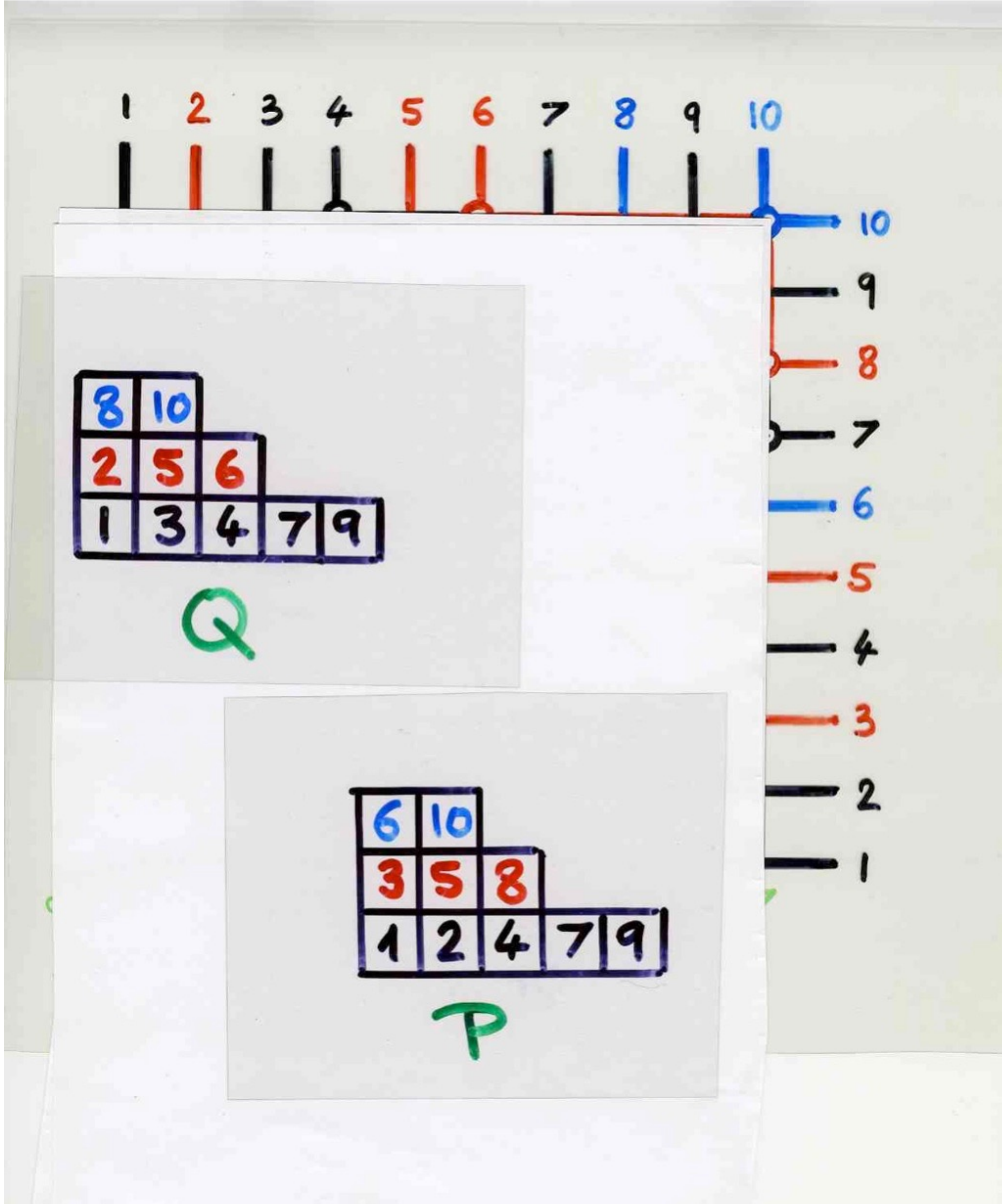
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

no green points



end of the construction

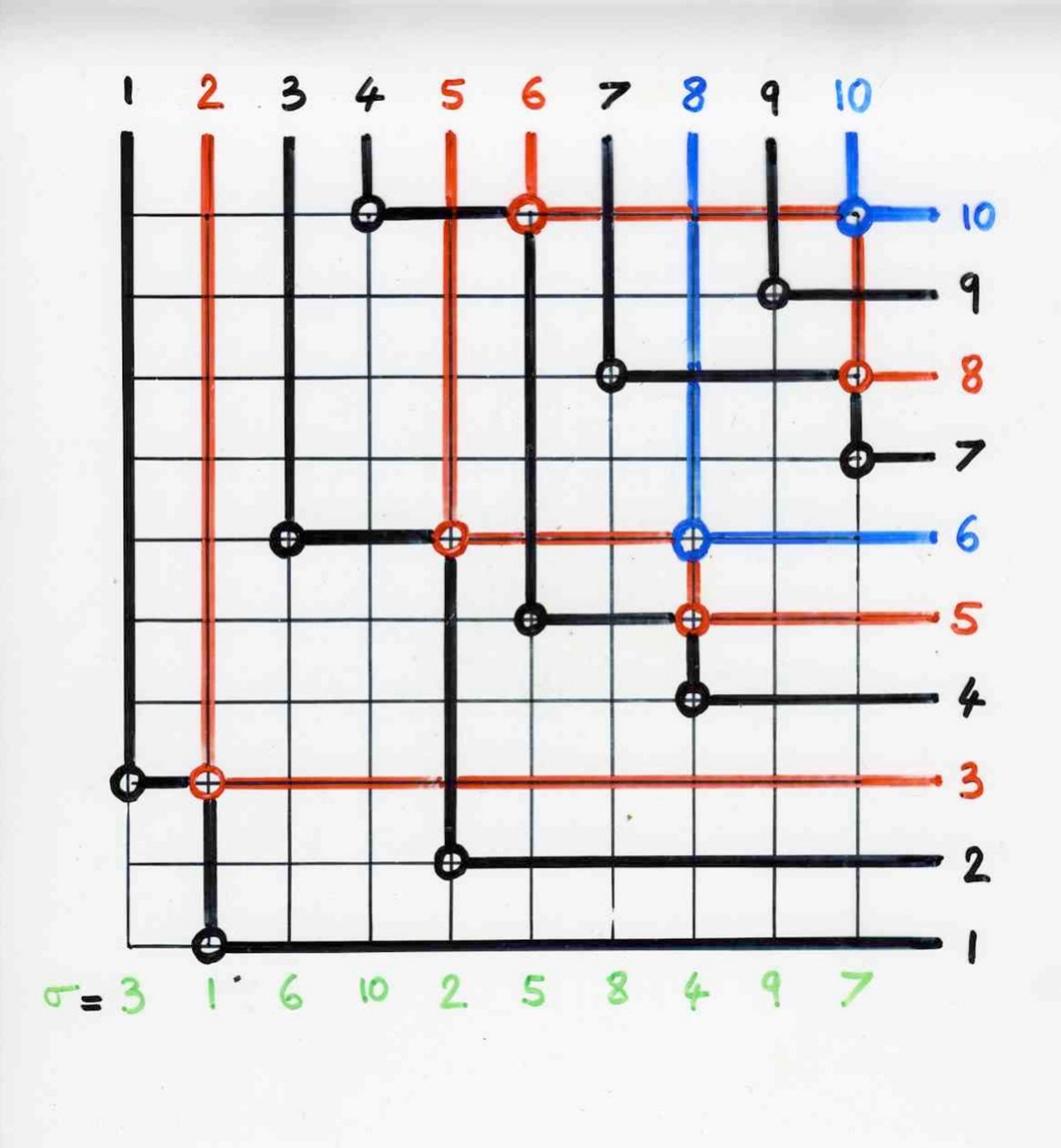

1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
what you see is
a "coding"
of the permutation


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P

proof of the equivalence

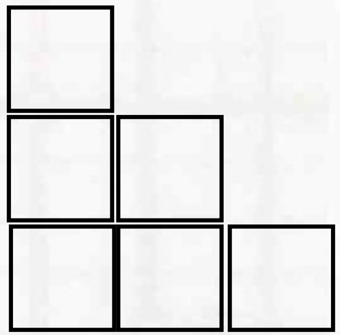
growth diagrams

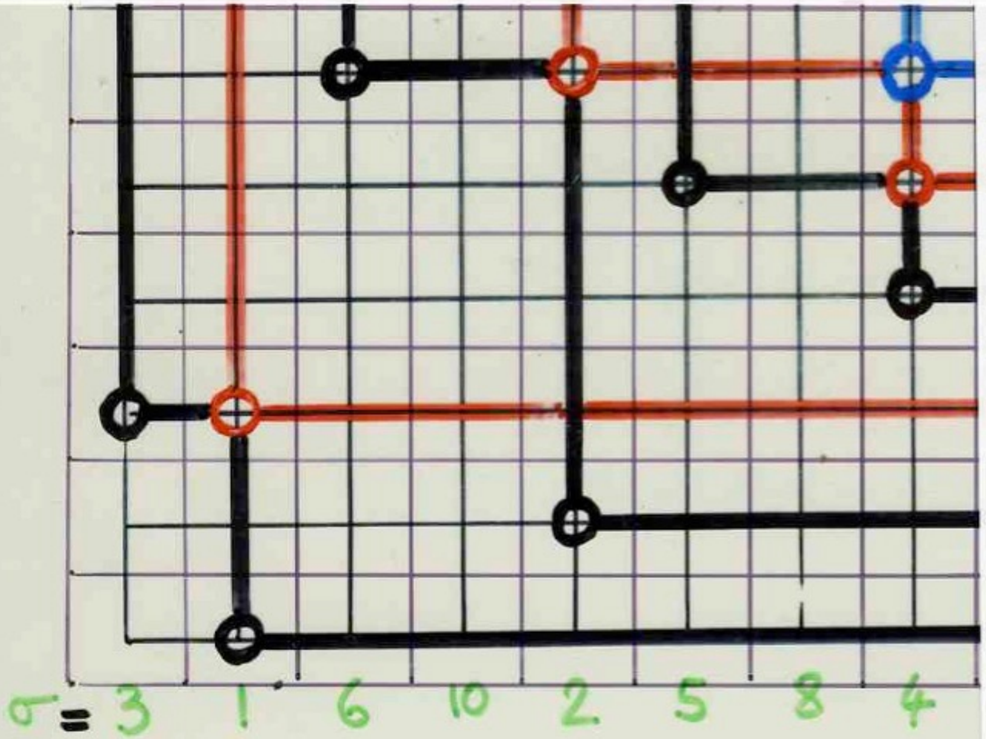
edge local rules



For any vertex of the grid translated by 1/2 we define a Ferrers diagram in the following Way

We get a tableau of Ferrers diagrams





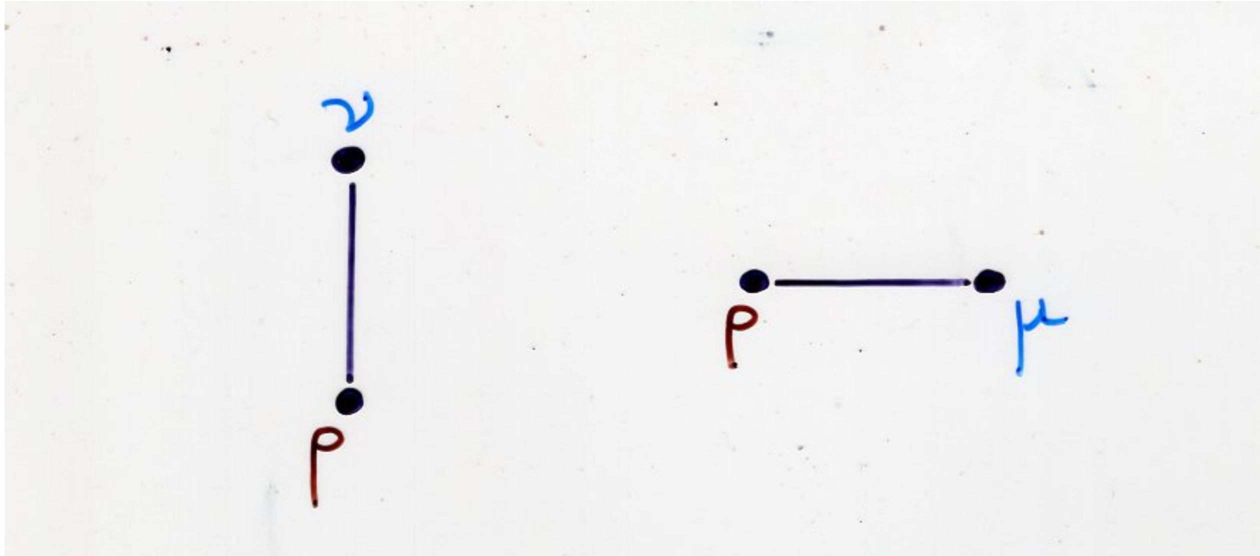
I claim that this tableau is the same as the one we get from Fomin growth diagrams

- label the first set of "shadow lines" of the permutation σ by (1)
  (black lines on the figure)

- then by (2) the second set, i.e. the "shadow lines" of the skeleton
  $Sq(\sigma)$
  (the red lines)

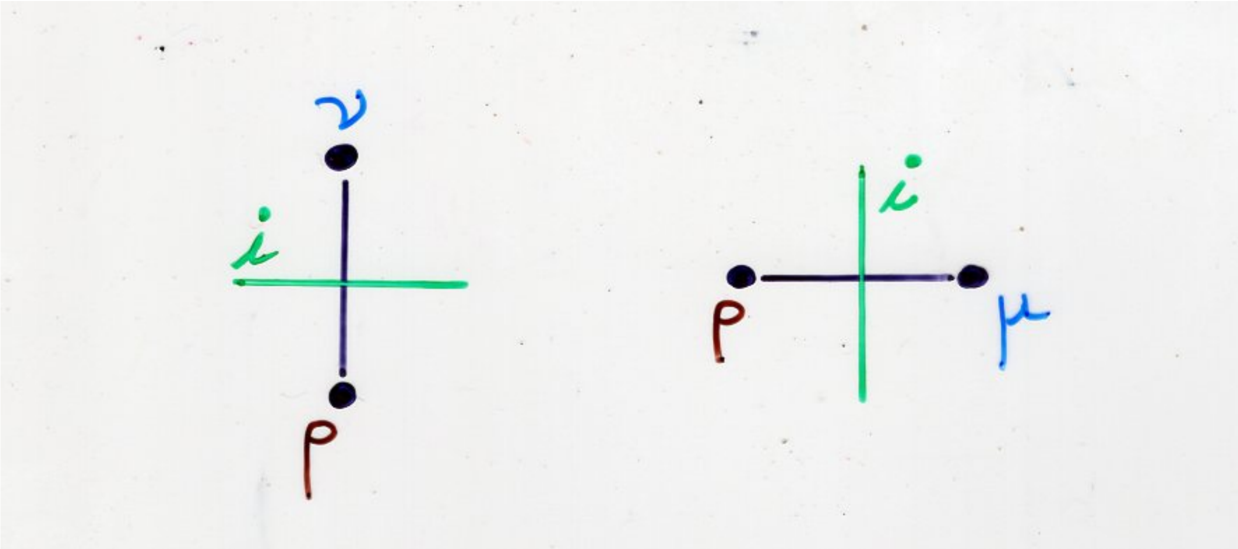
- etc., (3) the blue lines of $Sq(Sq(\sigma))$

- ...



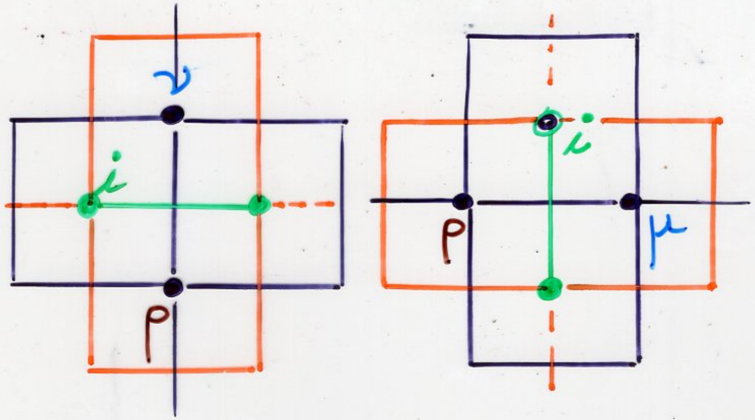
if no shadow lines are crossing, then

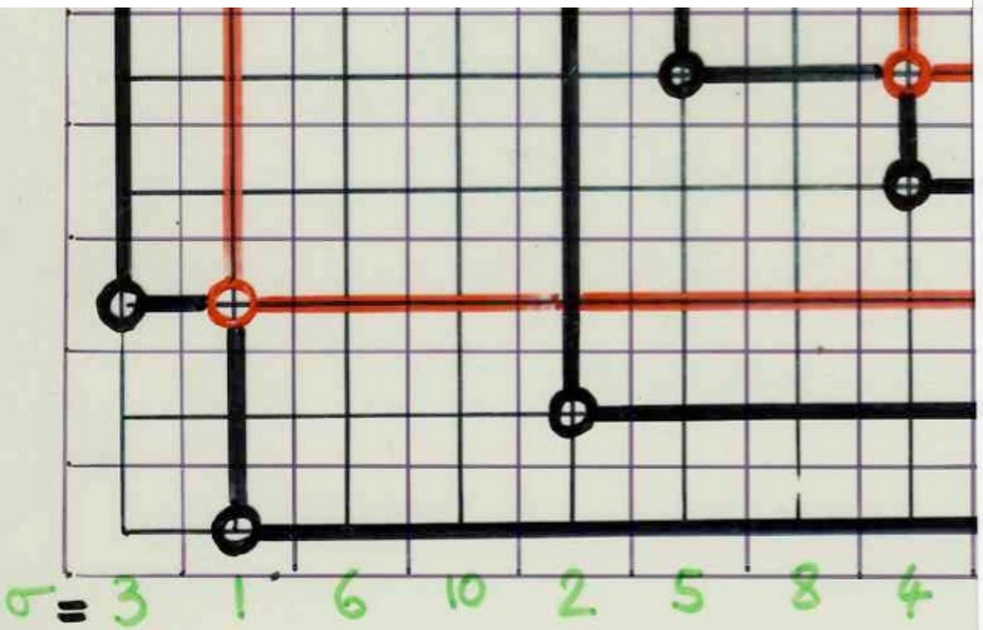
$$\genfrac{}{}{0pt}{}{\mu}{\nu} = \rho .$$

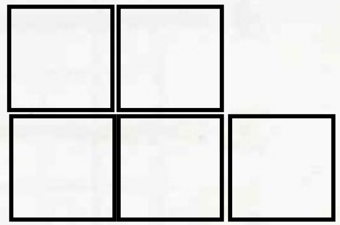


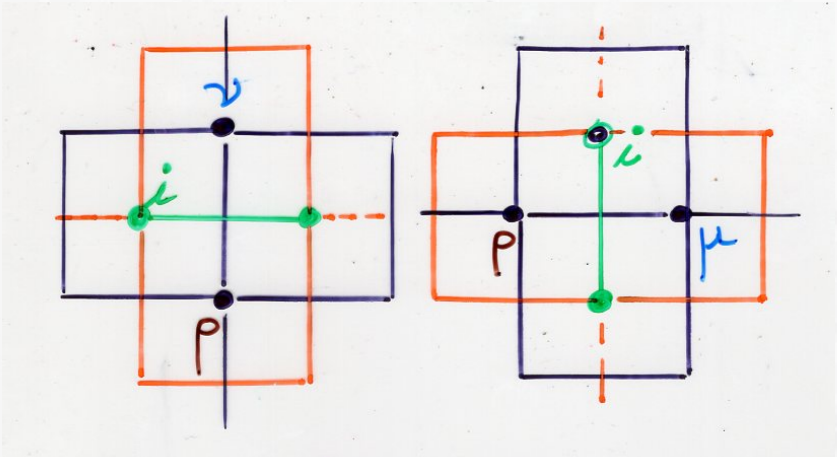
if a shadow line with label i is crossing, then

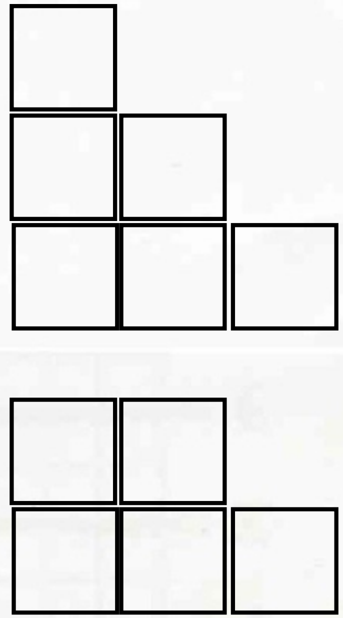
$$\genfrac{}{}{0pt}{}{\mu}{\nu} = \rho + (i)$$

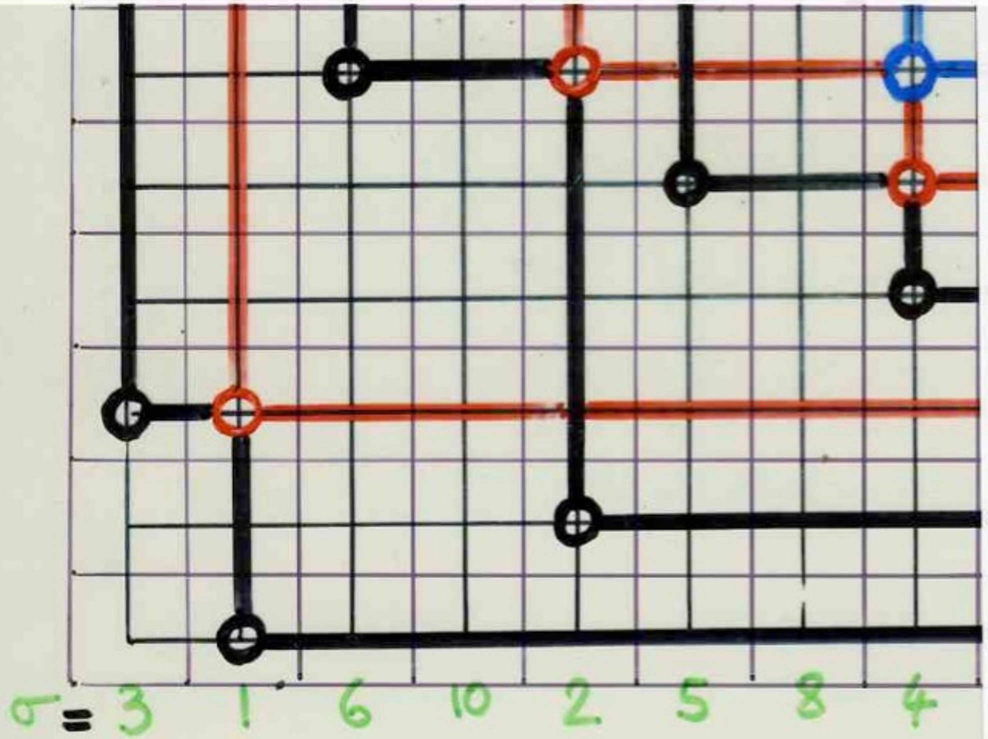

ν
i
ρ
i
ρ
μ


σ = 3 1 6 10 2 5 8 4

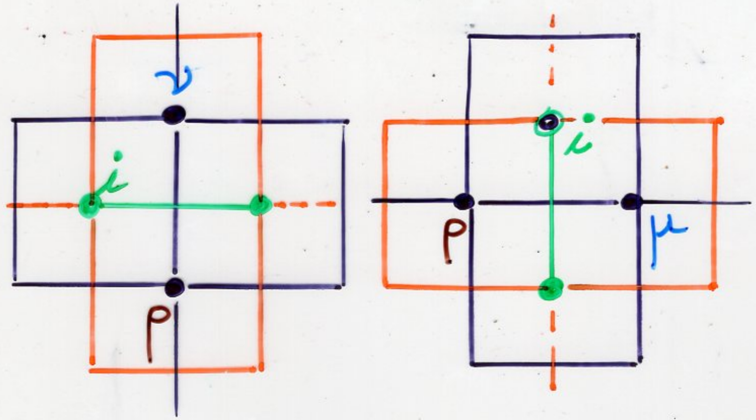


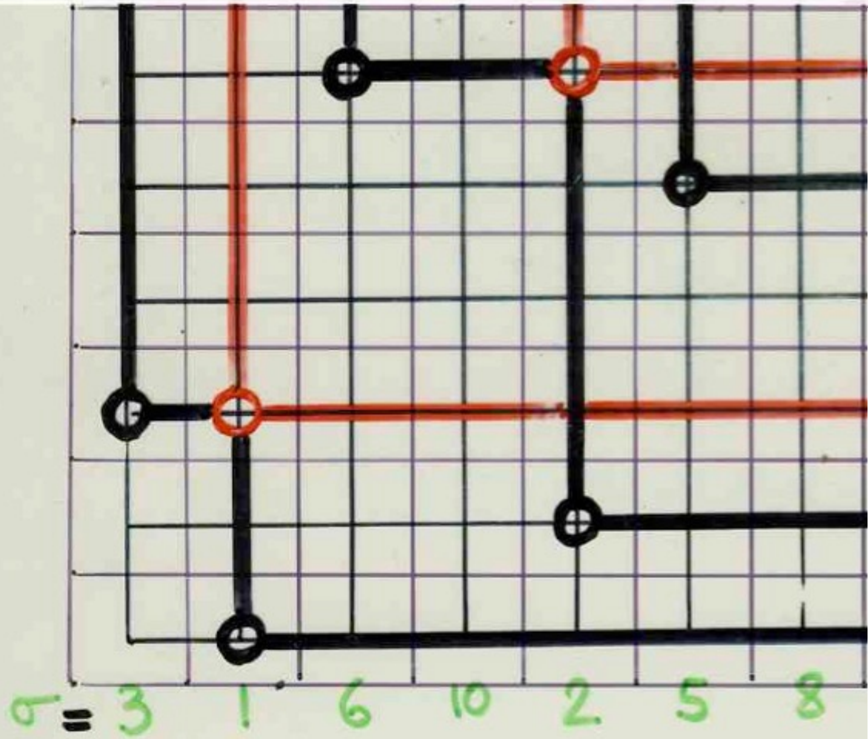


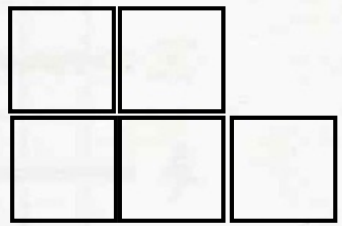




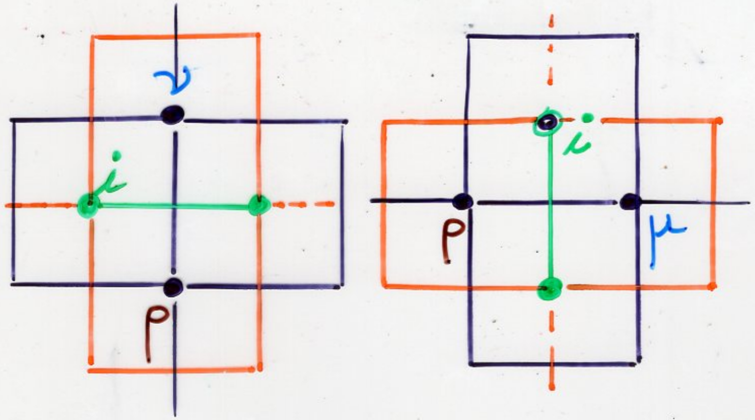
$$\substack{\mu \\ \nu} = \rho + (i)$$

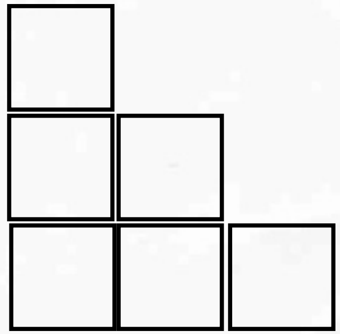


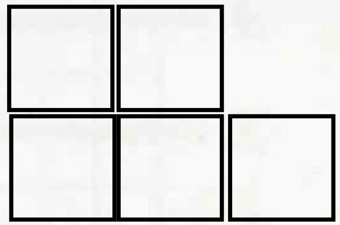


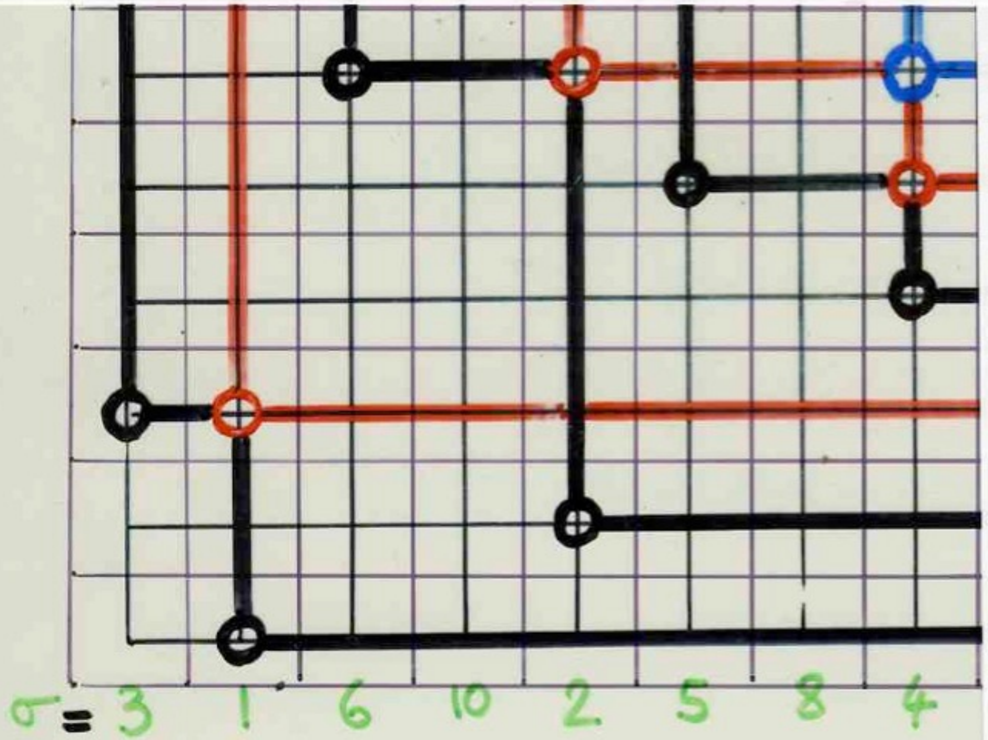


$$\mu / \nu = \rho + (i)$$

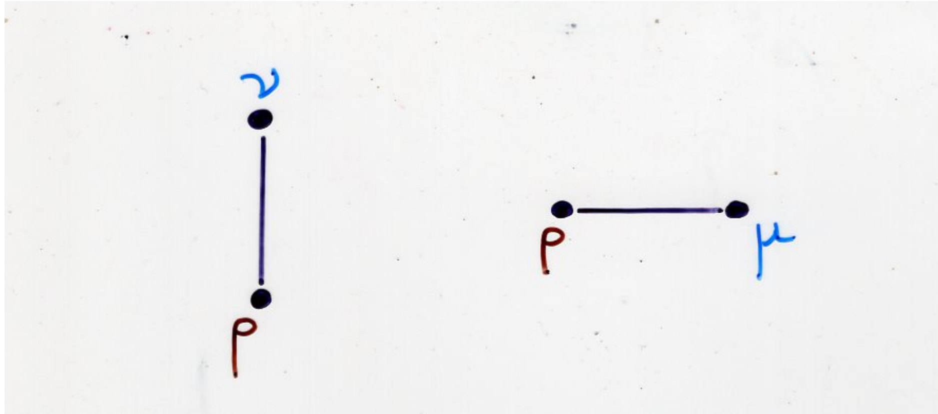


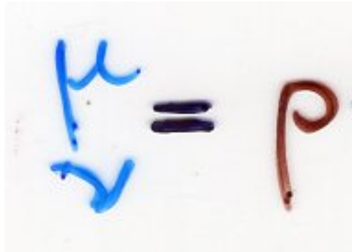


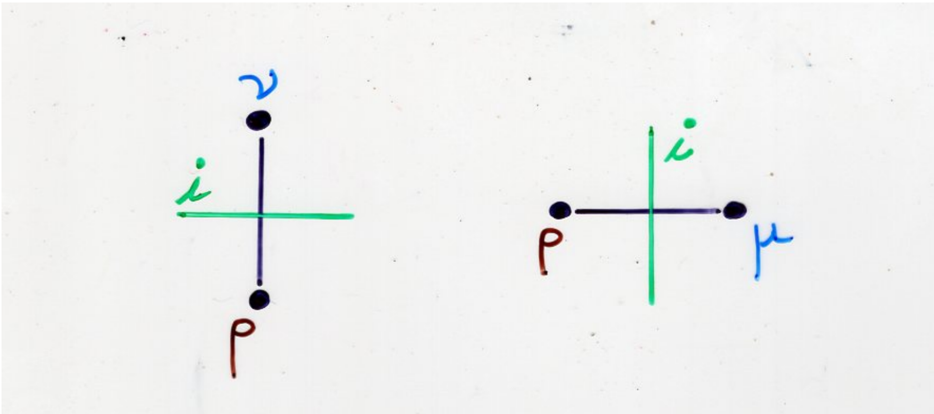


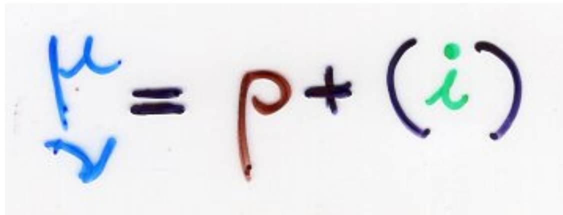


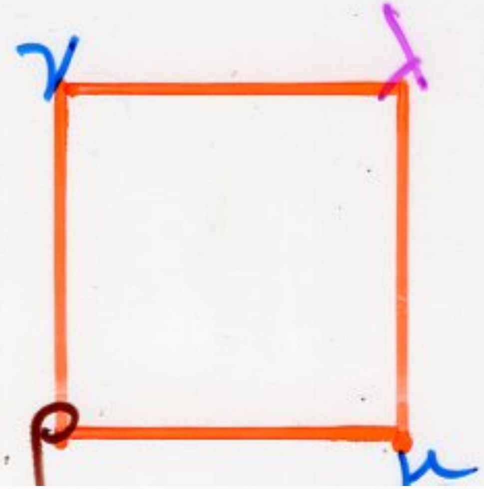
$$\frac{\mu}{\nu} = \rho + (i)$$


ν
ρ
ρ
μ

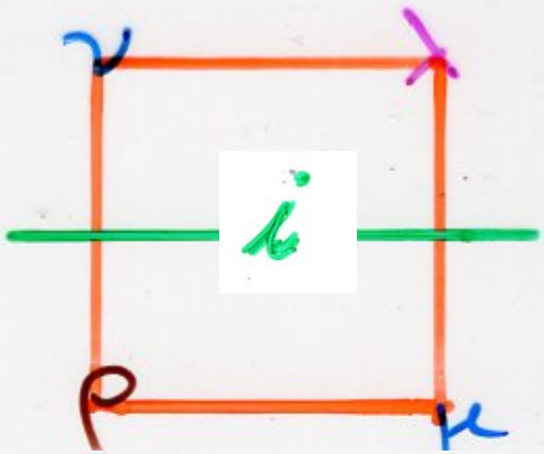

μ
ν
= ρ


ν
i
ρ
i
ρ
μ


μ
ν
= ρ + (i)

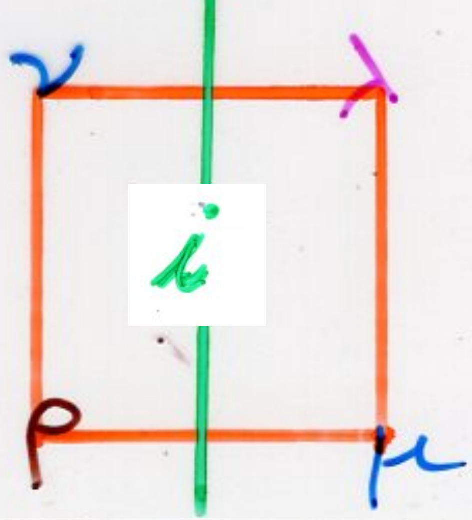


$\lambda = \rho = \mu = \nu$



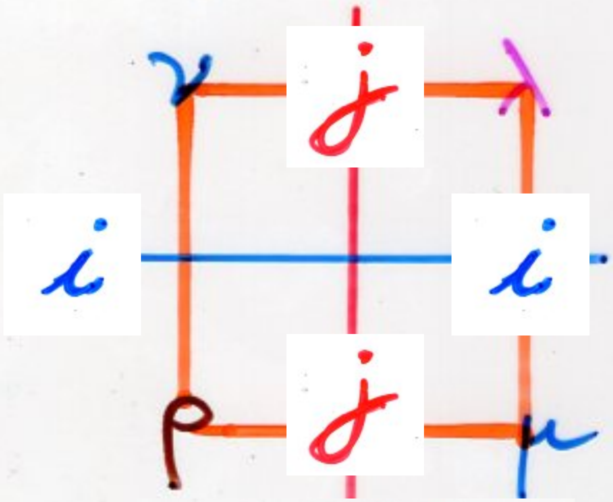
$\rho = \mu$

$\lambda = \nu = \rho + (i)$



$\rho = \nu$

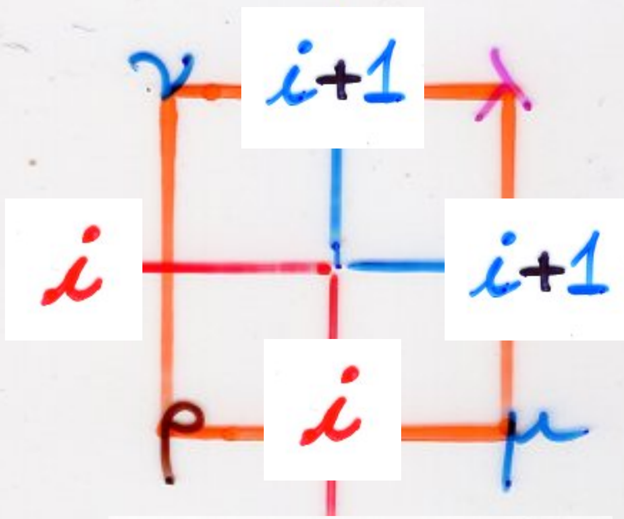
$\lambda = \mu = \rho + (j)$



$\nu = \rho + (i)$

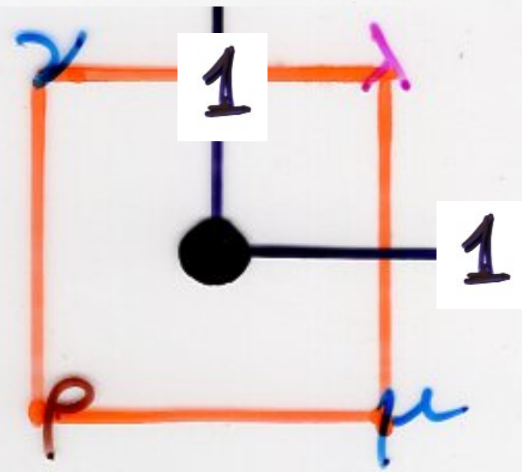
$\mu = \rho + (j)$

$\lambda = \rho + (i) + (j)$

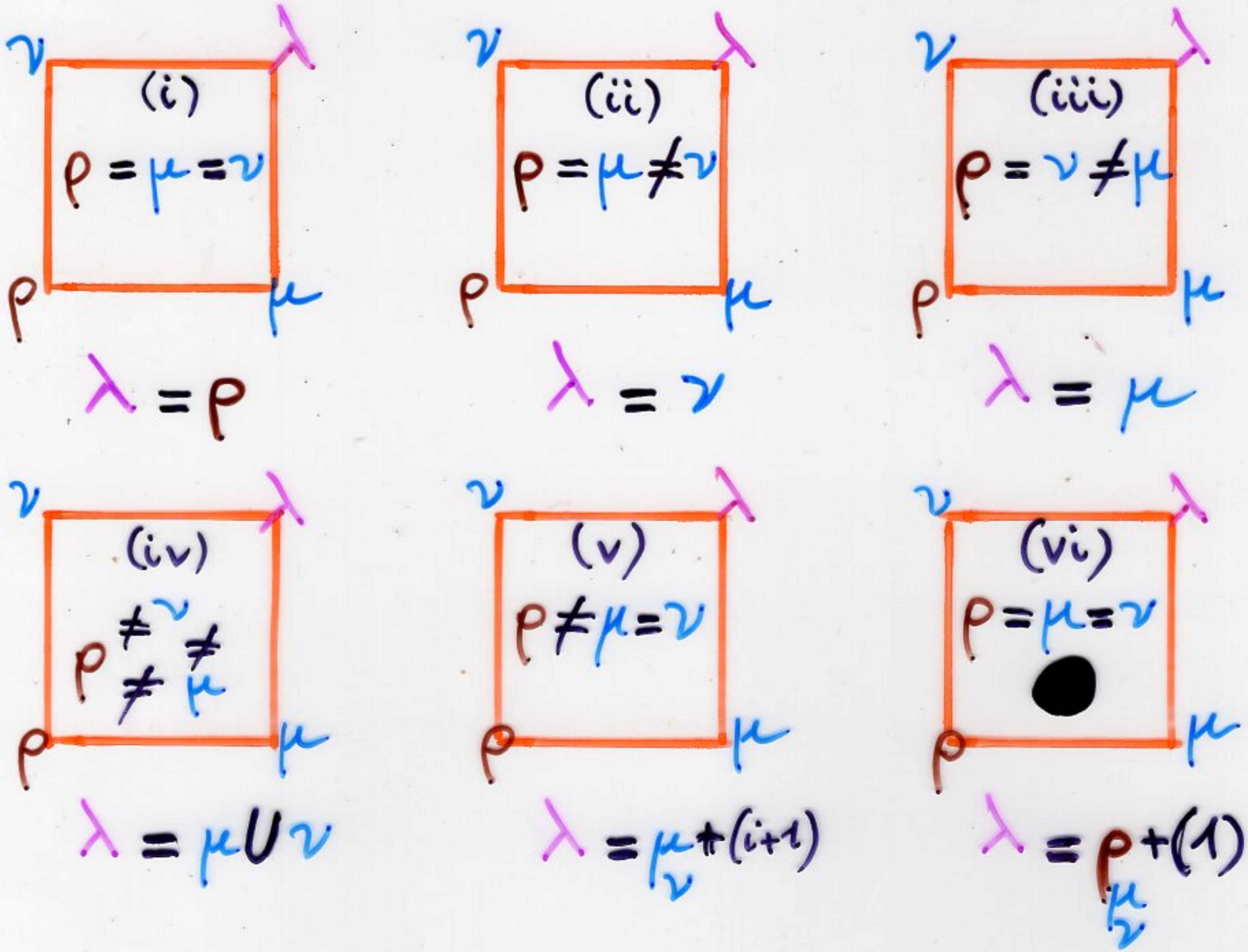


$\mu = \nu = \rho + (i)$

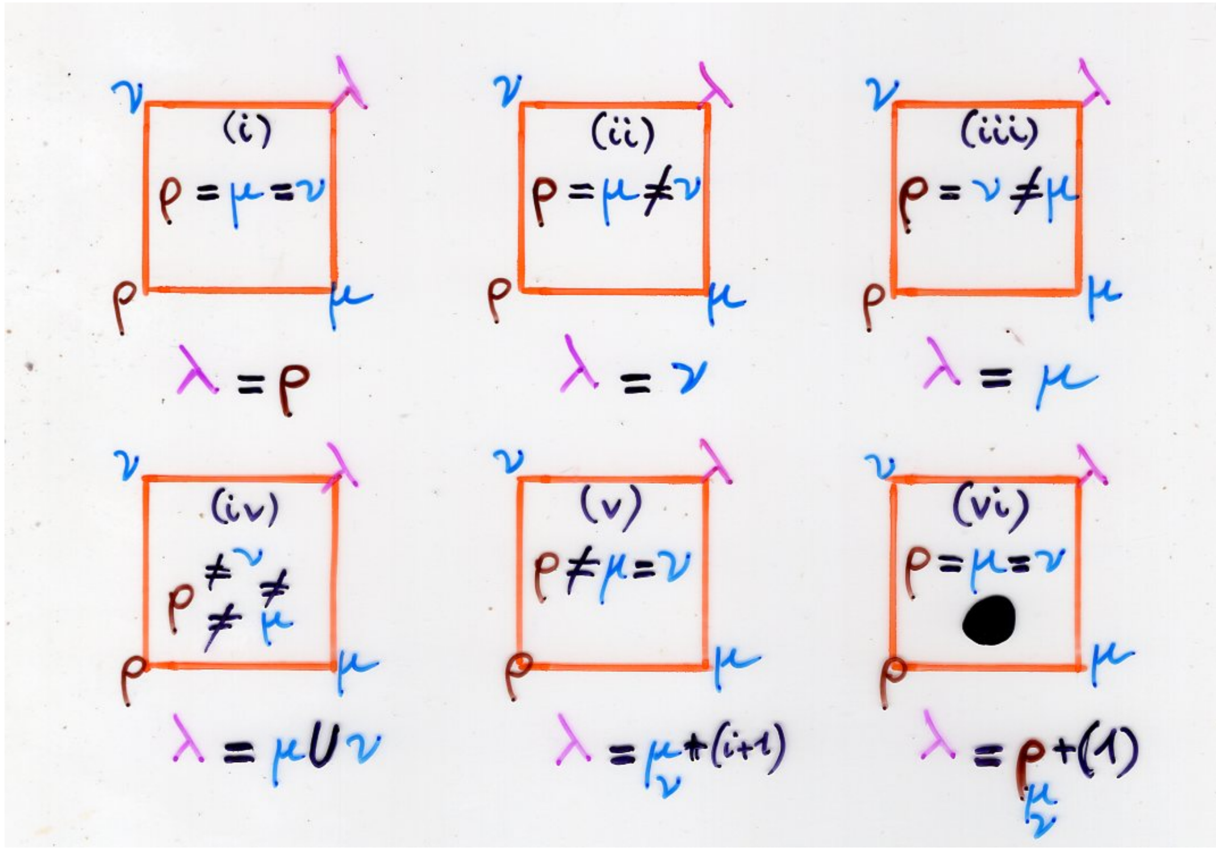
$\lambda = \mu + (i+1)$



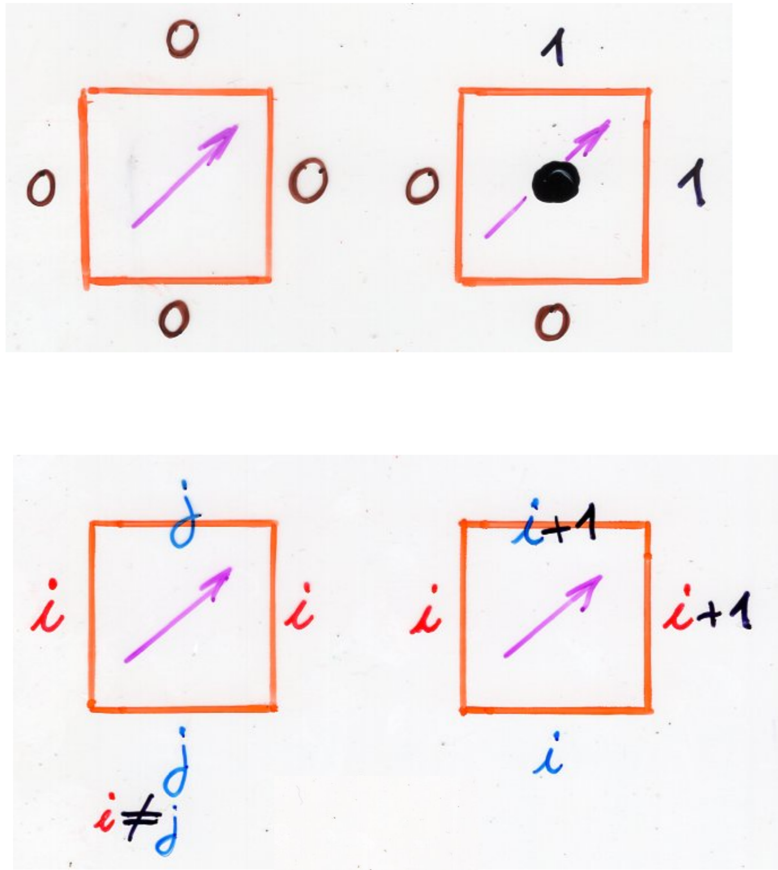
$\lambda = \left\{ \begin{matrix} \rho \\ \mu \\ \nu \end{matrix} \right. + (1)$


(i)
ρ = μ = ν
λ = ρ
(ii)
ρ = μ ≠ ν
λ = ν
(iii)
ρ = ν ≠ μ
λ = μ
(iv)
λ = μ ∪ ν
(v)
ρ ≠ μ = ν
λ = μ + (i+1)
(vi)
ρ = μ = ν
λ = ρ + (1)

"local rules" on the vertices



"local rules" on the edges



« local rules on vertices »

Marc A. A. van Leeuwen (1996)

The Robinson-Schensted and Schützenberger algorithms, an elementary approach

C.Krattentheler, (2006).

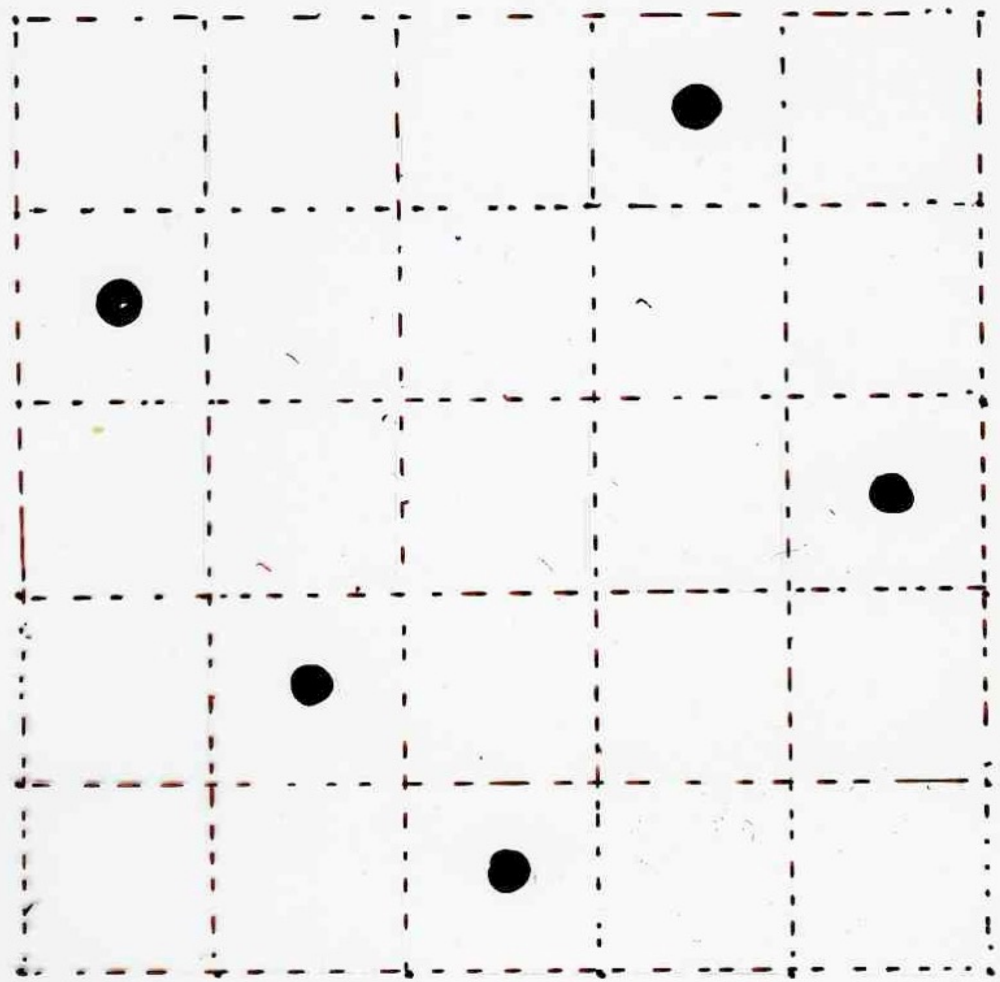
GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

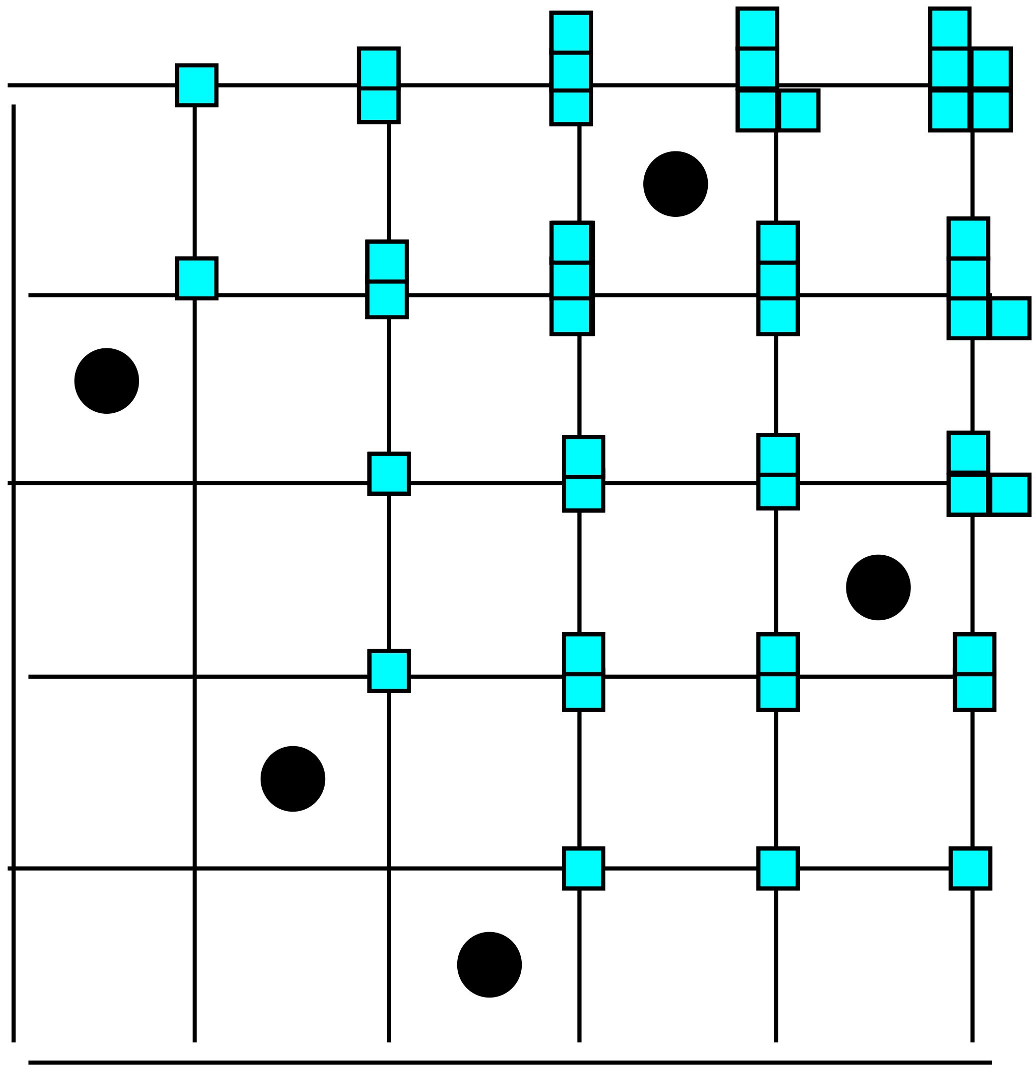
M.Rubey. (2007)

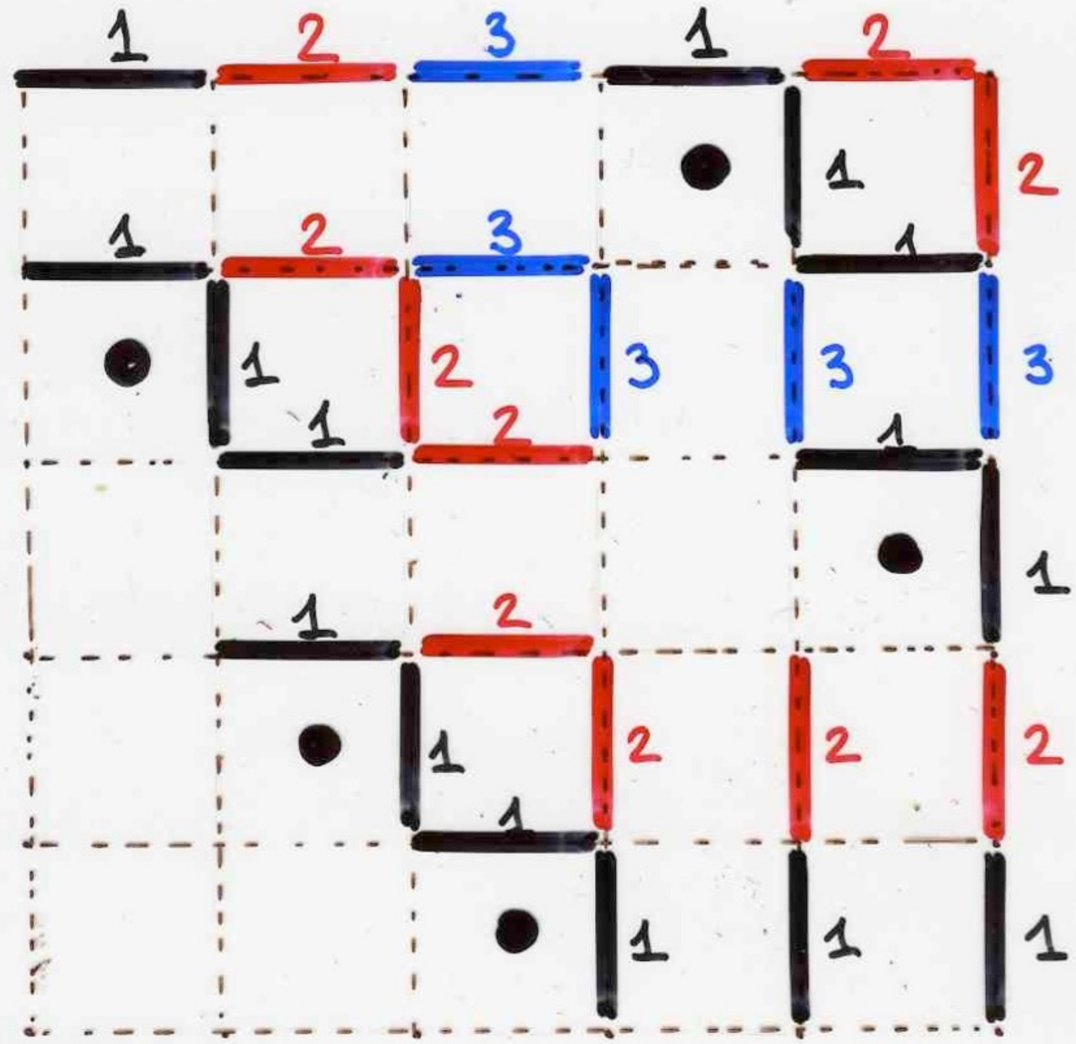
Increasing and Decreasing Sequences in Fillings of Moon Polyominoes

I claim that much attention should be given to the « local rules on edges » rather that « local rules on vertices ».

This is part of the philosophy of the « cellular ansatz »







1
2
3
1
2
1
2
1
2
3
1
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2
3
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3
1
1
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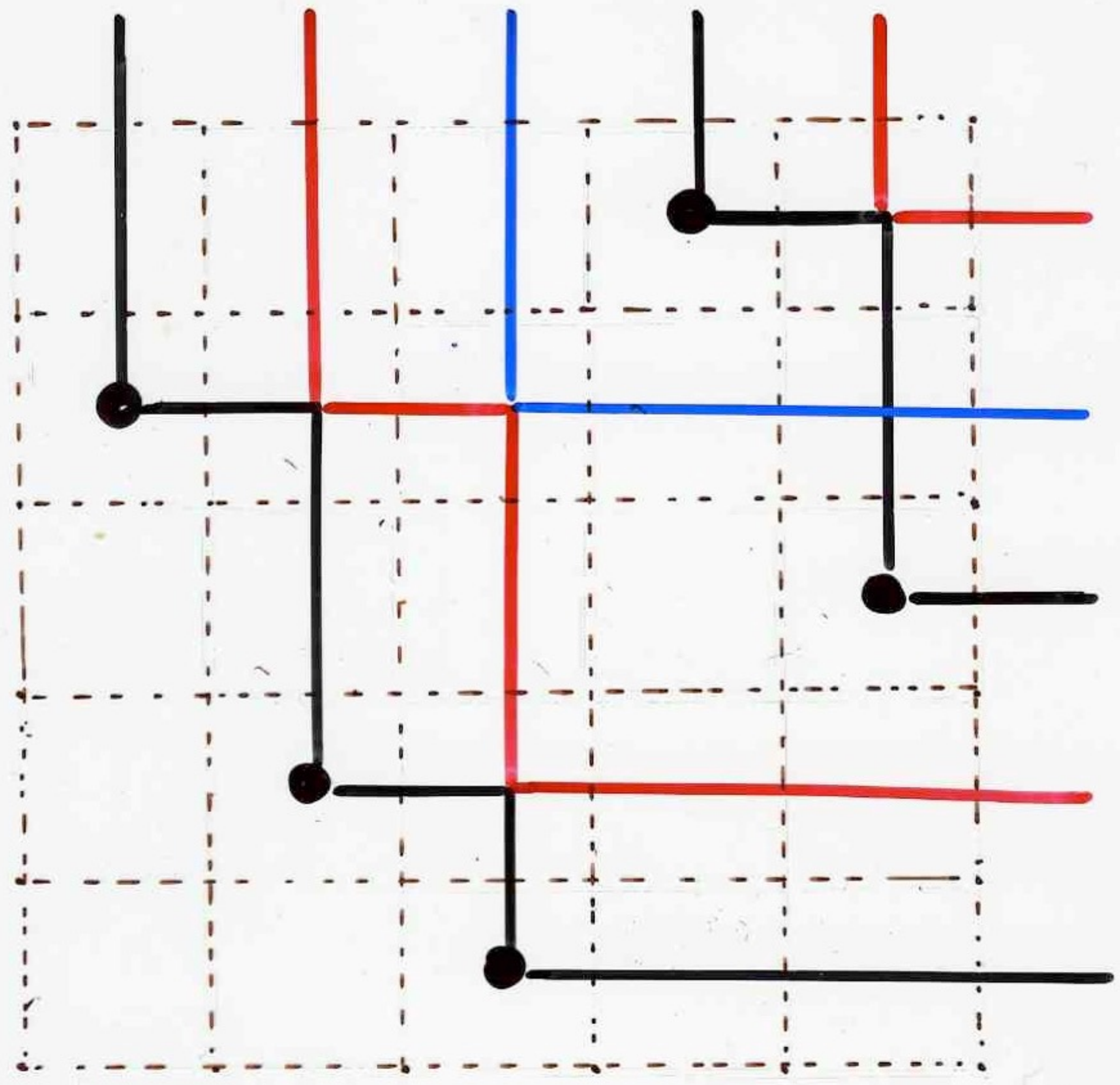
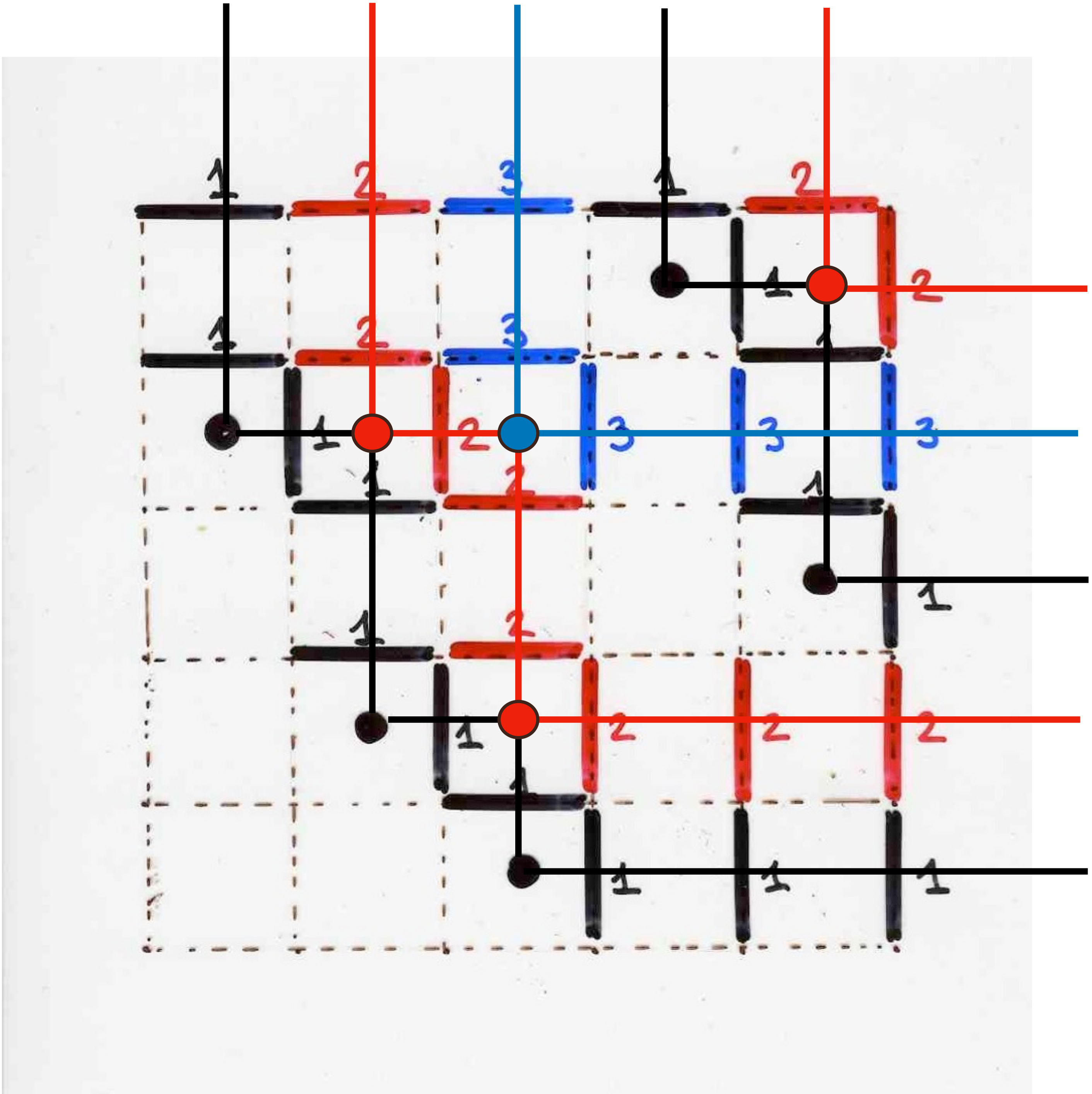
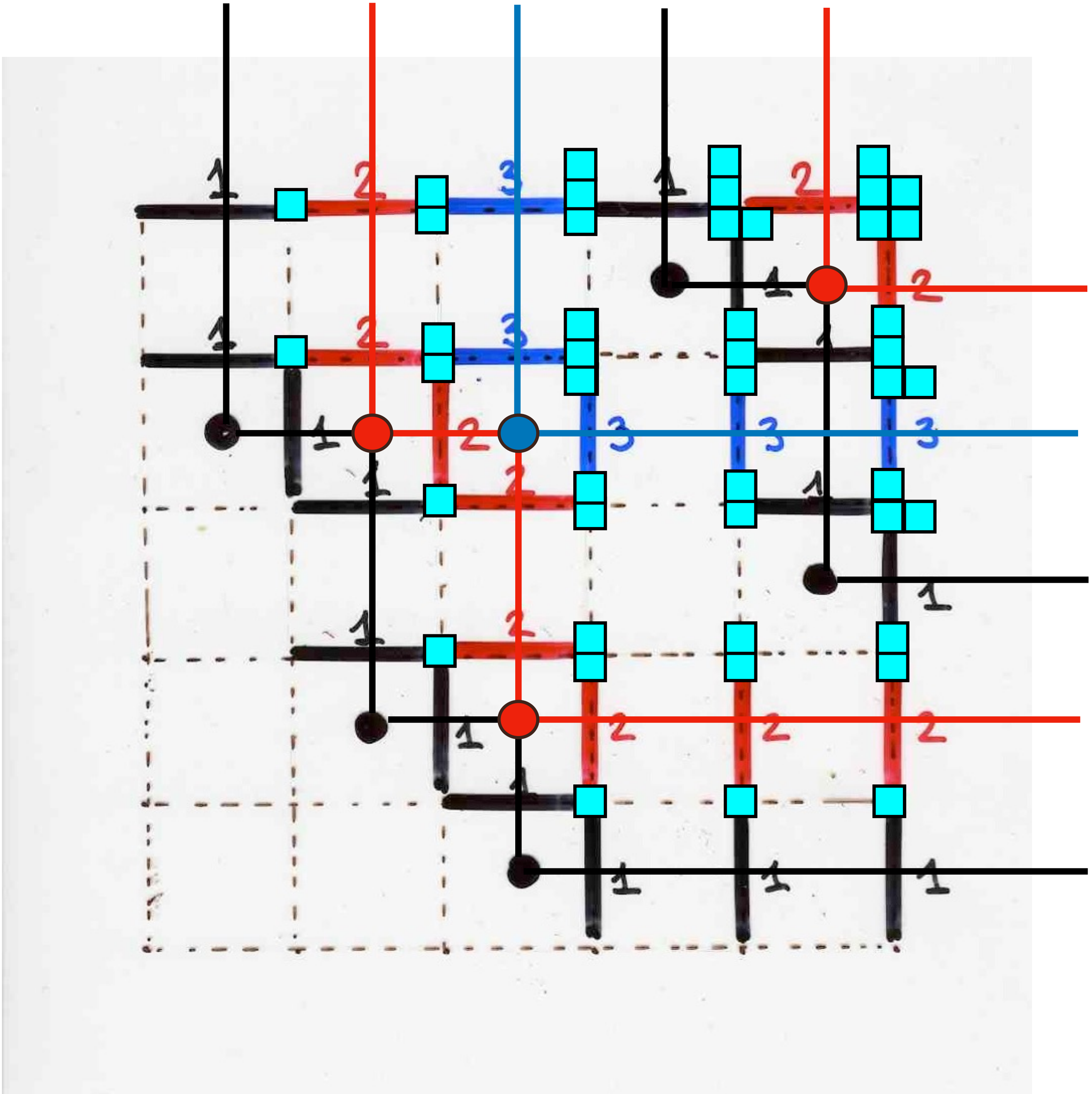
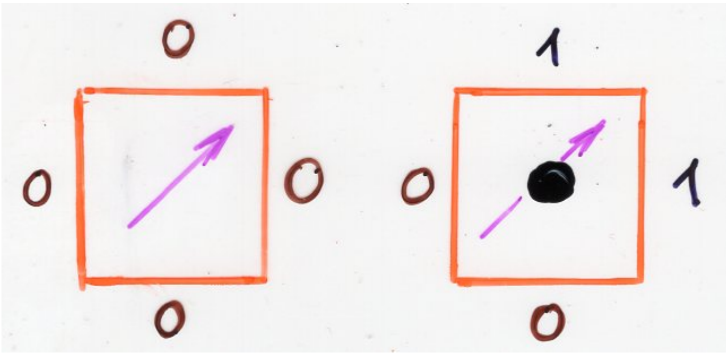
2
1
1
1
1

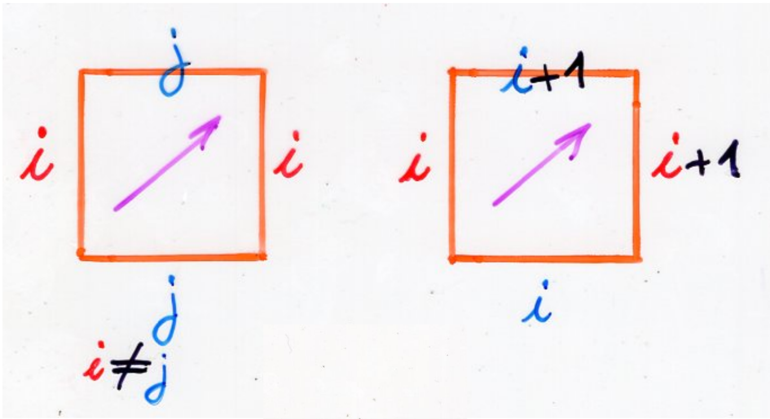






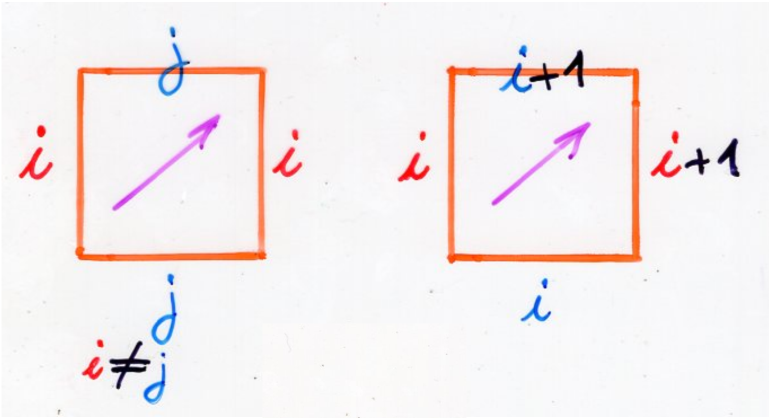
The RSK bilateral edge local rules





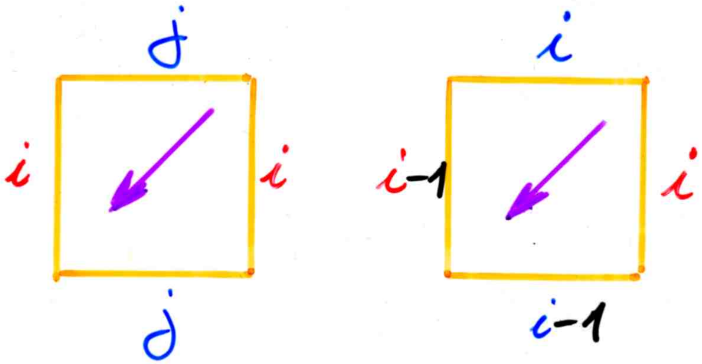
"local rules" on the edges

$$i, j \geq 0$$



"local rules" on the edges

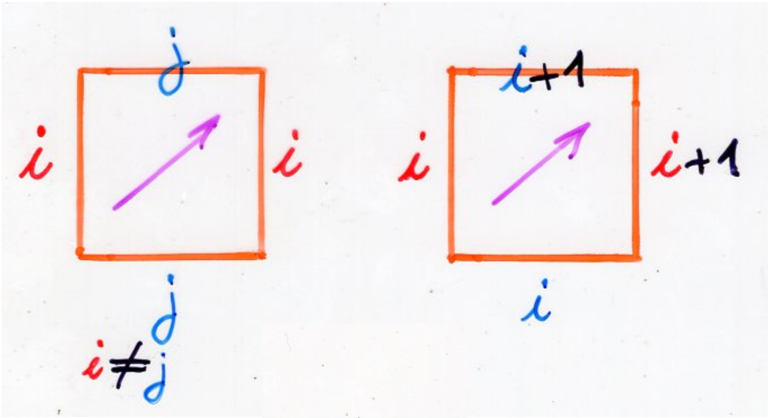
$$i, j \geq 0$$

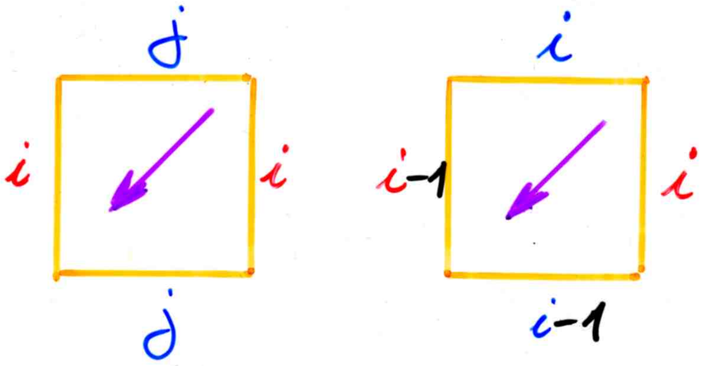


$i \neq j$

$i, j \in \mathbb{Z} - \{0\}$

bilateral local rules on edges

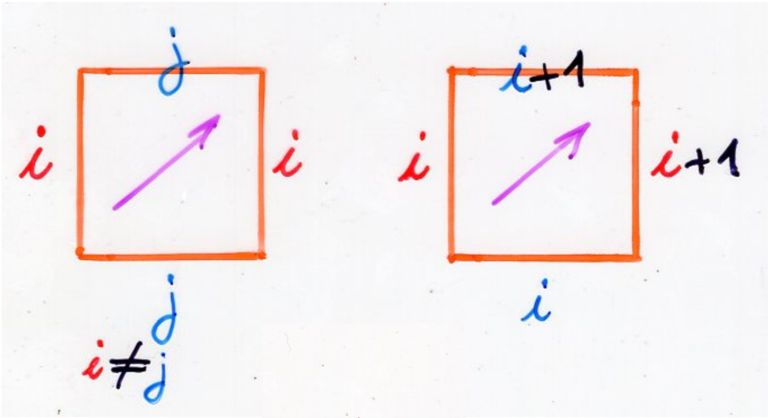


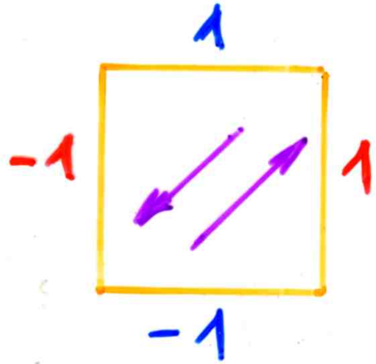


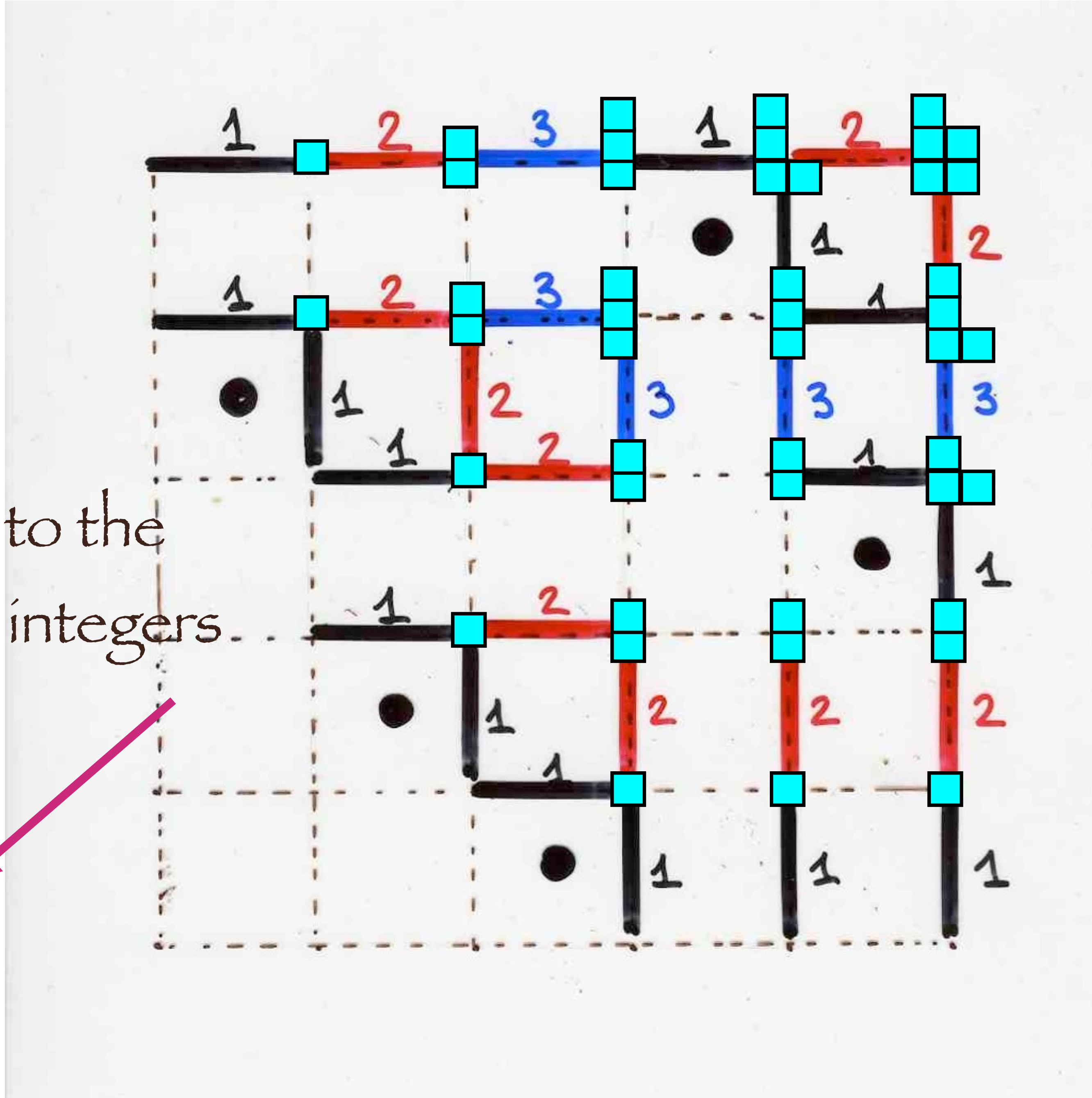
$i \neq j$

bilateral local rules on edges

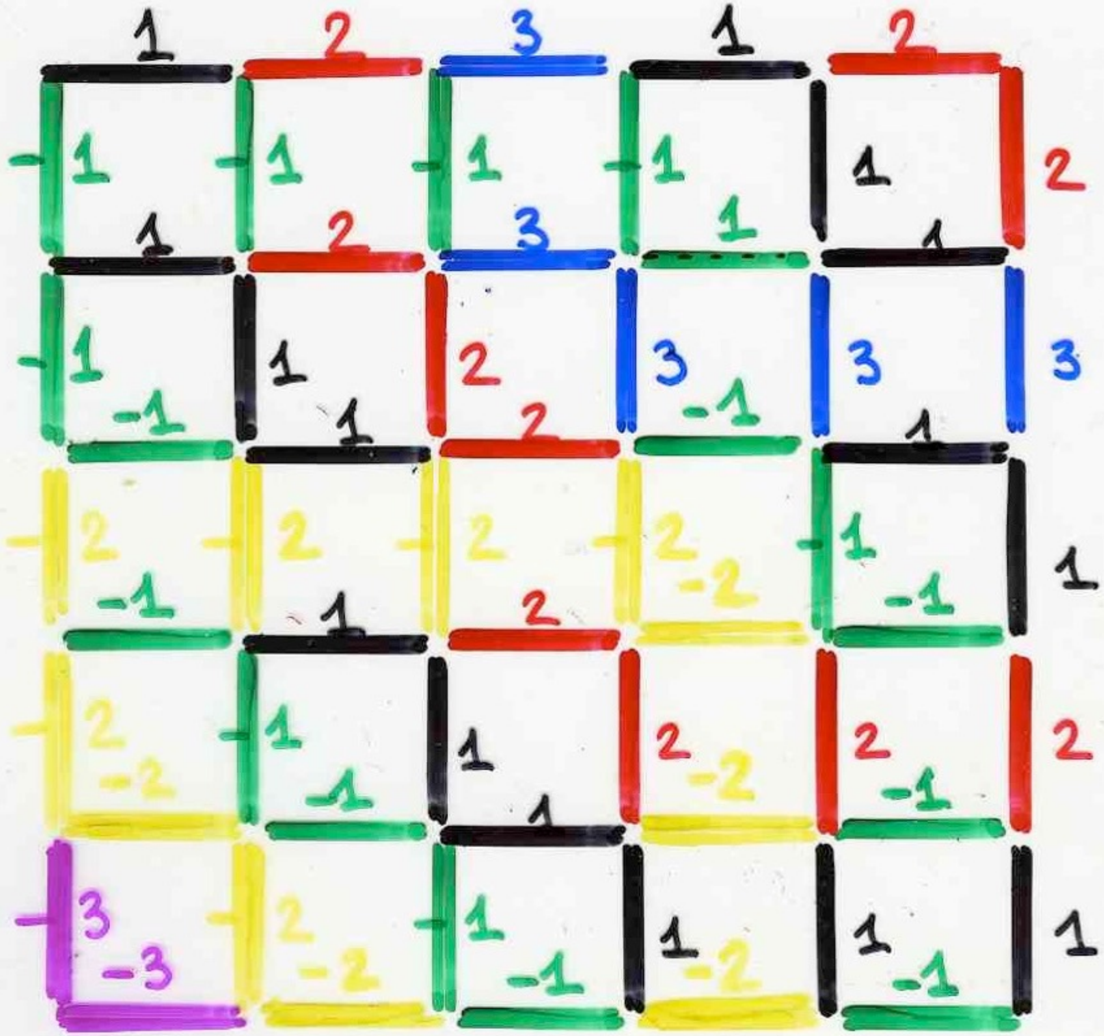
$i, j \in \mathbb{Z} - \{0\}$



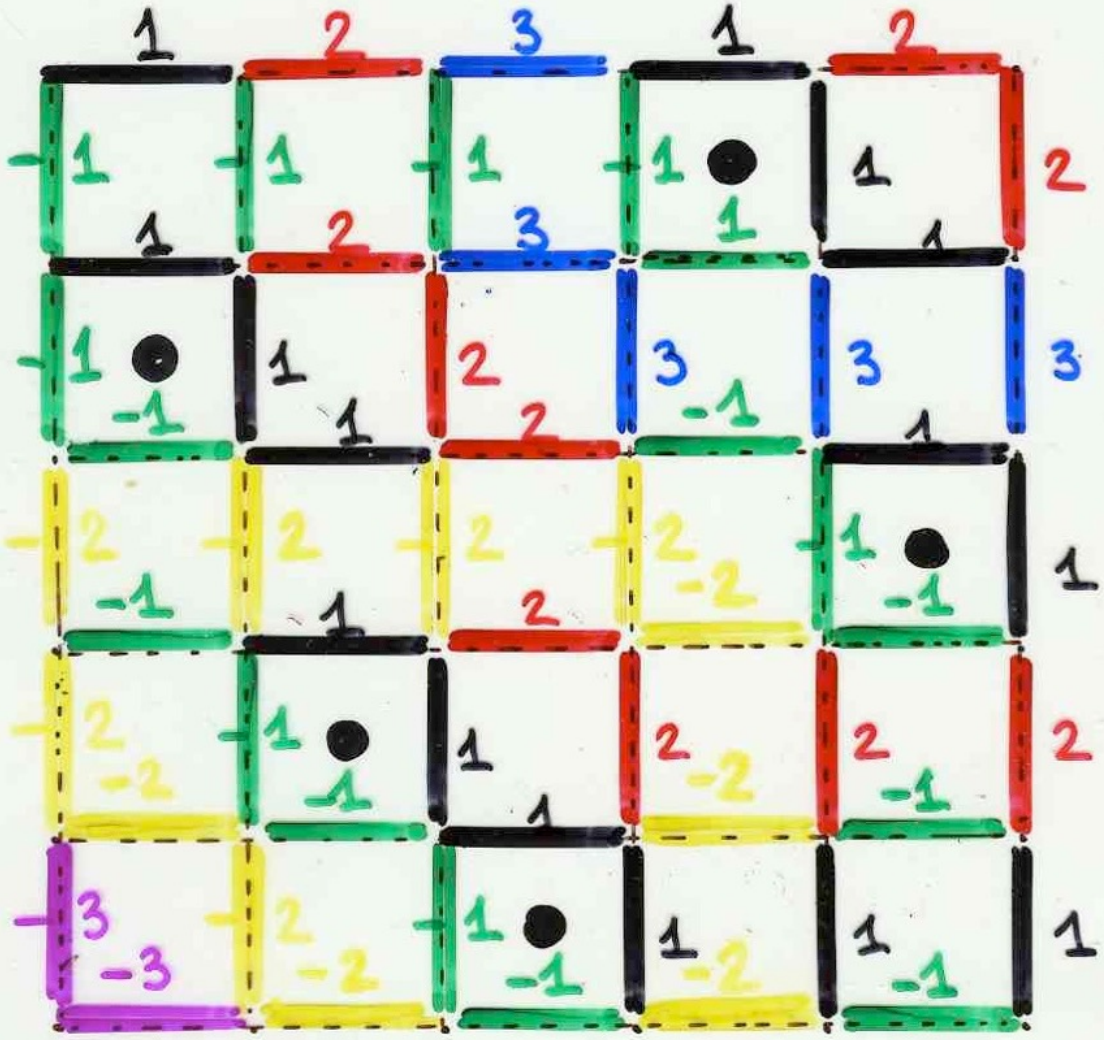




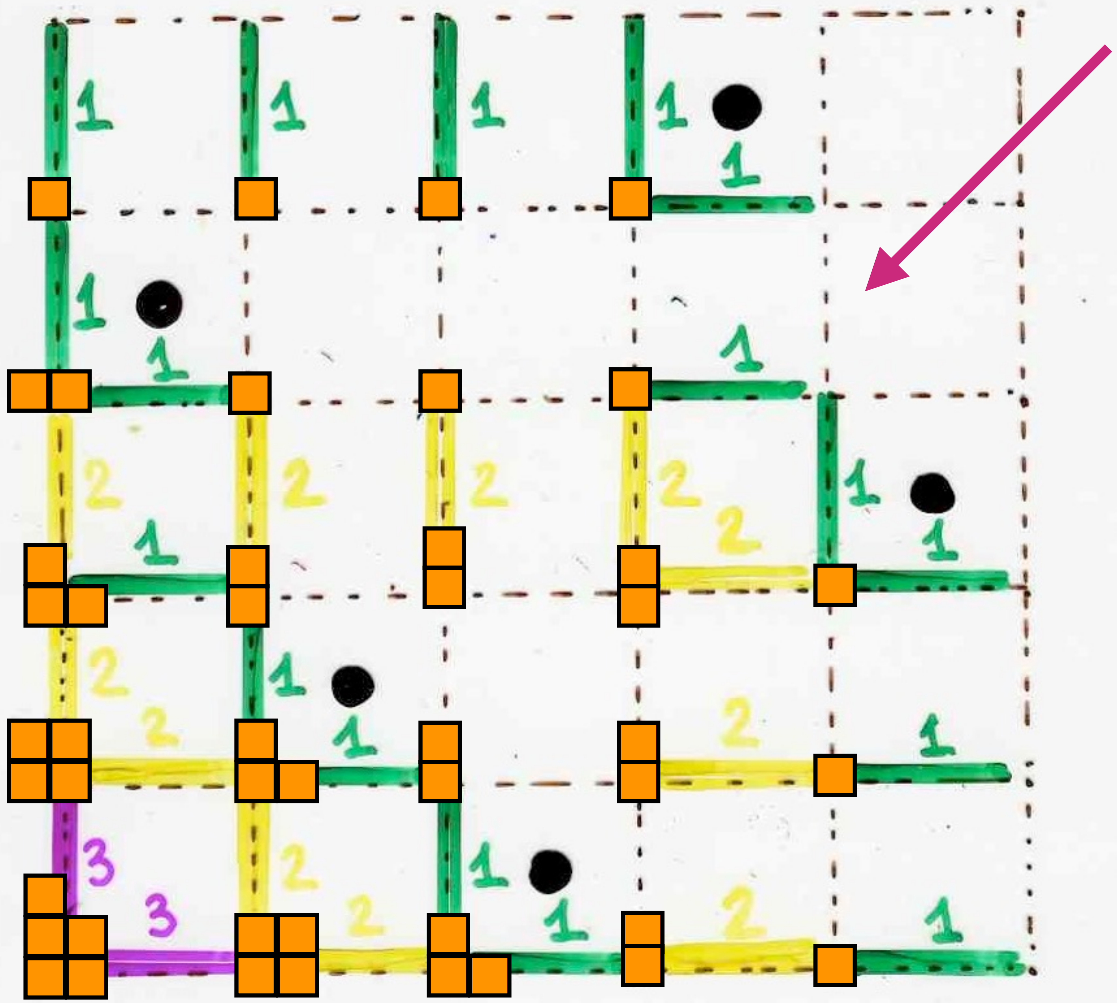
Going to the negative integers

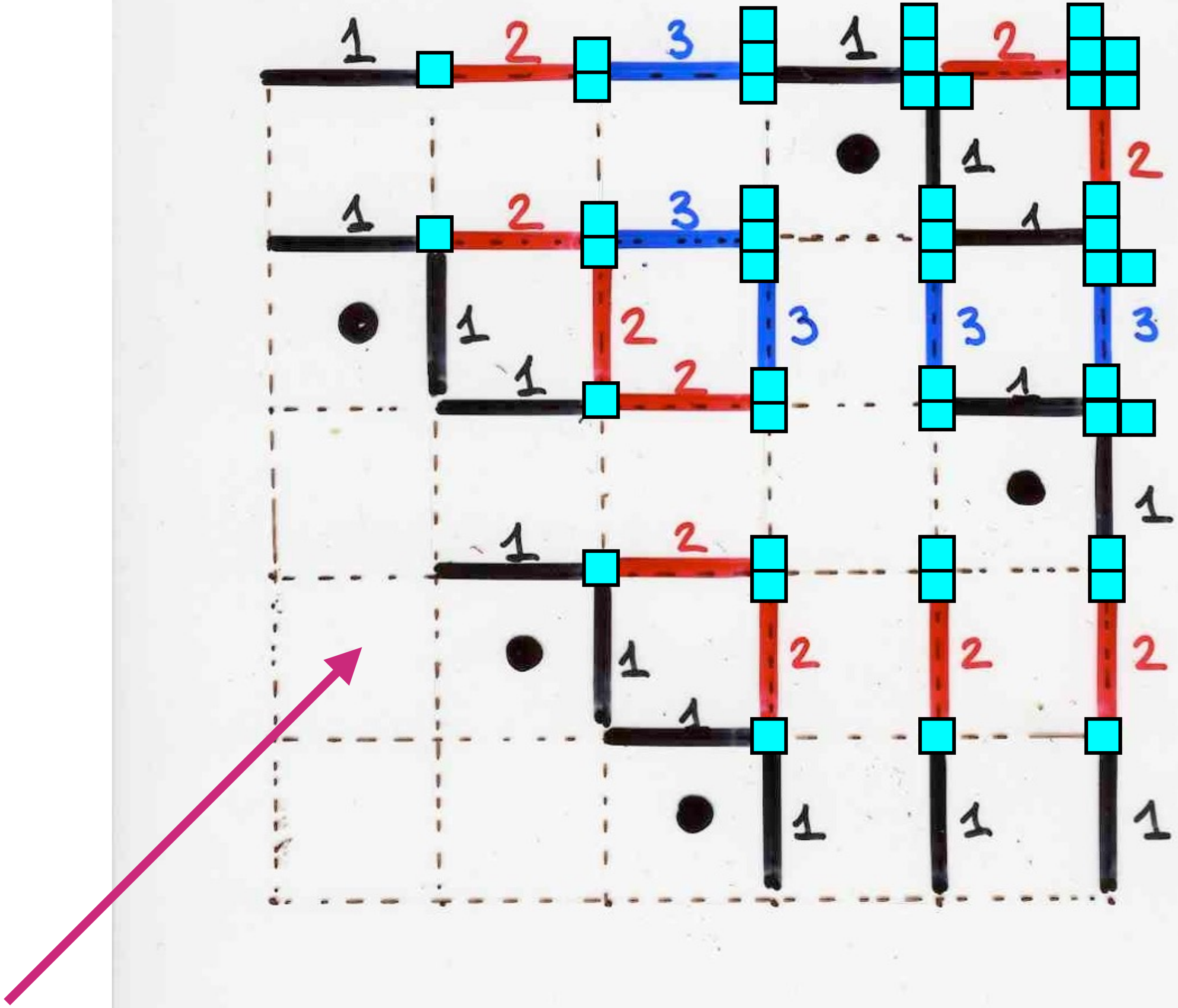


1 2 3 1 2
-1 -1 -1 -1 1 2
1 2 3 1 1
-1 1 2 3 3 3
-1 1 2 -1 1
-2 -2 -2 -2 -1
-1 -2 -1 1
1 2
-2 -1 1 2 2 2
-2 -1 -2 -1
1
-3 -2 -1 1 1 1
-3 -2 -1 -2 -1

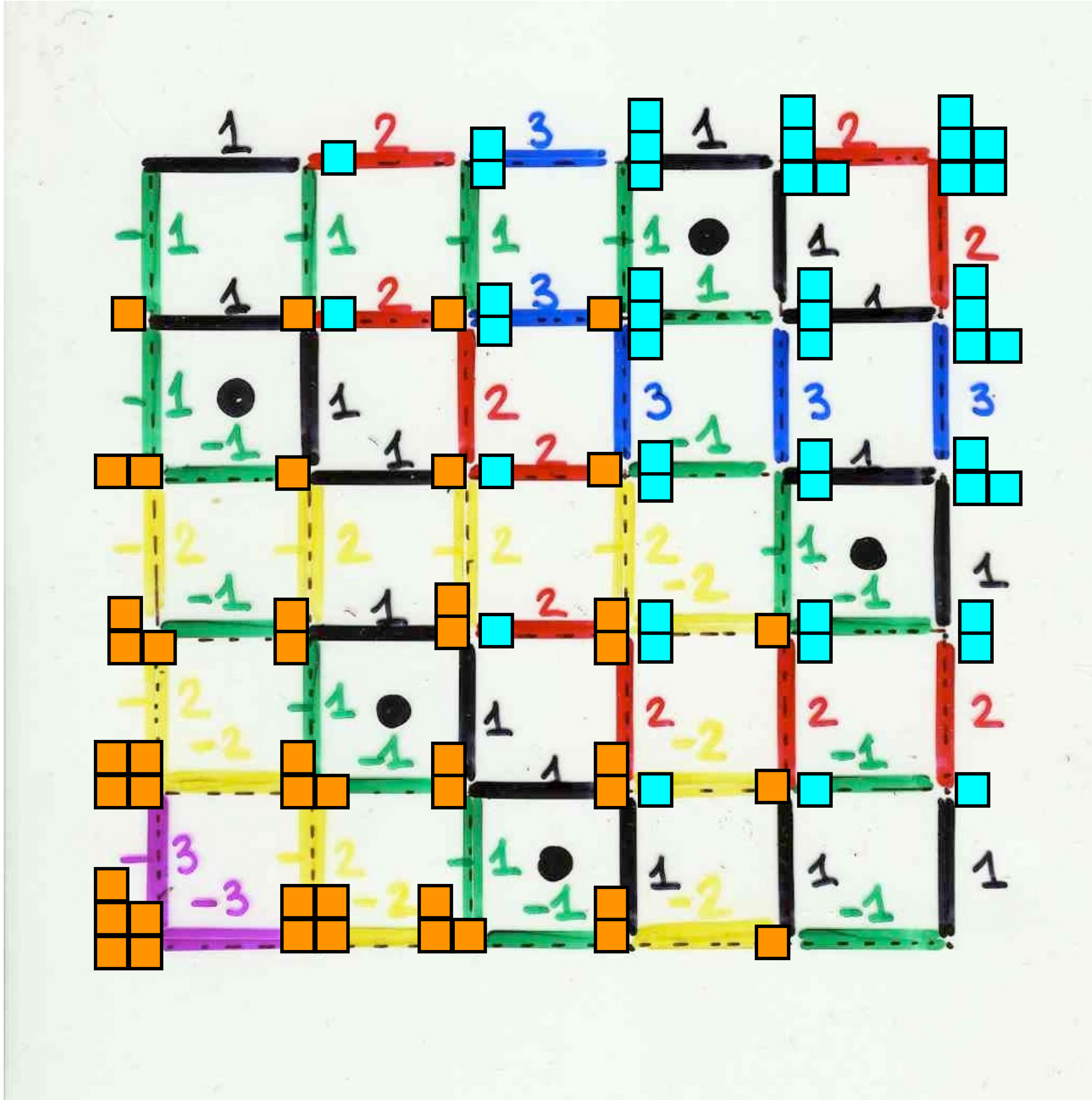


1 2 3 1 2
-1 -1 -1 -1 1 1 2
1 2 3 1 1
-1 1 -1 1 2 3 -1 3 3
1 2 1
-2 -1 -2 -2 -2 -2 -1 1 -1 1
1 2
-2 2 -2 -1 1 -1 1 -2 2 -2 2 -1 2
1
-3 3 -3 -2 2 -2 -1 1 -1 1 -2 1 -1 1

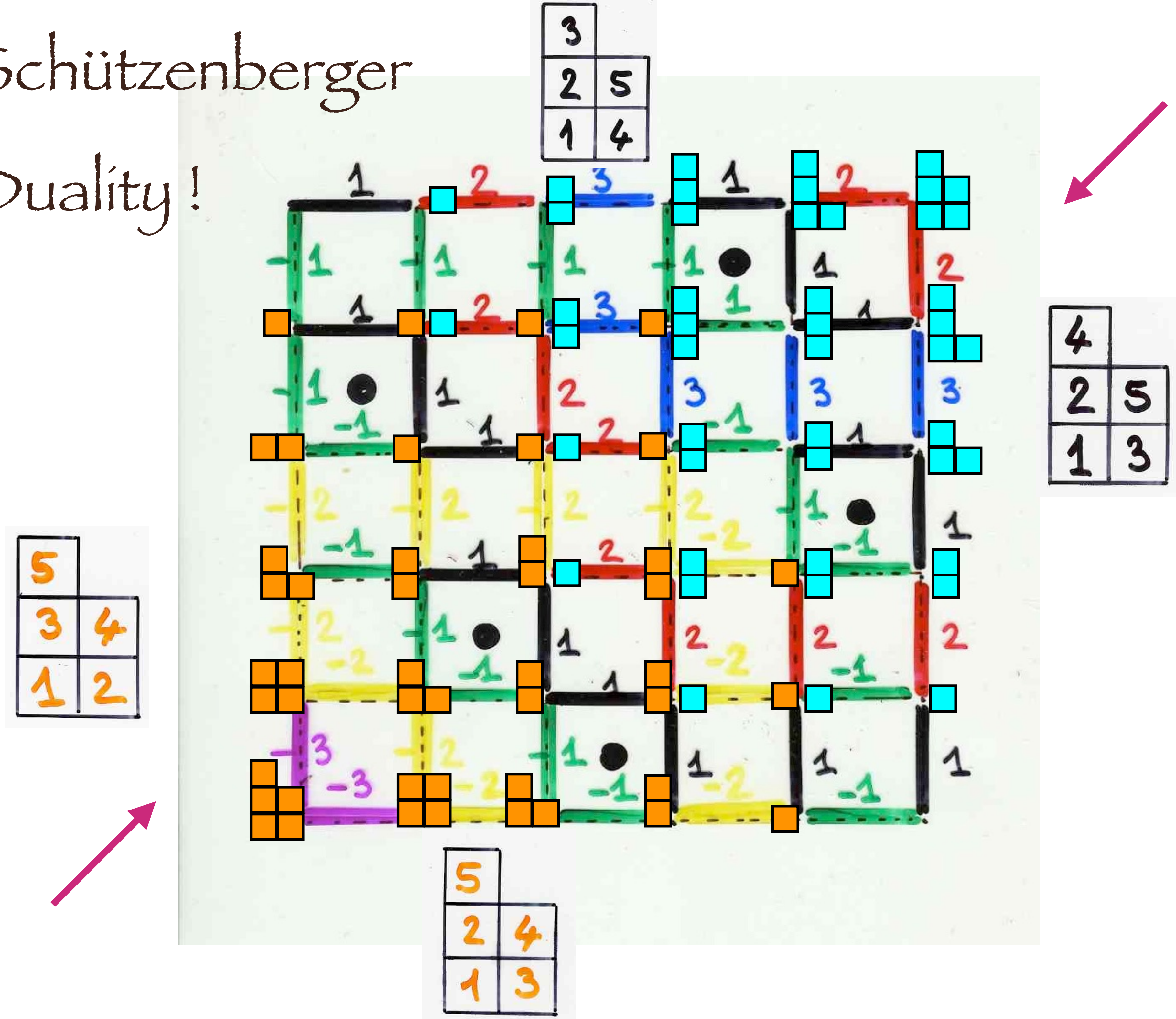




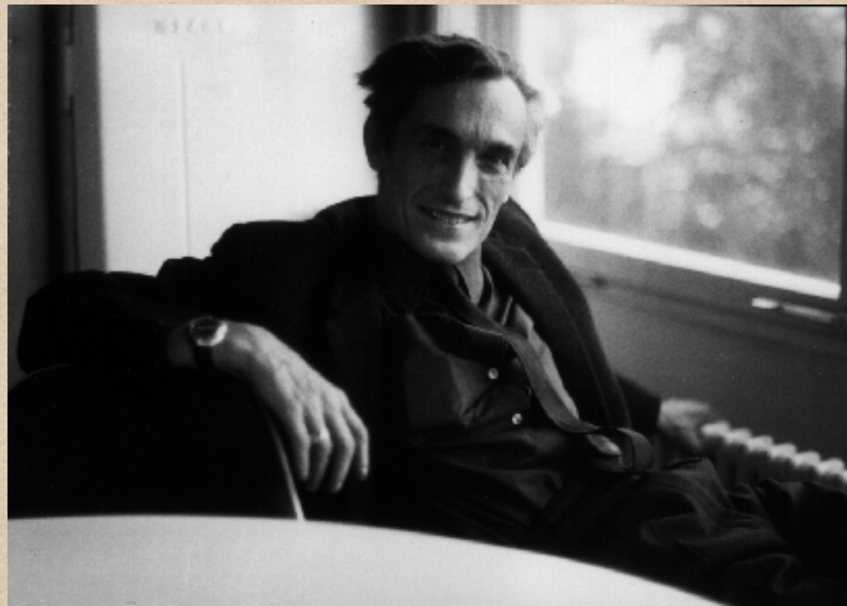
1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1



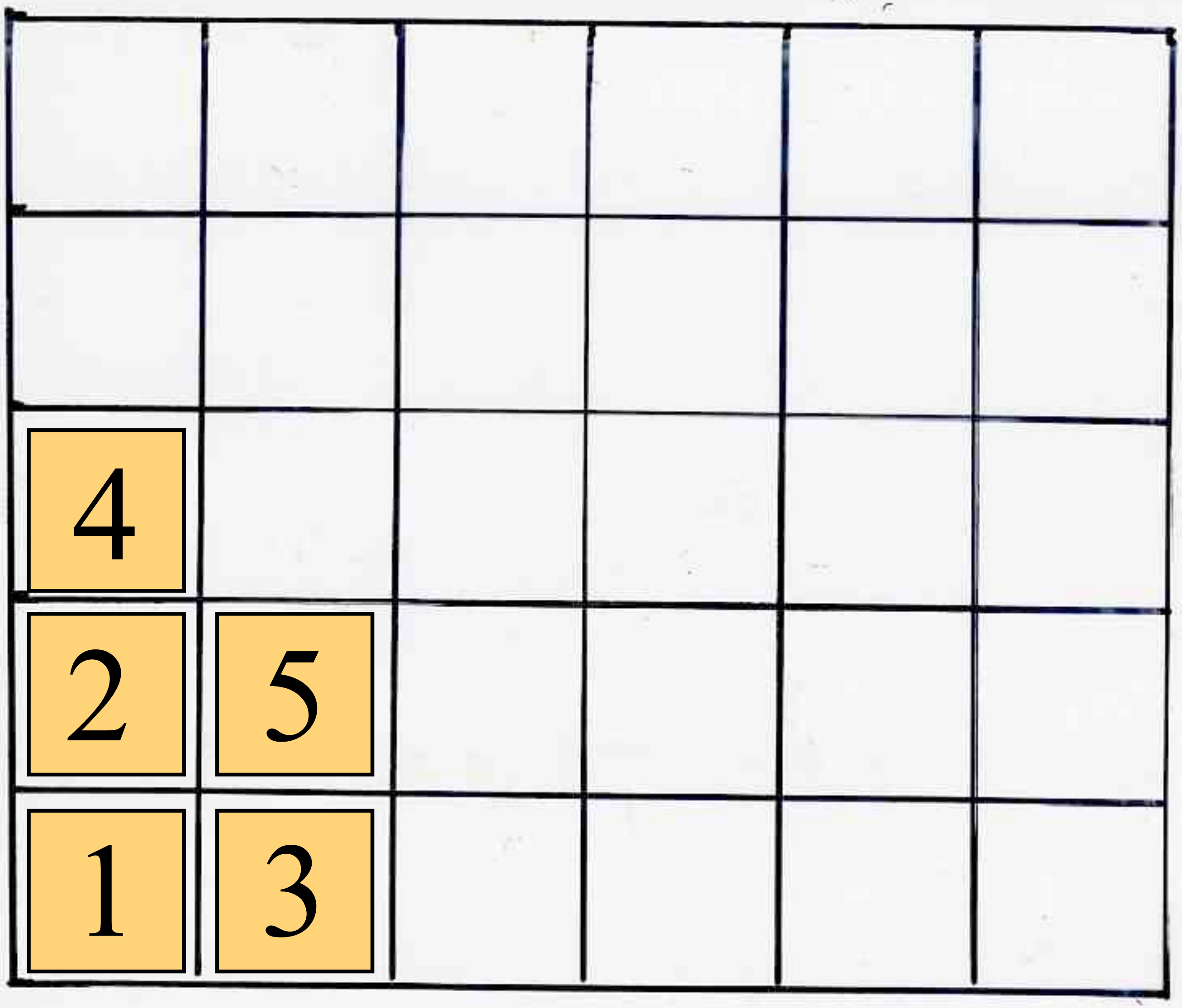
Schützenberger Duality !

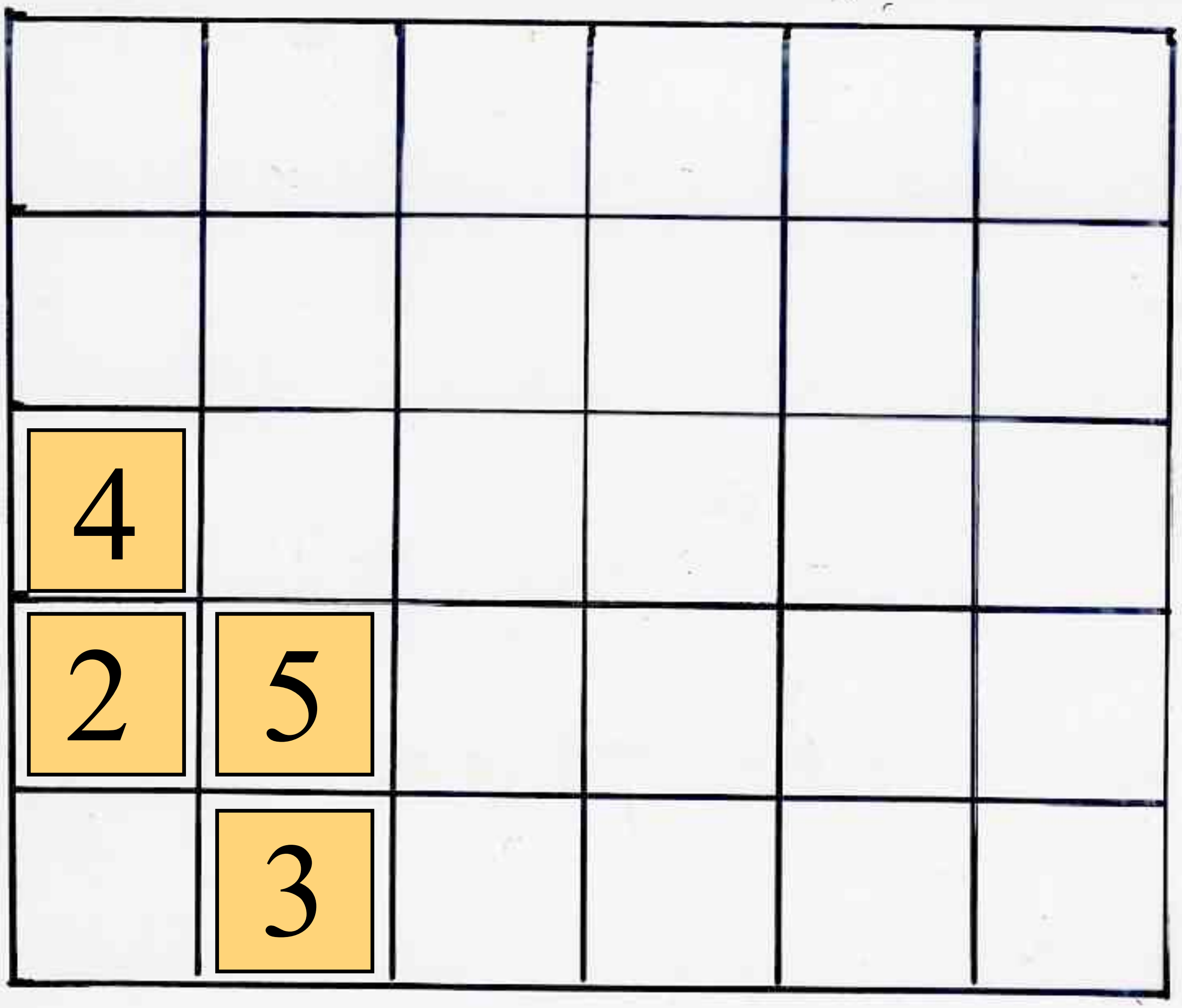


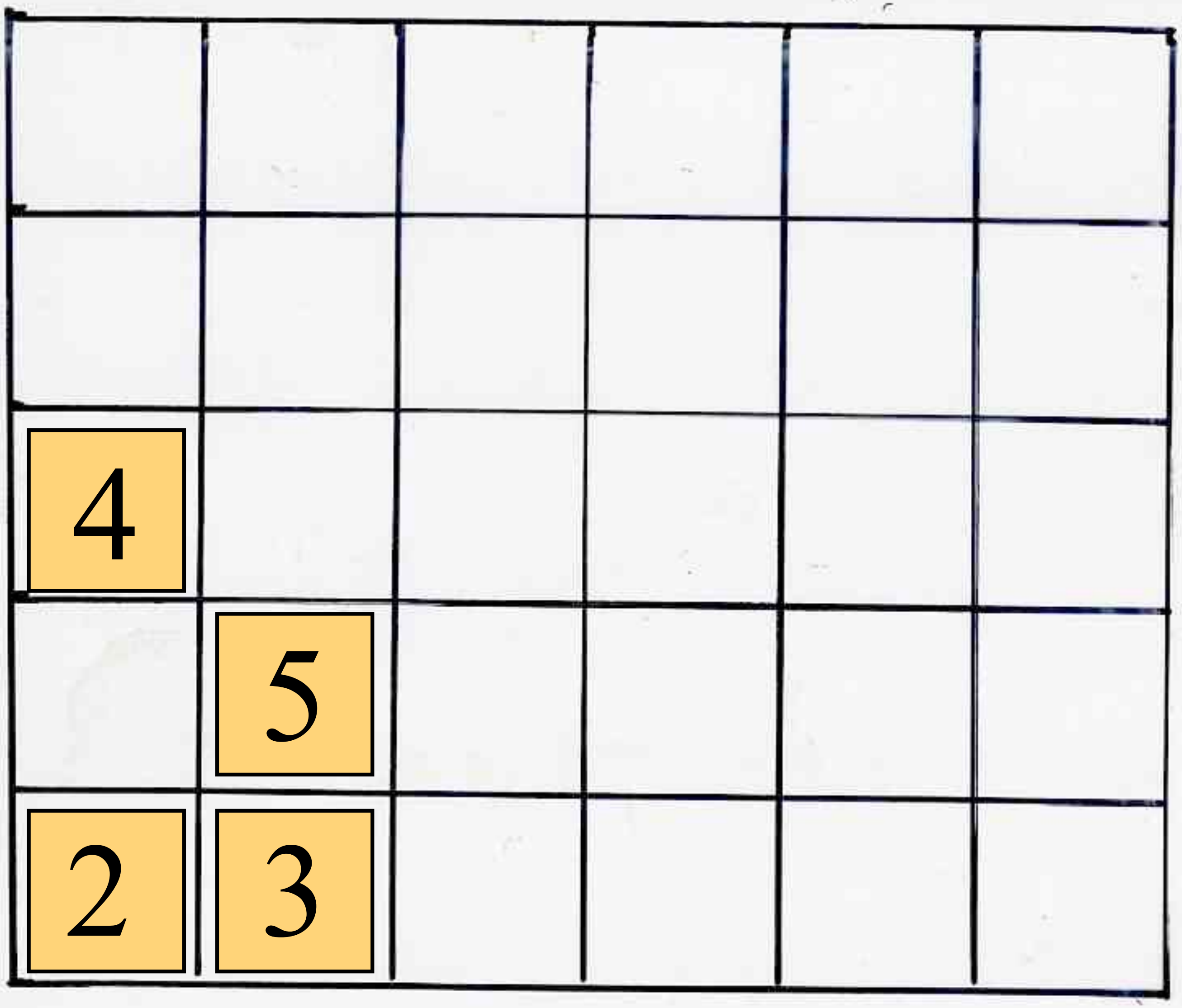
dual of a Young tableau

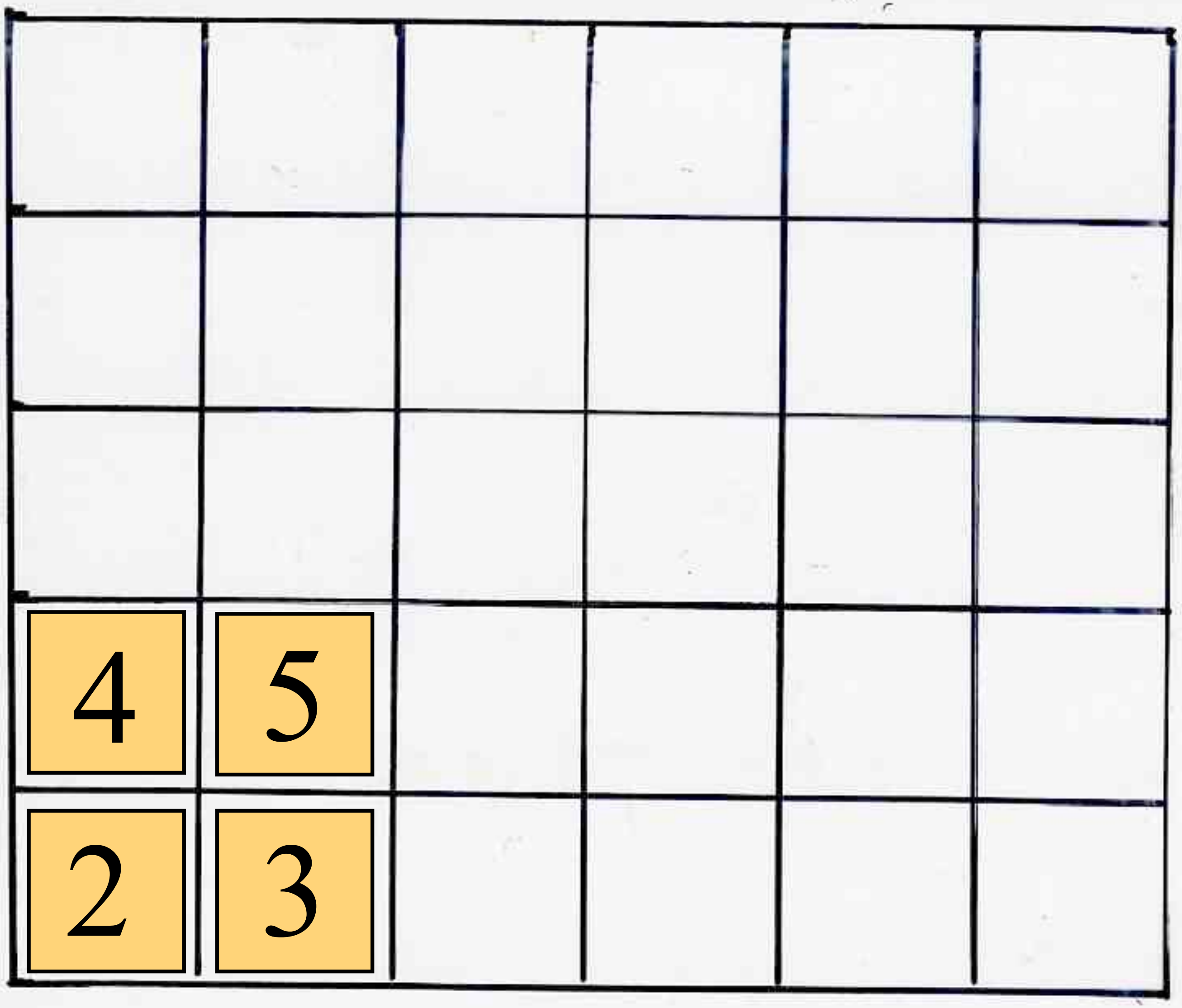


M.P. Schützenberger


4
2
5
1
3


4
2
5
3


4
5
2
3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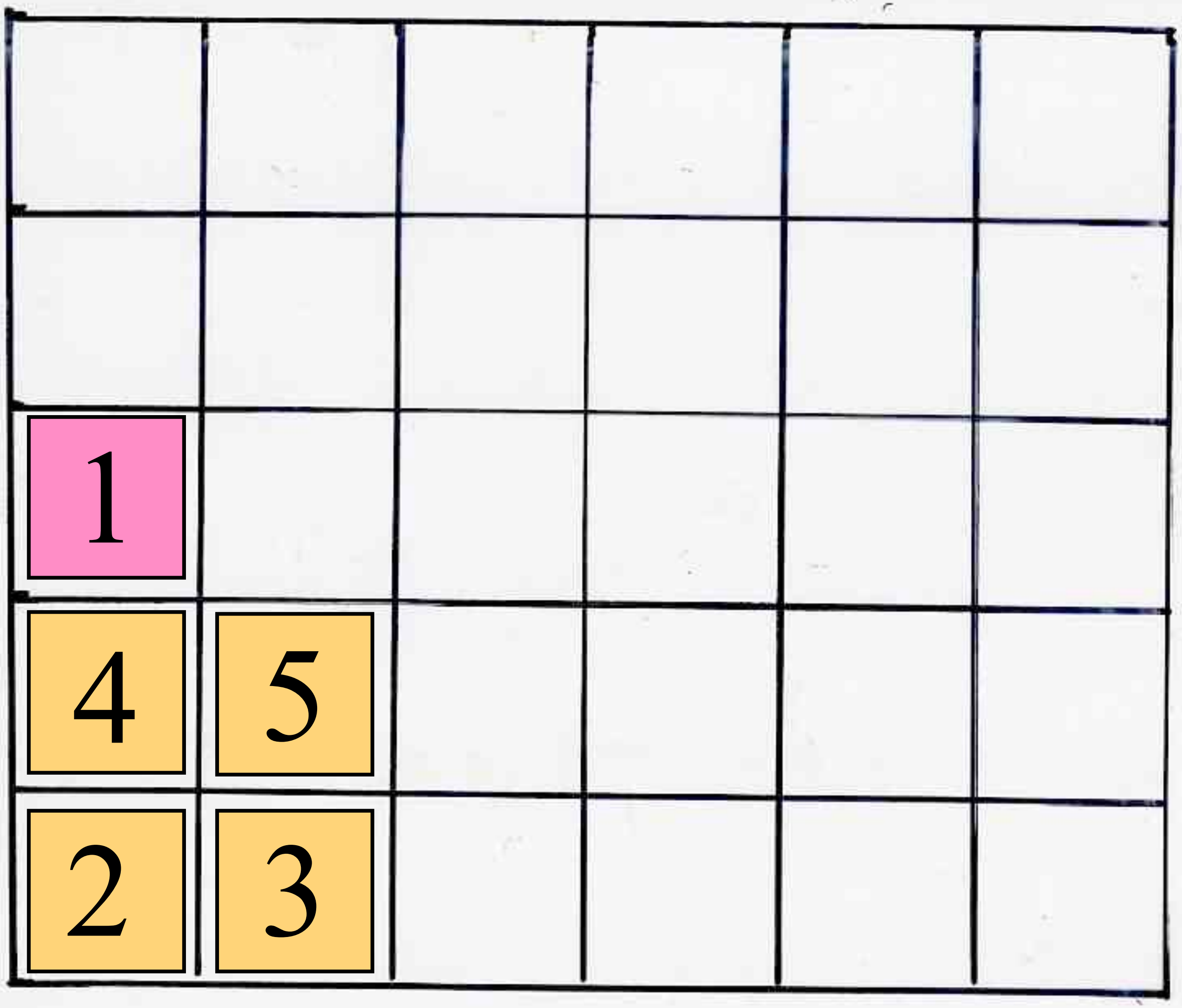
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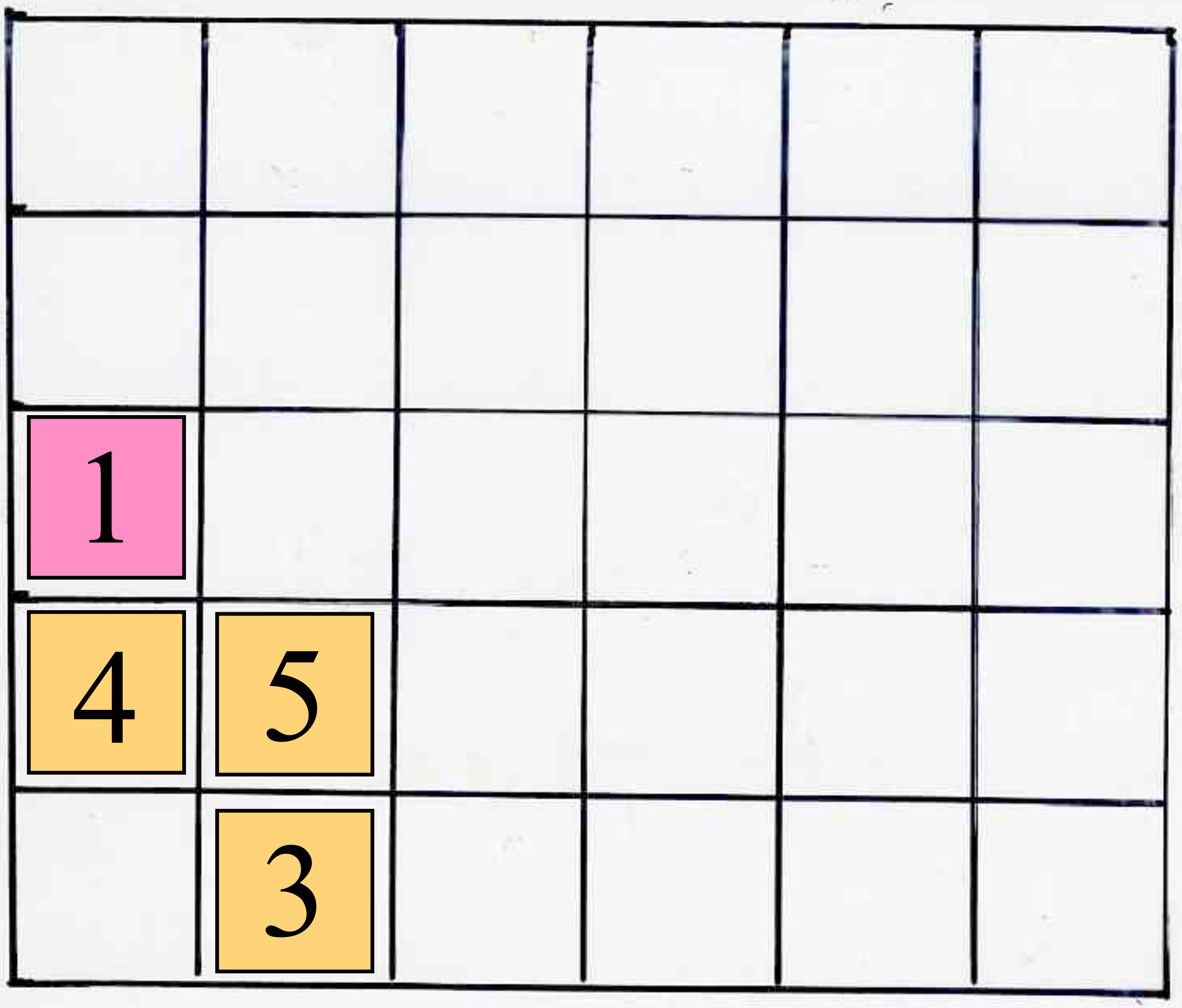
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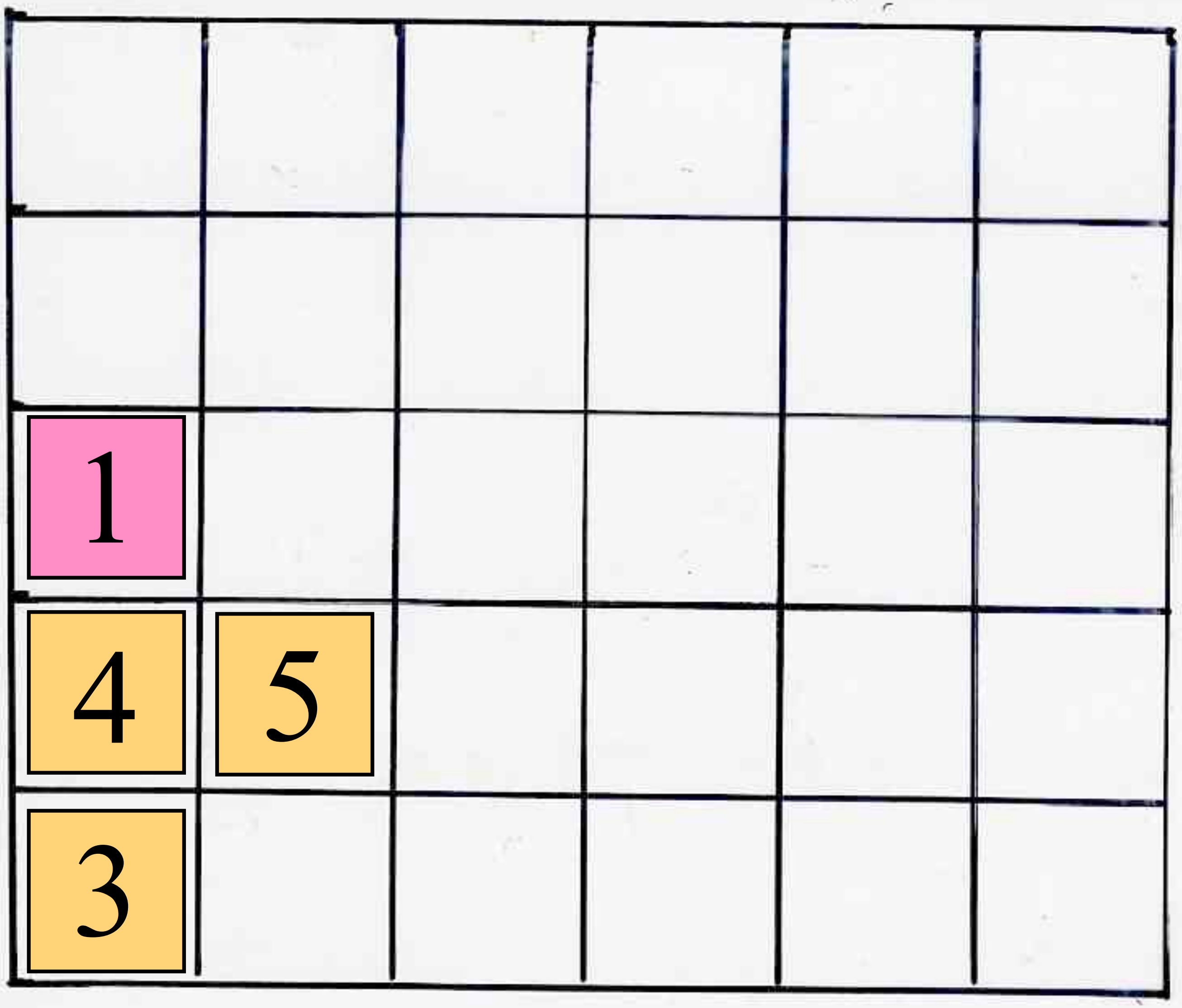
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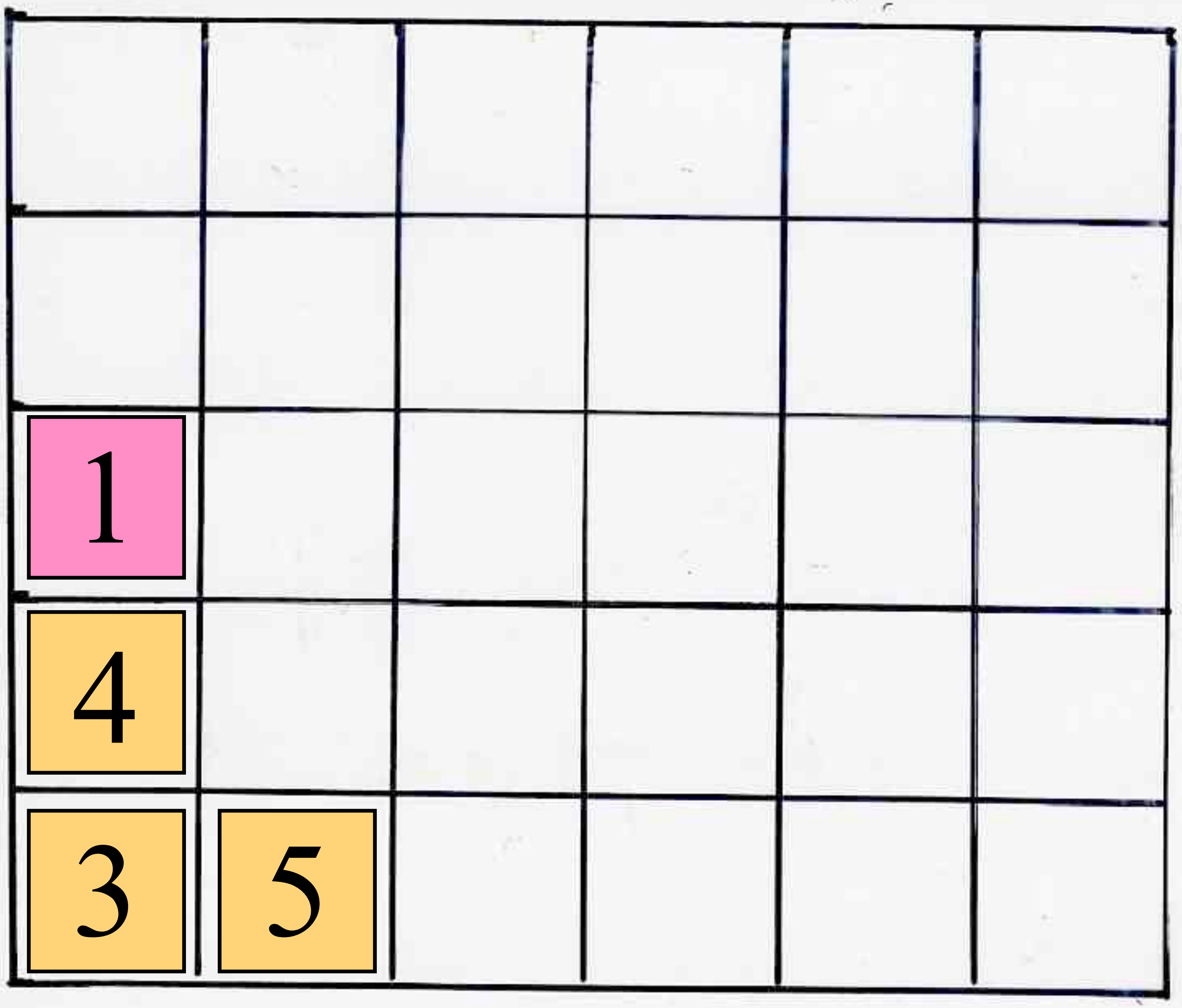
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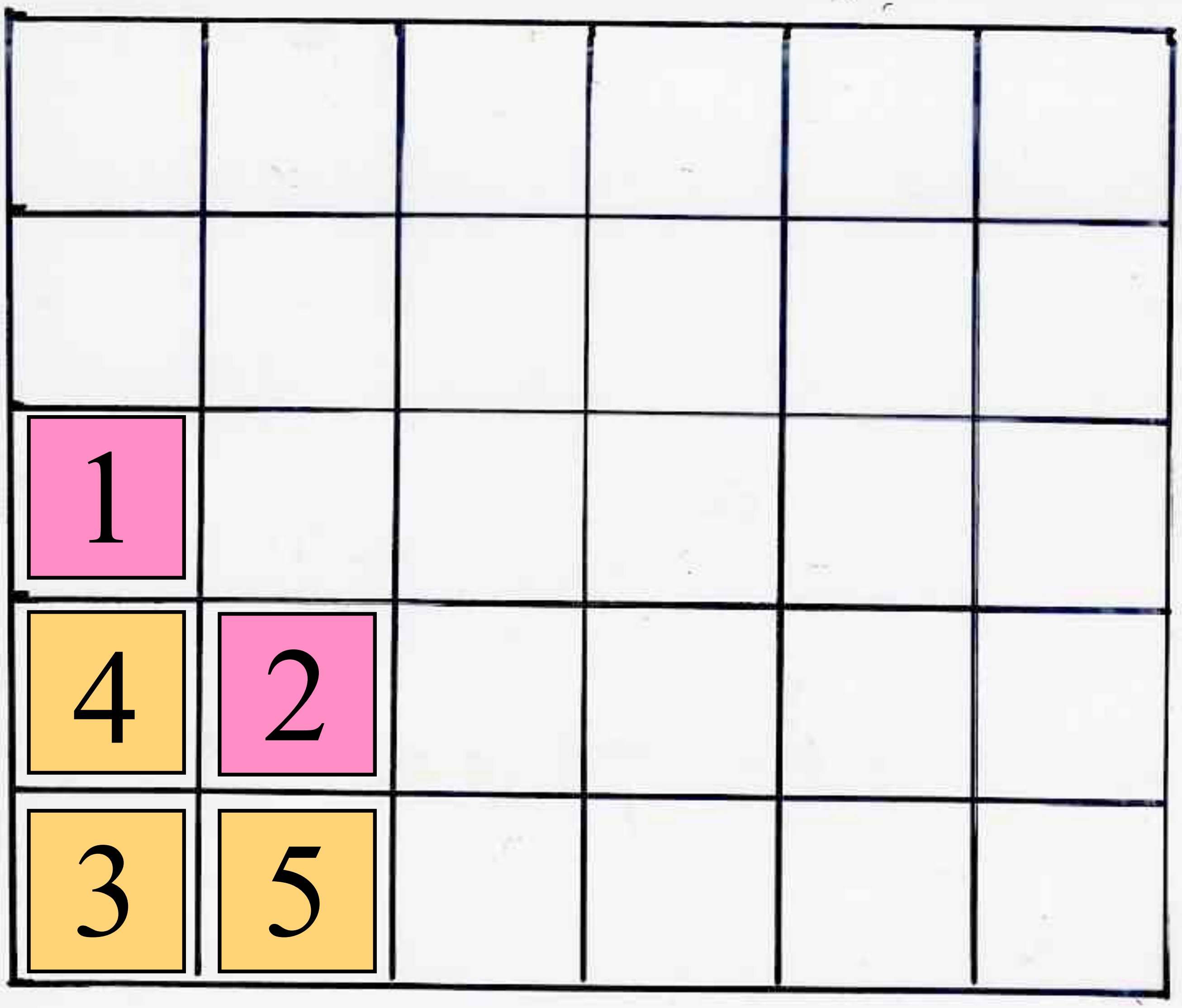
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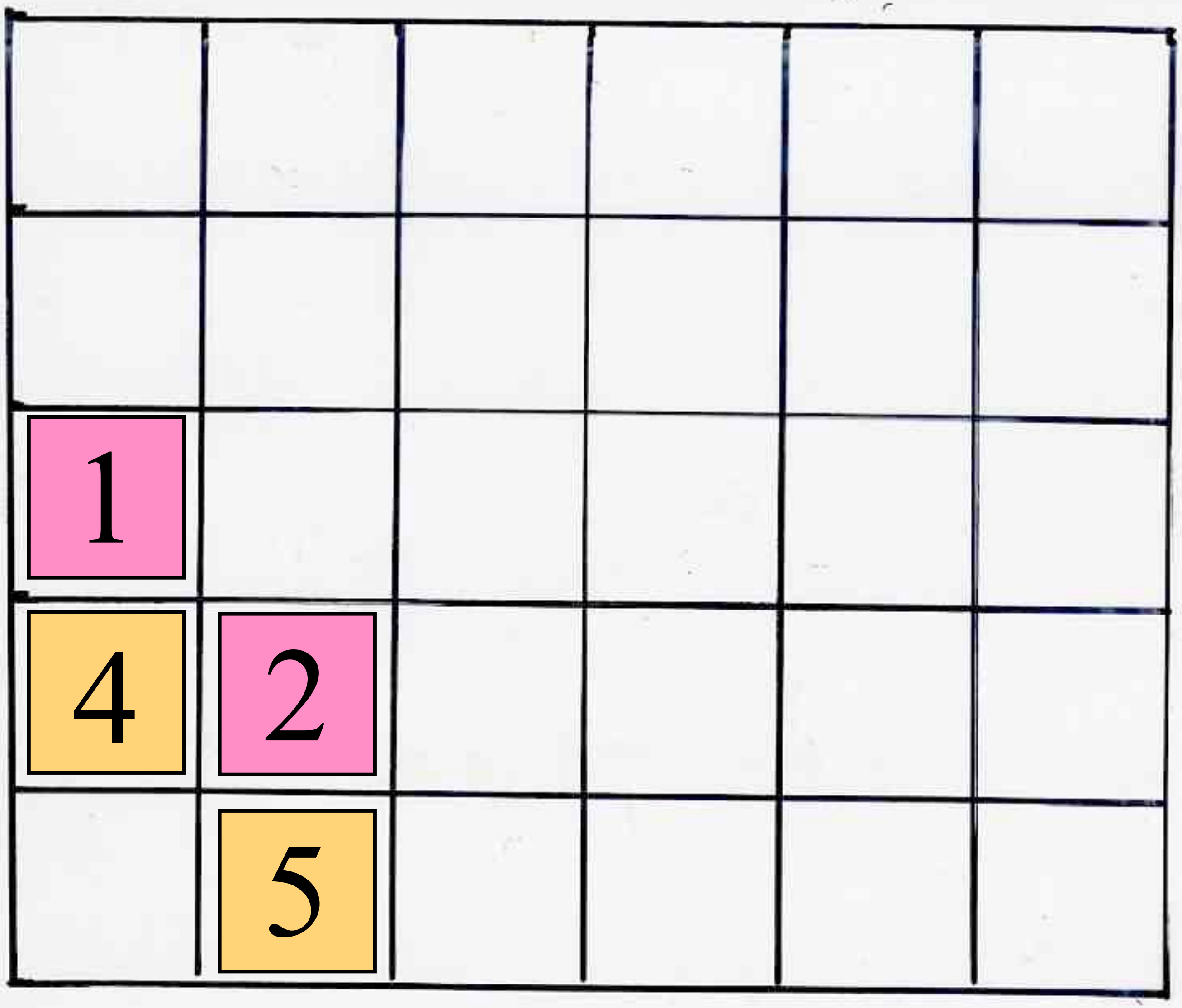
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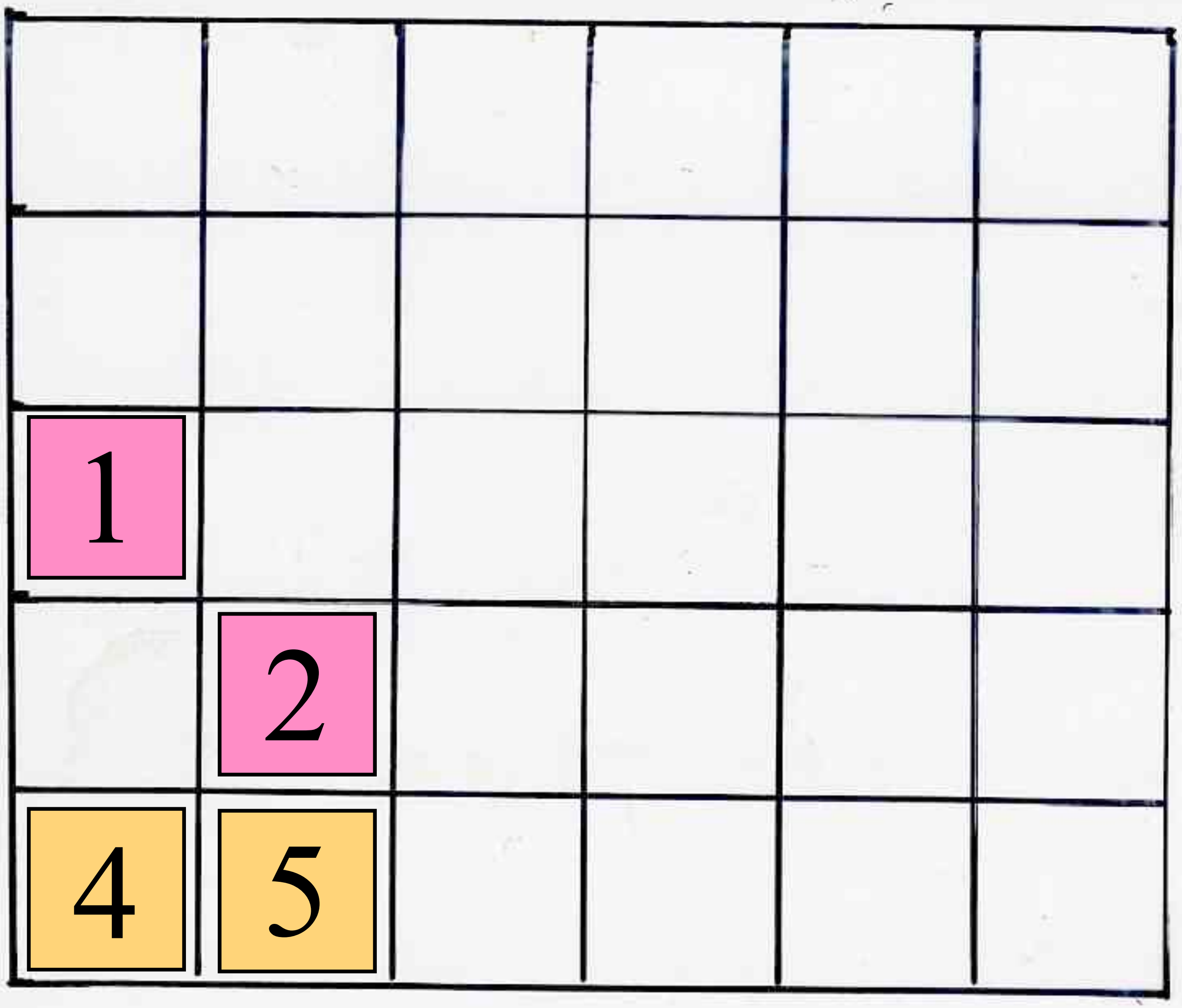
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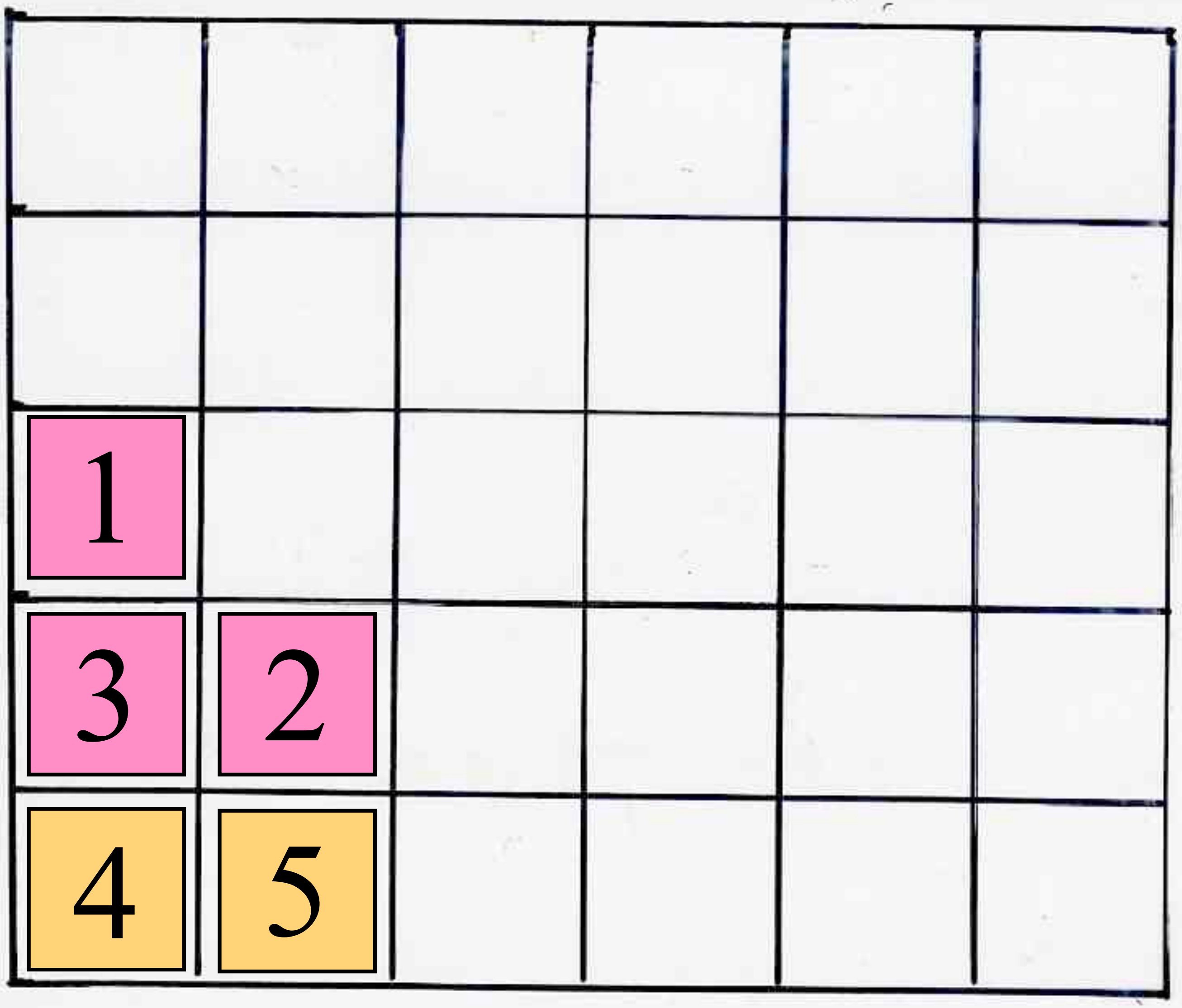
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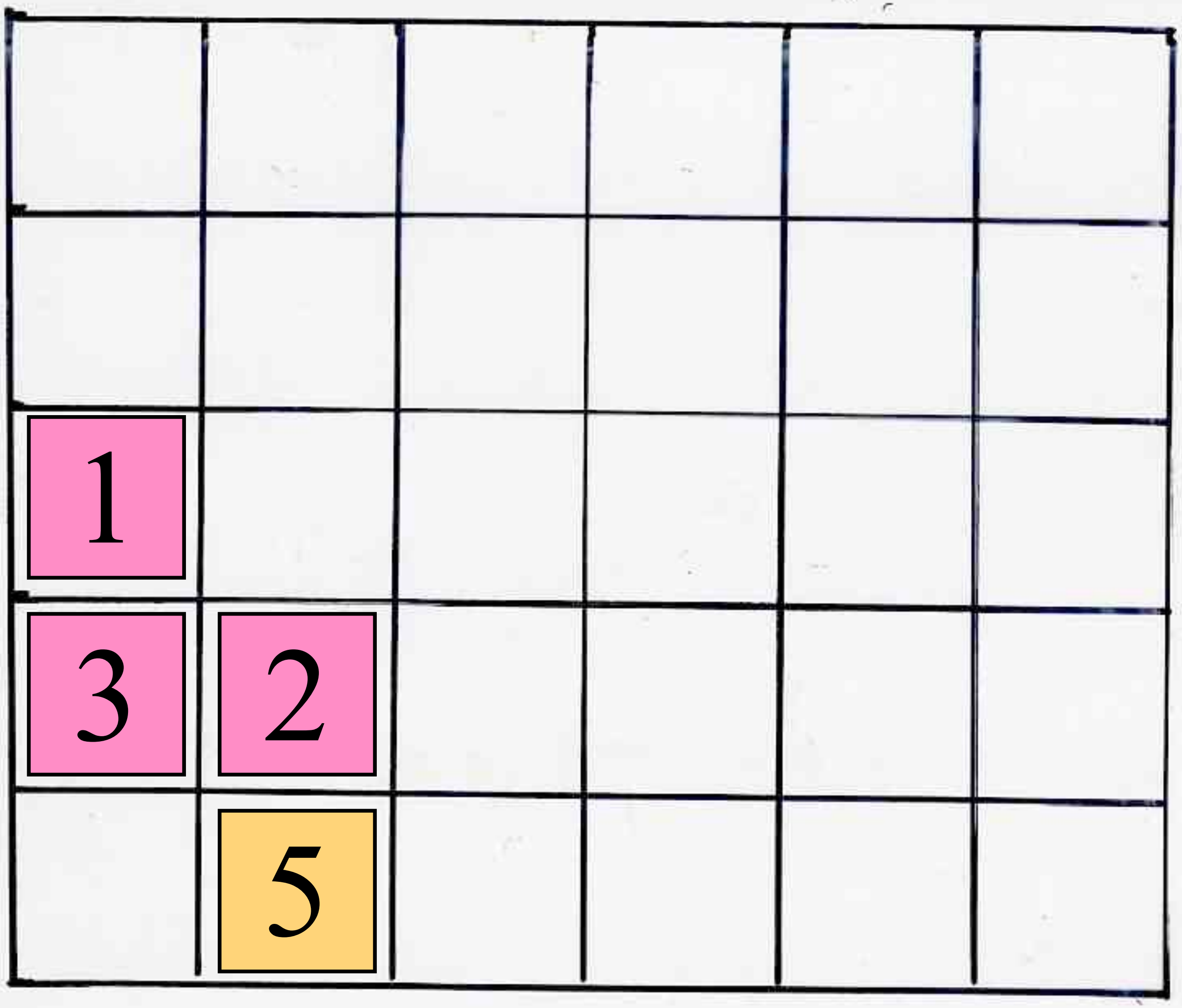
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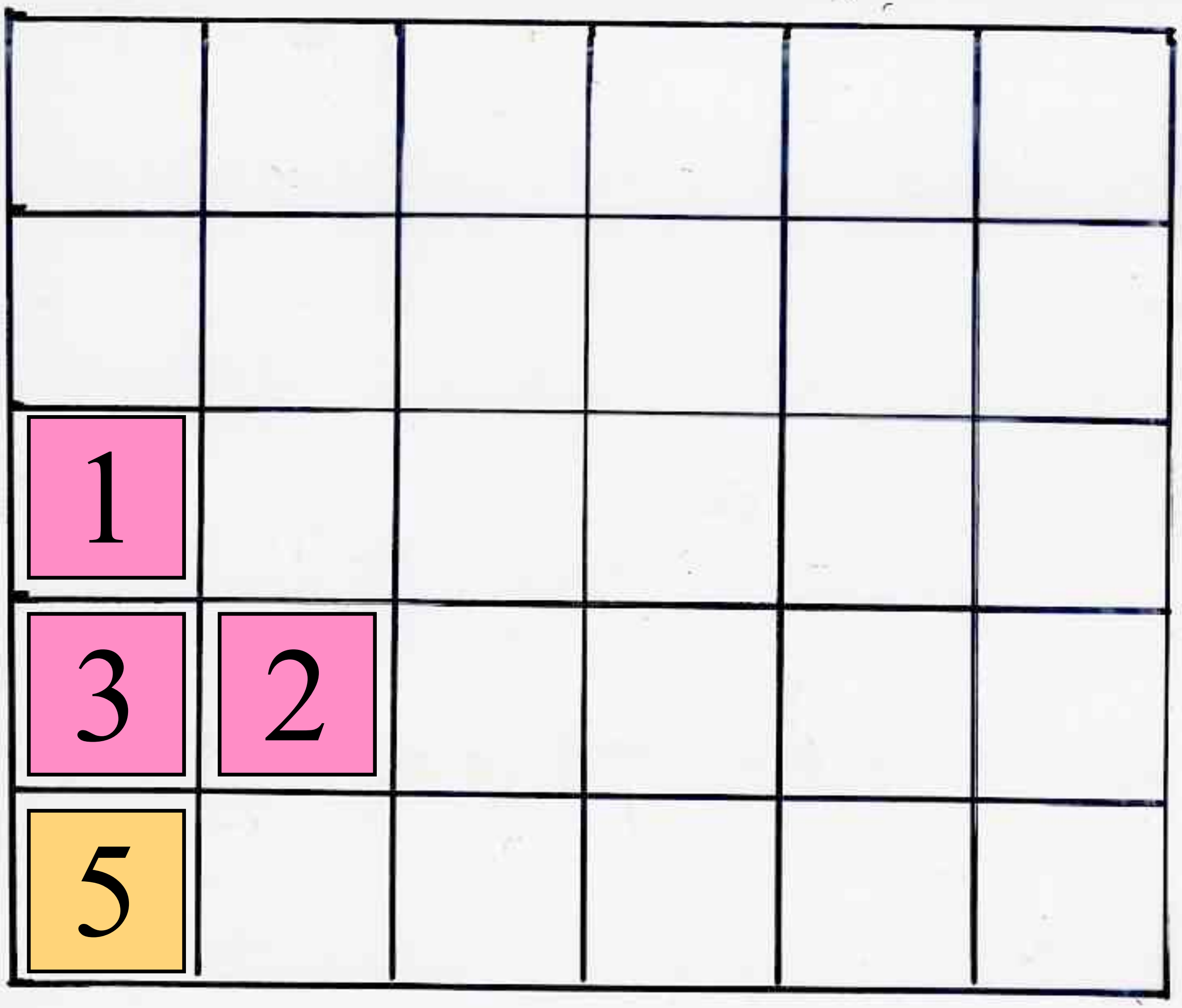
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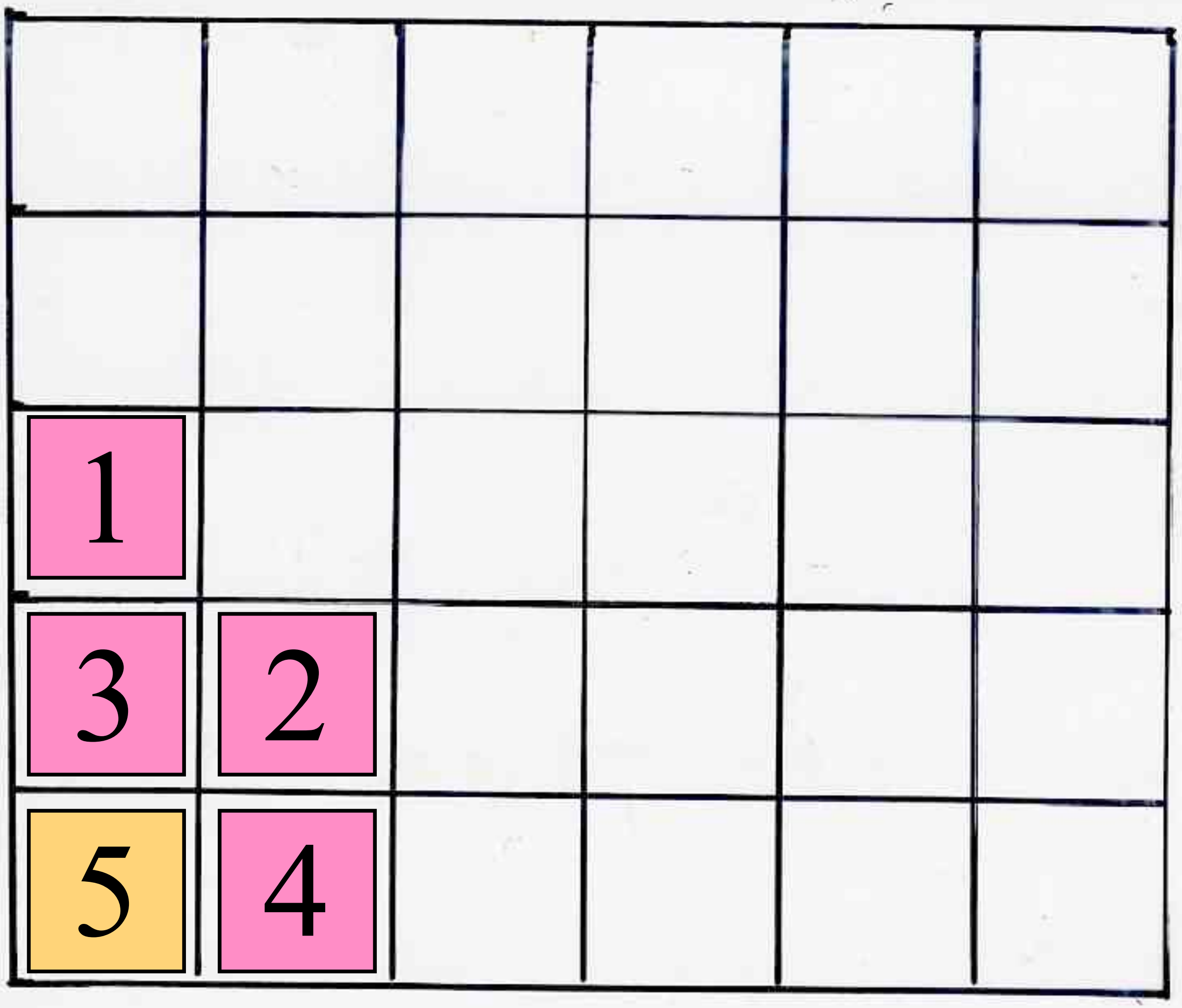
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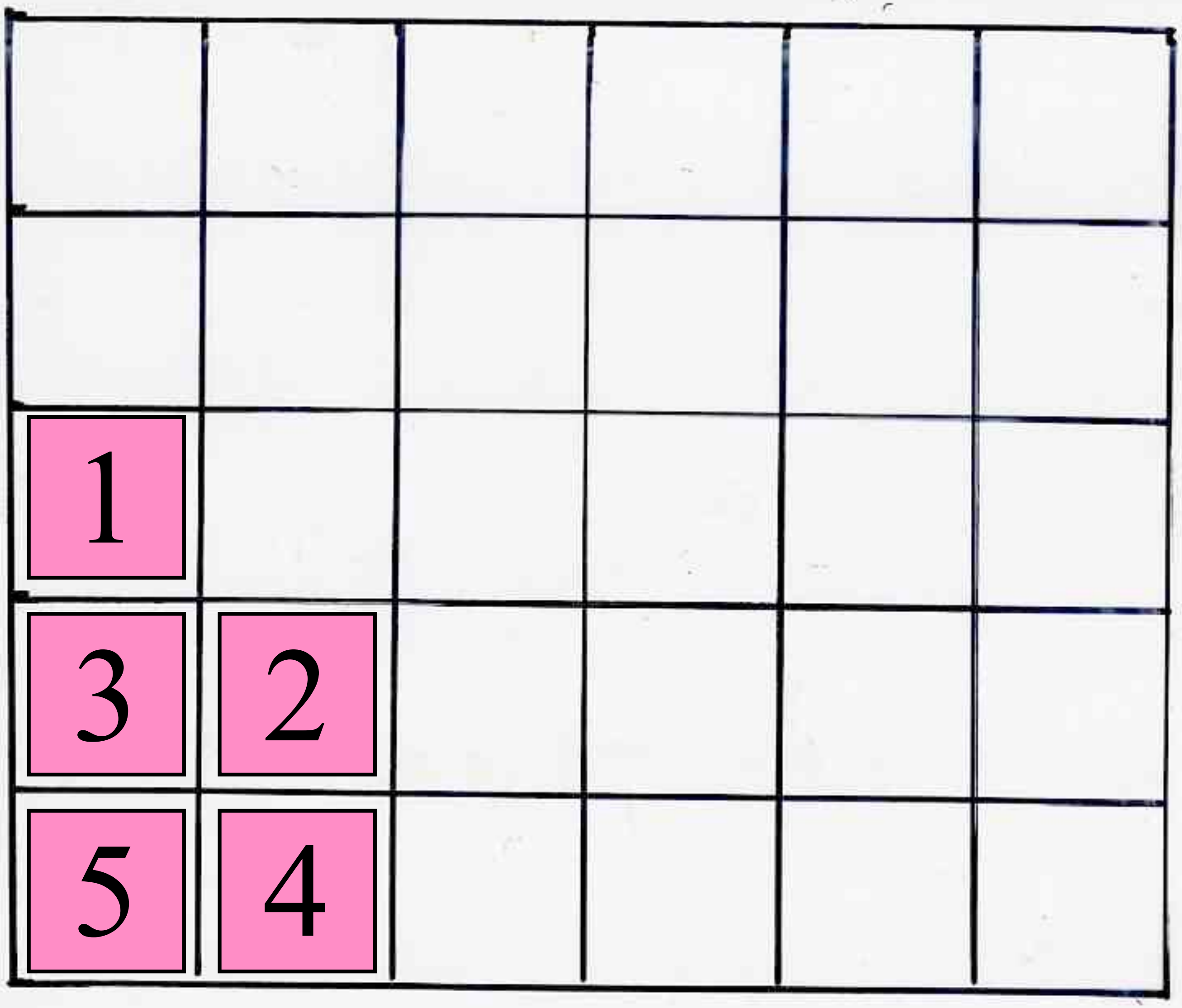
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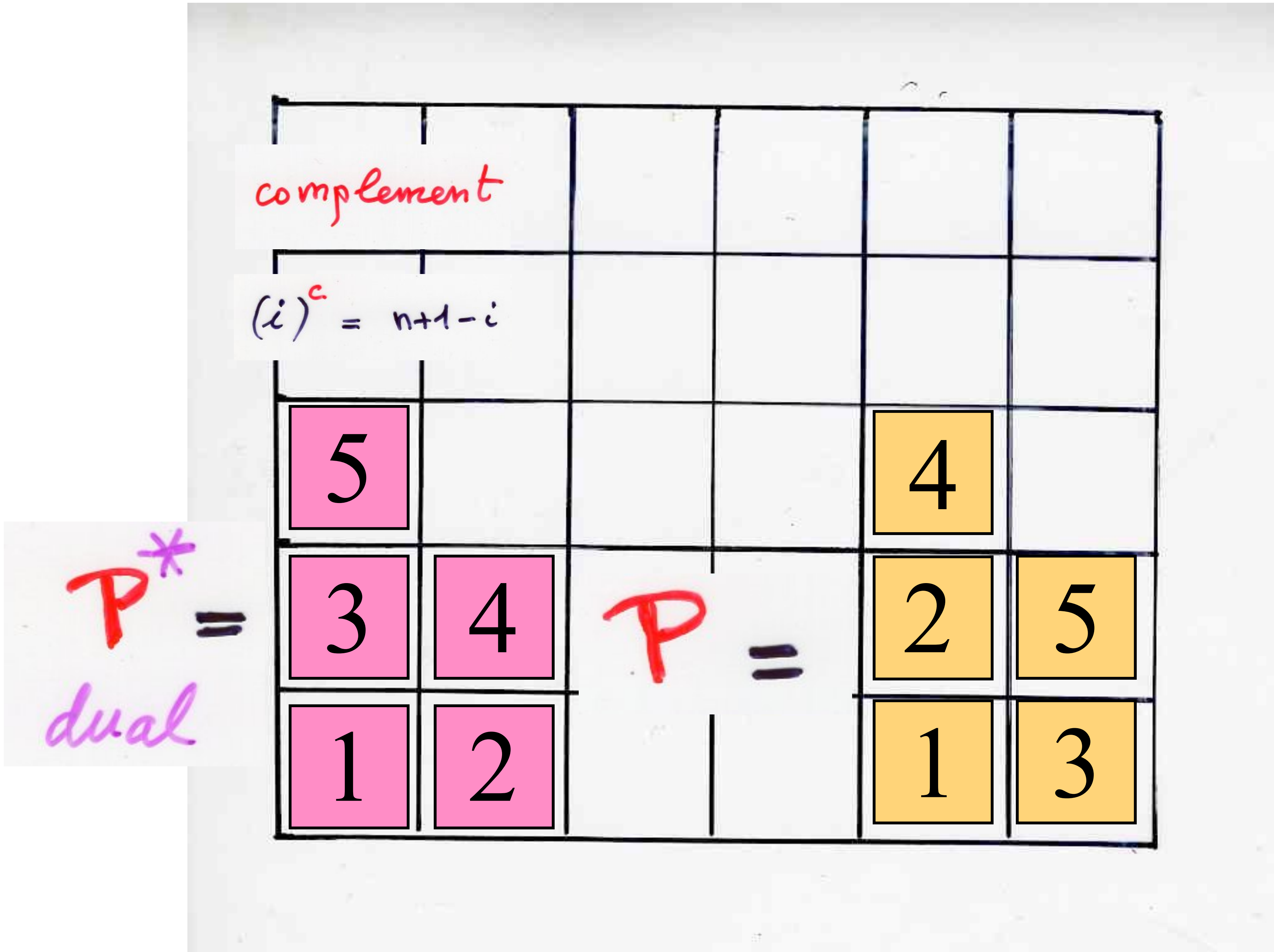

complement
$(i)^c = n+1-i$
$P^* =$
dual
$P =$
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3 4
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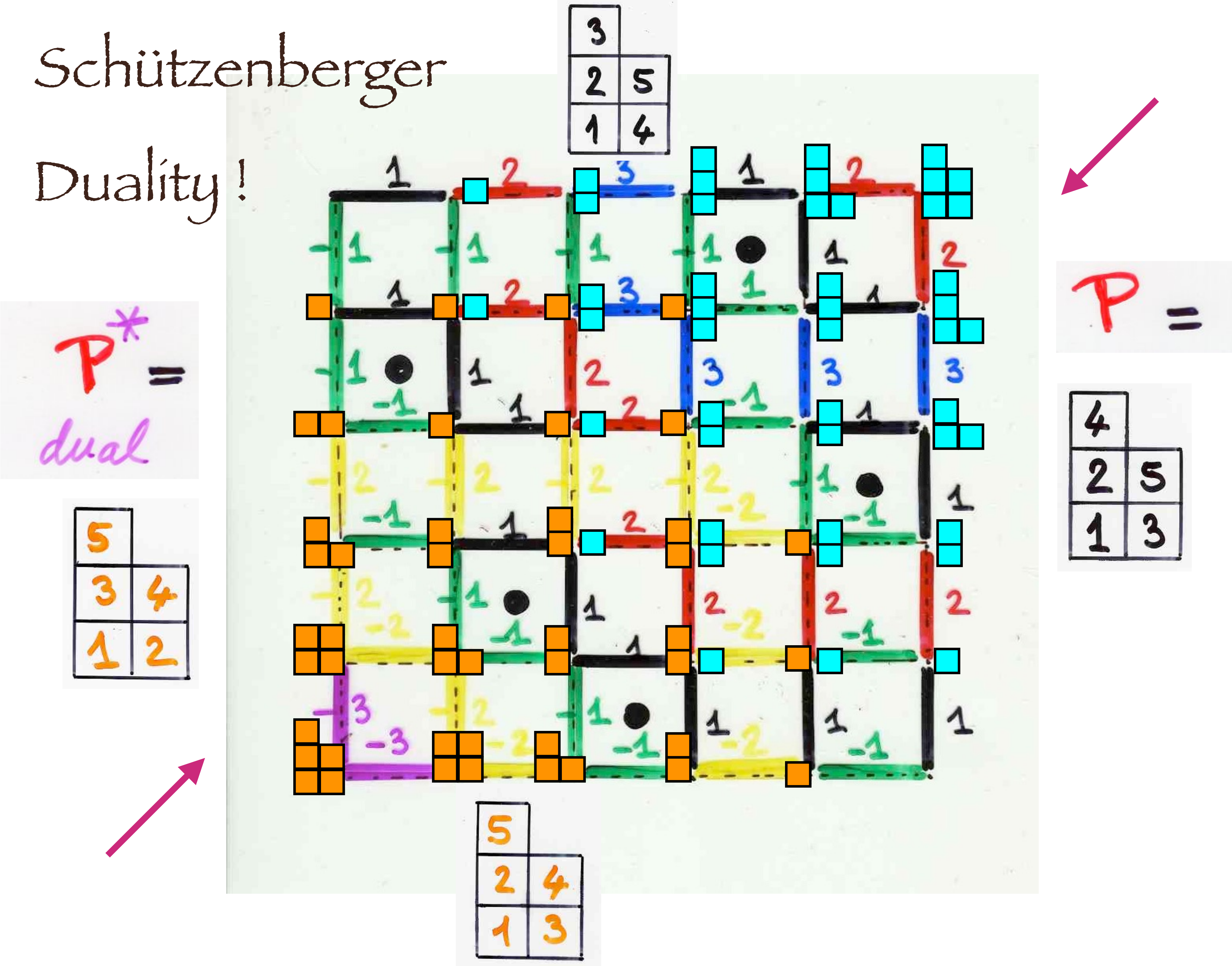
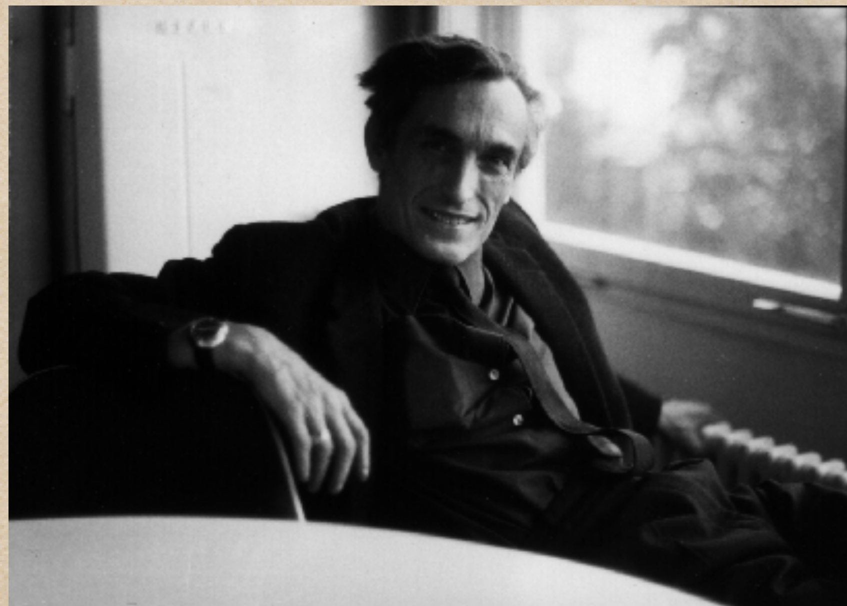
Schützenberger Duality !

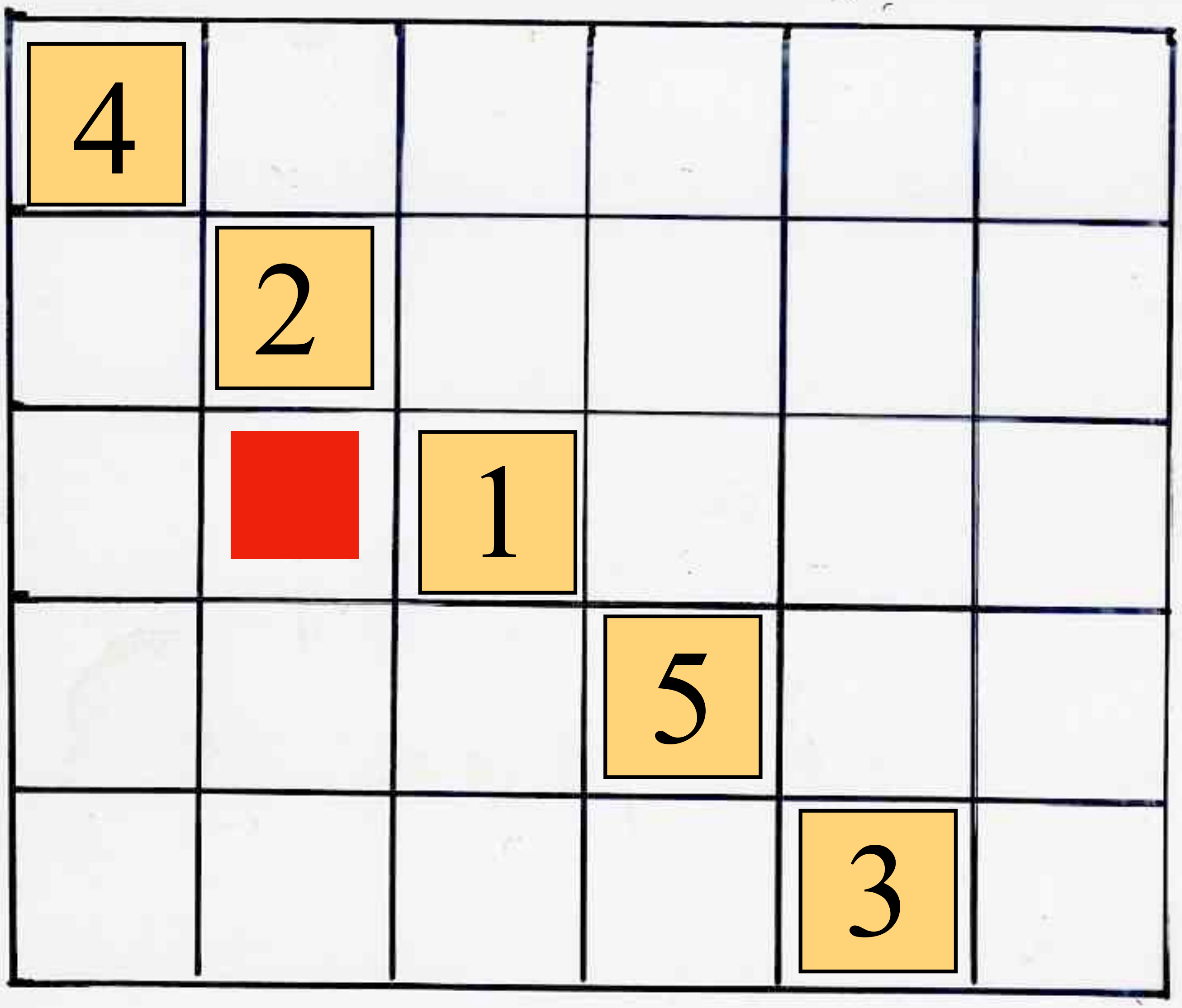


Jeu de taquin

M.P. Schützenberger

(1976)





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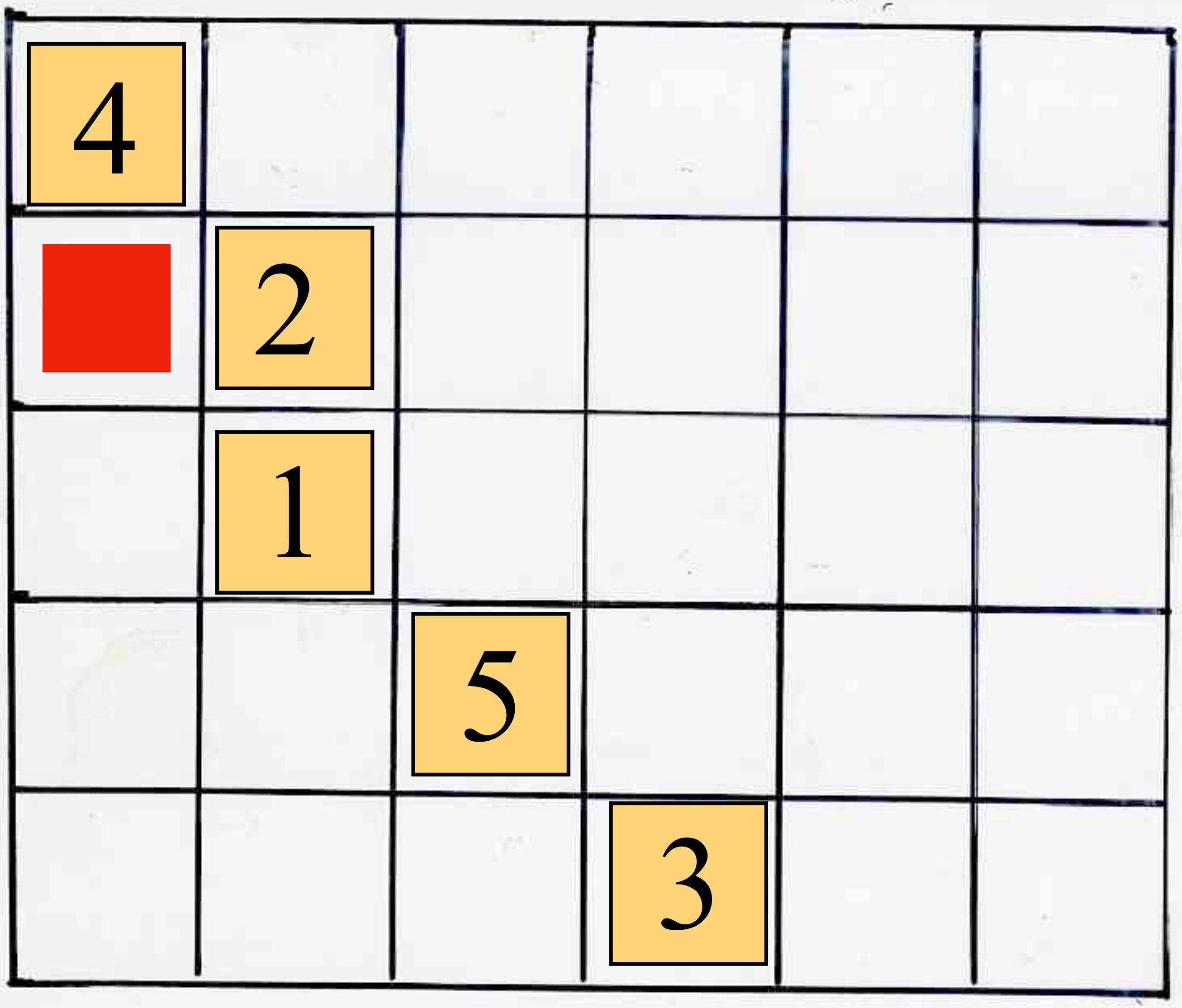
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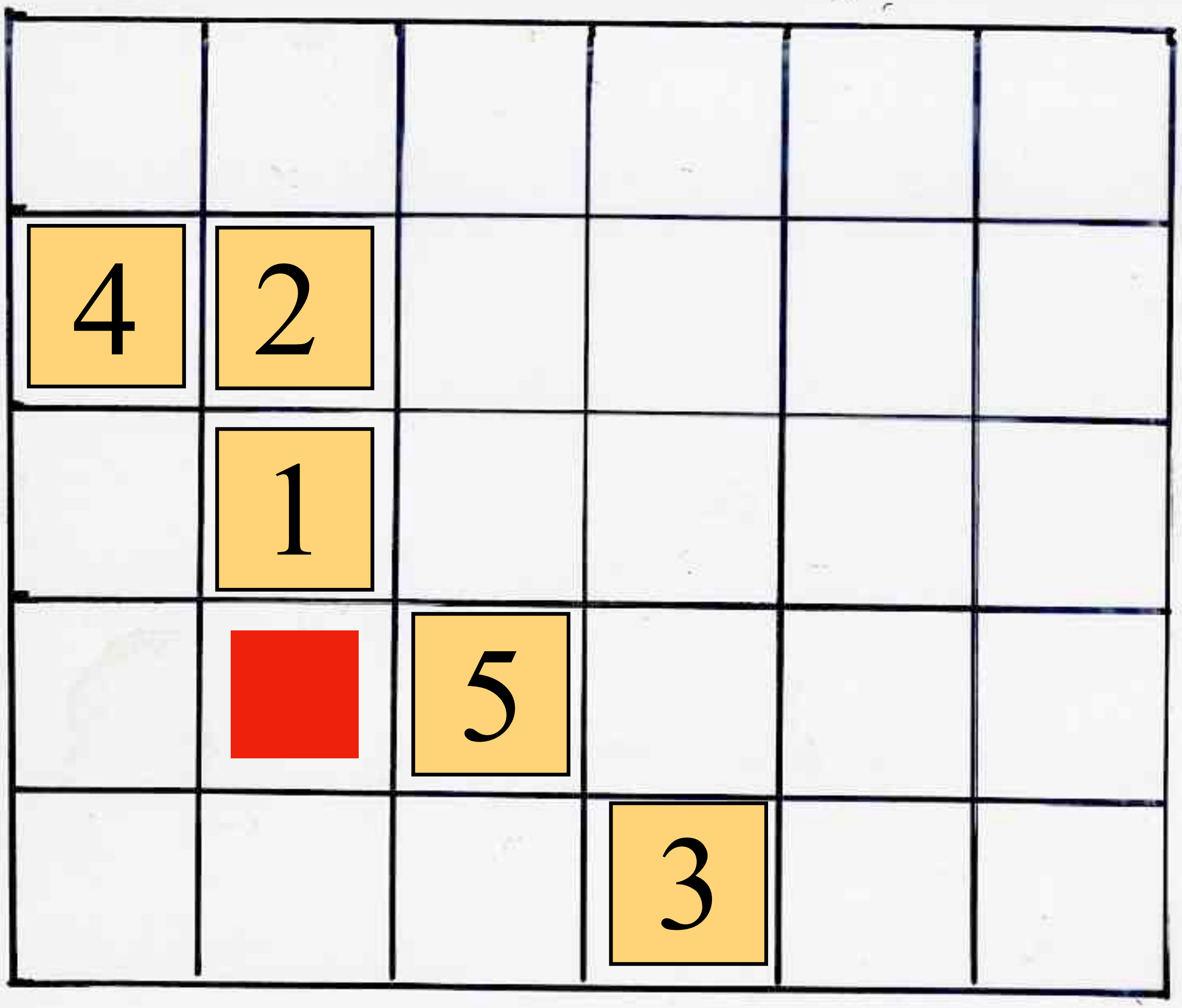
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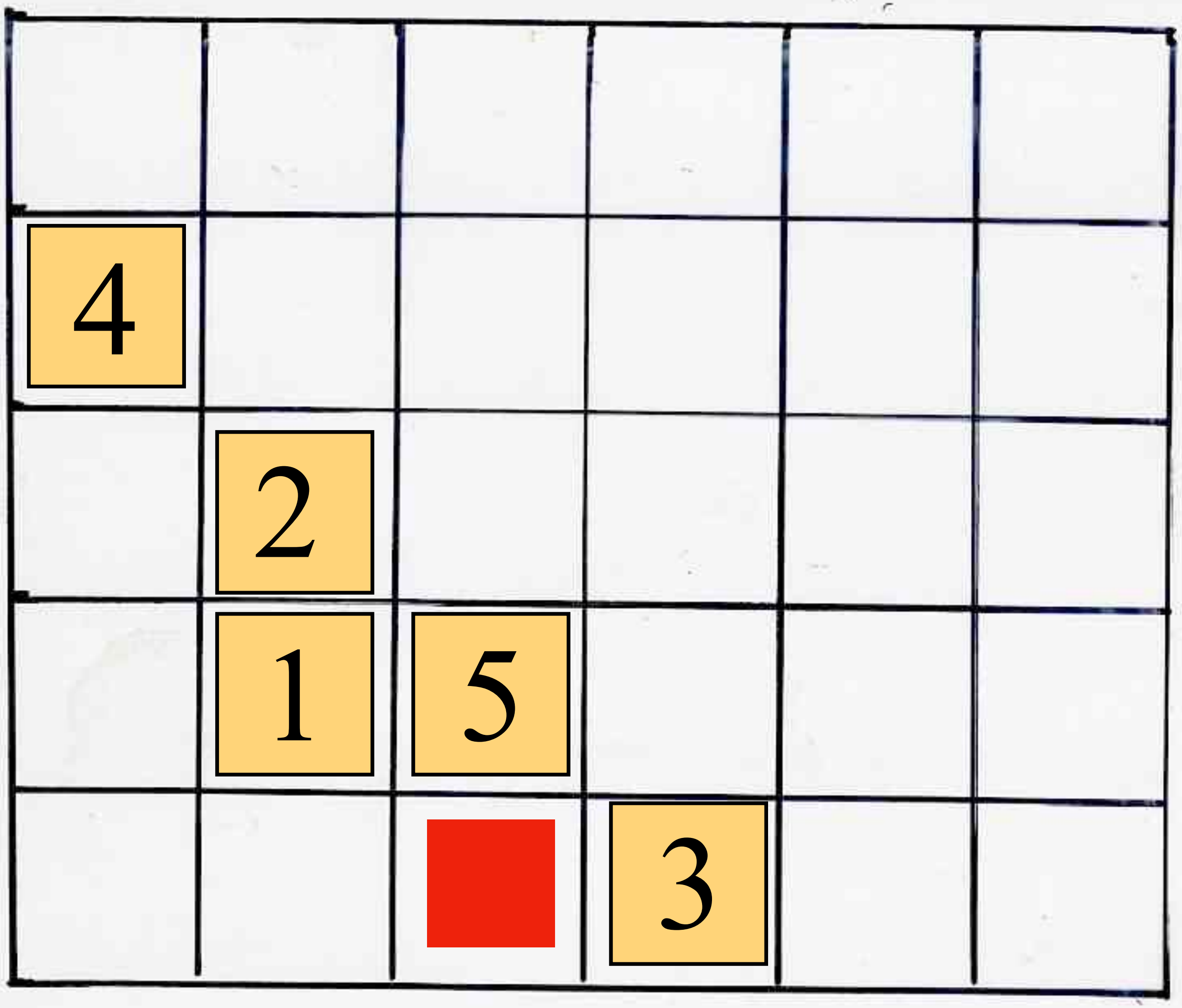
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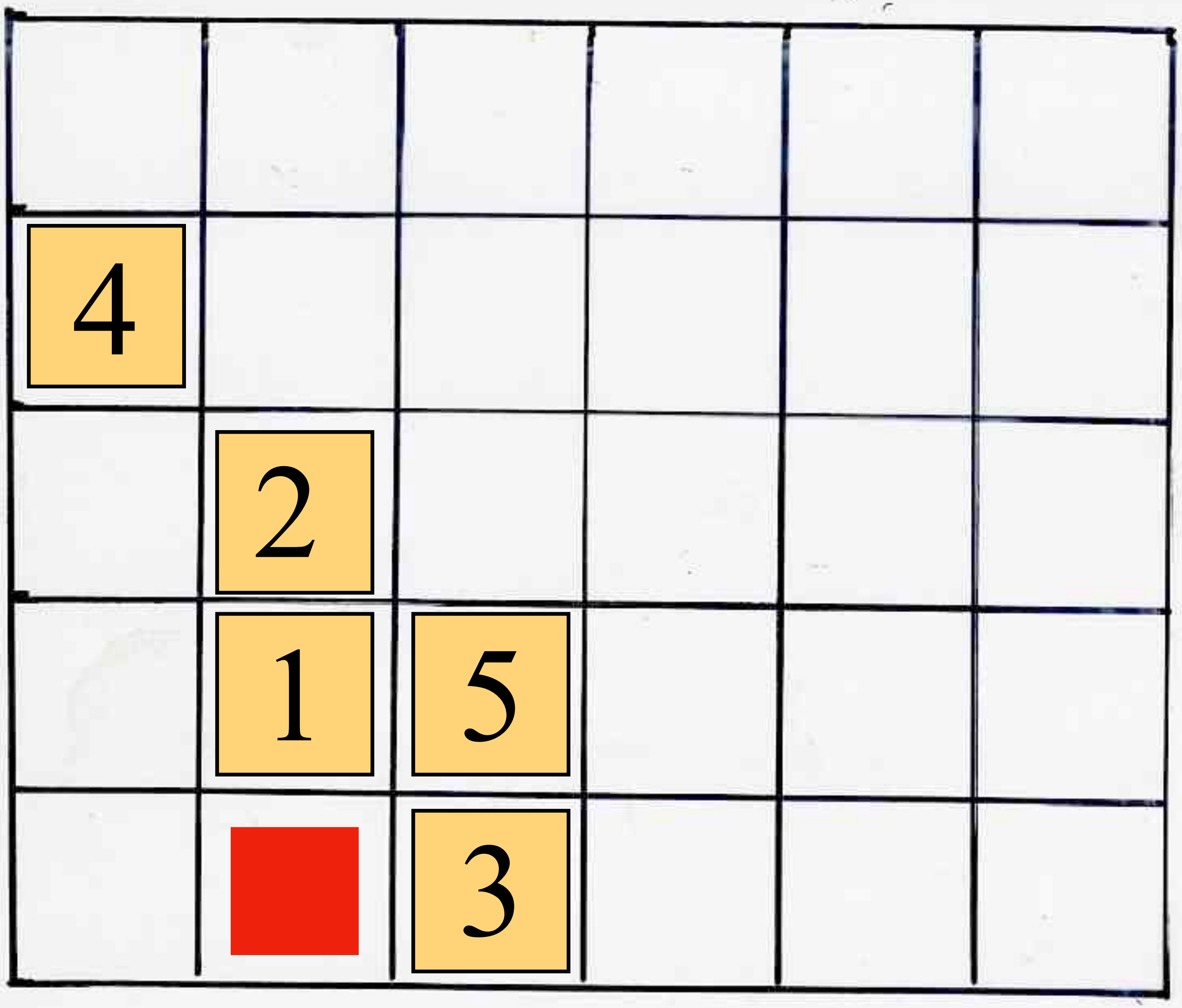
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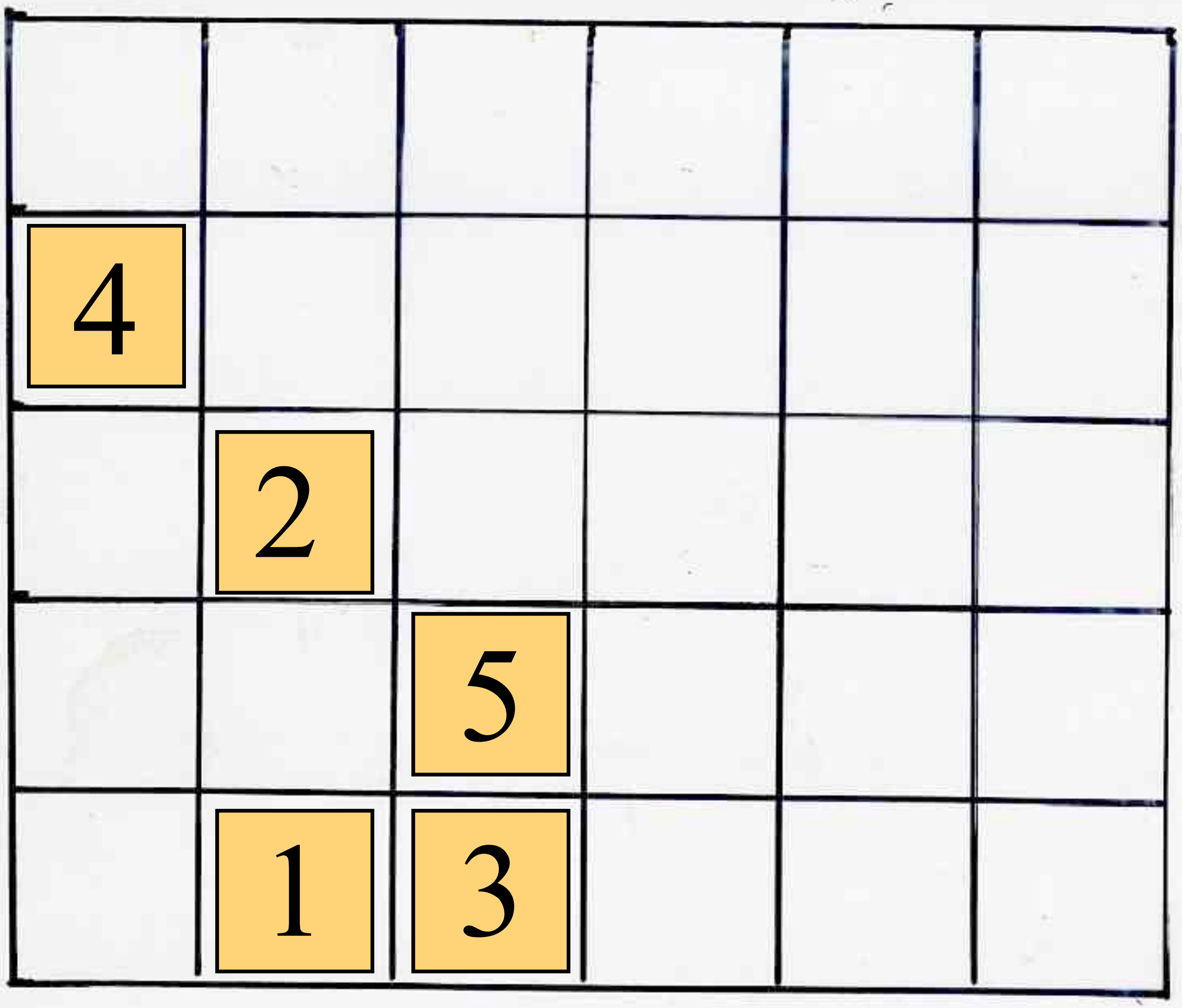
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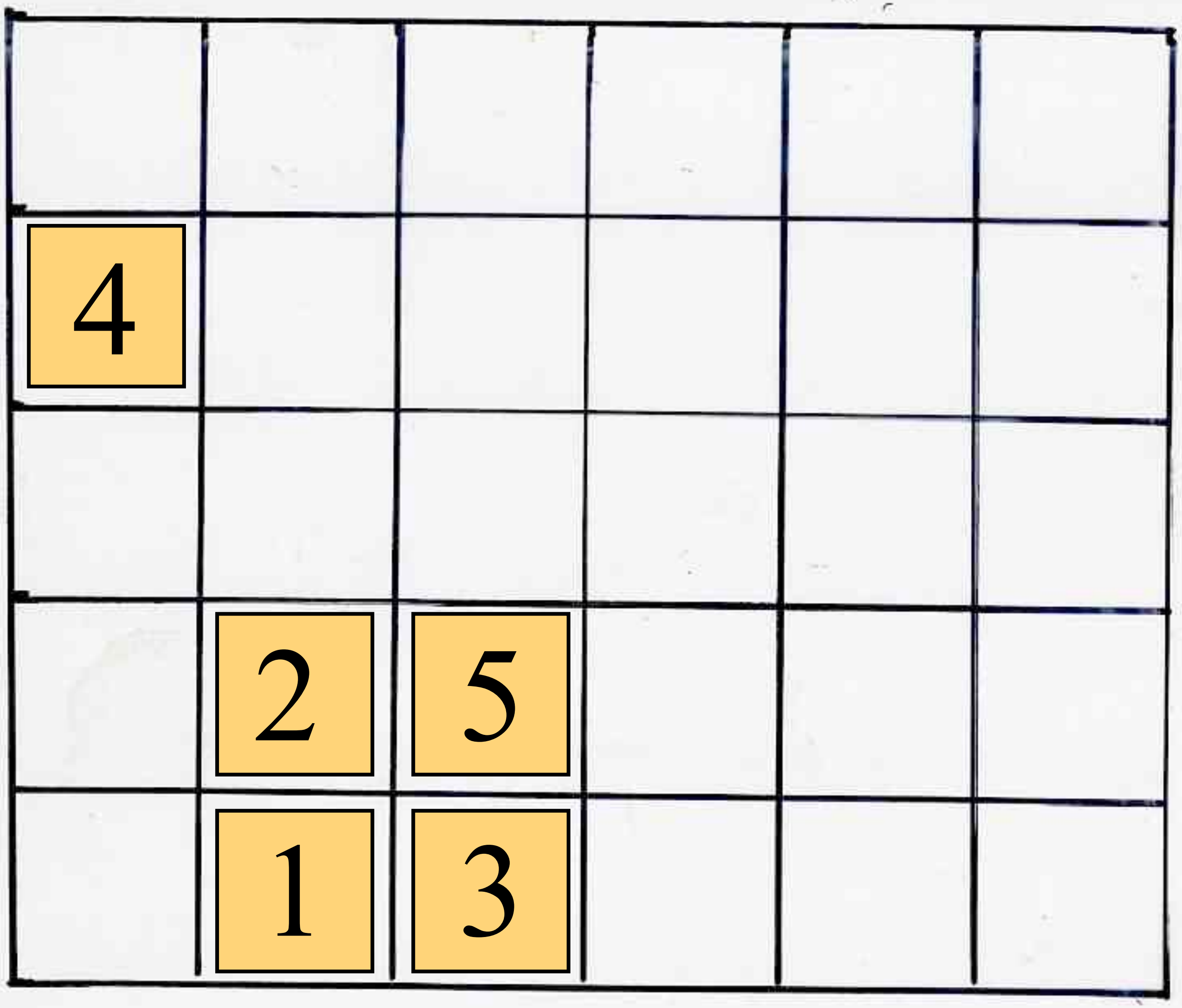
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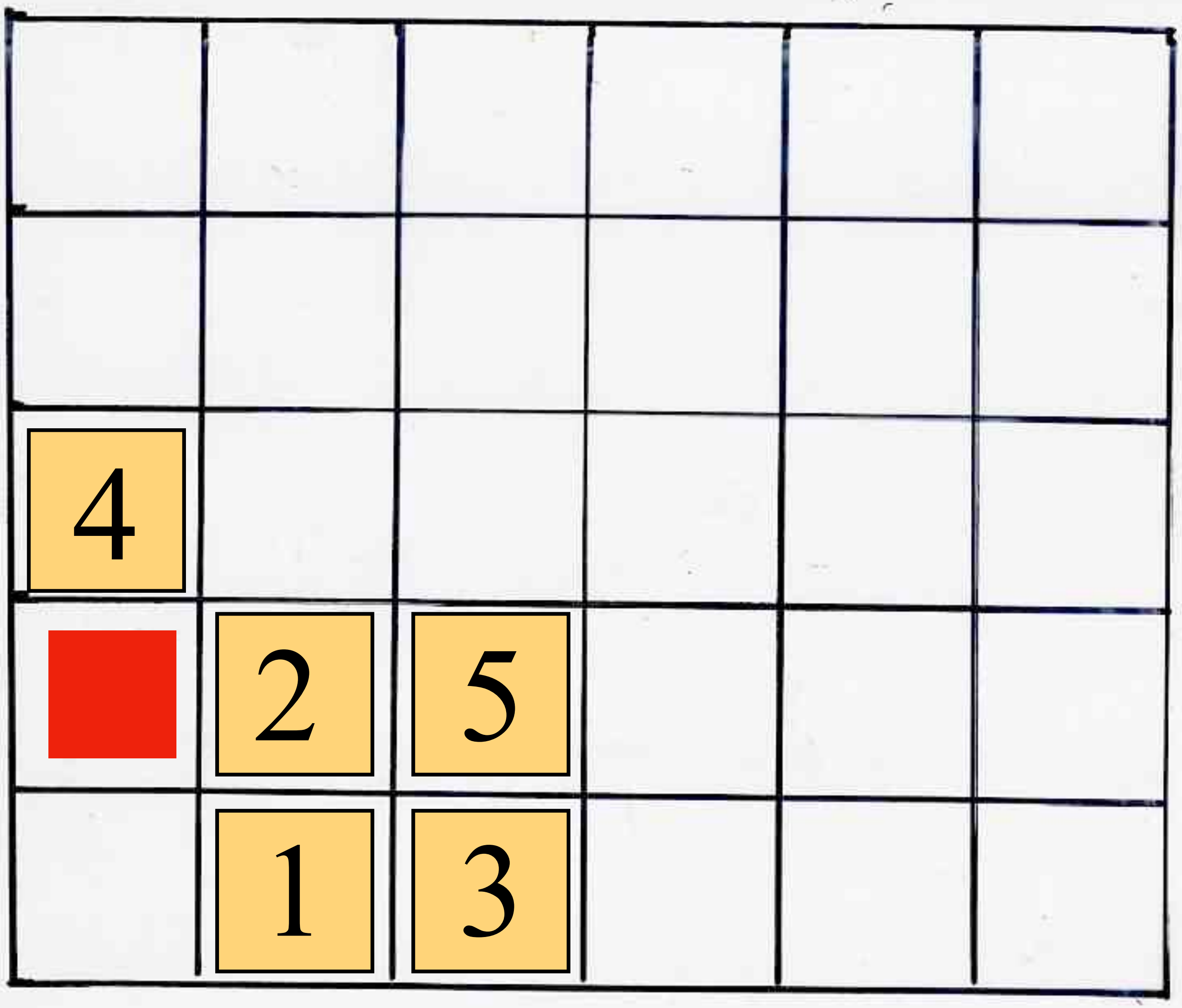
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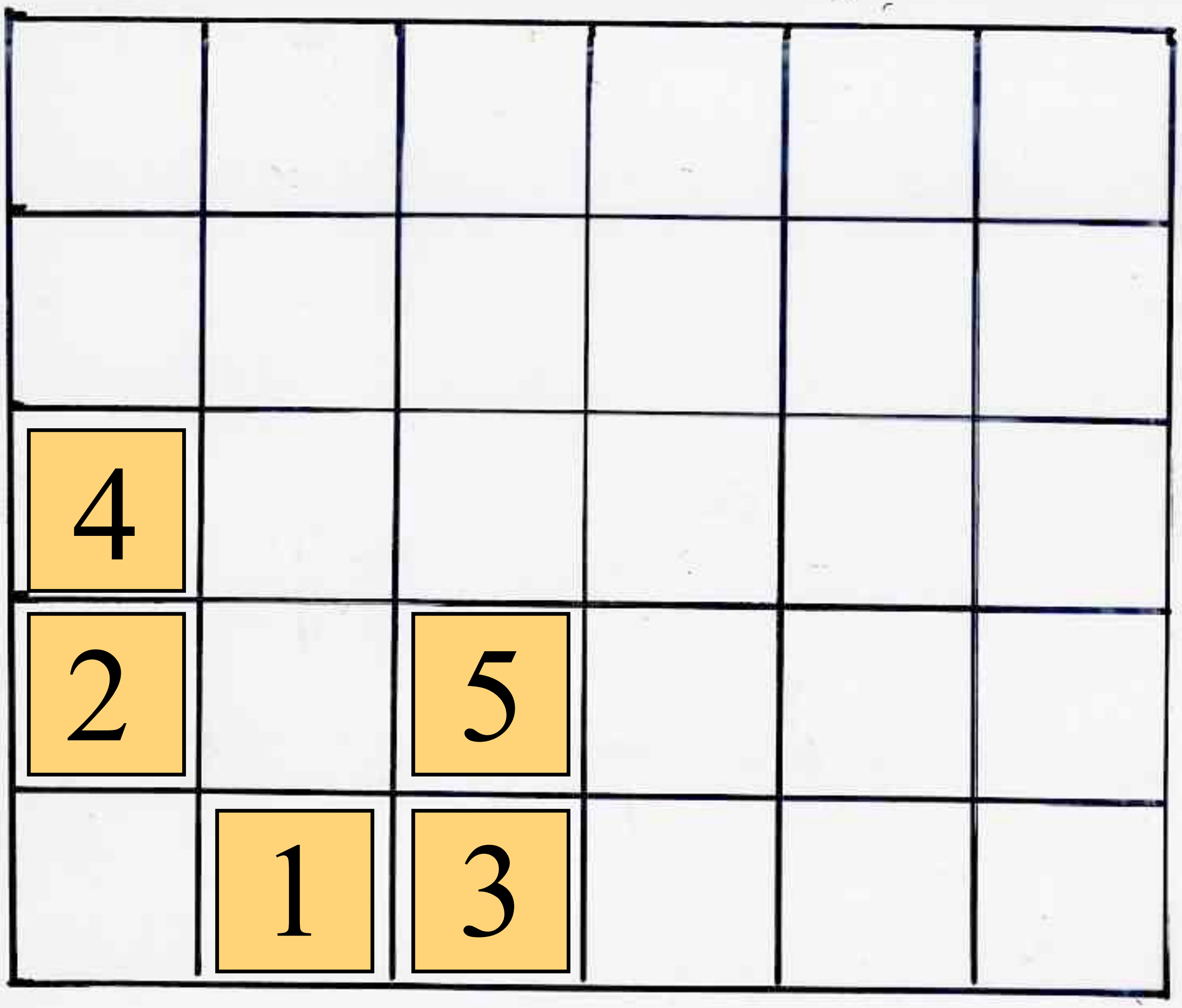
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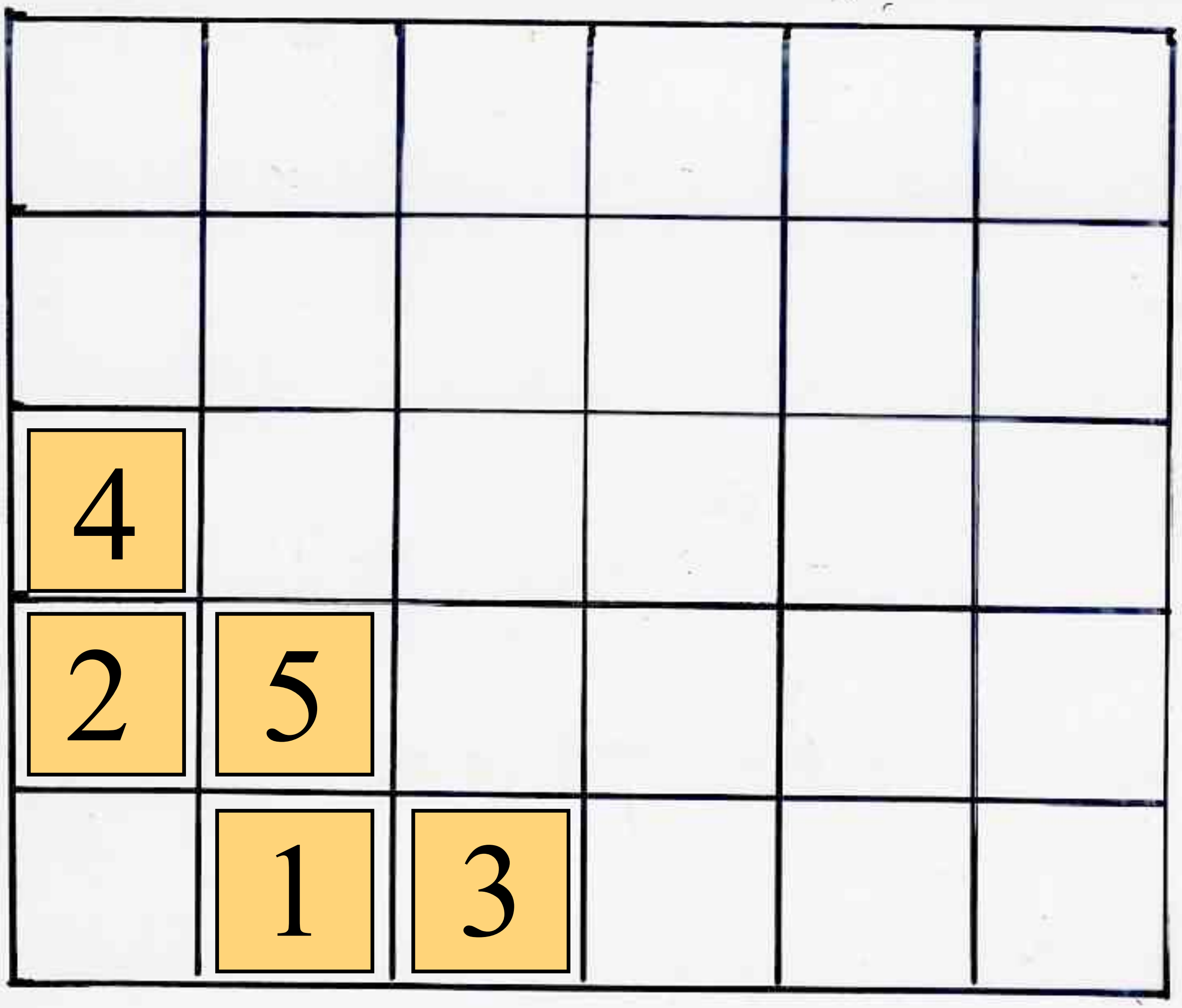
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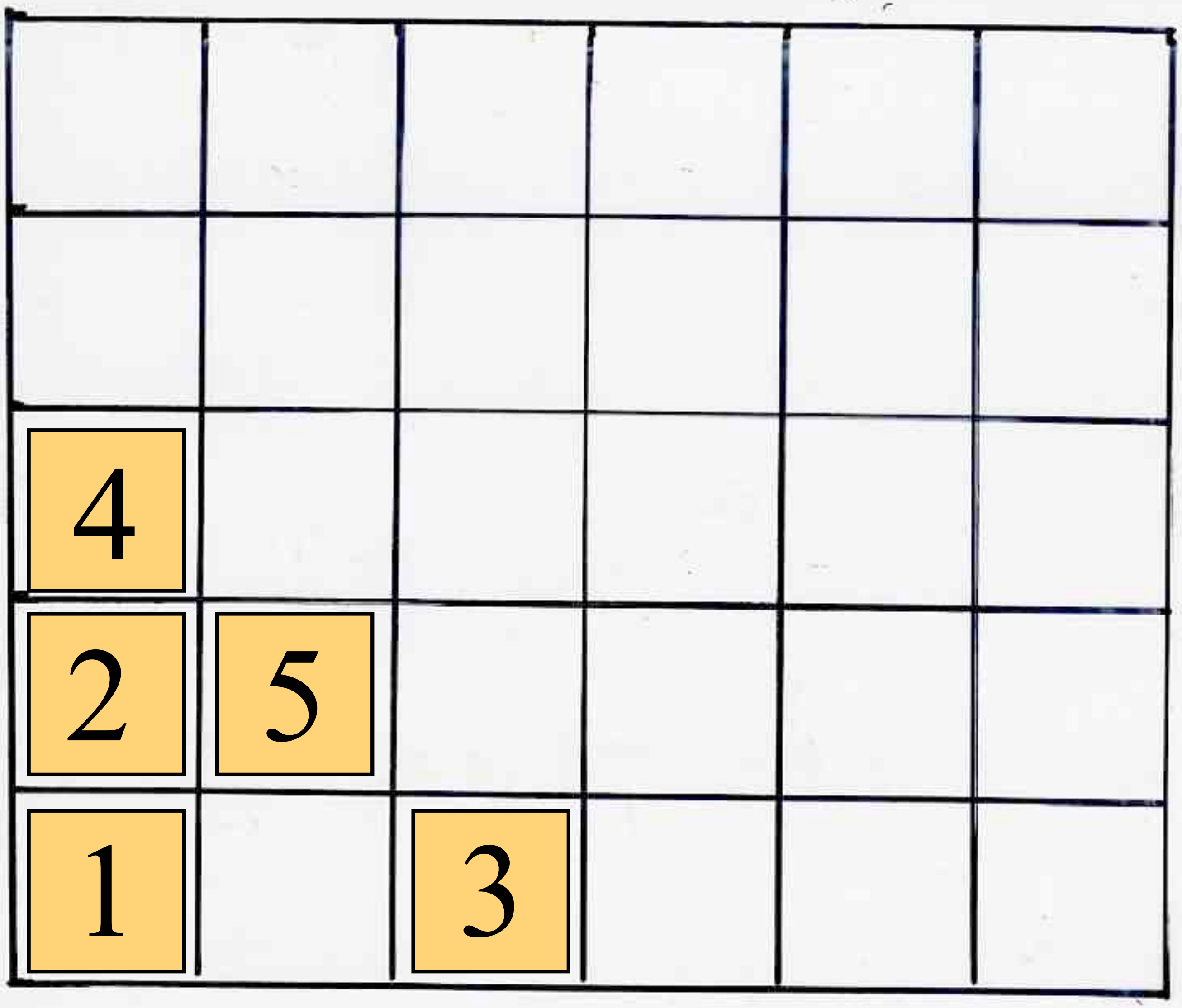
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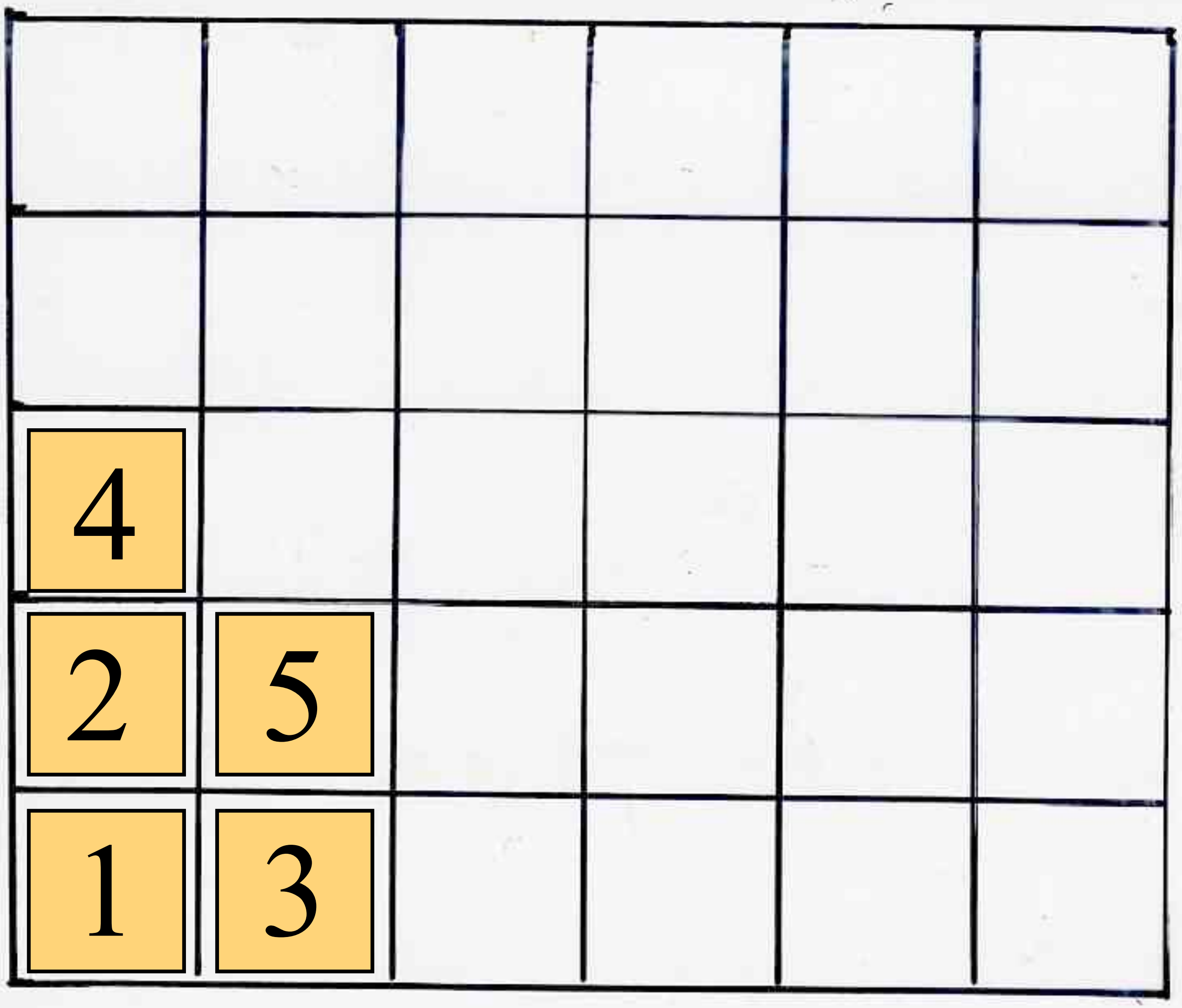
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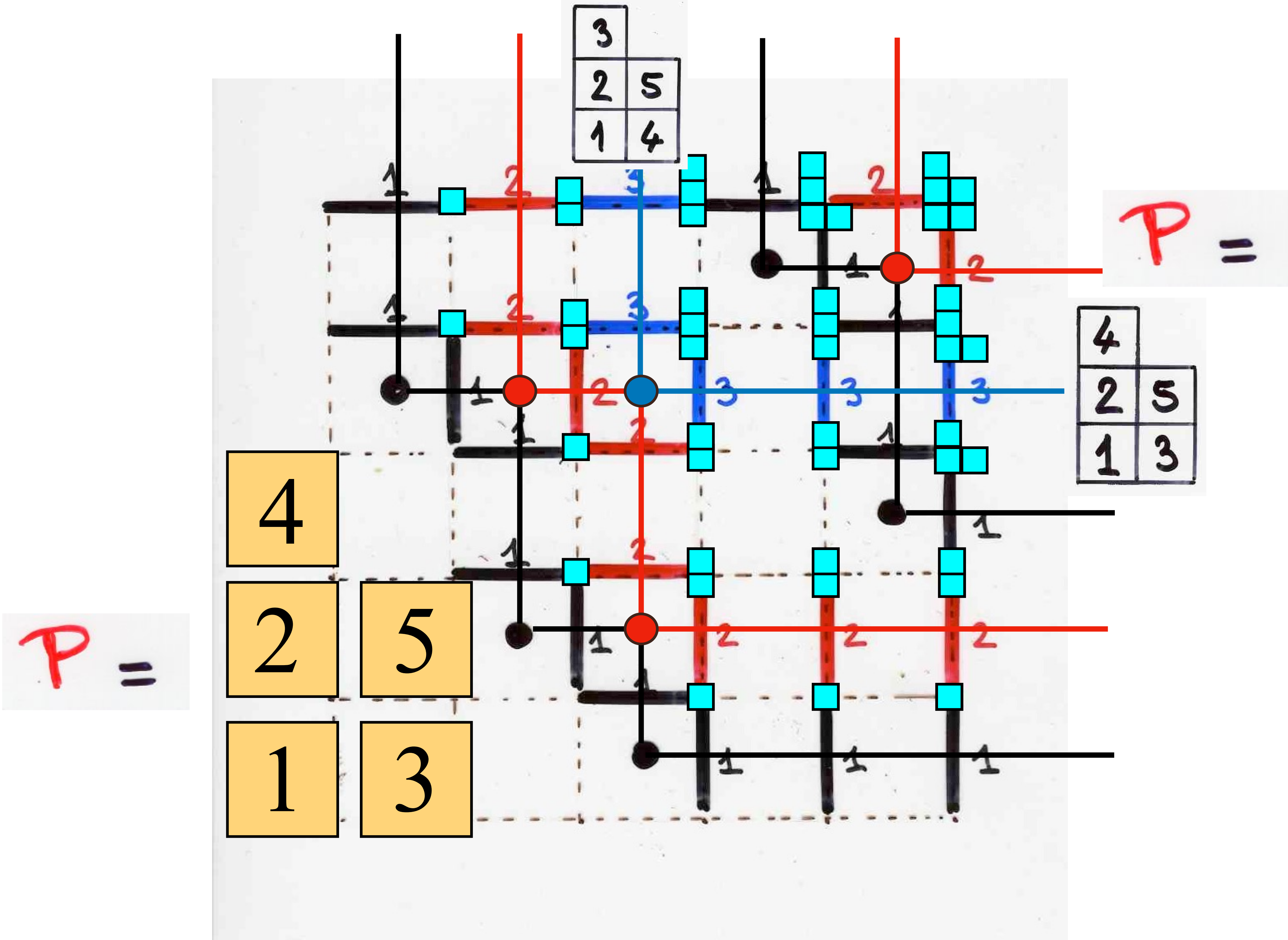
P =
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P =
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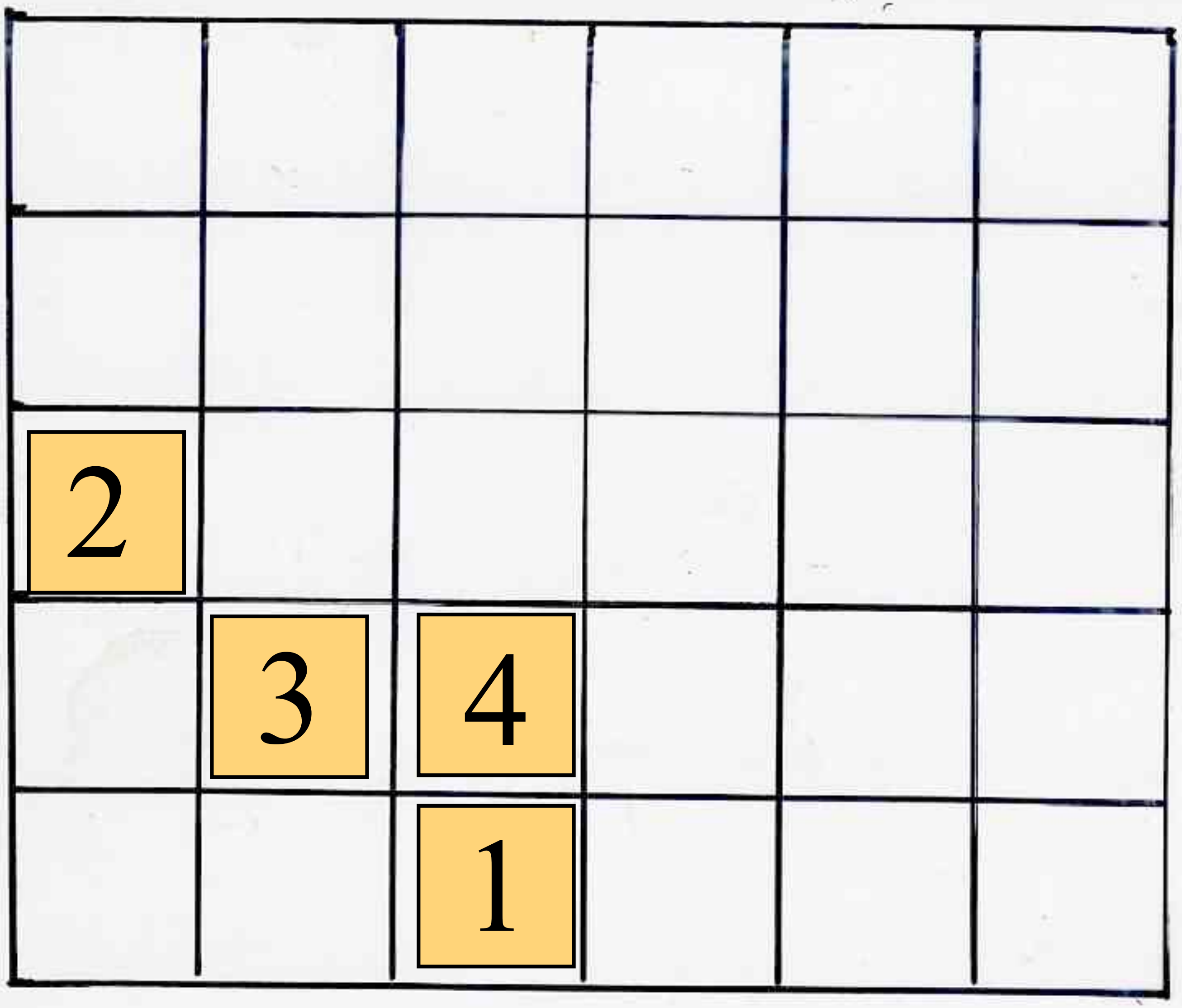
Jeu de taquin

with growth diagrams

S. Fomin, 1986, 1994

Сергей Владимирович Фомин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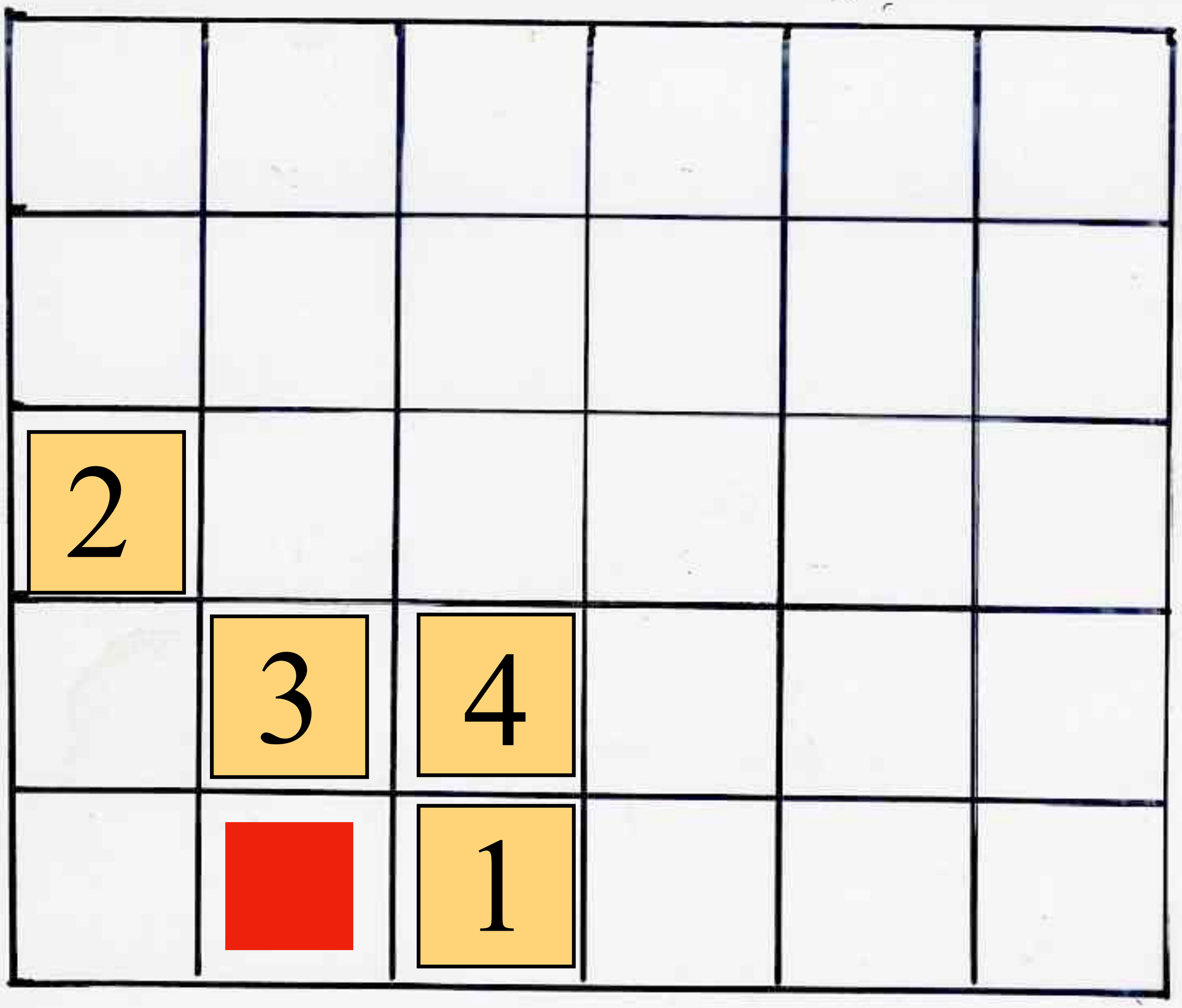
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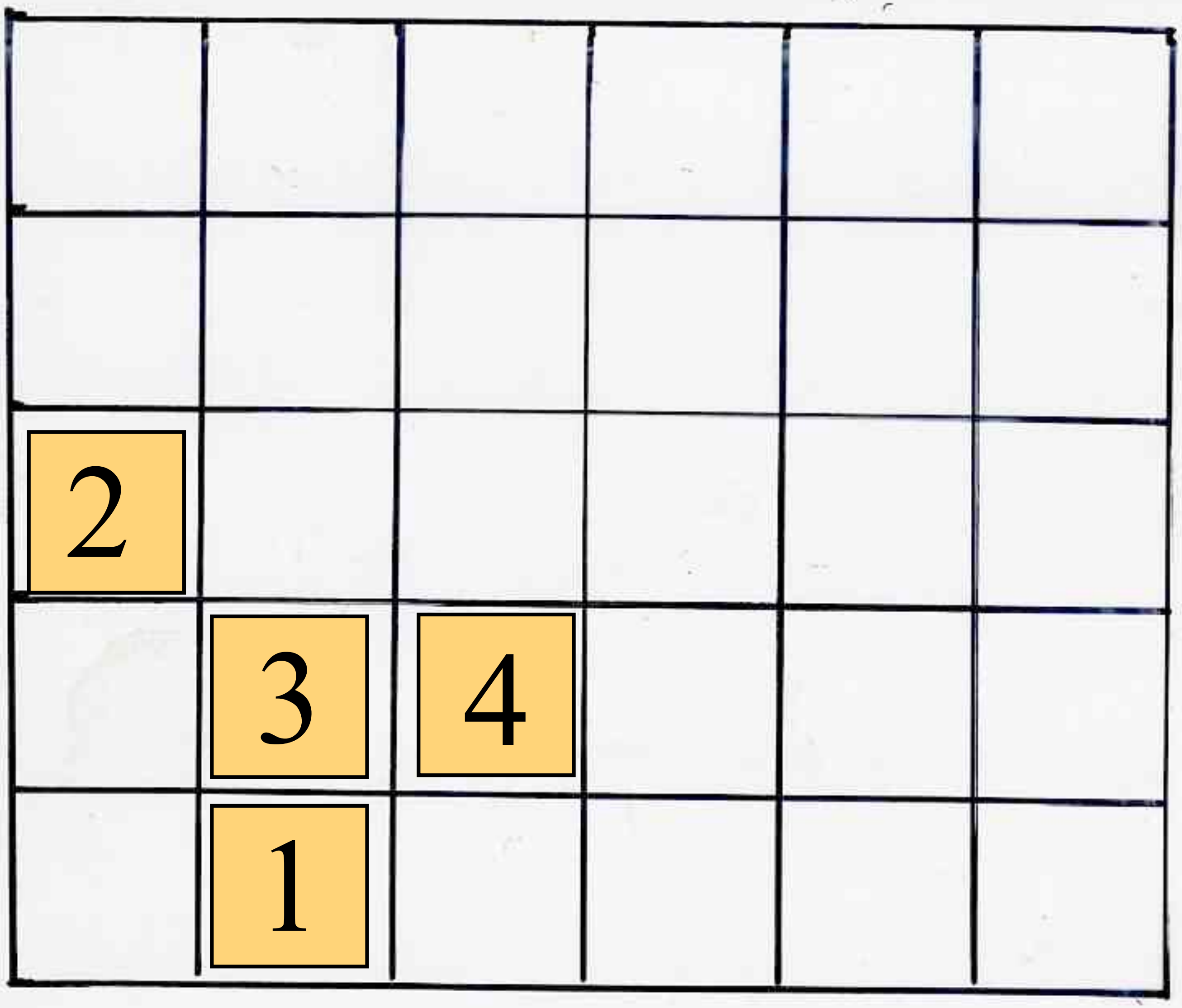
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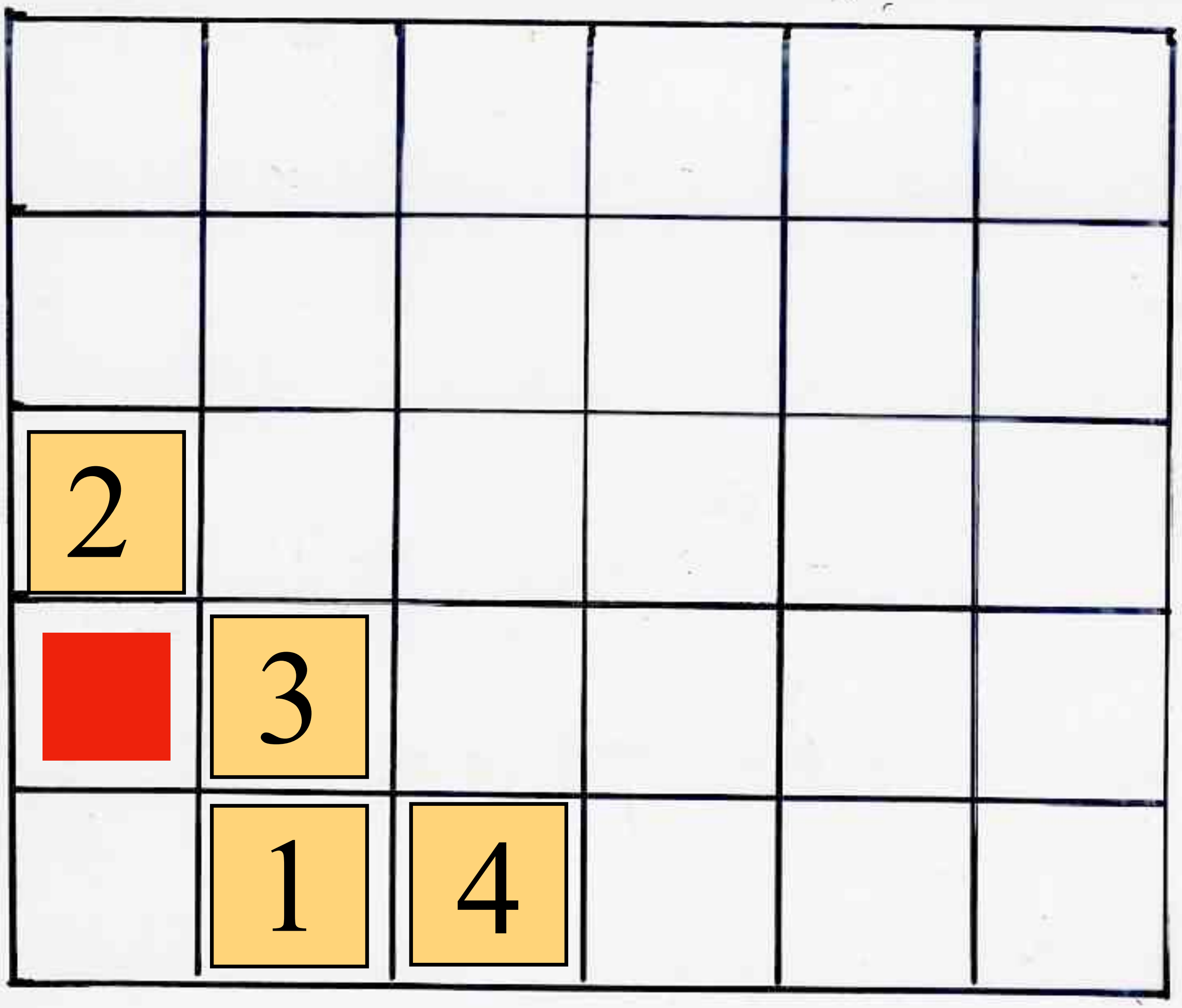

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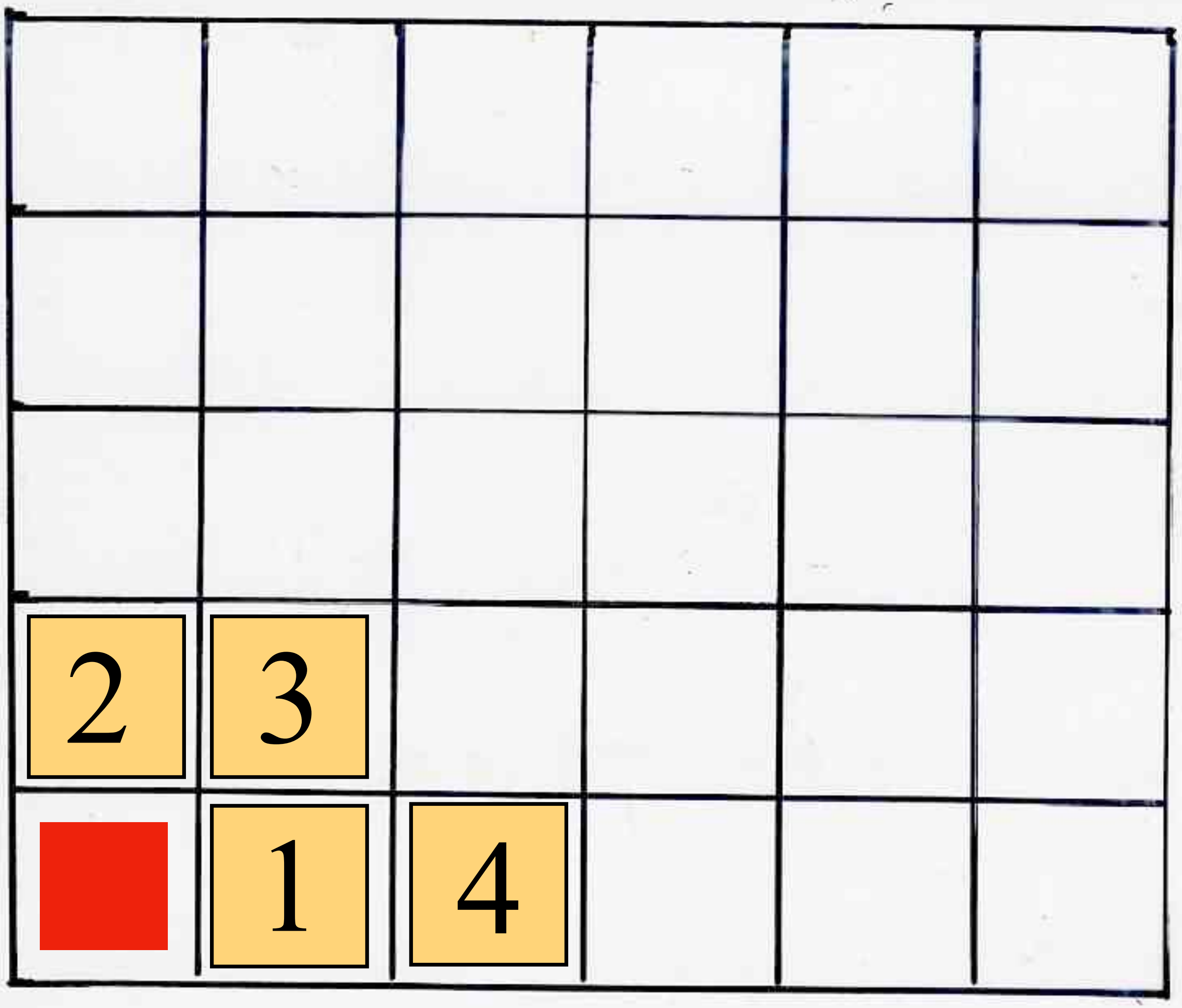
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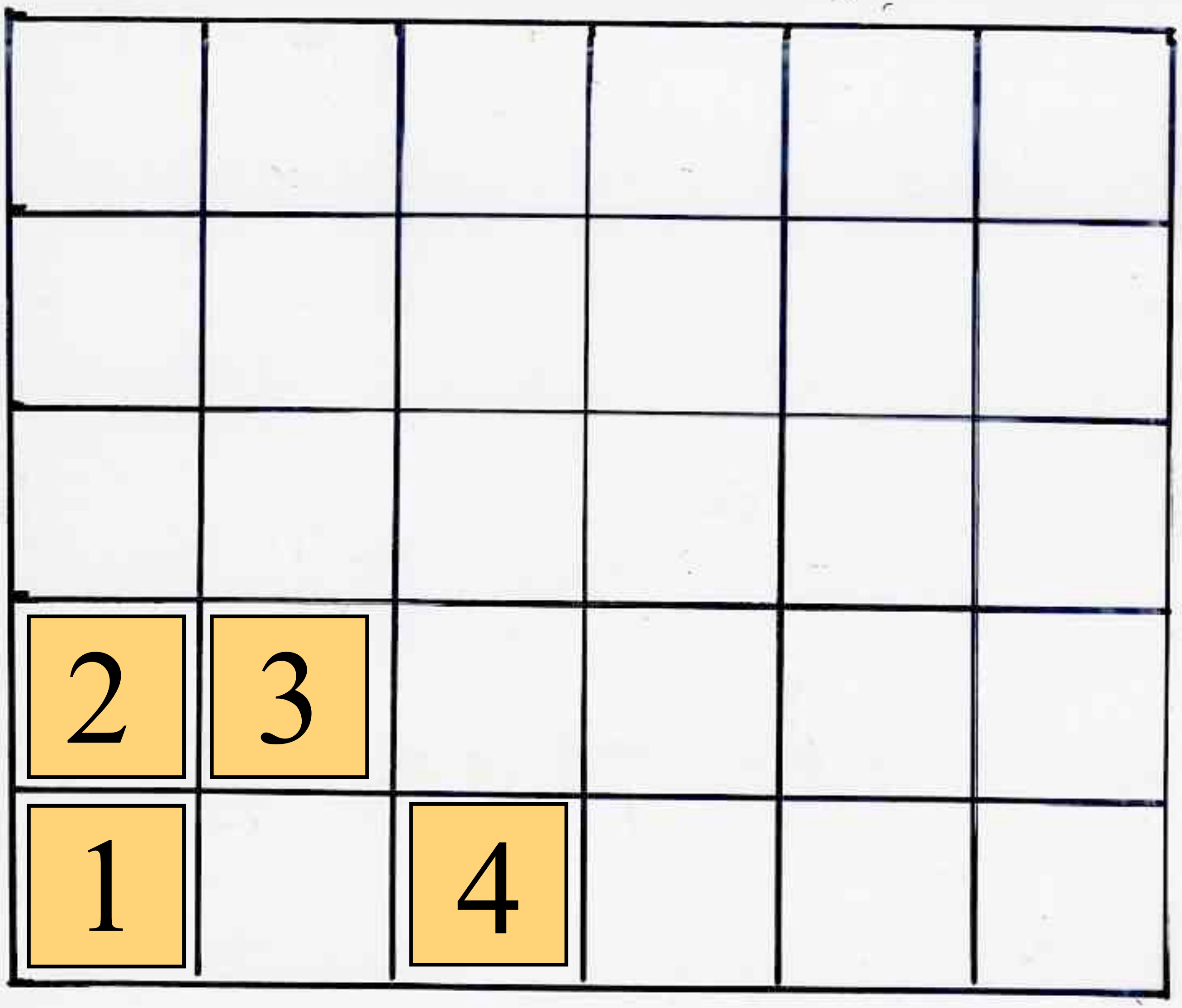
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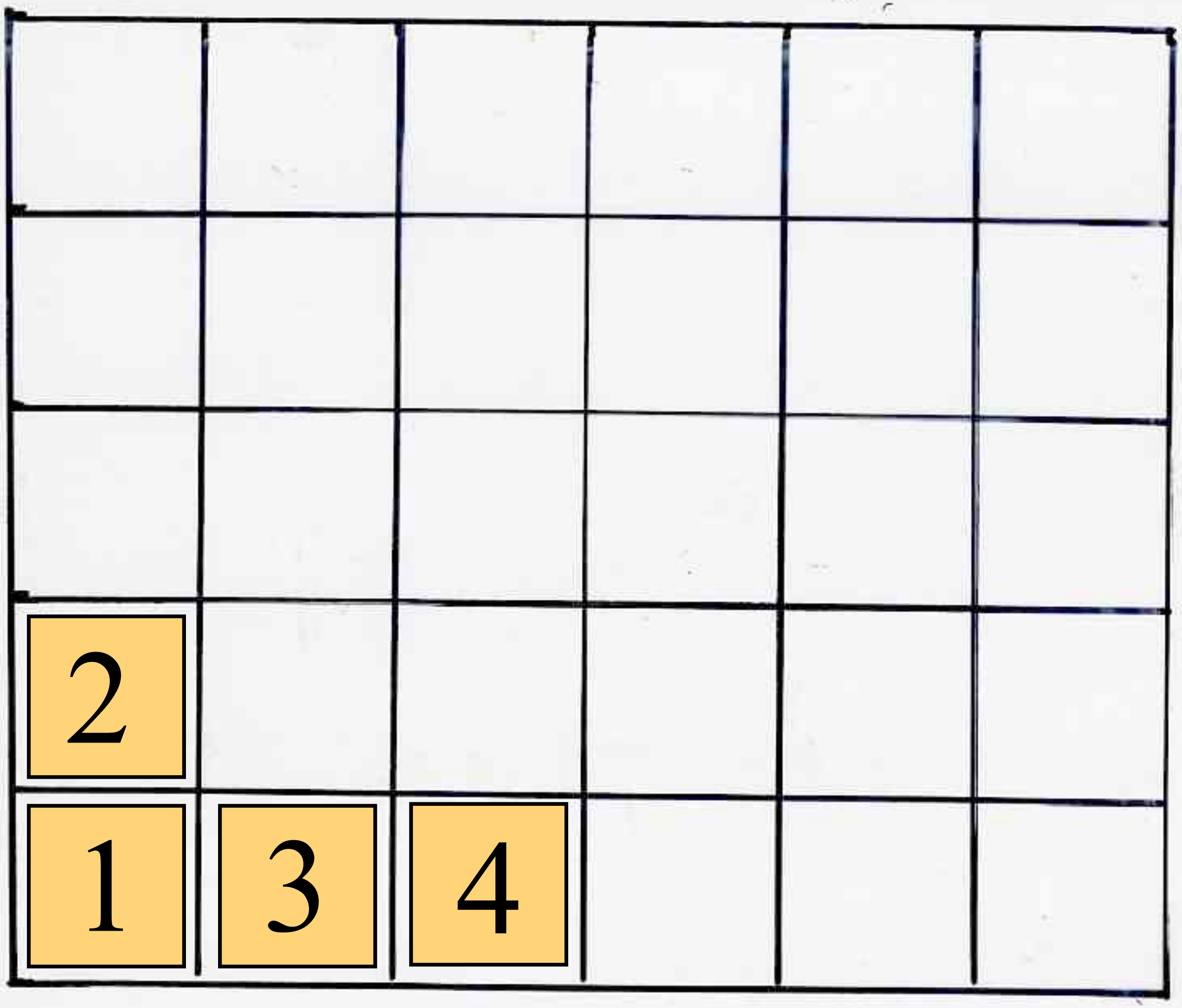
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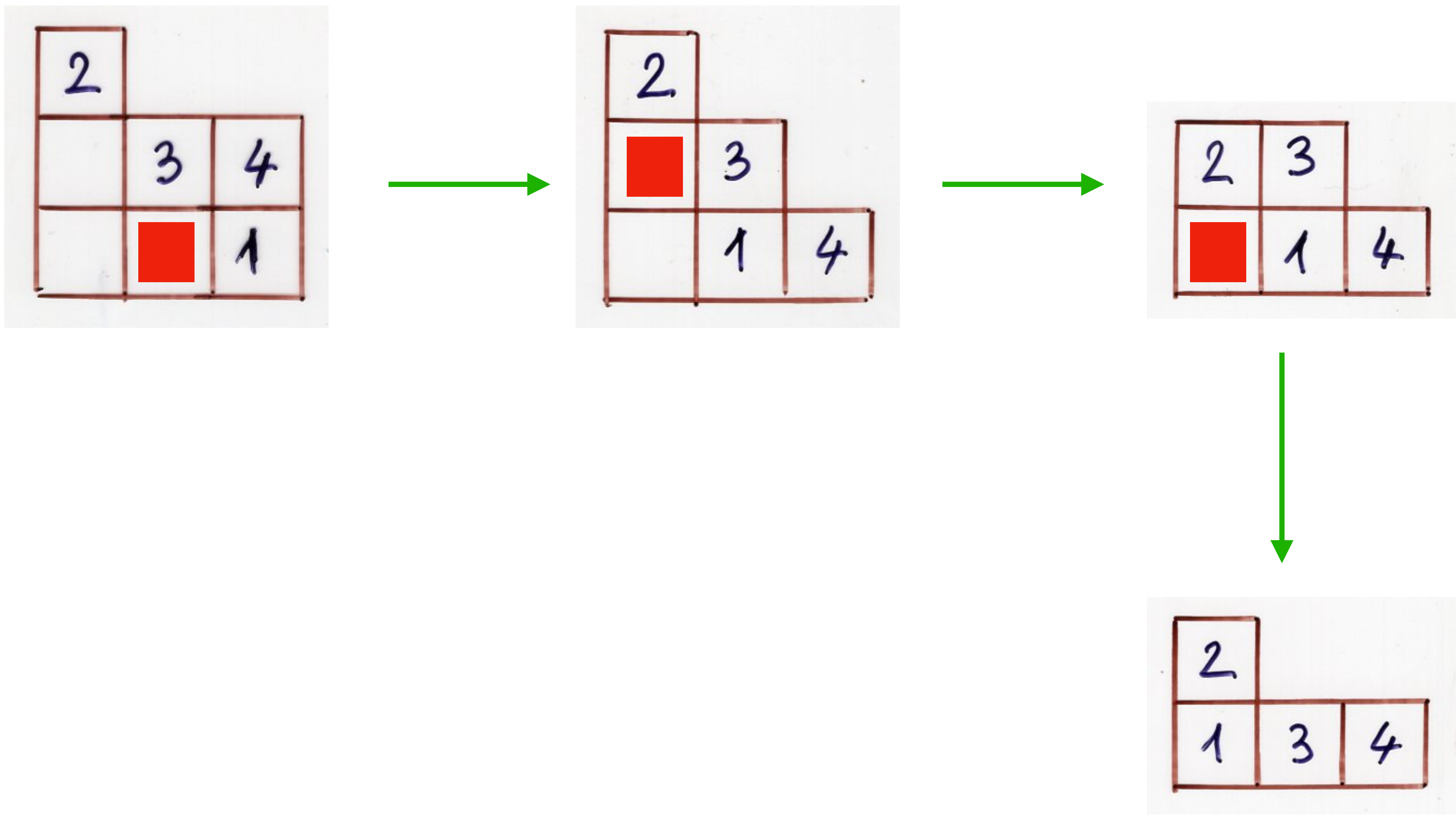

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1 4
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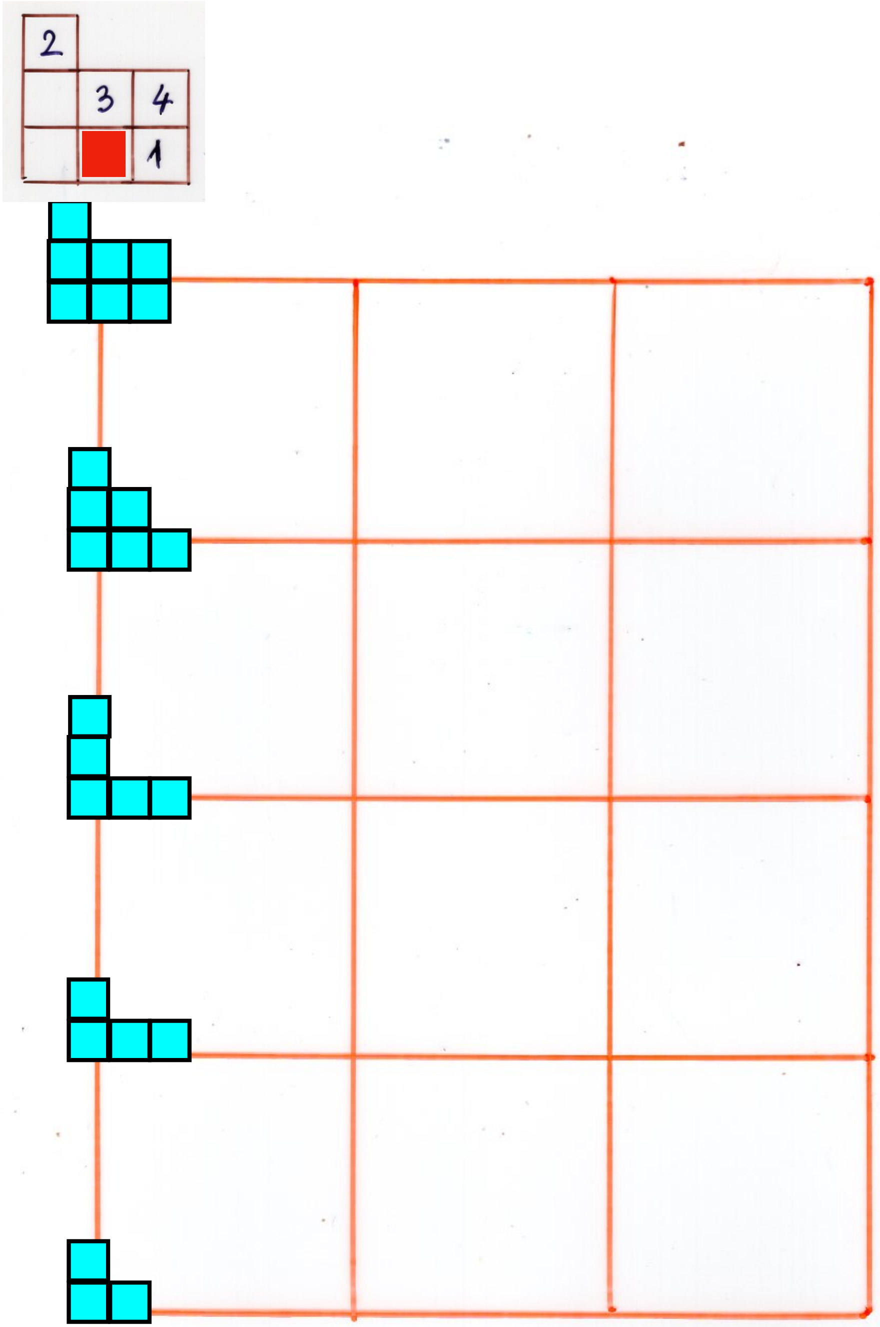
1 3 4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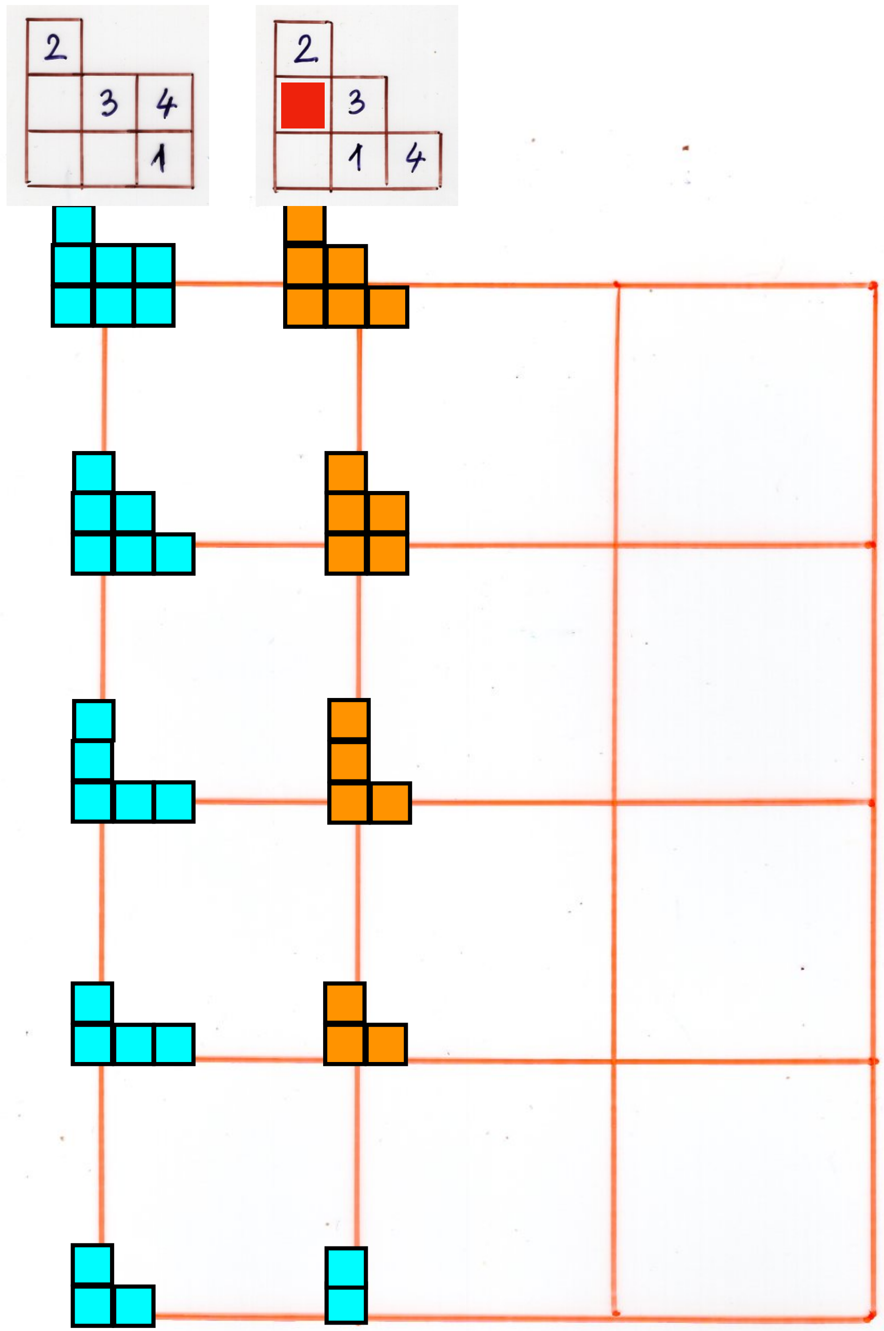
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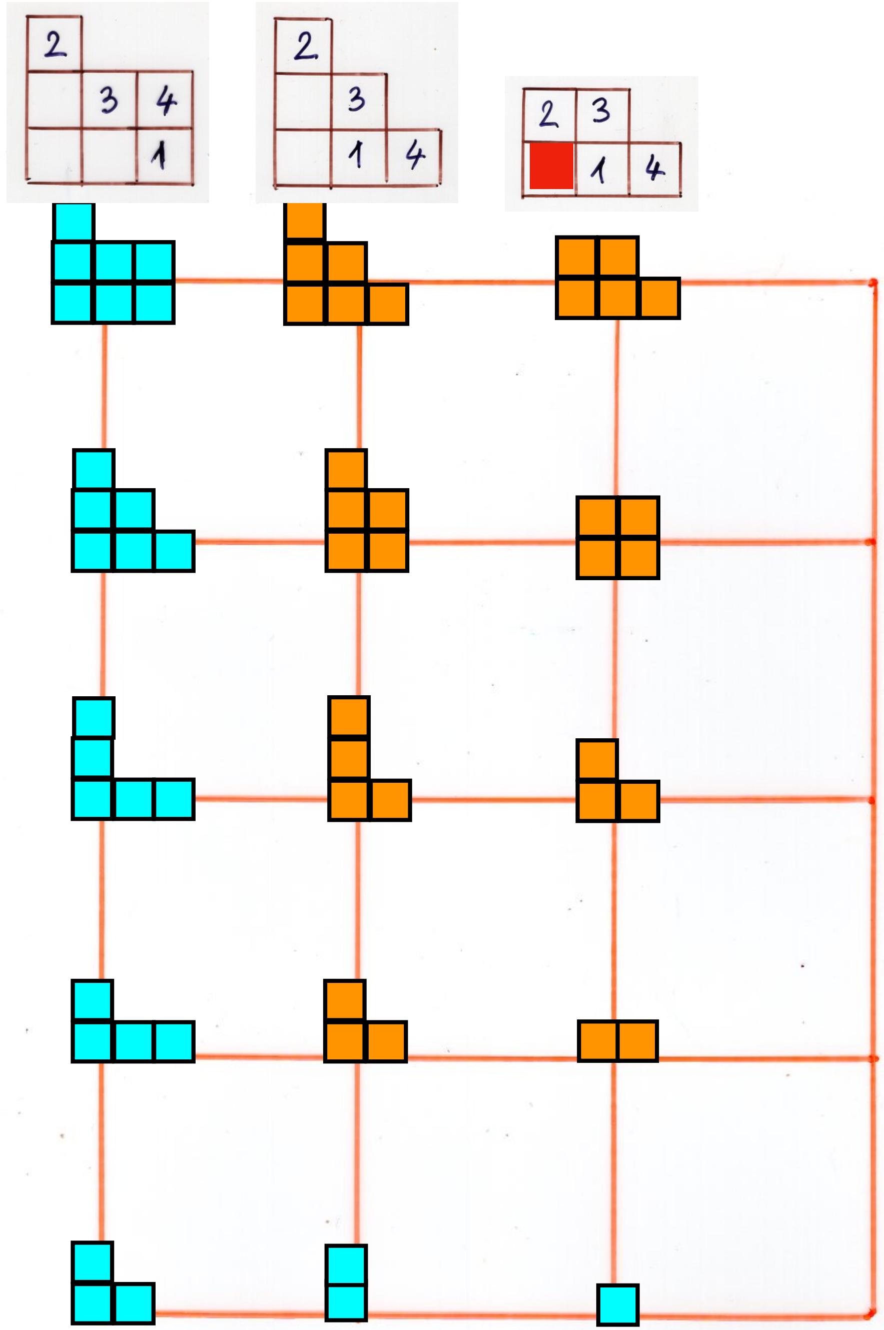
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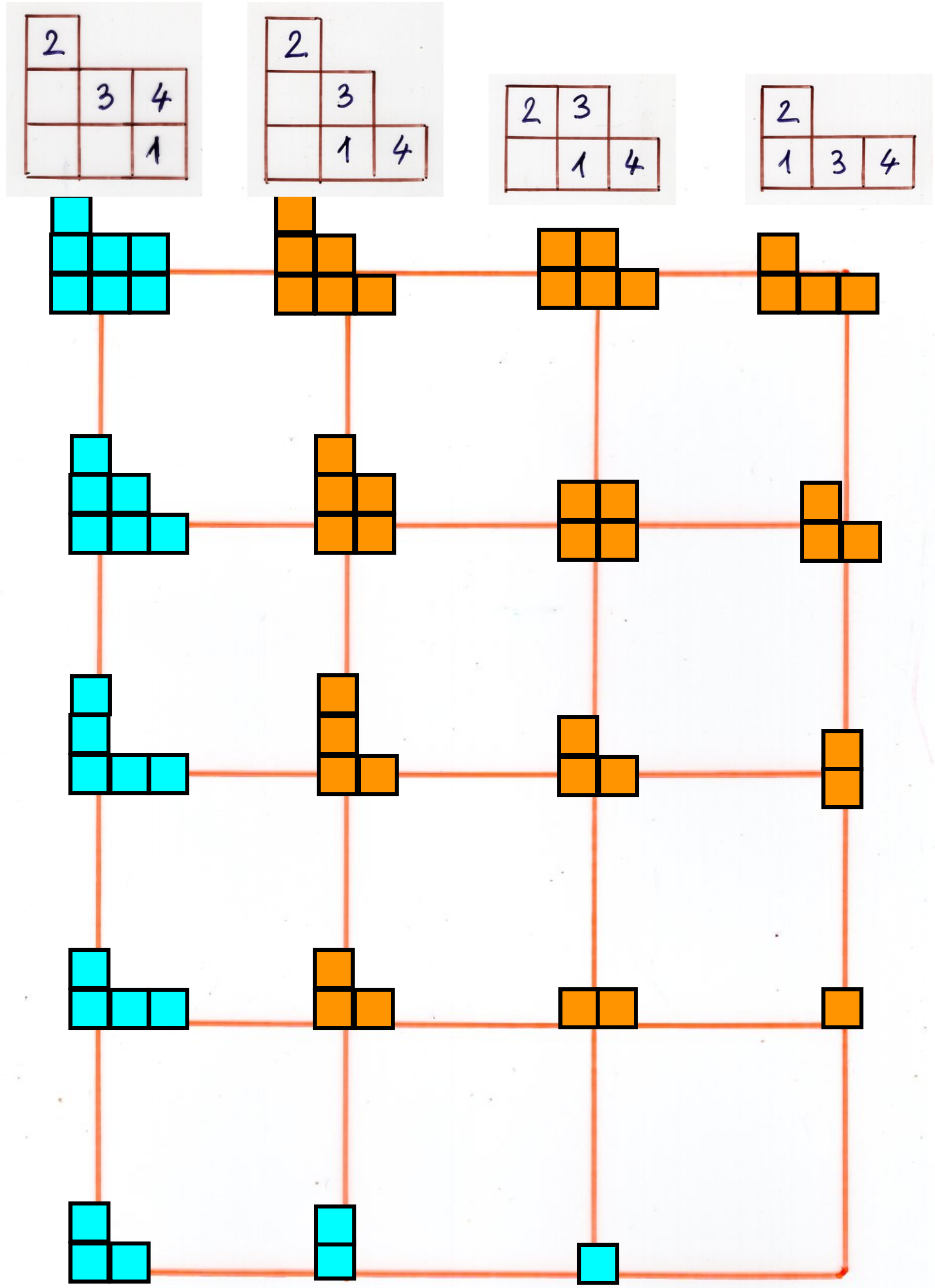
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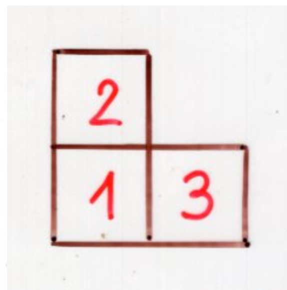
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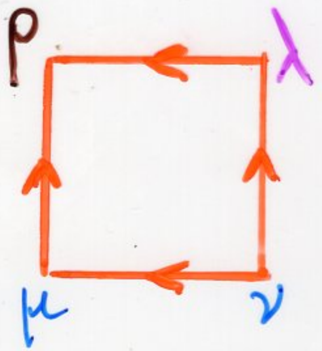
Proposition jeu de taquin local rules (Fomin)

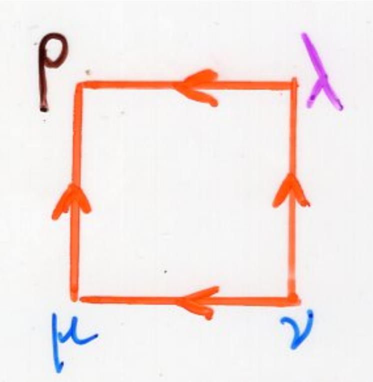


cell of the jeu de taquin growth diagram
(ρ covers μ and λ, μ and λ cover ν)

Then λ is uniquely determined from μ, ν, ρ by the following "local rule":

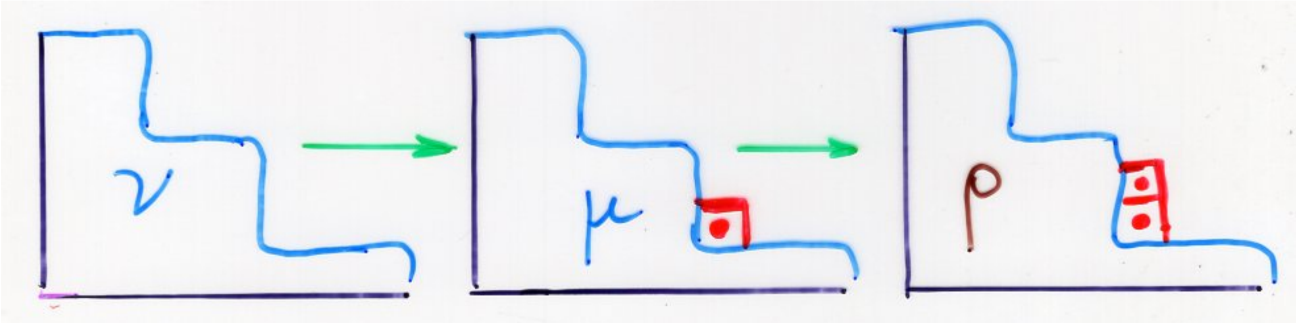
(i) • if μ is the only shape of its size that contains ν and is contained in ρ then $\lambda = \mu$

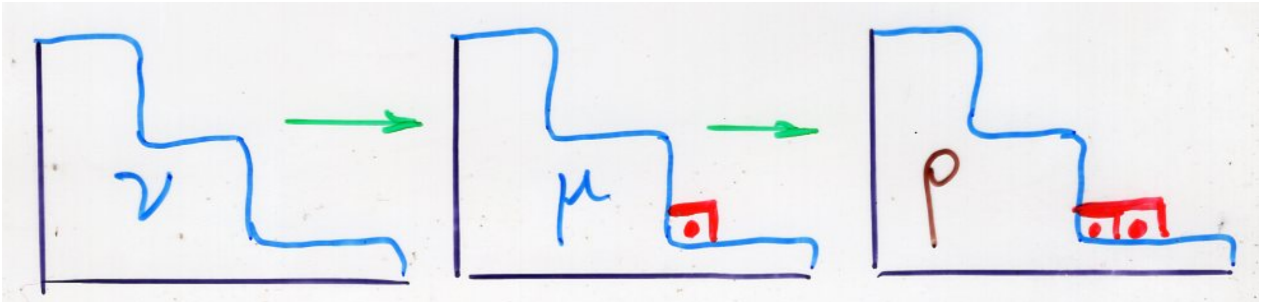
(ii) • otherwise there is a unique such shape different from μ, and this is λ

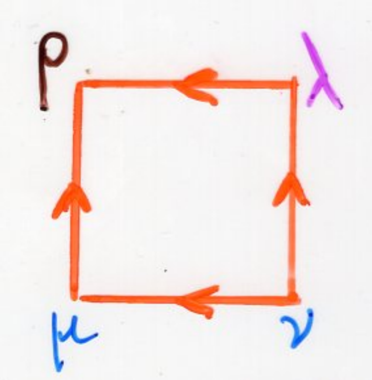


jeu de taquin local rules

(i) • if μ is the only shape of its size that contains ν and is contained in ρ then $\lambda = \mu$

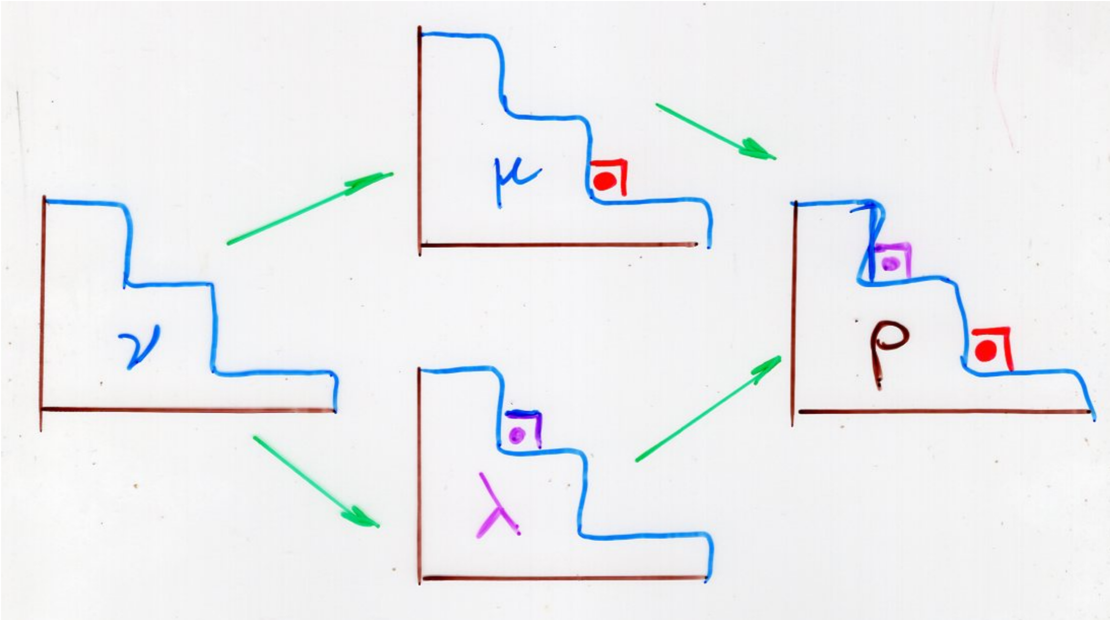


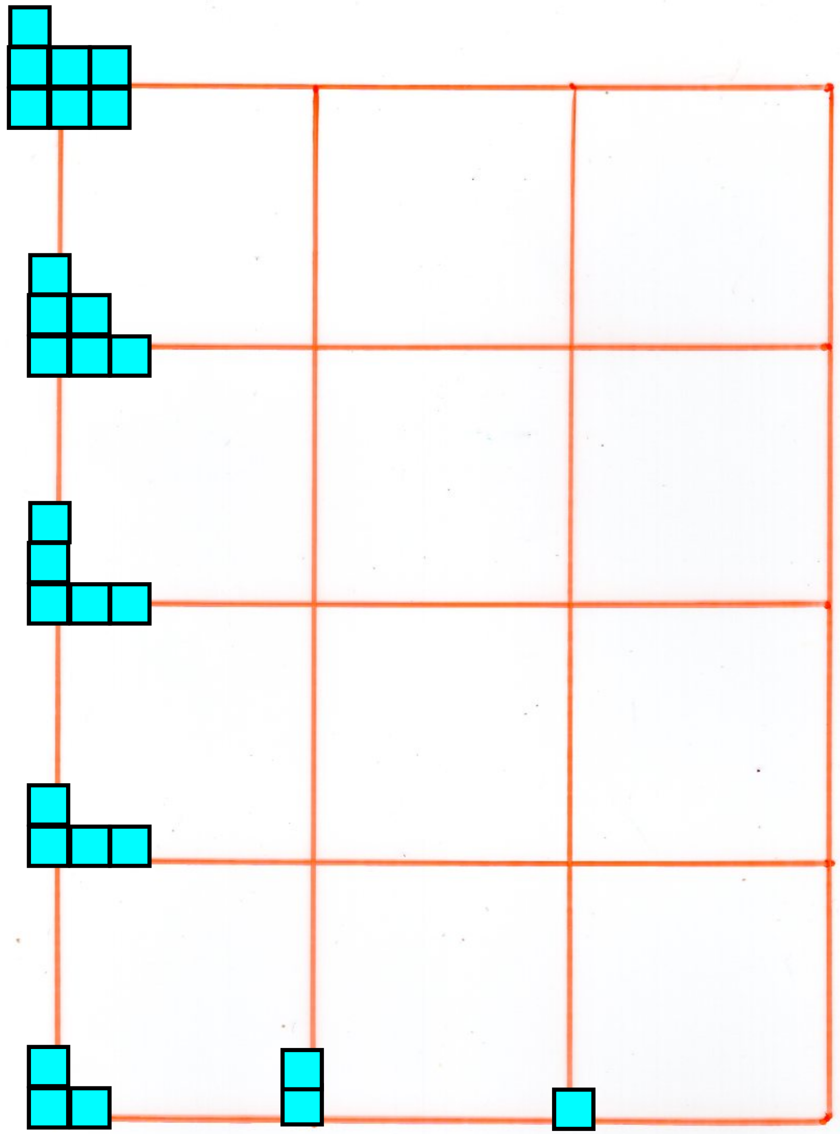


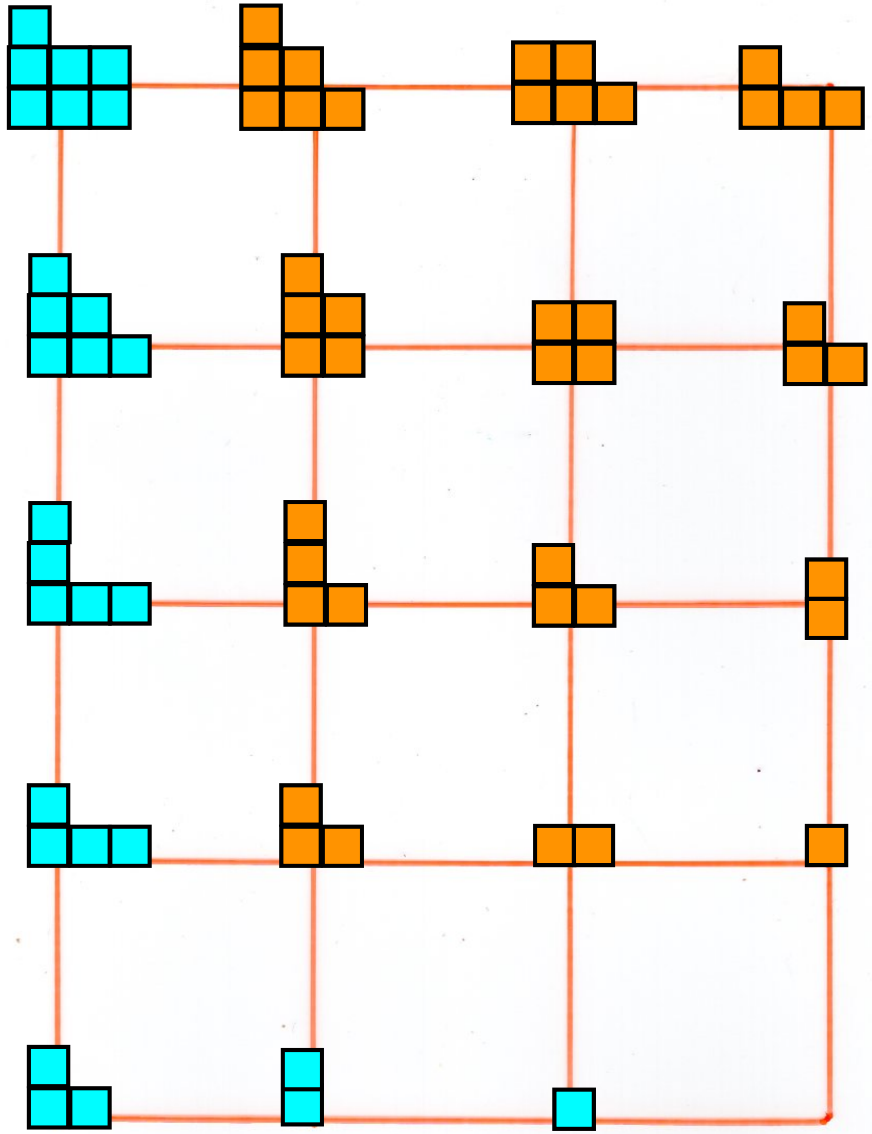


jeu de taquin
local rules

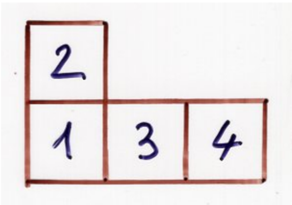
(ii) • otherwise there is a unique such shape different from μ, and this is λ



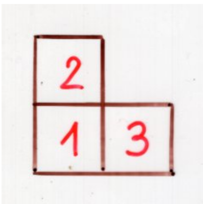


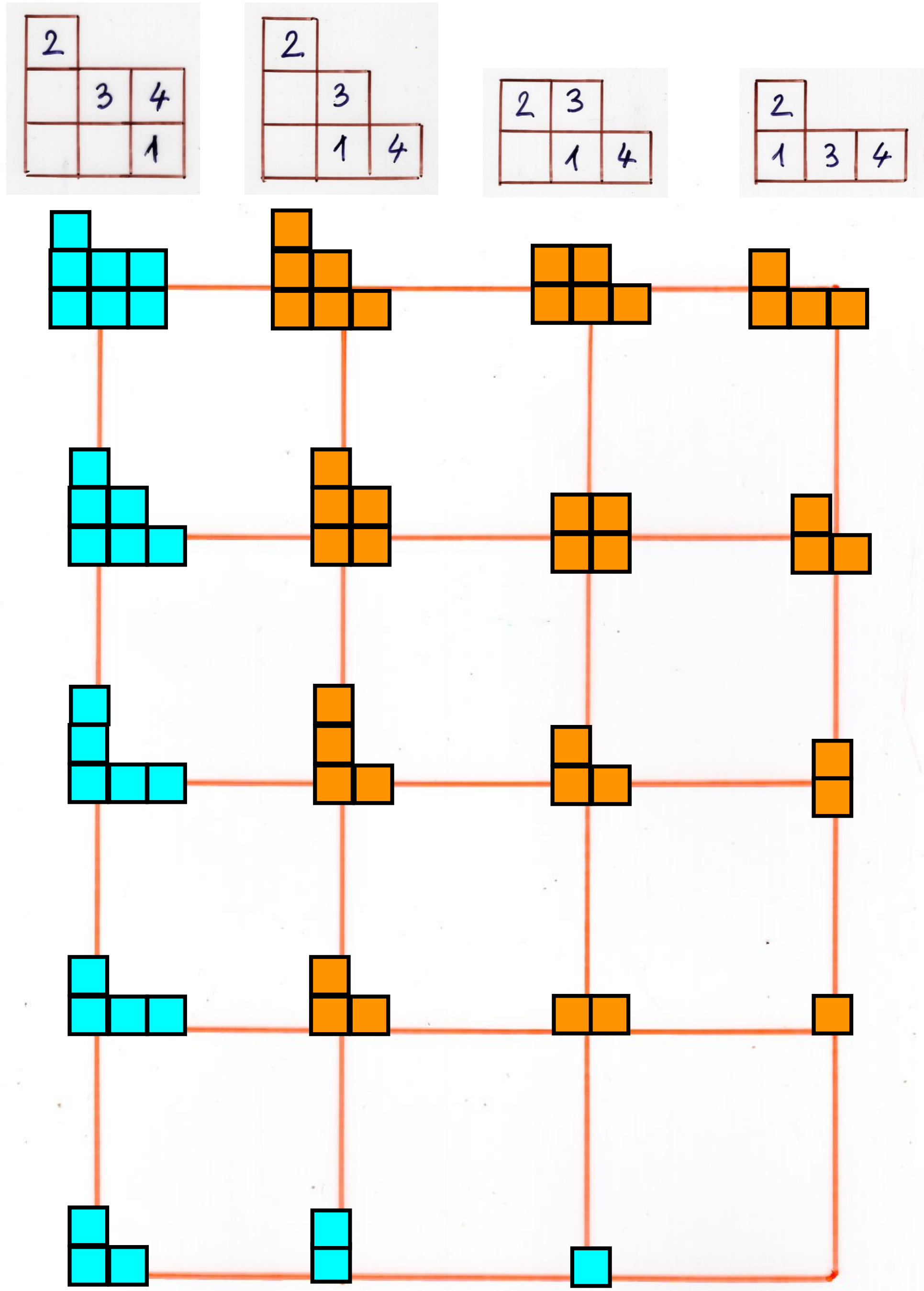


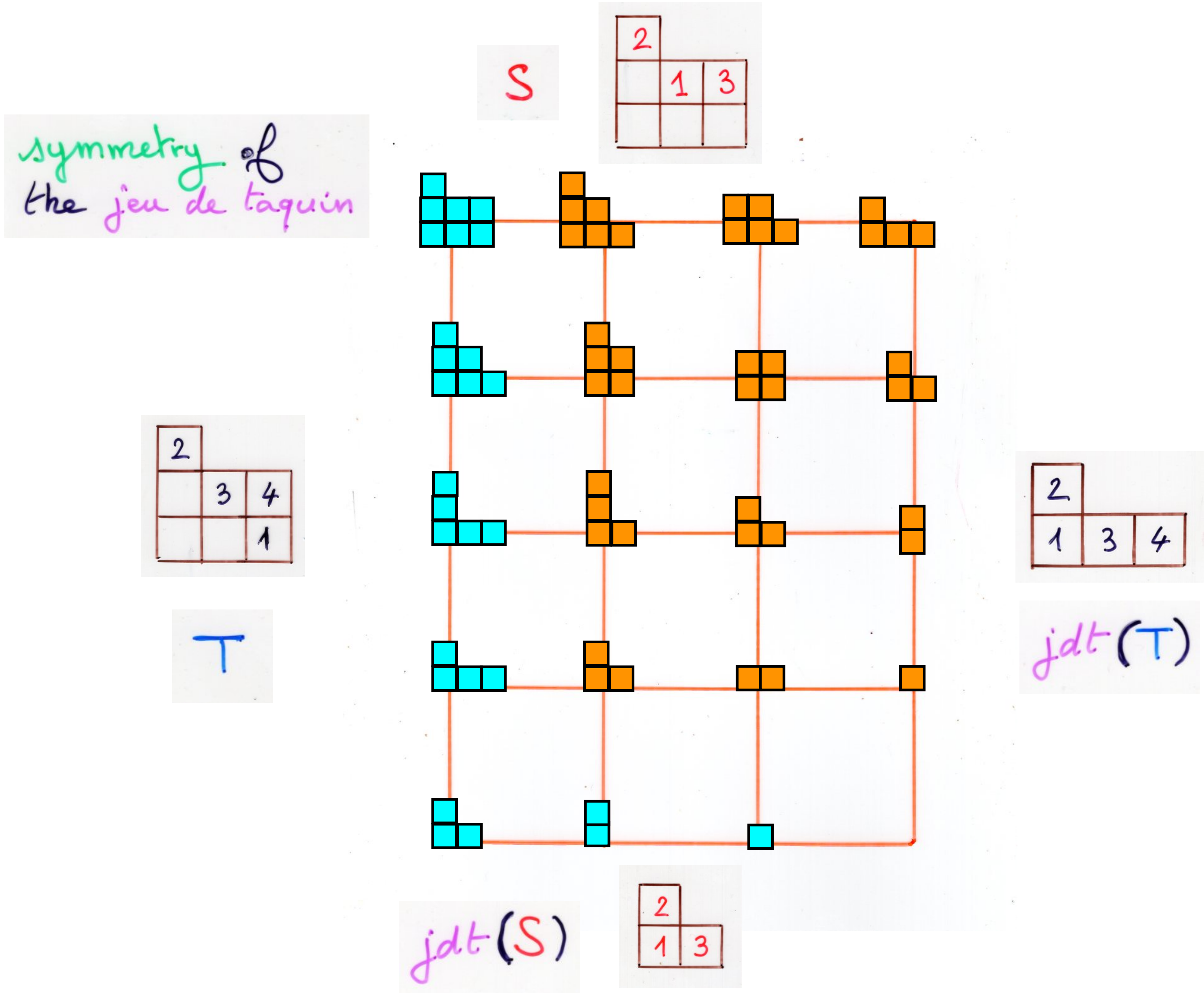
the tableau



is independant of the choice of the tableau





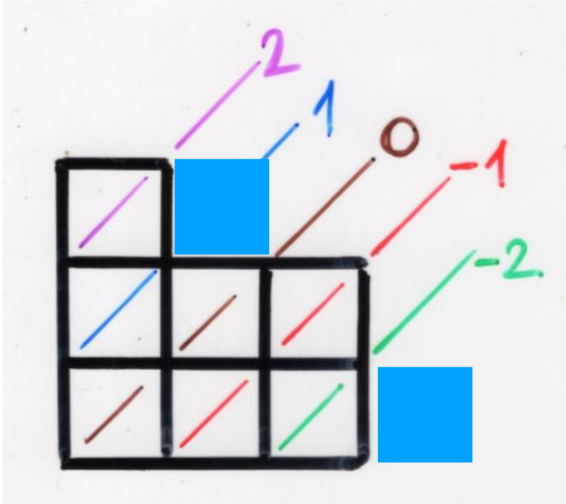

symmetry of the jeu de taquin
S
2
1 3
T
2
3 4
1
jdt(T)
2
1 3 4
jdt(S)
2
1 3

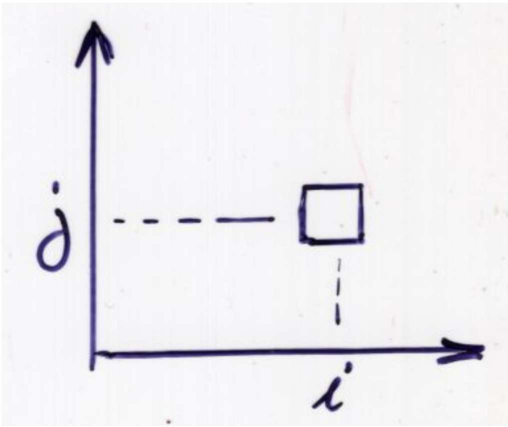
Jeu de taquin

with local rules on edges ?

diagonal operators

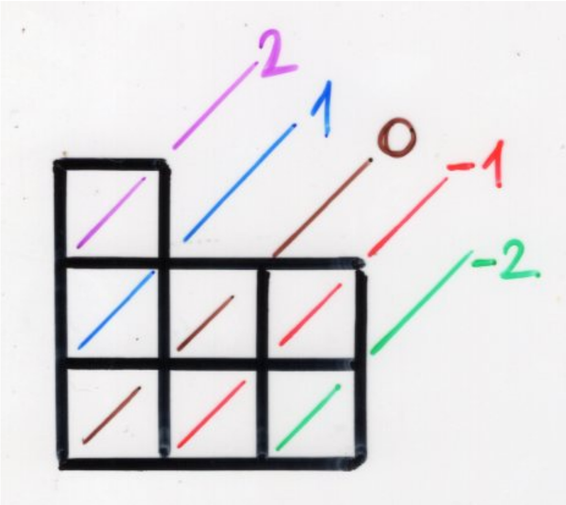
$\Delta_i \quad i \in \mathbb{Z}$





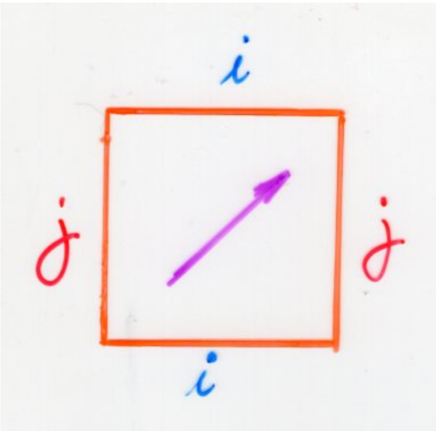
$(i, j) \longrightarrow j - i$

content

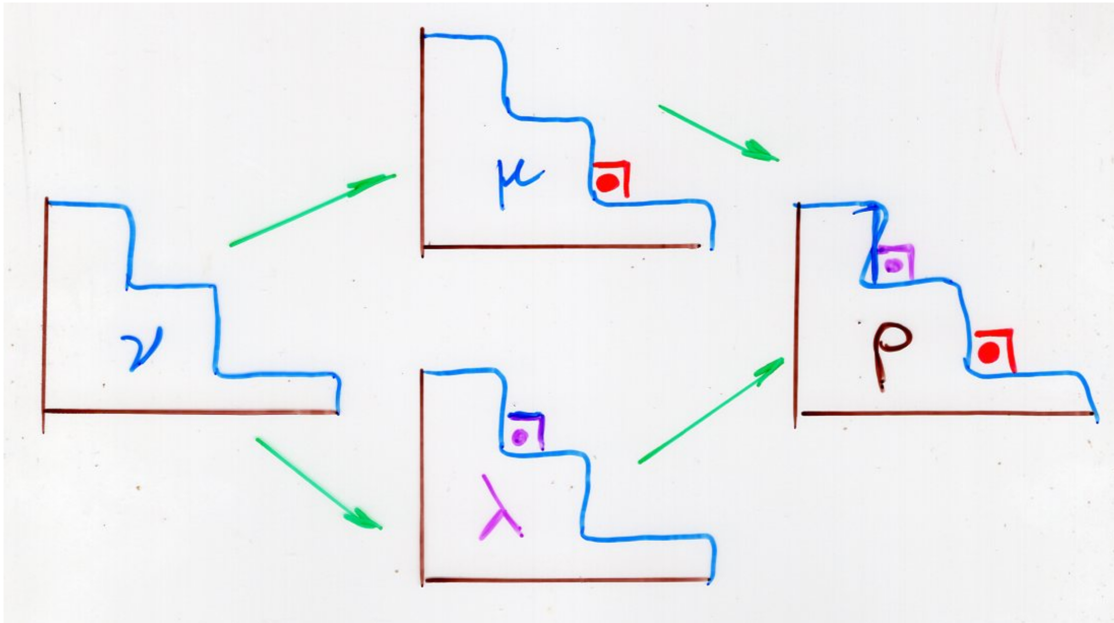


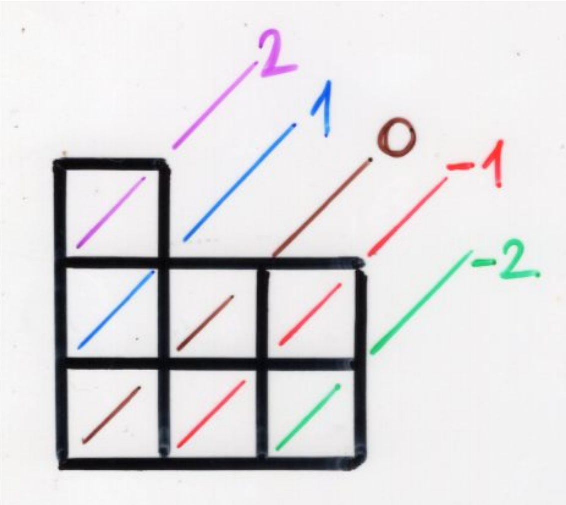
jeu de taquin

local rules on edges



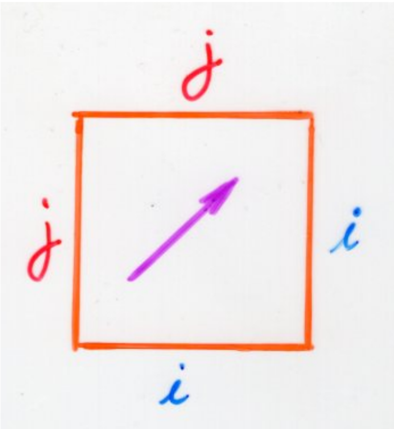
$$|i-j| \geq 2$$





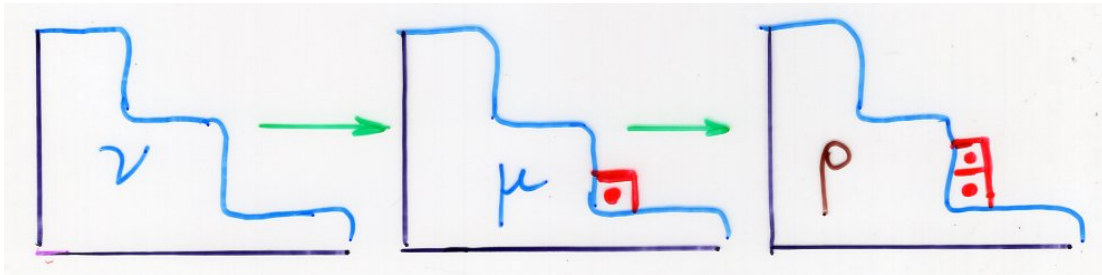
jeu de taquin

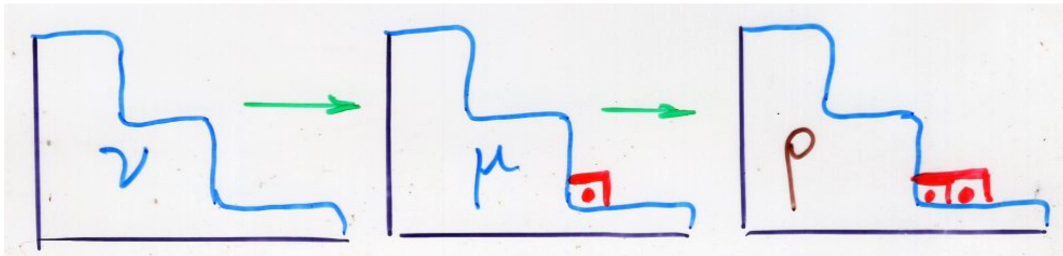
local rules on edges

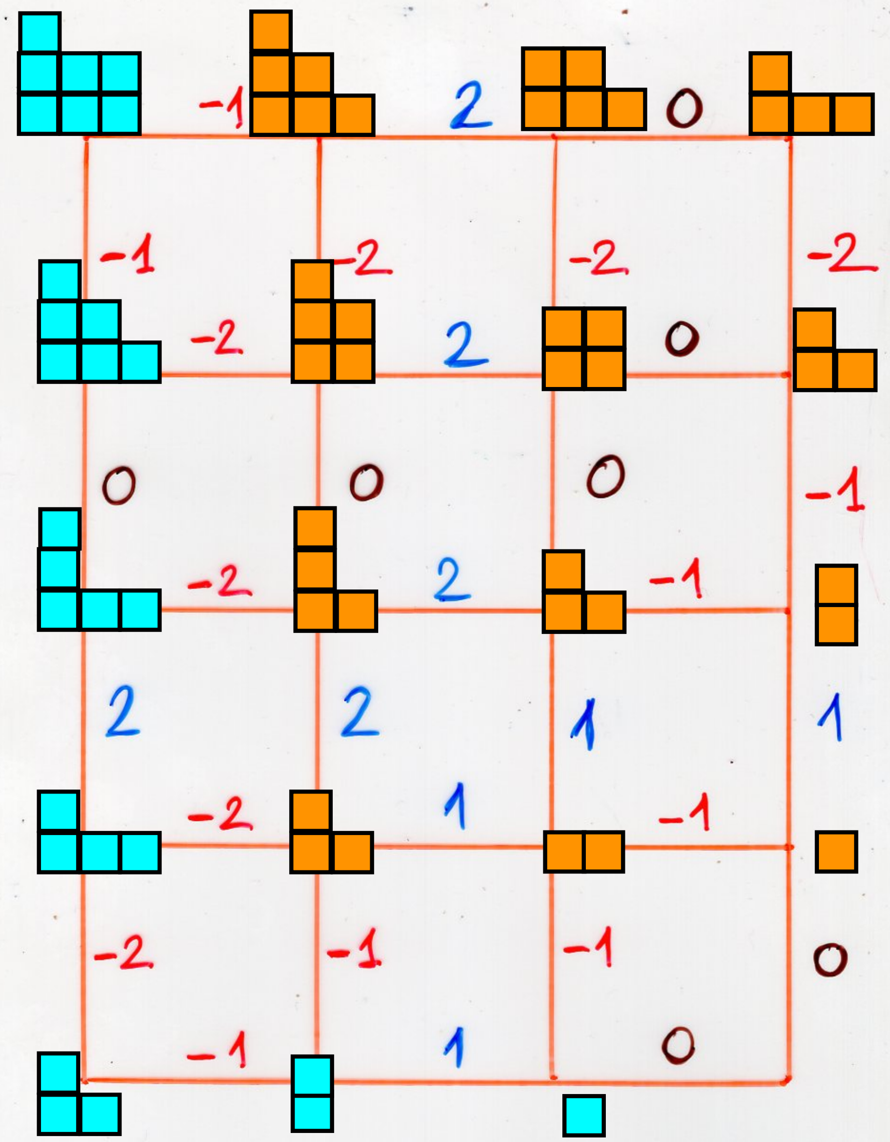


$|i-j| \leq 1$

or







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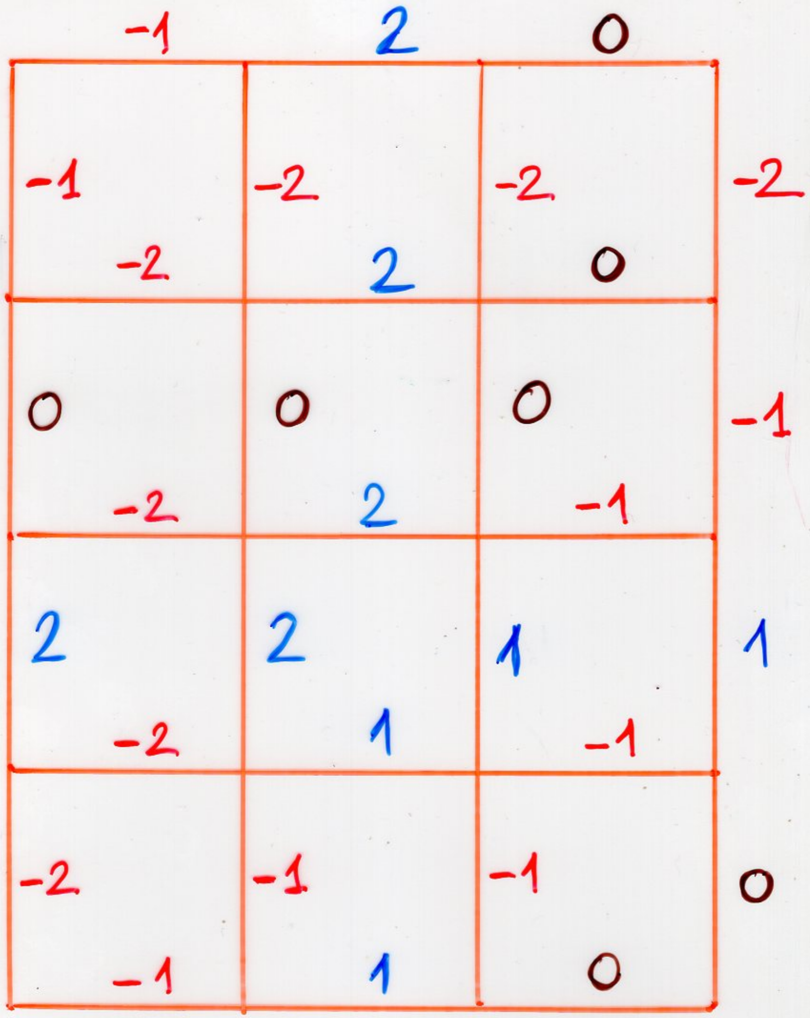
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-2 2 0
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-2 2 -1
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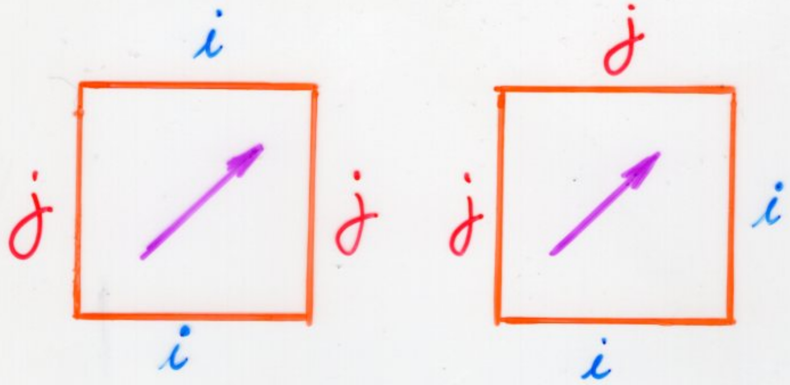
-2 1 -1
-2 -1 -1 0
-1 1 0

jeu de taquin

local rules on edges



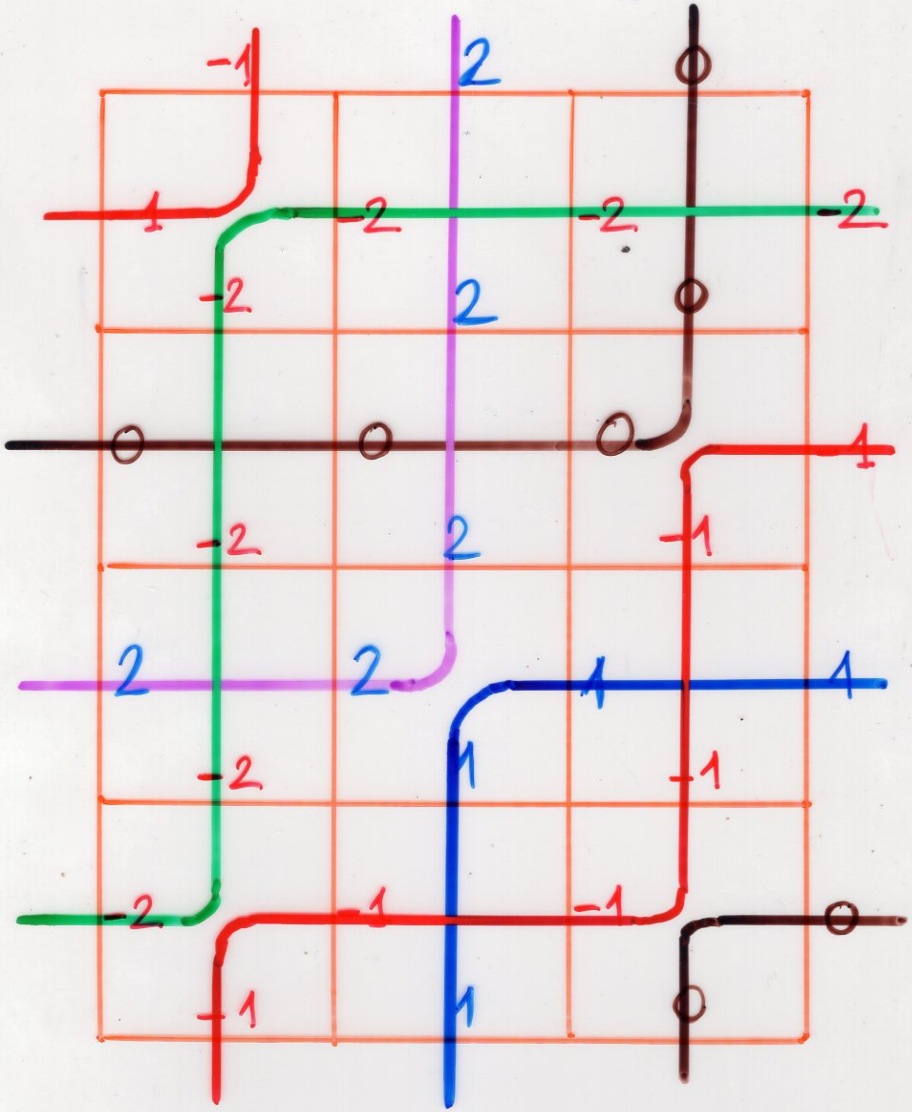
$i, j \in \mathbb{Z}$

$|i-j| \geqslant 2$ $|i-j| \leqslant 1$

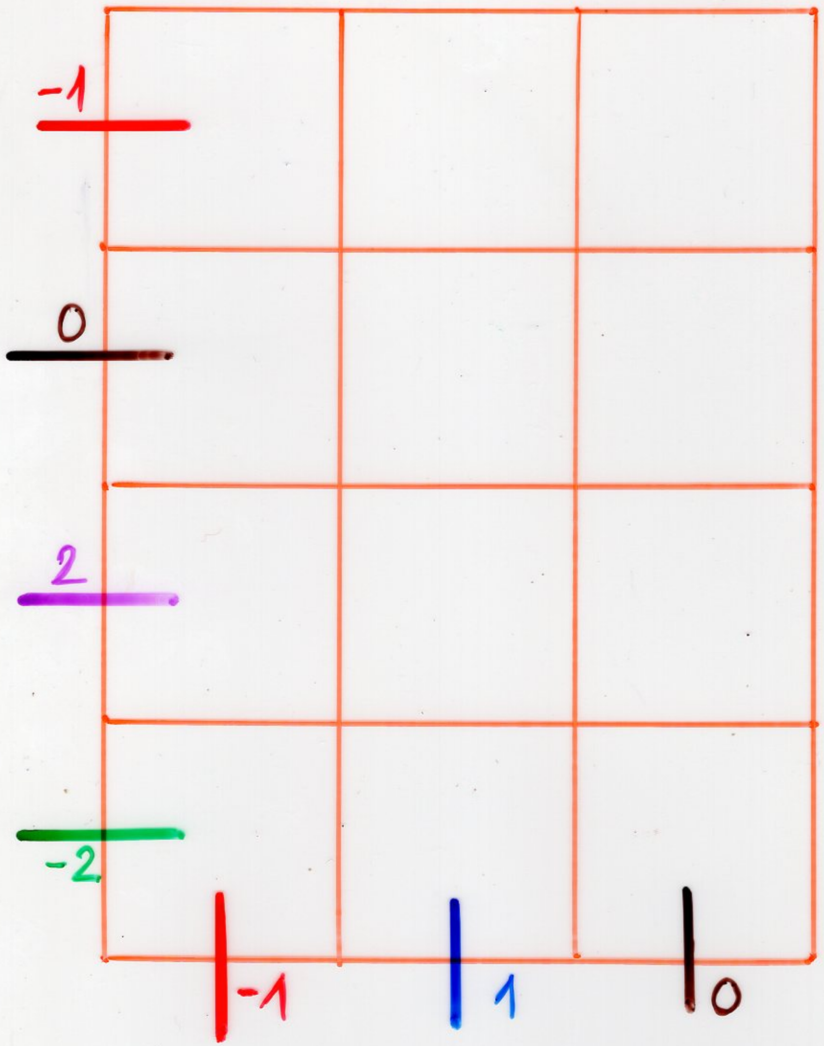
in fact here $i=j$ impossible

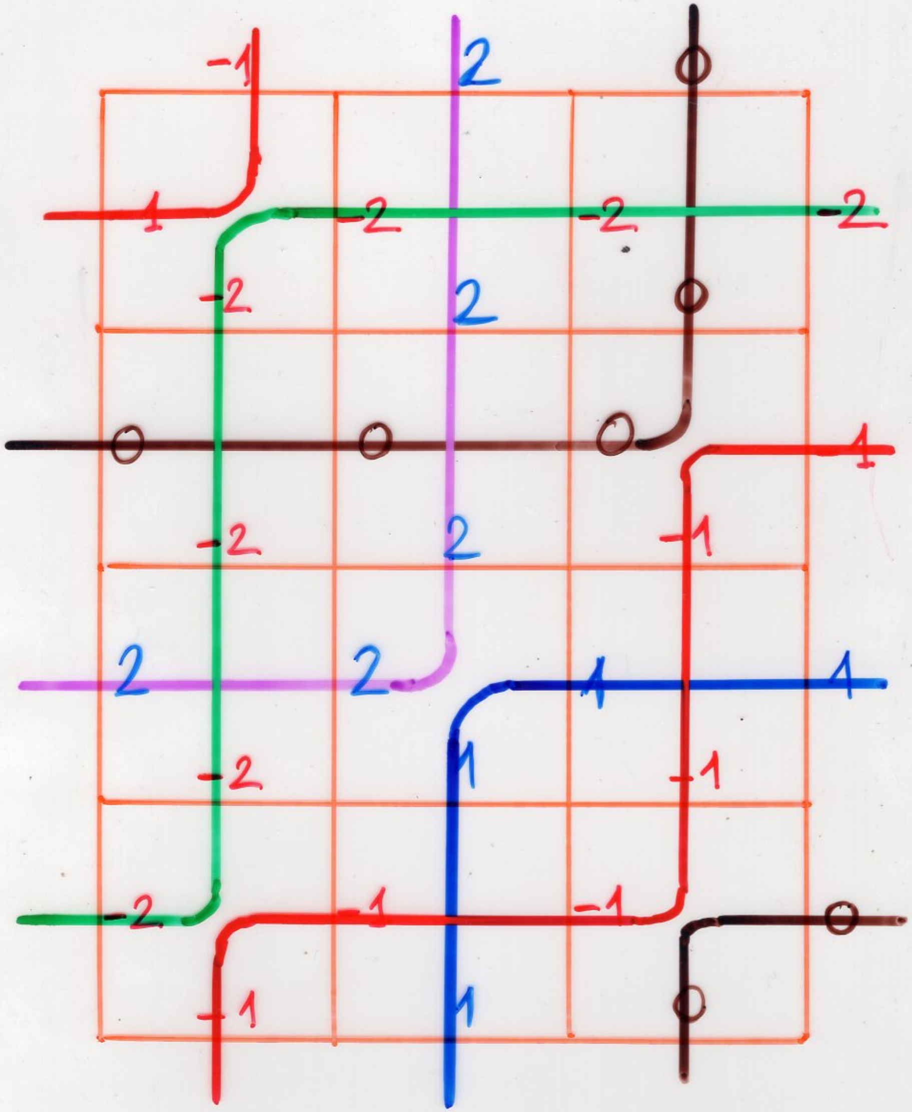
nil-Temperley-Lieb

planar automaton



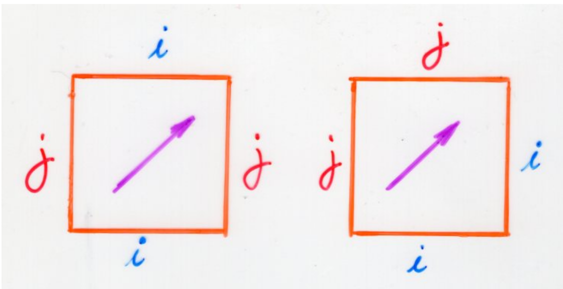
-1
2
0
1
-2
-2
-2
-2
2
0
0
0
0
1
-2
2
-1
2
2
2
1
4
-2
1
-1
-2
-1
-1
0
-1
1
0


-1
0
2
-2
-1
1
0


-1
2
0
1
-2
-2
-2
-2
2
0
0
0
0
1
-2
2
-1
2
2
1
4
1
-2
-1
-2
-1
-1
0
-1
1
0

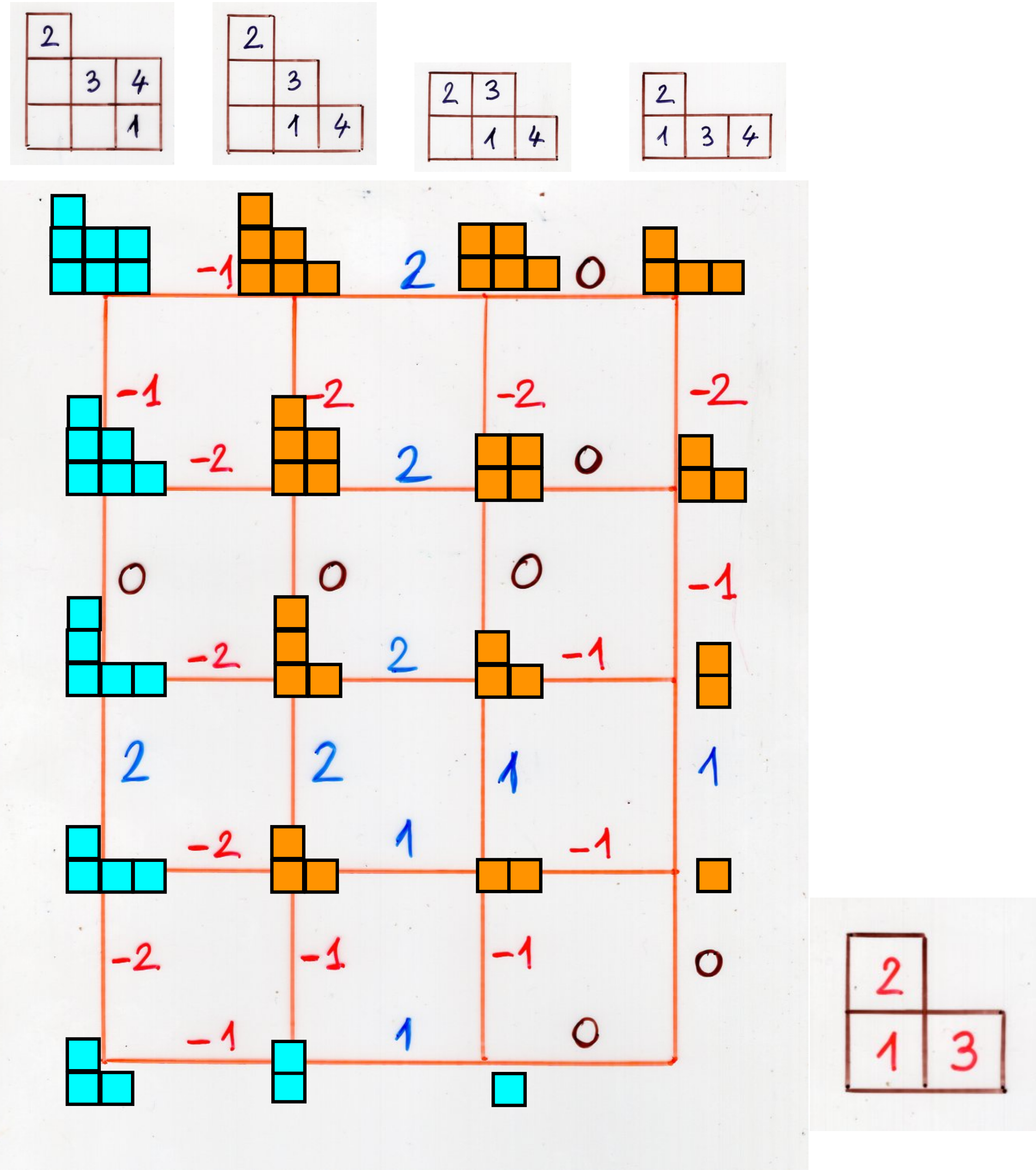
diagonal operators Δ_i $i \in \mathbb{Z}$

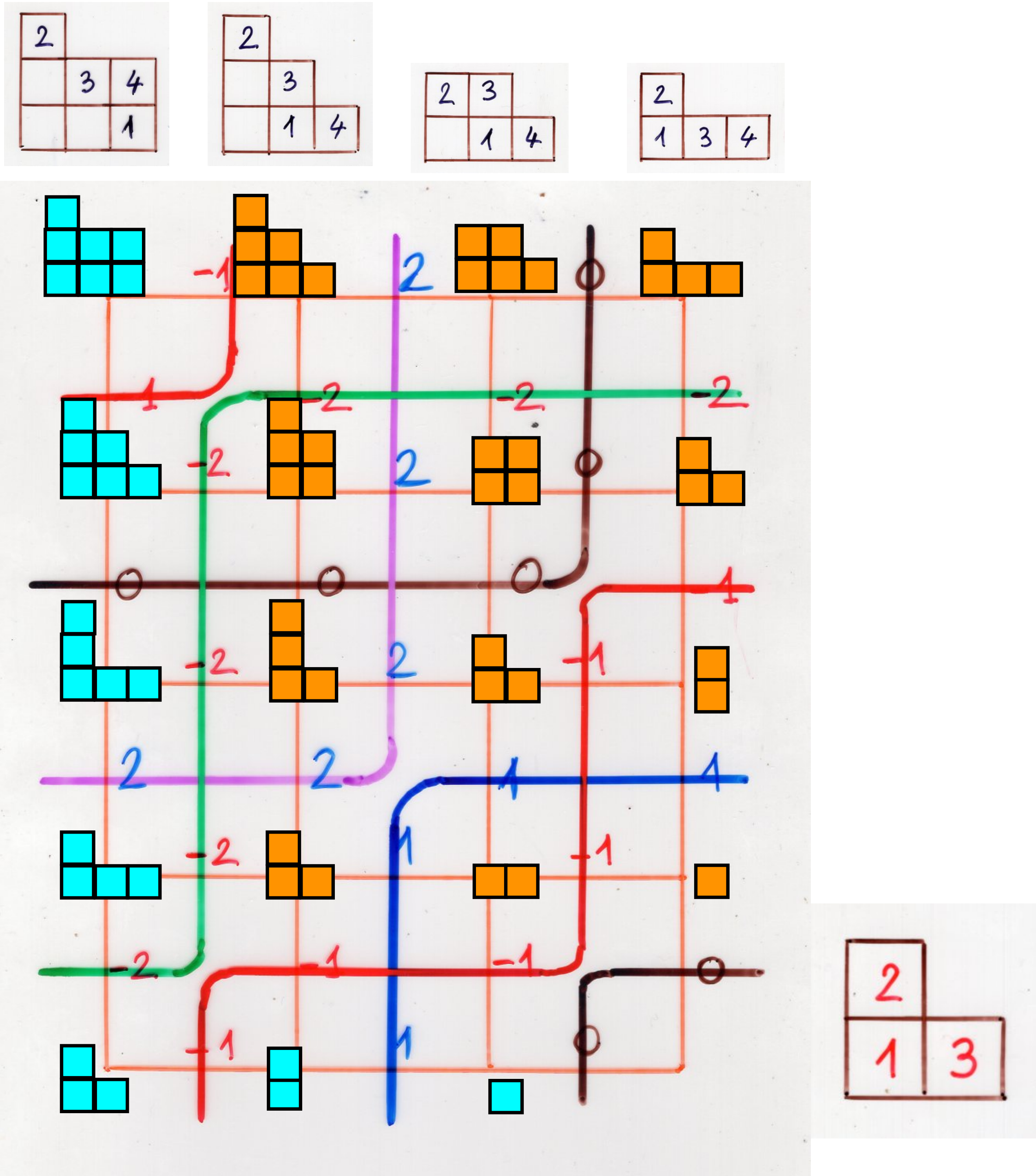
jeu de taquin
local rules on edges



$|i-j| \geq 2$ $|i-j| \leq 1$

$i, j \in \mathbb{Z}$





Jeu de taquin

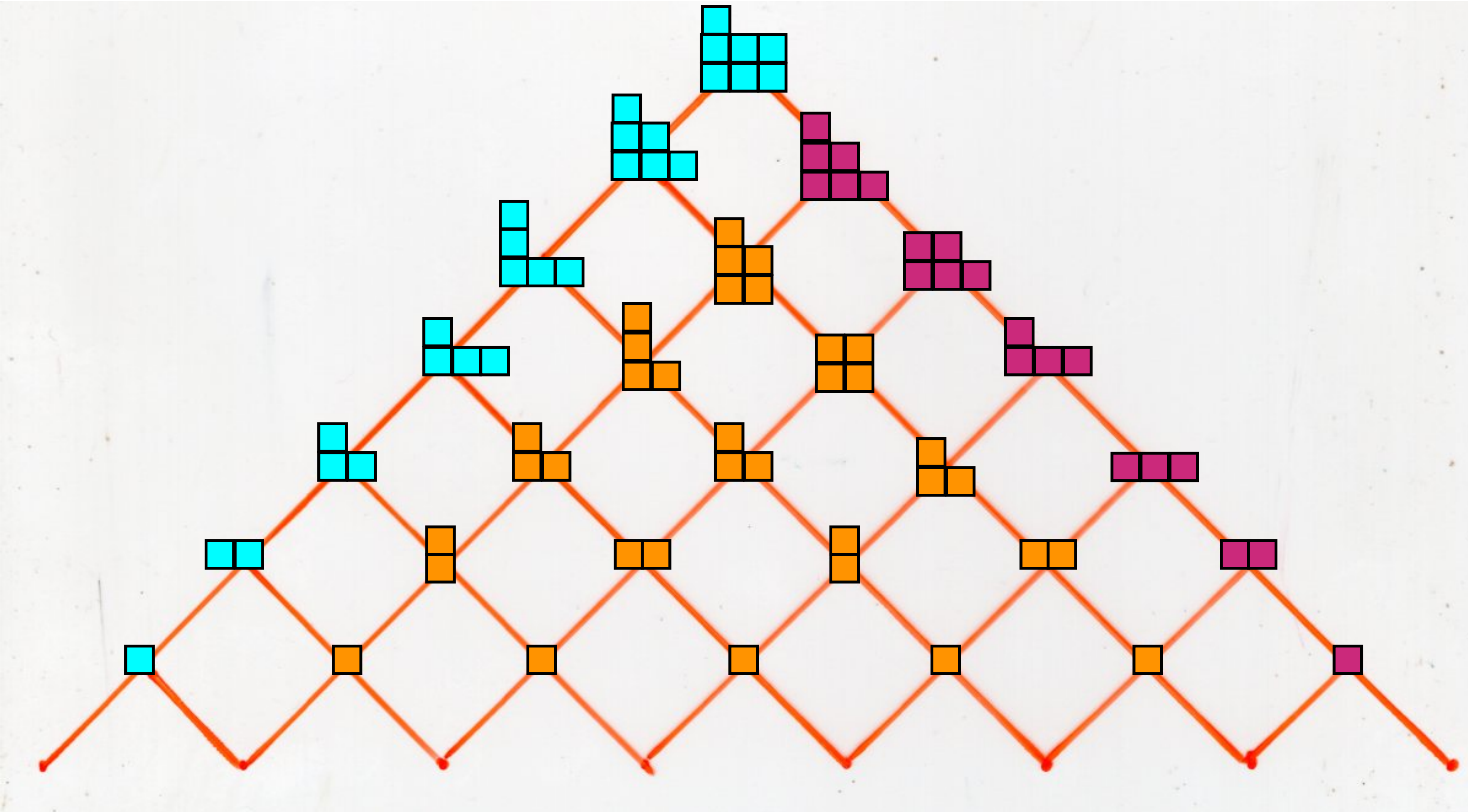
with growth diagrams

S. Fomin, 1986, 1994

Сергей Владимирович Фомин

for the dual of a Young tableaux

dual of a tableau



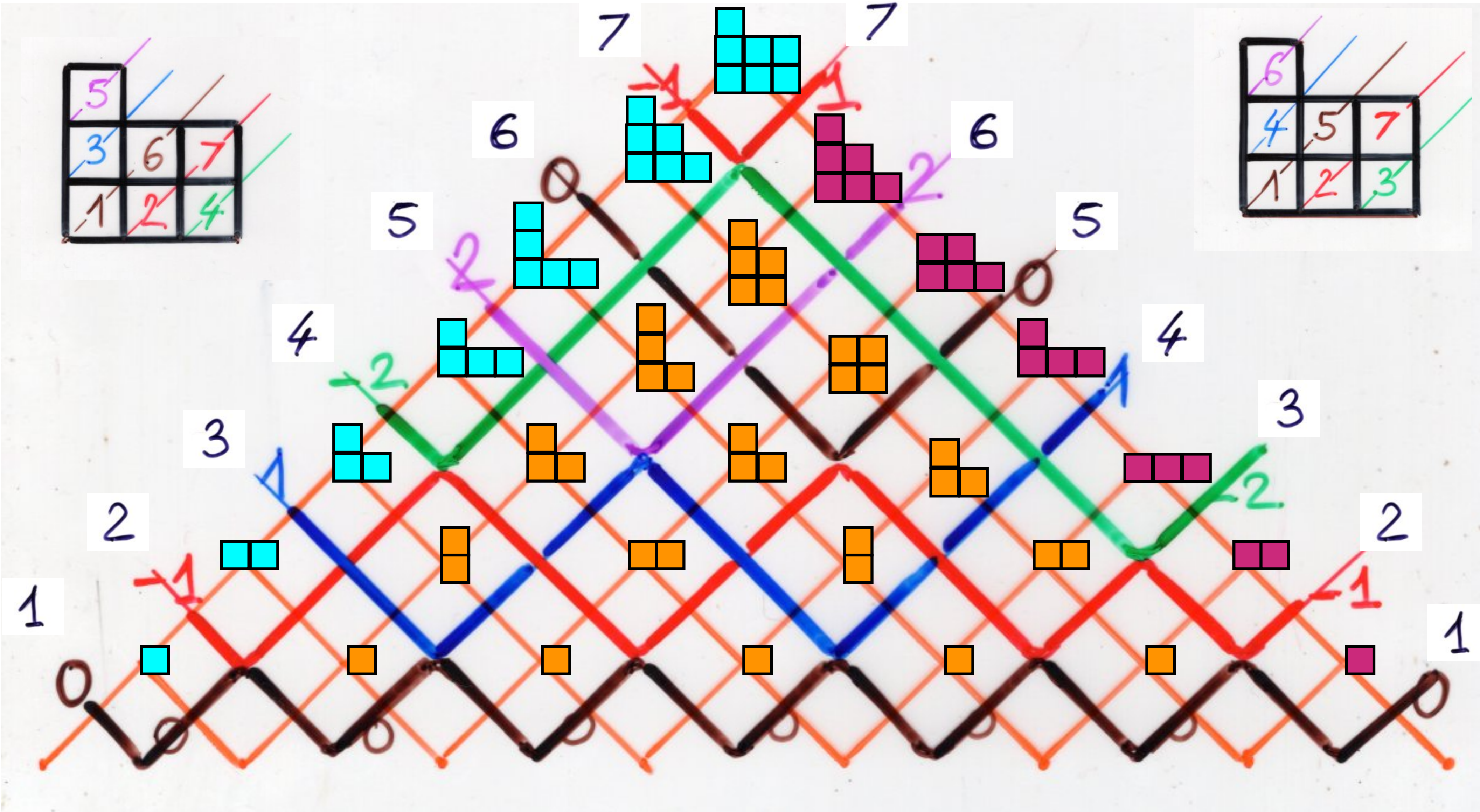
Schützenberger involution

Jeu de taquin

with local rules on edges

for the dual of a Young tableaux

dual of a tableau



Schützenberger involution

Proposition The map $T \longrightarrow T^*$ is an involution $(T^*)^* = T$

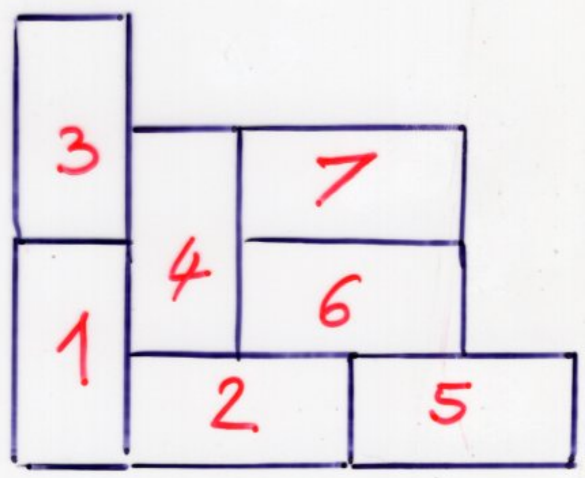
T Young tableau

T^* dual tableau

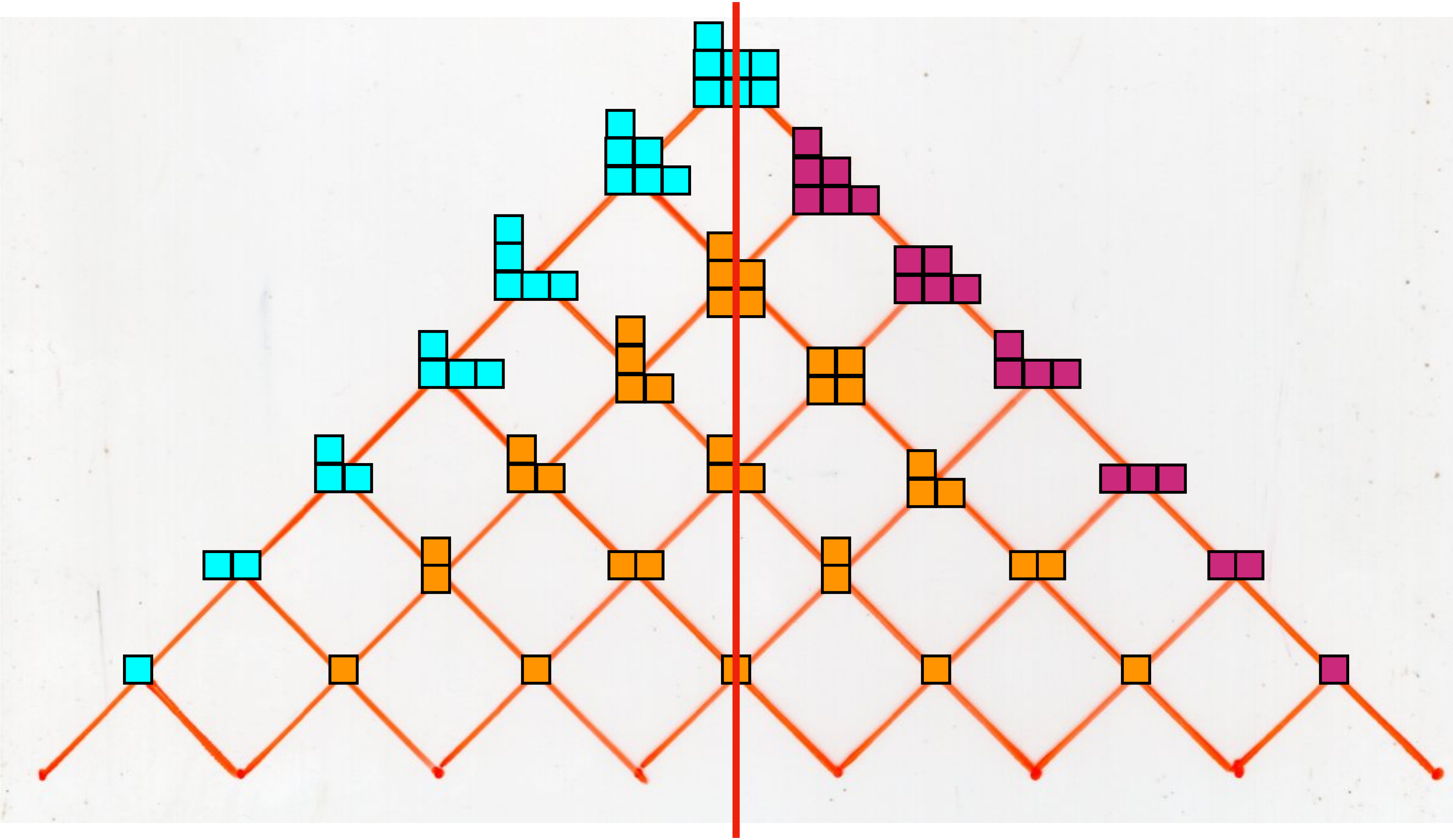
evac(T) other notation

Proposition

tableaux such that $T = T^*$ are in bijection with domino tableaux



dual of a tableau



Schützenberger involution

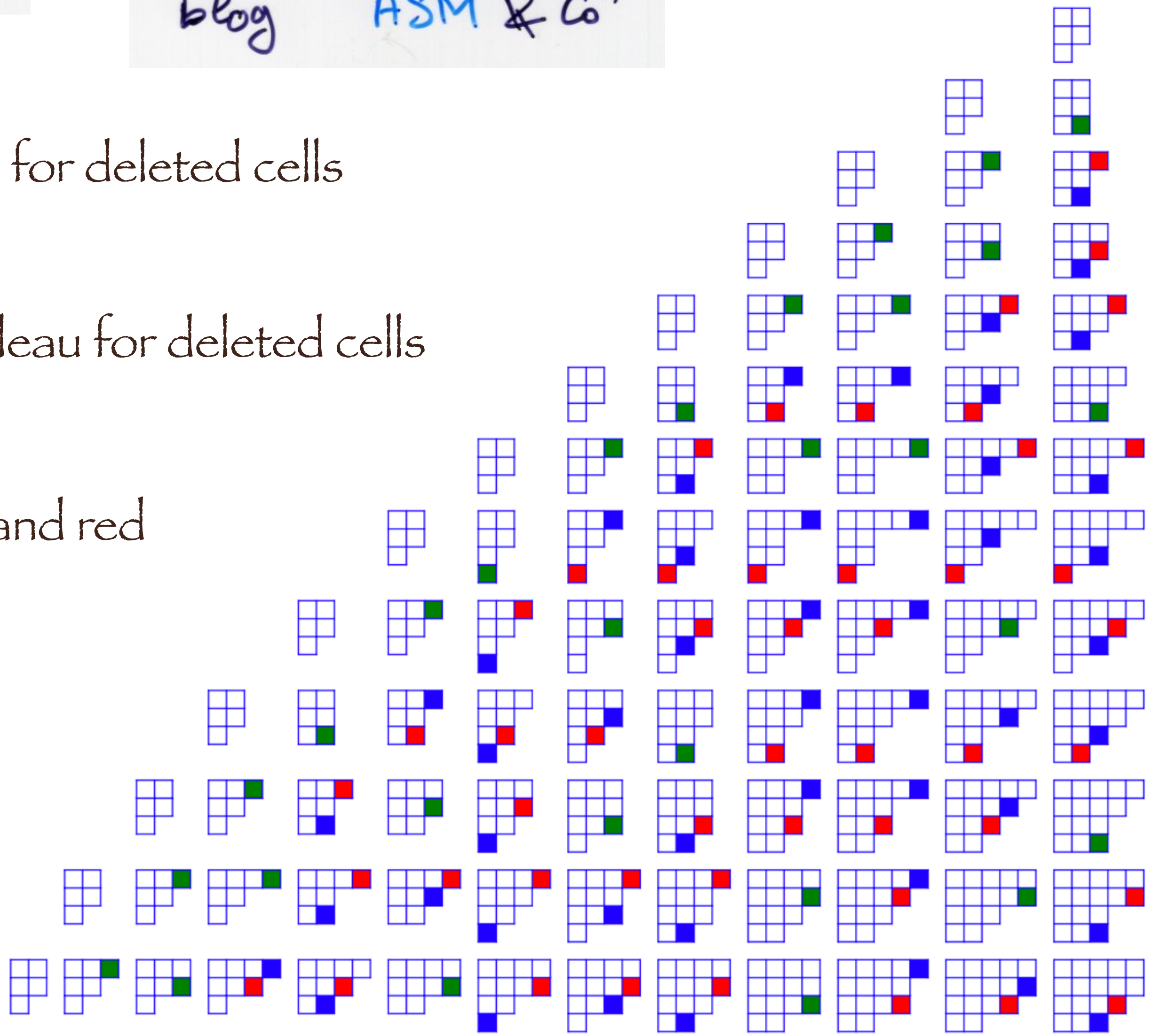
Betrema

website "Tableaux"
blog "ASM & Co"

blue cells:
in each row of the tableau for deleted cells

red cells:
in each column of the tableau for deleted cells

greene cells:
cells which are both blue and red



Schur functions

and

jeu de taquin

Schur Functions

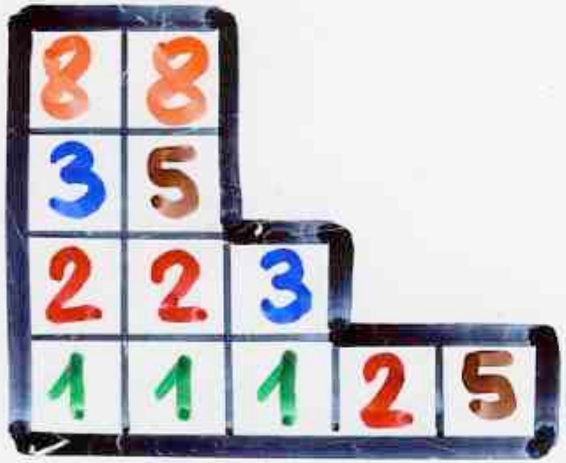
$$s_\lambda(x_1, x_2, \dots, x_m) = \sum_{T} v(T)$$

T Young tableau
shape λ
entries $1, 2, \dots, m$

Jacobi (1841)
Schur (1901)

Littlewood-Richardson (1934)

basis of **symmetric** functions

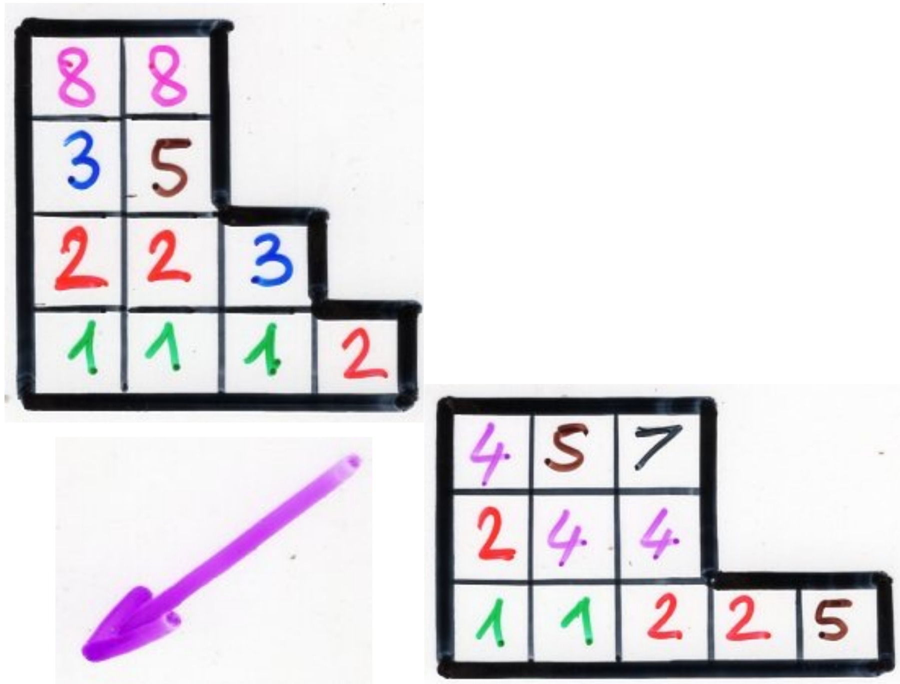


Schur functions

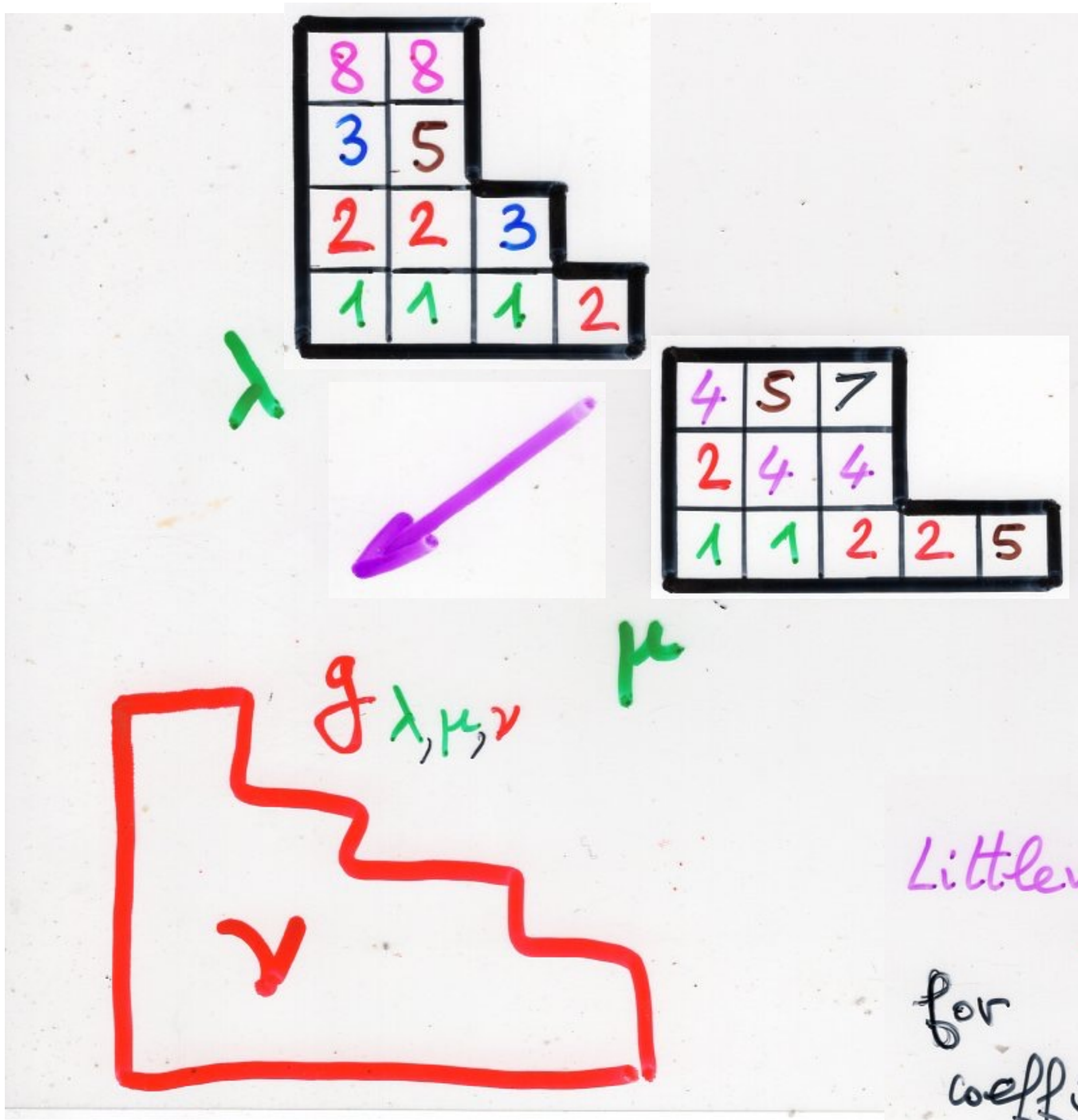
$$s_\lambda s_\mu = \sum_\nu g_{\lambda,\mu,\nu} s_\nu$$

$s_\lambda(x_1, \ldots, x_m)$

Littlewood-Richardson



Jeu de taquin



Jeu de taquin

Littlewood-Richarson rule (1934) for computing the coefficients $g_{\lambda,\mu,\nu}$

jeu de taquin in recent research work

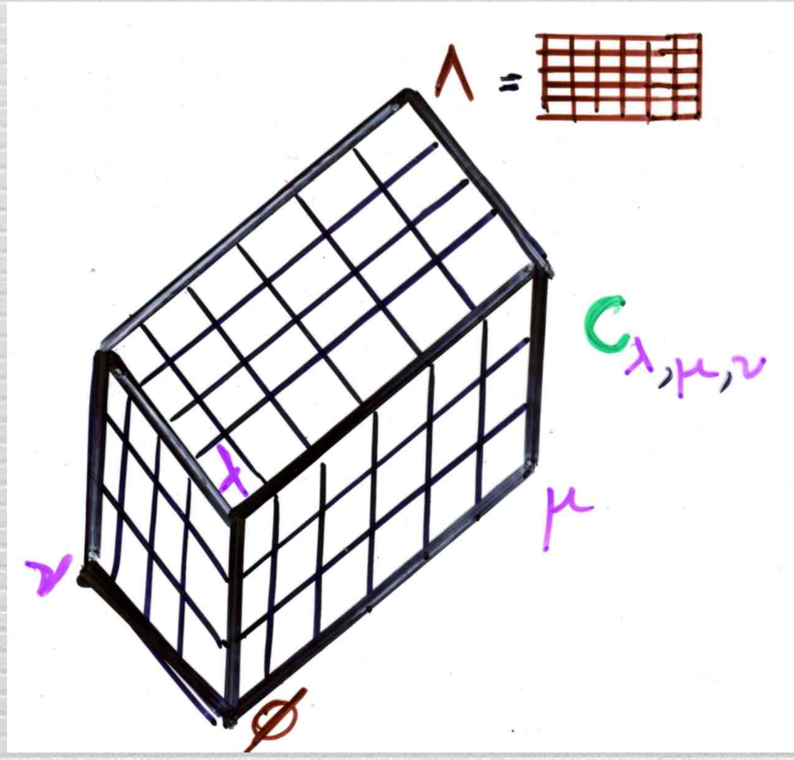
- algebraic combinatorics

  Pechenik, Yong (2015)

  analogue of Littlewood-Richardson coefficients
  in the "equivariant K-theory"
  of the Grassmannian

Thomas, Yong (2007), cartons

3D symmetries for Littlewood-Richardson coefficients



- bijective combinatorics

  Fang (2015)

  - bijective proof of a character identity
    (Frobenius, Murnaghan-Nakayama)

  Krattenthaler (2016)

  - bijection between oscillating tableaux
    (Burrill conjecture)

- probabilistic combinatorics

  Romik, Śniady (2015)

  random infinite tableaux

Thank you !