Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 6
q-analogues

Ch6b

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reminding Ch 6a

q-analogue

$$[i]_q = 1 + q + \dots + q^{i-1} = \frac{1-q^i}{1-q}$$

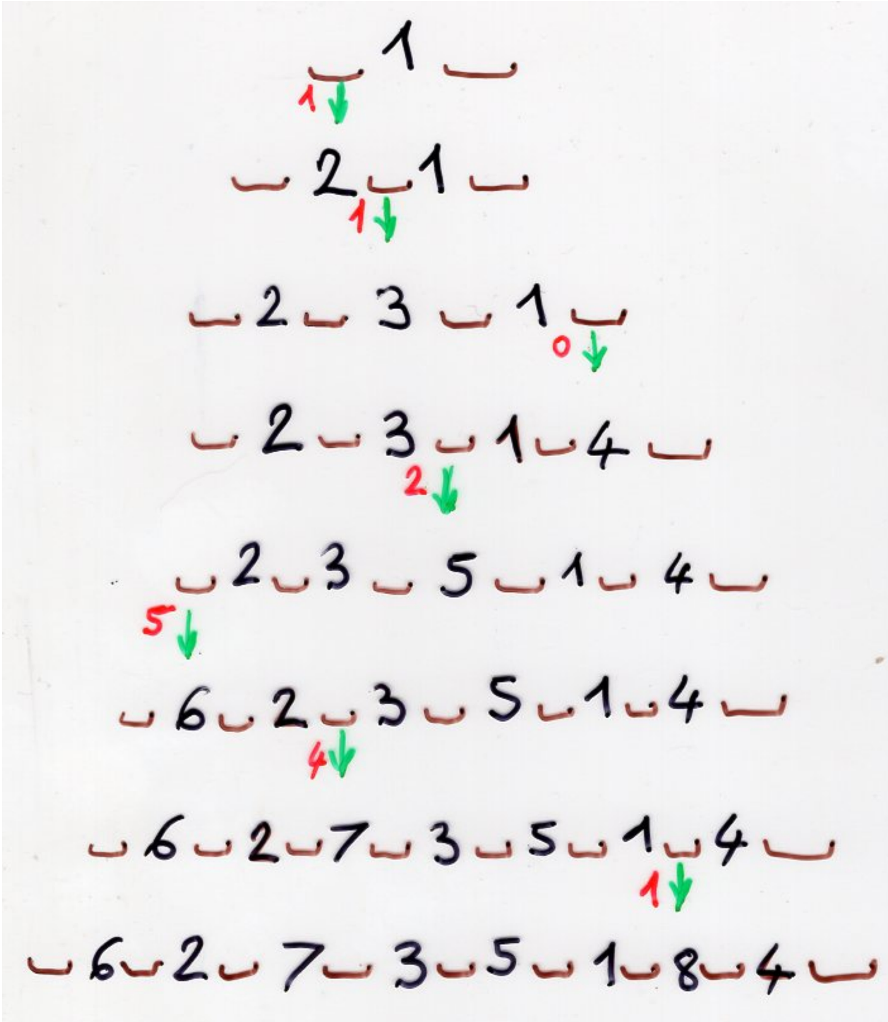
$$[n!]_q = [1]_q \times [2]_q \times \dots \times [n]_q$$

$$= \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}$$

$$[n!]_q = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$

Inv

number of inversions



Maj

Major index

σ = 6 2 7 3 5 1 8 4

q-Hermite I
(continuous)

$$\lambda_k = [k]_q$$

q-Hermite II
discrete

$$\lambda_k = q^{k-1}[k]_q$$

"continuous version"

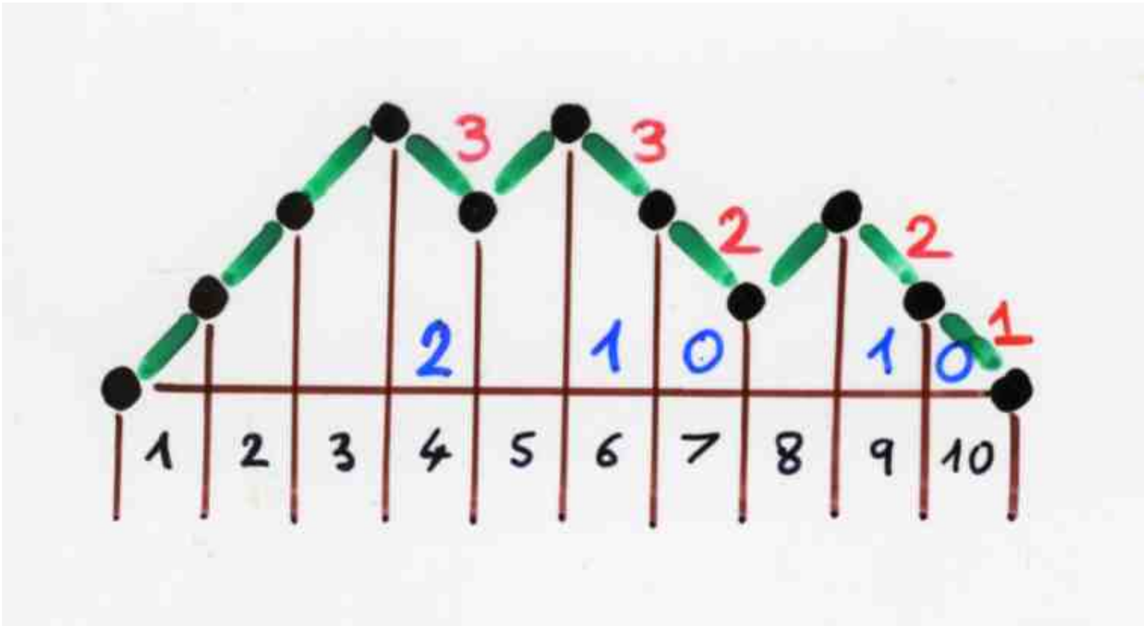
q-Charlier I

$$\begin{cases} b_k = a + [k]_q \\ \lambda_k = a[k]_q \end{cases}$$

discrete

q-Charlier II

$$\begin{cases} b_k = aq^k + [k]_q \\ \lambda_k = aq^{k-1}[k]_q \end{cases}$$


3
3
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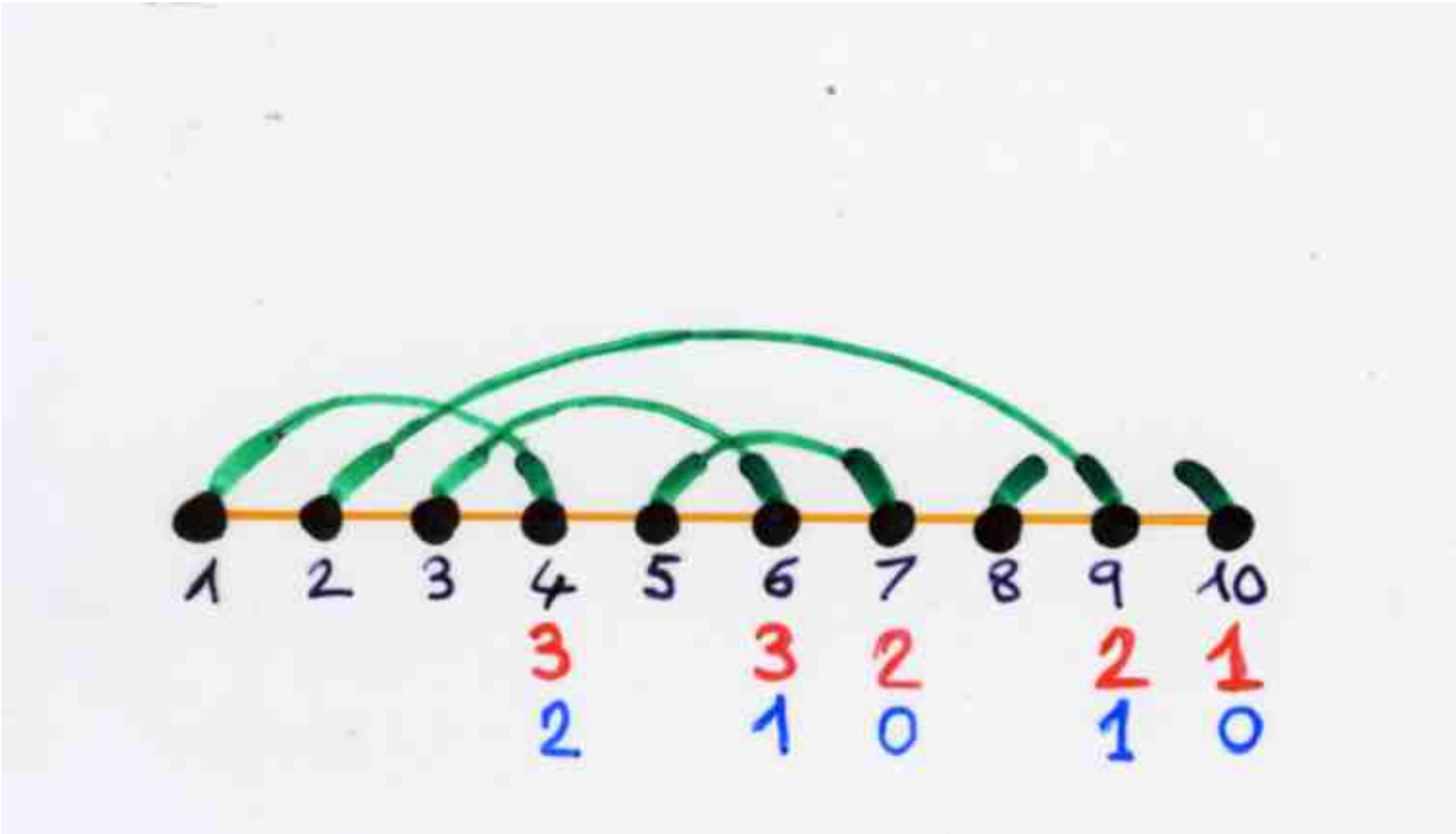
10


1
2
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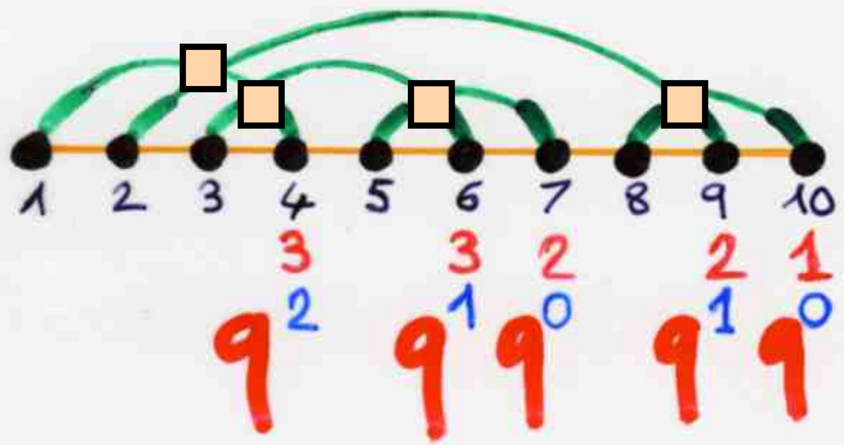
4
5
6
7
8
9
10
3
3
2
2
1
2
1
0
1
0

q-weight of an Hermite history

$$v_q(h) = q^{\left(\sum_{i=1}^{n} p_i\right)}$$

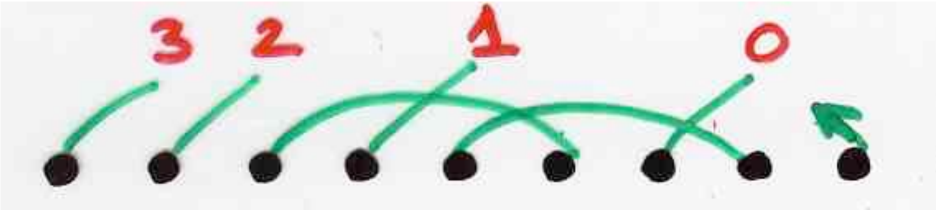


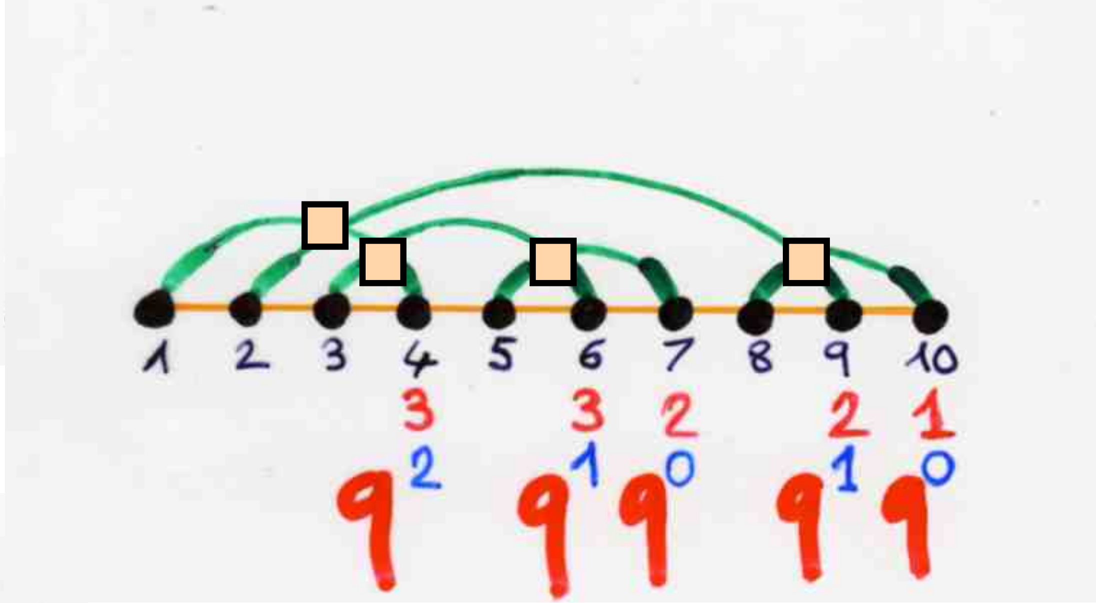
crossing 

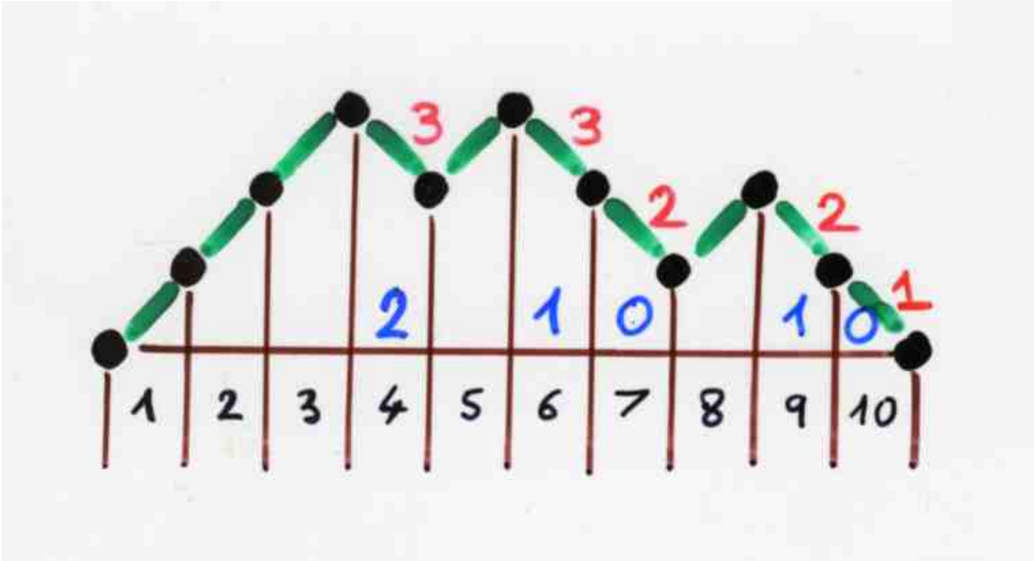
$$q^{2+1+0+1+0} = q^4$$

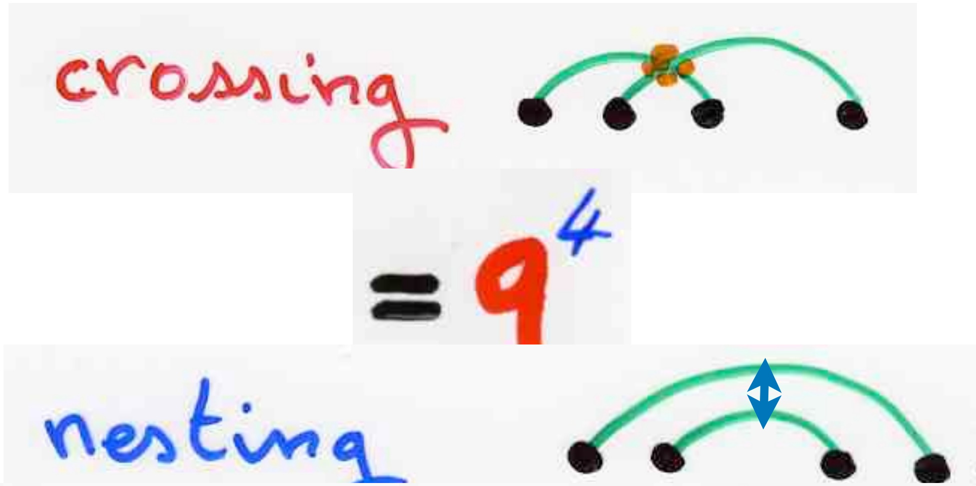
Proposition

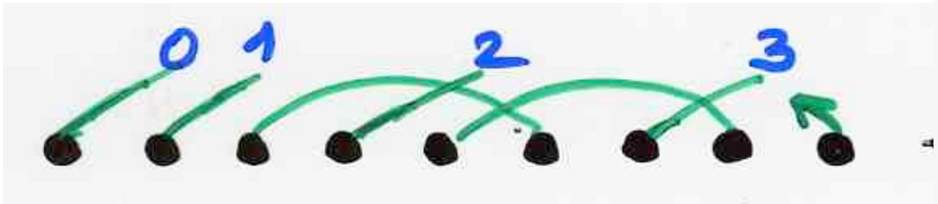
$$\mu^{I}_{2n}(q) = \sum_{\substack{I \\ \text{chord diagrams} \\ \text{on } [1,2n]}} q^{cr(I)}$$

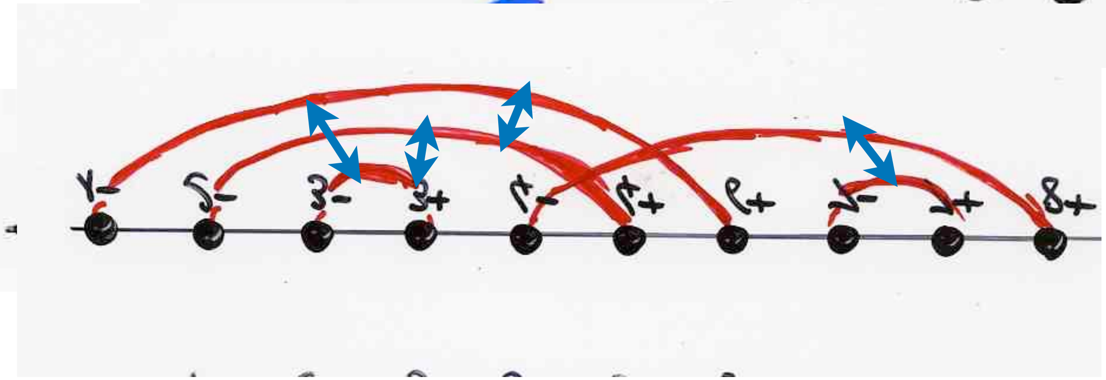

3 2 1 0


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
q^2 q^1 q^0 q^1 q^0


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10


crossing
$= q^4$
nesting


0 1 2 3



q-Hermite $\overline{\mathrm{II}}$
(discrete I)

$$\lambda_k = q^{k-1} [k]_q$$

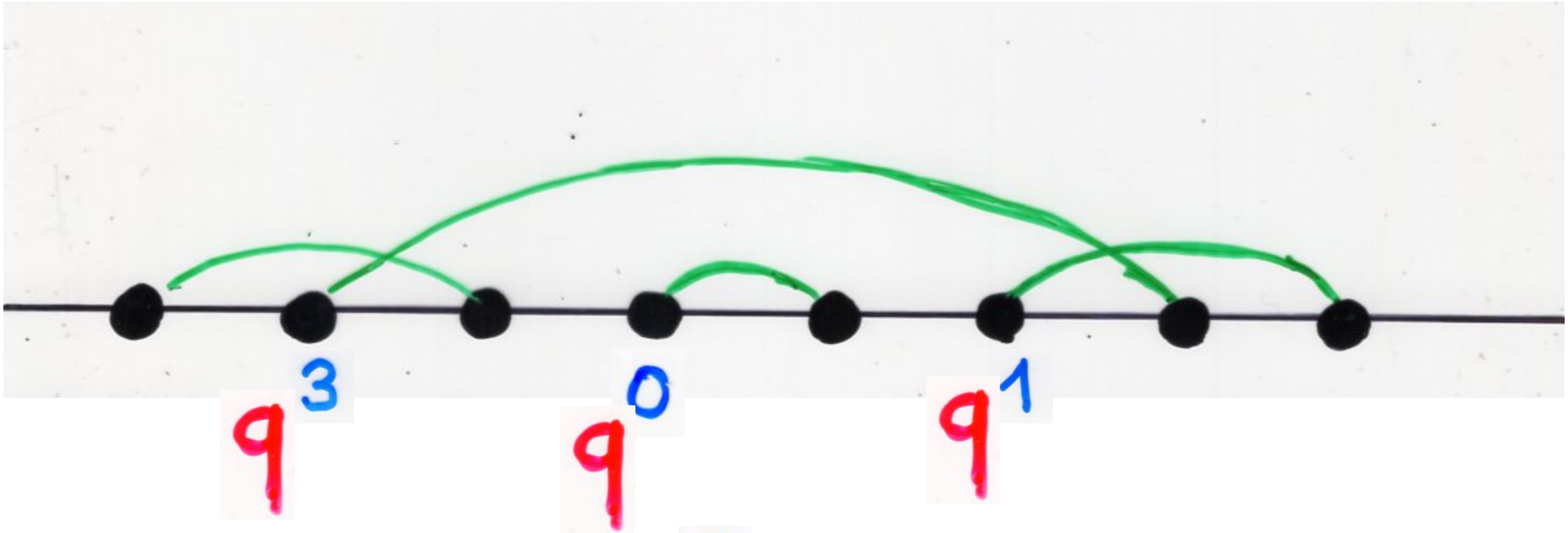
Proposition

moments

$$\mu^{\overline{\mathrm{II}}}_{2n}(q) = [1]_q \cdot [3]_q \cdots [2n-1]_q$$

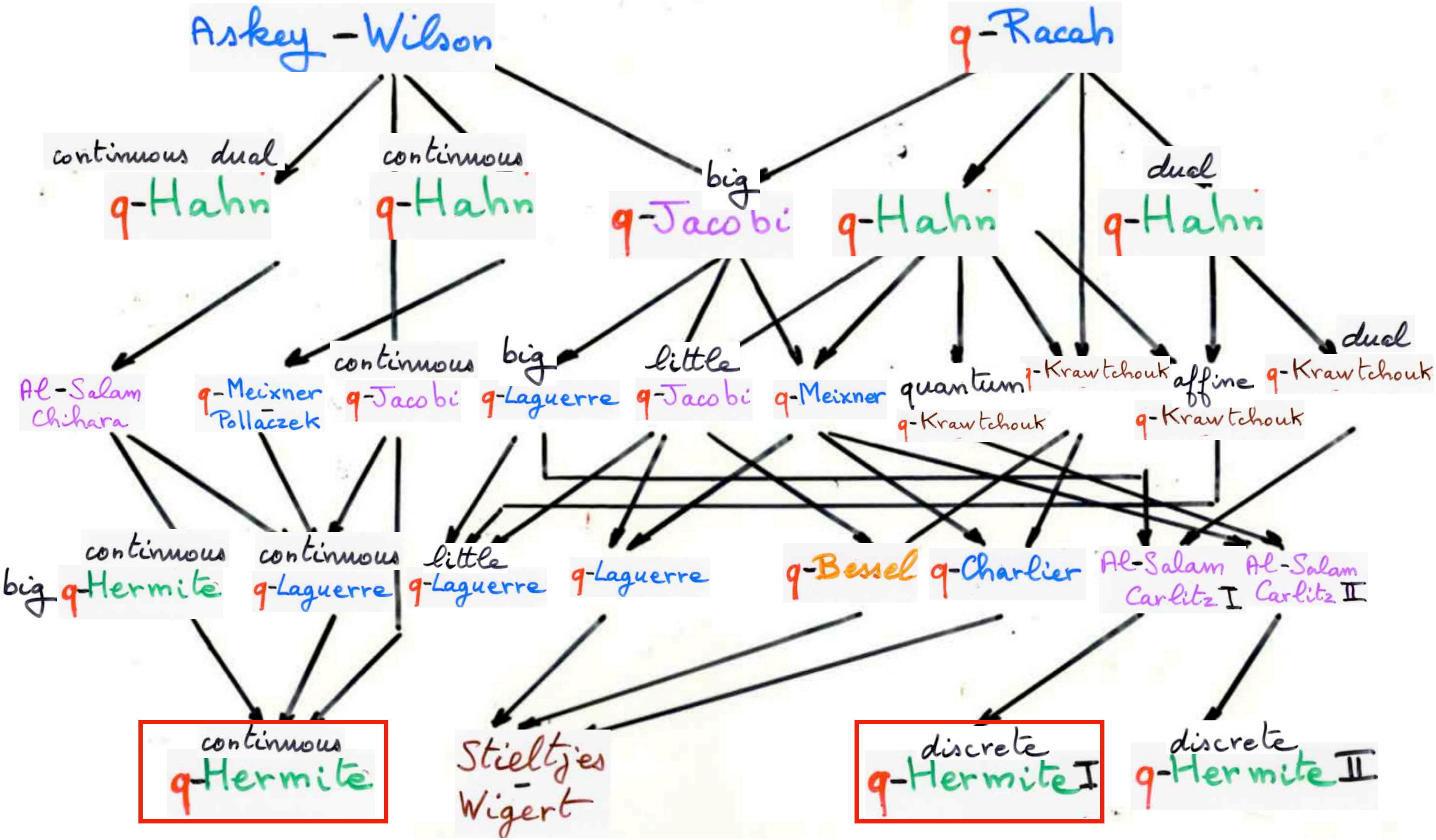
Proposition

$$\mathrm{Inv}(I) = \mathrm{cr}(I) + 2\,\mathrm{nest}(I)$$

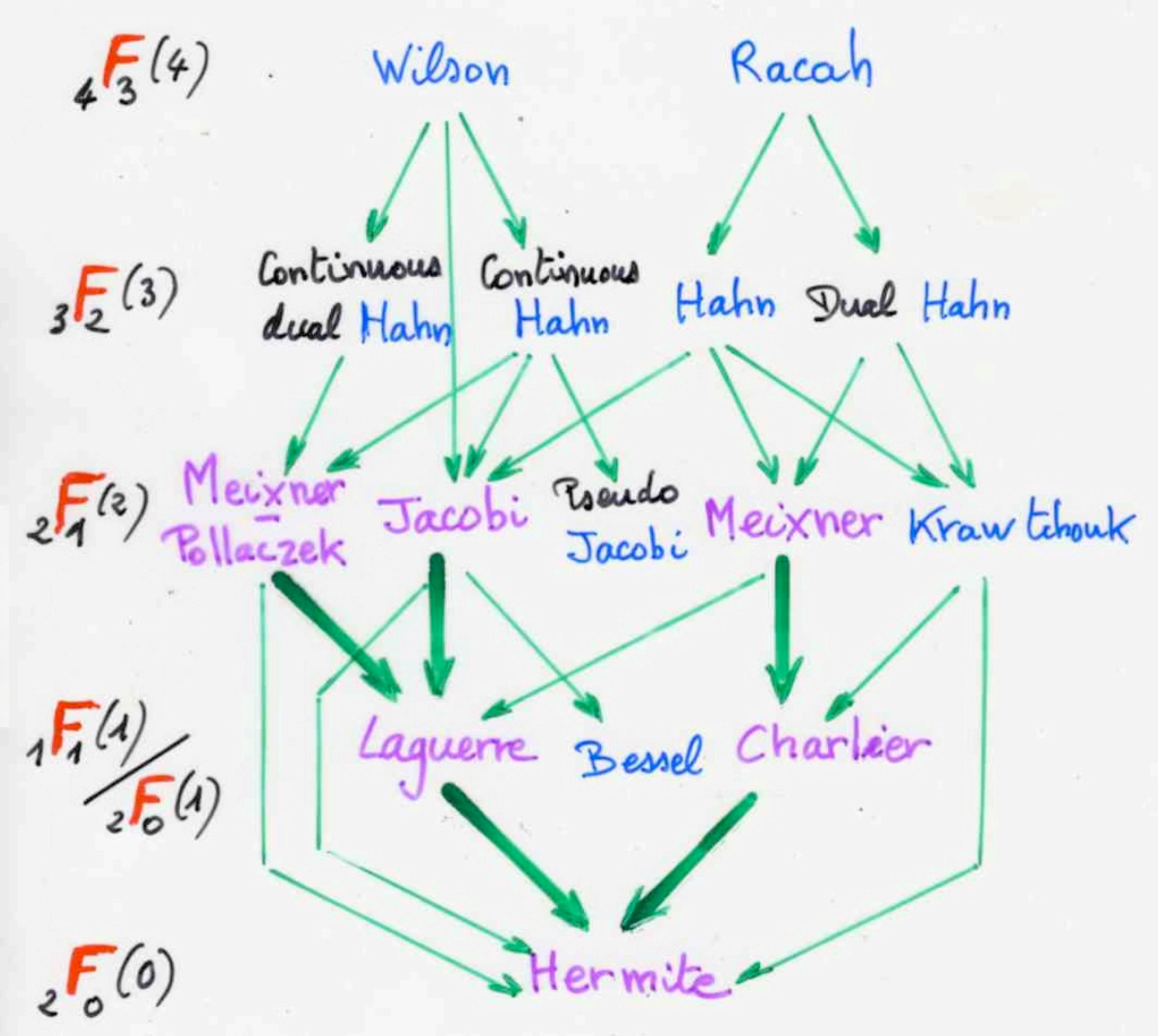


$$\mathrm{Inv}(I) = q^0\, q^3\, q^1$$

scheme of basic hypergeometric orthogonal polynomials



Askey scheme of hypergeometric orthogonal polynomials



orthogonal Sheffer polynomials

Basic hypergeometric series

basic hypergeometric series

Gauss (1812) $\quad {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right]$

$$1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)\, b(b+1)}{1.2 \;\; c(c+1)} z^2 + \cdots$$

Heine (1846, 1847, 1878)

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})\,(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)\;\;(1-q^c)(1-q^{c+1})} z^2 + \cdots$$

$$\lim_{q \to 1} \frac{1-q^a}{1-q} = a$$

$$ {}_2\phi_1(a,b;c;q,z) \equiv {}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,z\right] $$

$$ = \sum_{n\geqslant 0} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n $$

$$ (a;q)_n = \begin{cases} 1 & n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2\ldots \end{cases} $$

$$ (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \qquad (a;q)_\infty = \prod_{k\geqslant 0}(1-aq^k) $$

$_r\phi_s$ basic hypergeometric series

$$_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots b_s; q, z)$$

$$\equiv {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right]$$

$$= \sum_{n \geqslant 0} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n \; (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n$$

$|q| < 1$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$|q| > 1$

$$(a;q)_n = (a^{-1};p)_n (-a)^n p^{-\binom{n}{2}}$$

$p = q^{-1}$.

q-Laguerre polynomials

q-Laguerre I
(continuous)

"continuous version"

discrete

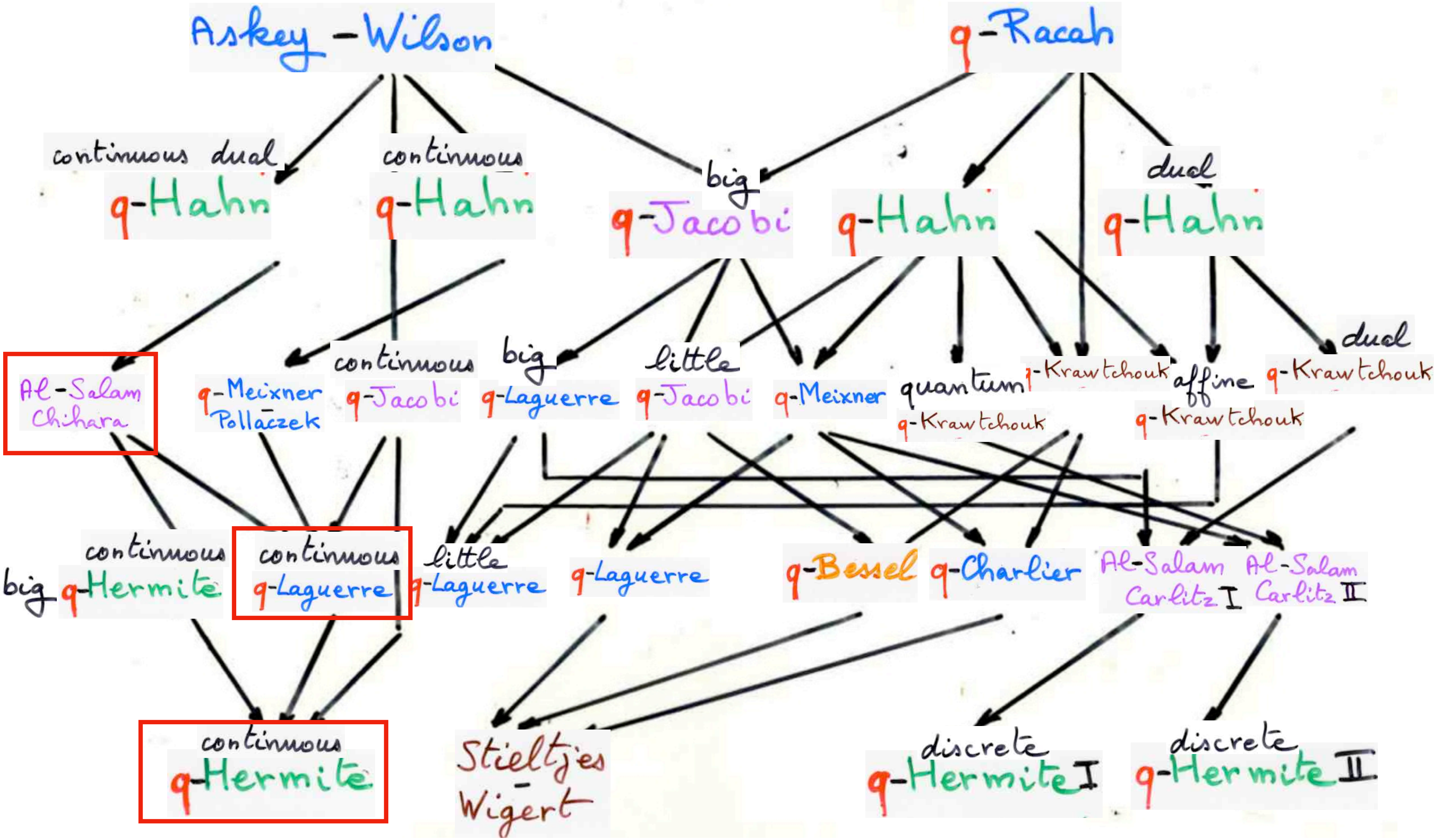
q-Laguerre I

q-Laguerre II

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

scheme of basic hypergeometric orthogonal polynomials



Al-Salam – Chihara polynomials

$$Q_{n+1}(x) = \left(2x - (\alpha+\beta)q^n\right) Q_n(x) - (1-q^n)(1-\alpha\beta q^{n-1}) Q_{n-1}(x)$$

$$x = \frac{u+u^{-1}}{2} \qquad x = \cos\theta \,, \quad u = e^{i\theta}$$

$$Q_n(x;\alpha,\beta;q) = \frac{(\alpha\beta;q)_n}{\alpha^n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix}; q, q\right]$$

$$= (\alpha u; q)_n u^{-n}\, {}_2\phi_1\left[\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1} q^{-n+1} u^{-1} \end{matrix}; q, \alpha^{-1} q u\right]$$

Kasraoui, Stanton, Zeng (2011)

Al-Salam – Chihara q-Laguerre polynomials

$$L_n(x;q) = \frac{1}{(q-1)^n} Q_n\left(\frac{(q-1)x}{2}+1; 1, q; q\right)$$

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$L_n(x;q) =$$

$$\sum_{k\geq 0} (-1)^{n-k} \frac{[n!]_q}{[k!]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \prod_{j=0}^{k-1} \left(x - (1-q^{-j})[j]_q \right)$$

Simion, Stanton (1996) $\begin{cases} a = s = u = 1 \\ r = t = q \end{cases}$

Octabasic Laguerre polynomials

q-Laguerre I
(continuous)

Moments

q-Laguerre I

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\mu_n = (n+1)!$$

q-Laguerre restricted histories

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\mu_n = n!$$

q-Laguerre I

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$

bijection

$$h = (\omega_c ; \underbrace{(p_1, \dots, p_n)}_{P})$$

$|\omega| = n$

$$\longleftrightarrow$$

permutations

$\sigma \in \mathfrak{S}_{n+1}$

Laguerre histories

$(n+1)!$

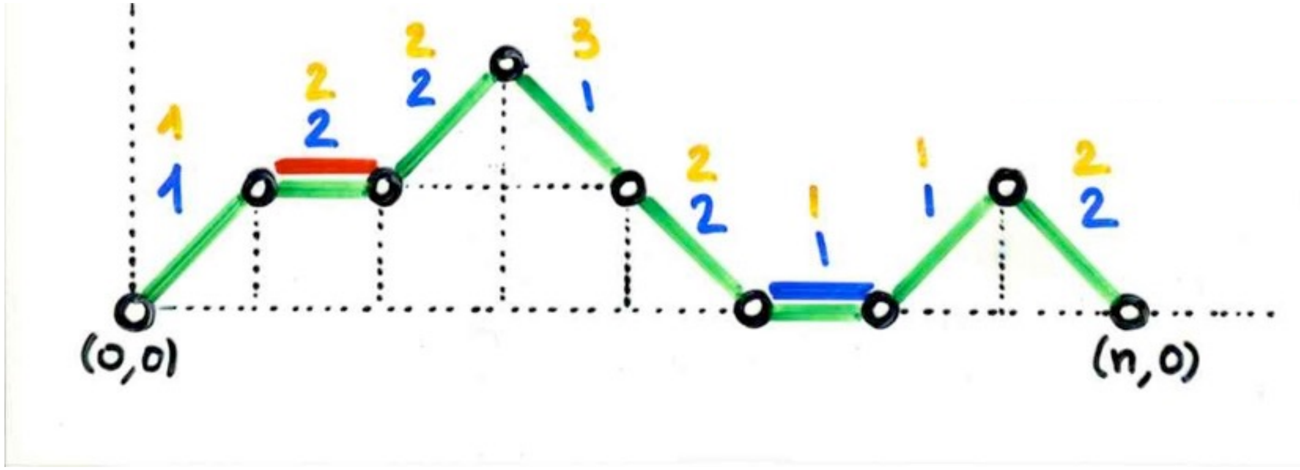
$|h| = |\omega|$

length of the history

J. Françon, X.V. (1979)

see Ch2a, p56-66

Laguerre history



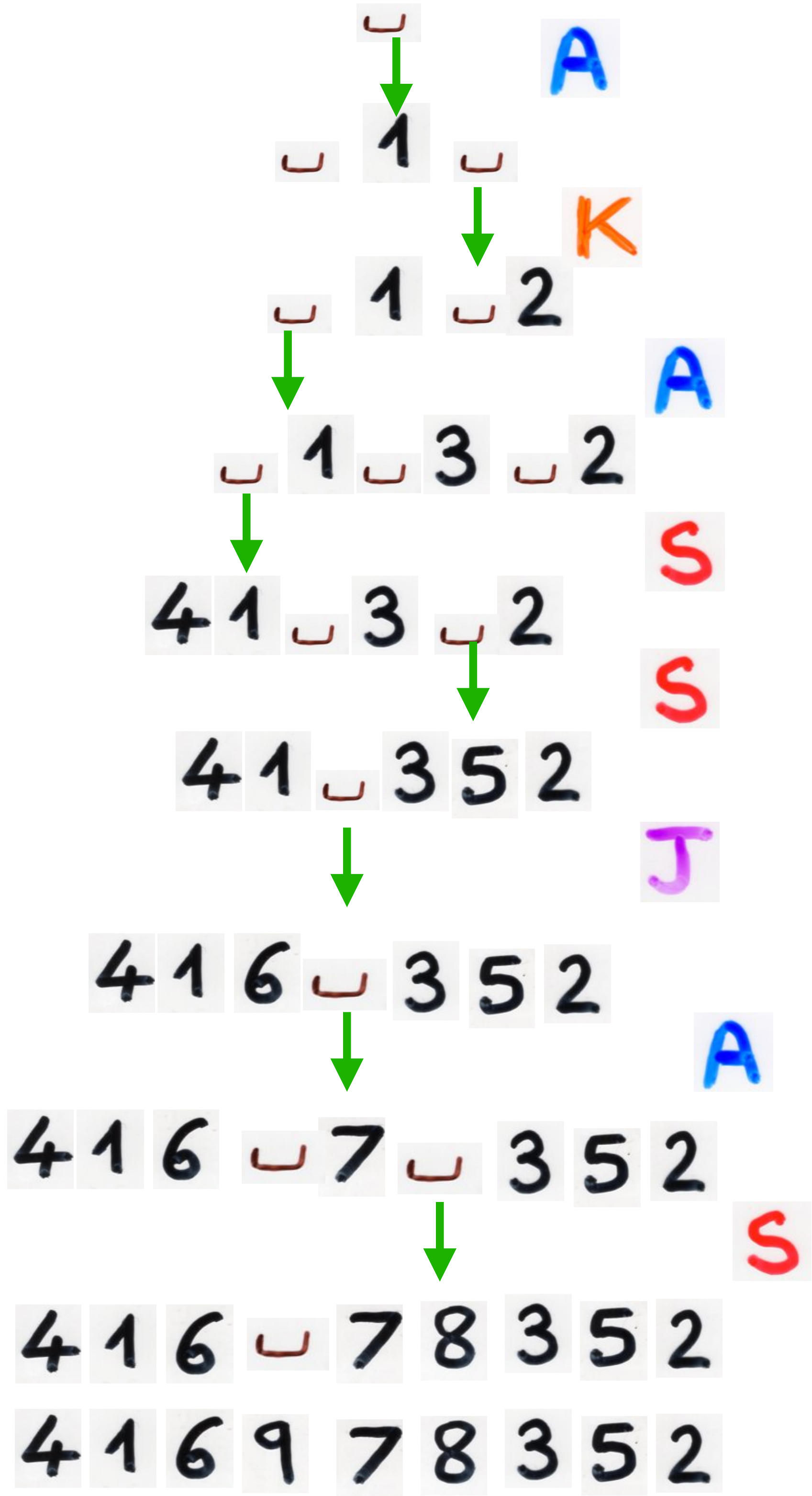
$$\langle k |\, A = (k+1)\, \langle (k+1) |$$

$$\langle k |\, K = (k+1)\, \langle k |$$

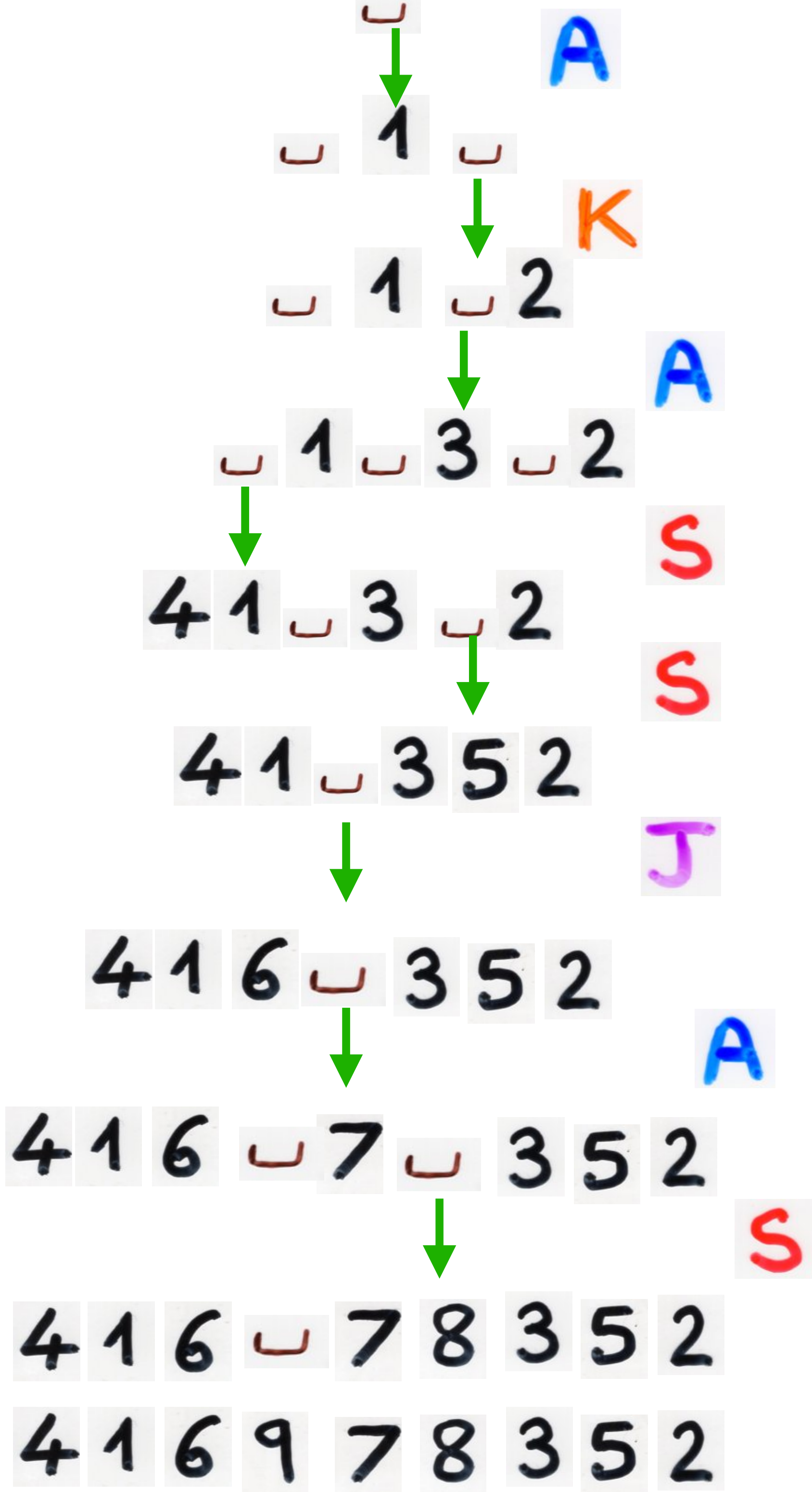
$$\langle k |\, J = (k+1)\, \langle k |$$

$$\langle k |\, S = (k+1)\, \langle (k-1) |$$

1 1 1
2 2 2
3 2 2
4 3 1
5 2 2
6 1 1
7 1 1
8 2 2



␣ A
␣ 1 ␣ K
␣ 1 ␣ 2 A
␣ 1 ␣ 3 ␣ 2 S
4 1 ␣ 3 ␣ 2 S
4 1 ␣ 3 5 2 J
4 1 6 ␣ 3 5 2 A
4 1 6 ␣ 7 ␣ 3 5 2 S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2



"q-analogue" of Laguerre histories

choice function

$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories $\quad q^4 = v_q(h)$

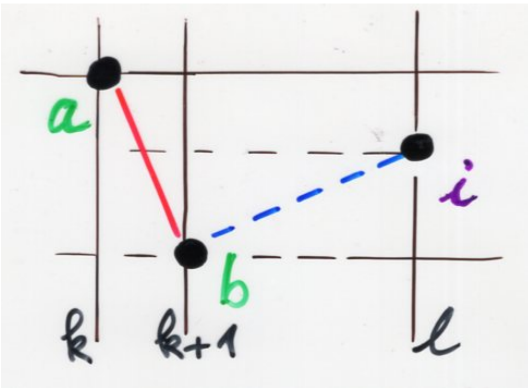
Lemma

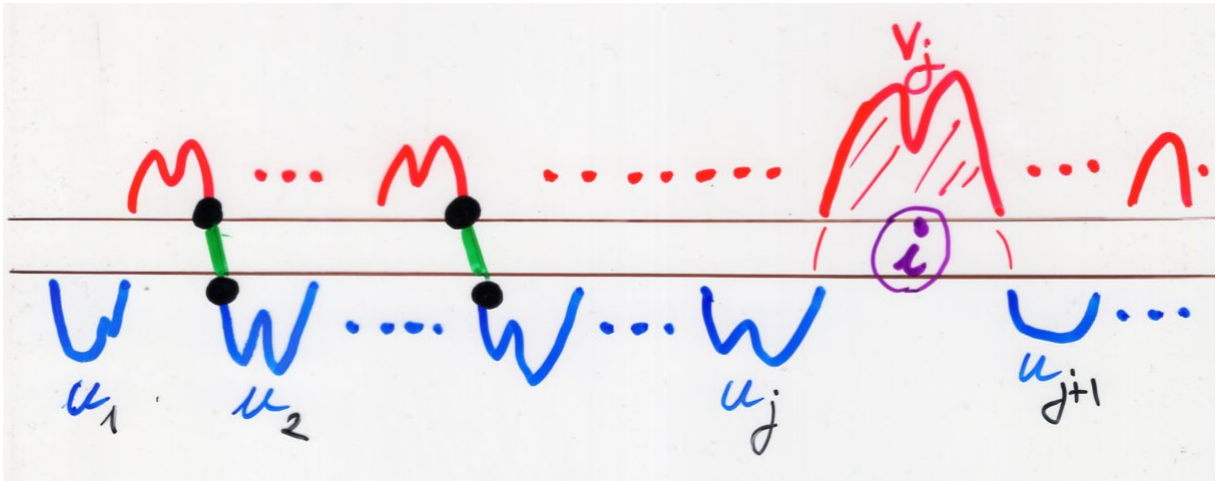
$p_i = j$ is also defined by:

$j = 1 +$ number of triples (a, b, i) having the patter $(31-2)$, that is:

$a = \sigma(k)$, $b = \sigma(k+1)$, $i = \sigma(\ell)$

with $k < k+1 < \ell$ and $b < i < a$



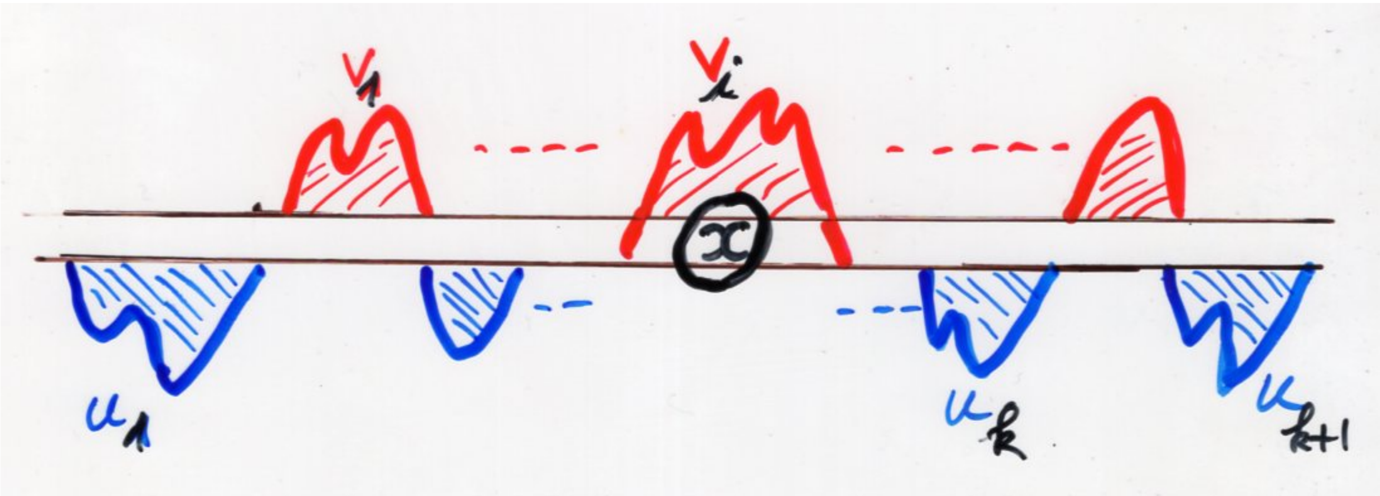


$4\ 1\ 6 \sqcup 7 \sqcup 3\ 5\ 2$

$\downarrow$ **S**

$4\ 1\ 6 \sqcup 7\ 8\ 3\ 5\ 2$

- $\sigma = u_1 \, v_1 \cdots u_k \, v_k \, u_{k+1}$
- letters $(u_i) < x$
- letters $(v_j) \geqslant x$
- words $v_1, u_2, \ldots, u_k, v_k$

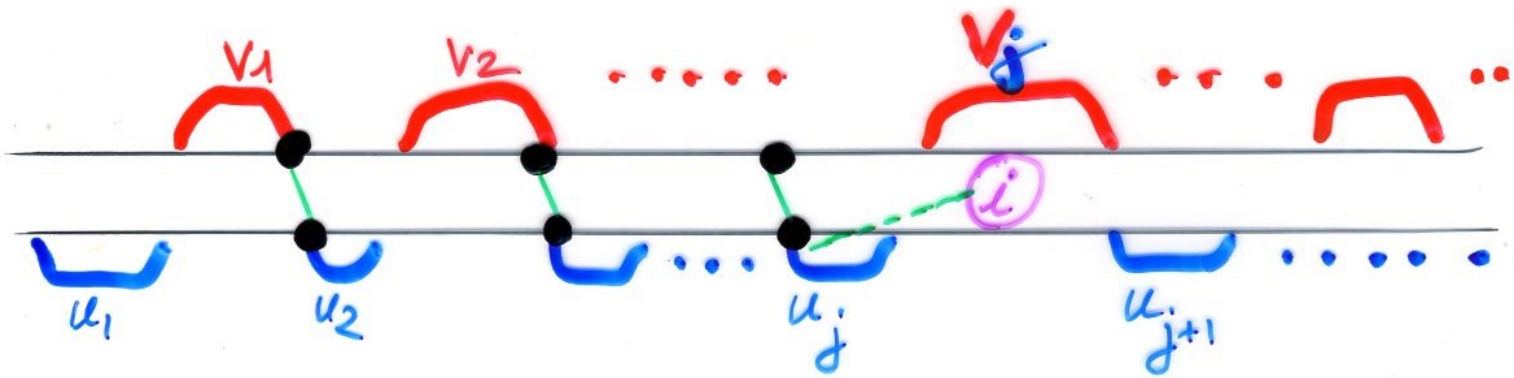


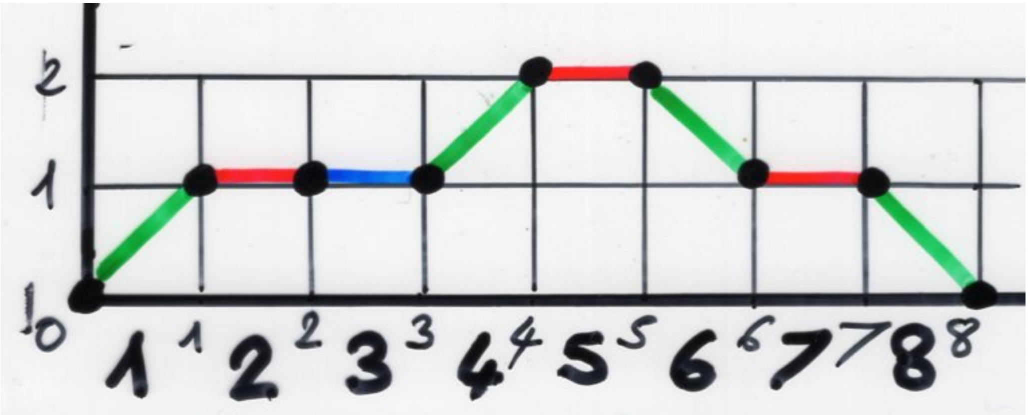
see Ch2a, p70-75

weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]} = v_q(h)$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern (31-2)





1	2	1	1	1	1	1

restricted
Laguerre
histories

see Ch2b, p19-23

see Ch2c, p3-15

$\sqcup$	
$\sqcup 1 \quad \sqcup$	1
$\sqcup 1 \sqcup 2$	2
$\sqcup 1\,3 \sqcup 2$	1
$\sqcup 4 \sqcup 1\,3 \sqcup 2$	1
$\sqcup 5\,4 \sqcup 1\,3 \sqcup 2$	1
$\sqcup 5\,4\,6\,1\,3 \sqcup 2$	1
$\sqcup 7\,5\,4\,6\,1\,3 \sqcup 2$	1
$\sqcup 7\,5\,4\,6\,1\,3\,8\,2$	1

Proposition

$$\mu_n(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{31-2(\sigma)}$$

$$\beta = \alpha + 1$$

$\alpha = 1$ $\quad$ $\sigma \in \mathfrak{S}_{n+1}$ Laguerre histories

$\alpha = 0$ $\quad$ $\sigma \in \mathfrak{S}_n$ $\rightarrow$ restricted Laguerre histories

q- Laguerre I

q - Laguerre
restricted
histories

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008)

q-Hermite I
(continuous)

$$\lambda_k = [k]_q$$

$$\mu^{I}_{2n}(q) = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{\binom{k}{2}}$$

Touchard (1952)
Riordan (1975)
Read (1979)

Penaud (1995)
bijective proof

q-Laguerre I
(continuous)

with parameter beta

restricted
Laguerre
histories

$$\begin{cases} a_k = k+\beta \\ b'_k = k+\beta \\ b''_k = k \\ c_k = k \end{cases} \qquad \begin{matrix} (k \geq 0) \\ \\ (k \geq 1) \end{matrix}$$

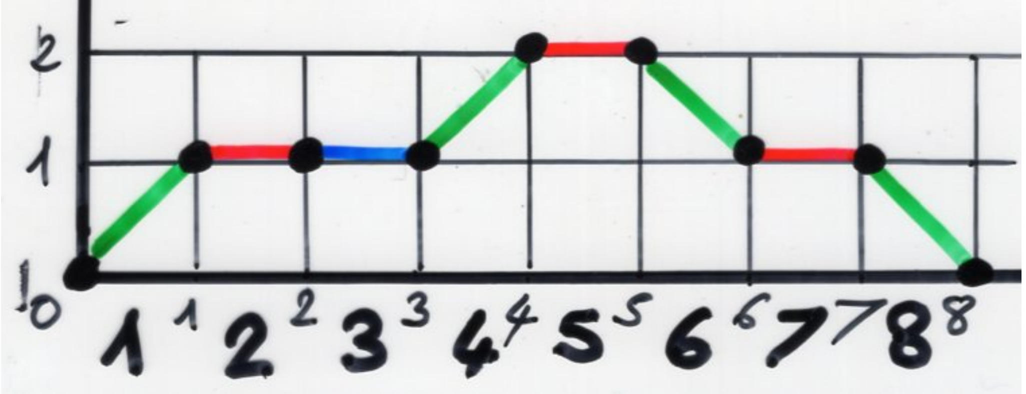
$$\begin{cases} \lambda_k = a_{k-1} c_k \\ b_k = b'_k + b''_k \end{cases}$$

$$\mu_n = \beta(\beta+1) \cdots (\beta+n-1)$$

$$[k;\beta]_q = (\beta + q + q^2 + \ldots + q^{k-1})$$

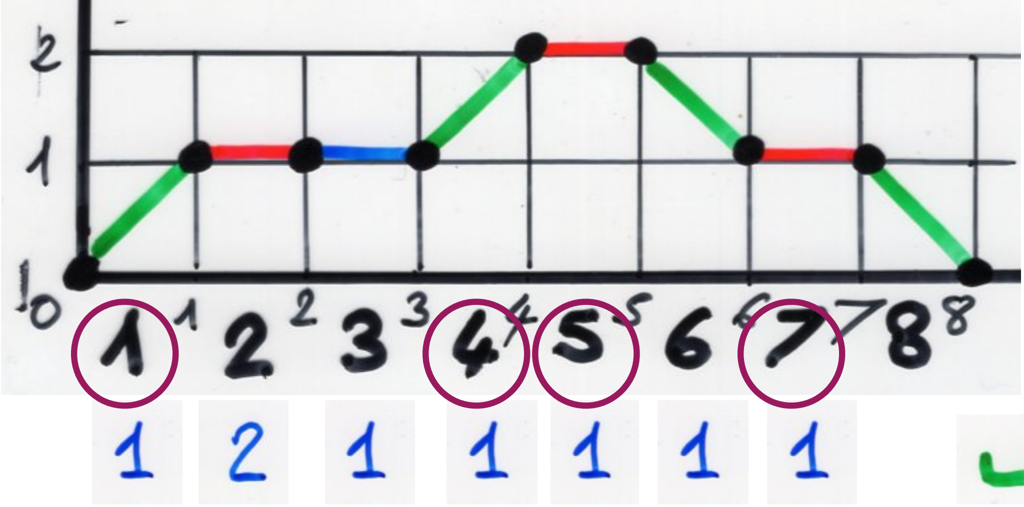
$$\begin{cases} a_k = [k+1;\beta]_q \\ b'_k = [k+1;\beta]_q \\ b''_k = [k]_q \\ c_k = [k]_q \end{cases}$$

$$\begin{cases} b_k = [k]_q + [k+1;\beta]_q \\ \lambda_k = [k;\beta]_q \, [k]_q \end{cases}$$



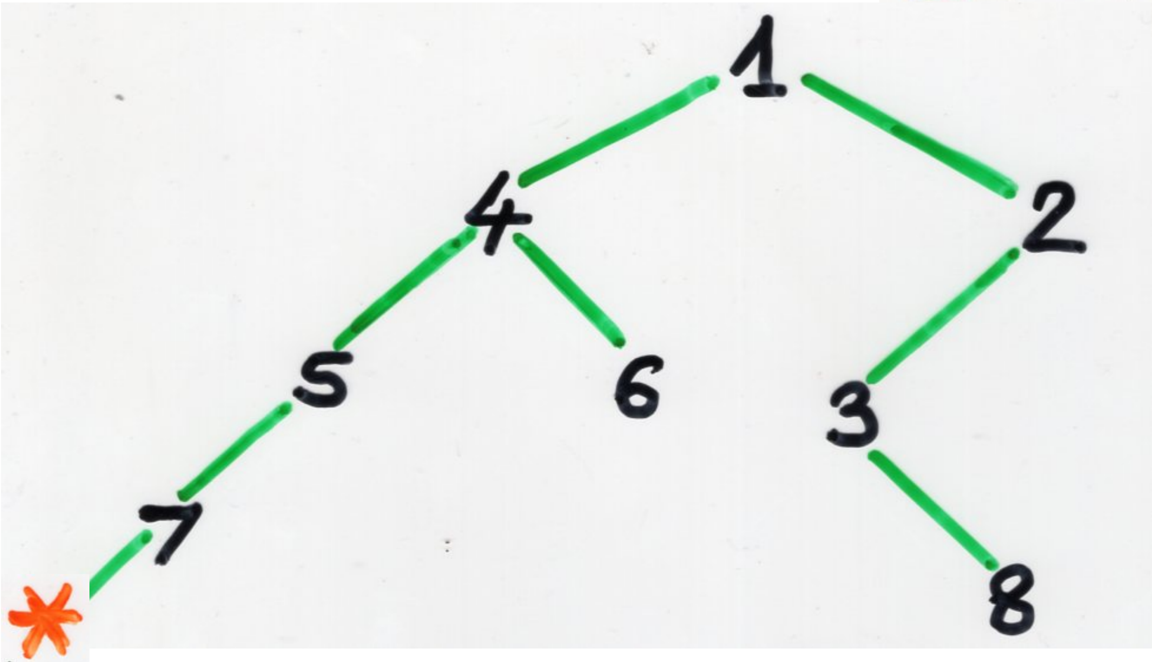
1	2	1	1	1	1	1

⊔	
⊔ 1 ⊔	1
⊔ 1 ⊔ 2	2
⊔ 1 3 ⊔ 2	1
⊔ 4 ⊔ 1 3 ⊔ 2	1
⊔ 5 4 ⊔ 1 3 ⊔ 2	1
⊔ 5 4 6 1 3 ⊔ 2	1
⊔ 7 5 4 6 1 3 ⊔ 2	1
⊔ 7 5 4 6 1 3 8 2	1



$\sqcup$	
$\sqcup\, 1 \,\sqcup$	1
$\sqcup\, 1 \sqcup 2$	2
$\sqcup\, 1\, 3 \sqcup 2$	1
$\sqcup\, 4 \sqcup 1\, 3 \sqcup 2$	1
$\sqcup\, 5\, 4 \sqcup 1\, 3 \sqcup 2$	1
$\sqcup\, 5\, 4\, 6\, 1\, 3 \sqcup 2$	1
$\sqcup\, 7\, 5\, 4\, 6\, 1\, 3 \sqcup 2$	1
$\sqcup\, 7\, 5\, 4\, 6\, 1\, 3\, 8\, 2$	1

see Ch2b, p19-23



lr-min elements

$\sigma = 7 / 5 / 4\ 6 / 1\ 3\ 8\ 2$

$$\begin{cases} b_k = [k]_q + [k+1;\beta]_q \\ \lambda_k = [k;\beta]_q \, [k]_q \end{cases}$$

<u>Proposition</u> $$\mu_n^{(\beta)}(q) = \sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} q^{31\text{-}2(\sigma)}$$

subdivided Laguerre histories

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

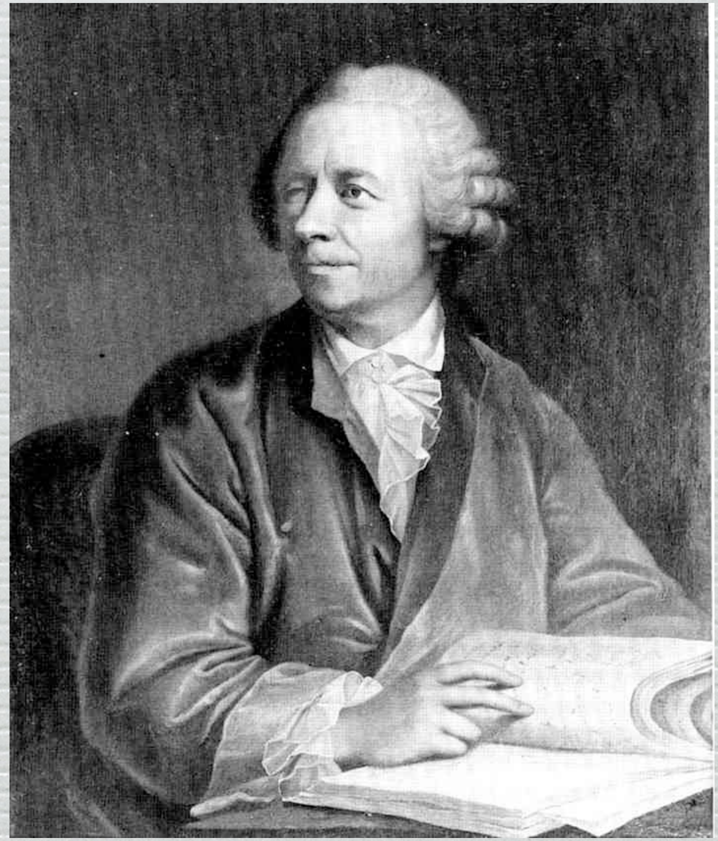
AVCTORE

Leonh. Euler.

§. 1.

Varii in Analysin recepti sunt modi quantitates, quae alias difficulter assignari queant, commode exprimendi. Quantitates scilicet irrationales et transcendentes, cuiusmodi sunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per series infinitas exhiberi solent, quae, cum terminis constent cognitis, valores illarum quantitatum satis distincte indicant. Series autem istae duplicis sunt generis, ad quorum prius pertinent illae series, quarum termini additione subtractioneue sunt connexi; ad posterius vero referri possunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter est $=1$, exprimi solet; priore nimirum area circuli aequalis dicitur $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-$etc. in infinitum; posteriore vero modo eadem area aequatur huic expressioni $\frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\cdot\frac{8.10}{9.9}\cdot\frac{10}{11}$ etc. in infinitum. Quarum serierum illae reliquis merito praeferuntur, quae maxime conuergant, et paucissimis sumendis terminis valorem quantitatis quaesitae proxime praebeant.

§. 2. His duobus serierum generibus non immerito superaddendum videtur tertium, cuius termini continua diui-



§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

9

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

$$\gamma_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \dots}}}}}}$$

$$\sum_{n\geqslant 0} n!\, t^n = \cfrac{1}{1-1t-\cfrac{1t}{1-2t-\cfrac{2t}{1-2t-\cfrac{2t}{1-3t-\cfrac{3t}{1-\cdots}}}}}$$

$$\gamma_k = \left\lceil \frac{k}{2} \right\rceil$$

$$S(t;\gamma) = J(t;b,\lambda)$$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\gamma_{2k-1} \end{cases}$$

q-Laguerre
restricted
histories

$$\sum_{n\geqslant 0} n!\, t^n = \cfrac{1}{1-1t-\cfrac{1^2t^2}{1-3t-\cfrac{2^2t^2}{1-5t-\cfrac{3^2t^2}{\cdots\cdots}}}}$$

$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\dots}}}}}}$$

$$\mu_n(q)$$

$$\gamma_k = \left[\left\lceil \frac{k}{2} \right\rceil\right]_q$$

$$S(t;\gamma) = J(t;b,\lambda)$$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$

q-Laguerre
restricted
histories

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-1t-\cfrac{1^2t^2}{1-3t-\cfrac{2^2t^2}{1-5t-\cfrac{3^2t^2}{\dots\dots}}}}$$

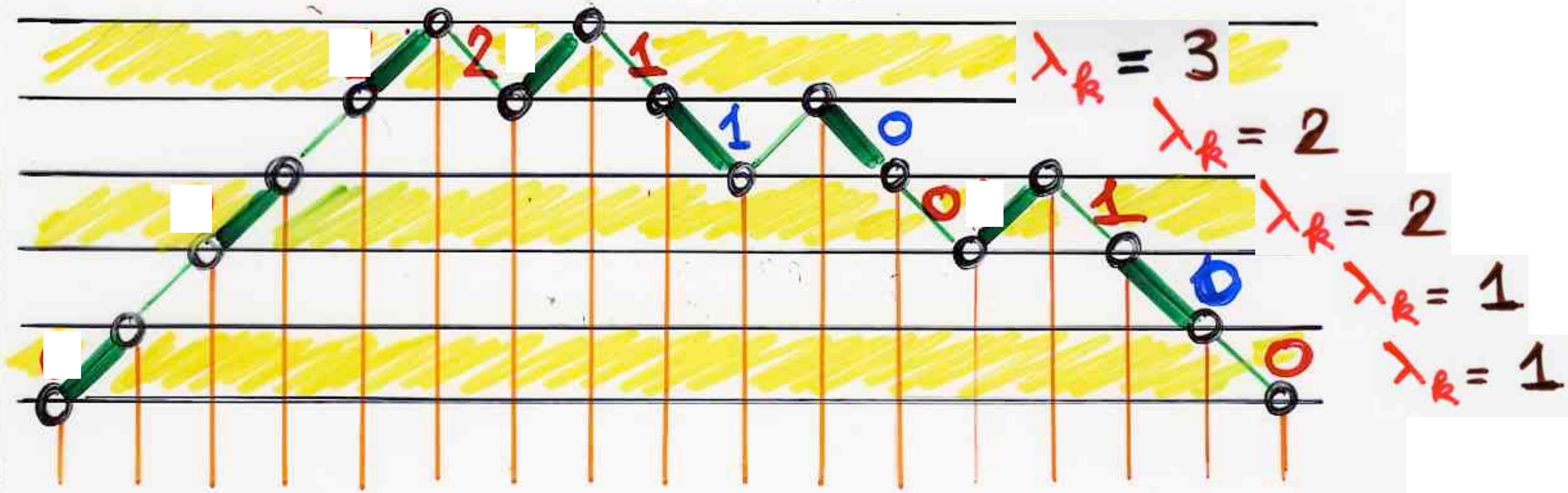
Corollary

$$\gamma_k = \left[\left\lceil \frac{k}{2} \right\rceil\right]_q$$

$$\sum_{n\geq 0} \mu_n(q)\, t^n = \cfrac{1}{1-(1)t - \cfrac{}{1-(1)t - \cfrac{}{1-(1+q)t - \cfrac{}{1-(1+q)t - \cfrac{}{1-(1+q+q^2)t - \cfrac{}{1- \cdots}}}}}}$$

$\gamma_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$

$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$



subdivided Laguerre history

H

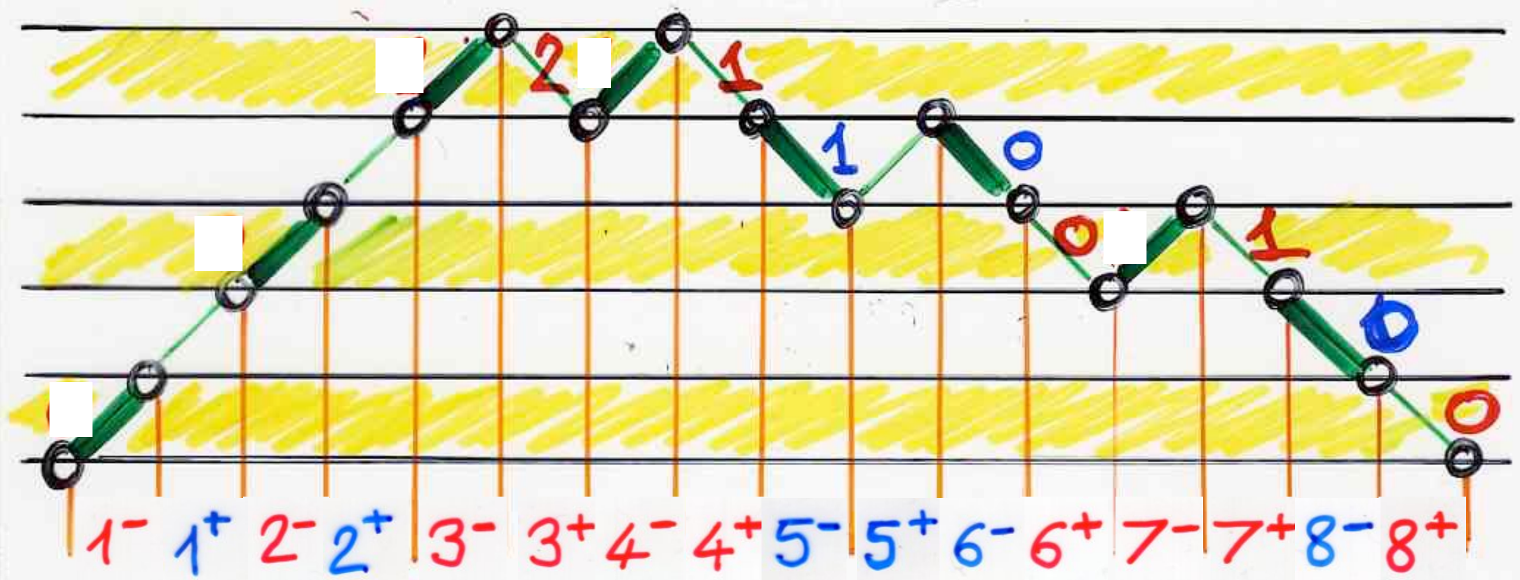
Bijection

subdivided Laguerre histories 

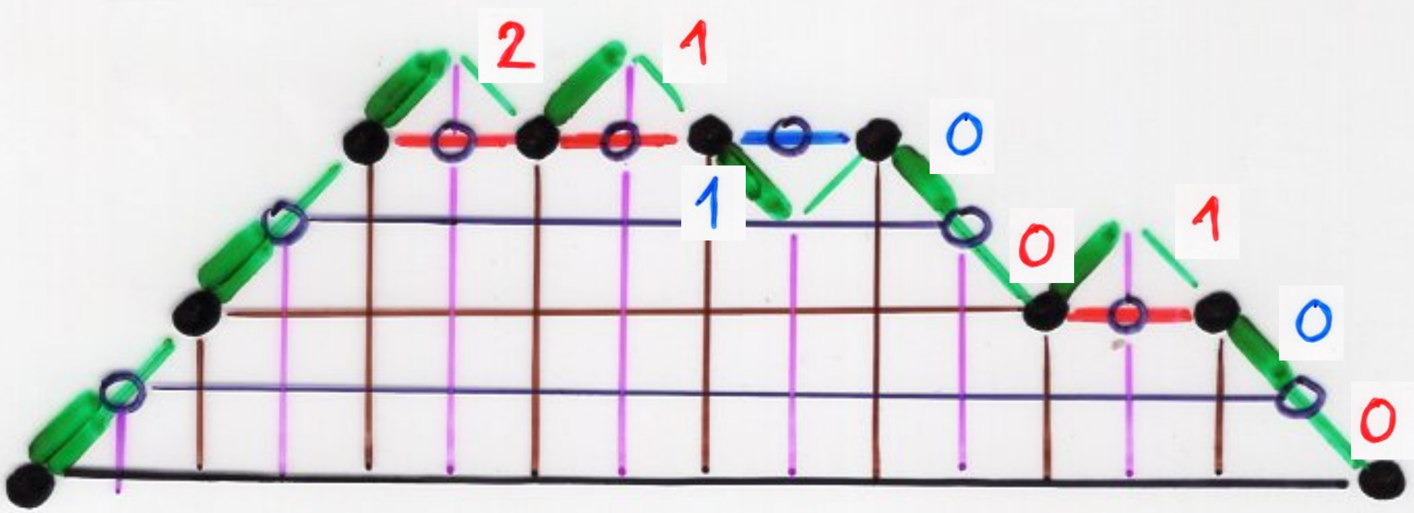
$\updownarrow$

(restricted) Laguerre histories 

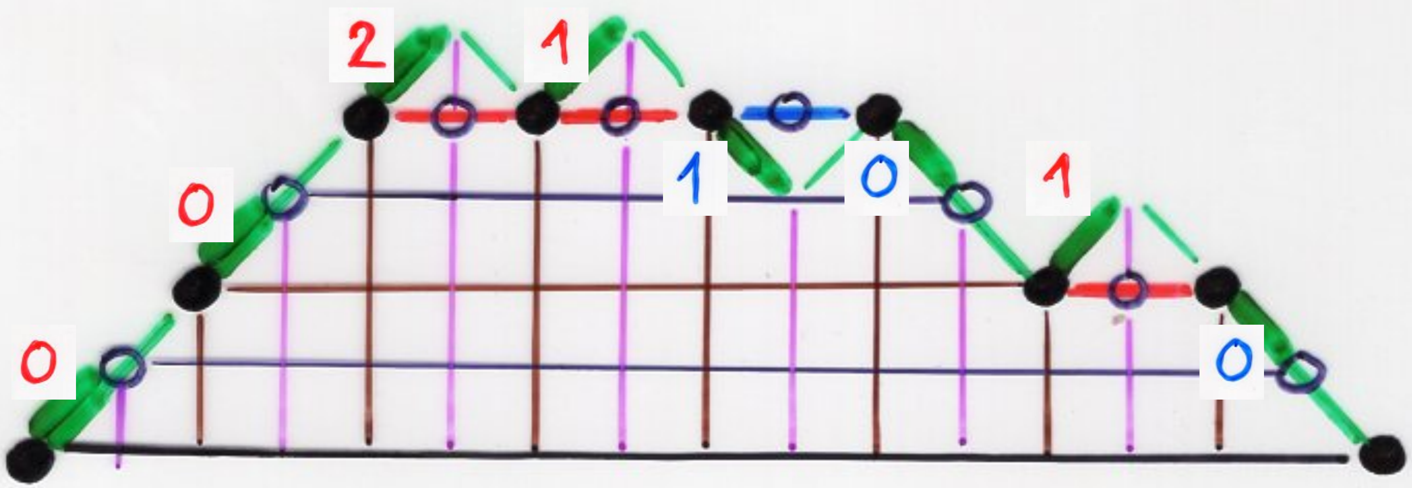
see Ch3b, p82-91



subdivided Laguerre history H



subdivided Laguerre history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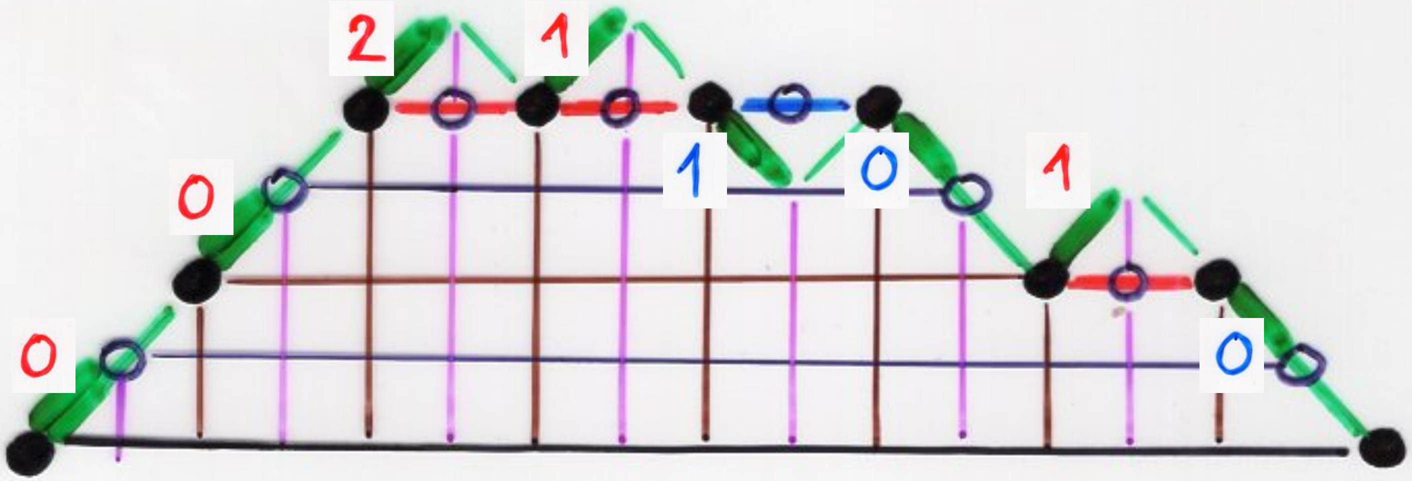
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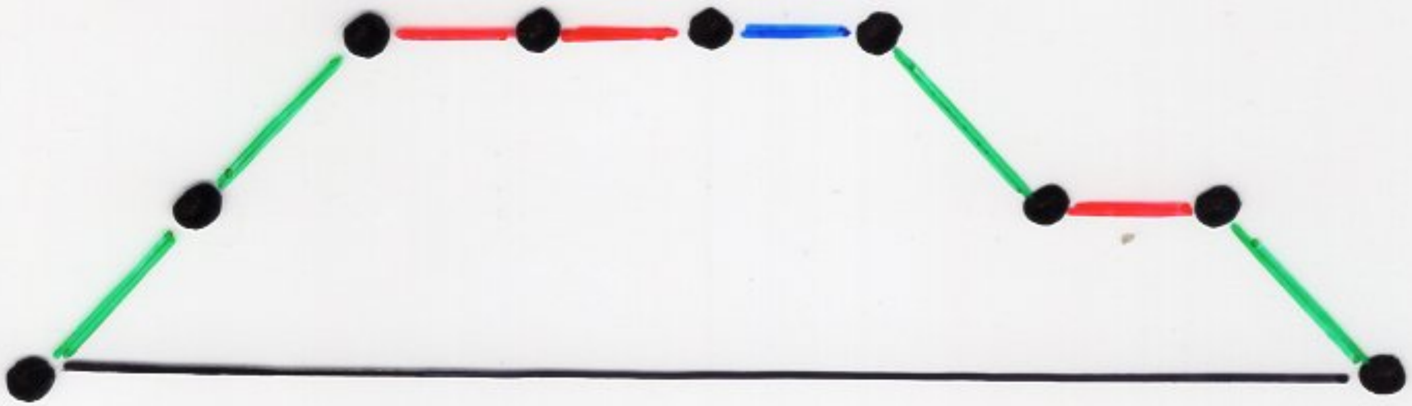

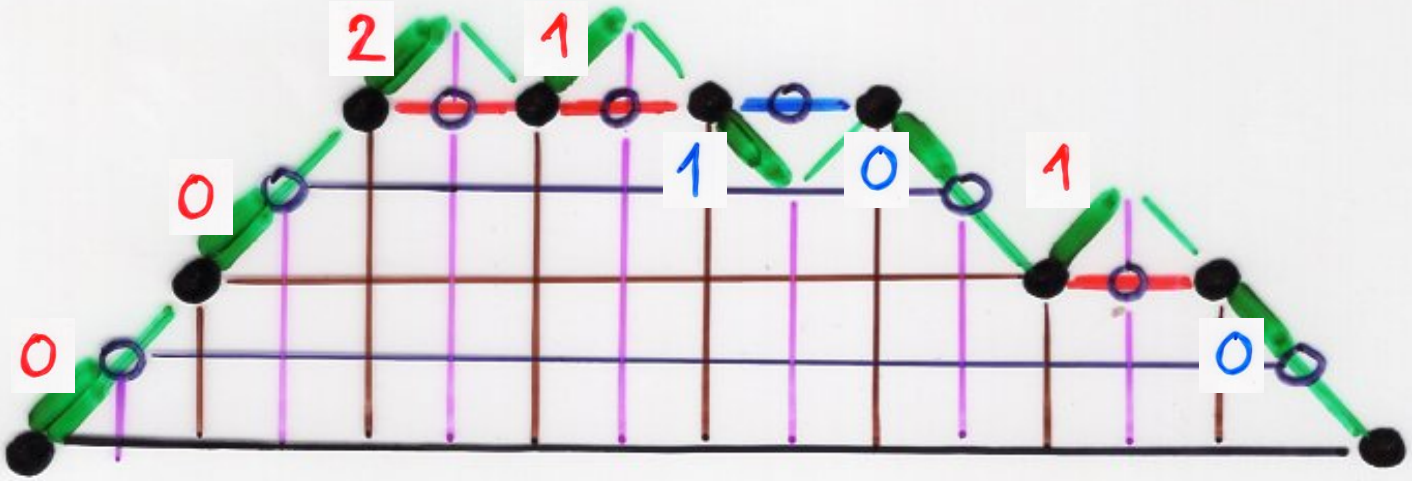
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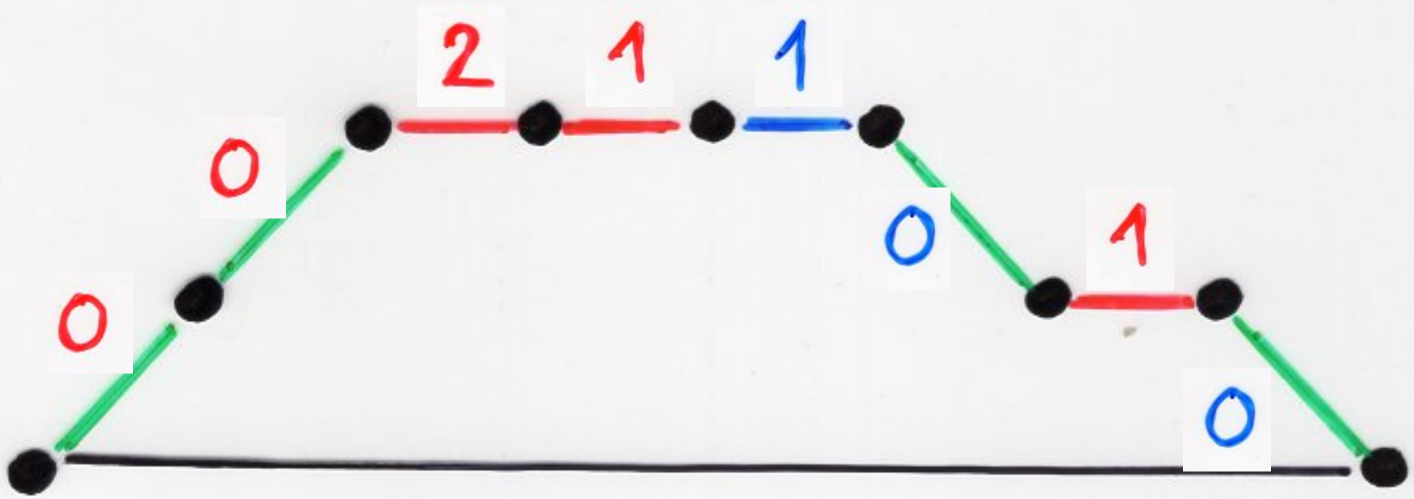
2
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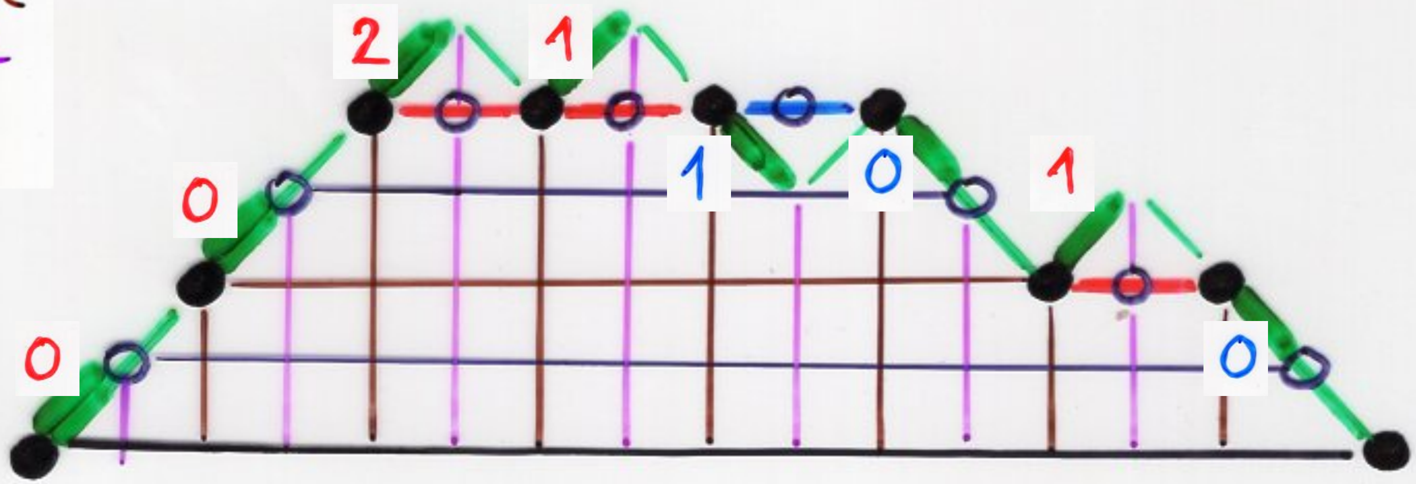
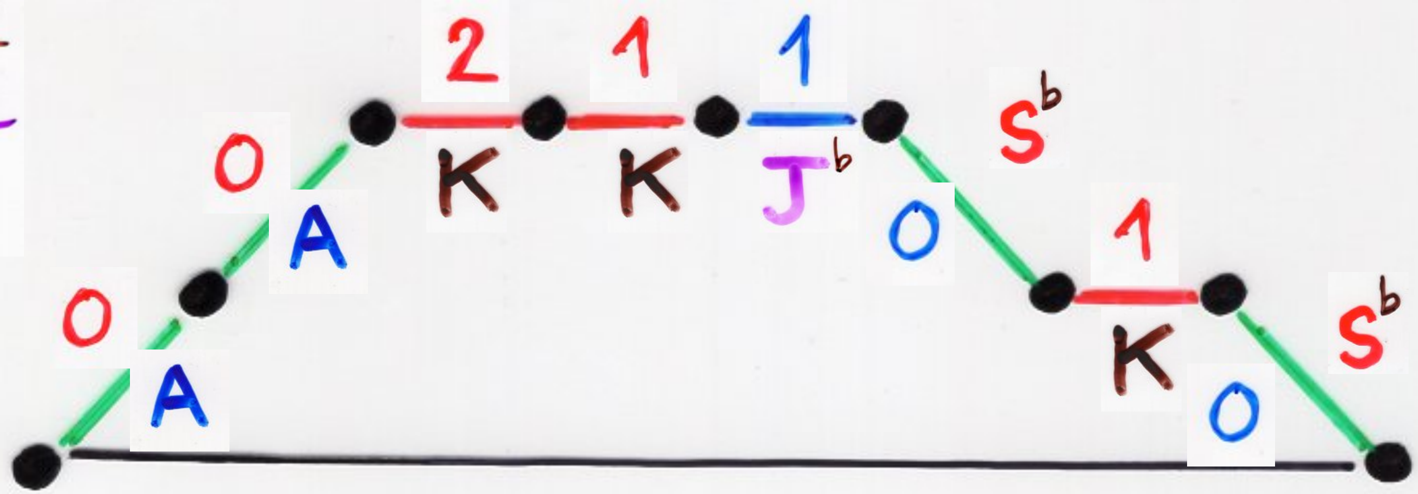
subdivided Laguerre history

H



restricted Laguerre history

h



$$v_q(H) = v_q(h)$$

Bijection

Permutations σ

$\updownarrow$

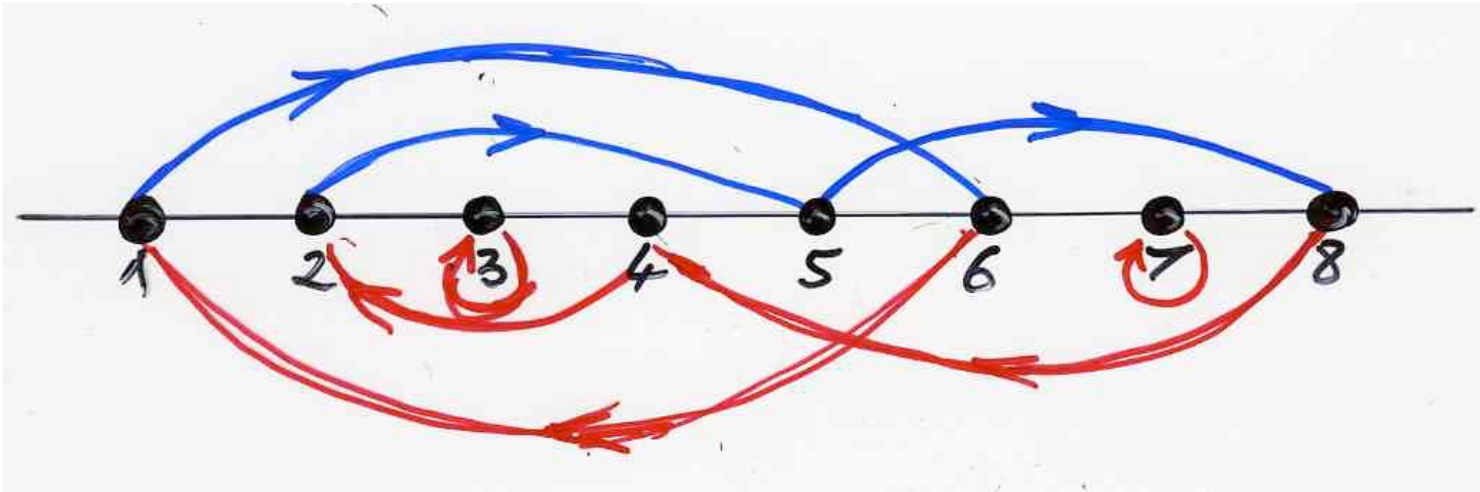
subdivided Laguerre histories H

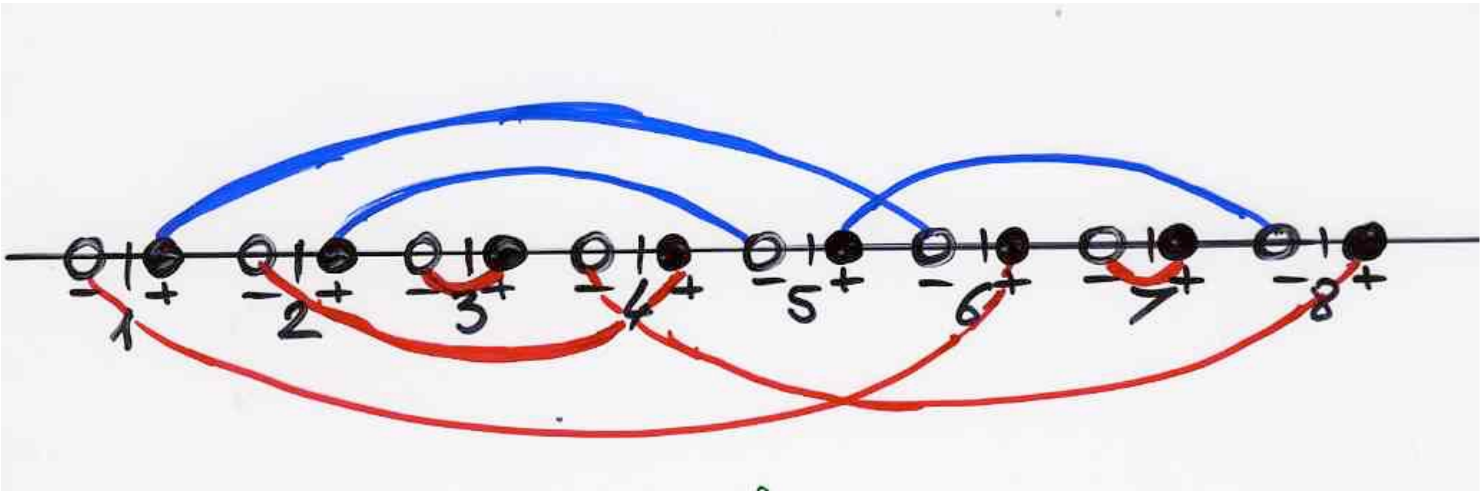
see Ch3b, p43-72

A. de Médicis, X.V.
(1994)

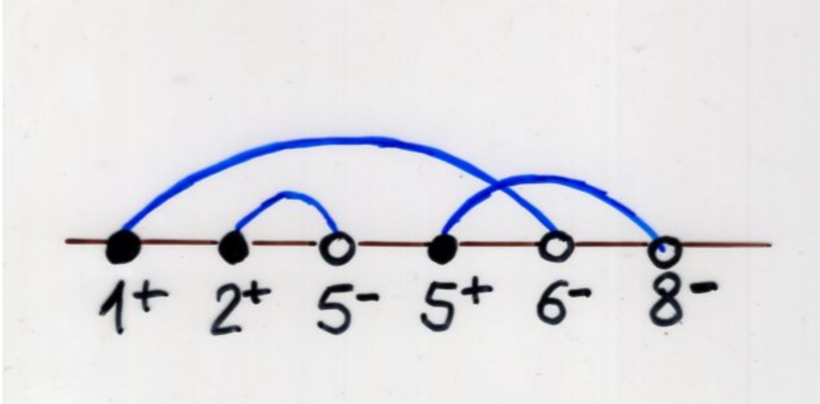
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

σ





τ_{exc}

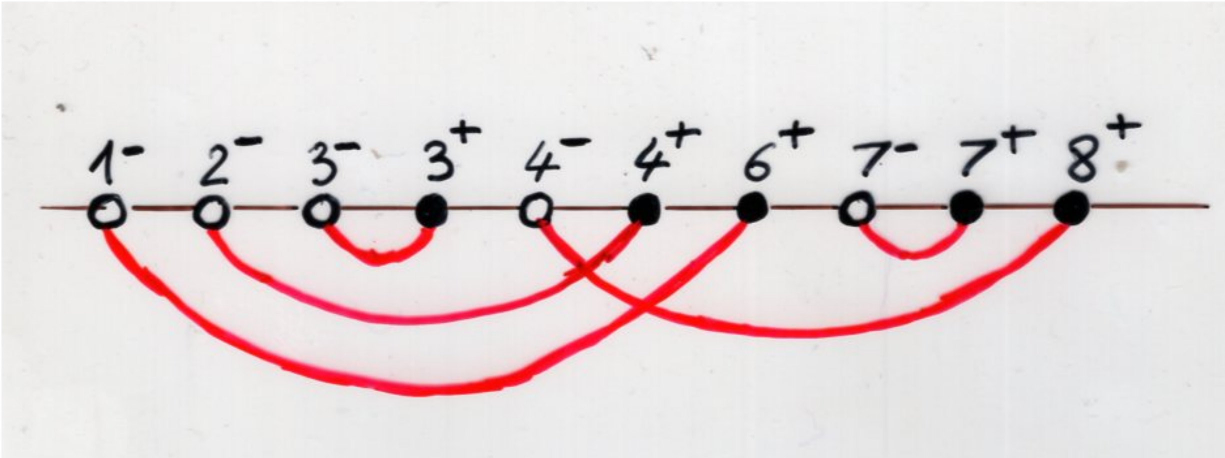


chord diagram

(strict) exceedances

$i < \sigma(i)$

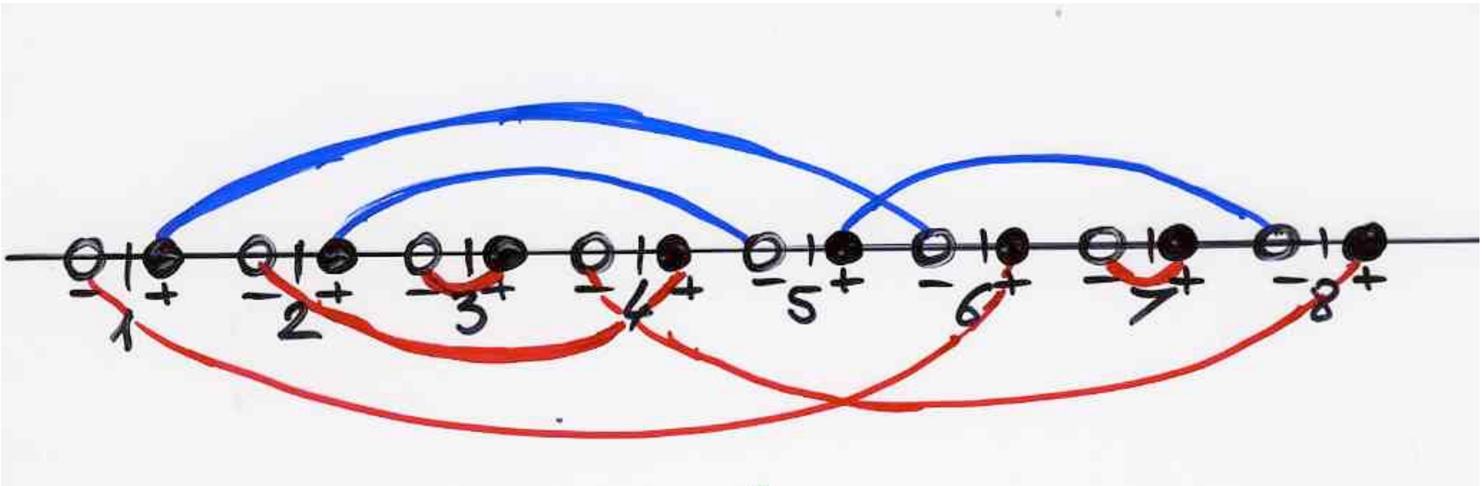
τ_{nexc}

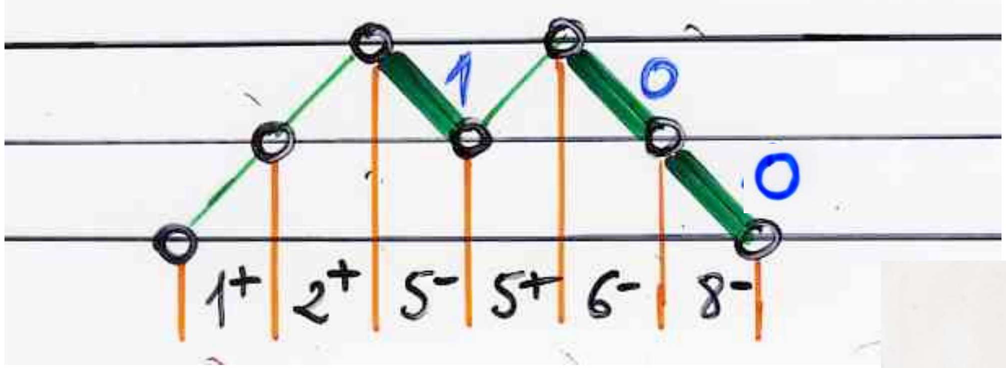


chord diagram

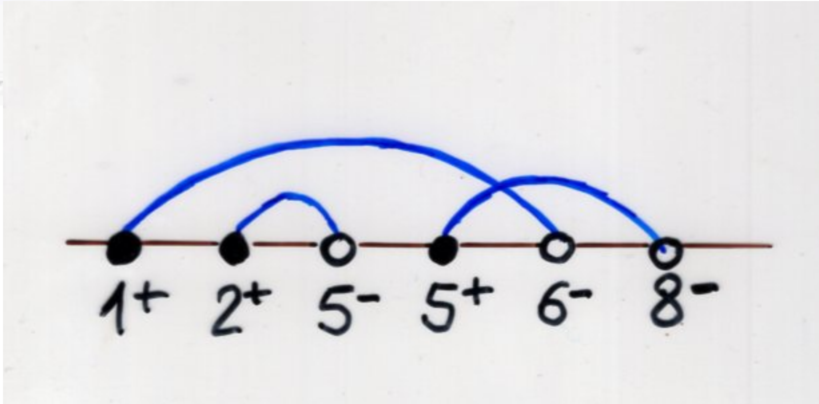
non-exceedances

$i \geqslant \sigma(i)$



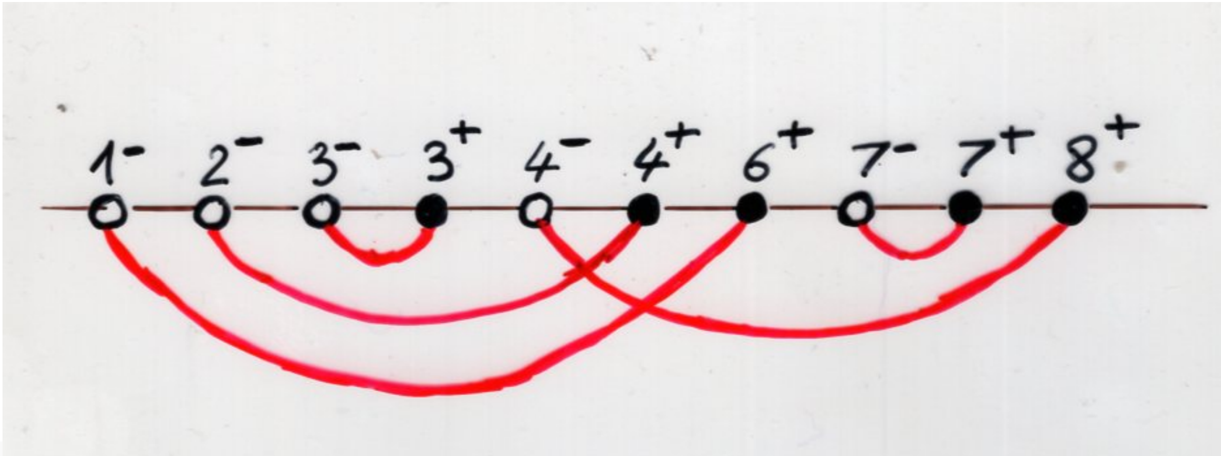


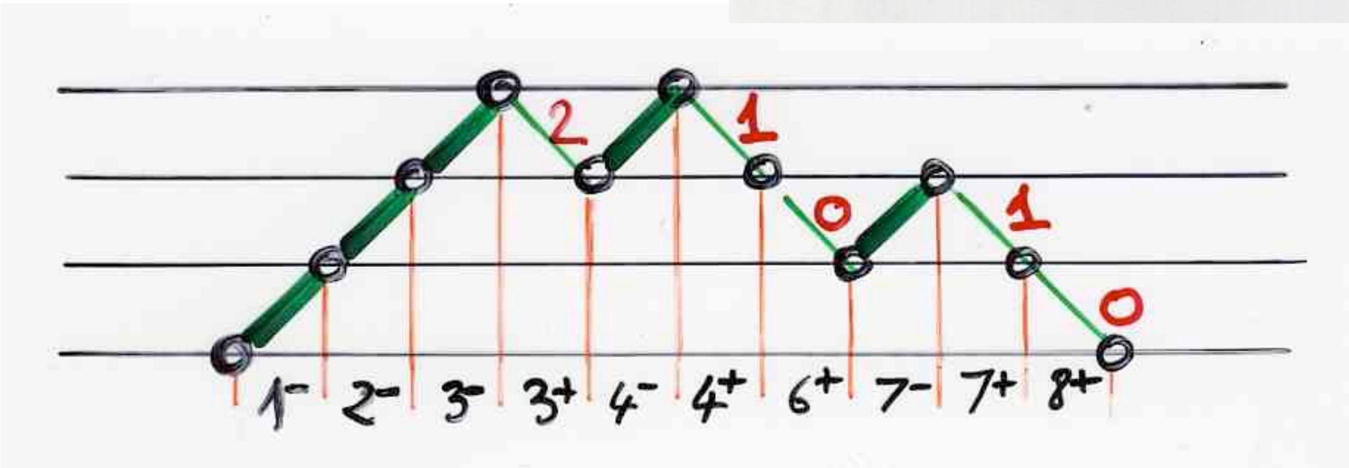
τ_{exc}

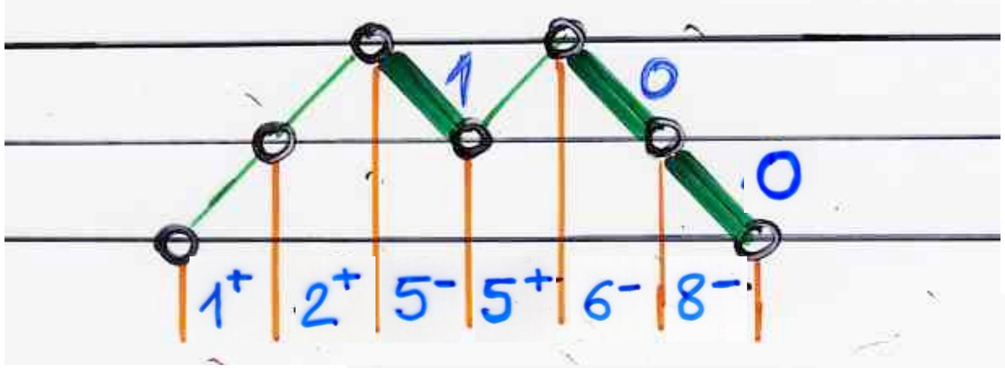


pair of two Hermite histories ("shuffle")

τ_{nexc}

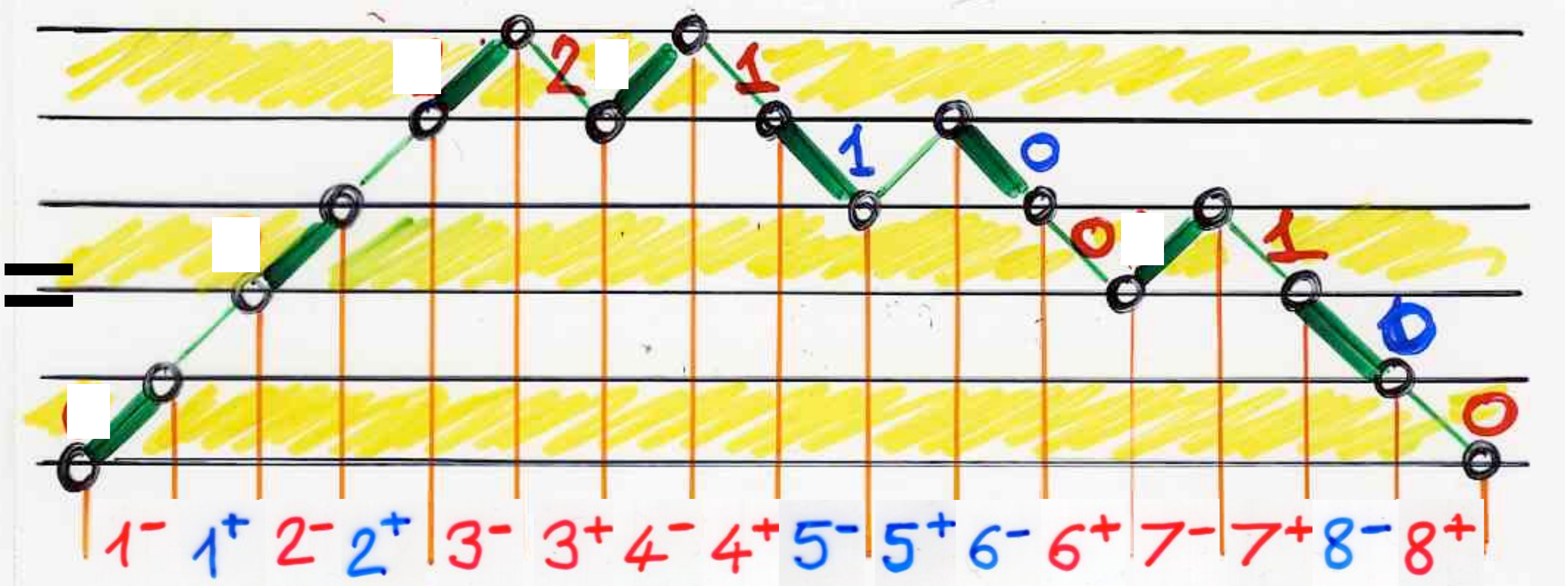




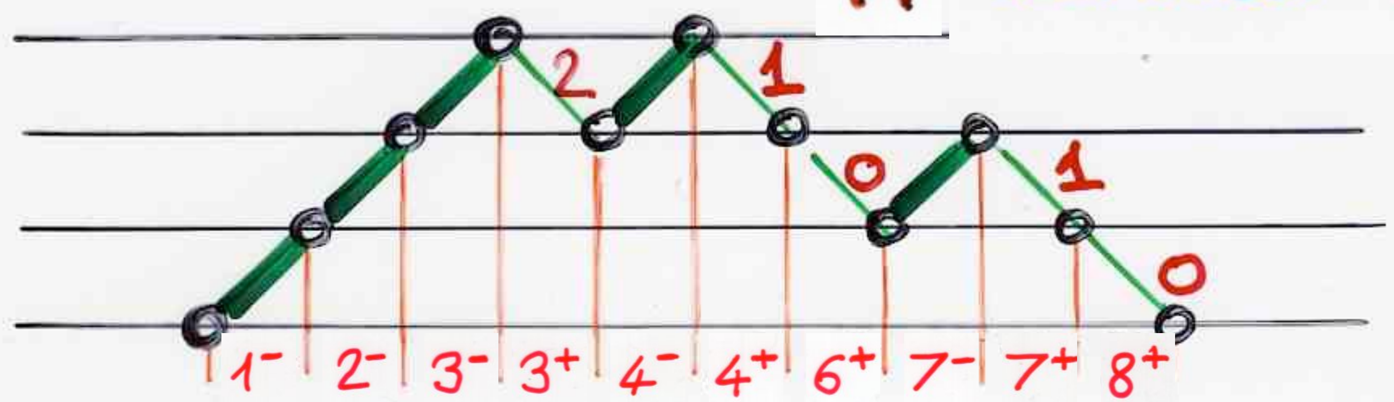


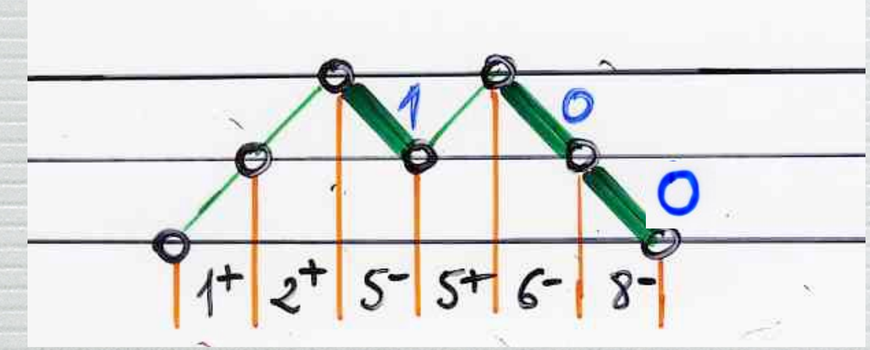
pair of two
Hermite histories
("shuffle")

=



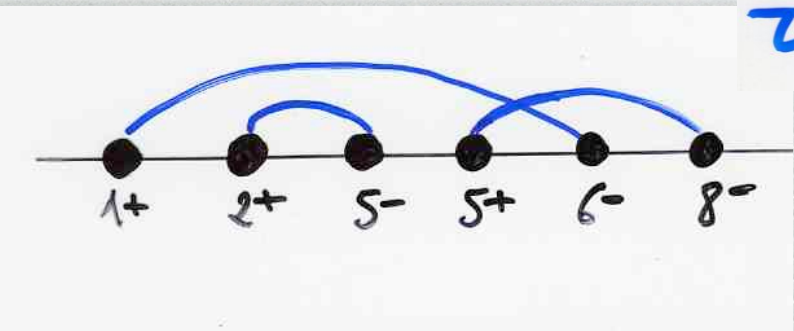
H subdivided Laguerre history

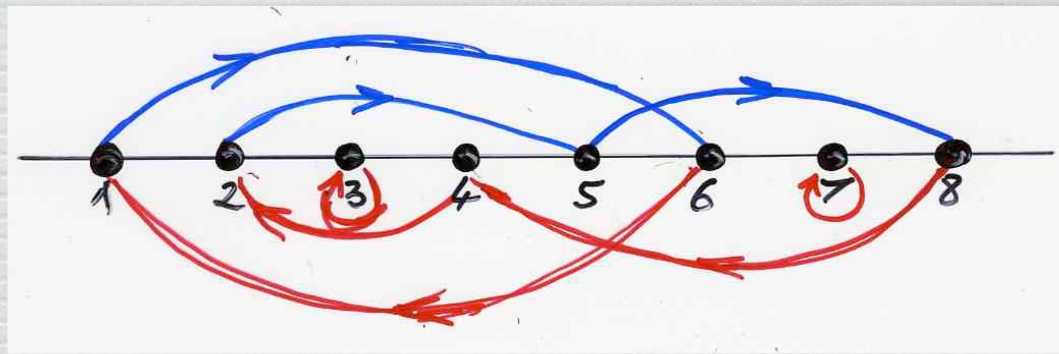


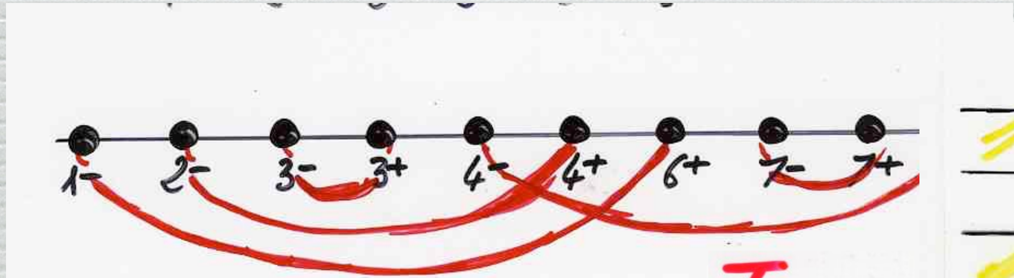


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

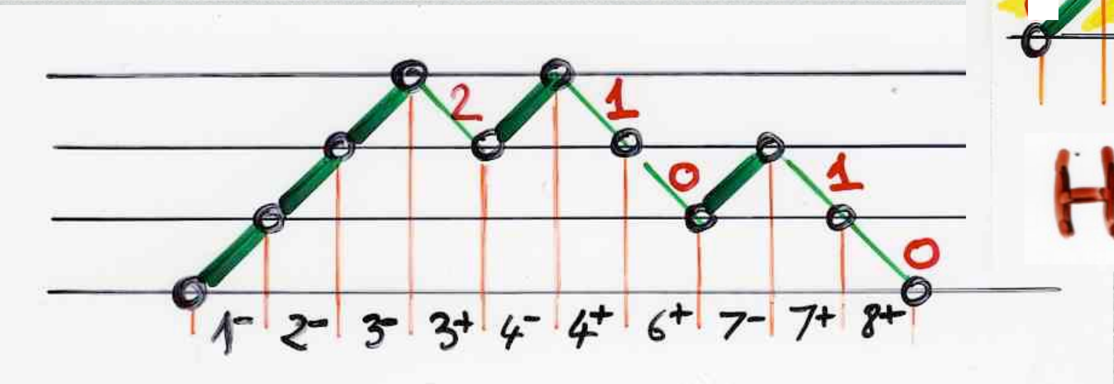
τ_{exc}

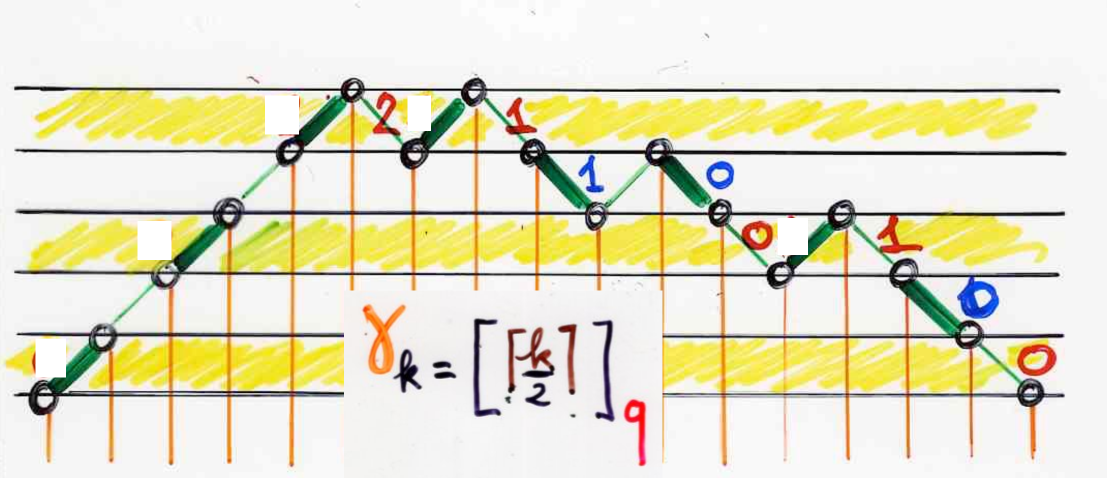






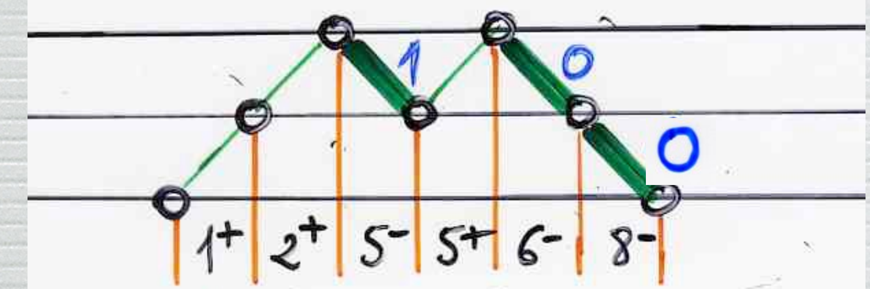
τ_{nexc}



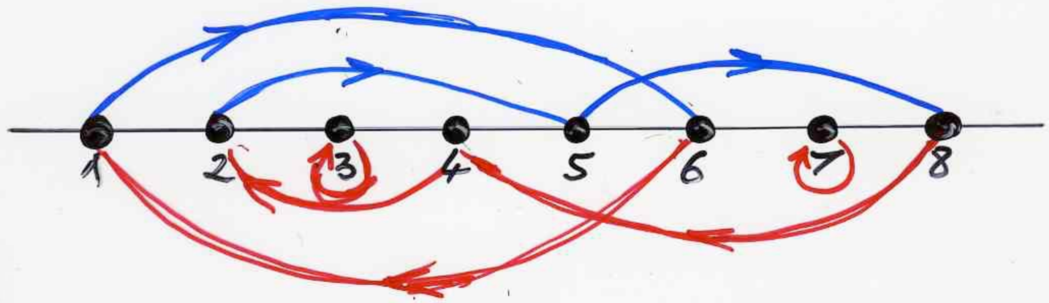


$$\gamma_k = \left[\left\lceil \frac{k}{2} \right\rceil\right]_q$$

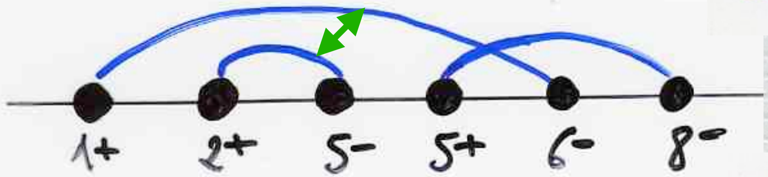
H subdivided Laguerre history



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

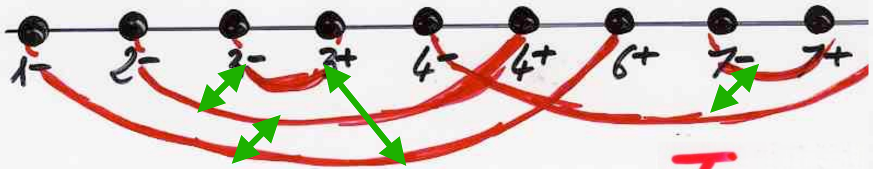


τ_{exc}

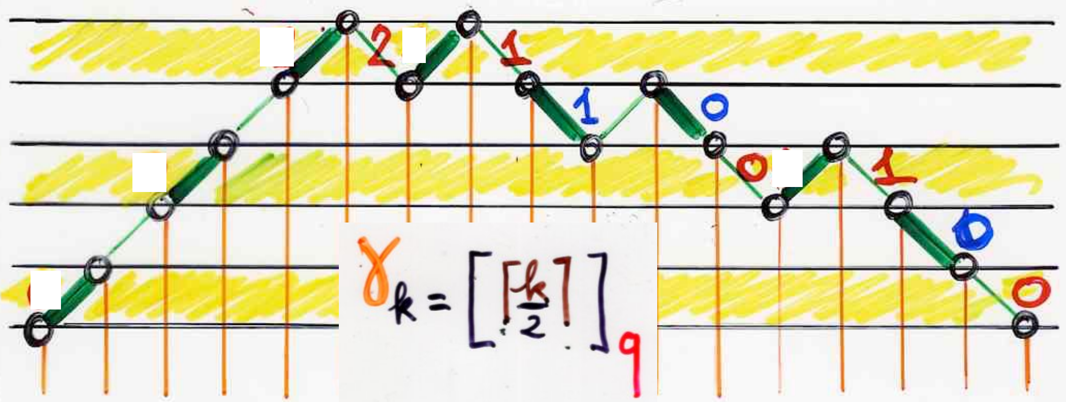


nb of nestings q

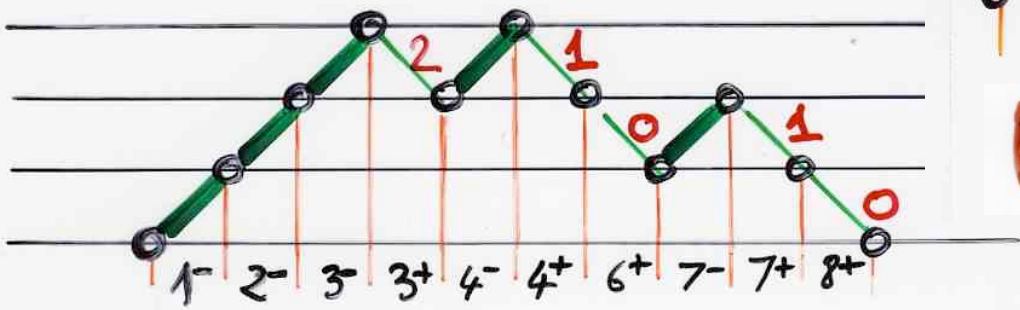
$$v_q(H) = \mathrm{nest}(\tau_{exc}) + \mathrm{nest}(\tau_{nexc})$$



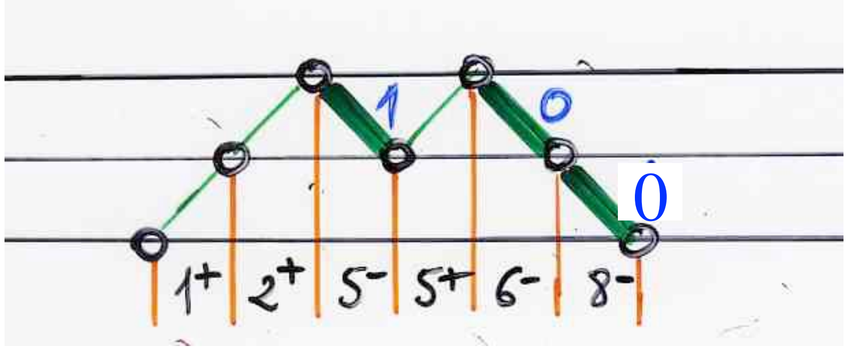
τ_{nexc}



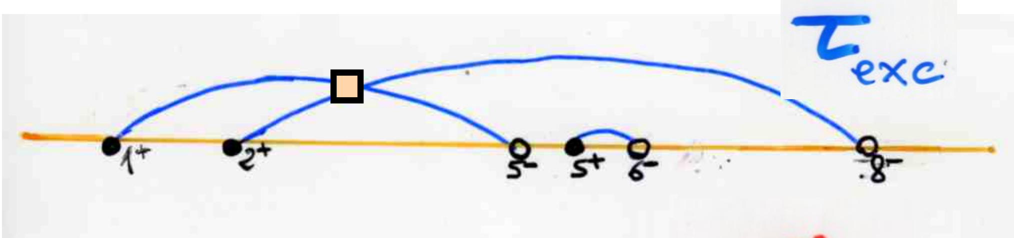
$$\gamma_k = \left[\left\lceil \frac{k}{2} \right\rceil\right]_q$$

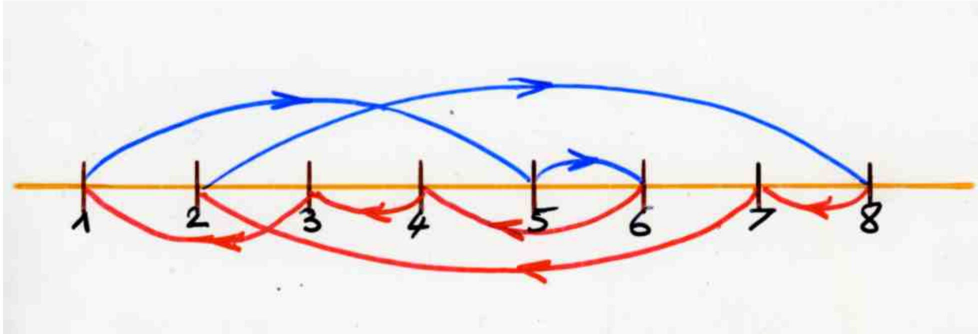


H subdivided Laguerre history



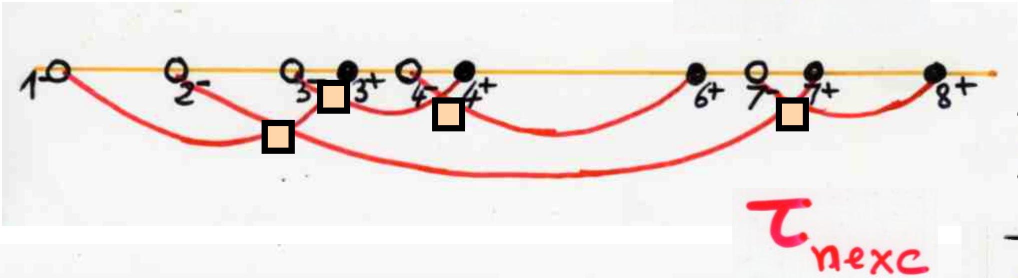
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

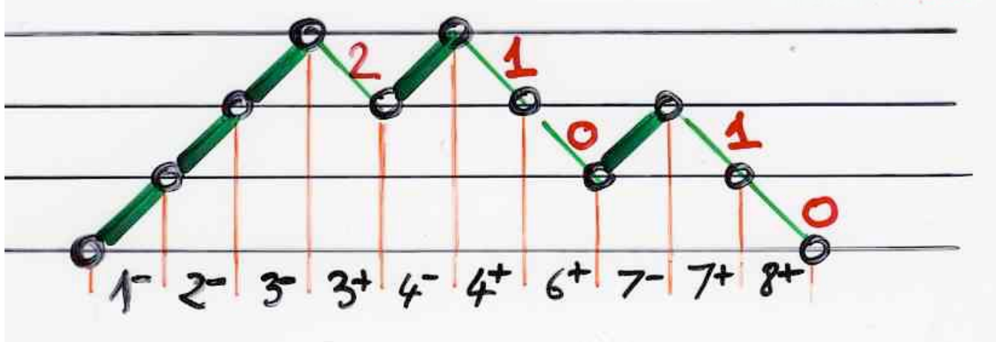


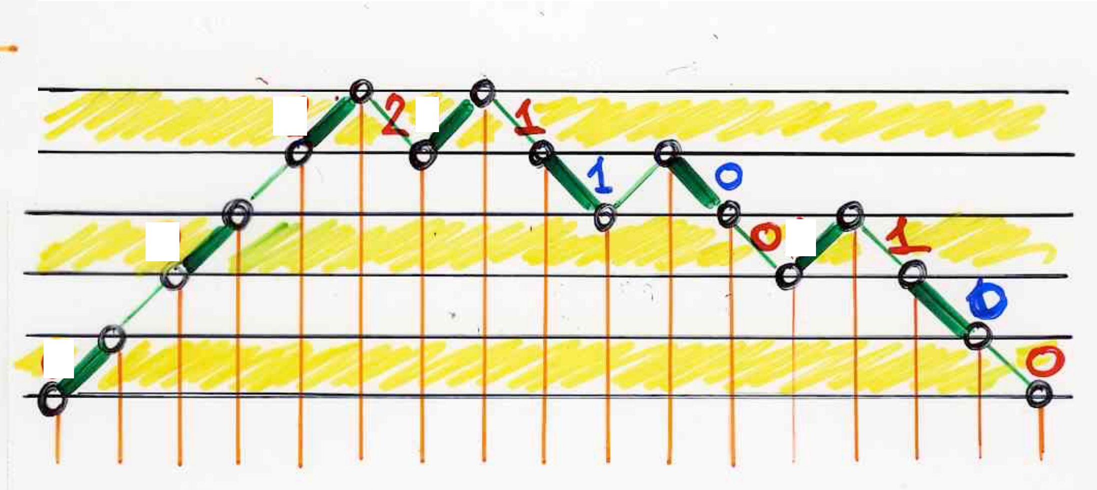


nb of crossings q

$$v_q(H) = cr(\tau_{exc}) + cr(\tau_{nexc})$$







H subdivided Laguerre history

Definition

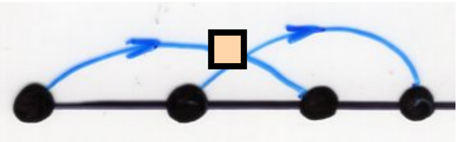
Corteel (2007)

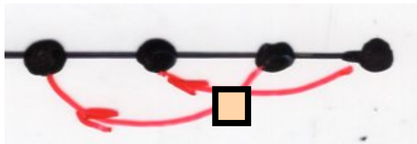
number of crossings of a permutation

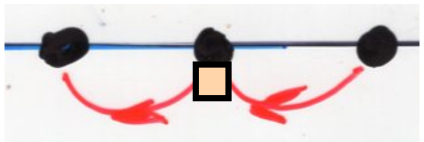
9

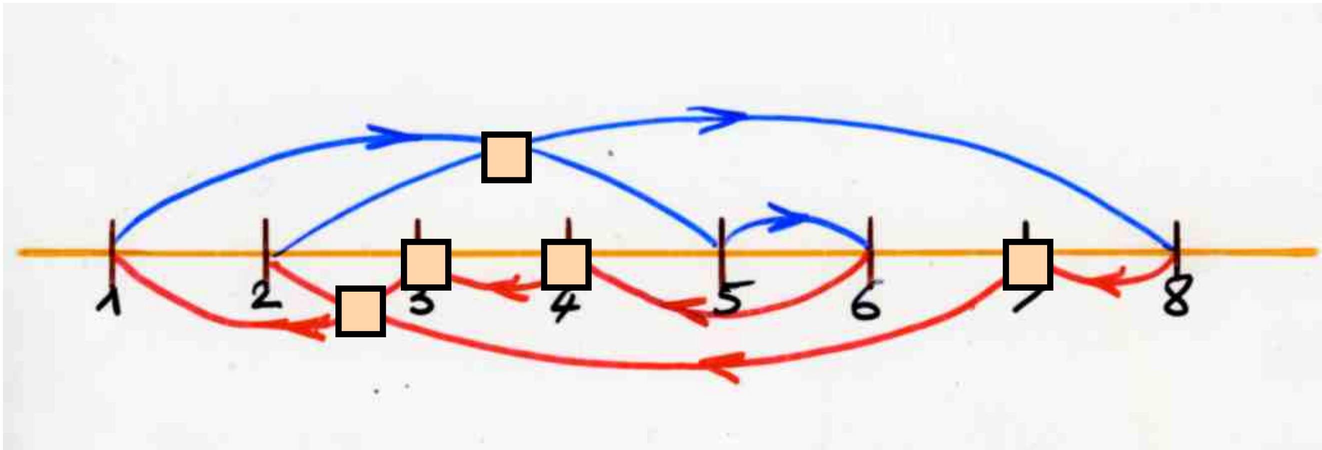
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

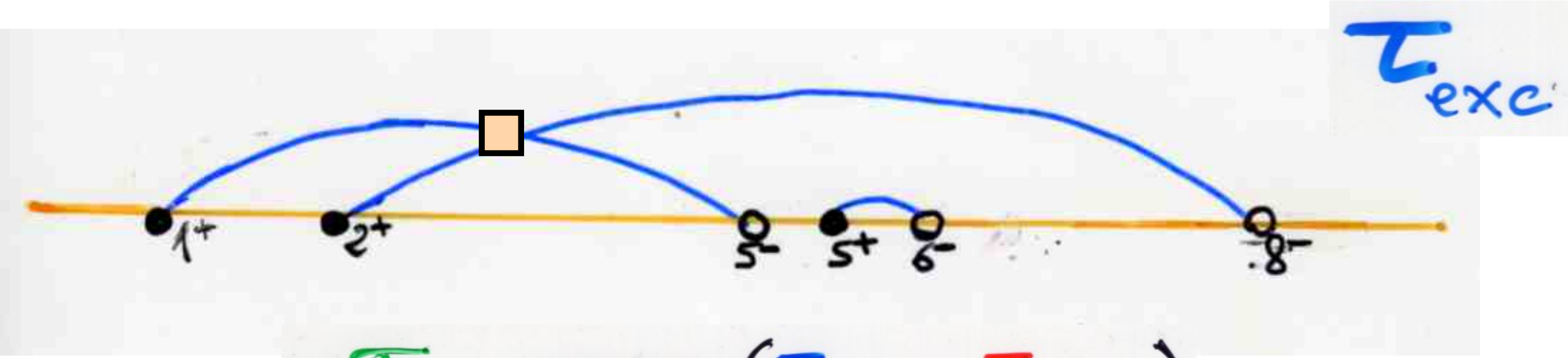
(strict) exceedances





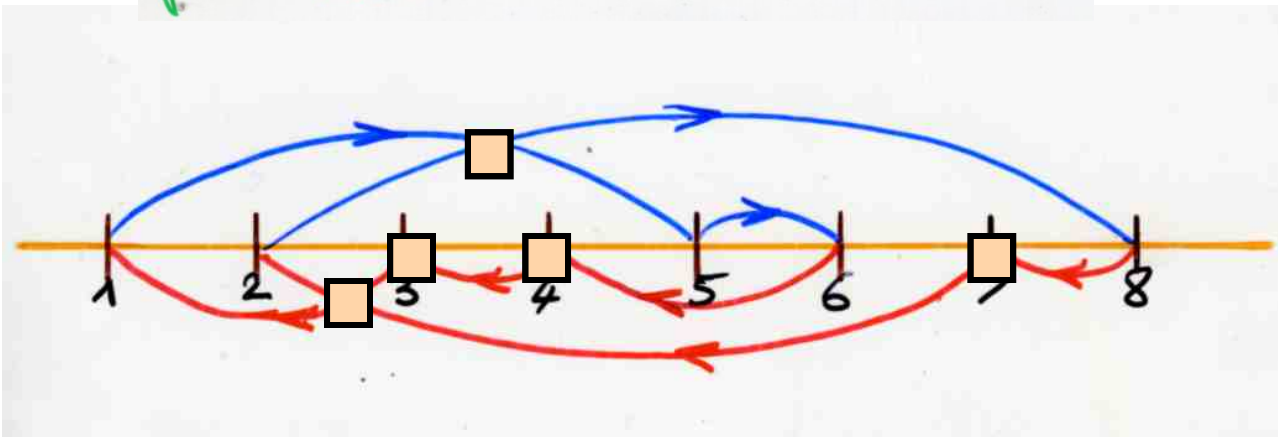




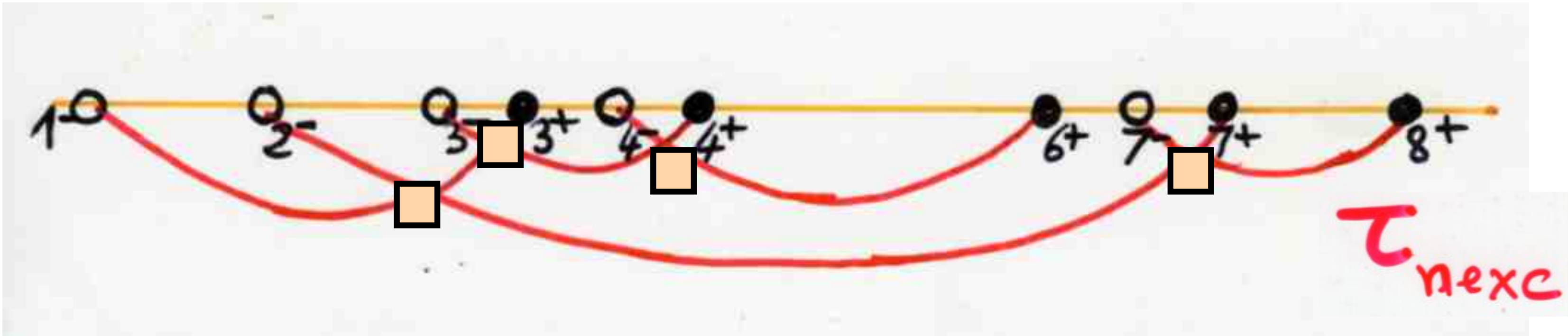


σ $\longleftrightarrow$ $(\tau_{exc}, \tau_{nexc})$

permutation



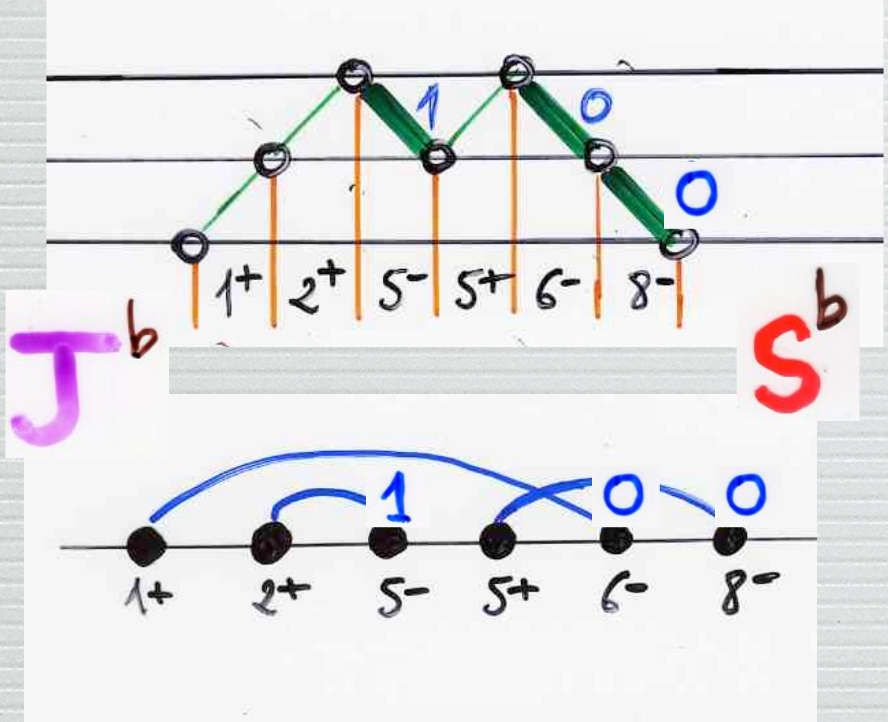
Lemma $cr(\sigma) = cr(\tau_{exc}) + cr(\tau_{nexc}) = v_q(H)$

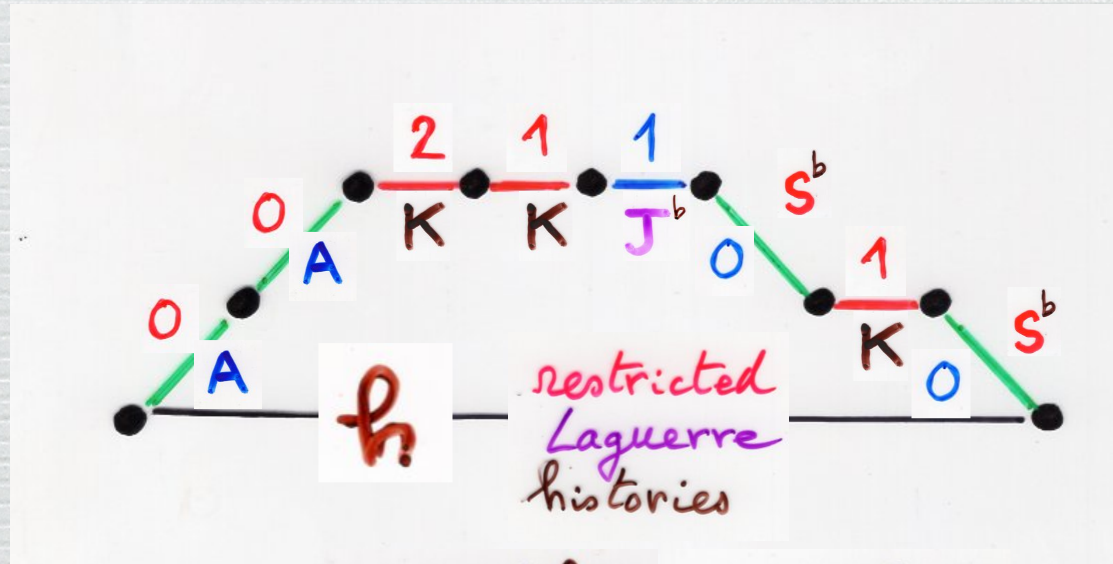


Corollary

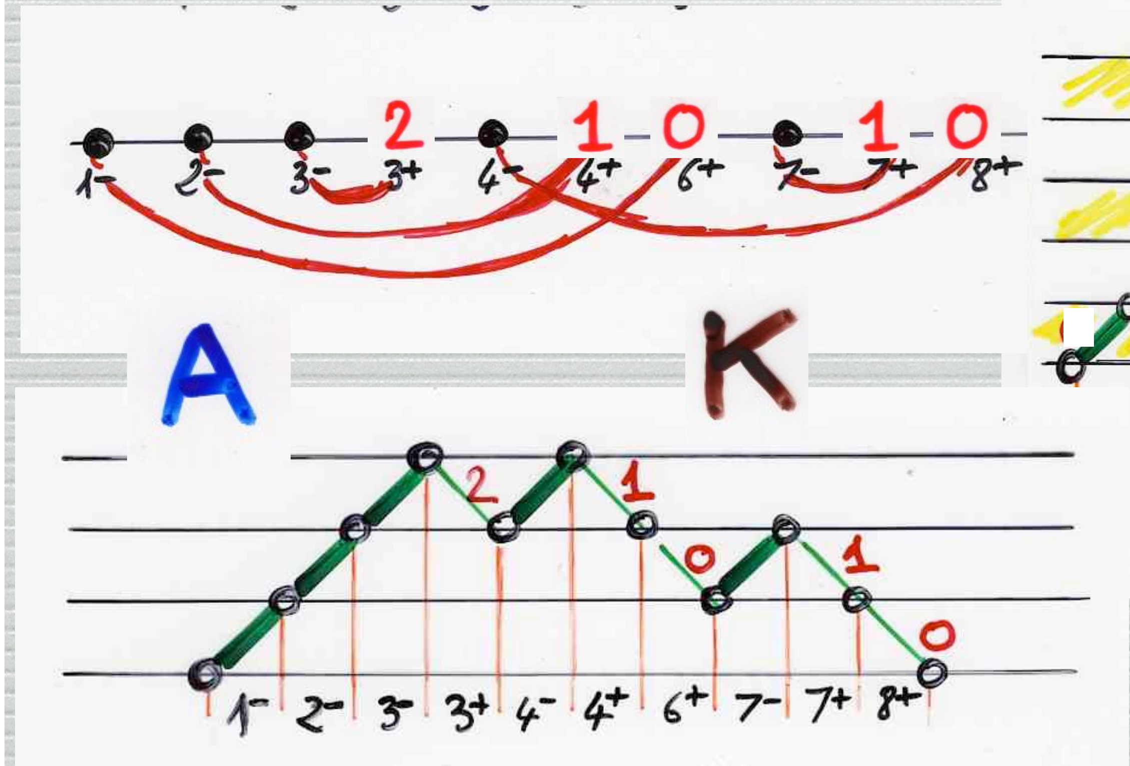
$$\mu_n(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{cr(\sigma)}$$

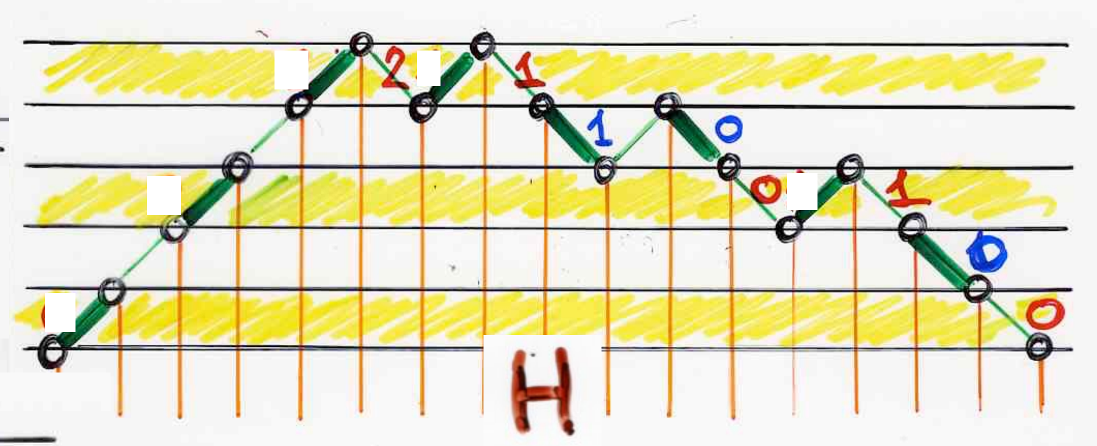
number of crossings
of a permutation



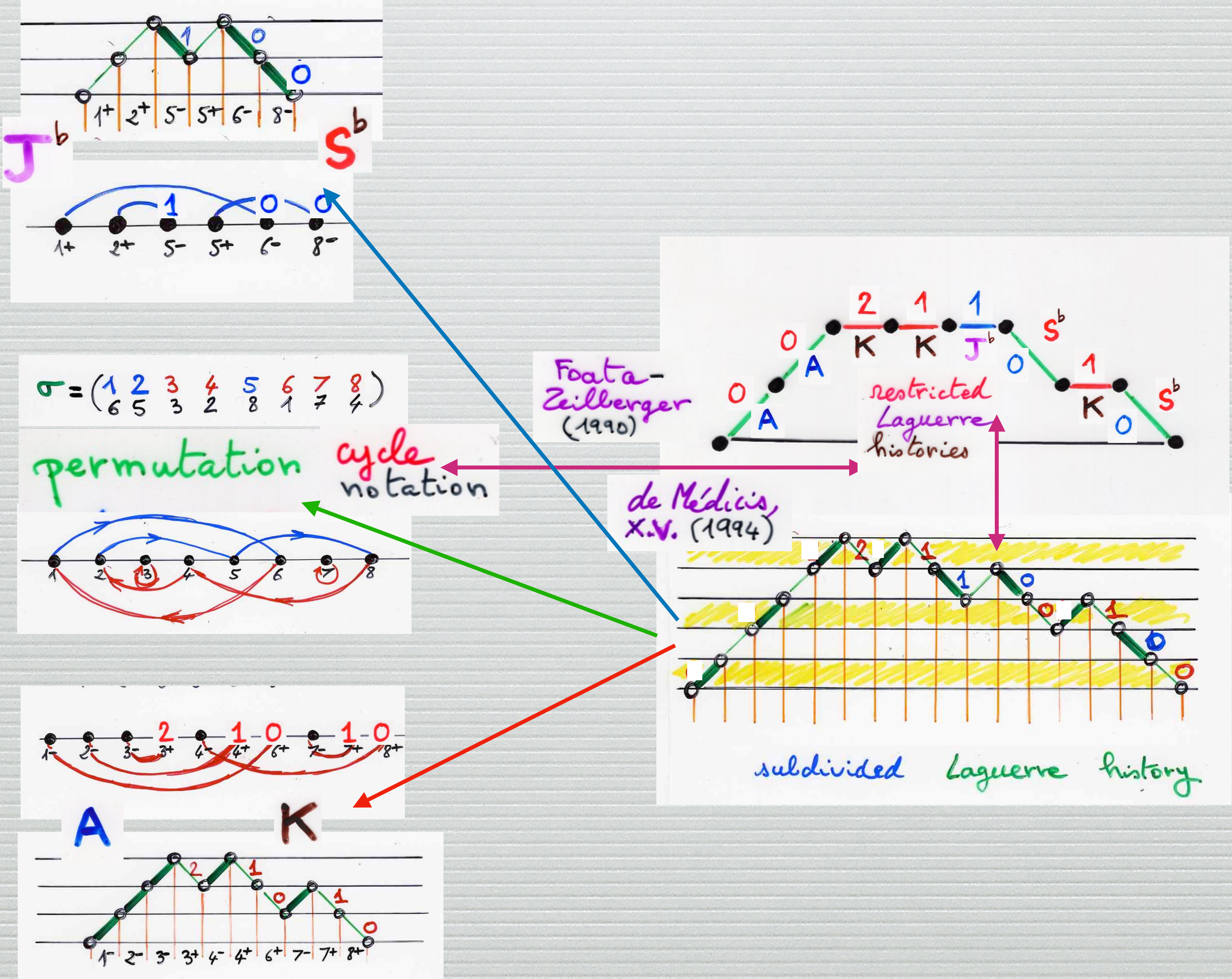


$$v_q(h) = v_q(H)$$





subdivided Laguerre history



J^b
S^b
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
permutation
cycle notation
Foata-Zeilberger (1990)
restricted Laguerre histories
de Médicis, X.V. (1994)
subdivided Laguerre history
A
K

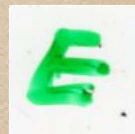
$$\begin{cases} b_k = y[k+1]_q + [k]_q \\ \lambda_k = y[k]_q^2 \end{cases}$$

$$\mu_n(y,q) = \sum_{\sigma \in \mathfrak{S}_n} y^{\mathrm{wex}(\sigma)} q^{\mathrm{cr}(\sigma)}$$

$$\mathrm{wex}(\sigma) = \left\{ \begin{array}{c} \text{number of } i,\ 1 \le i \le n \\ i \le \sigma(i) \end{array} \right\}$$

Interpretation with Laguerre heaps of segments

Bijection (restricted) Laguerre histories 

Laguerre heaps of segments 

see Ch2c, p95-104, p113-133

$$A|k\rangle = (k+1)\,|(k+1)\rangle$$

$$K|k\rangle = (k+1)\,|k\rangle$$

restricted Laguerre histories

$$J^b|k\rangle = k\,|k\rangle$$

$$S^b|k\rangle = k\,|(k-1)\rangle$$

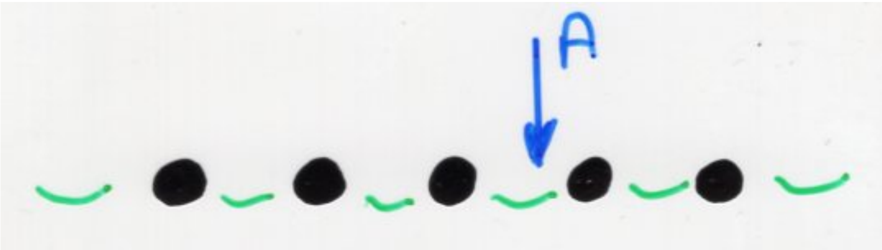
dictionnary data structure

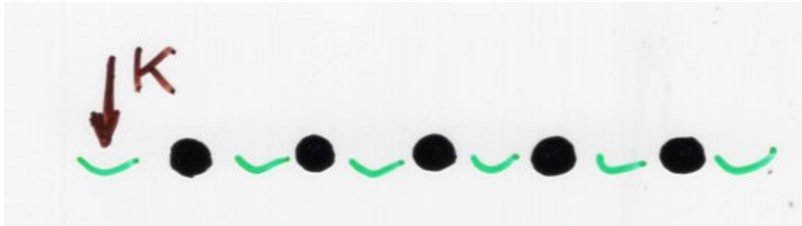
add or delete any element

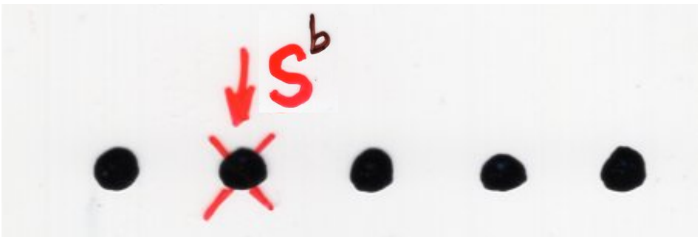
ask questions

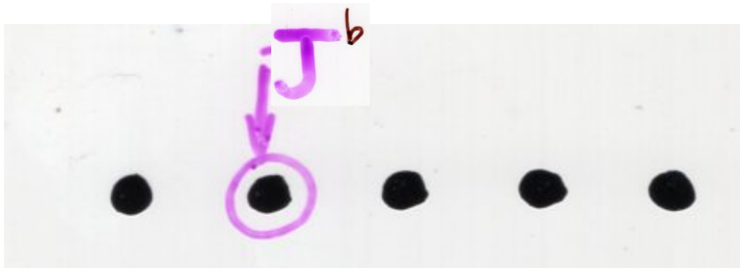
J^b positive

K negative







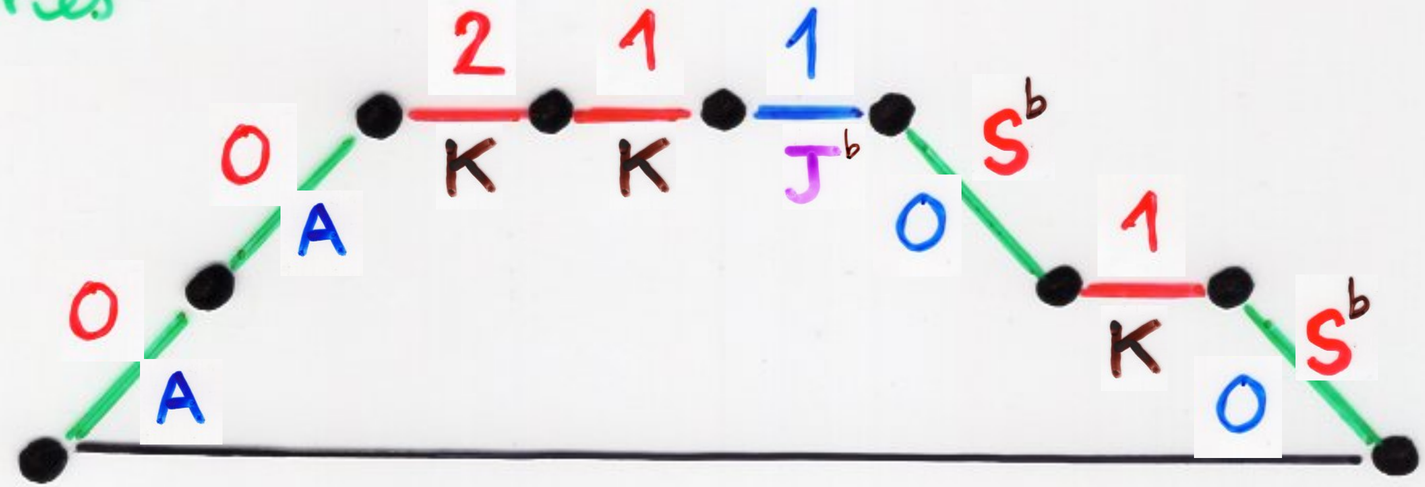


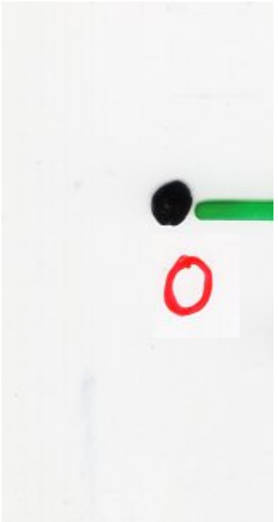
$$A|k\rangle = (k+1)\,|(k+1)\rangle \qquad J^b|k\rangle = k\,|k\rangle$$

$$K|k\rangle = (k+1)\,|k\rangle \qquad S^b|k\rangle = k\,|(k-1)\rangle$$

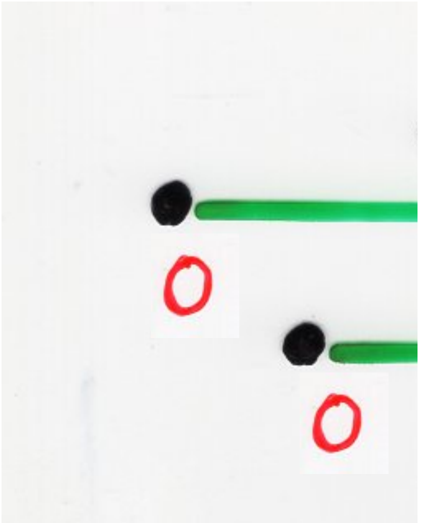
restricted
Laguerre
histories

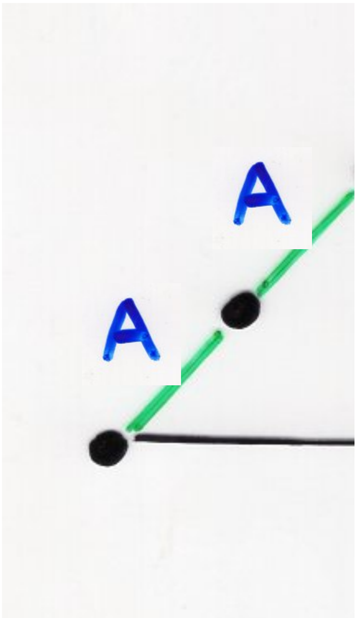
h

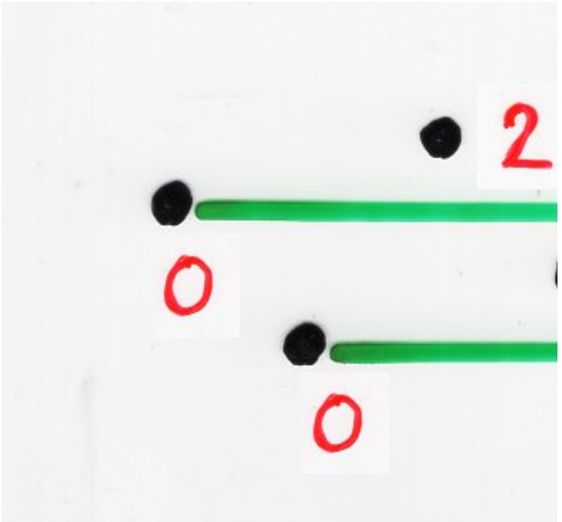


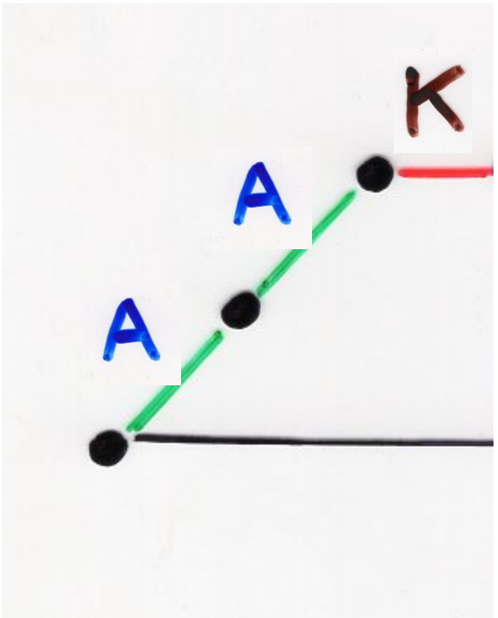

O

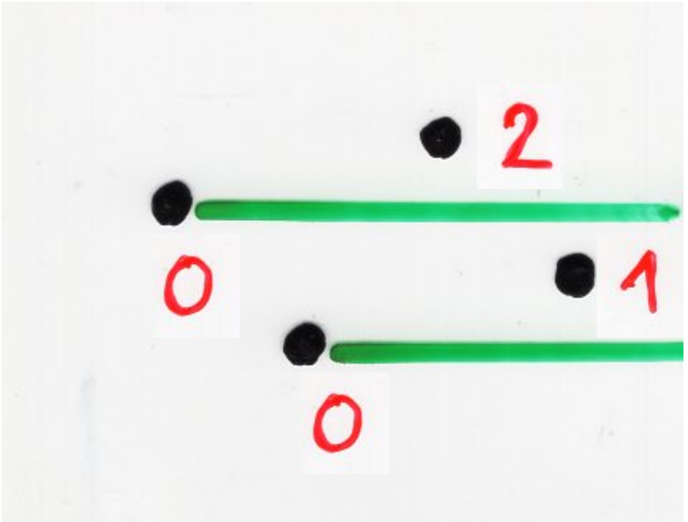

A

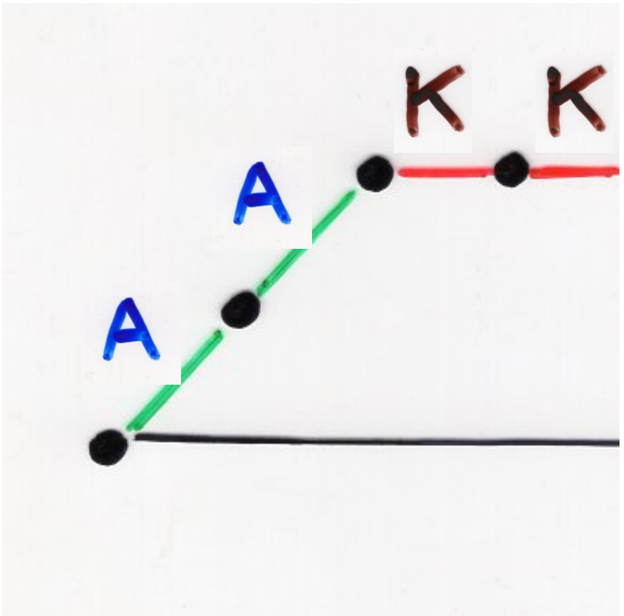

O
O

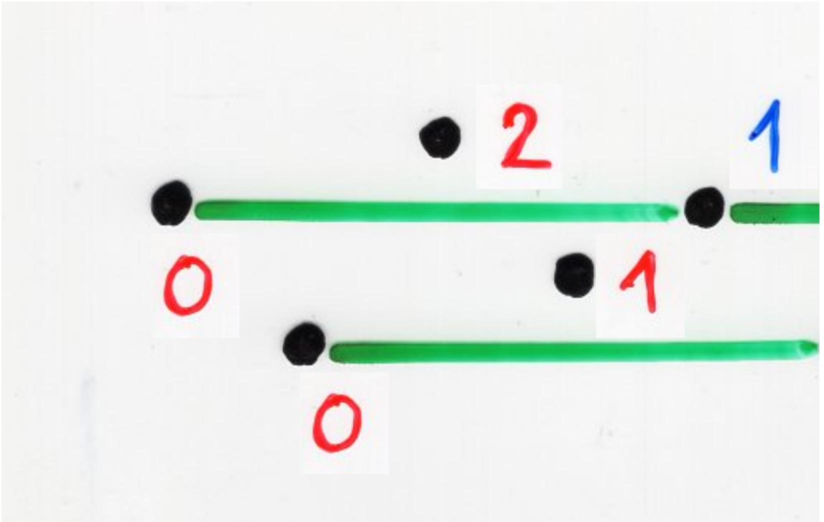

A
A


2
O
O

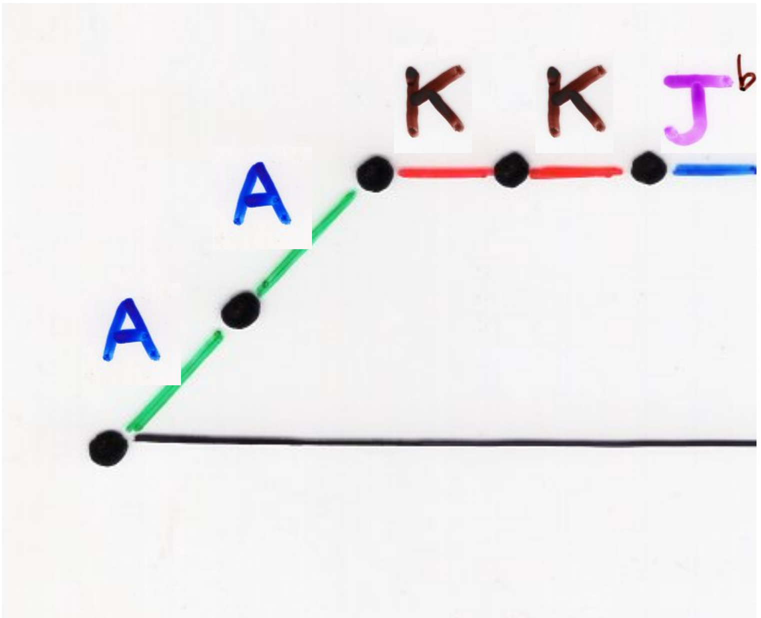

K
A
A


2
0
1
0

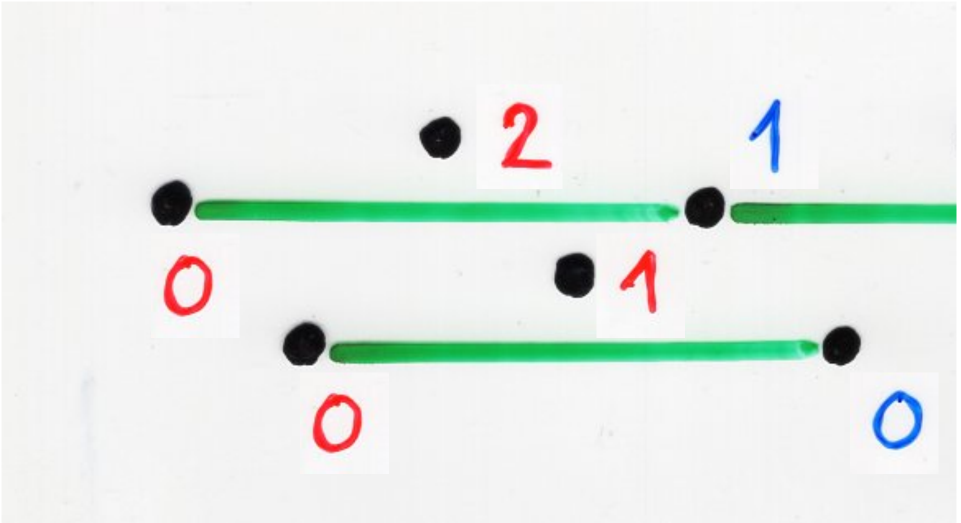

K
K
A
A

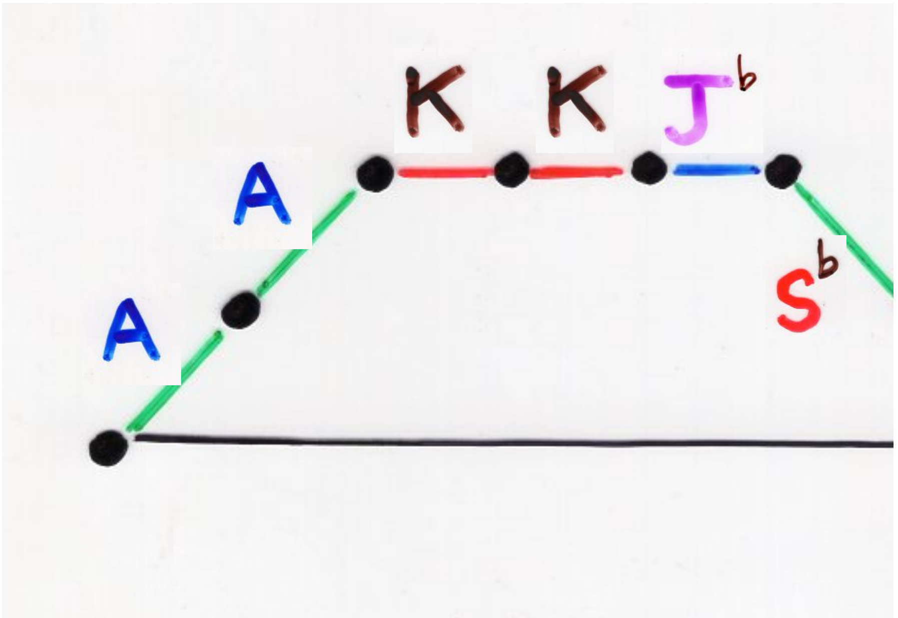


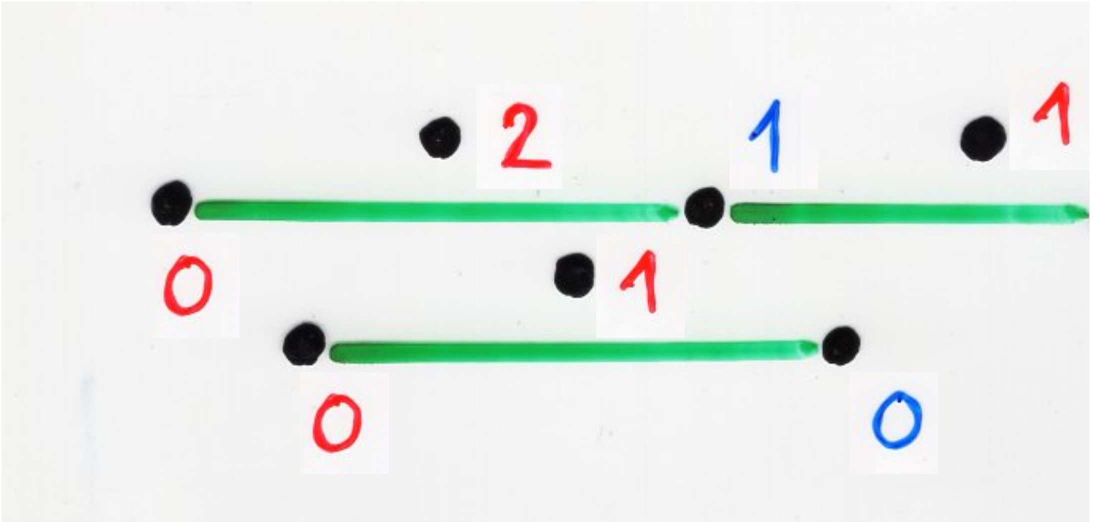
2
1
0
1
0



K
K
J
b
A
A


2
1
0
1
0
0


K
K
J b
A
A
S b


2
1
1
0
1
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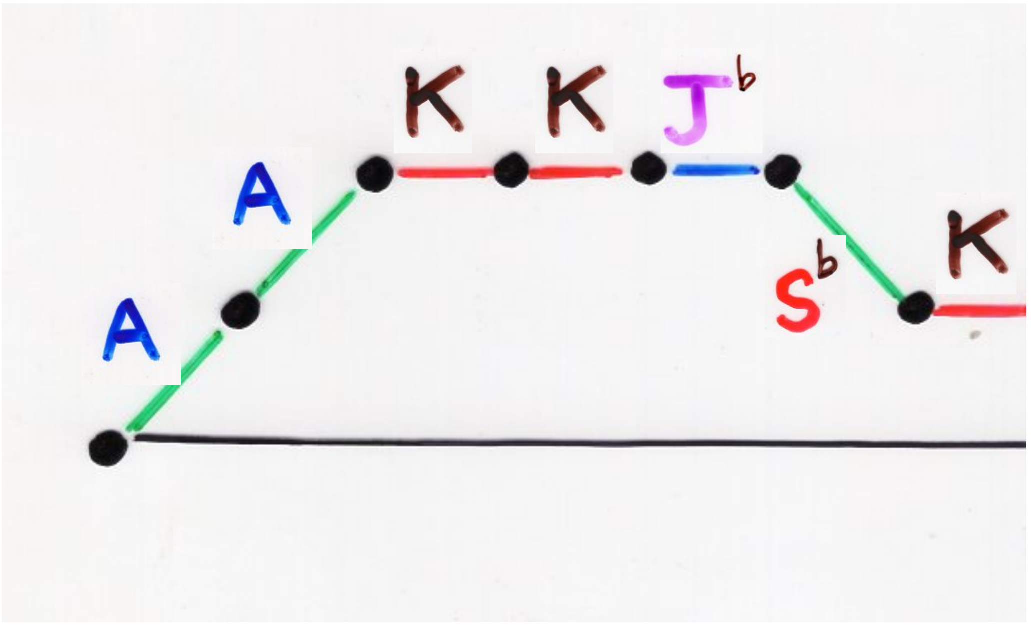
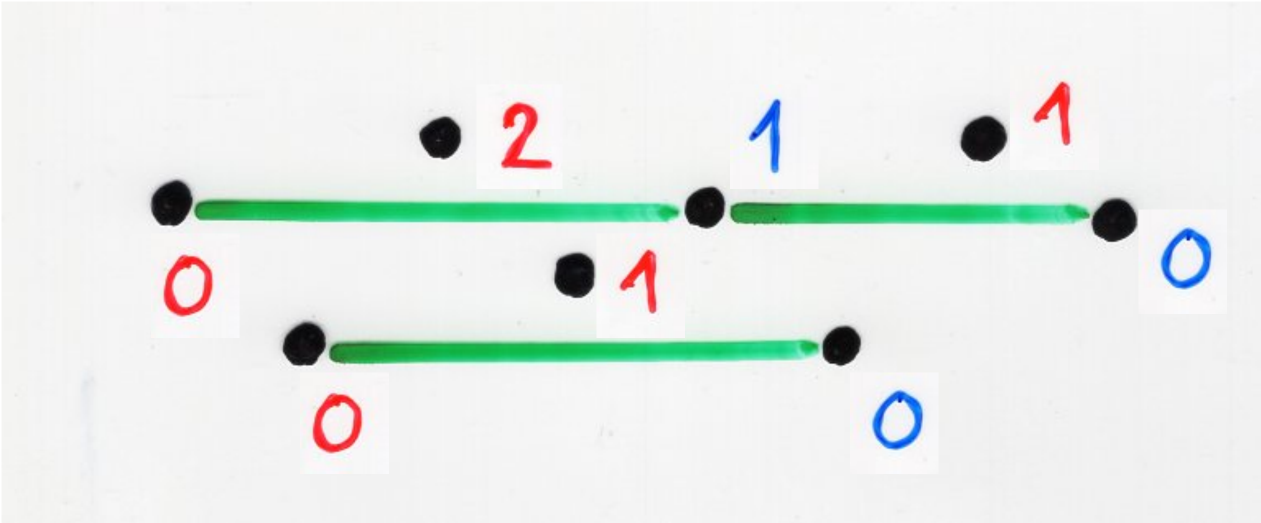
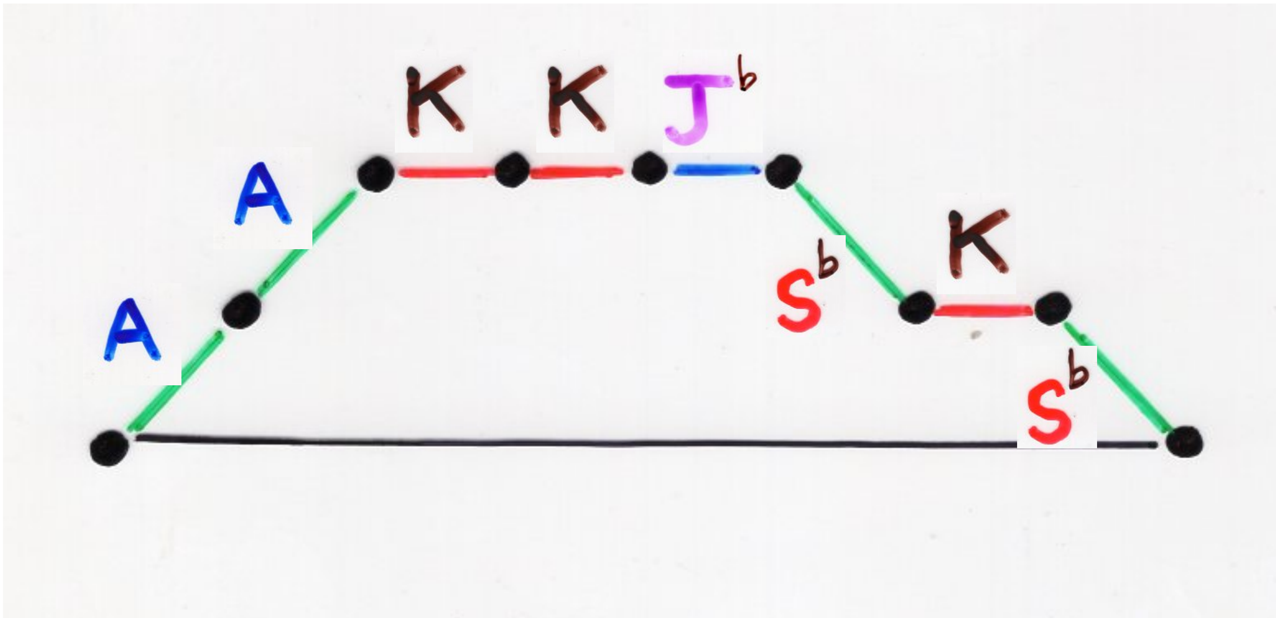
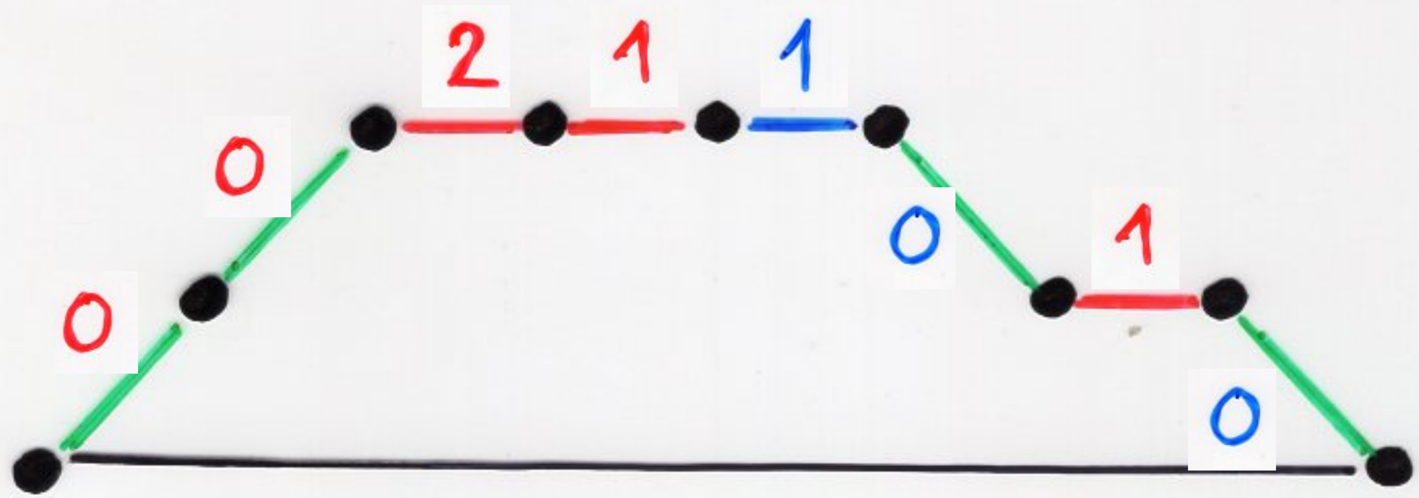
0


K
K
J b
A
A
S b
K

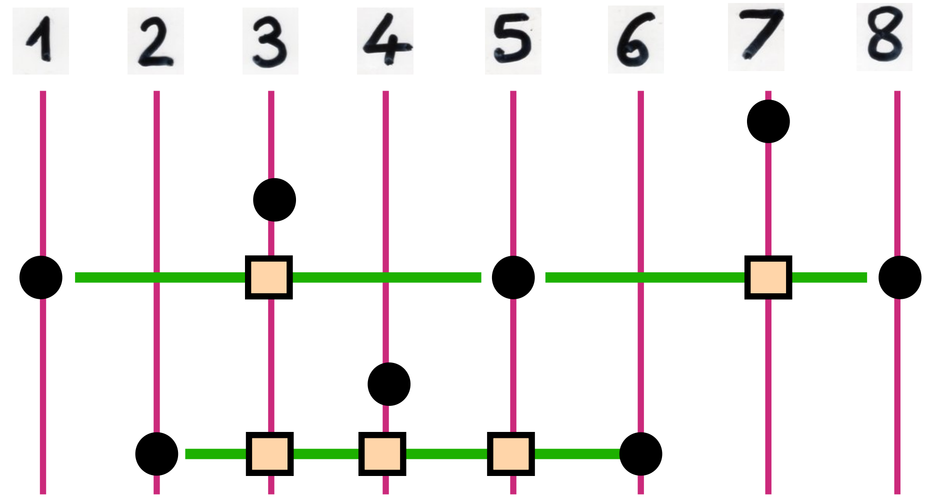

2
1
1
0
0
1
0
0


K
K
J^b
A
A
S^b
K
S^b

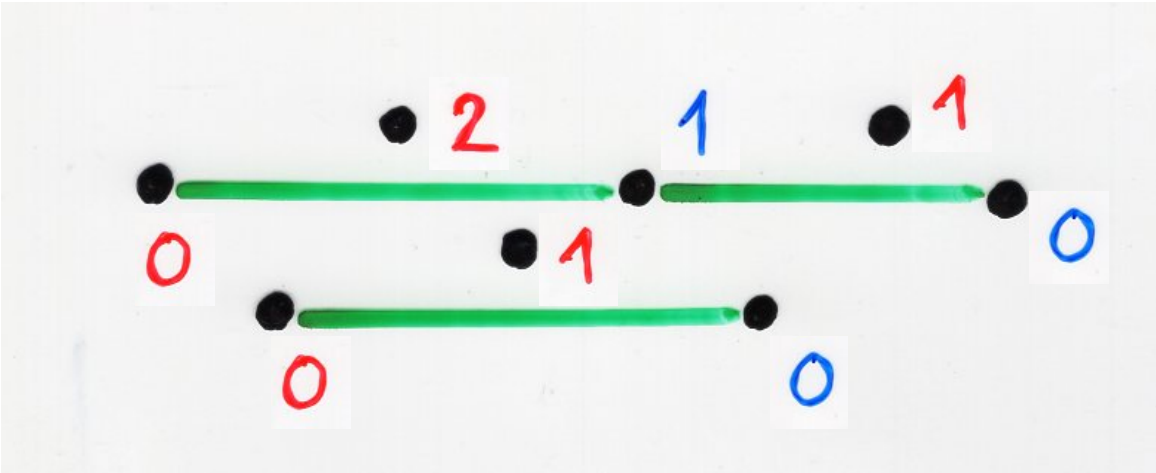


0
0
2
1
1
0
1
0

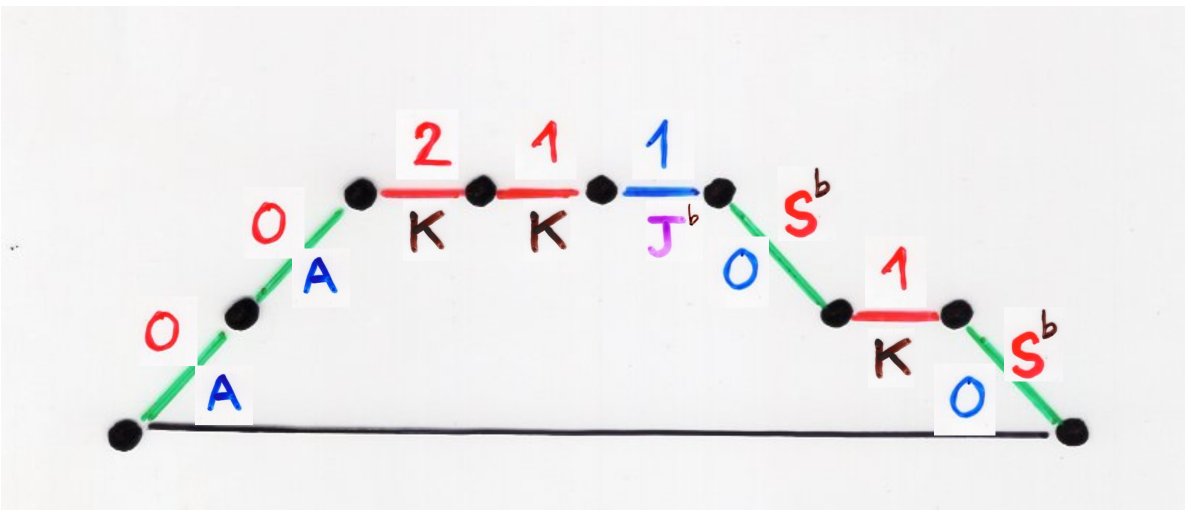
9



E



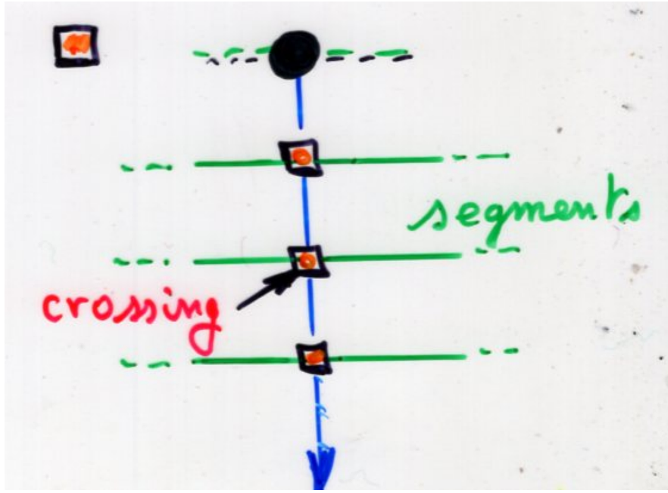
E

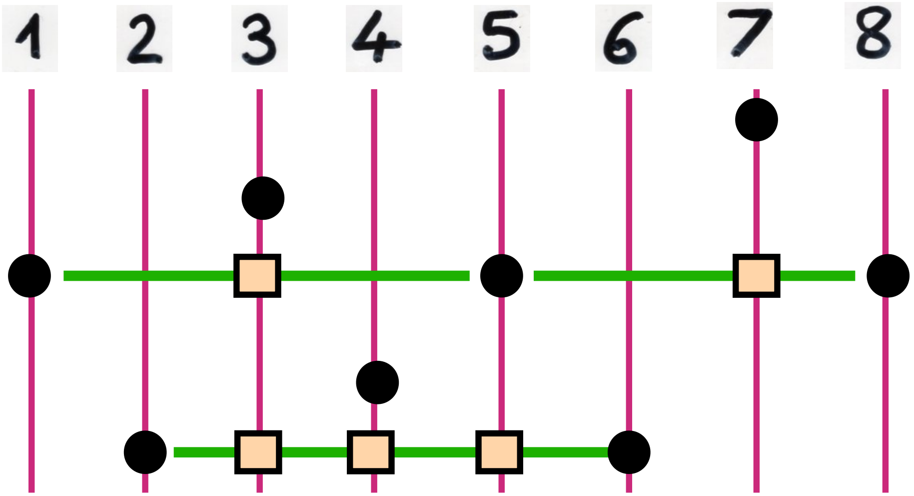


h

Definition

$cr(E)$ = number of $\square$





Definition

$s(E)$ = number of points $\bullet$

$\bullet$ — - - - or $\bullet$

with no $\bullet$ below

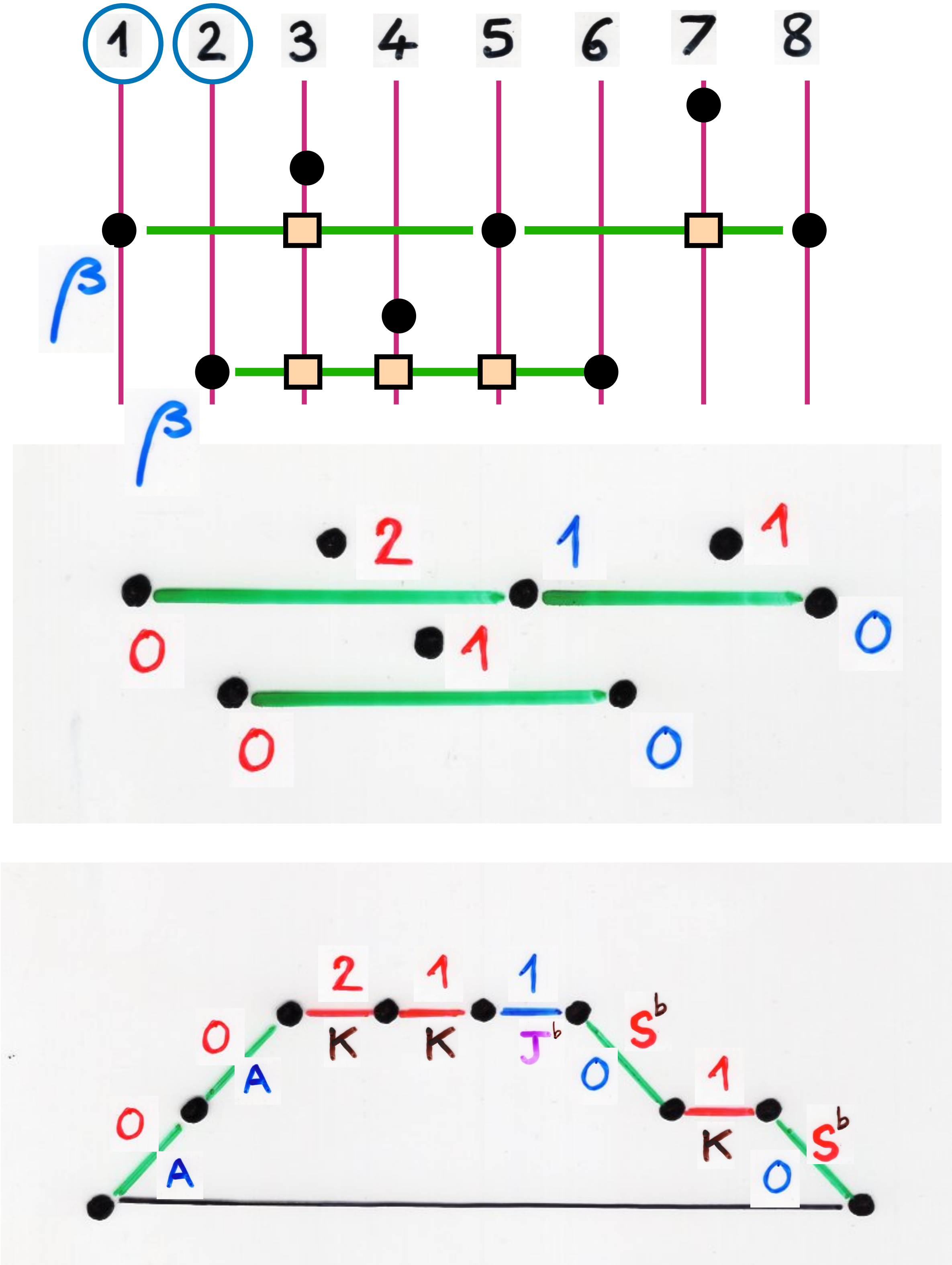
(operator A) (operator K)

Proposition

$$\mu_n^{(\beta)}(q) = \sum_{E} \beta^{s(E)} q^{cr(E)}$$

E Laguerre heaps of segments on $[1,n]$

9



Bijection

(restricted) Laguerre histories h

$\updownarrow$

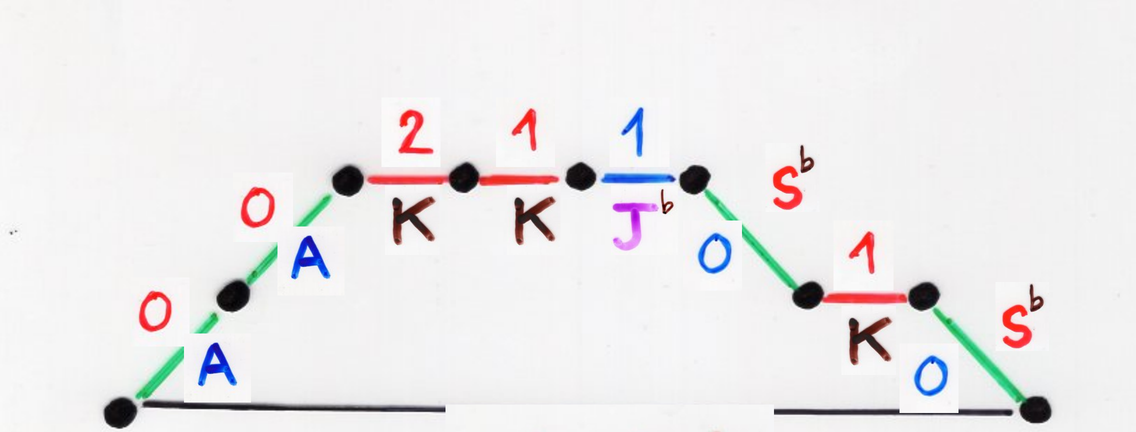
Permutations τ
(word notation)

see Ch3b, p127-129

$$
\begin{array}{ll}
 & \sqcup \\
A & \sqcup 1 \sqcup \\
A & \sqcup 2 \sqcup 1 \sqcup \\
K & \sqcup 2 \sqcup 1\,3 \sqcup \\
K & \sqcup 2\,4 \sqcup 1\,3 \sqcup \\
J^b & \sqcup 2\,4 \sqcup 5\,1\,3 \sqcup \\
S^b & 6\,2\,4 \sqcup 5\,1\,3 \sqcup \\
K & 6\,2\,4 \sqcup 5\,1\,3\,7 \sqcup \\
S^b & 6\,2\,4\,8\,5\,1\,3\,7 \sqcup
\end{array}
$$

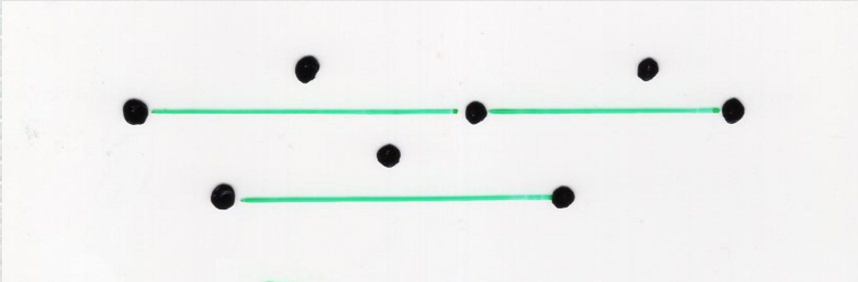
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$K\,|k\rangle = (k+1)\,|k\rangle$$

$$J^b\,|k\rangle = k\,|k\rangle$$
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$



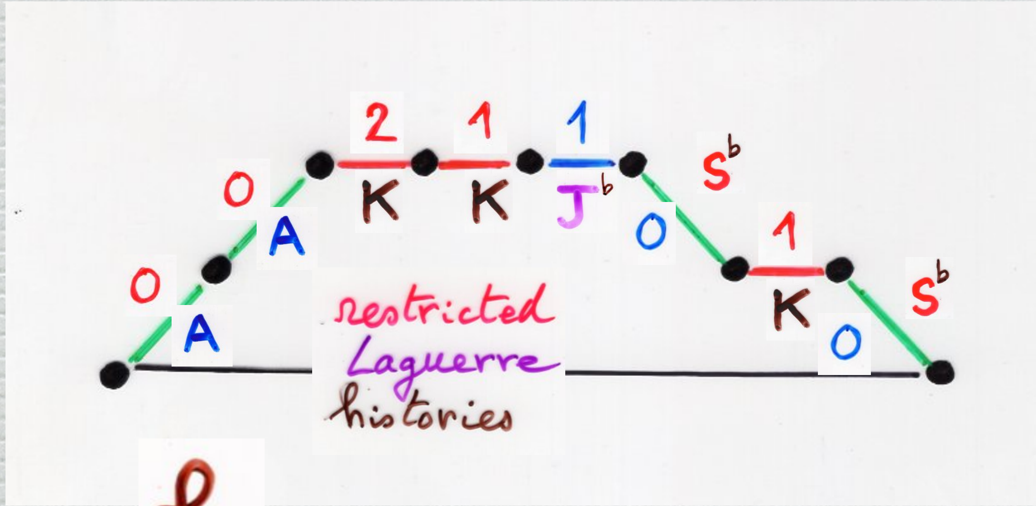
h restricted Laguerre histories

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$



E

Laguerre heap of segment



h

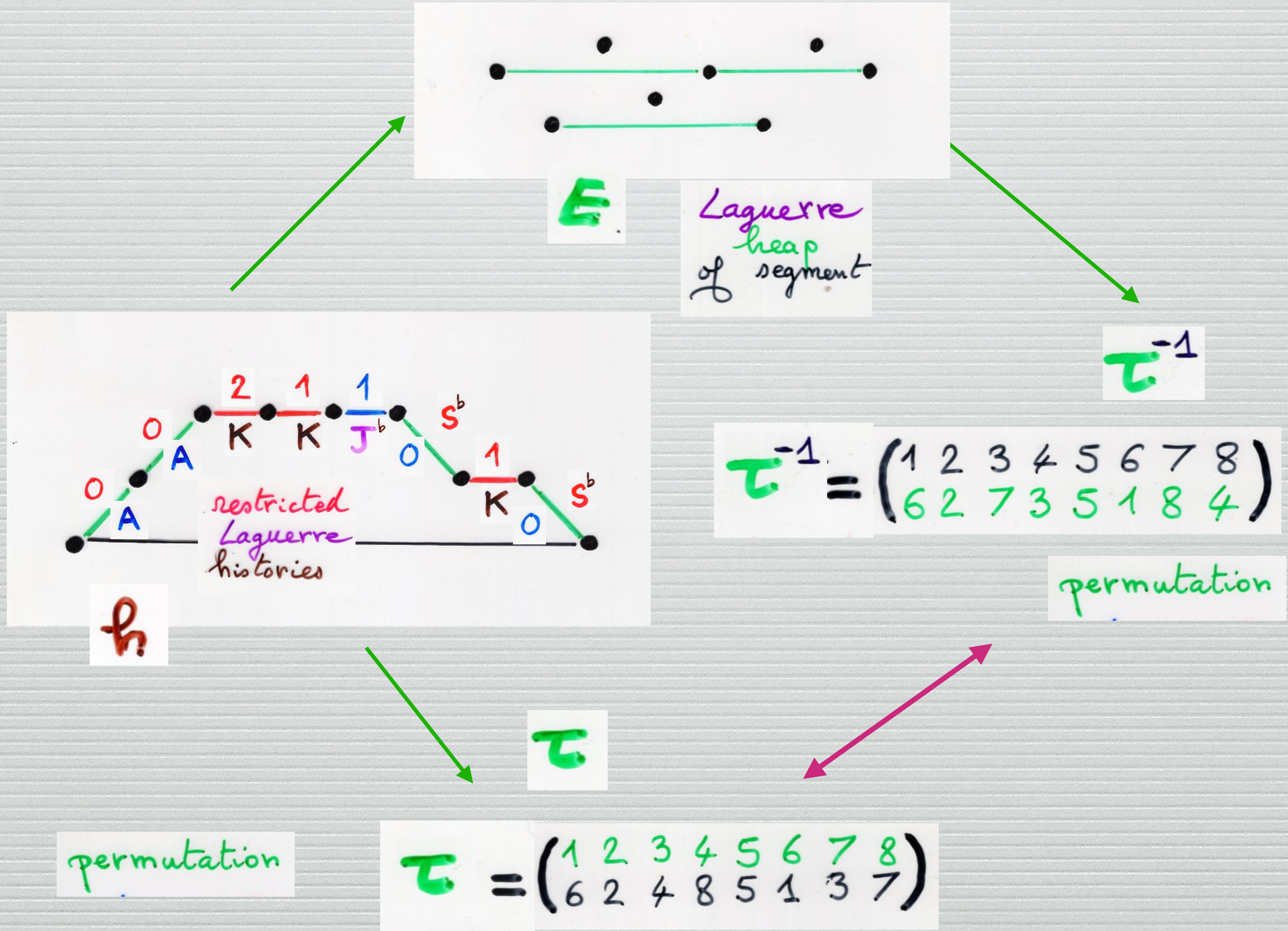
$$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&2&7&3&5&1&8&4\end{pmatrix}$$

permutation

τ

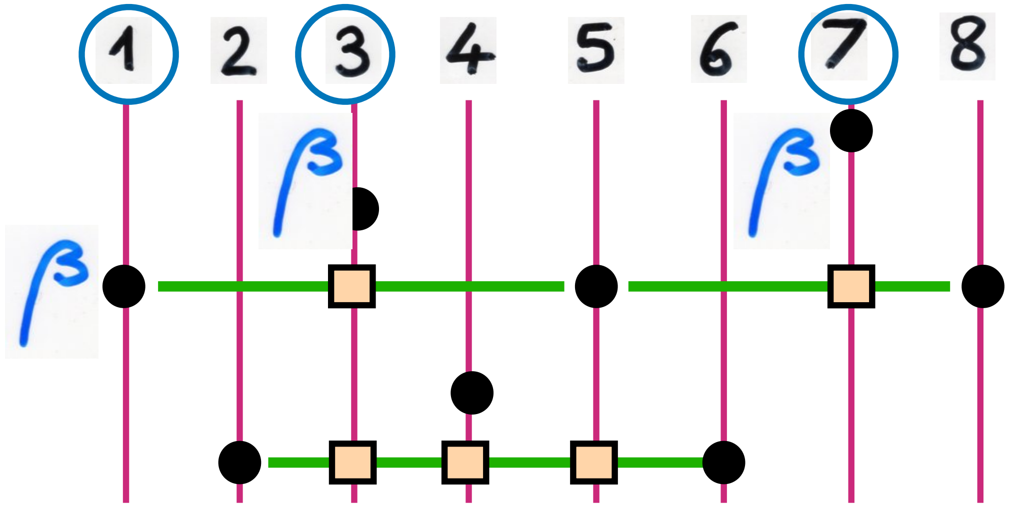
permutation

$$\tau = \begin{pmatrix}1&2&3&4&5&6&7&8\\6&2&4&8&5&1&3&7\end{pmatrix}$$

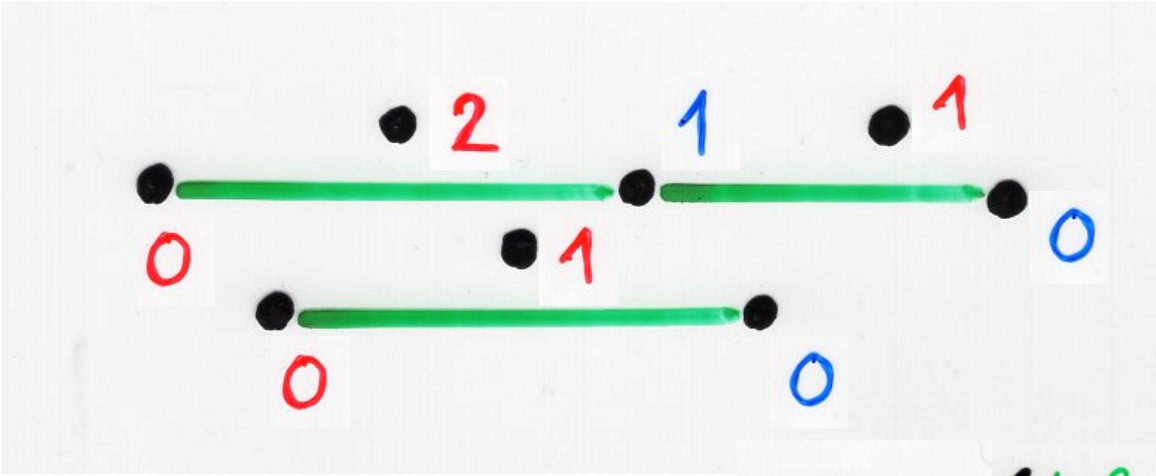


E
Laguerre heap of segment
h
restricted Laguerre histories
O A O A 2 K 1 K 1 J^b S^b O 1 K S^b O
τ^{-1}
$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 7 & 3 & 5 & 1 & 8 & 4 \end{pmatrix}$
permutation
τ
permutation
$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$

9

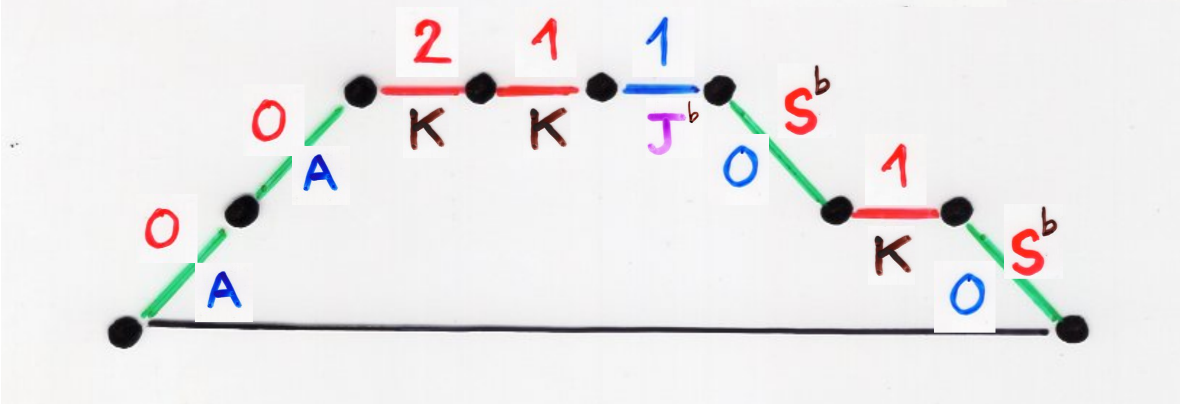


E



E

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$



h

q-analogue of Euler 's continued fractions

see Ch3b, p92-96

$$z = 1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}$$

reperietur enim iisdem operationibus inſtitutis:

$$z = \cfrac{1}{1+\cfrac{mx}{1+\cfrac{nx}{1+\cfrac{(m+n)x}{1+\cfrac{2nx}{1+\cfrac{(m+2n)x}{1+\cfrac{3nx}{1+\cfrac{(m+3n)x}{1+\cfrac{4nx}{1+\cfrac{(m+4n)x}{1+\cfrac{5nx}{1+\text{etc.}}}}}}}}}}}}$$

Eadem vero expreſſio, aliaeque ſimiles facile erui pos-

$$\sum_{n\geqslant 0} \beta(\beta+1)(\beta+2)\cdots(\beta+n-1) = \cfrac{1}{1-\beta t - \cfrac{1t}{1-(\beta+1)t - \cfrac{2t}{1-(\beta+2)t - \cfrac{3t}{\cdots}}}}$$

$$\begin{cases} \gamma_{2k} = k & (k\geqslant 1) \\ \gamma_{2k+1} = k+\beta & (k\geqslant 0) \end{cases}$$

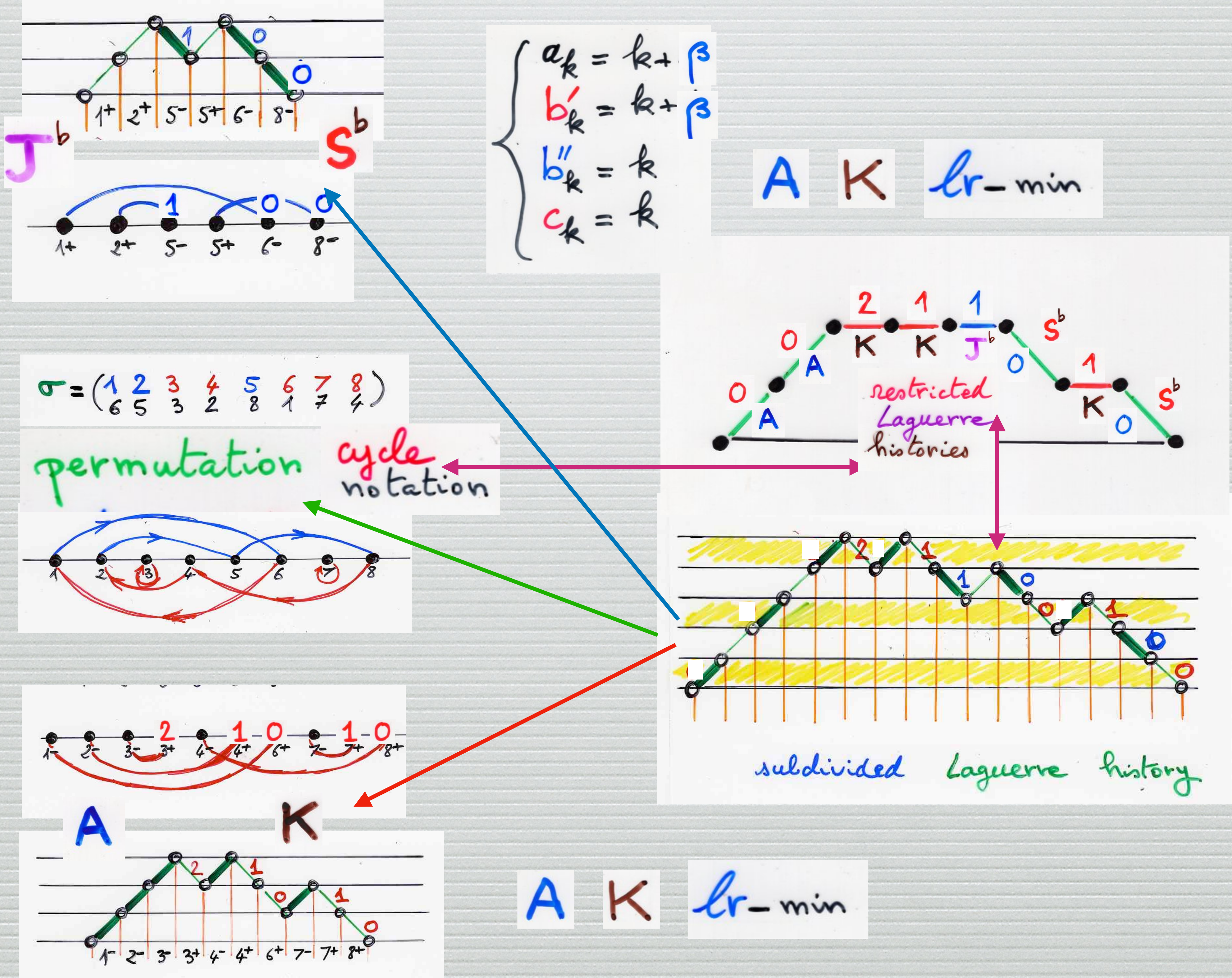
$$\begin{cases} a_k = k+\beta \\ b'_k = k+\beta & (k\geqslant 0) \\ b''_k = k \\ c_k = k & (k\geqslant 1) \end{cases}$$

$$\begin{cases} b'_k = \gamma_{2k+1} \\ b''_k = \gamma_{2k} \end{cases} \qquad \begin{cases} a_{k-1} = \gamma_{2k-1} \\ c_k = \gamma_{2k} \end{cases}$$

restricted
Laguerre
histories

$$\begin{cases} b_k = 2k+\beta \\ \lambda_k = (k-1+\beta)k \end{cases}$$

$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$



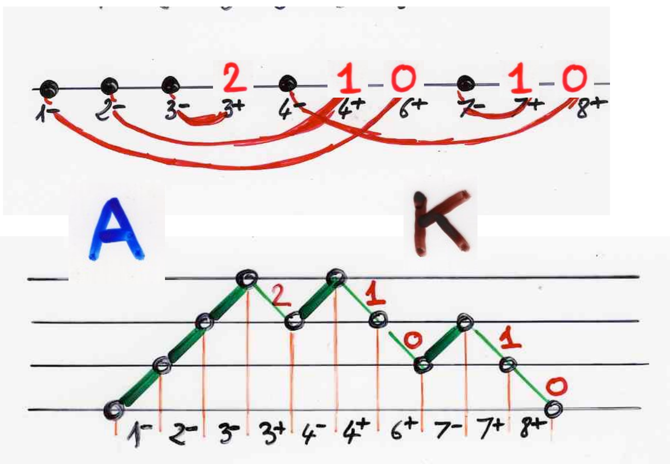
$a_k = k + \beta$
$b'_k = k + \beta$
$b''_k = k$
$c_k = k$
A K lr-min
J^b S^b
1+ 2+ 5- 5+ 6- 8-
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$
permutation
cycle notation
restricted Laguerre histories
subdivided Laguerre history
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+
A K
A K lr-min

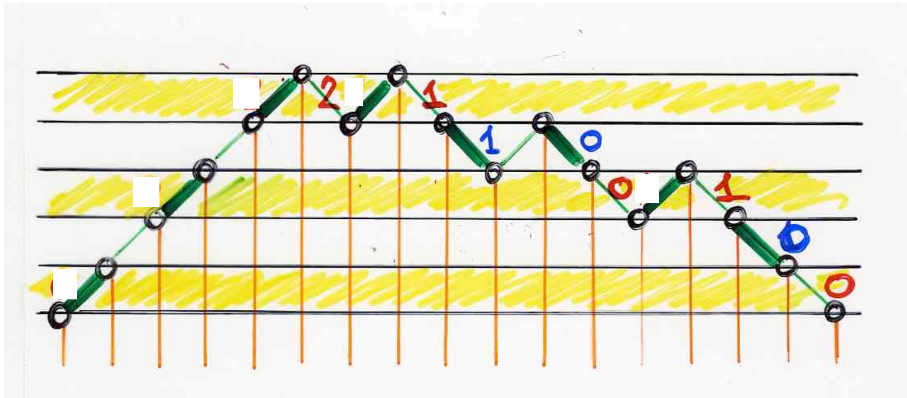
$$\sum_{n\geq 0} \mu_n^{(\beta)}(q) = \cfrac{1}{1-\cfrac{\beta t}{1-\cfrac{1t}{1-\cfrac{(\beta+1)t}{1-\cfrac{2t}{1-\cfrac{(\beta+2)t}{1-\cfrac{3t}{\cdots}}}}}}}$$

$$\begin{cases} \gamma_{2k} = [k]_q \\ \gamma_{2k+1} = [k;\beta]_q \end{cases}$$

A K lr_min

$$[k;\beta]_q = (\beta + q + q^2 + \cdots + q^{k-1})$$



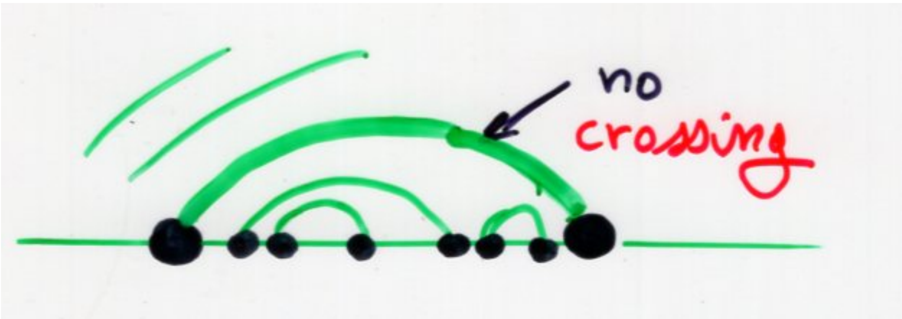


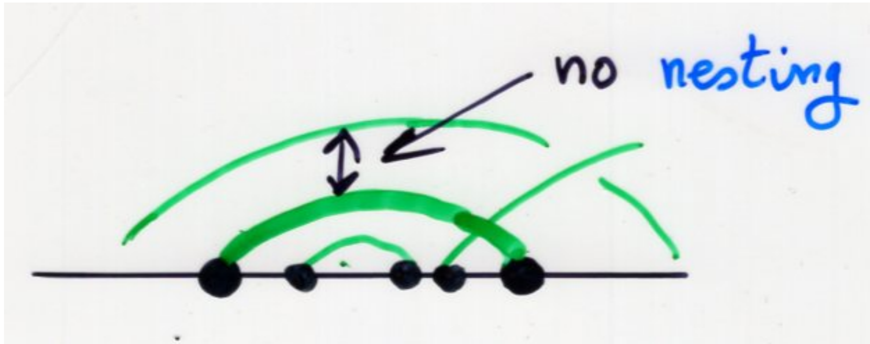
subdivided Laguerre history

$$[k;\beta]_q = (\beta + q + q^2 + \dots + q^{k-1})$$

β interpretation :

first (resp. last) choice





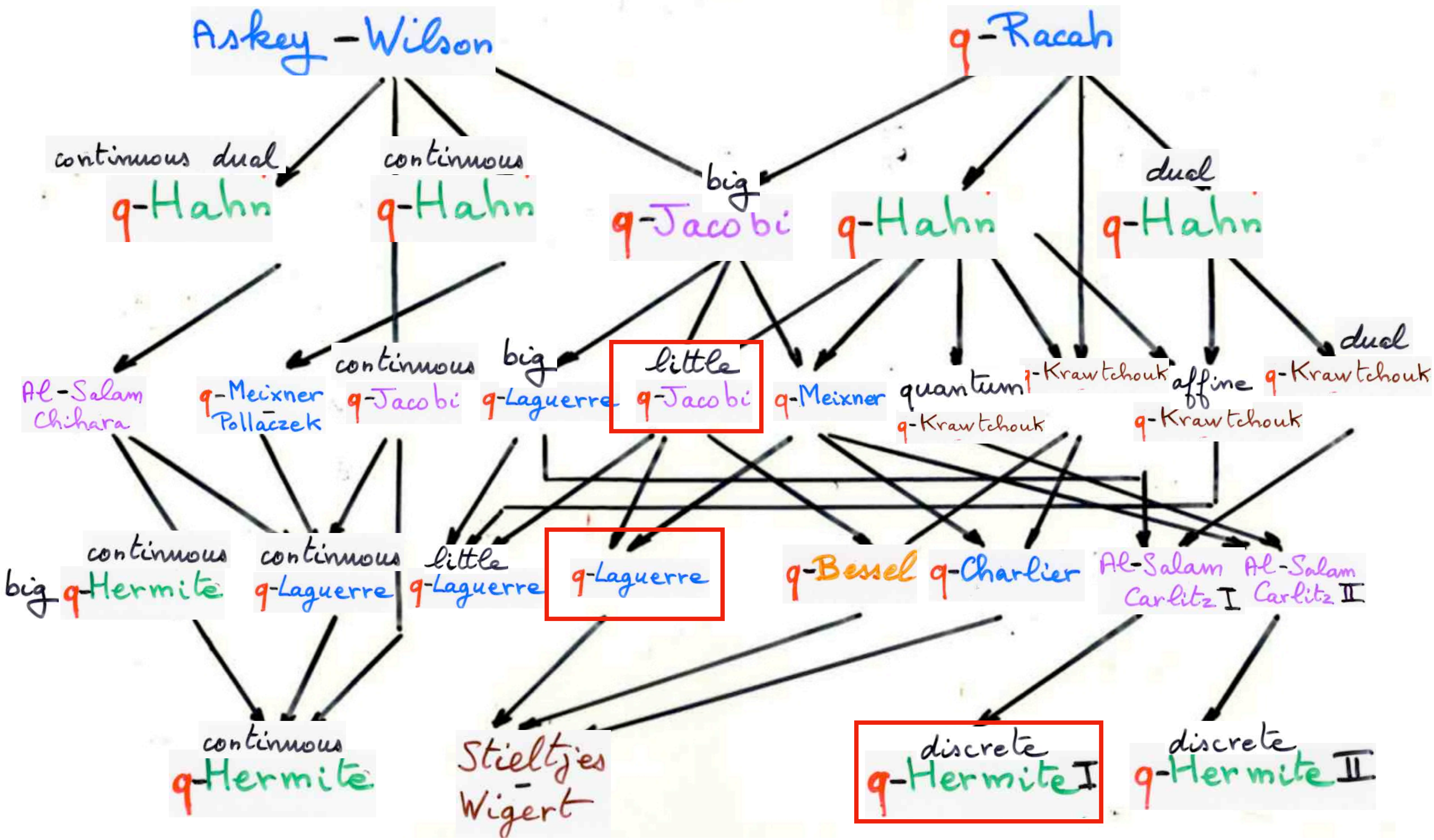
q-Laguerre II

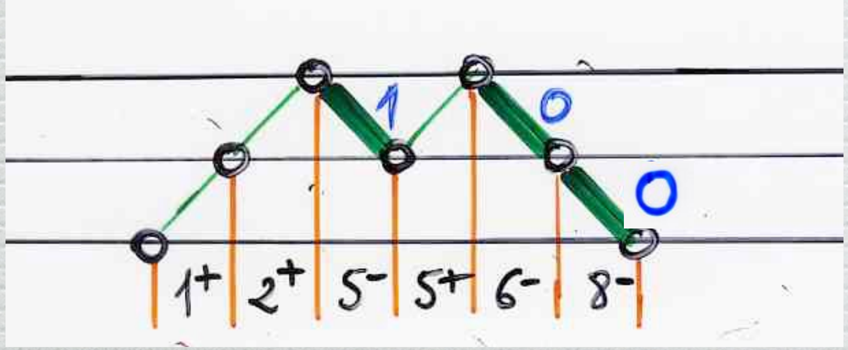
discrete

q-Laguerre II

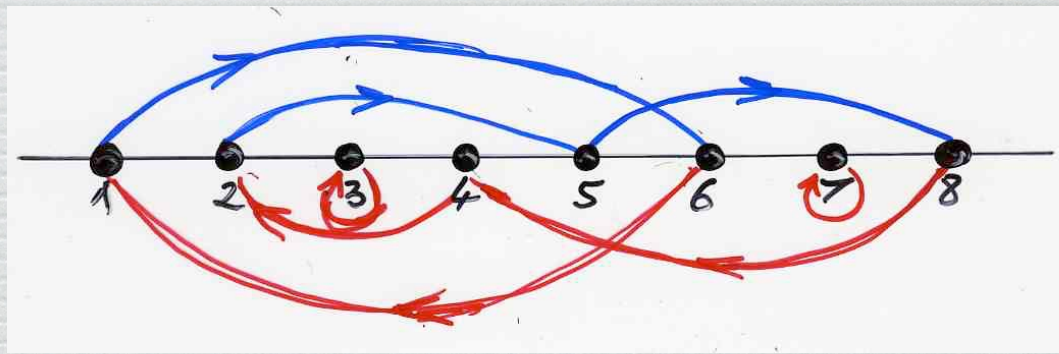
$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

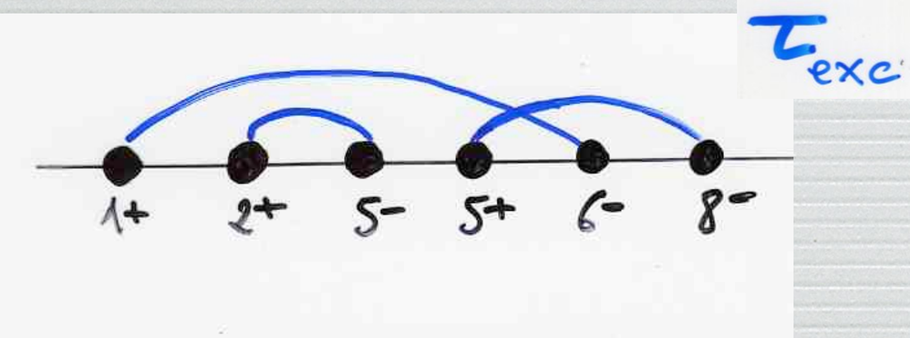
scheme of basic hypergeometric orthogonal polynomials

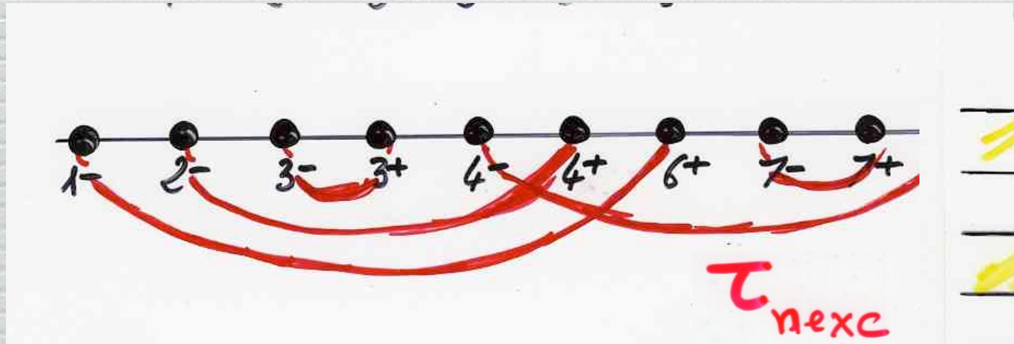


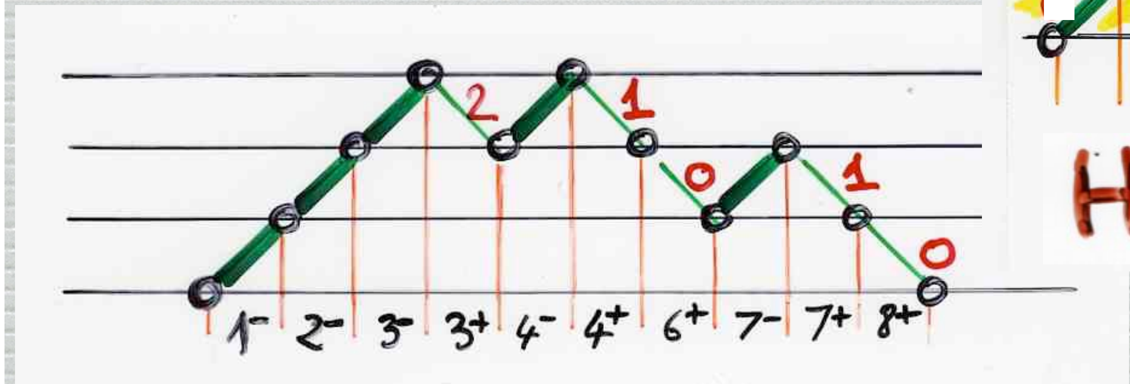


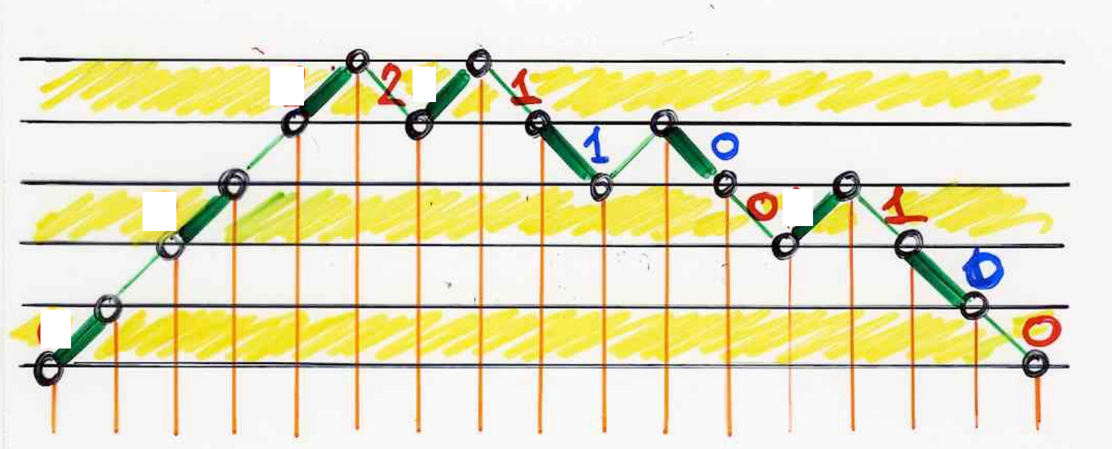
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$











H subdivided Laguerre history

Proposition

$$\mathrm{Inv}(\sigma) = \mathrm{exc}(\sigma) + \mathrm{Inv}(\tau_{exc}) + \mathrm{Inv}(\tau_{nex})$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

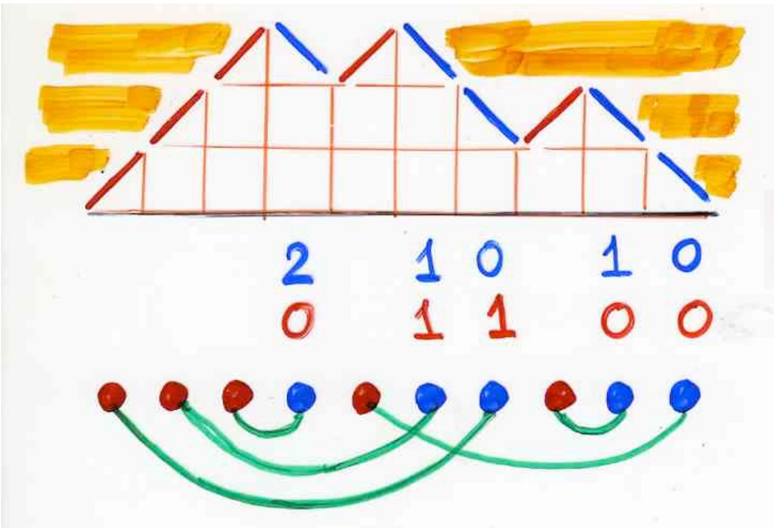
$$\begin{array}{lcccccccccc} \mathrm{Inv} & 5 & 4 & 2 & 1 & 3 & & 1 & & \longrightarrow 16 \end{array}$$

$$\mathrm{Inv}(\sigma) = \mathrm{exc}(\sigma) + \mathrm{Inv}(\tau_{exc}) + \mathrm{Inv}(\tau_{nex})$$

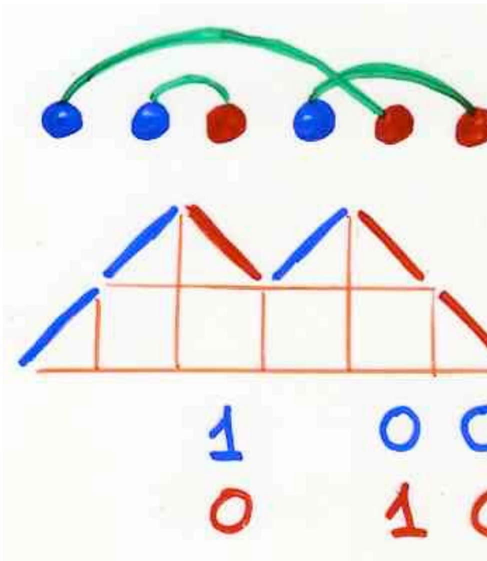
$$3 \qquad\qquad 1+2 \qquad\qquad 2+8$$

$$\longrightarrow 16$$

$$\mathrm{Inv}(\tau) = \mathrm{cr}(\tau) + 2\,\mathrm{nest}(\tau)$$



τ_{nexc}



τ_{exc}

discrete q-Laguerre II

$$\begin{cases} b_k = q^k\left([k]_q + [k+1]_q\right) \\ \lambda_k = q^{2k-1}[k]_q \times [k]_q \end{cases}$$

Proposition $\mu_n(q) = [n!]_q$

Heine continued fraction
(J-fraction)

Biane (1993)

$$\begin{cases} b_k = [k]_q + [k+1;\beta]_q \\ \lambda_k = [k;\beta]_q \, [k]_q \end{cases}$$

q-Laguerre I

q-Laguerre II

$$\begin{cases} b_k^{(\beta)}(q) = q^k \left([k+1;\beta]_q + [k]_q \right) \\ \lambda_k^{(\beta)}(q) = q^{2k-1} [k]_q [k;\beta]_q \end{cases}$$

Proposition

$$\mu_n^{(\beta)}(q) = [n;\beta\,!]_q$$

$$= [1;\beta]_q [2;\beta]_q \cdots [n;\beta]_q$$

q-Hermite II
(discrete I)

$$\lambda_k = q^{k-1}[k]_q$$

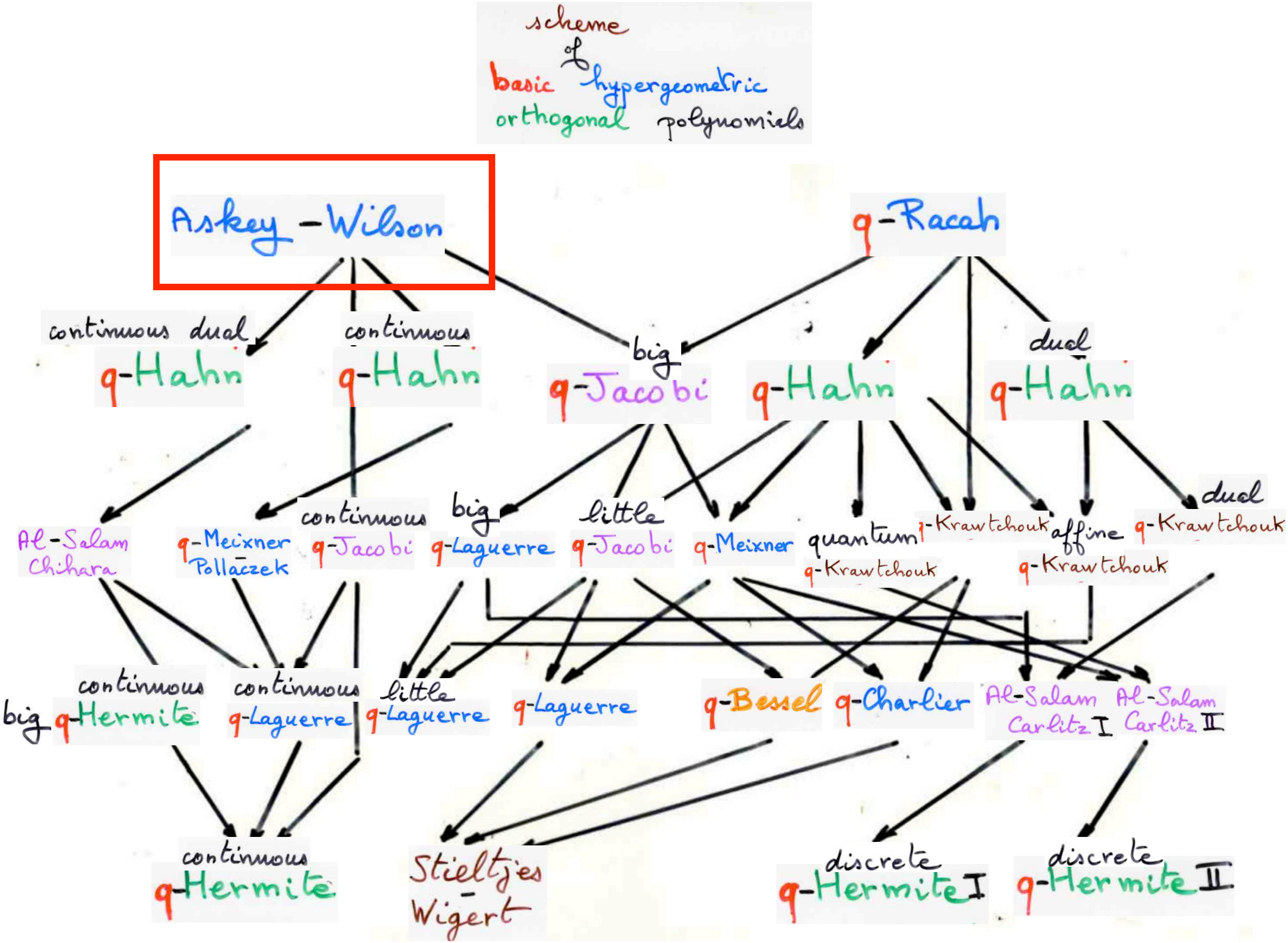
moments

Proposition

$$\mu^{II}_{2n}(q) = [1]_q \cdot [3]_q \cdots [2n-1]_q$$

The power of bijective proof:

The Askey-Wilson integral


scheme of basic hypergeometric orthogonal polynomials
Askey-Wilson
q-Racah
continuous dual q-Hahn
continuous q-Hahn
big q-Jacobi
q-Hahn
dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
continuous q-Jacobi
big q-Laguerre
little q-Jacobi
q-Meixner
quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes-Wigert
discrete q-Hermite I
discrete q-Hermite II

Askey-Wilson polynomials

$$P_n(x) = P_n(x; a, b, c, d \mid q)$$

$$P_n(x) = a^{-n} (ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k}$$

$$(a_1, a_2, \dots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k)$$

${}_4\phi_3$ basic hypergeometric function

Askey-Wilson polynomials

$$\int_0^{\pi} p_n(\cos\theta, a, b, c, d, q)\, p_m(\cos\theta, a, b, c, d, q)\, w(\cos\theta, a, b, c, d, q)\, d\theta$$

$$= 0 \quad n \neq m$$

$$w(\cos\theta, a, b, c, d \mid q) =$$

$$\frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$$

The Askey-Wilson integral

$$\frac{(q)_\infty}{2\pi}\int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta =$$

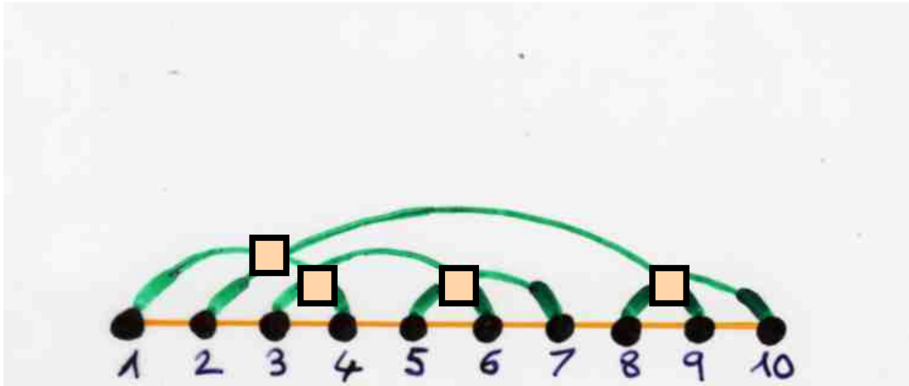
$$\frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

integral of the product
of 4 q-Hermite polynomials

q-analogue of Hermite polynomials

$$H_k(x;q)$$

$$H_{k+1}(x) = xH_k(x) - k\,H_{k-1}(x)$$



$$[k]_q = 1+q+q^2+\dots+q^{k-1}$$

$$(a)_\infty = \prod_{i\geq 0}(1-aq^i)$$

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_k(\cos\theta|q)\,H_l(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty = (q)_k\,\delta_{kl}$$

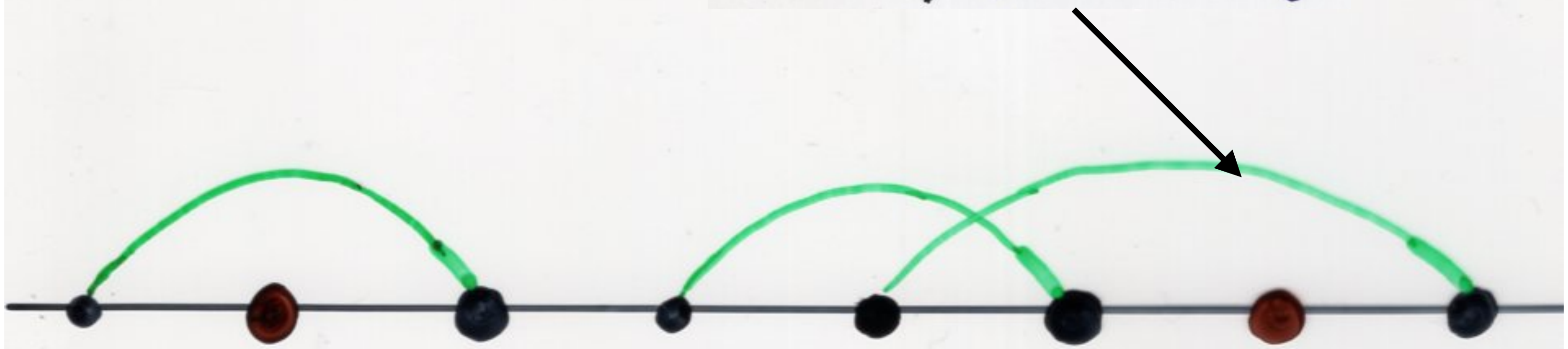
$$H_n(x;q) = \sum_{\substack{\alpha \\ \text{matching of } [1,n]}} (-1)^{d(\alpha)} x^{n-2d(\alpha)} q^{s(\alpha)}$$

$$s(\sigma) = \sum_{\substack{e \\ \text{edges (cycles length 2)}}} s(e)$$

q-Hermite I
(continuous)

$$\lambda_k = [k]_q$$

$e = \{i, j\}$, $i<j$



$$s(e) = \left\{ \begin{array}{l} \text{number of indices } k, \\ i<k<j,\ \sigma(k)<j \end{array} \right\}$$

The Askey-Wilson integral

integral of the product
of 4 q-Hermite polynomials

Ismail, Stanton, X.V. (1987)

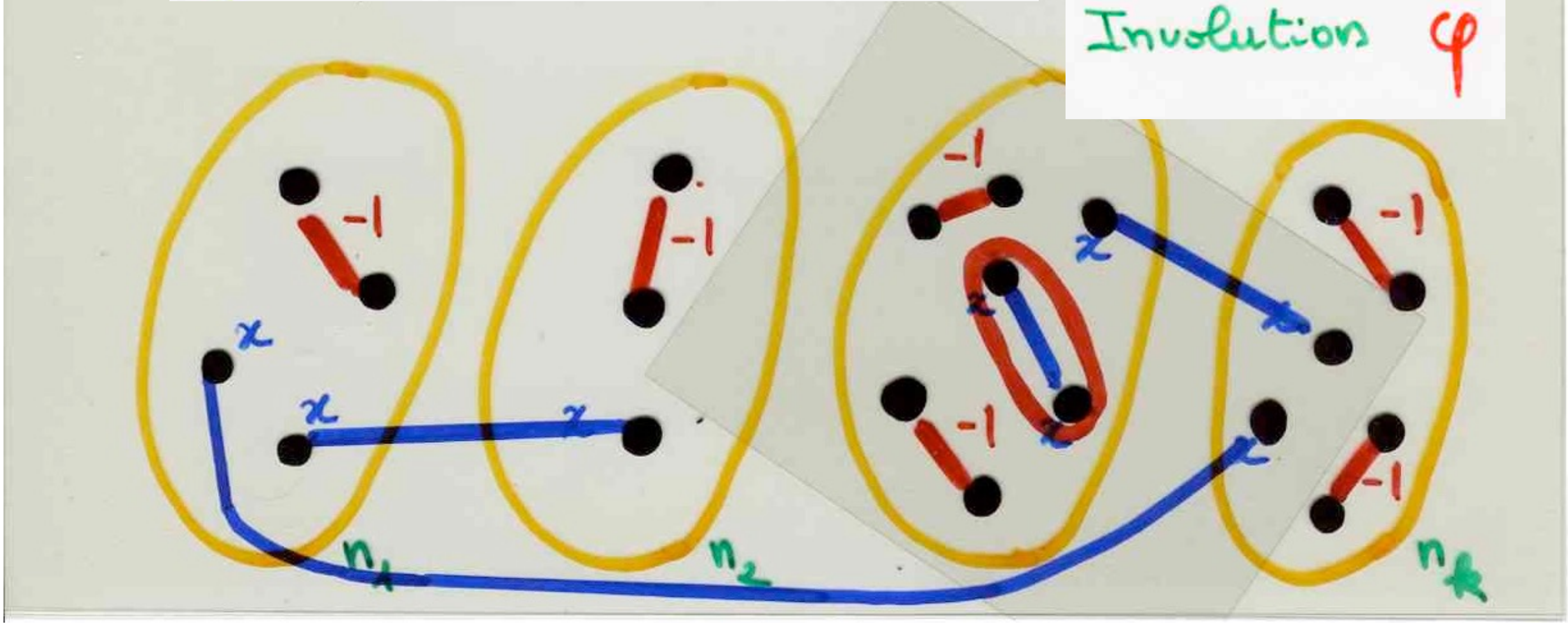
$$\int_{-\infty}^{+\infty} H_{n_1}(x;q)\, H_{n_2}(x;q)\, H_{n_3}(x;q)\, H_{n_4}(x;q)\, v(x;q)\, dx$$

$$v(\cos\theta;q) = \frac{(q)_\infty}{2\pi} (e^{2i\theta})_\infty (e^{-2i\theta})_\infty$$

de Sainte-Catherine, X.V. (1985)

$$f\left(H_{n_1}(x)\, H_{n_2}(x) \cdots H_{n_k}(x)\right) =$$

Involution φ



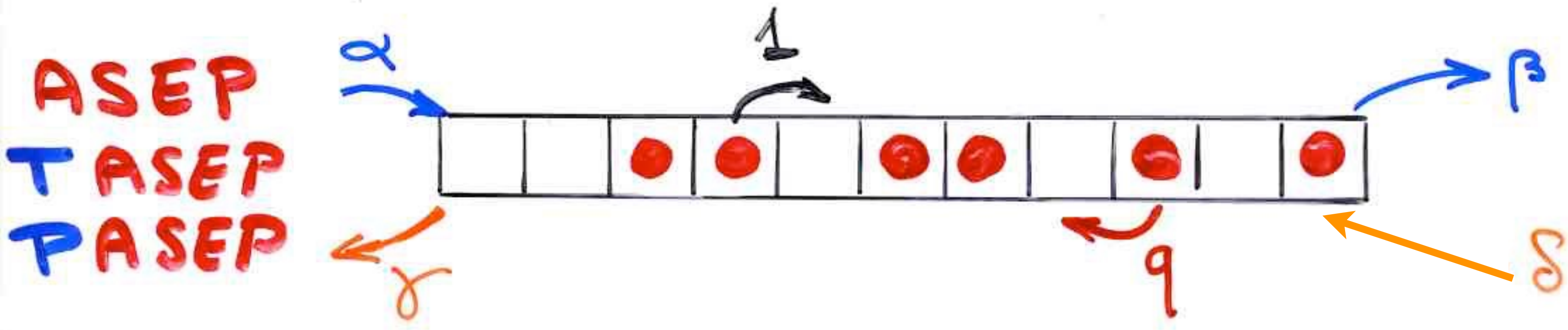
φ dimer of α_i $\longrightarrow$ internal edge of α

PASEP

and

orthogonal polynomials

toy model in the physics of
dynamical systems far from equilibrium



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

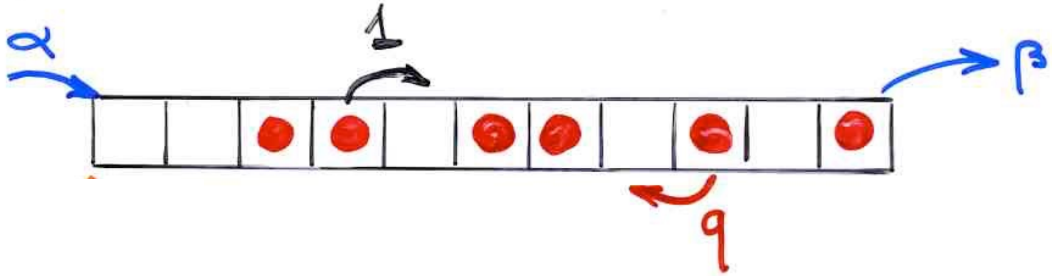
$$\begin{cases} DE = qED + E + D \\ \langle W| (\alpha E - \gamma D) = \langle W| \\ (\beta D - \delta E) |V\rangle = |V\rangle \end{cases}$$

colum vector V

row vector W

PASEP with 3 parameters

$\gamma = \delta = 0$ $\quad q, \alpha, \beta$

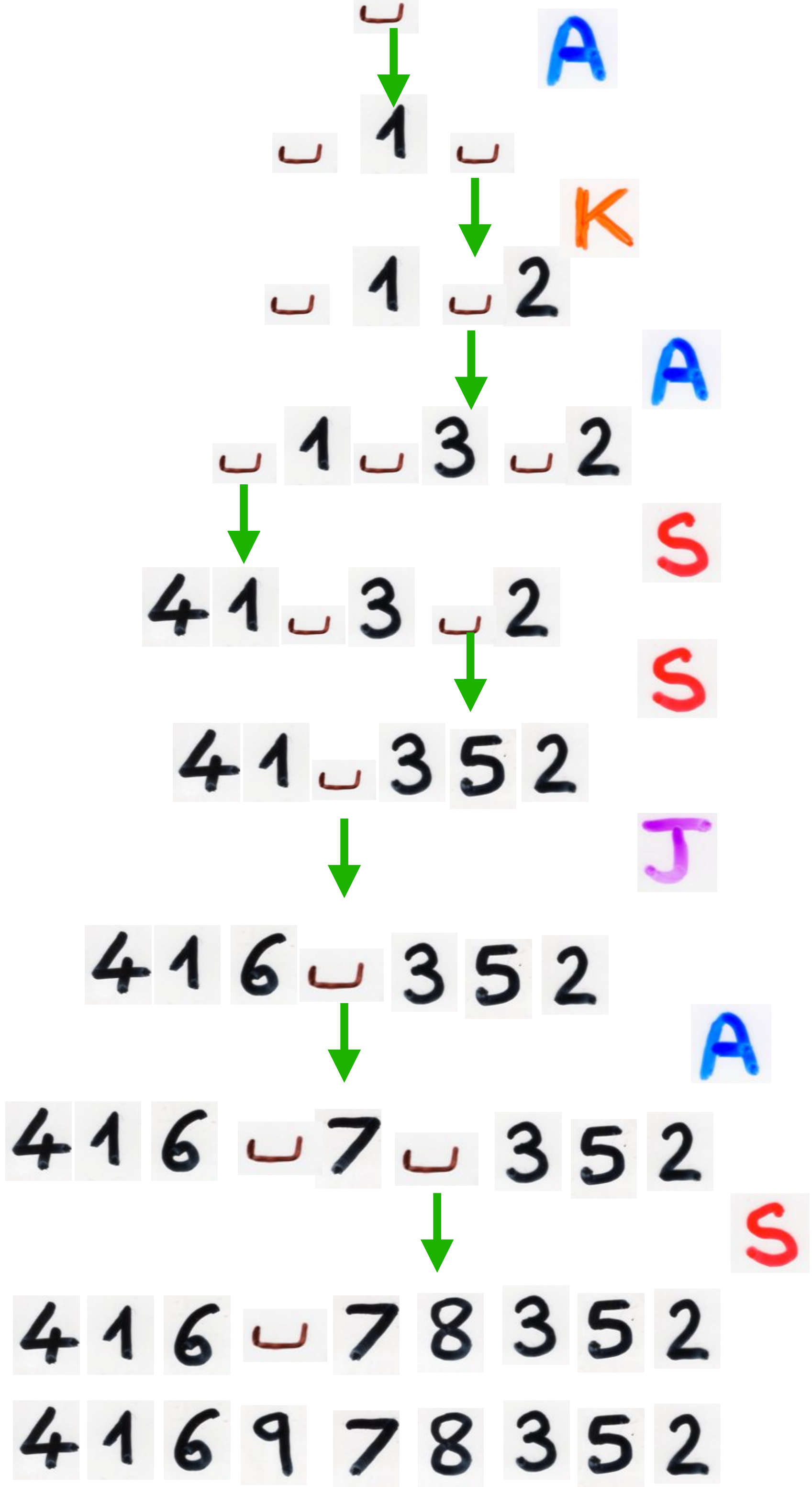


PASEP

$$\begin{cases} DE = qED + E + D \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

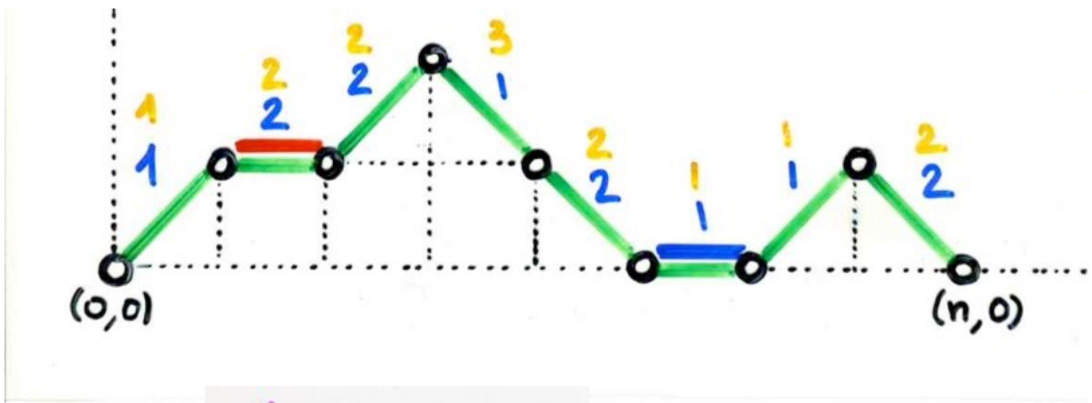
$\bar{\beta} = \dfrac{1}{\beta}$

$\bar{\alpha} = \dfrac{1}{\alpha}$



$$D = A + K$$
$$E = S + J$$

$$DE = ED + E + D$$



Laguerre histories

PASEP with 3 parameters

"continuous version"

Z_n partition function = moments of q-Laguerre I

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q, α, β

→ Uchiyama, Sasamoto, Wadati (2003)

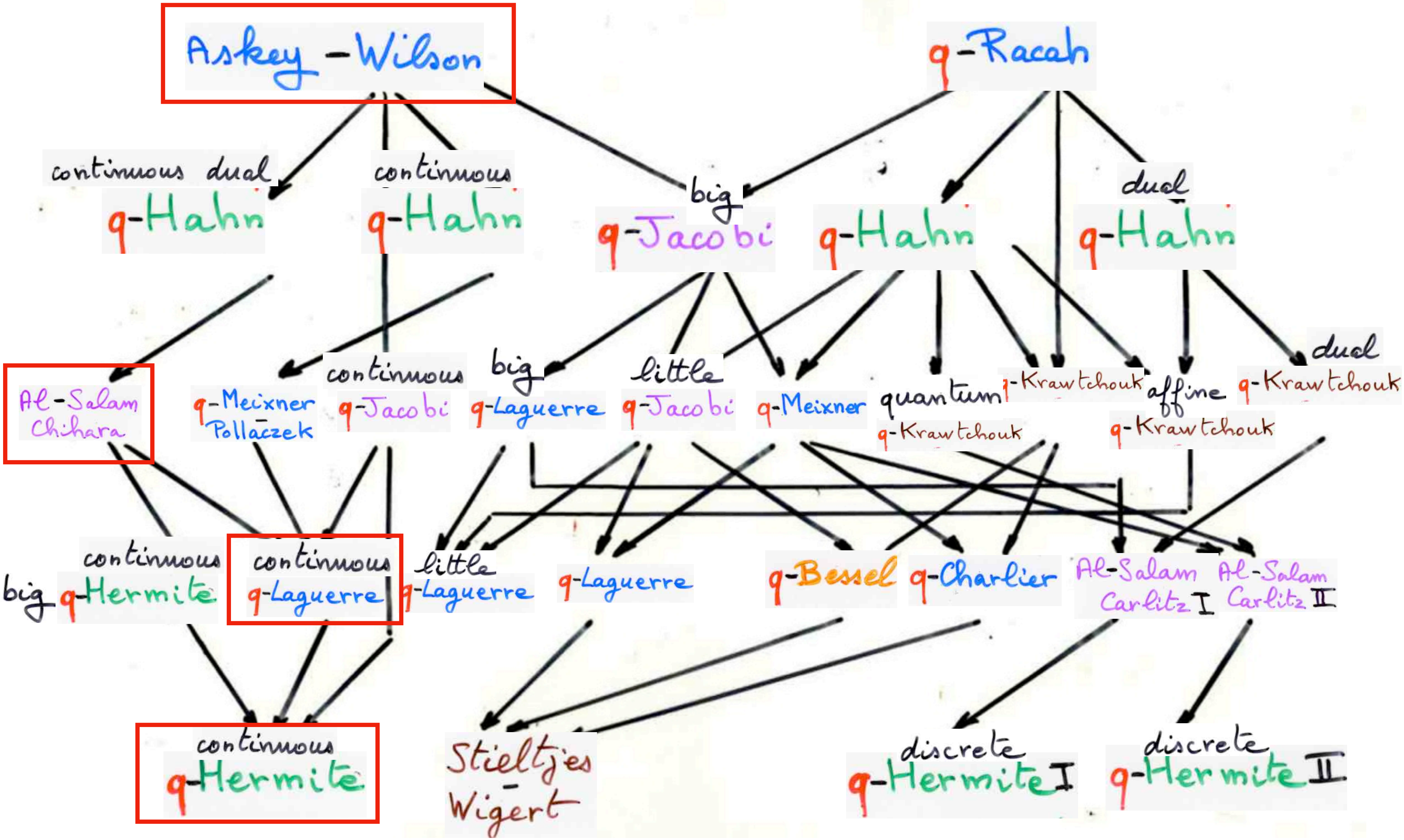
$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009) staircase tableaux

scheme of basic hypergeometric orthogonal polynomials



Epilogue

Part I, II, III, IV of ABjC

Thursday 14, March