Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 3
Continued fractions

Ch 3a

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February 11, 2019

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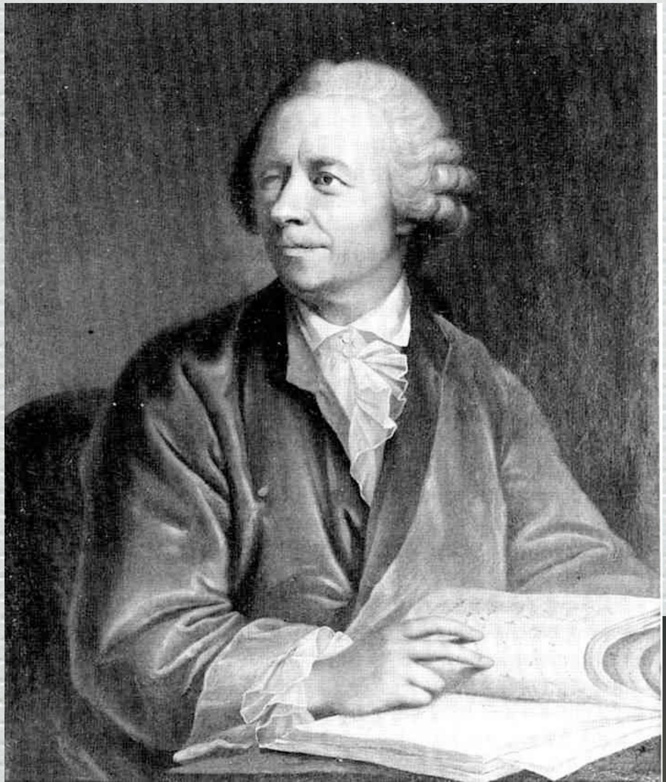
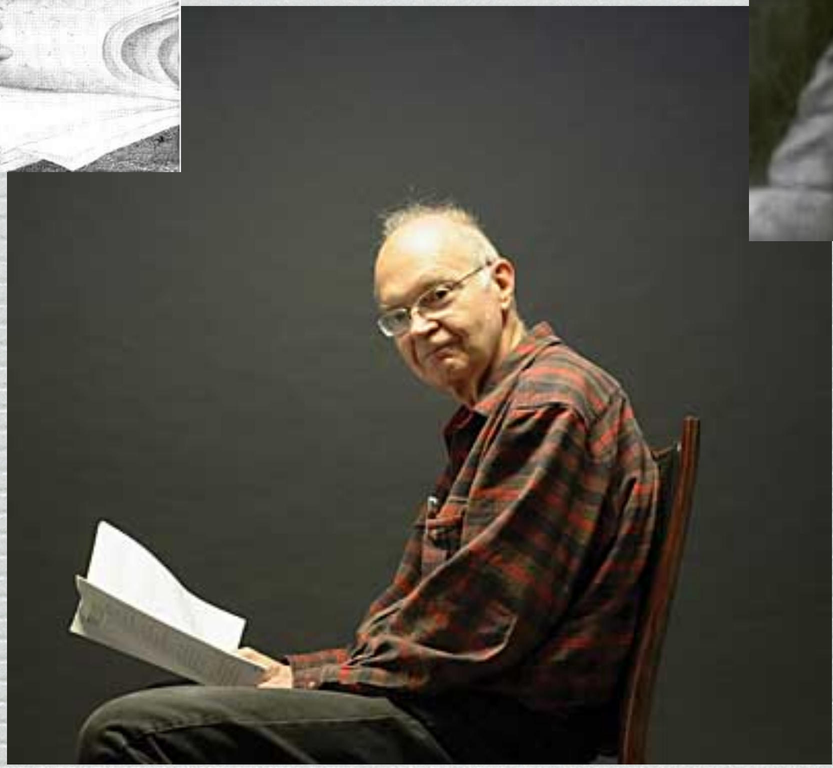
This lecture is dedicated to my friend

Philippe Flajolet

(1948-2011)





Happy New Year 2009

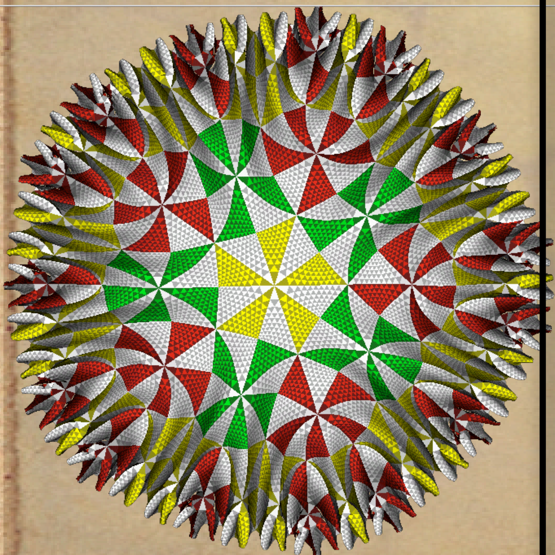
GIFT. *Define the "equiharmonic numbers" by*

$$K_\nu := \frac{(6\nu)!}{\Omega^{6\nu}} \sum_{(n_1,n_2)\in(\mathbb{Z}\times\mathbb{Z})\setminus\{(0,0)\}} \frac{1}{(n_1 e^{-2i\pi/3} + n_2 e^{2i\pi/3})^{6\nu}}, \qquad \Omega := \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)^3.$$

The generating function of (K_ν) *admits the continued fraction representation*

$$\frac{7}{36}\sum_{\nu\geq 1} K_\nu z^{\nu-1} = \cfrac{1}{1-\cfrac{d_1\cdot z}{1-\cfrac{d_2\cdot z}{\ddots}}},$$

where $d_1 = \frac{10880}{13}$, $d_2 = \frac{13810240}{247}$, $d_n = \frac{1}{4}\frac{(3n)(3n+1)^2(3n+2)^2(3n+3)^2(3n+4)}{(6n+1)(6n+7)}$.



27 Dec 2008

Maths and Computer Science

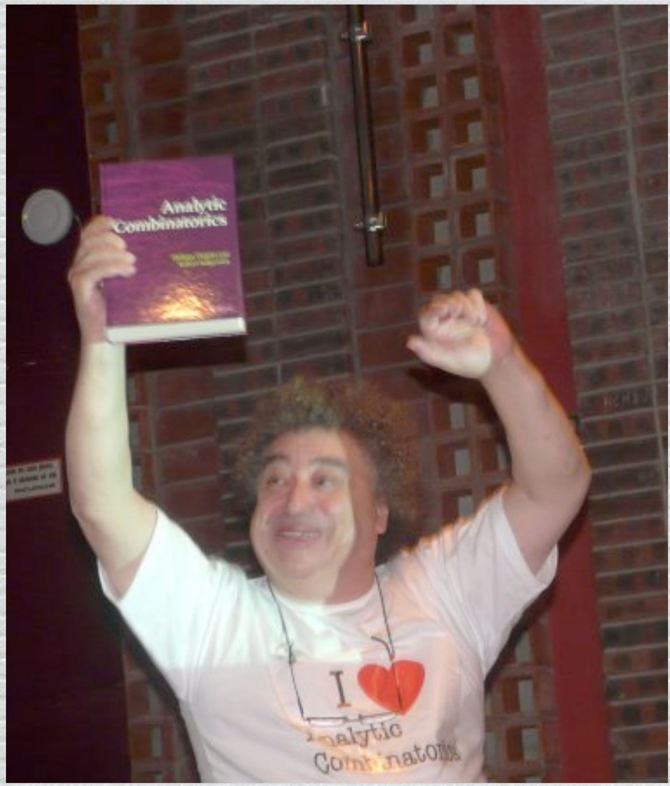
ALEA

Séminaire Flajolet

X.V.: Survey of 16 papers of P.F. on continued fractions

Collected works, Vol 5, Ch 3.

Analytic Combinatorics

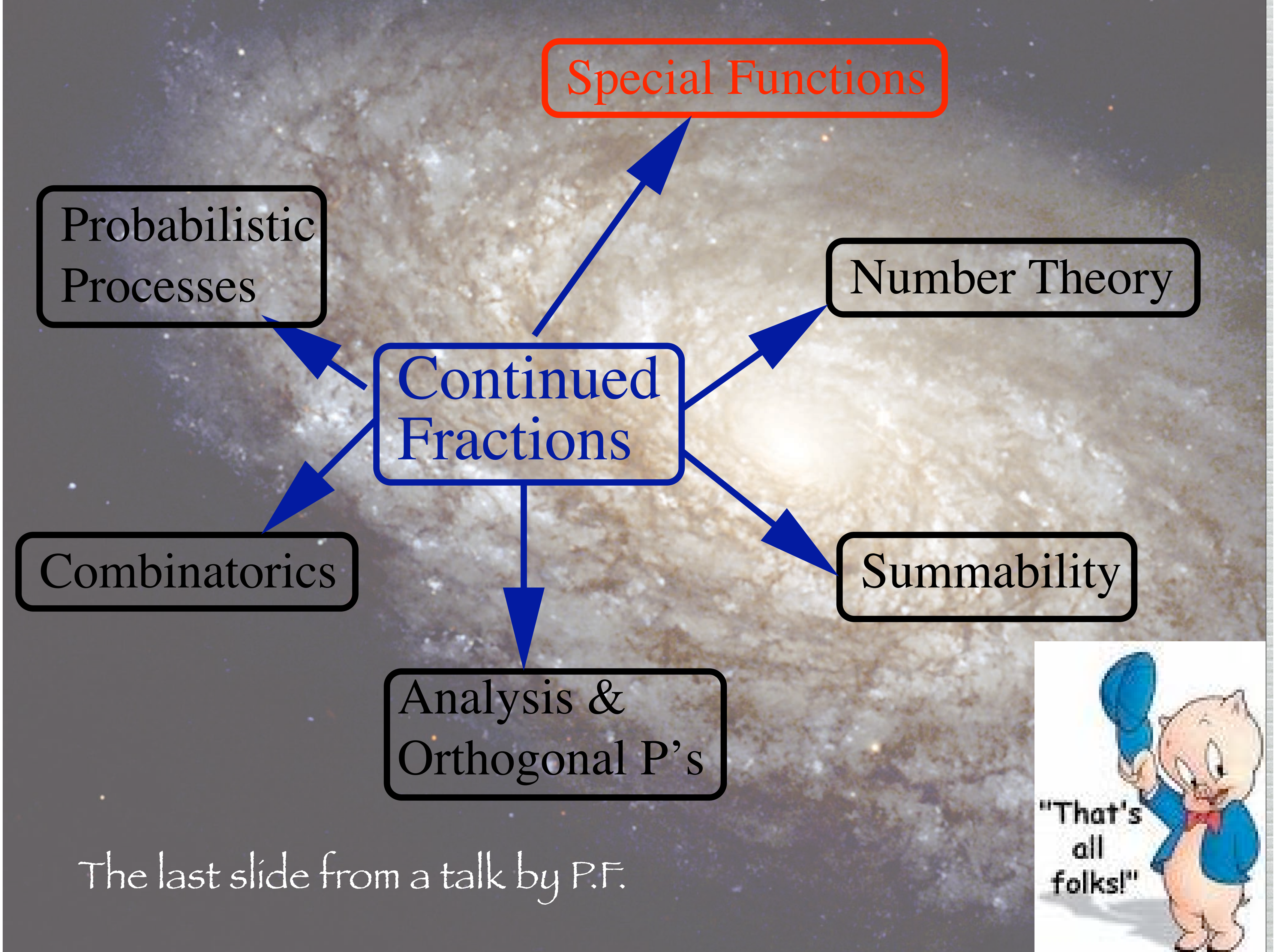


With Robert Sedgewick

Price Leroy P.Steele (2019)

60th birthday

Photo M. Soria, Paris, 1-2 Dec.
2008


Special Functions
Probabilistic Processes
Number Theory
Continued Fractions
Combinatorics
Summability
Analysis & Orthogonal P's
"That's all folks!"
The last slide from a talk by P.F.

arithmetic continued fractions

continued fraction in number theory

$$\phi - 1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}$$

ϕ golden number

$$\frac{1+\sqrt{5}}{2}$$

convergents

$$\cfrac{1}{1+\cfrac{1}{1+1}} = \frac{2}{3} \quad (2)$$

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}} = \frac{3}{5} \quad (3)$$

$$\underbrace{\cfrac{1}{1+\cfrac{1}{1+\cdots\cfrac{1}{1+1}}}}_{k} = \frac{F_k}{F_{k+1}}$$

Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$
$$F_0\ F_1\ F_2\ F_3\ F_4\ F_5$$

$$\frac{F_k}{F_{k+1}} \longrightarrow \phi - 1 = \frac{\sqrt{5}-1}{2}$$

Apéry

$$\zeta(3) = \sum 1/n^3$$

irrational

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\cdots}}}}$$

$$\varpi(n) = (2n+1)\left(17n(n+1)+5\right)$$

Some analytic continued fractions …

DE FRACTIONIBVS CONTINVIS.
DISSERTATIO.

AVCTORE
Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \cdot \frac{10}{11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

atque series infinita ita se habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis est huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

Euler

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

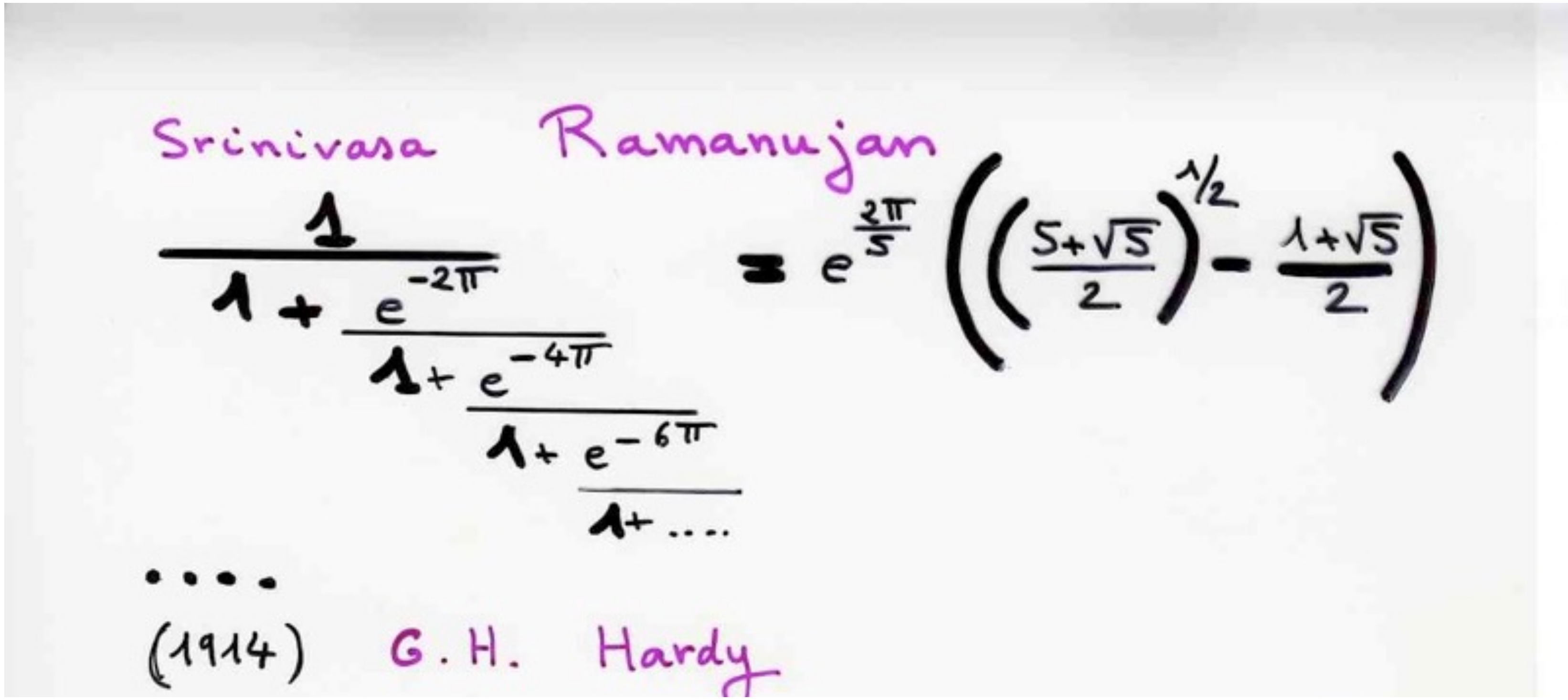
$$z = 1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}$$

reperietur enim iisdem operationibus inſtitutis :

$$z = \cfrac{1}{1+\cfrac{mx}{1+\cfrac{nx}{1+\cfrac{(m+n)x}{1+\cfrac{2nx}{1+\cfrac{(m+2n)x}{1+\cfrac{3nx}{1+\cfrac{(m+3n)x}{1+\cfrac{4nx}{1+\cfrac{(m+4n)x}{1+\cfrac{5nx}{1+\text{etc.}}}}}}}}}}}$$

Eadem vero expreſſio, aliæque ſimiles facile erui pos-

Srinivasa Ramanujan
1887 - 1920



[These formulas] defeated me completely. I had never seen anything in the least like this before. A single look at them is enough to show they could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them.

Srinivasa Ramanujan
1887 - 1920

Godfrey Harold Hardy
1877 - 1947

Ramanujan continued fraction

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cdots\cdots \atop \overline{1+\cfrac{q^k}{\cdots\cdots}}}}}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots}{1+\cfrac{q^k}{\cdots}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

Reminding Part I, Ch 1a, 29-46

formal power series algebra

formalisation

Formal power series algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \dots]$

$$a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$\mathbb{K}[t]$ polynomials algebra

$$(a_0, a_1, a_2, \dots, a_n, \dots)$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable t and coefficients in $\mathbb{K}$)

algebra of formal power series

- sum $\quad f+g=h, \quad a_n+b_n=c_n$
- product $\quad fg=h, \quad c_n=\sum_{\substack{p+q=n \\ p,q\geq 0}} a_p b_q$
- product (by a scalar) $\quad \lambda f=h, \quad c_n=\lambda a_n$

$$f=\sum_{n\geq 0} a_n t^n, \quad g=\sum_{n\geq 0} b_n t^n, \quad h=\sum_{n\geq 0} c_n t^n$$

- **Inverse**

$$\frac{1}{1-f} = 1 + f + f^2 + \dots + f^n + \dots$$

(si ord$(f) \geq 1$)

- **derivative**

$$f' \quad \frac{df}{dt} = \sum_{n \geq 1} n\, a_n t^{n-1}$$

generating power series
of the coefficients (numbers a_n)

$$\sum_{n\geq 0} a_n t^n = f(t)$$

(ordinary generating function)

exponential generating function

$$\sum_{n\geq 0} a_n \frac{t^n}{n!}$$

summable family

example

$$\sum_{i \geq 0} (t + t^2)^i =$$

$1 + (t + t^2)$

$(t^2 + 2t^3 + t^4)$

$(t^3 + 3t^4 + 3t^5 + t^6)$

$(t^4 + 4t^5 + 6t^6 + \ldots$

$+ (t^5 - - - - -$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

Reminding Part I, Ch 1a, 55-62, 63-75

operations on combinatorial objects

formalisation

symbolic method

Philippe Flajolet (1948–2011)

(with Robert Sedgewick)

Analytic Combinatorics

(Cambridge Univ. Press, 2008)

Operations on combinatorial objects

Def. class of valued combinatorial objects

$\mathcal{A} = (A, v)$ A finite or enumerable set

$v : A \longrightarrow \mathbb{K}[X]$
valuation

Def. $f_{\mathcal{A}} = \sum_{\alpha \in A} v(\alpha)$

generating power series
of objects $\alpha \in A$ weighted by v

ex: objects of size n

$X = \{t\}$ $v(\alpha) = t^n$

n is the size of α, $|\alpha| = n$

$a_n = |A_{t^n}|$ (finite set)

$=$ number of objects $\alpha \in A$ of size n

$$f_A = \sum a_n t^n$$

$\alpha = (A, v_A)$ $\beta = (B, v_B)$

- **sum** $\alpha + \beta = \mathcal{C}$

$A \cap B = \emptyset$ $= (C, v_C)$

- $C = A \cup B$ (disjoint union)
- $v_C / A = v_A$ $v_C / B = v_B$

Lemma $f_{\mathcal{C}} = f_\alpha + f_\beta$

• **product** $\mathcal{A} \cdot \mathcal{B} = \mathcal{C}$

$= (C, v_C)$

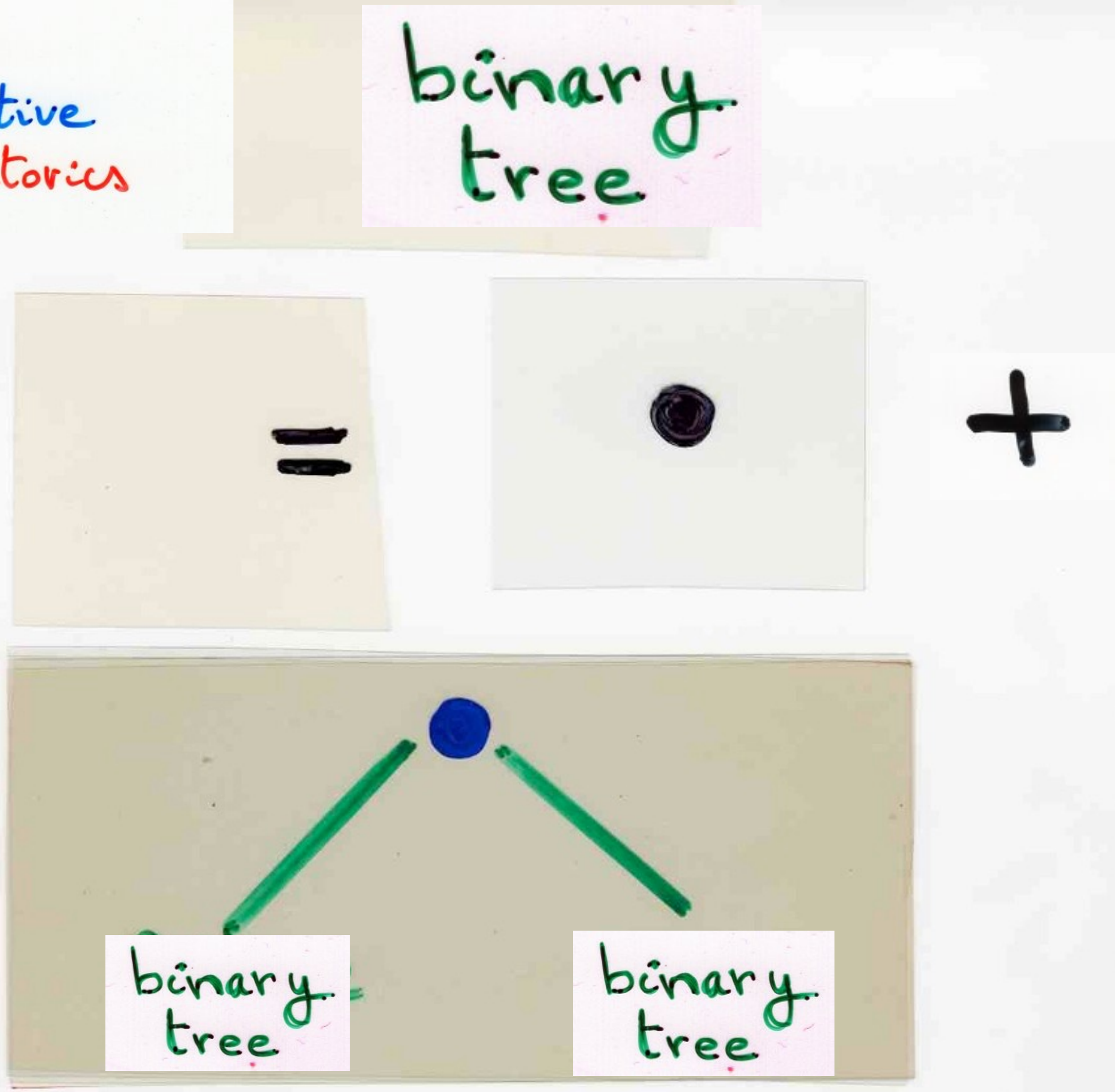
- $C = A \times B$
- $(\alpha, \beta) \in C$ $\quad v_C(\alpha, \beta) = v_A(\alpha) v_B(\beta)$

ex: "size" $|(\alpha, \beta)| = |\alpha| + |\beta|$

ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

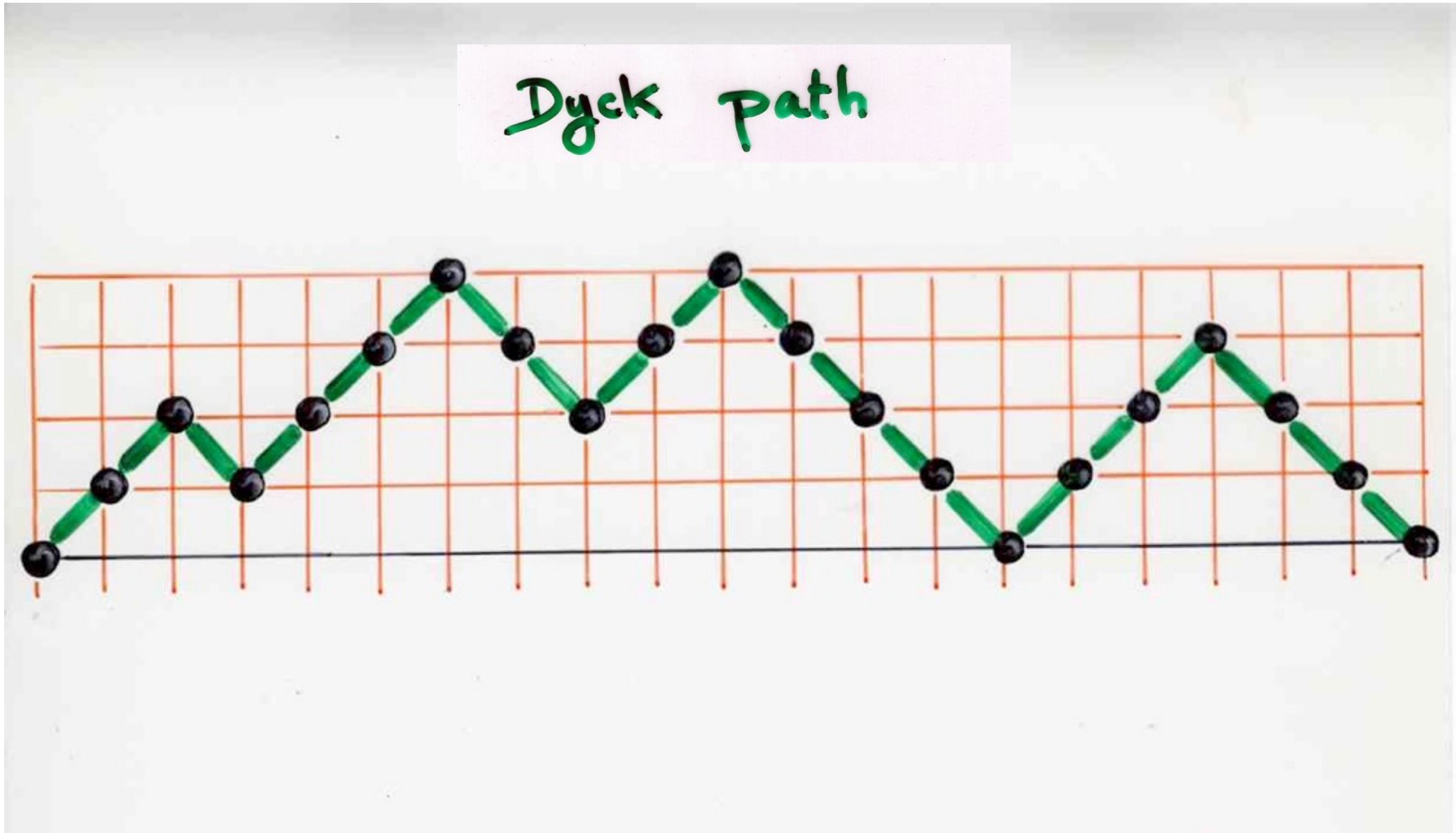
modern
enumerative
combinatorics



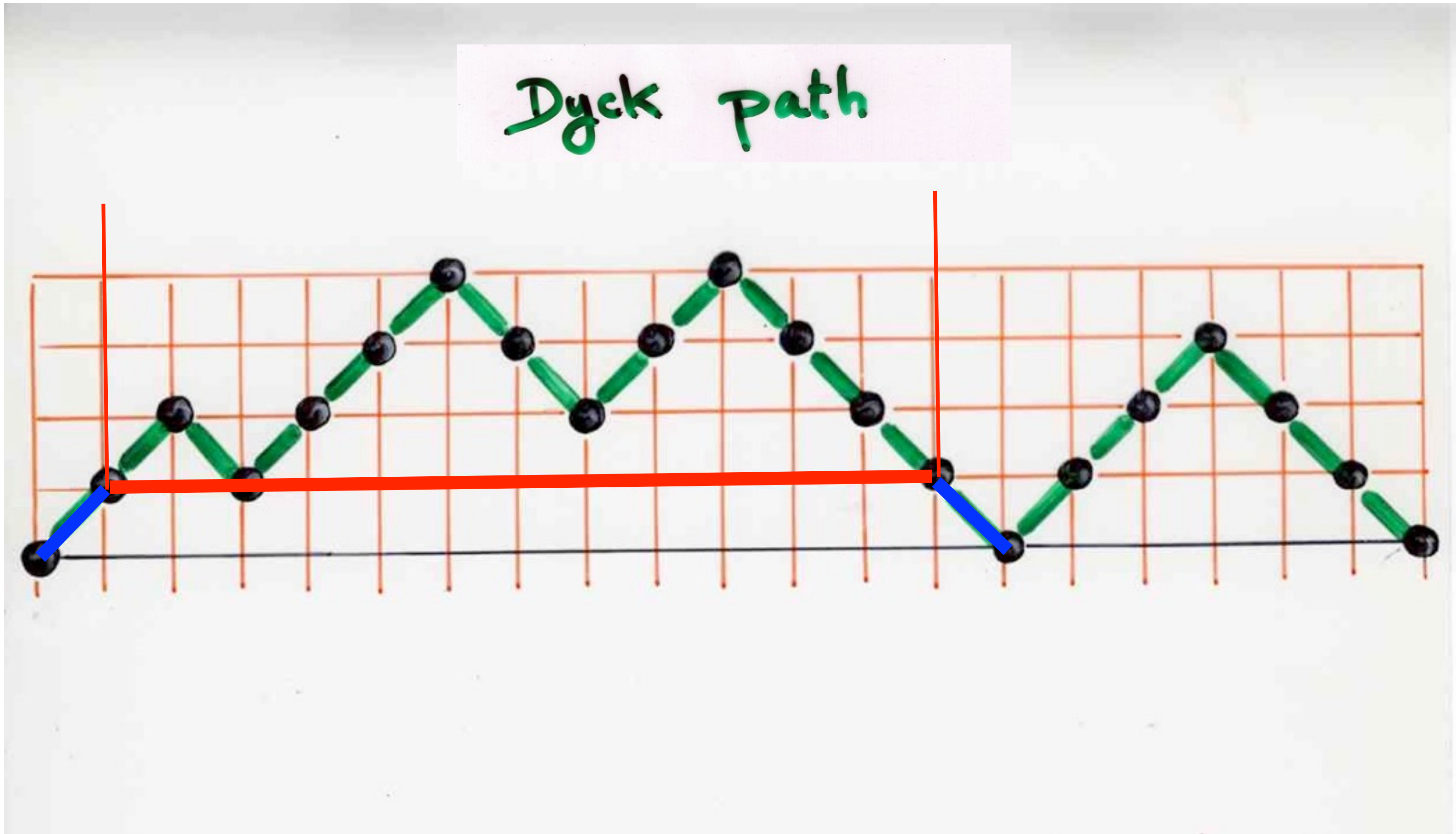
$$\underset{\text{binary tree}}{B} = \{\bullet\} + (B \times \underset{\text{root}}{\bullet} \times B)$$

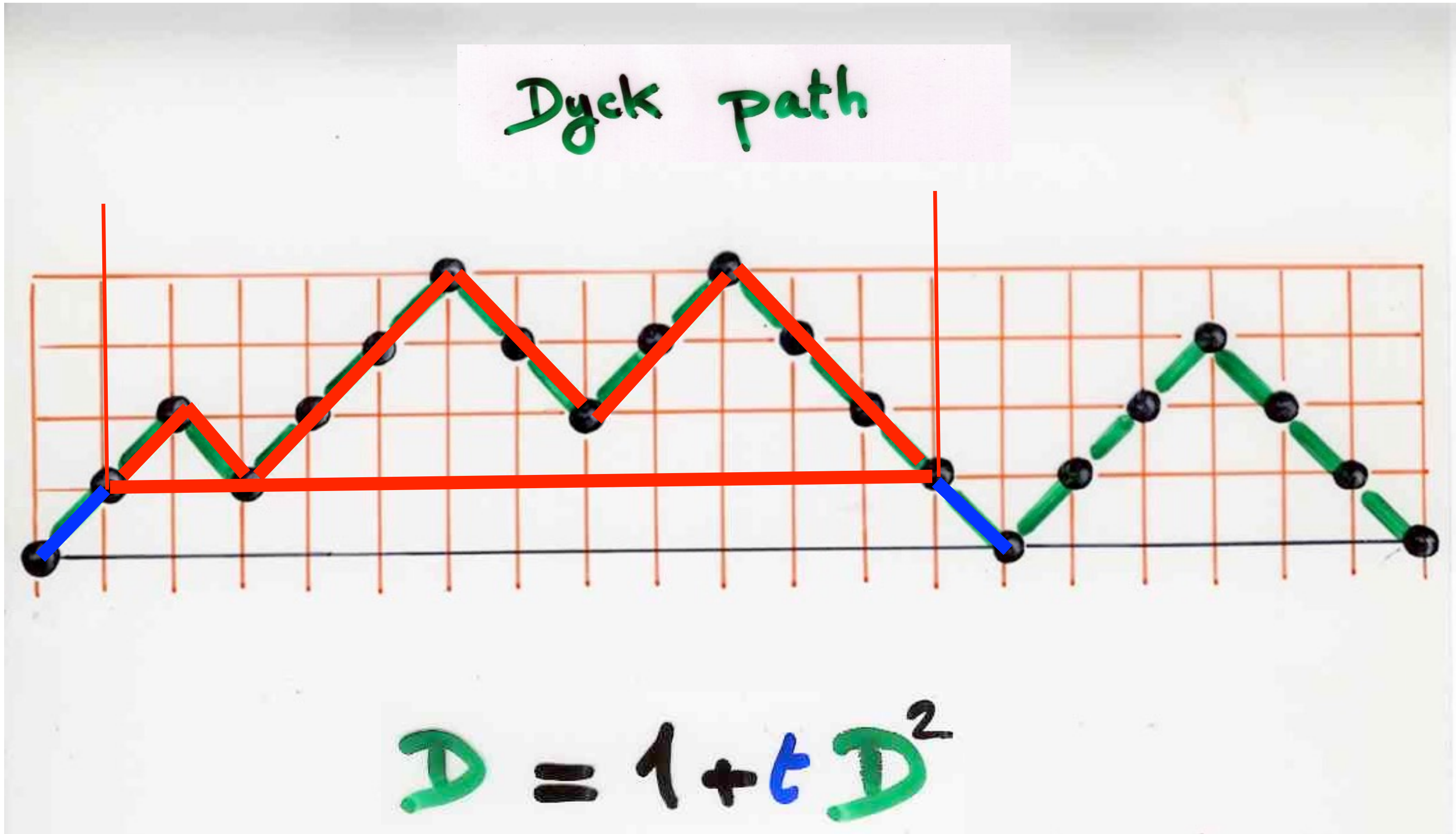
$$y = 1 + t\, y^2$$

algebraic equation

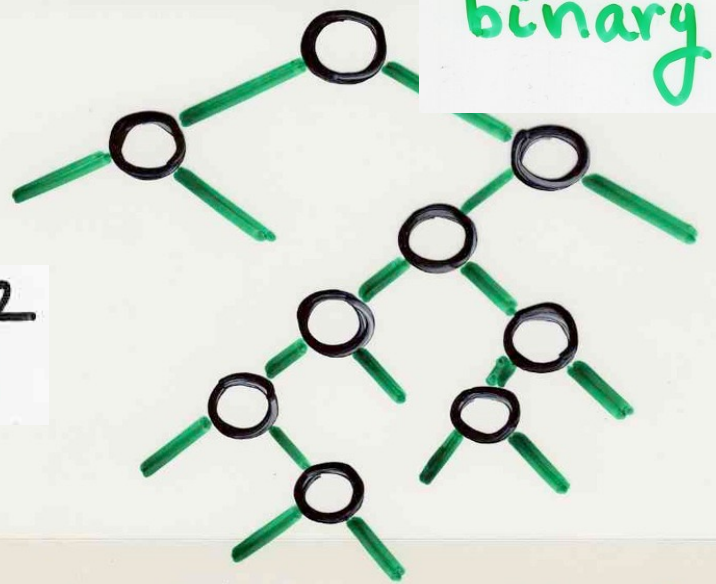


Dyck path


Dyck path


Dyck path
D = 1 + t D²

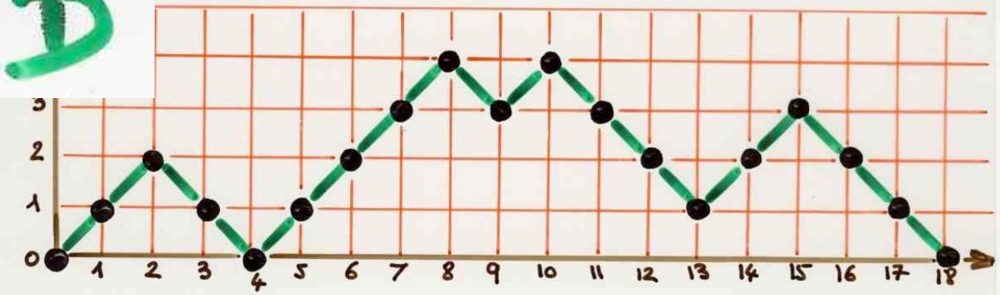
binary tree



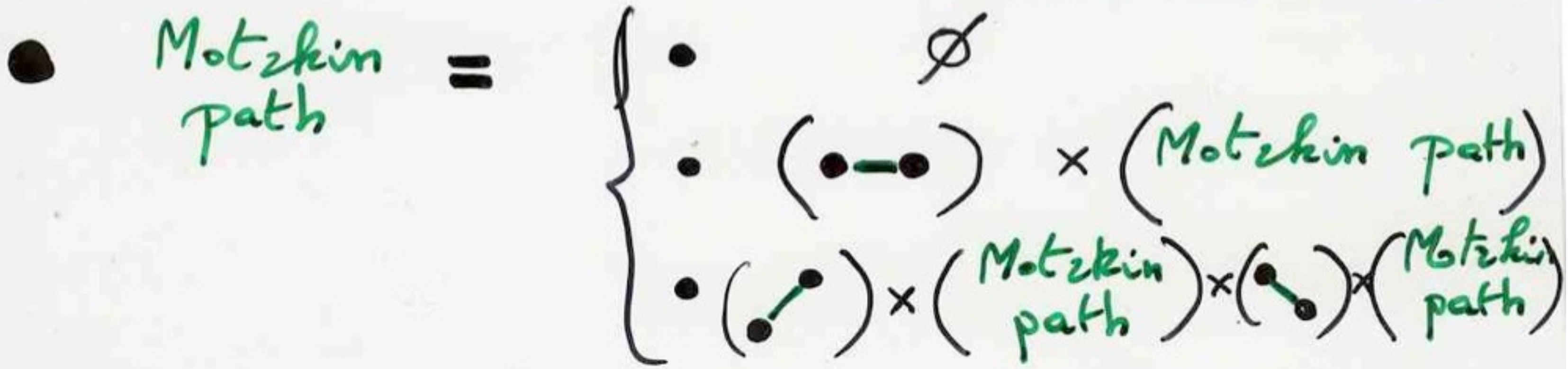
$$A = 1 + tA^2$$

Dyck path

$$D = 1 + tD^2$$




Motzkin path


Motzkin path =
∅
× (Motzkin path)
× (Motzkin path) × × (Motzkin path)

$$m = 1 + t\,m + t^2 m^2$$

sequence

- $\mathcal{A} = (A, v_A)$ $\quad\quad$ $\mathcal{C} = (C, v_C)$

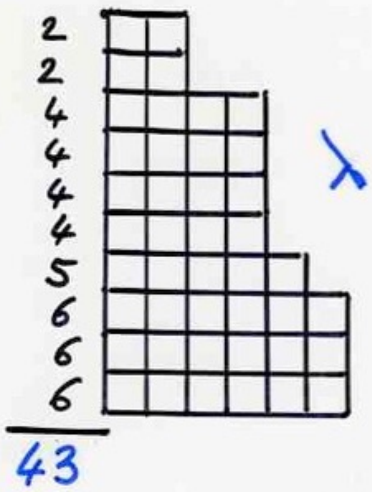
$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \dots + \mathcal{A}^n + \dots$$
$$= \mathcal{A}^*$$

Lemma $$f_{\mathcal{A}^*} = \frac{1}{1 - f_{\mathcal{A}}}$$

partition of an integer n

$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$

$n = 43 = 6+6+6+5+4+4+4+4+2+2$



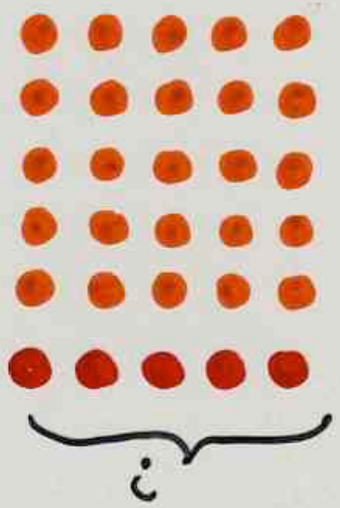
Ferrers diagram

generating function

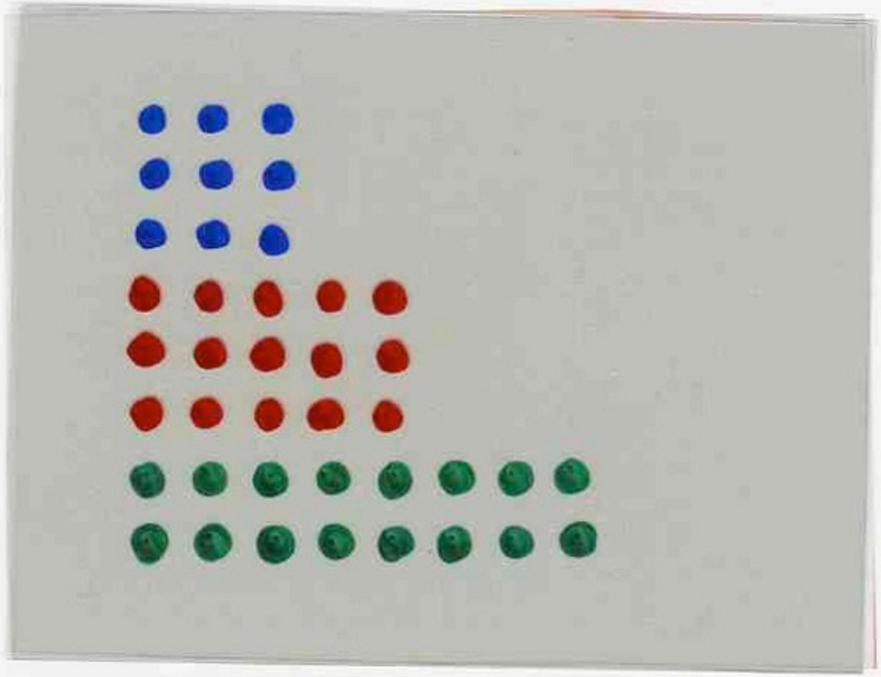
for (integer) partitions

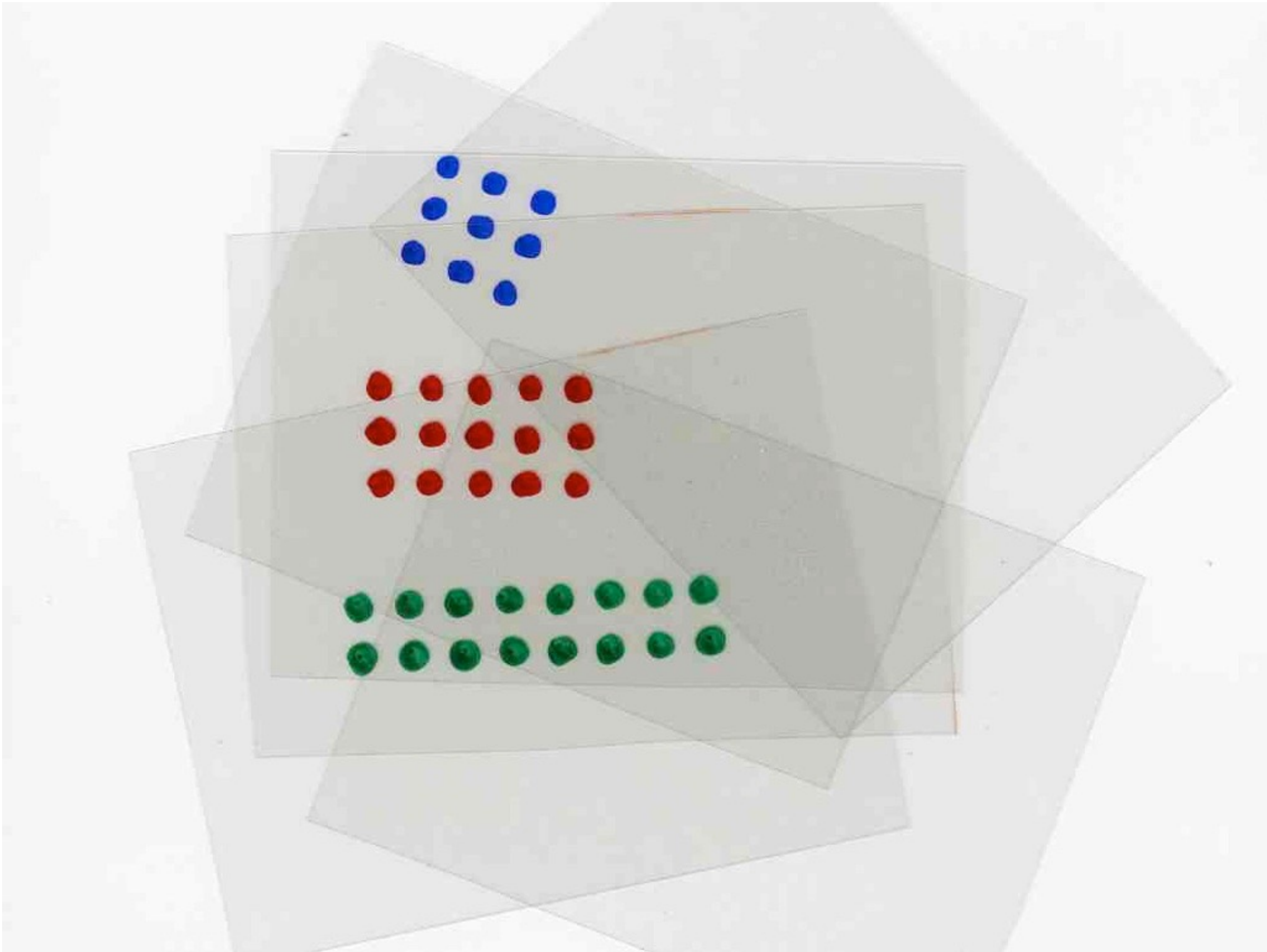
$$\sum_{n \geq 0} a_n q^n$$

q^i



$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\ldots\cdot(1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

generating function for the number of partitions of an integer n

analytic continued fractions

continued fractions

Stieltjes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots}}}}}$$

$$\underbrace{\qquad\qquad}_{S(t;\lambda)}$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\lambda_2t^2\over \cdots\cdots \over 1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}$$

$J(t;b,\lambda)$

Jacobi continued fraction

$b=\{b_k\}_{k\geq 0}$ $\lambda=\{\lambda_k\}_{k\geq 1}$

classical theory

continued fractions

J-fraction

$$J(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

orthogonal polynomials

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

classical theory

continued fractions

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{\cdots\cdots\cfrac{}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments

generating function

orthogonal polynomials

$$P_{k+1}(x) = (x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

classical theory

continued fractions | **orthogonal polynomials**

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t \cdots}}}$$

moments

generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

The fundamental Flajolet Lemma

The fundamental Flajolet Lemma

combinatorial interpretation of a
continued fraction with weighted paths

Discrete Mathematics 32 (1980) 125–161

Discrete Maths (1980)

COMBINATORIAL ASPECTS OF CONTINUED FRACTIONS

P. FLAJOLET

IRIA, 78150 Rocquencourt, France

Received 23 March 1979
Revised 11 February 1980

We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers.... We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

Introduction

In this paper we present a geometrical interpretation of continued fractions together with some of its enumerative consequences. The basis is the equivalence

From chapter 29 of the book of Aigner and Ziegler “Proof from the BOOK”

(about the LGV Lemma)

The essence of Mathematics is proving theorems - and so that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside - Frobenius Lemma in combinatorics.

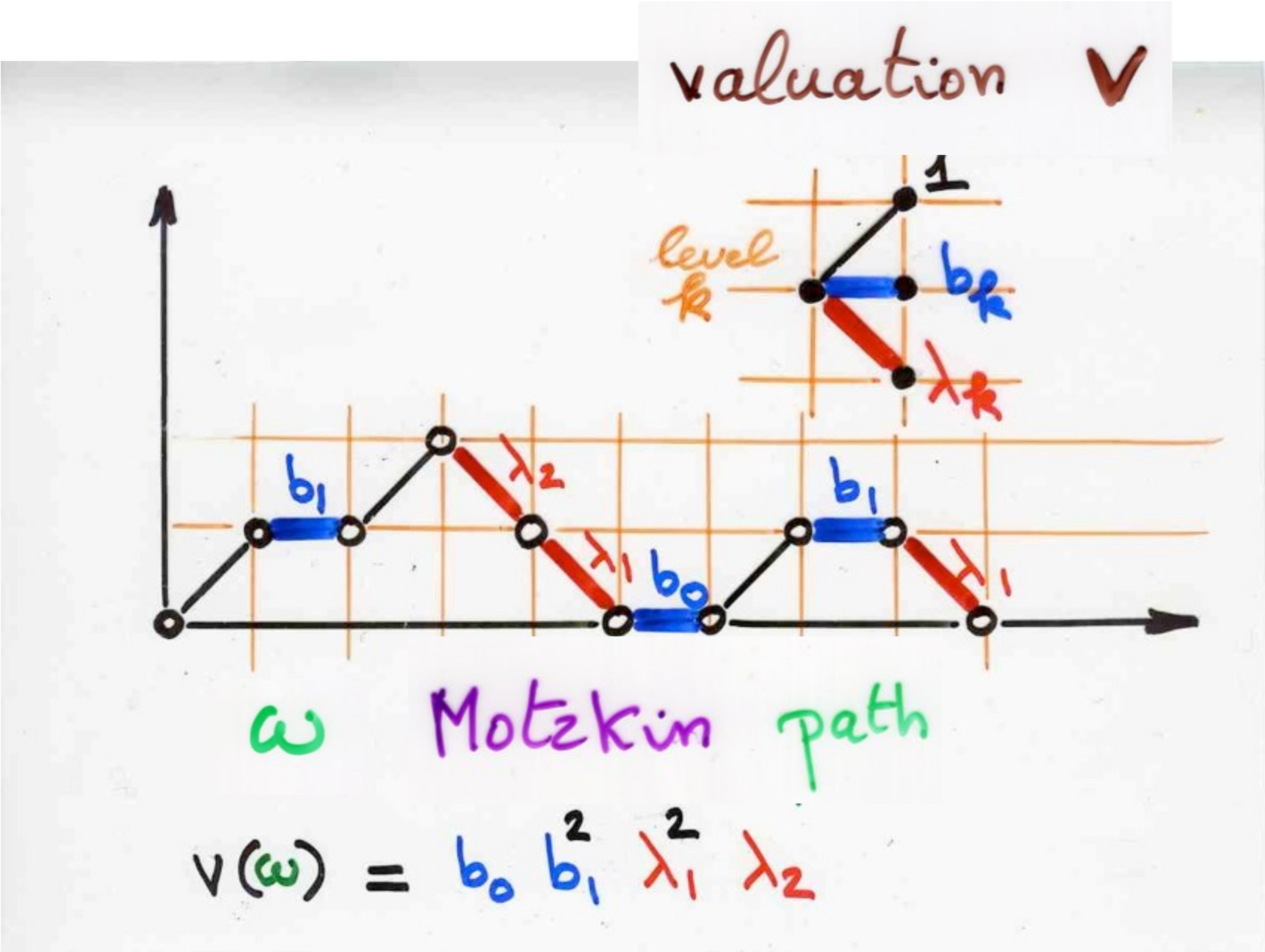
Now what makes a mathematical statement a true Lemma ? First it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be of faint envy: Why haven’t I noticed this before ? And thirdly, on an esthetic level, the Lemma - including the proof - should be beautiful.

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$J(t;\mathbf{b},\boldsymbol{\lambda})$

Jacobi continued fraction

$\mathbf{b}=\{b_k\}_{k\geq 0}$ $\boldsymbol{\lambda}=\{\lambda_k\}_{k\geq 1}$



valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 b_1^2 \lambda_1^2 \lambda_2$

$$\sum_{\omega} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

ω Motzkin path

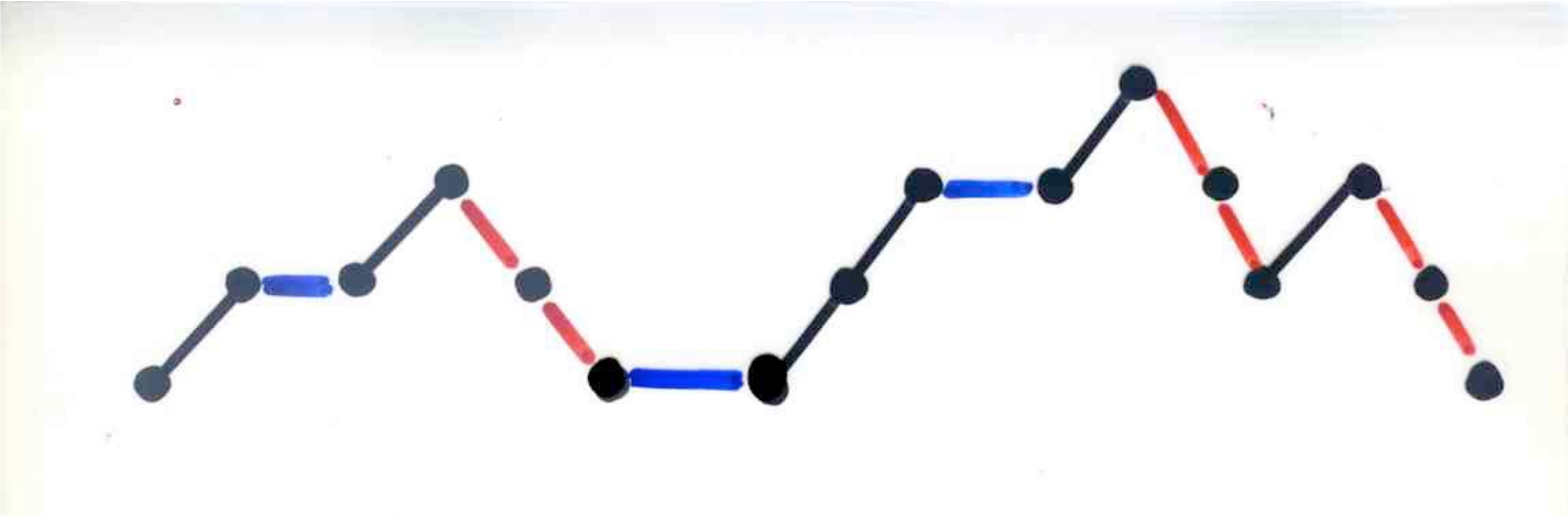
$$J(t; \mathbf{b}, \boldsymbol{\lambda})$$

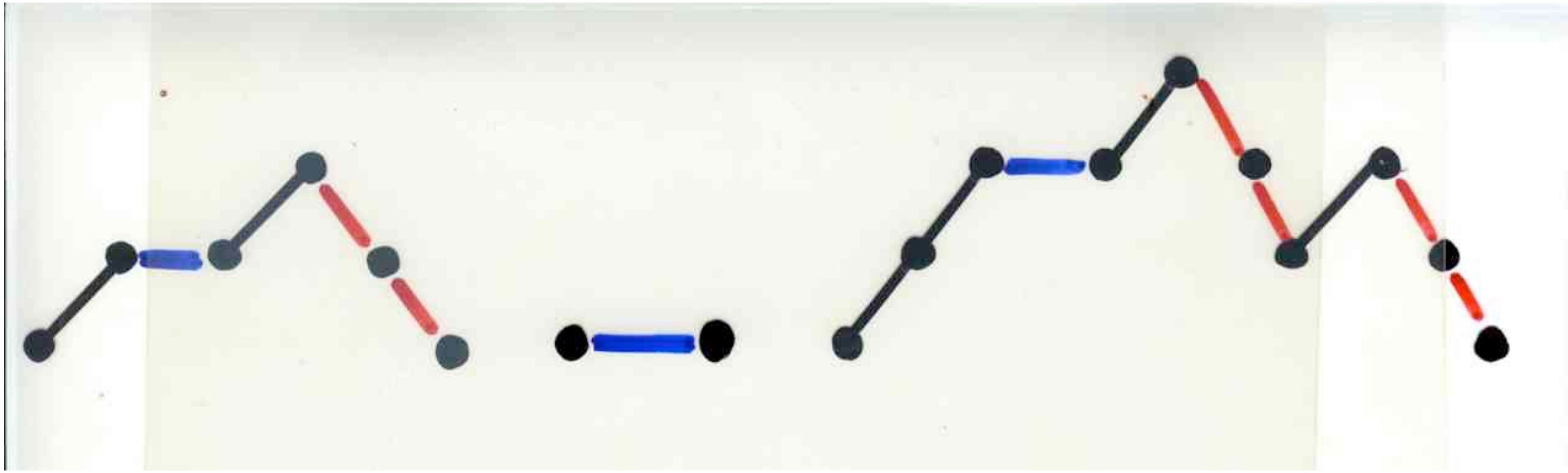
Jacobi continued fraction

$$\mathbf{b} = \{b_k\}_{k \geq 0} \qquad \boldsymbol{\lambda} = \{\lambda_k\}_{k \geq 1}$$

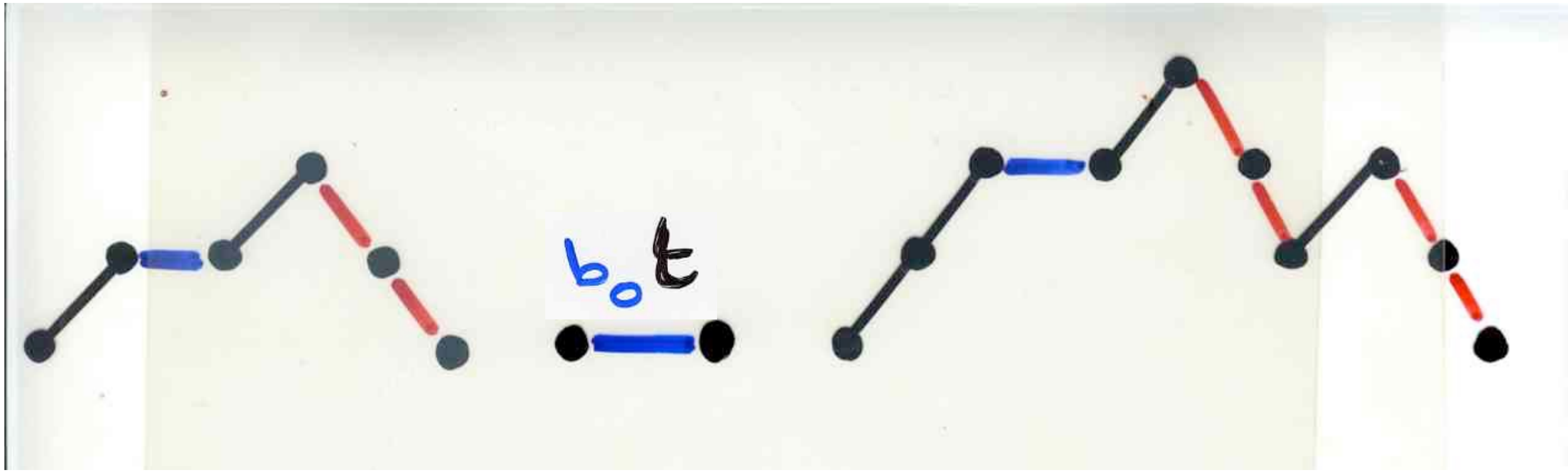
Philippe Flajolet fundamental Lemma

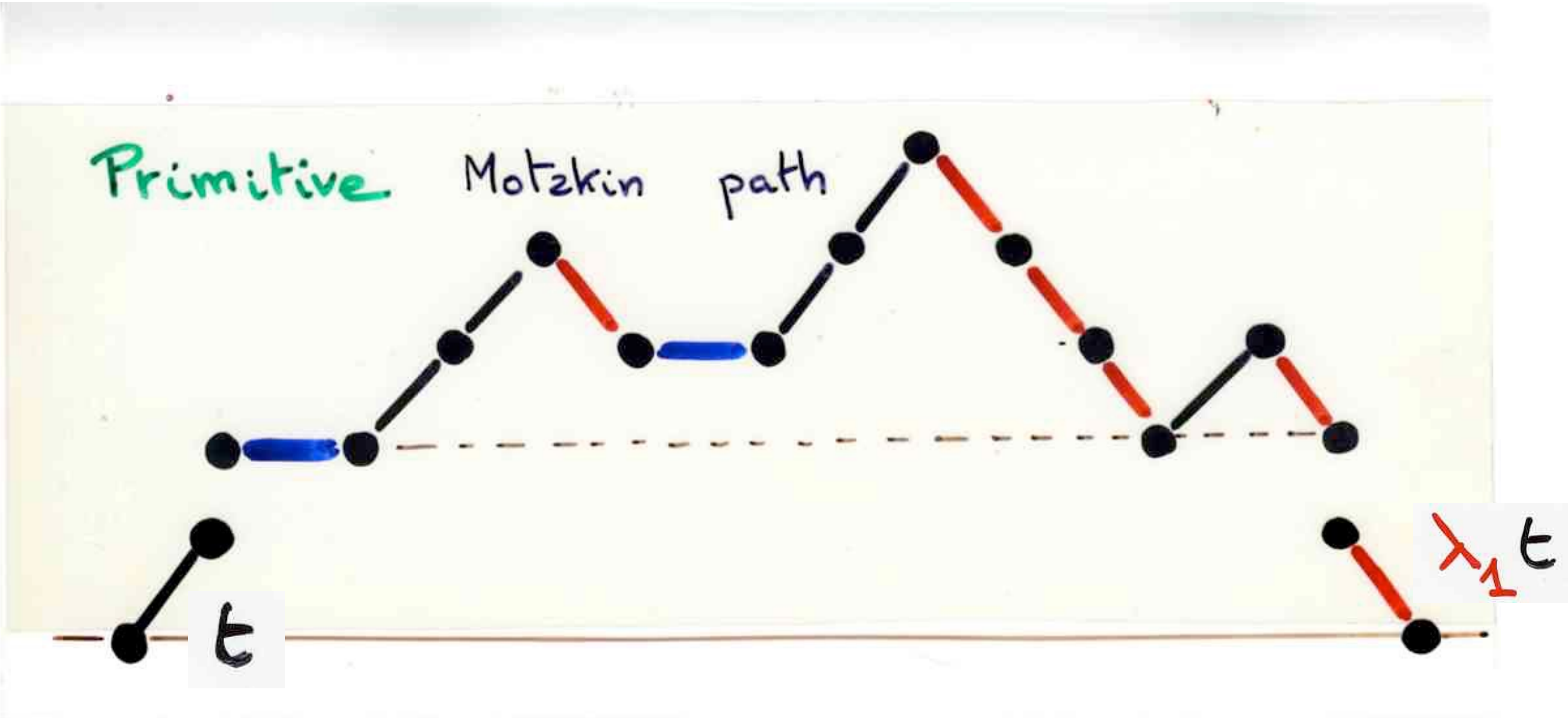
proof:





$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \frac{1}{1 - \sum_{\substack{\omega \\ \text{primitive} \\ \text{Motzkin} \\ \text{path}}} v(\omega)}$$

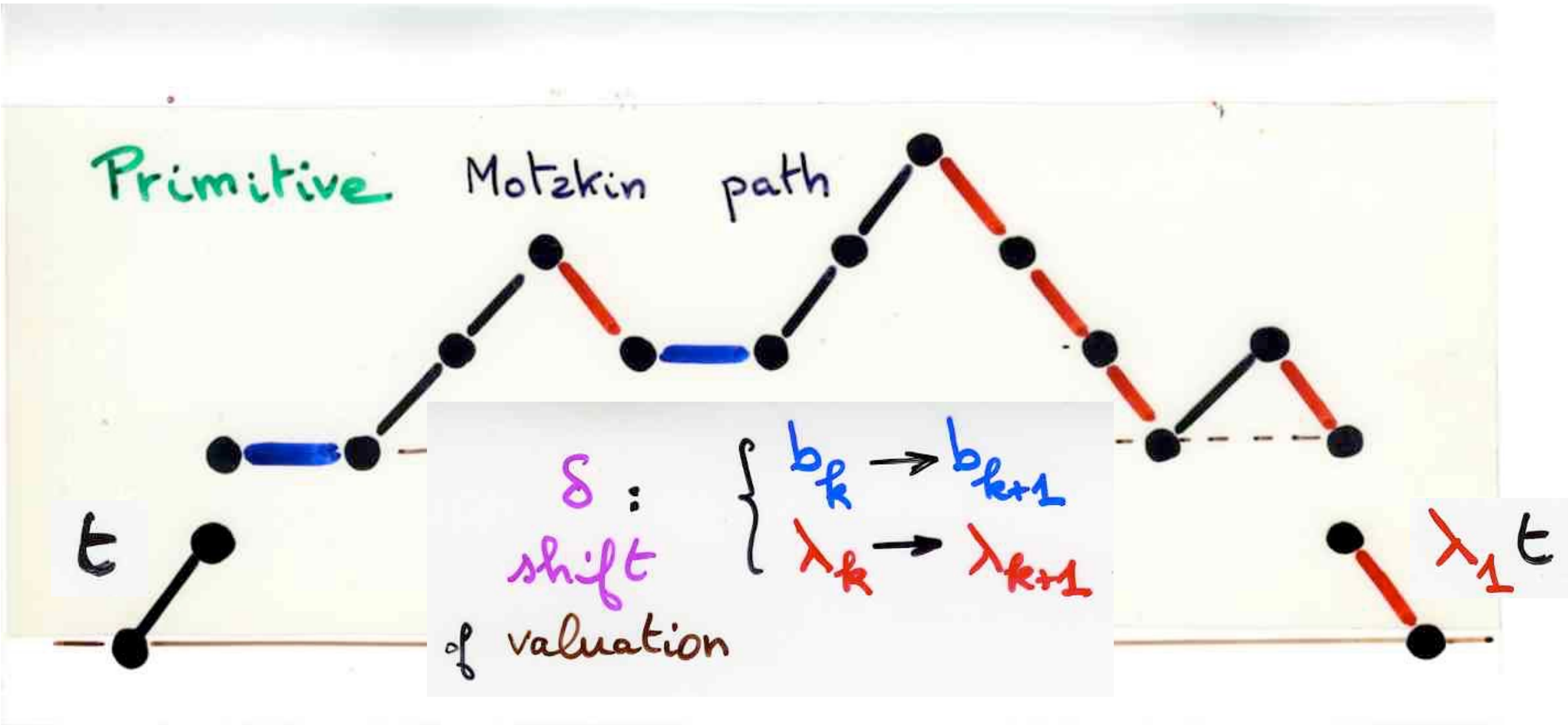

b0t


Primitive Motzkin path
t
λ1t

$$\sum_{\omega} v(\omega)\, t^{|\omega|} = \frac{1}{1 - b_0 t - \lambda_1 t^2 \,(\text{same})}$$

Motzkin path

δ : shift of valuation $\begin{cases} b_k \to b_{k+1} \\ \lambda_k \to \lambda_{k+1} \end{cases}$


Primitive Motzkin path
t
δ :
shift
of valuation
$b_k \to b_{k+1}$
$\lambda_k \to \lambda_{k+1}$
$\lambda_1 t$

$$\sum_{\substack{\omega \\ \text{Motzkin path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \lambda_2 t^2 \left(\,'' \, \right)}}$$

δ^2

$$\sum_{\omega} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \lambda_2 t^2 \cfrac{\cdots\cdots}{\cdots\cdots\; 1 - b_k t - \lambda_{k+1} t^2 \cfrac{}{\cdots\cdots}}}}$$

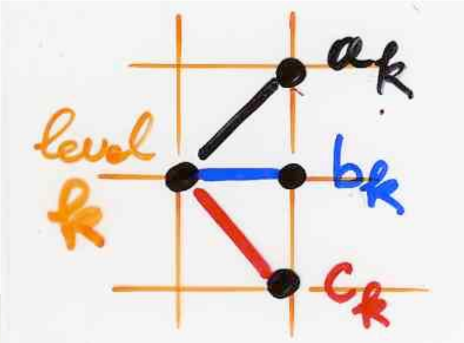
ω Motzkin path

$$J(t; \mathbf{b}, \boldsymbol{\lambda})$$

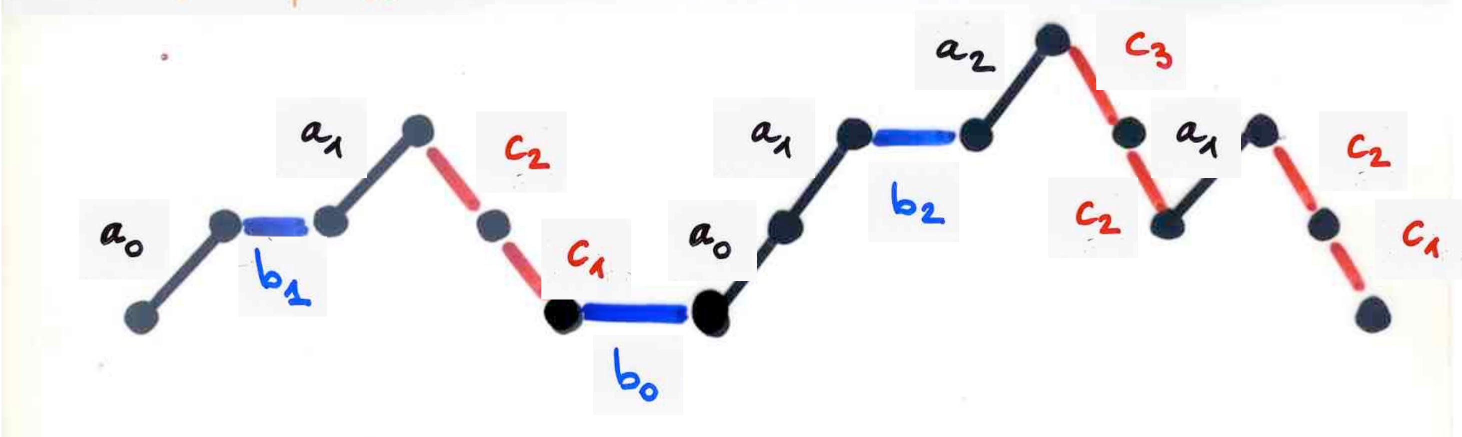
Jacobi continued fraction

$$\mathbf{b} = \{b_k\}_{k\geq 0} \qquad \boldsymbol{\lambda} = \{\lambda_k\}_{k\geq 1}$$

Philippe Flajolet fundamental Lemma



non-commutative power series



$$\left[b_0 + a_0 \left(b_1 + a_1 (\cdots\cdots)^* c_2 \right)^* c_1 \right]^*$$

Continued fractions
and
orthogonal polynomials

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

continued fractions

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

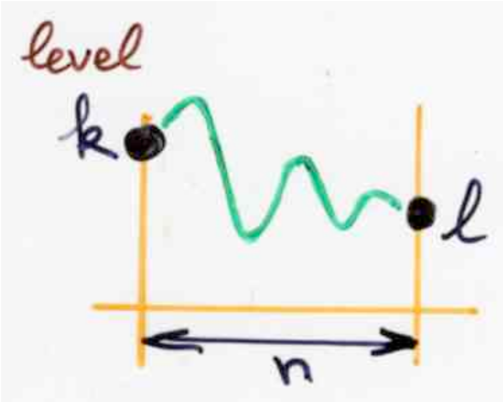
Motzkin path
$|\omega| = n$

Philippe Flajolet
fundamental
Lemma

(main) **Theorem** Ch 1

$$f\left(P_k P_\ell x^n\right) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_\ell$$

"Motzkin path"
$|\omega| = n$ level $k \rightsquigarrow \ell$



classical theory

orthogonal polynomials

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

classical theory

continued fractions ⟷ **orthogonal polynomials**

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cdots\cdots \cfrac{}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

Motzkin path $|\omega| = n$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path $|\omega| = n$

example:

Laguerre polynomials
and
continued fractions

Laguerre polynomials

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

Laguerre history

$$\mu_n = (n+1)!$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots}}}}$$

$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$

$$\mu_n = n!$$

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

Euler

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n! \, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cdots}}}}}}$$

Euler

Ch 3b

subdivided Laguerre history

A. de Médicis, X.V. (1994)

- contraction of continued fractions

example with subdivided Laguerre histories

(Euler continued fraction)

convergents

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$$J(t; b, \lambda)$$

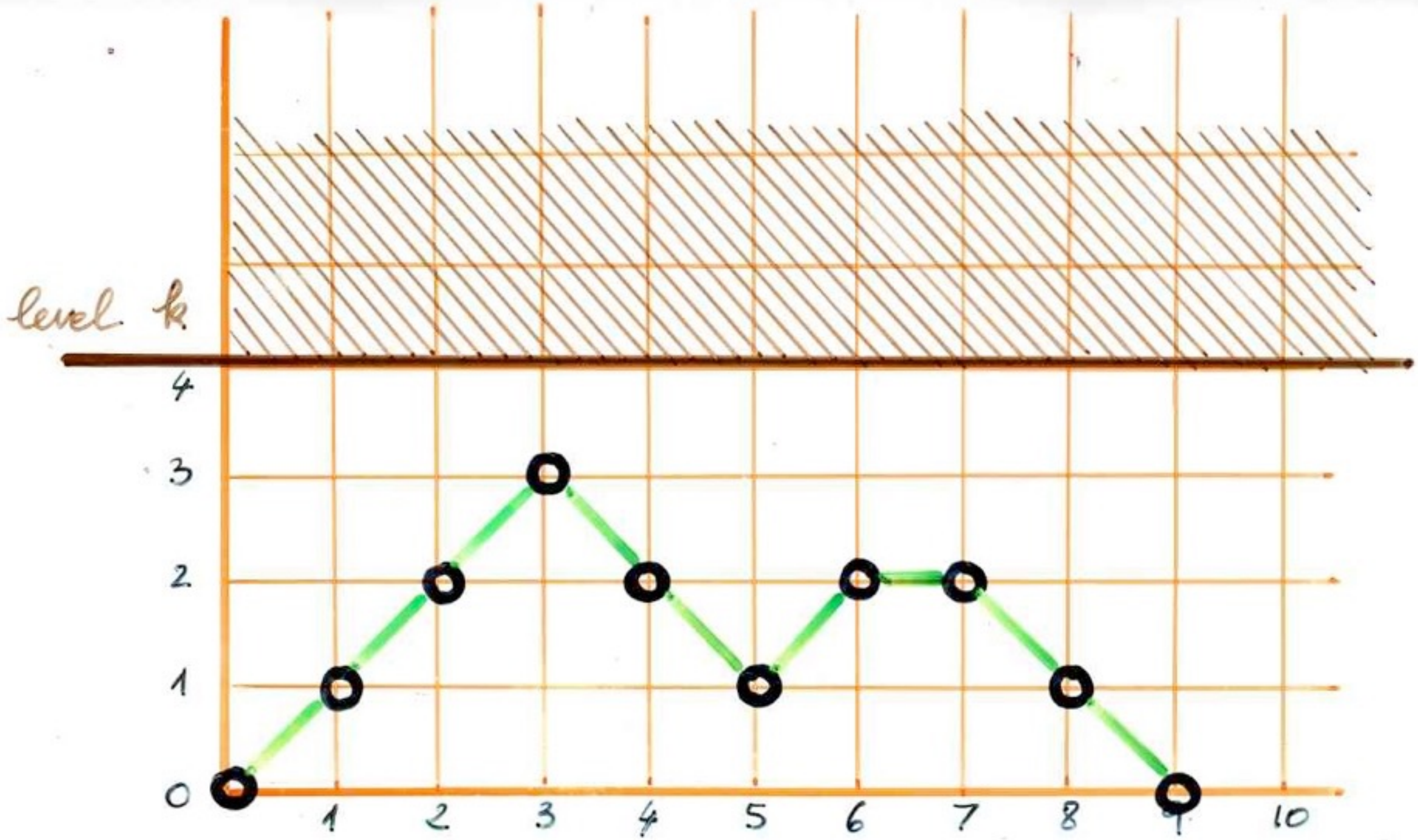
Jacobi continued fraction

$$b = \{b_k\}_{k\geq 0} \qquad \lambda = \{\lambda_k\}_{k\geq 1}$$

$$J_k(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t \ldots}}}}$$

convergents

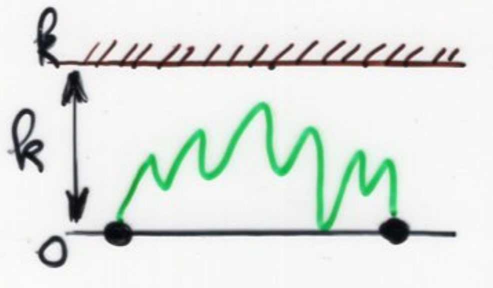
$$J(t; b, \lambda)$$



level k
4
3
2
1
0
1
2
3
4
5
6
7
8
10

Proposition

$$J_k(t) = \sum_{\substack{\omega \\ \text{Motzkin path} \\ \text{height} \leq k}} v(\omega)\, t^{|\omega|}$$



Proposition

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

reciprocal

$$P_k^*(t) = t^k P_k(1/t)$$

$$\deg(P_k(t)) = k$$

$$\{\delta P_n(x)\}_{n \geq 0}$$

$$\{P_n(x)\}_{n\geq 0}$$

sequence of polynomials

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = (x - b_0)$$

$$\{\delta P_n(x)\}_{n\geq 0} \qquad (\delta b_k)_{k\geq 0}\,,\ (\delta \lambda_k)_{k\geq 1}$$

$$\delta b_k = b_{k+1} \qquad \delta \lambda_k = \lambda_{k+1}$$

Convergents:

Linear algebra proof

Part I, Ch 1b, 79-91

linear algebra
proof

<u>Lemma</u> $S = \{1, 2, \ldots, n\}$

$A = (a_{i,j})$ $n \times n$ matrix

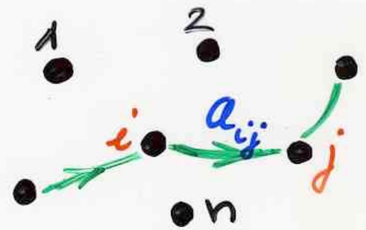
$$(I - A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

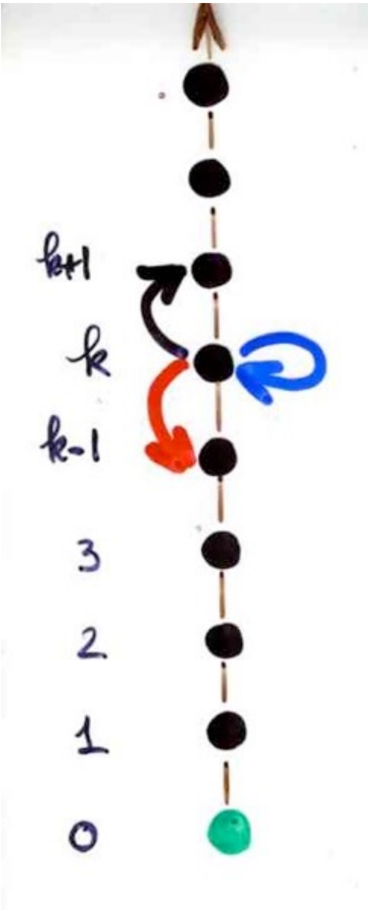
with $v(i,j) = a_{i,j}$

$$(I_n - A)^{-1}_{ij} = \frac{cof_{ji}(I_n - A)}{det(I_n - A)}$$

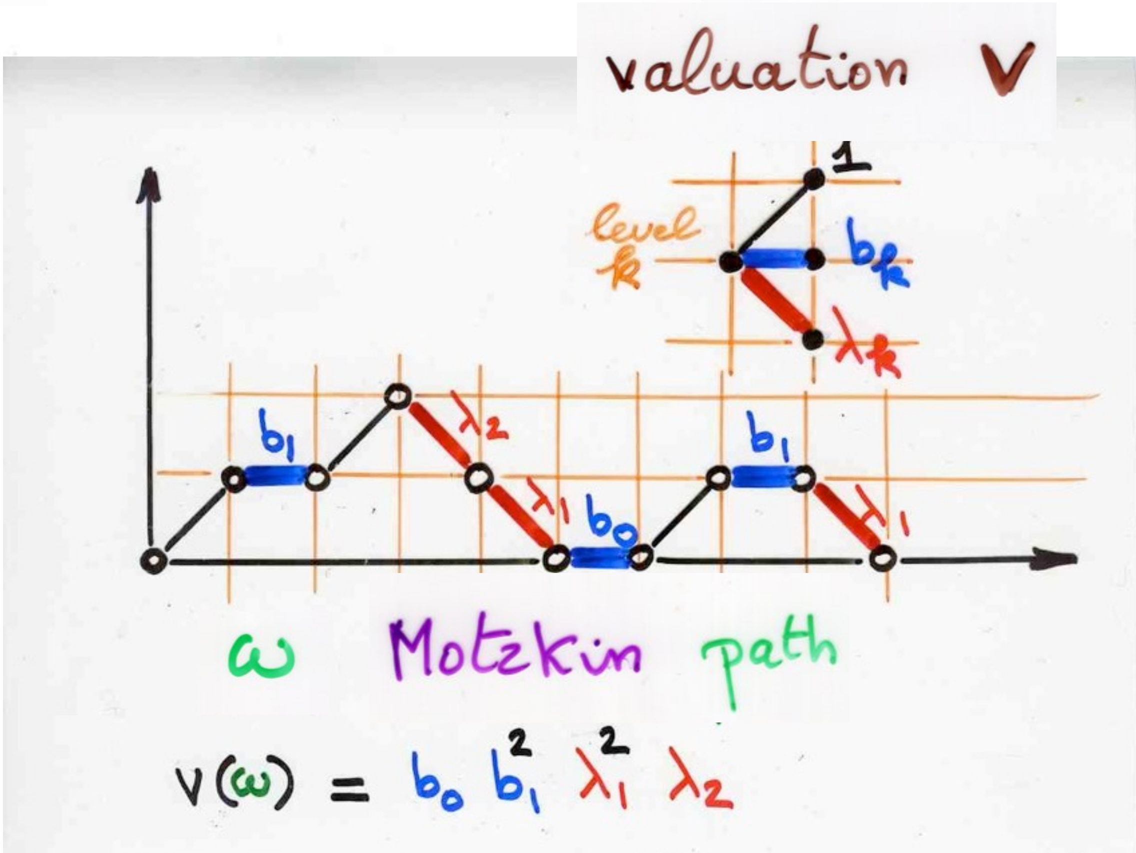
$$I_n + A + A^2 + \ldots + A^n + \ldots$$

$$A = (a_{ij})$$

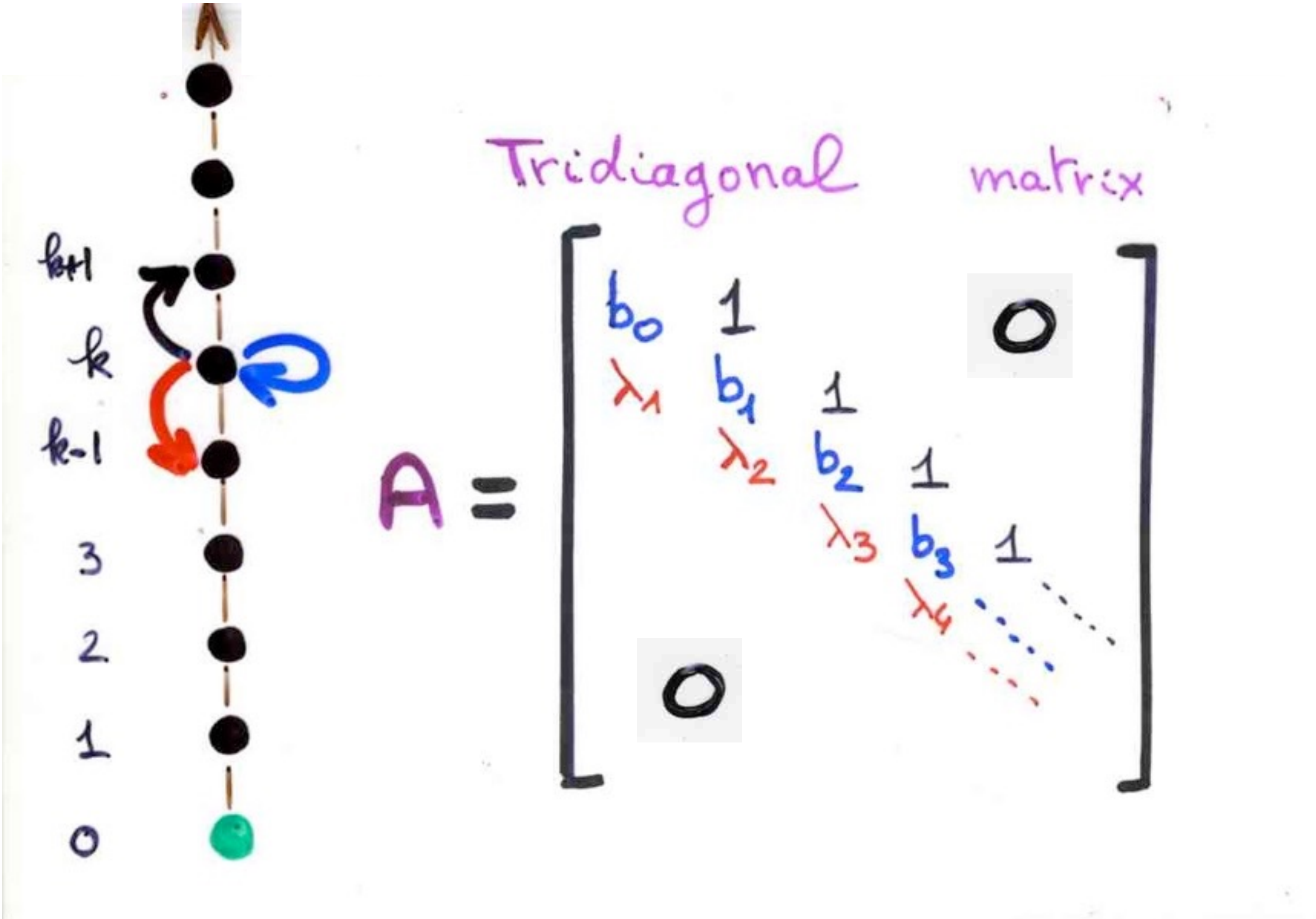




k+1
k
k-1
3
2
1
0



valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 b_1^2 \lambda_1^2 \lambda_2$



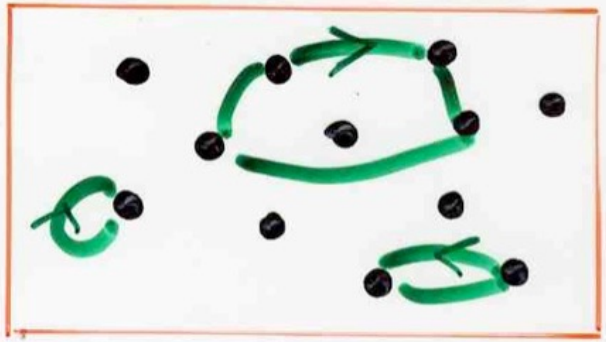
From Part IV, Ch 1c, 92-98

$$\det(A) = \sum_{\sigma} (-1)^{inv(\sigma)} \, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

σ permutations of $\mathfrak{S}_n$

$$\det(I_n - A) = \sum_{\{\gamma_1, \ldots, \gamma_r\}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$

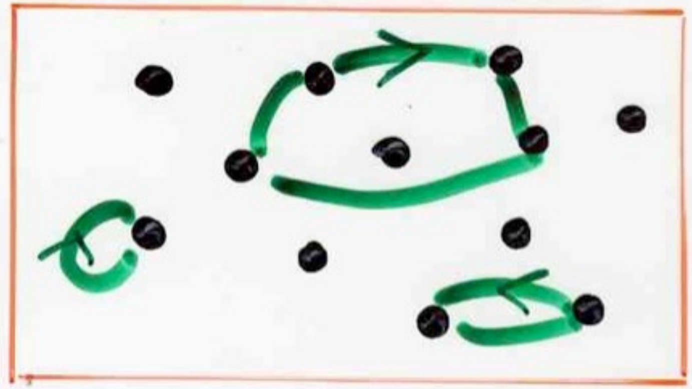
$\{\gamma_1, \ldots, \gamma_r\}$ 2 by 2 disjoint cycles



Proposition

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

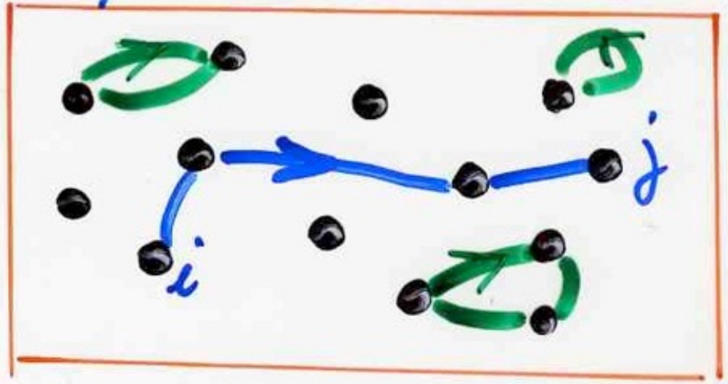
$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$

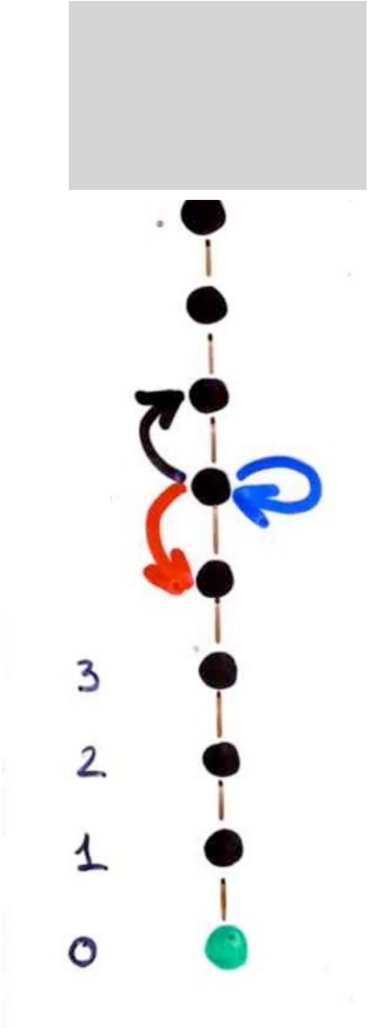


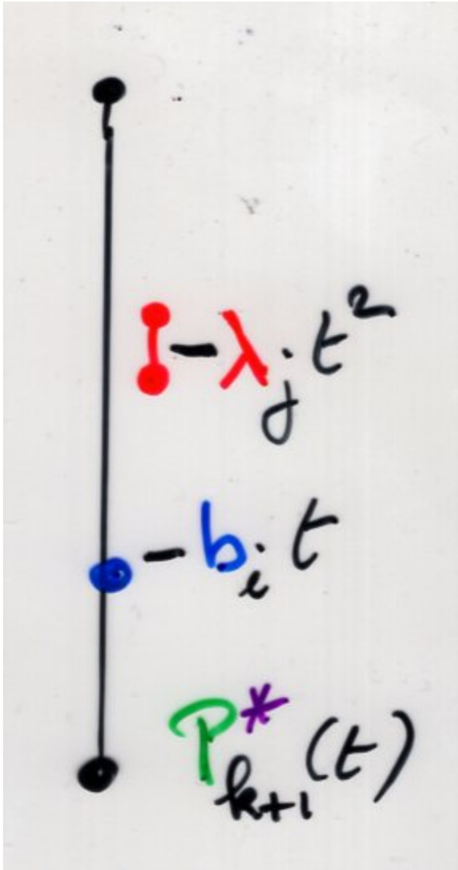
$$N_{ij} = \sum_{\{\eta; \gamma_1, \dots, \gamma_r\}} (-1)^r v(\eta) v(\gamma_1) \cdots v(\gamma_r)$$

η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η







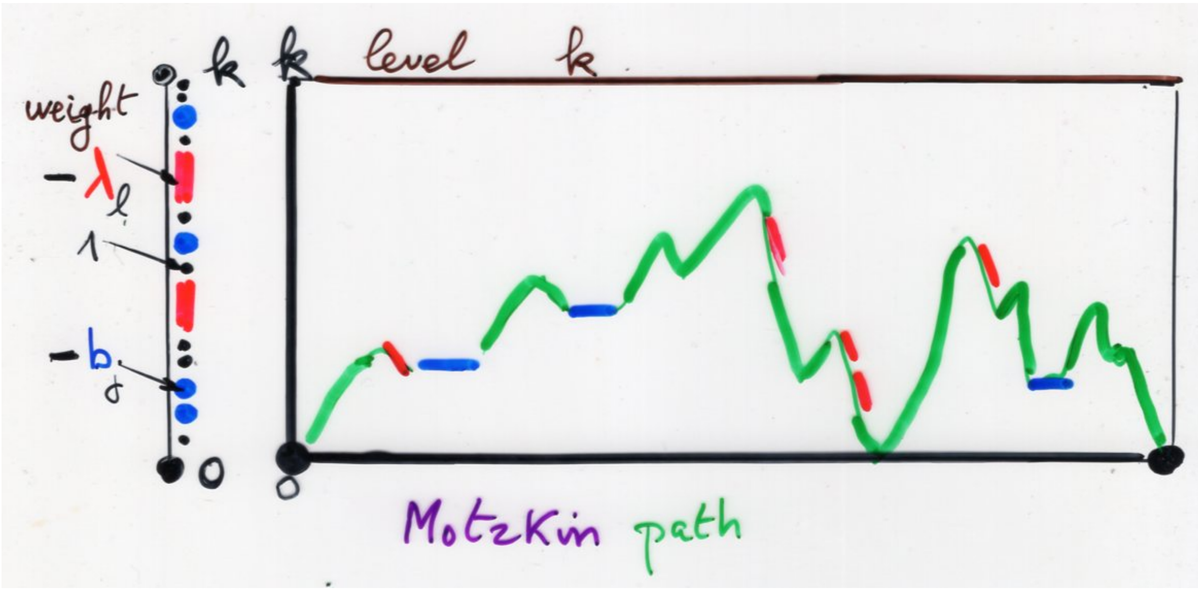
convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

Convergents:

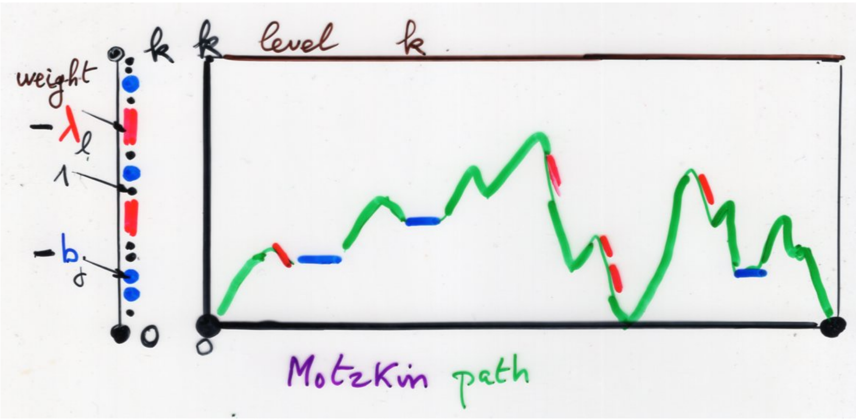
Bijective proof

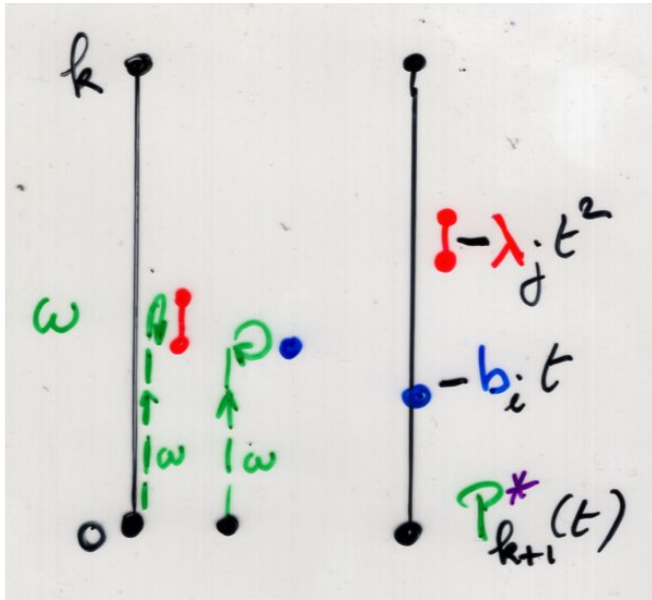
$$P^{*}_{k+1}(t) \left[\sum_{\substack{\omega \\ \text{Motzkin path} \\ \text{height} \leq k}} v(\omega)\, t^{|\omega|} \right] = \delta P^{*}_{k}(t)$$



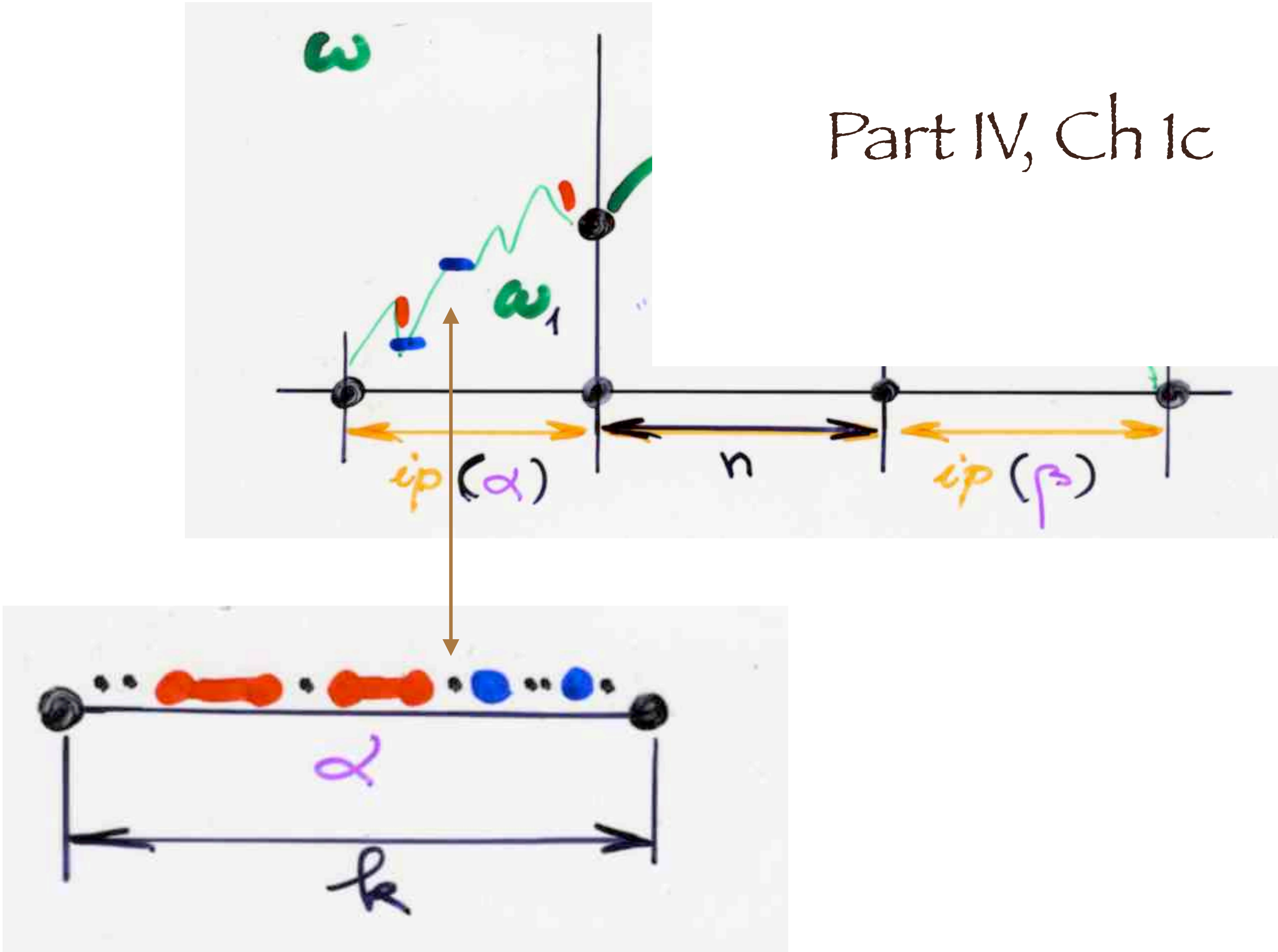
sign-reversing involution

$$P^*_{k+1}(t)\left[\sum_{\substack{\omega\\ \text{Motzkin path}\\ \text{height}\leq k}} v(\omega)\, t^{|\omega|}\right] = \delta P^*_k(t)$$





Part IV, Ch 1c



Back to:

(direct) bijective proof of the identity

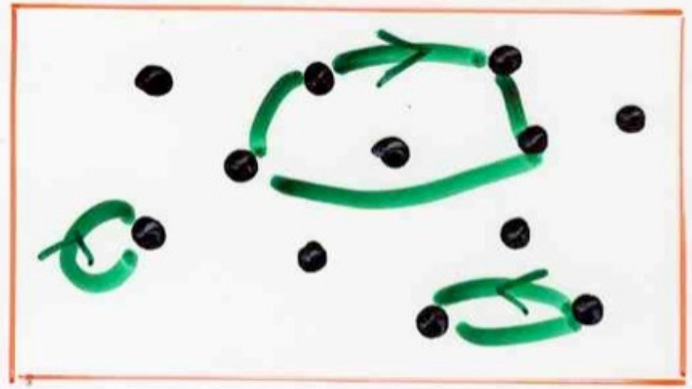
$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

Part 1, Ch 1c, p 10-18

Proposition

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

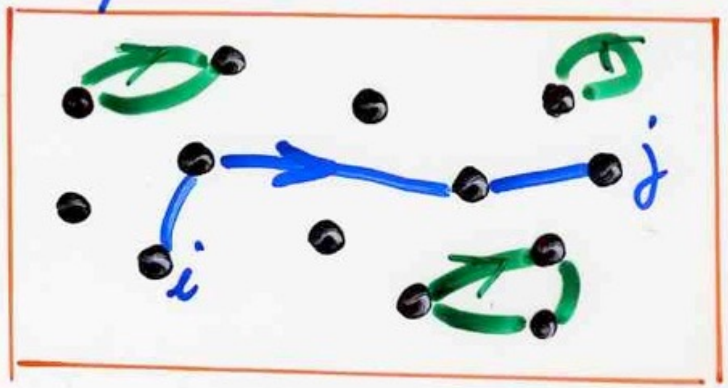
$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta;\, \gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$

η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η



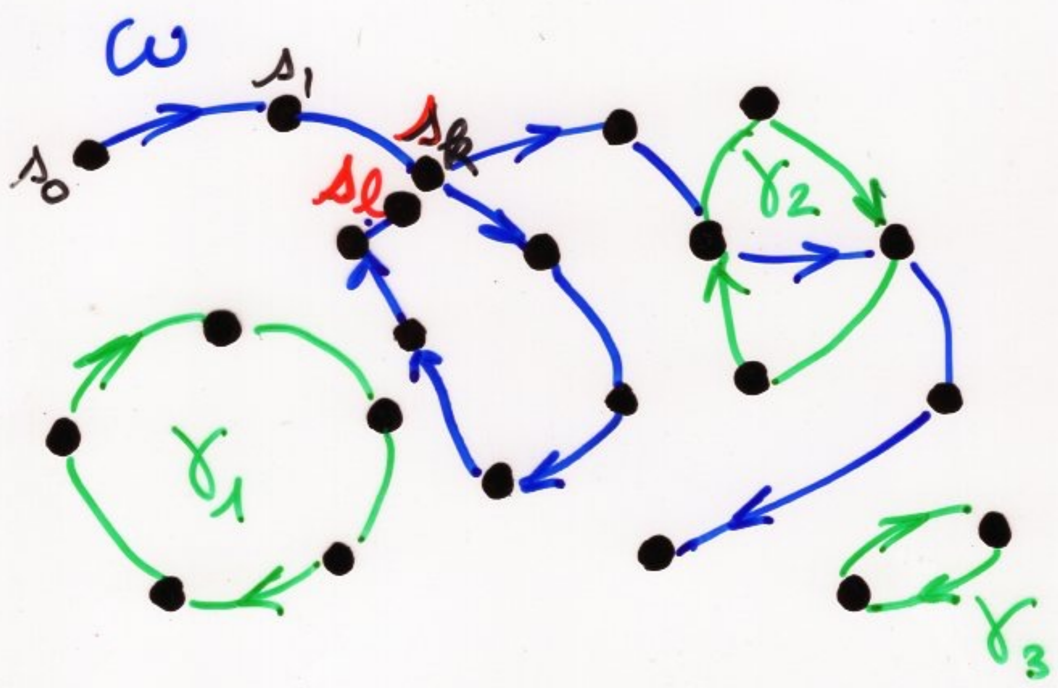
(direct) bijective proof of

$$\left(\sum_{\substack{\omega \\ i \leadsto j}} v(\omega)\right) D = N_{ij}$$

case (i) $\varphi(\xi) = (\omega' ; \{\gamma_1, \dots, \gamma_r, \gamma\})$

with $\omega' = (s_0, \dots, s_{k-1}, s_l, \dots, s_n)$

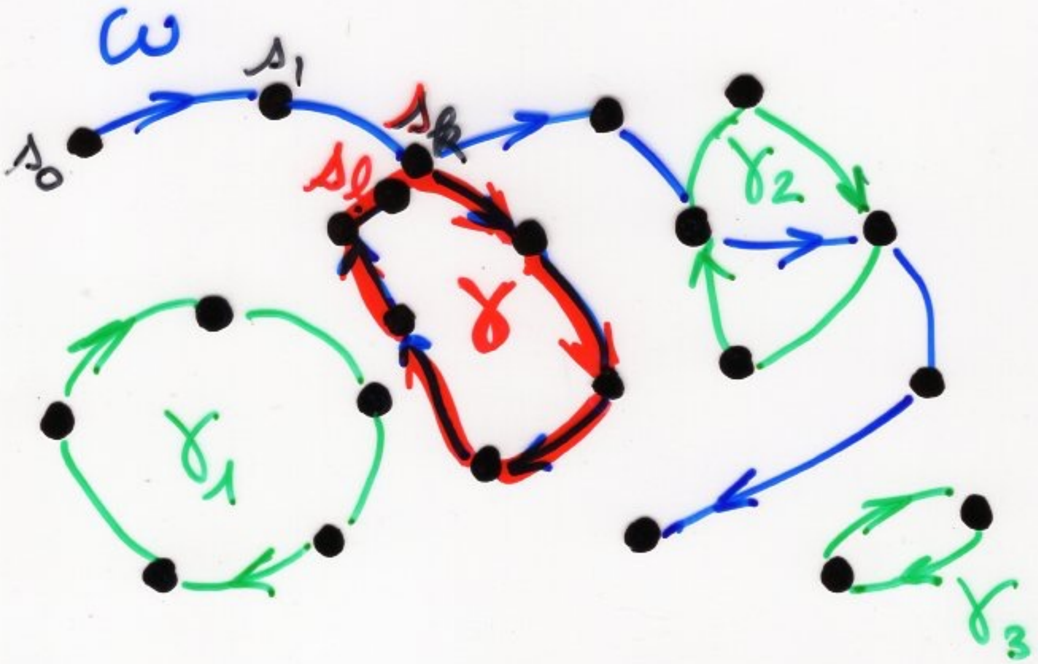
$\gamma = (s_k, s_{k+1}, \dots, s_{l-1})$



case (ii) $s_\ell \in \gamma_j = (s_\ell, y_1, \dots, y_p)$

then $\varphi(\xi) = (\omega' ; \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots \gamma_r\})$

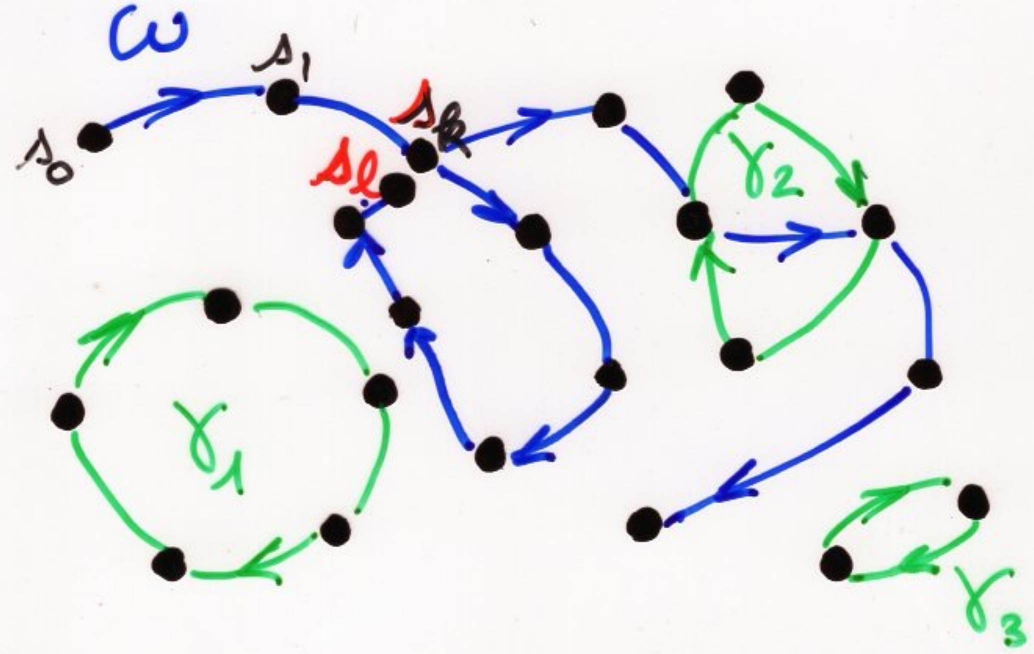
with $\omega' = (s_0, \dots, s_\ell, y_1, \dots, y_p, s_\ell, s_{\ell+1}, \dots, s_n)$



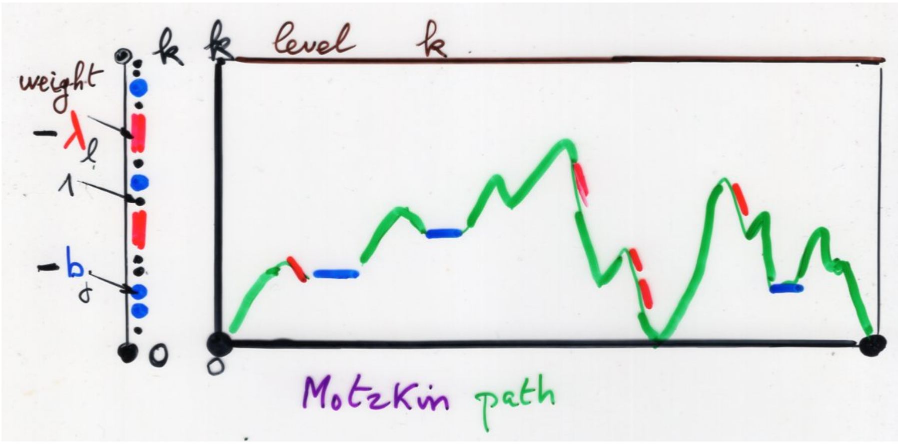
case (ii) $s_\ell \in \gamma_j = (s_\ell, y_1, \dots, y_p)$

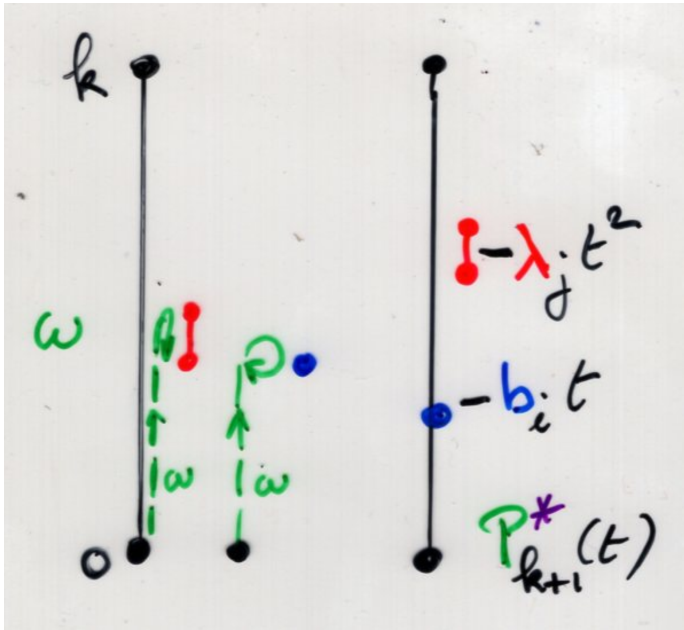
then $\varphi(\xi) = \left(\omega', \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots \gamma_r\}\right)$

with $\omega' = (s_0, \dots, s_\ell, y_1, \dots, y_p, s_\ell, s_{\ell+1}, \dots, s_n)$



$$P^*_{k+1}(t) \left[\sum_{\substack{\omega \\ \text{Motzkin path} \\ \text{height} \leq k}} v(\omega)\, t^{|\omega|} \right] = \delta P^*_k(t)$$





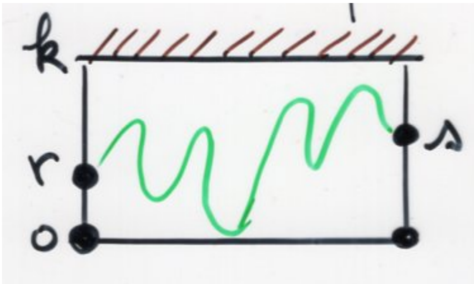
Some extensions of

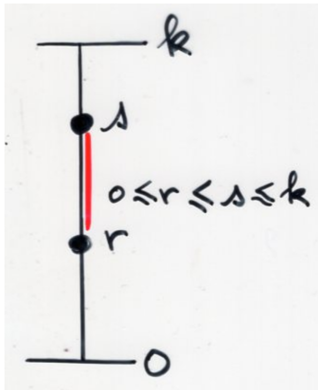
convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

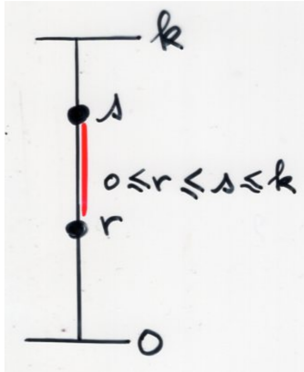
$0 \leq r, s \leq k$

$$\mu^{\leq k}_{n,r,s} = \sum_{\substack{|\omega| = n \\ \text{"Motzkin path" } r \rightsquigarrow s}} v(\omega)$$





$$\sum \mu^{\leq k}_{n,r,s} t^n = \frac{t^{s-r} P^*_r(t) \, \delta^{s+1} P^*_{k-s}(t)}{P^*_{k+1}(t)}$$



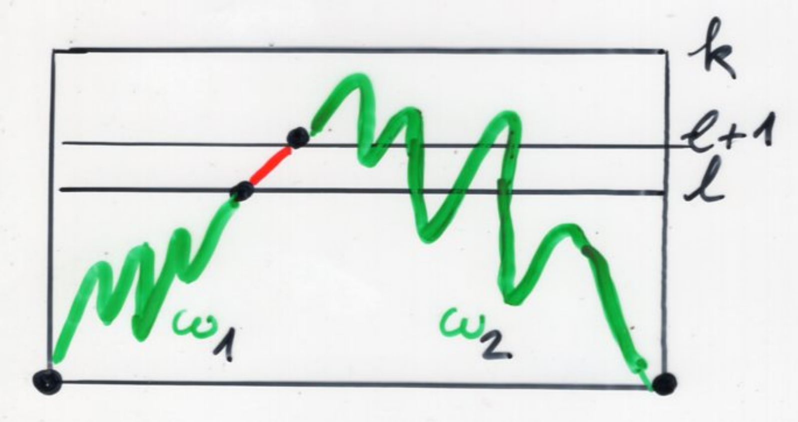
$$\sum \mu^{\leq k}_{n,r,s}\, t^n = \frac{t^{s-r}\, P^*_r(t)\, \delta^{s+1} P^*_{k-s}(t)}{P^*_{k+1}(t)}$$

$$= (\lambda_r \cdots \lambda_{s+1})\, \frac{t^{r-s}\, P^*_s(t)\, \delta^{r+1} P^*_{k-r}(t)}{P^*_{k+1}(t)}$$

$r = s = 0$

$$J_k(t) = \frac{\delta P^*_k(t)}{P^*_{k+1}(t)}$$

$$\sum_{n\geq 0} \mu^{\leq k}_{n,0,k}\, t^n = \frac{t^k}{P^*_{k+1}(t)}$$



$0 \leq \ell < k$

$$\frac{\delta P^*_k(t)}{P^*_{k+1}(t)} - \frac{\delta P^*_\ell(t)}{P^*_{\ell+1}(t)} = \frac{(\lambda_1 \cdots \lambda_{\ell+1})\, t^{2\ell+2}\, \delta^{\ell+2}\, P^*_{k-\ell-1}(t)}{P^*_{\ell+1}(t)\, P^*_{k+1}(t)}$$

$$\delta P^*_k(t) P^*_{\ell+1}(t) - \delta P^*_\ell(t) P^*_{k+1}(t) = (\lambda_1 \cdots \lambda_{\ell+1})\, t^{2\ell+2}\, \delta^{\ell+2}\, P^*_{k-\ell-1}$$

$\ell = k-1$

$$\delta P^*_k(t) P^*_k(t) - \delta P^*_{k-1}(t) P^*_{k+1}(t) = \lambda_1 \cdots \lambda_k\, t^{2k}$$

Contraction of continued fractions

continued fractions

Stieltjes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots \over 1-\cfrac{\lambda_k t}{\cdots\cdots}}}}$$

$$\underbrace{\qquad\qquad}_{S(t;\lambda)}$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$J(t;b,\lambda)$

Jacobi continued fraction

$b=\{b_k\}_{k\geq 0}$ $\lambda=\{\lambda_k\}_{k\geq 1}$

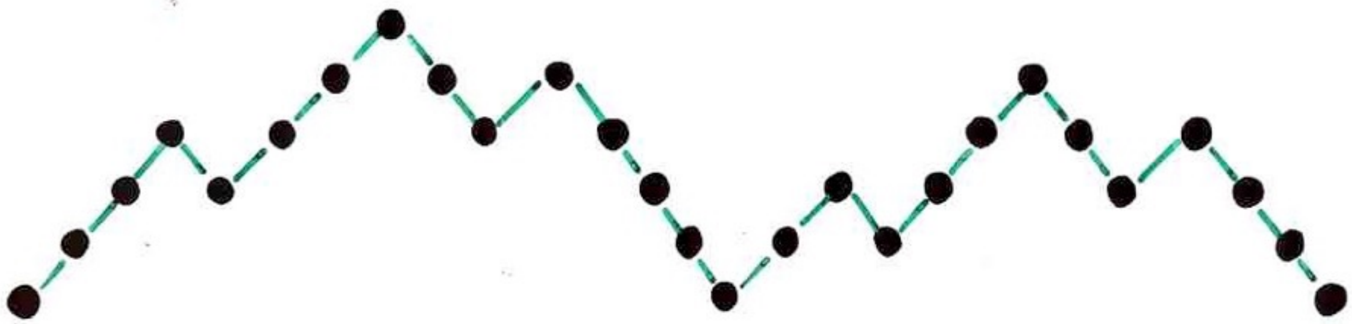
$S(t;\gamma)=J(t;b,\lambda)$

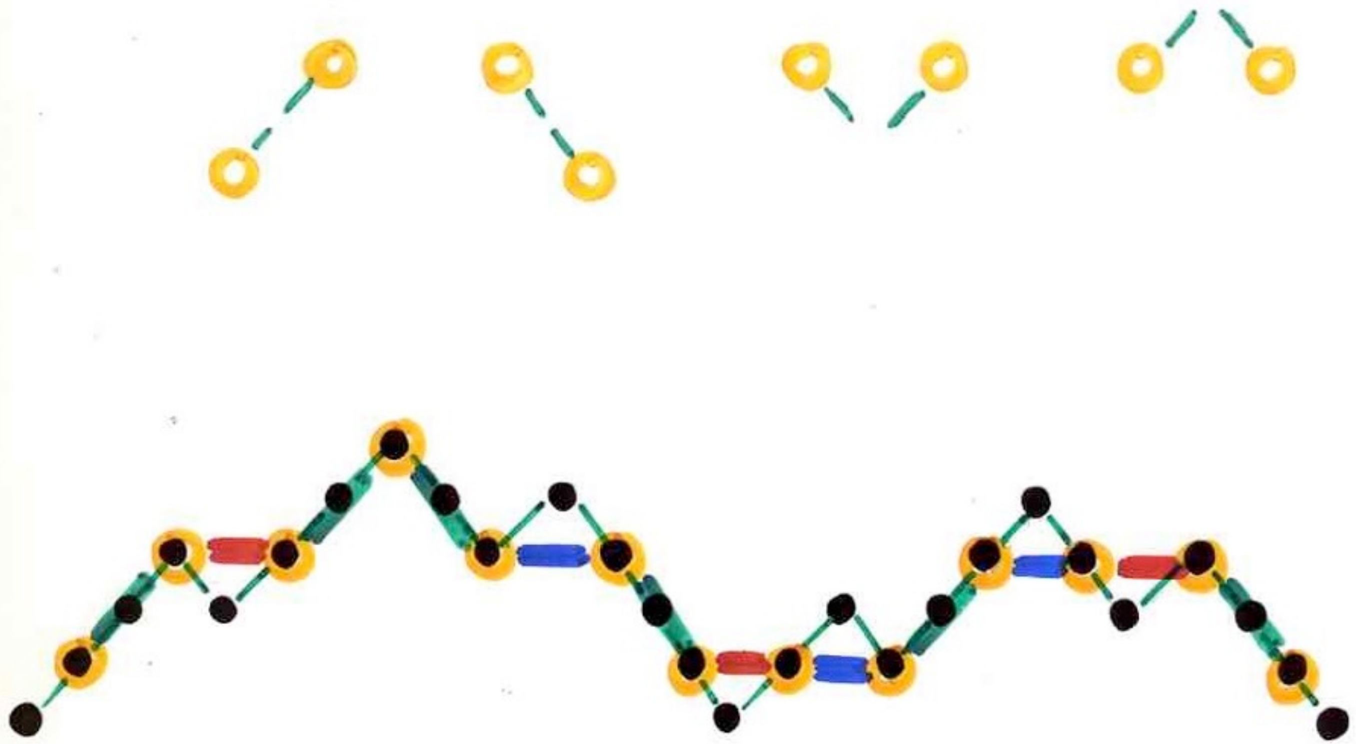
Part I, Ch 2a, 55-58

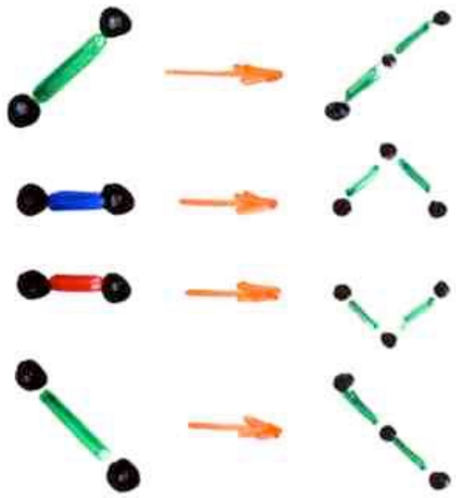
bijection

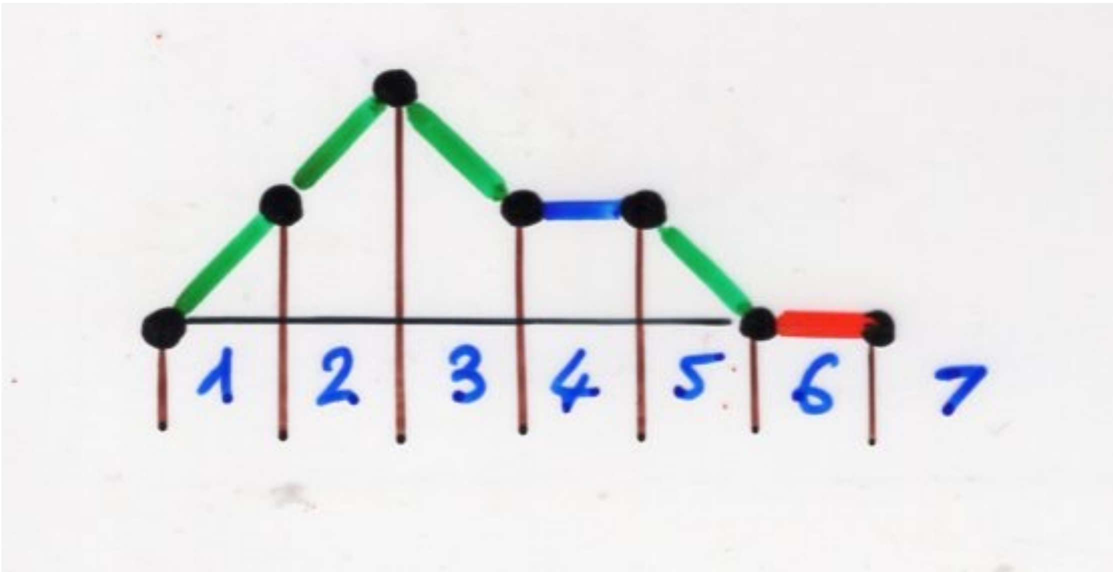
Dyck paths

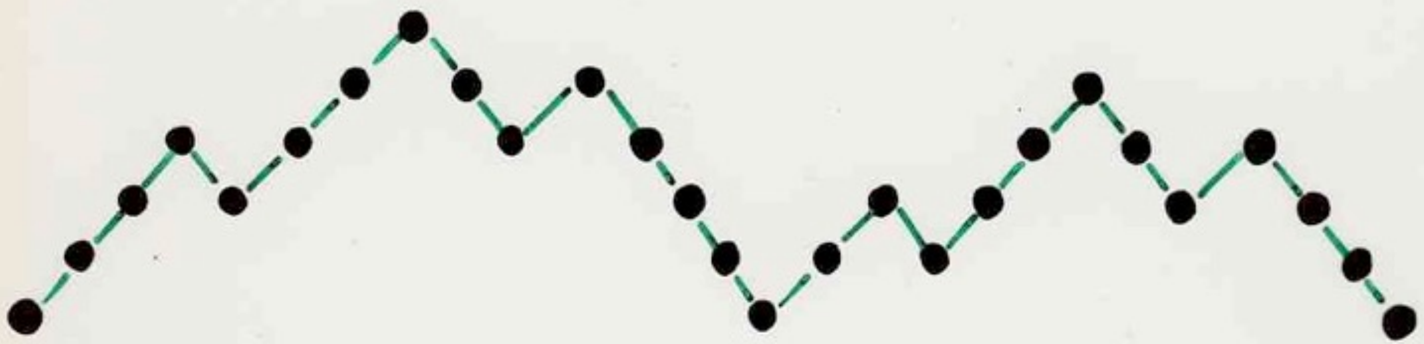
2-colored Motzkin paths

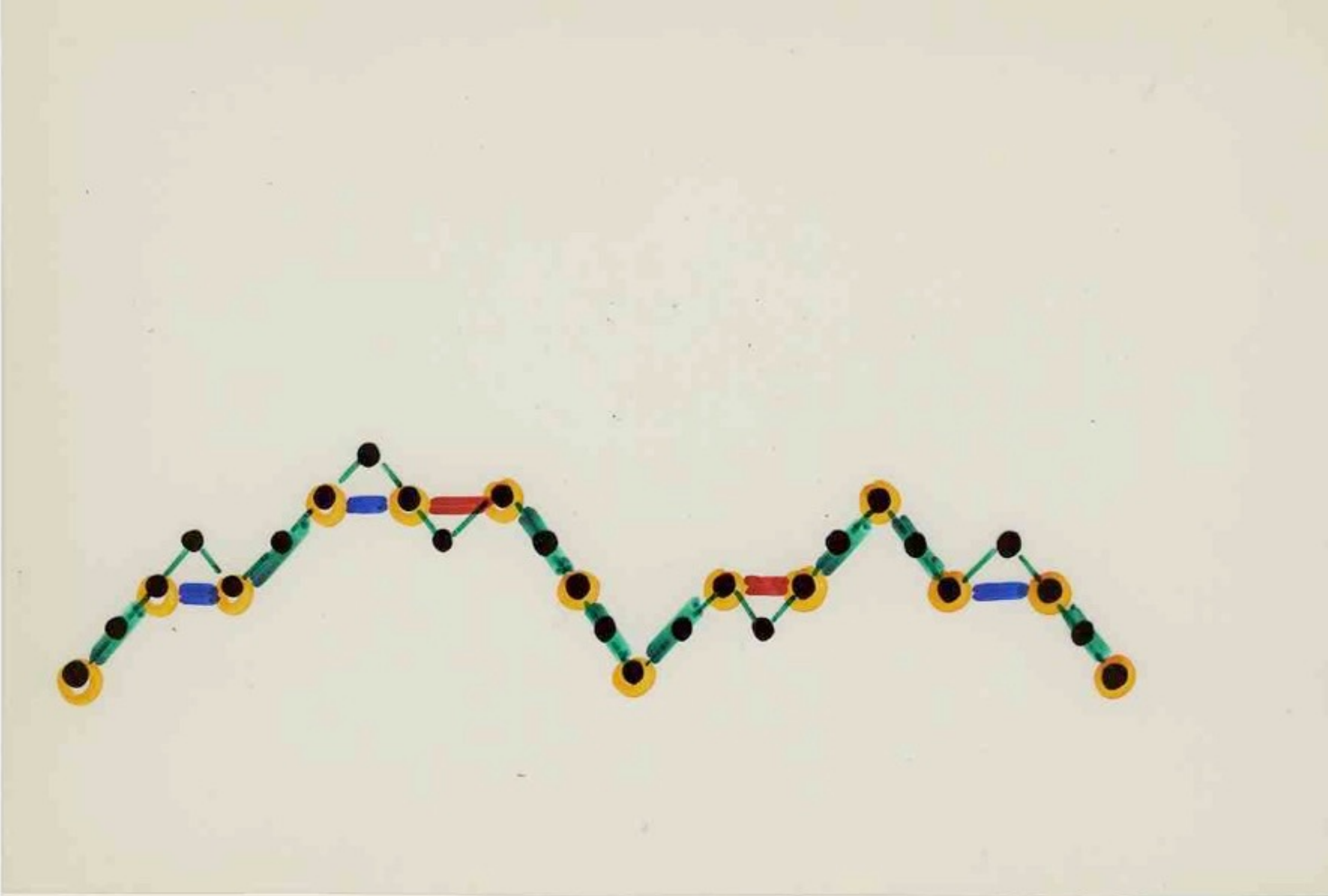


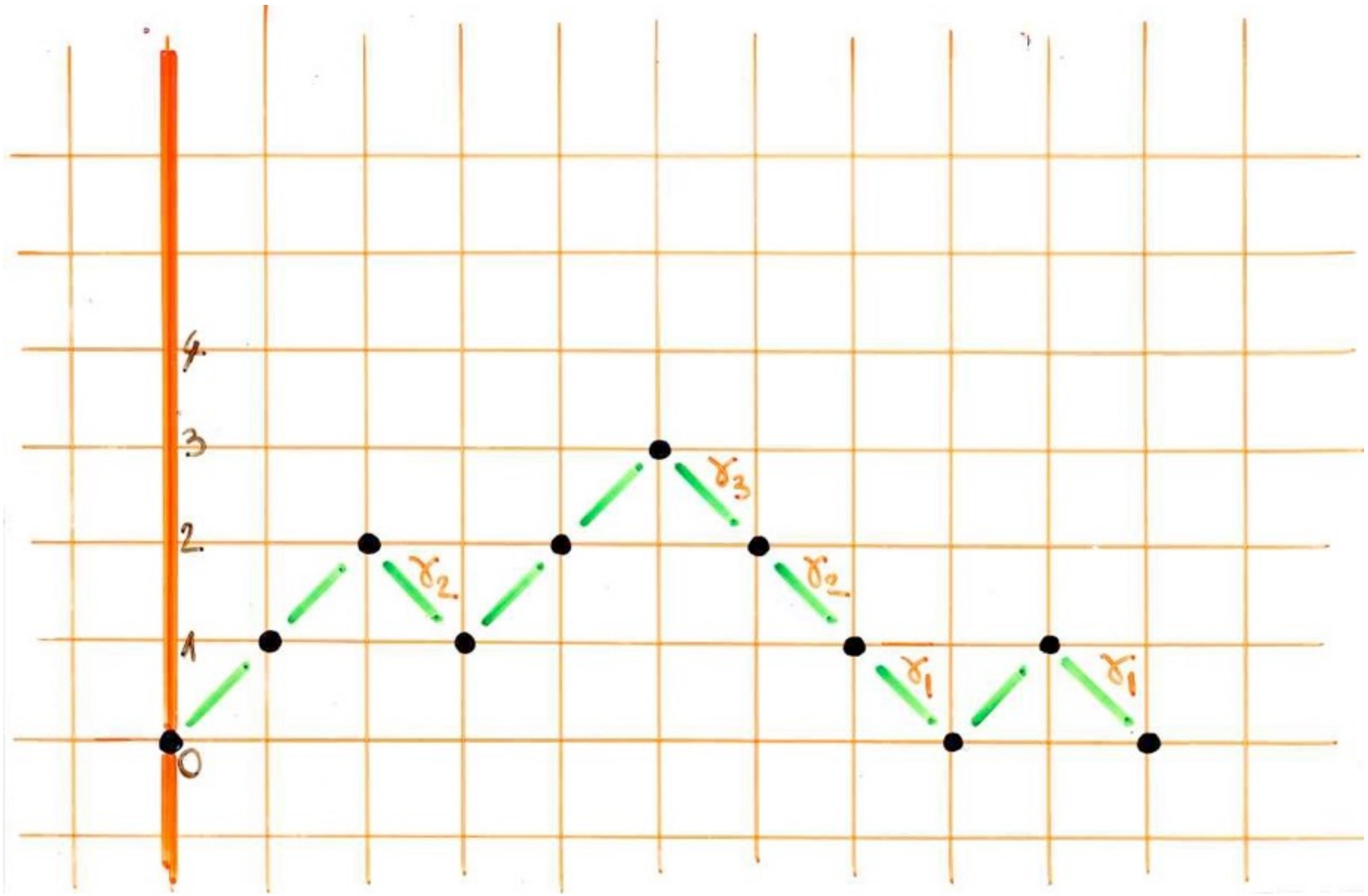




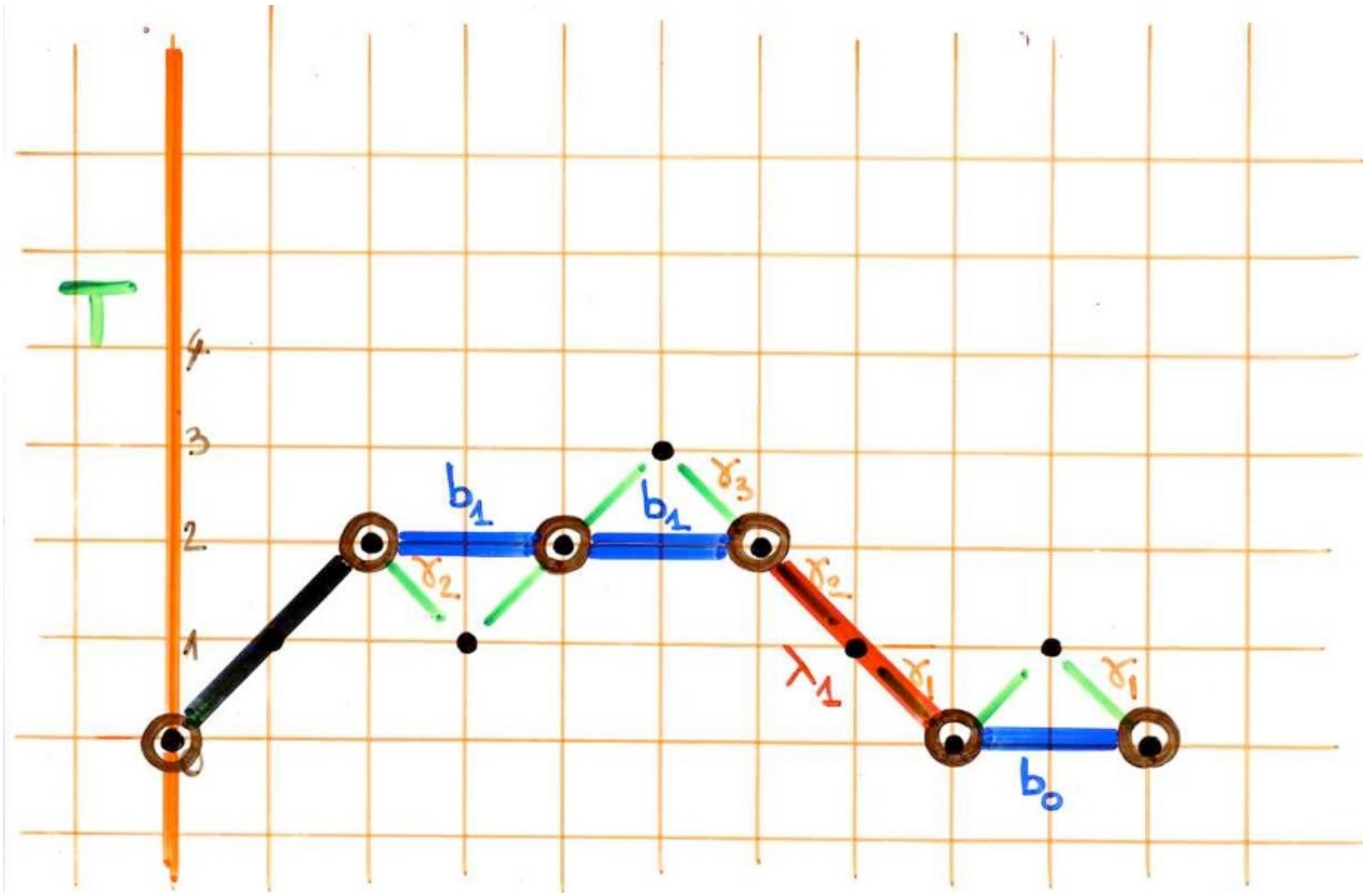

1
2
3
4
5
6
7

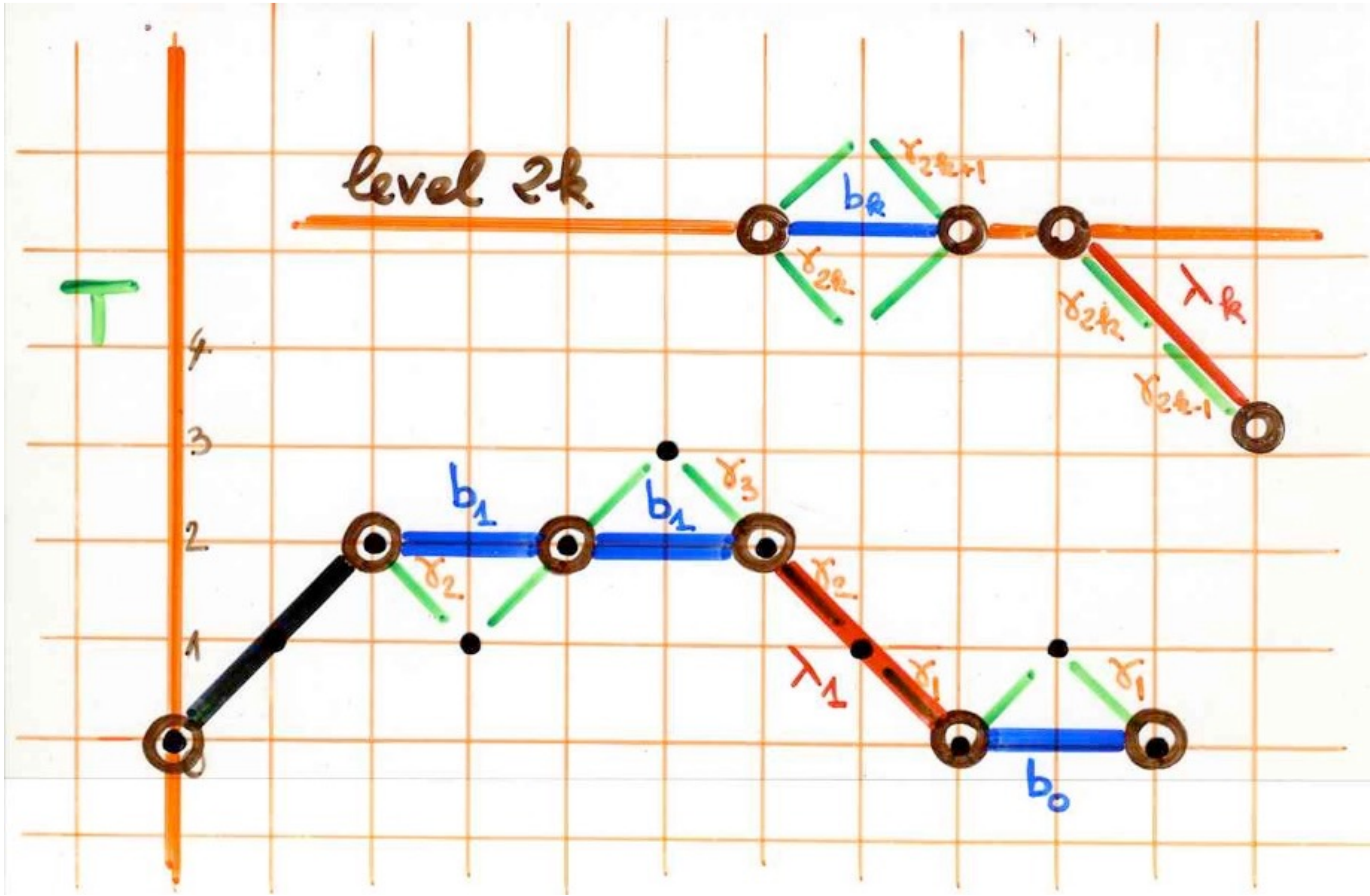






4
3
2
1
0
γ2
γ3
γ2
γ1
γ1

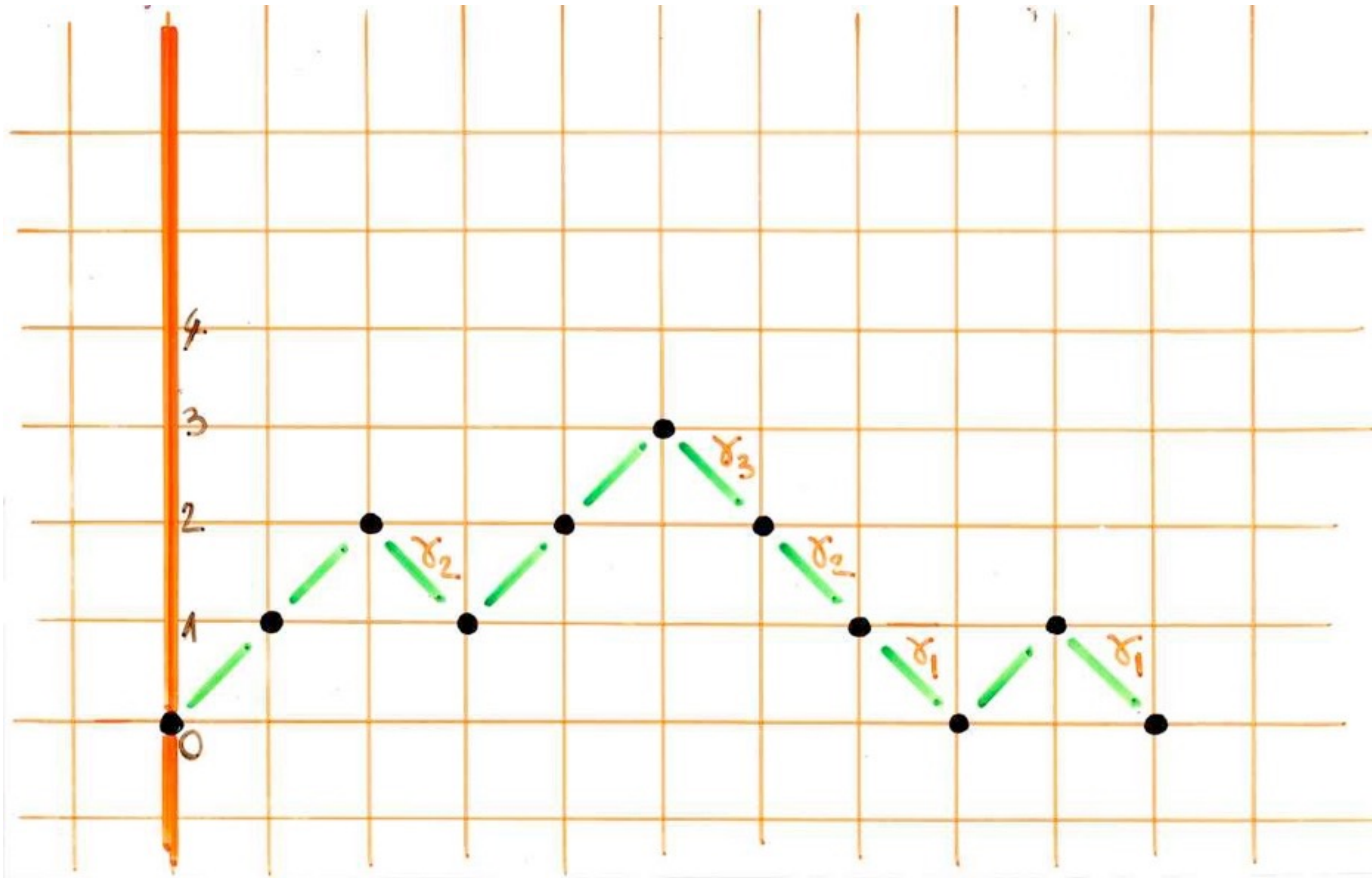

T
4
3
2
1
0
b1
b1
γ2
γ3
γ2
γ1
γ1
τ1
b0



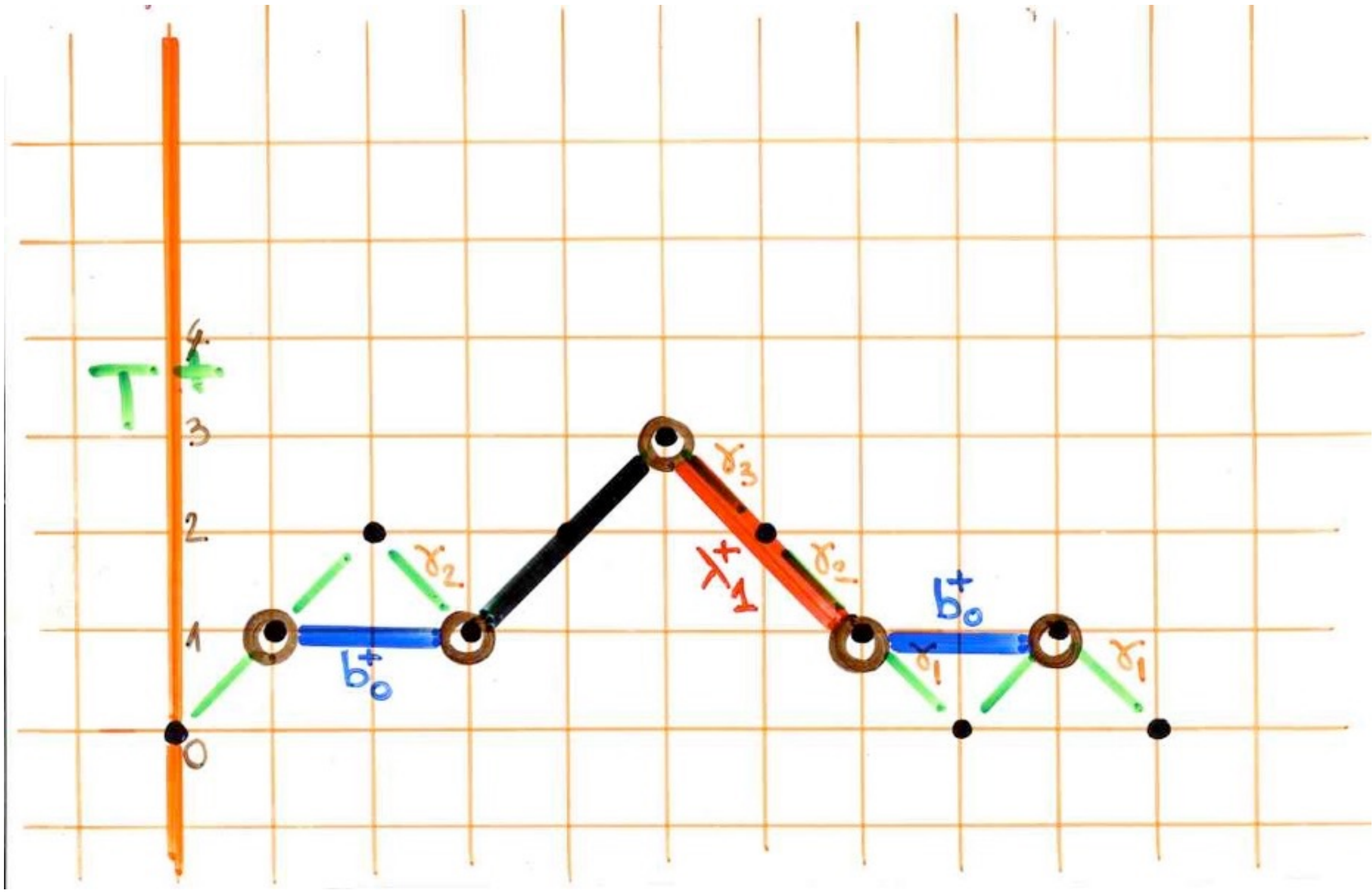
level 2k
T
b_k
γ_{2k+1}
γ_{2k}
γ_{2k}
λ_k
γ_{2k-1}
4
3
2
1
0
b_1
b_1
γ_3
γ_2
γ_2
λ_1
γ_1
γ_1
b_0

$$S(t;\gamma) = J(t;b,\lambda)$$

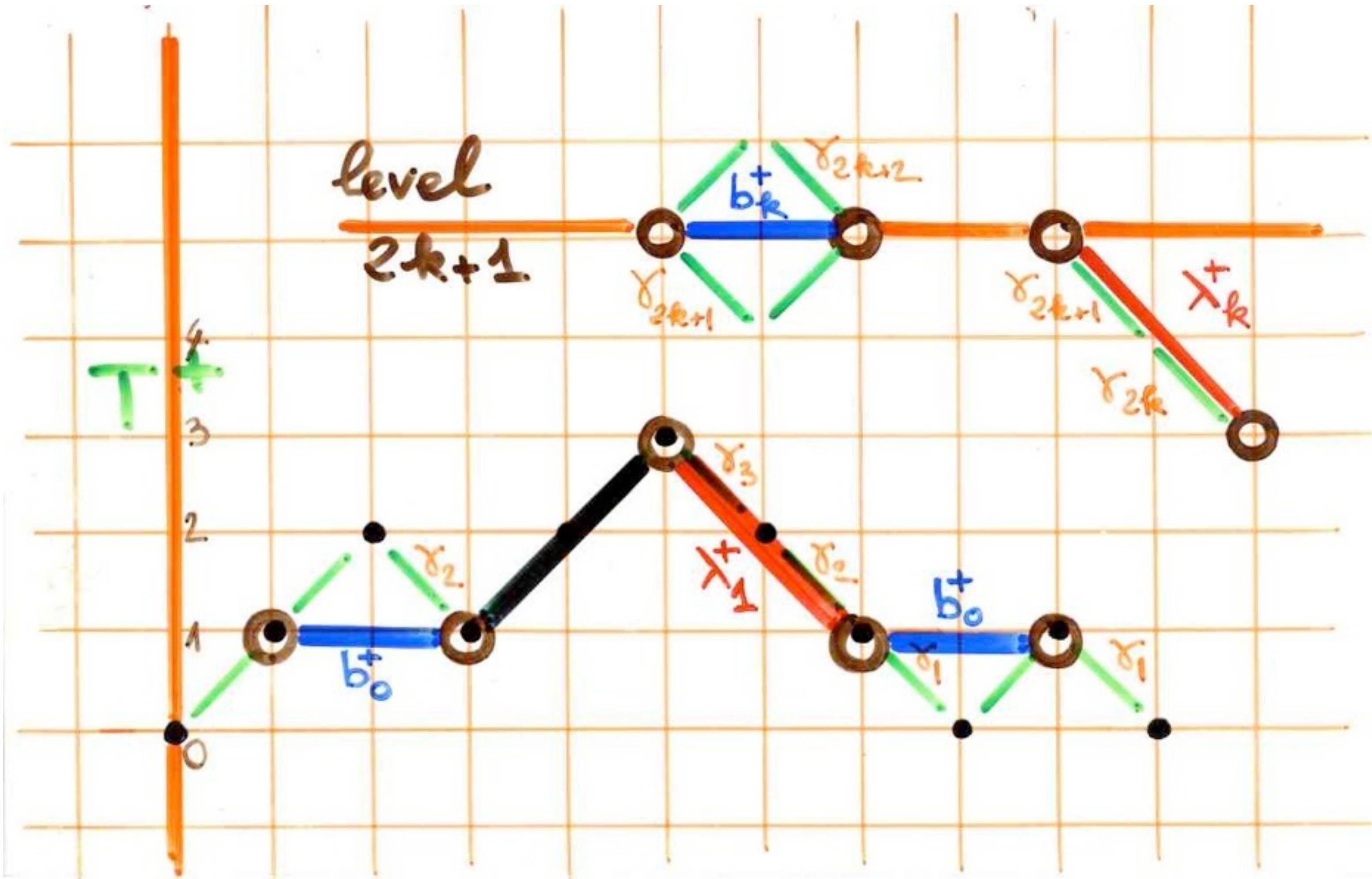
$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$



4
3
2
1
0
γ_2
γ_3
γ_2
γ_1
γ_1



T+
4
3
2
1
0
b+0
γ2
γ3
λ+1
γ0
b+0
γ1
γ1



level
2k+1
b_k^+
γ_{2k+2}
γ_{2k+1}
γ_{2k+1}
λ_k^+
γ_{2k}
T+
4
3
2
1
0
γ_3
γ_2
λ_1^+
γ_2
b_0^+
b_0^+
γ_1
γ_1

$$S(t;\gamma) = 1 + \gamma_1 t \, J(t; b^+, \lambda^+)$$

$$\begin{cases} b_k^+ = \gamma_{2k+1} + \gamma_{2k+2} \\ \lambda_k^+ = \gamma_{2k+1}\gamma_{2k} \end{cases}$$

Some examples

$$\tan t = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

D. André (1880)

$$\frac{1}{\cos t} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

$$\sec t = \frac{1}{\cos t}$$

Part I, Ch 3b, 61-79

Part I, Ch 3b, complements

$$\tan t = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

D. André (1880)

$$\frac{1}{\cos t} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

$$\sec t = \frac{1}{\cos t}$$

E_{2n} $\{1, 5, 61, 1385, \ldots\}$

secant numbers (Euler numbers)

alternating permutations

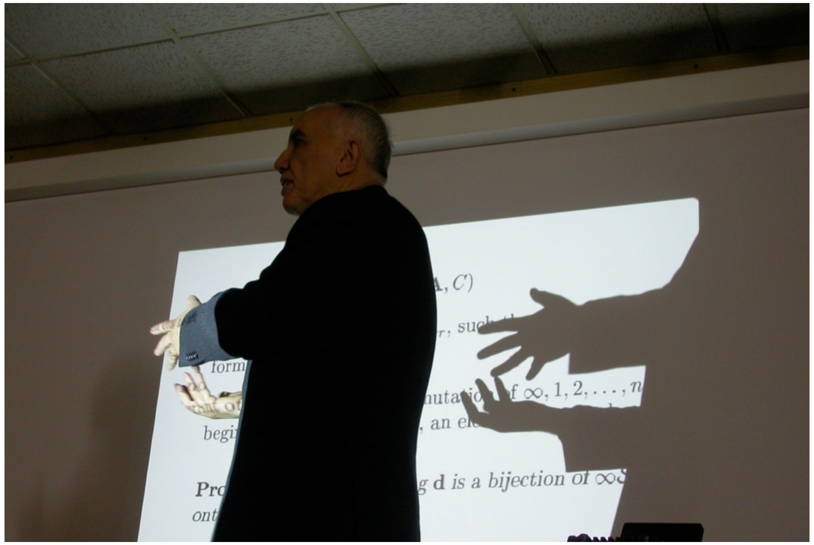
T_{2n+1} $\{1, 2, 16, 272, 7936, \ldots\}$

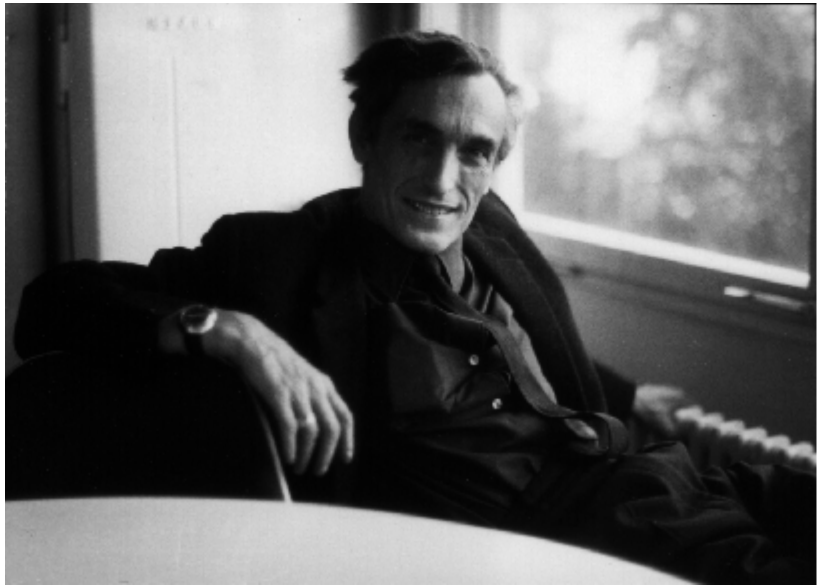
tangent numbers

Permutations alternantes

D. André (1880)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$





D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens"
(1970)

Euler

erit: $\alpha = 1$

$\beta = 1$

$\gamma = 5$

$\delta = 61$

$\varepsilon = 1385$

$\zeta = 50521$

$\eta = 2702765$

$\theta = 199360981$

$\iota = 19391512145$

$\varkappa = 2404879661671$

&c.

ex hisque valoribus obtinebitur:

$$\sec x = \alpha + \frac{\beta}{1.2} xx + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2 \ldots 6} x^6 + \frac{\varepsilon}{1.2 \ldots 8} x^8 + \&c.$$

Euler

erit hanc feriem ab illa fubtrahendo :

$$\mathrm{tg}\,x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots.6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots.8} + \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots.8} - \&c.$$

Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi;

erit : $\mathrm{tang}\,x = \frac{2\mathrm{A}x}{1.2} + \frac{2^3\mathrm{B}x^3}{1.2.3.4} + \frac{2^5\mathrm{C}x^5}{1.2\ldots6} + \frac{2^7\mathrm{D}x^7}{1.2\ldots8} + \&c.$

Laplace transform

$$\int_0^\infty e^{-u} \tan(ut)\, du = \cfrac{1}{1 - \cfrac{1\times 2\, t^2}{1 - \cfrac{2\times 3\, t^2}{1 - \cfrac{3\times 4\, t^2}{\cfrac{\cdots}{1 - \cfrac{k(k+1)\, t^2}{\cdots}}}}}}$$

$$\int_0^{\infty} e^{-u} \frac{1}{\cos(ut)}\, du = \cfrac{1}{1 - \cfrac{1 \times 1\, t^2}{1 - \cfrac{2 \times 2\, t^2}{1 - \cfrac{3 \times 3\, t^2}{\cfrac{\cdots\cdots}{1 - \cfrac{k^2\, t^2}{\cdots\cdots}}}}}}$$

Bernoulli numbers

B_{2n} $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, \ldots$

Genocchi numbers

$$G_{2n} = 2(2^{2n}-1)\, B_{2n}$$

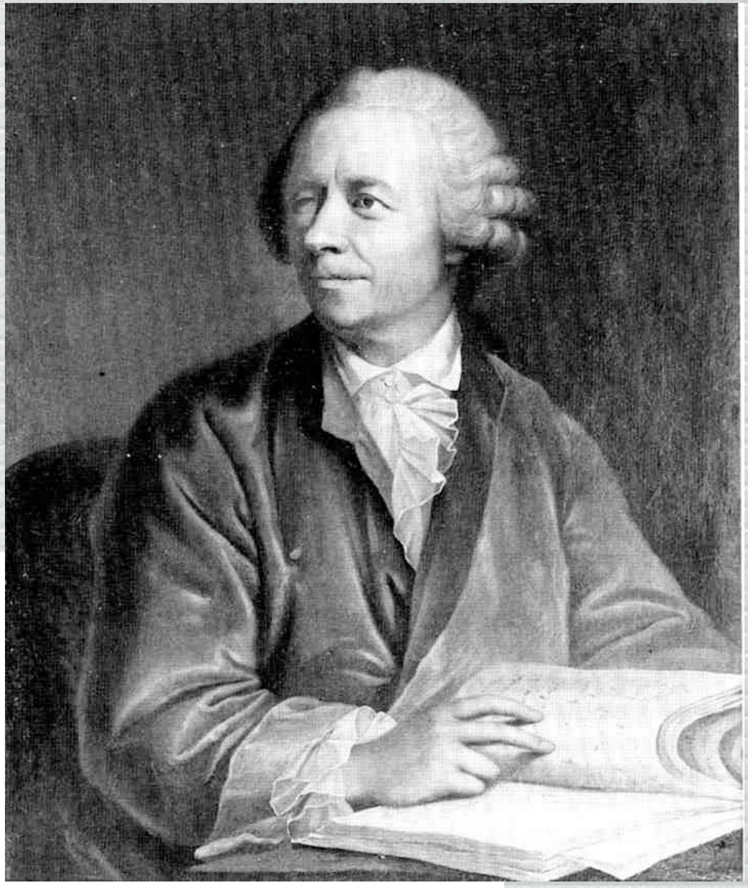
Bernoulli

$$2^{2n} G_{2n+2} = (n+1)\, T_{2n+1}$$

G_{2n} $\{1, 1, 3, 17, 155, 2073, \ldots\}$

G_{2n} $\{1, 1, 3, 17, 155, 2073, \ldots\}$

Angelo Genocchi
1817 - 1889



Hinc igitur calculo inſtituto reperietur:

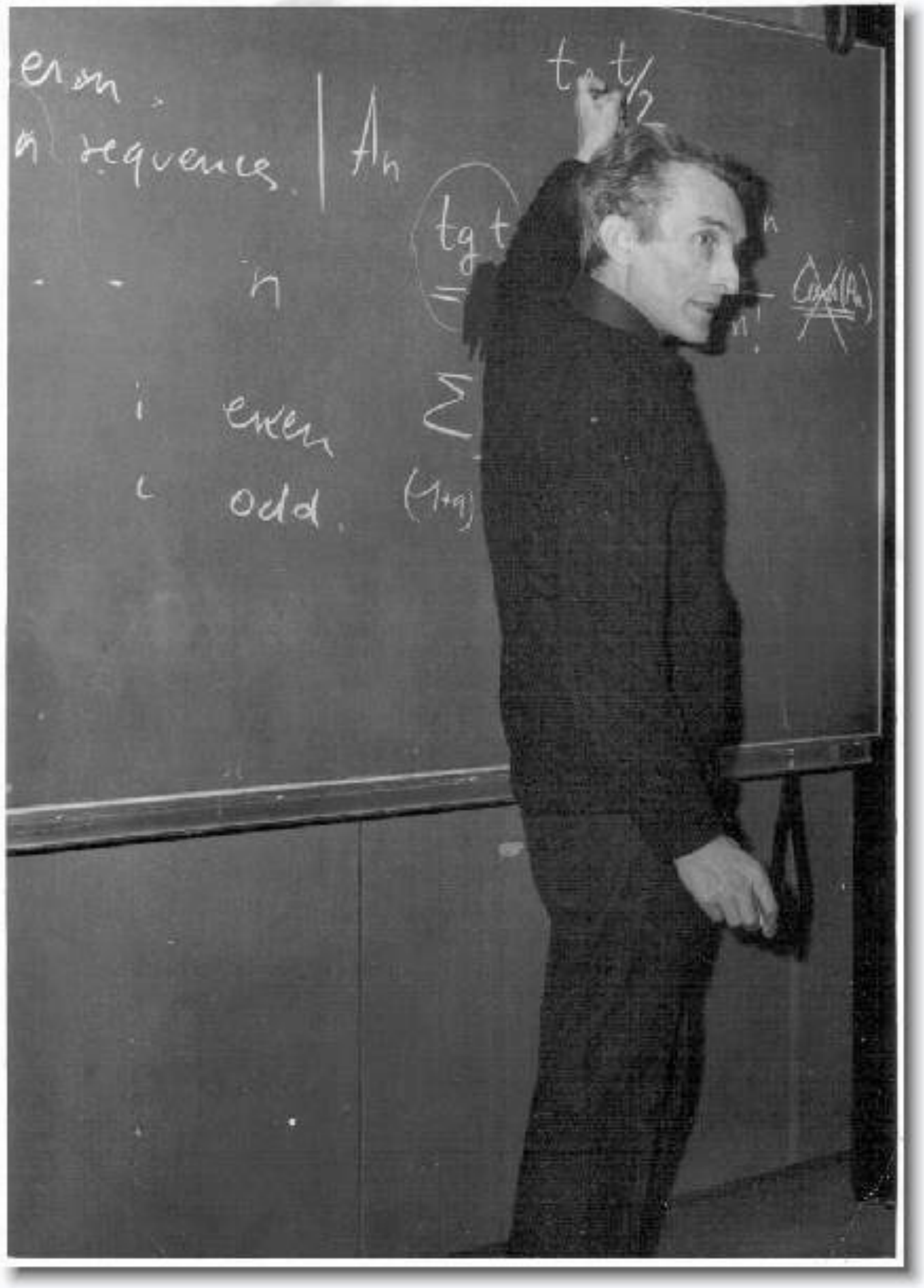
$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5.31$$
$$F = 2073 = 691.3$$
$$G = 38227 = 7.5461 = 7.\frac{127.129}{3}.$$
$$H = 929569 = 3617.257$$
$$I = 28820619 = 43867.9.73 \quad \&c.$$

Genocchi numbers

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$



our Master

Marcel Paul
Schützenberger

1920 - 1996

André permutations,

non-commutative
differential equations

Genocchi numbers

$$\sum_{n \geqslant 0} G_{2n+2}\, t^{2n} = \cfrac{1}{1 - \cfrac{1\times 1\, t^2}{1 - \cfrac{1\times 2\, t^2}{1 - \cfrac{2\times 2\, t^2}{1 - \cfrac{2\times 3\, t^2}{1 - \cfrac{3\times 3\, t^2}{\cfrac{\cdots\cdots}{1 - \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{k+1}{2} \right\rceil}}}}}}}$$

alternating permutations

$\sigma \in \mathfrak{S}_{2n-1}$

associated Laguerre history

p_i is odd for $1 \leq i \leq 2n-2$

$h = (\omega ; p_1, \ldots, p_{2n-1})$

« Alternative pistols » D. Dumont, X.V. 1978

Complements

elliptic functions ....

Jacobi elliptic functions

$$\begin{cases} sn' = cn \cdot dn, & sn(0) = 0 \\ cn' = - dn \cdot sn, & cn(0) = 1 \\ dn' = -k^2 sn \cdot cn, & dn(0) = 1 \end{cases}$$

Dumont, X.V., Flajolet 80's

3 different combinatorial interpretations

$$\int_0^\infty e^{-u}\,\mathrm{cn}(ut)\,du = \cfrac{1}{1-\cfrac{1^2t^2}{1-\cfrac{2^2\alpha^2t^2}{1-\cfrac{3^2t^2}{1-\cfrac{4^2\alpha^2t^2}{\cdots}}}}}$$

Séminaire Lotharingien de Combinatoire **54** (2006), Article B54g

THE FERMAT CUBIC, ELLIPTIC FUNCTIONS, CONTINUED FRACTIONS, AND A COMBINATORIAL EXCURSION

ERIC VAN FOSSEN CONRAD AND PHILIPPE FLAJOLET

Kindly dedicated to Gérard···Xavier Viennot on the occasion of his sixtieth birthday.

Abstract. Elliptic functions considered by Dixon in the nineteenth century and related to Fermat's cubic, $x^3+y^3=1$, lead to a new set of continued fraction expansions with sextic numerators and cubic denominators. The functions and the fractions are pregnant with interesting combinatorics, including a special Pólya urn, a continuous-time branching process of the Yule type, as well as permutations satisfying various constraints that involve either parity of levels of elements or a repetitive pattern of order three. The combinatorial models are related to but different from models of elliptic functions earlier introduced by Viennot, Flajolet, Dumont, and Françon.

In 1978, Apéry announced an amazing discovery: "$\zeta(3) \equiv \sum 1/n^3$ *is irrational*". This represents a great piece of Eulerian mathematics of which van der Poorten has written a particularly vivid account in [59]. At the time of Apéry's
was known about the arithmetic nature of the zeta values at odd i
unnaturally his theorem triggered interest in a whole range of proble
recognized to relate to much "deep" mathematics [38, 51]. Apéry's orig
proof crucially depends on a continued fraction representation of $\zeta(3)$.

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\ddots}}}}, \tag{1}$$

where $\quad \varpi(n) = (2n+1)(17n(n+1)+5).$

Lucelle (2005)
Séminaire Lotharingien de Combinatoire
54th SLC

Apéry $\zeta(3) = \sum 1/n^3$

irrational

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\cdots}}}}$$

$$\varpi(n) = (2n+1)\left(17n(n+1)+5\right)$$

$$\mathrm{sm}(z) = \mathrm{Inv} \int_0^z \frac{dt}{(1-t^2)^{2/3}}$$

$$\begin{cases} \mathrm{sm}' = \mathrm{cm}^2 \\ \mathrm{cm}' = -\mathrm{sm}^2 \end{cases} \qquad \begin{matrix} \mathrm{sm}(0) = 0 \\ \mathrm{cm}(0) = 1 \end{matrix}$$

$$\mathrm{sm}(z)^3 + \mathrm{cm}(z)^3 = 1$$

Dixon (1890)

Conrad (2002)

$$\int_0^\infty \mathrm{sm}(u)\, e^{-u/x}\, du = \cfrac{x^2}{1 + b_0 x^3 - \cfrac{1.2^2.3^2.4\, x^6}{1 + b_1 x^3 - \cfrac{4.5^2.6^2.7\, x^6}{1 + b_2 x^3 - \cdots}}}$$

$$b_n = 2(3n+1)((3n+1)^2+1)$$

Polya urn model

y ○ → ● ●

x ● → ○ ○

history

$x \to yy \to yxx \to yyyx \to xxyyx \to xyyyyx$

nb of **histories** total $n!$

- starting ●, ending ○○○..○ — $sm(-z)$
- starting ● ending ●●●....● $cm(-z)$

$$sm(z) = z - 4\frac{z^4}{4!} - 160\frac{z^7}{7!} - 20800\frac{z^{10}}{10!} - \cdots$$

$$cm(z) = z - 2\frac{z^3}{3!} - 40\frac{z^6}{6!} - 3680\frac{z^9}{9!} - \cdots$$

nb of histories total $n!$

- starting ●, ending ○○○..○ $-sm(-z)$
- starting ● ending ●●●....● $cm(-z)$

Jacobian elliptic functions

sn, cn, dn

X.V. (1980) Jacobi permutations

Dumont (1979) Flajolet alternating

Schett generation

cycle structure

- class of permutations

  based on parity

- 2-repeated permutations

  (with J. Françon) → sn, cn, dn
  (1989)

  Jacobian elliptic

  3-repeated (*) permutations

  → $-sm(-z)$ continued fraction

P. F. with R. Bacher (2010)

pseudofactorial

$$a_{n+1} = (-1)^{n+1} \sum \binom{n}{k} a_k a_{n-k}$$

$$\sum_{n \geq 0} a_n z^n = \cfrac{1}{1 + 1z - \cfrac{3 \cdot 1^2 z^2}{1 - 1z + \cfrac{2^2 z^2}{1 + 3z + \cfrac{3 \cdot 3^2 z^2}{1 - 3z + \cfrac{4^2 z^2}{\cdots}}}}}$$

Weierstraß function $\wp$

lattice sum

A Happy New Year 2010

Consider the **integer sequence** (p_n), which starts as

$$2, 144, 96768, 268240896, 2111592333312, 37975288540299264, \ldots$$

and is defined by sums over the **square lattice**,

$$p_n := (-1)^{n+1}(4n{+}3)! \left[\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right]^{-4n-4} \sum_{a,b=-\infty}^{+\infty} \left[(2a+1)+(2b+1)\sqrt{-1}\right]^{-4n-4}.$$

The following **continued fraction** expansion holds:

$$\sum_{n=0}^{\infty} p_n z^n = \cfrac{2}{1-2\cdot 2^2(2^2+5)\,z-\cfrac{2\cdot 3^2\cdot 4^2\cdot 5^2\cdot 6\,z^2}{1-2\cdot 6^2(6^2+5)\,z-\cfrac{6\cdot 7^2\cdot 8^2\cdot 9^2\cdot 10\,z^2}{1-2\cdot 10^2(10^2+5)\,z-\ddots}}}.$$

[A follow up to R. Bacher and P. Flajolet, *The Ramanujan Journal*, 2010, in press.]

Philippe Flajolet

ॐ भूर्भुवः स्वः
तत्सवितुर्वरेण्यं ।
भर्गो देवस्य धीमहि,
धीयो यो नः
प्रचोदयात् ॥