Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 3
Continued fractions

Ch 3b

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Reminding Ch 3 a

continued fractions

Stieljes

$$\underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}}_{S(t;\lambda)}$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$J(t;\mathbf{b},\boldsymbol{\lambda})$

Jacobi

continued fraction

$\mathbf{b}=\{b_k\}_{k\geq 0}$ $\quad \boldsymbol{\lambda}=\{\lambda_k\}_{k\geq 1}$

$$\sum_{\omega \text{ Motzkin path}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{1-b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

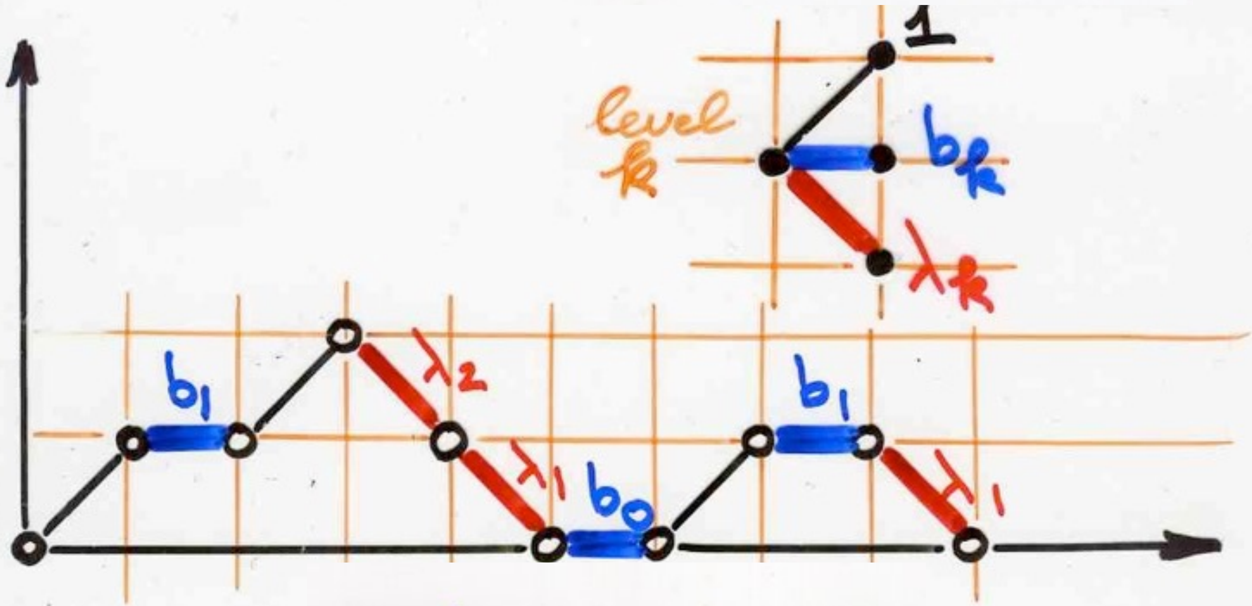
$J(t;\mathbf{b},\boldsymbol{\lambda})$

Jacobi continued fraction

$\mathbf{b}=\{b_k\}_{k\geq 0}$ $\boldsymbol{\lambda}=\{\lambda_k\}_{k\geq 1}$

Philippe Flajolet fundamental Lemma

valuation v



ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

classical theory

continued fractions ⟷ **orthogonal polynomials**

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \lambda_1 t^2 \cfrac{}{\cdots \cfrac{}{1 - b_k t - \lambda_{k+1} t^2 \cfrac{}{\cdots}}}}$$

Motzkin path
$|\omega| = n$

$P_{k+1}(x) =$
$(x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$

$f(x^n) = \mu_n$
moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path
$|\omega| = n$

$$J_k(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t}}}}$$

$$J(t; b, \lambda)$$

convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

Various extensions

Contraction of continued fraction

$$S(t;\gamma) = \cfrac{1}{1-\cfrac{\gamma_1 t}{1-\cfrac{\gamma_2 t}{\cfrac{\cdots}{1-\cfrac{\gamma_k t}{\cdots}}}}}$$

$\gamma = \{\gamma_k\}_{k\geq 1}$

Jacobi

$$J(t;b,\lambda) = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{1-b_1 t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}}$$

$b = \{b_k\}_{k\geq 0}$ $\lambda = \{\lambda_k\}_{k\geq 1}$

$$S(t;\gamma) = J(t;b,\lambda)$$

$$\int_0^\infty e^{-u}\tan(ut)\,du = \cfrac{1}{1-\cfrac{1\times 2\,t^2}{1-\cfrac{2\times 3\,t^2}{1-\cfrac{3\times 4\,t^2}{\cfrac{\cdots}{1-\cfrac{k(k+1)\,t^2}{\cdots}}}}}}$$

Laplace transform

$$\sum_{n\geq 0} G_{2n+2}\,t^{2n} = \cfrac{1}{1-\cfrac{1\times 1\,t^2}{1-\cfrac{1\times 2\,t^2}{1-\cfrac{2\times 2\,t^2}{1-\cfrac{2\times 3\,t^2}{1-\cfrac{3\times 3\,t^2}{\cfrac{\cdots}{1-\left\lceil\frac{k}{2}\right\rceil\left\lceil\frac{k+1}{2}\right\rceil}}}}}}}$$

Genocchi numbers

Jacobi elliptic functions

$$\int_0^\infty e^{-u}\,\mathrm{cn}(ut)\,du = \cfrac{1}{1-\cfrac{1^2t^2}{1-\cfrac{2^2\alpha^2t^2}{1-\cfrac{3^2t^2}{1-\cfrac{4^2\alpha^2t^2}{\cdots}}}}}$$

$$\int_0^\infty e^{-u}\frac{1}{\cos(ut)\,du} = \cfrac{1}{1-\cfrac{1\times1\,t^2}{1-\cfrac{2\times2\,t^2}{1-\cfrac{3\times3\,t^2}{\cfrac{\cdots}{1-\cfrac{k^2\,t^2}{\cdots}}}}}}$$

Dixon (1890)

Fermat cubic

$$x^3+y^3=1$$

Dixonian elliptic functions

$$\begin{cases} sm' = cm^2, & sm(0)=0 \\ cm' = -sm^2, & cm(0)=1 \end{cases}$$

Conrad (2002)

$$\int_0^\infty sm(u)\, e^{-u/x}\, du = \cfrac{x^2}{1+b_0x^3-\cfrac{1.2^2.3^2.4\,x^6}{1+b_1x^3-\cfrac{4.5^2.6^2.7\,x^6}{1+b_2x^3-\cdots}}}$$

$$b_n = 2(3n+1)((3n+1)^2+1)$$

Van Fossen Conrad, Flajolet (2006)

Consider the **integer sequence** (p_n), which starts as

$$2, 144, 96768, 268240896, 2111592333312, 37975288540299264, \ldots$$

and is defined by sums over the **square lattice**,

$$p_n := (-1)^{n+1}(4n+3)! \left[\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right]^{-4n-4} \sum_{a,b=-\infty}^{+\infty} \left[(2a+1)+(2b+1)\sqrt{-1}\right]^{-4n-4}.$$

The following **continued fraction** expansion holds:

$$\sum_{n=0}^{\infty} p_n z^n = \cfrac{2}{1 - 2\cdot 2^2(2^2+5)\,z - \cfrac{2\cdot 3^2\cdot 4^2\cdot 5^2\cdot 6\,z^2}{1-2\cdot 6^2(6^2+5)\,z - \cfrac{6\cdot 7^2\cdot 8^2\cdot 9^2\cdot 10\,z^2}{1-2\cdot 10^2(10^2+5)\,z - \ddots}}}.$$

[A follow up to R. Bacher and P. Flajolet, *The Ramanujan Journal*, 2010, in press.]

Continued fractions

other examples

"β-analog" of Tchebychev 2nd kind

moments $\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \beta^k$

number of Dyck paths having k peaks

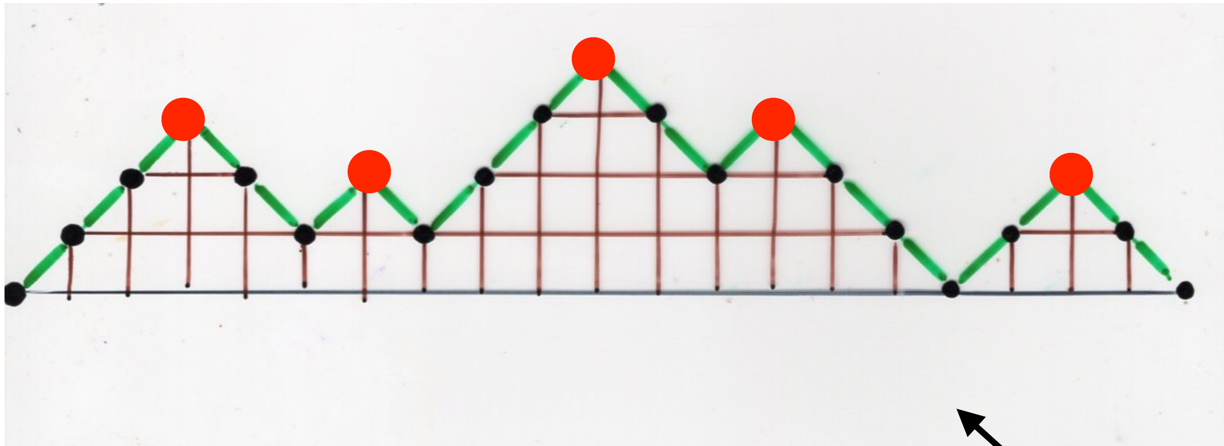
ω, $|\omega| = 2n$

$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \beta^k$$

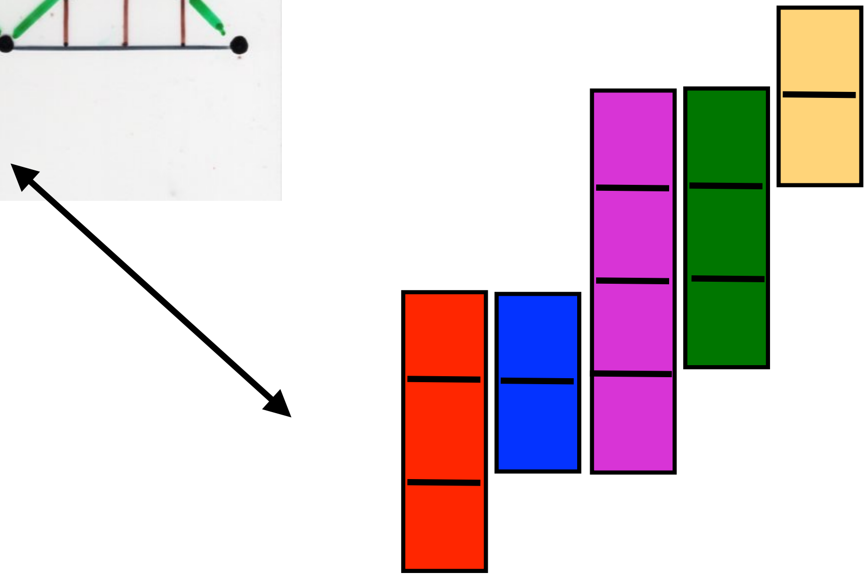
$$\sum_{n \geq 0} \mu_{2n}(\beta)\, t^n = \cfrac{1}{1 - \cfrac{\beta t}{1 - \cfrac{t}{1 - \cfrac{\beta t}{1 - \cfrac{t}{\cdots}}}}}$$

proof:

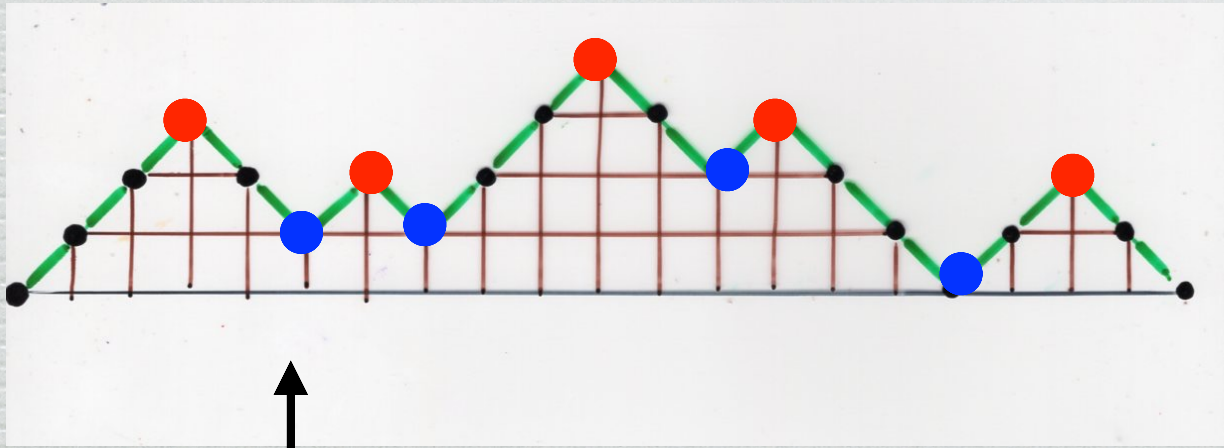
(β)- distribution on Catalan numbers



number of peaks in Dyck paths

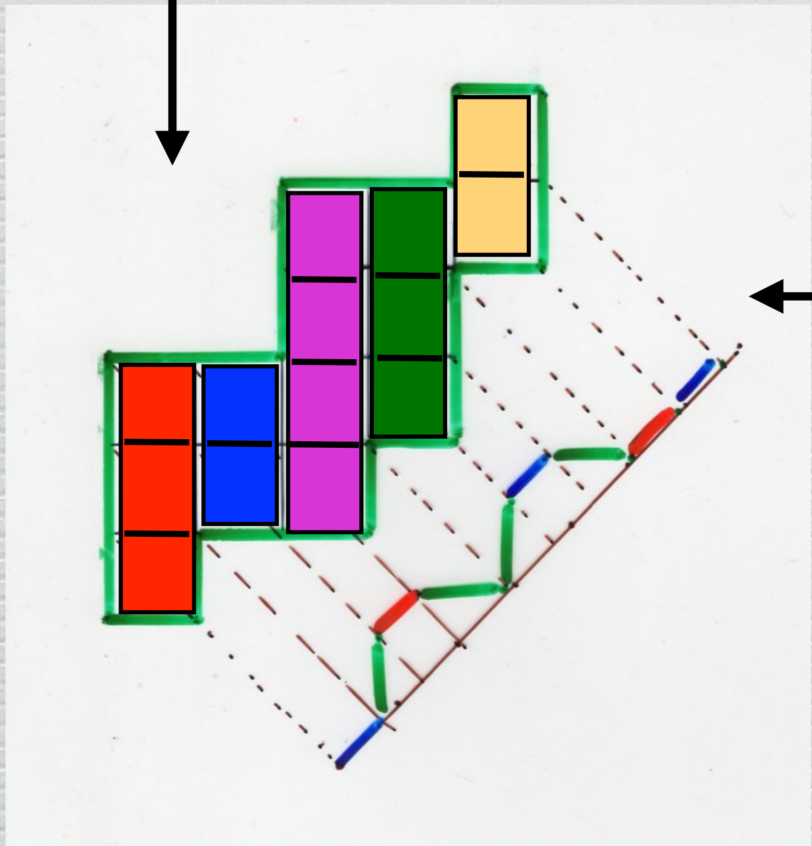


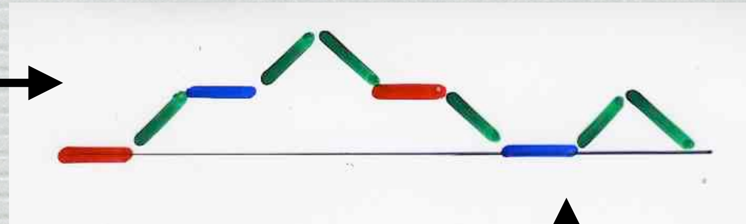
number of columns in staircase polygons

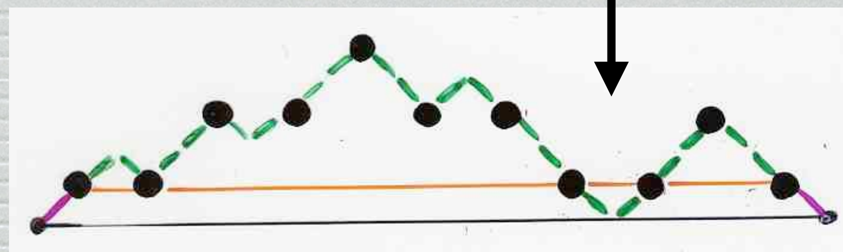


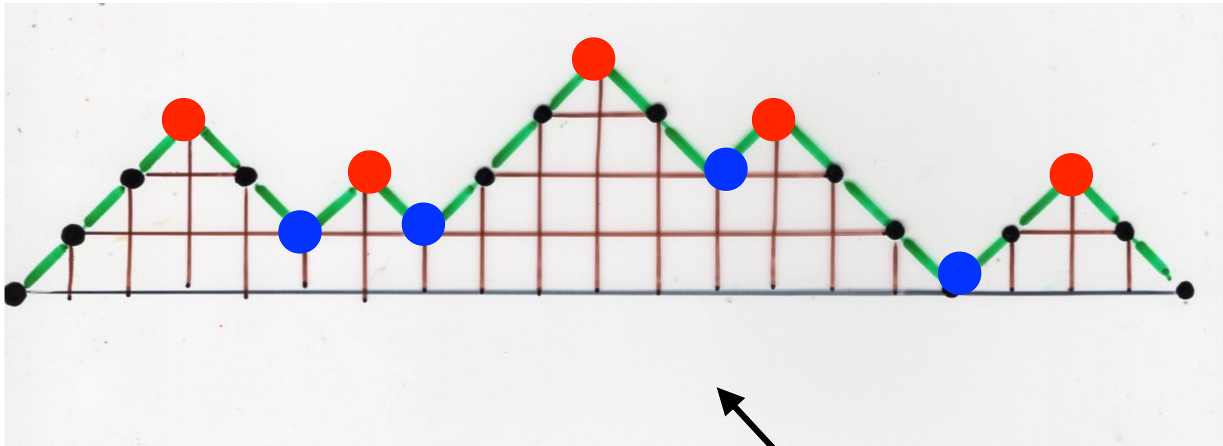
See ABjC, Part I, Ch2a

(β)- distribution $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$









moments

number of Dyck paths having k peaks

$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \beta^k$$

ω, $|\omega| = 2n$

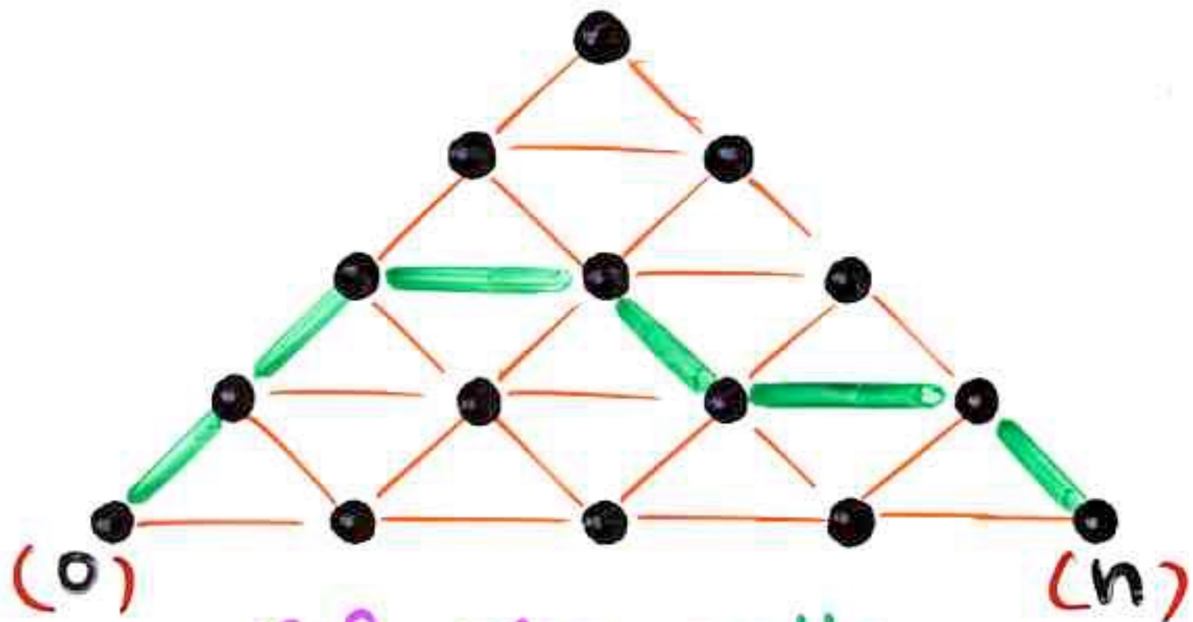
• Dyck path

number of SE steps from odd to even level

Proposition

$$\lambda_k = 1 \quad k \text{ even}$$
$$= \beta \quad k \text{ odd}$$

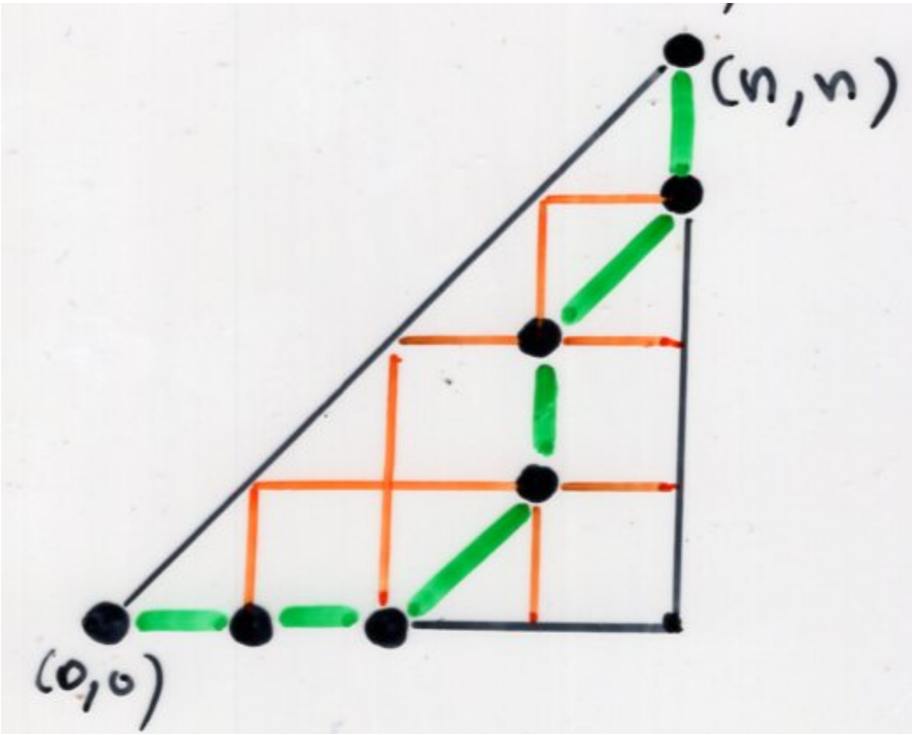
Schröder numbers



Schröder path

(large) Schröder numbers

1, 2, 6, 22, 90, ...



(large) Schröder numbers

$$S(t) = \cfrac{1}{1 - 2t \cfrac{}{1 - t \cfrac{}{1 - 2t \cfrac{}{1 - t \cfrac{}{\cdots}}}}}$$

$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \beta^k$$

number of Dyck paths having k peaks

ω, $|\omega| = 2n$

$$\sum_{n \geq 0} \mu_{2n}(\beta) t^n = \cfrac{1}{1 - \cfrac{\beta t}{1 - \cfrac{t}{1 - \cfrac{\beta t}{1 - \cfrac{t}{\cdots}}}}}$$

small Schröder numbers S^{-}_{n}

= number of Schröder path
$(0) \to (n)$ with no step 
at 0 level

1, 1, 3, 11, 45, ...

<u>exercise</u> $S^{-}_{n} = \frac{1}{2} S_{n}$

exercise

(small) Schröder

$$S^{-}(t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{2t}{1 - \cfrac{t}{1 - \cfrac{2t}{1 - \cfrac{t}{\cdots}}}}}}$$

Staircase polygons

Parallelogram polyominoes

$n = 6$

$a(P) = 10$

staircase polygons

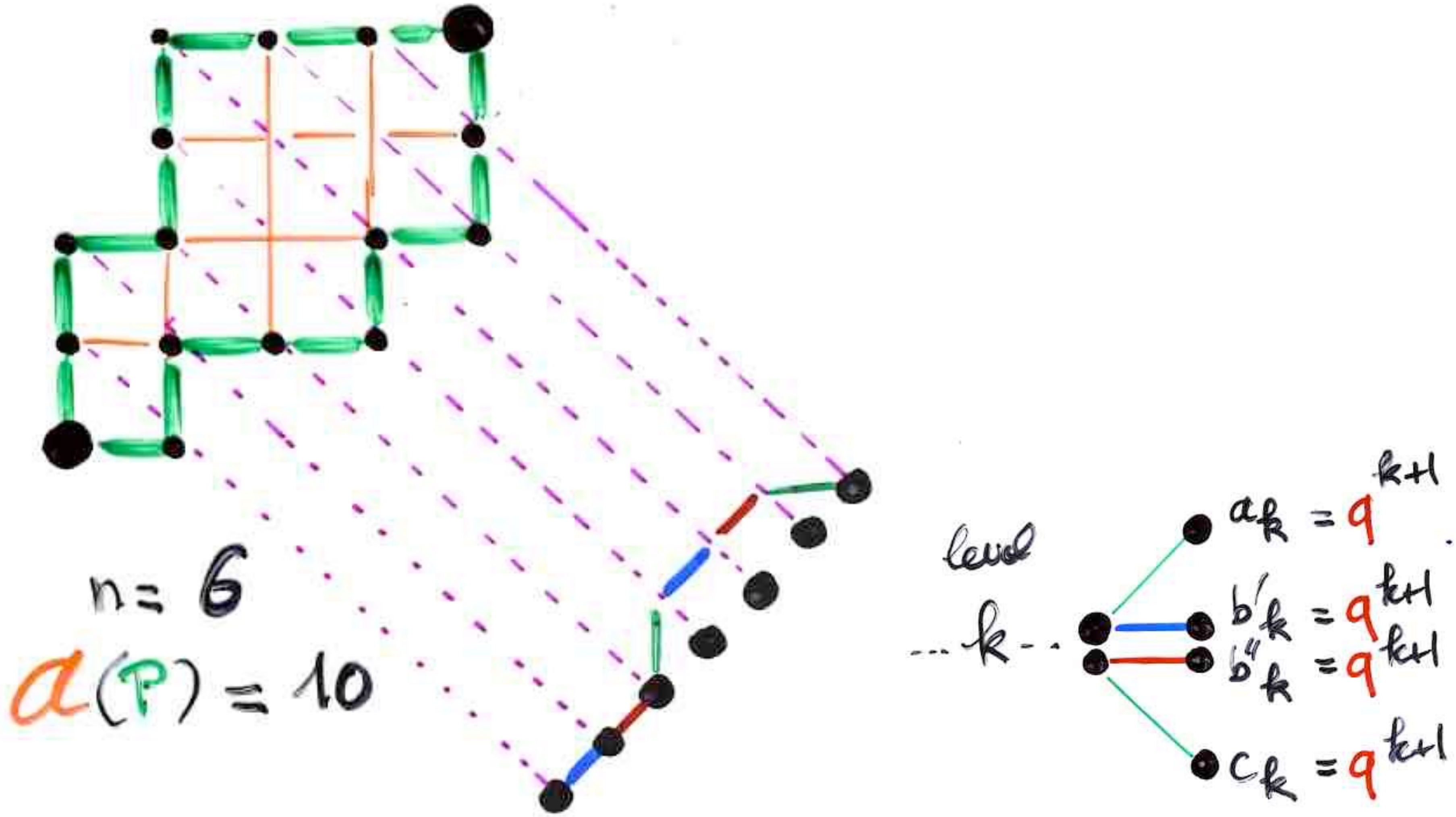
parallelogram polyominoes

$P_{n,j}$

perimeter $2n+4$

$a(P)$
area

area j



$$a(P) = q\,v(\omega_c)$$

2-colored Motzkin path

staircase polygons

parallelogram polyominoes

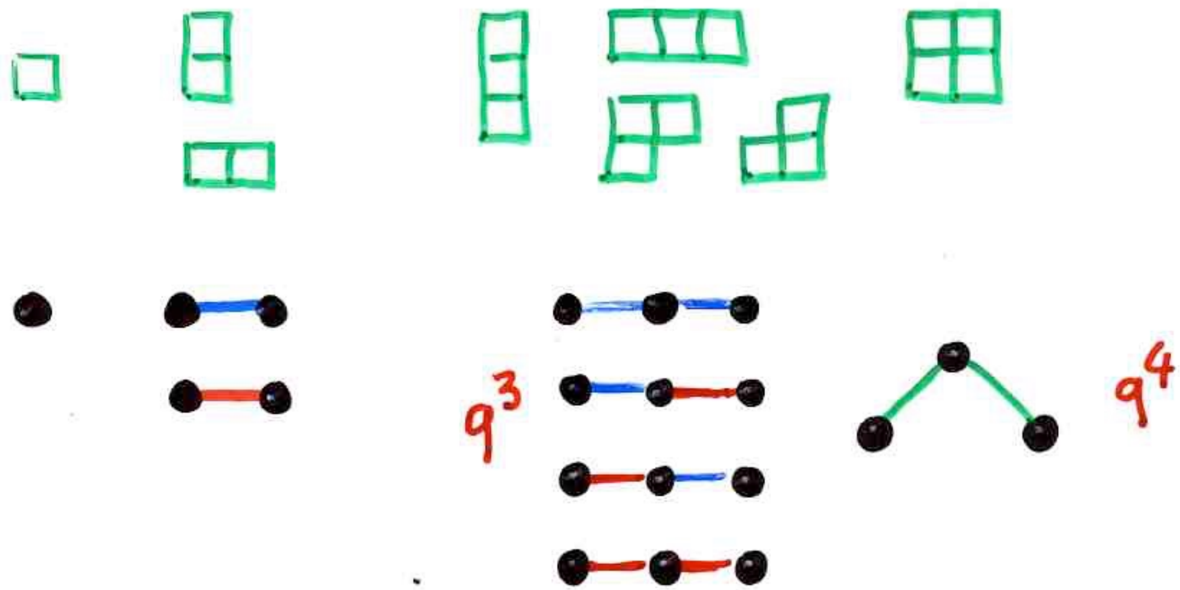
$P_{n,j}$ perimeter $2n+4$ area j

$$\sum_{0\leq j,n} P_{n,j}\, q^j t^n = \cfrac{q}{1-2qt-\cfrac{q^3t^2}{1-2q^2t-\cfrac{q^5t^2}{1-2q^3t-\cfrac{q^7t^2}{\cdots}}}}$$

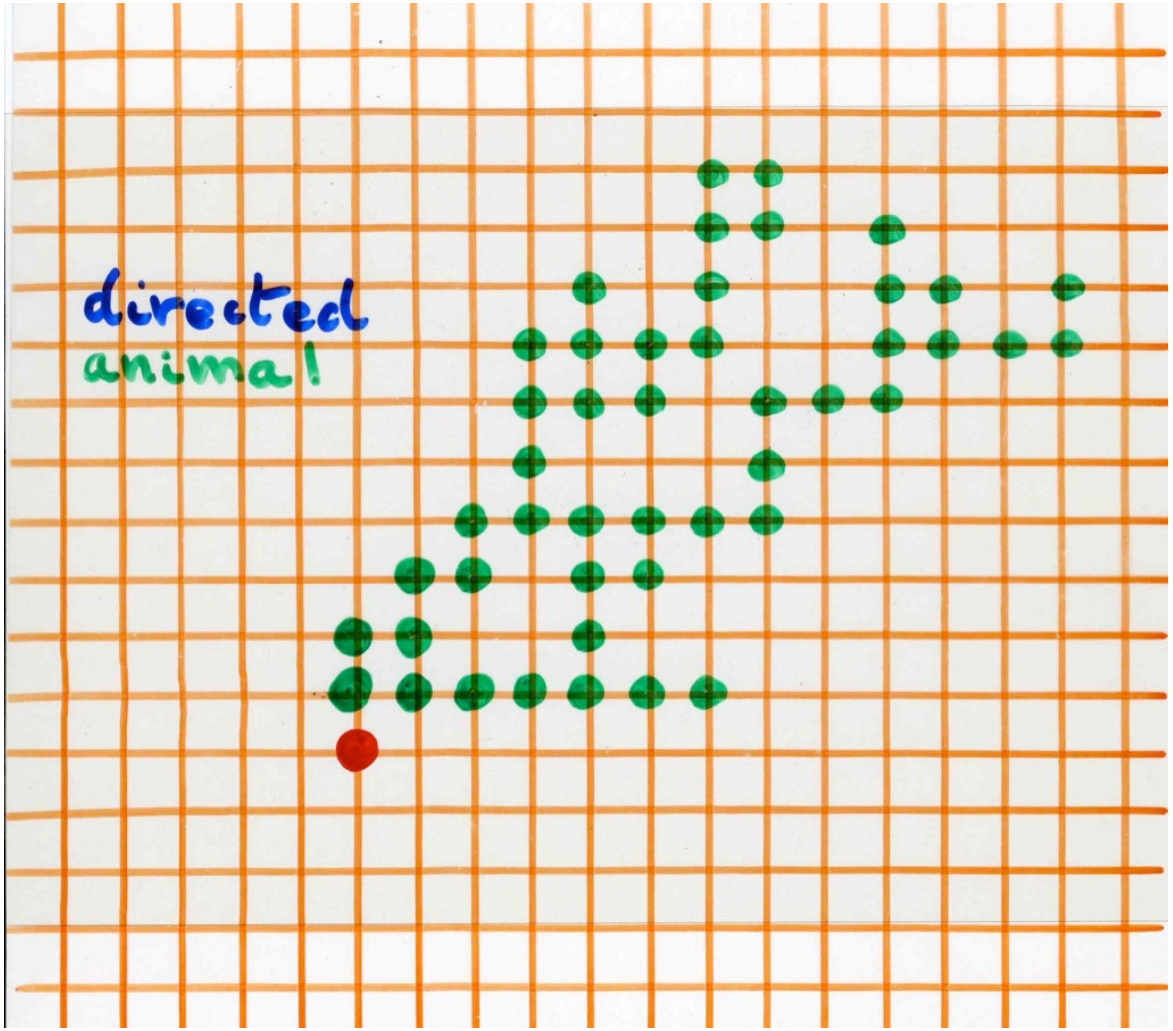
$$\begin{cases} a_k = q^{k+1} \\ b_k = 2q^{k+1} \\ c_k = q^{k+1} \end{cases} \qquad \lambda_k = q^{2k+1}$$

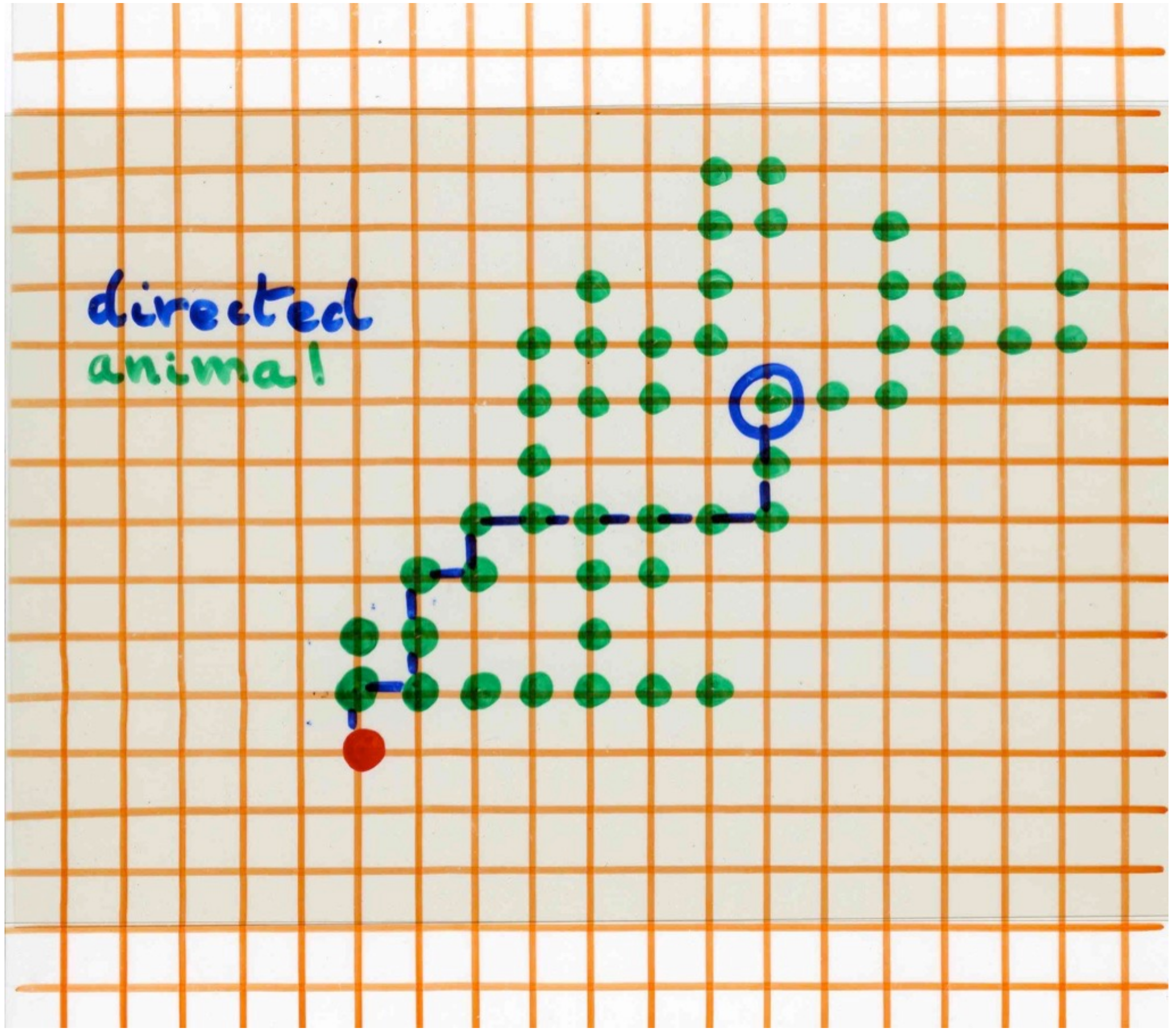
$$\sum_{0\le j,n} p_{n,j}\, q^j t^n = \cfrac{q}{1-2qt-\cfrac{q^3t^2}{1-2q^2t-\cfrac{q^5t^2}{1-2q^3t-\cfrac{q^7t^2}{\cdots}}}}$$

$$= q + 2q^2t + (4q^3+q^4)t^2 + \cdots$$



Directed animals


directed
animal


directed
animal

y generating function
for the number of
directed animal
with n points
satisfies the system of
algebraic equations :

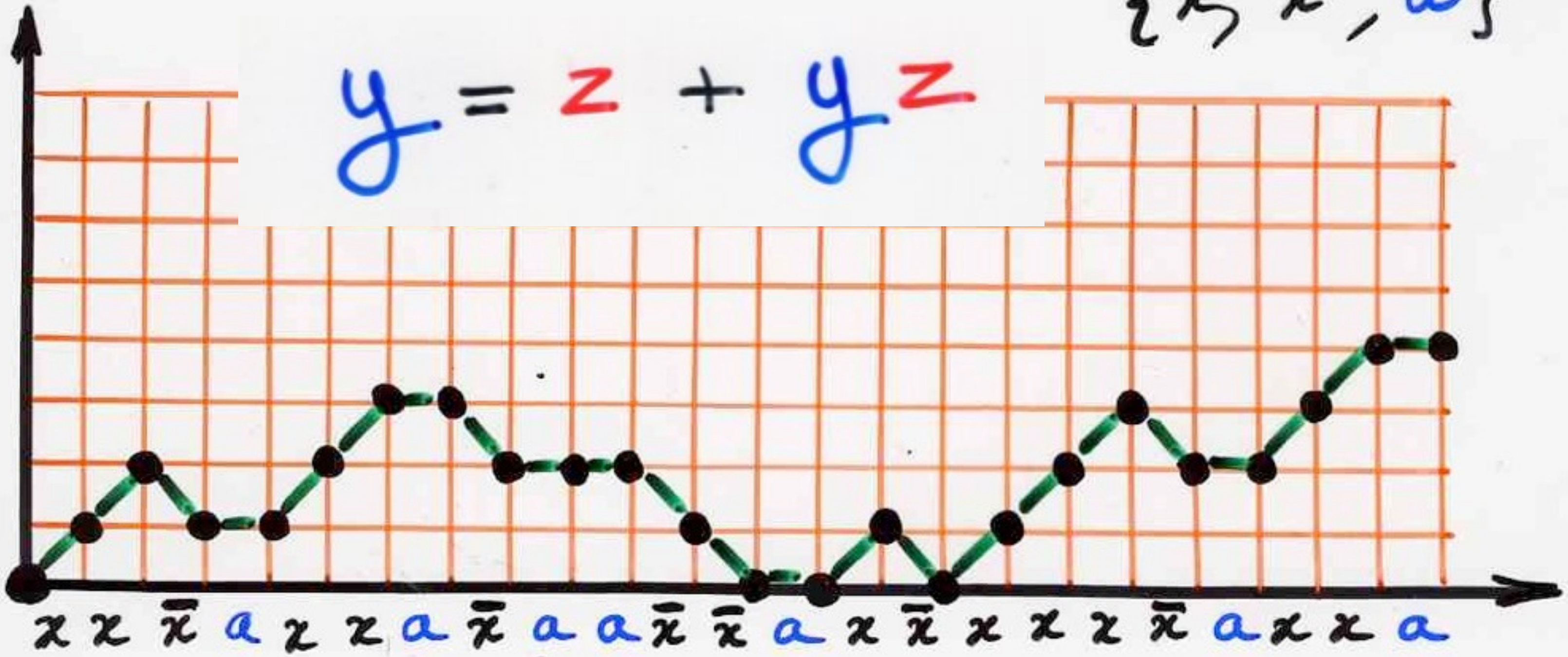
$$y = z + yz$$

$$z = t + tz + tz^2$$

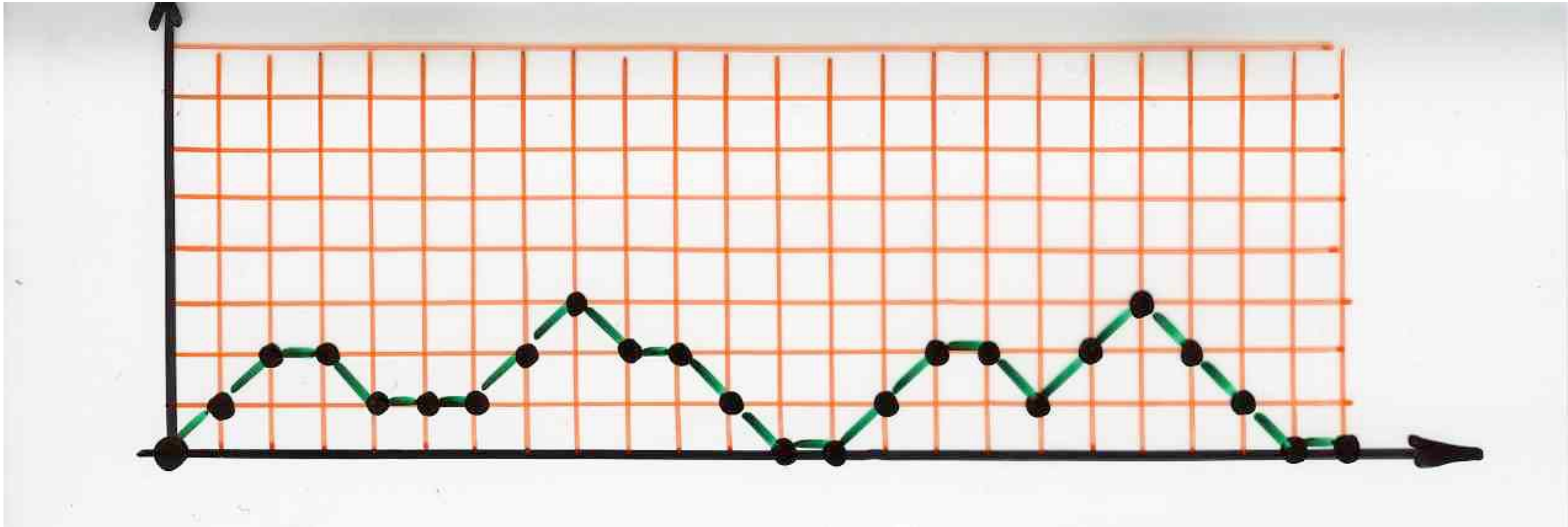
prefix (left factor) of a Motzkin path (word)

$\{x, \bar{x}, a\}$

$$y = z + yz$$



Motzkin path



$$z = t + tz + tz^2$$

bijection

directed animals size n ⟷ left factor of a Motzkin path length $(n-1)$

<u>exercise</u> (easy)

directed animal

$$\begin{cases} \lambda_k = 1 \\ b_k = 1, \quad k \geq 1 \\ b_0 = 2 \end{cases}$$

subdivided Laguerre histories

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

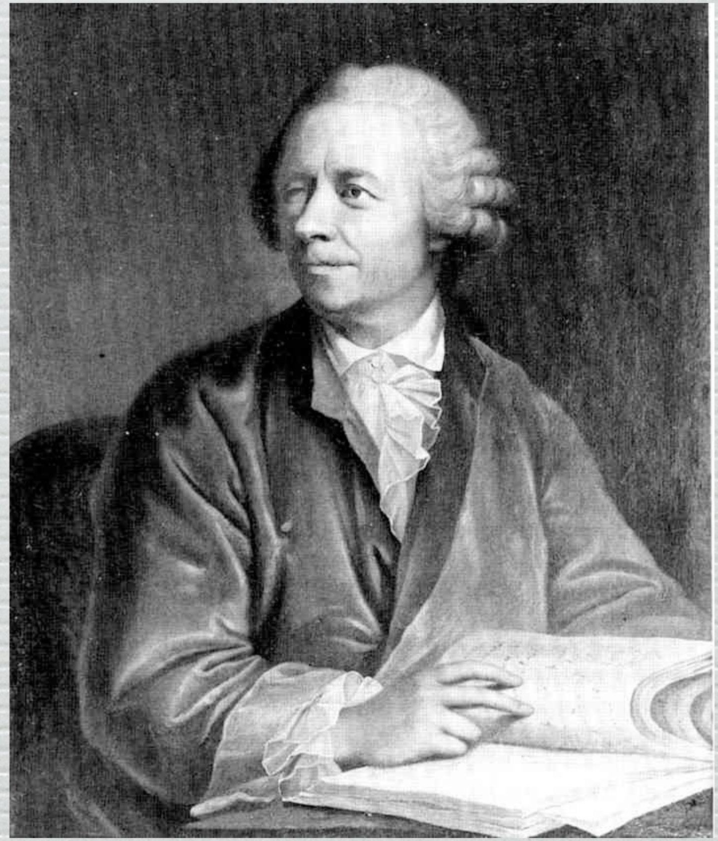
AVCTORE

Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4\ 4.6.6.8.8.10.10}{3.3.5.5.7.7\ 9.9.11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-



§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

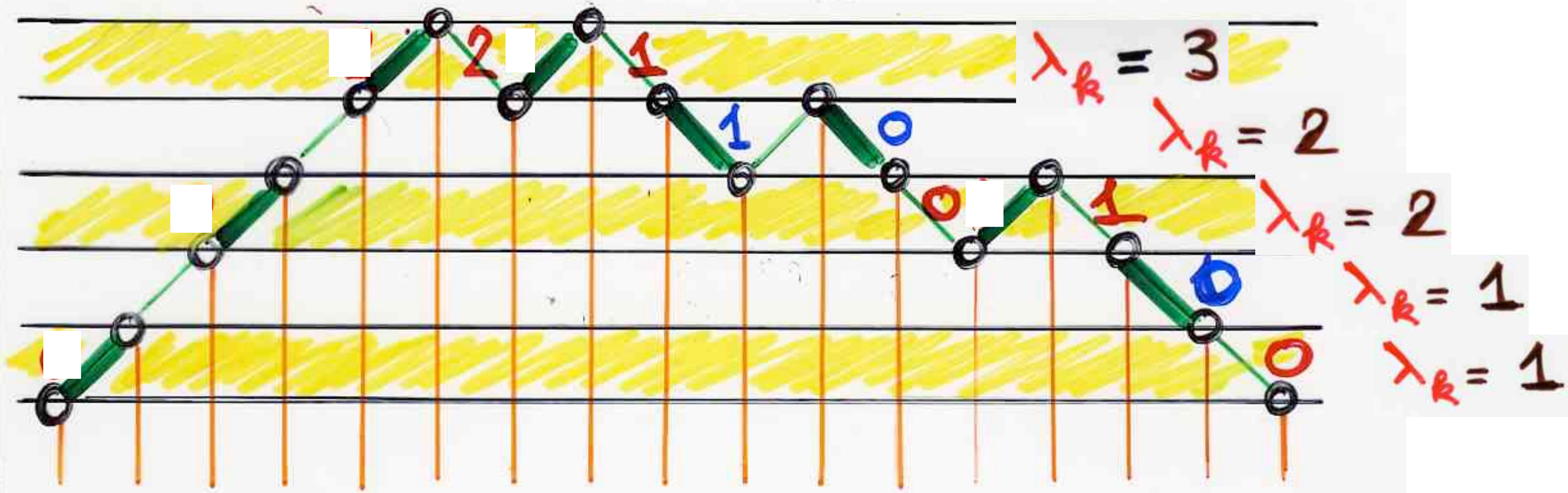
$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n! \, t^n = \cfrac{1}{1 - 1t \cfrac{}{1 - 1t \cfrac{}{1 - 2t \cfrac{}{1 - 2t \cfrac{}{1 - 3t \cfrac{}{1 - \dots}}}}}}$$

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$



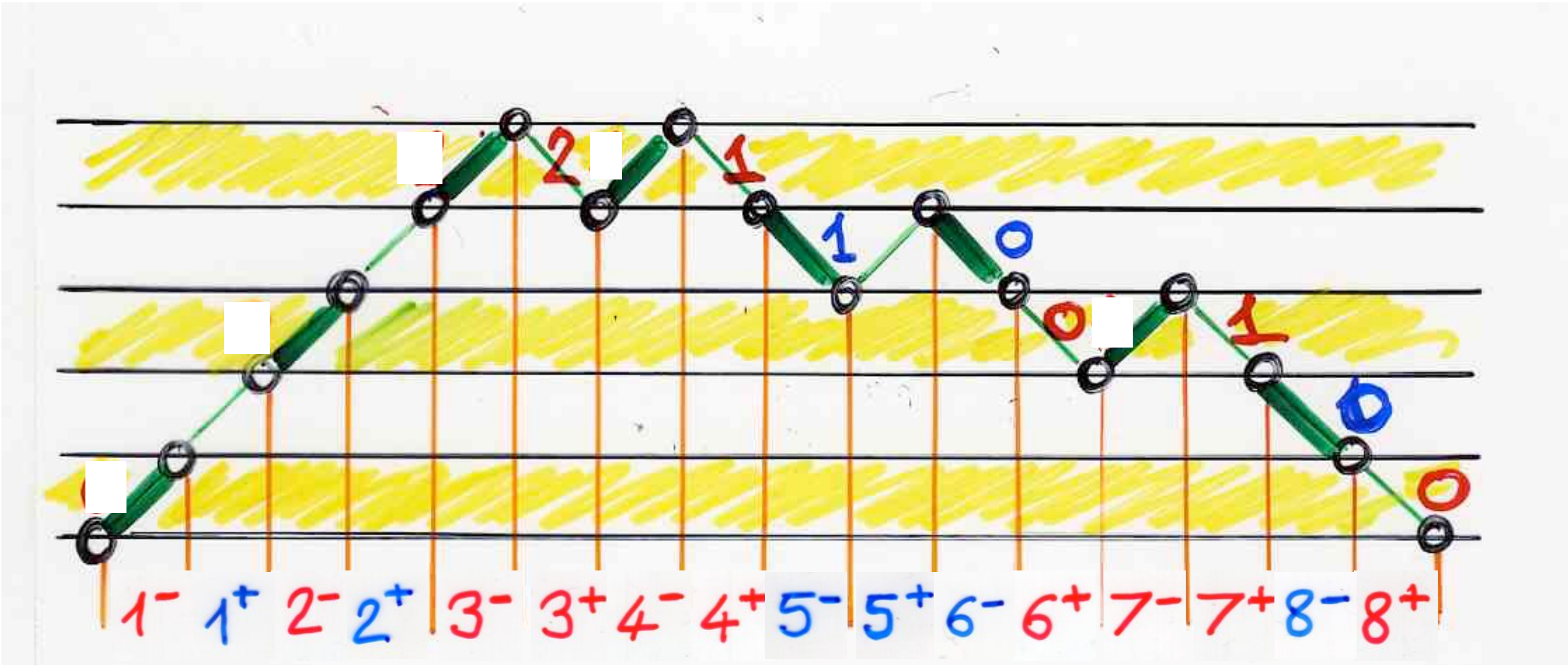
subdivided Laguerre history

bijection

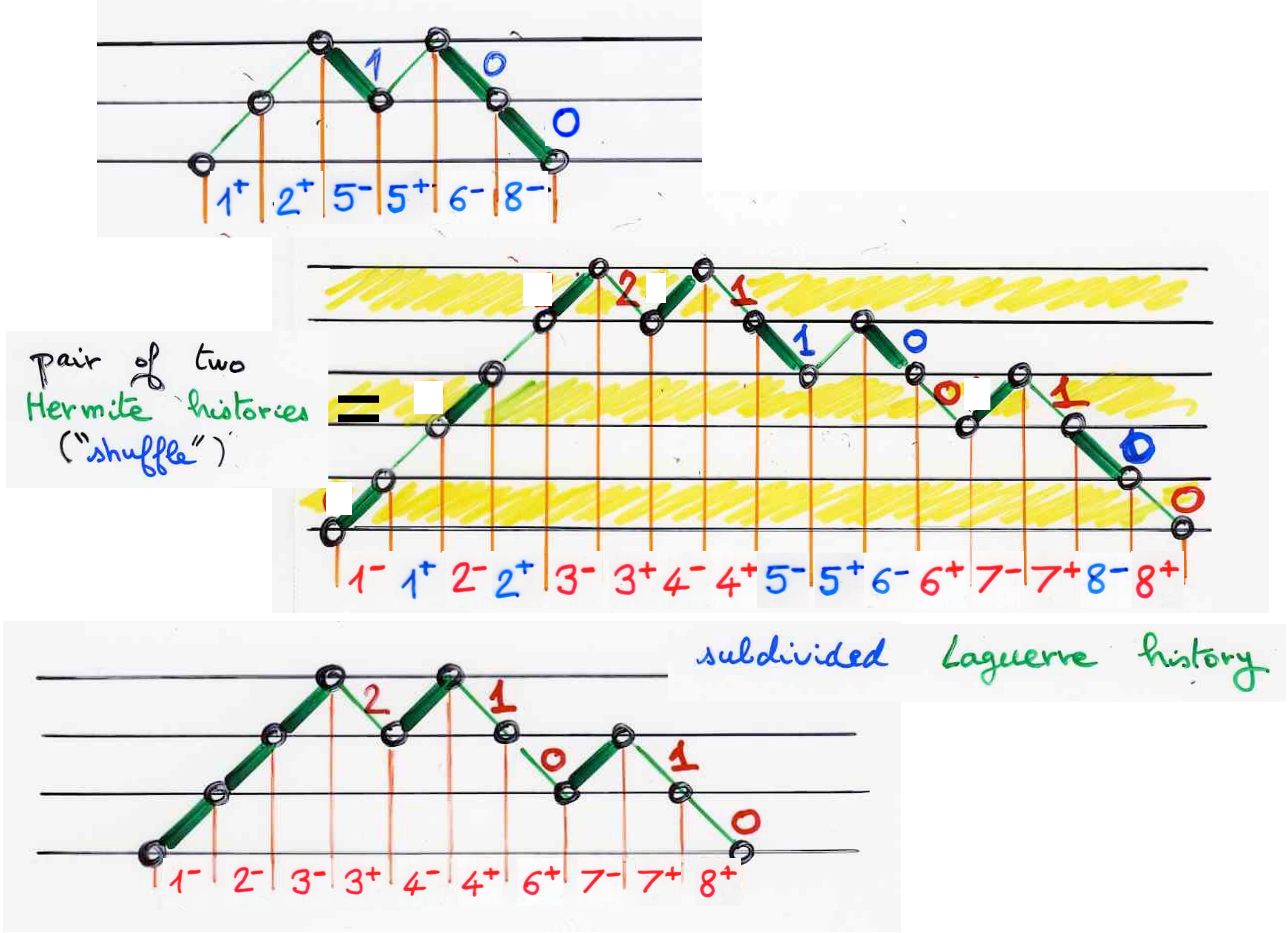
subdivided Laguerre histories

permutations

A. de Médicis, X.V.
(1994)



subdivided Laguerre history



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subdivided Laguerre history
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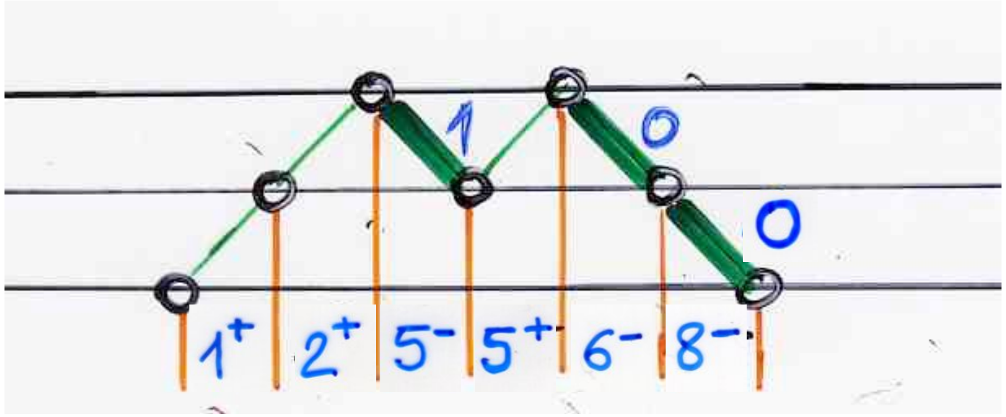
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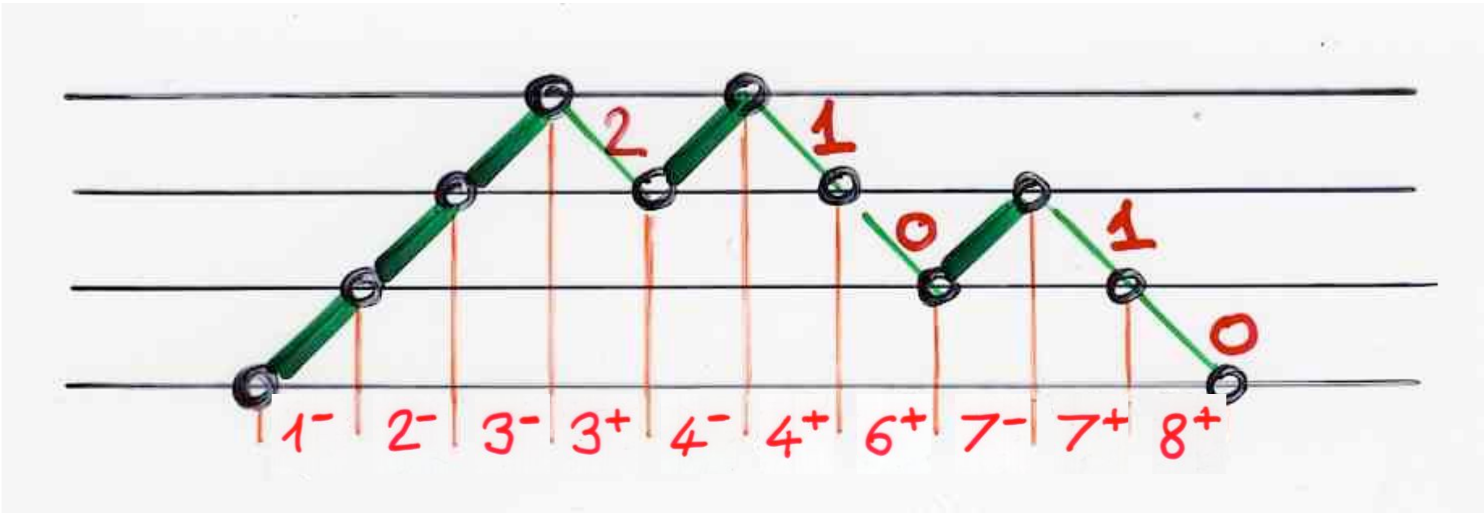
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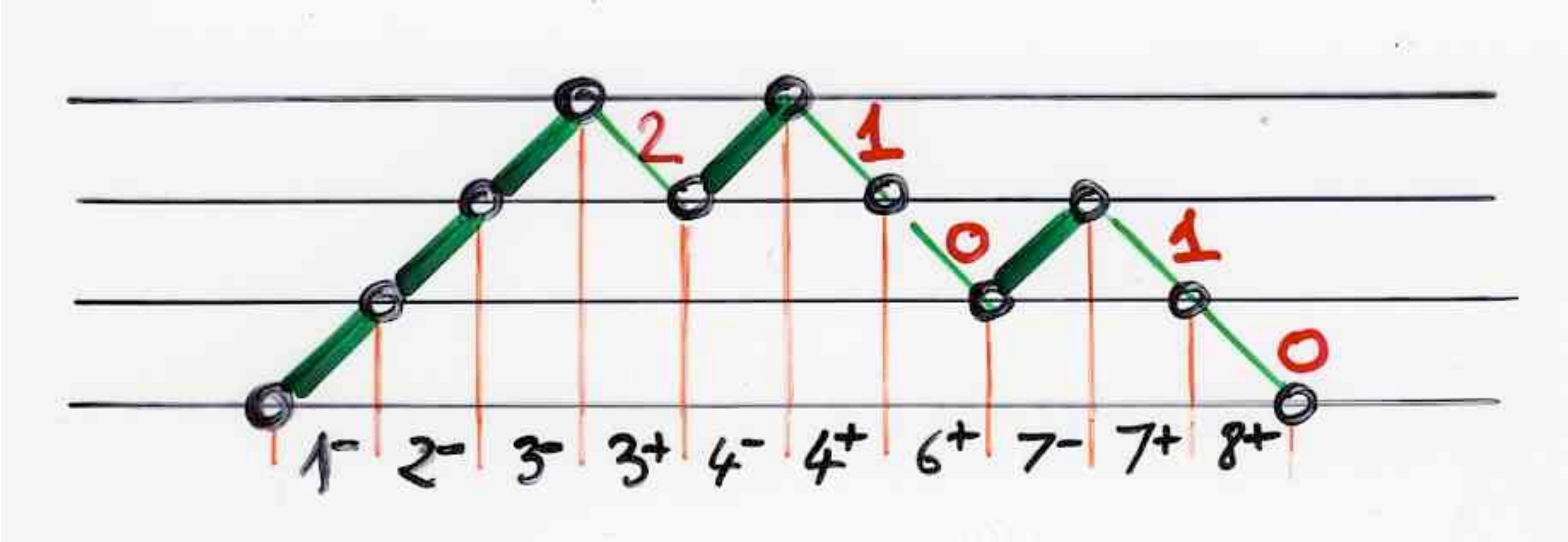
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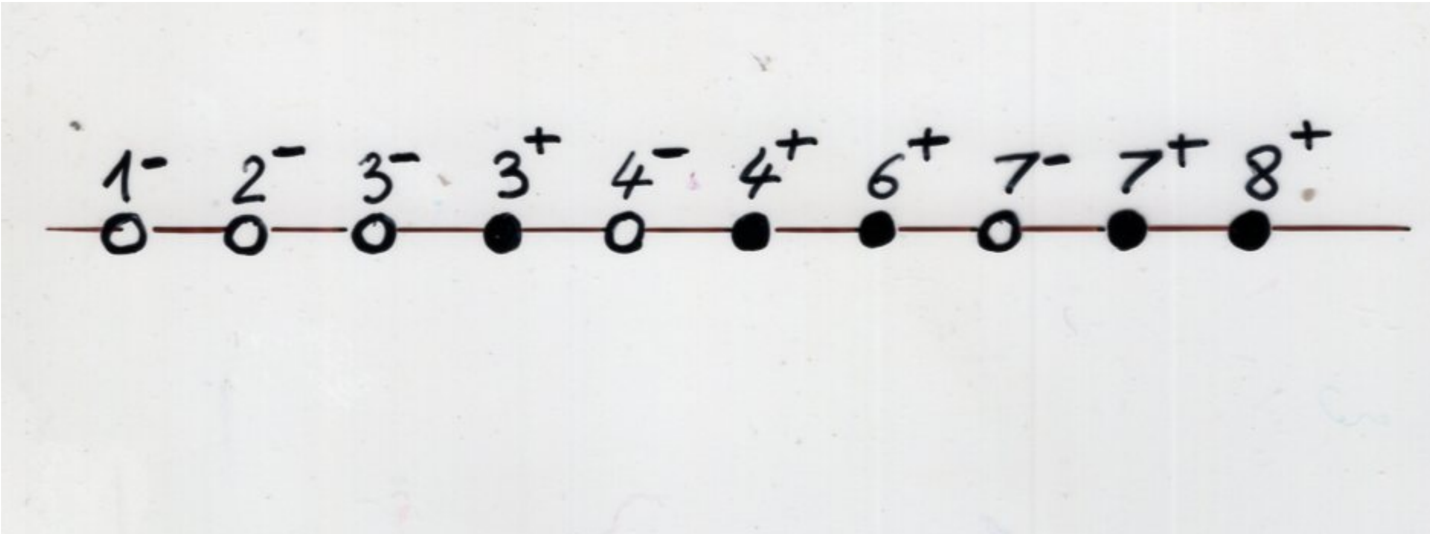
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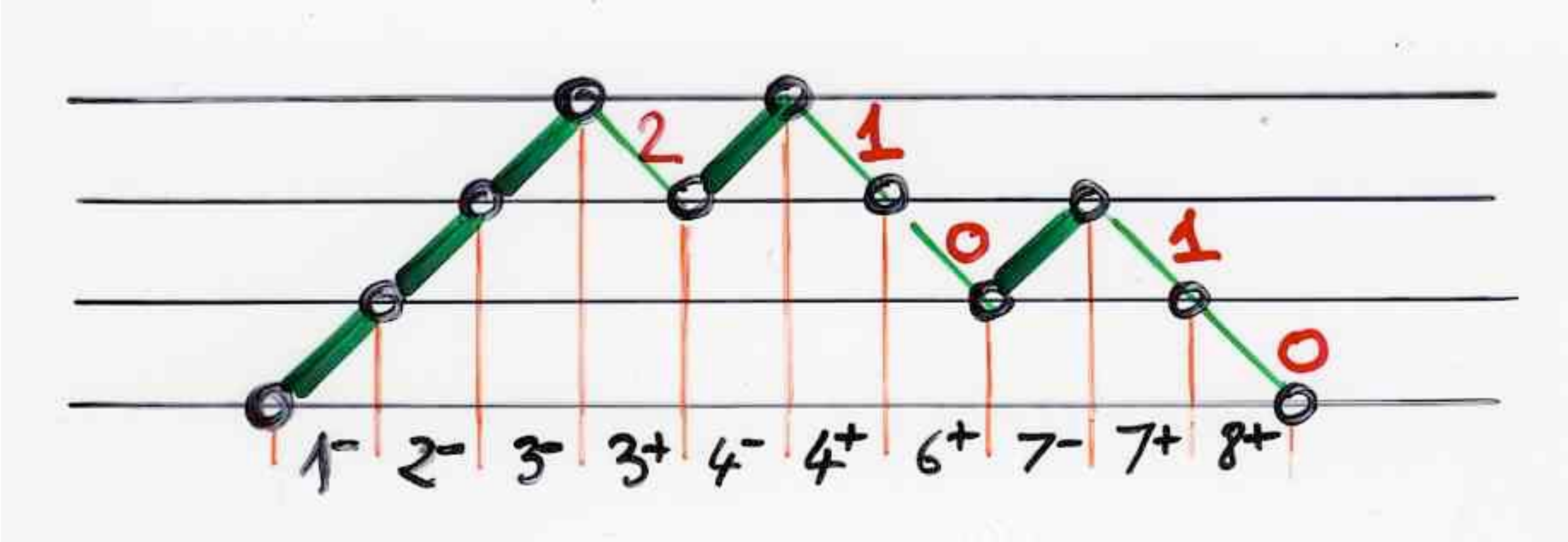
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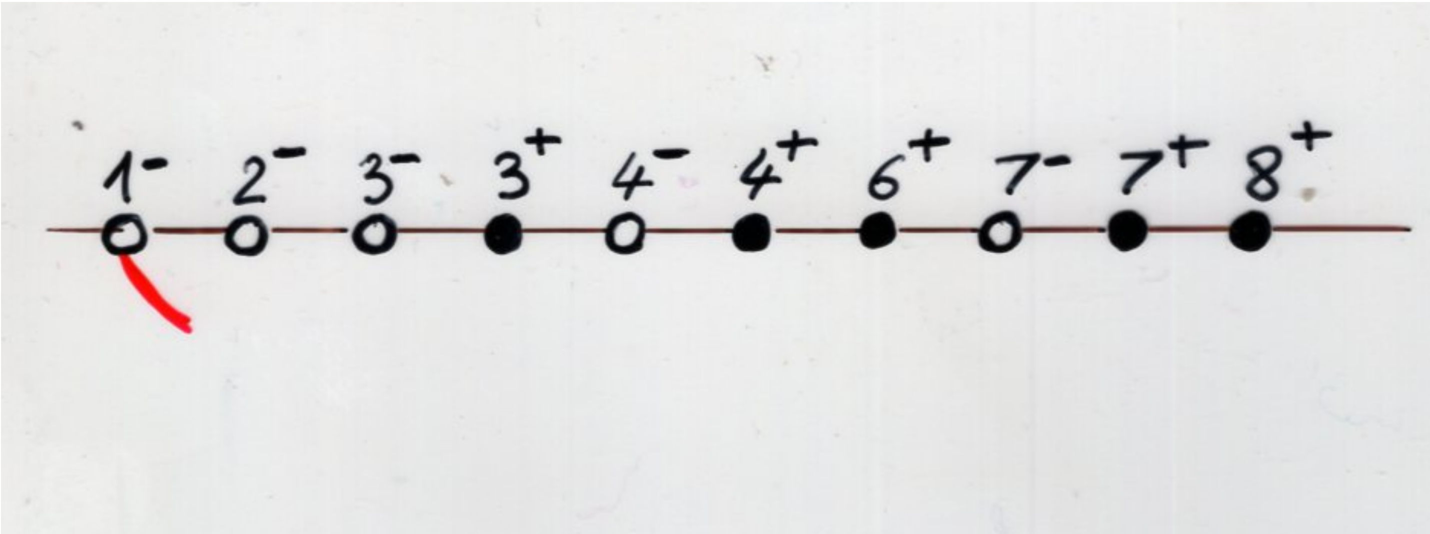
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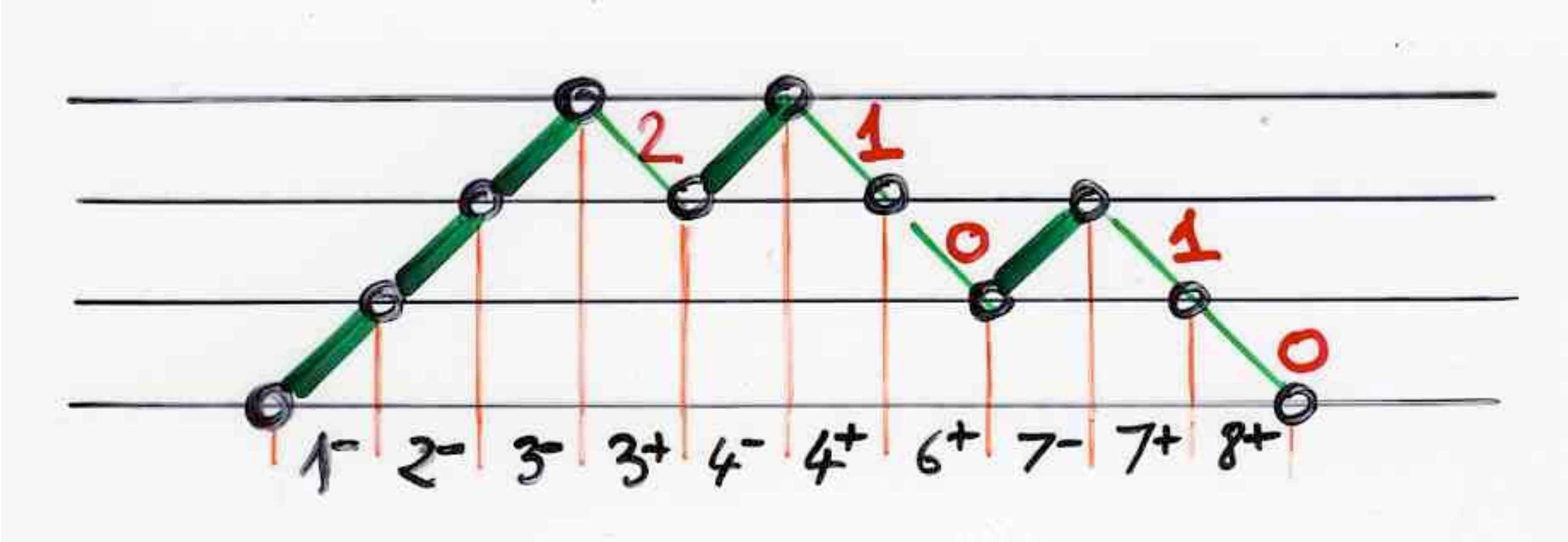
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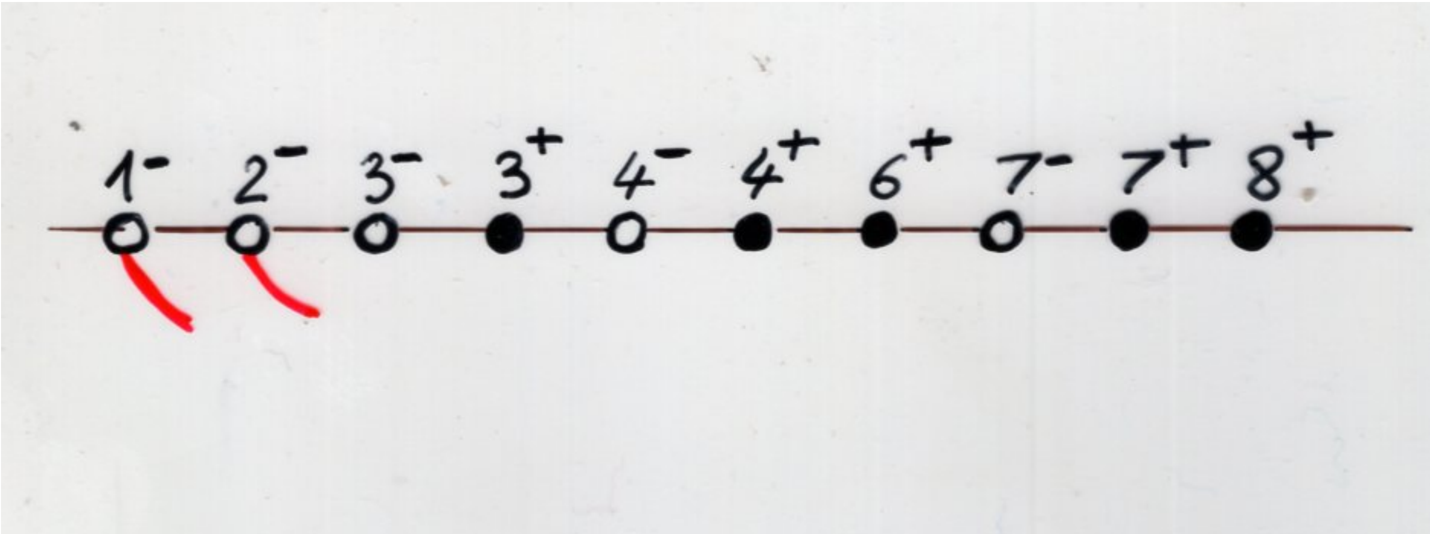
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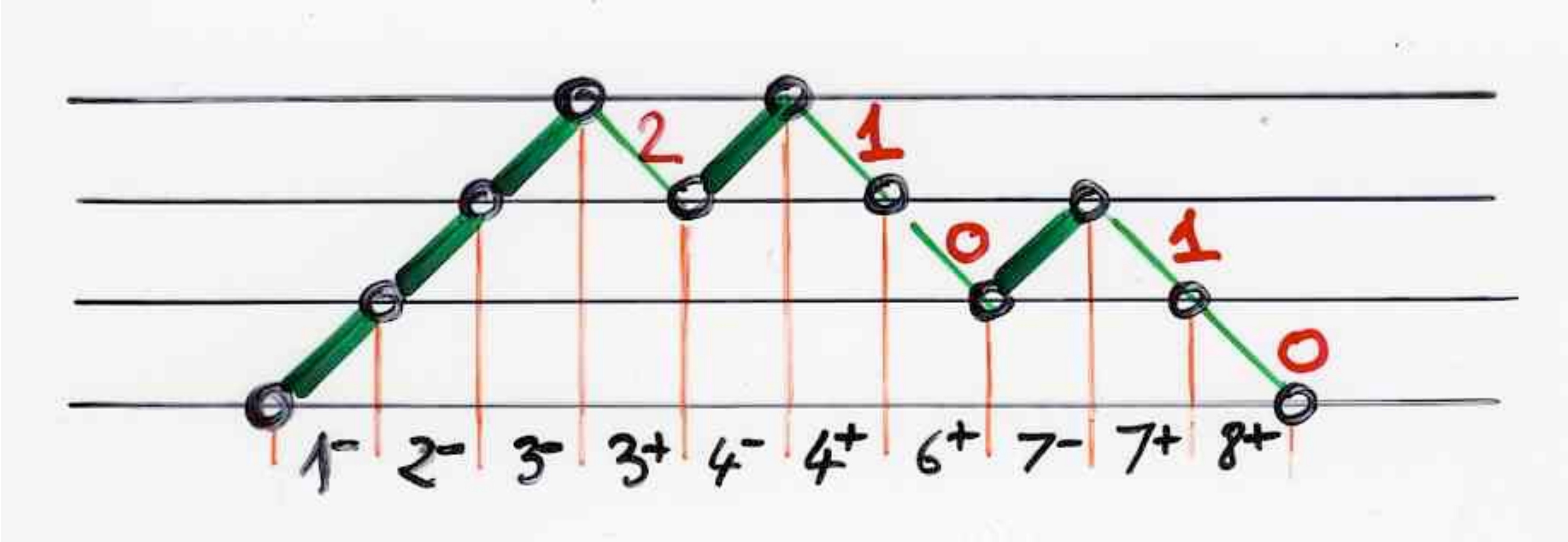
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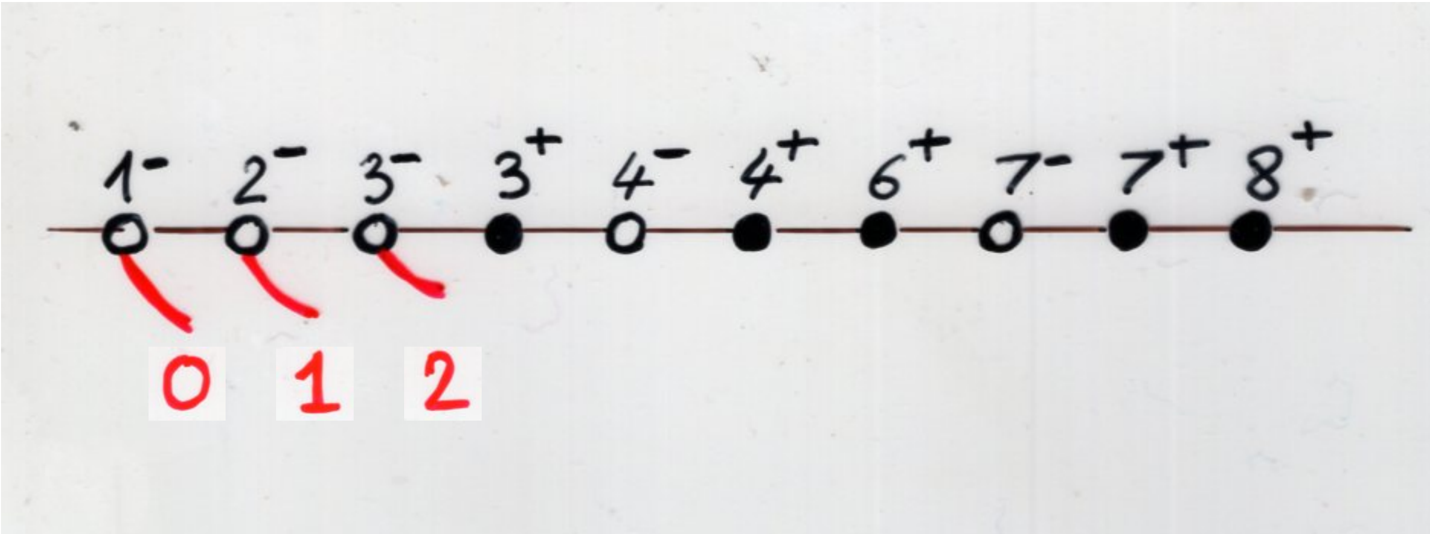
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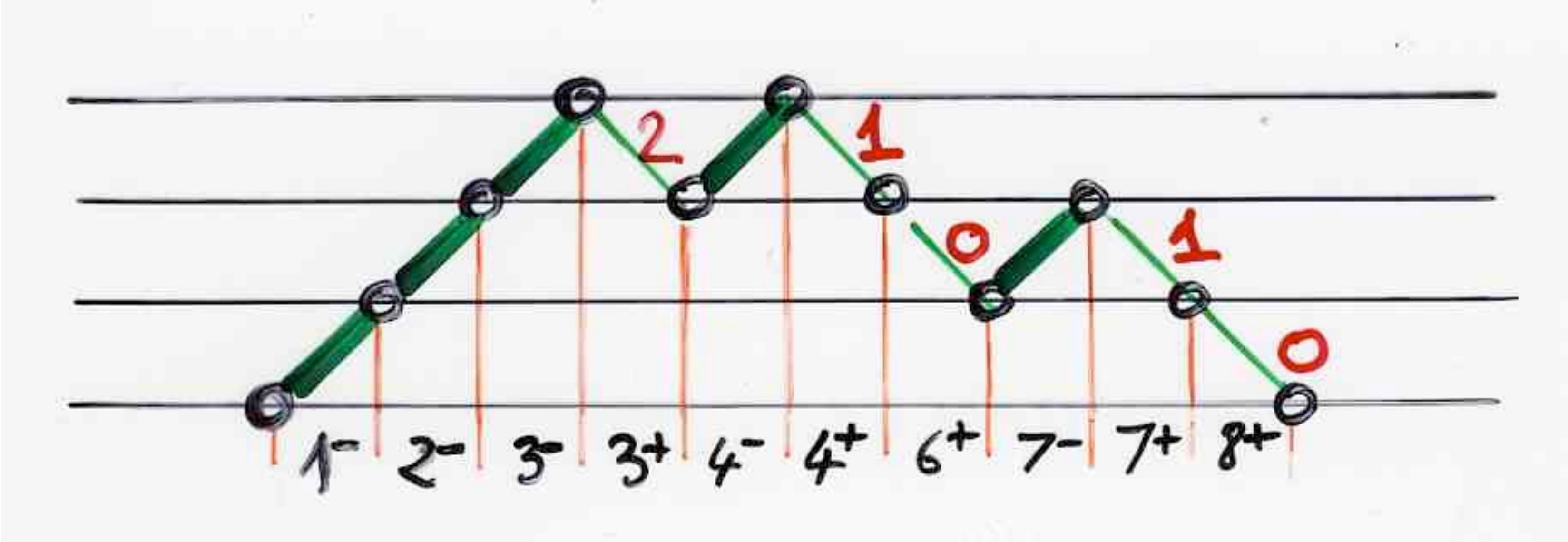
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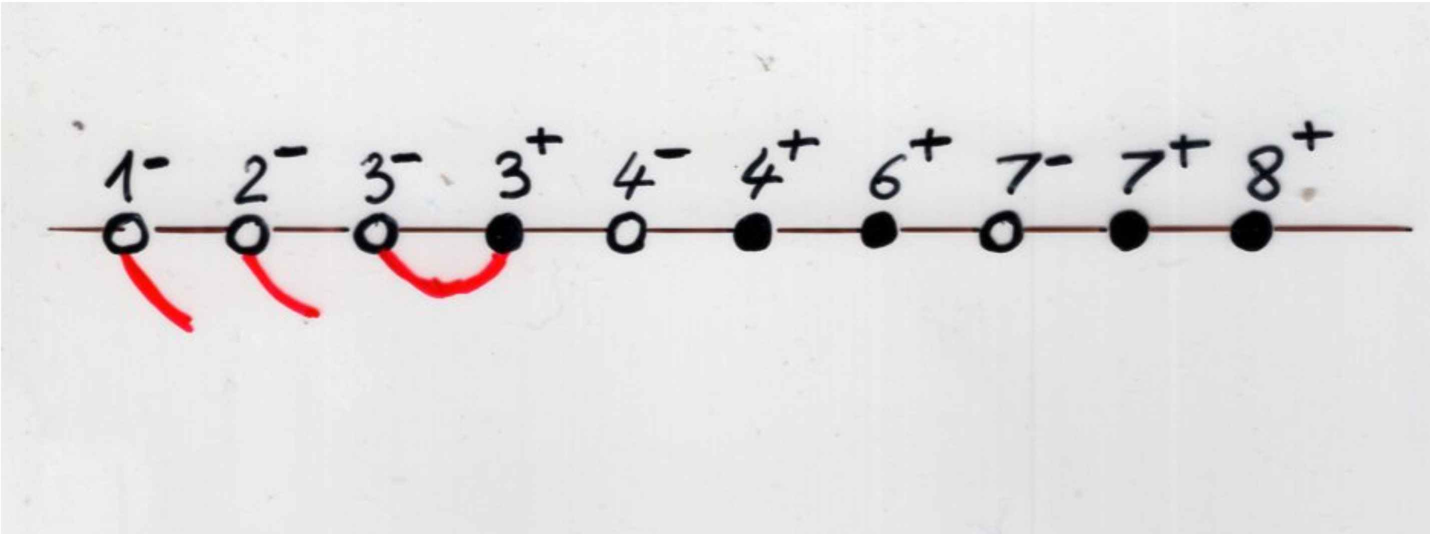
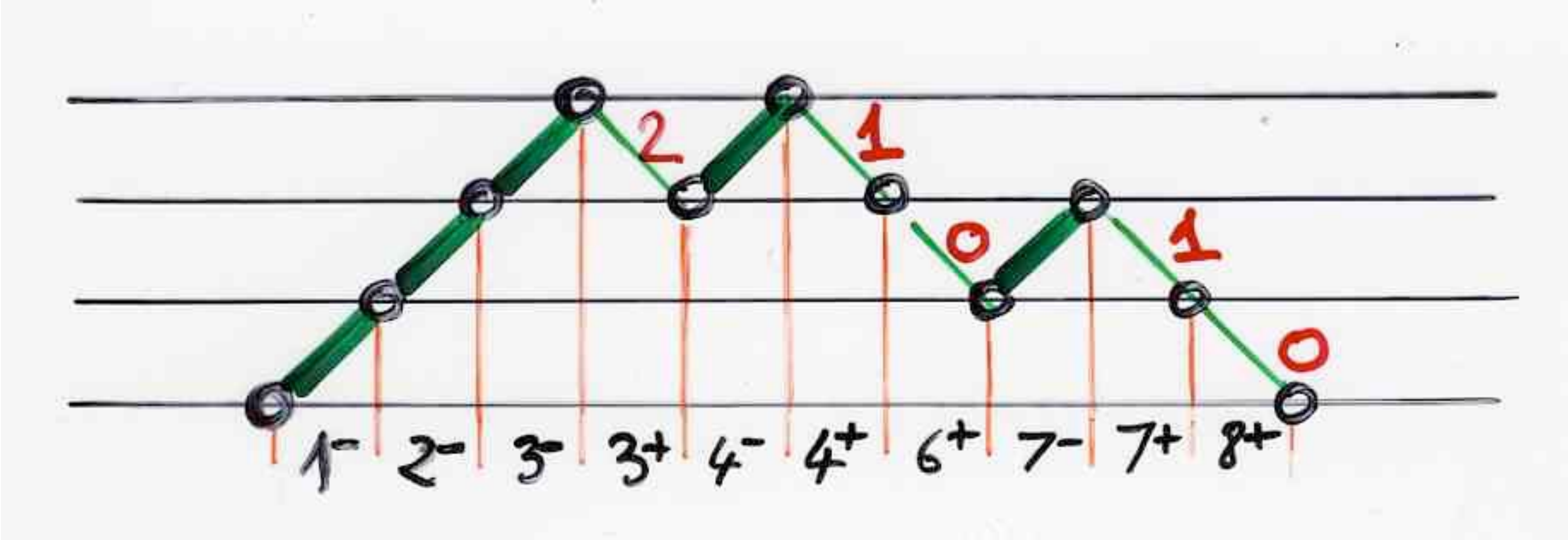
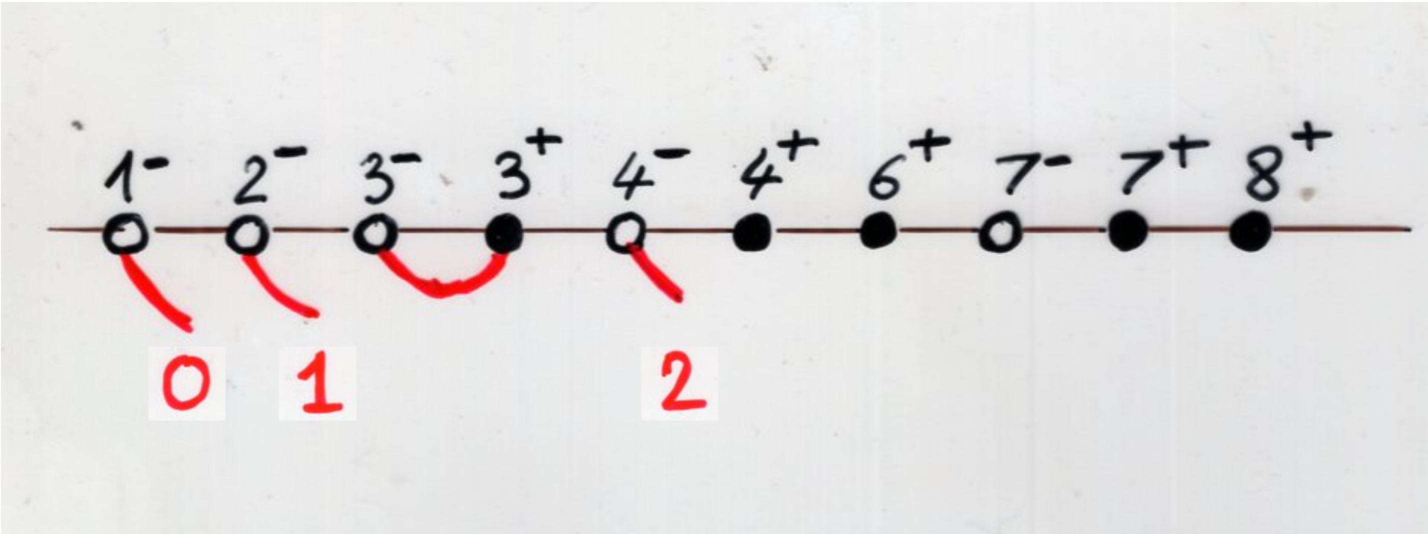
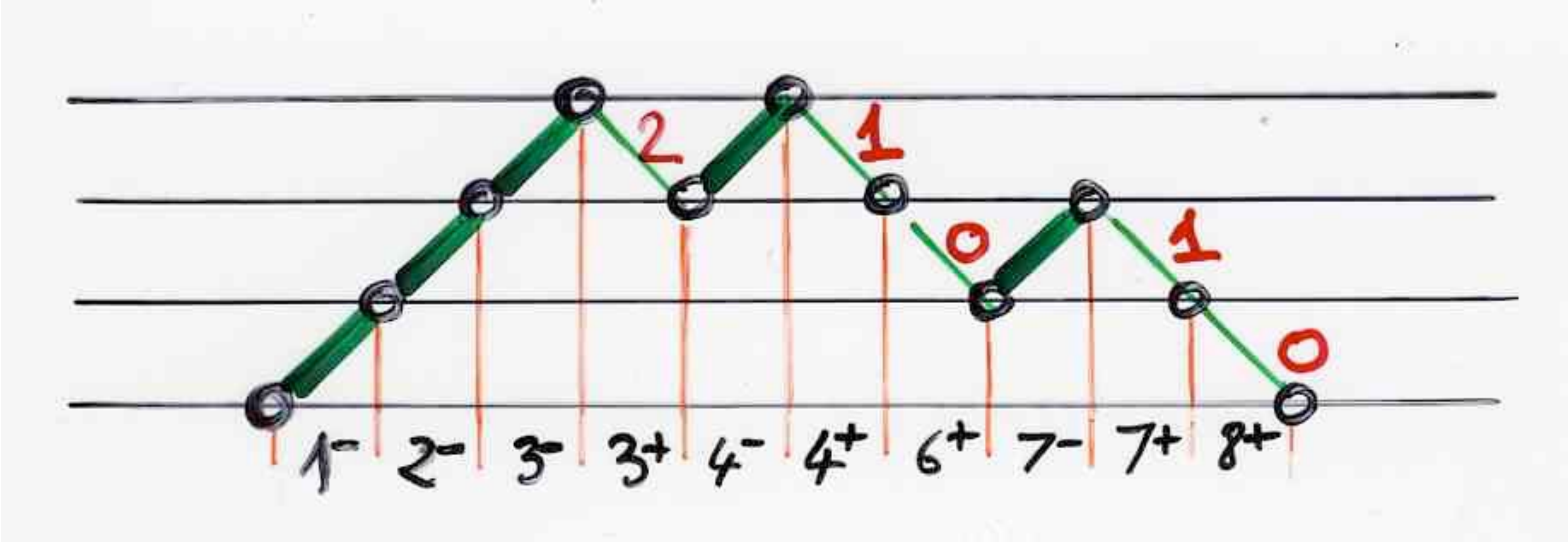
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

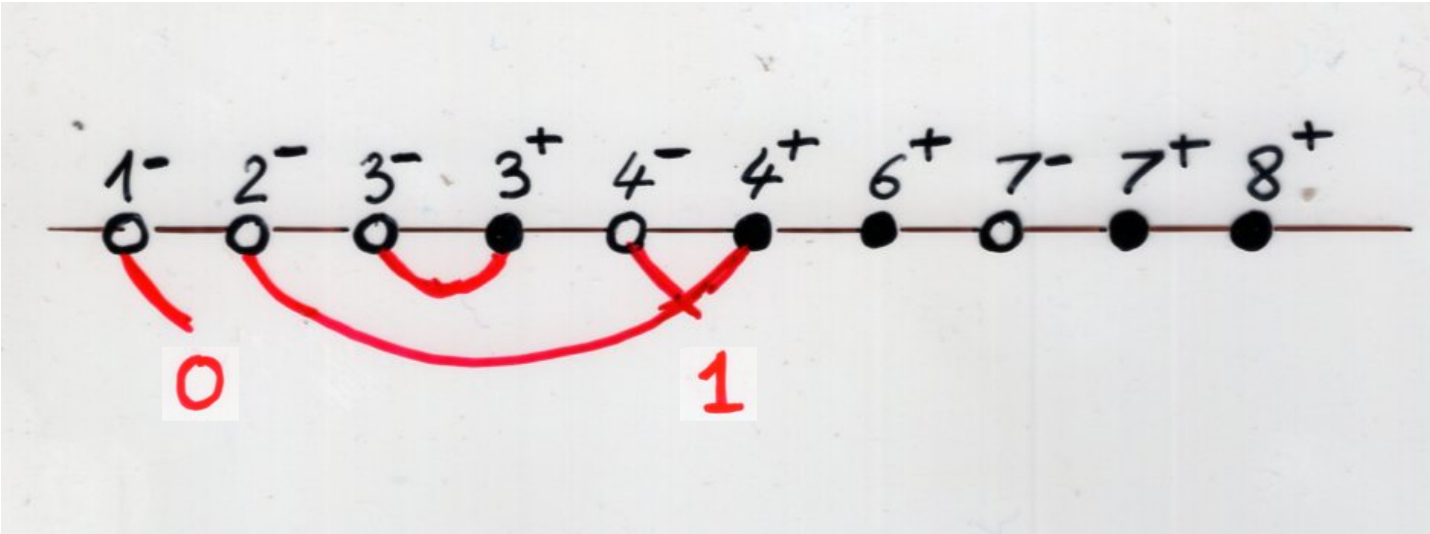

1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+


2
1
0
1
0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

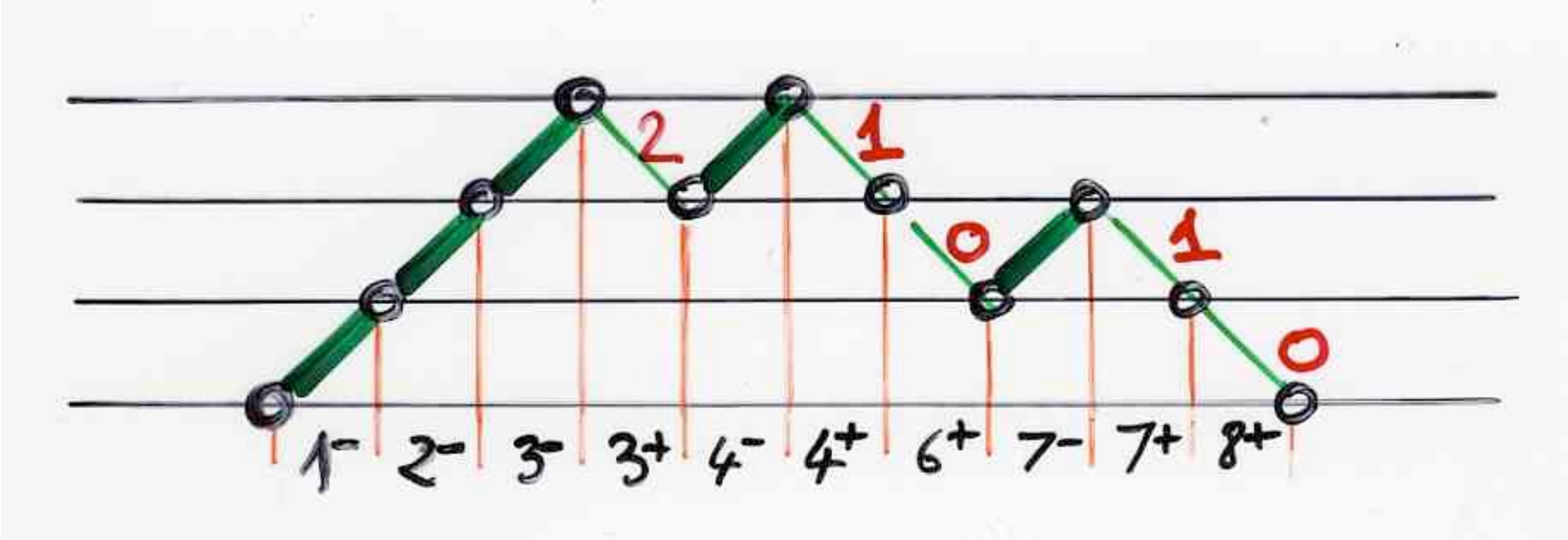

1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+
0 1 2

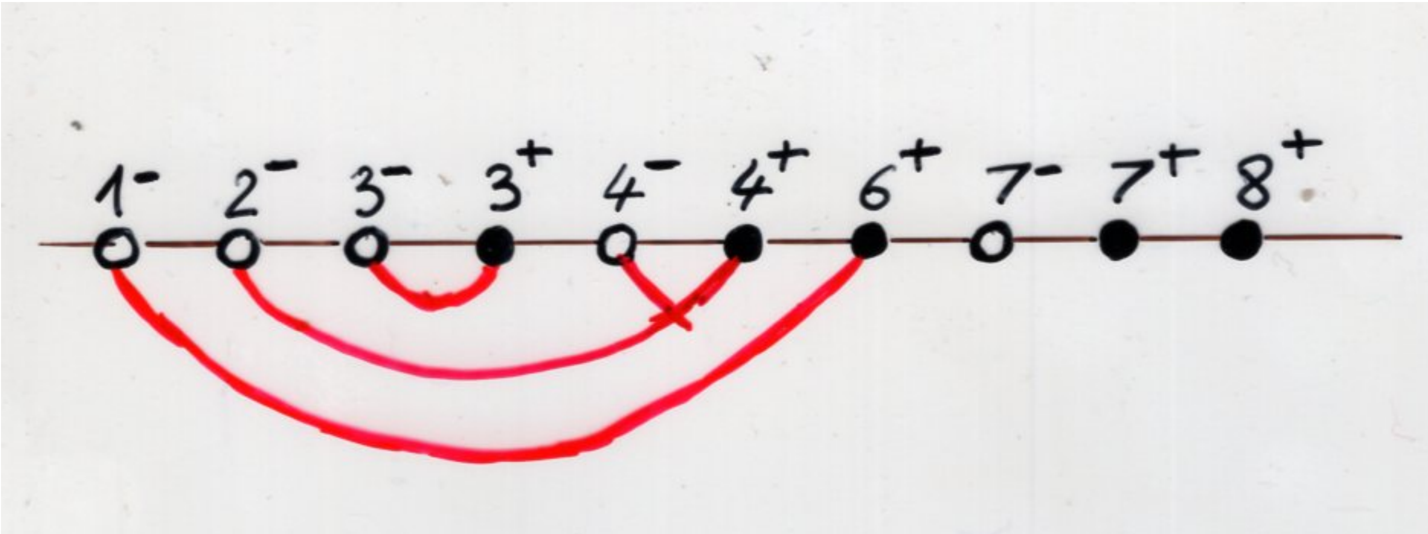


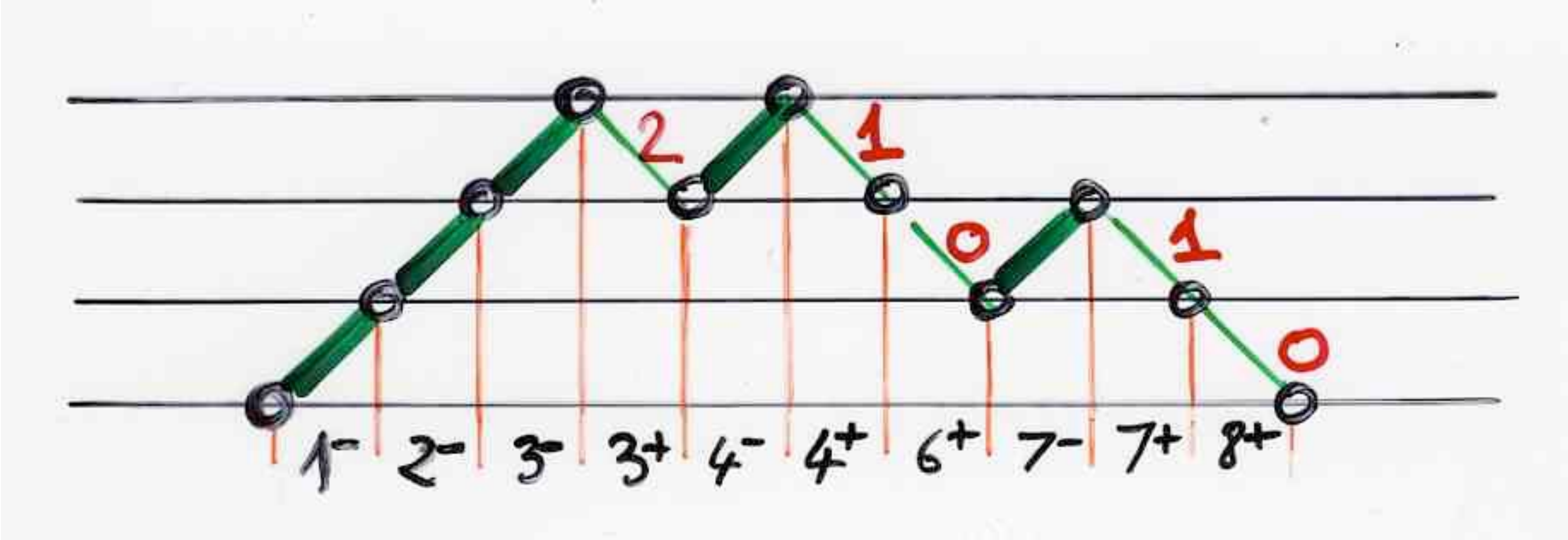
2
1
0
1
0
1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

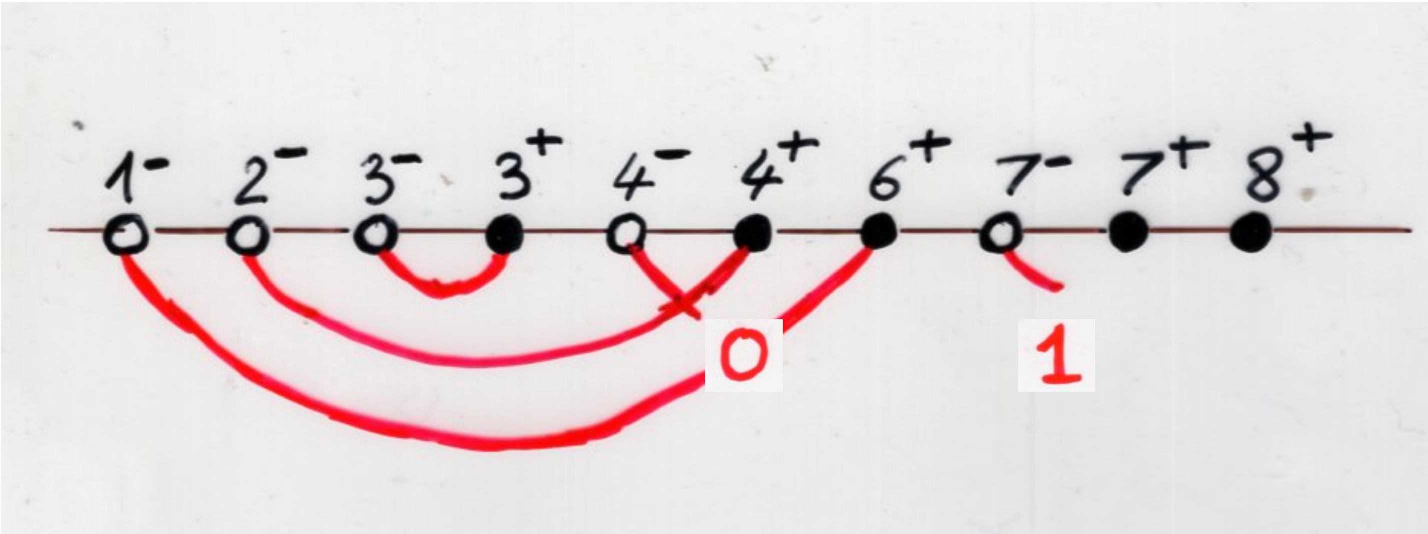


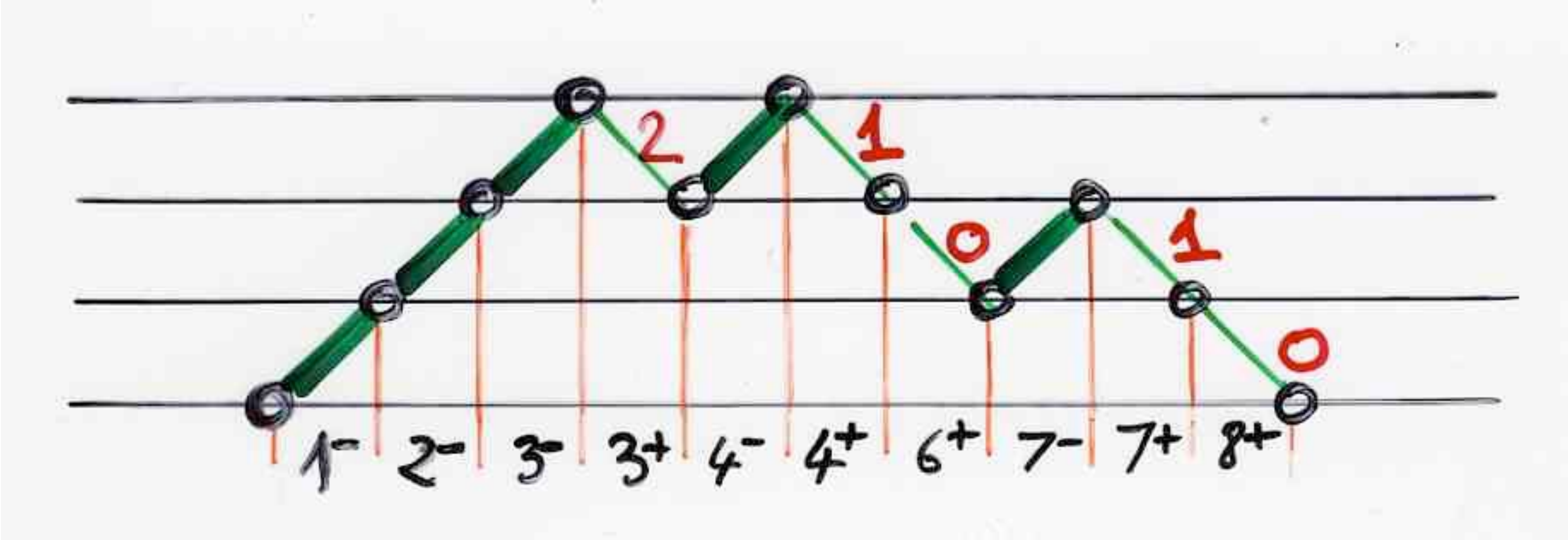
1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺
0
1

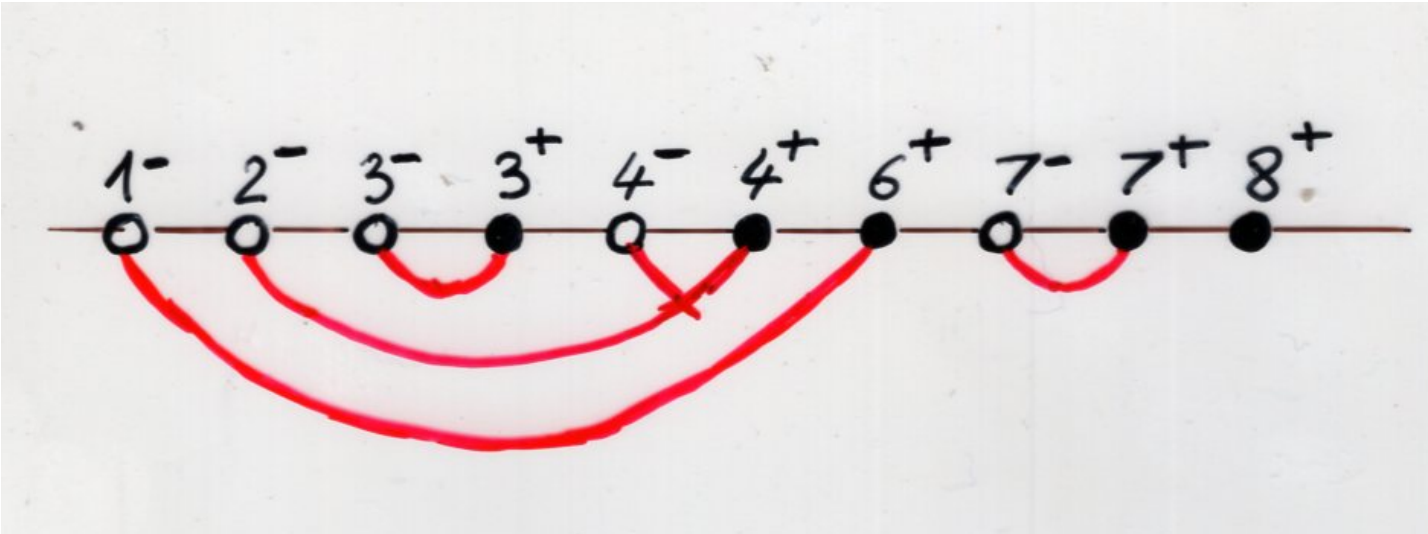

2
1
0
1
0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

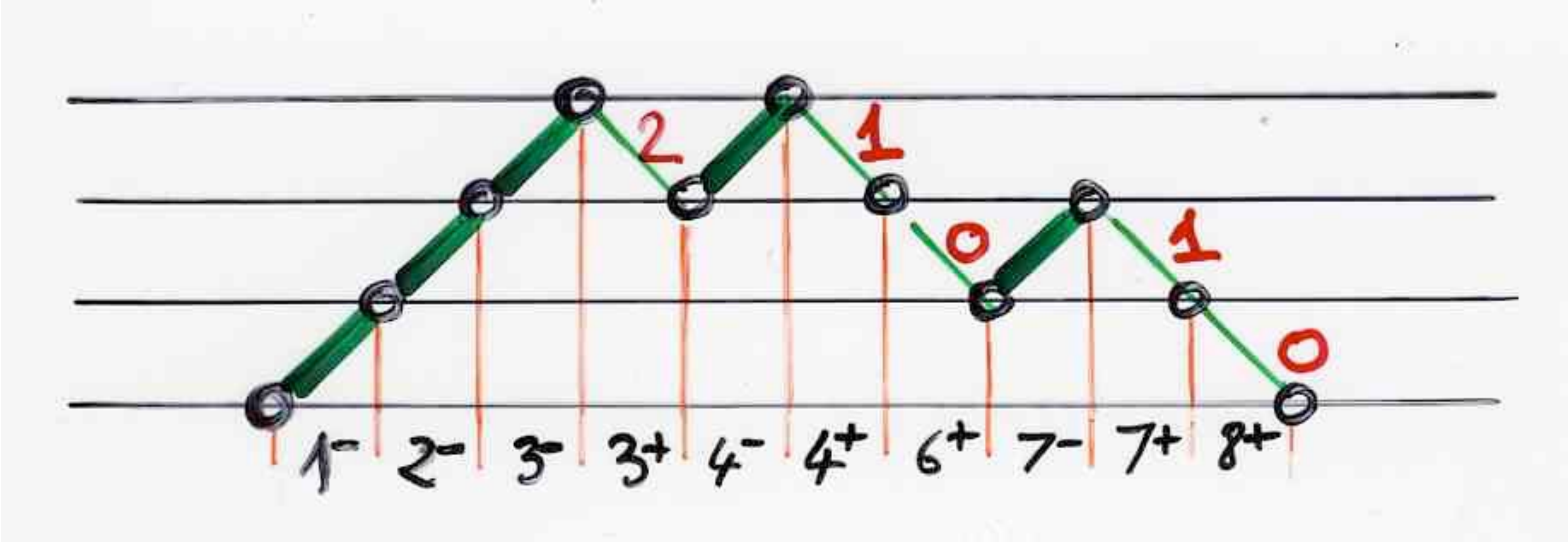

1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

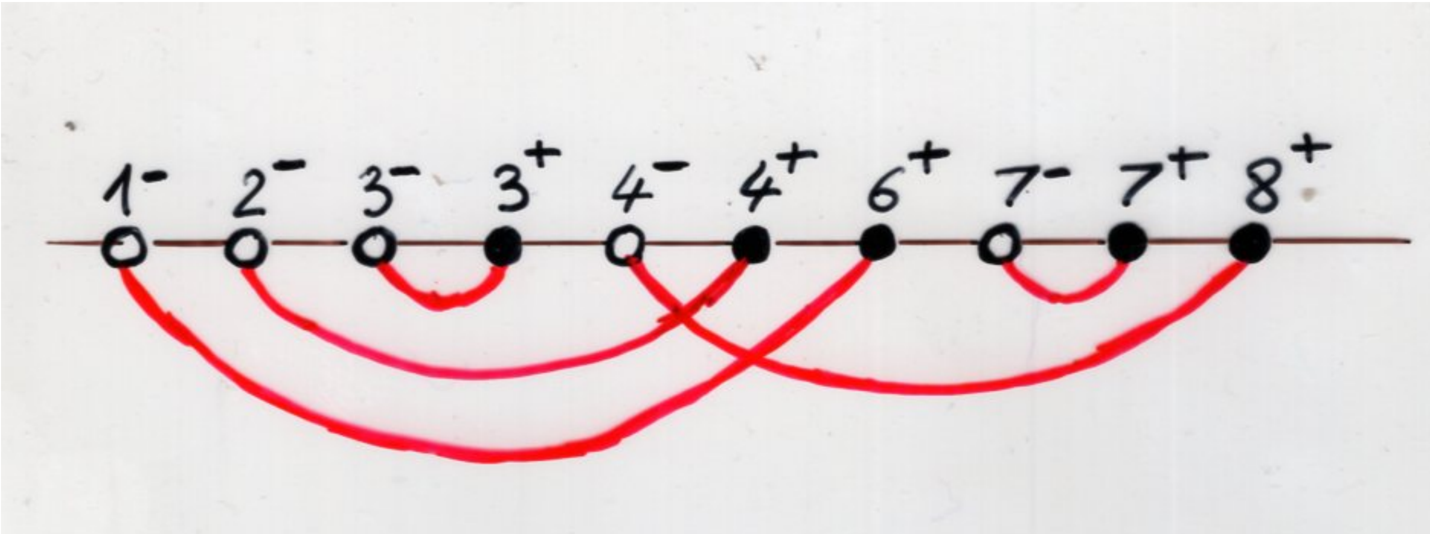

2
1
0
1
0
1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

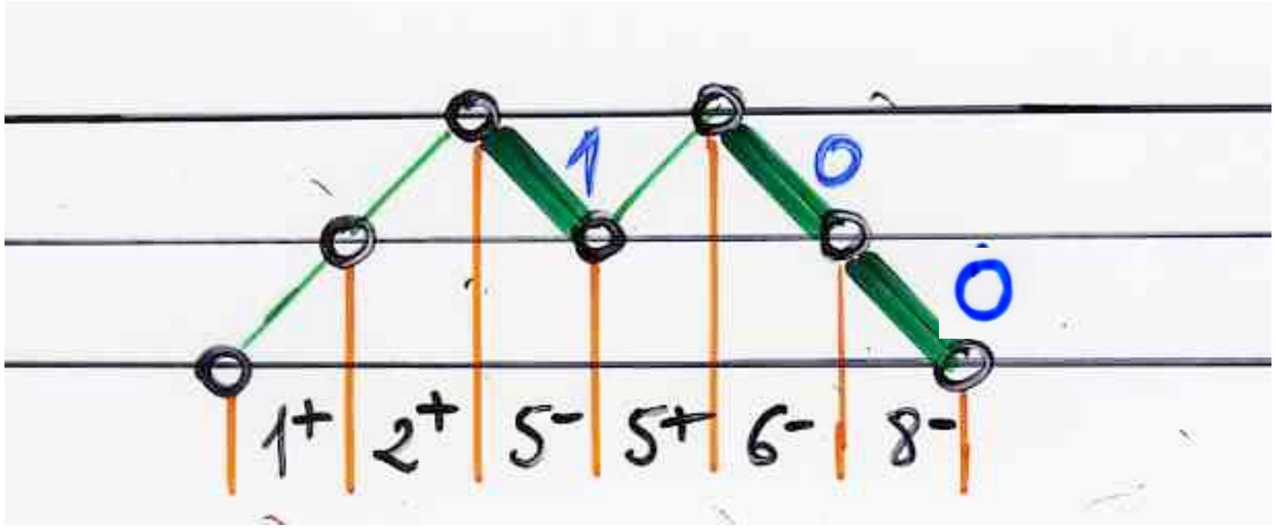

1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺
0
1

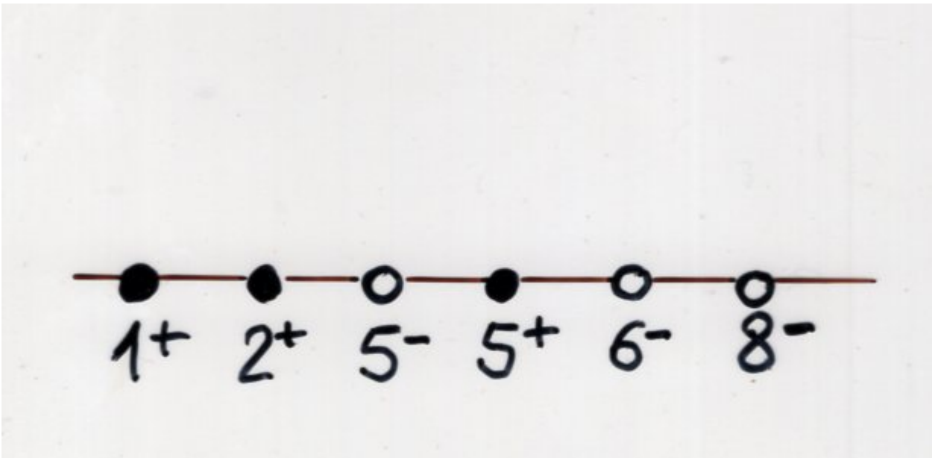

2
1
0
1
0
1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

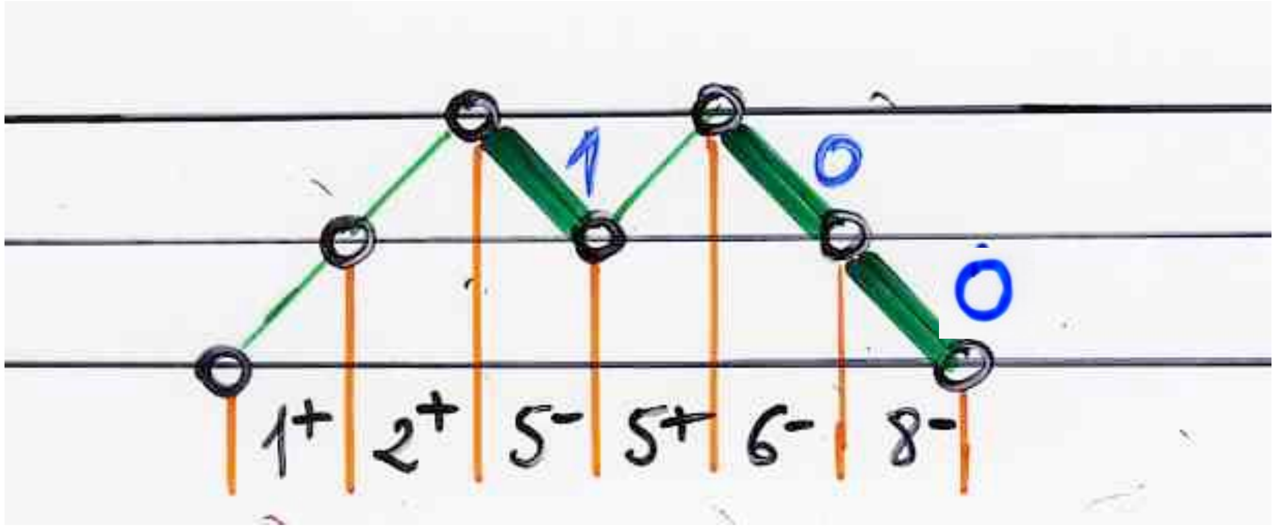

1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

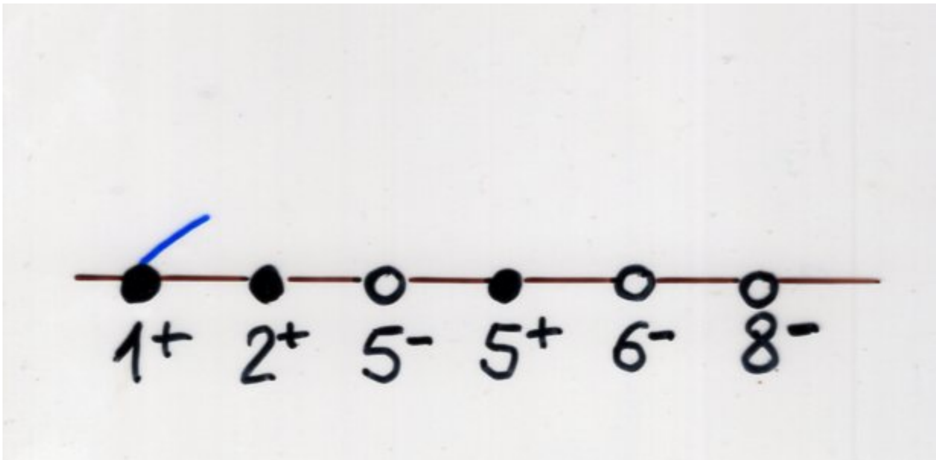

2
1
0
1
0
1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

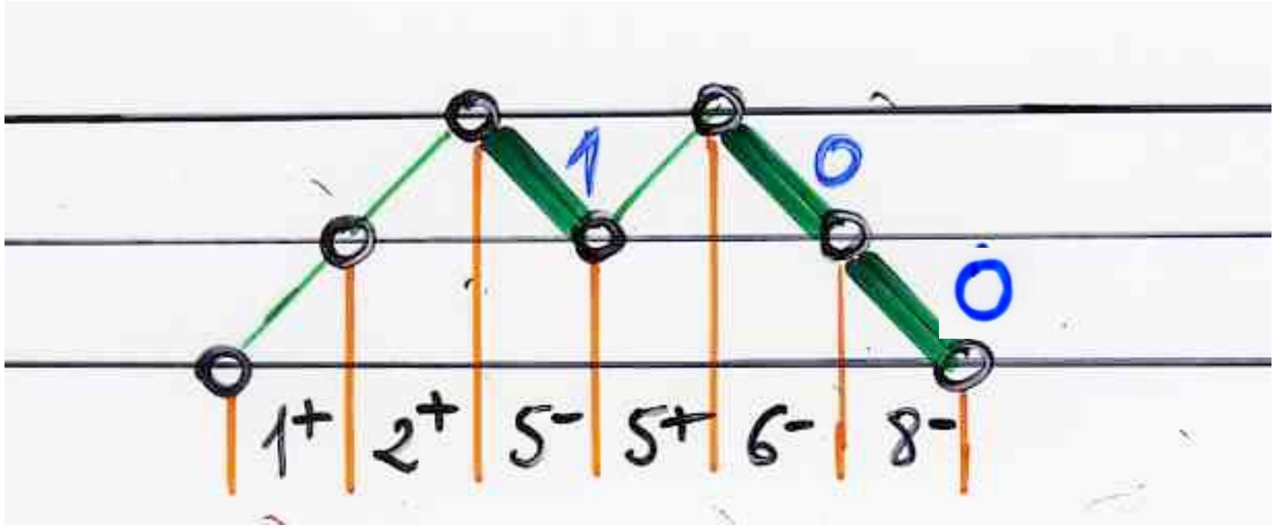

1⁻ 2⁻ 3⁻ 3⁺ 4⁻ 4⁺ 6⁺ 7⁻ 7⁺ 8⁺

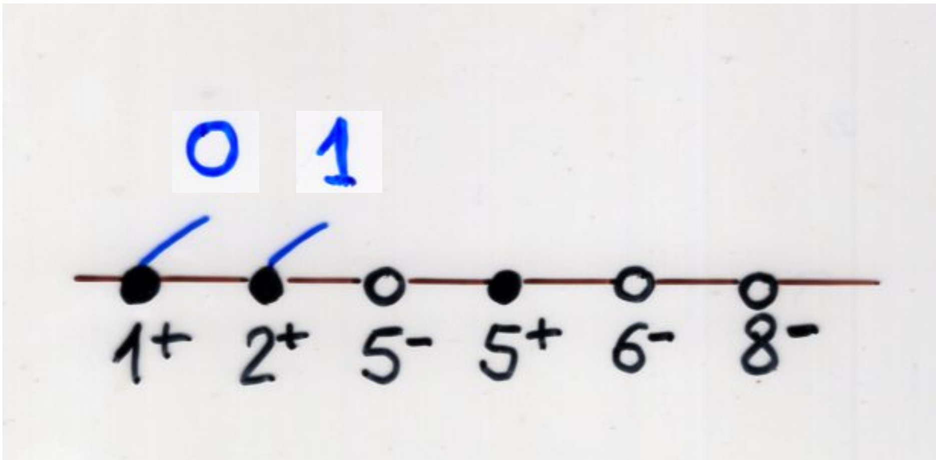

1
0
0
1+ 2+ 5- 5+ 6- 8-

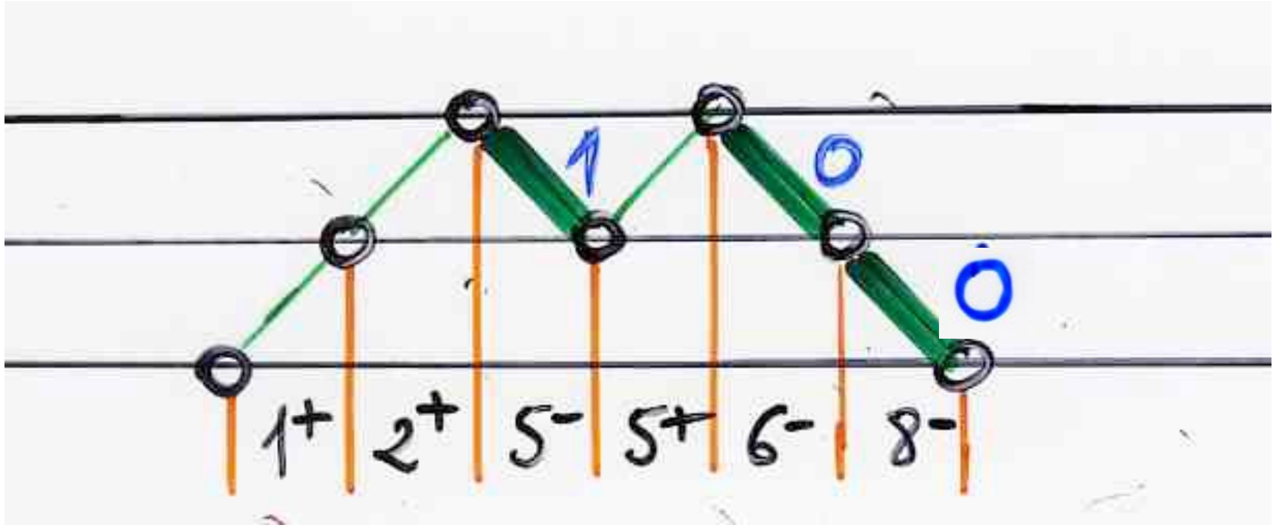

1+ 2+ 5- 5+ 6- 8-


1
0
0
1+ 2+ 5- 5+ 6- 8-

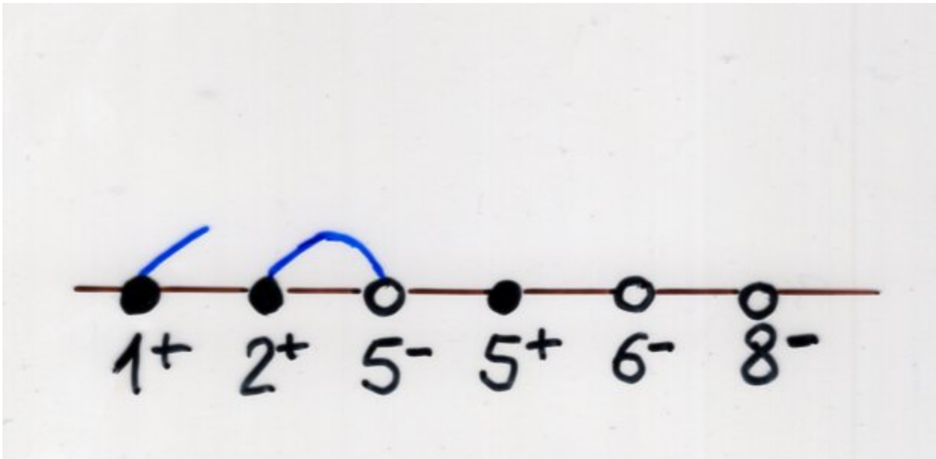

1+ 2+ 5- 5+ 6- 8-


1
0
0
1+ 2+ 5- 5+ 6- 8-

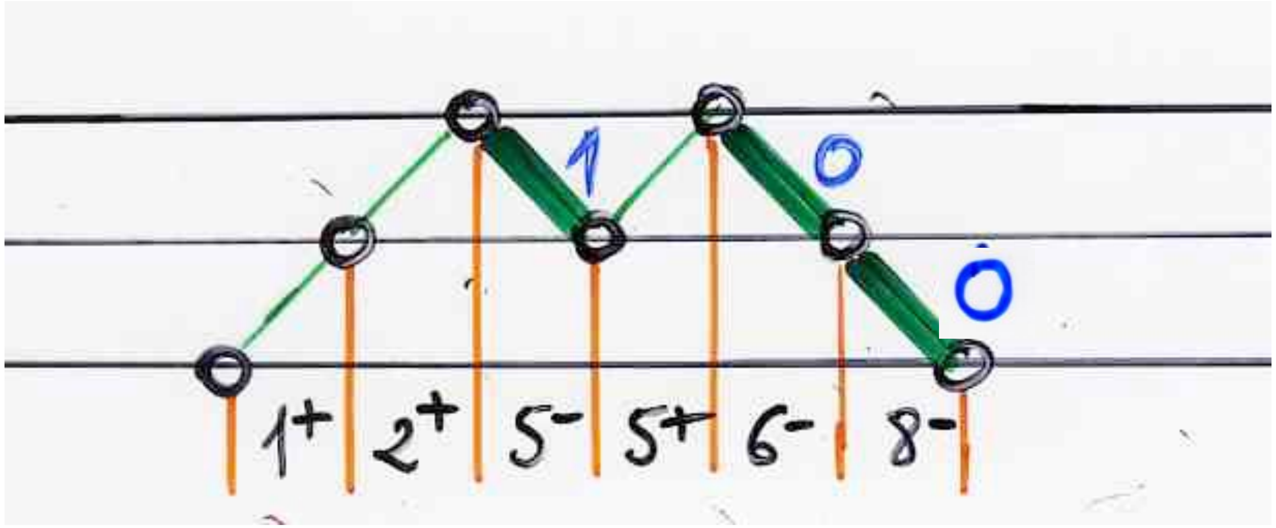

0 1
1+ 2+ 5- 5+ 6- 8-

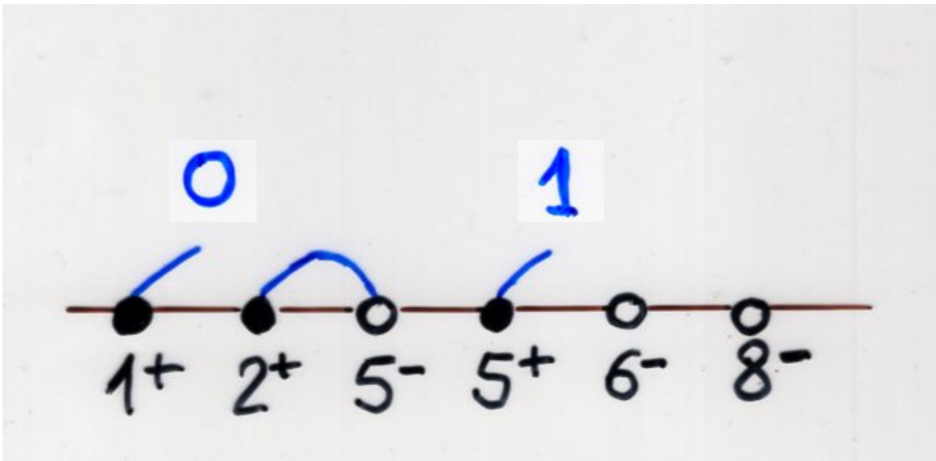


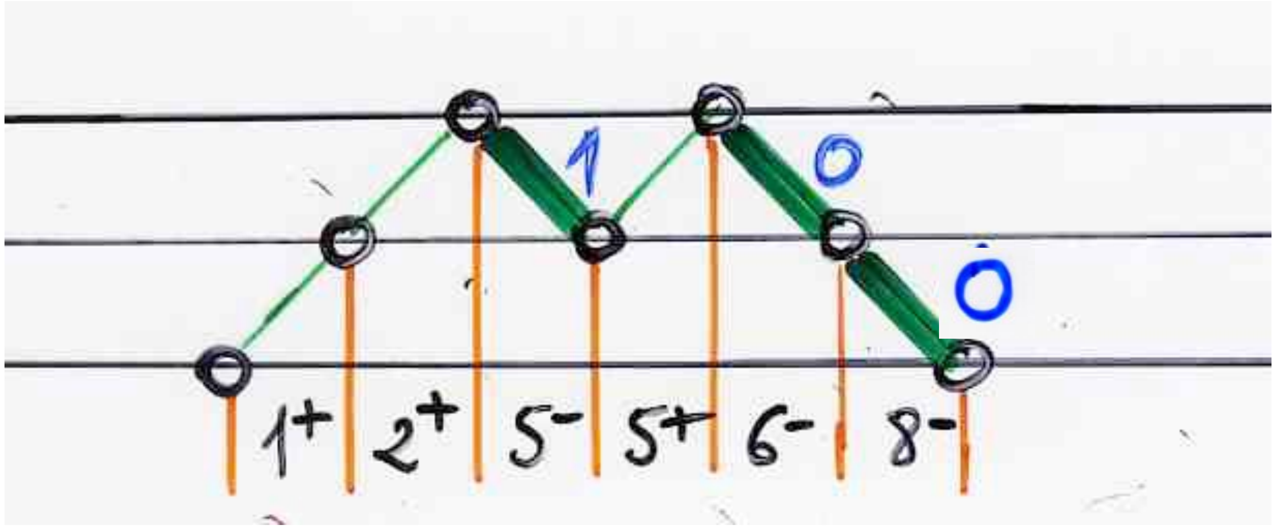
1
0
0
1+ 2+ 5- 5+ 6- 8-

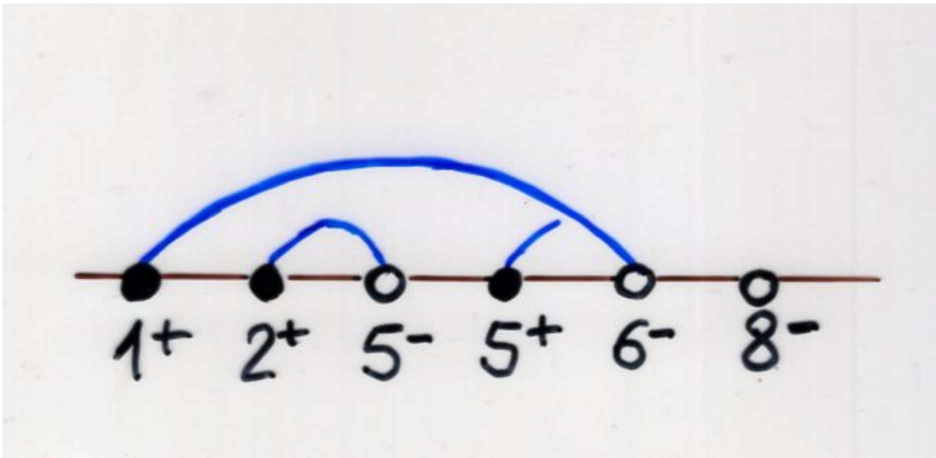


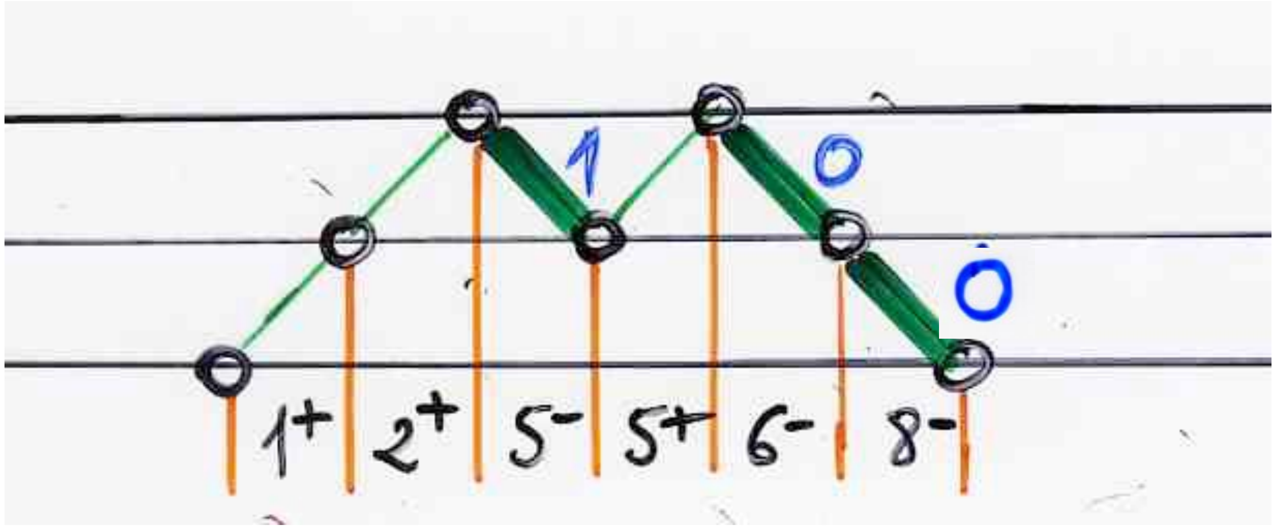
1+ 2+ 5- 5+ 6- 8-

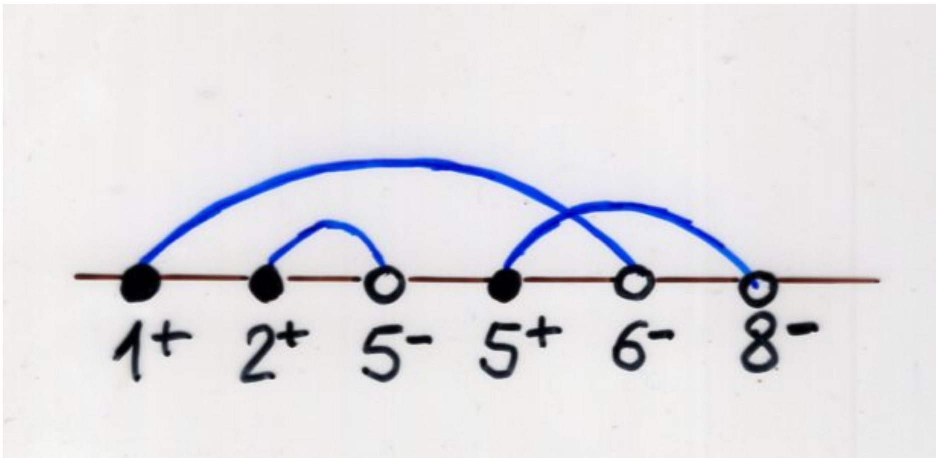

1
0
0
1+ 2+ 5- 5+ 6- 8-

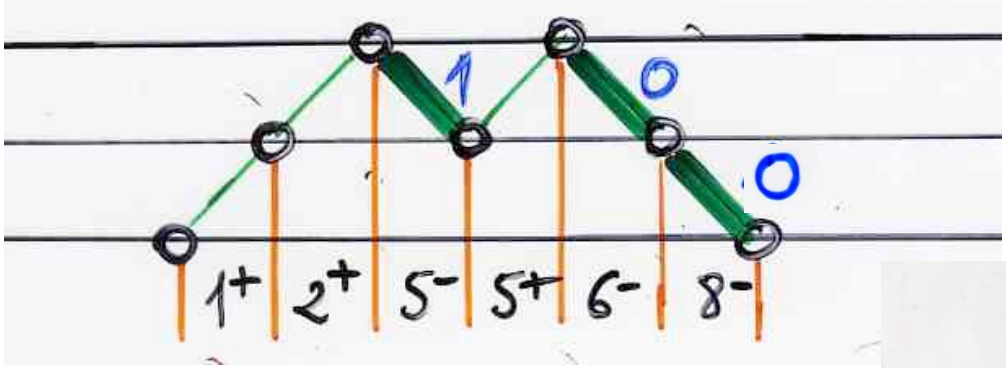

0
1
1+ 2+ 5- 5+ 6- 8-

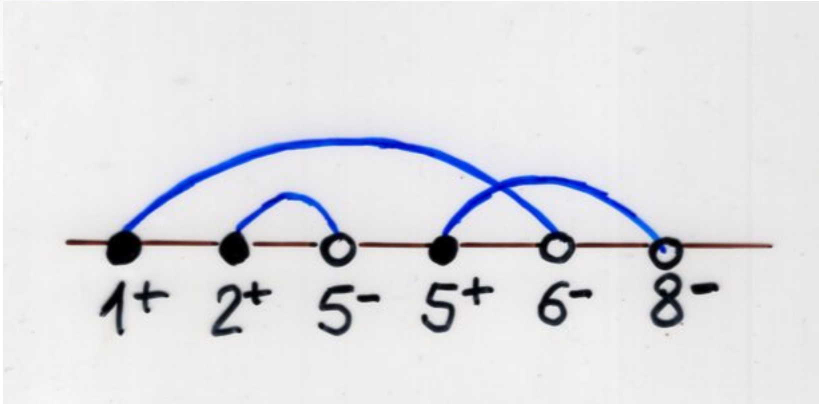

1
0
0
1+ 2+ 5- 5+ 6- 8-


1+ 2+ 5- 5+ 6- 8-

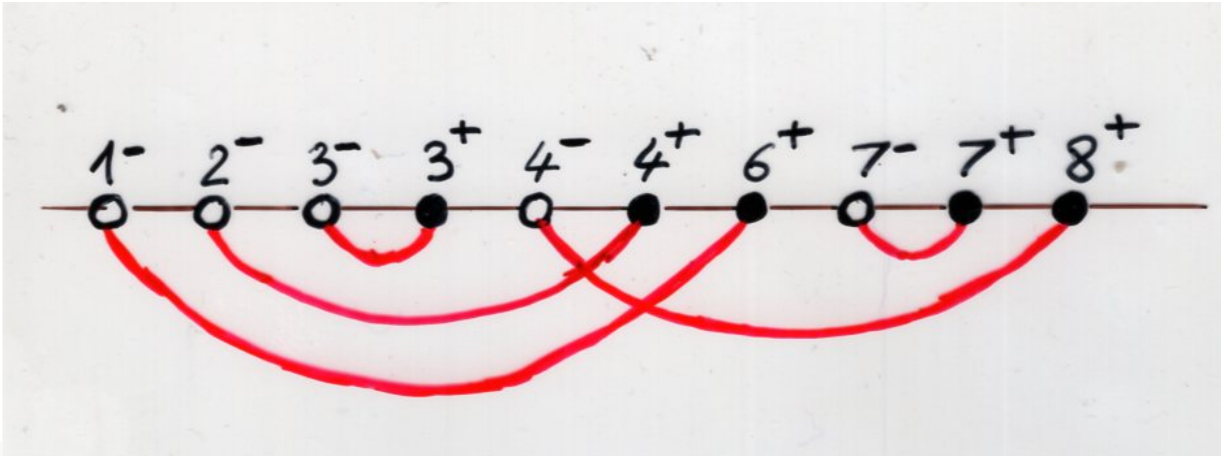

1
0
0
1+ 2+ 5- 5+ 6- 8-

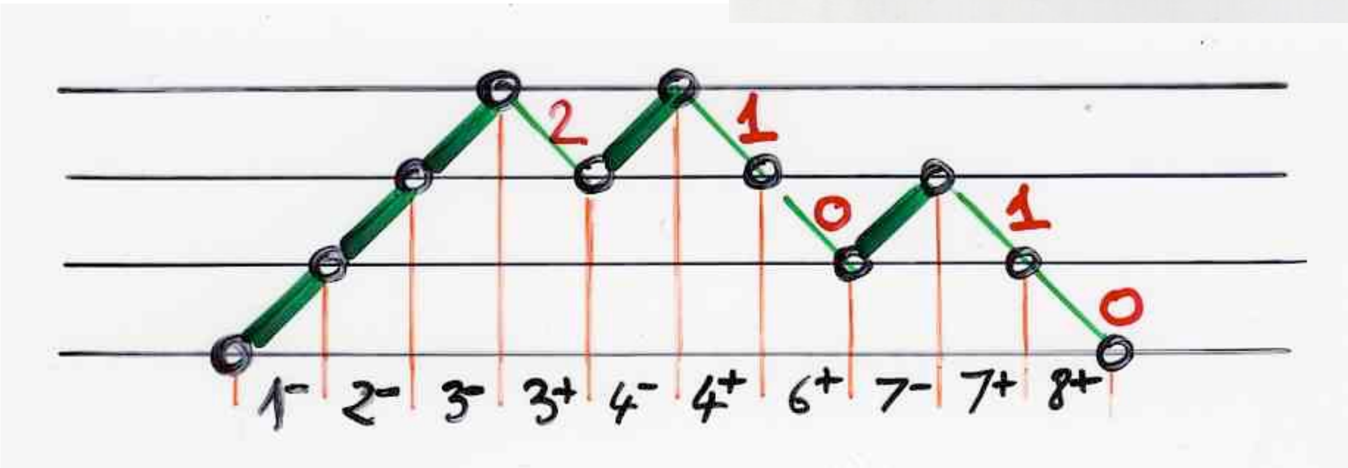

1+ 2+ 5- 5+ 6- 8-

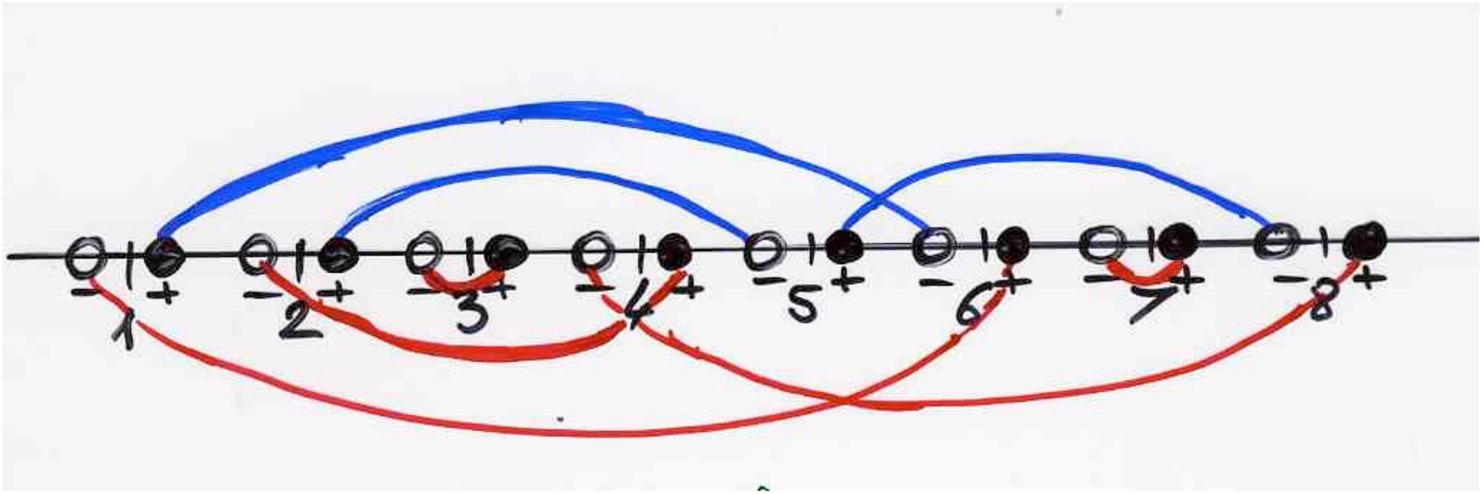




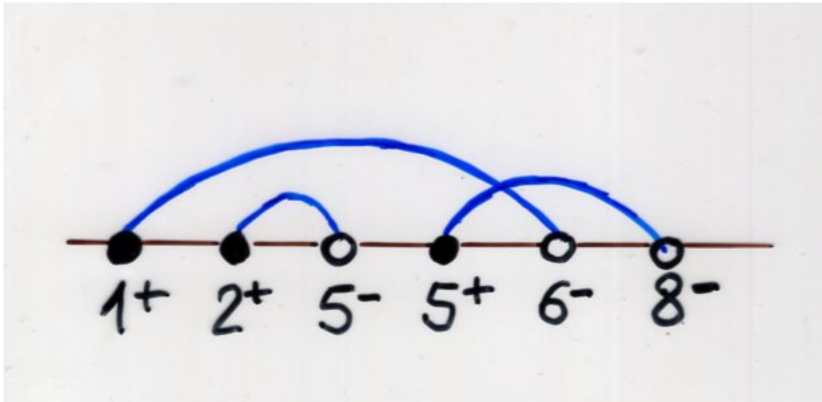
pair of two
Hermite histories
("shuffle")



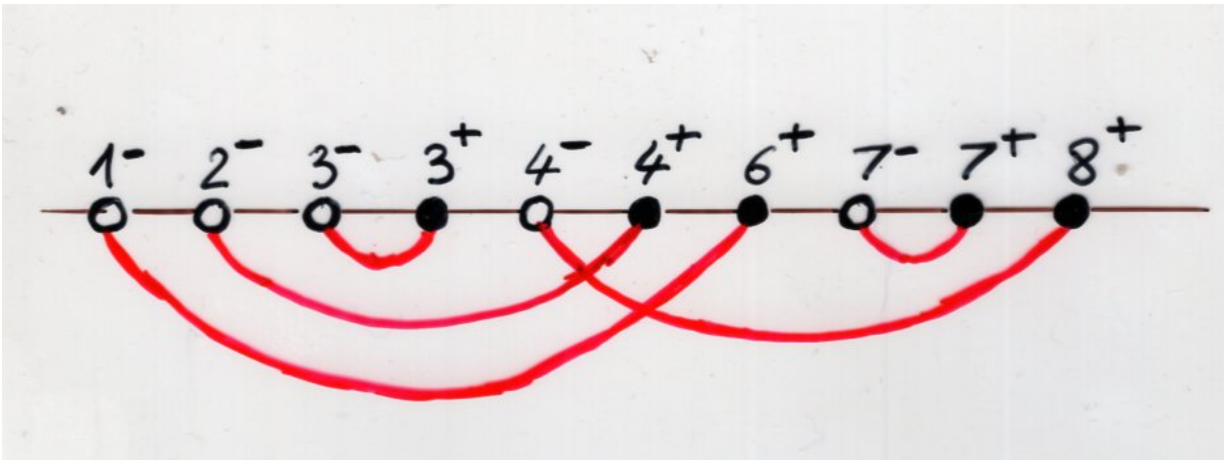




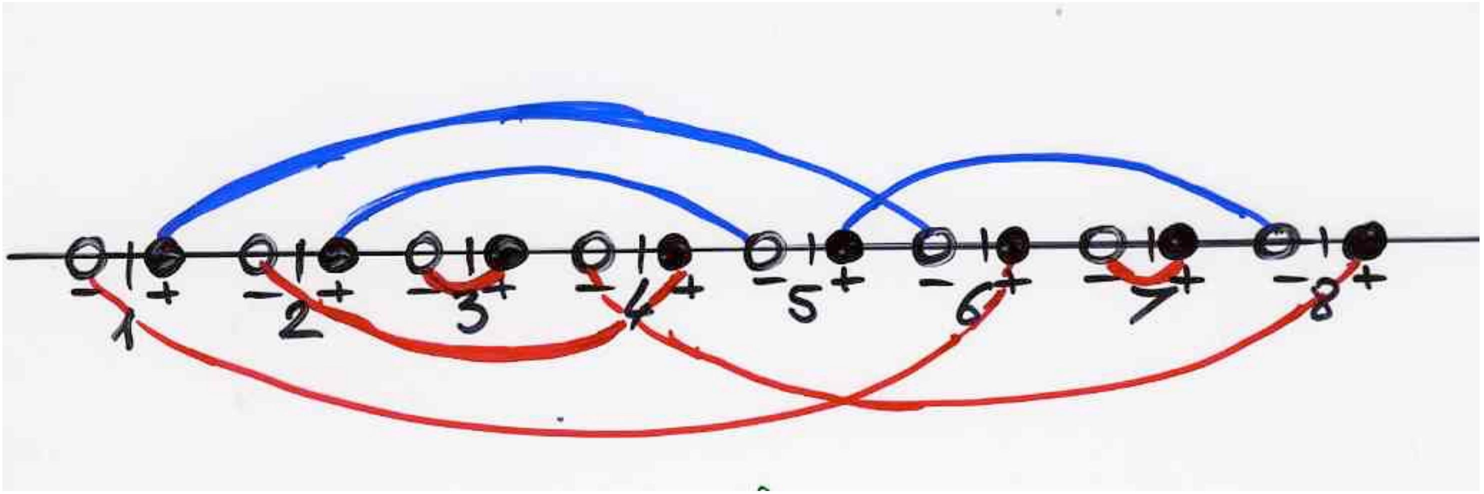
1
2
3
4
5
6
7
8

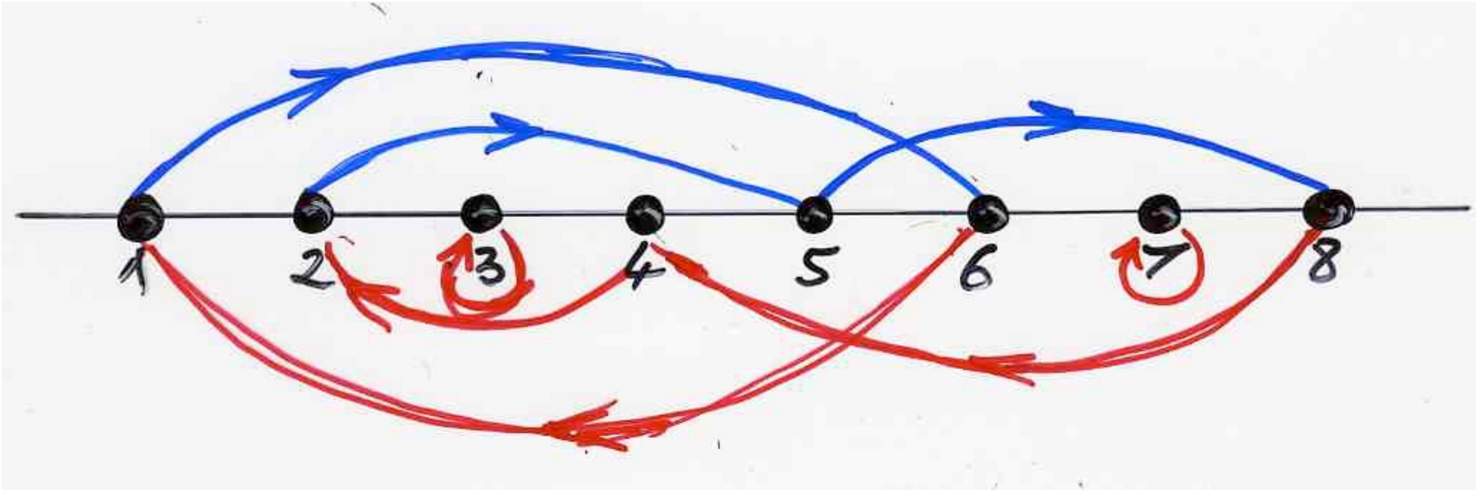


1+ 2+ 5- 5+ 6- 8-



1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+





$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

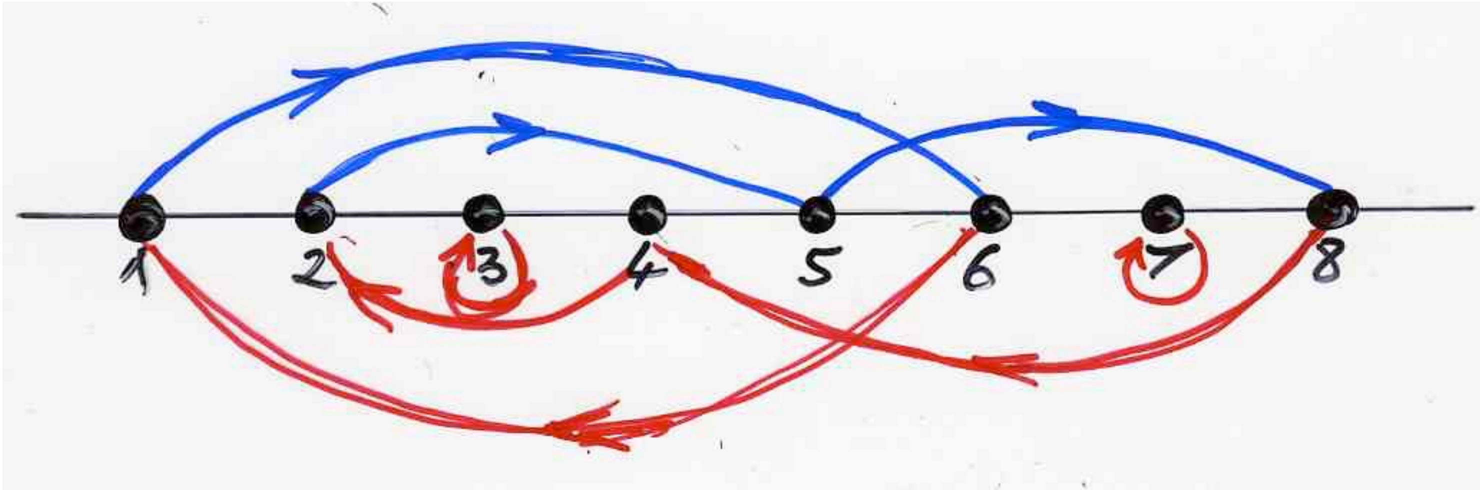
reverse bijection:

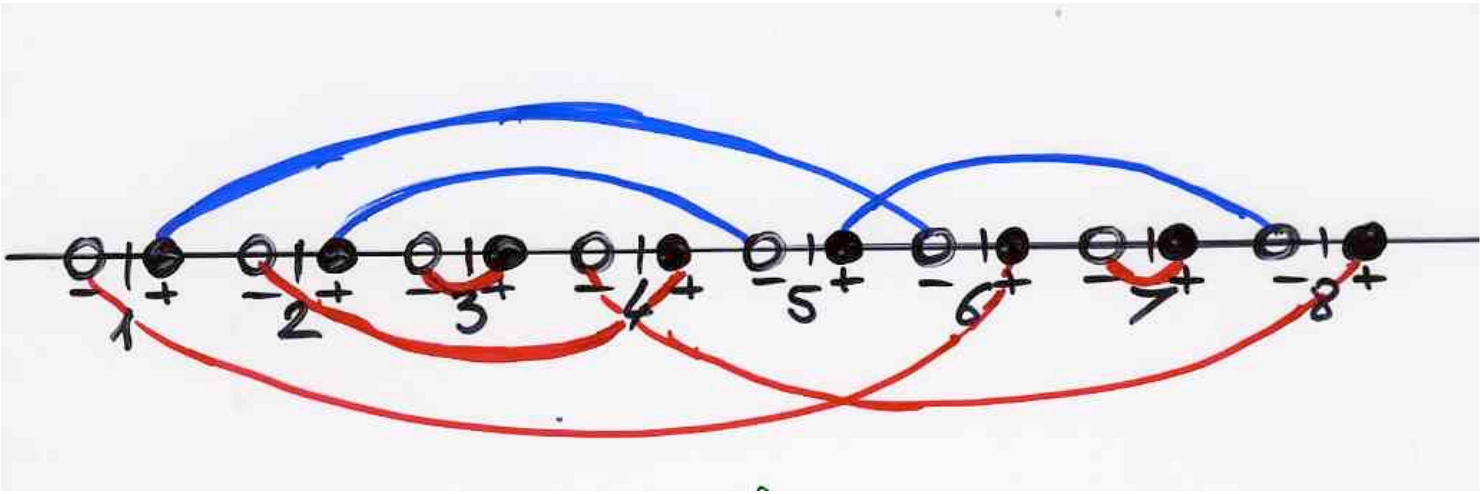
from permutations

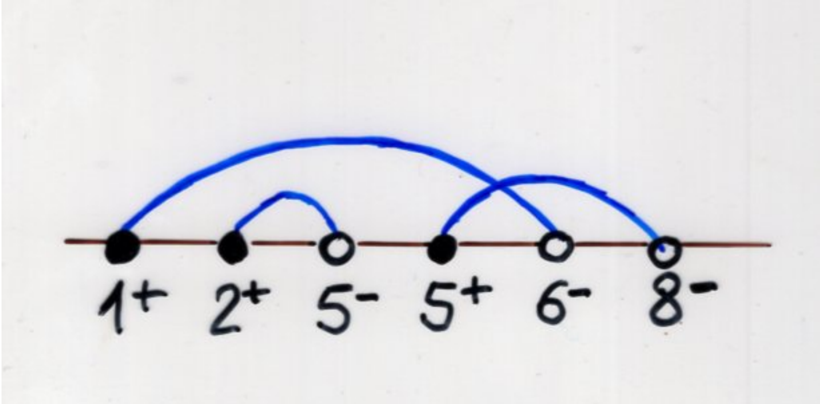
to

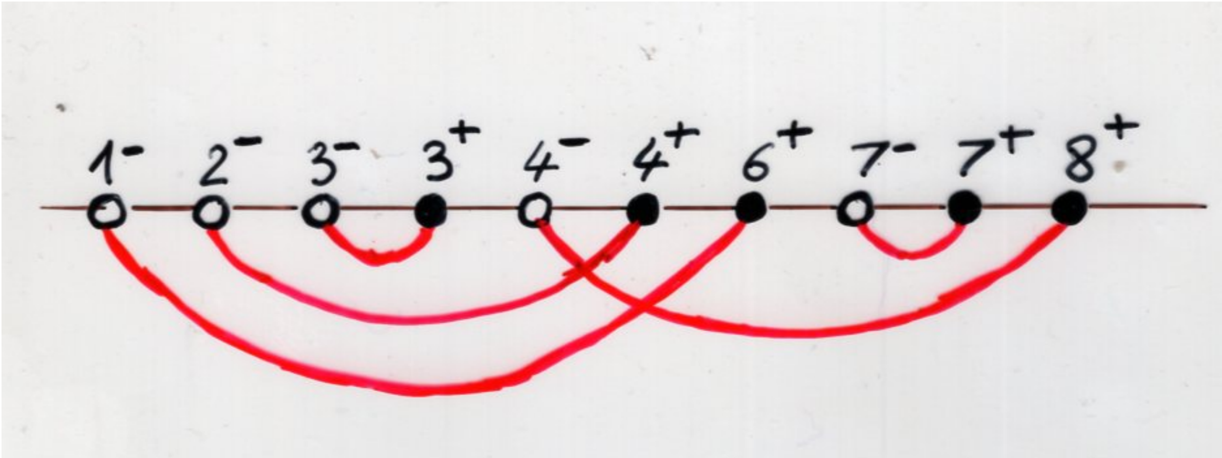
subdivided Laguerre histories

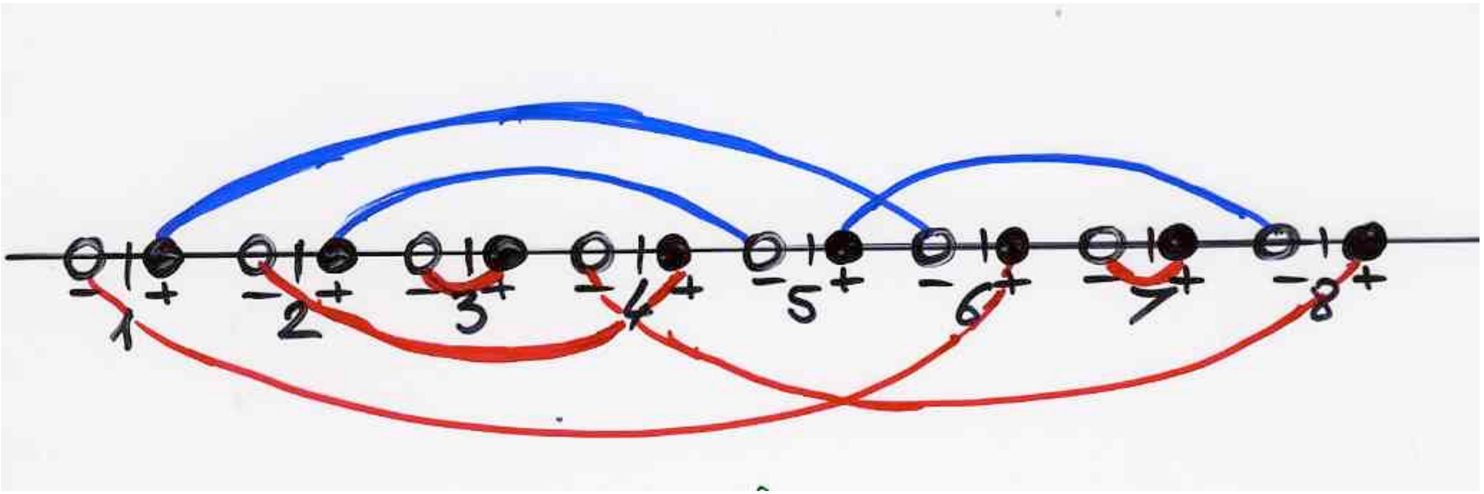
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

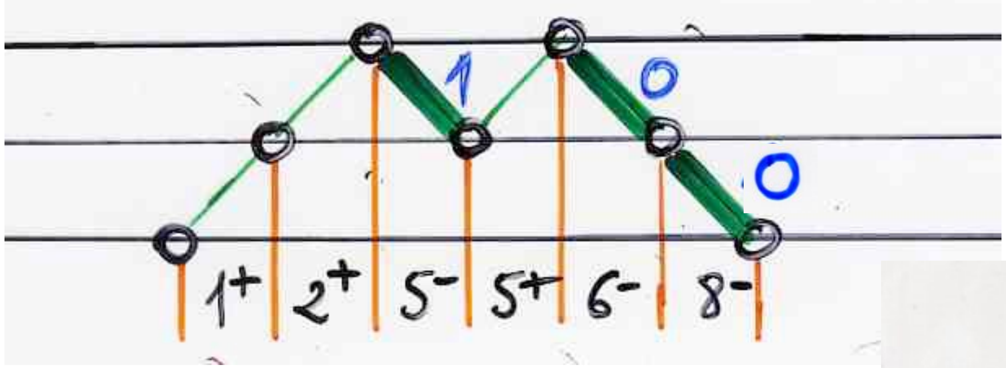


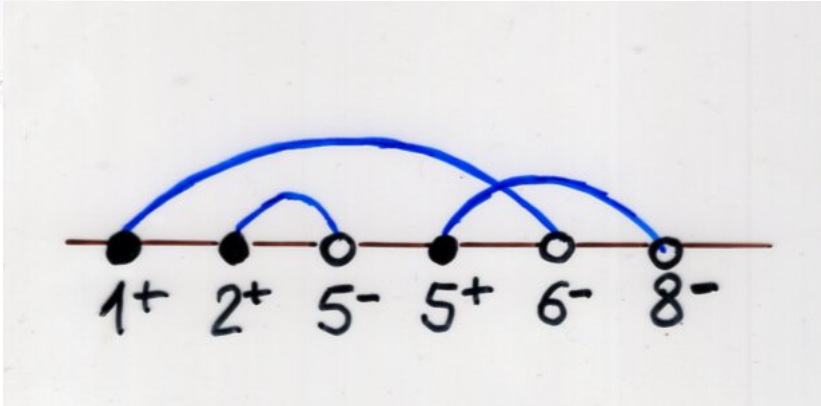



1+
2+
5-
5+
6-
8-

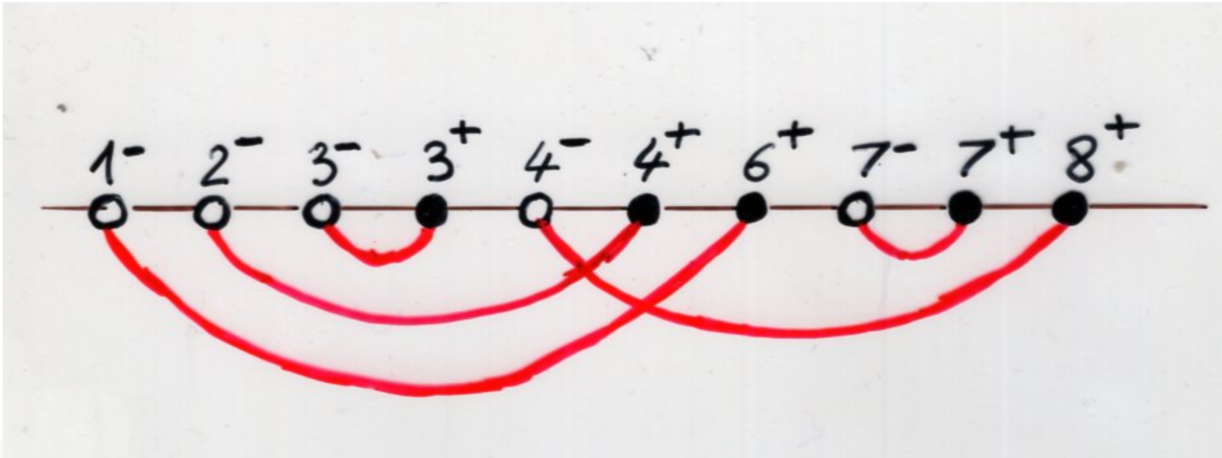

1-
2-
3-
3+
4-
4+
6+
7-
7+
8+

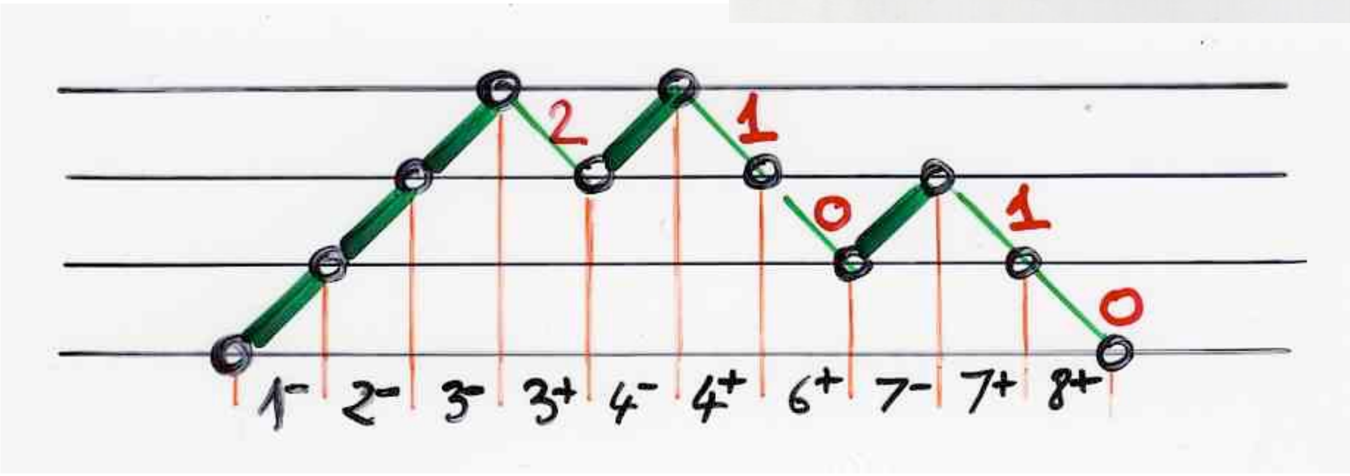

− 1 +
− 2 +
− 3 +
− 4 +
− 5 +
− 6 +
− 7 +
− 8 +

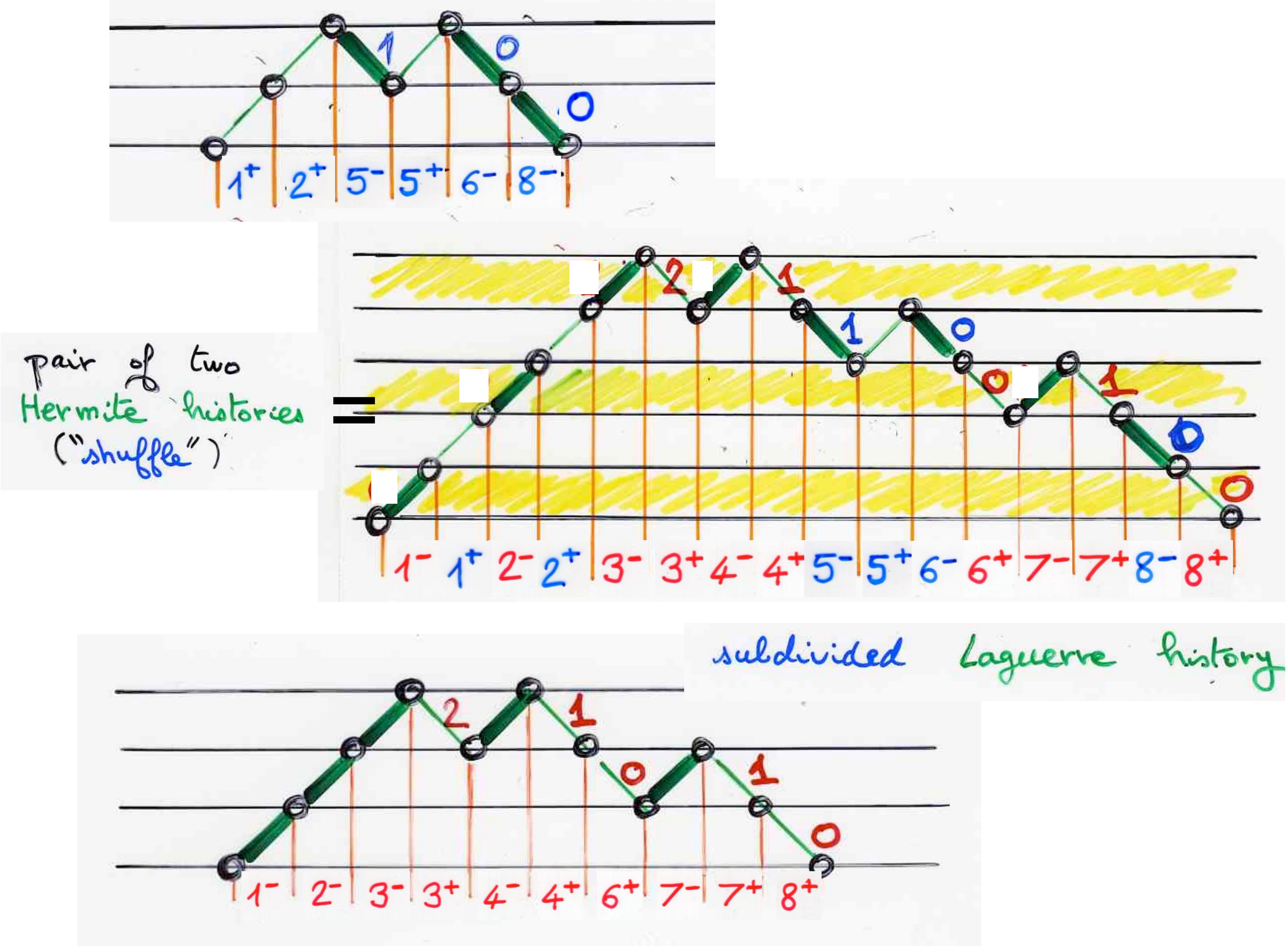




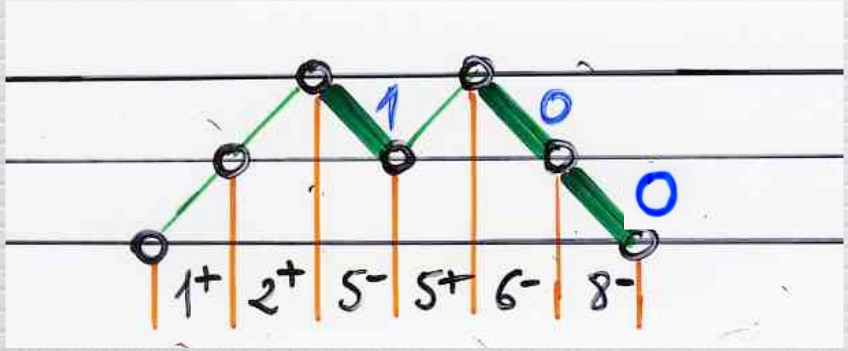
pair of two
Hermite histories
("shuffle")



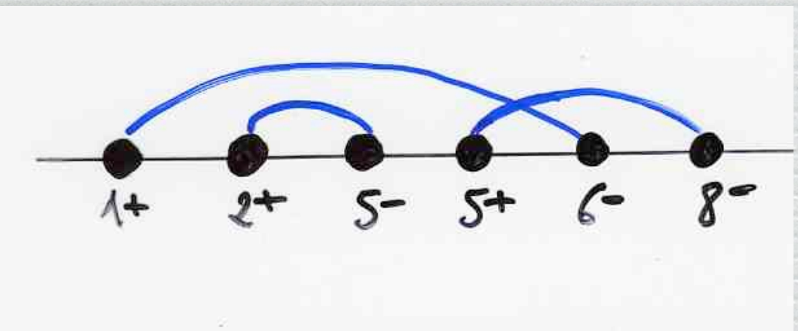


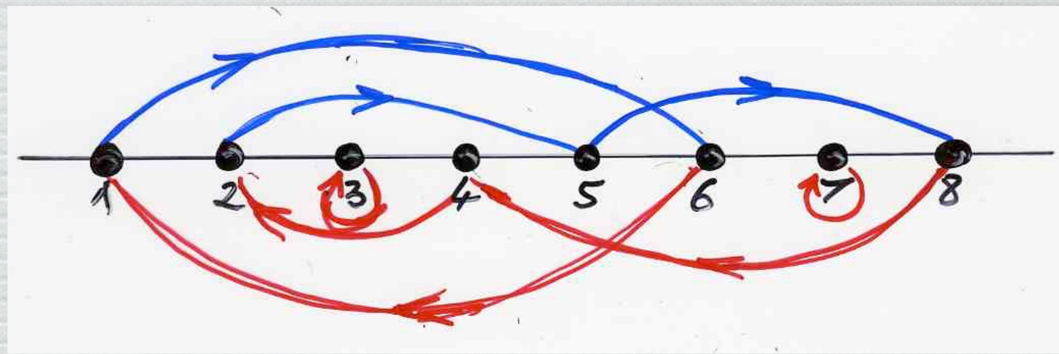


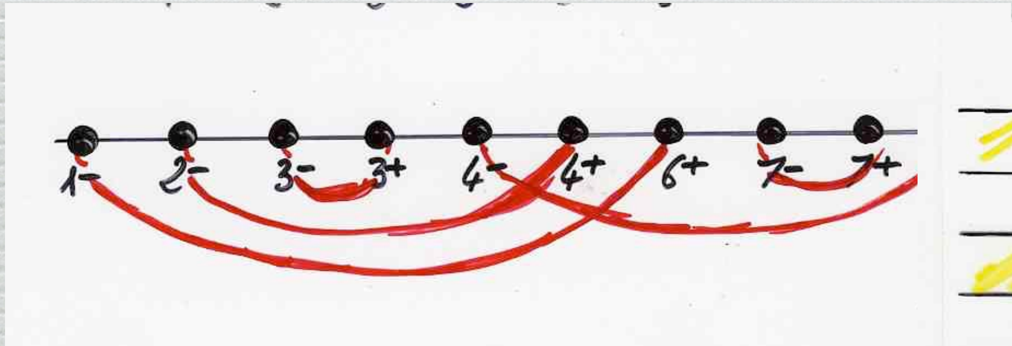
1+ 2+ 5- 5+ 6- 8-
pair of two Hermite histories ("shuffle")
=
1- 1+ 2- 2+ 3- 3+ 4- 4+ 5- 5+ 6- 6+ 7- 7+ 8- 8+
subdivided Laguerre history
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

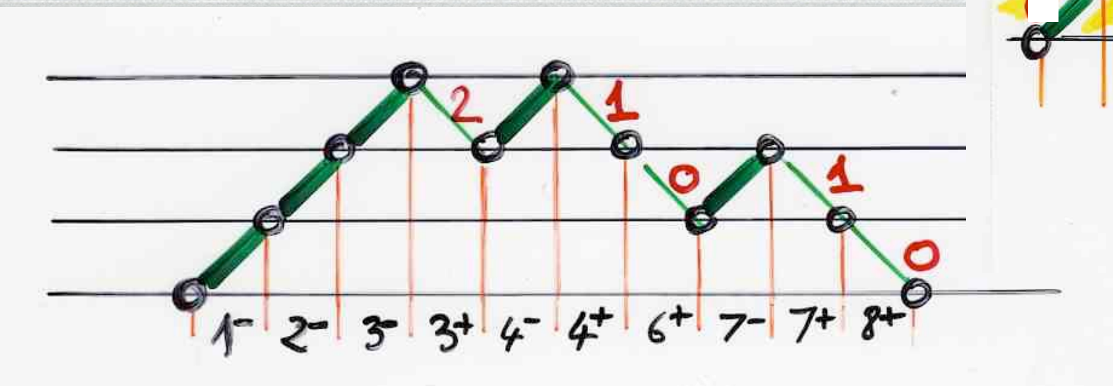


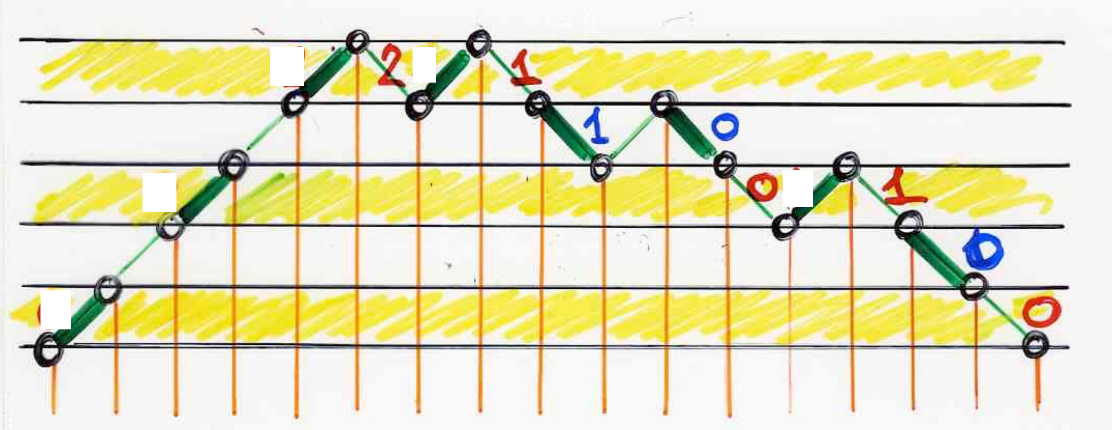
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$











subdivided Laguerre history

Contraction of continued fractions

Contraction of continued fraction

$$S(t;\gamma) = \cfrac{1}{1-\cfrac{\gamma_1 t}{1-\cfrac{\gamma_2 t}{\cdots \cfrac{}{1-\cfrac{\gamma_k t}{\cdots}}}}}$$

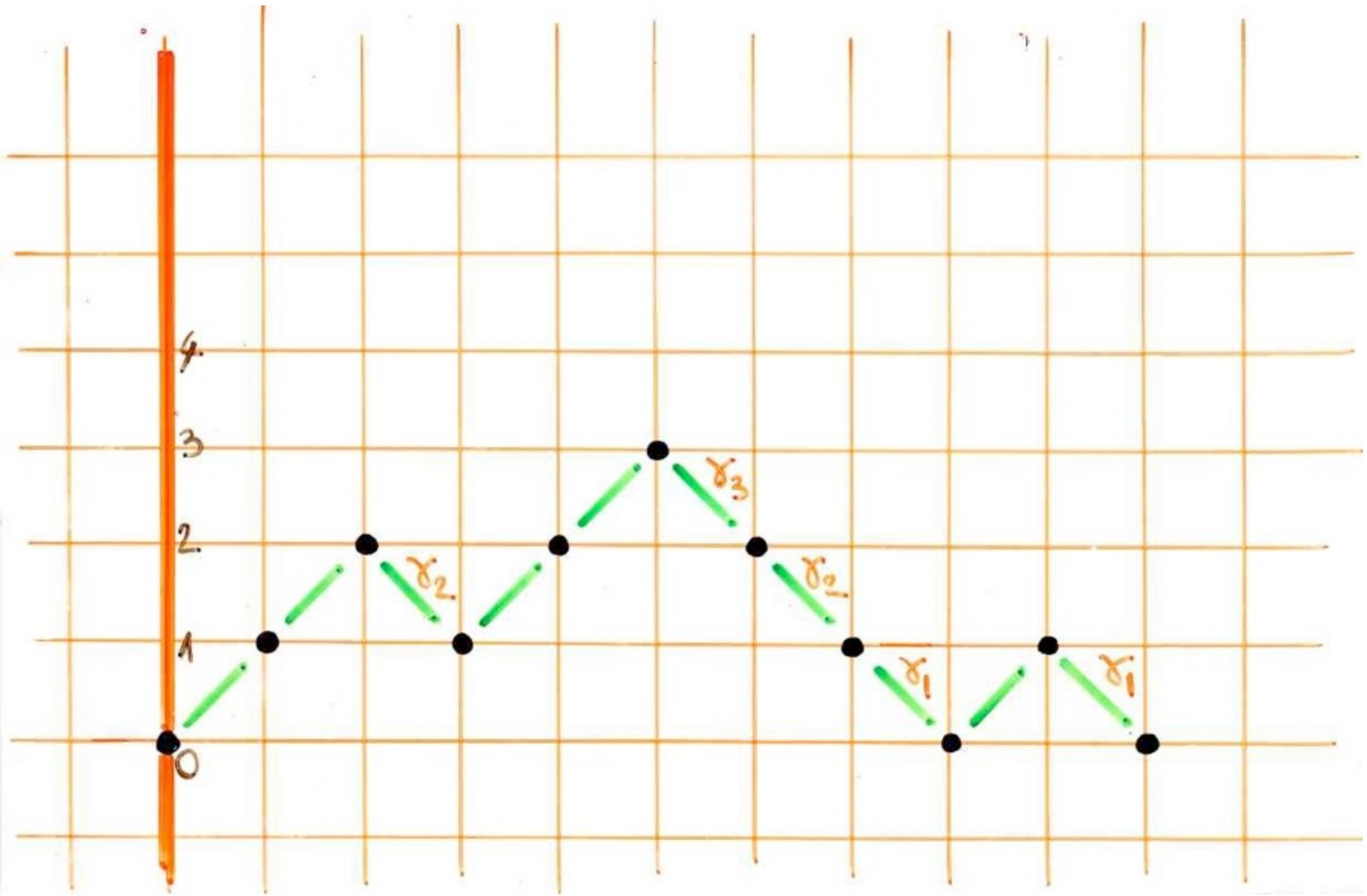
$$\gamma = \{\gamma_k\}_{k\geq 1}$$

Jacobi

$$J(t;b,\lambda) = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{1-b_1 t-\cfrac{\lambda_2 t^2}{\cdots \cfrac{}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}}$$

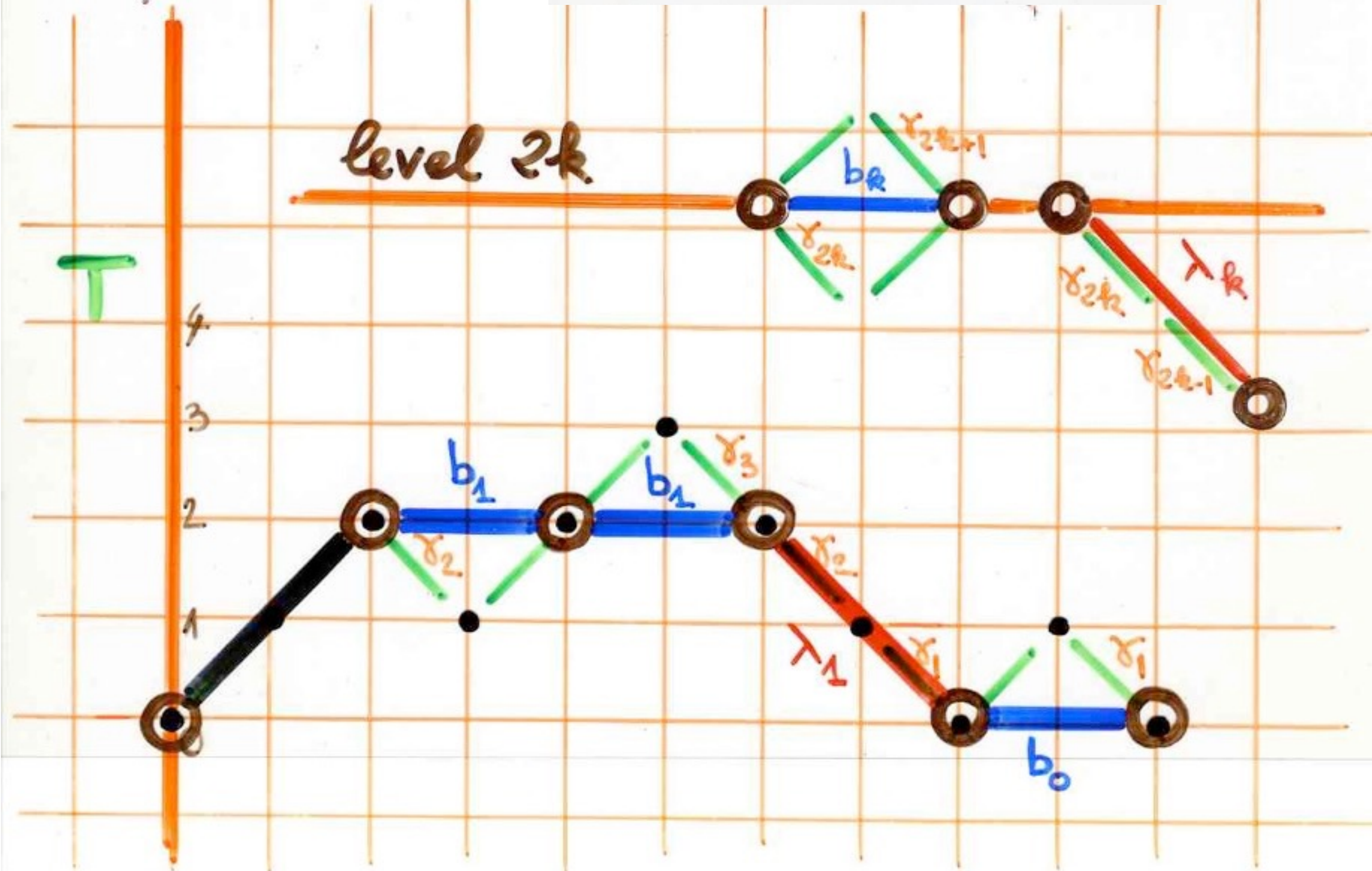
$$b = \{b_k\}_{k\geq 0} \qquad \lambda = \{\lambda_k\}_{k\geq 1}$$

$$S(t;\gamma) = J(t;b,\lambda)$$


4
3
2
1
0
γ2
γ3
γ2
γ1
γ1

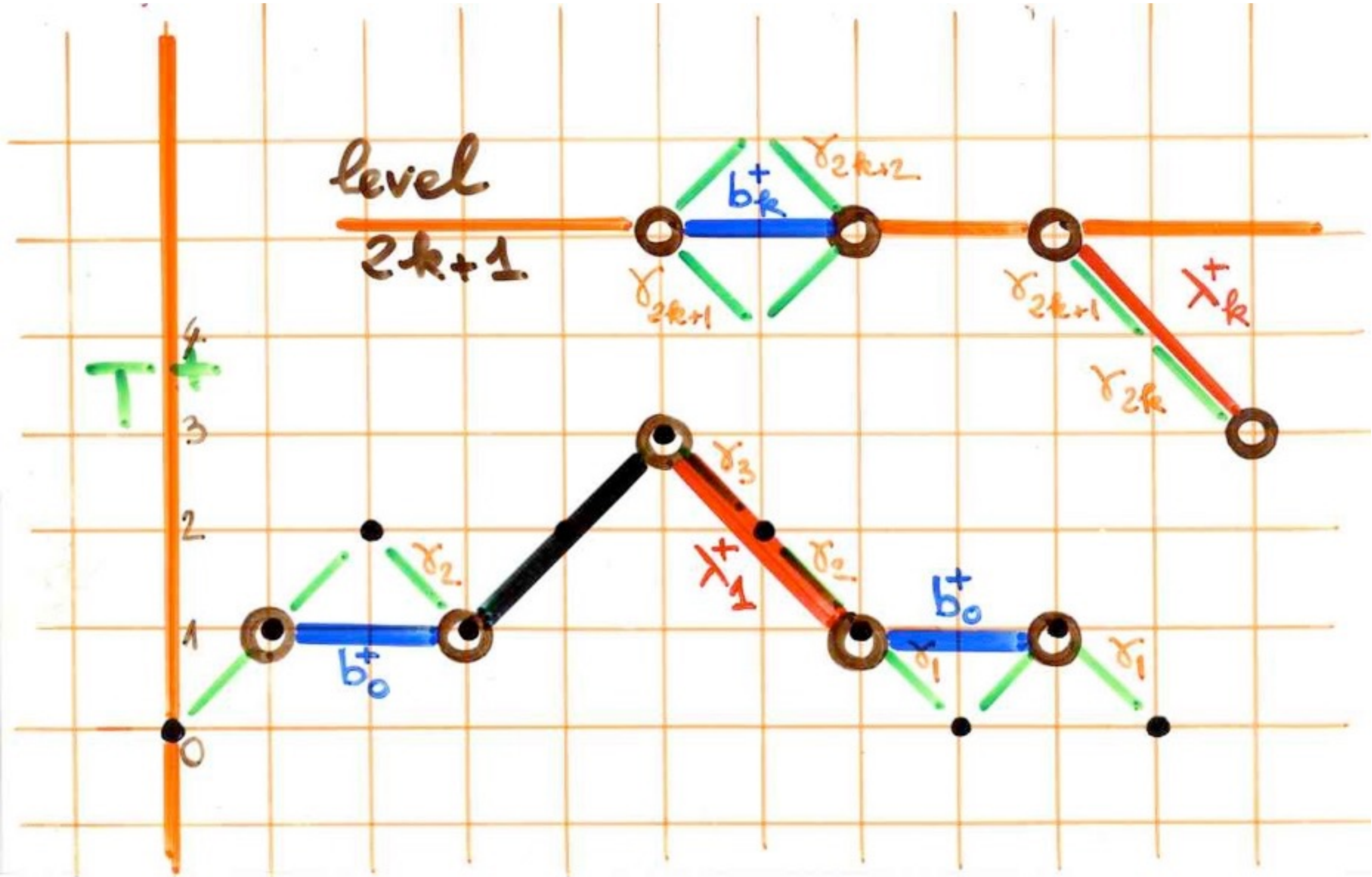
$$S(t;\gamma) = J(t;b,\lambda)$$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$



$$S(t;\gamma) = 1 + \gamma_1 t J(t; b^+, \lambda^+)$$

$$\begin{cases} b^+_k = \gamma_{2k+1} + \gamma_{2k+2} \\ \lambda^+_k = \gamma_{2k+1}\gamma_{2k} \end{cases}$$



$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-1t\cfrac{}{1-1t\cfrac{}{1-2t\cfrac{}{1-2t\cfrac{}{1-3t\cfrac{}{1-\dots}}}}}}$$

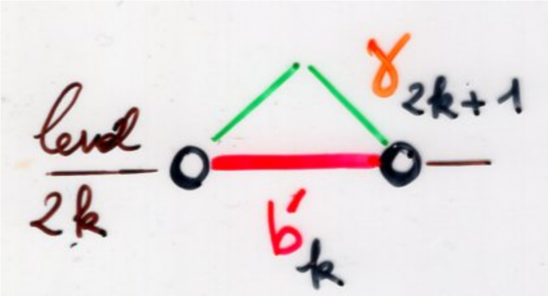
$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

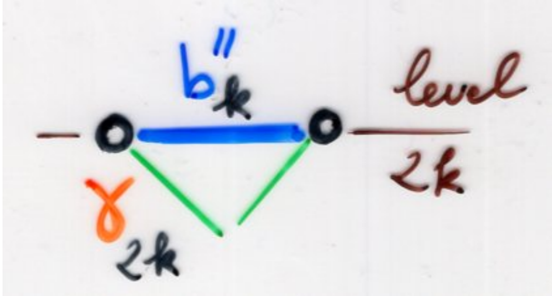
$$S(t;\gamma) = J(t;b,\lambda)$$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$

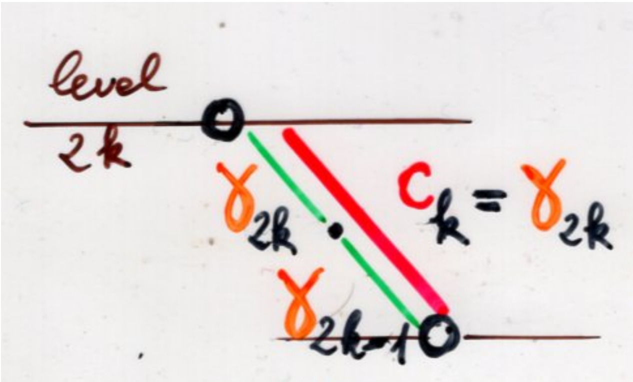
$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-1t-1^2t^2\cfrac{}{1-3t-2^2t^2\cfrac{}{1-5t-3^2t^2\cfrac{}{\dots\dots}}}}$$

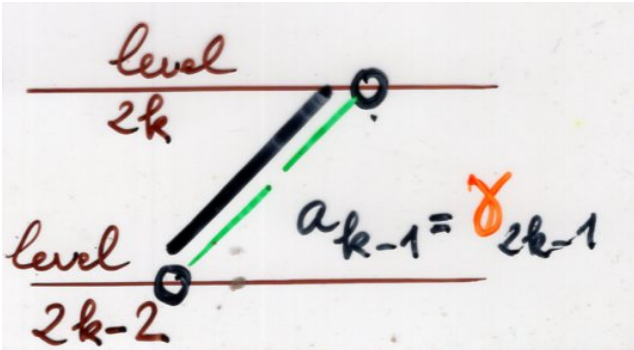
$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$





$$\begin{cases} b'_k = \gamma_{2k+1} \\ b''_k = \gamma_{2k} \end{cases} \quad (k \geq 0)$$





$$\lambda_k = a_{k-1}\, c_k$$

$$a_k = \gamma_{2k+1}$$

$$\begin{cases} a_{k-1} = \gamma_{2k-1} \\ c_k = \gamma_{2k} \end{cases} \quad (k \geq 1)$$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\, \gamma_{2k-1} \end{cases}$$

$$\gamma_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\begin{cases} \gamma_{2k-1} = k \\ \gamma_{2k} = k \end{cases} \quad (k \geq 1)$$

$$\begin{cases} b'_k = \gamma_{2k+1} \\ b''_k = \gamma_{2k} \end{cases}$$

$$\begin{cases} a_{k-1} = \gamma_{2k-1} \\ c_k = \gamma_{2k} \end{cases}$$

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases} \quad \begin{matrix} (k \geq 0) \\ \\ (k \geq 1) \end{matrix}$$

$$\mu_n = n!$$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

restricted
Laguerre
histories

From subdivided Laguerre histories

to

(restricted) Laguerre histories

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\cdots}}}}}}$$

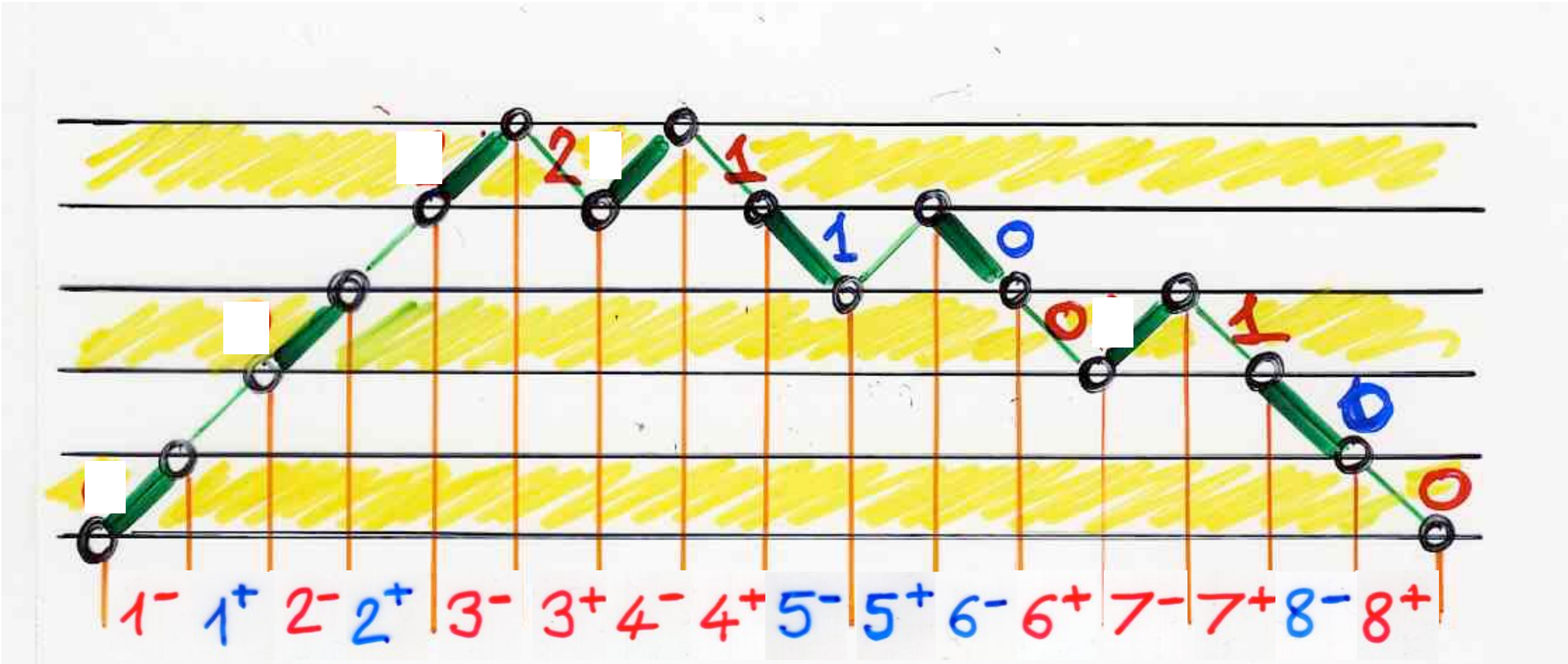
$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$S(t;\gamma) = J(t;b,\lambda)$$

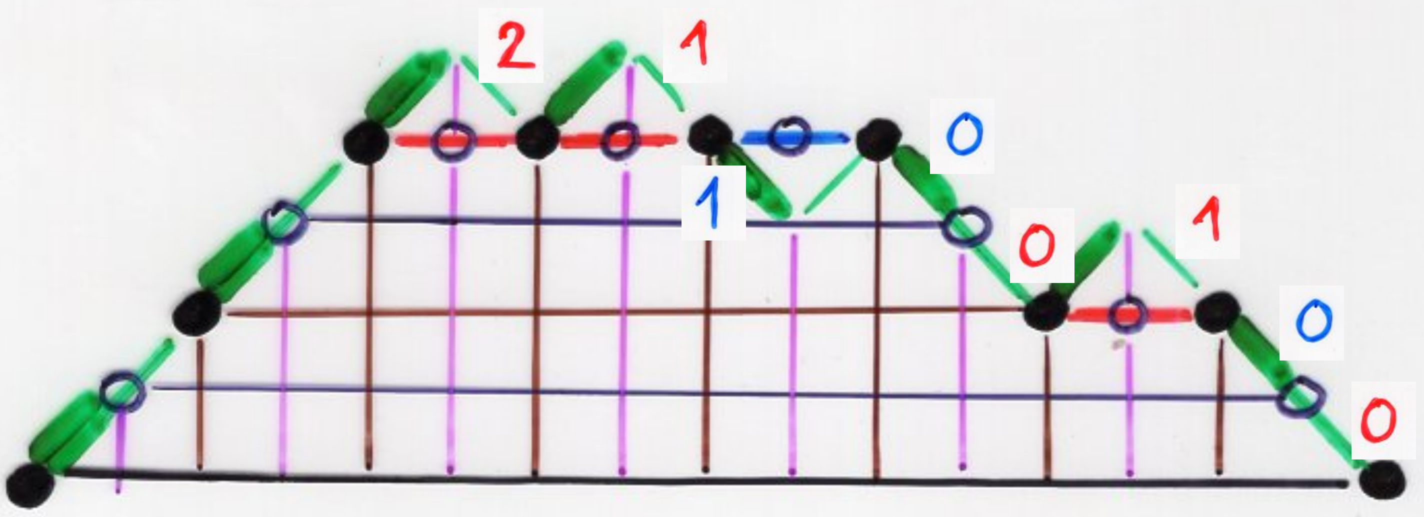
$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-1t-\cfrac{1^2t^2}{1-3t-\cfrac{2^2t^2}{1-5t-\cfrac{3^2t^2}{\cdots}}}}$$

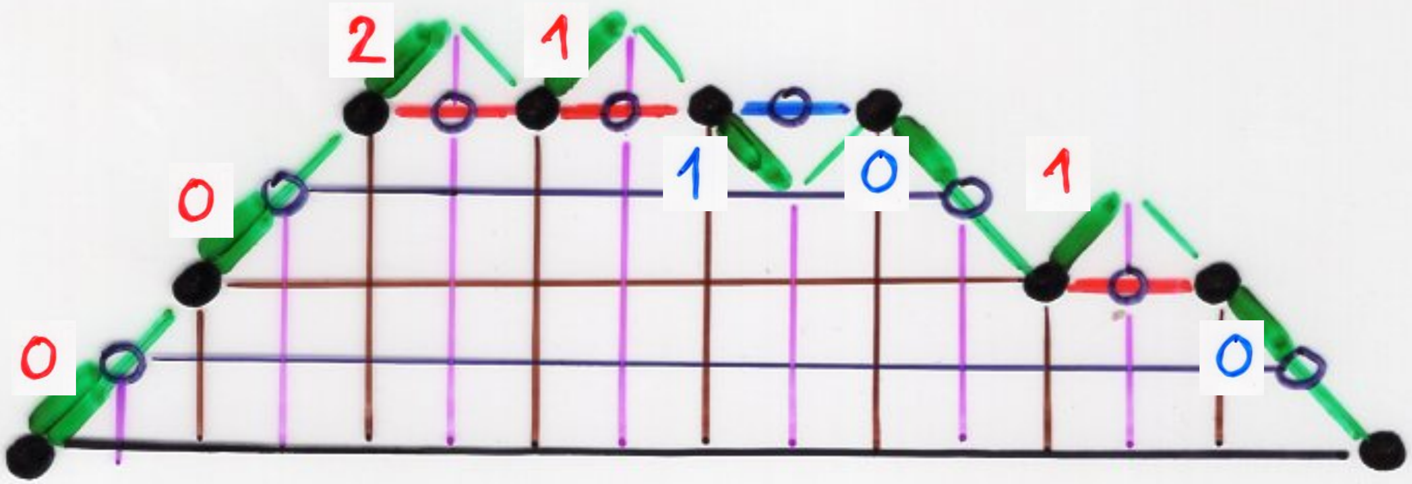
$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$



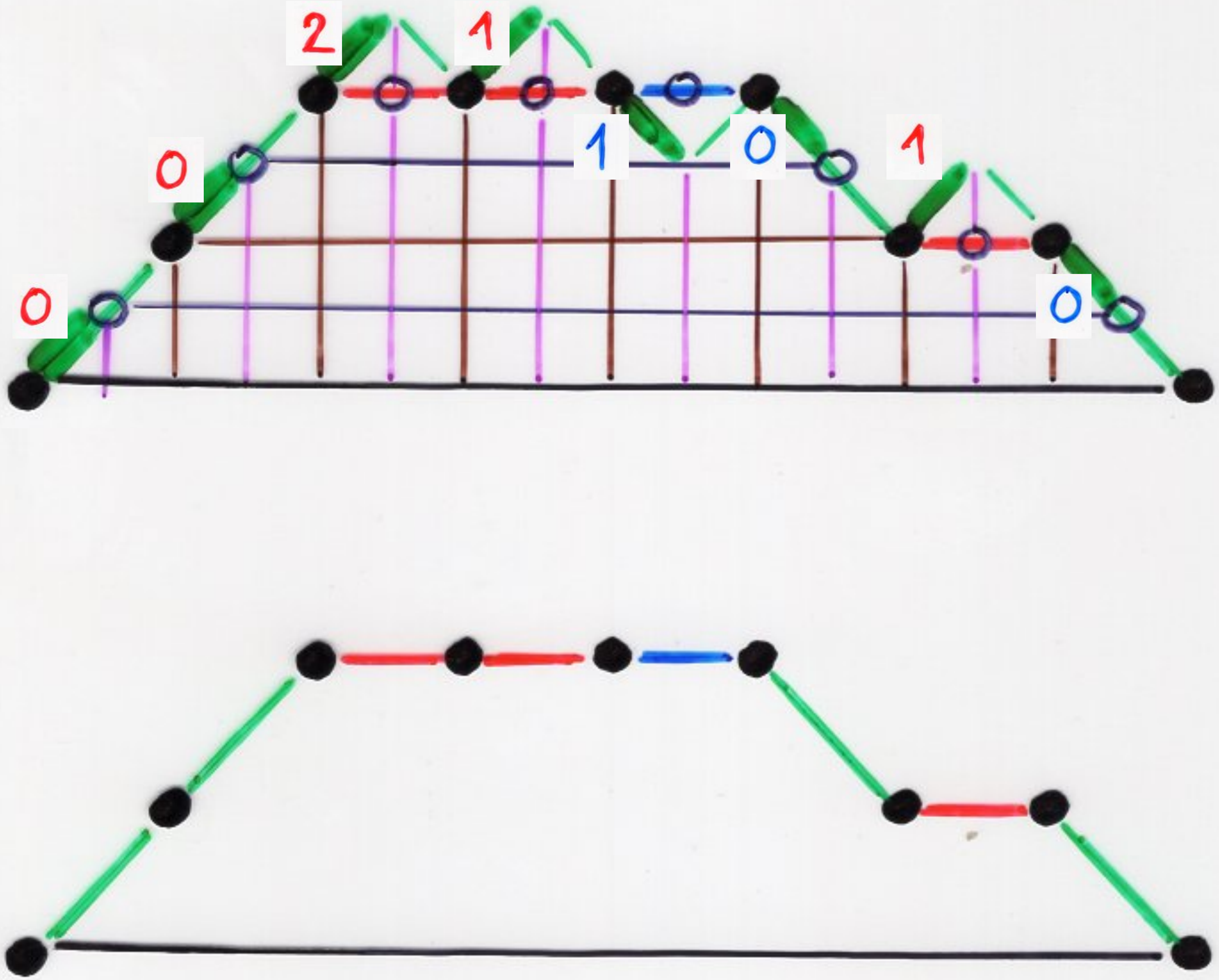
subdivided Laguerre history



subdivided Laguerre history



2
1
0
1
0
1
0
0



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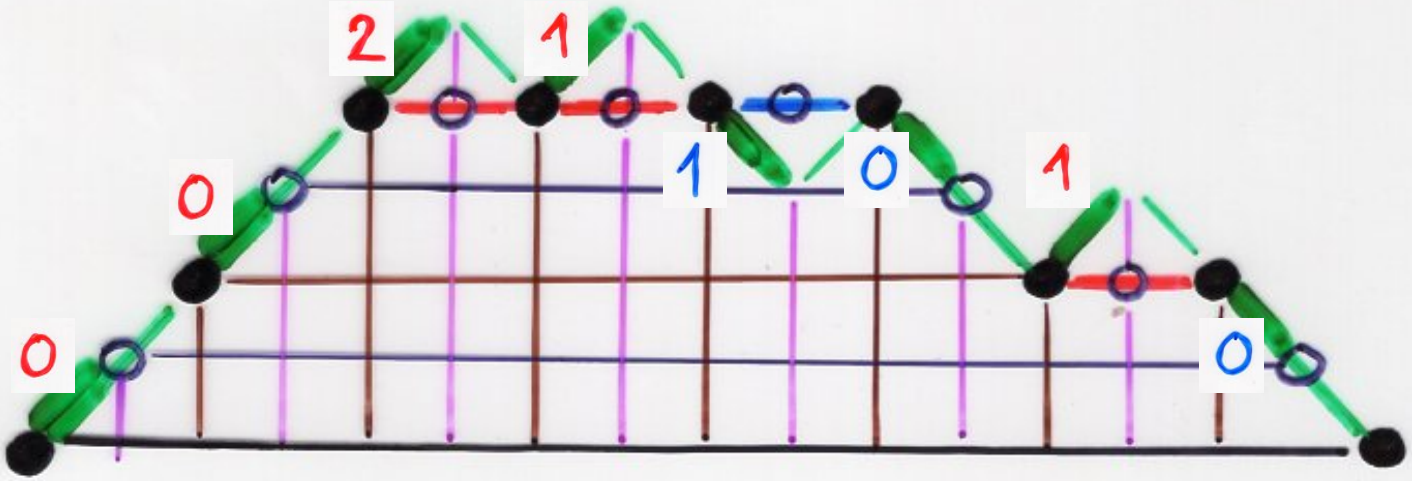
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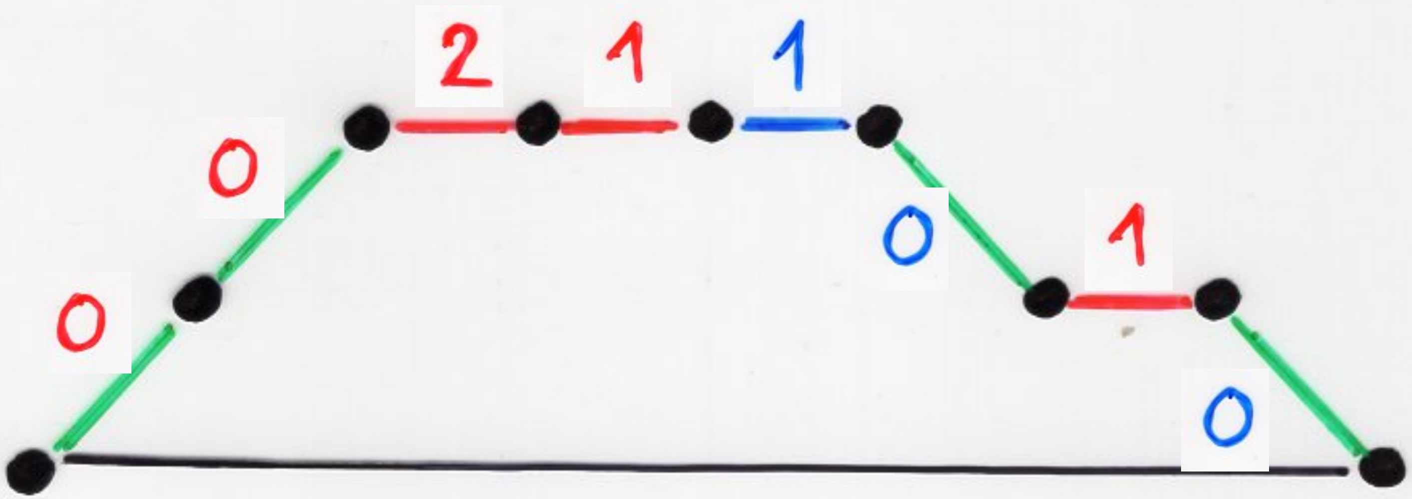
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0
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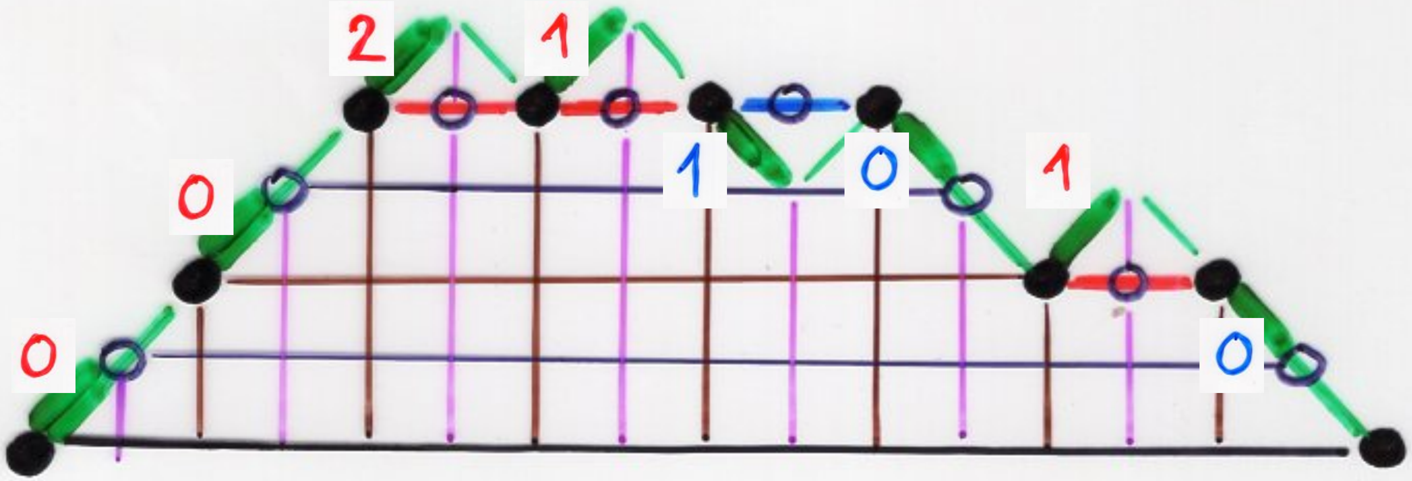
0
0



2
1
1
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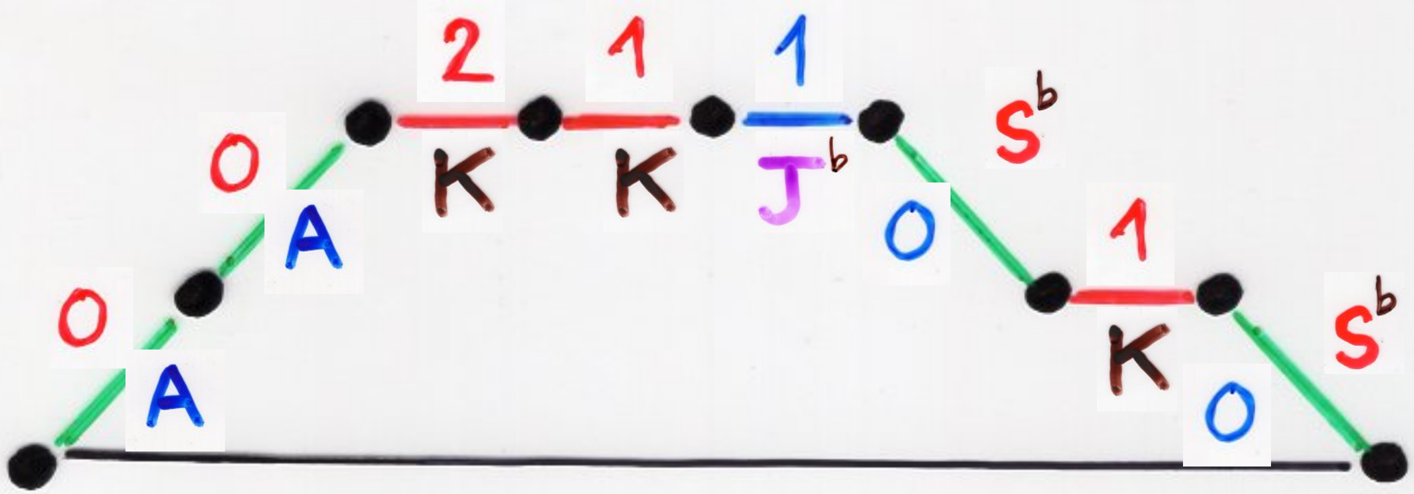
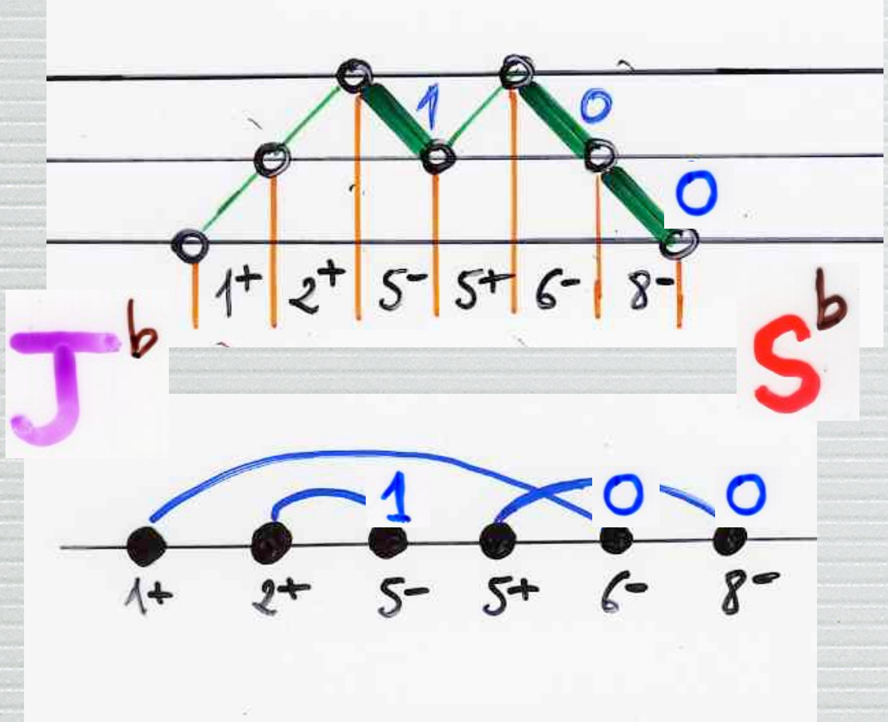
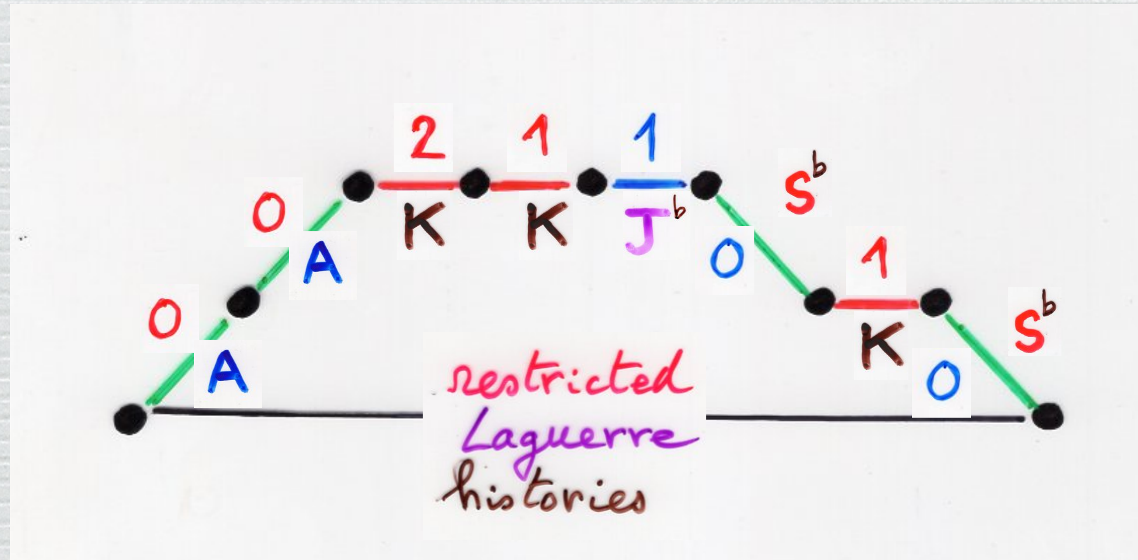
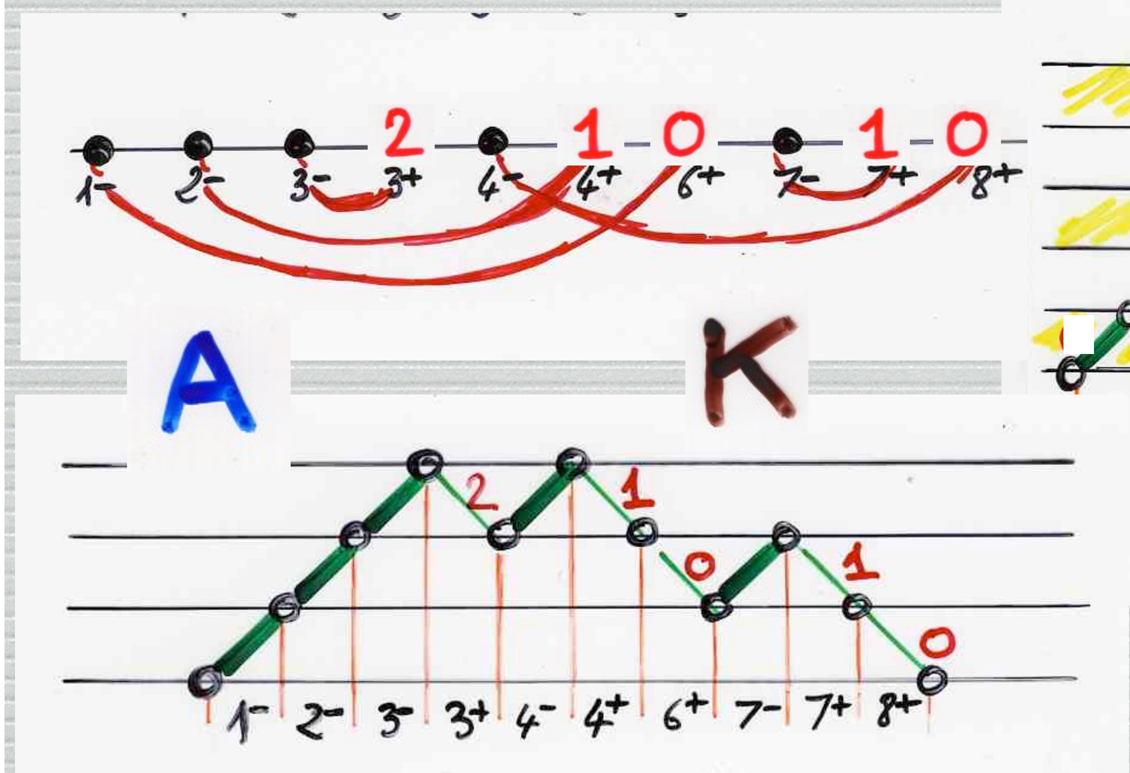
1
0
0
0

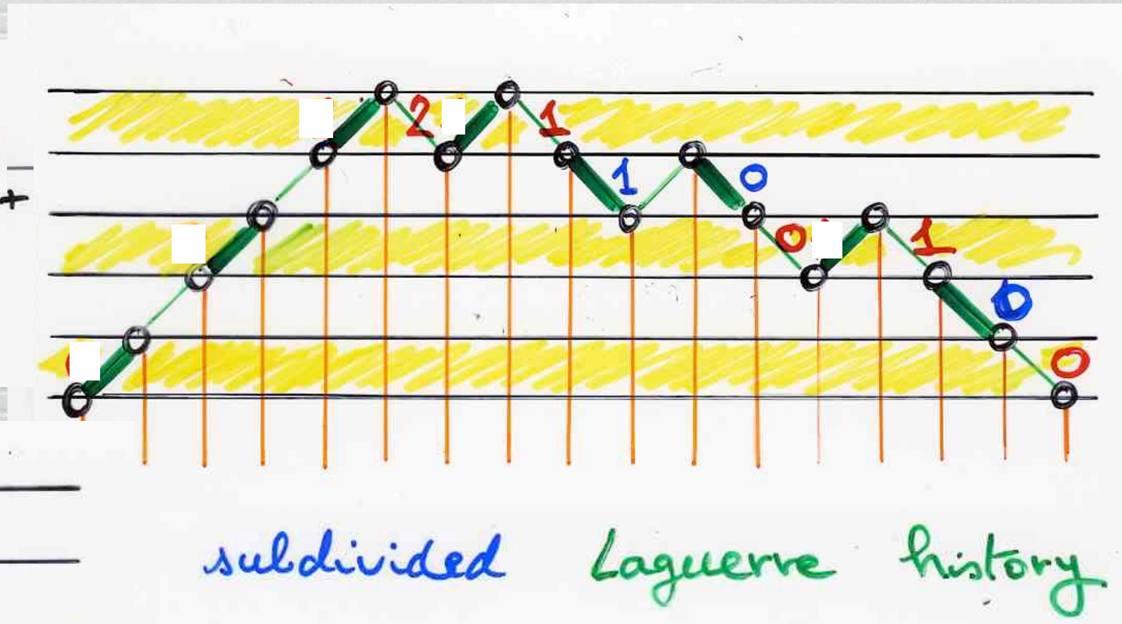


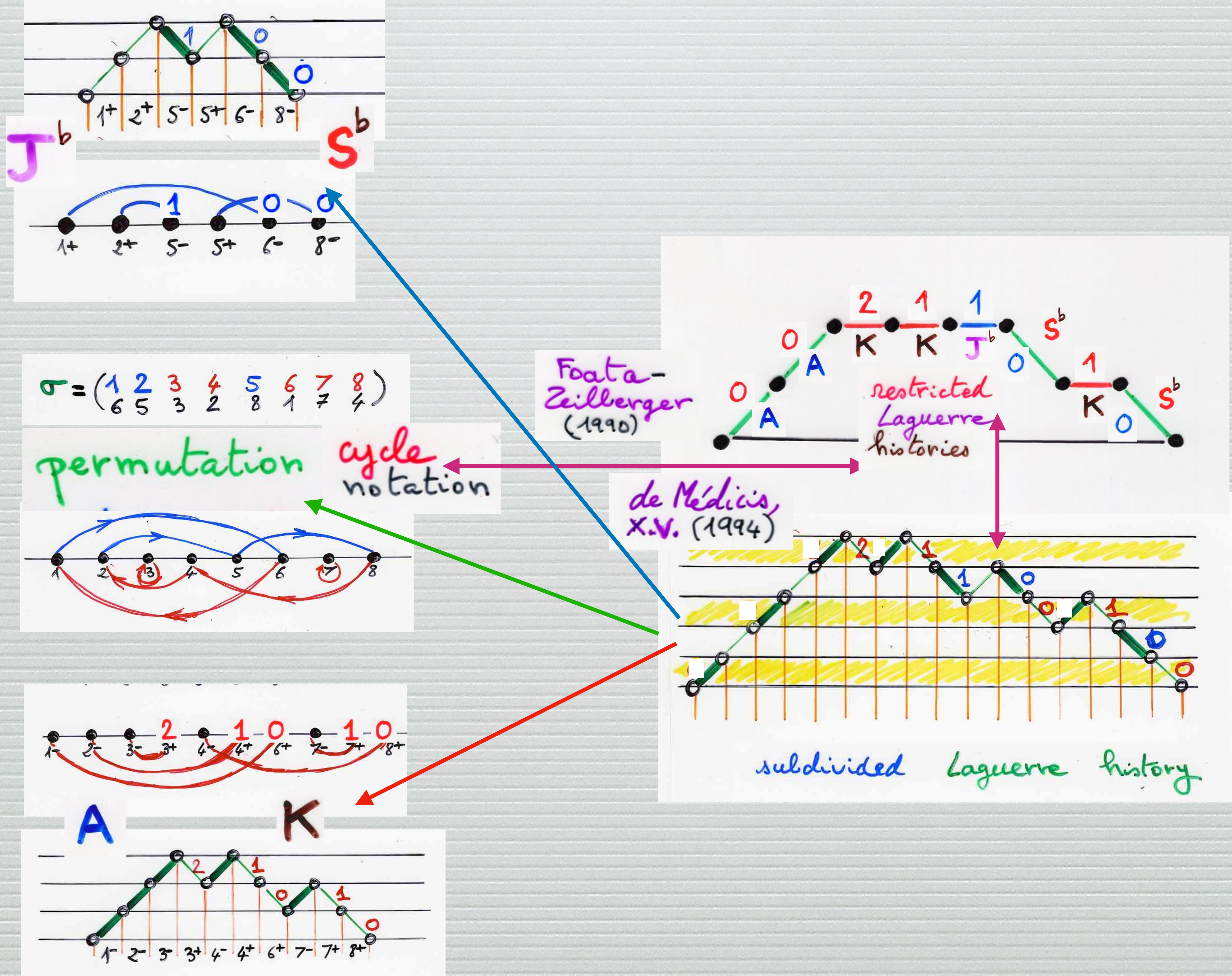
2
1
1
K
K
J^b
S^b
0
A
0
A
0
1
K
0
S^b


1 0 0
1+ 2+ 5- 5+ 6- 8-
J b
S b
1 0 0
1+ 2+ 5- 5+ 6- 8-


2 1 1
K K J b
0 A 0 A
S b 0 1 K S b 0
restricted Laguerre histories


2 1 0 1 0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+
A
K
2 1 0 1 0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+


2 1 1 0 0 1 0 0
subdivided Laguerre history



J^b
S^b
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
permutation
cycle notation
Foata-Zeilberger (1990)
restricted Laguerre histories
de Médicis, X.V. (1994)
subdivided Laguerre history
A
K

Combinatorial proof of Euler 's continued fractions

$$z = 1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}$$

reperietur enim iisdem operationibus inſtitutis:

$$z = \cfrac{1}{1+\cfrac{mx}{1+\cfrac{nx}{1+\cfrac{(m+n)x}{1+\cfrac{2nx}{1+\cfrac{(m+2n)x}{1+\cfrac{3nx}{1+\cfrac{(m+3n)x}{1+\cfrac{4nx}{1+\cfrac{(m+4n)x}{1+\cfrac{5nx}{1+\text{etc.}}}}}}}}}}}$$

Eadem vero expreſſio, aliaeque ſimiles facile erui poſ-

$$\sum_{n\geq 0} \beta(\beta+1)(\beta+2)\cdots(\beta+n-1) = \cfrac{1}{1-\beta t - \cfrac{1t}{1-(\beta+1)t - \cfrac{2t}{1-(\beta+2)t - \cfrac{3t}{\cdots}}}}$$

$$\begin{cases} \gamma_{2k} = k & (k \geq 1) \\ \gamma_{2k+1} = k+\beta & (k \geq 0) \end{cases}$$

$$\begin{cases} a_k = k+\beta \\ b'_k = k+\beta \end{cases} \quad (k \geq 0)$$

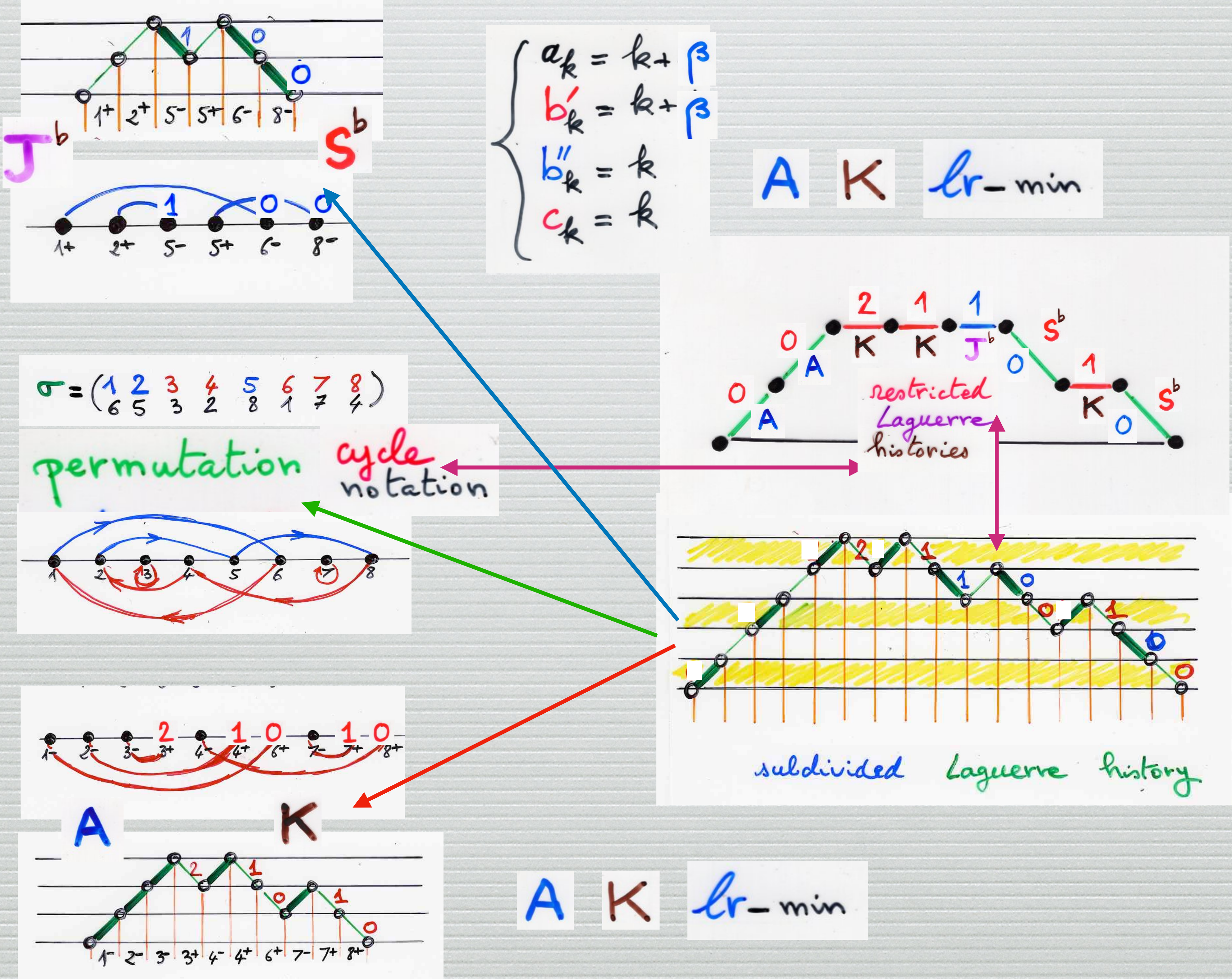
$$\begin{cases} b''_k = k \\ c_k = k \end{cases} \quad (k \geq 1)$$

$$\begin{cases} b'_k = \gamma_{2k+1} \\ b''_k = \gamma_{2k} \end{cases} \qquad \begin{cases} a_{k-1} = \gamma_{2k-1} \\ c_k = \gamma_{2k} \end{cases}$$

restricted
Laguerre
histories

$$\begin{cases} b_k = 2k+\beta \\ \lambda_k = (k-1+\beta)k \end{cases}$$

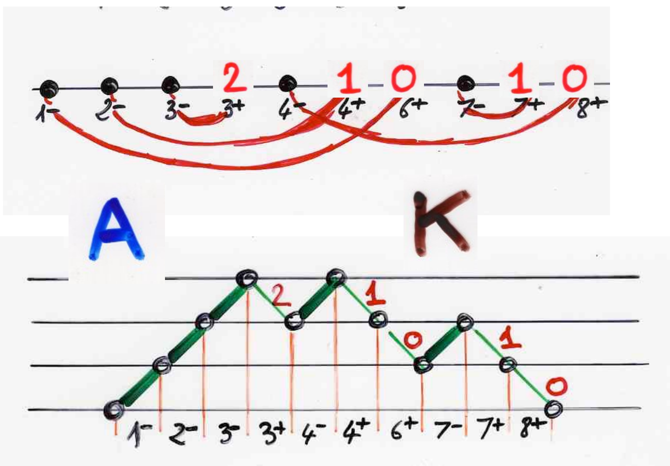
$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$

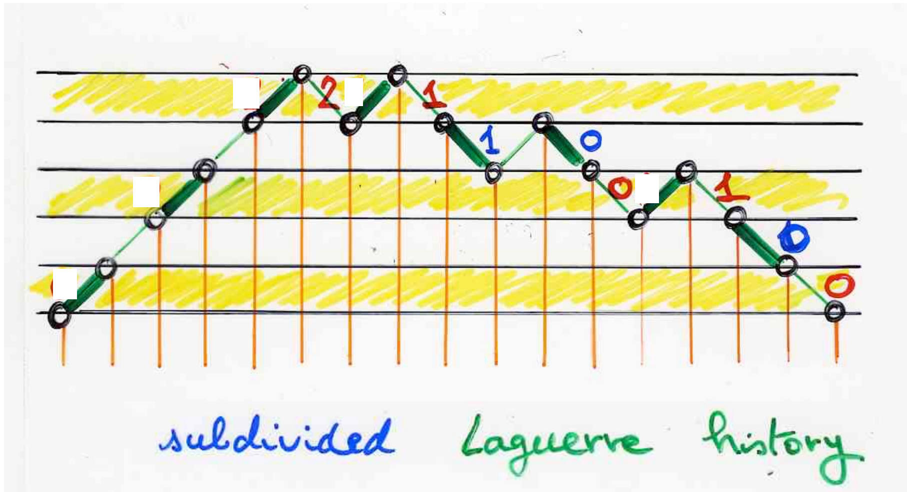


$a_k = k + \beta$
$b'_k = k + \beta$
$b''_k = k$
$c_k = k$
A K lr-min
J^b
S^b
1+ 2+ 5- 5+ 6- 8-
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$
permutation
cycle notation
restricted Laguerre histories
subdivided Laguerre history
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+
A K lr-min

$$\sum_{n\geq 0} \beta(\beta+1)(\beta+2)\cdots(\beta+n-1) = \cfrac{1}{1-\beta t - \cfrac{}{1-1t-\cfrac{}{1-(\beta+1)t-\cfrac{}{1-2t-\cfrac{}{1-(\beta+2)t-\cfrac{}{1-3t-\cdots}}}}}}$$

A K $lr_{-}min$

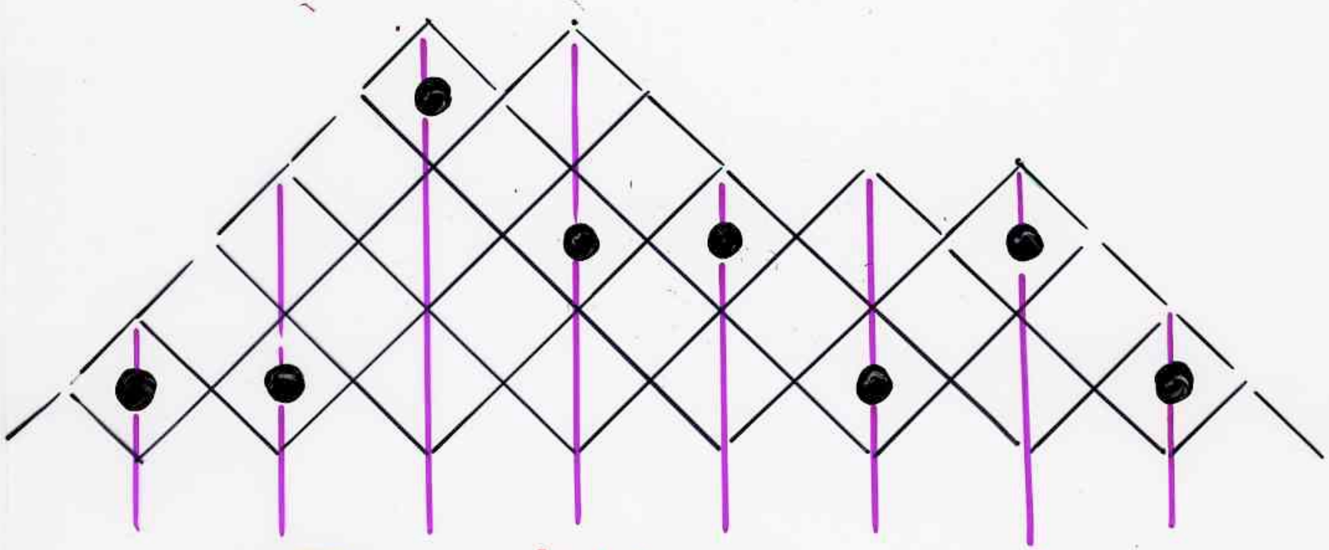




Bijection

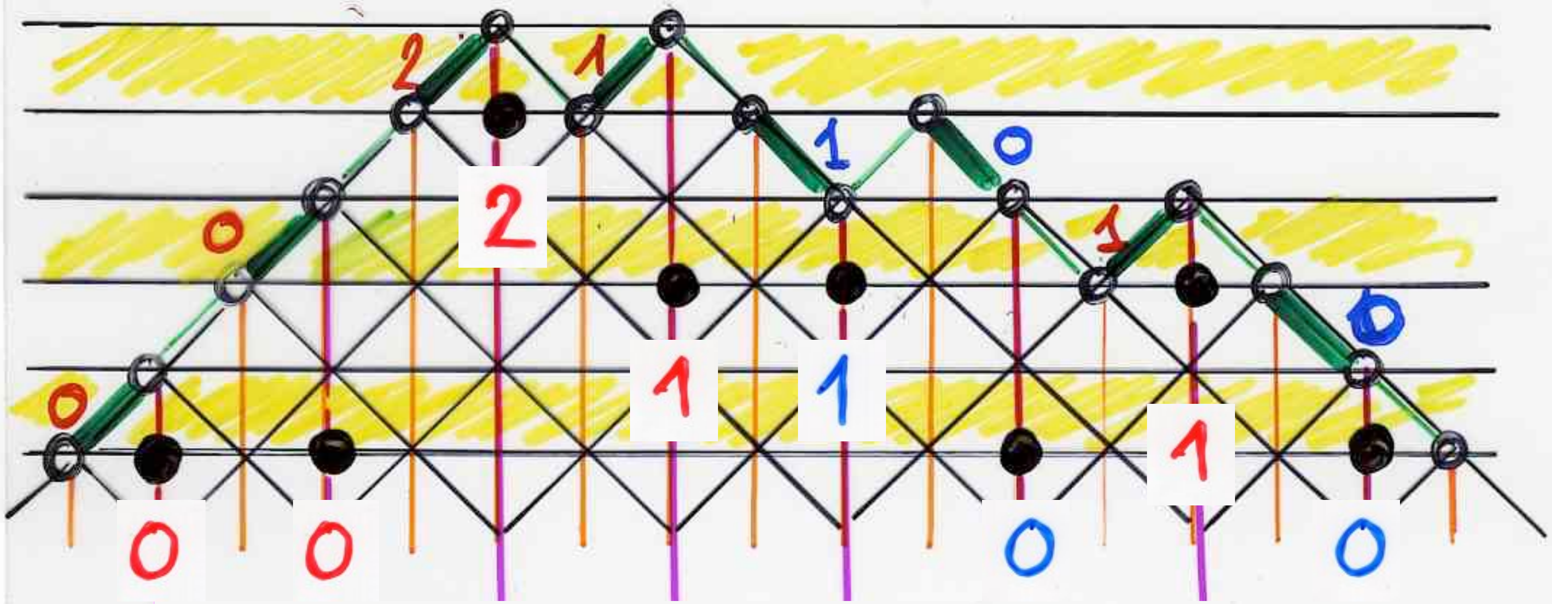
subdivided Laguerre histories

Dyck tableaux

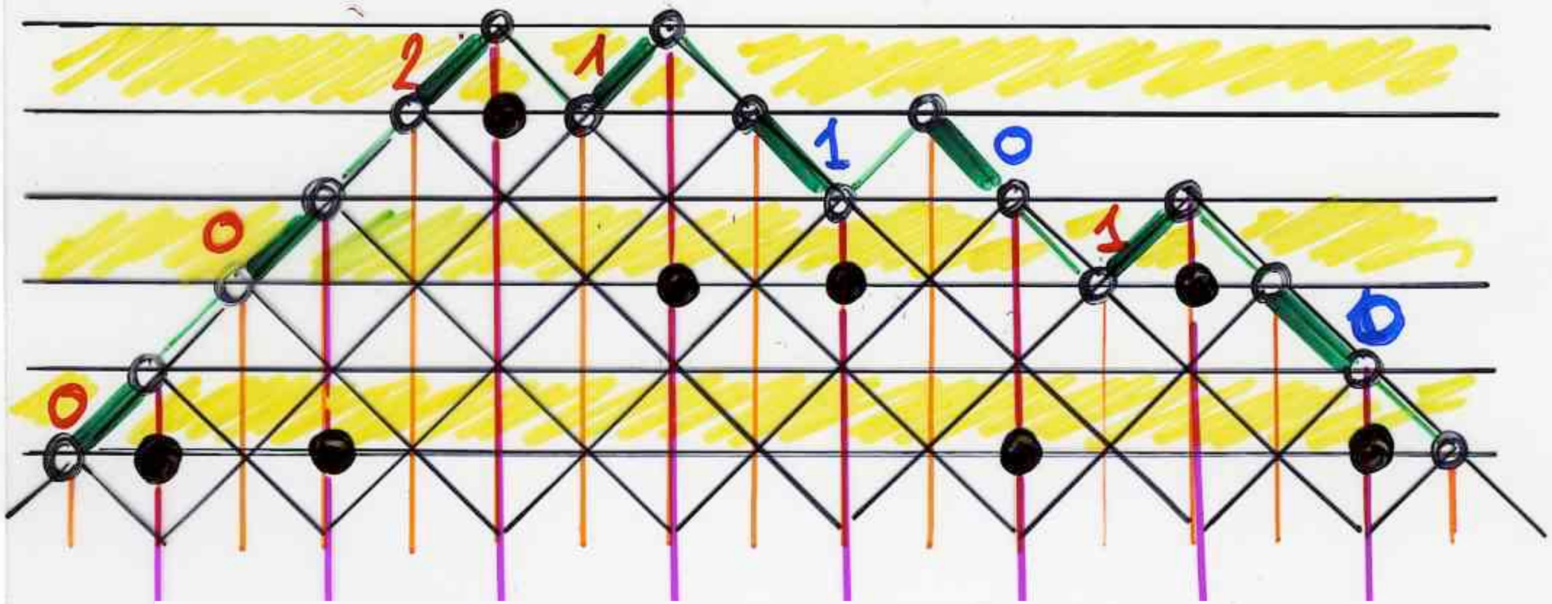


Dyck tableau

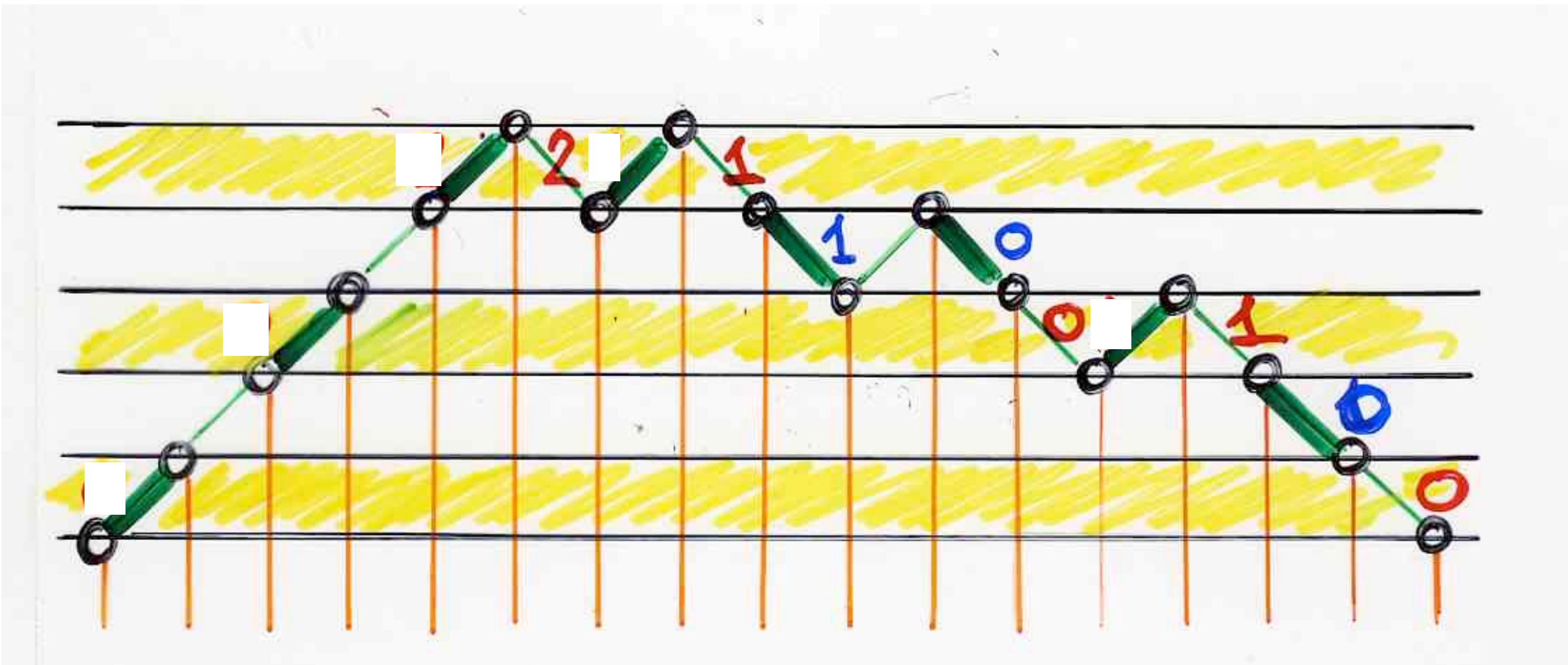
J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)



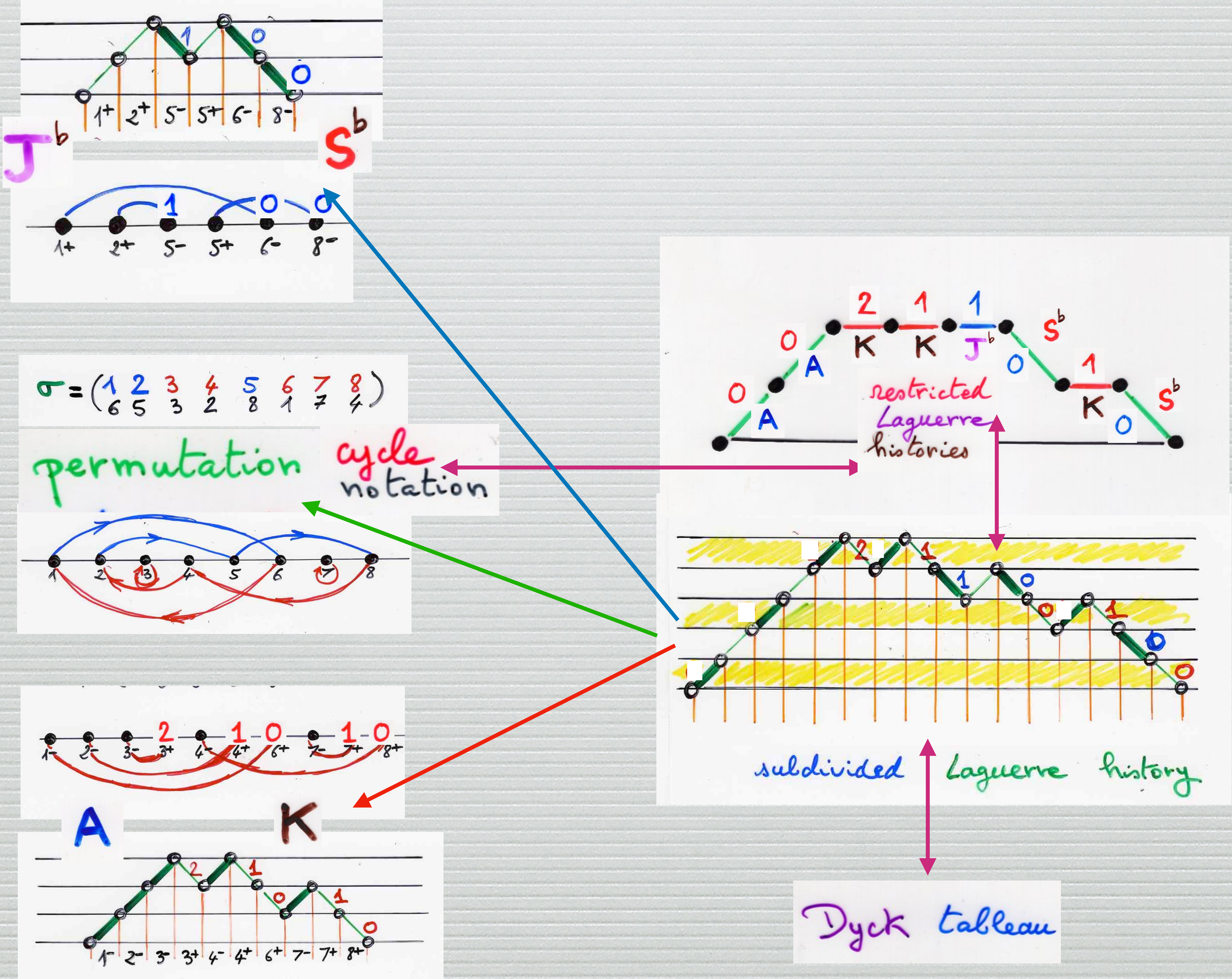
Dyck tableau as a subdivided Laguerre history



Dyck tableau as a subdivided Laguerre history



subdivided Laguerre history

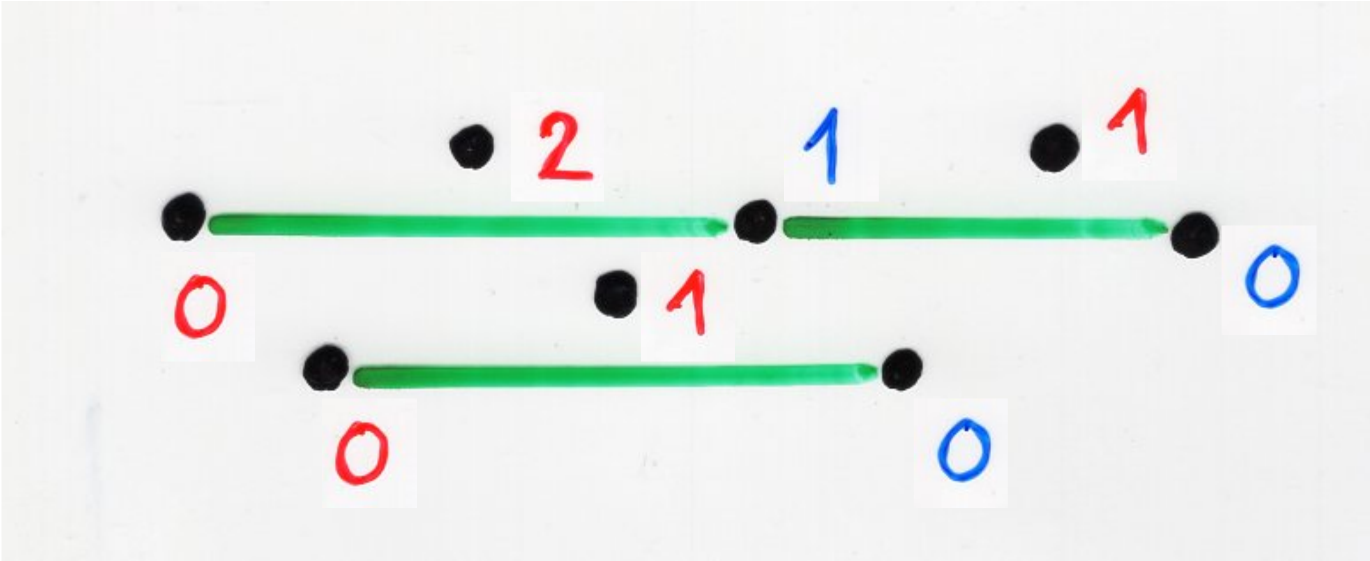


J^b
S^b
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
permutation
cycle notation
restricted Laguerre histories
subdivided Laguerre history
Dyck tableau
A
K

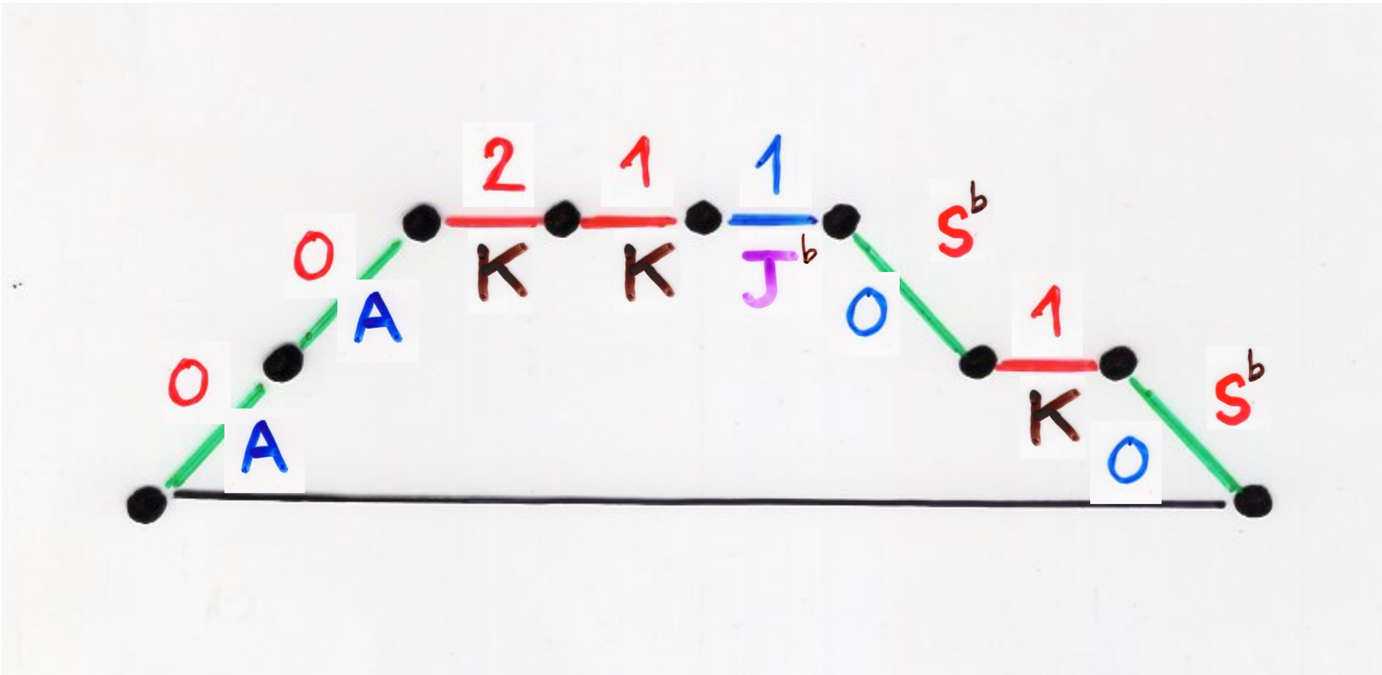
From (restricted) Laguerre histories

to

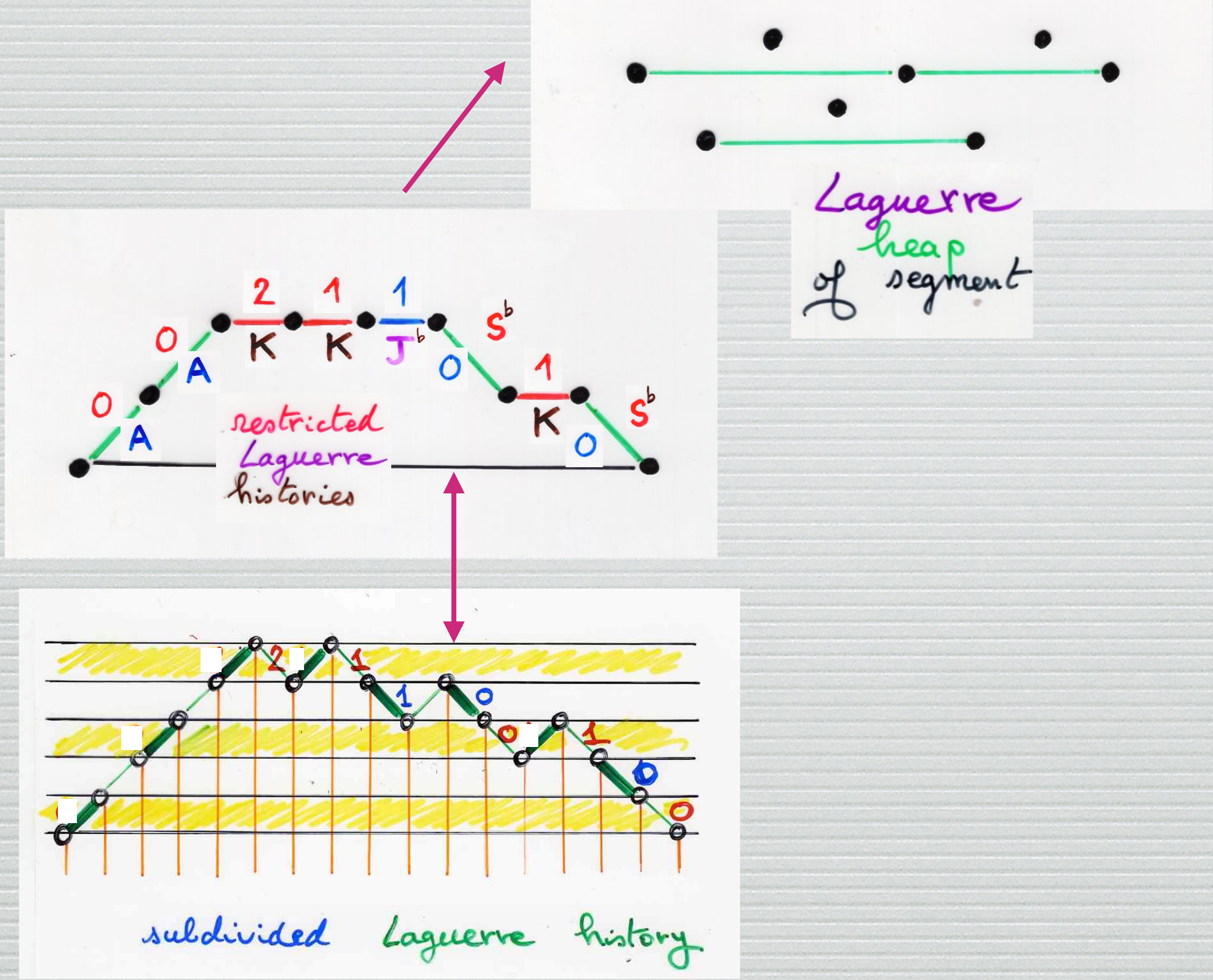
Laguerre heaps



2
1
1
0
0
1
0
0



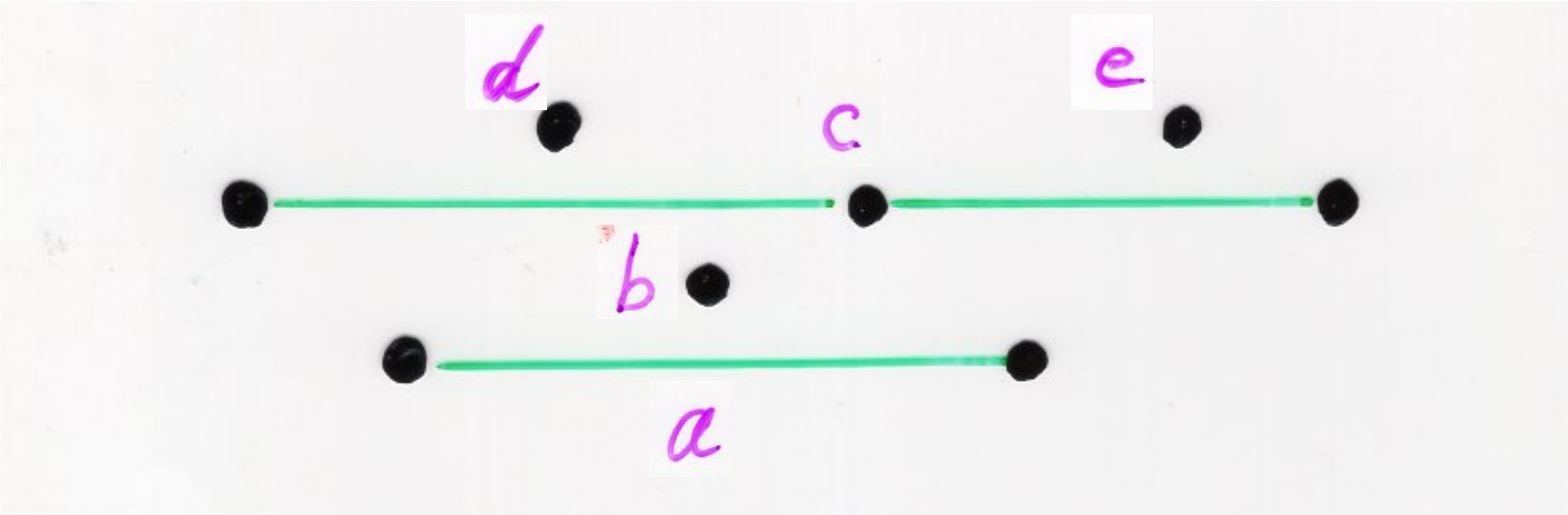
2
1
1
K
K
J^b
0
A
0
A
S^b
0
1
K
0
S^b

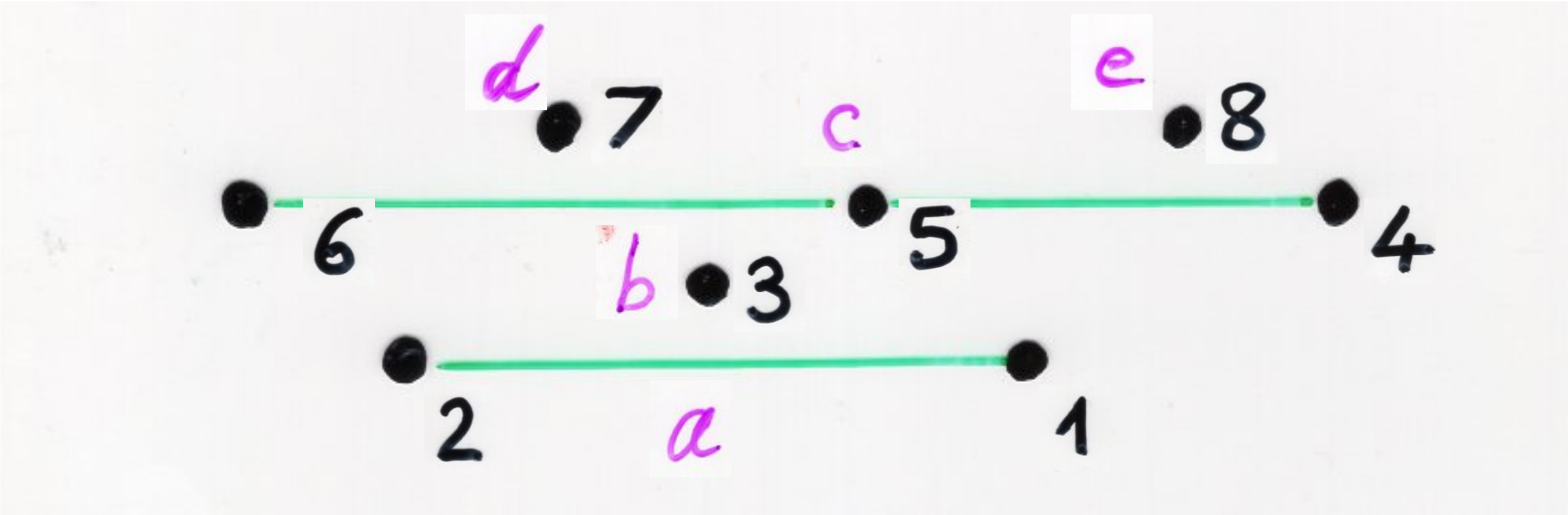

Laguerre heap of segment
restricted Laguerre histories
subdivided Laguerre history

Laguerre heaps

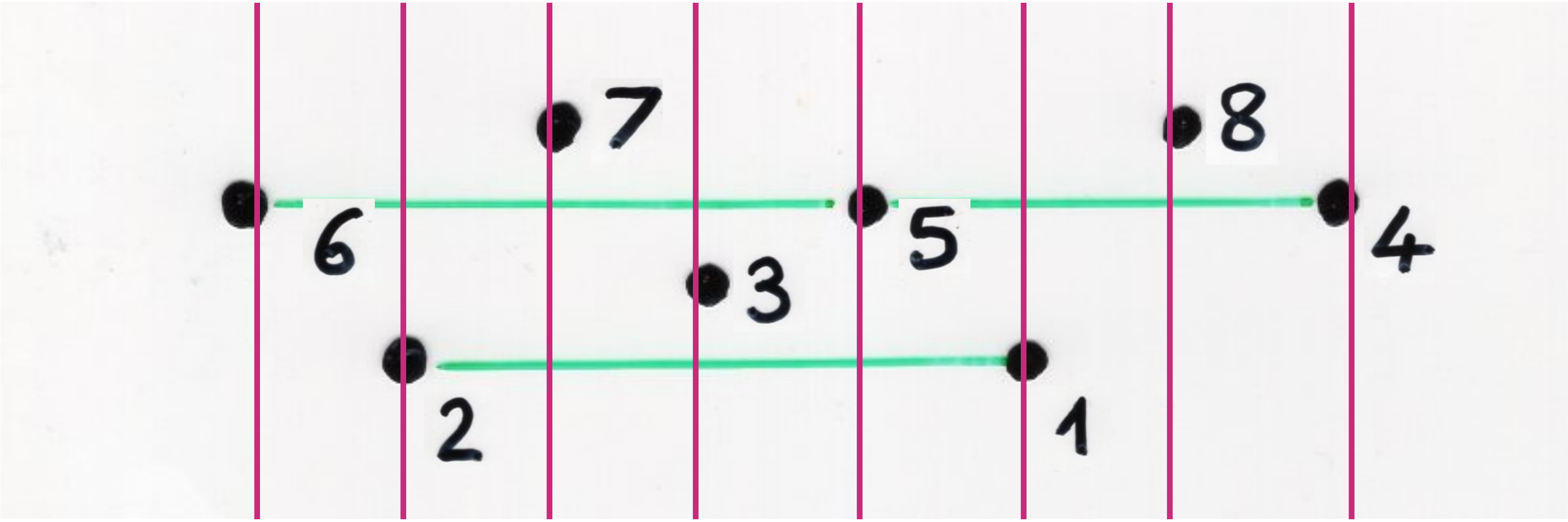
to

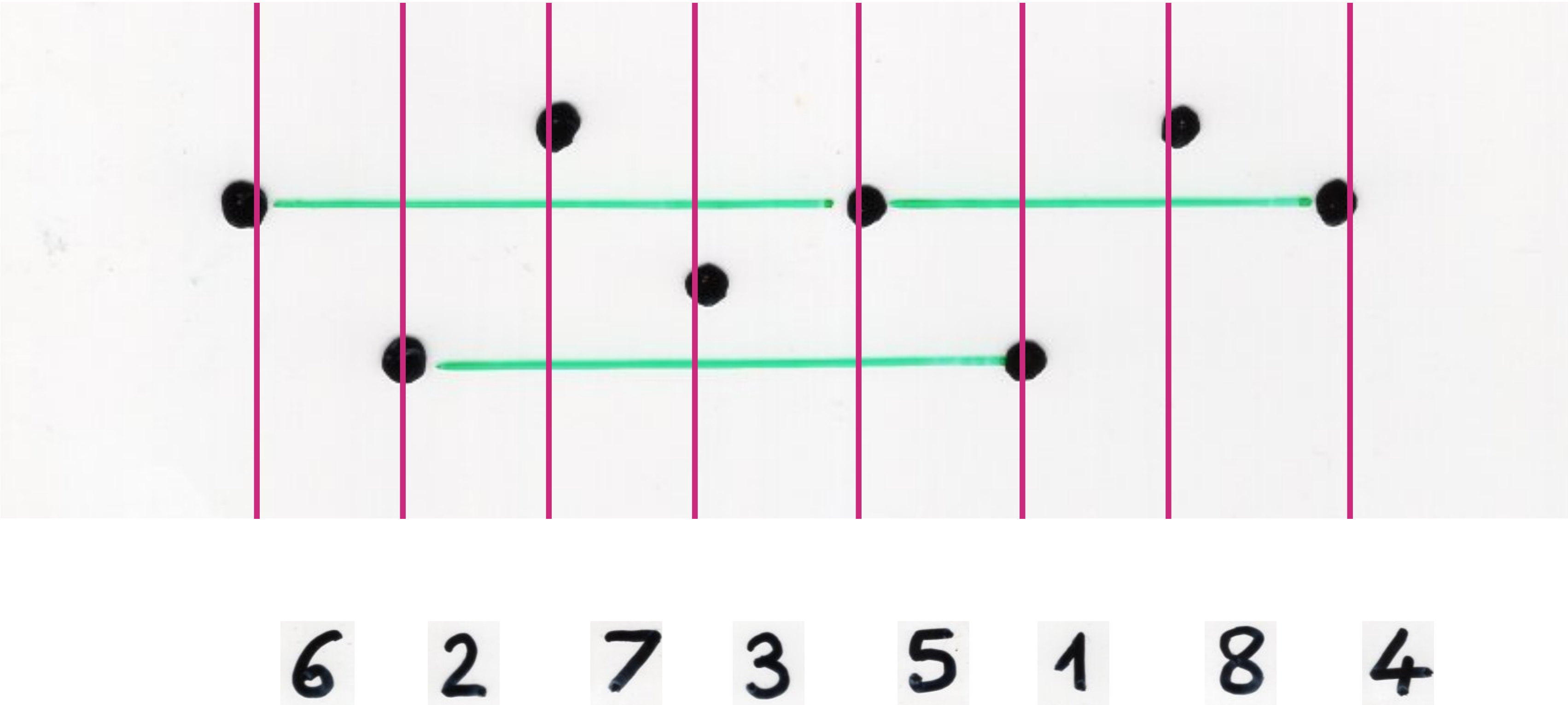
permutations

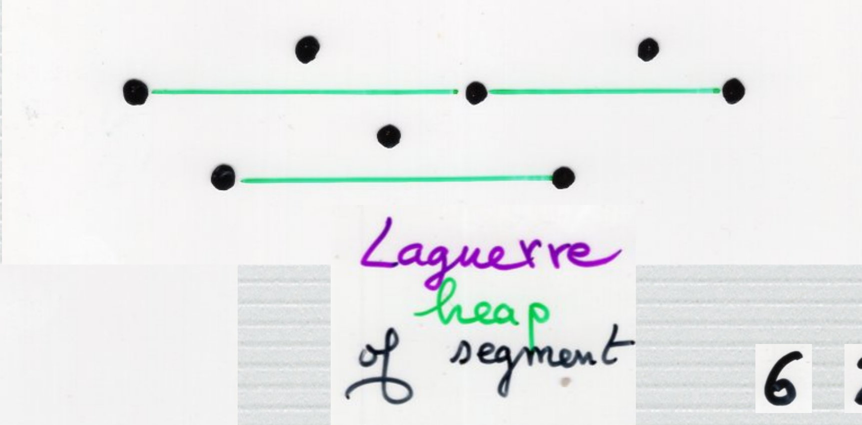

d
e
c
b
a



d
7
c
e
8
6
5
4
b
3
2
a
1


7
8
6
5
4
3
2
1

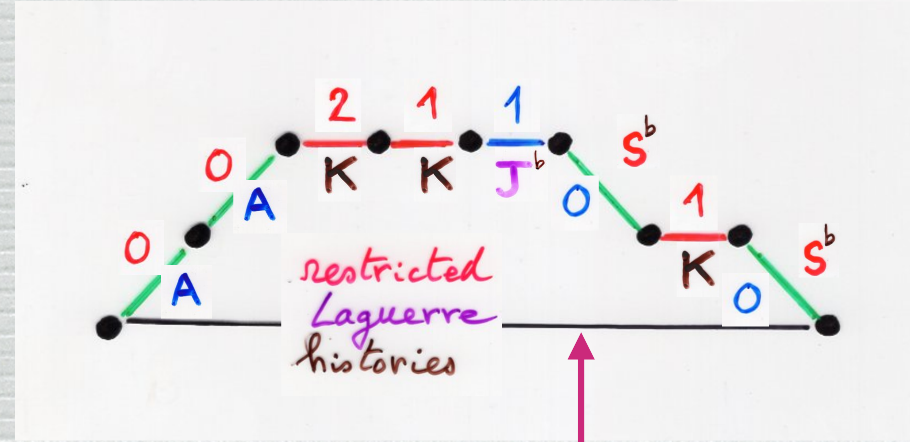

6
2
7
3
5
1
8
4

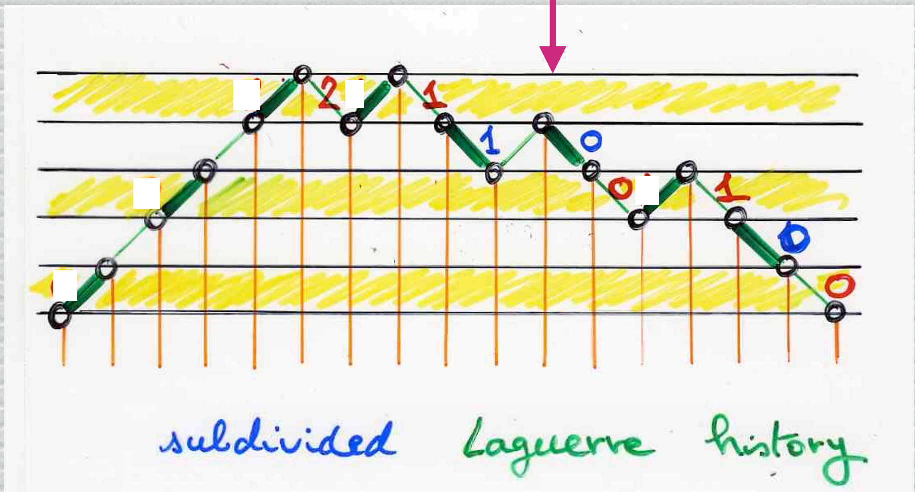


σ

6 2 7 3 5 1 8 4

permutation

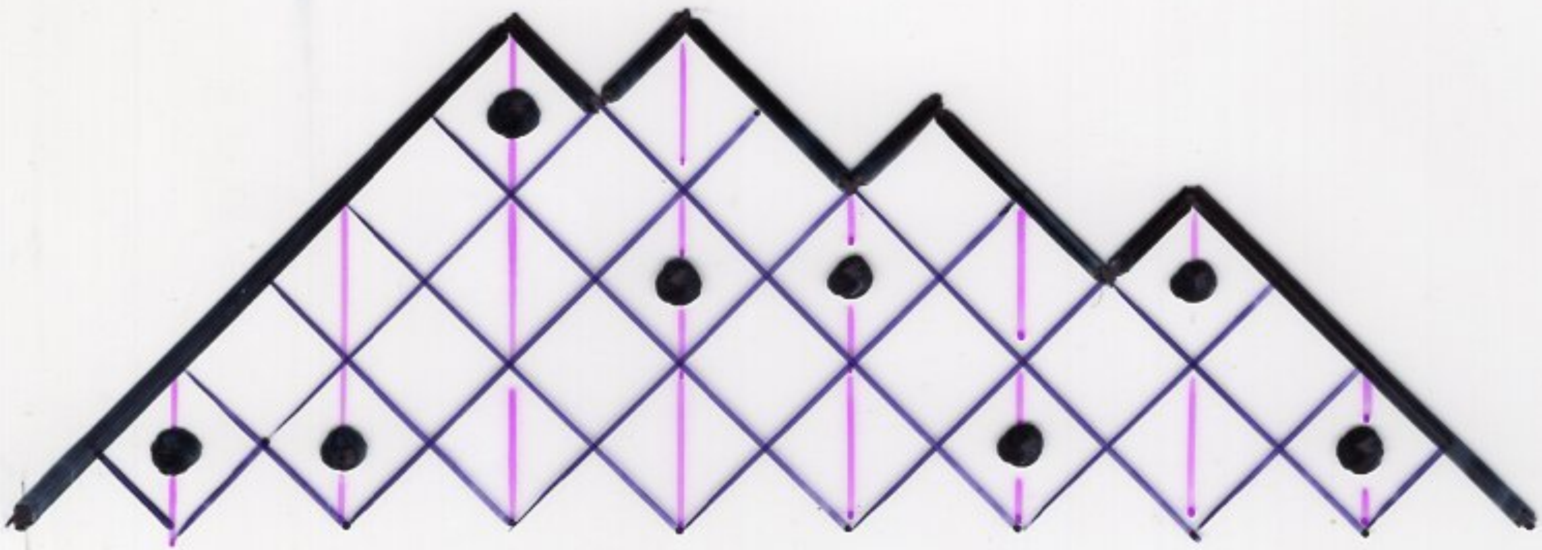


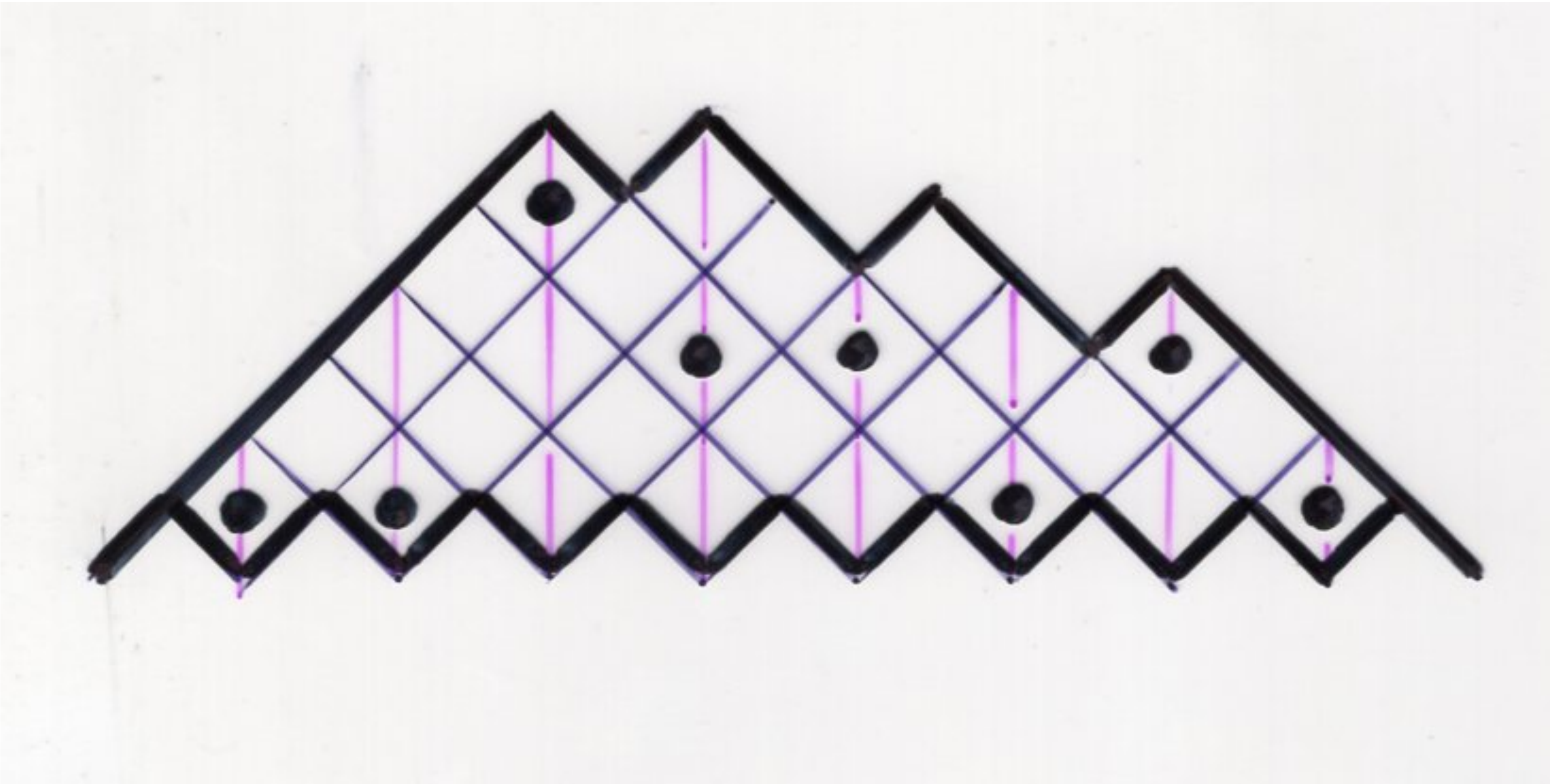


permutations

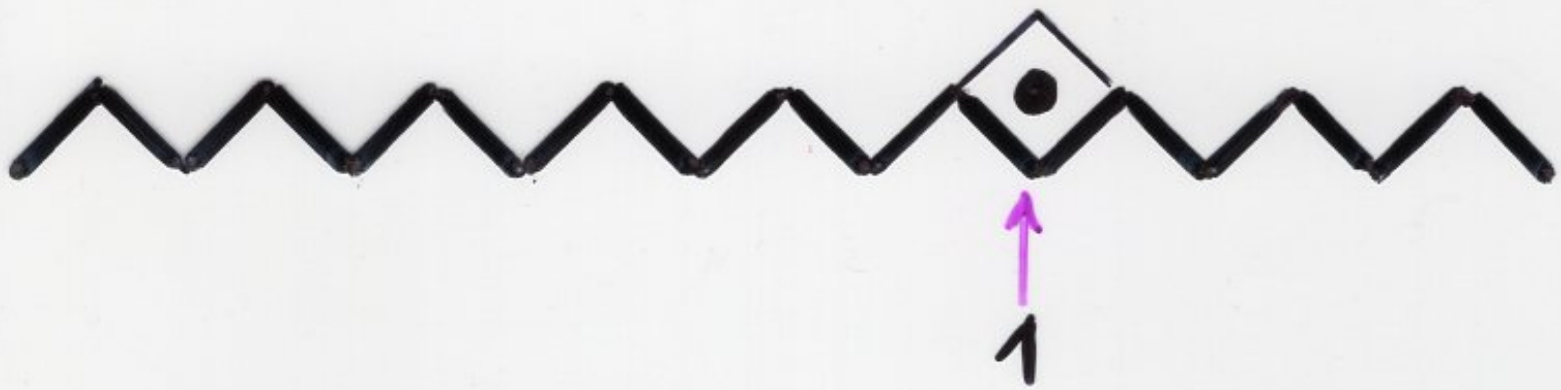
to

Dyck tableaux

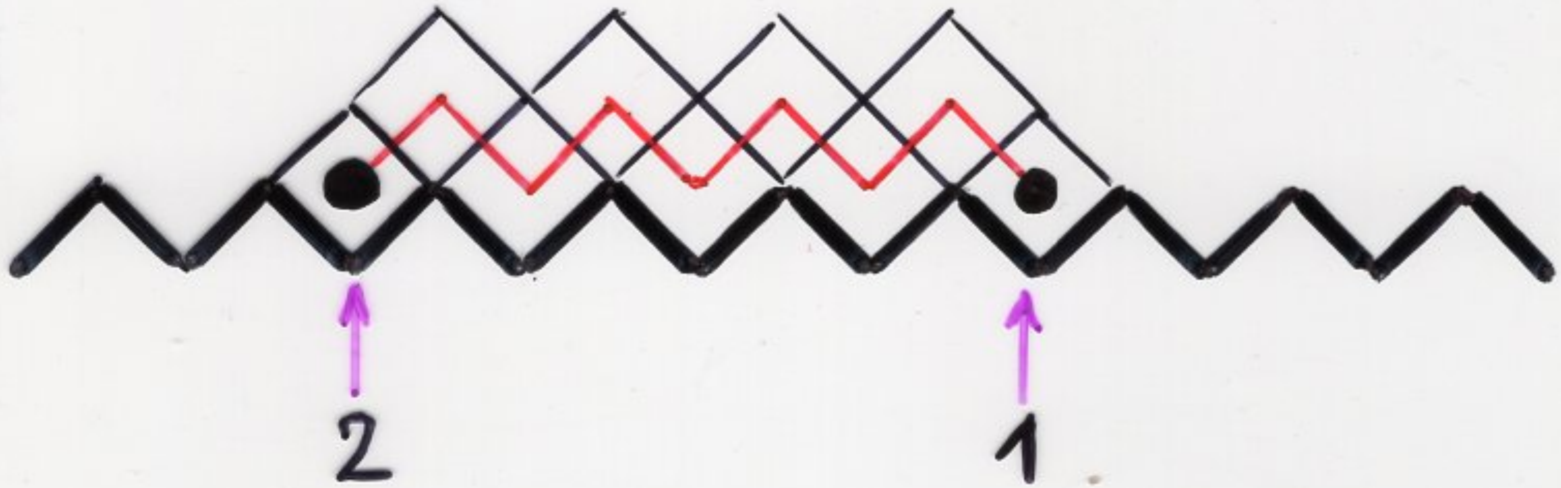




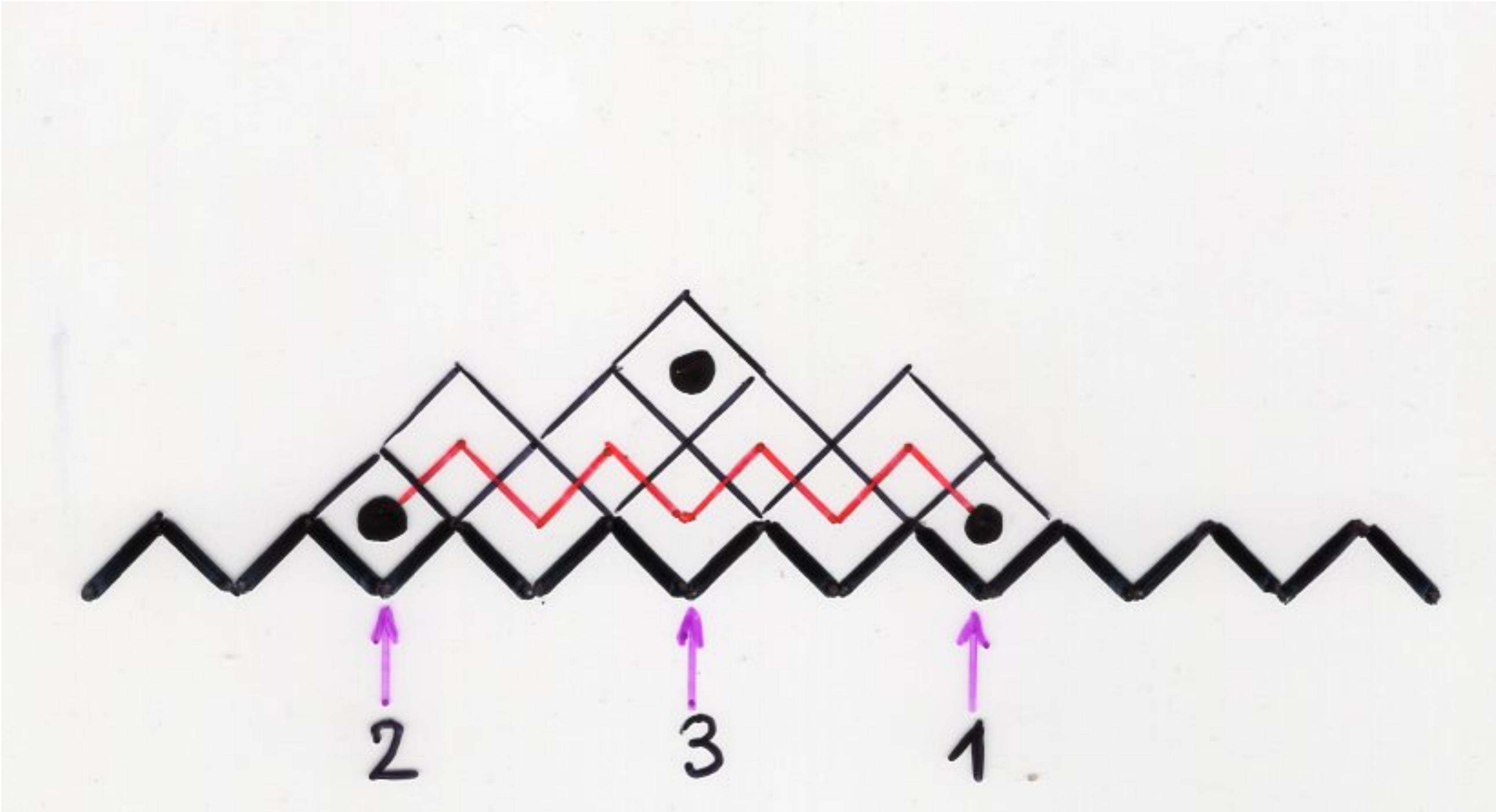
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



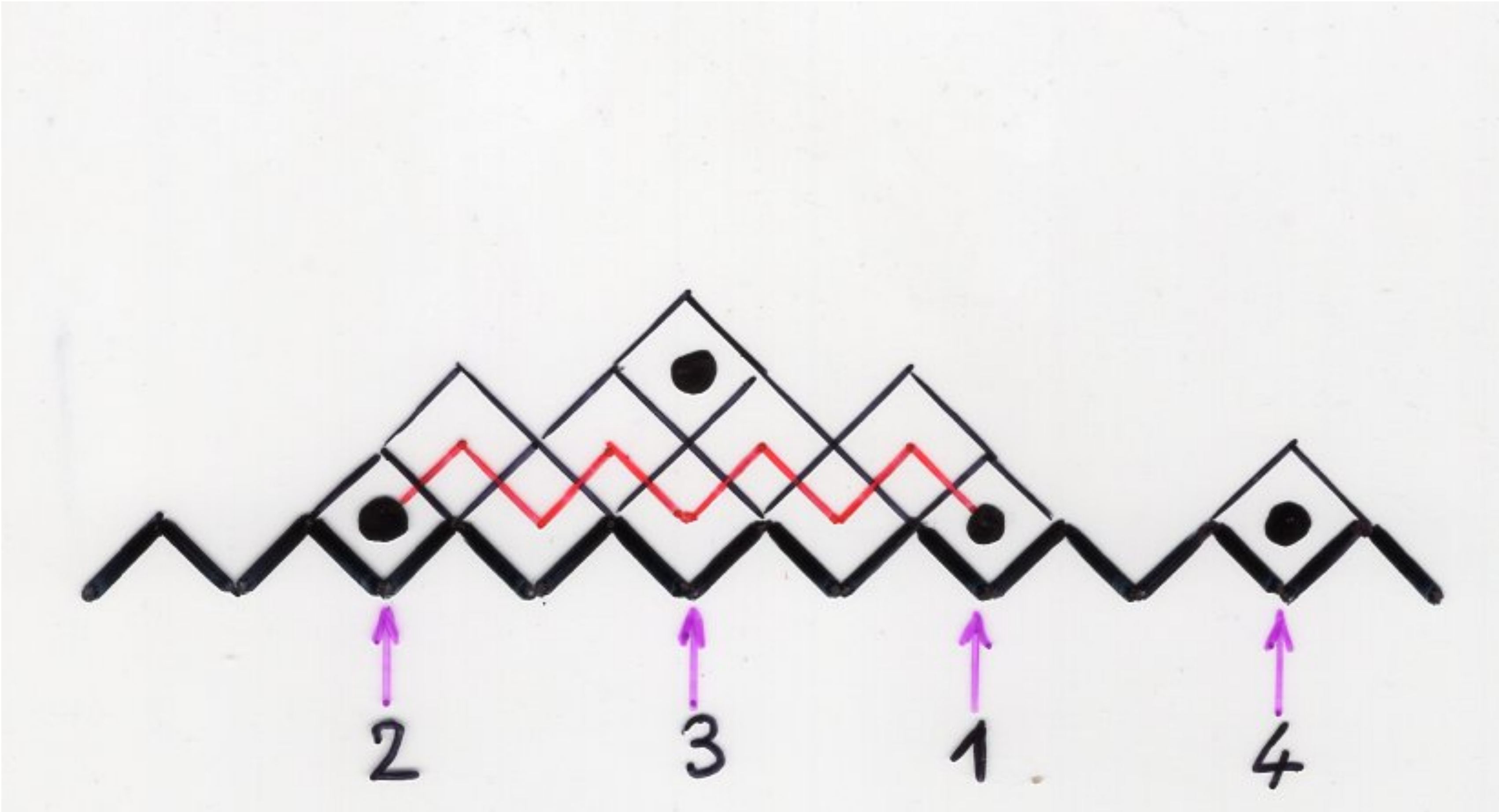
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



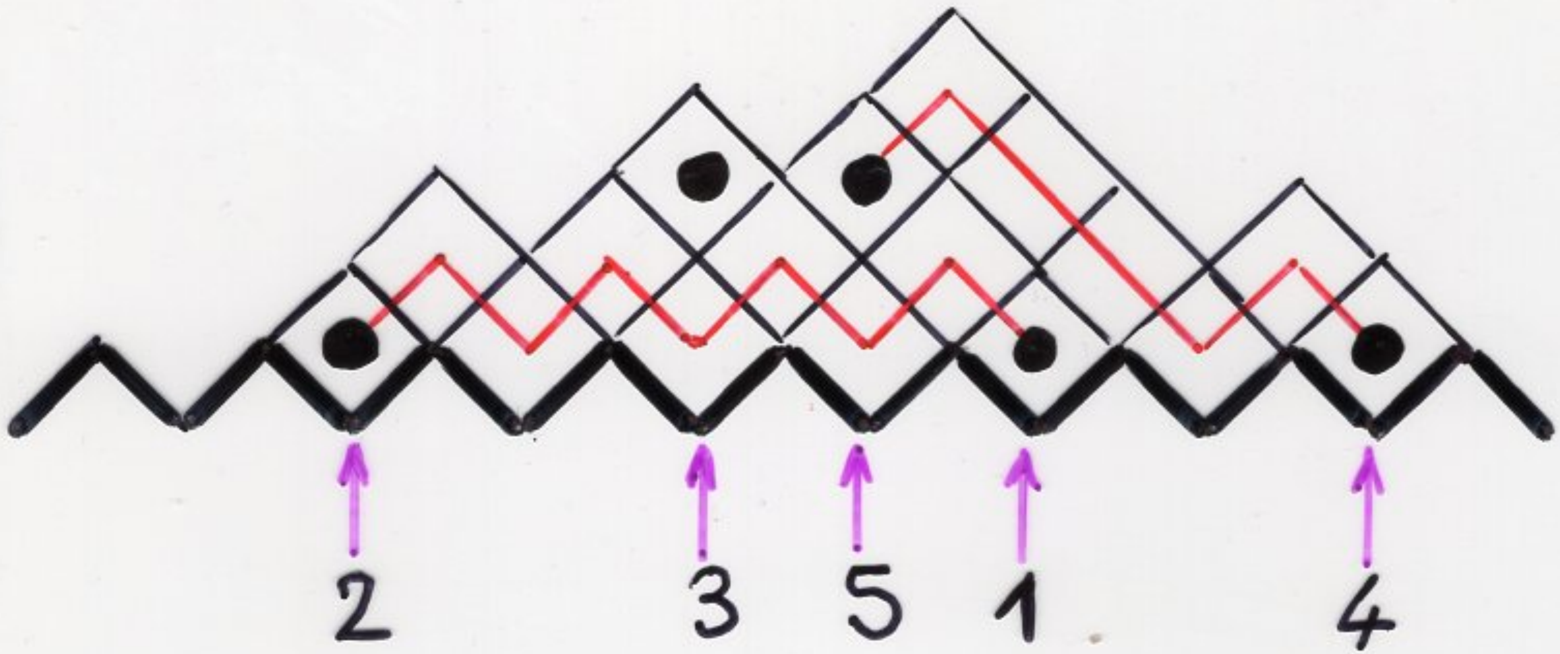
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



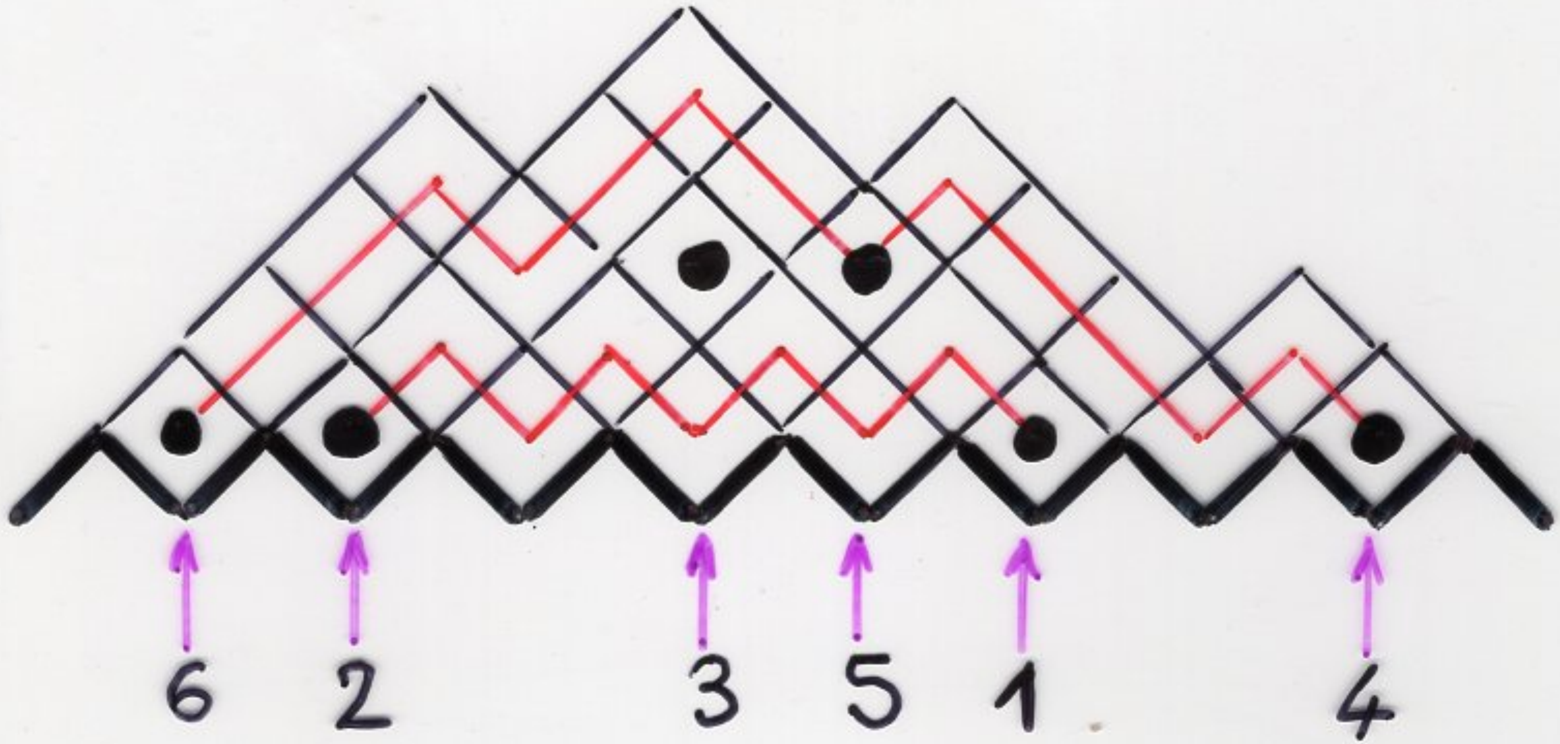
$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$



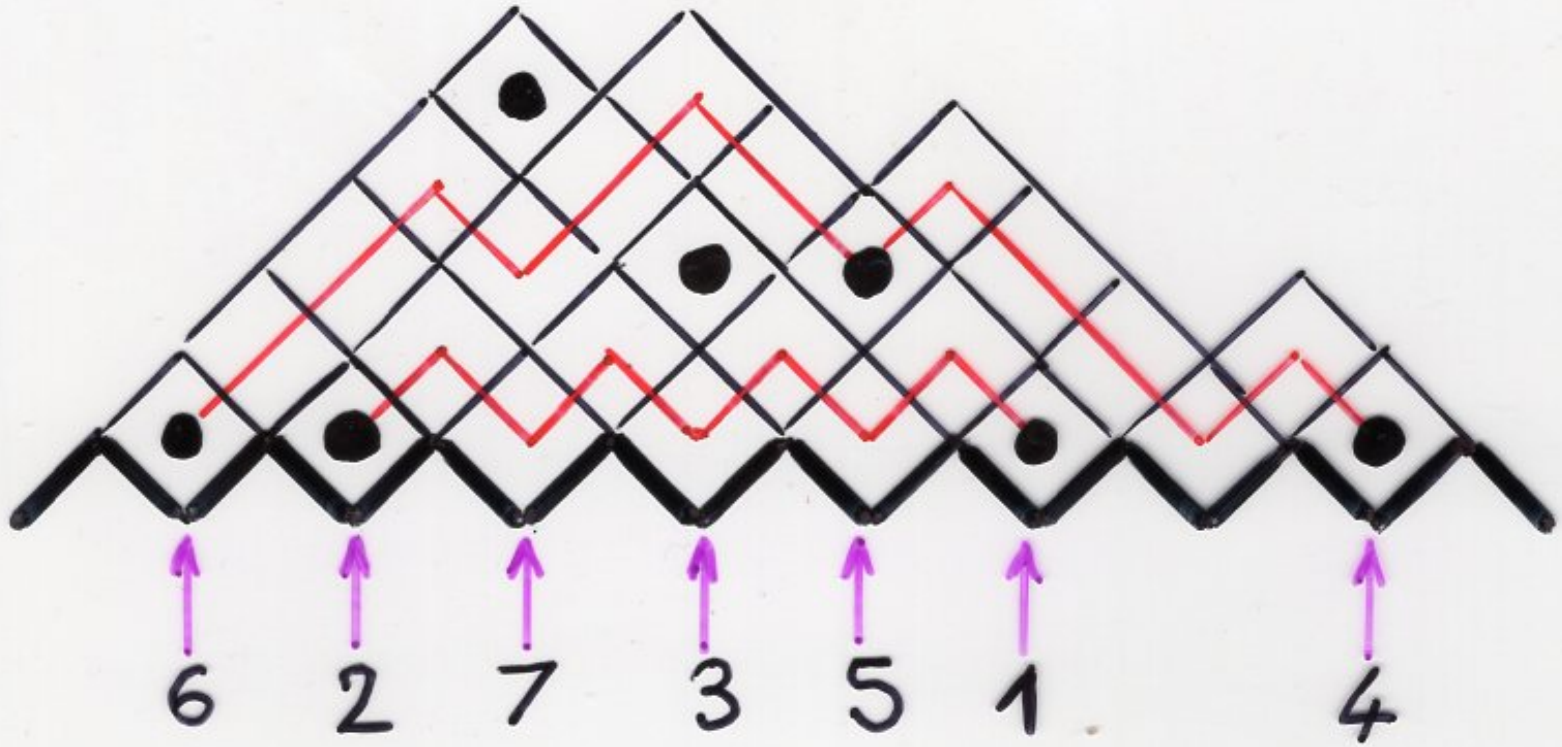
$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$



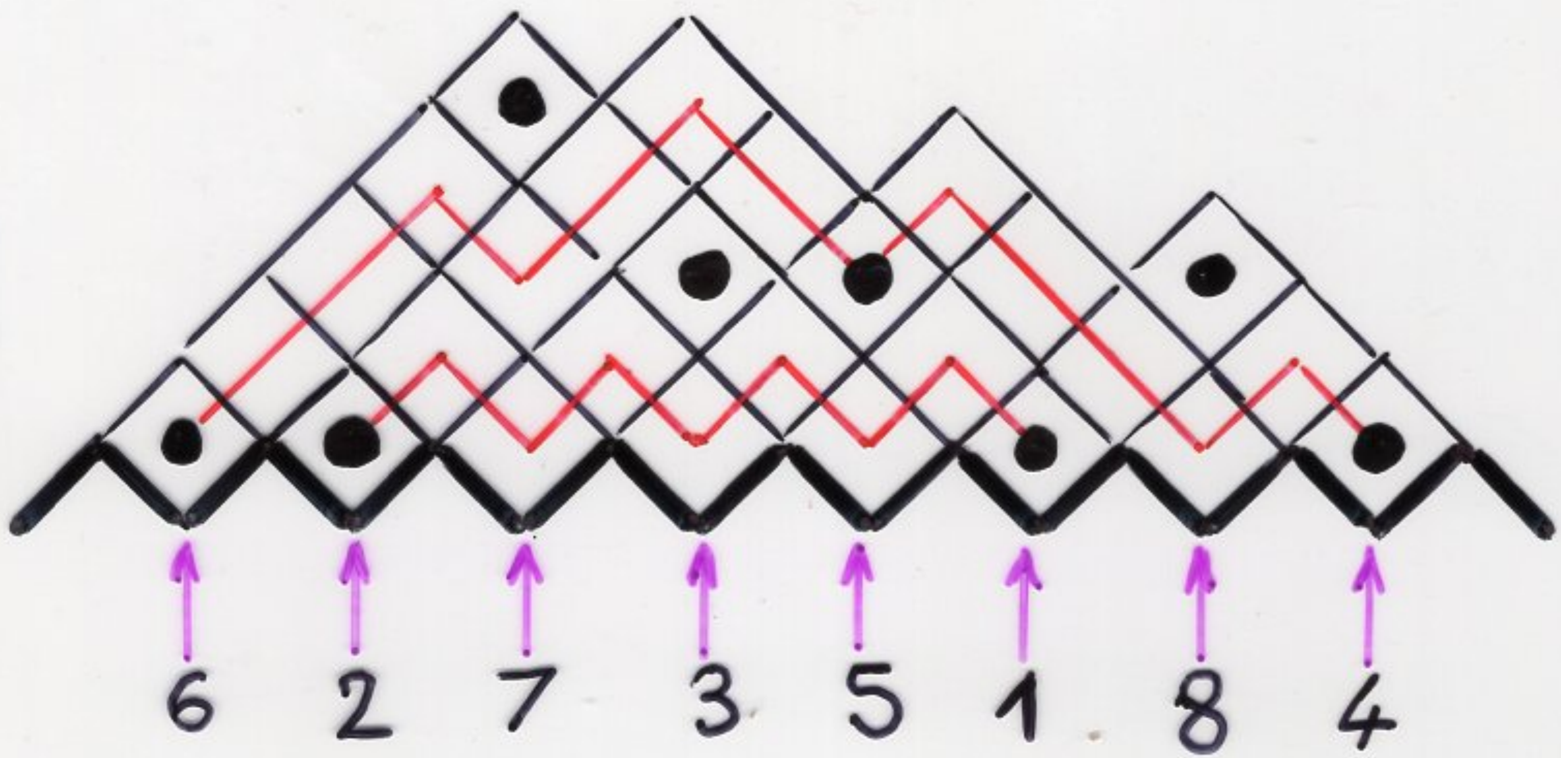
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

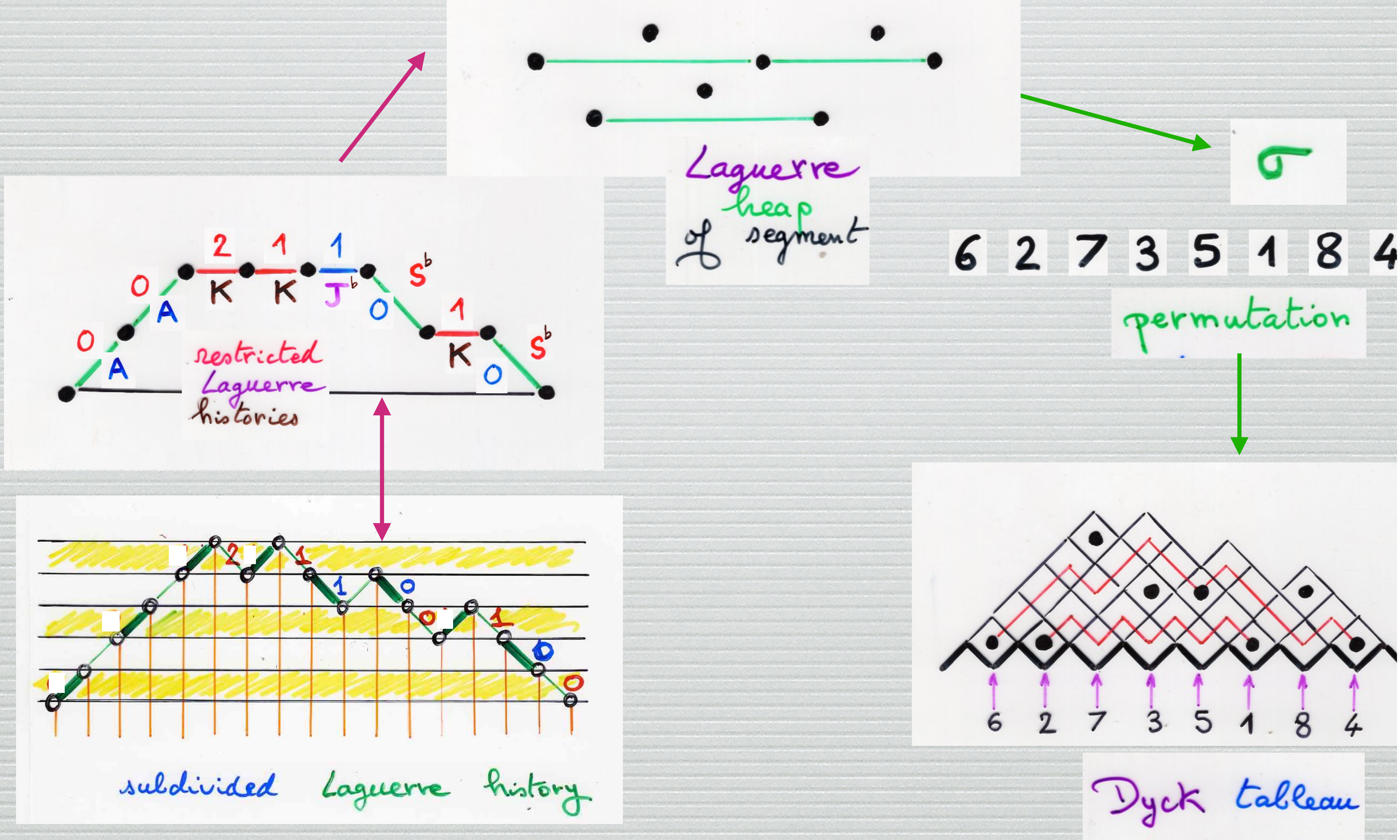


$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$


6 2 7 3 5 1 8 4

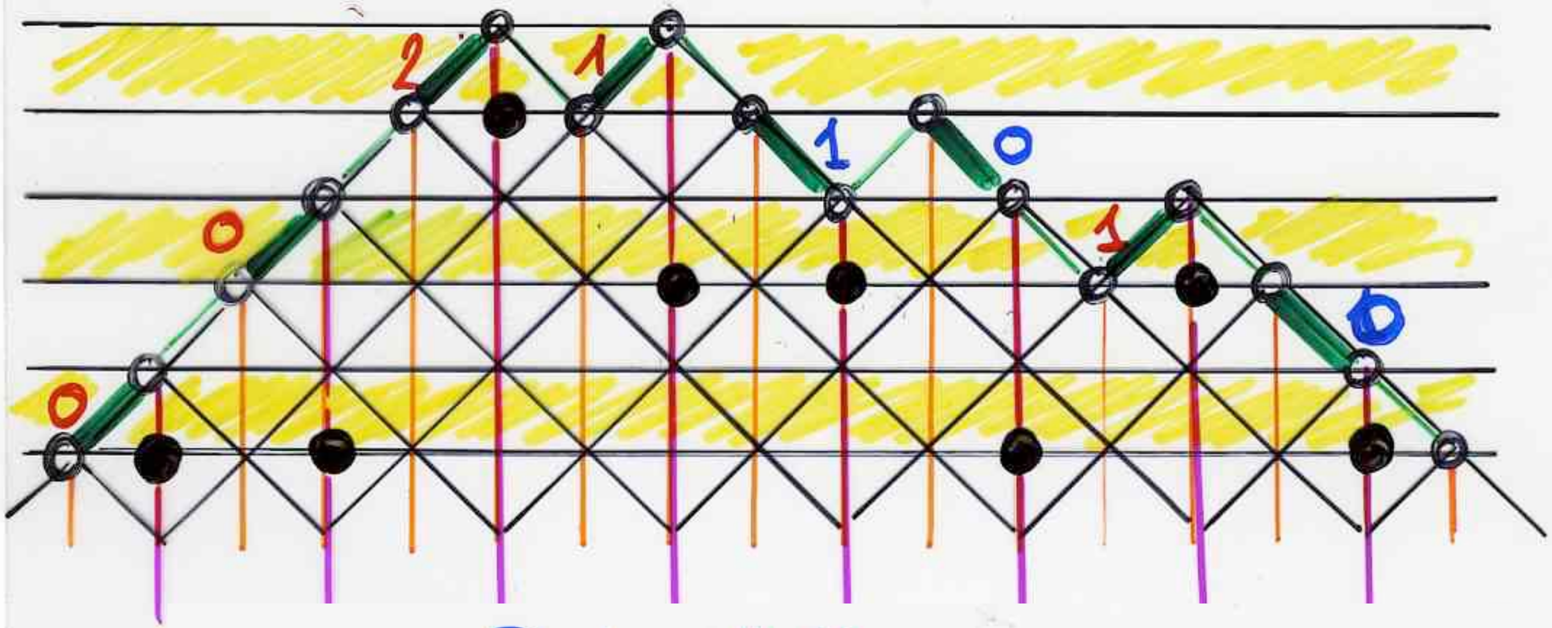


Laguerre heap of segment
σ
6 2 7 3 5 1 8 4
permutation
restricted Laguerre histories
subdivided Laguerre history
6 2 7 3 5 1 8 4
Dyck tableau

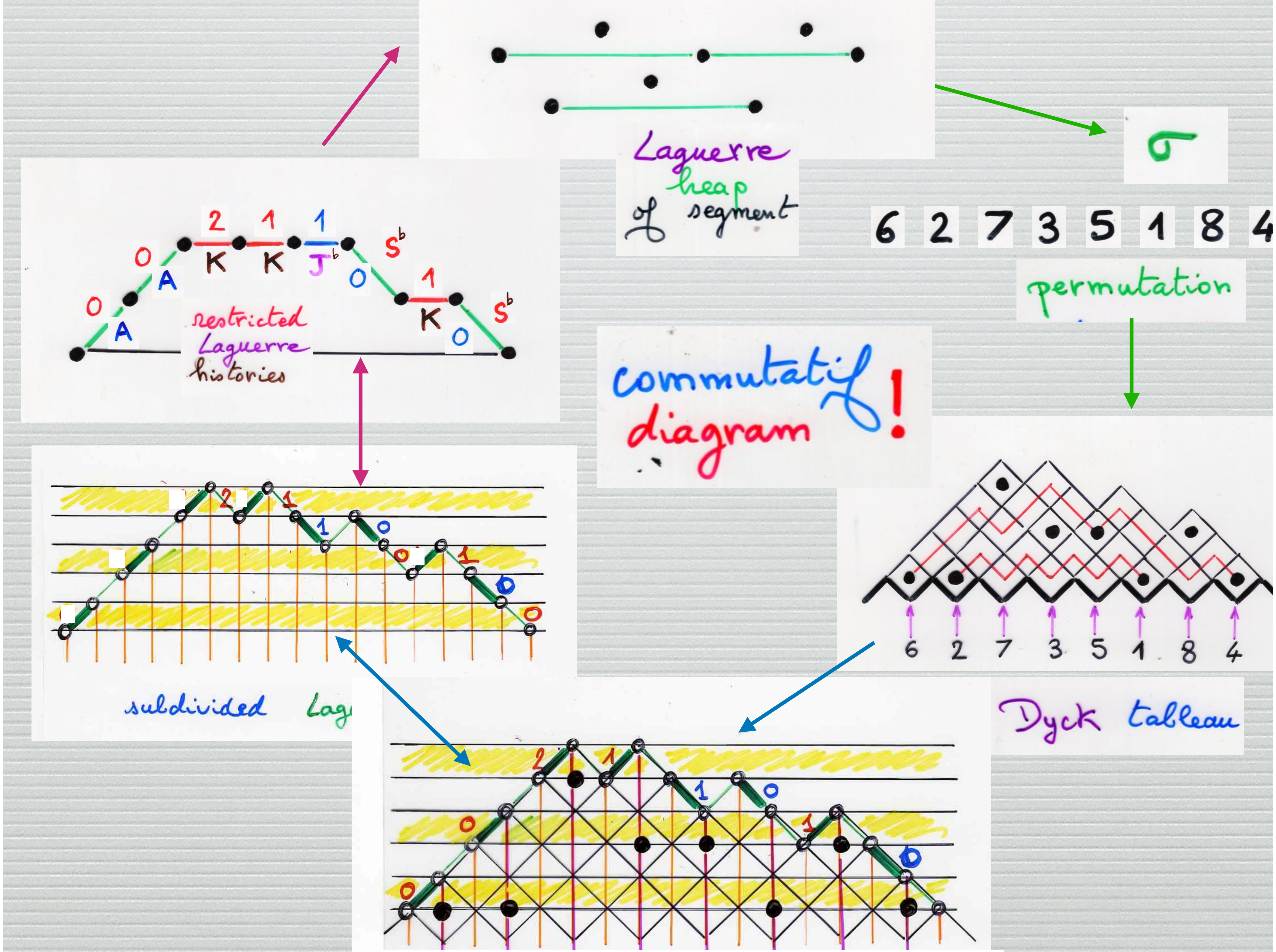
Dyck tableaux

to

subdivided Laguerre histories



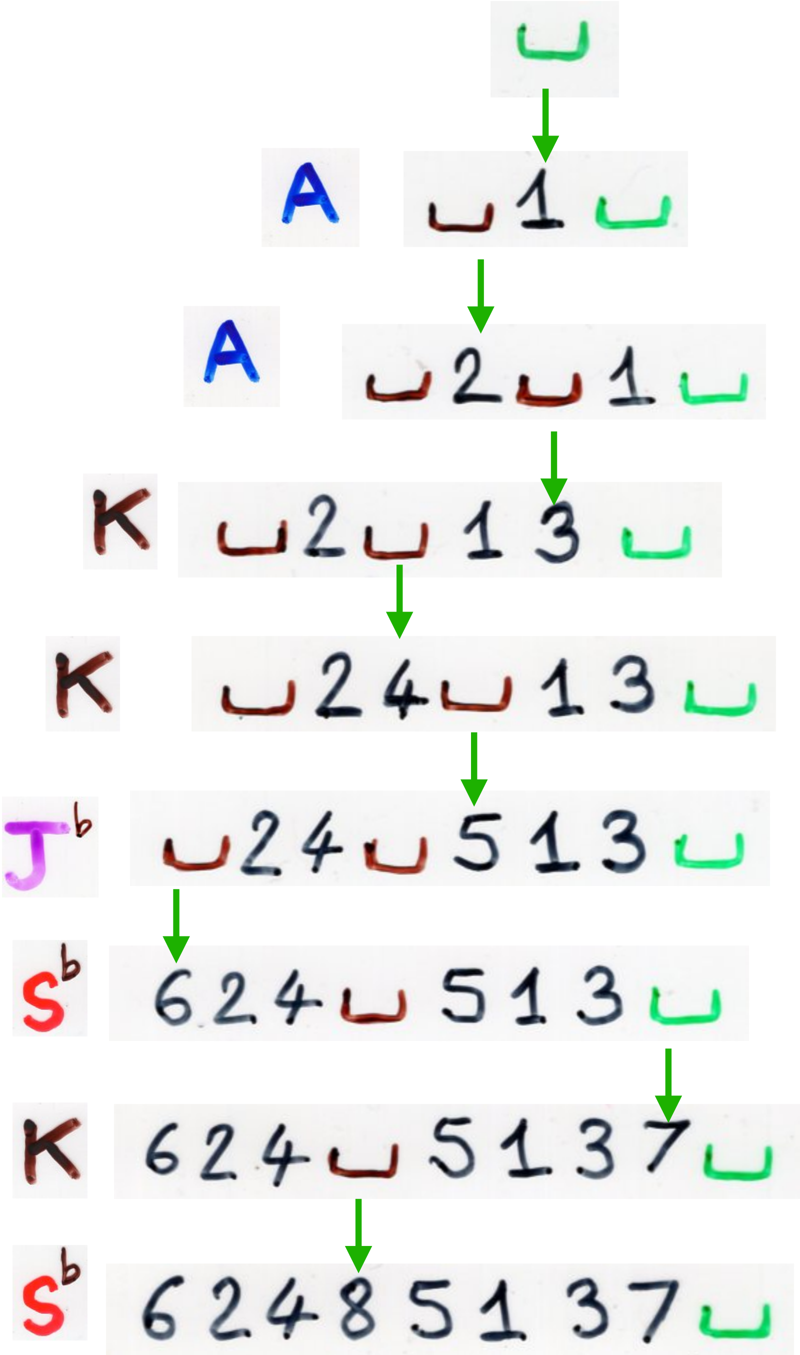
Dyck tableau as a subdivided Laguerre history

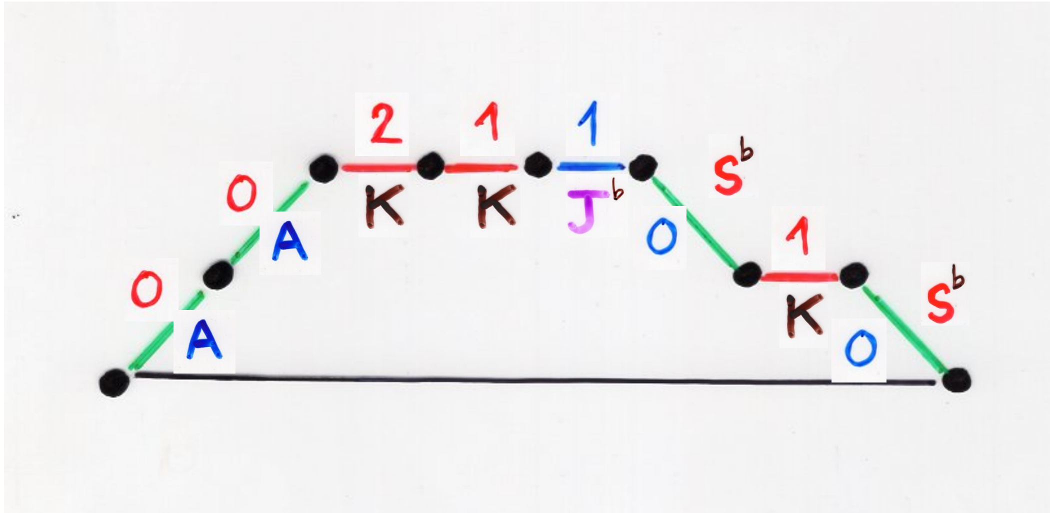

Laguerre heap of segment
σ
6 2 7 3 5 1 8 4
permutation
restricted Laguerre histories
O A O A 2 K 1 K 1 J^b O S^b 1 K O S^b
commutatif diagram !
2 1 1 0 0 1 0 0
subdivided Lag
6 2 7 3 5 1 8 4
Dyck tableau
2 1 0 1 0 1 0 0

From (restricted) Laguerre histories

to

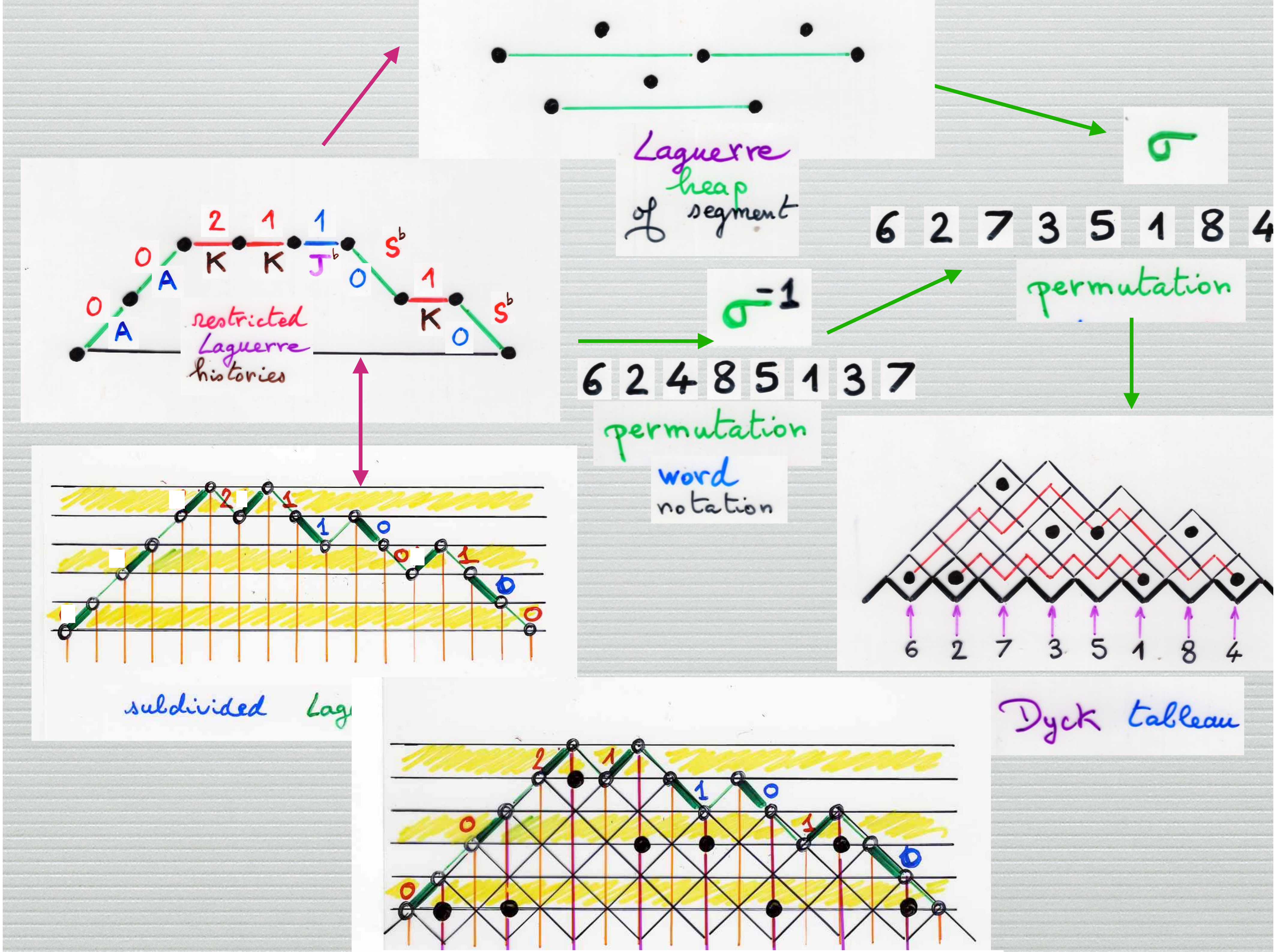
Permutations
(word notation)

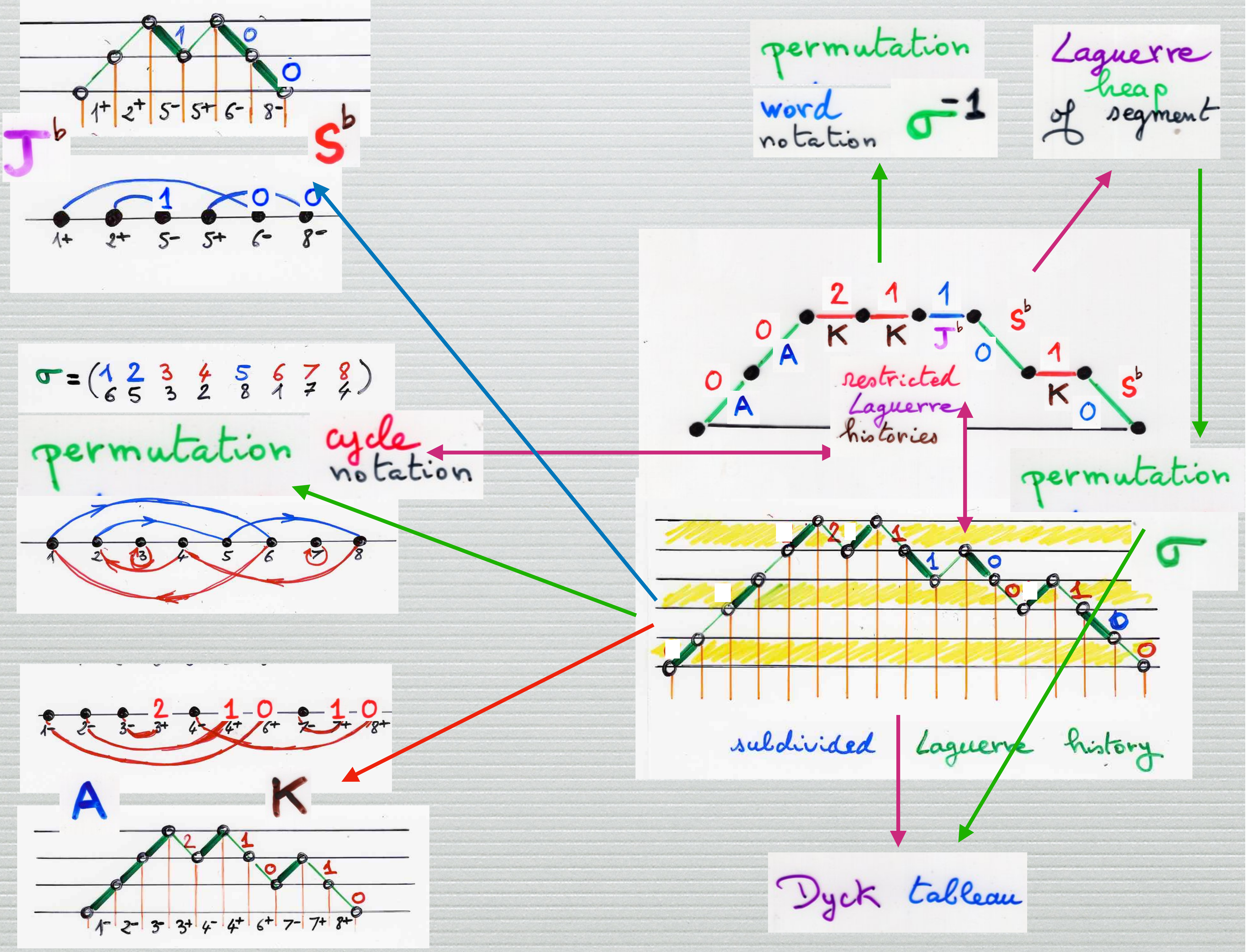


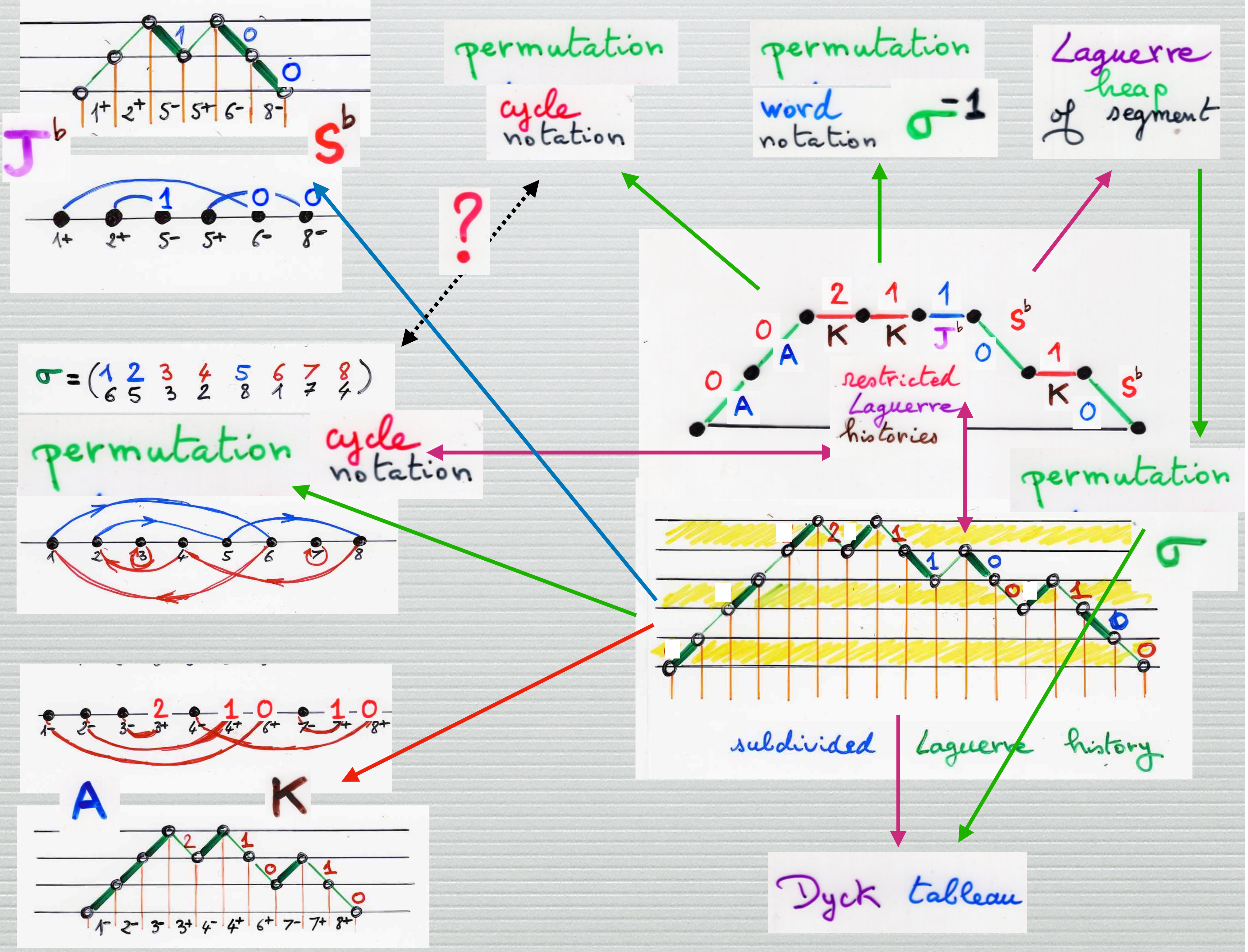


$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 7 & 3 & 5 & 1 & 8 & 4 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$


Laguerre heap of segment
σ
6 2 7 3 5 1 8 4
permutation
restricted Laguerre histories
σ⁻¹
6 2 4 8 5 1 3 7
permutation word notation
Dyck tableau
subdivided


J^b
S^b
permutation
word notation
σ^-1
Laguerre heap of segment
restricted Laguerre histories
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
permutation cycle notation
permutation
σ
subdivided Laguerre history
A
K
Dyck tableau



J^b
S^b
permutation
cycle notation
permutation
word notation
σ^-1
Laguerre heap of segment
?
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
permutation
cycle notation
restricted Laguerre histories
permutation
σ
subdivided Laguerre history
A
K
Dyck tableau

Sokal, Zeng
talk SLC 81 (2018)

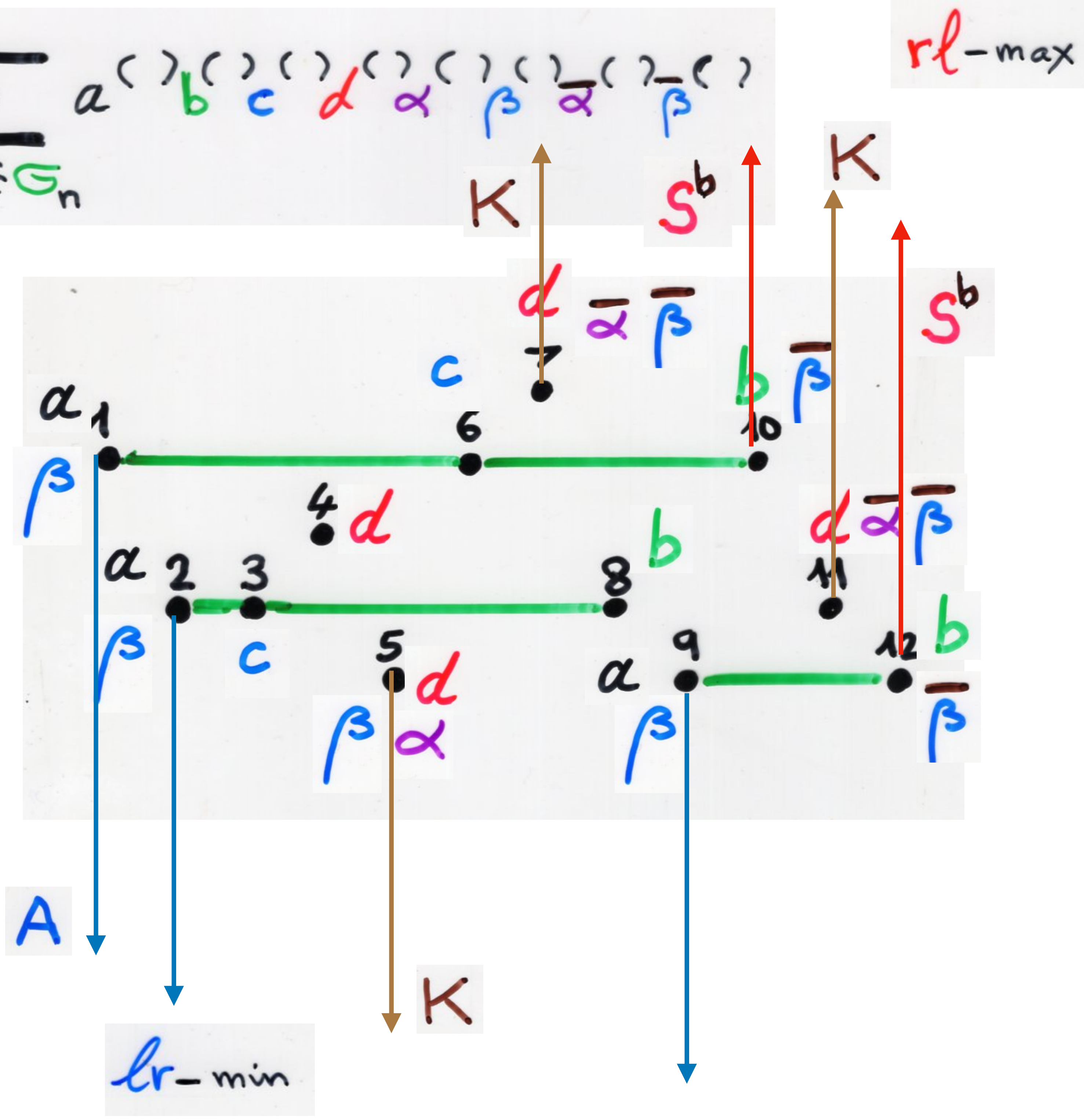
S. Yan, H. Zhou, Z. Lin
arXiv, 5 Feb 2019

"A new encoding of permutations by Laguerre histories"

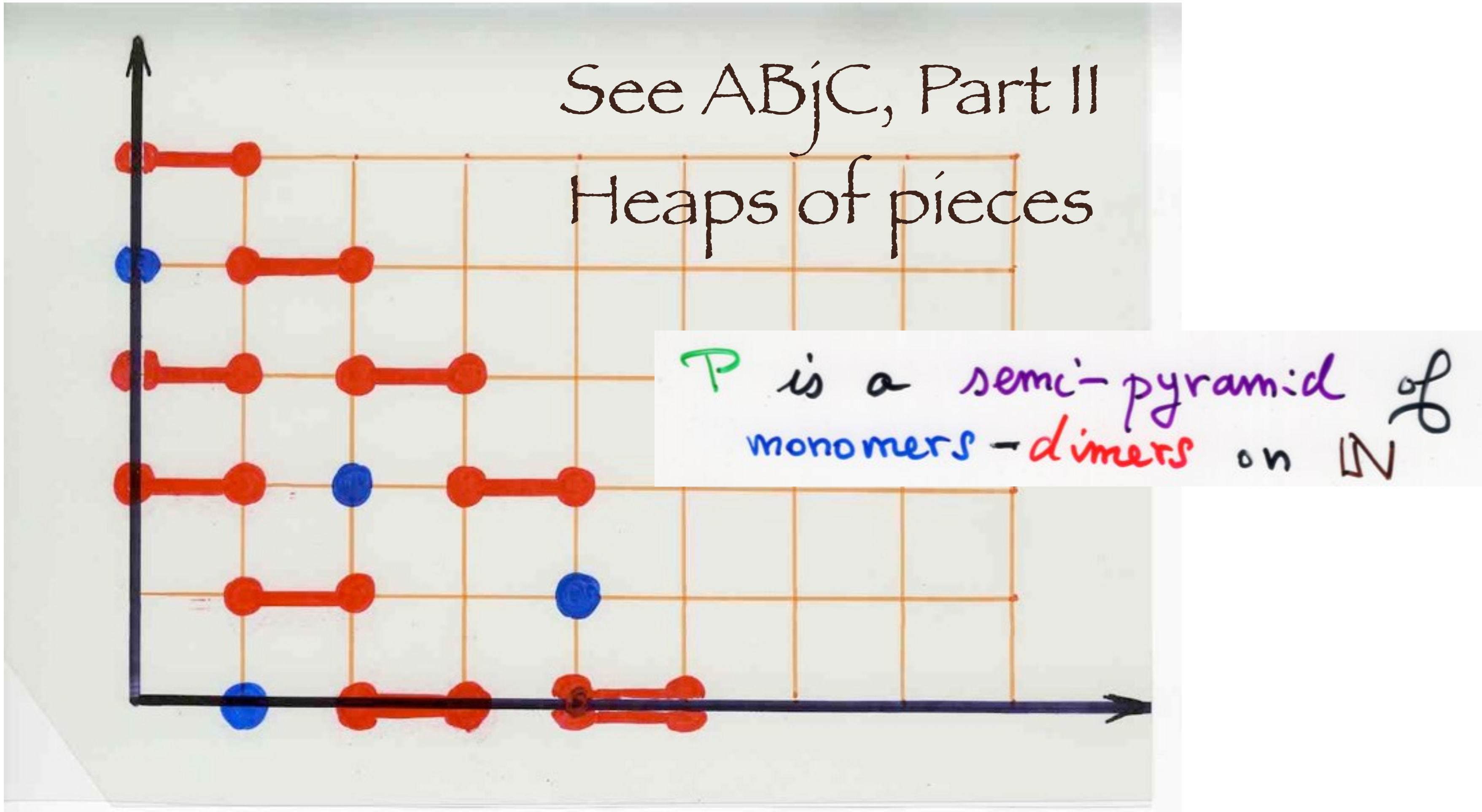
$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{(\ )} b^{(\ )} c^{(\ )} d^{(\ )} \alpha^{(\ )} \beta^{(\ )} \bar{\alpha}^{(\ )} \bar{\beta}^{(\ )}$$

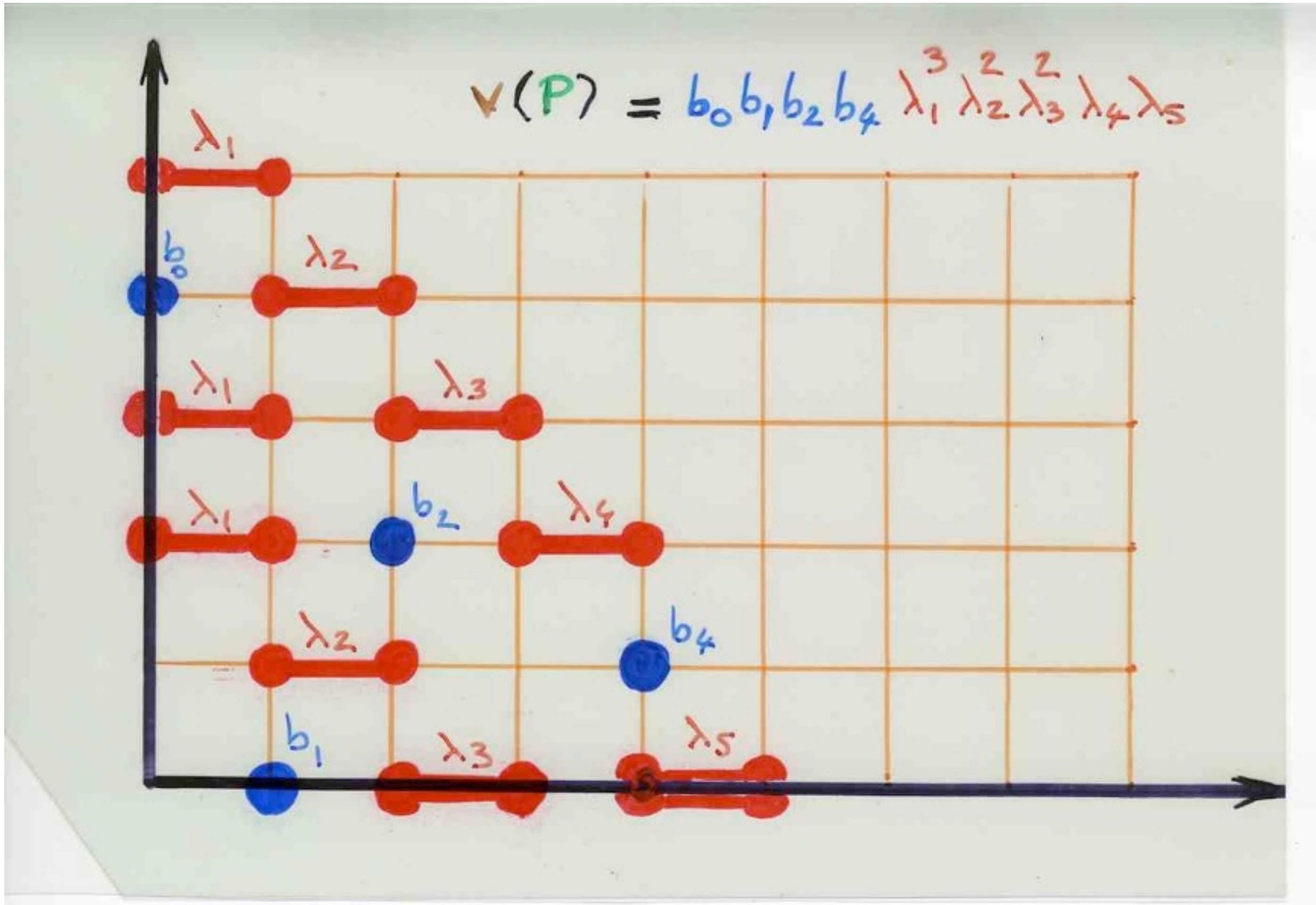
12 parameters

a	A
b	S^b
c	J^b
d	K
α	$\bar{\alpha}$
β	$\bar{\beta}$

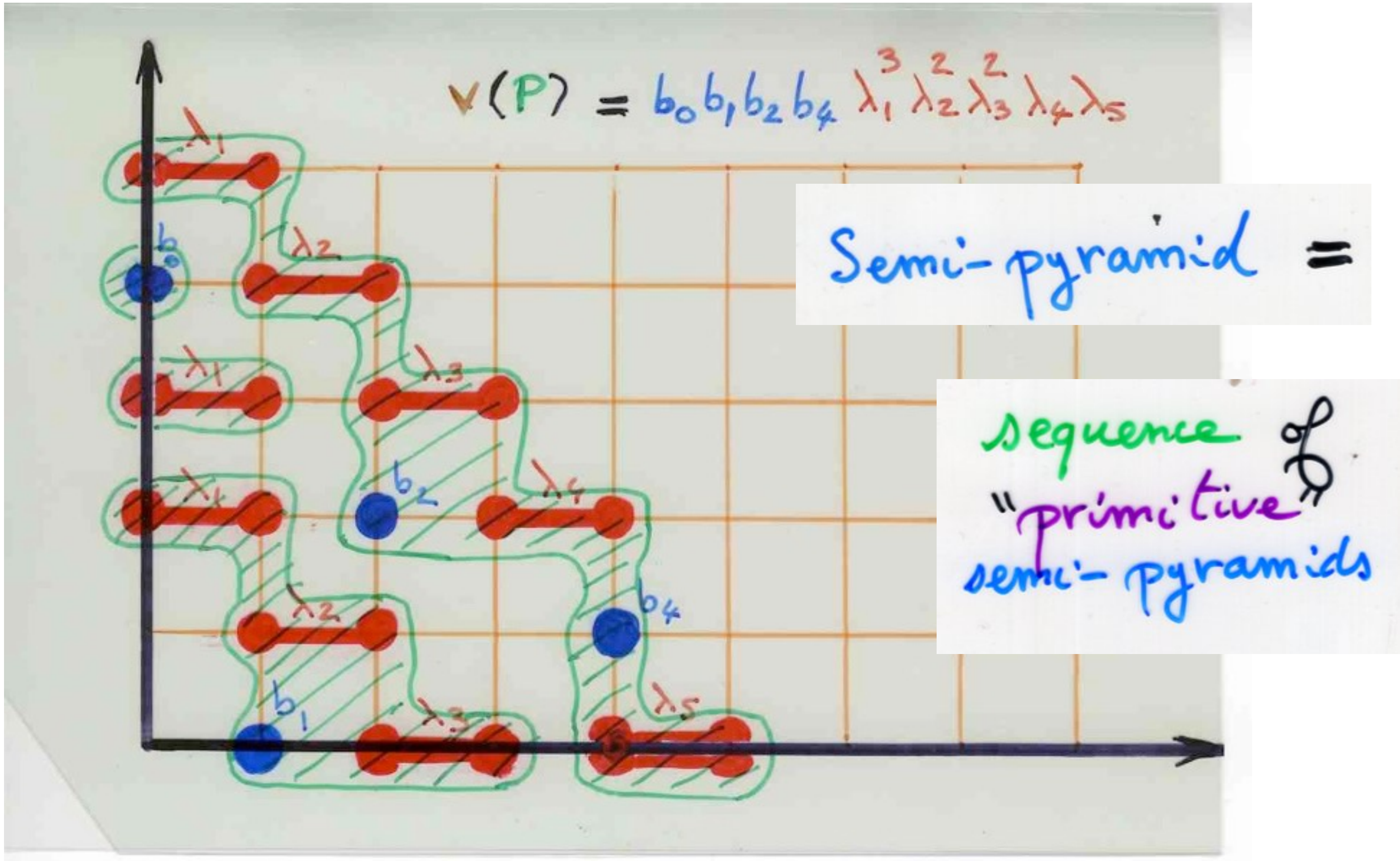


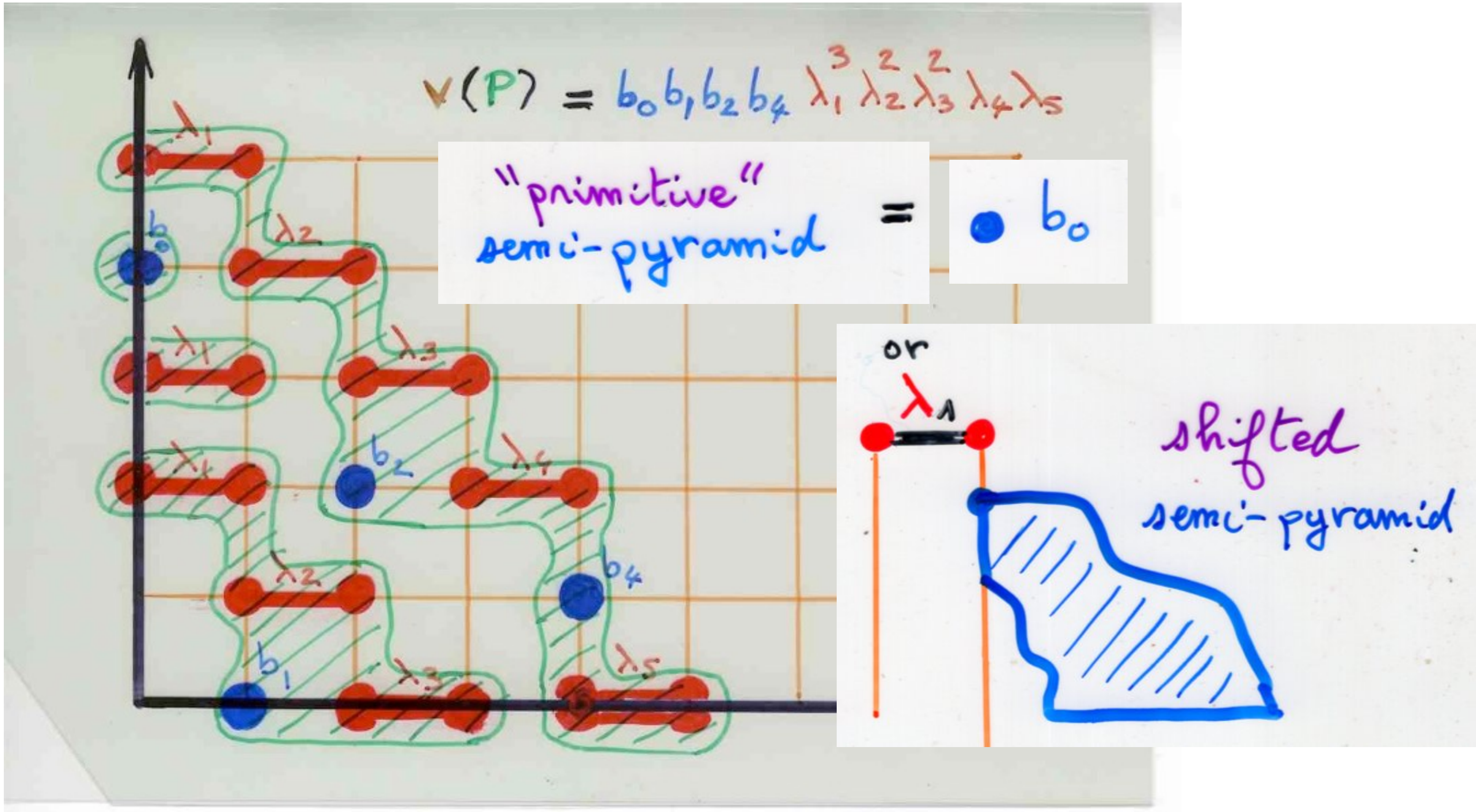
Interpretation with heaps of monomers and dimers


See ABjC, Part II
Heaps of pieces
$\mathcal{P}$ is a semi-pyramid of monomers-dimers on $\mathbb{N}$



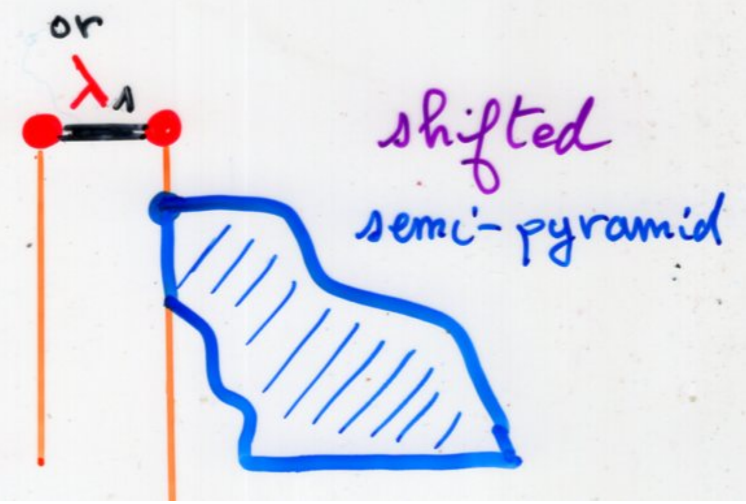
$v(P) = b_0 b_1 b_2 b_4 \lambda_1^3 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5$
λ_1
b_0
λ_2
λ_1
λ_3
λ_1
b_2
λ_4
λ_2
b_4
b_1
λ_3
λ_5


$v(P) = b_0 b_1 b_2 b_4 \lambda_1^3 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5$
Semi-pyramid =
sequence of "primitive" semi-pyramids
λ_1
λ_2
λ_3
λ_4
λ_5
b_0
b_1
b_2
b_4



$v(P) = b_0 b_1 b_2 b_4 \lambda_1^3 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5$
"primitive" semi-pyramid
=
b_0
or
λ_1
shifted semi-pyramid

"primitive" semi-pyramid = $\bullet\ b_0$ or



$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E)\, t^{|E|} = \frac{1}{1 - b_0 t - \lambda_1 t^2 \left(\sum_{\substack{E \\ \text{semi-pyramid}}} \delta v(E)\, t^{|E|} \right)}$$

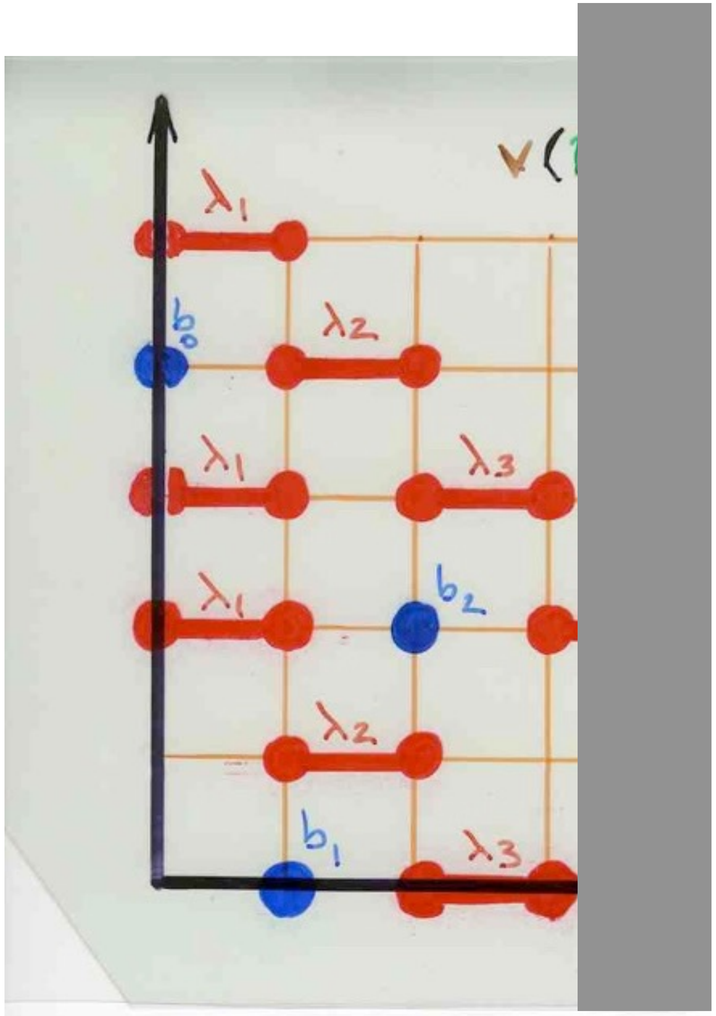
$$|E| = m(E) + 2\, d(E)$$

$m(E)$: number of monomers

$d(E)$: number of dimers

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E)\, t^{|E|} = \frac{1}{1 - b_0 t - \lambda_1 t^2 \left(\sum_{\substack{E \\ \text{semi-pyramid}}} \delta\, v(E)\, t^{|E|} \right)}$$

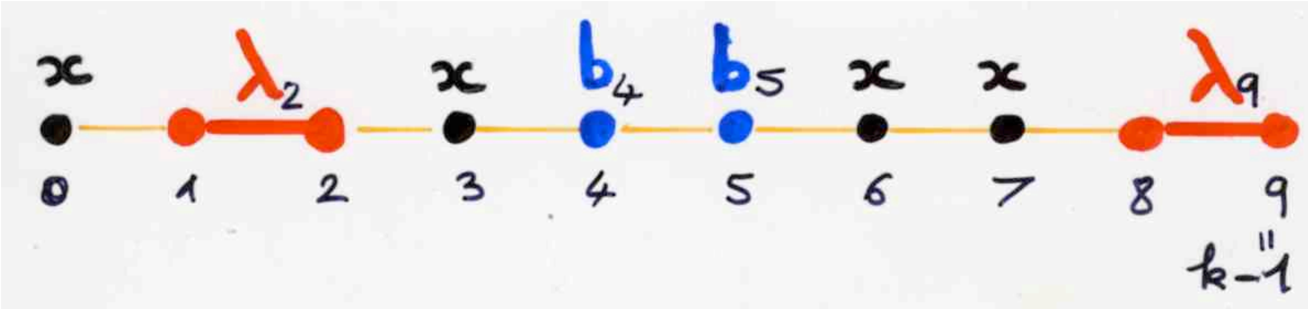
$$J(t; b, \lambda) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cdots\cdots \cfrac{}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$



inversion theorem

$$\frac{N}{D}$$

Pavage = trivial heap (→ Part II)
of monomers, dimers

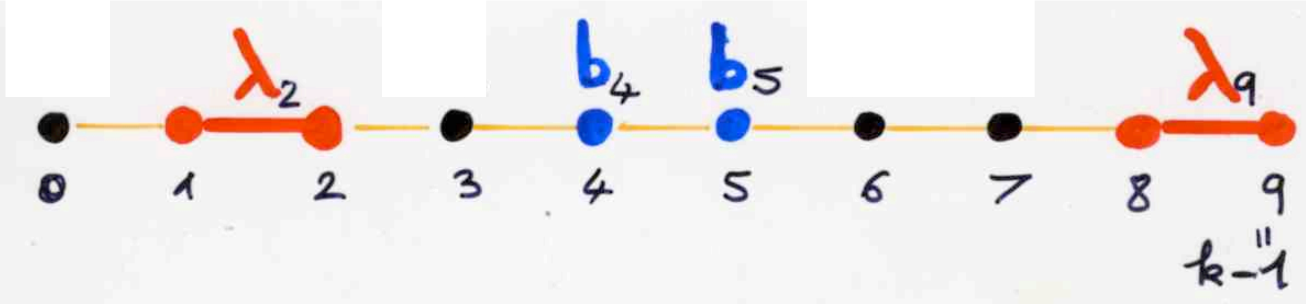


$$v(\alpha) = b_4 \, b_5 \, \lambda_2 \, \lambda_9 \qquad\qquad (-1)^4 \, b_4 \, b_5 \, \lambda_2 \, \lambda_9 \, x^4$$

$ip(\alpha)$ = number of isolated points of α

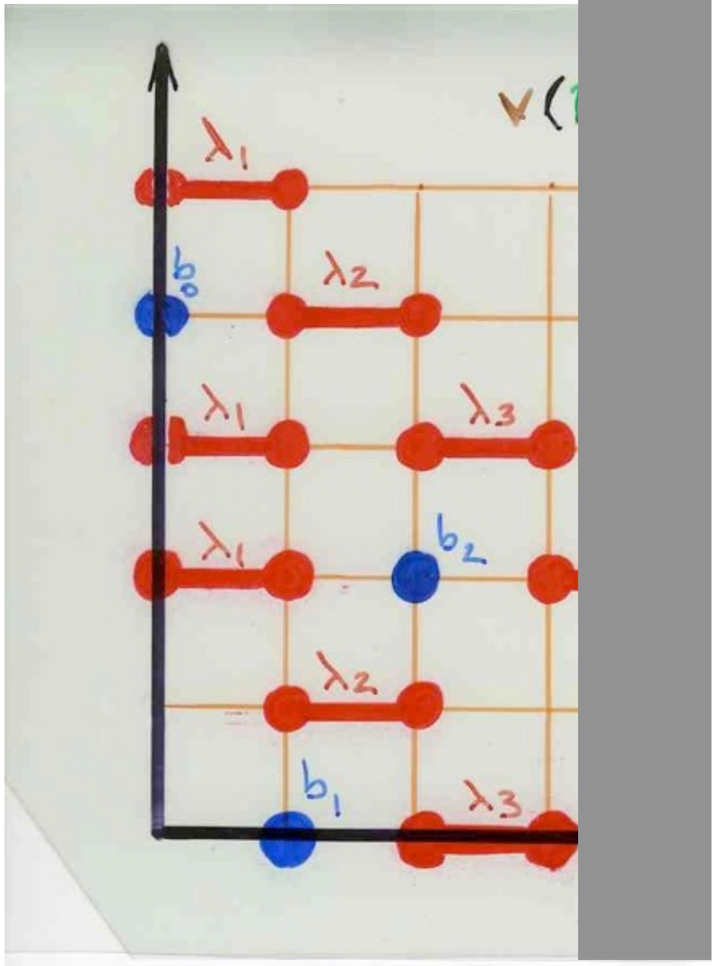
$|\alpha|$ = number of pieces (monomers − dimers) of the pavage α

$$P_n(x) = \sum_{\substack{\alpha \\ \text{pavage of } [0, n-1]}} (-1)^{|\alpha|} \, v(\alpha) \, x^{ip(\alpha)}$$



$$P^*_n(x) = \sum_{\substack{\alpha \\ \text{pavage of } [0,n-1]}} (-1)^{|\alpha|} v(\alpha)\, x^{m(\alpha)+2d(\alpha)}$$

(= trivial heap)



convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

inversion theorem

$$\frac{N}{D}$$