Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

The Art of Bijective Combinatorics

Part I. An introduction to enumerative, algebraic
and bijective combinatorics (2016)

Part II. Commutations and heaps of pieces
with interaction in physics, mathematics and computer science. (2017)

Part III. The cellular ansatz:
bijective combinatorics and quadratic algebra (2018)
Robinson-Schensted-Knuth, PASEP, Tilings, Alternating Sign Matrices ...
under the same roof

Part IV. Combinatorial theory of orthogonal polynomials
and continued fractions (2019)

The Art of Bijective Combinatorics

« ABjC »

« Video-book »

- videos
- slides
- www.viennot.org

mirror website

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Each course can be followed independantly

Two levels:

- for master and graduate students
- for professors and more advanced students

under the name « complements »

sometimes no proof

Chapter 0
Overview of the course

IMSc, Chennai
10 January 2019

Xavier Viennot
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Orthogonal polynomials

classical analysis

special functions

trigonometric
hypergeometric
Bessel, elliptic
) functions

numerical analysis

interpolation
mechanical quadrature
differential and integral equations

Probalities
theory

quantum
statistical
mechanics

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

$$\int_{-1}^{+1} U_m(x)\, U_n(x)\, (1-x^2)^{1/2} = \frac{\pi}{2}\, \delta_{m,n}$$

$$\cos(n\theta) = T_n(\cos\theta)$$

$T_n(x)$ Tchebychef polynomial 1st kind

$\{P_n(x)\}_{n\geq 0}$

sequence of polynomials

$P_n(x) \in \mathbb{R}[x]$

$\deg(P_n(x)) = n$

degree

$f(P(x)Q(x)) = \int_{\mathbb{R}} P(x)Q(x)\, d\mu(x)$

measure μ on $\mathbb{R}$

book
G. Szegö
(1938)

Orthogonal polynomials

reed. (1958, 1966, 1975)

book T. Chihara (1978) reed. 2011

origin: continued fractions

Euler

§. 21. Datur vero alius modus in summam huius seriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: sit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}.$$

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

DE FRACTIONIBVS CONTINVIS.
DISSERTATIO.

AVCTORE
Leonh. Euler.

§. I.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $=1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-$etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4\ 4.6.\ 6.8.\ 8.10.\ 10}{3.3.5.5.7.7\ 9.9.\ 11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

continued fractions

Stieltjes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots \cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$S(t;\lambda)$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cdots\cfrac{}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}}$$

$$J(t;b,\lambda)$$

Jacobi continued fraction

$$b=\{b_k\}_{k\geq 0} \qquad \lambda=\{\lambda_k\}_{k\geq 1}$$

equivalence

orthogonal polynomials ⟷ continued fractions

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

reciprocal of orthogonal polynomials

$$P^*_k(x) = x^k P_k(1/x)$$

books:

G. Andrews, R. Askey, R. Roy (1999) special functions

M. Ismail (2005) classical orthogonal polynomials

S. Khrushchev (2008) orthogonal polynomials and continued fractions from Euler's point of view

X.V., "Une théorie combinatoire des polynômes orthogonaux généraux" Lecture Note, UQAM, Montréal, (1983)

« Video-book »

Part IV. Combinatorial theory of orthogonal polynomials and continued fractions (2019)

W. Jones, W. Thron (1980, 1984)

continued fractions
analytic theory and applications

R. Koekoek, P. Lesky, R. Swarttow (2010)

Hypergeometric orthogonal polynomials
and their q-analogues

late 70's, early 80's

combinatorial interpretations

of classical orthogonal polynomials

Hermite, Laguerre, Jacobi

combinatorial interpretations

of linearization coefficients

$$P_k(x)\,P_l(x) = \sum_n a_{kl}^{n}\,P_n(x)$$

positivity

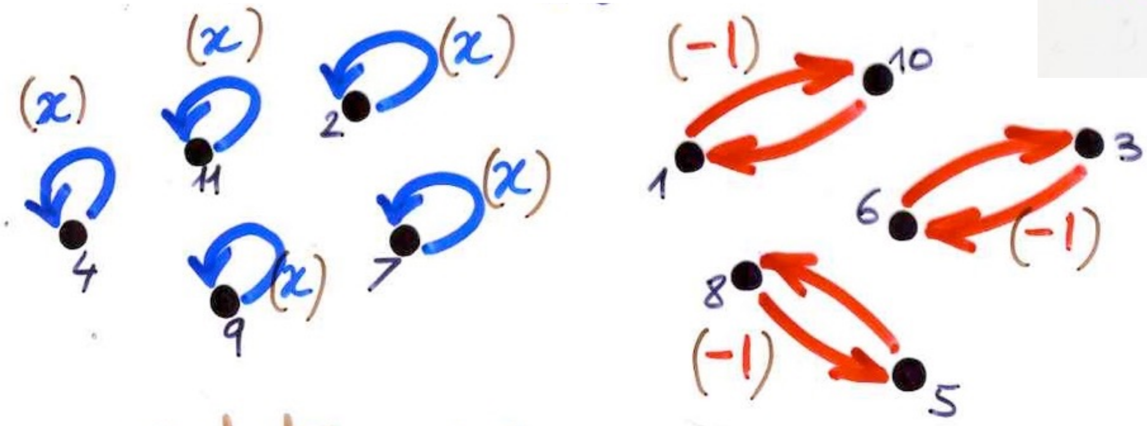
Combinatorial interpretation of Hermite polynomials

Hermite polynomial

$$H_n(x)$$

$$\int_{-\infty}^{+\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = \sqrt{\pi}\, 2^n\, n!\, \delta_{nm}$$

Hermite configuration



weight (x)
(-1)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 2 & 6 & 4 & 8 & 3 & 7 & 5 & 9 & 1 & 11 \end{pmatrix}$$

(combinatorial)
Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$

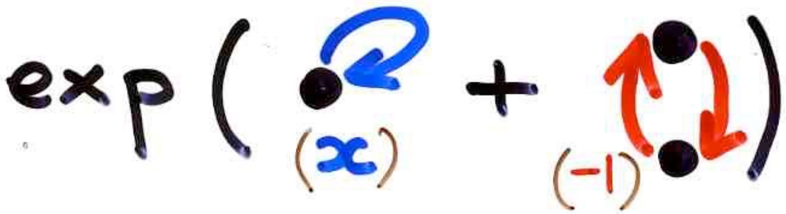
$$H_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \frac{n!}{2^k \, k! \, (n-2k)!} x^{n-2k}$$

(combinatorial)
Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$

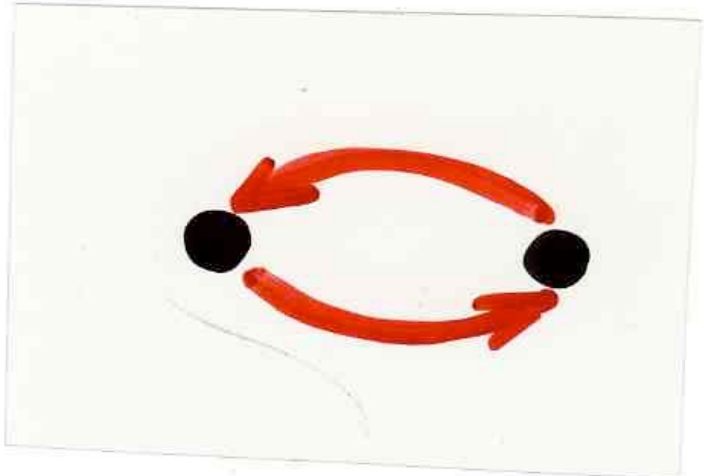
$$\sum_{n \geqslant 0} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

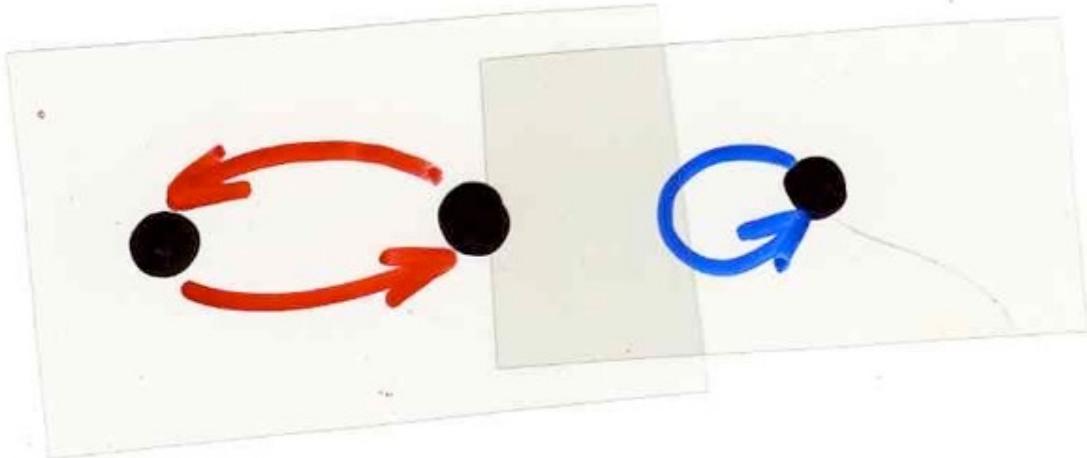
ABjC, part I, Ch3

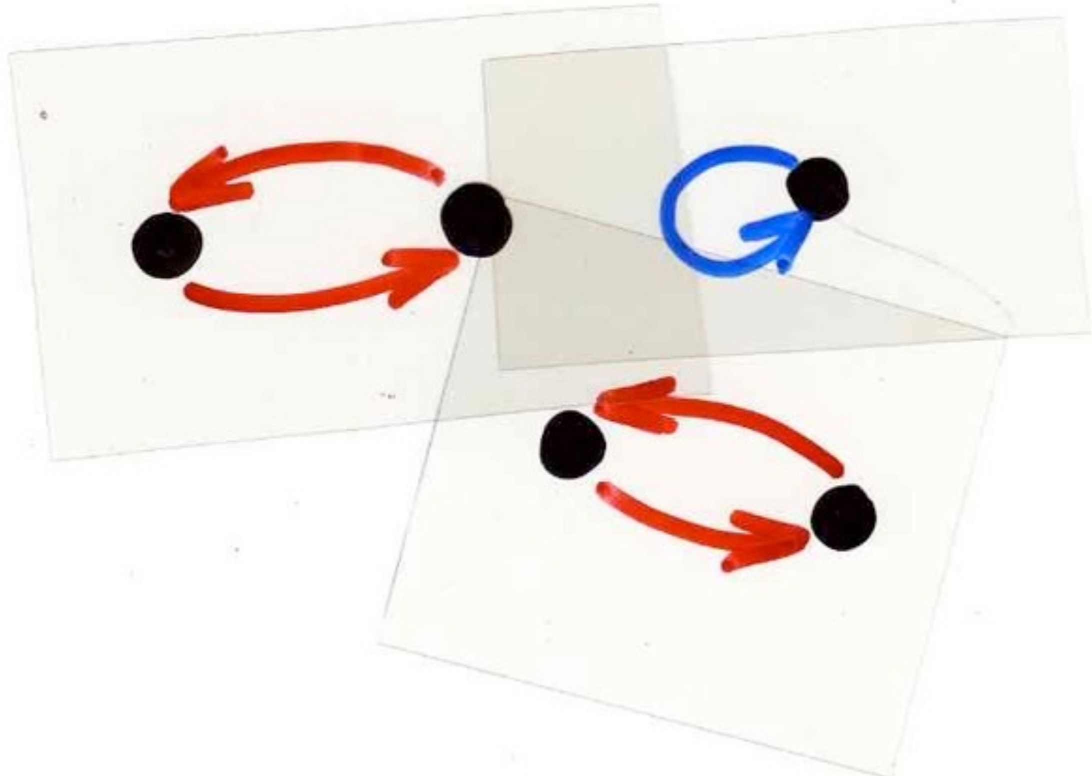


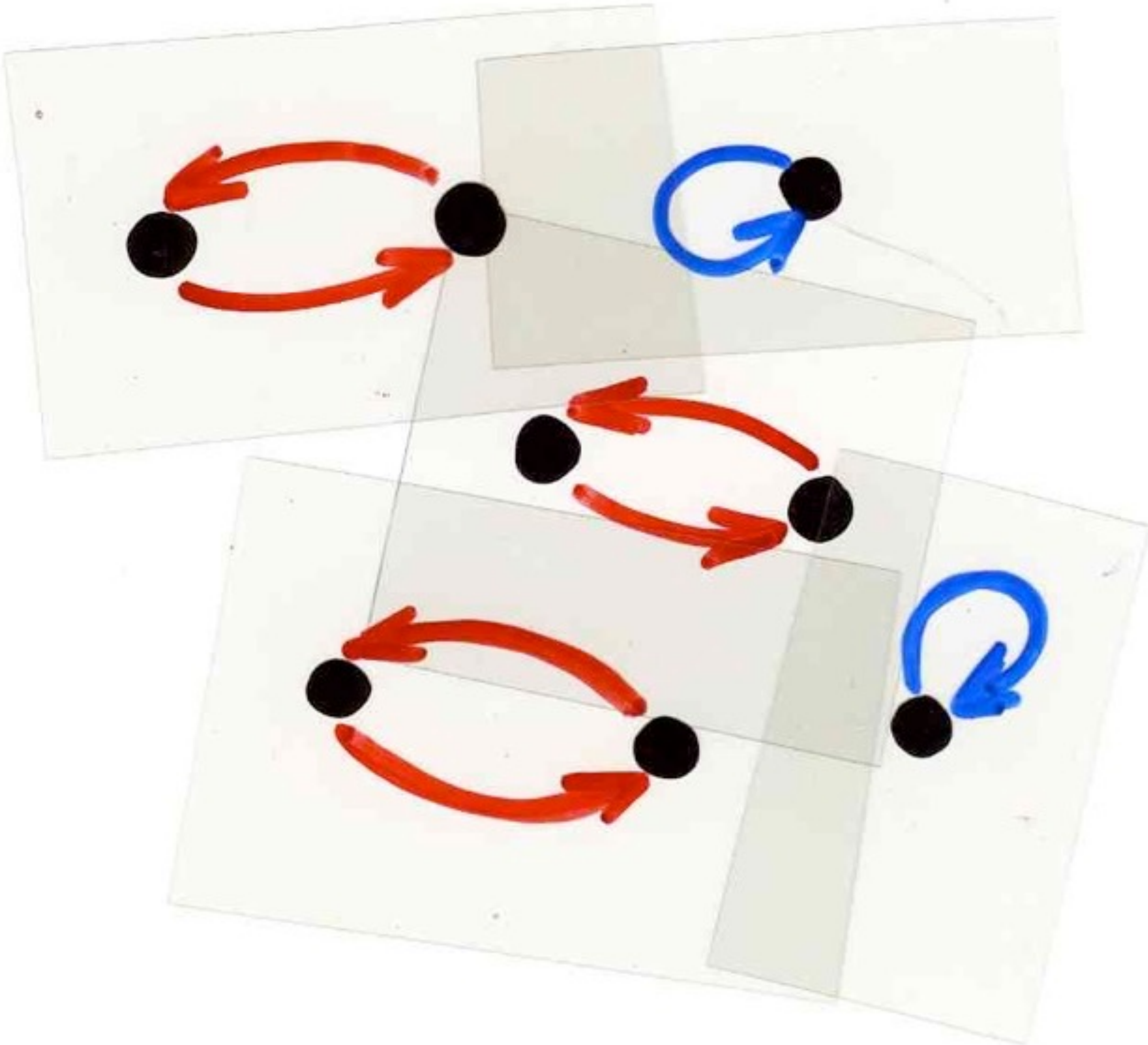
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

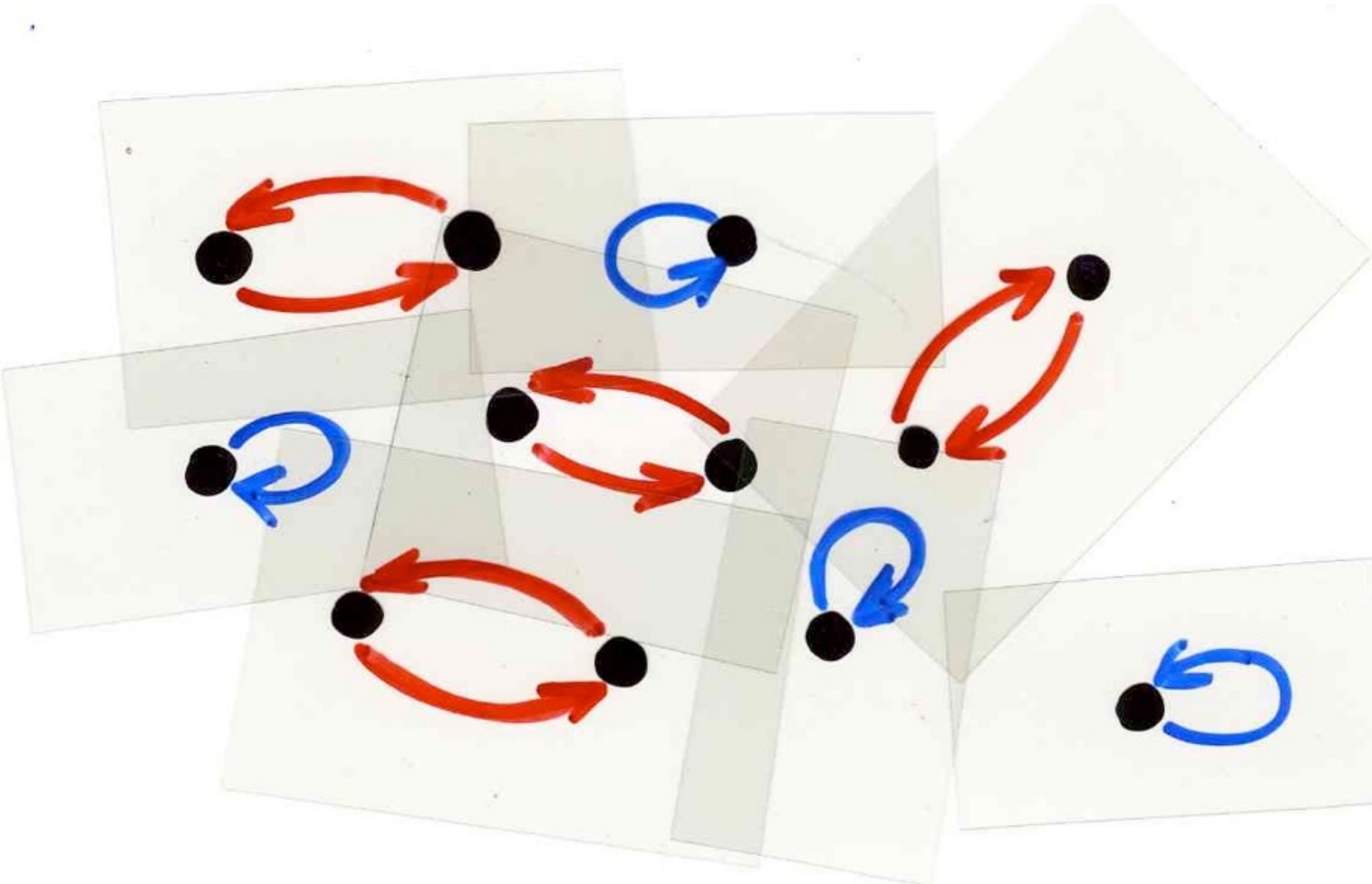
(combinatorial) Hermite polynomials

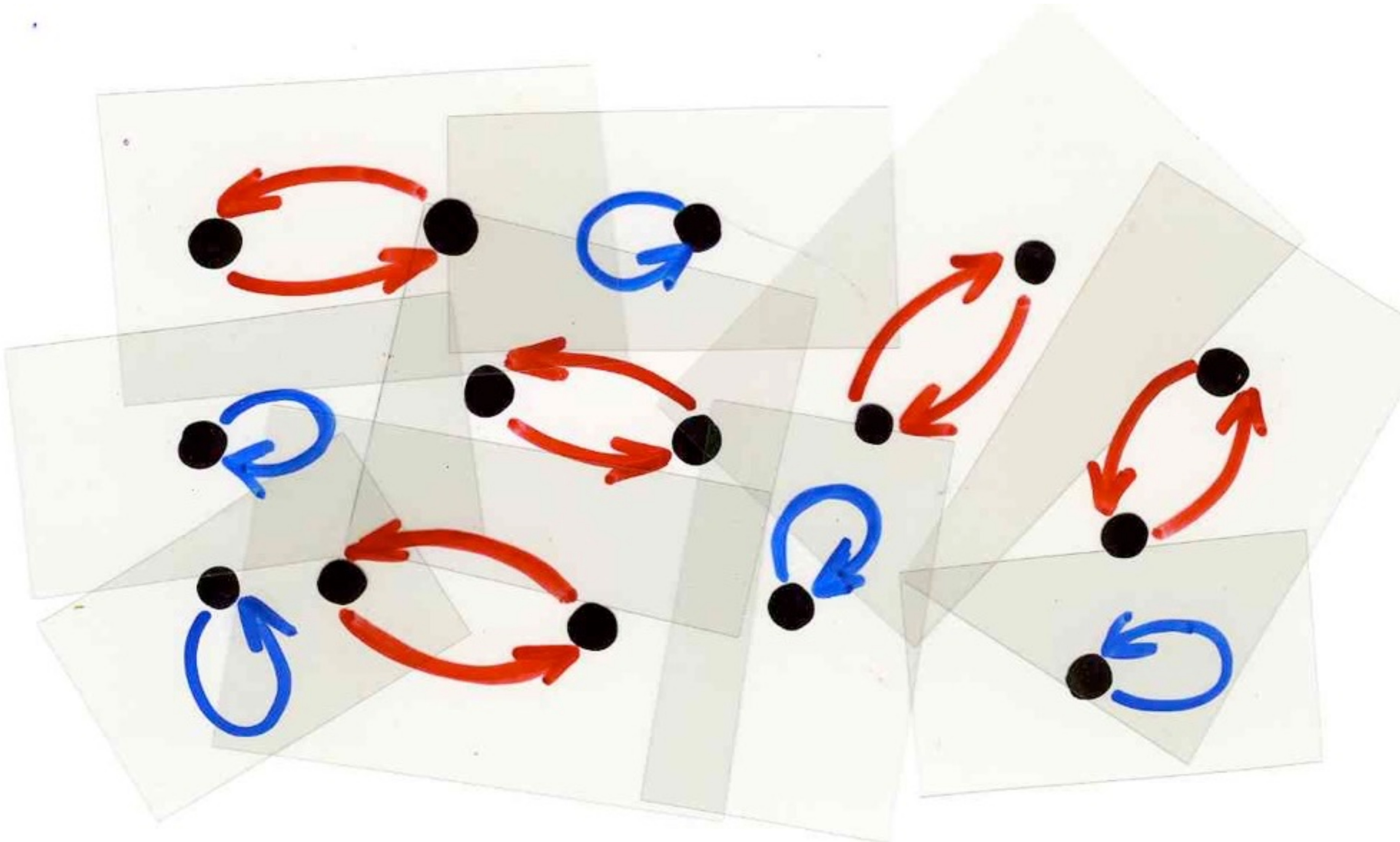




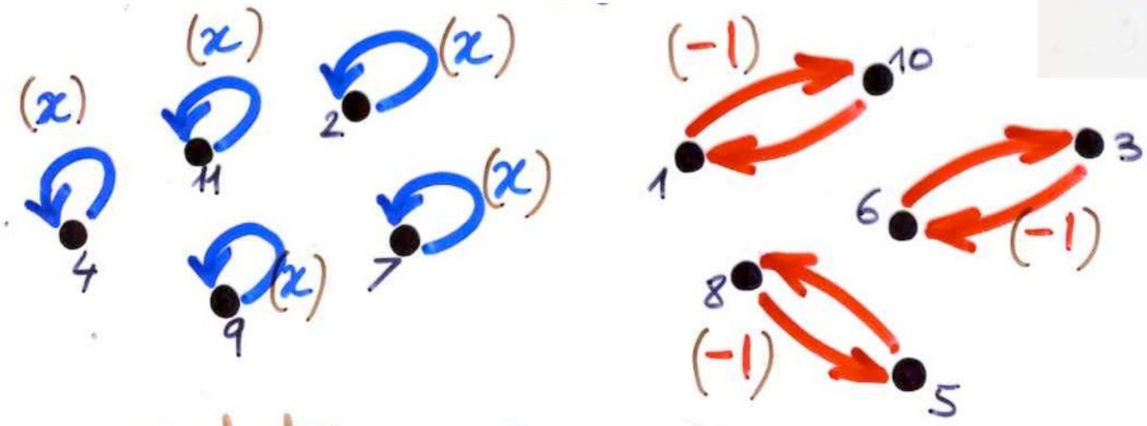








Hermite configuration



weight (x) (-1)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 2 & 6 & 4 & 8 & 3 & 7 & 5 & 9 & 1 & 11 \end{pmatrix}$$

(combinatorial)
Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$

Combinatorial proof of formulae

Mehler identity

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

ABjC, Part I, Ch3b, p.26

valued combinatorial objects

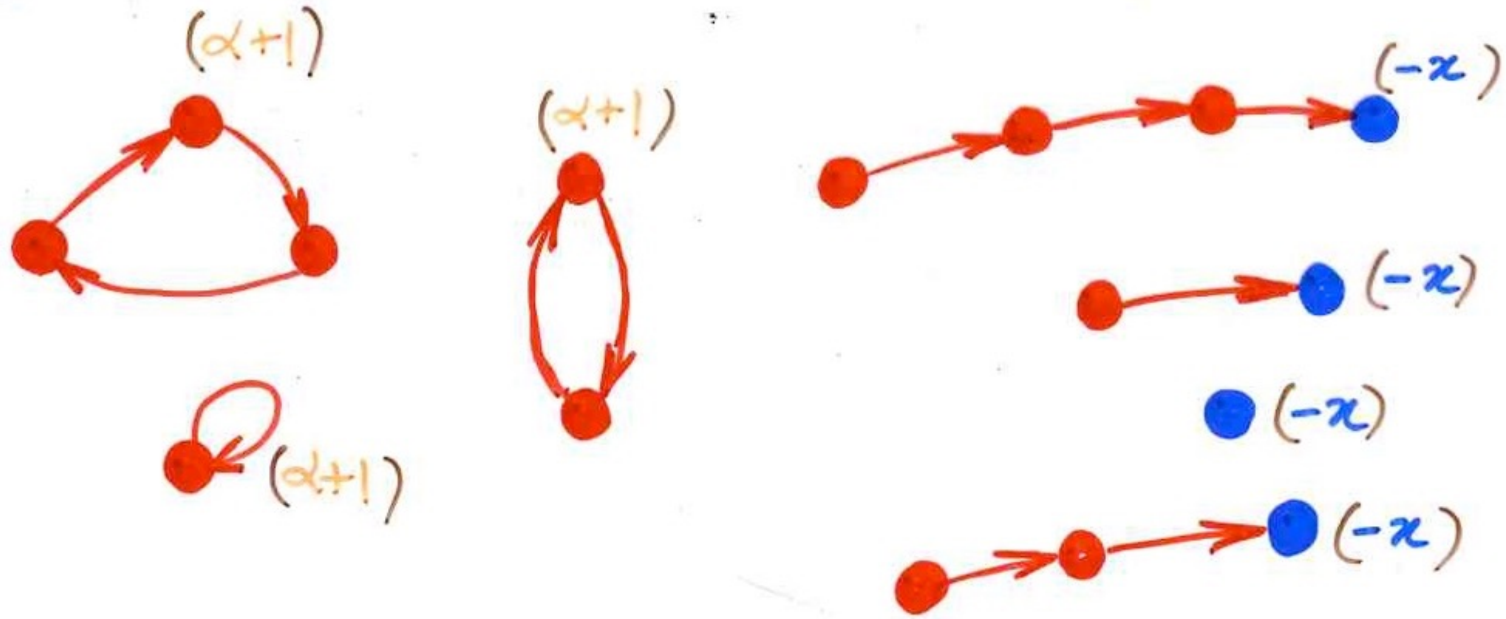
weight function

Laguerre polynomials

$$\int_0^\infty L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, e^{-x} x^{\alpha}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{mn}$$

$$\sum_{n \geq 0} \tilde{L}_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



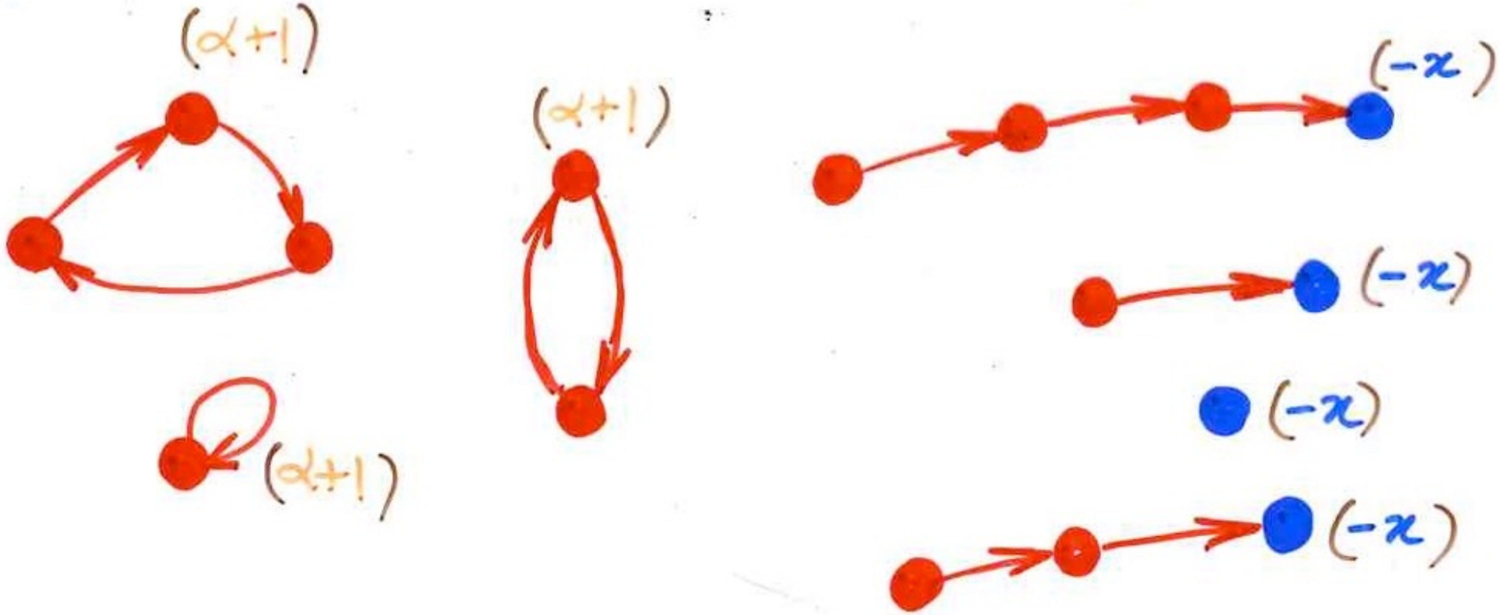
$$L_n^{\alpha}(x) = \sum_{LC} v(LC)$$

Laguerre configurations on $[1,n]$

$$v(LC) = (\alpha+1)^{i}(-x)^{j}$$

i = number cycles 

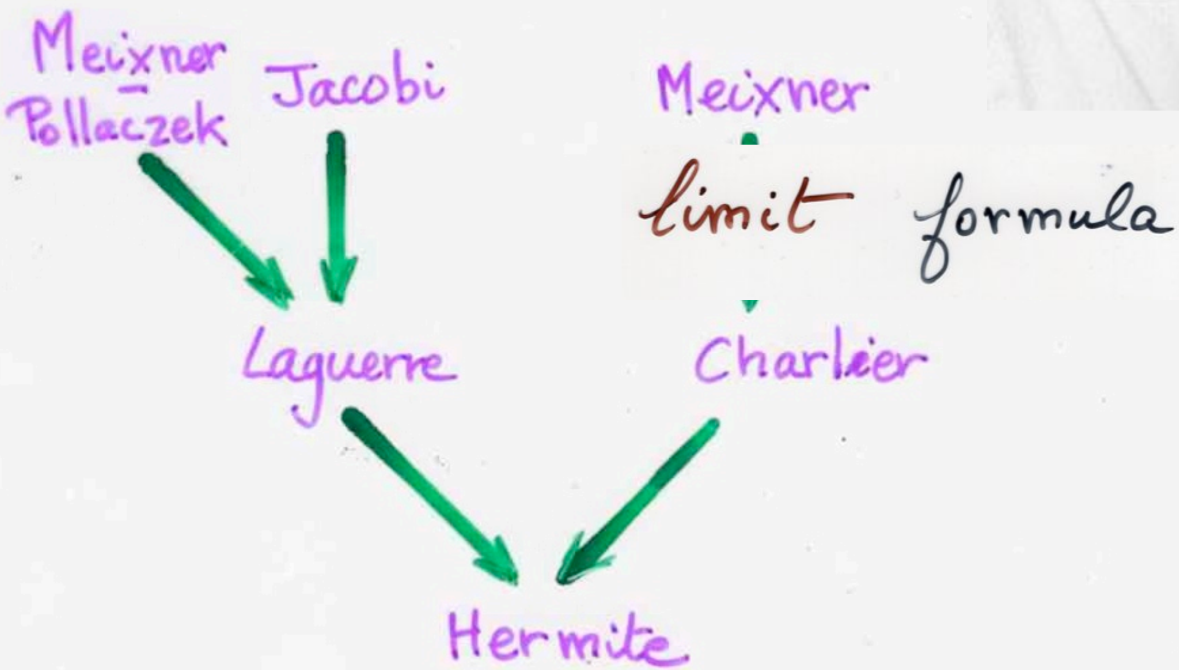
j = number of chains 

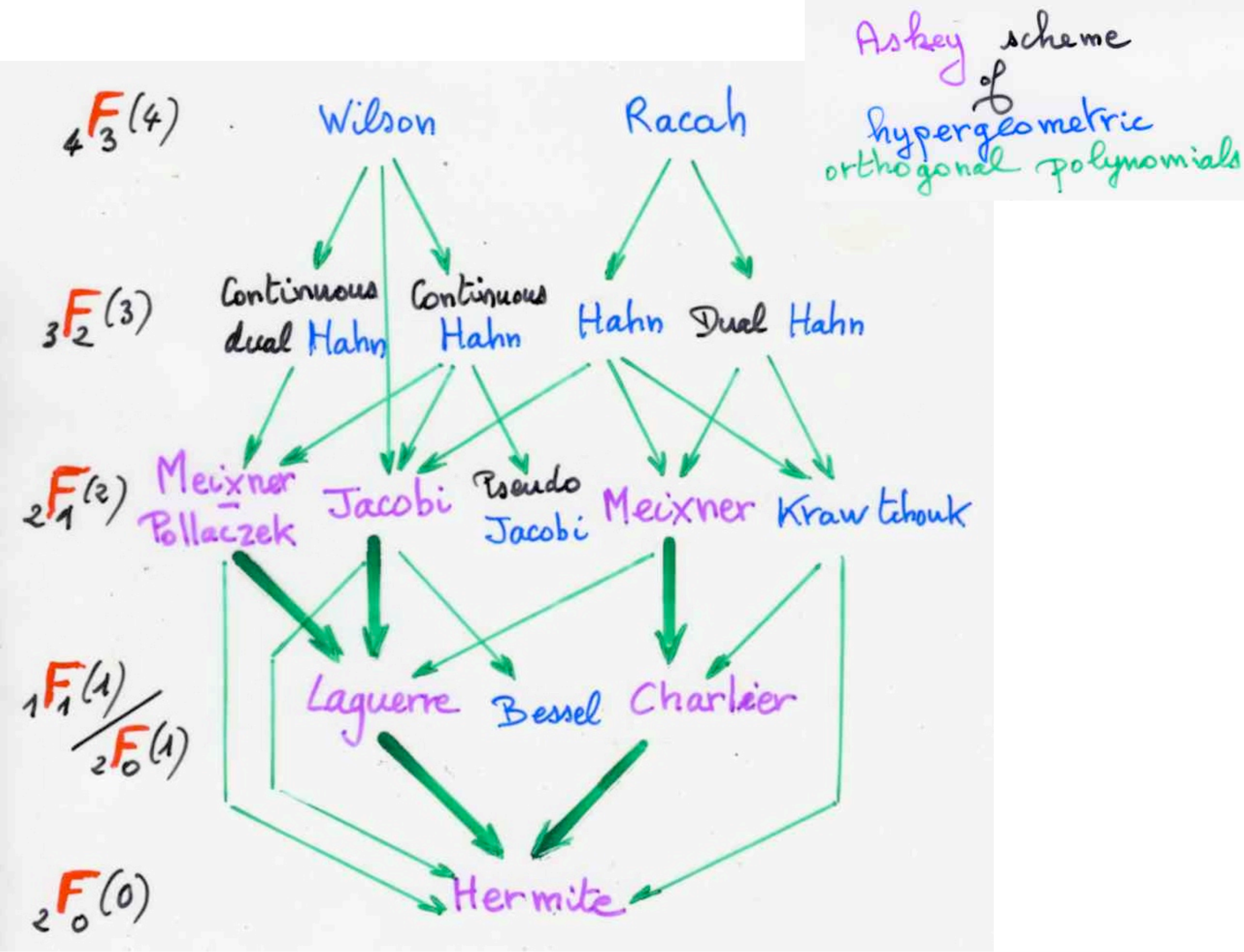


Chapter 5 Orthogonality and exponential structures

→ ABjC, Part I, Ch3

Askey scheme





Askey scheme of hypergeometric orthogonal polynomials
$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
$_1F_1(1)$ / $_2F_0(1)$
Laguerre
Bessel
Charlier
$_2F_0(0)$
Hermite

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

Tchebychef polynomials (Chebyshev)

1st kind $\alpha = \beta = -\frac{1}{2}$

2nd kind $\alpha = \beta = \frac{1}{2}$

Legendre polynomials $\alpha = \beta = 0$

Gegenbauer (ultraspherical) polynomials $\alpha = \beta = \lambda - \frac{1}{2}$

limit formula

example

Jacobi ⟶ Laguerre

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

(formal) orthogonal polynomials

$$\mathfrak{f}\big(P(x)\,Q(x)\big) = \int_{\mathbb{R}} P(x)\,Q(x)\,d\mu(x)$$

measure μ on $\mathbb{R}$

$$\mathfrak{f}(x^n) = \int_{\mathbb{R}} x^n\,d\mu(x)$$

moments problem

$$\mathfrak{f}(x^n) = \mu_n$$

moments

$\mathbb{K}$ ring

field $\mathbb{R}$, $\mathbb{C}$
or $\mathbb{Q}[\alpha, \beta, \ldots]$

$\mathbb{K}[x]$
polynomials in x

$\{P_n(x)\}_{n\geq 0}$
sequence of polynomials

$P_n(x) \in \mathbb{K}[x]$.

Definition $\{P_n(x)\}_{n \geq 0}$ sequence of polynomials orthogonal iff $\exists$ $f : \mathbb{K}[x] \to \mathbb{K}$ linear functional

(i) $\deg(P_n) = n$, for $n \geq 0$ (deg: degree)

(ii) $f(P_k P_l) = 0$, for $k \neq l \geq 0$

(iii) $f(P_k^2) \neq 0$, for $k \geq 0$

$$f(x^n) = \mu_n$$

moments

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

moments of (Tchebychev) 1 st kind / 2 nd kind

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan number

$$\frac{2}{\pi}\int_{-1}^{+1} x^{2n} (1-x^2)^{1/2} dx = \frac{1}{4^n} C_n$$

Catalan

moments of Hermite polynomial

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \times 3 \times \ldots \times (2n-1)$

number of involutions on $\{1, \ldots, 2n\}$ with no fixed points

moments Laguerre polynomials

$\mu_n = n!$

Combinatorial theory
of orthogonal polynomials

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k\geq 1$

(formal) Favard's Theorem

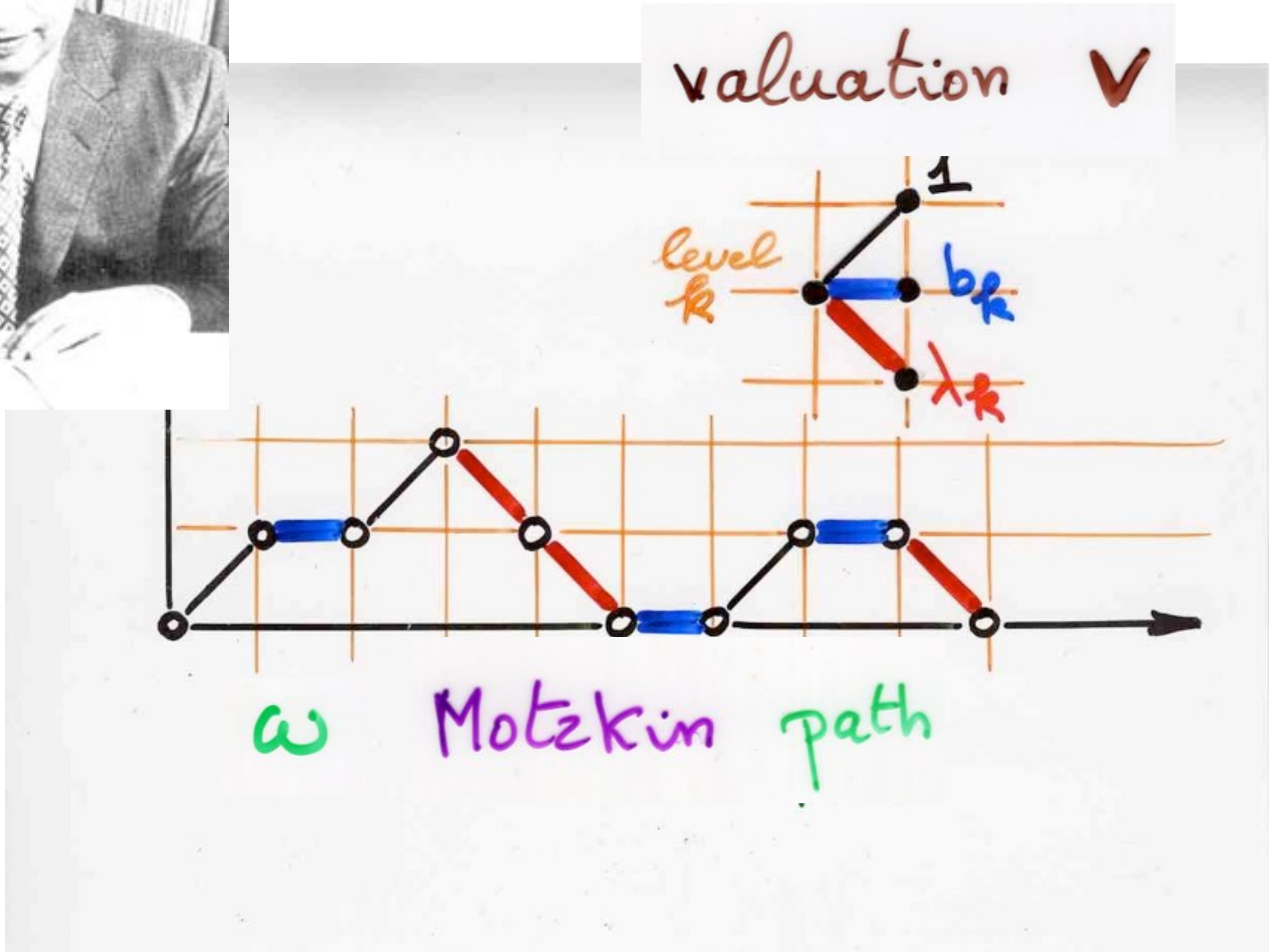
3-terms linear recurrence relation

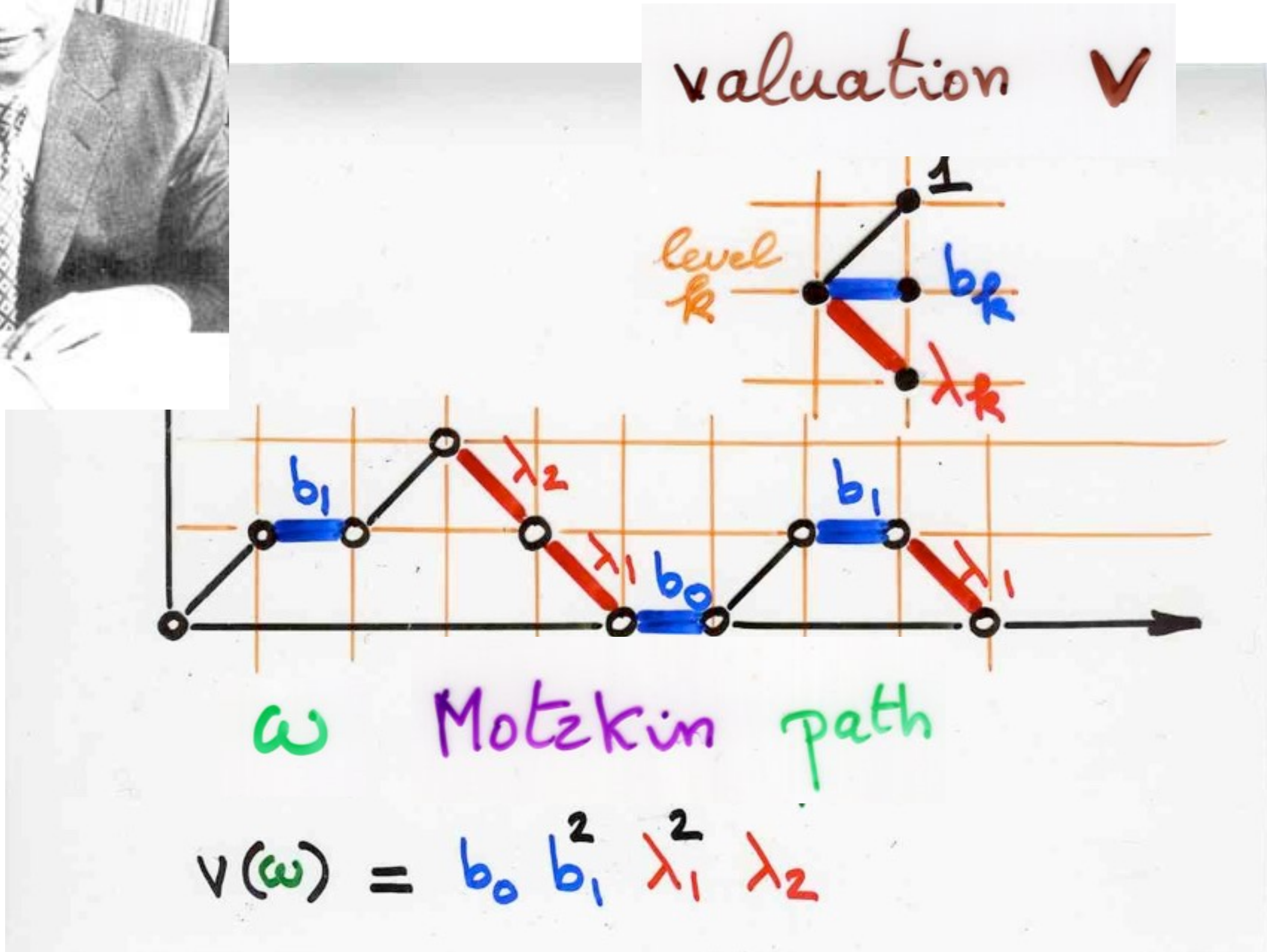
$\Rightarrow$ orthogonality

$\{b_k\}_{k \geqslant 0}$ $\{\lambda_k\}_{k \geqslant 1}$

$b_k, \lambda_k \in \mathbb{K}$
ring

μ_n ?


valuation v
1
level k
b_k
λ_k
ω Motzkin path


valuation v
level k
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λ_1
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b_1
λ_1
ω Motzkin path
v(ω) = b_0 b_1^2 λ_1^2 λ_2

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k \geq 1$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

length

$$f(x^n) = \mu_n$$

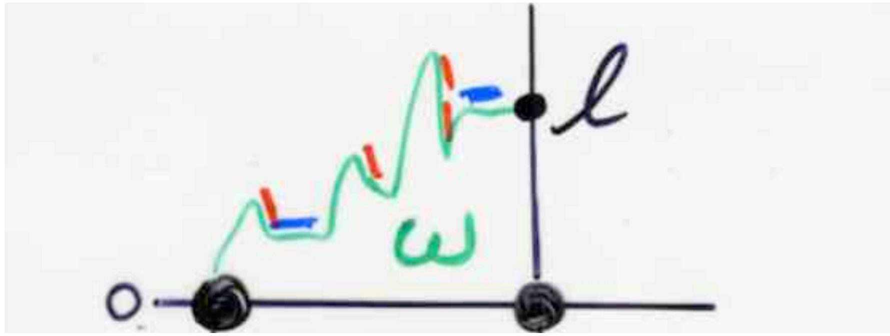
Chapter 1 Paths and moments

- Tchebychev, Hermite, Laguerre $(\alpha=0)$
- (formal) orthogonal polynomials
- moments μ_n as weighted Motzkin paths

3 bijective proofs:

- 3-term recurrence $\Rightarrow$ orthogonality
  (Favard theorem)
- inverse polynomials
- positivity of some linearization coefficients

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$



$$Q_n(x) = \sum_{i=0}^{n} q_{n,i} x^i$$

inverse sequence

$$\{Q_n(x)\}_{n\geq 0}$$

linearization coefficients

Lemma

$$P_k(x)\, P_l(x) = \sum_n a_{kl}^n\, P_n(x)$$

$$a_{kl}^n = \frac{f(P_k P_n P_l)}{f(P_n^2)}$$

positivity

The notion of histories

example: Hermite histories

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

moments

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

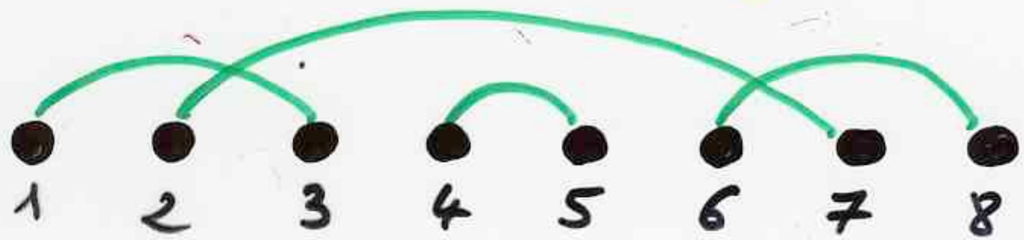
$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

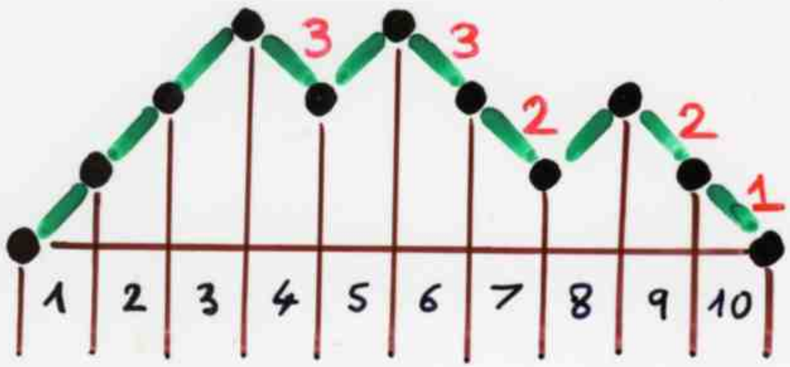
chord diagrams

perfect matching



moments

Hermite polynomials



$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

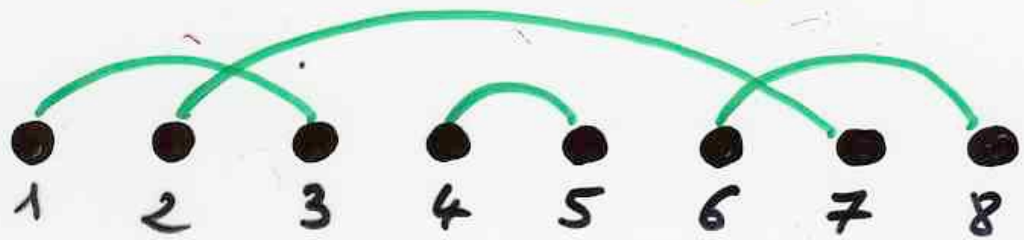
$$\mu_{2n+1} = 0$$

$$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

chord diagrams

perfect matching



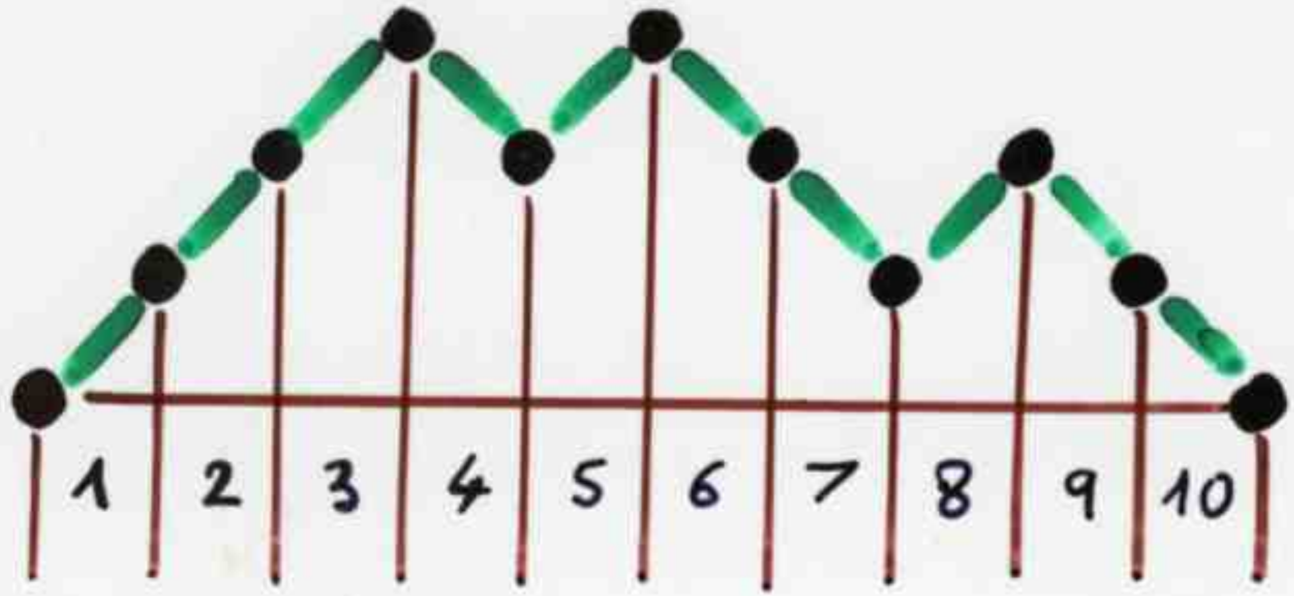
history

Françon (1978)

data structures
in
computer science

sequence
of
primitive
operations

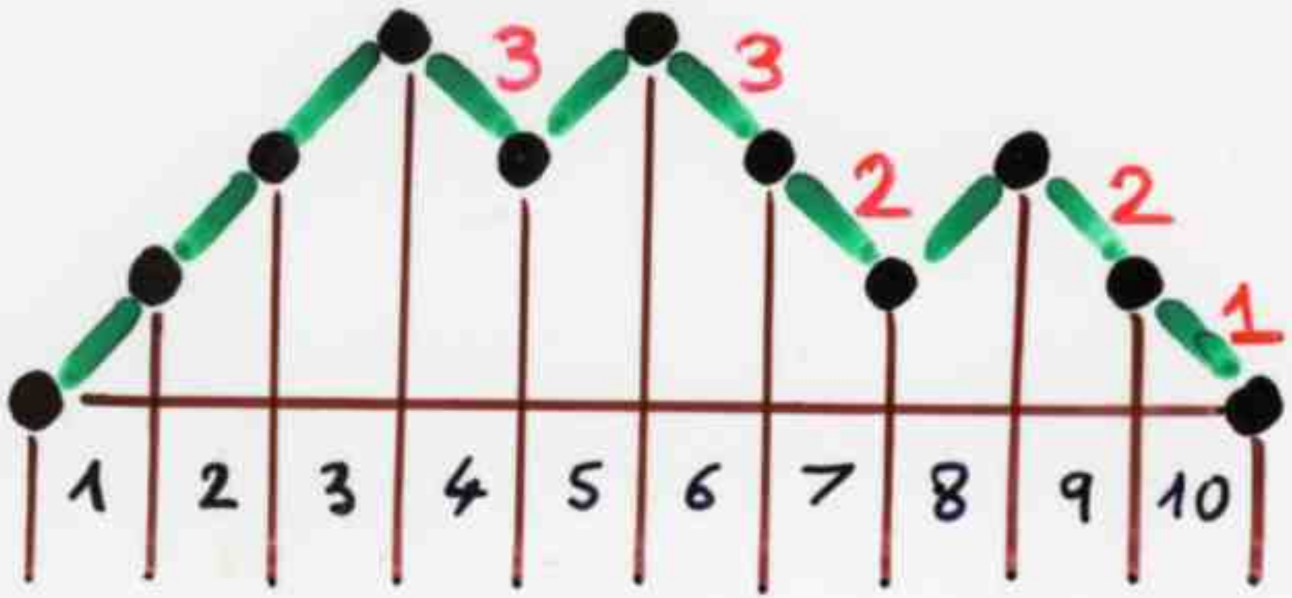
Hermite history



Hermite history

Hermite polynomials

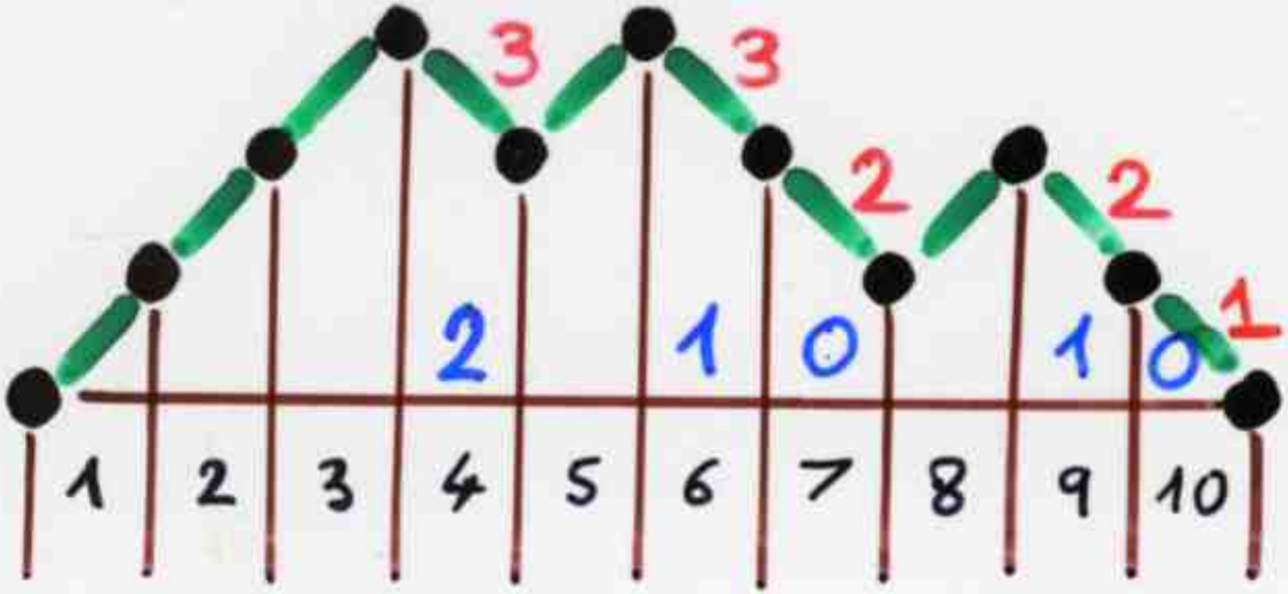
$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$



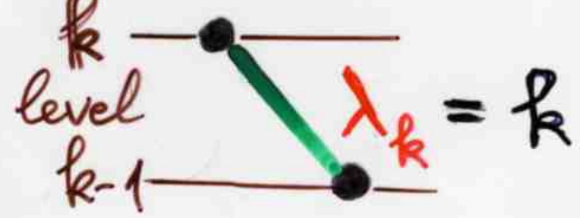
Hermite history

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

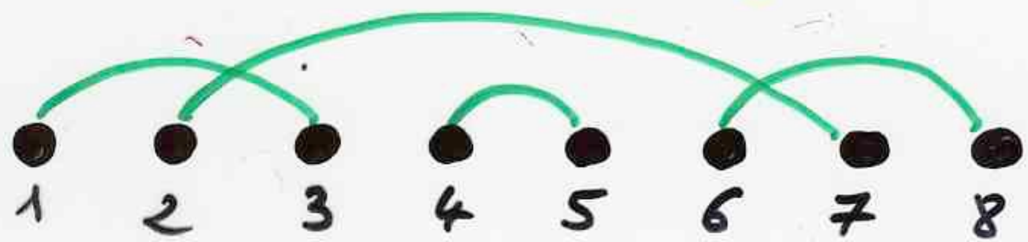


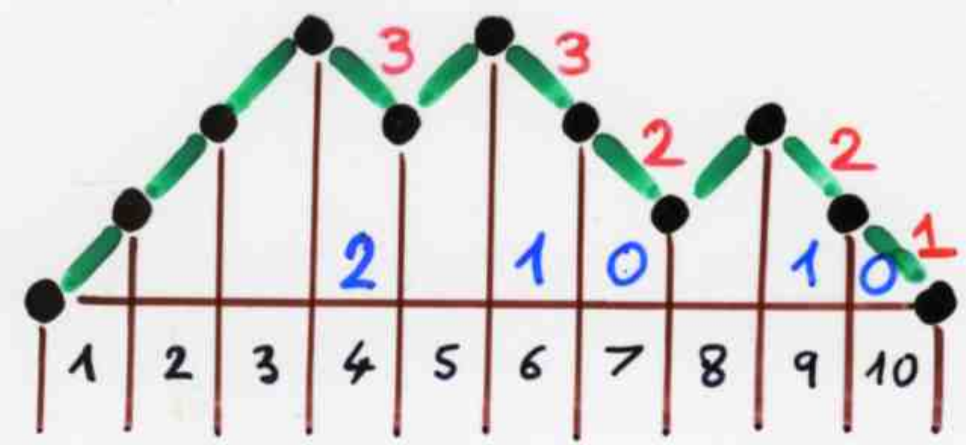
$$0 \leq i < \lambda_k = k$$



bijection

Hermite history ⟷ chord diagrams
perfect matching

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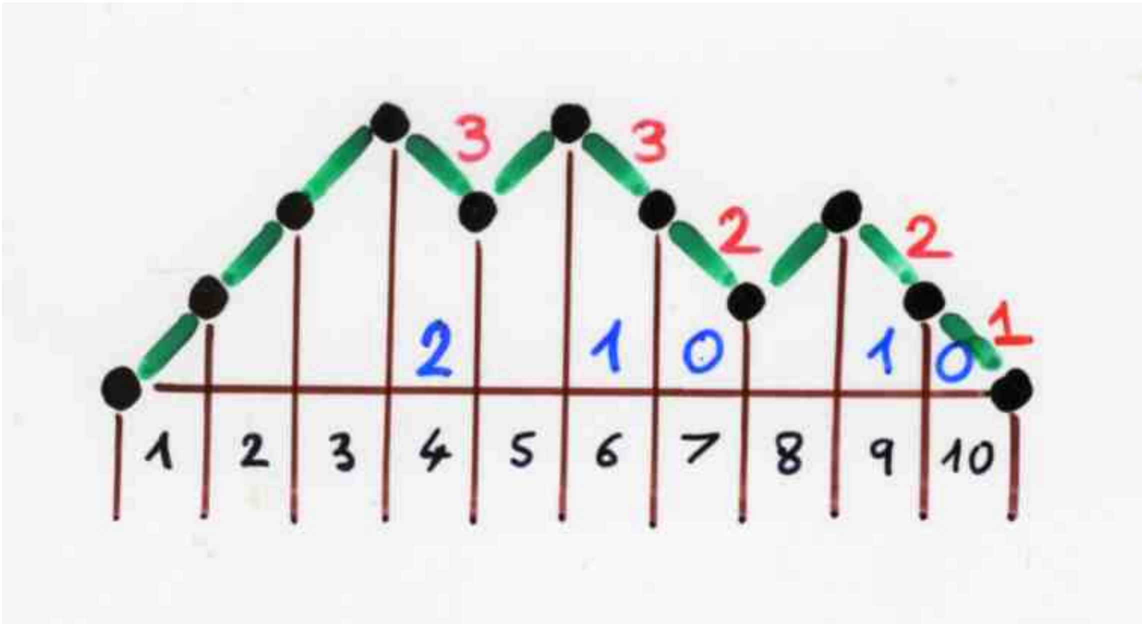
1 2 3 4 5 6 7 8 9 10



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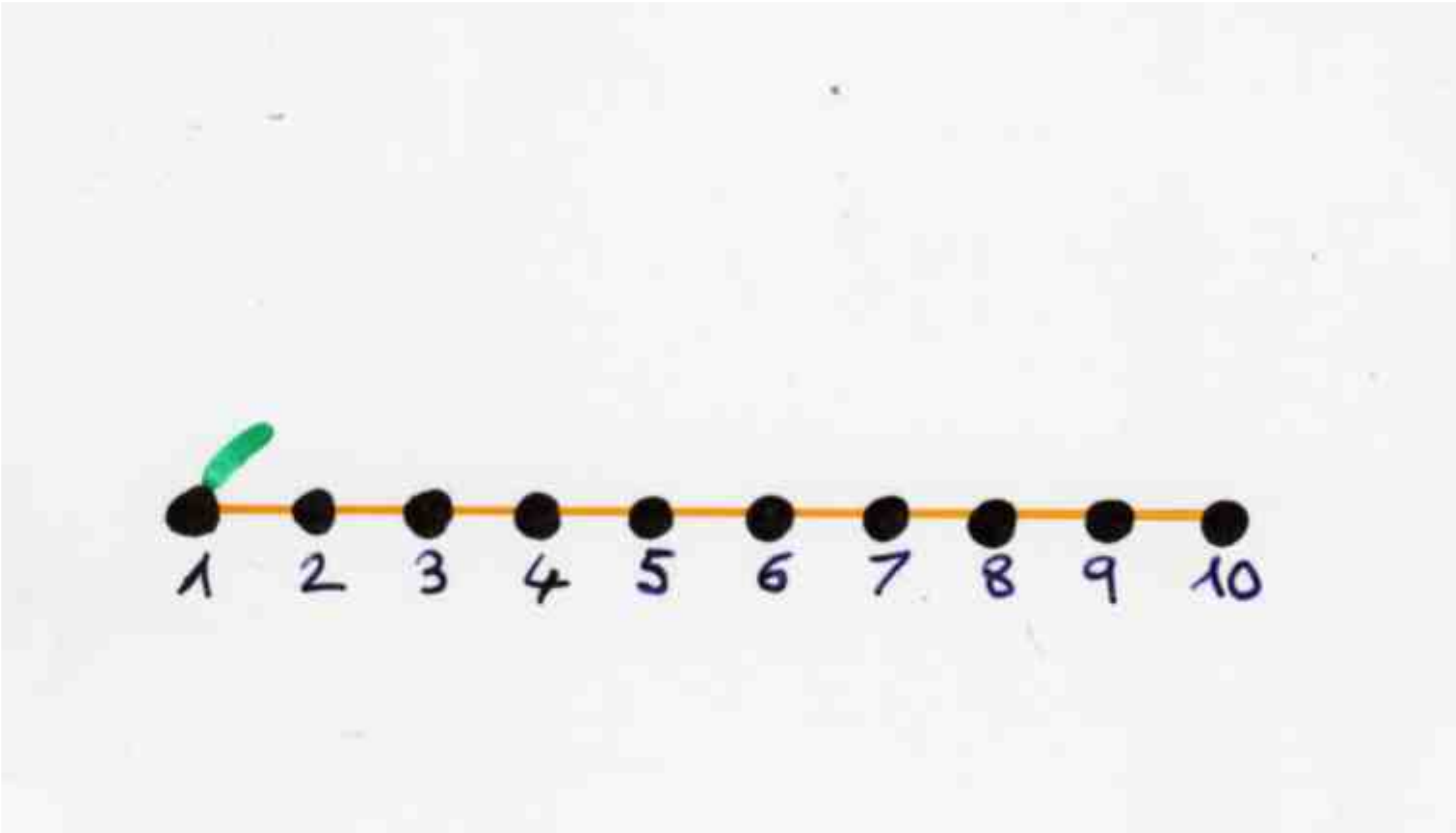

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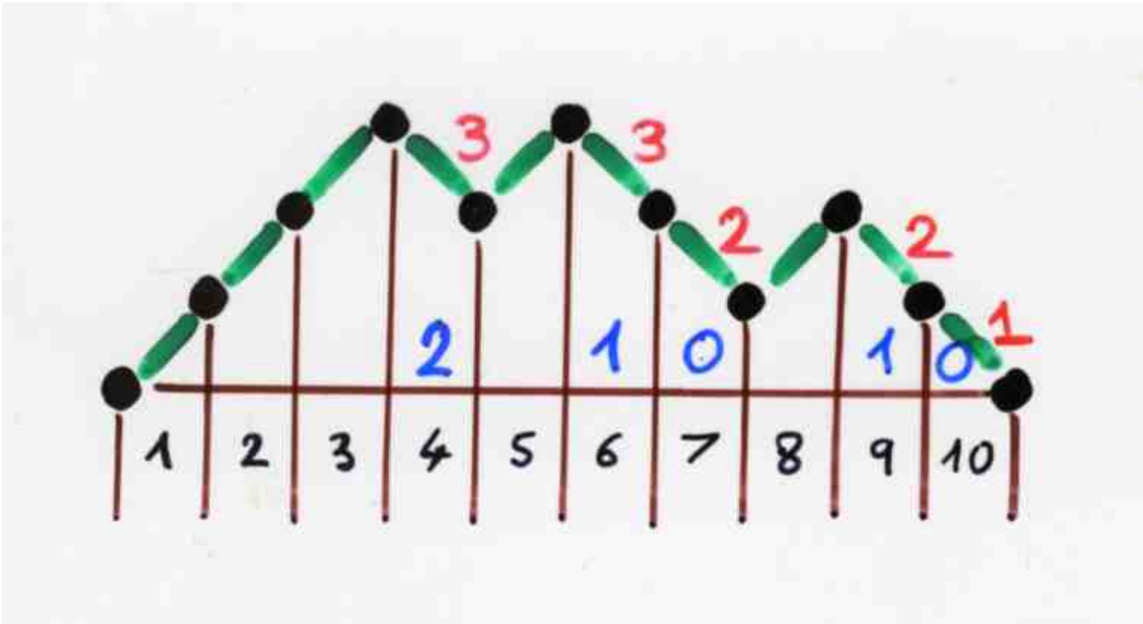
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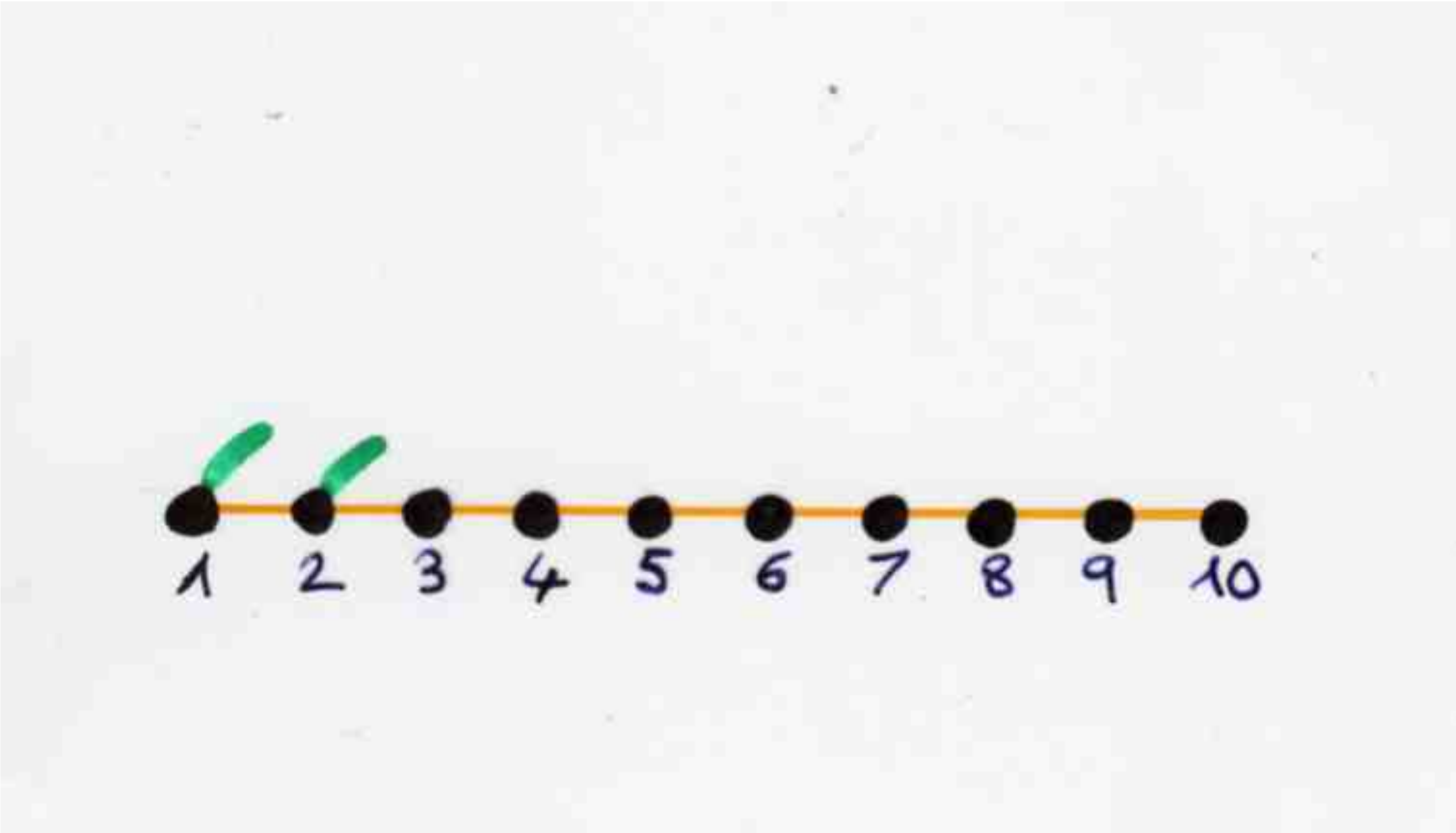
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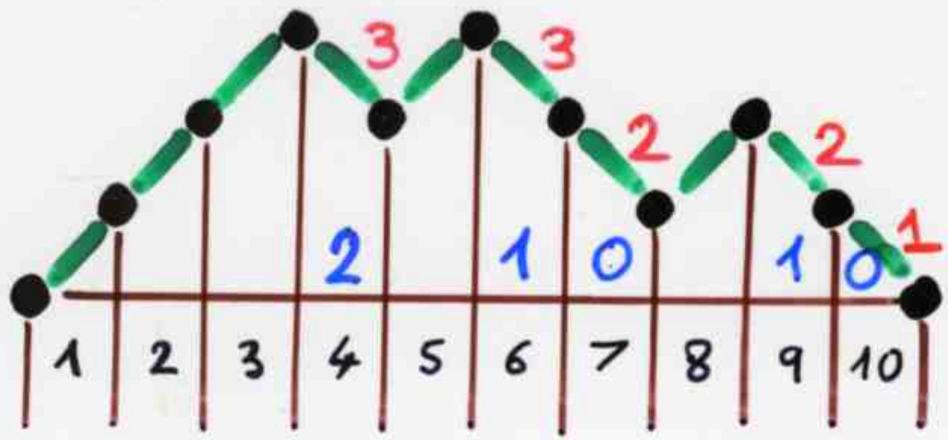
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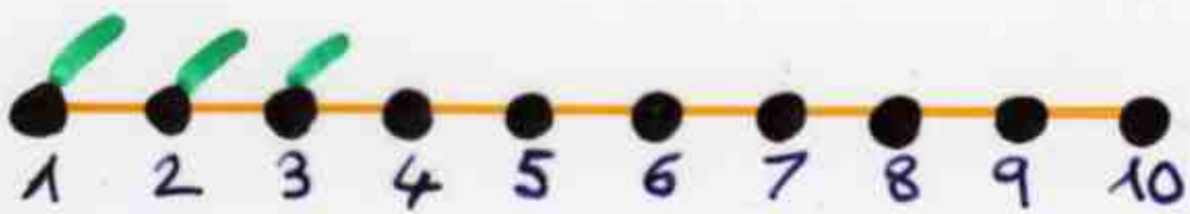
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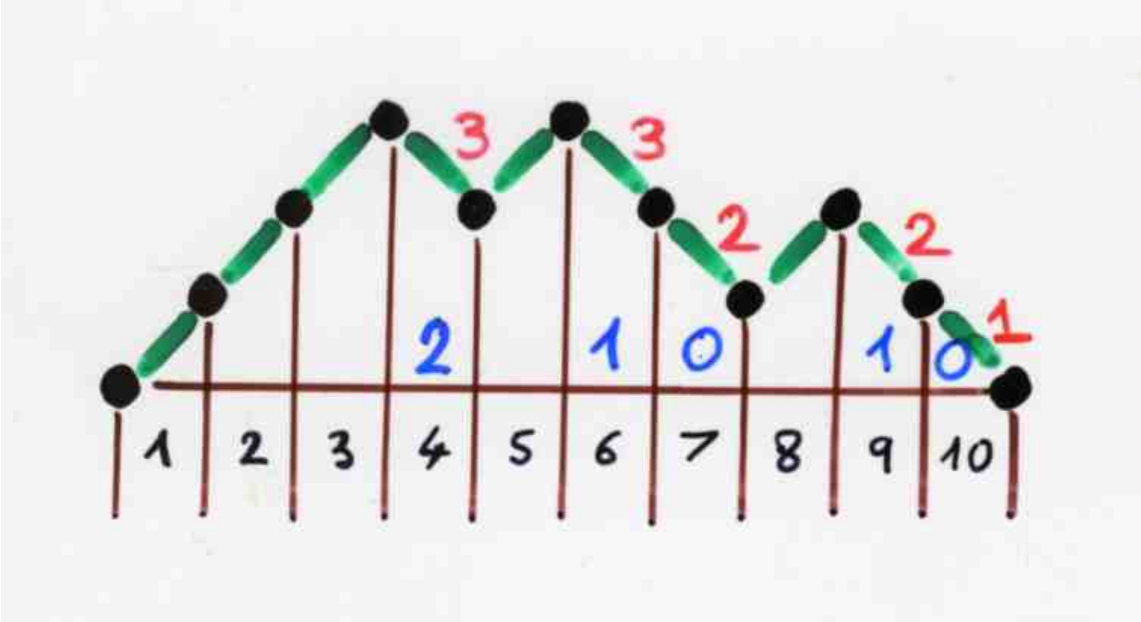
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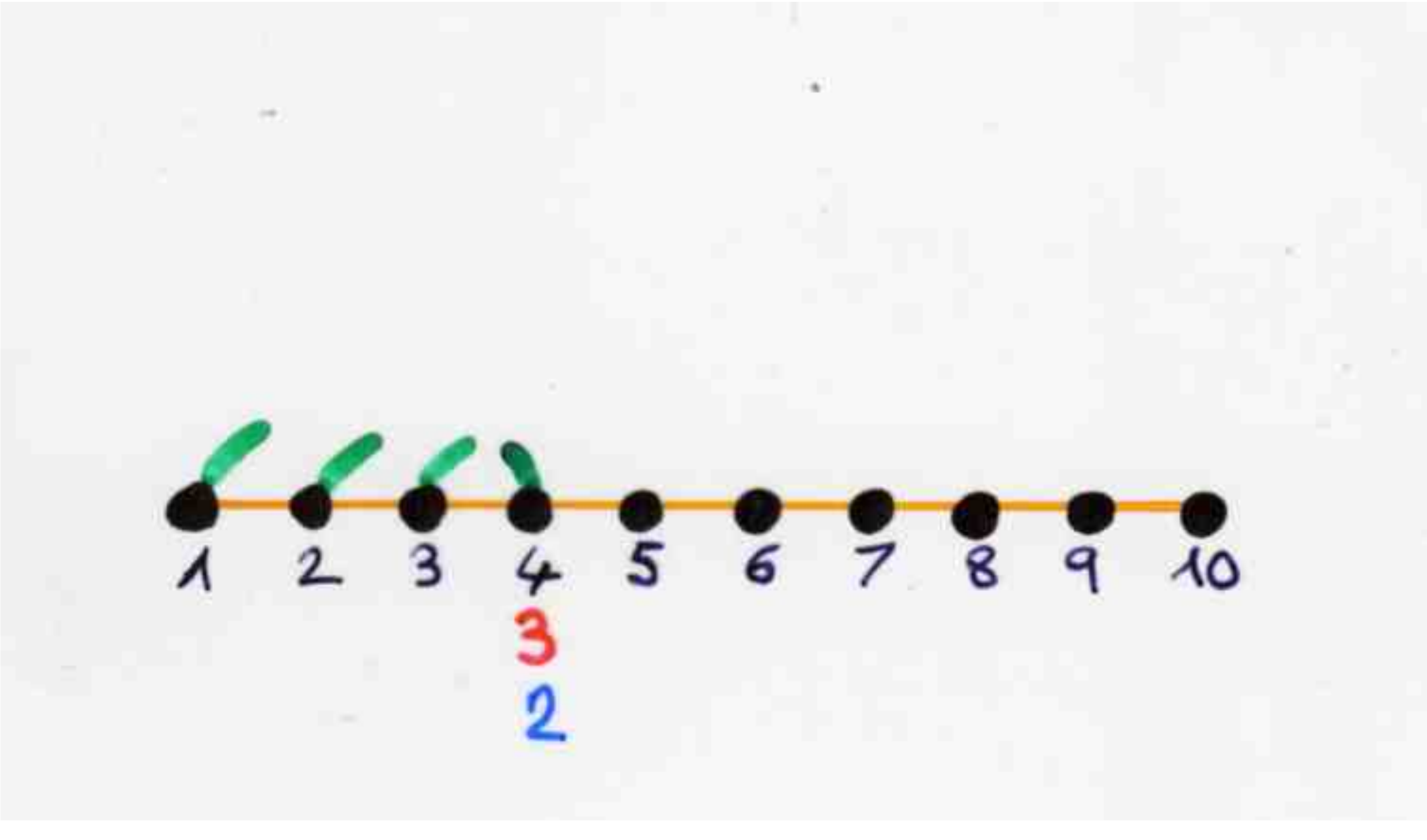
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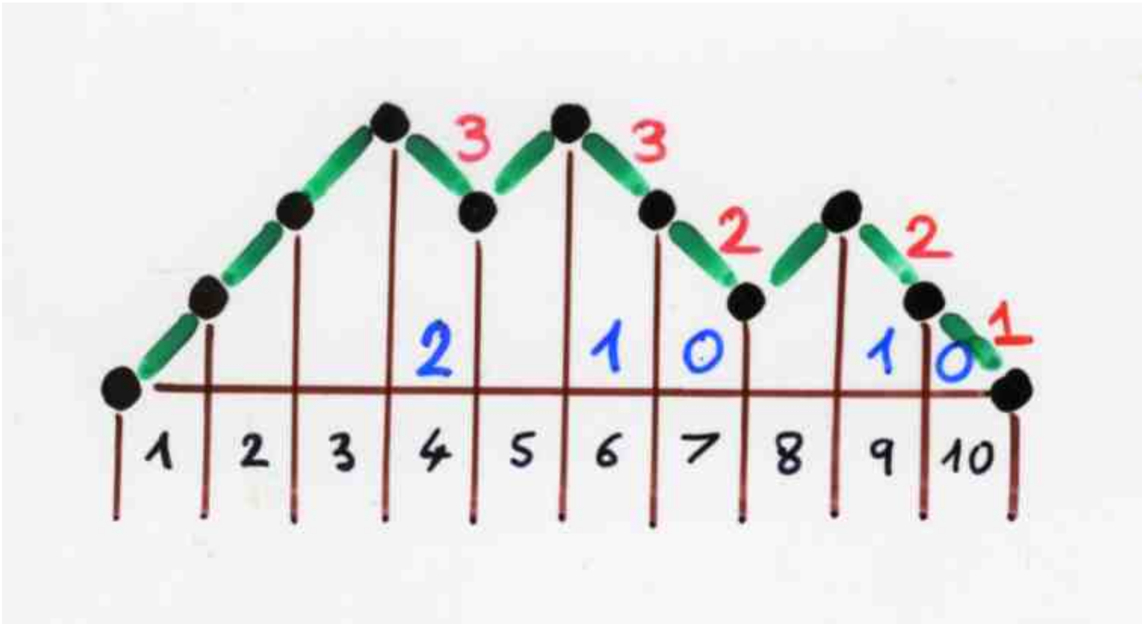
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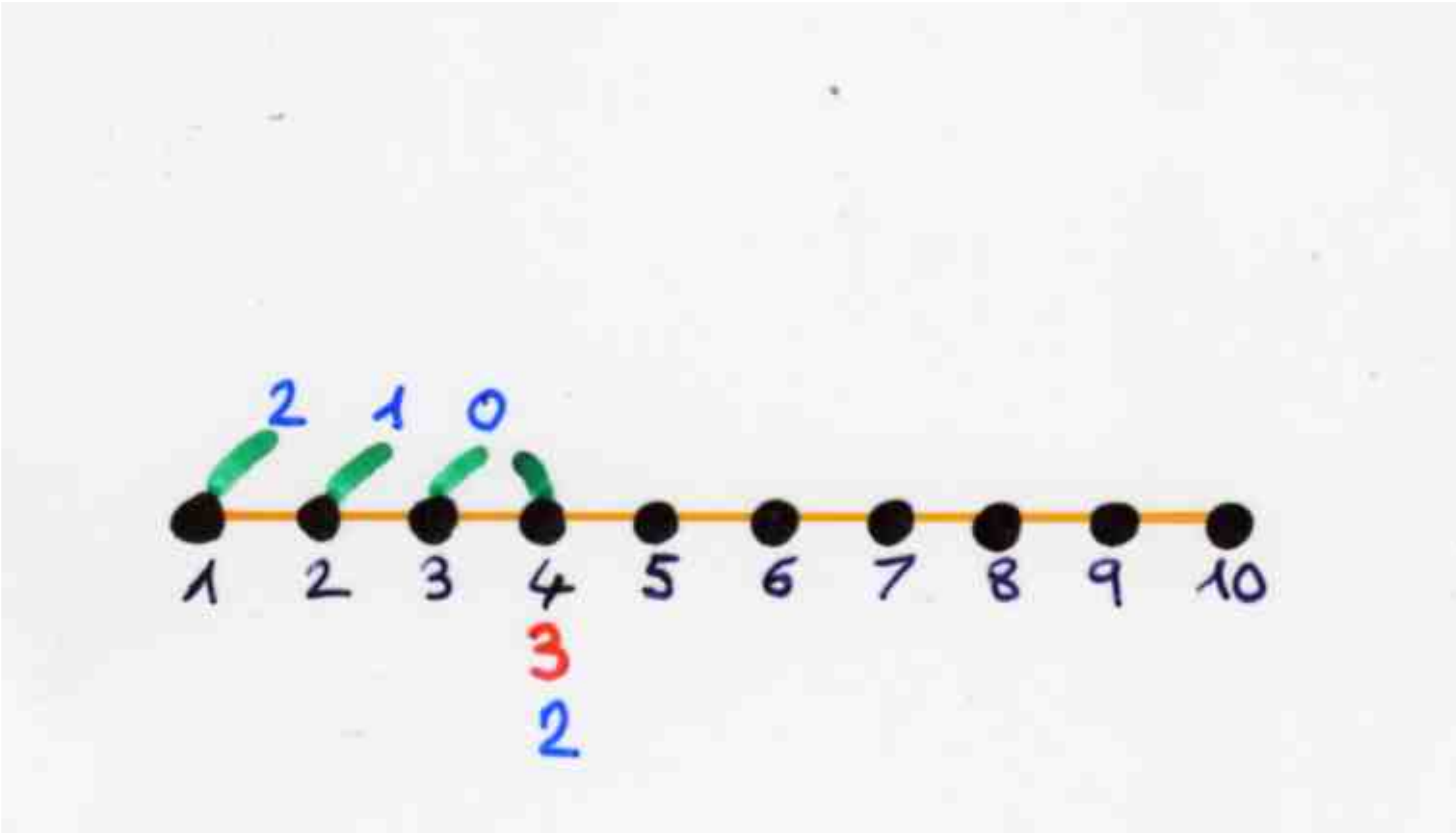
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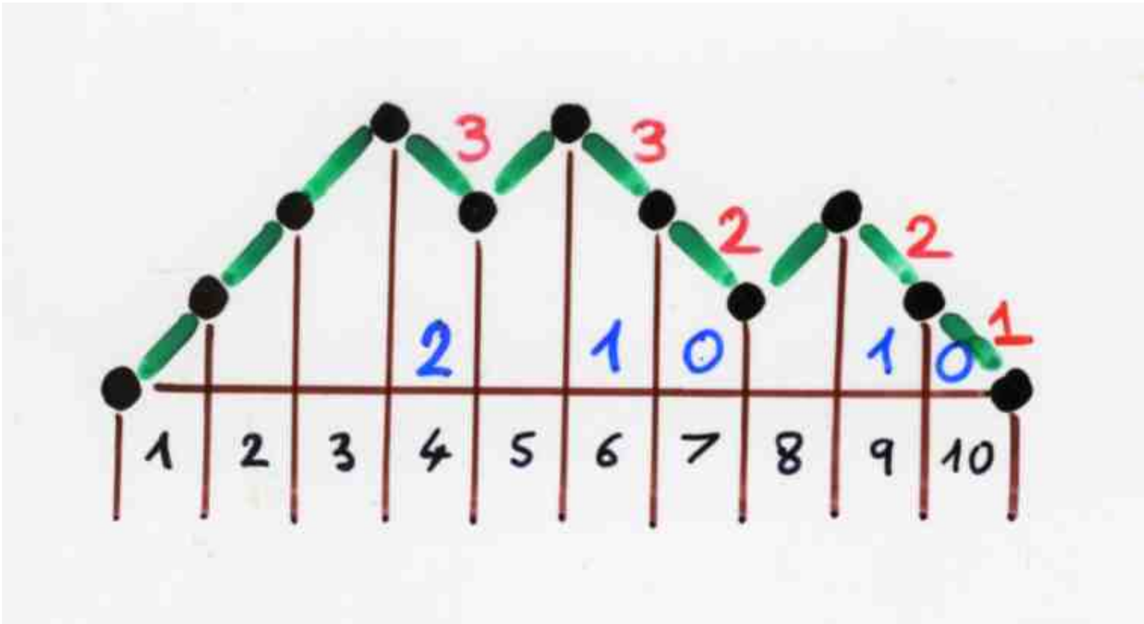
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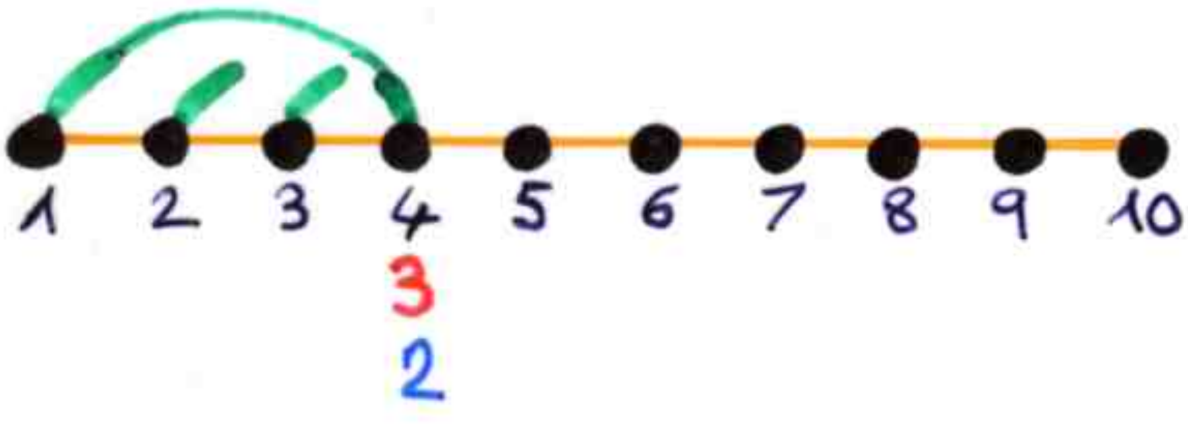
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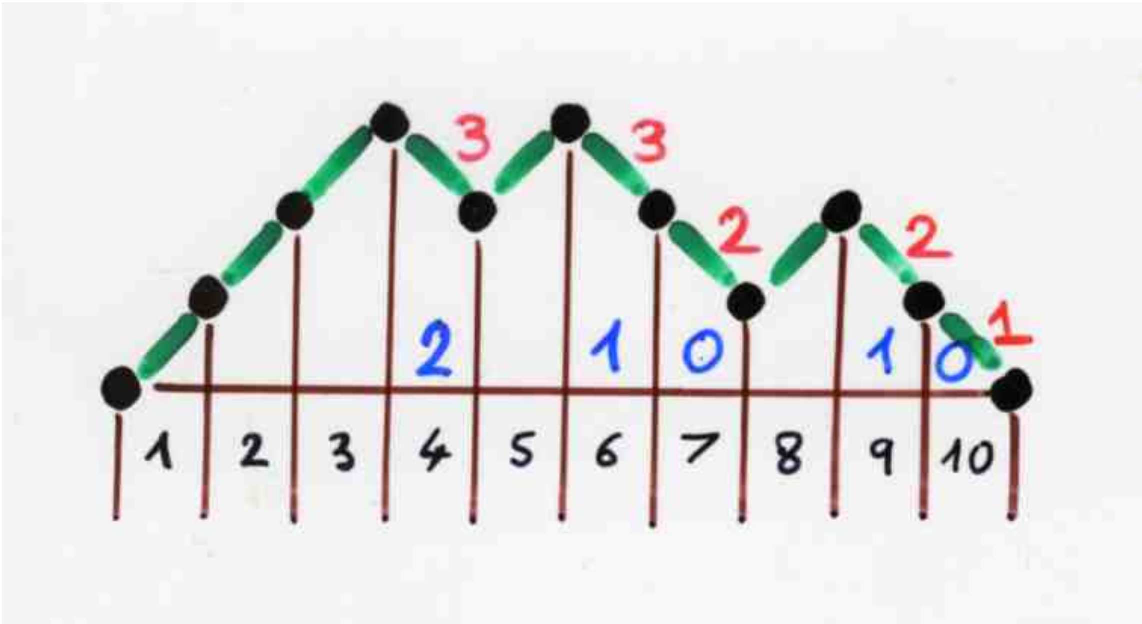
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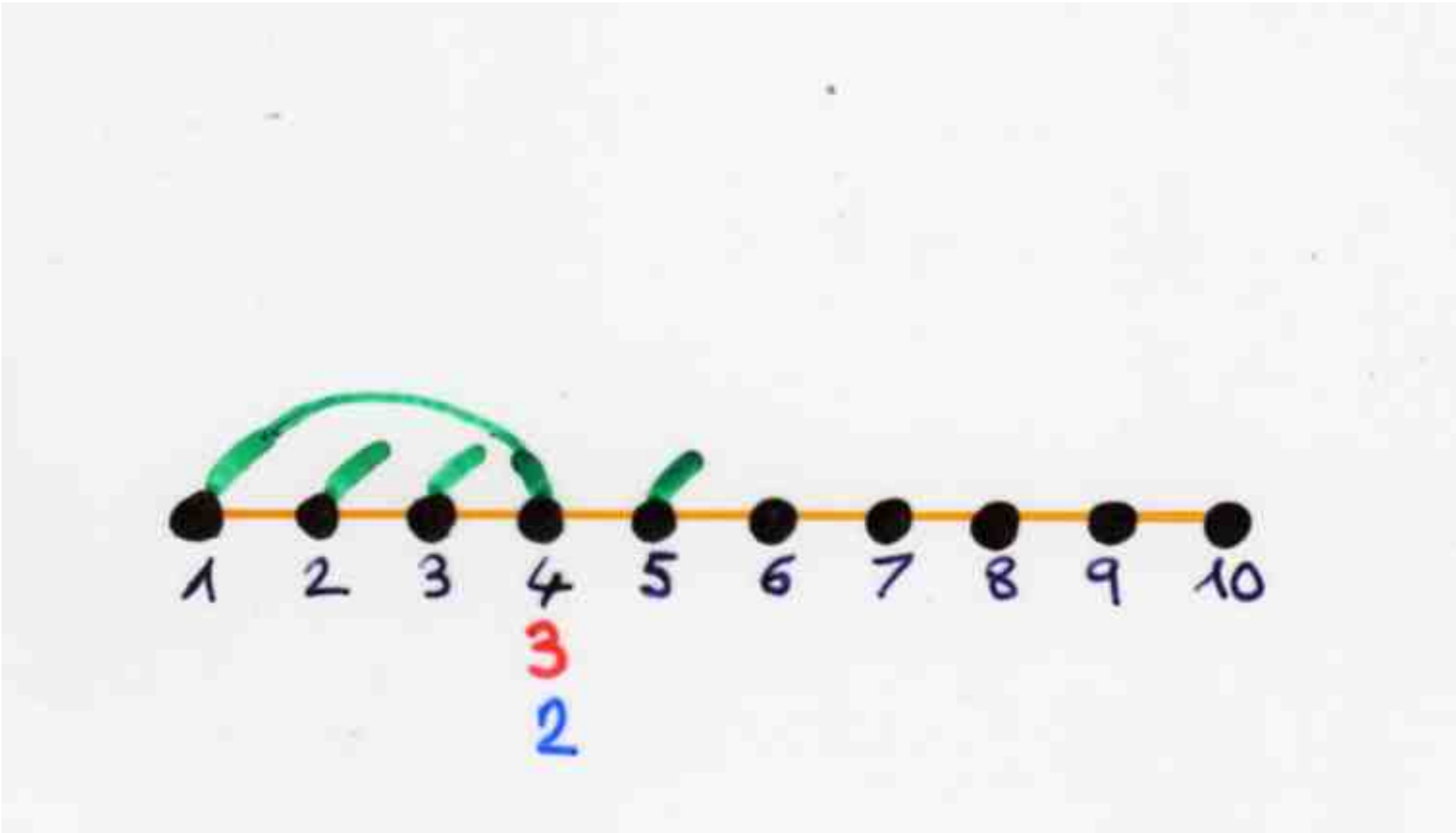
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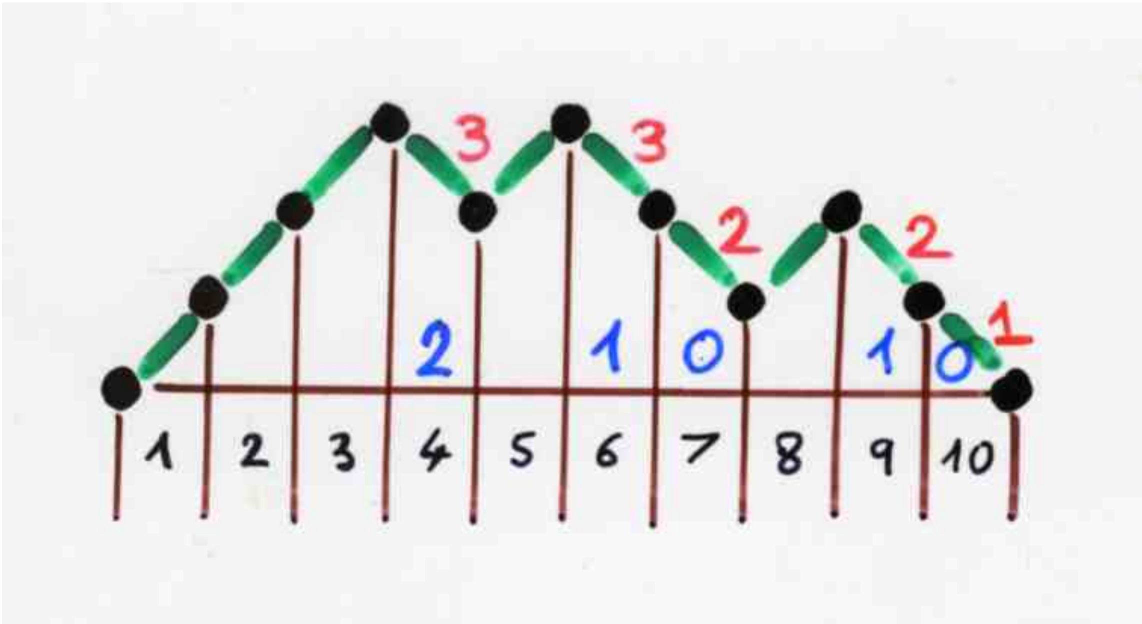
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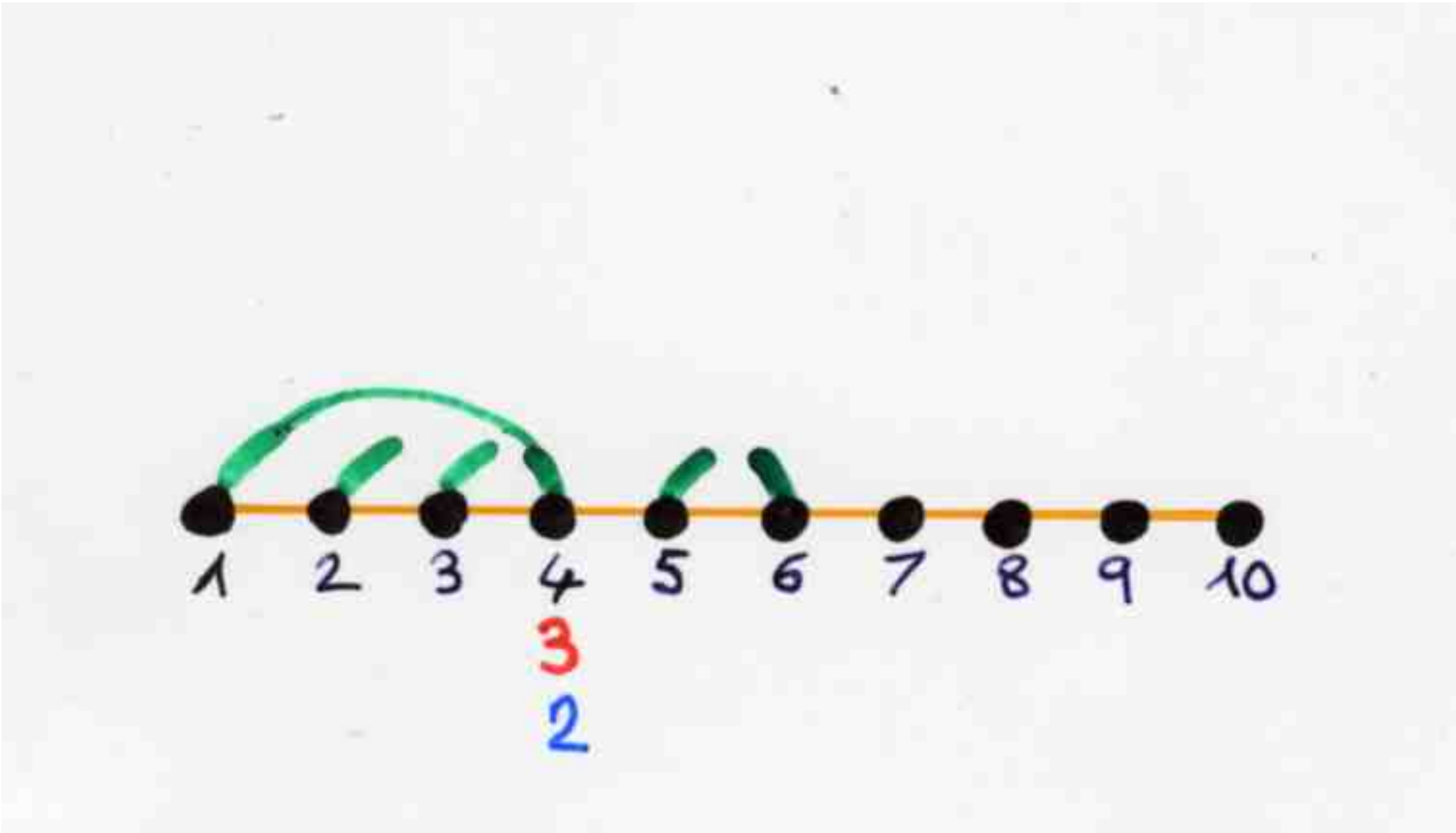
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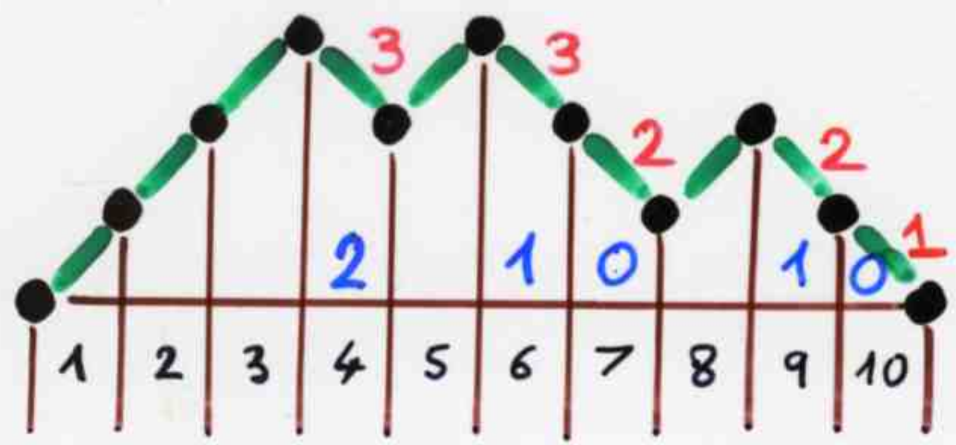
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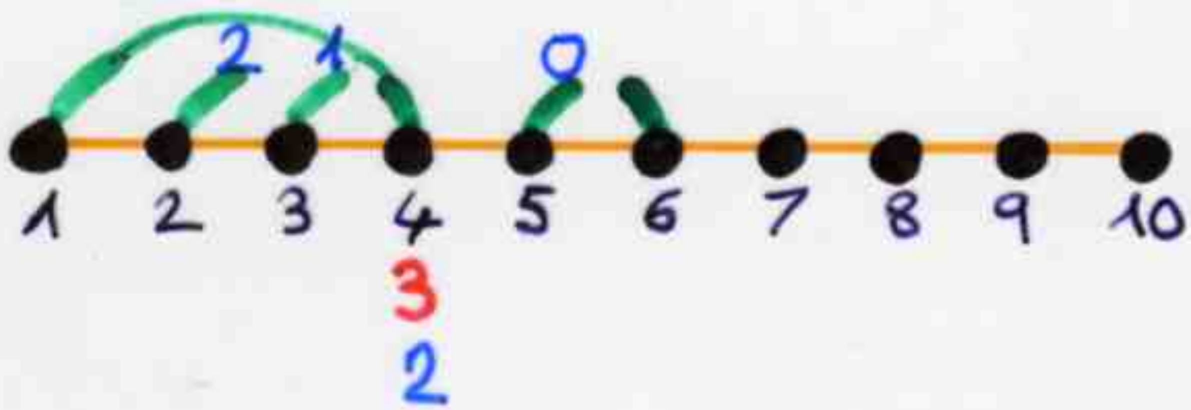
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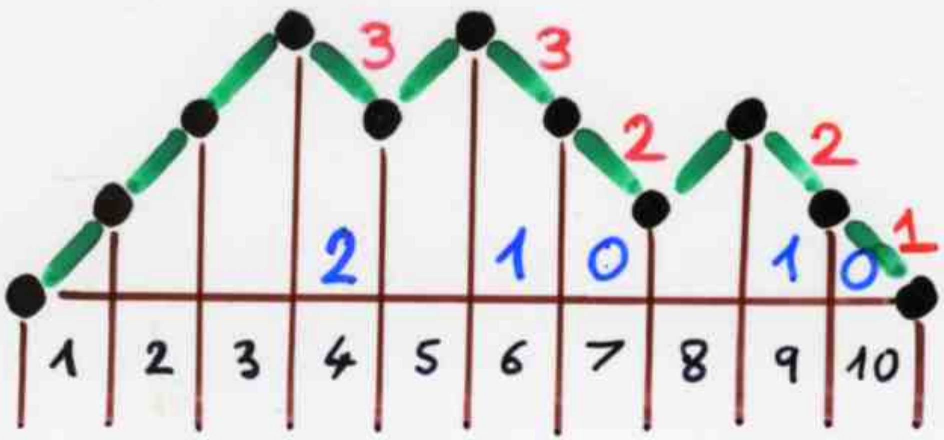
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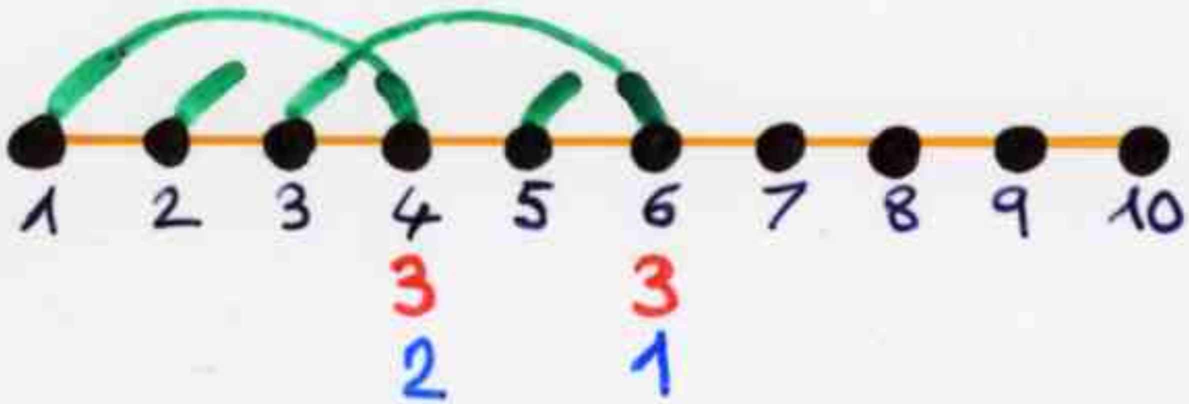
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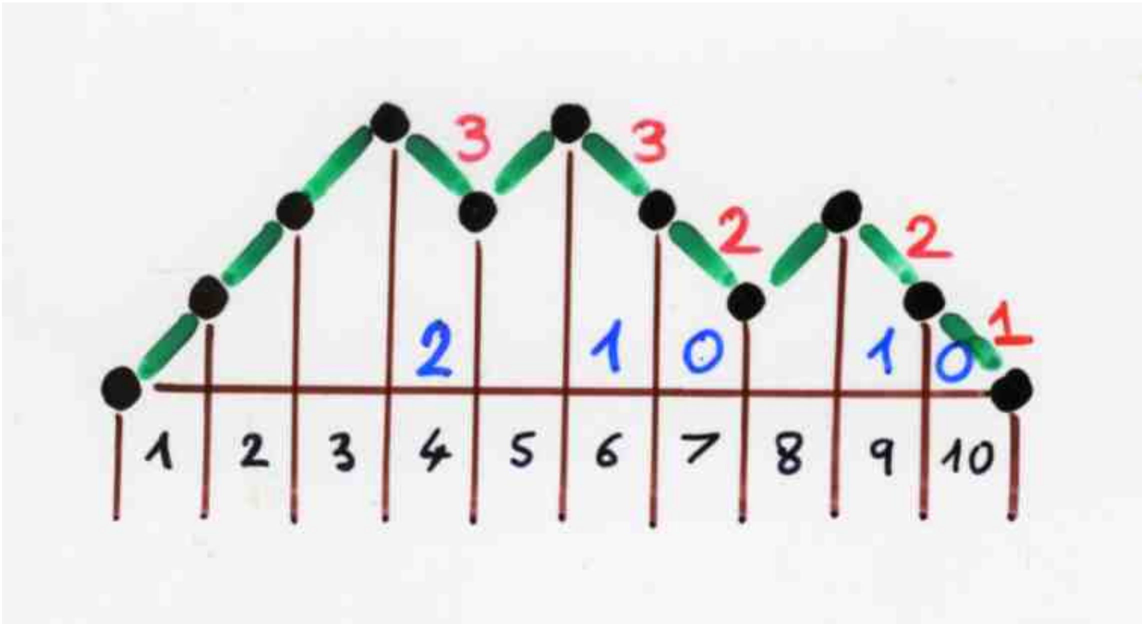
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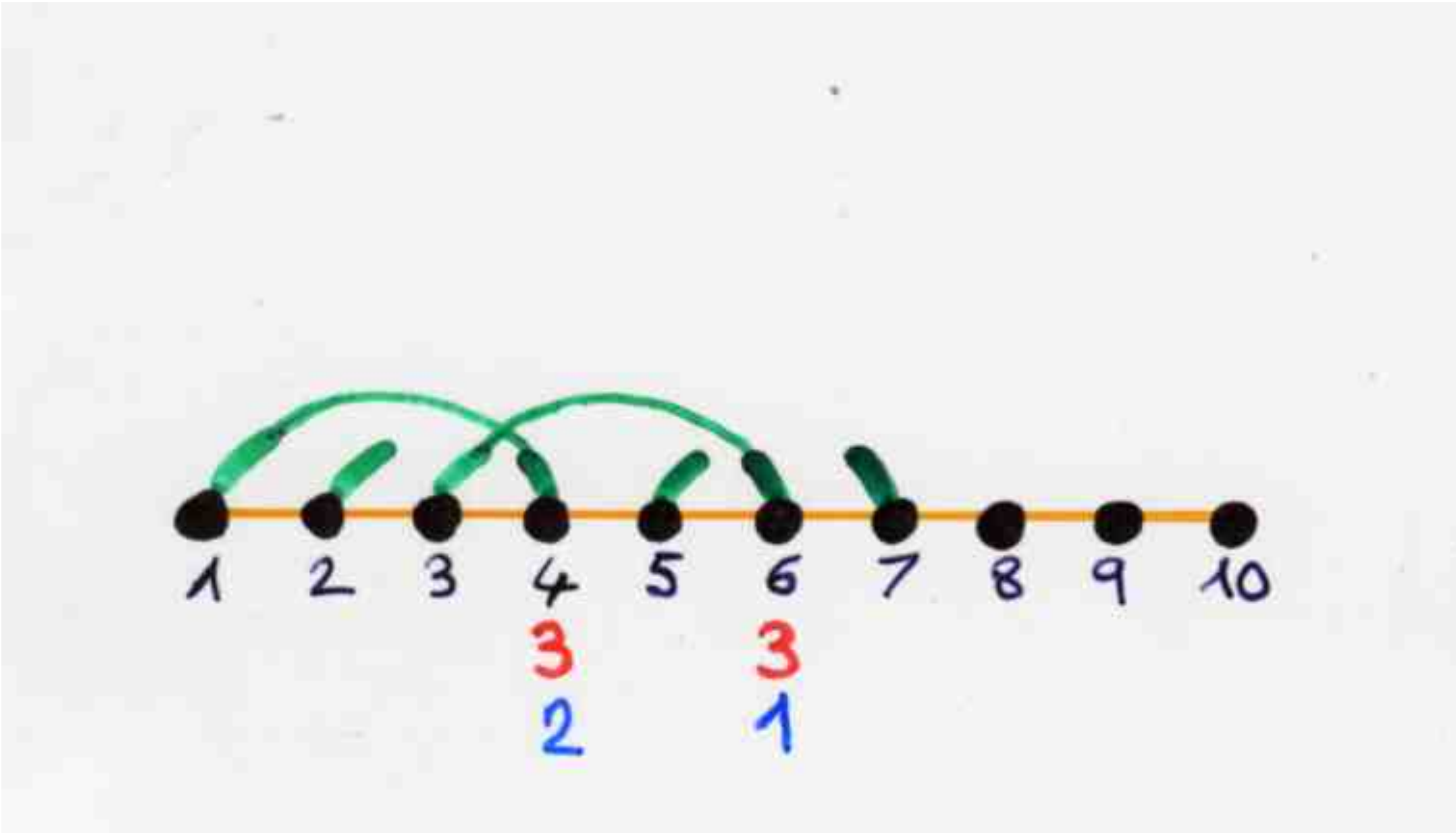
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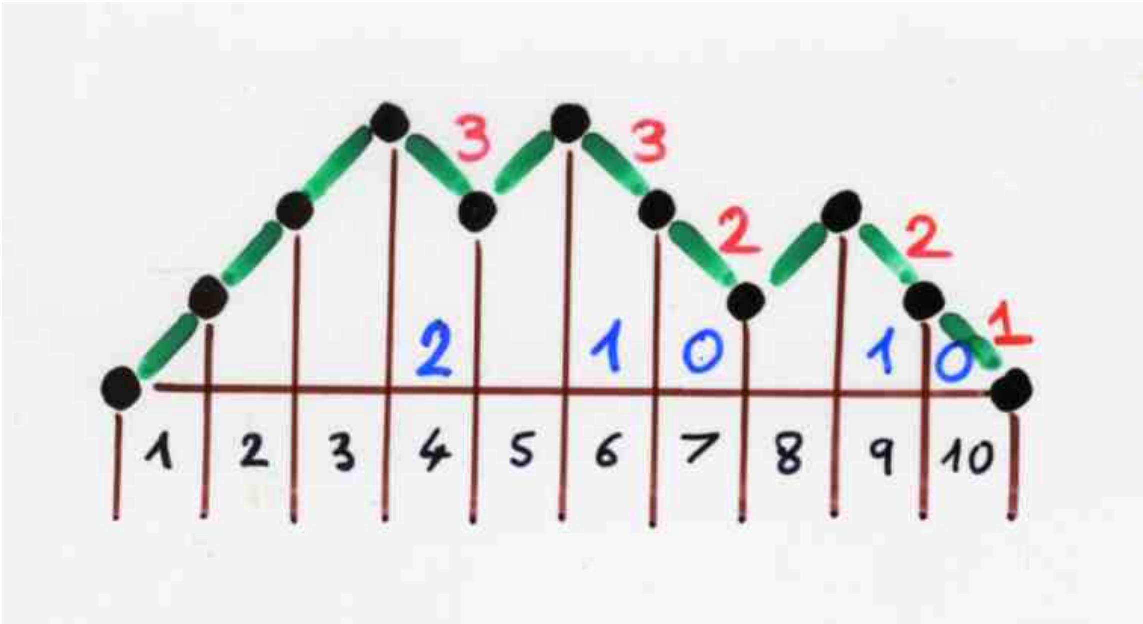
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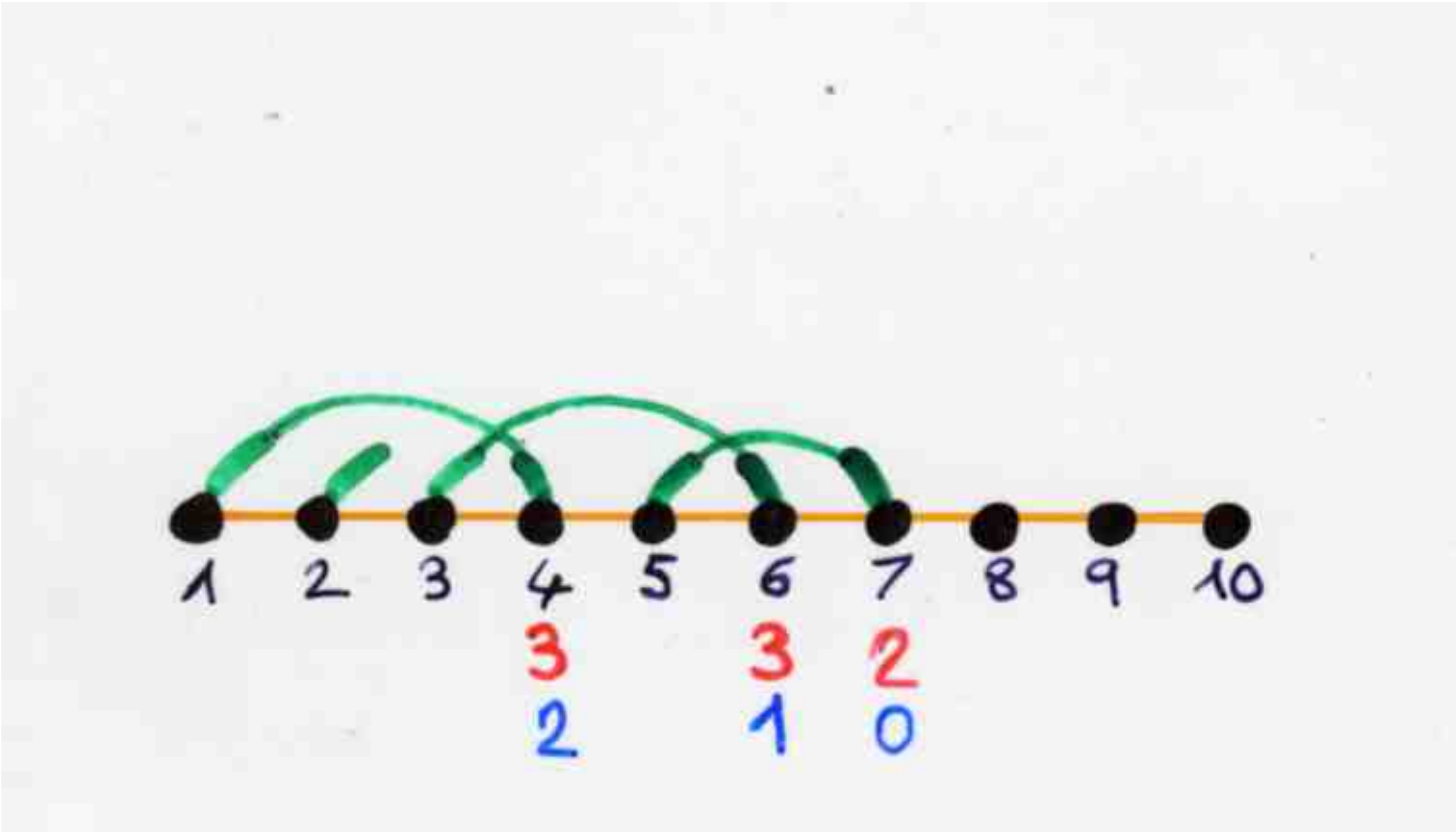
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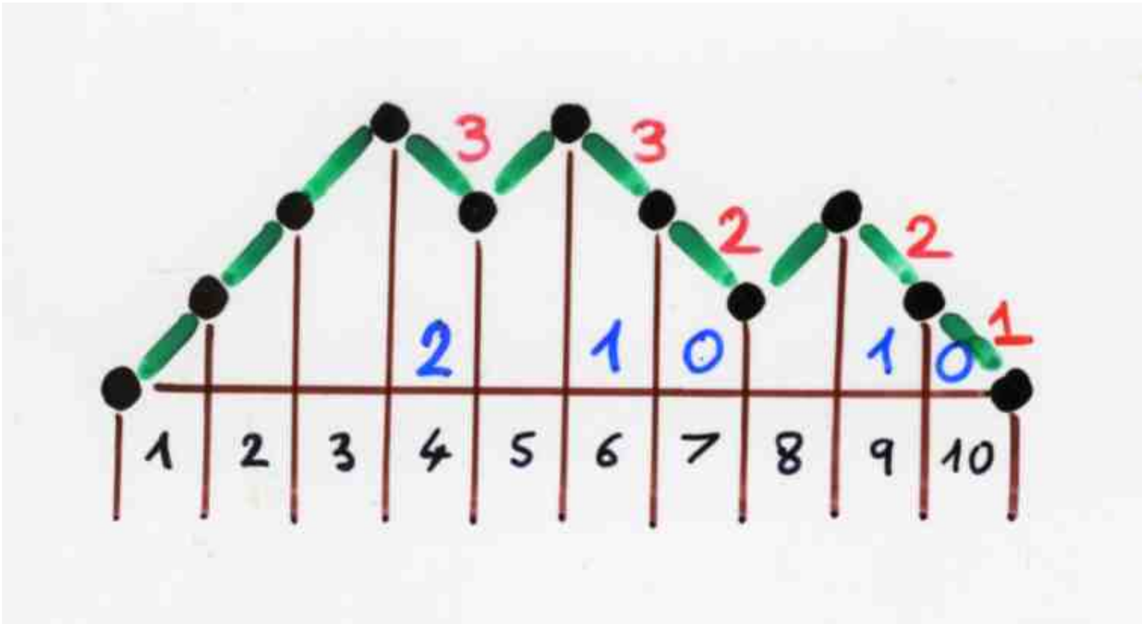

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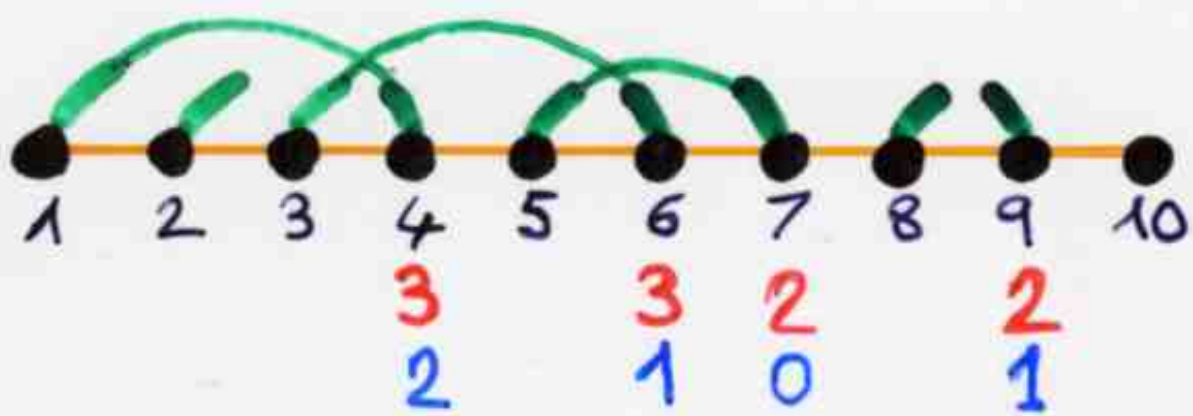

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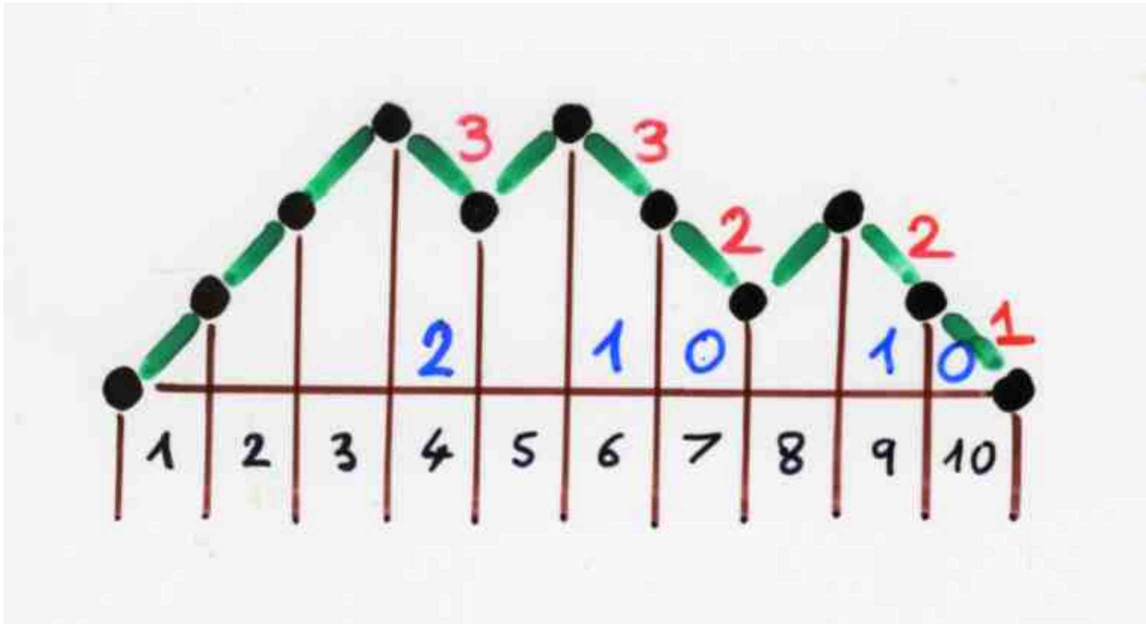
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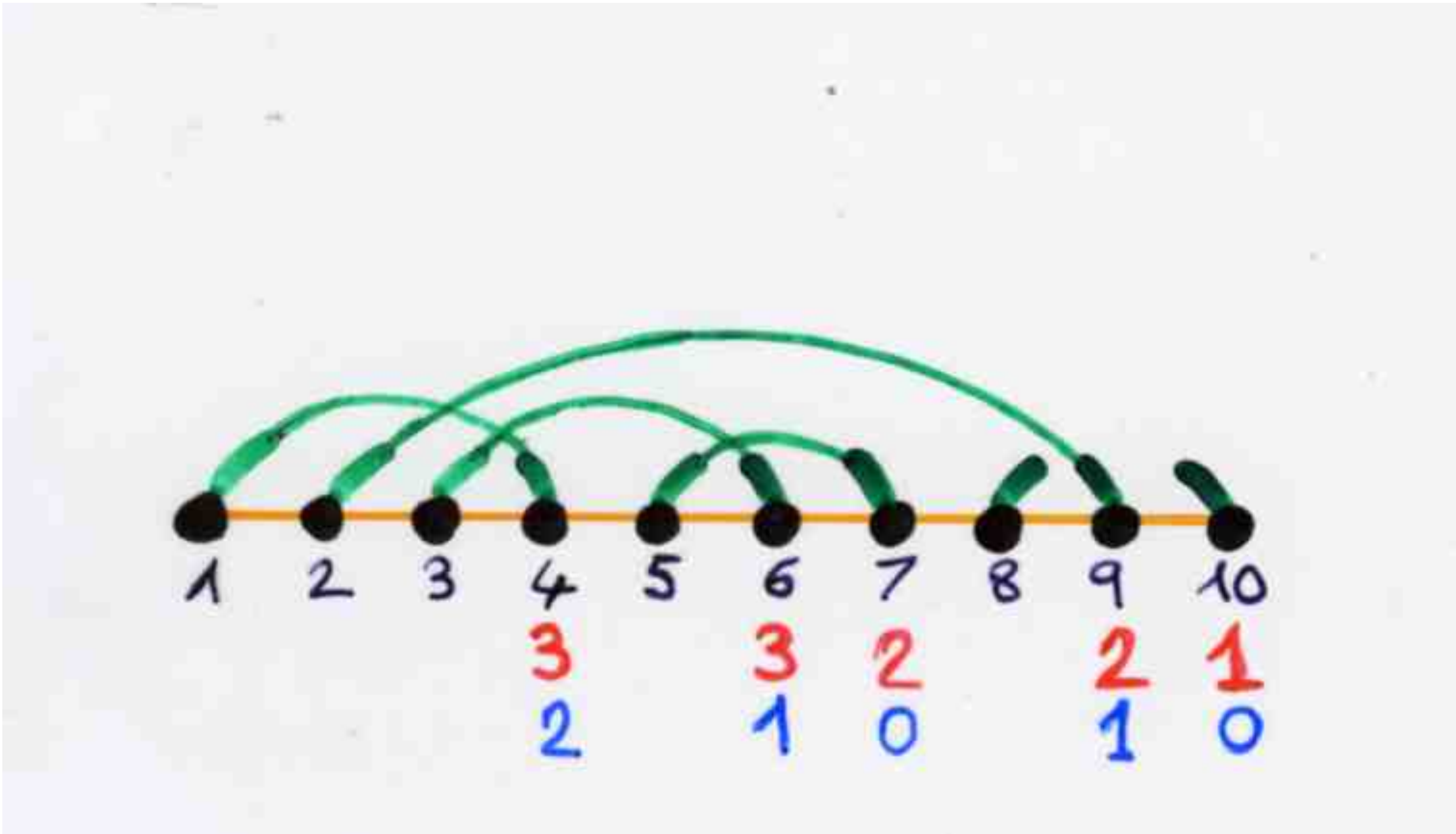
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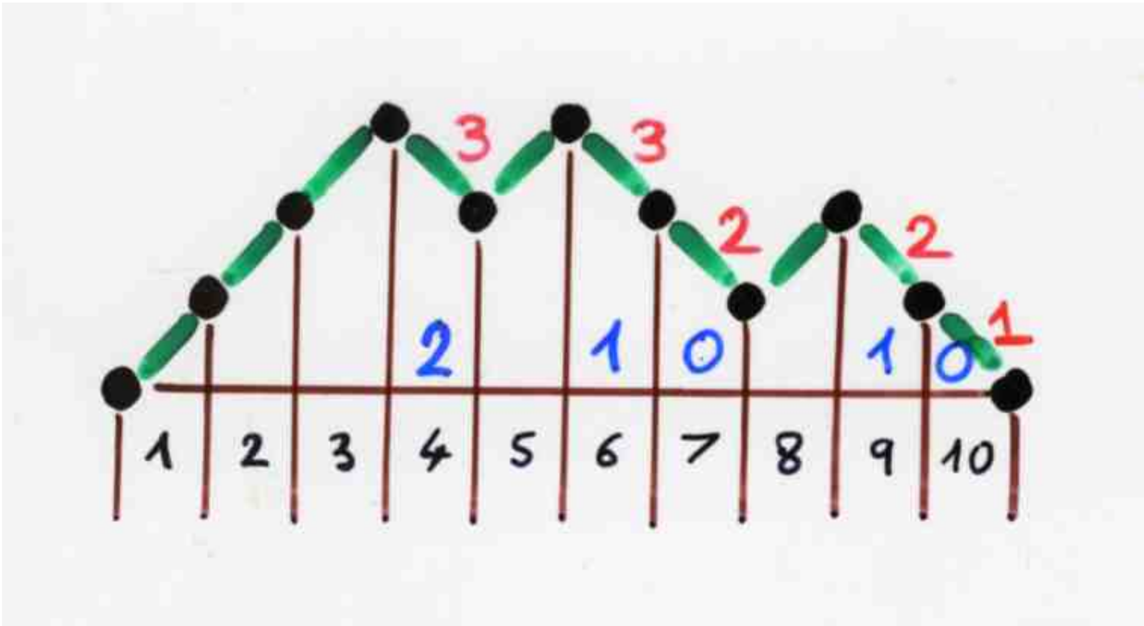

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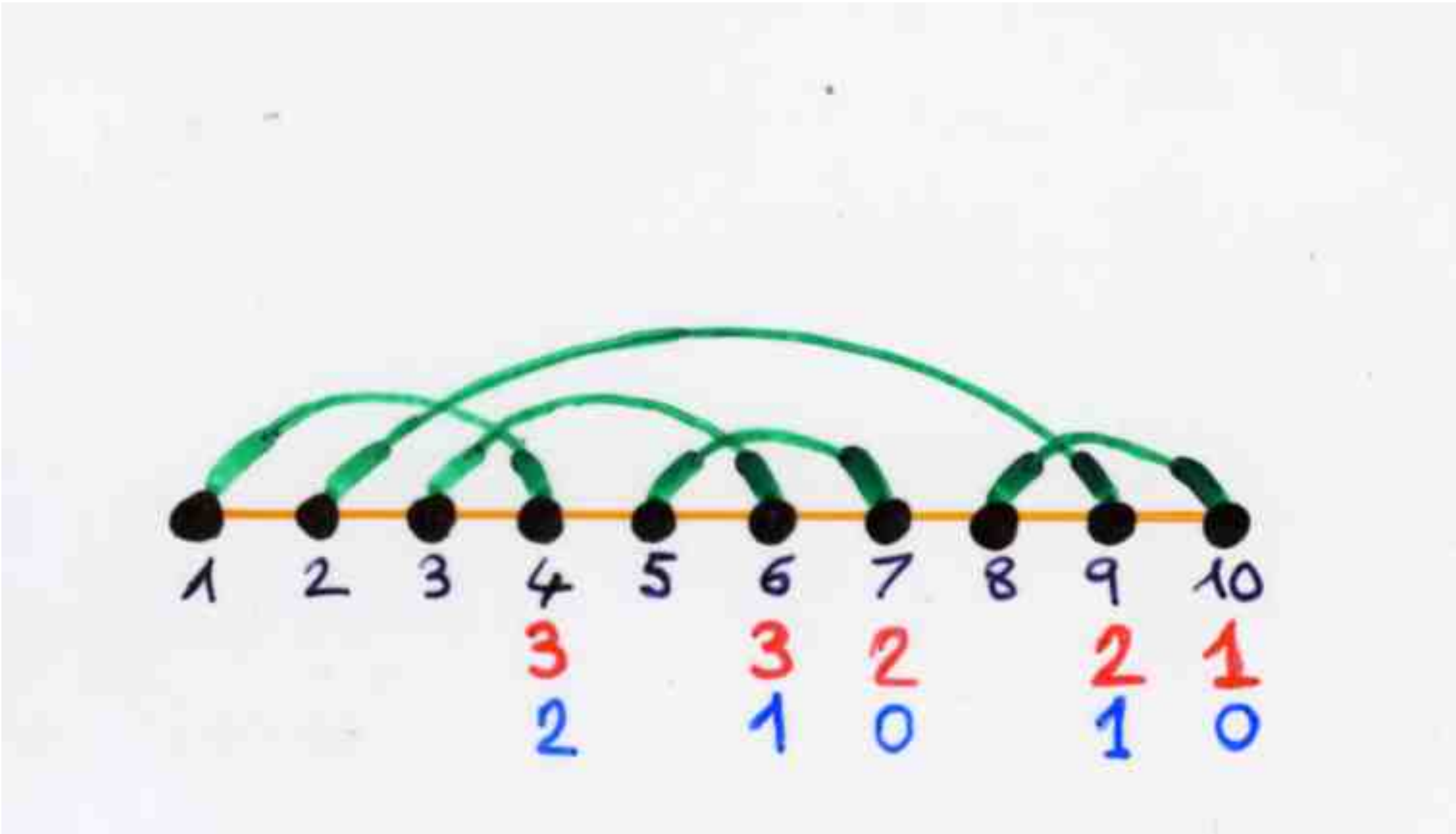


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

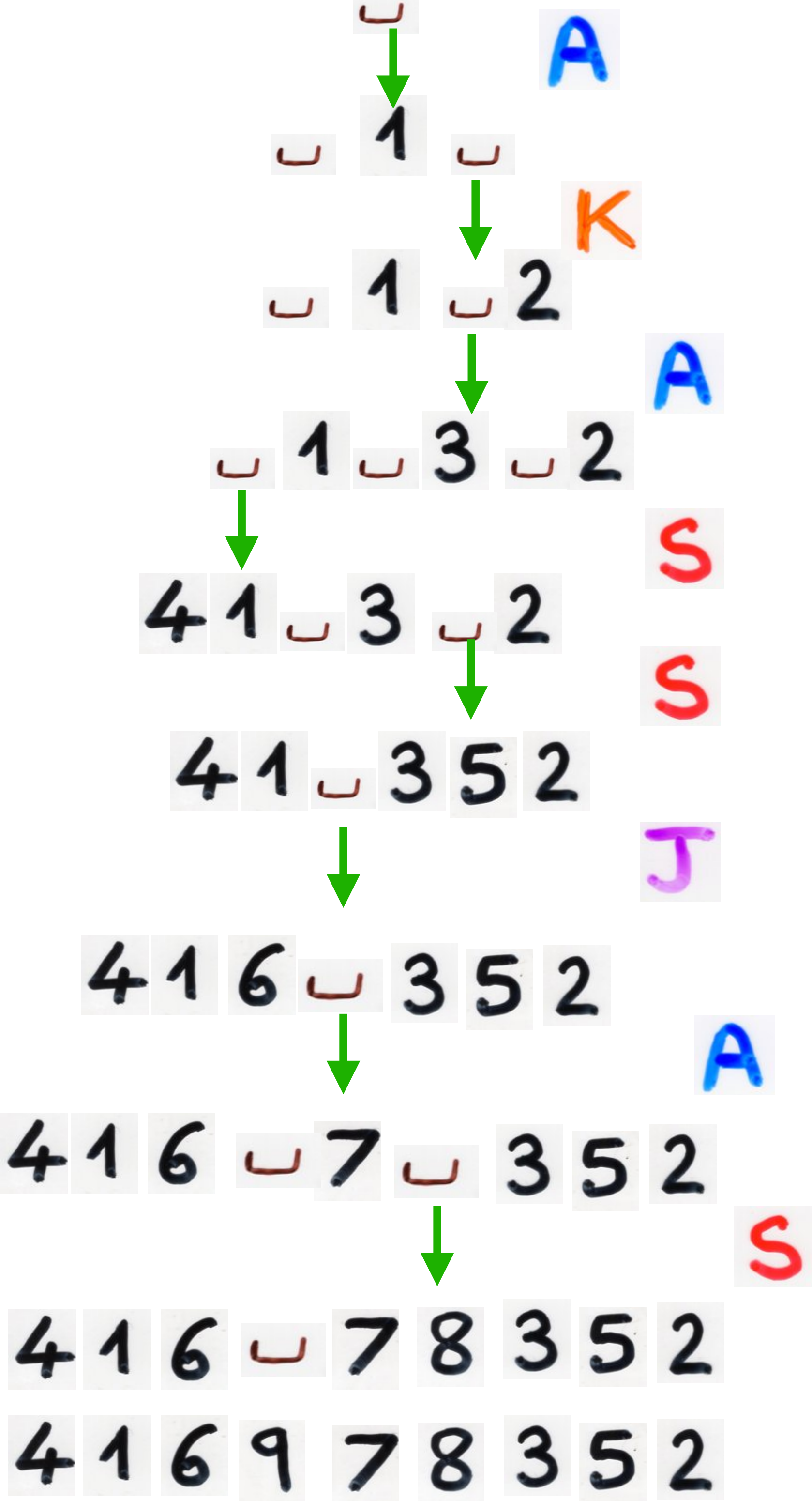
Laguerre histories

The FV bijection

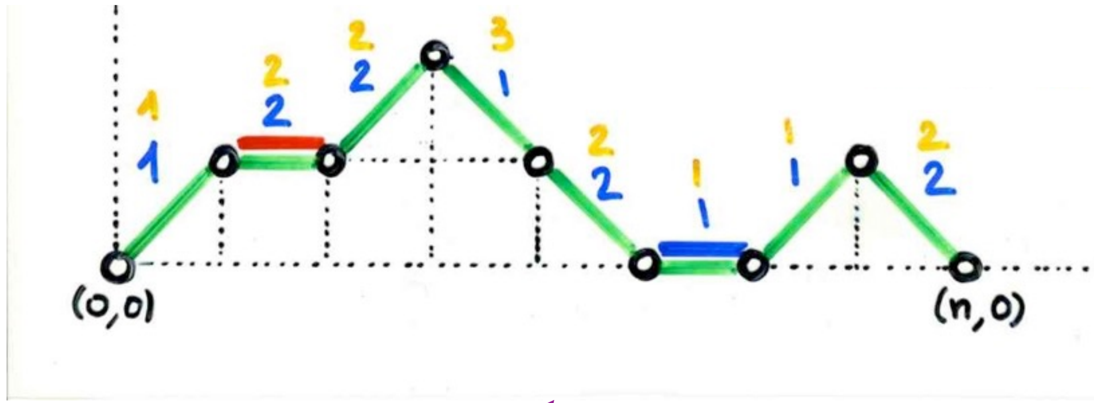
Laguerre polynomials

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$



Laguerre history



Françon, X.V. (1979)

ABjC, Part I, Ch4

ABjC, Part III, Ch5

Sheffer polynomials

$$\sum_{n\geqslant 0} P_n(x) \frac{t^n}{n!} = g(t) \exp\left(x f(t)\right)$$

Rota
umbral calculus

delta operator Q

$$D x^n = n x^{(n-1)}$$

$\{P_n(x)\}_{n\geqslant 0}$ orthogonal polynomials

Meixner (1934)

are Sheffer polynomials

$\Longleftrightarrow$ $\{P_n(x)\}_{n\geqslant 0}$ are one of the 5 possible types:

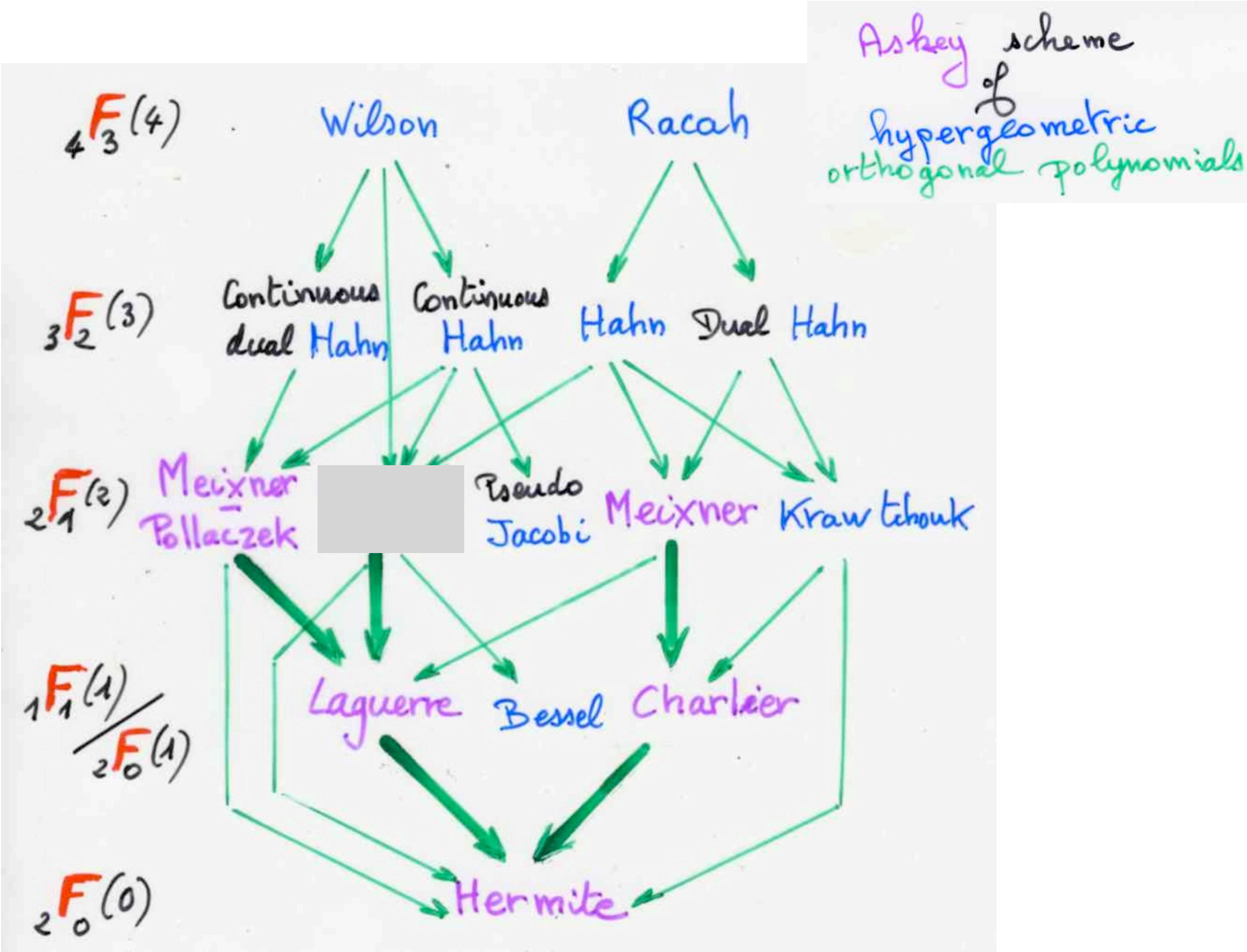
Hermite

Laguerre

Charlier

Meixner

Meixner – Pollaczek



Askey scheme of hypergeometric orthogonal polynomials
$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner Pollaczek
Pseudo Jacobi
Meixner
Krawtchouk
$_1F_1(1)$ / $_2F_0(1)$
Laguerre
Bessel
Charlier
$_2F_0(0)$
Hermite

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1}=0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\frac{(1+c)k+\beta c}{(1-c)}$	$\frac{ck(k-1+\beta)}{(1-c)^2}$	$\sum_{\sigma\in\mathfrak{S}_n} \frac{\beta^{s(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

Chapter 5 Orthogonality and exponential structures

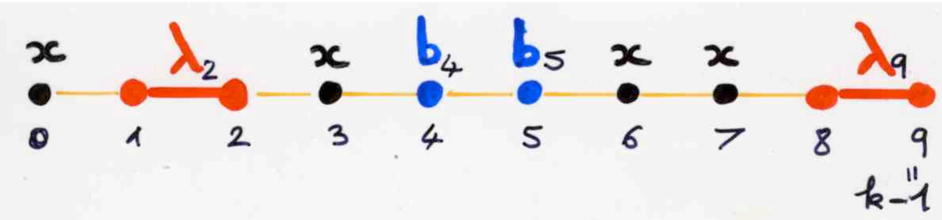
- the 5 orthogonal Sheffer polynomials
- introduction to Rota umbral calculus
- 5 interpretations of the S and Q delta operators

Duality

duality

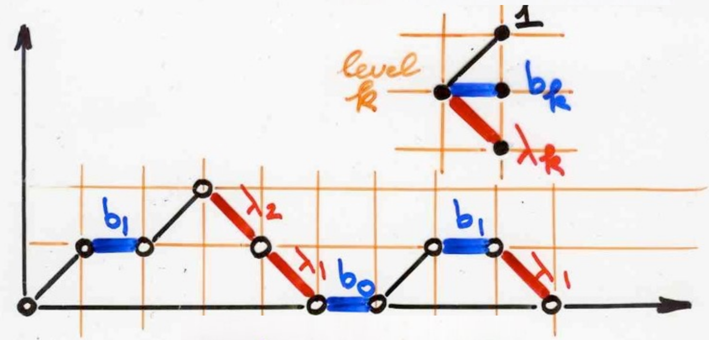
orthogonal polynomial ⟷ moments μ_n

$$\{P_n(x)\}_{n\geq 0}$$



weighted Motzkin paths

valuation v



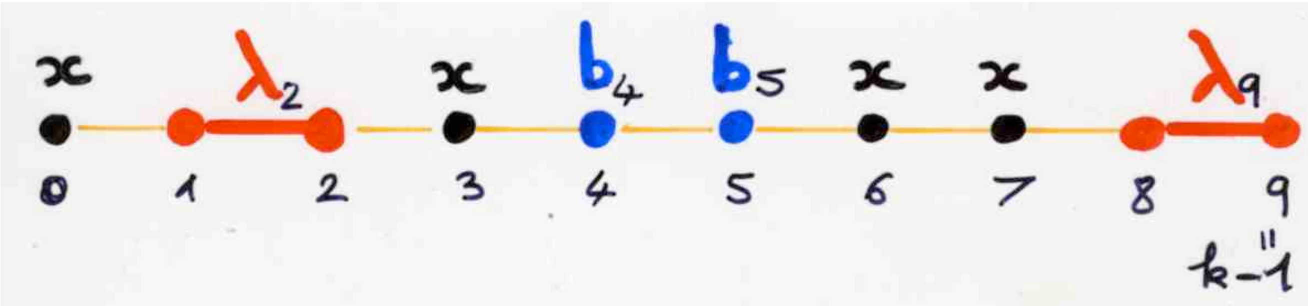
ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

3-terms linear recurrence relation

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

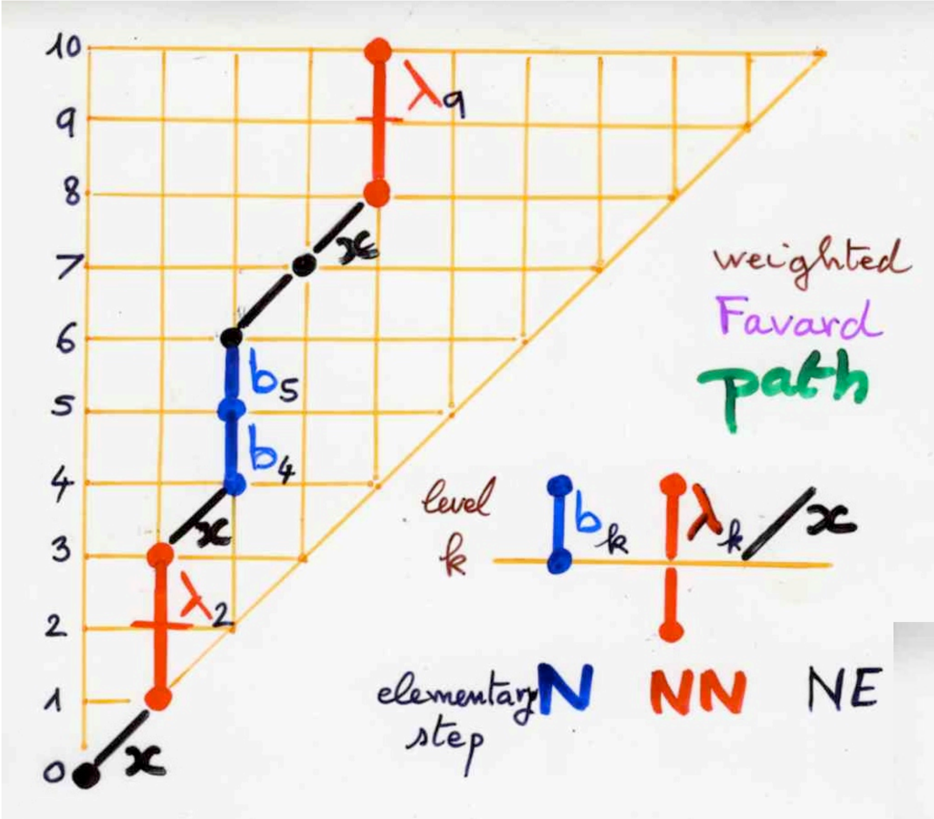
for every $k \geq 1$



$$(-1)^4\, b_4 b_5 \lambda_2 \lambda_9\, x^4$$

"pavage"

monomer, dimer

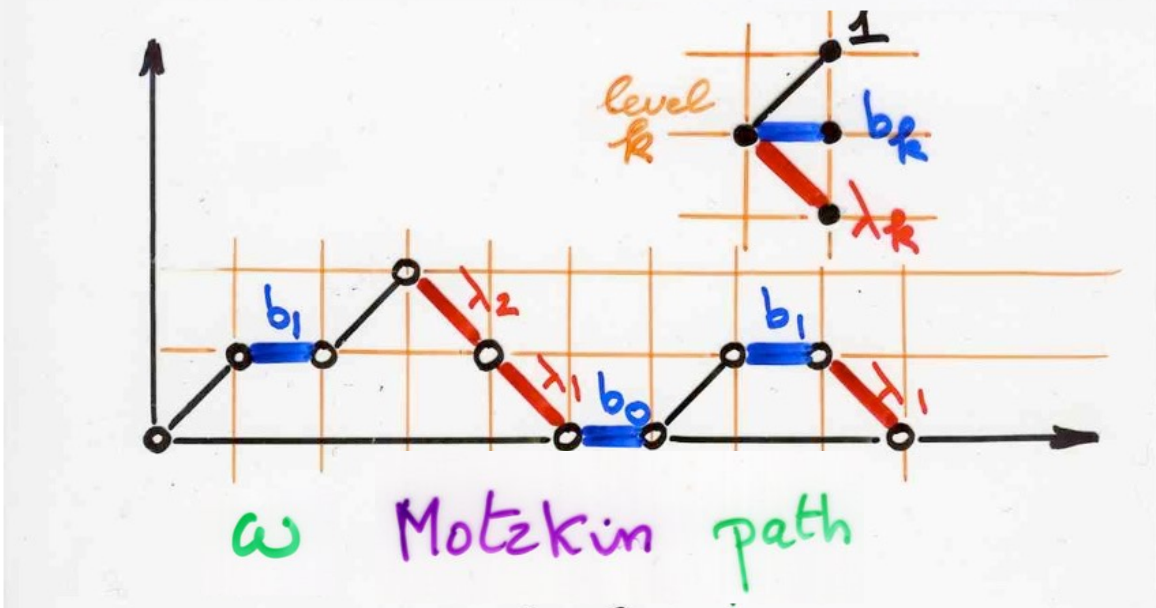


$$(-1)^4\, b_4 b_5 \lambda_2 \lambda_9 x^4$$

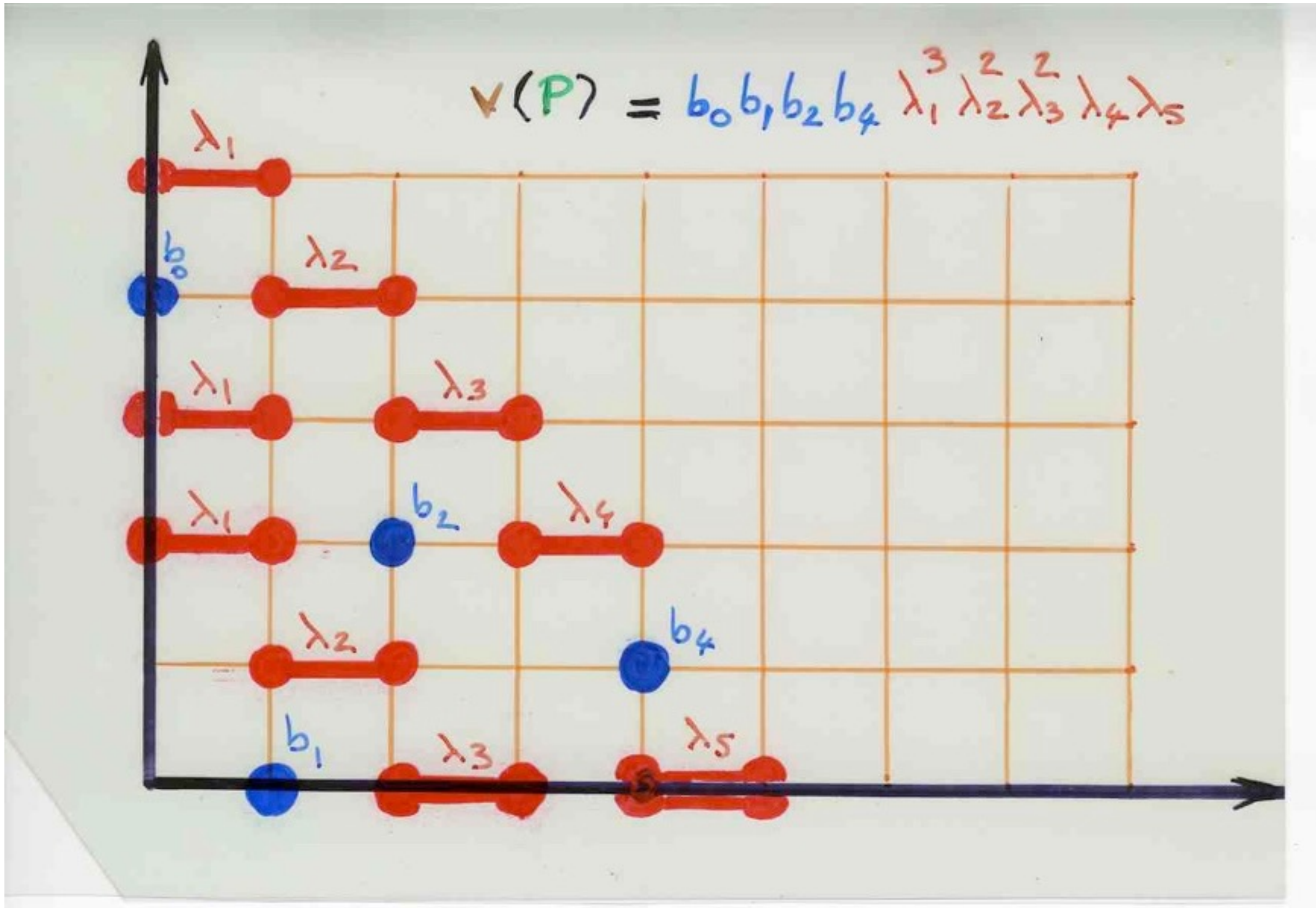
Favard paths

duality

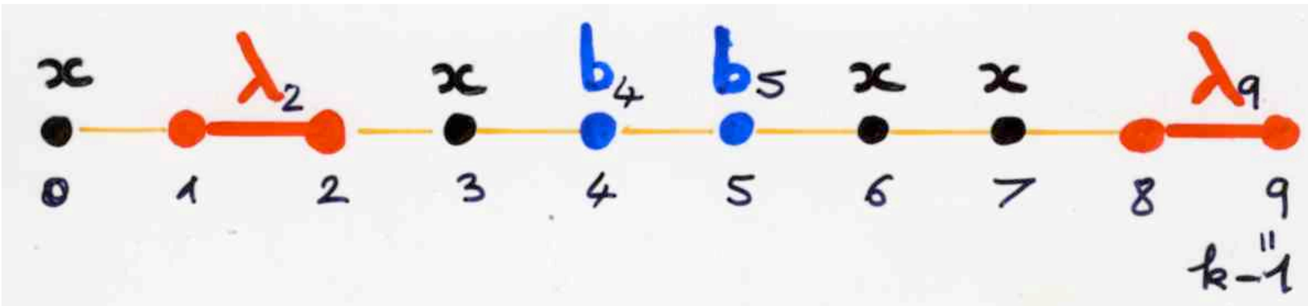
valuation v



$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

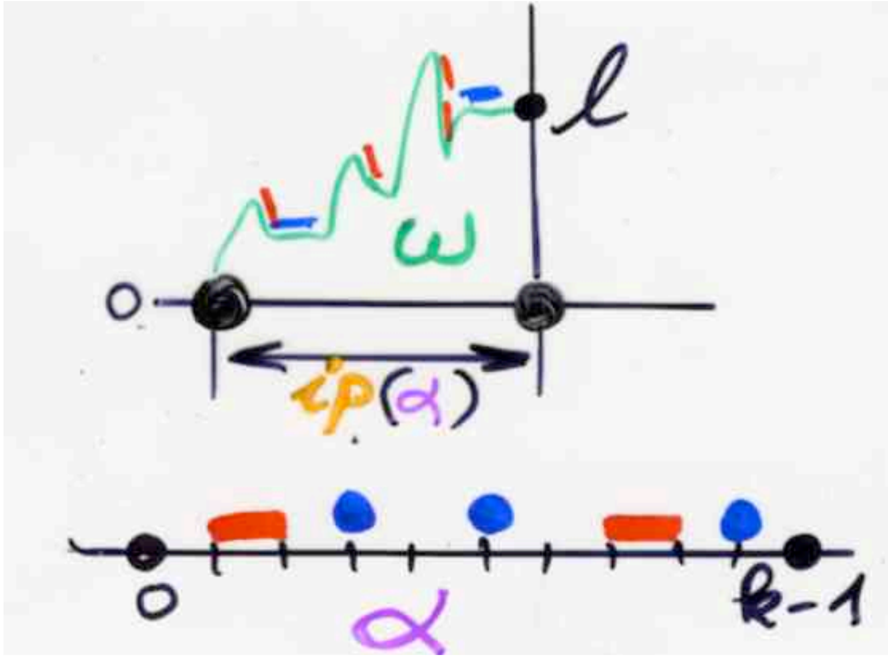


Pavage = trivial heap (→ Part II)
of monomers, dimers



$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

$$Q_n(x) = \sum_{i=0}^{n} q_{n,i}\, x^i$$



inverse sequence

$$\{Q_n(x)\}_{n\geq 0}$$

duality

$\lambda_k = 0$ $b_k = k$ $k \geq 0$

$\mu_{n,i} = s_{n,i}$ Stirling numbers 1st kind

duality

= number of (set) partitions of $\{1, \dots, n\}$ into i blocks

$p_{n,i} = (-1)^i S_{n,i}$ Stirling numbers 2nd kind

= number of permutations of $\{1, \dots, n\}$ having i cycles

$$\begin{cases} \lambda_k = 0, & k \geq 1 \\ b_k = x_k, & k \geq 0 \end{cases}$$

duality

homogeneous
(or complete)
elementary

$h_p(x_1, \ldots, x_m)$
$e_p(x_1, \ldots, x_m)$

symmetric
functions

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\exp\left(x\, f(t)\right)$$

Lagrange inversion

duality

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{1}{s\left(q^{<-1>}(t)\right)}\exp\left(x\, q^{<-1>}(t)\right)$$

Rota
umbral calculus

S , Q

delta
operators

analytic continued fractions

continued fractions

Stieljes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cdot\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$$S(t;\lambda)$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$$J(t;b,\lambda)$$

Jacobi continued fraction

$$b=\{b_k\}_{k\geq 0} \qquad \lambda=\{\lambda_k\}_{k\geq 1}$$

classical theory

continued fractions — **orthogonal polynomials**

J-fraction

$$J(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cdots\cdots \cfrac{}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

classical theory

continued fractions | **orthogonal polynomials**

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

moments generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments

generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

Motzkin path
$|\omega| = n$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path
$|\omega| = n$

The fundamental Flajolet Lemma

combinatorial interpretation of a
continued fraction with weighted paths

continued fractions

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cdots\cdots\cfrac{}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

Motzkin path
$|\omega| = n$

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

DE FRACTIONIBVS CONTINVIS. DISSERTATIO.

AVCTORE
Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \cdot \frac{10}{11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

Hermite histories

moments

atque series infinita ita se habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9$$

quae aequalis est huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

combinatorial
theory of
orthogonal polynomials

moments X.V. (1983)
Françon, X.V. (1978)

and
continued fractions
Flajolet (1980)

Chapter 3 Contimued fractions

- Jacobi, Stieltjes continued fractions
- Flajolet seminal Lemma
- convergents and orthogonal polynomials
- contraction of continued fractions
  example with subdivided Laguerre histories
  (Euler continued fraction)

§. 21. Datur vero alius modus in summam huius seriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: sit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

Euler

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

Laguerre polynomials

Laguerre history

$$\begin{cases} b_k = (2k+2) \\ \lambda_k = k(k+1) \end{cases}$$

$$\mu_n = (n+1)!$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots}}}}$$

$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$

$$\mu_n = n!$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t\cfrac{}{1 - 1t\cfrac{}{1 - 2t\cfrac{}{1 - 2t\cfrac{}{1 - 3t\cfrac{}{\cdots}}}}}}$$

Euler

subdivided Laguerre history

A. de Médicis, X.V. (1994)

- contraction of continued fractions
  example with subdivided Laguerre histories
  (Euler continued fraction)

q-d algorithm

combinatorial proof

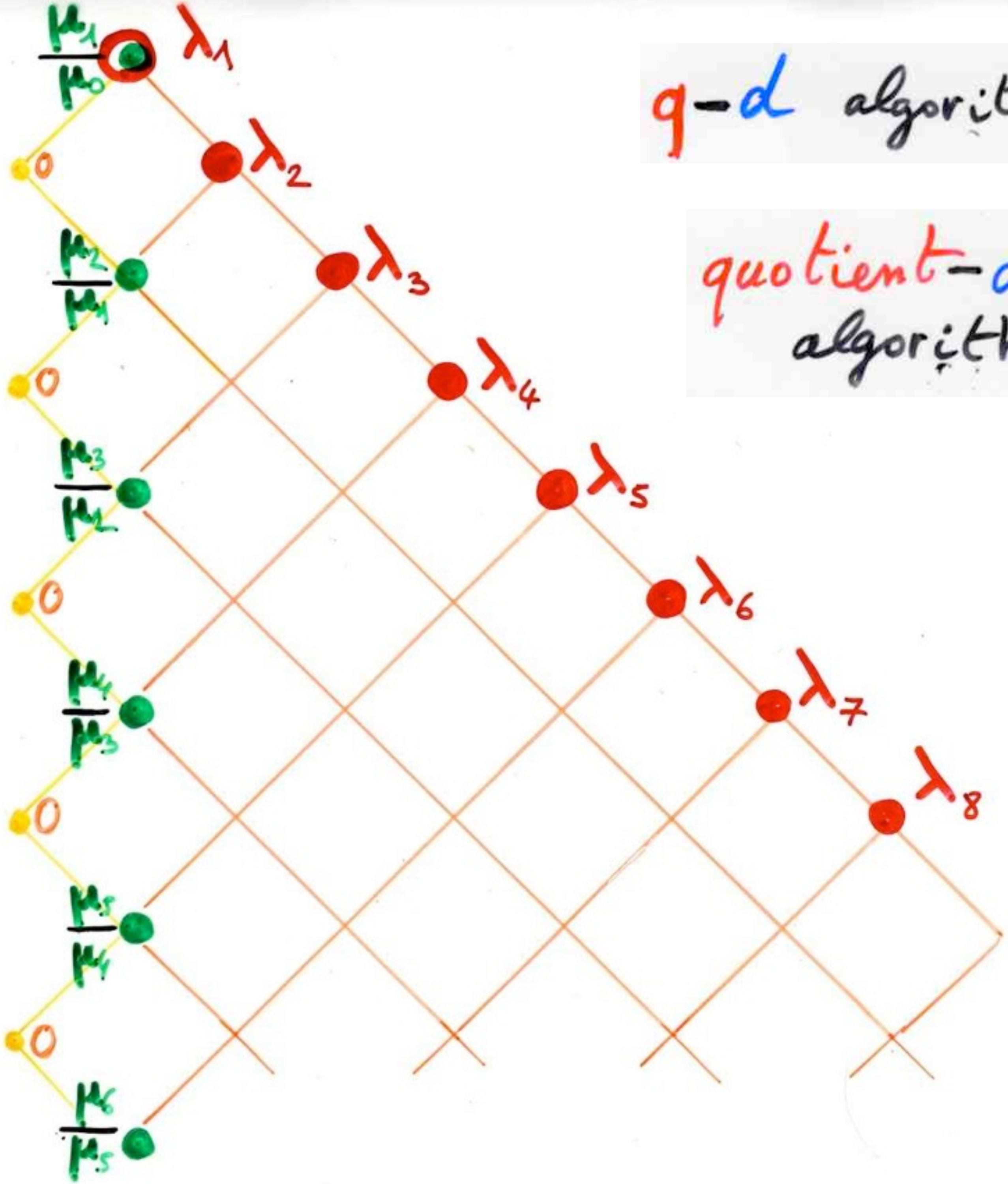
quotient-difference algorithm

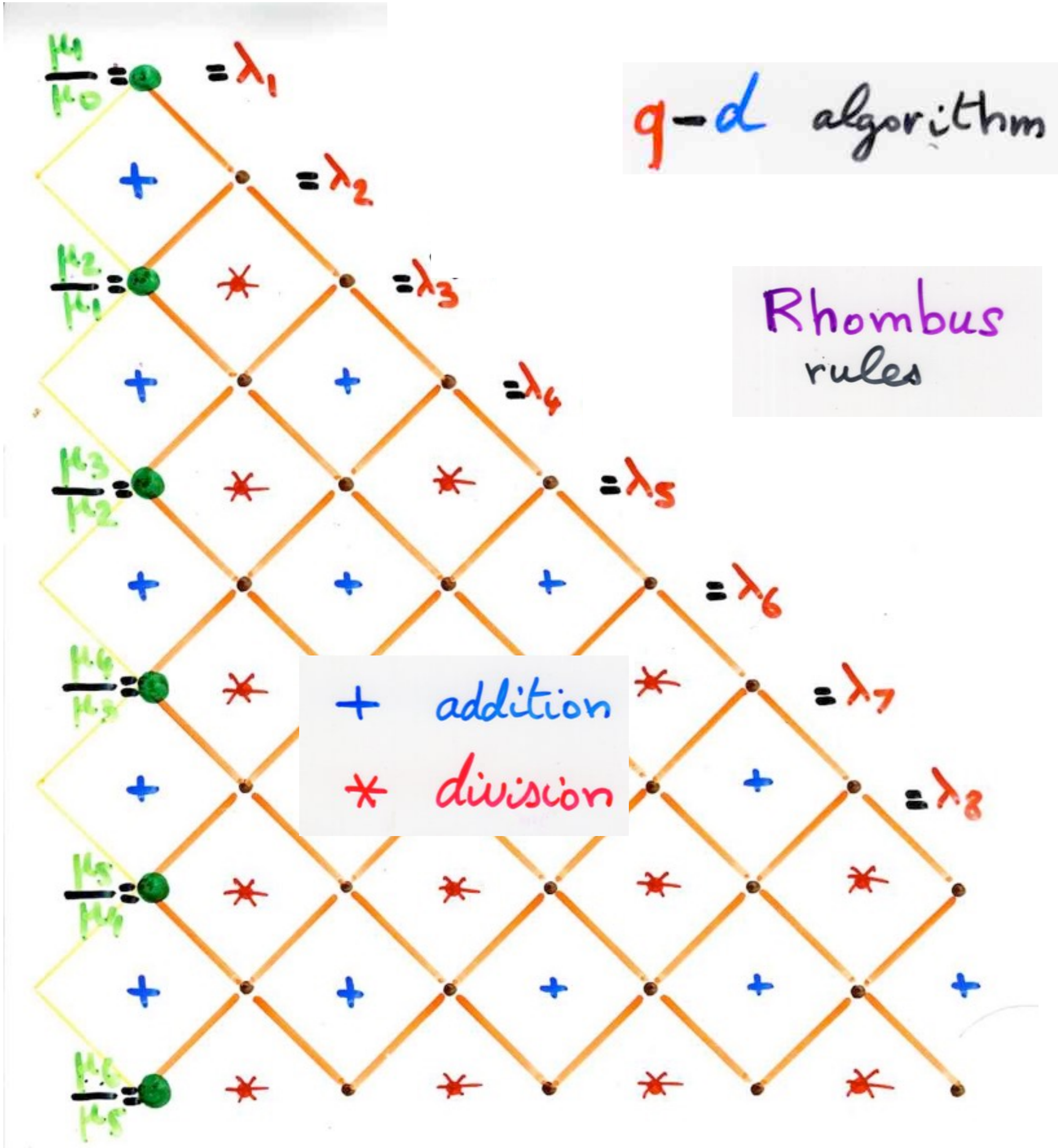
<u>Chapter 4</u> Computation of $\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$
(expanding a power series into Jacobi continued fraction)

q-d algorithm

q-d algorithm

quotient-difference algorithm





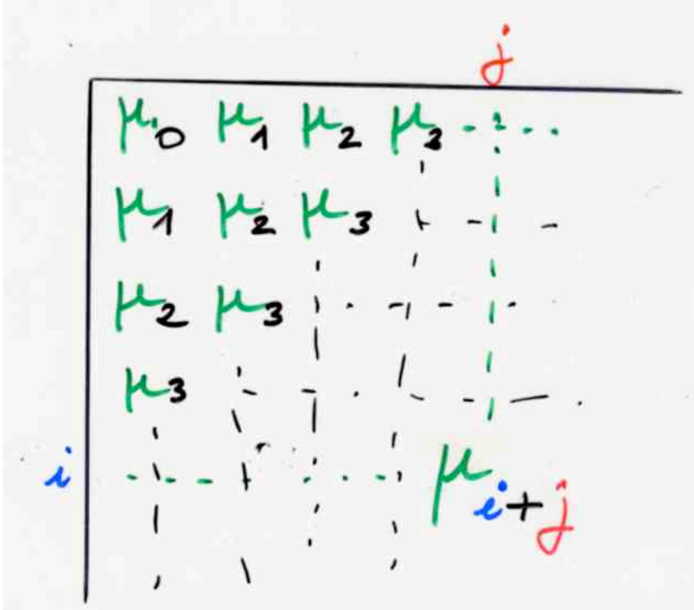
q-d algorithm
Rhombus rules
+ addition
* division
μ1/μ0 =
μ2/μ1 =
μ3/μ2 =
μ4/μ3 =
μ5/μ4 =
μ6/μ5 =
= λ1
= λ2
= λ3
= λ4
= λ5
= λ6
= λ7
= λ8

Hankel determinants

Hankel determinant

any minor of the matrix

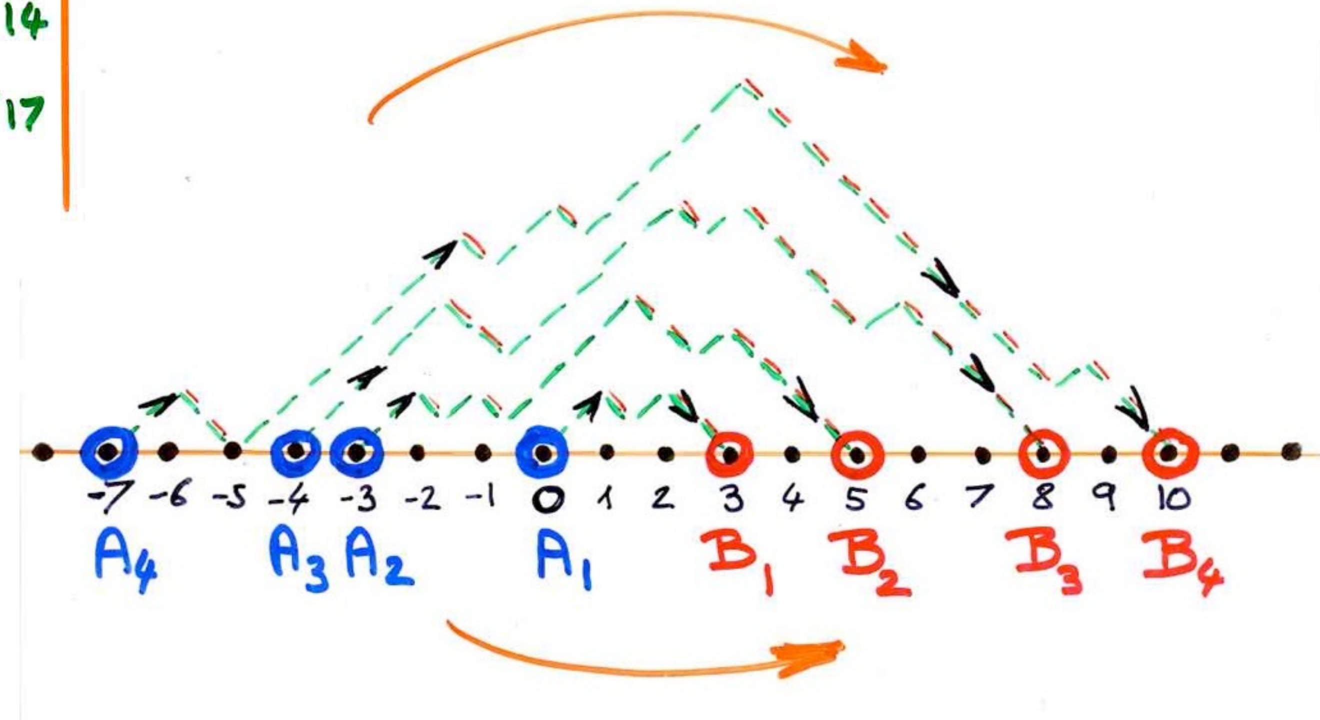
$\mathbf{H}\left(\{\mu_n\}_{n\geq 0}\right)$



LGV Lemma

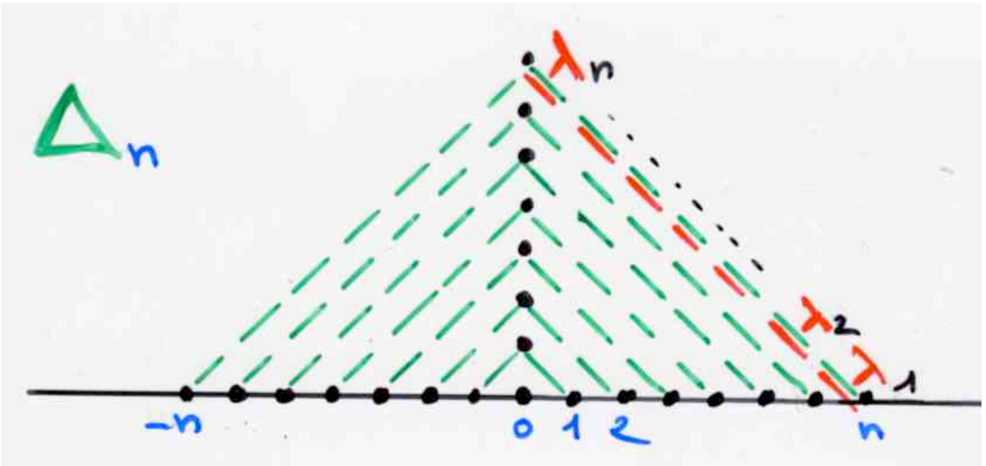
determinant ↔ configuration of non-intersecting paths

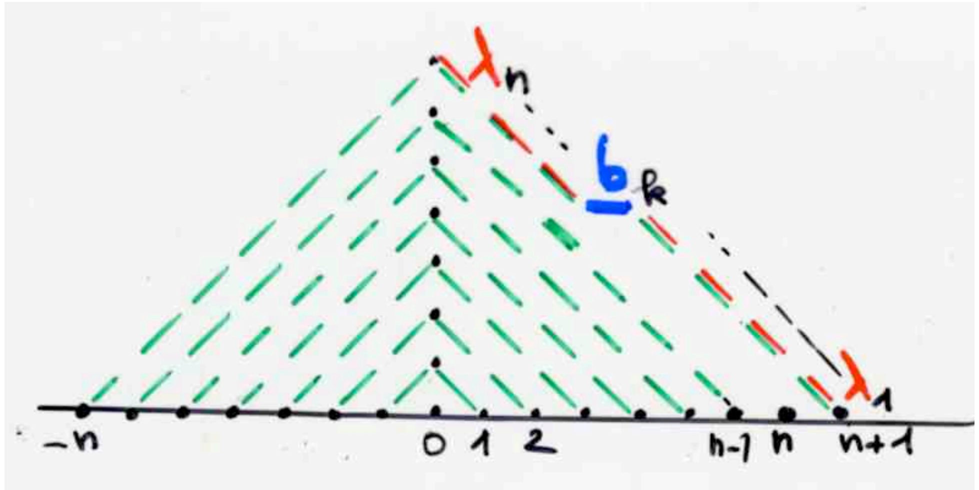
$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$



$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\chi_n = \det \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}$$





Chapter 4 Computation of $\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$
(expanding a power series into Jacobi continued fraction)

- Hankel determinant, LGV lemma
- qd-algorithm (quotient-difference)
- Ramanujan algorithm

Ramanujan algorithm

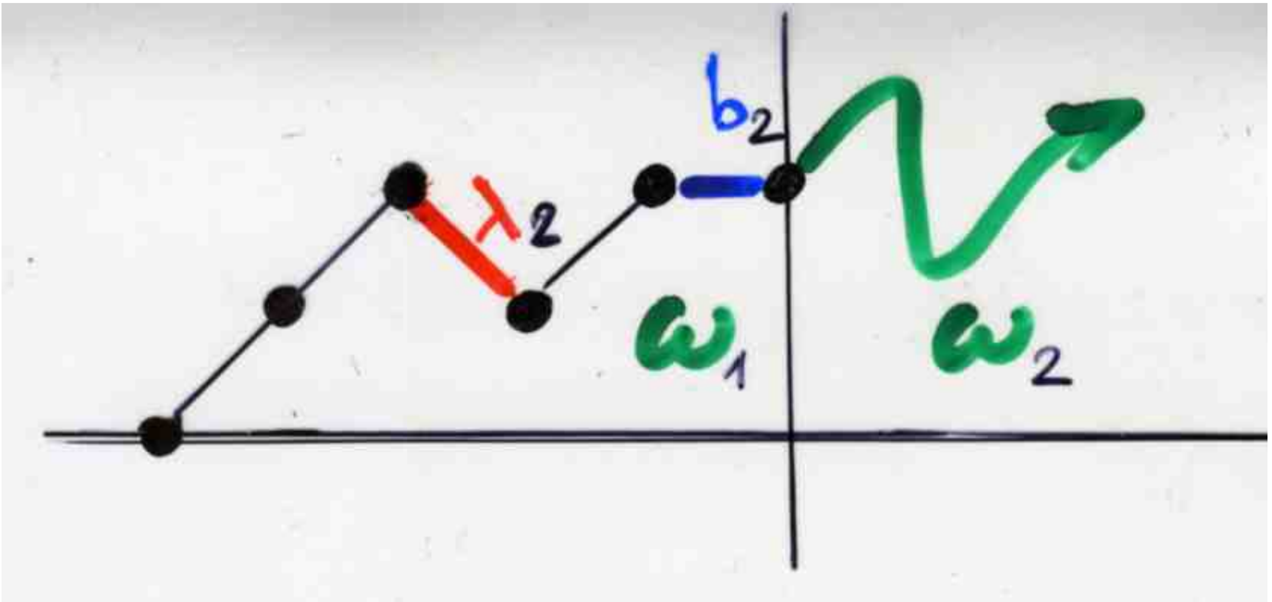
Notebook
Chapter 12, entry 17

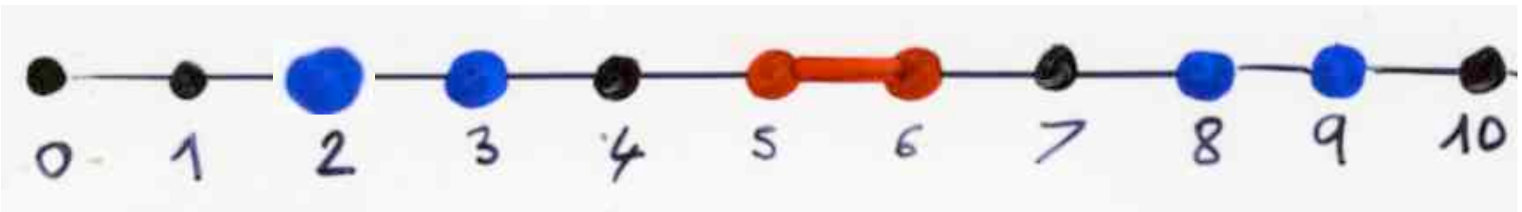
same ("essence" of) bijection

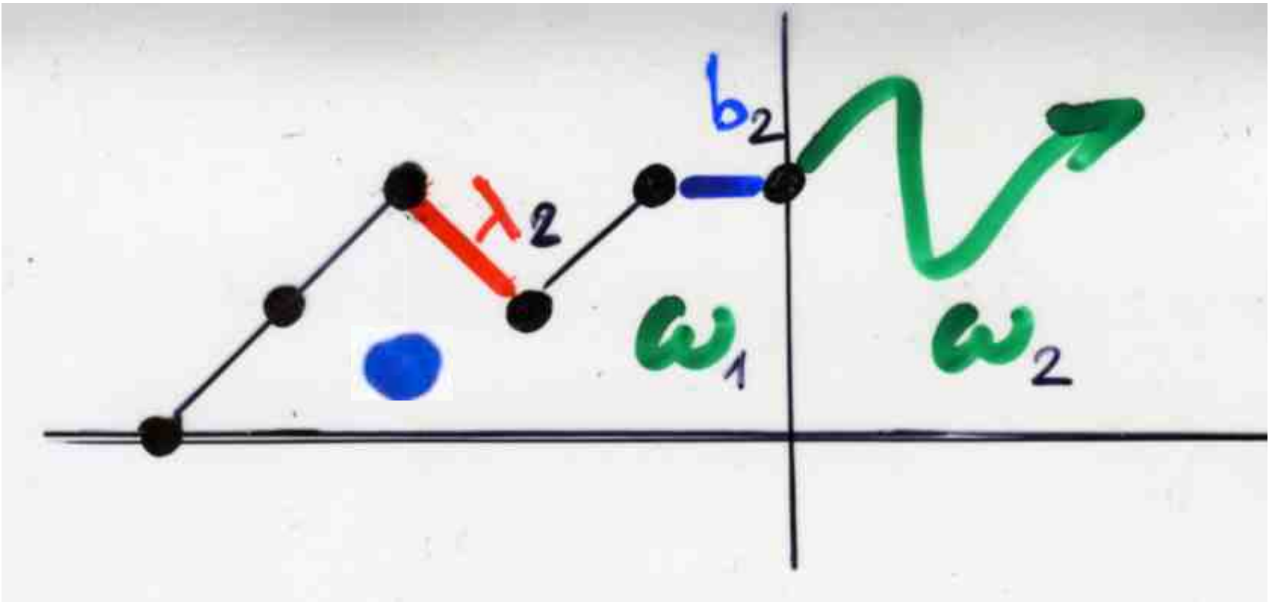
- 3 bijective proofs Ch 1
- convergents of continued fractions and orthogonal polynomial
- Ramanujan algorithm

3 bijective proofs:

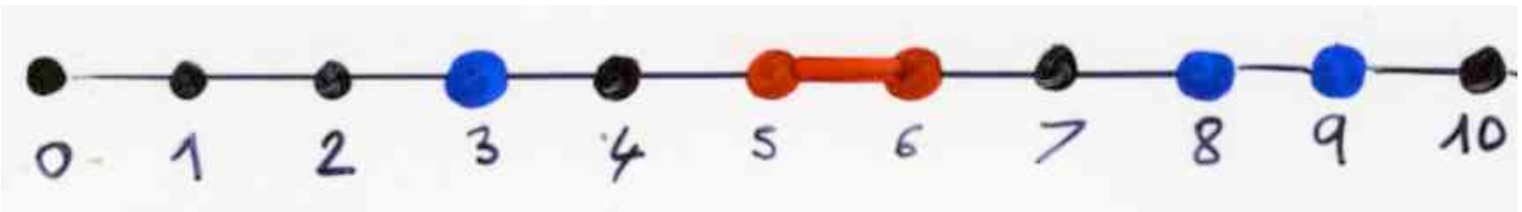
- 3-term recurrence ⇒ orthogonality (Favard theorem)
- inverse polynomials
- positivity of some linearization coefficients


b2
λ2
ω1
ω2

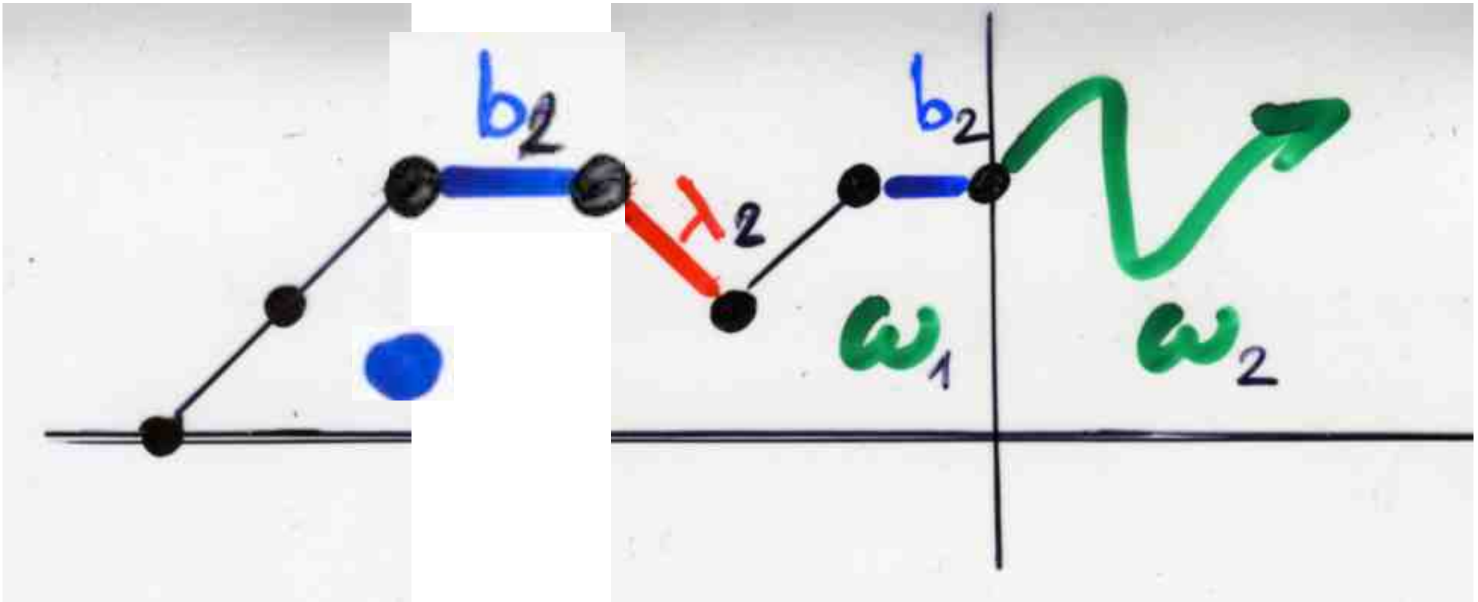

0
1
2
3
4
5
6
7
8
9
10



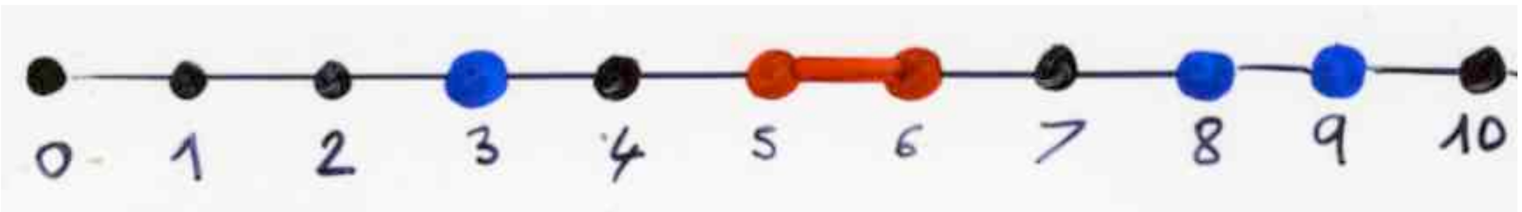
b_2
λ_2
ω_1
ω_2



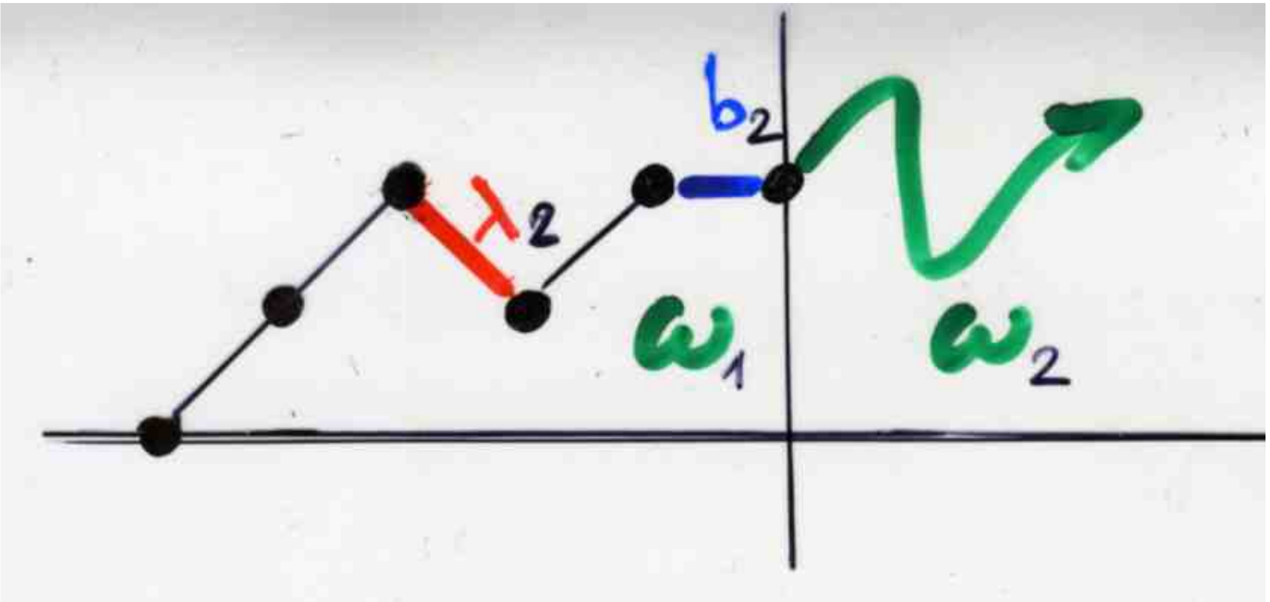
0 1 2 3 4 5 6 7 8 9 10

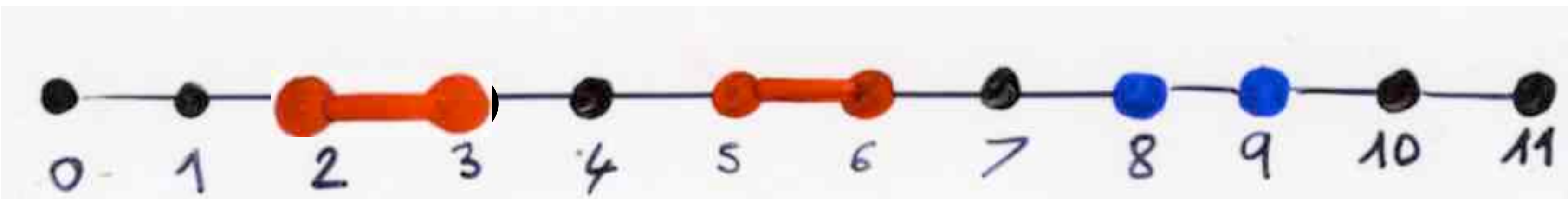


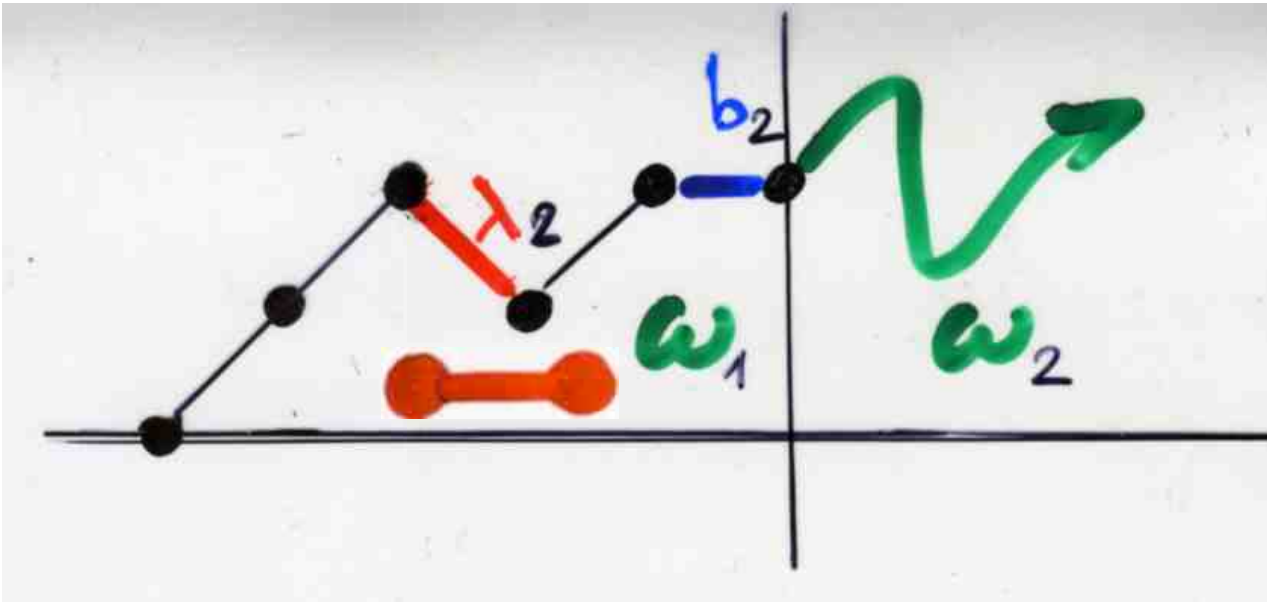
b2
λ2
b2
ω1
ω2

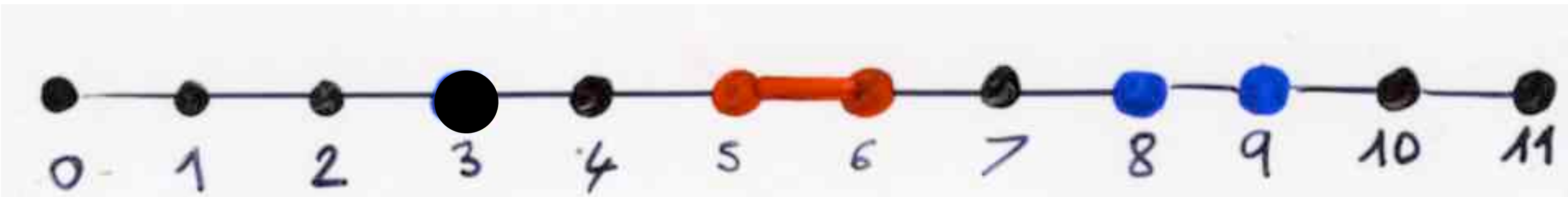


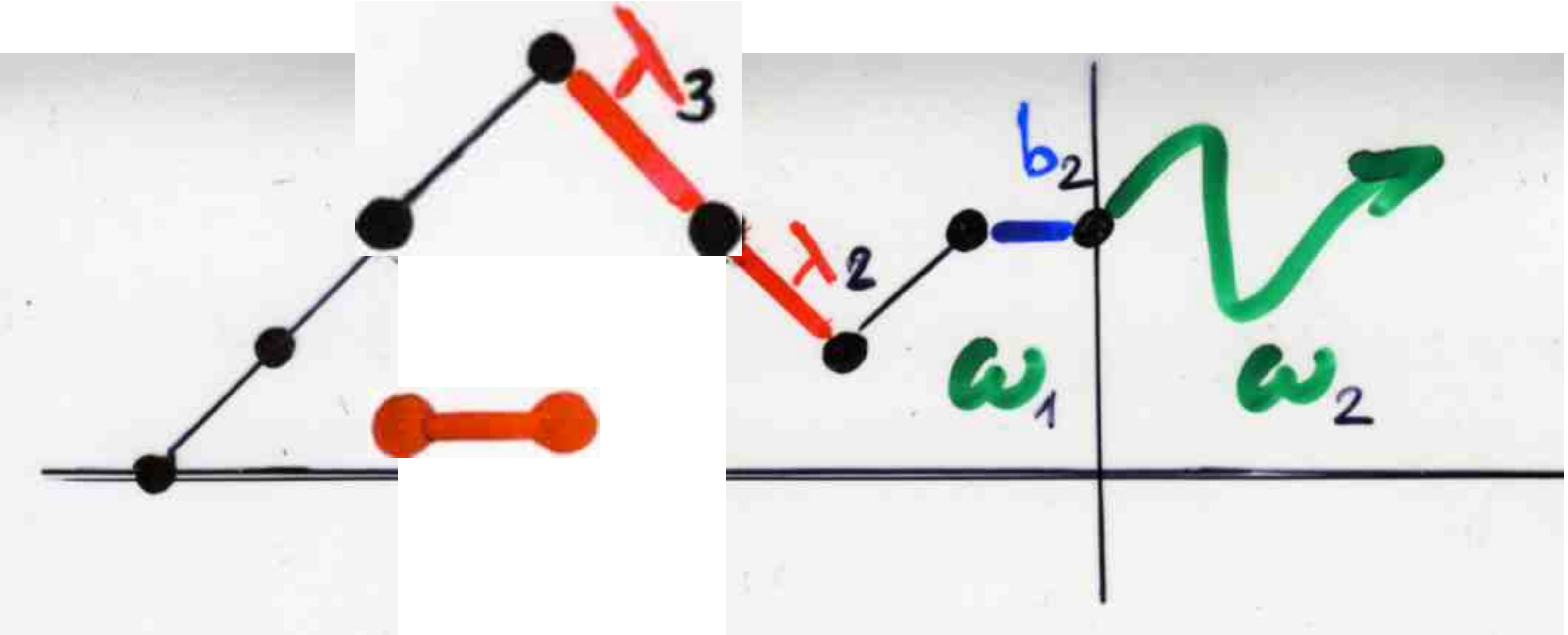
0
1
2
3
4
5
6
7
8
9
10


b2
λ2
ω1
ω2

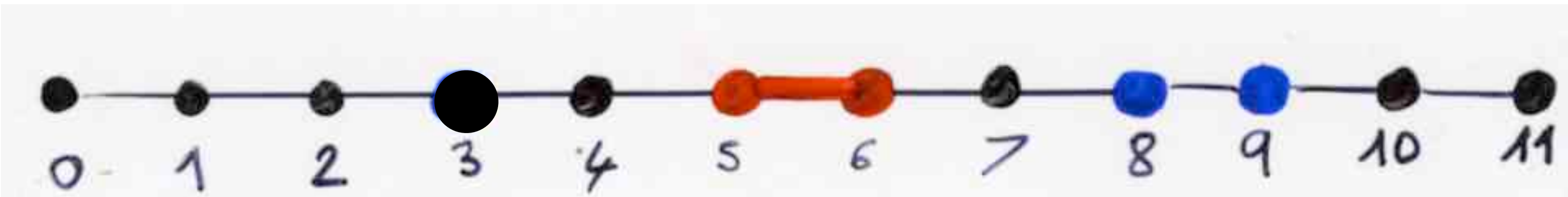

0 1 2 3 4 5 6 7 8 9 10 11


b_2
λ_2
ω_1
ω_2


0 1 2 3 4 5 6 7 8 9 10 11



λ_3
λ_2
b_2
ω_1
ω_2

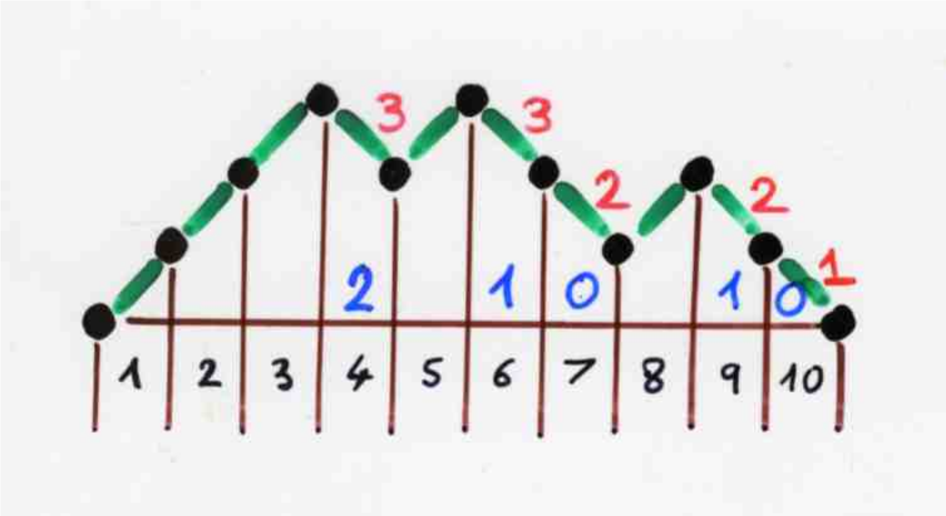


0 1 2 3 4 5 6 7 8 9 10 11

Some q-analogues of orthogonal polynomials

$\lambda_k = [k]_q$

$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$



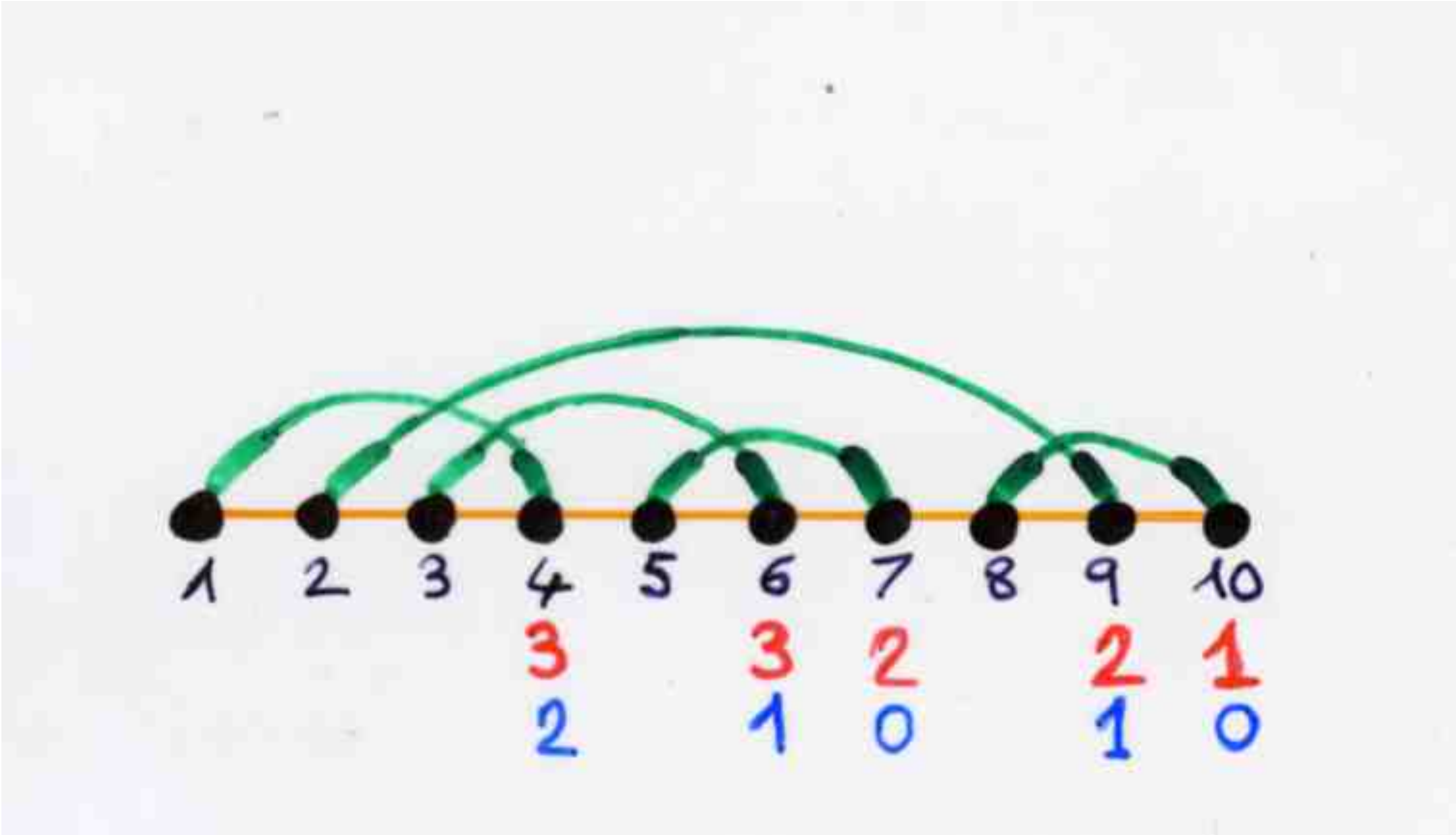
Hermite history related to ω

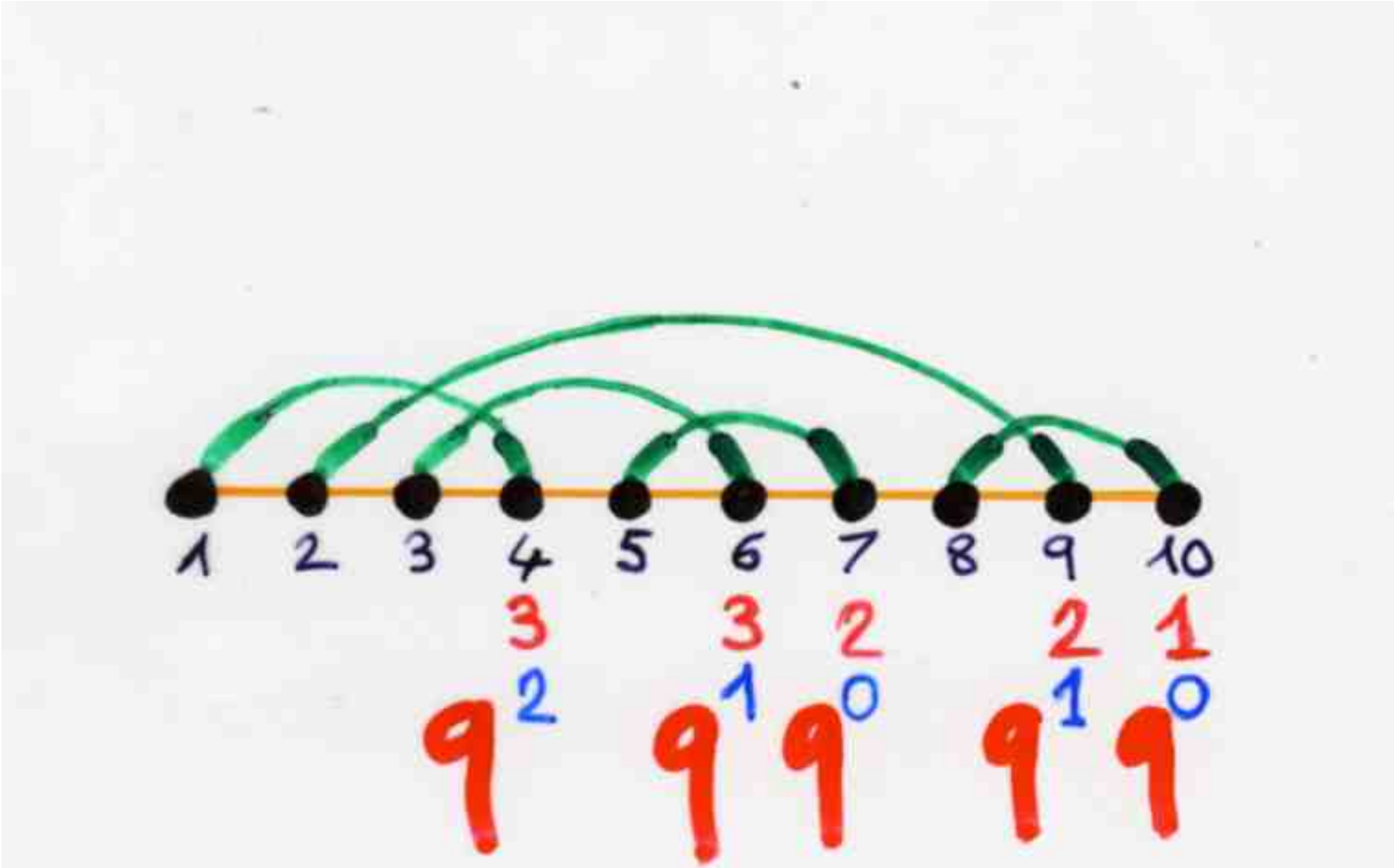
ω
Dyck path

$v_q(h)$

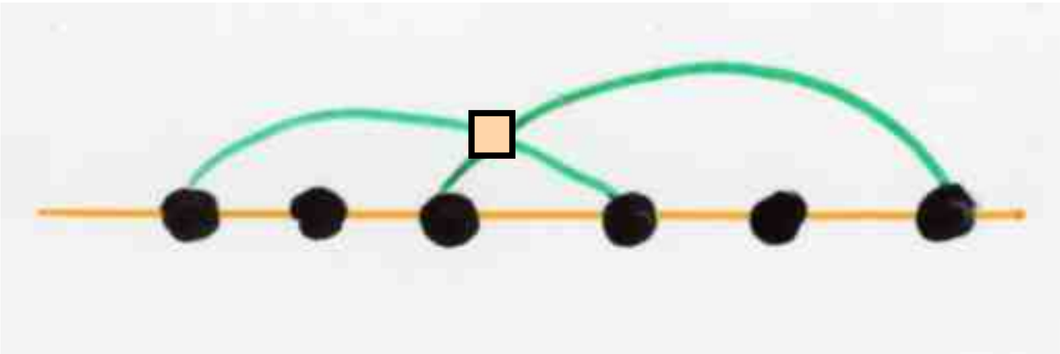
$q^{2+1+0+1+0}$

$= q^4$

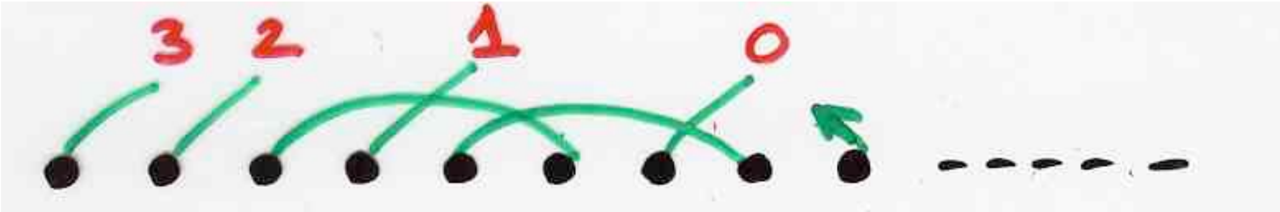

1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

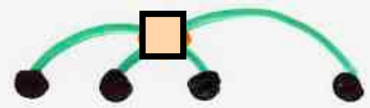


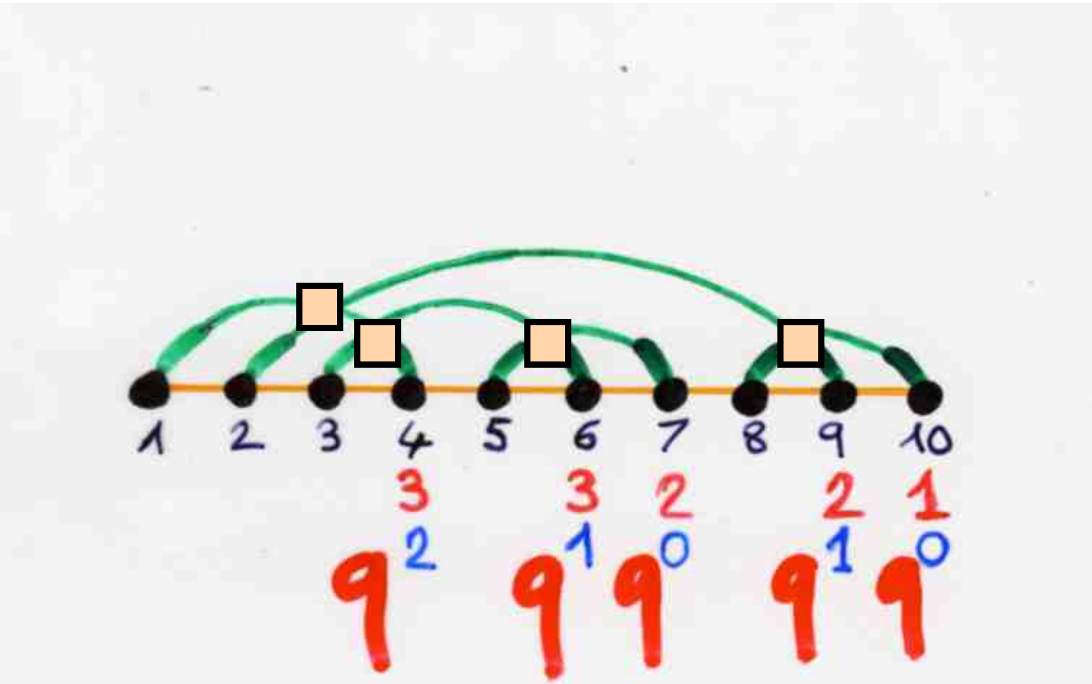
$$q^{2+1+0+1+0} = q^{4} \qquad v_q(h)$$



crossing



crossing 



$v_q(h)$

$q^{2+1+0+1+0} = q^4$

q-Hermite II

$$\begin{cases} \mu^{II}_{2n+1,q} = 0 \\ \mu^{II}_{2n,q} = [1]_q \cdot [3]_q \cdots [2n-1]_q \end{cases}$$

$H^{II}_n(x;q)$ $\quad b_k = 0 \quad$ $\lambda_k = q^{k-1}[k]_q$

q- Laguerre I

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

q - Laguerre restricted histories

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q- Laguerre II

$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

$$\mu_n = [n!]_q$$

subdivided Laguerre history

A. de Médicis, X.V. (1994)

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

Euler

9

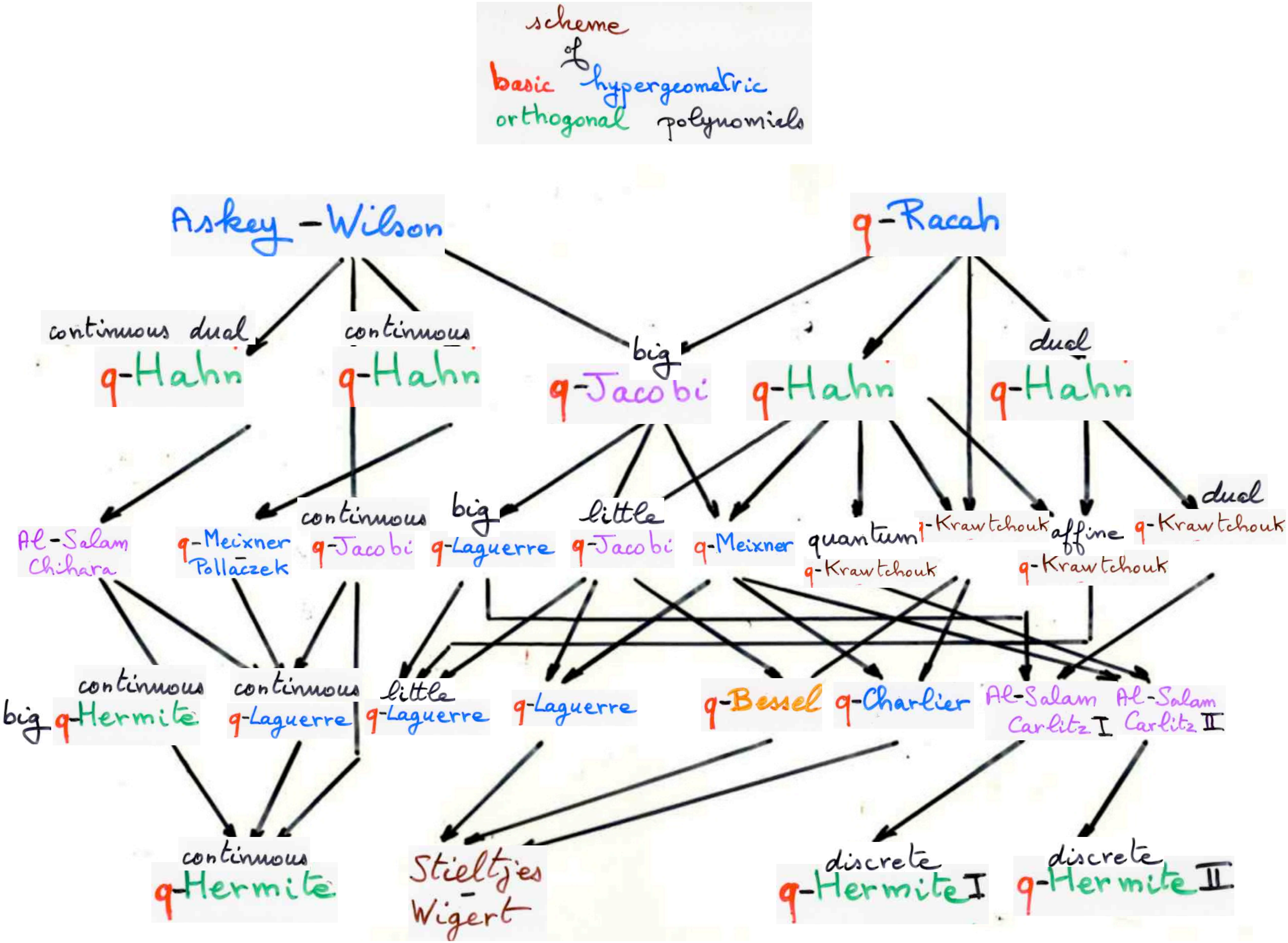
$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

Chapter 6 q-analogues

- Two q-Hermite and two q-Laguerre with their q-histories
- q-Charlier, Al-Salam-Chihara, polynomials
- Askey-Wilson polynomials

$$p_n(a,b,c,d;q|x)$$


scheme of basic hypergeometric orthogonal polynomials
Askey-Wilson
q-Racah
continuous dual q-Hahn
continuous q-Hahn
big q-Jacobi
q-Hahn
dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
continuous q-Jacobi
big q-Laguerre
little q-Jacobi
q-Meixner
quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes - Wigert
discrete q-Hermite I
discrete q-Hermite II

Chapter 1 Paths and moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path
$|\omega| = n$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

Chapter 2 Moments and histories

$$\mu_n = n!$$

moments
Laguerre polynomials

Laguerre polynomials

Hermite polynomials

Meixner polynomials

Charlier polynomials

Meixner - Pollaczek polynomials

Chapter 3 Continued fractions

Chapter 4 Computation of $\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

(expanding a power series into Jacobi continued fraction)

Chapter 5 Orthogonality and exponential structures

Chapter 6 q- analogues

Linearization coefficients

Further chapters .....

Chapter 7 Linearization coefficients

Chapter 8 Operators, quadratic algebra and orthogonal polynomials

Chapter 9 Applications and interactions

Chapter 10 Extensions

Chapter 7 Linearization coefficients

combinatorial interpretation of the linearization coefficients of :

- the 5 orthogonal Sheffer polynomials
- q-Hermite, q-Laguerre, q-Charlier polynomials
- combinatorial proof of the Askey-Wilson integral with a product of 4 q-Hermite polynomials

linearization coefficients

Lemma

$$P_k(x)\,P_\ell(x) = \sum_n a_{k\ell}^{n}\, P_n(x)$$

$$a_{k\ell}^{n} = \frac{f(P_k P_n P_\ell)}{f(P_n^2)}$$

positivity

orthogonality

$$f\big(H_m(x)\,H_n(x)\big) = n!\,\delta_{m,n}$$

The Askey-Wilson integral

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta =$$

$$\frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

The Askey-Wilson integral

integral of the product
of 4 q-Hermite polynomials
(type II)

Ismail, Stanton, X.V. (1986)

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_k(\cos\theta|q)\,H_l(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty = (q)_k\,\delta_{kl}$$

Chapter 8 Operators, quadratic algebra and orthogonal polynomials

- q- Hermite and (Weyl-Heisenberg) algebra defined by $UD = qDU + Id$
- Rook placements, q-Hermite, operators U, D, and Al-Salam-Chihara polynomials
- q-Laguerre and the (TASEP) quadratic algebra $DE = qED + E + D$

quantum
mechanics

a	annihilation	D
$a^\dagger$	creation	U

$$[a, a^\dagger] = 1$$

$\leftarrow$

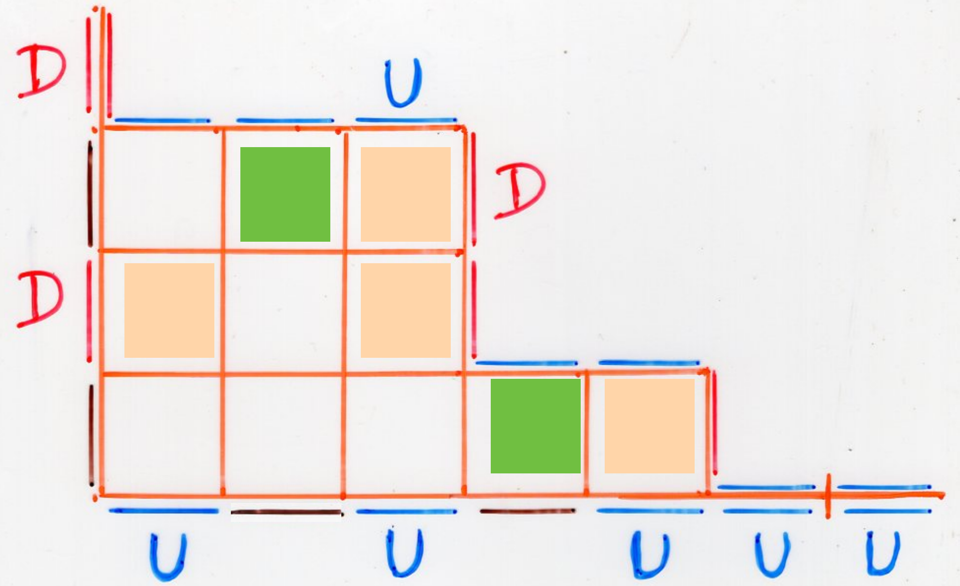
$$a|n> = \sqrt{n}\,|(n-1))$$

$$a^\dagger|n> = \sqrt{n+1}\,|(n+1))$$

$$UD = qDU + Id$$

$$UD = qDU + Id$$

Rooks placement



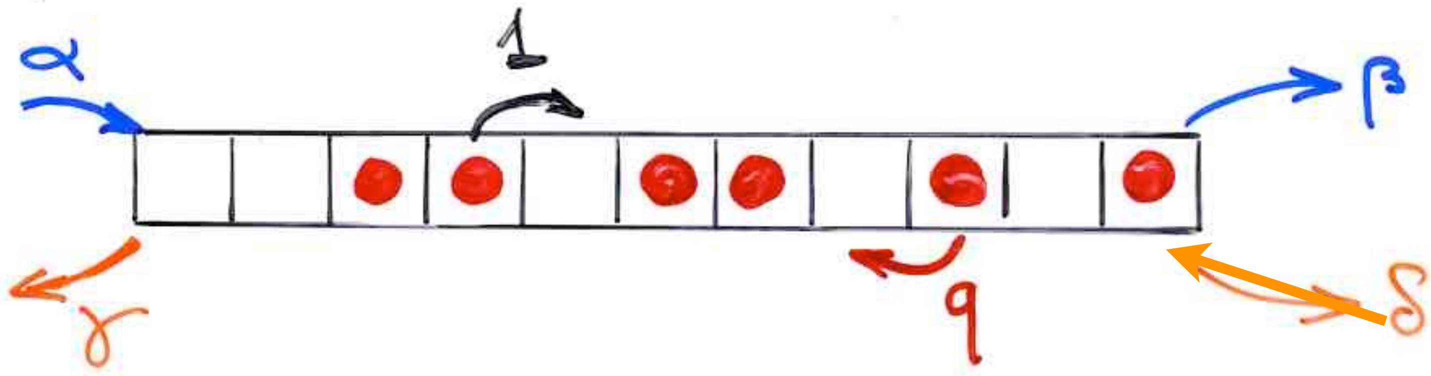
Josuat-Vergès (2011)

PASEP

Partialy asymmetric exclusion process

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\quad\quad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

Pairs of Hermite histories

↕

subdivided Laguerre histories

⤡

restricted Laguerre histories

↕

permutations

↕

Laguerre histories

$$UD = qDU + I$$

Hermite polynomials

$$DE = qED + E + D$$

Laguerre polynomials

→ Uchiyama, Sasamoto, Wadati (2003)

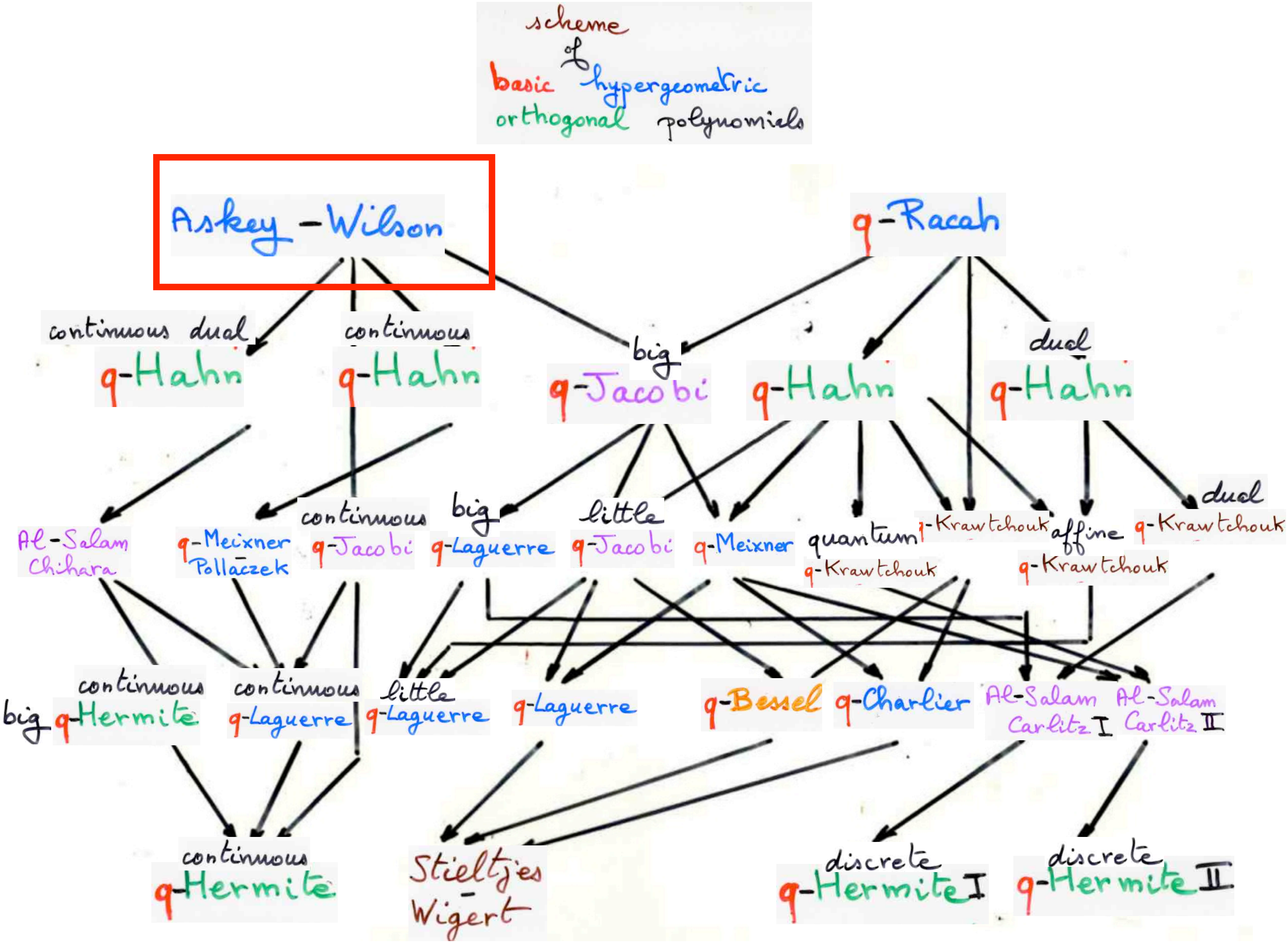
$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009)

staircase tableaux


scheme of basic hypergeometric orthogonal polynomials
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quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes Wigert
discrete q-Hermite I
discrete q-Hermite II

Chapter 9 Applications and interactions

- birth and death process in probability theory (Karlin, Mc Gregor)
- Computing integrated cost for data structures in computer science
- Polya urns in probability theory
- the PASEP model in physics
- orthogonal polynomials and Smith normal form

Data structures

Integrated cost

computer science

data structure

integrated cost

Priority queue

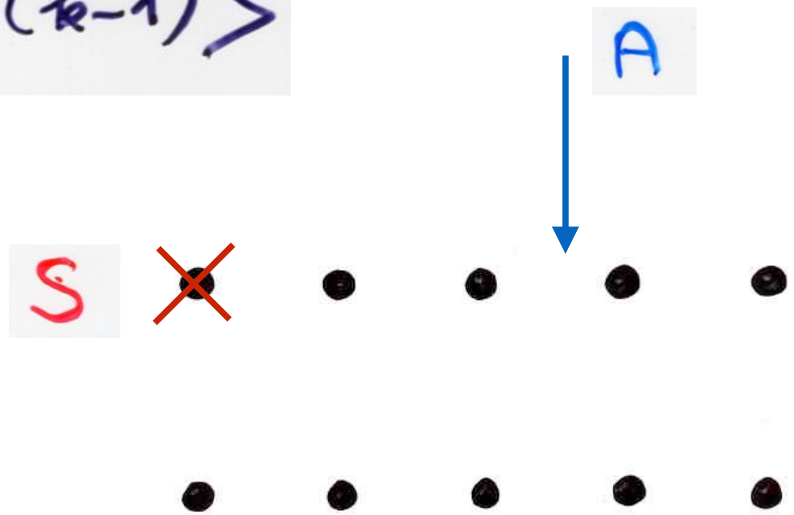
data structure history

Françon (1976)

Françon, Flajolet, Vuillemin,
(1980)

Priority queue

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$S\,|k\rangle = |(k-1)\rangle$$



data structures

Computer Science

$$AS - SA = I$$

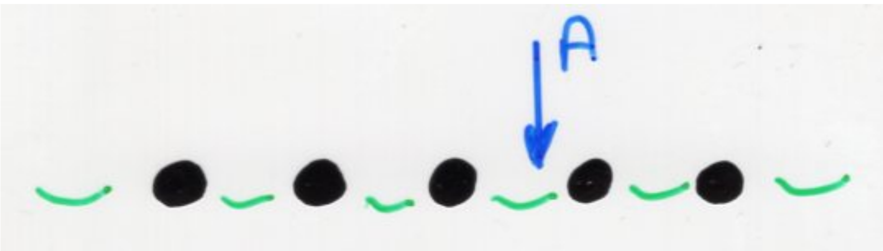
dictionnary data structure

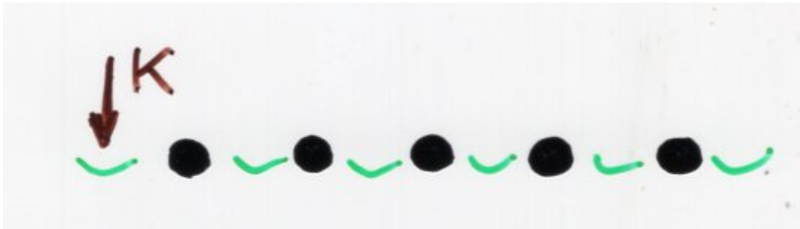
add or delete any element

ask questions

J positive

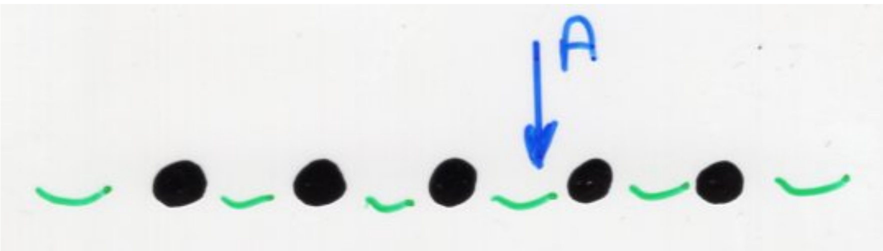
K negative

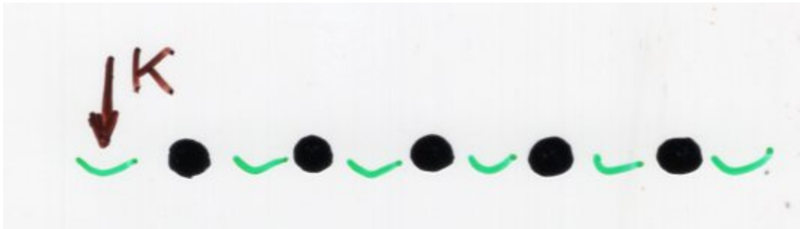




$$\begin{cases} D = A + K \\ E = S + J \end{cases}$$

$$DE = ED + E + D$$





computation of the integrated cost
of a data structure
for a random sequence
of primitive operations,
knowing the average cost
of a single primitive operation

Françon, Flajolet, Vuillemin (1980, ...)

("abstract") data structures	Possibility functions			number of histories (moments)	orthogonal polynomials
	a_k	q_k	s_k		
stack	1	0	1	C_n Catalan number	
Priority queue	k+1	0	1	$1 \cdot 3 \cdots (2n-1)$ involutions no fixed points	
Dictionary	k+1	2k+1	k	$n!$ permutations	
linear list	k+1	0	k	E_{2n} alternating permutations	
symbols Table	k+1	k	1	$B_n^{(2)}$ partitions	

("abstract") data structures	Possibility functions			number of histories (moments)	orthogonal polynomials
	a_k	q_k	s_k		
stack	1	0	1	C_n Catalan number	Tchebychev 2nd kind $\tilde{U}_n(x)$
Priority queue	$k+1$	0	1	$1 \cdot 3 \cdots (2n-1)$ involutions no fixed points	Hermite $H_n(x)$
Dictionary	$k+1$	$2k+1$	k	$n!$ permutations	Laguerre $L_n^{(0)}(x)$
linear list	$k+1$	0	k	E_{2n} alternating permutations	Meixner-Pollaczek $P_n(0,1;x)$
symbols Table	$k+1$	k	1	$B_n^{(2)}$ partitions	Charlier $C_n^{(1)}(x)$

Chapter 10 Extensions

- Biorthogonality
- L- fractions, extension of the matrix inversion theorem (Ch1)
- multicontinued fractions, T- fractions, tree-like fractions, examples
- combinatorial theory of Padé approximants and P- fraction

Padé approximants

Padé
(1863 – 1953)

type $[p/q]$

Padé approximants

$$f(t) = \sum_{n \geq 0} a_n t^n \simeq \frac{N_p(t)}{D_q(t)}$$

$$f(t) = \frac{N_p(t)}{D_p(t)} + o\left(t^{p+q}\right)$$

$\deg(N_p(t)) \leq p$

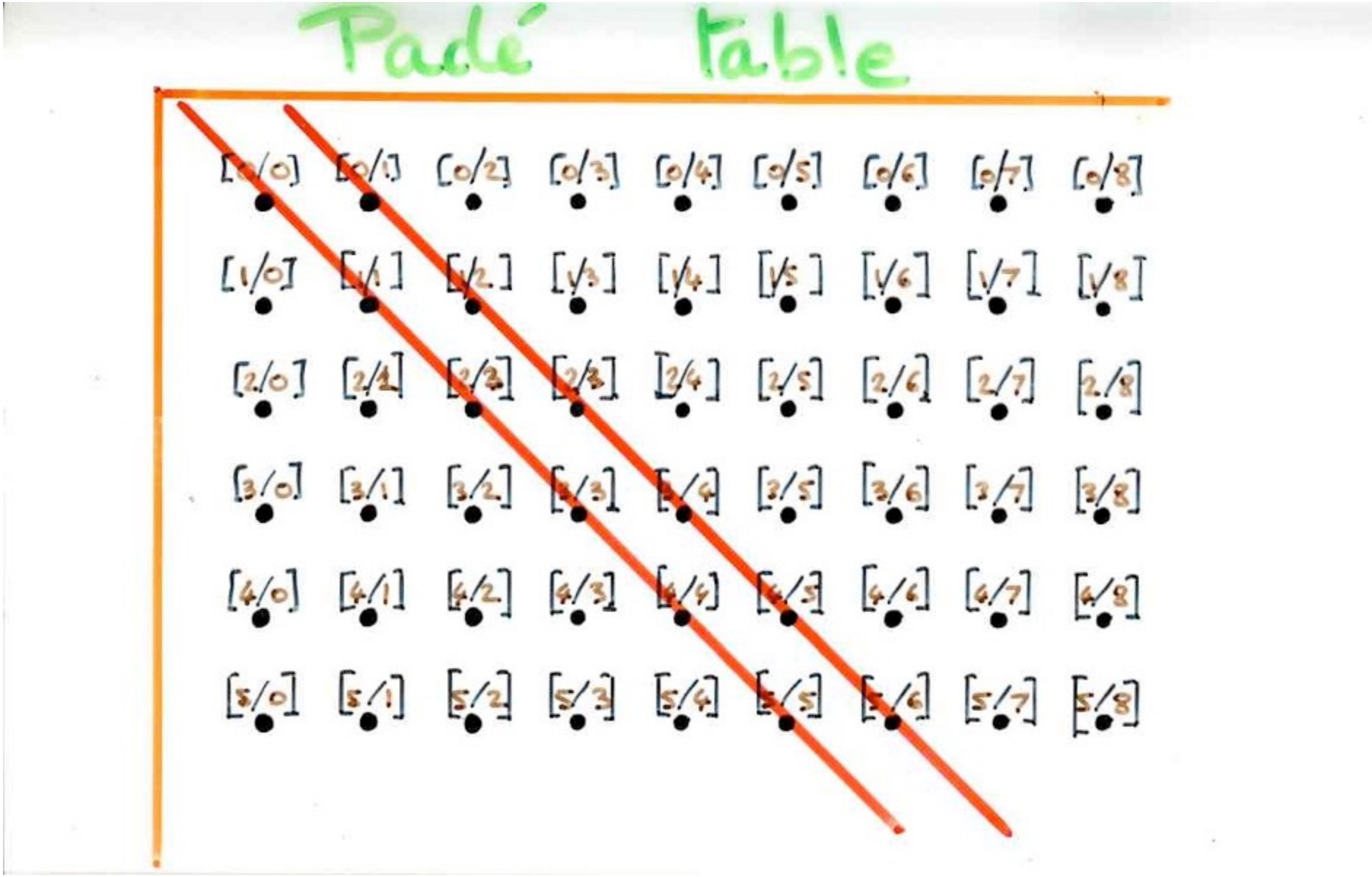
$\deg(D_q(t)) \leq q$

Taylor expansion of N_p/D_q coincides with f until the degree $p+q$

Roblet (1994)

Padé table

[0/0]	[0/1]	[0/2]	[0/3]	[0/4]	[0/5]	[0/6]	[0/7]	[0/8]
[1/0]	[1/1]	[1/2]	[1/3]	[1/4]	[1/5]	[1/6]	[1/7]	[1/8]
[2/0]	[2/1]	[2/2]	[2/3]	[2/4]	[2/5]	[2/6]	[2/7]	[2/8]
[3/0]	[3/1]	[3/2]	[3/3]	[3/4]	[3/5]	[3/6]	[3/7]	[3/8]
[4/0]	[4/1]	[4/2]	[4/3]	[4/4]	[4/5]	[4/6]	[4/7]	[4/8]
[5/0]	[5/1]	[5/2]	[5/3]	[5/4]	[5/5]	[5/6]	[5/7]	[5/8]


Padé table
[0/0] [0/1] [0/2] [0/3] [0/4] [0/5] [0/6] [0/7] [0/8]
[1/0] [1/1] [1/2] [1/3] [1/4] [1/5] [1/6] [1/7] [1/8]
[2/0] [2/1] [2/2] [2/3] [2/4] [2/5] [2/6] [2/7] [2/8]
[3/0] [3/1] [3/2] [3/3] [3/4] [3/5] [3/6] [3/7] [3/8]
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[5/0] [5/1] [5/2] [5/3] [5/4] [5/5] [5/6] [5/7] [5/8]

$$J^{\leq k}(\{b_k\}, \{\lambda_k\}; t) = \frac{N_k(t)}{D_{k+1}(t)}$$

convergent of a
Jacobi continued fraction
order k

=

Padé approximant
order $[k/k+1]$

Padé table


[0/0] [0/1] [0/2] [0/3] [0/4] [0/5] [0/6] [0/7] [0/8]
[1/0] [1/1] [1/2] [1/3] [1/4] [1/5] [1/6] [1/7] [1/8]
[2/0] [2/1] [2/2] [2/3] [2/4] [2/5] [2/6] [2/7] [2/8]
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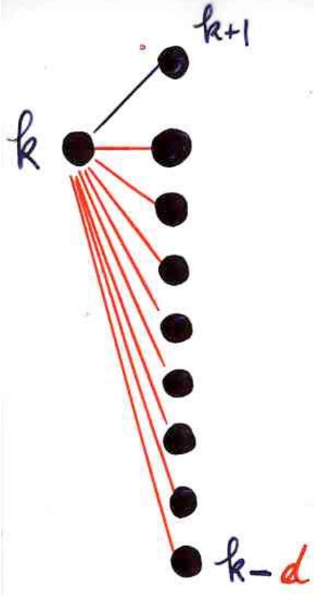
[5/0] [5/1] [5/2] [5/3] [5/4] [5/5] [5/6] [5/7] [5/8]

$$H_{n,k} = \det \begin{bmatrix} \mu_n & \mu_{n+1} & \cdots & \mu_{n+k-1} \\ \mu_n & \mu_{n+2} & & \vdots \\ \vdots & \vdots & & \vdots \\ \mu_{n+k-1} & \cdots & \cdots & \mu_{n+2k-2} \end{bmatrix}$$

Hankel determinant

$$[p/q] \text{ exists} \iff H_{p-q+1,\,q} \neq 0$$

L - fractions, T - fractions, ...



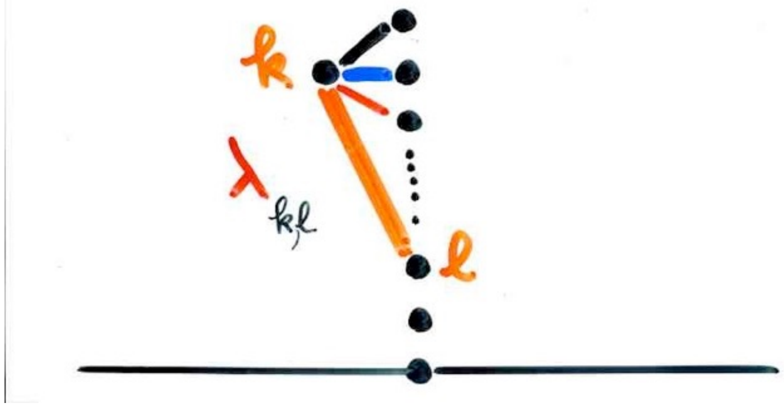
d-orthogonality

Maroni

Lukasiewicz paths

$P_k(x) = x^k + \dots$

any sequence



Padé approximants

C-, P-, L-, ; T- continued fractions

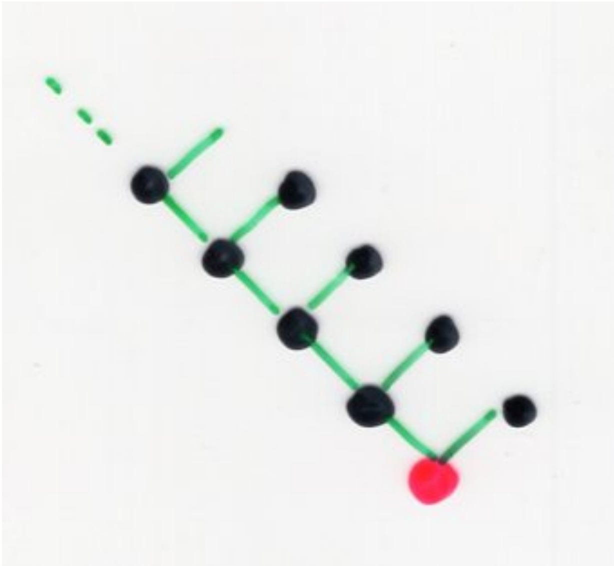
Roblet (1994)

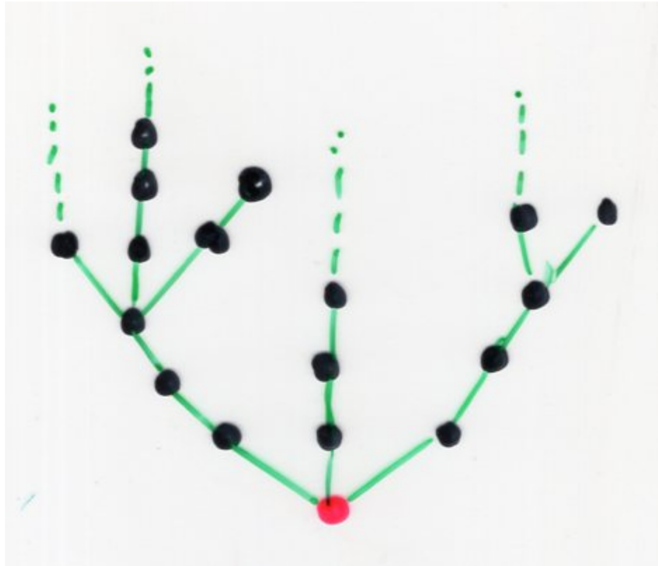
J- continued
S- fraction

T- fraction

Tree-like continued fraction

Jacobi
Stieltjes





W. Jones, W. Thron : (1980, 1984)

continued fractions
analytic theory and applications

The Art of Bijective Combinatorics

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- www.viennot.org

mirror website

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IMSc, Chennai, India

Part I (2016)

Part II (2017)

Part III (2018)

Part IV (2019)