Course IMSc, Chennai, India

January-March 2018

The cellular ansatz:
bijective combinatorics and quadratic algebra

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Chapter 4
Trees and tableaux
Ch4b

The Loday-Ronco algebra of binary trees

IMSc, Chennai
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3 graded Hopf algebras:
Malvenuto-Reutenauer,
Loday-Ronco,
Solomon algebra

graded Hopf algebra

- product $*$
- coproduct Δ
- antipode
- graded $H = \bigoplus_{n \geqslant 1} H_n$

$$H_k \otimes H_\ell \xrightarrow{*} H_{k+\ell}$$

$$Q_n \xleftarrow{c} Y_n \xleftarrow{\Psi} S_n$$

2^{n-1} C_n $n!$

Catalan number

$$\frac{1}{n+1}\binom{2n}{n}$$

permutations

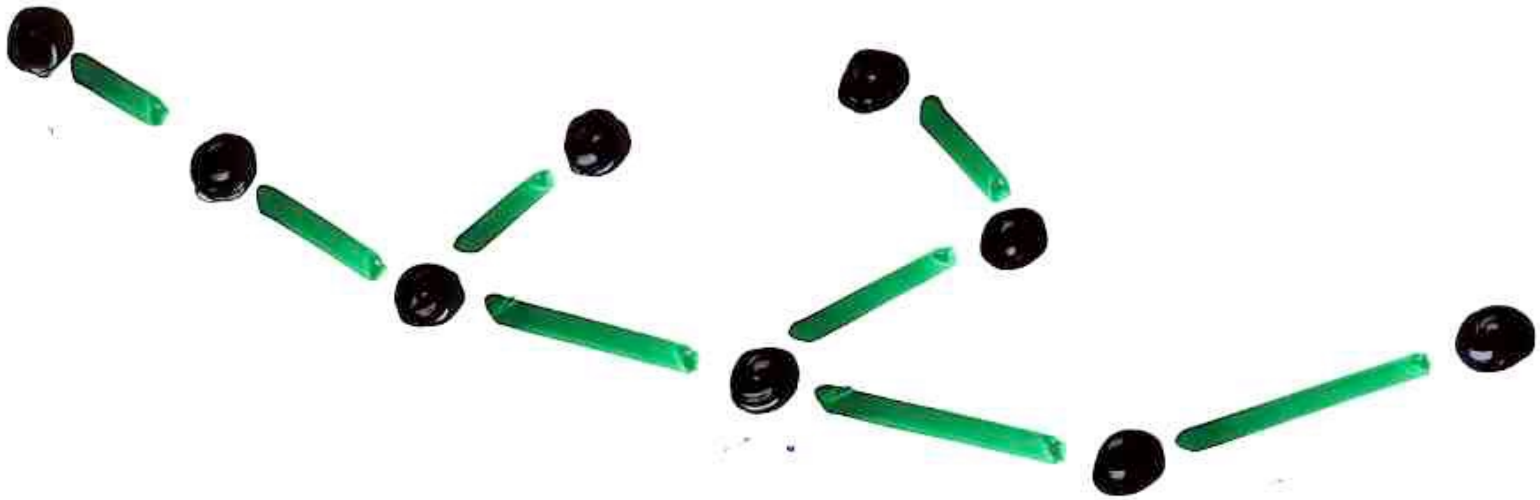
increasing binary trees

canopy

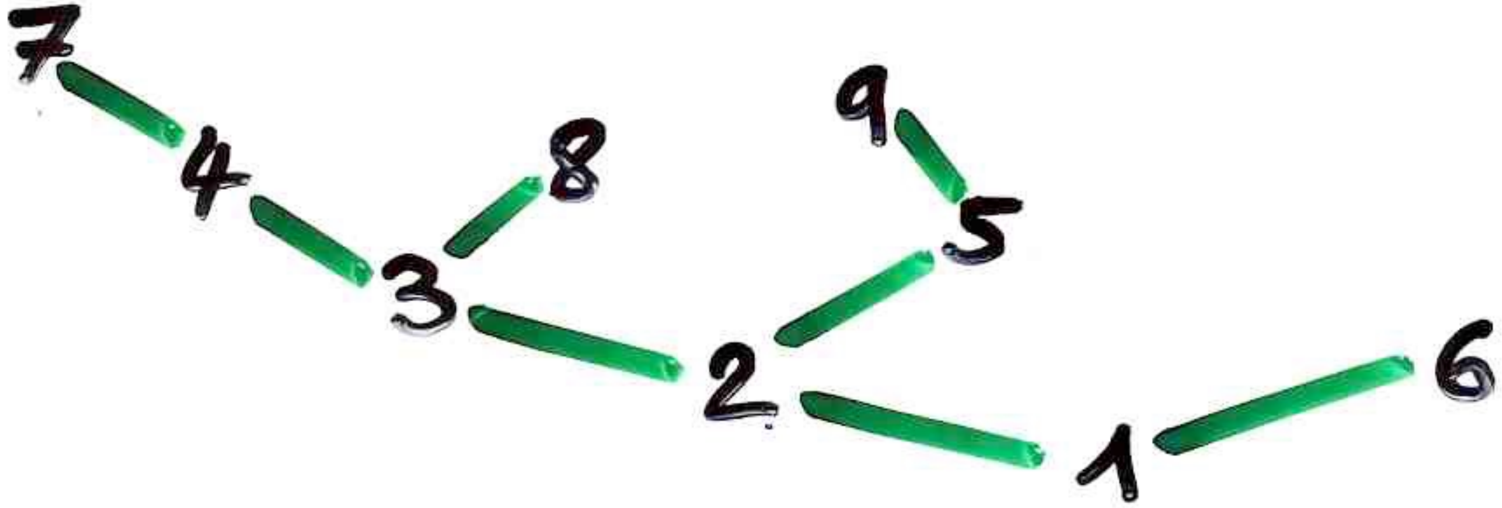
up-down sequence

See BJC I, Ch 4a, 74-94

Definition Increasing binary tree

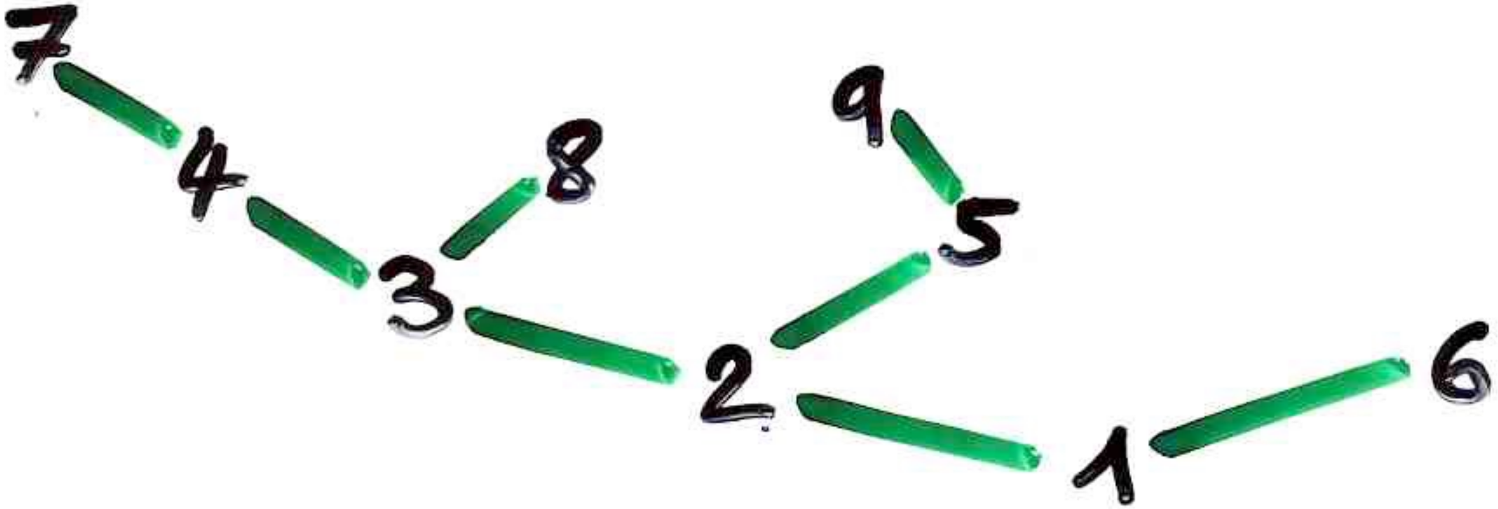


Definition Increasing binary tree



Bijection increasing binary tree ⟷ permutation

T σ



σ = 7 4 3 8 2 9 5 1 6

Bijection   increasing binary tree $\longleftrightarrow$ permutation

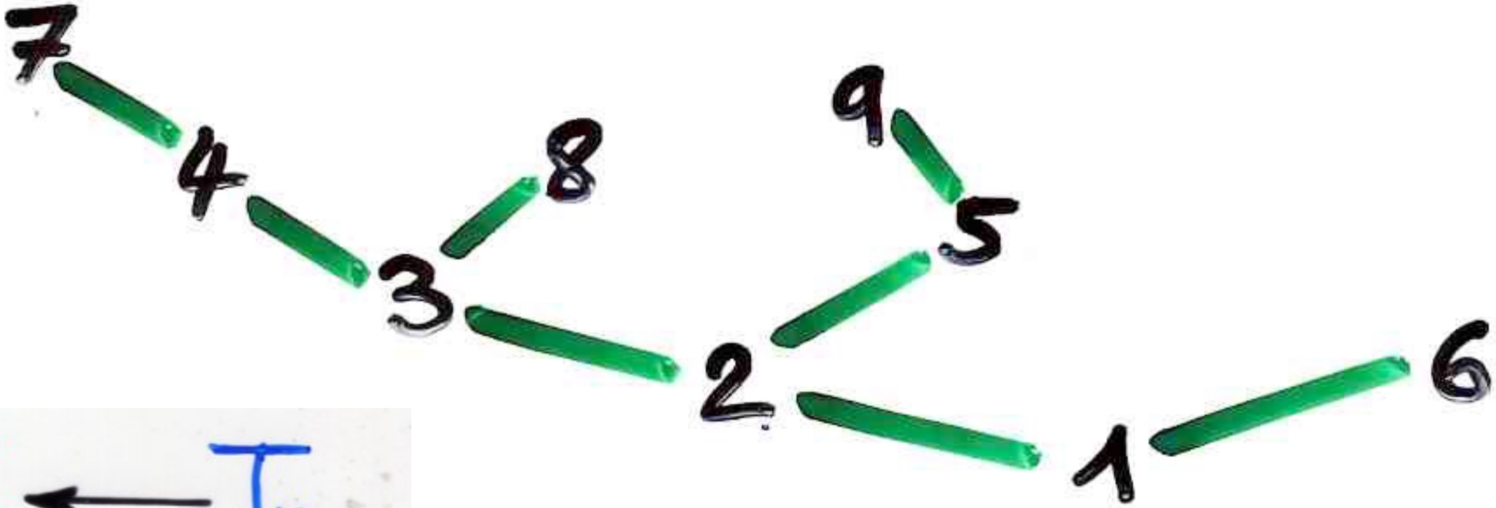
T $\qquad\qquad$ σ

$T \xrightarrow{\pi} \sigma$ $\qquad$ symmetric order of vertices (or "projection")

$\sigma \xrightarrow{\delta} T$ $\qquad$ "déployé"

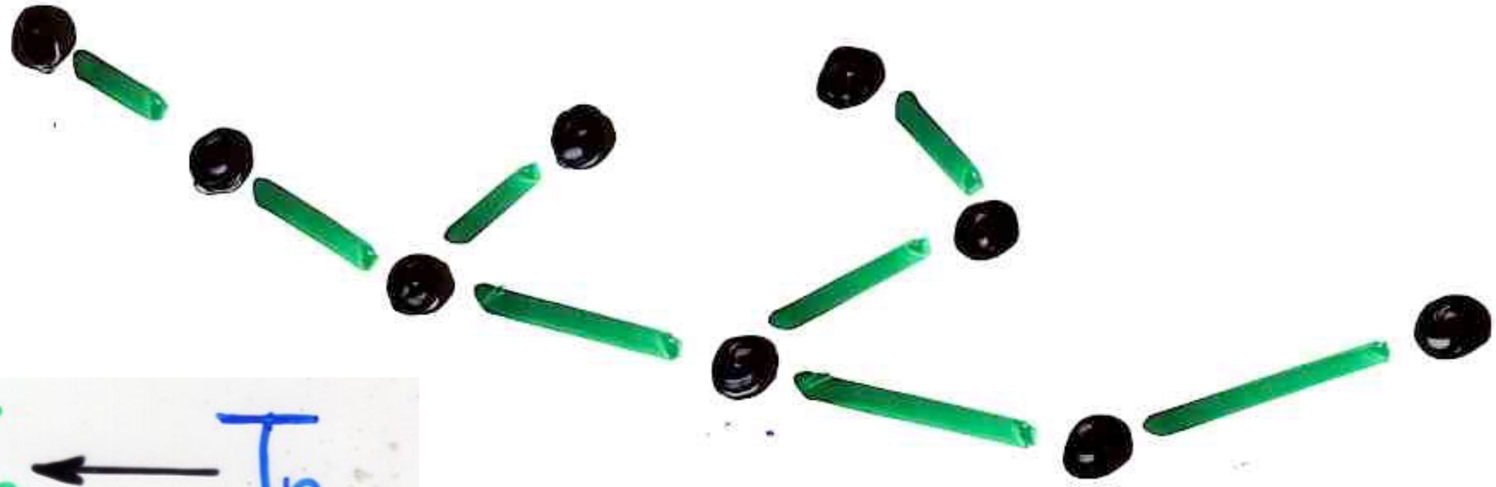
word $w = u\,m\,v$ $\qquad$ m (unique) minimum letter

$\delta(w) =$



$$\begin{tikzcd} Y_n & T_n \arrow[l] \\ & S_n \arrow[u, "\delta"'] \arrow[ul, "\psi"] \end{tikzcd}$$

$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$



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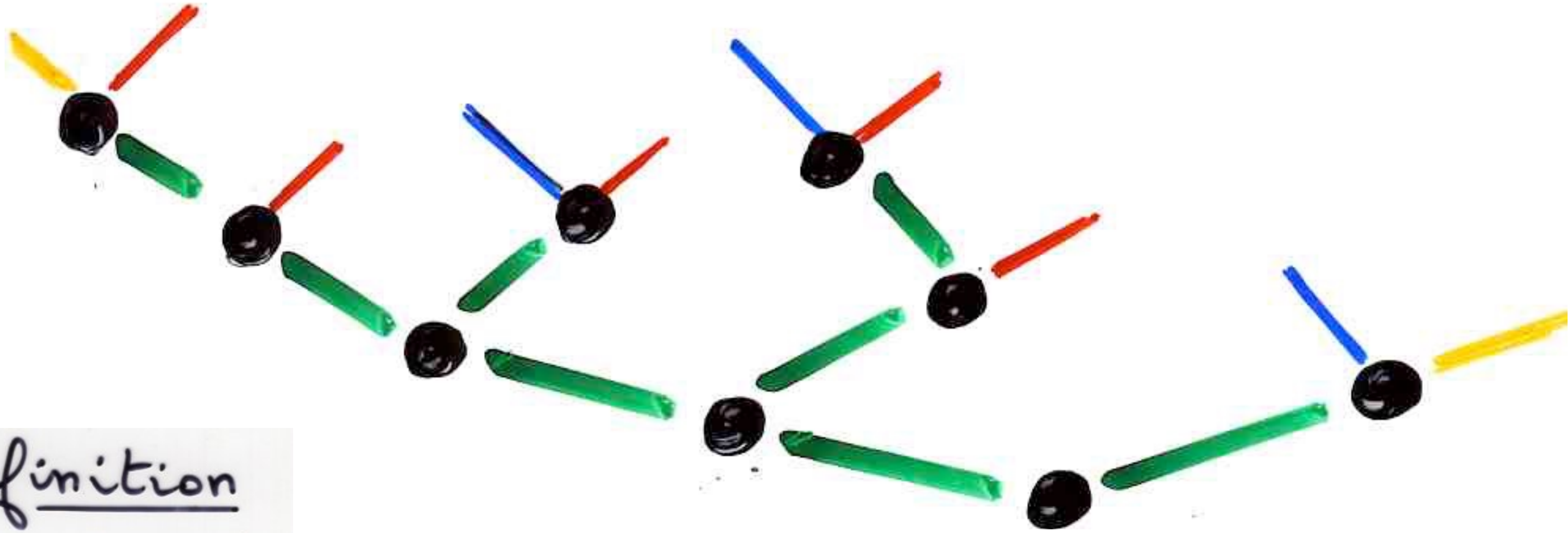


Definition

canopy of a binary tree

$c(B) = / \; / \; \backslash \; / \; \backslash \; / \; / \; \backslash$

$$Q_n \xleftarrow{c} Y_n$$

$$2^{n-1} \qquad C_n$$

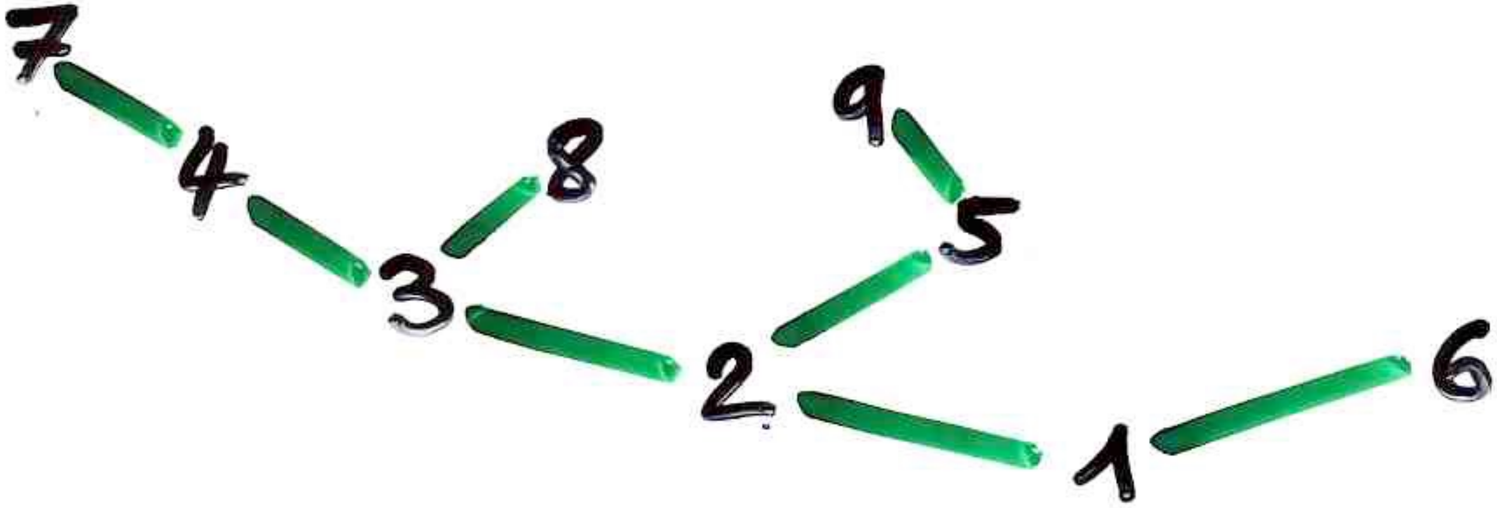
$$\begin{tikzcd}
Q_n & Y_n \arrow[l, "c"'] & S_n \arrow[l, "\Psi"'] \\
 & Y_n & T_n \arrow[l] \\
 & & S_n \arrow[u, "\delta"'] \arrow[ul, "\psi"] \arrow[uull]
\end{tikzcd}$$

Definition

$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ \ldots$

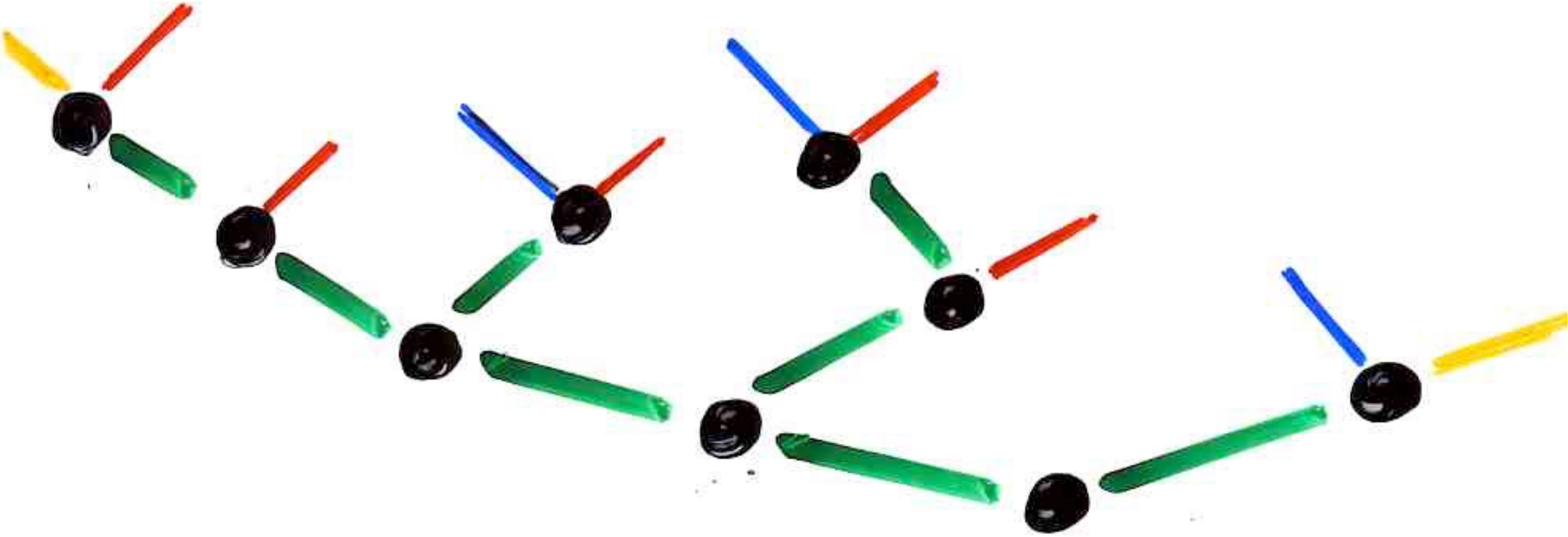
up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$

$$\begin{tikzcd}
\underset{\text{permutation}}{\sigma} \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] & \begin{array}{l}\text{increasing}\\ \text{binary}\\ \text{tree}\end{array} \\
 & B \arrow[d] & \text{binary tree} \\
\underset{\text{up-down sequence}}{UD(\sigma)} \arrow[r, phantom, "="] & \underset{\text{canopy}}{c(B)} &
\end{tikzcd}$$



σ = 7 4 3 8 2 9 5 1 6 ...

up-down sequence − − + − + − − +



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6\ \ldots$

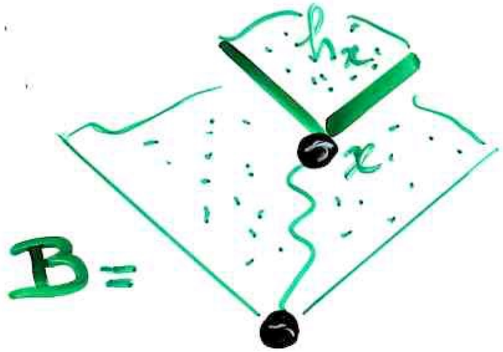
up-down sequence $-\ -\ +\ -\ +\ -\ -\ +$

"hook-length formula"

$$\frac{n!}{\prod_{x} h_x}$$

n nb of vertices

product of size of sub-trees

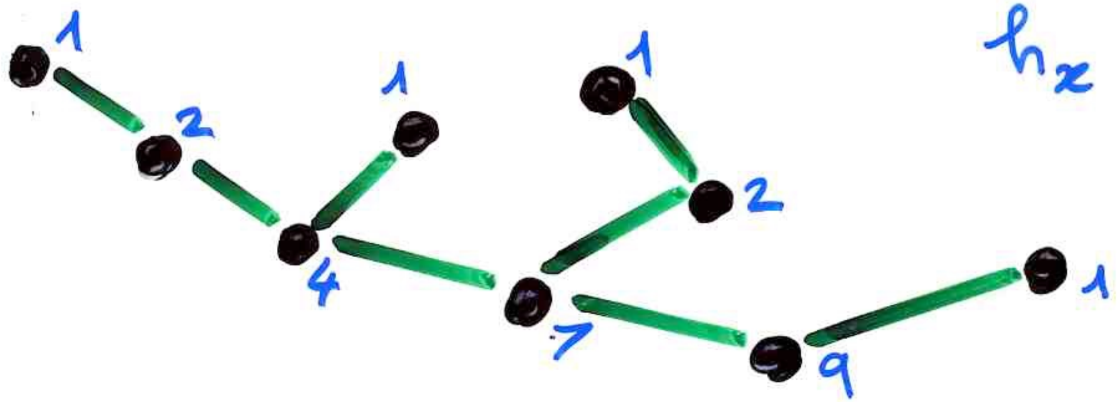


nb of increasing binary tree

for a binary tree B

example

"hook-length"
h_x



$$\frac{9!}{2^2 . 4 . 7 . 9} = 360$$

Hopf algebras:
Malvenuto-Reutenauer,
Loday-Ronco

graded Hopf algebra

- product $*$
- coproduct Δ
- antipod
- graded $H = \bigoplus_{n \geqslant 1} H_n$

$$H_k \otimes H_\ell \xrightarrow{*} H_{k+\ell}$$

Malvenuto-Reutenauer
Hopf algebra permutations

$\mathbb{K}[S_n]$ group algebra of S_n
symmetric group

$$\mathbb{K}[S_\infty] = \bigoplus_{n \geqslant 1} \mathbb{K}[S_n]$$

$$\begin{tikzcd} Y_n & S_n \arrow[l, "\psi"'] \end{tikzcd}$$

$$\begin{tikzcd} Y_n & T_n \arrow[l] \\ & S_n \arrow[ul, "\psi"] \arrow[u, "\delta"'] \end{tikzcd}$$

Definition

$$Y_n \xrightarrow{\psi^*} \mathbb{K}[S_n]$$

$$\psi^*(B) = \sum_{\substack{\sigma \in S_n \\ \psi(\sigma) = B}} \sigma$$

$$\mathbb{K}[Y_\infty] \xrightarrow{\psi^*} \mathbb{K}[S_\infty]$$

Standardisation

9	4	11	7	10
3	1	5	2	4

4	7	9	10	11
1	2	3	4	5

$$\mathrm{Std}\,(9\ \ 4\ \ 11\ \ 7\ \ 10) = 3\ \ 1\ \ 5\ \ 2\ \ 4$$

Definition

$\alpha \in S_k\ ,\ \beta \in S_\ell\ ,\quad n = k+\ell$

Malvenuto-Reutenauer product $*$ in $\mathbb{K}[S_\infty]$

$$\alpha * \beta = \sum_{\substack{u,v = \{1,..,n\} \\ \mathrm{Std}(u)=\alpha\ ,\ \mathrm{Std}(v)=\beta}} uv$$

example

$$12 * 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$$

Proposition The image of ψ^* is a sub-Hopf algebra of $\mathbb{K}[S_\infty]$.
Thus $\mathbb{K}[Y_\infty]$ inherits a structure of Hopf algebra.

Definition Loday-Ronco product of two binary trees B and B'

$$B * B' = (\psi^*)^{-1}\left(\psi^*(B) * \psi^*(B')\right)$$

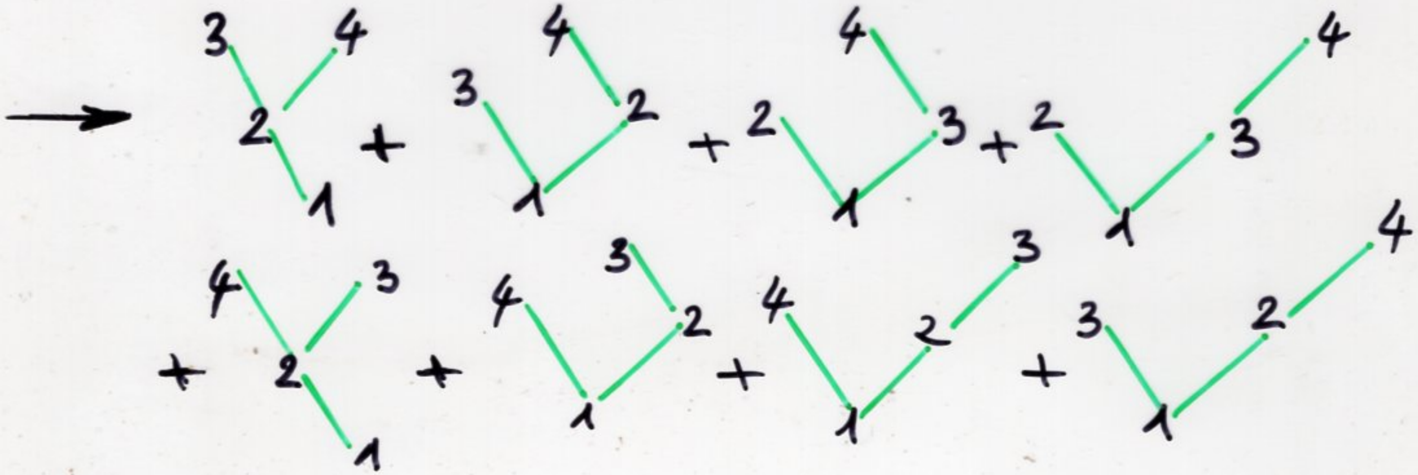
$$\psi^*(B * B') = \left(\psi^*(B) * \psi^*(B')\right)$$

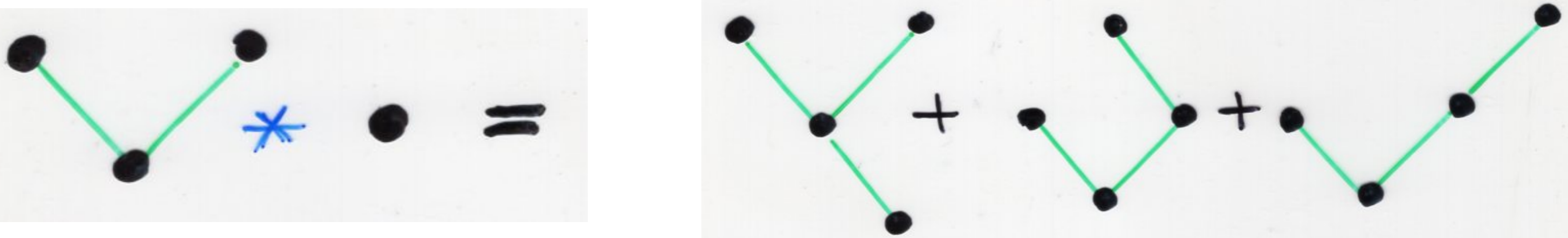
sum of binary trees

example

$(213 + 312) * (1)$

$$= (3241 + 3142 + 2143 + 2134) + (4231 + 4132 + 4123 + 3124)$$

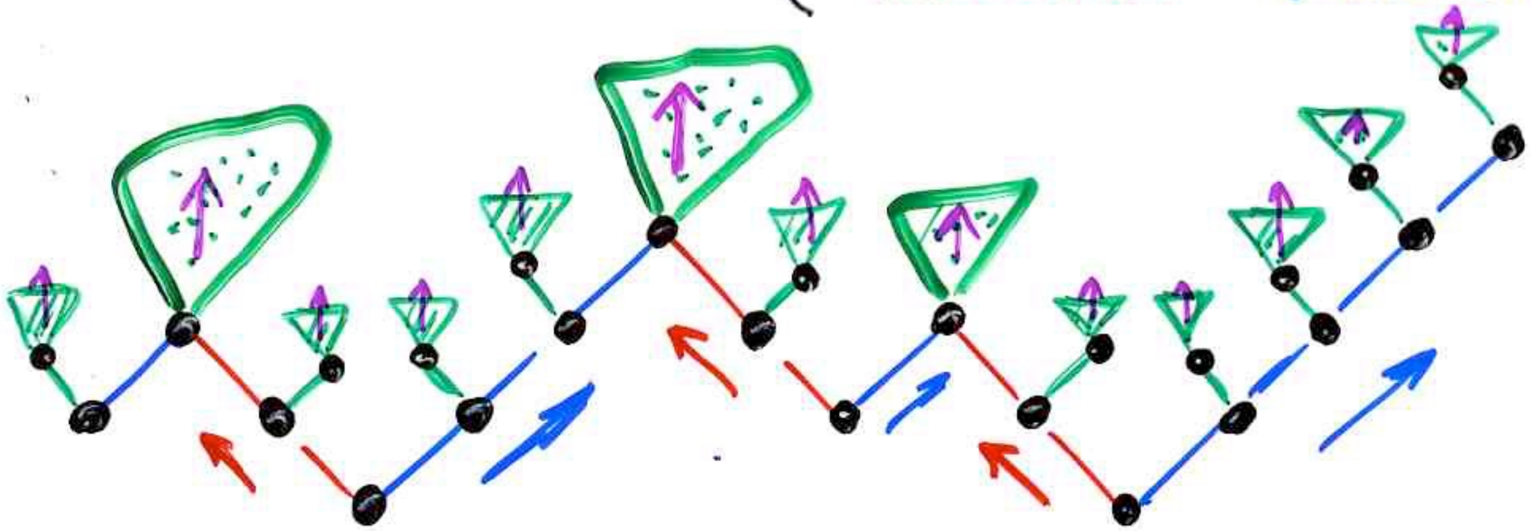


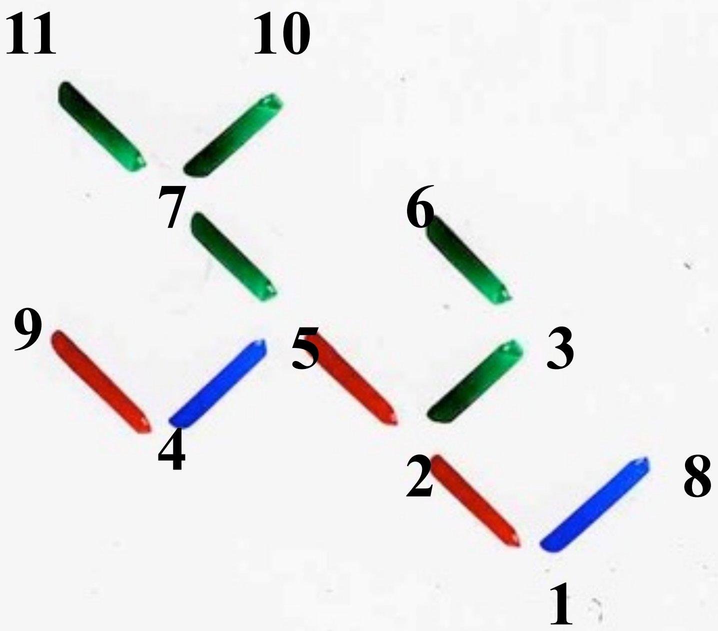


jeu de taquin
for

increasing binary trees

Definition Increasing woods ("buissons" croissants)

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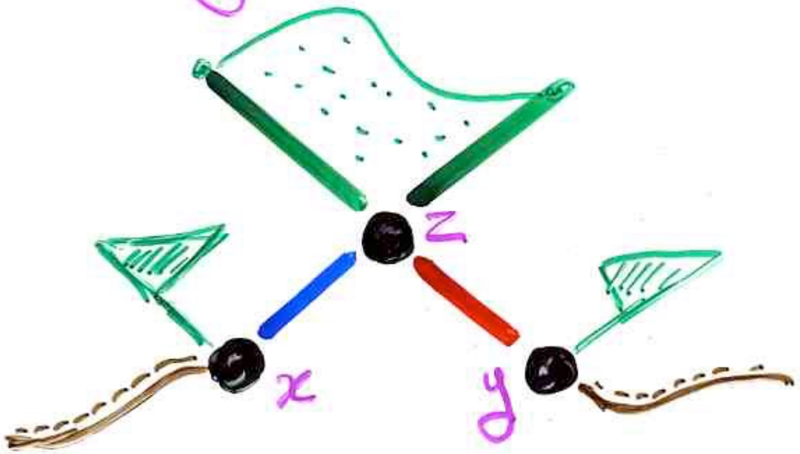
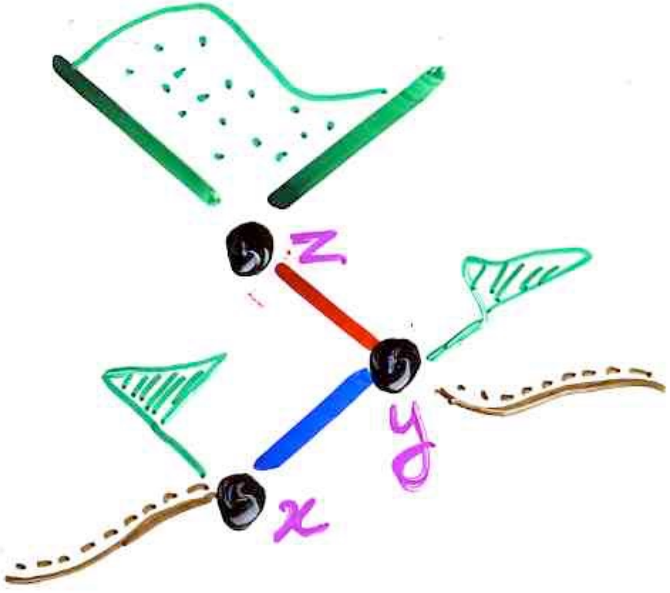
Definition Sliding in an increasing woods

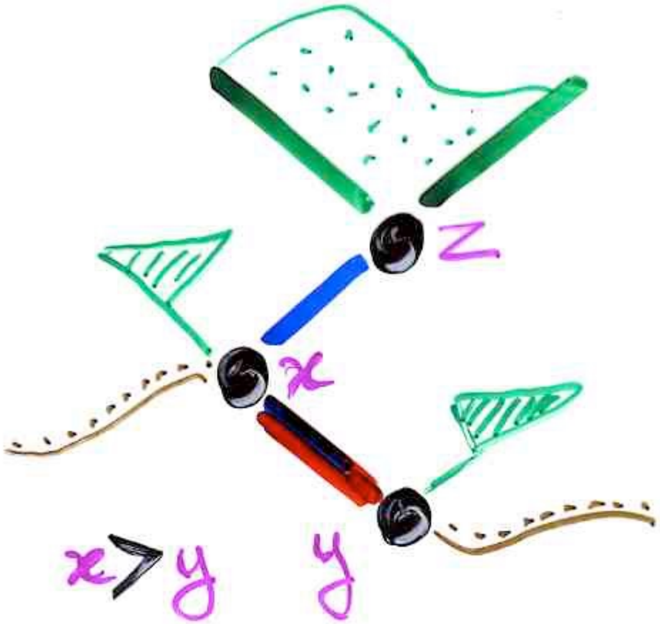


$z > x$

$z > y$

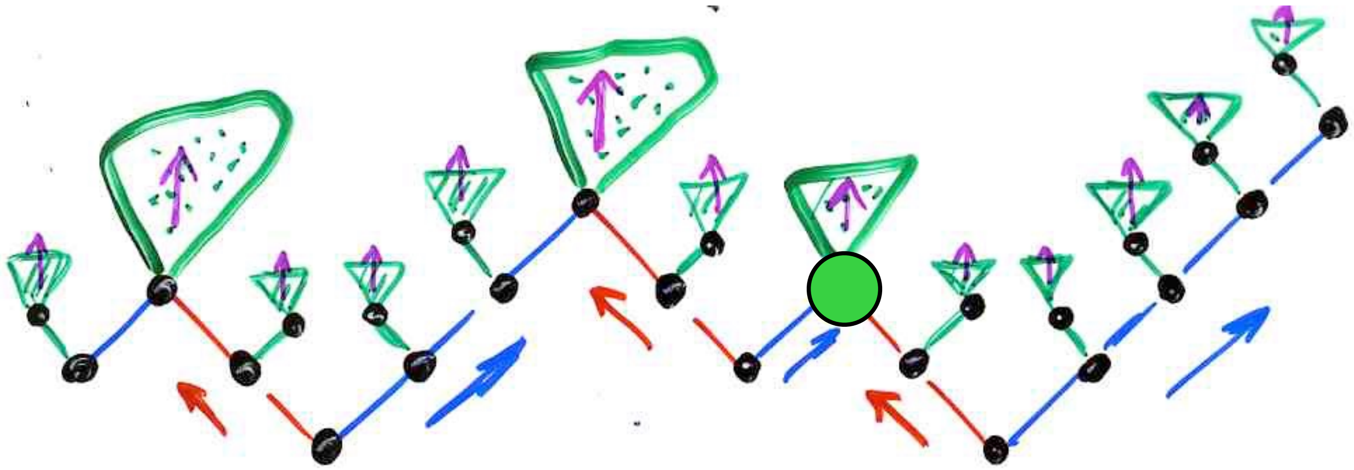


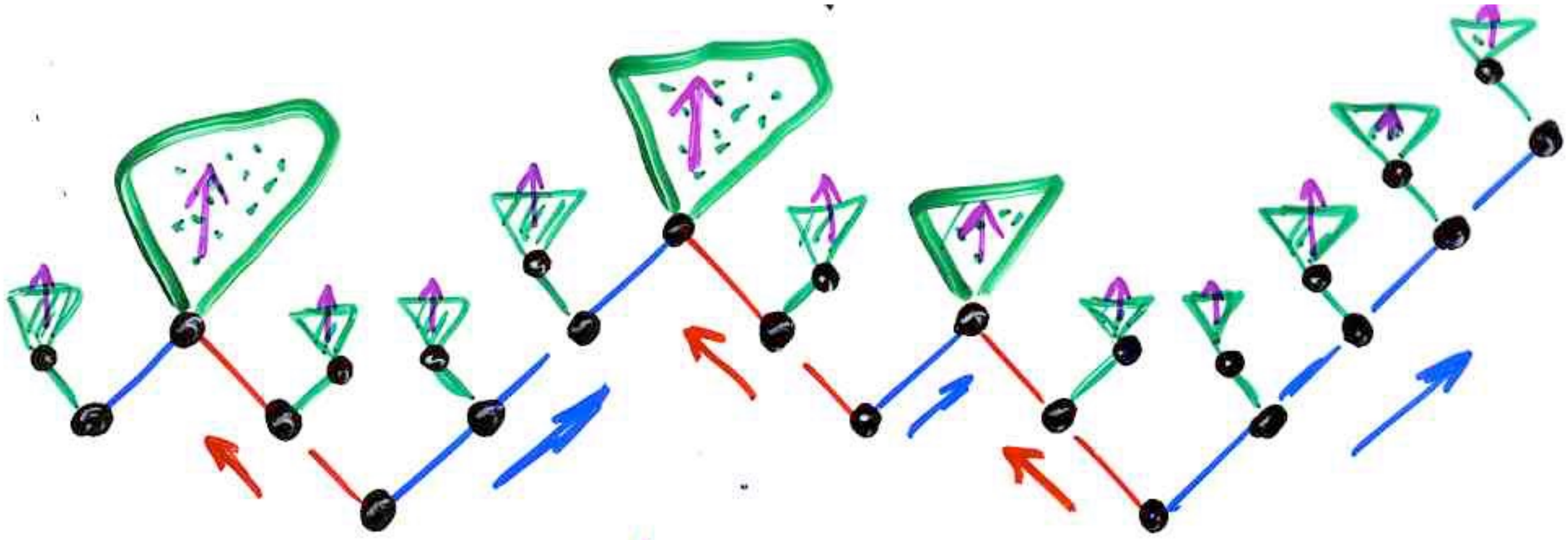
$x < y$



$x > y$

"jeu de taquin" for increasing woods





symmetric order for woods

Lemma invariance of the symmetric order through slidings of an increasing woods

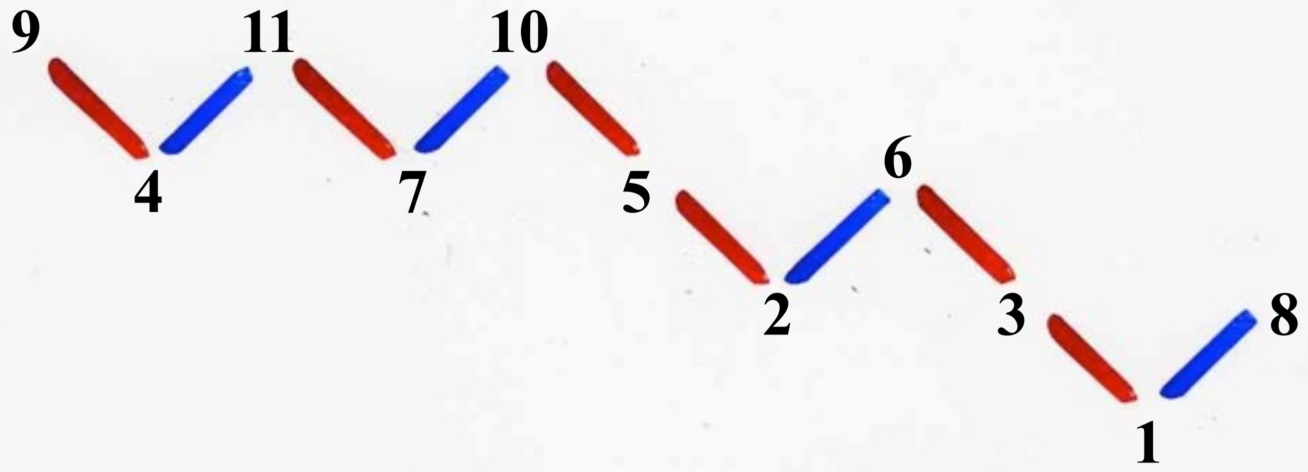
Corollary

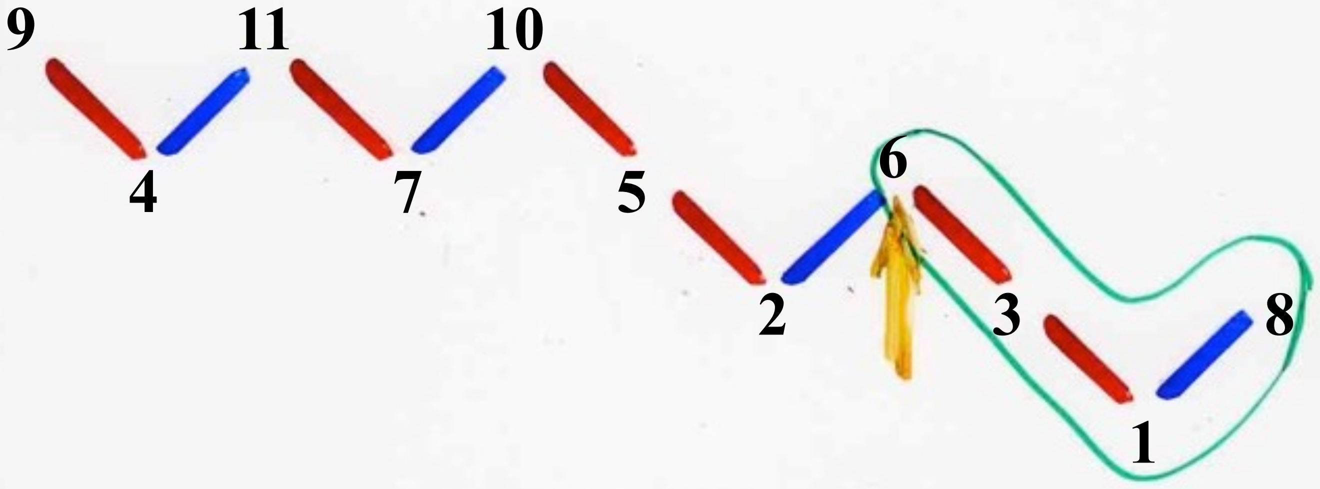
$\sigma \rightarrow$ UD-wood

$\downarrow$ "jeu de taquin"

$T(\sigma) = \delta(\sigma)$ déployé

increasing binary tree

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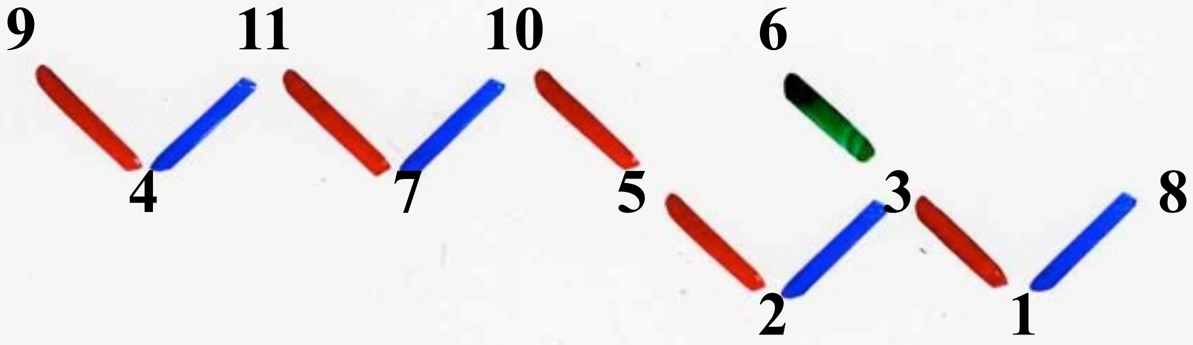
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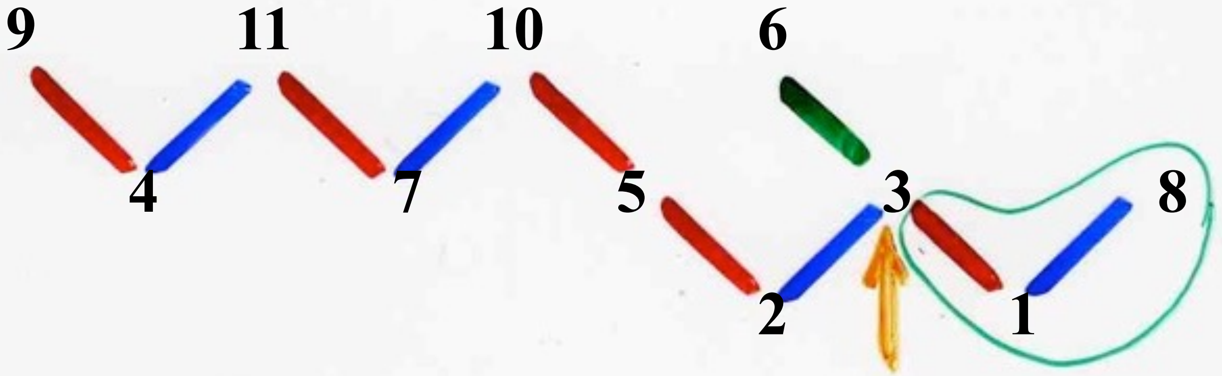
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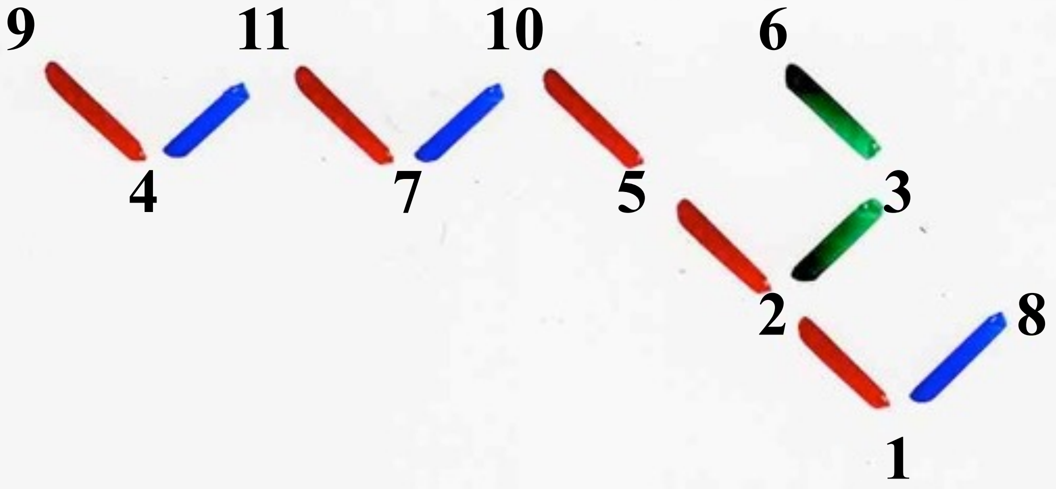
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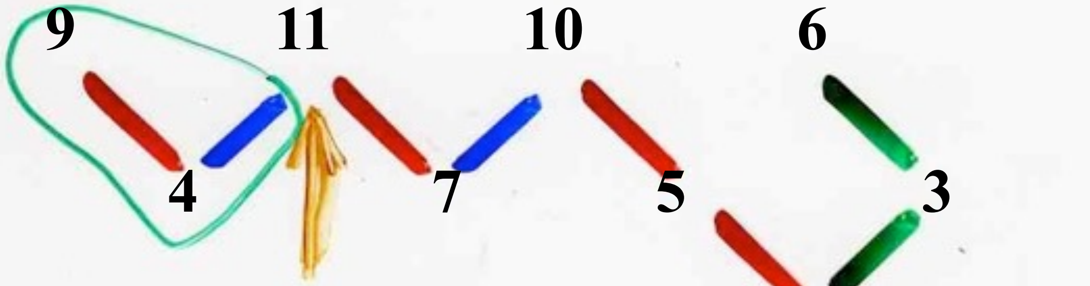
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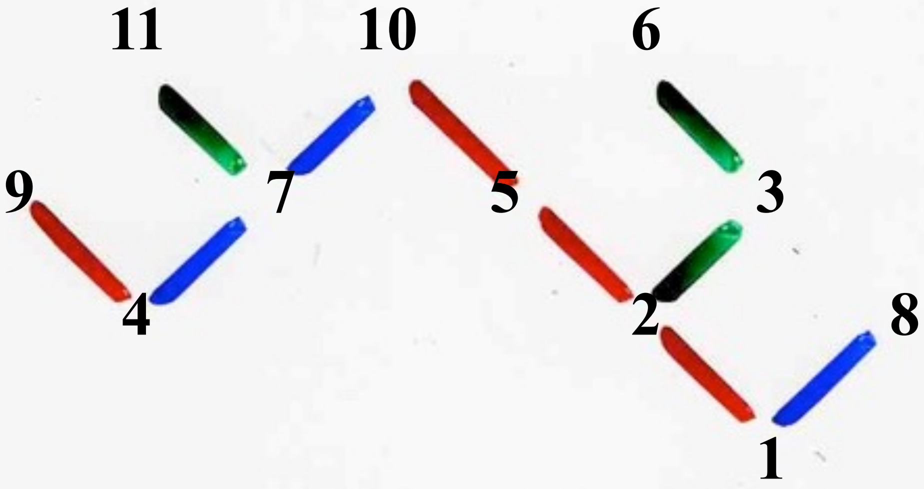
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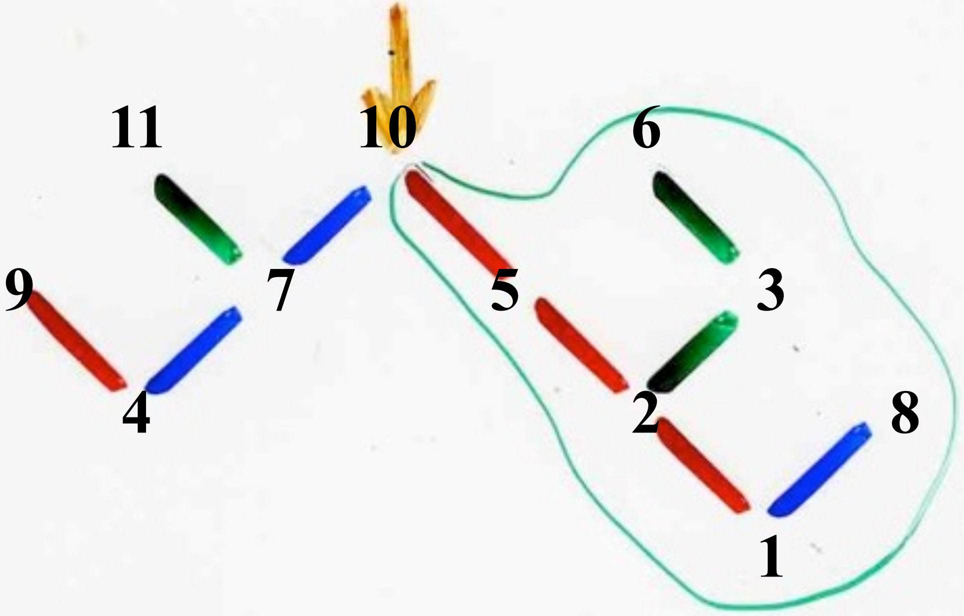
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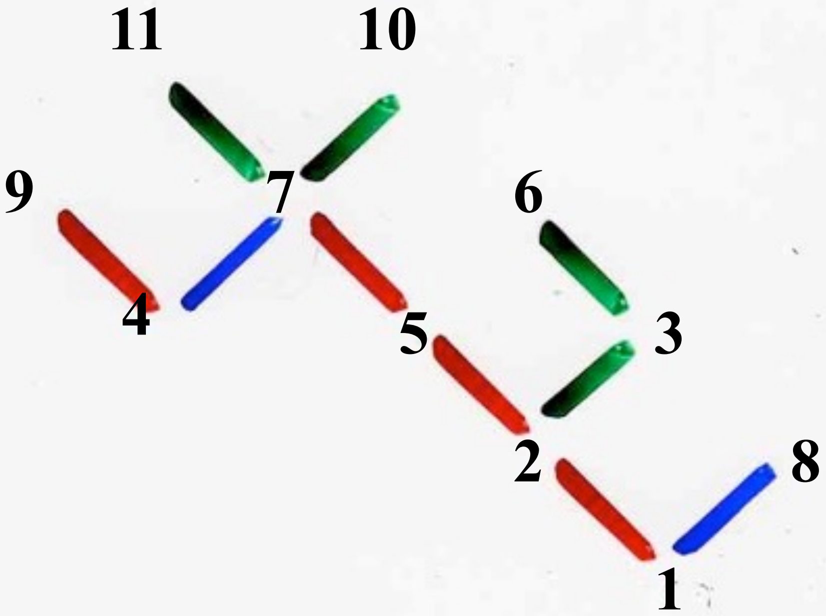
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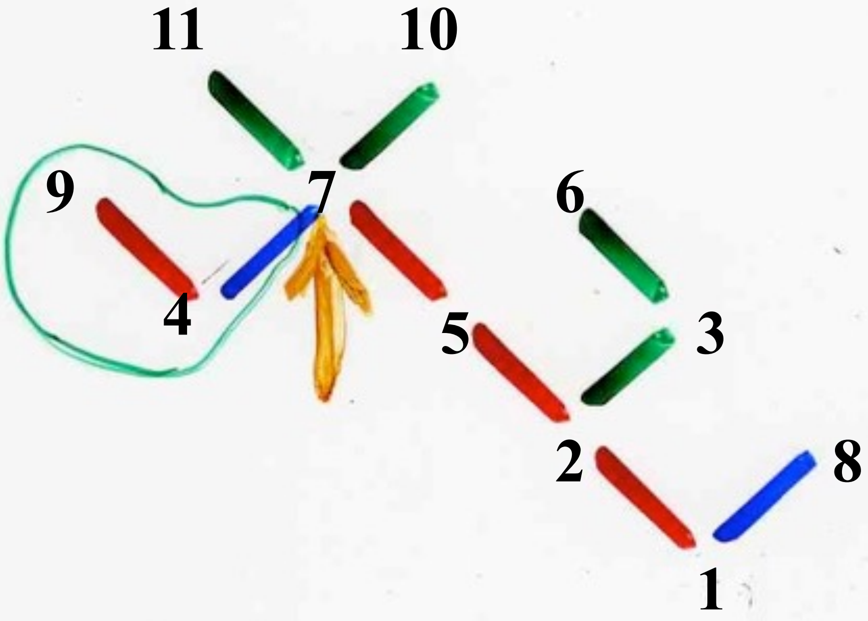
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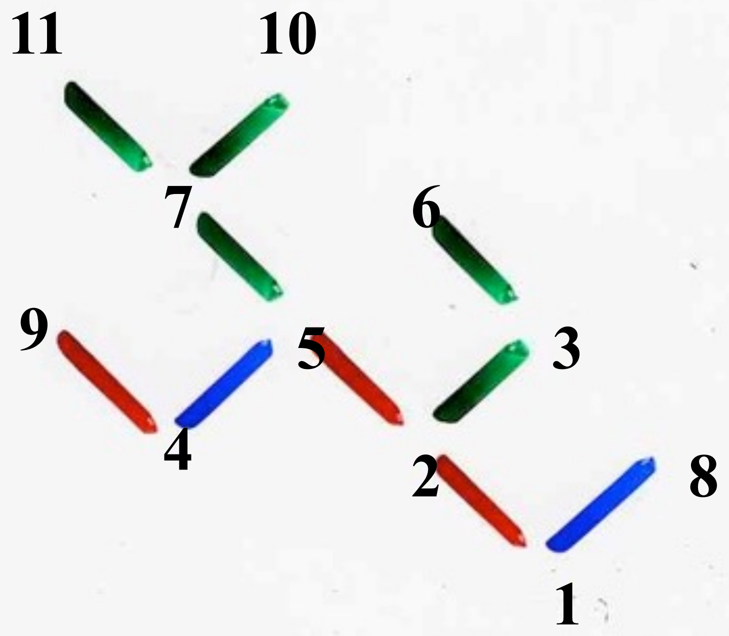
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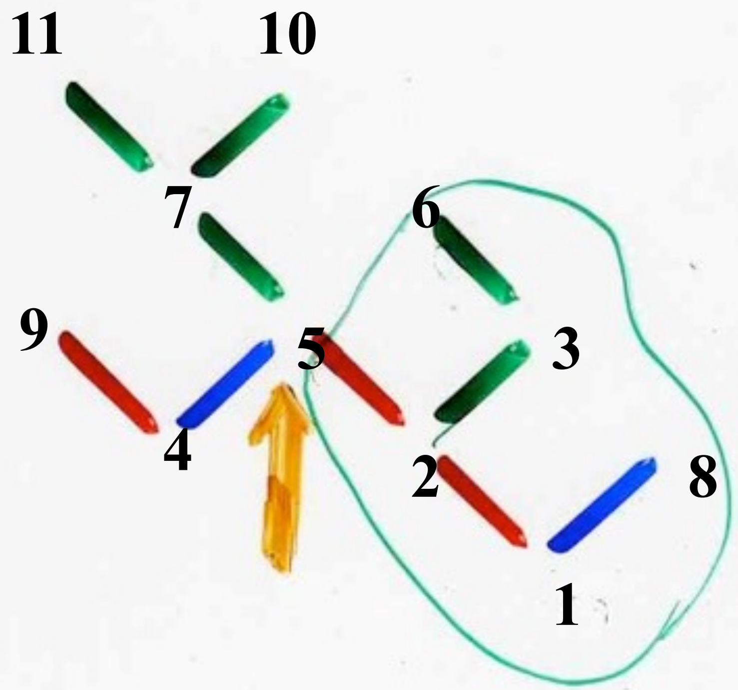
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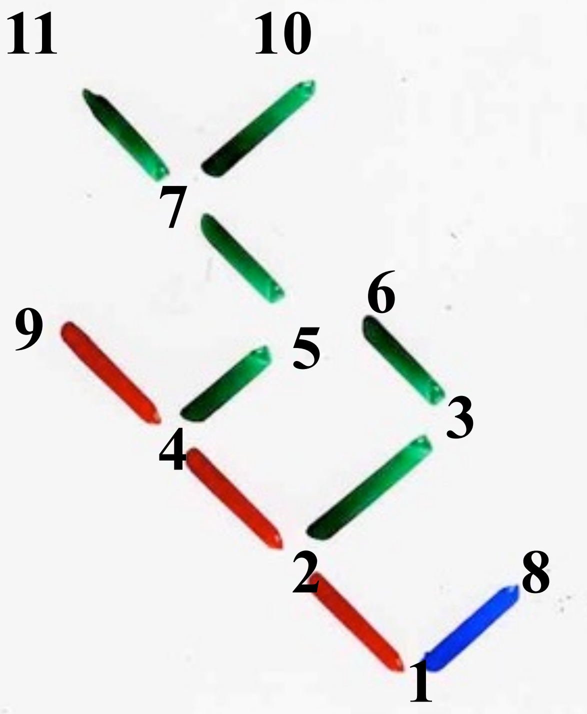
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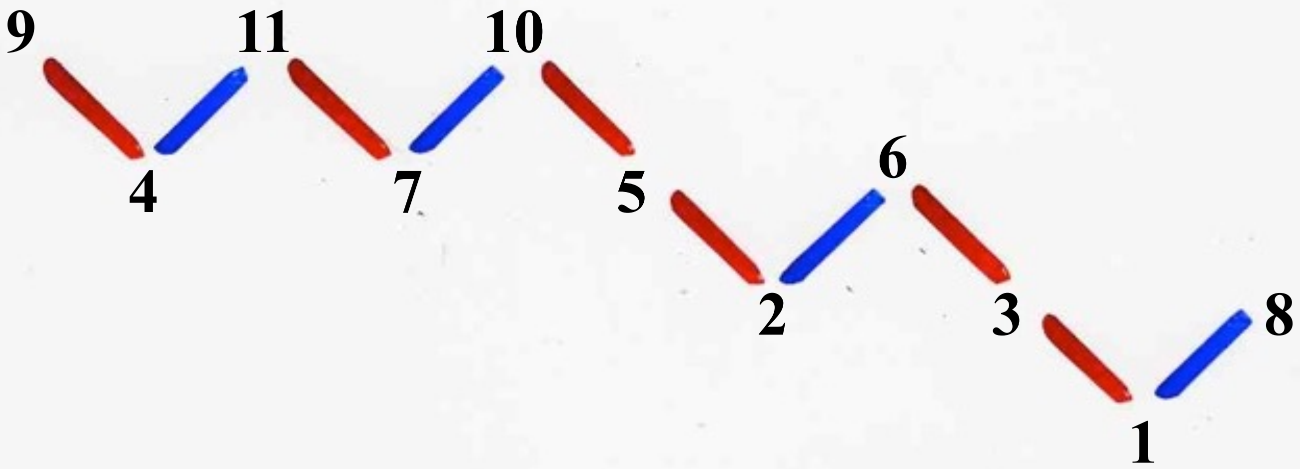
behind this "jeu de taquin"
there is a Catalan alternative tableau



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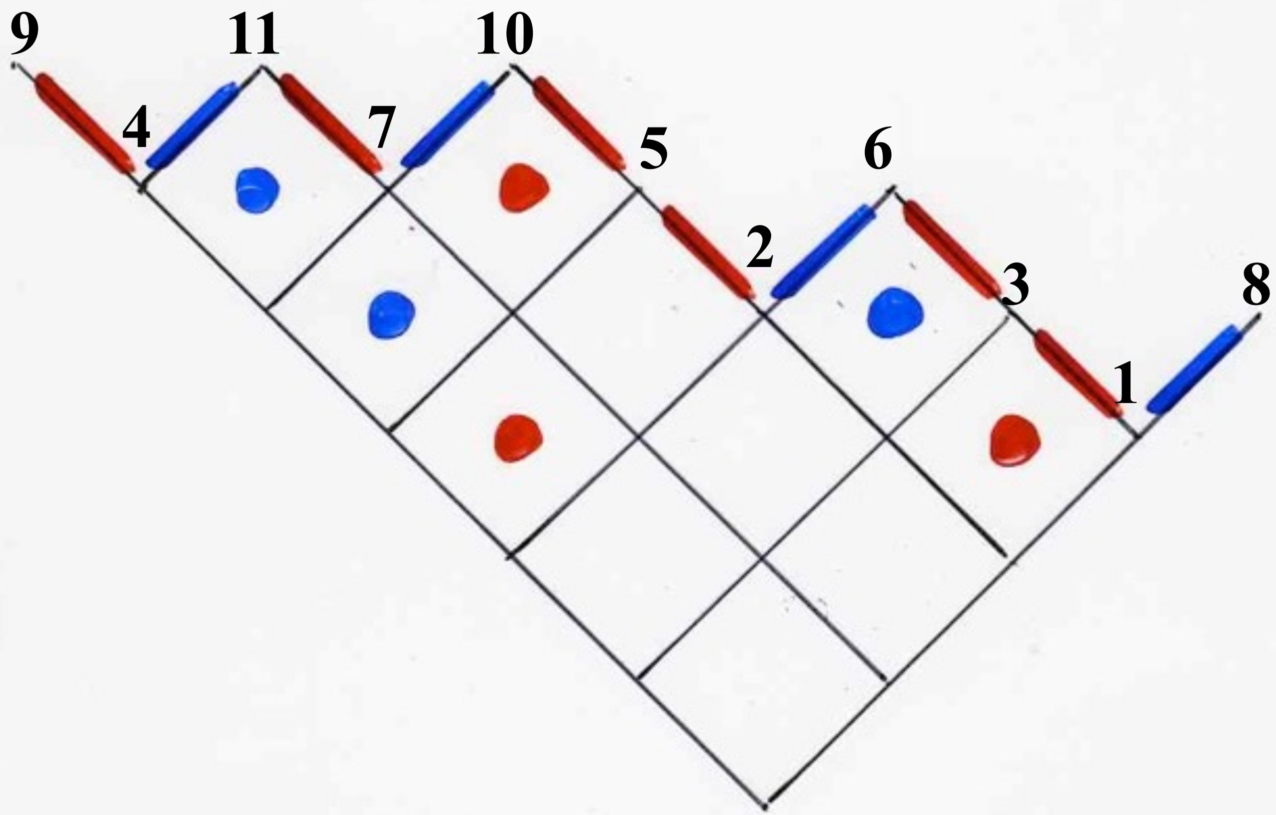


Catalan alternative tableaux

See BJC III, Ch 4a

alternative tableau

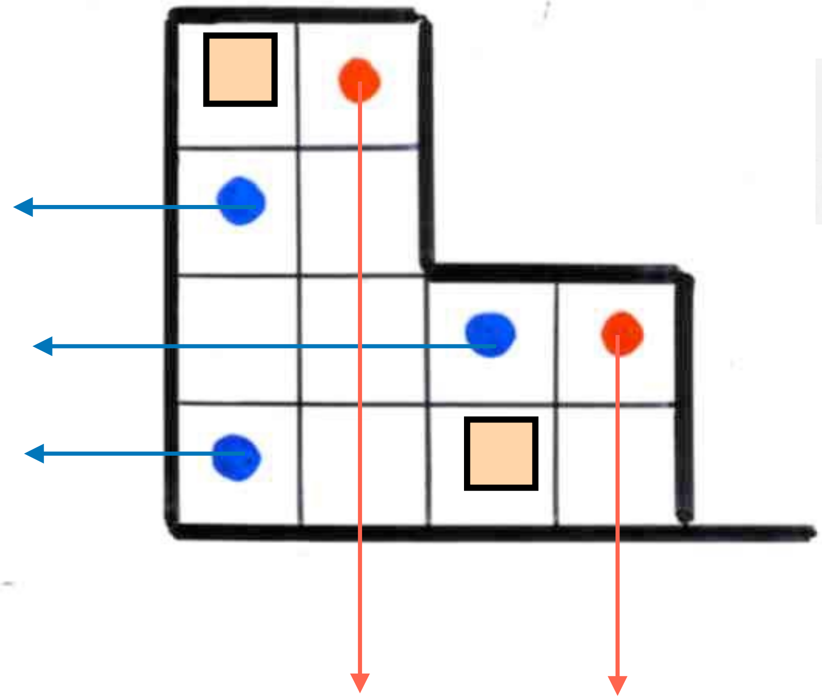
Definition

Ferrers diagram F

with possibly empty rows or columns

size of F

$n = (\text{number of rows}) + (\text{number of columns})$



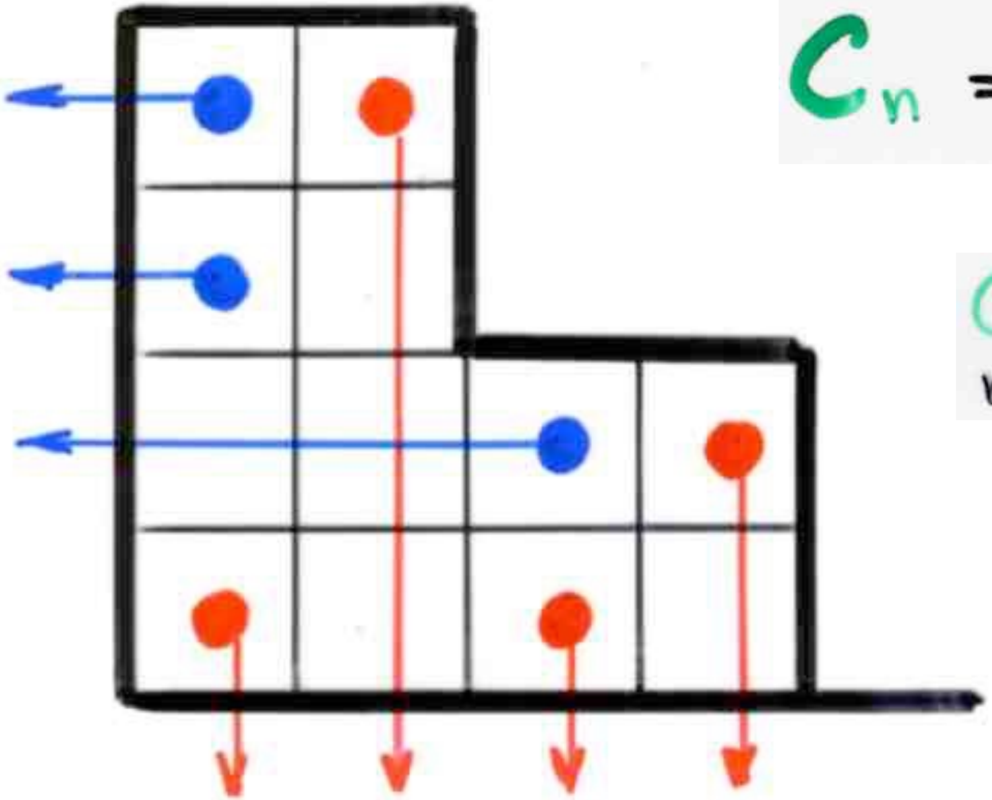
(i) some cells are coloured **red** or **blue**

(ii)
- no coloured cell at the left of a **blue** cell
- no coloured cell below a **red** cell

Def Catalan alternative tableau T

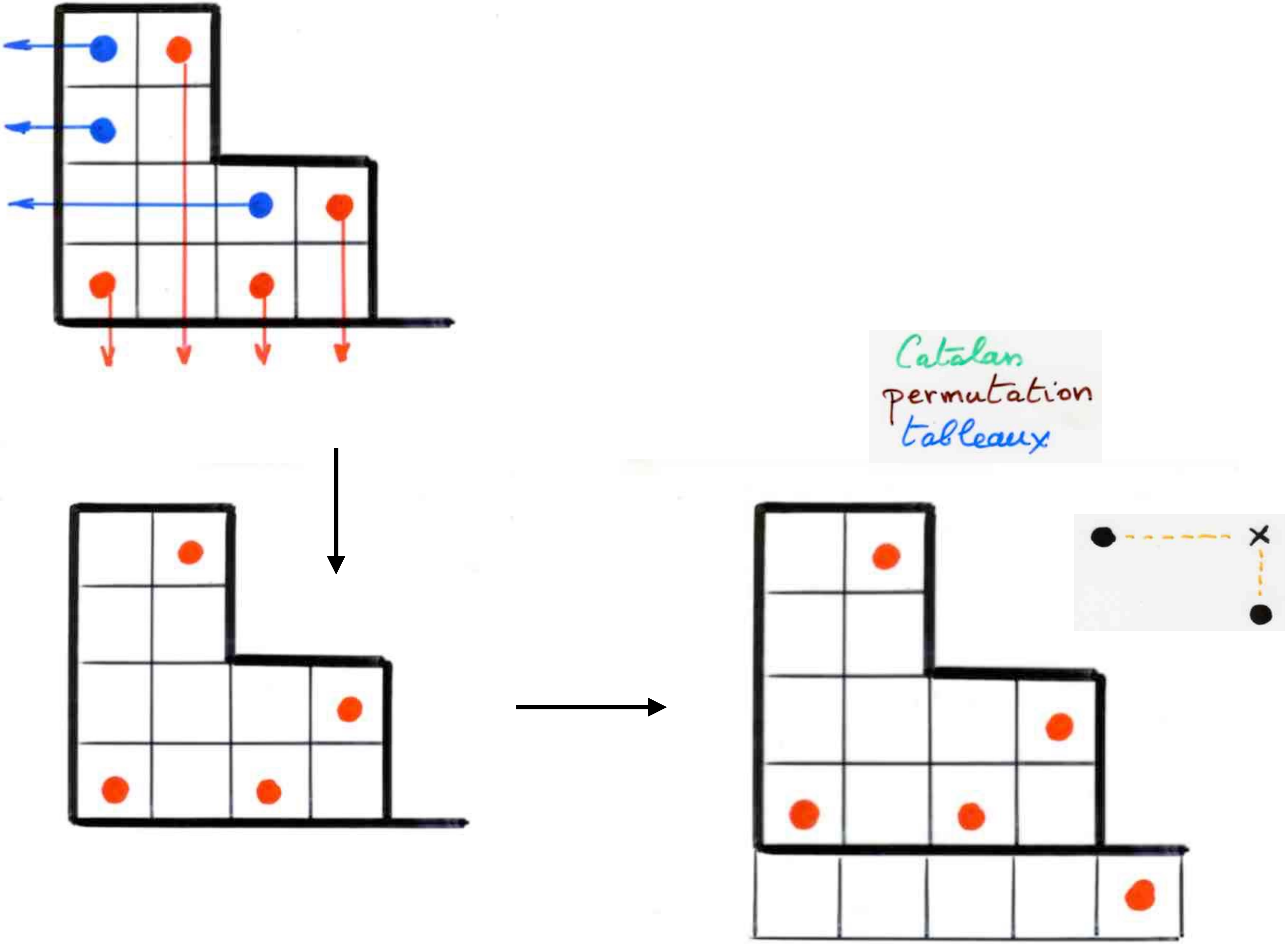
alt. tab. without cells 🟧

i.e: every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

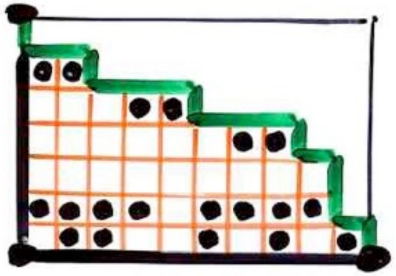
Catalan numbers



Catalan permutation tableaux

Definition Permutation Tableau

Ferrers diagram F $\subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

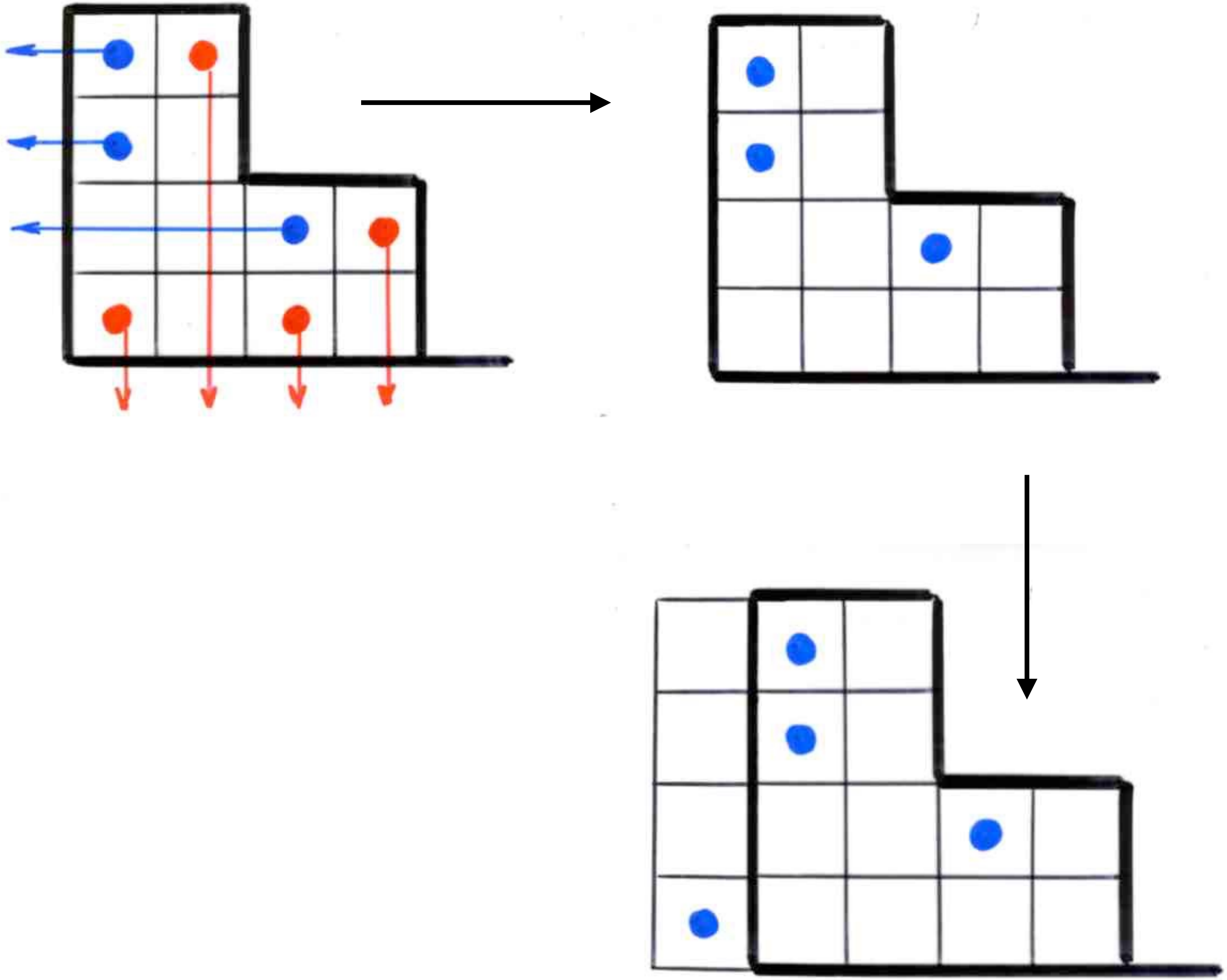
☐ = 0   ⊡ = 1

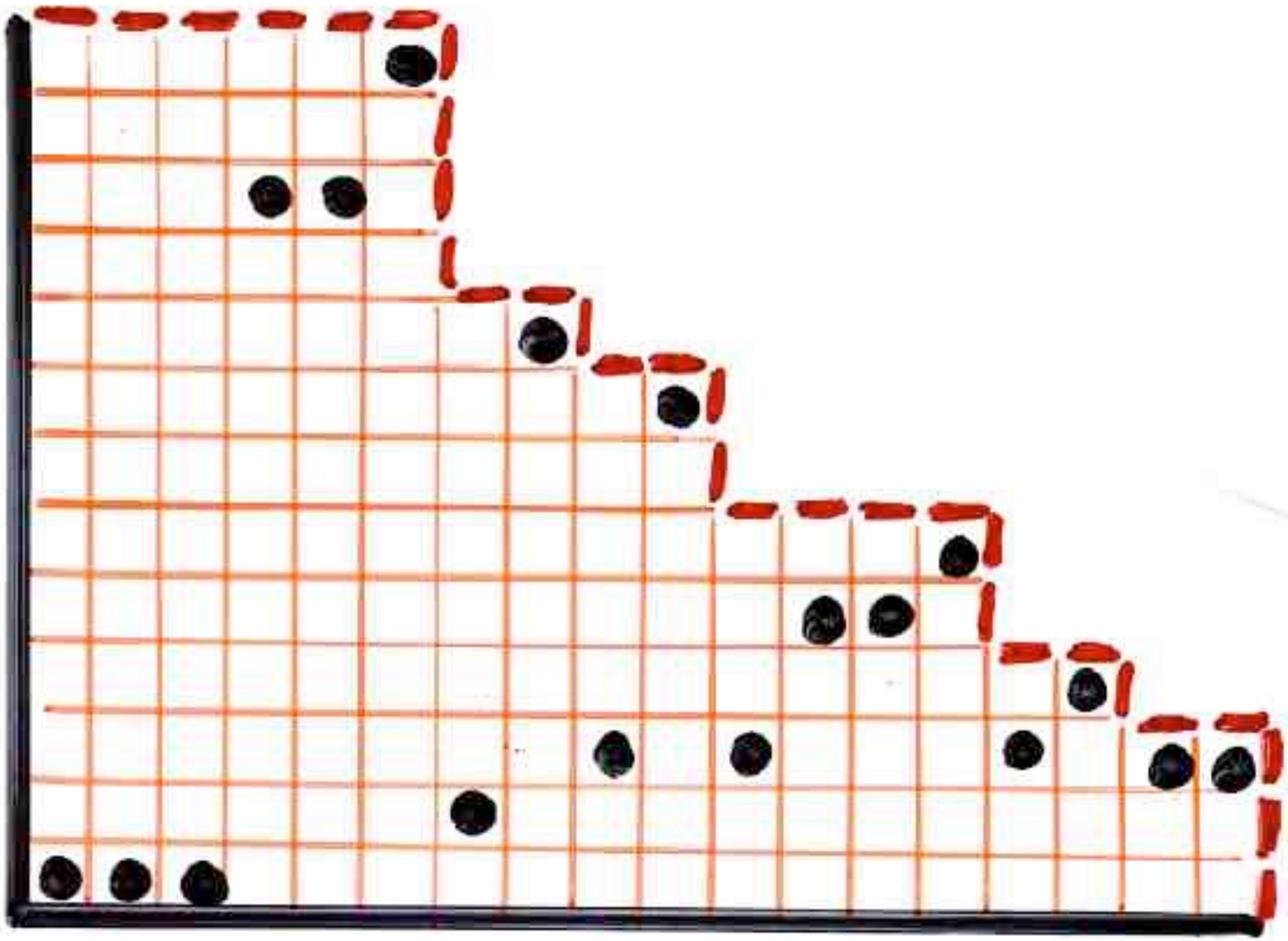
(ii)

forbidden

Definition Catalan permutation tableaux

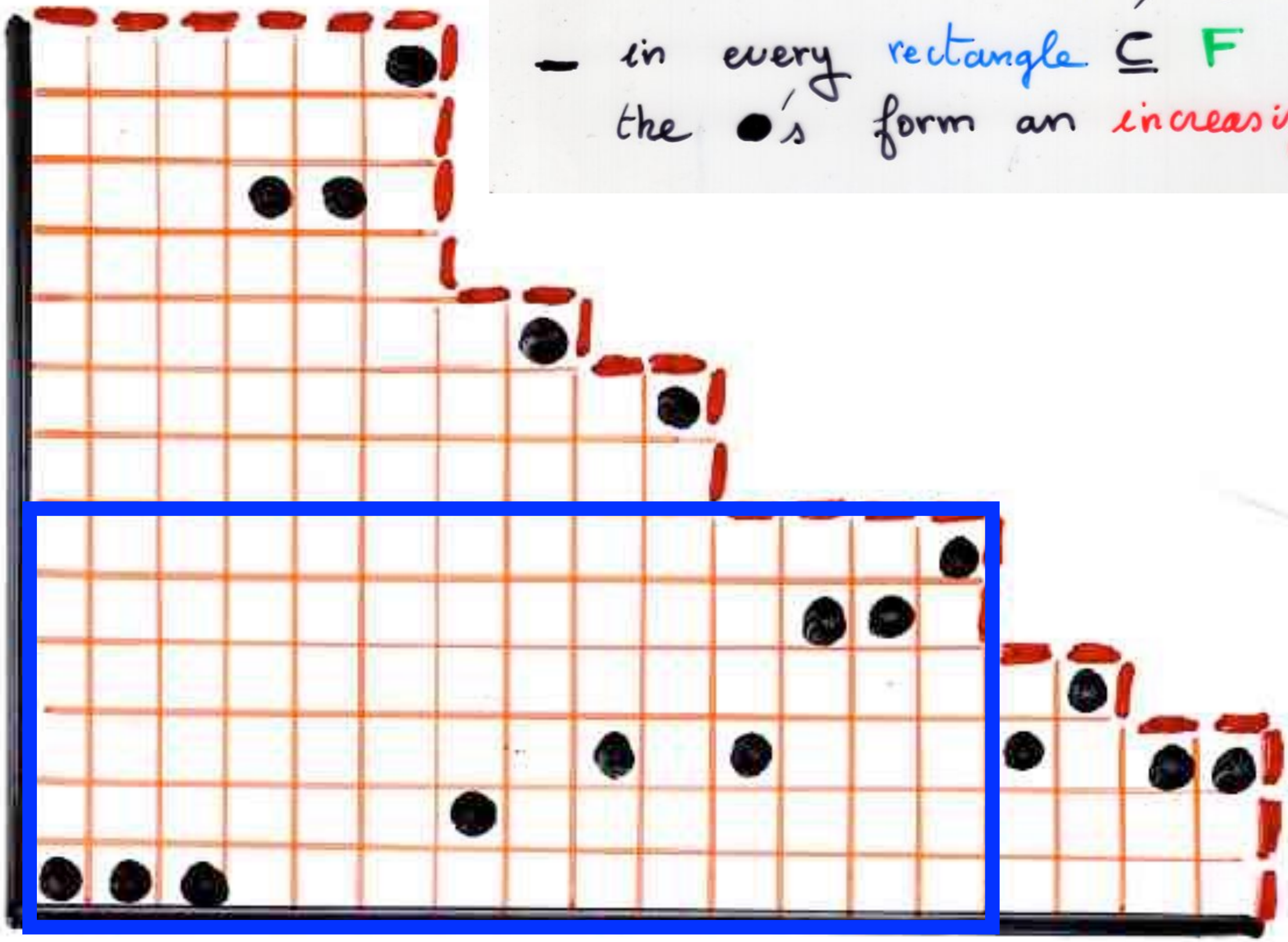
(iii) only one 1 in each column





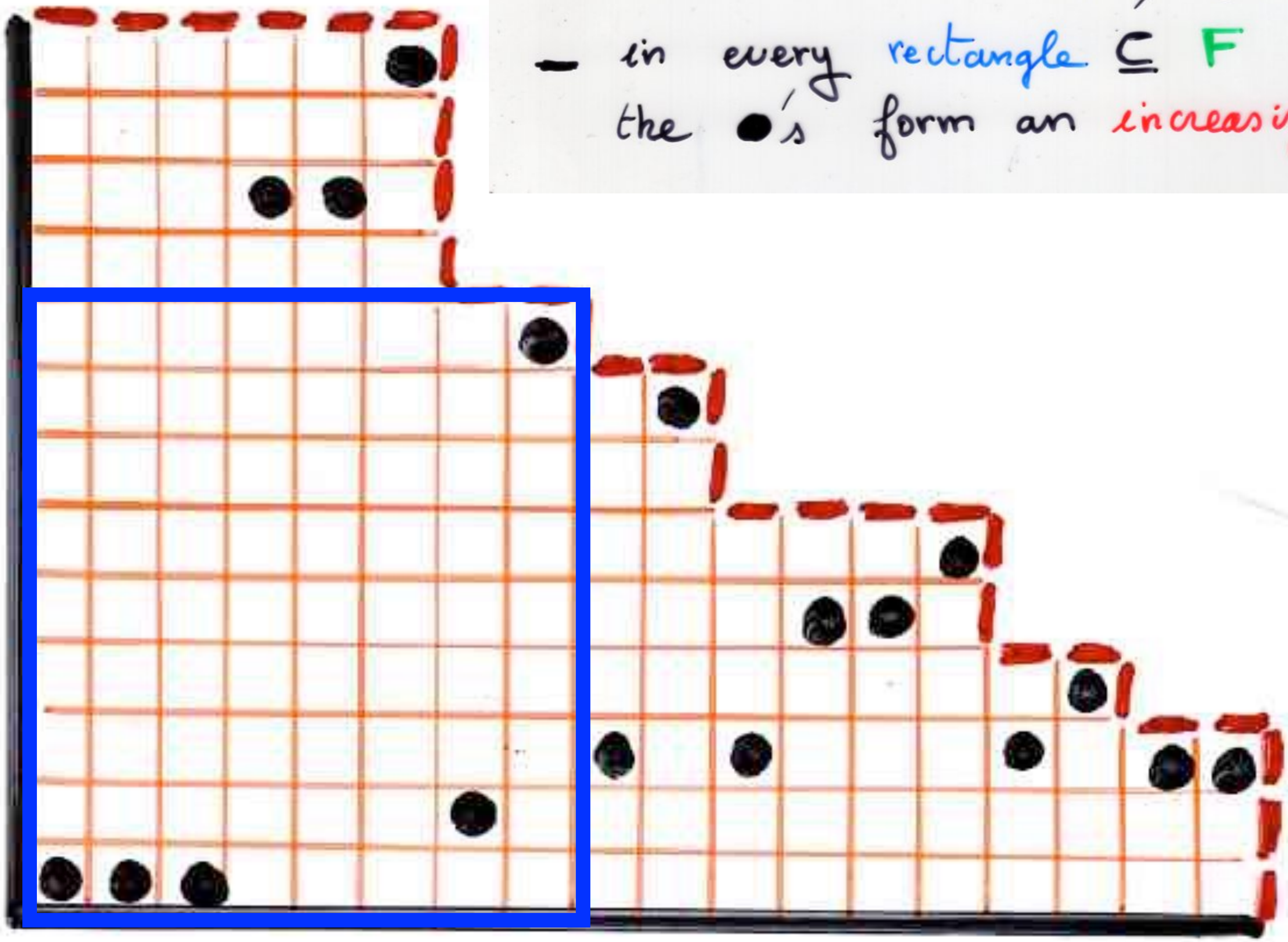
characterisation : (ii) $\Leftrightarrow$

- in every rectangle $\subseteq$ F the ●'s form an increasing sequence



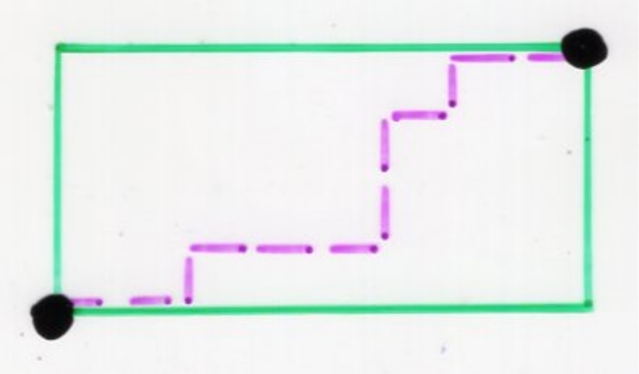
characterisation : (ii) ⟺

- in every rectangle ⊆ F the ●'s form an increasing sequence

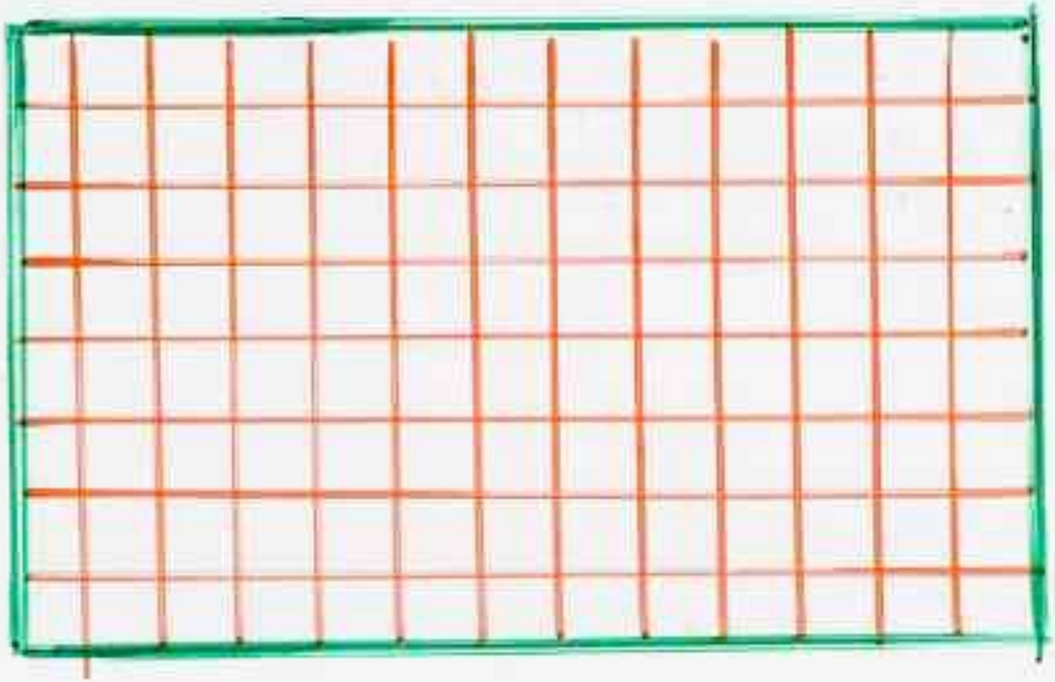


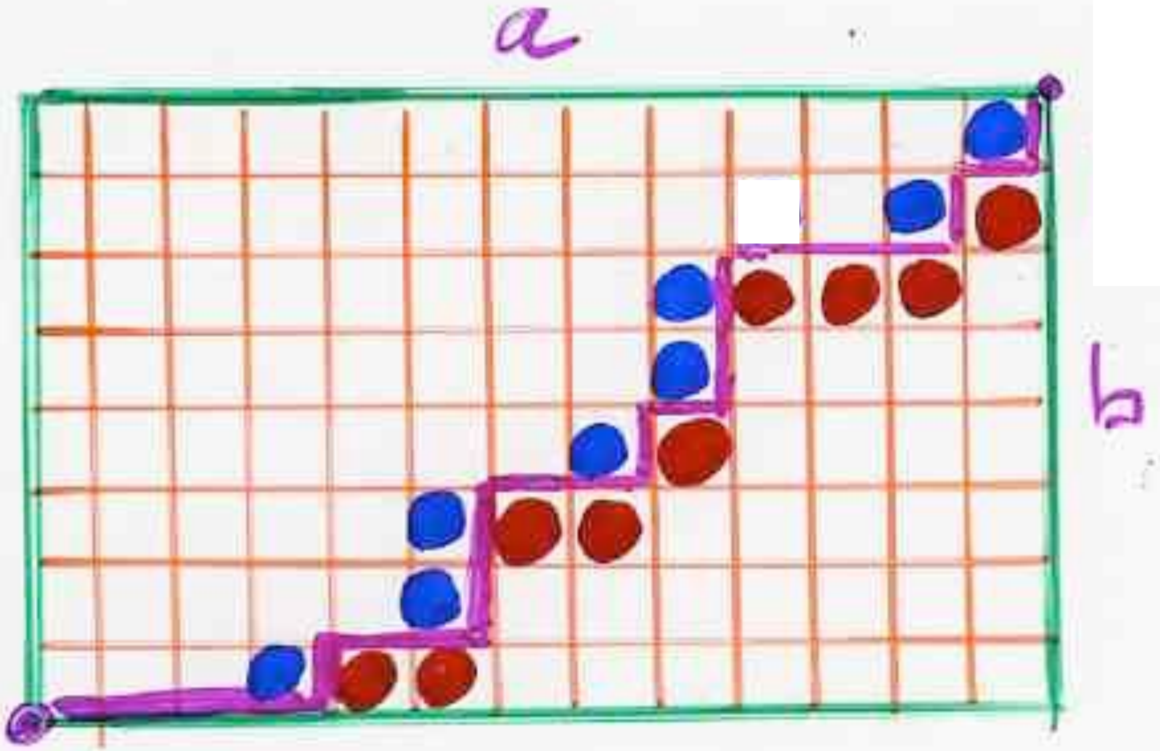
Corollary

If F is a rectangle $a \times b$
Catalan alternative tableaux
are in bijection with paths :



$$\binom{a+b}{a}$$

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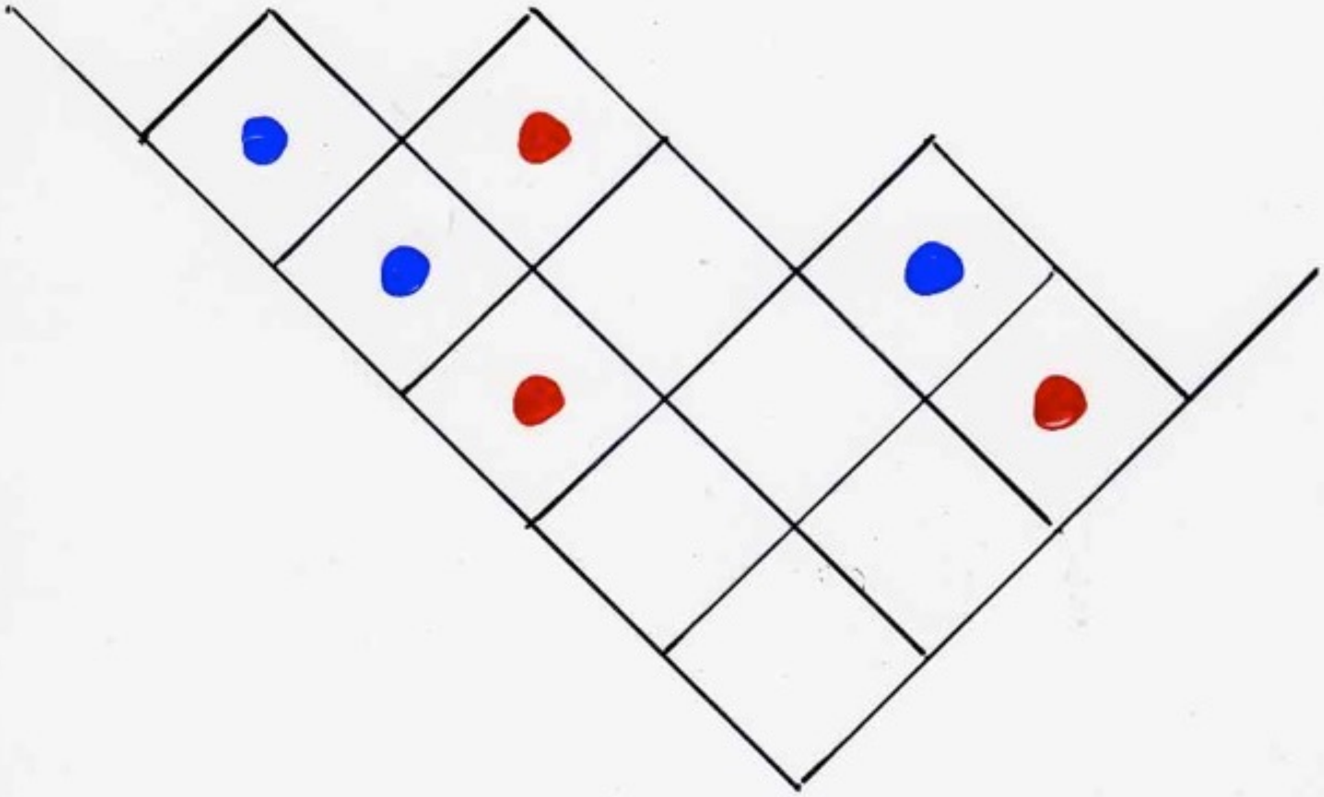
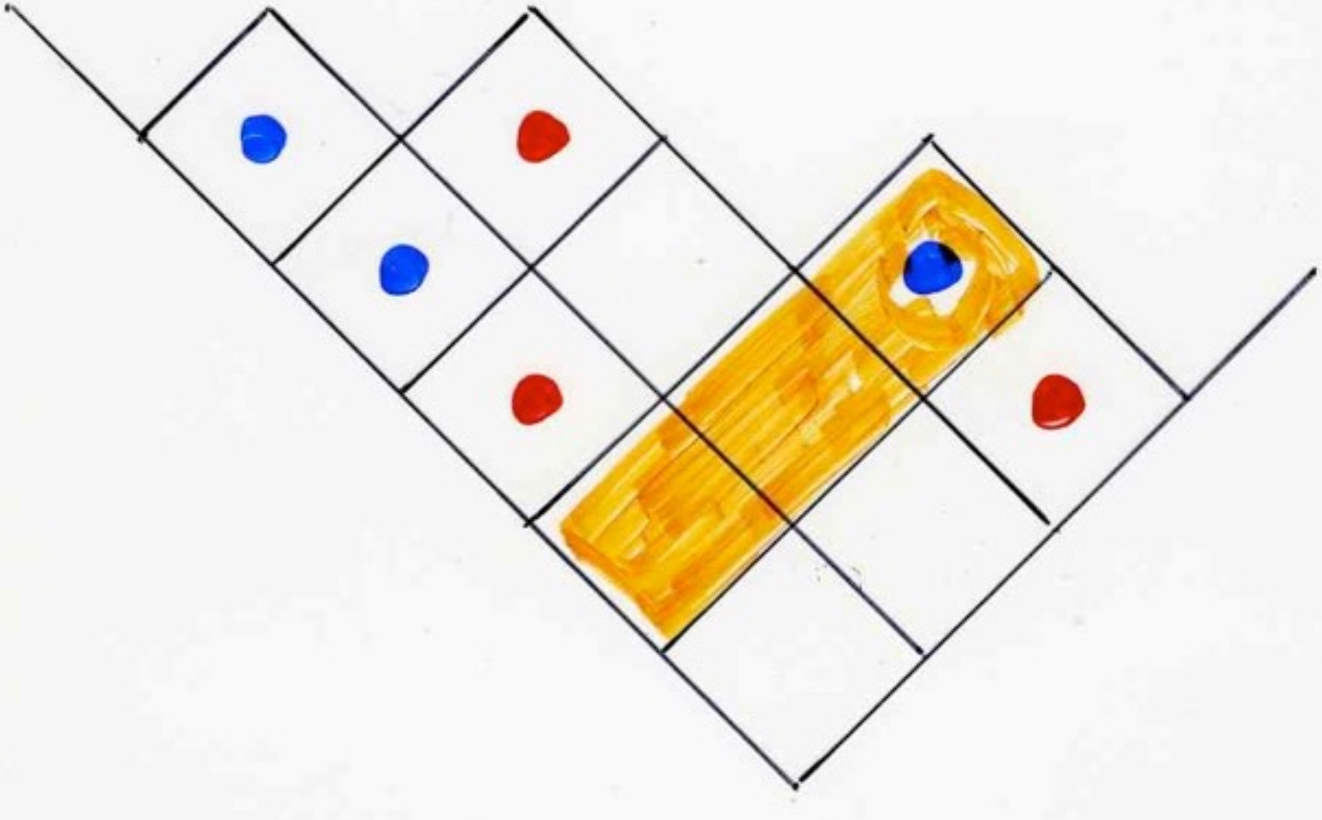
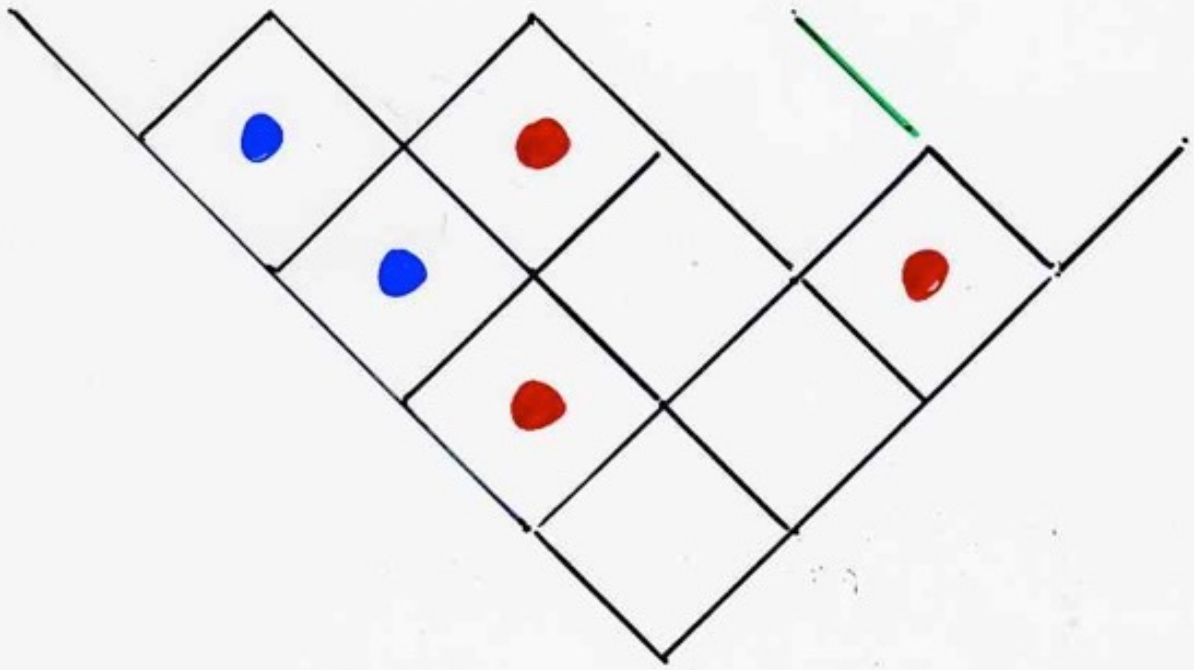
Bijection
Catalan alternative tableaux
binary trees

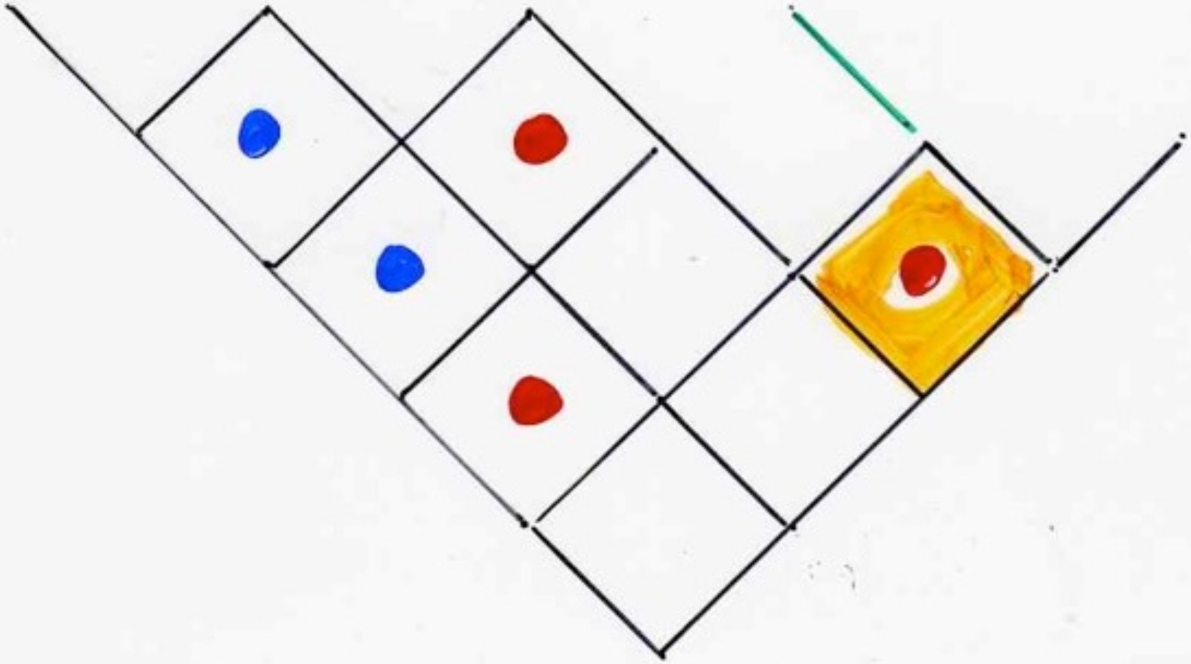
See BJC III, Ch 4a, 78-107

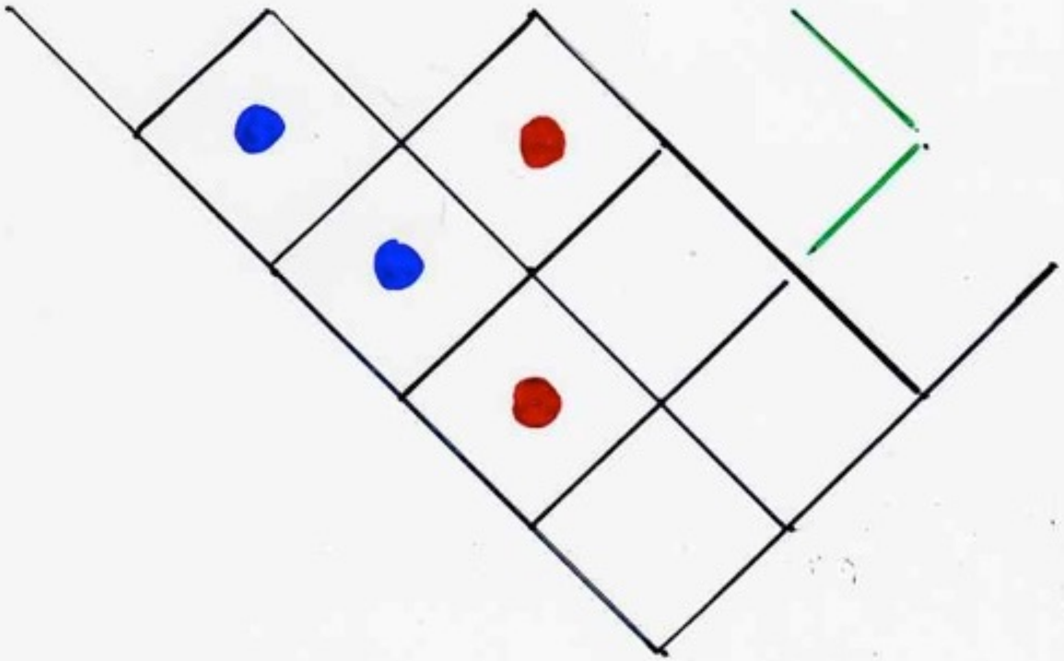
(second bijection Catalan alterntive tableaux — binary trees)

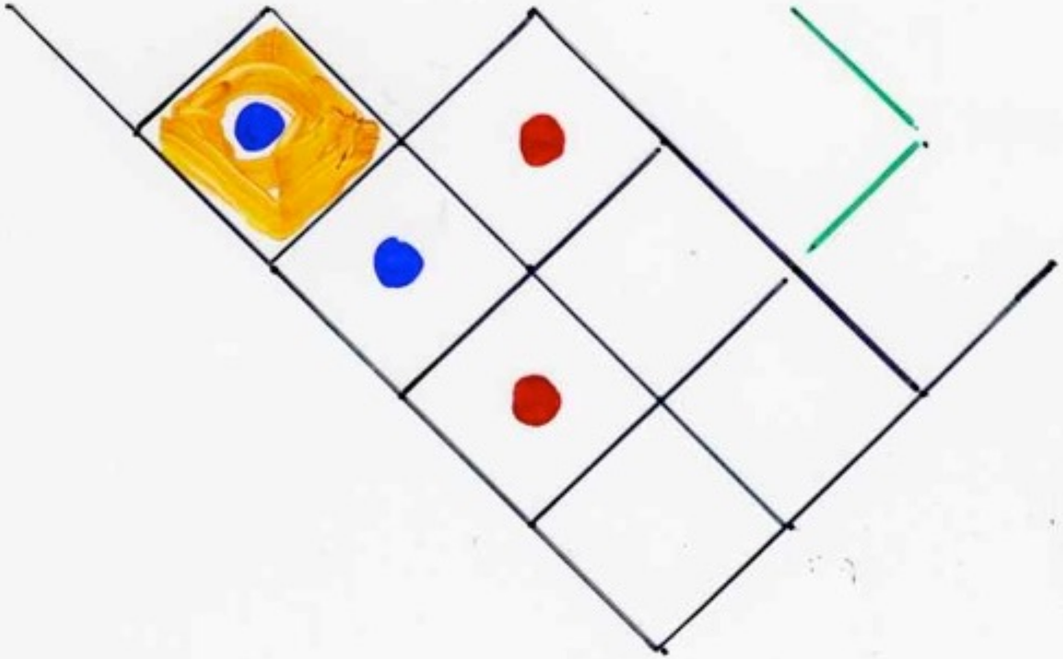


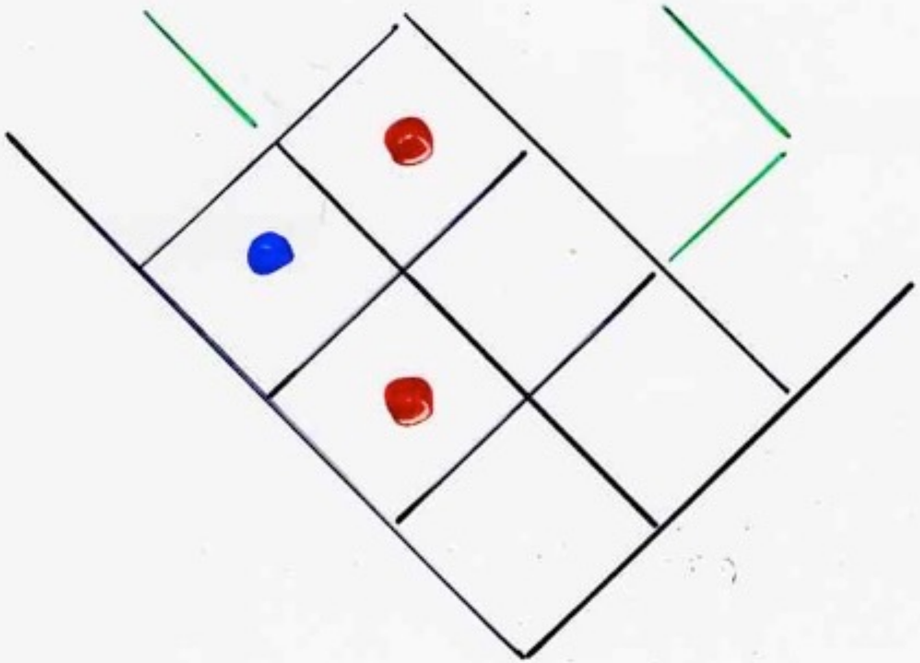


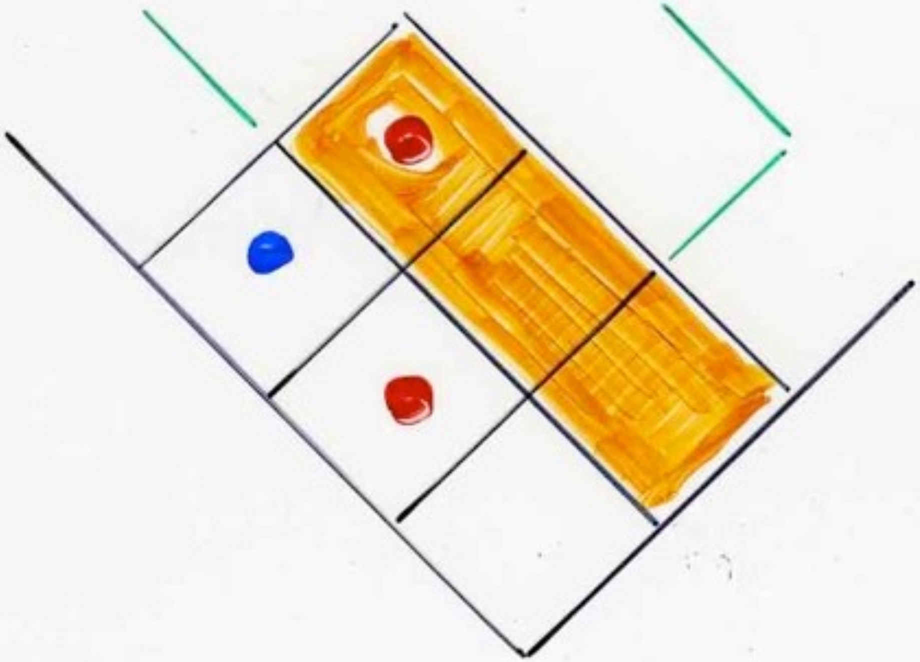


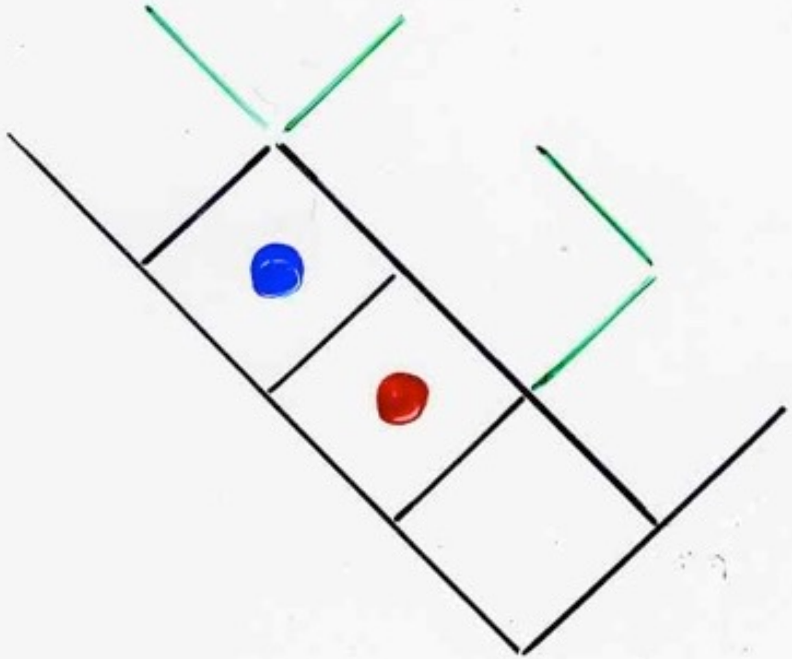


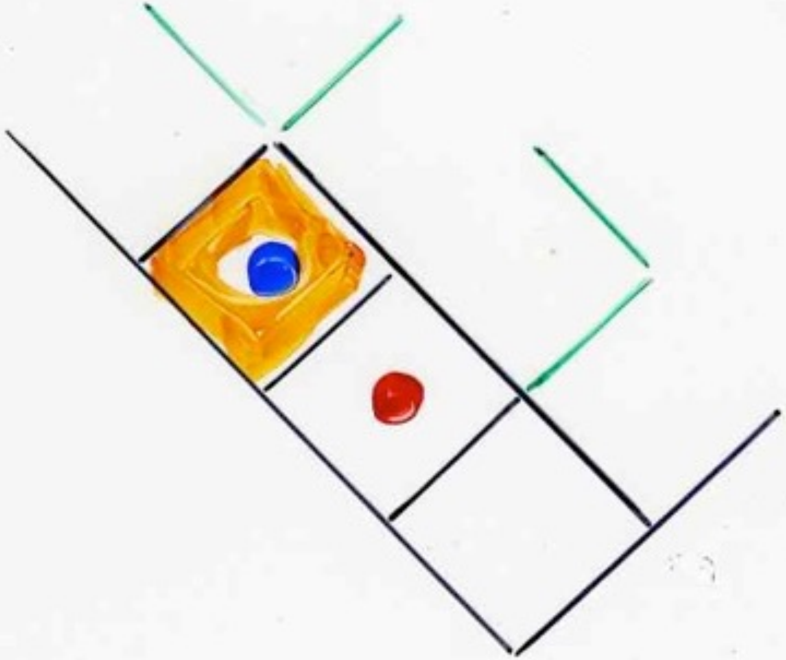


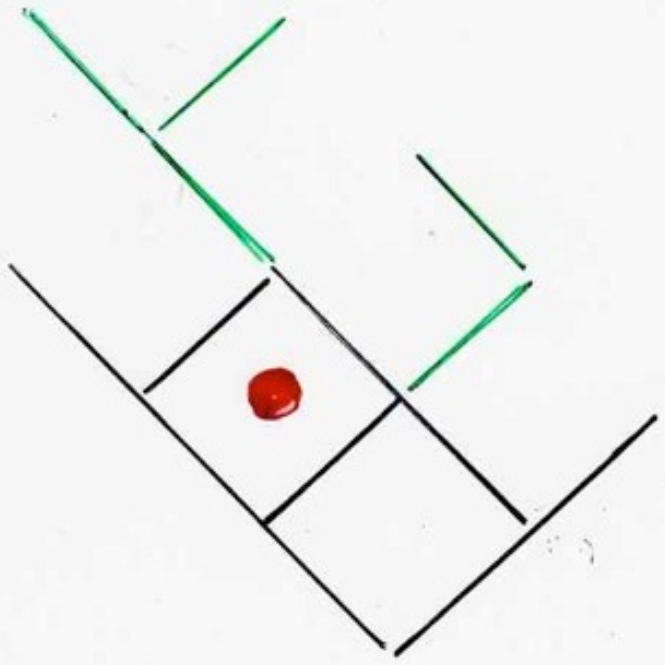


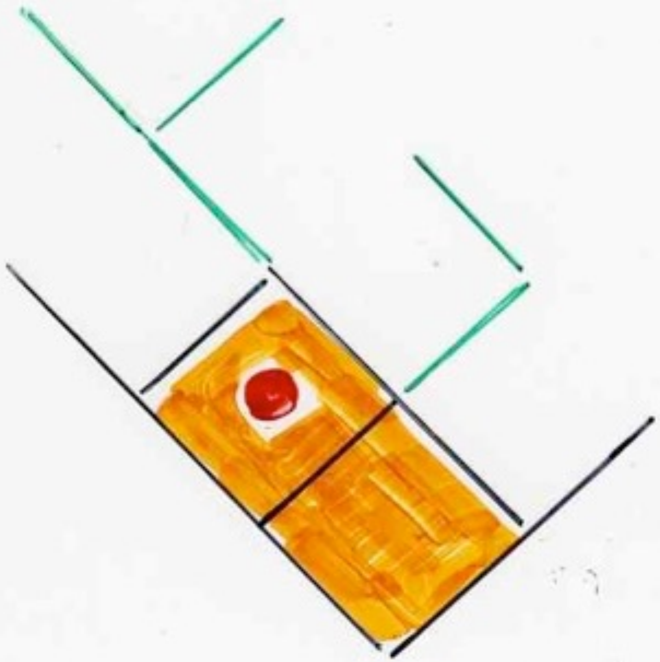


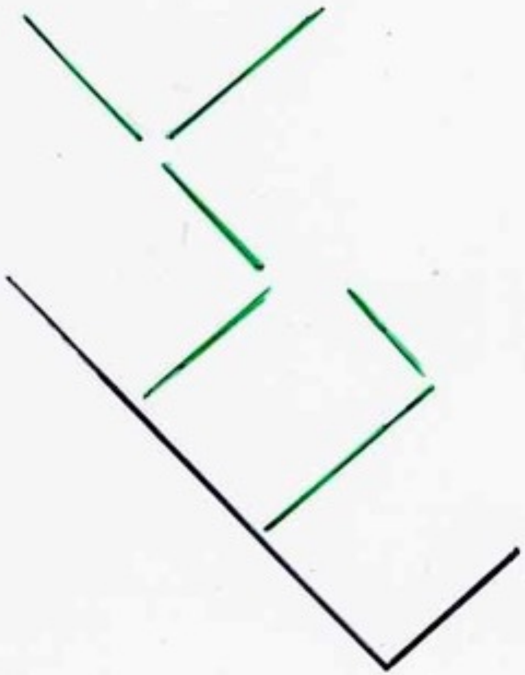






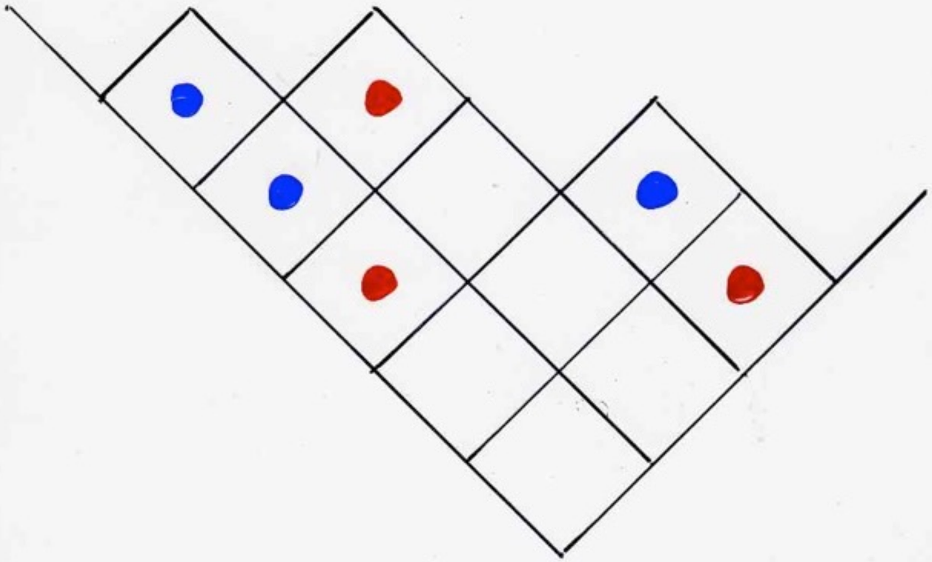


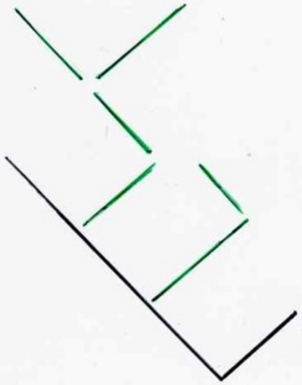




Proposition

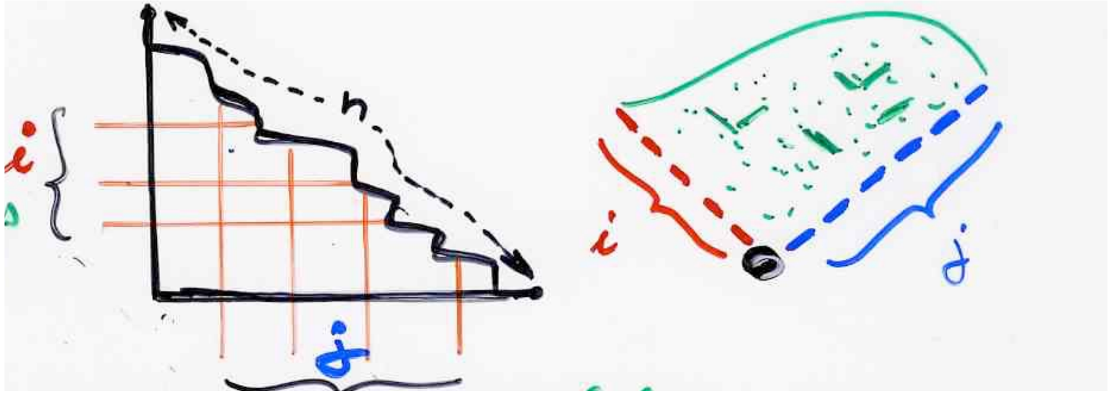
The map defined above is a bijection between alternative tableaux with profile v and binary trees with canopy v





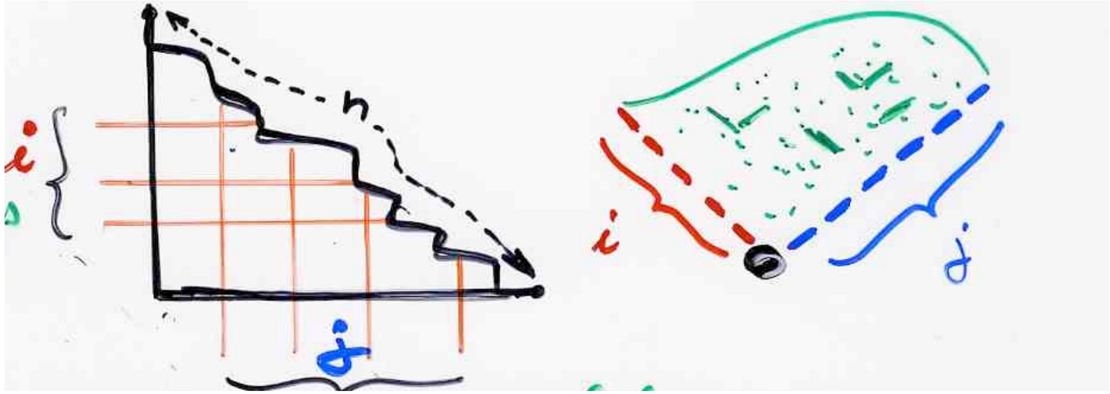
$i(T)$ = nb of rows without ●

$j(T)$ = nb of columns without ●



$i(T) = \mathrm{lpb}(B)$ length of left principal branch

$j(T) = \mathrm{rpb}(B)$ length of right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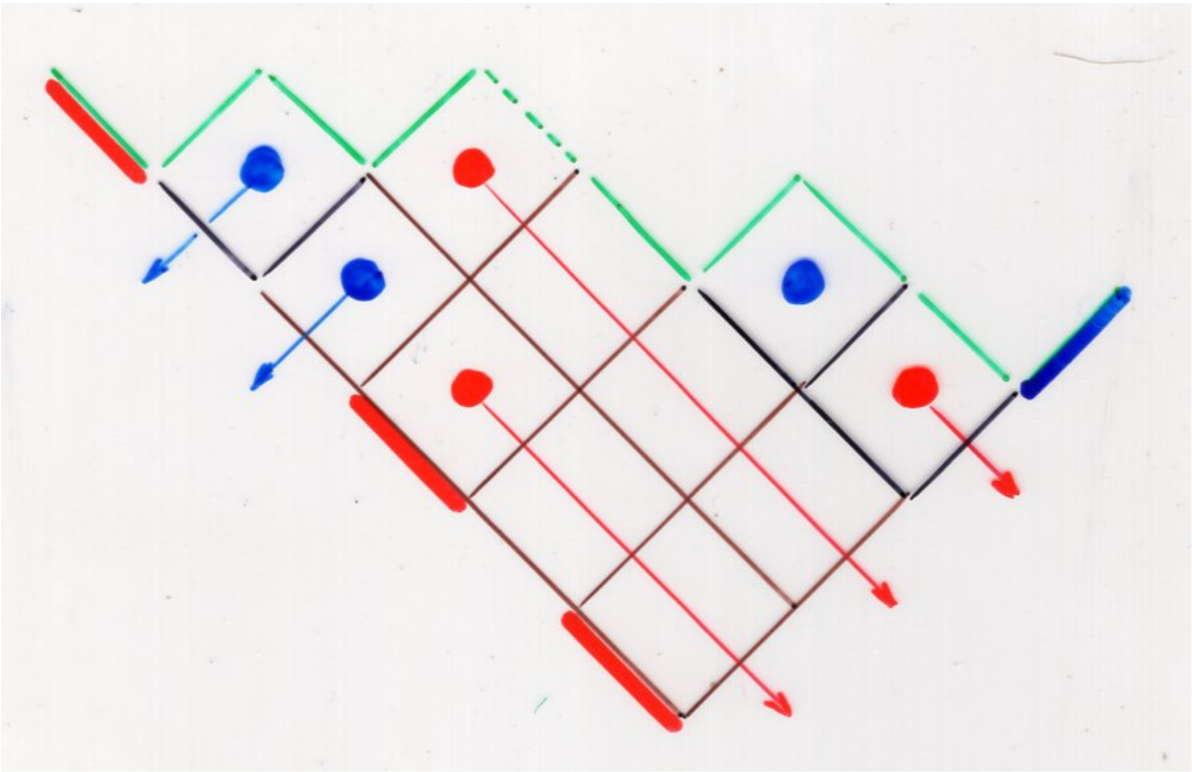
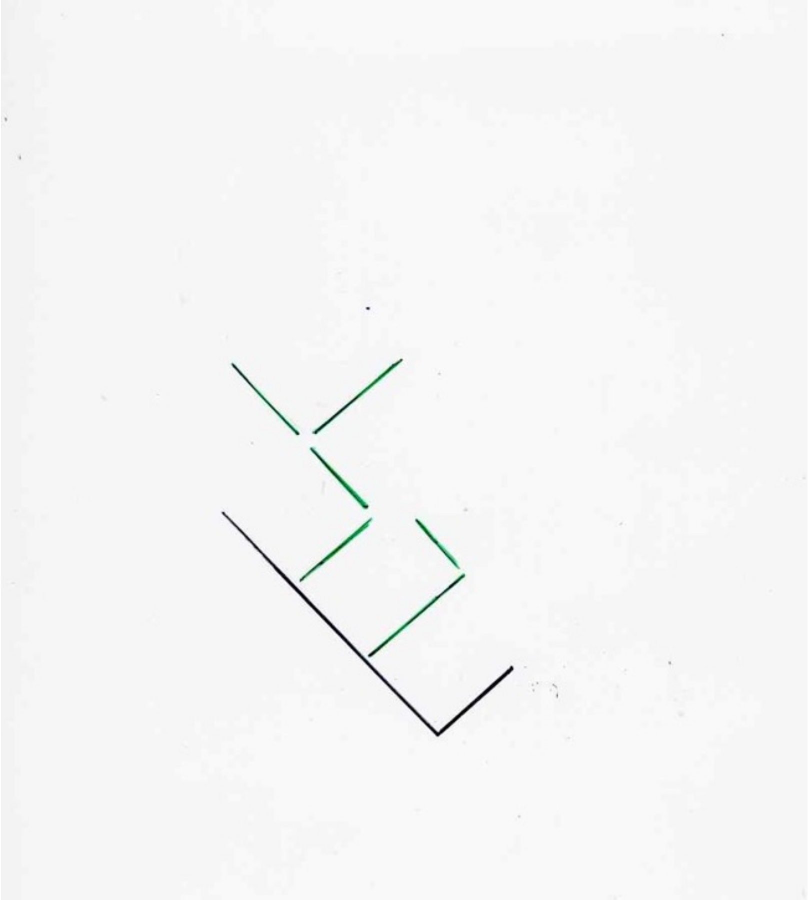


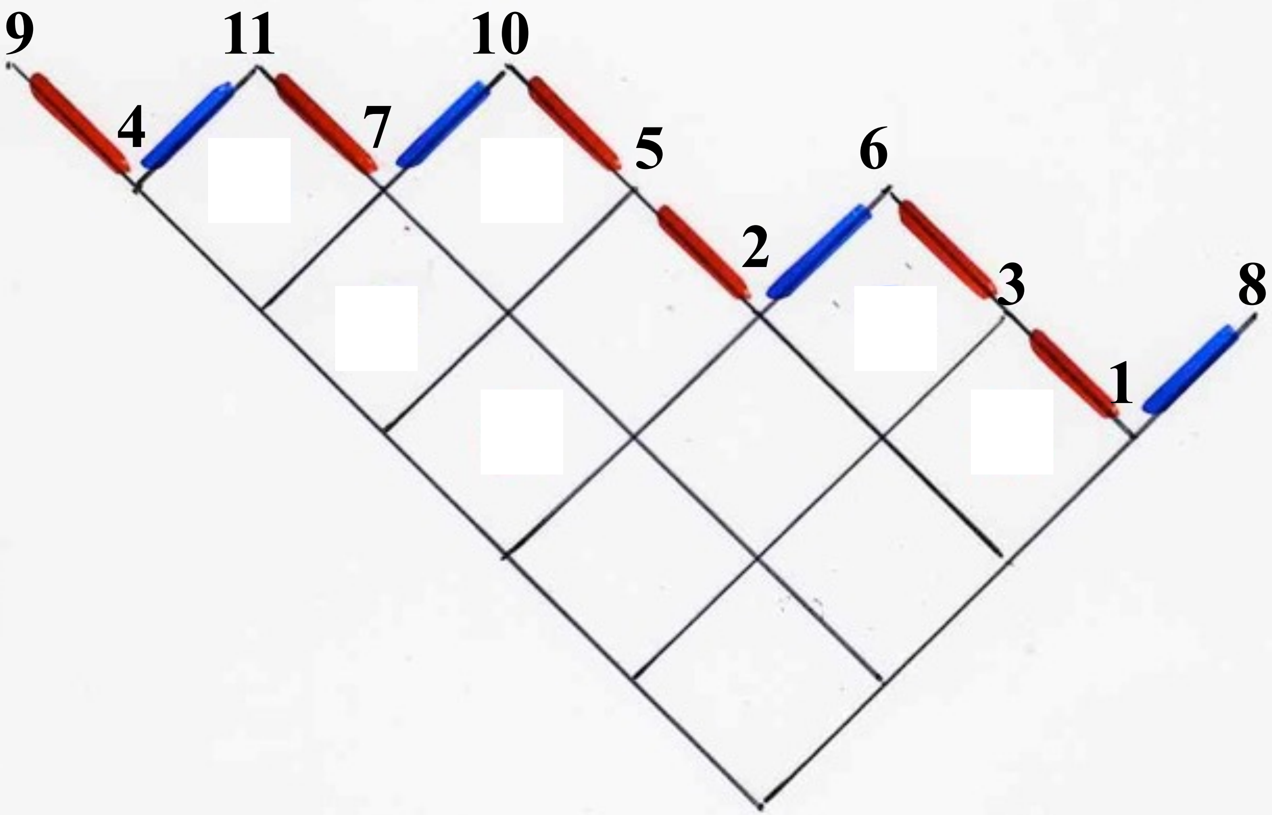


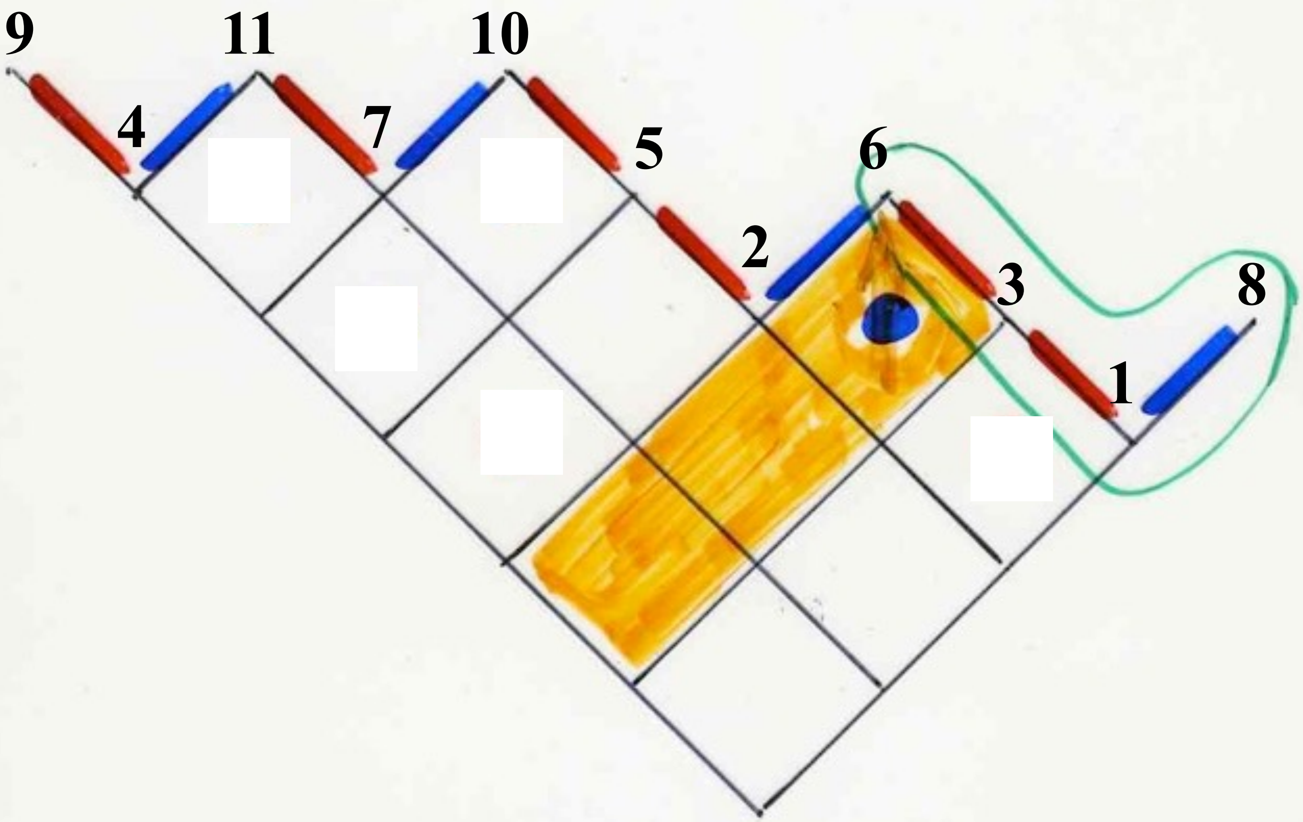
“jeu de taquin”
for an inceasing binary tree

$\downarrow$

Catalan alternative tableau

behind this "jeu de taquin"
there is a Catalan alternative tableau

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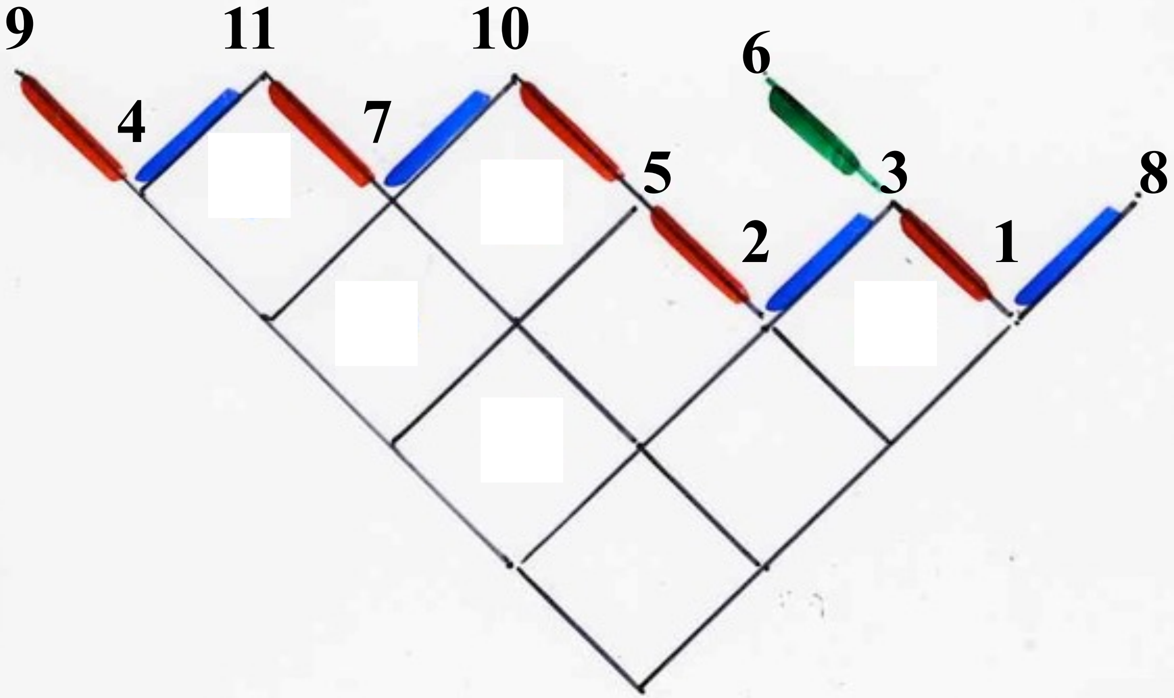
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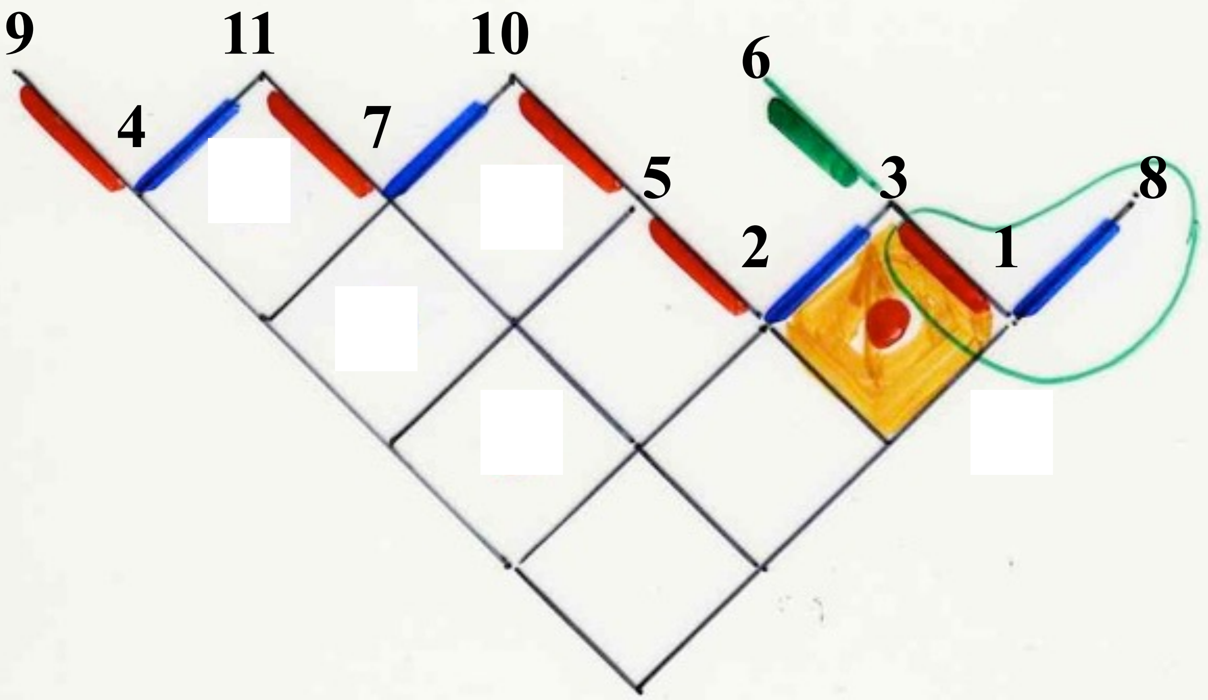
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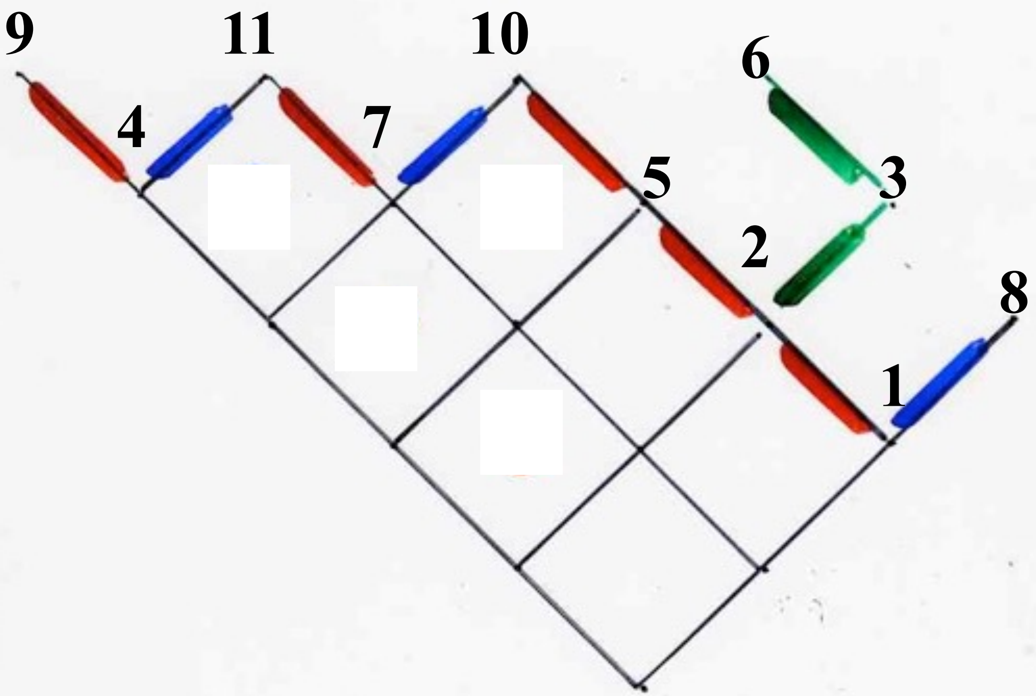
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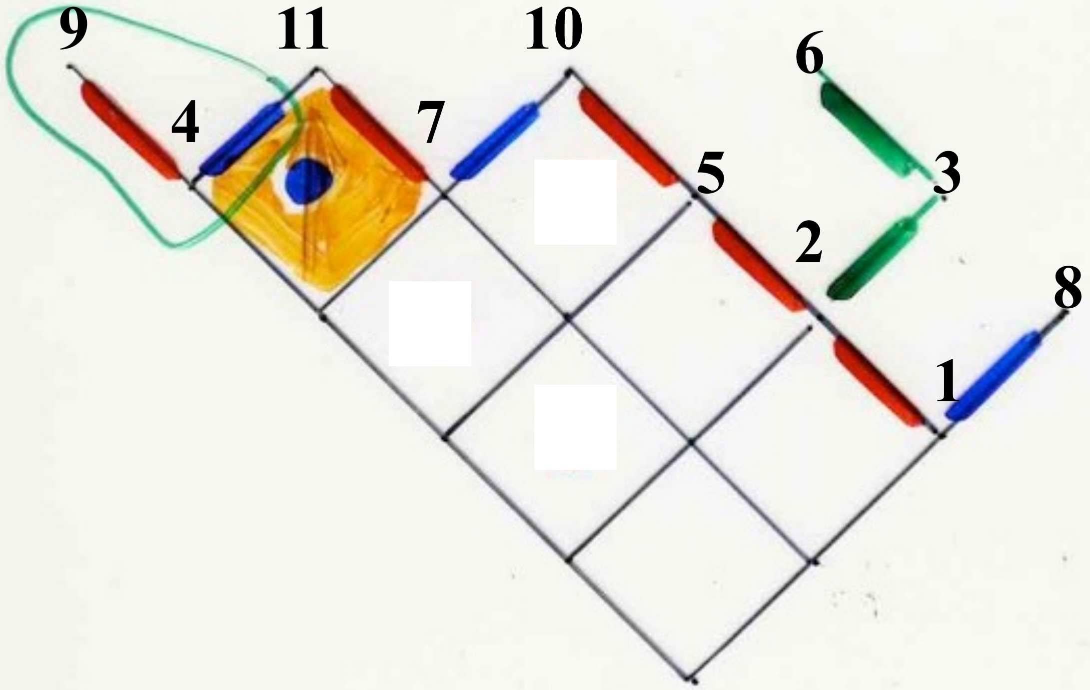
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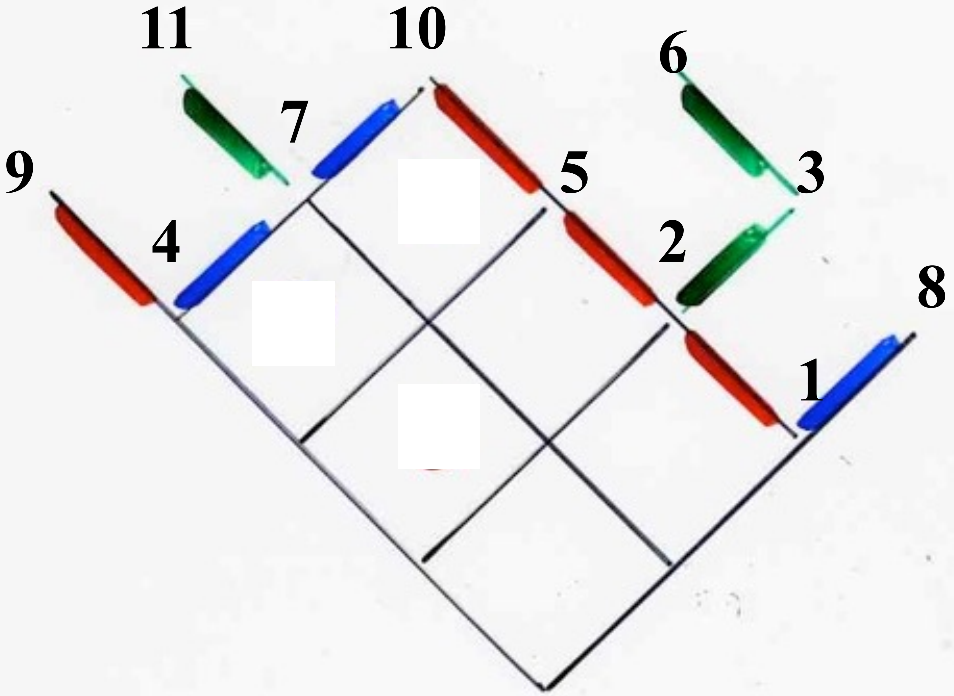
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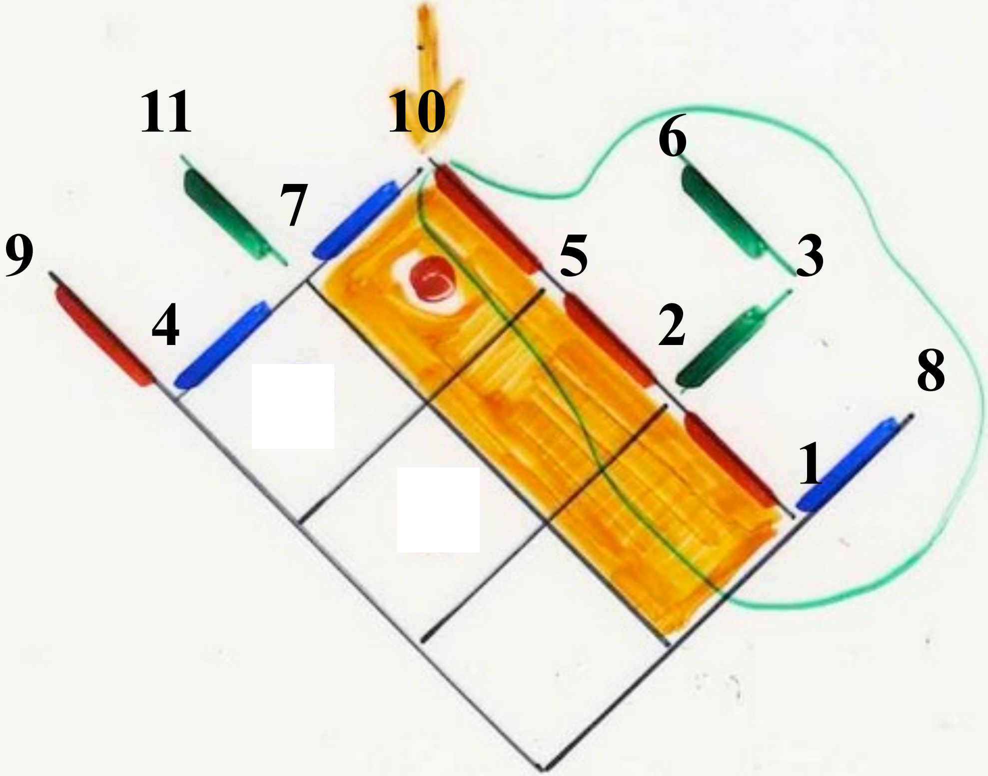
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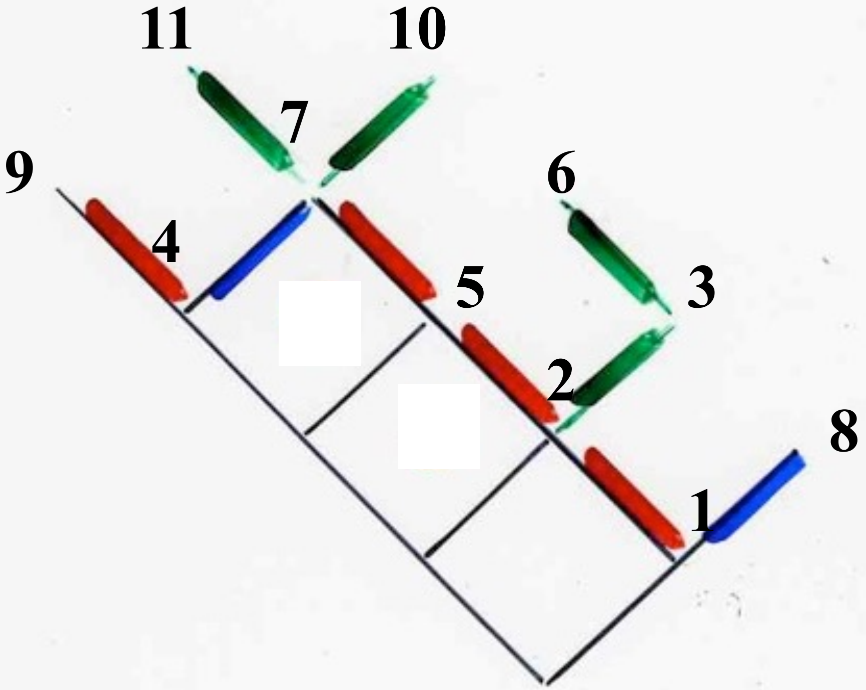
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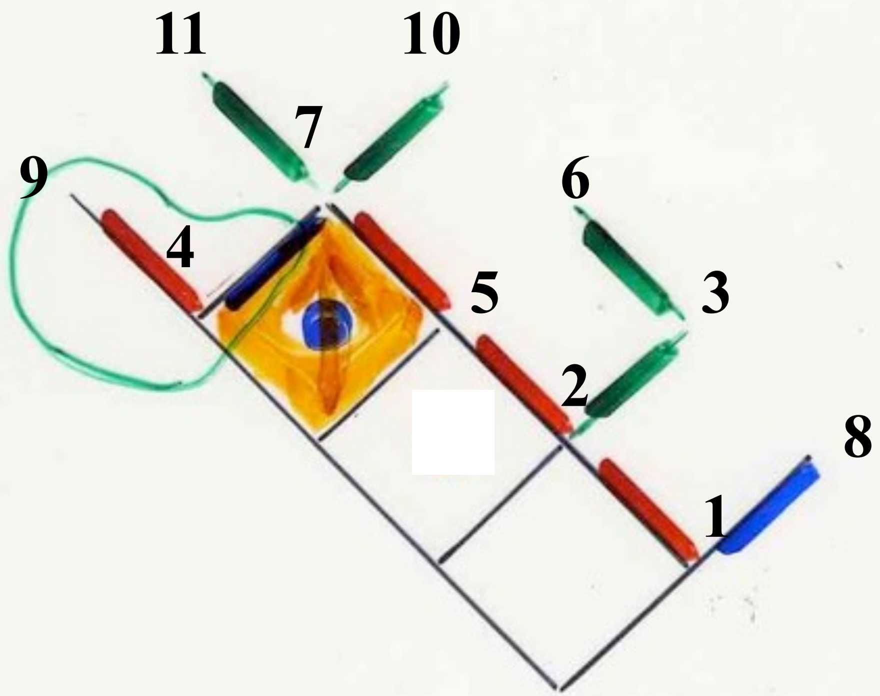
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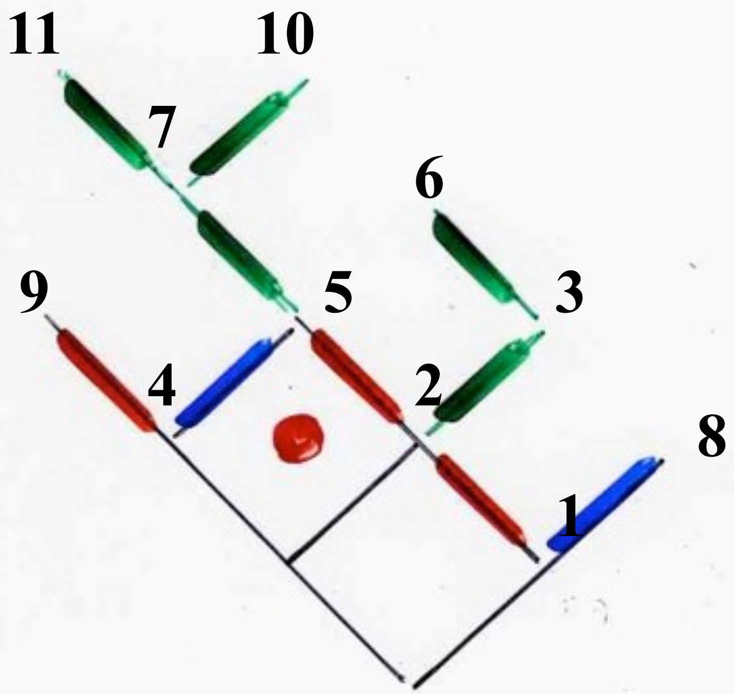
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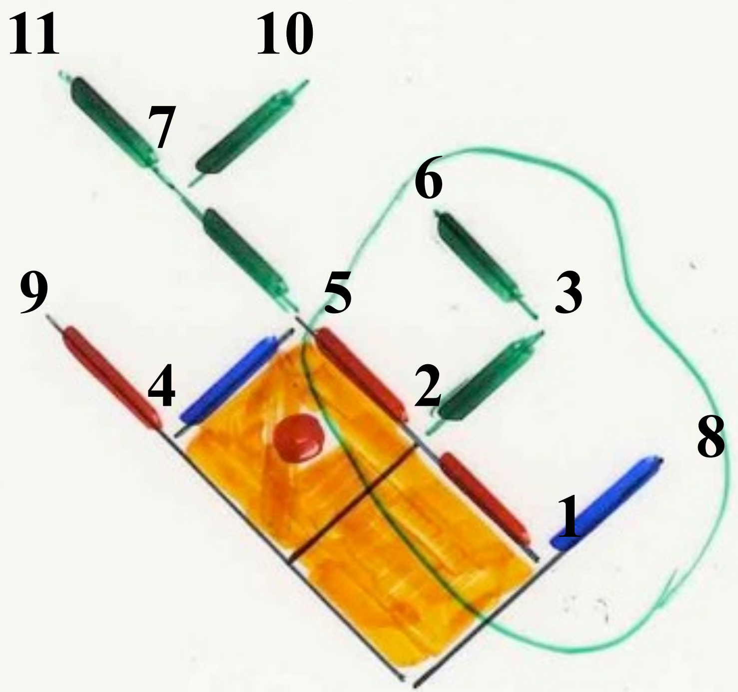
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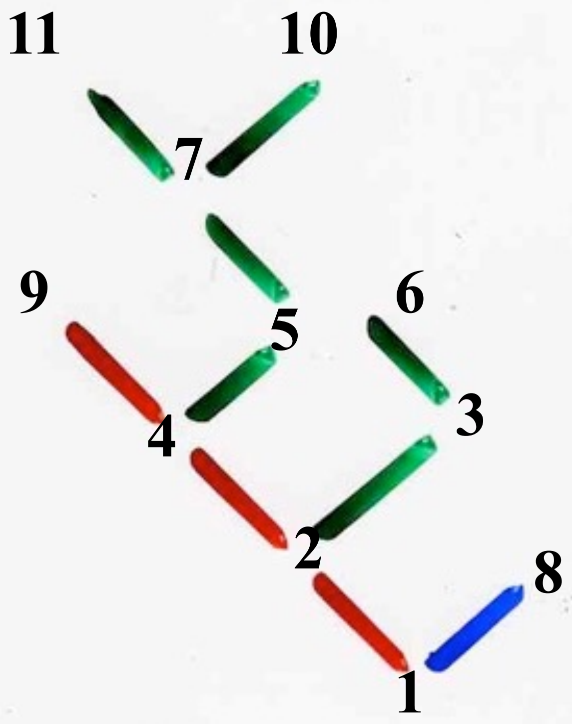
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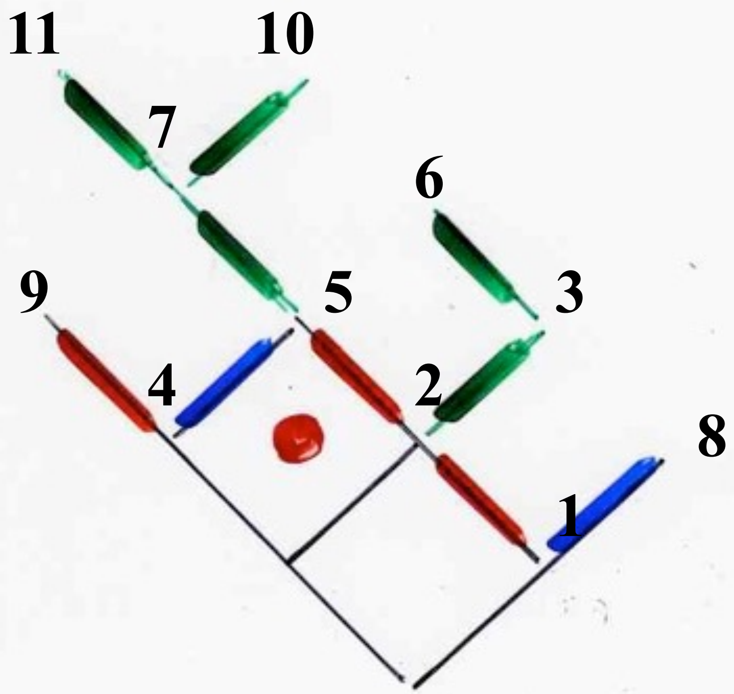
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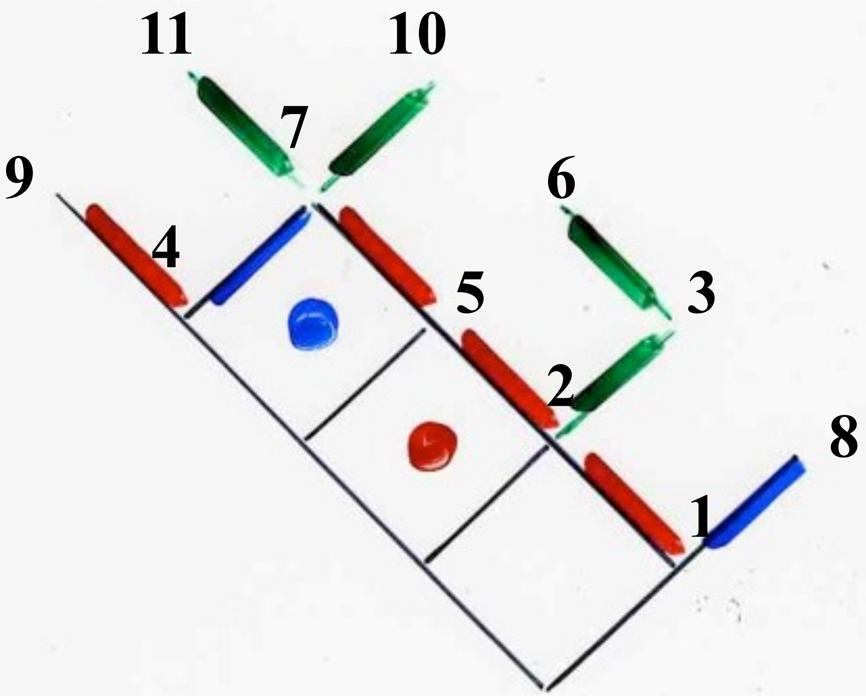
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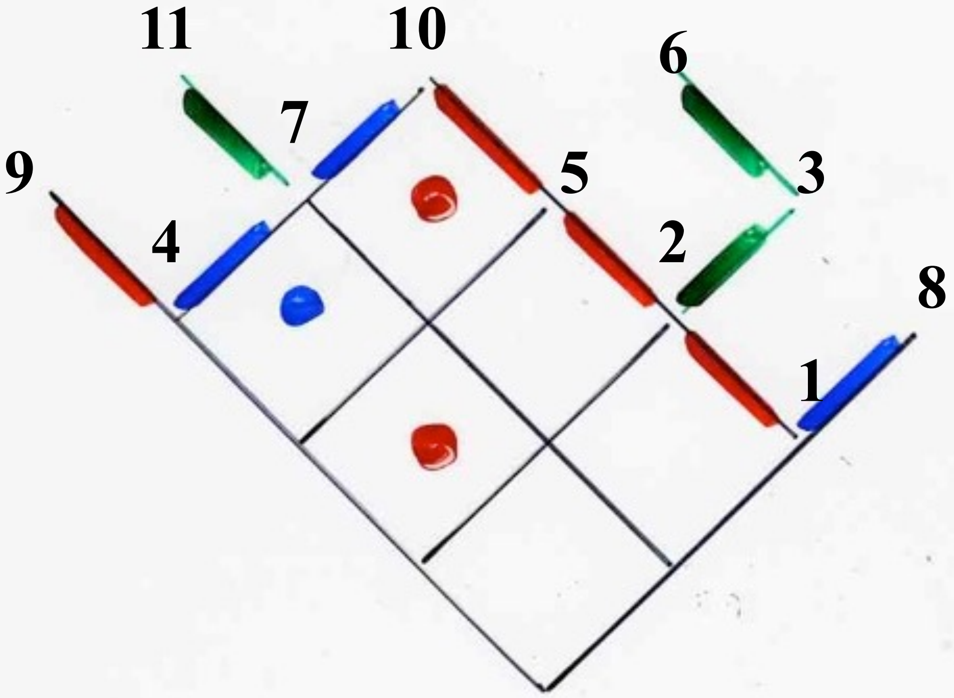
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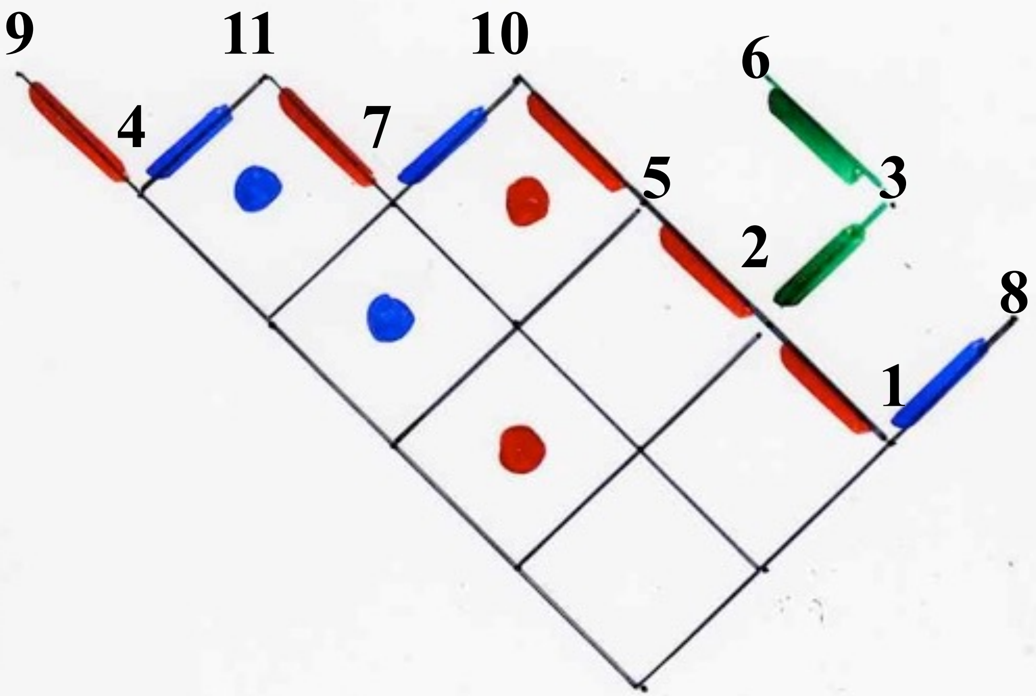
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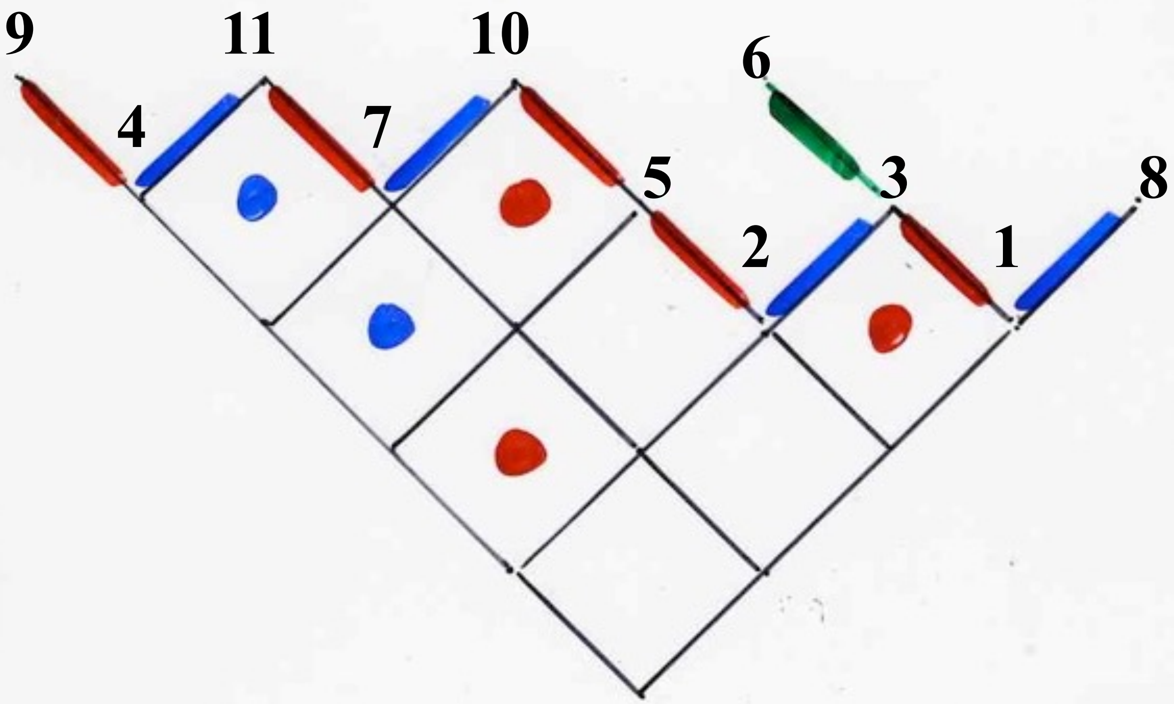
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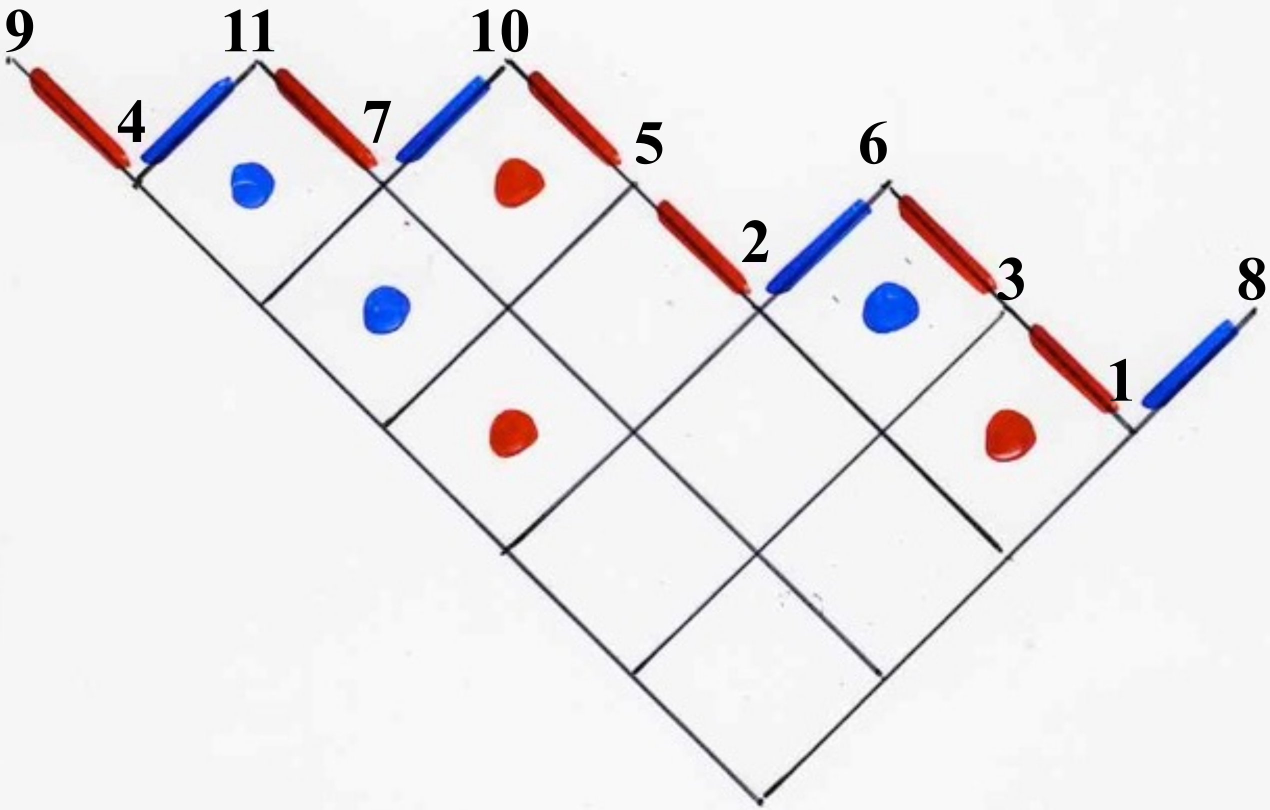
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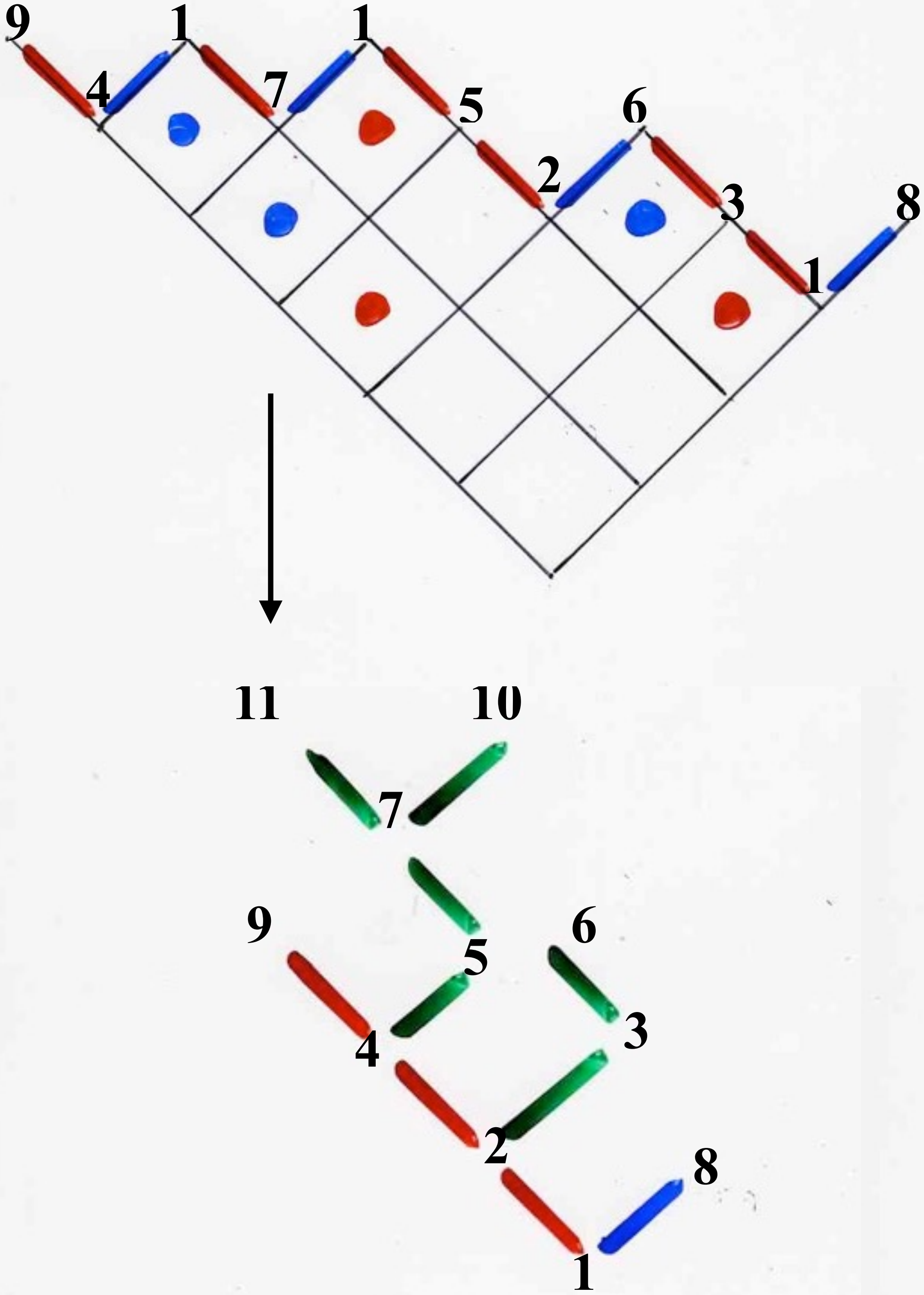
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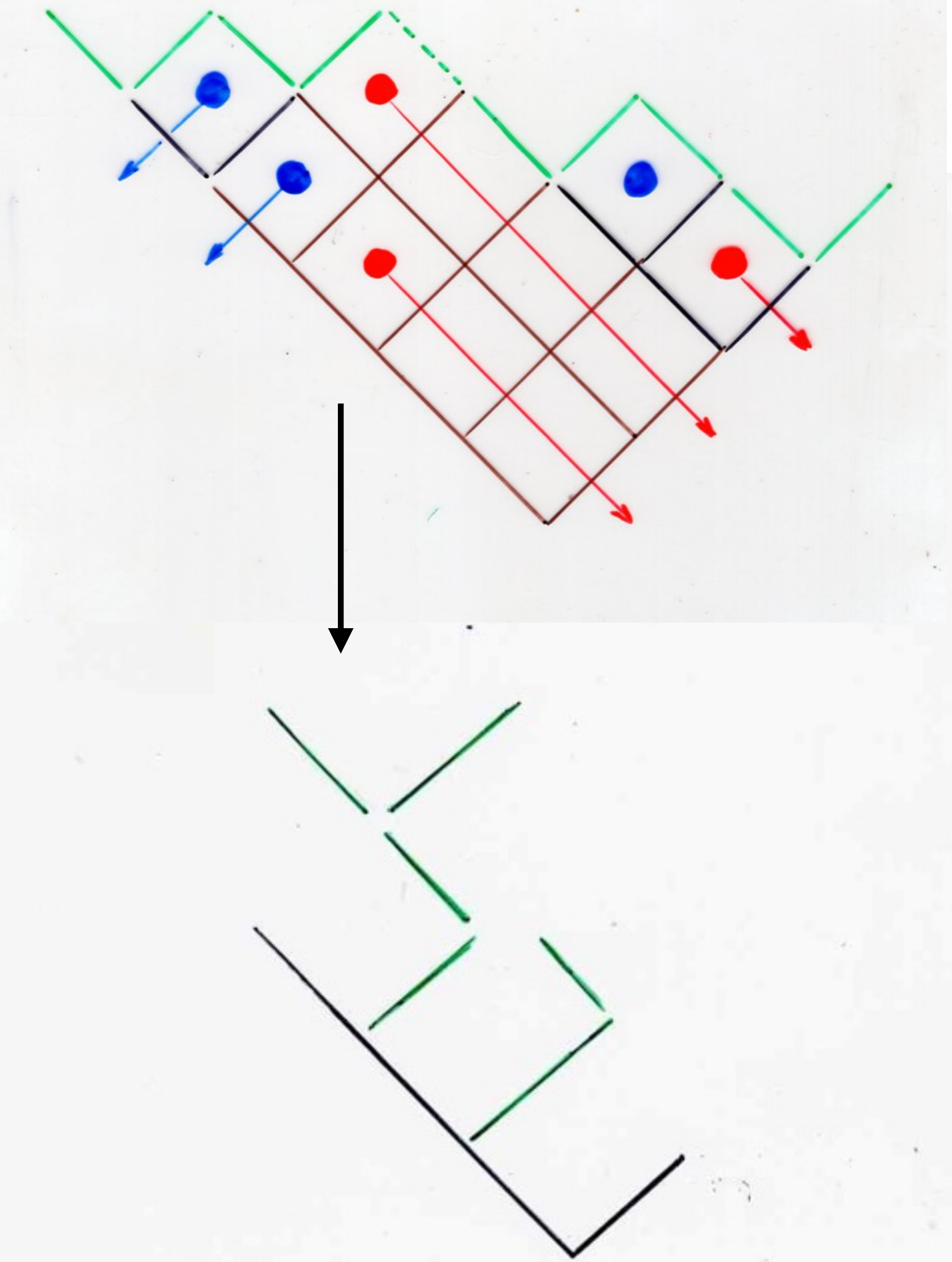
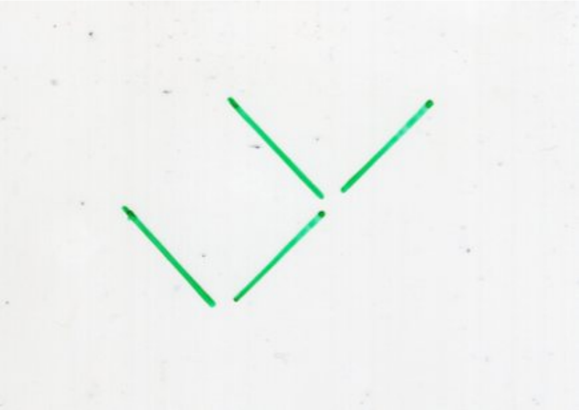
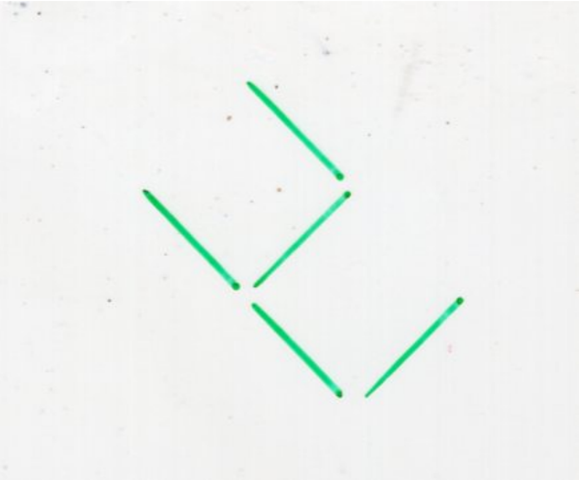


construction of

$$\mathcal{B} * \mathcal{B}' = (\psi^*)^{-1}\left(\psi^*(\mathcal{B}) * \psi^*(\mathcal{B}')\right)$$





Loday-Ronco
product $*$

$$B * B' = (\psi^*)^{-1}\left(\psi^*(B) * \psi^*(B')\right)$$

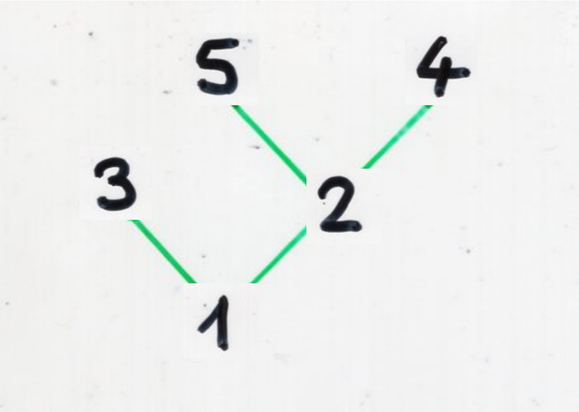
Malvenuto-Reutenauer
product $*$
in $\mathbb{K}[S_\infty]$

$$Y_n \xrightarrow{\psi^*} \mathbb{K}[S_n]$$

$$\psi^*(B) = \sum_{\substack{\sigma \in S_n \\ \psi(\sigma) = B}} \sigma$$

$$\left(\psi^*(B) * \psi^*(B') \right)$$

Malvenuto - Reutenauer
product $*$
in $\mathbb{K}[S_\infty]$



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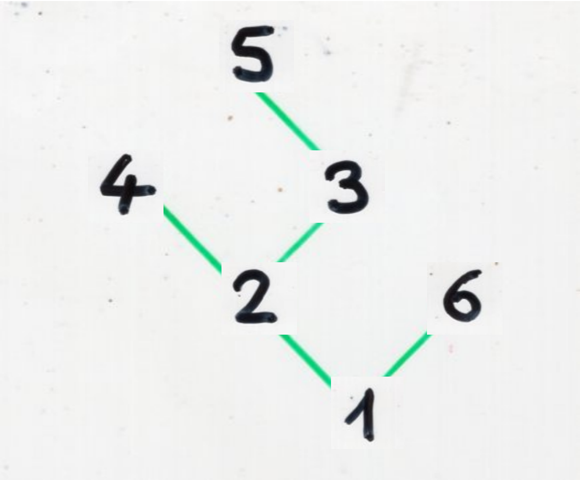

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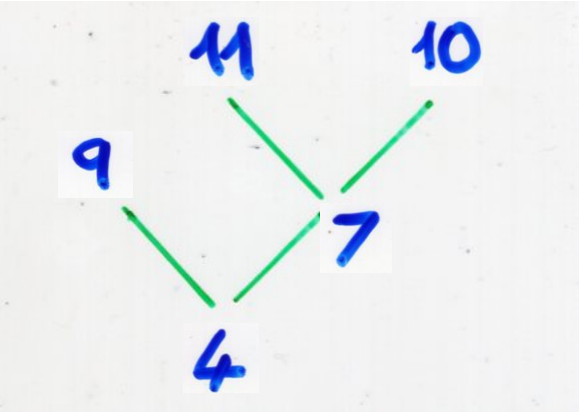

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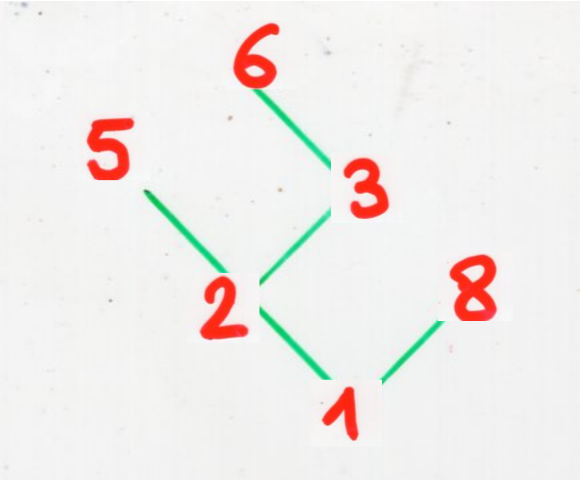

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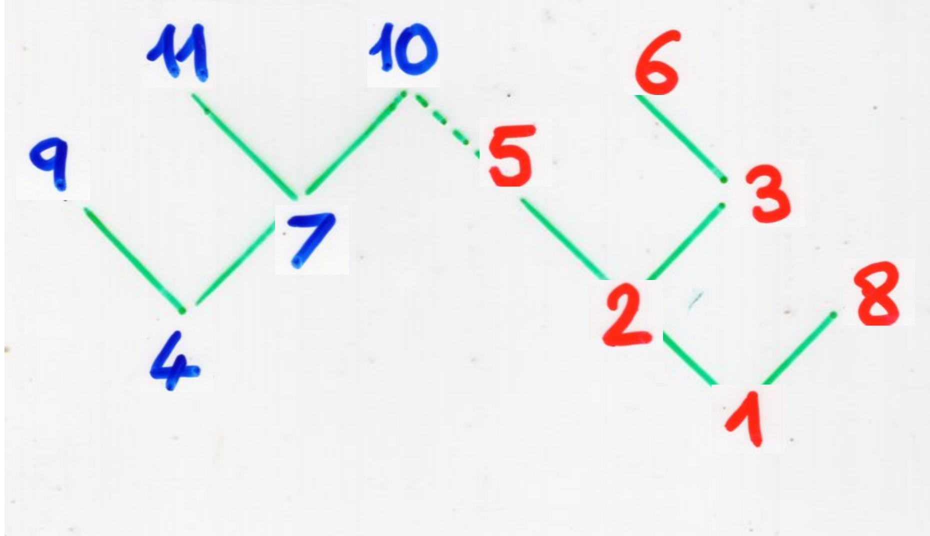

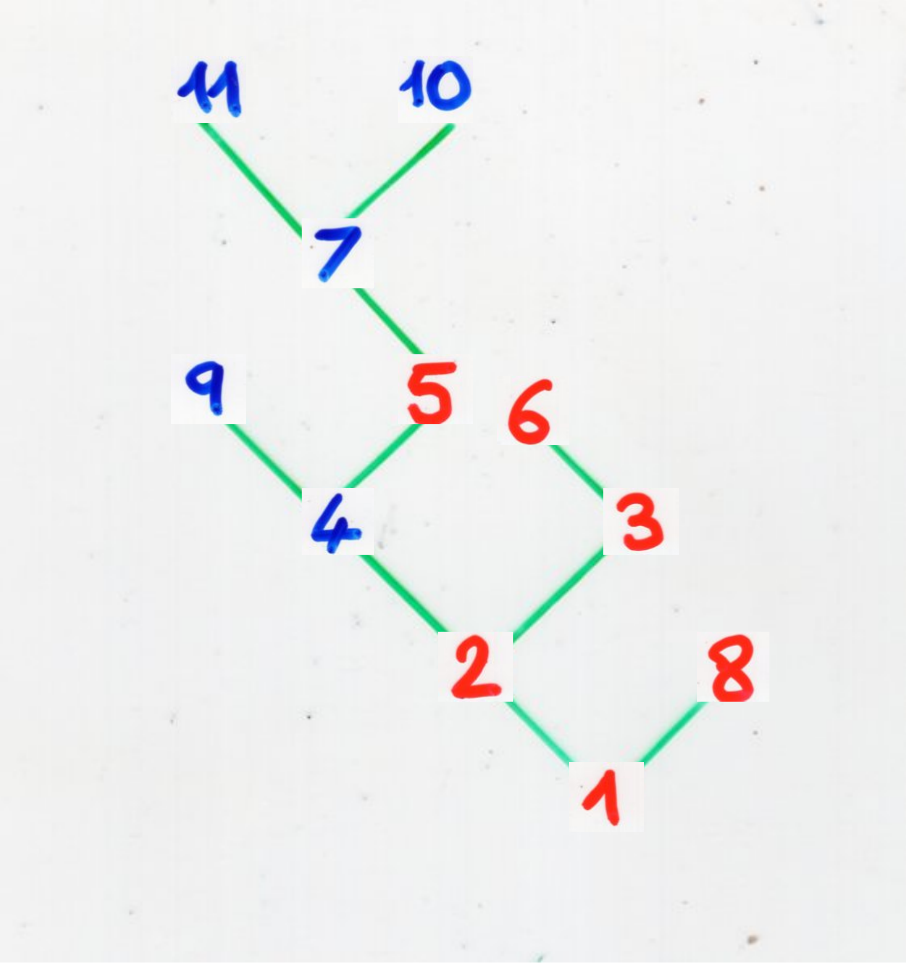

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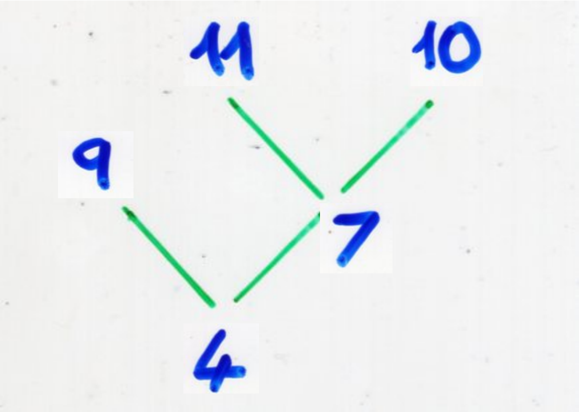

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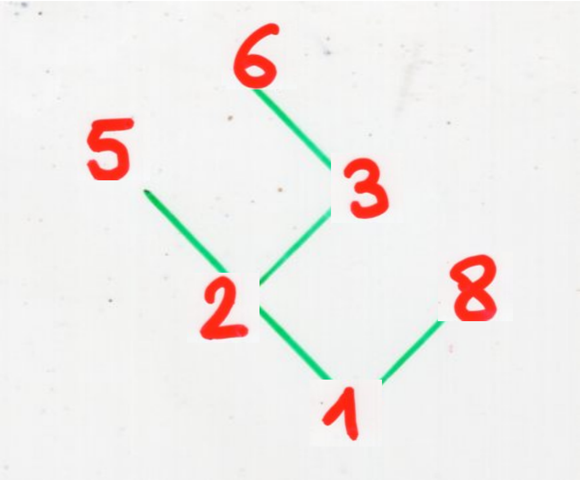

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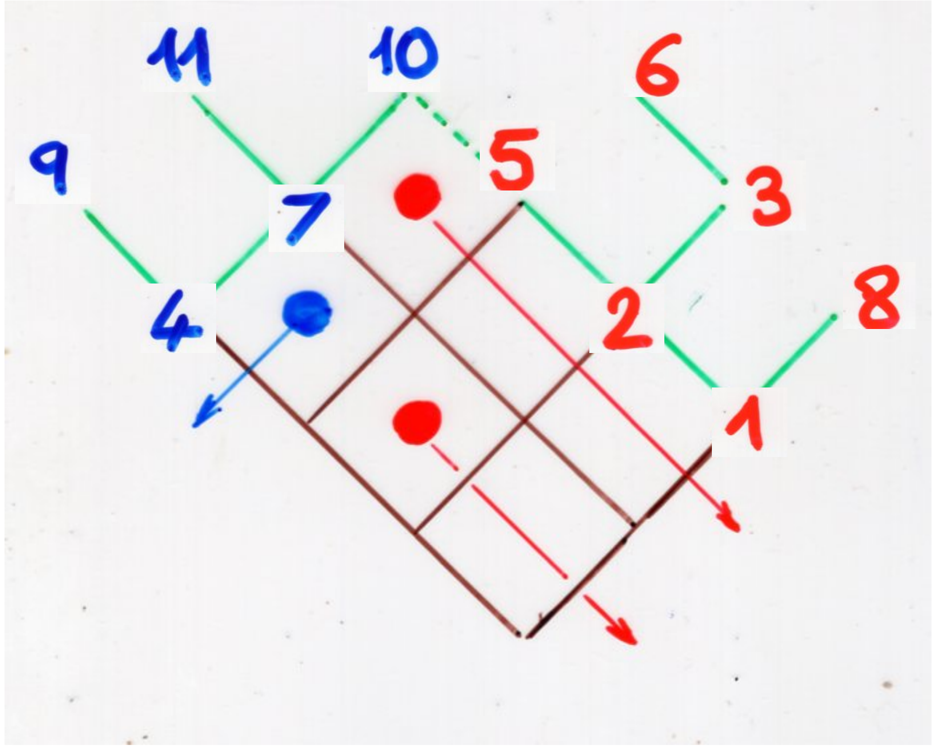

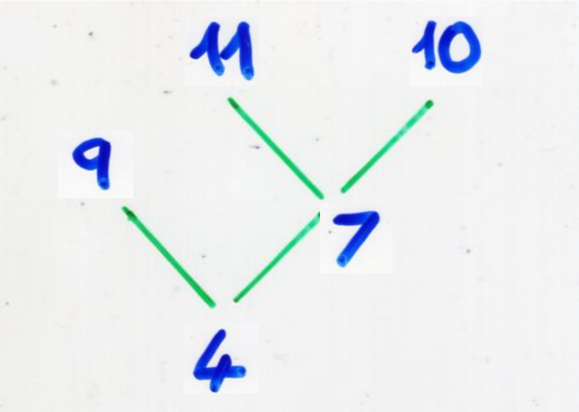

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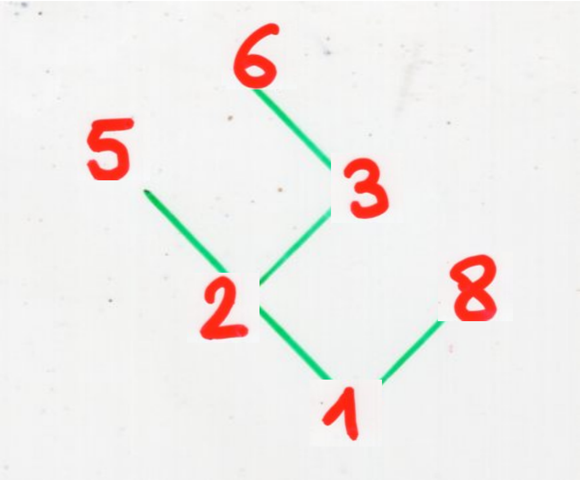

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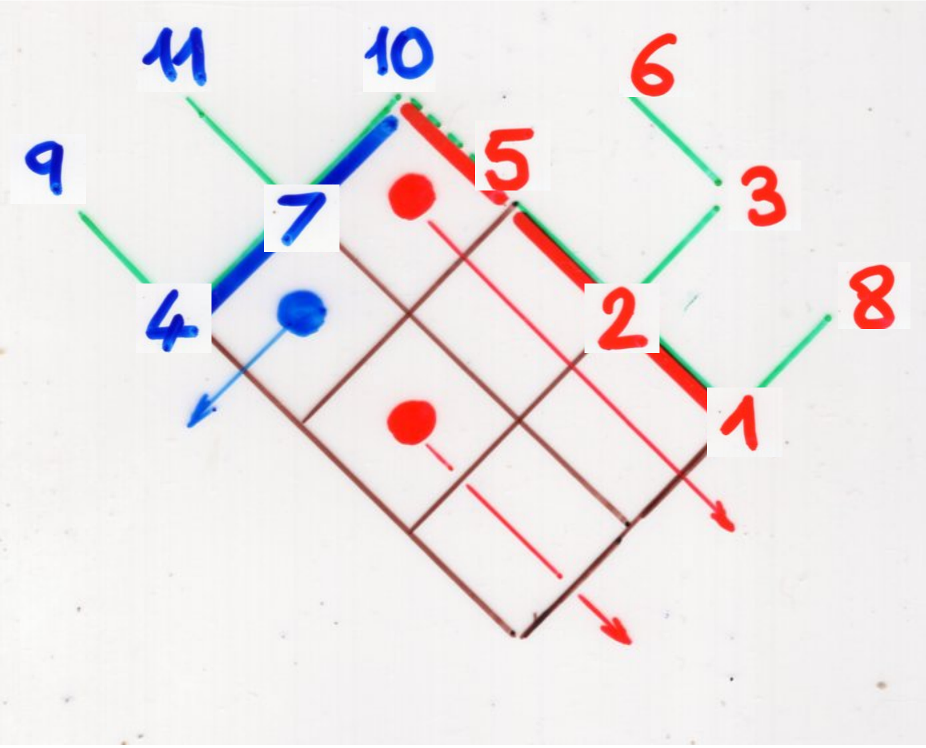
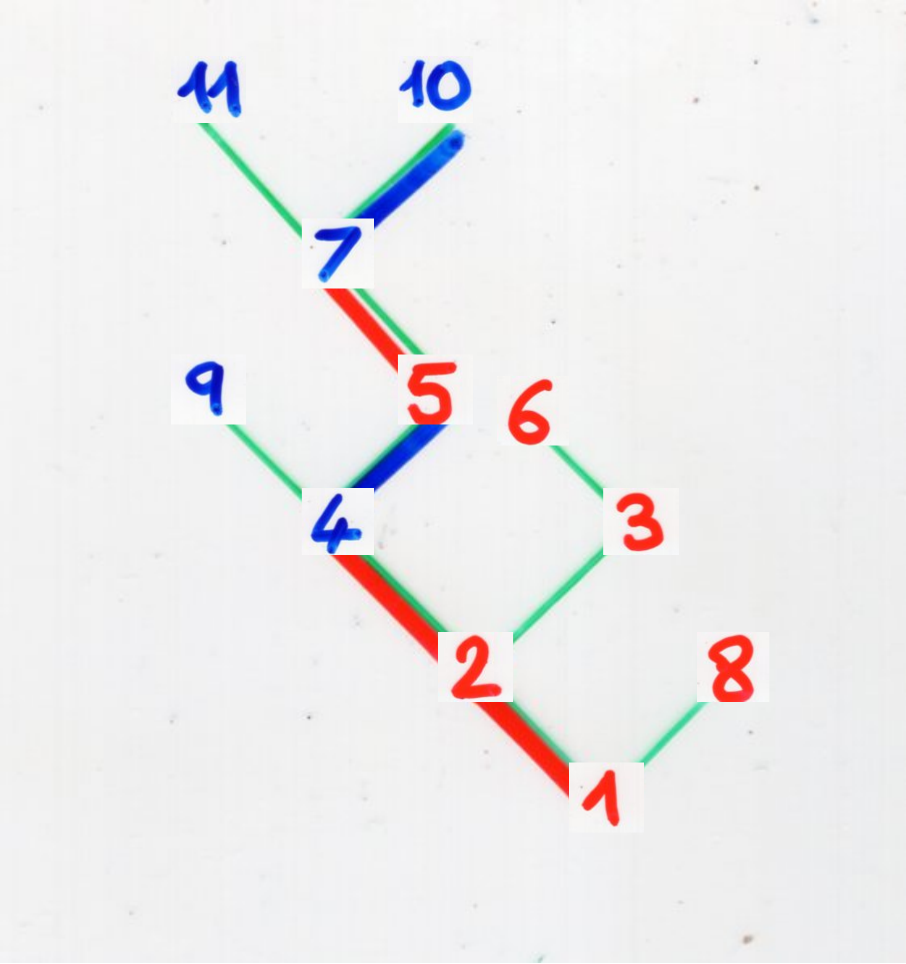




9 4 11 7 10 5 2 6 3 1 8

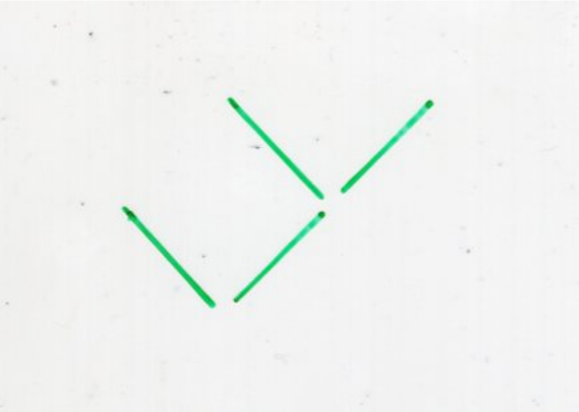
expression for
the Loday-Ronco product

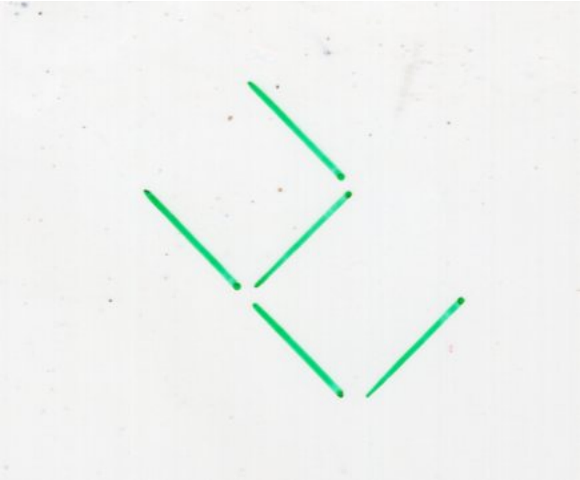
Loday-Ronco
product $*$

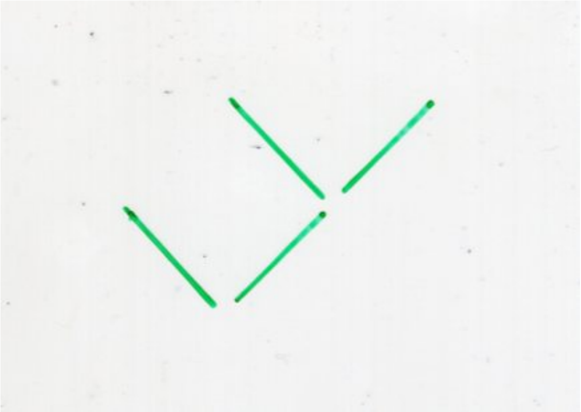
$$B' * B'' = \sum_{B} B$$

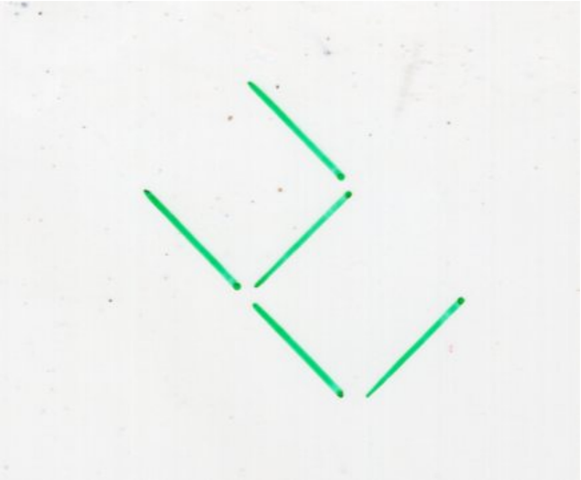
B binary tree obtained with jeu de taquin from all possible Catalan alternative tableaux

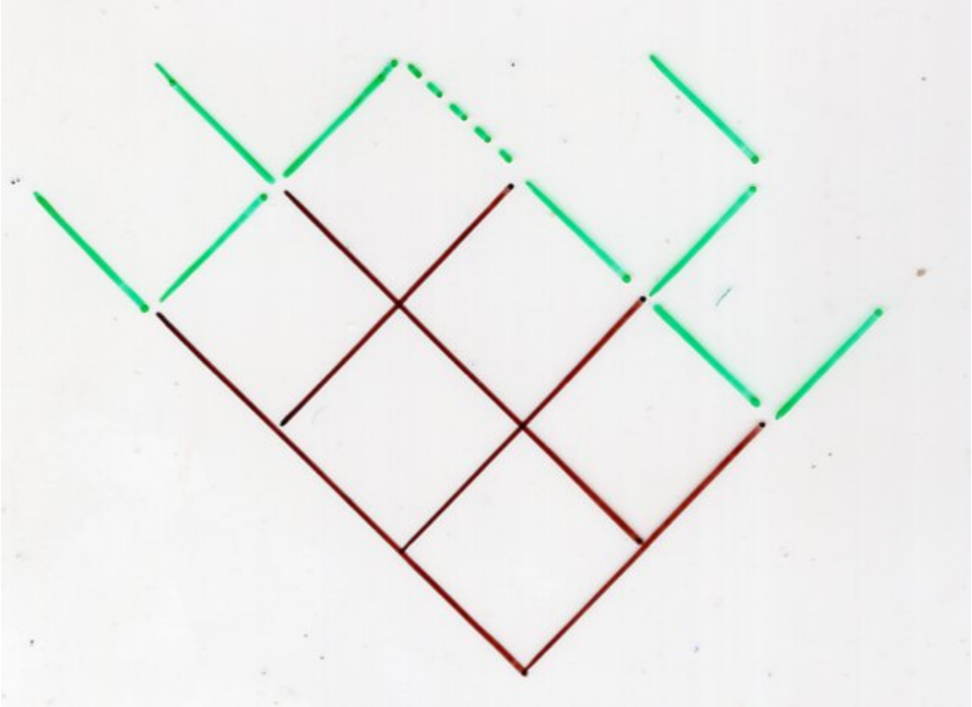
with one of the two rectangular shapes:

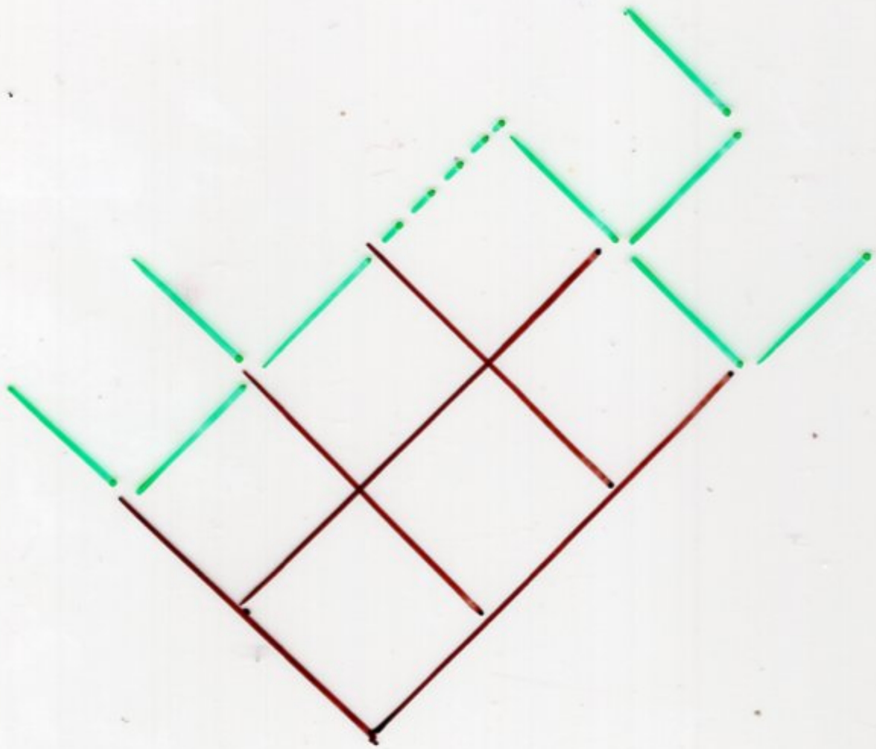


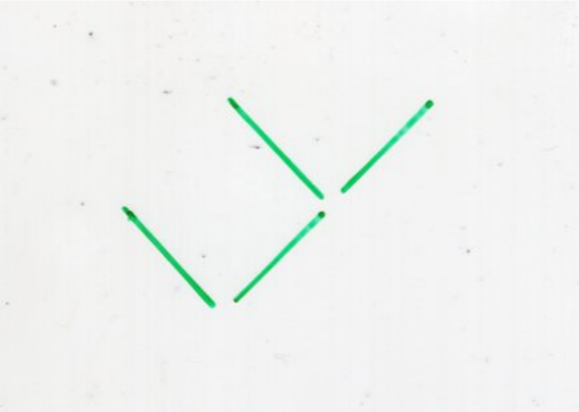


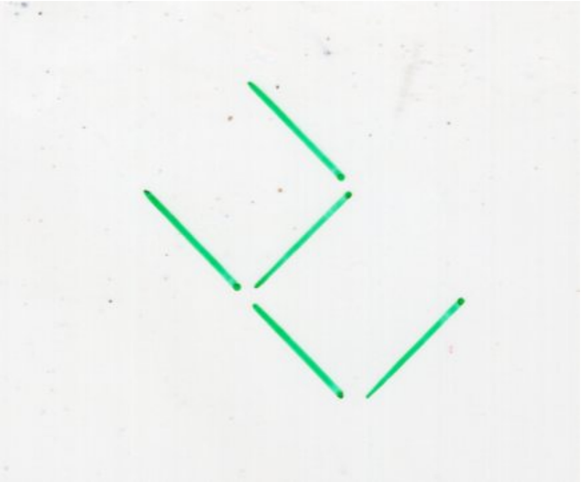




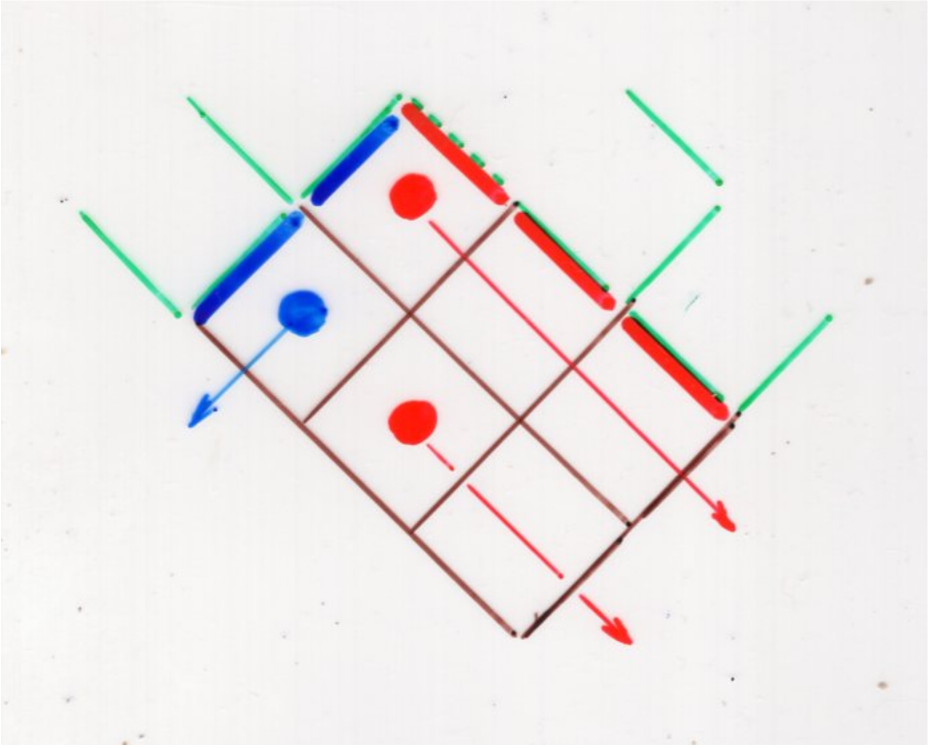


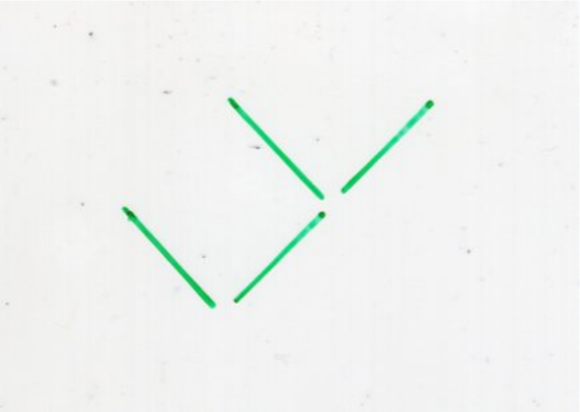


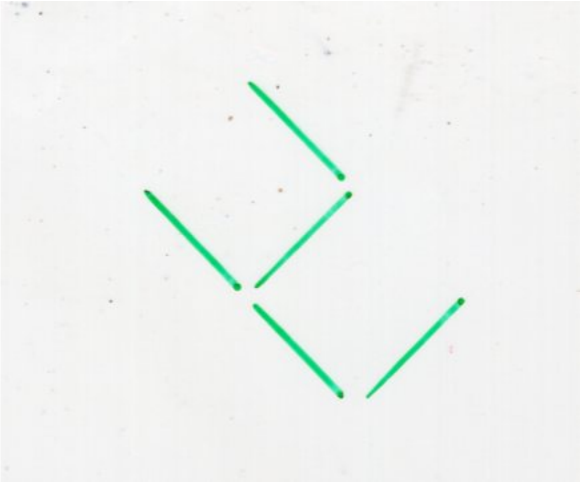




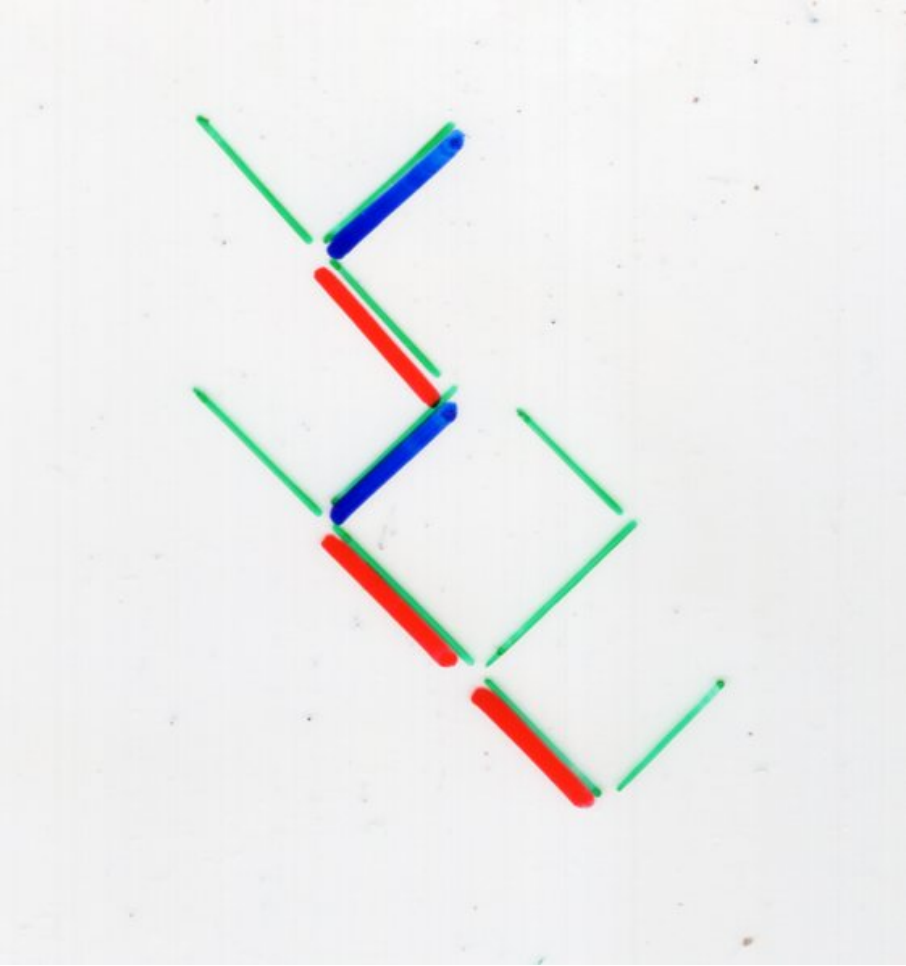
example

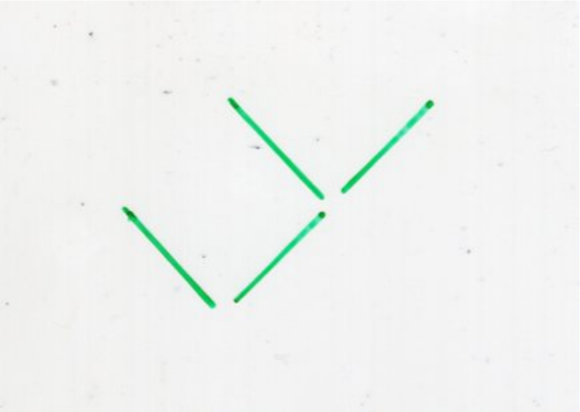


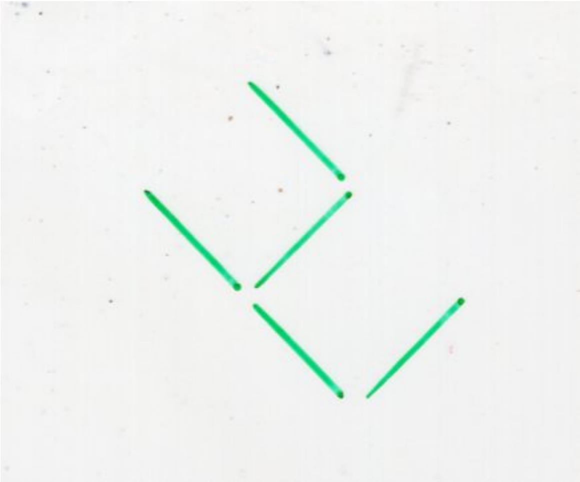




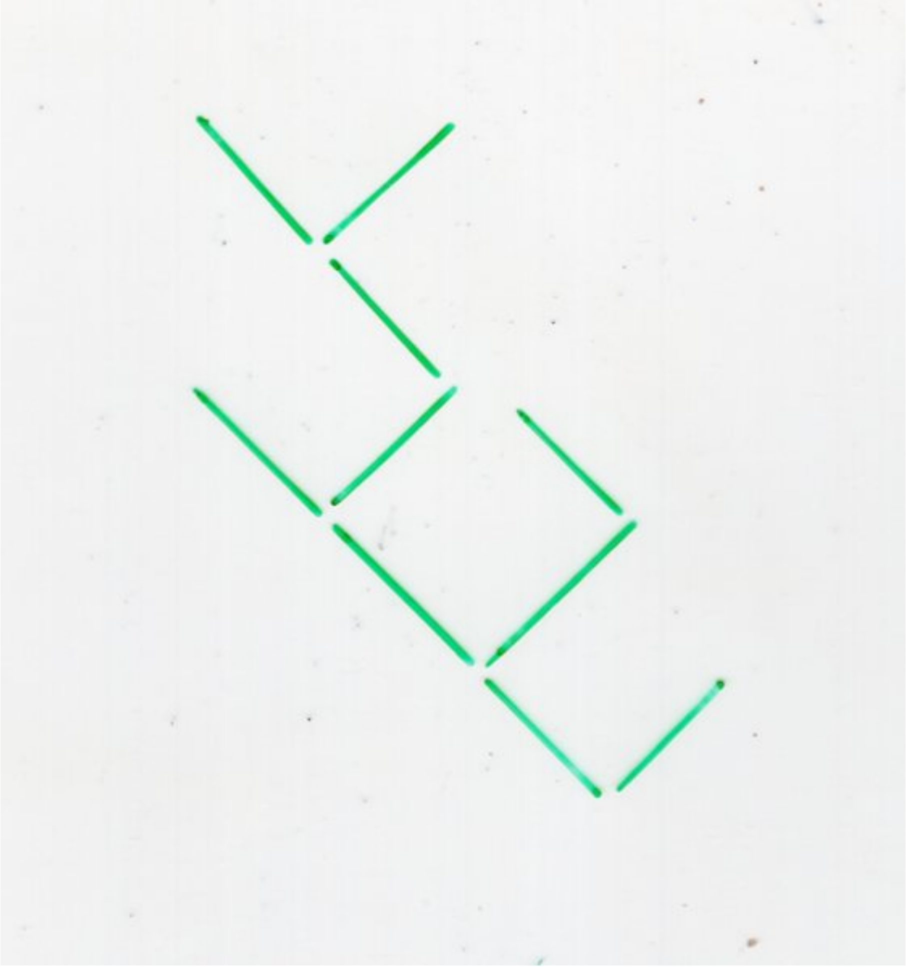
example



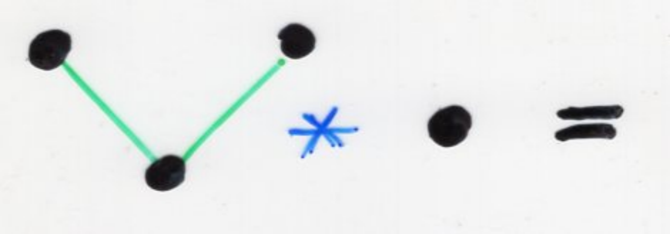


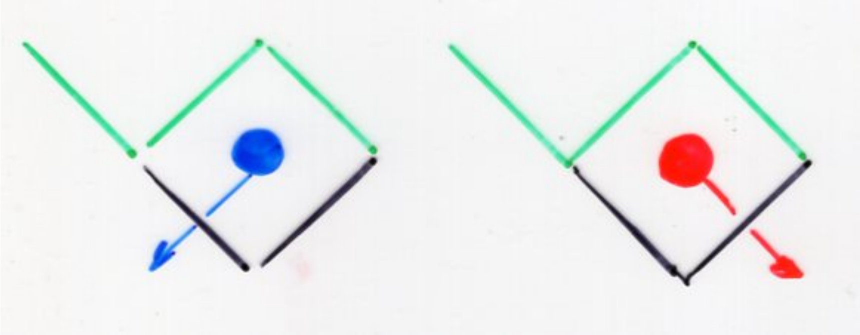


example



example





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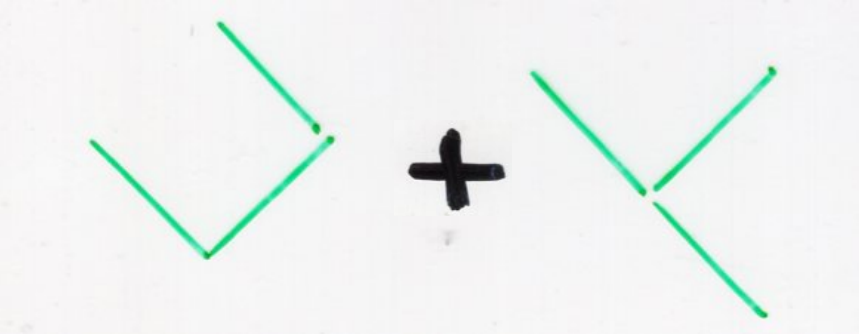

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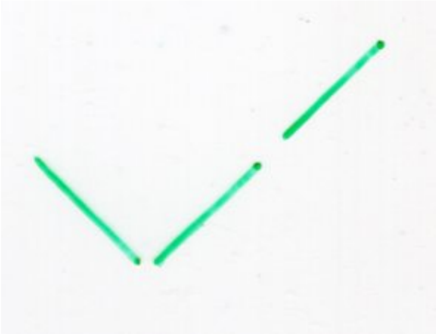
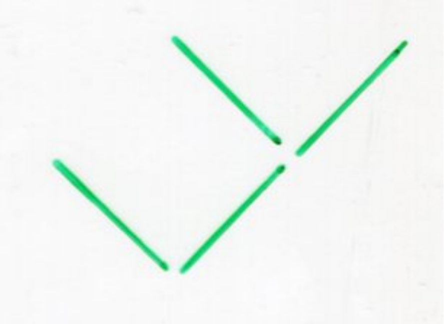
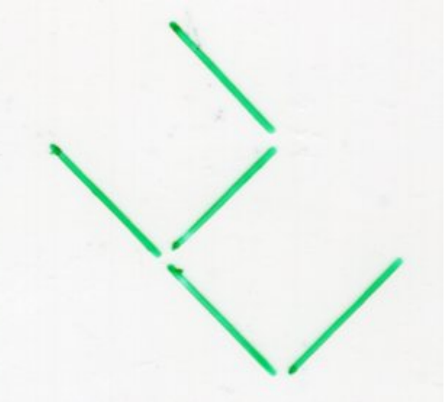
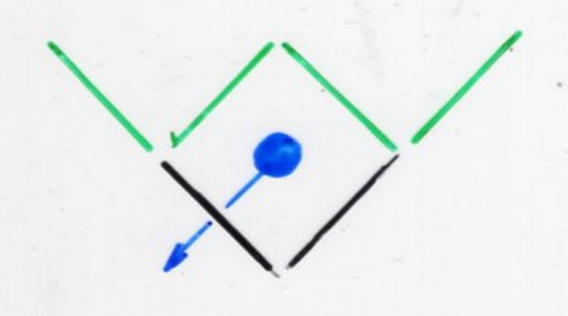


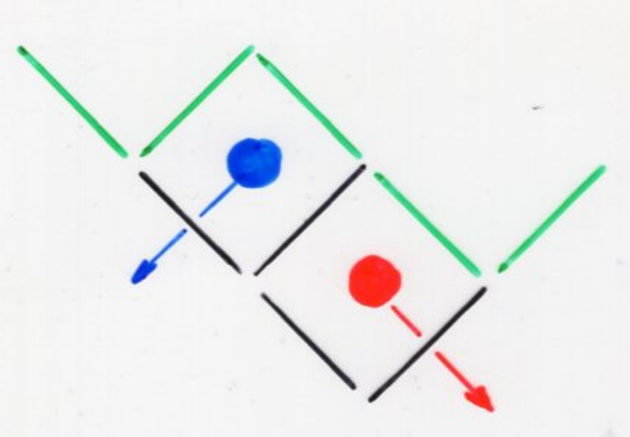
analog of the Loday-Ronco product

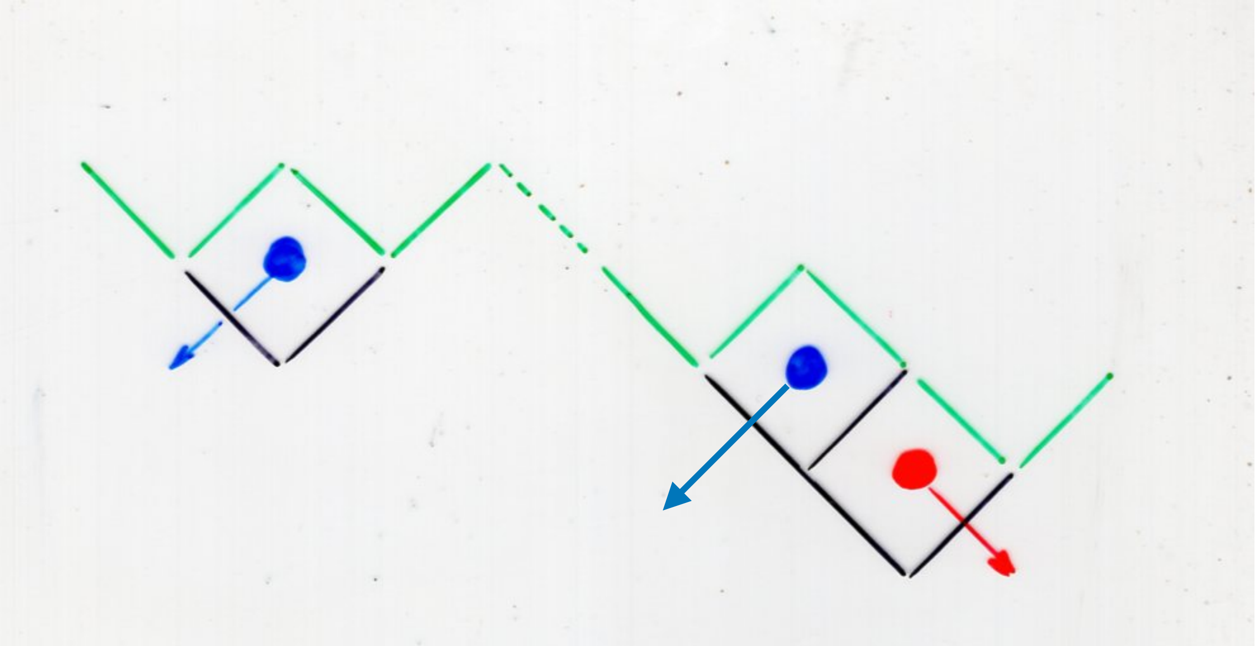
for Catalan alternative tableaux

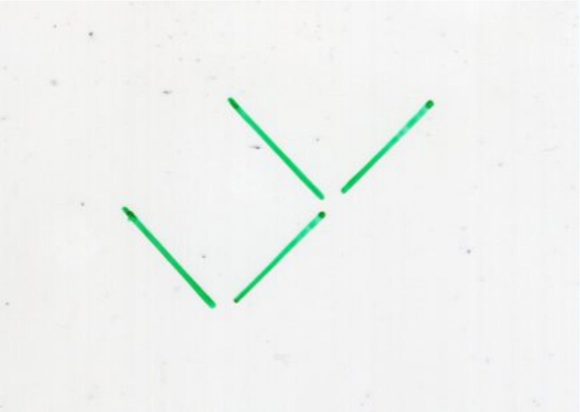


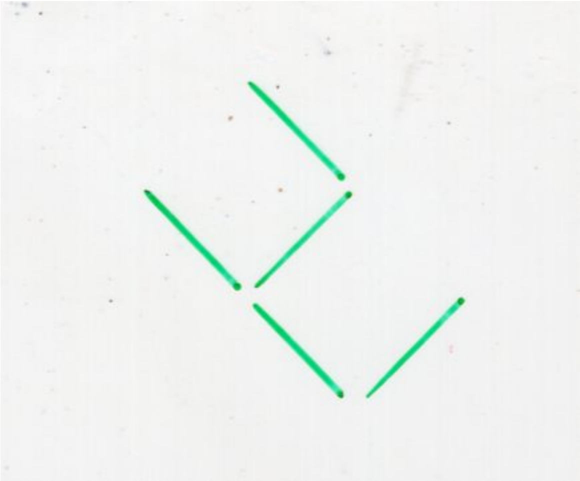


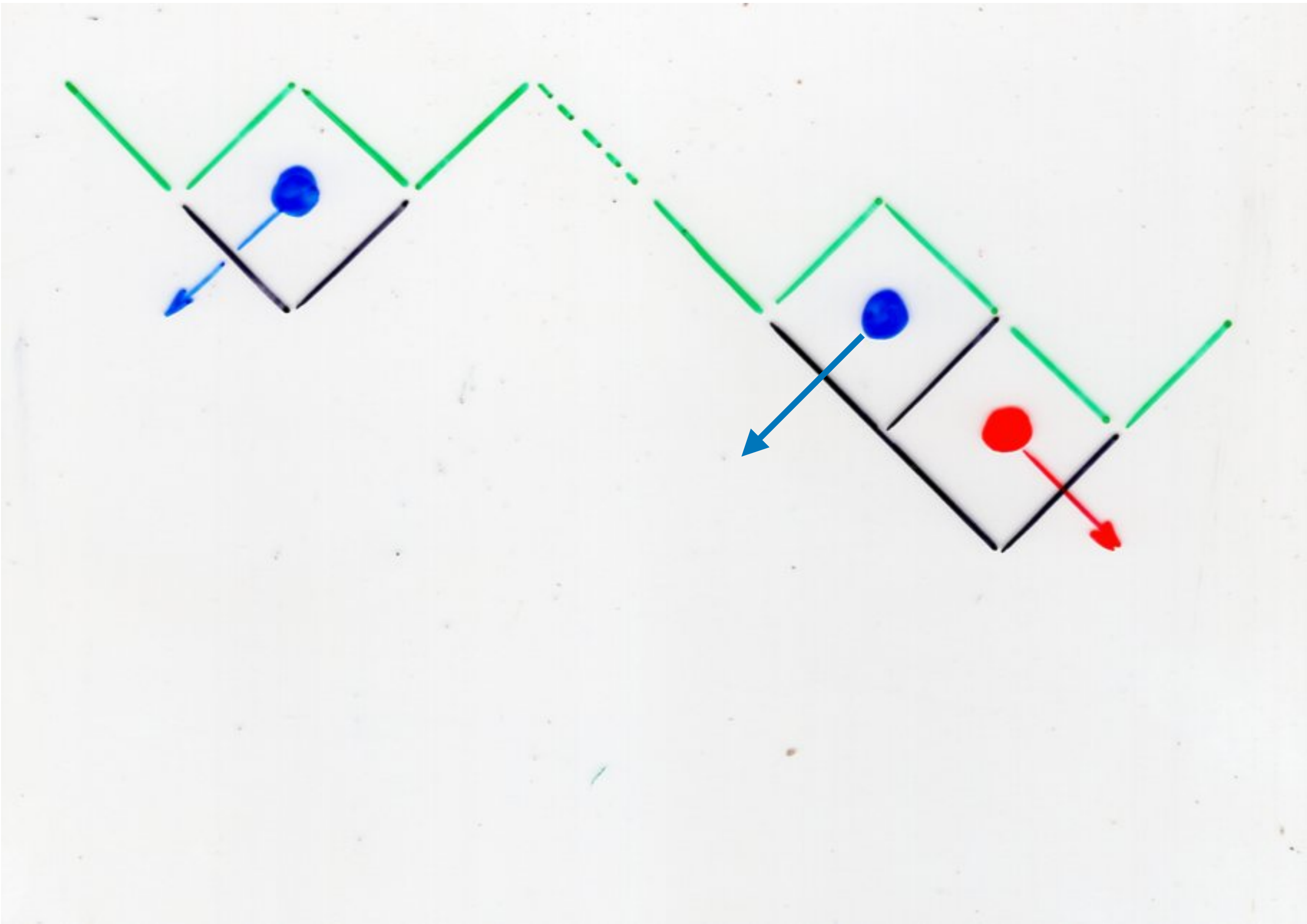


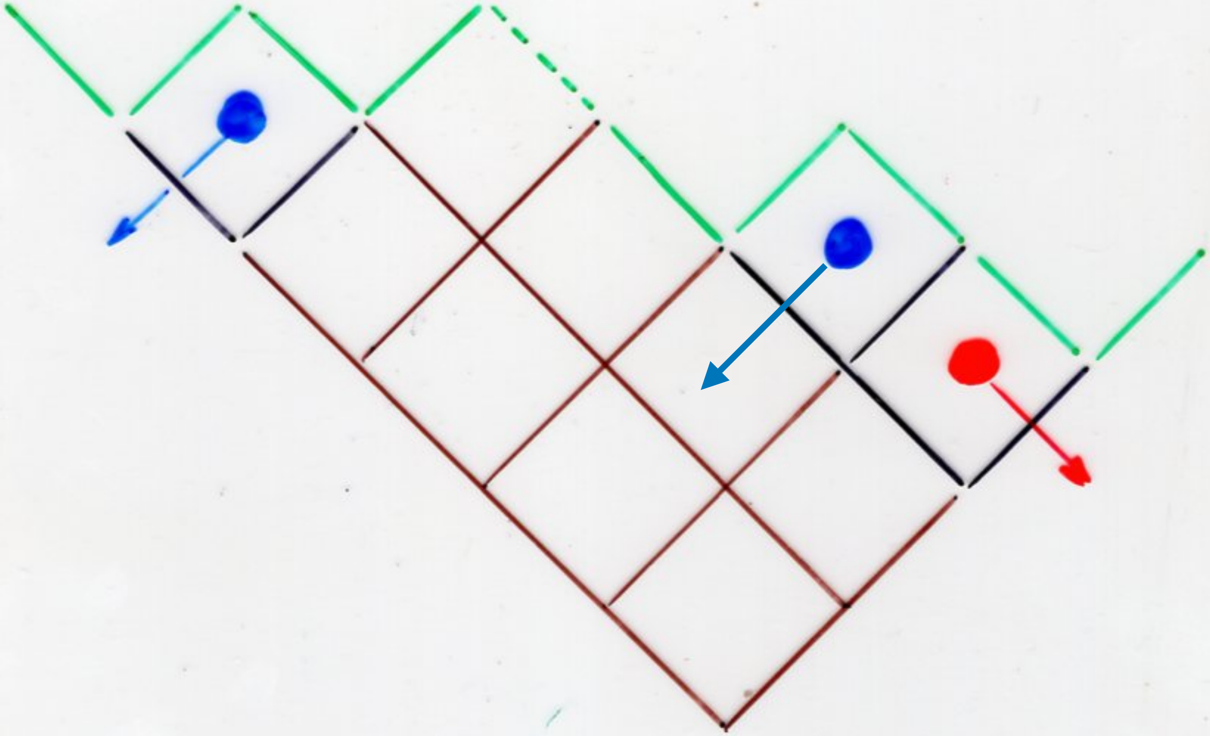


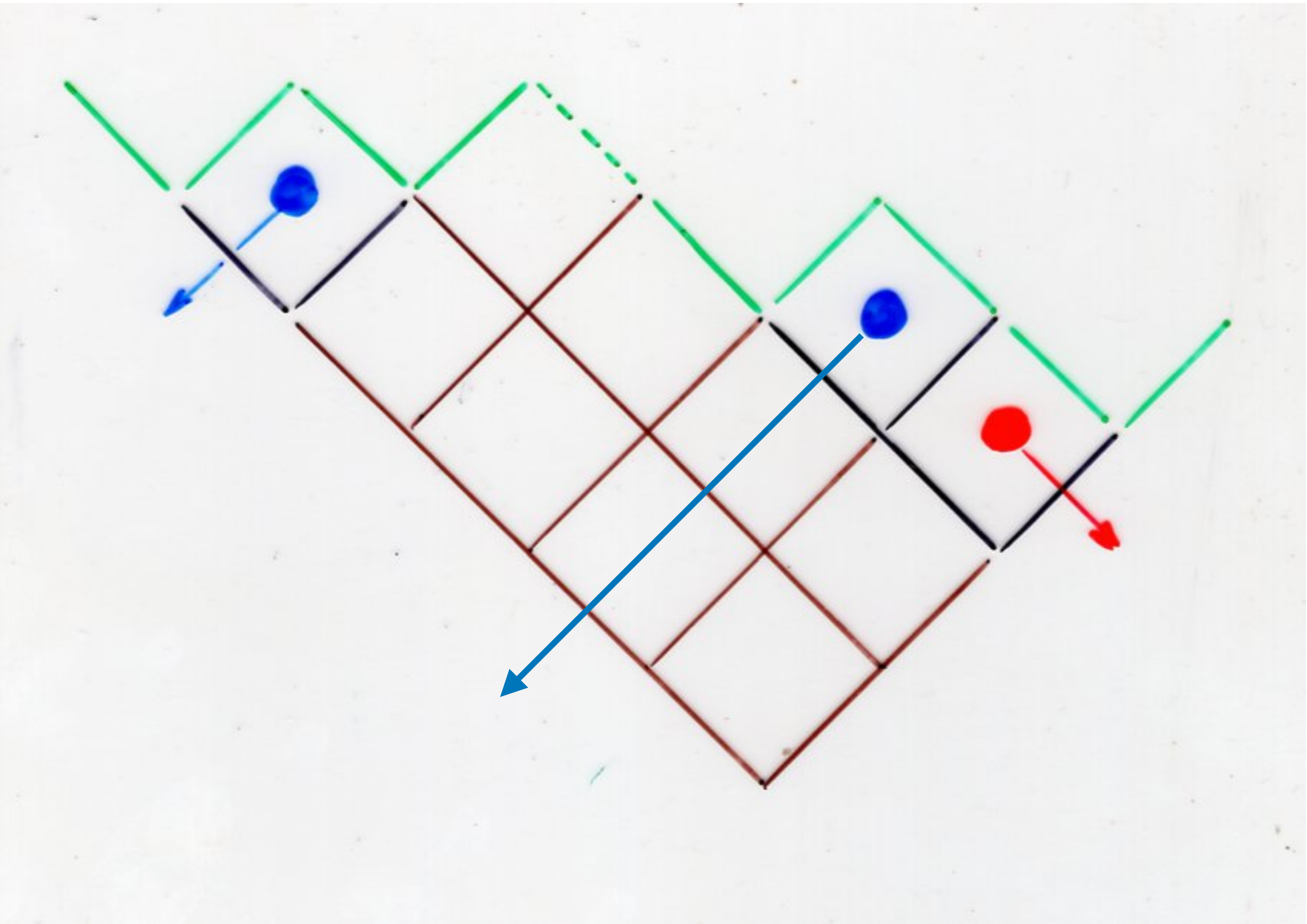


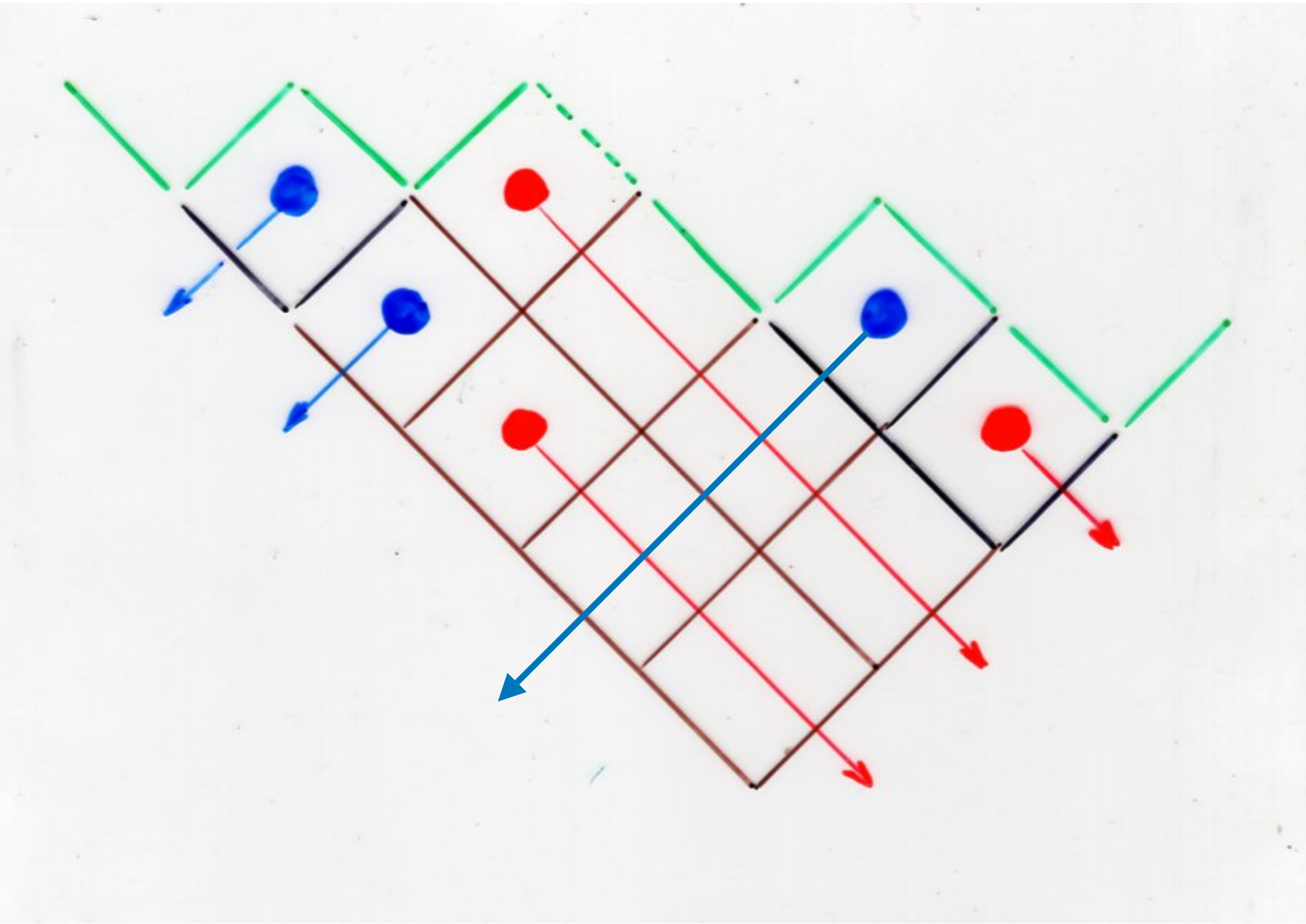


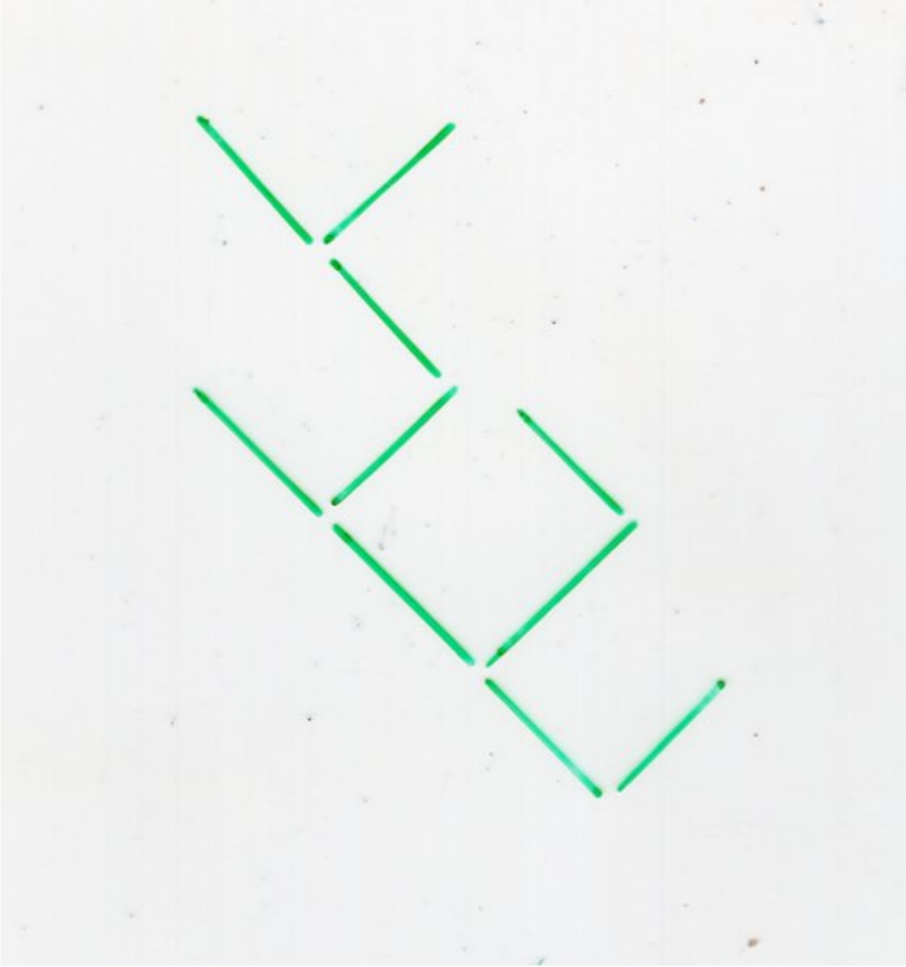


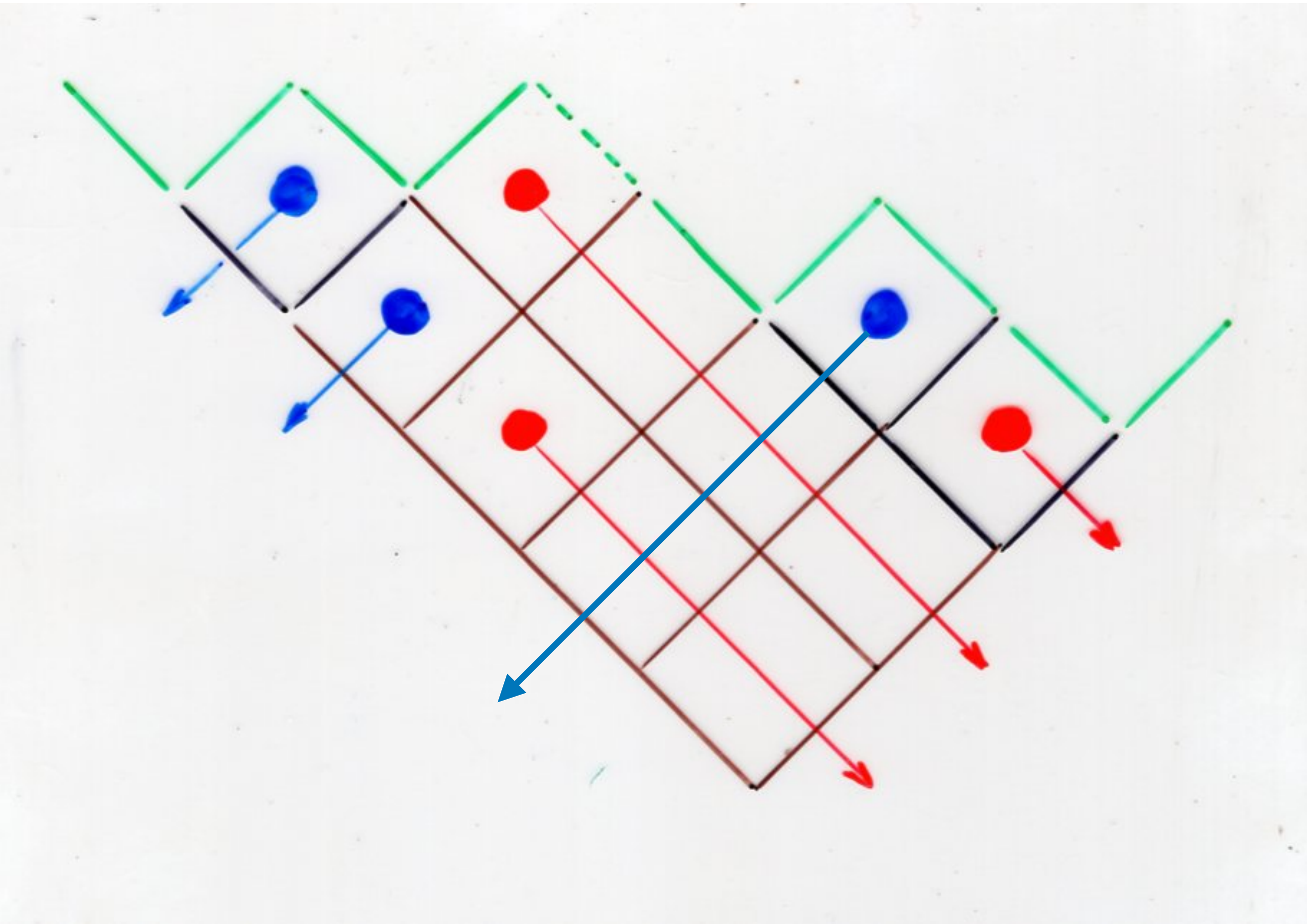


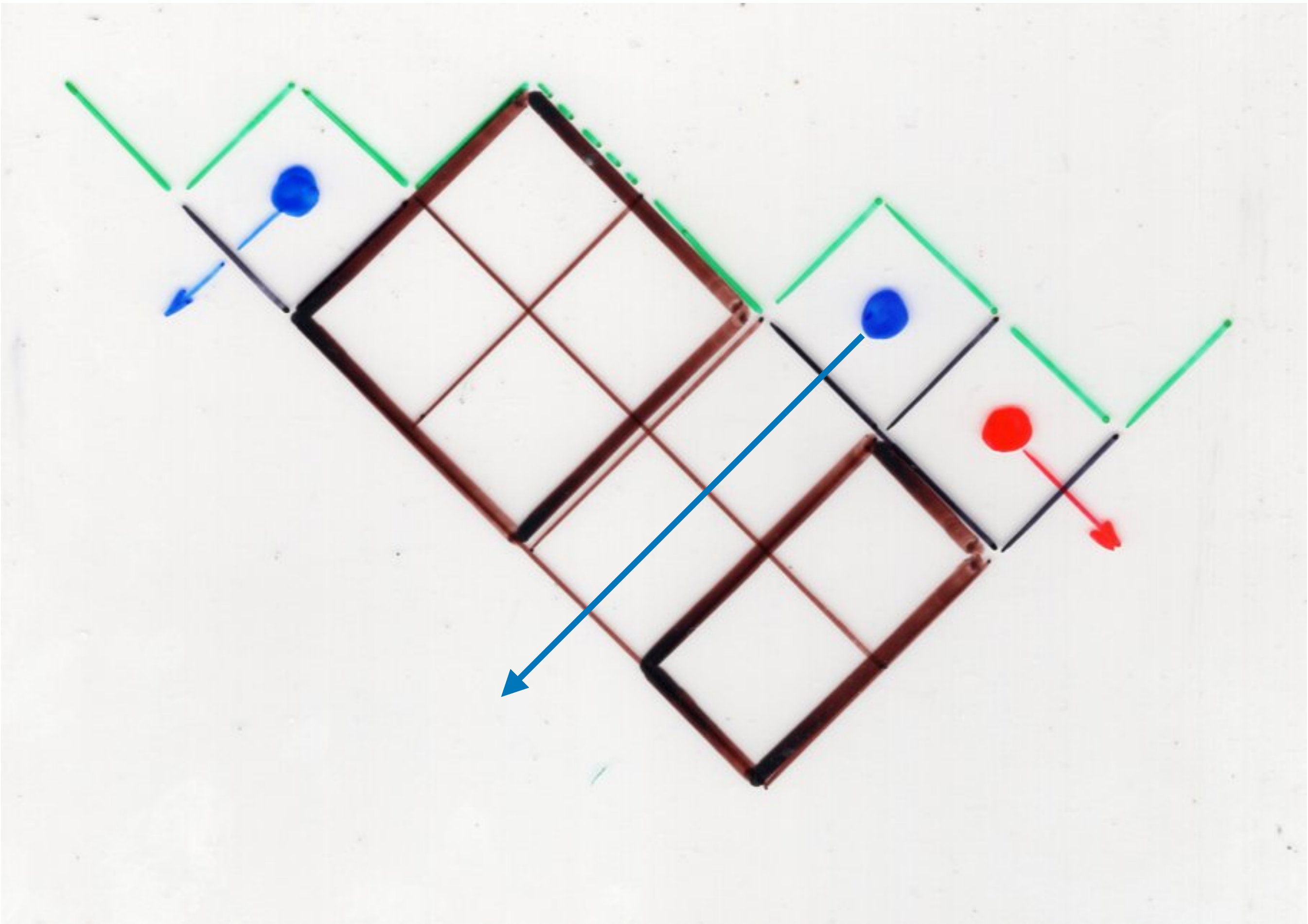


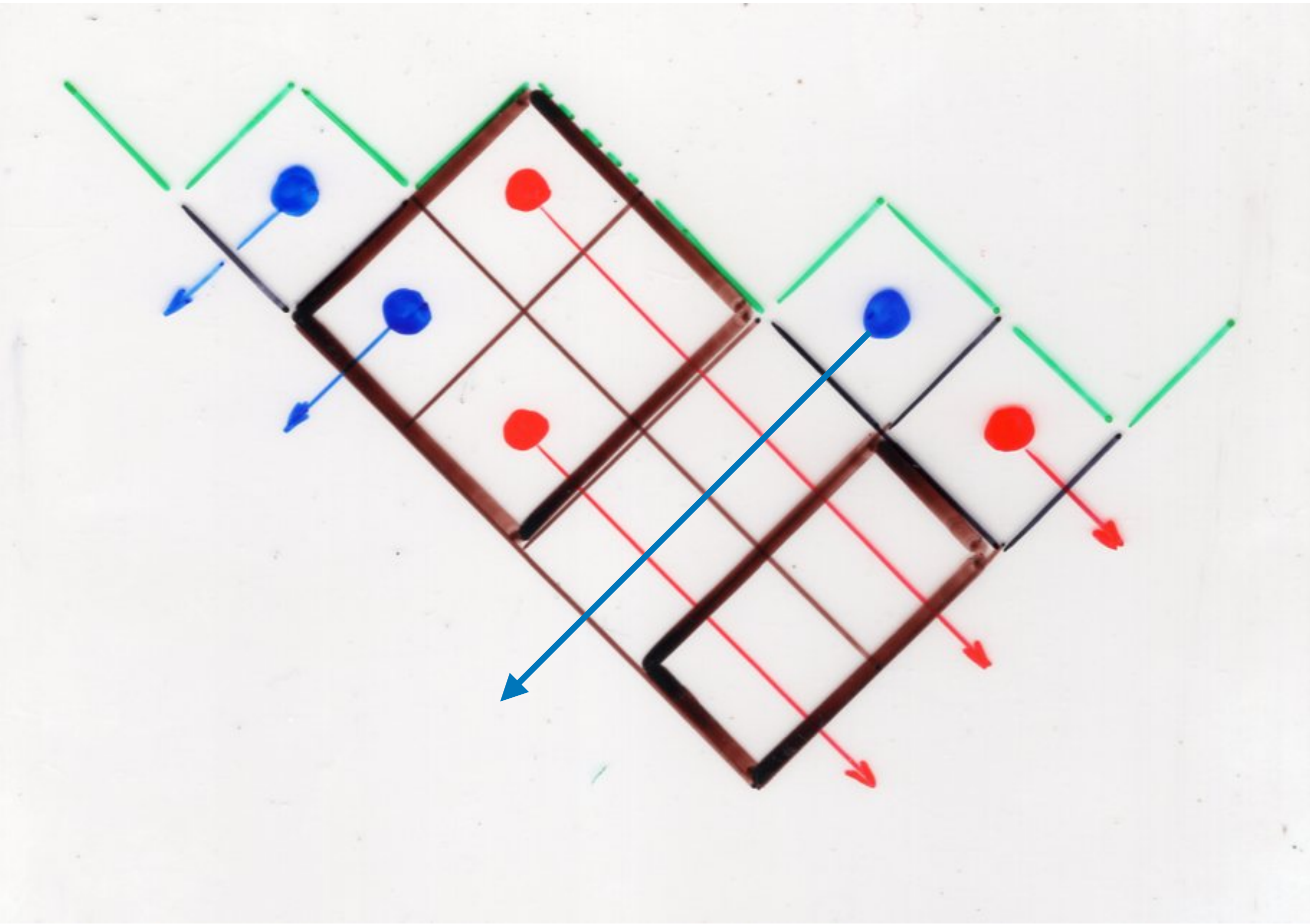


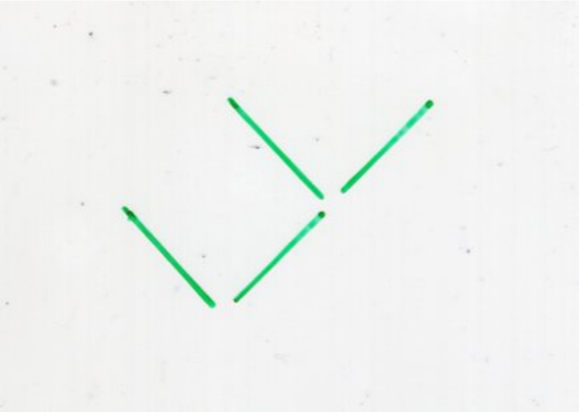


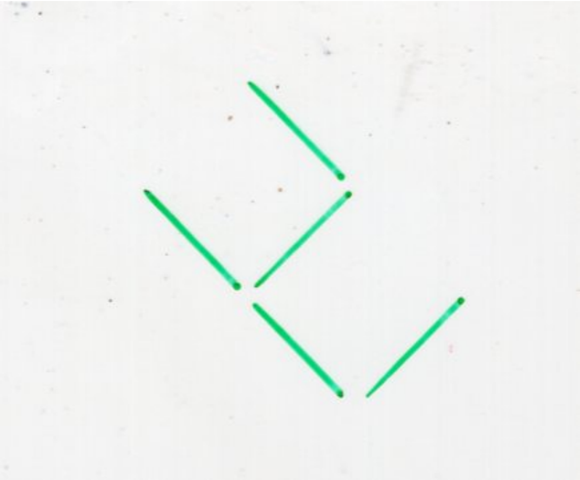


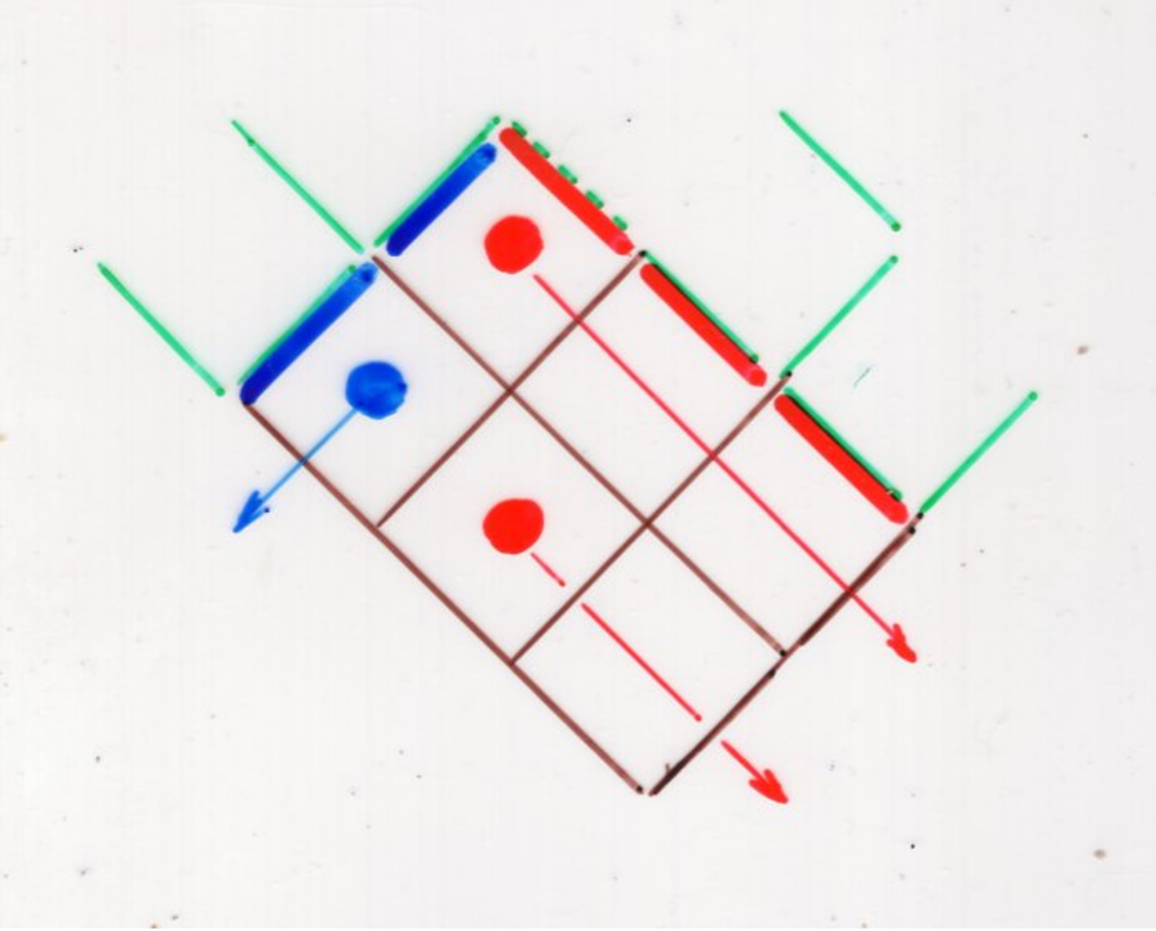


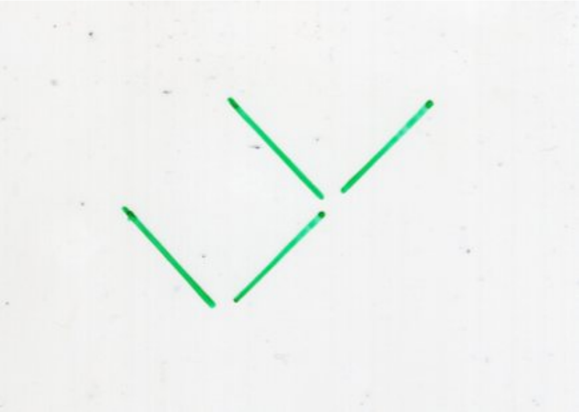


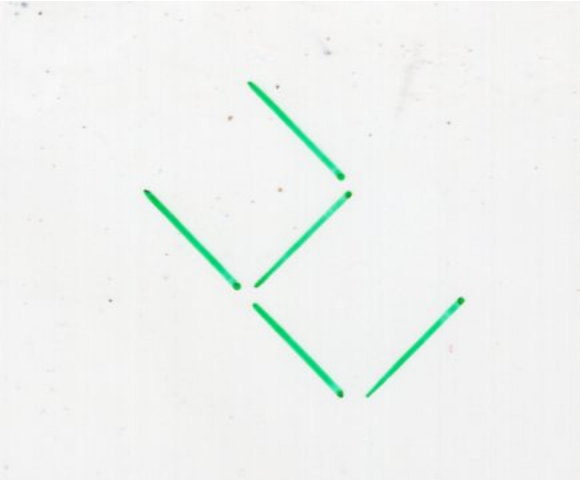


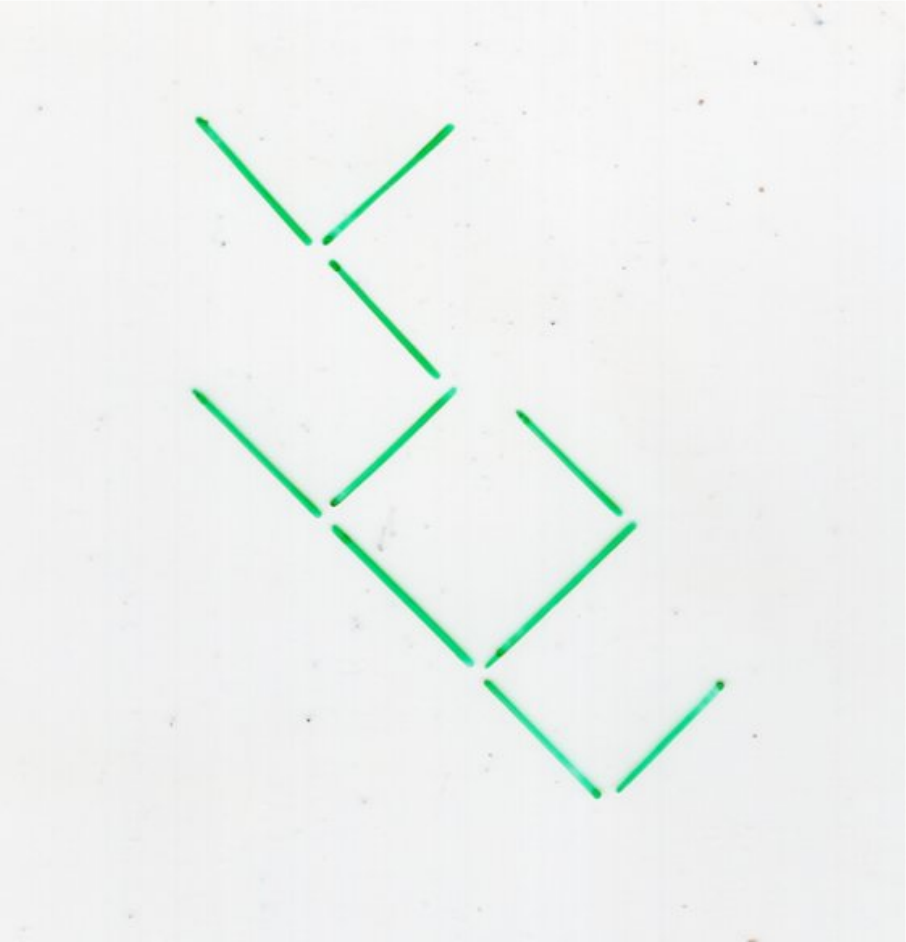


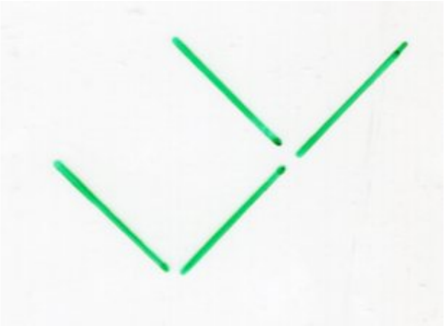


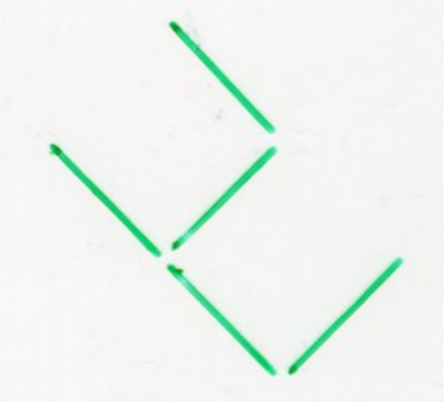


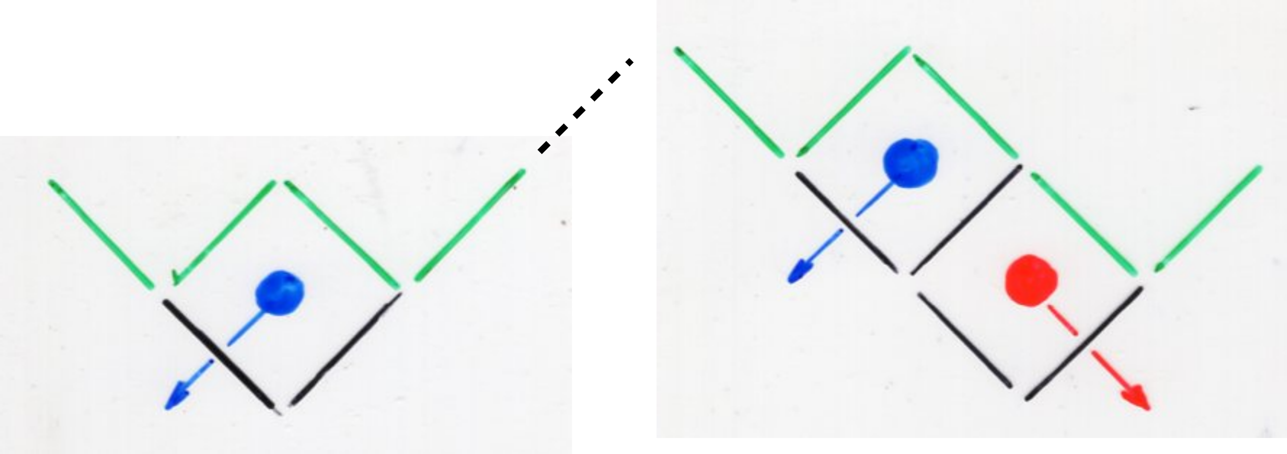


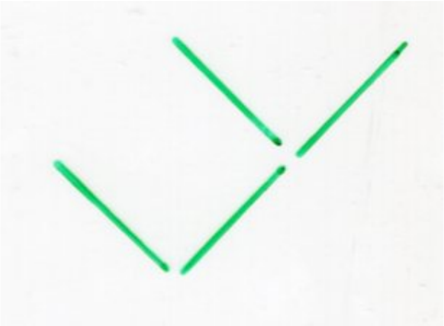


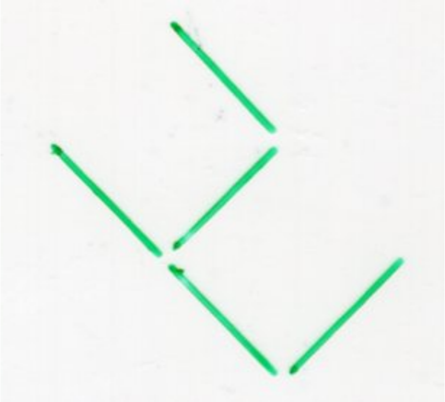


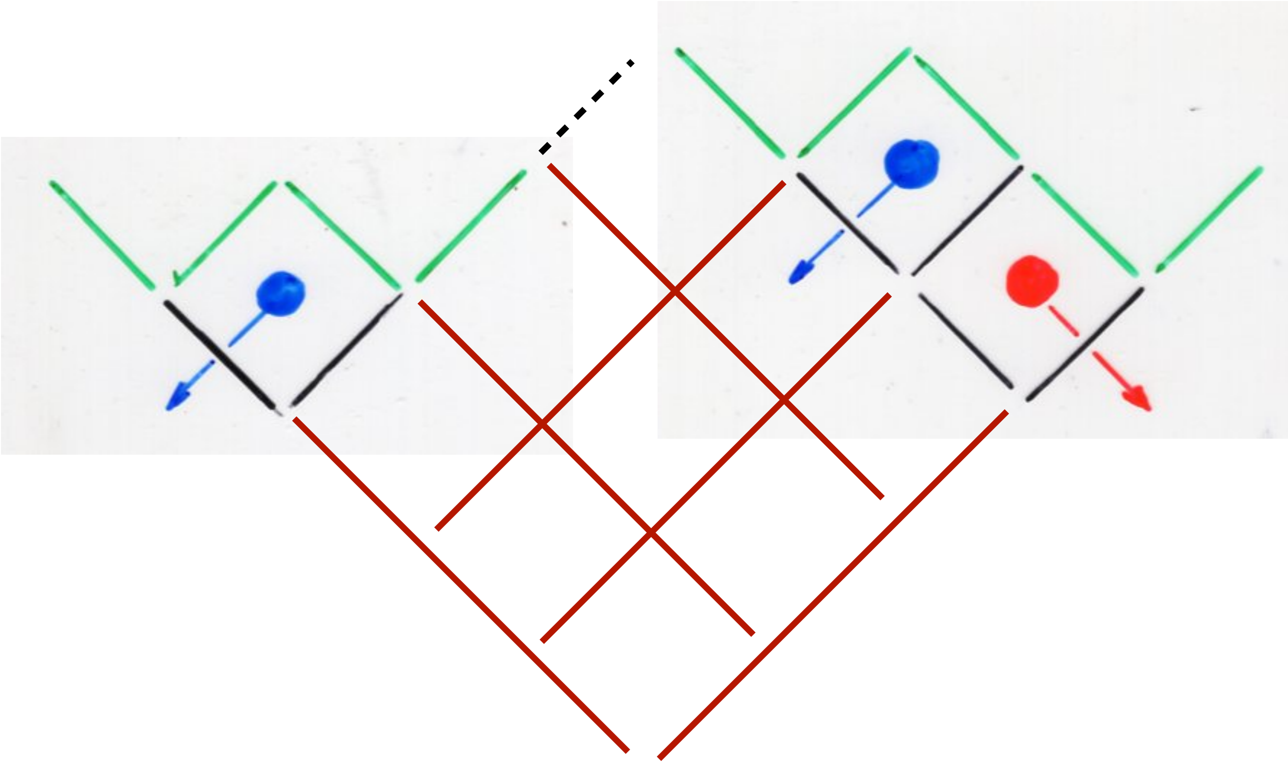




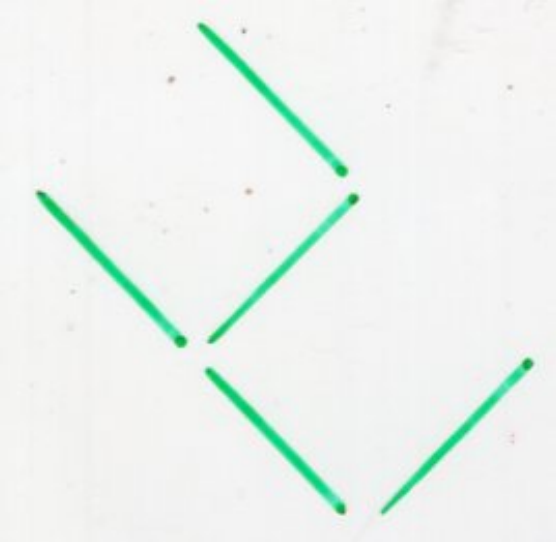








The # product



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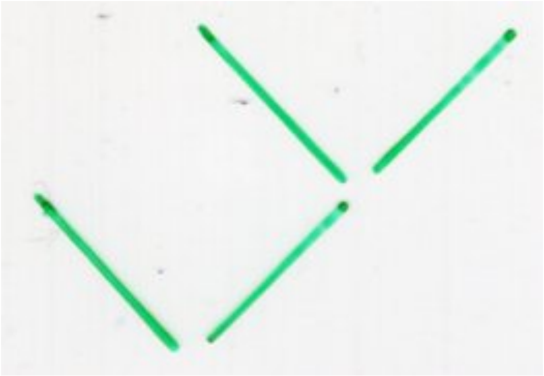
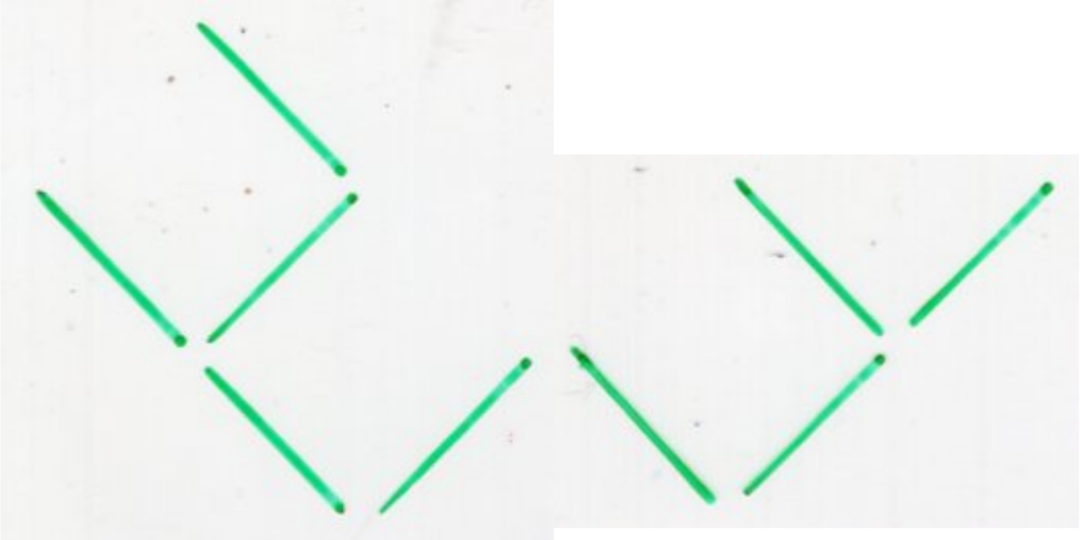
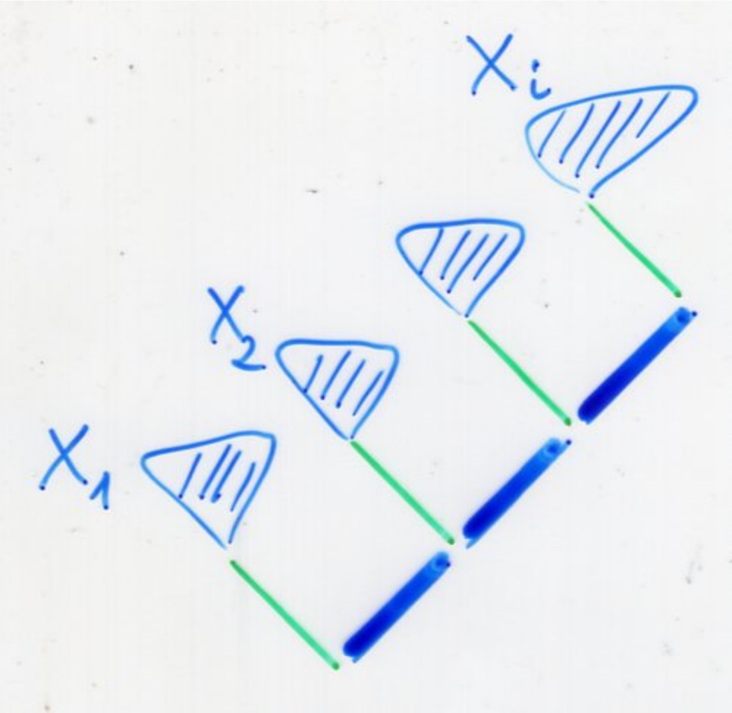
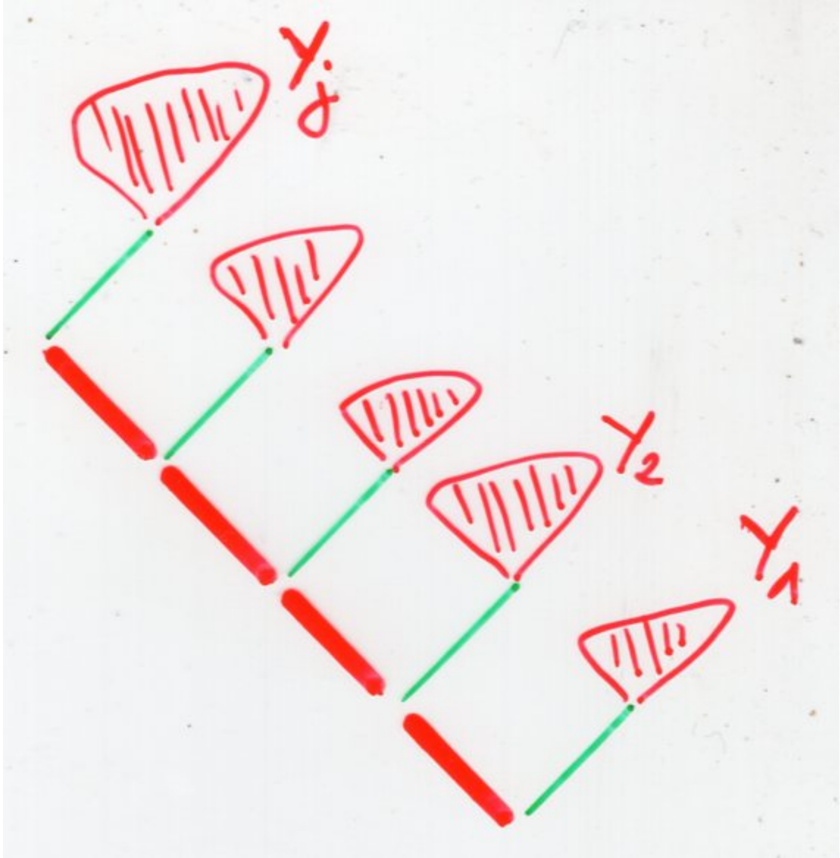


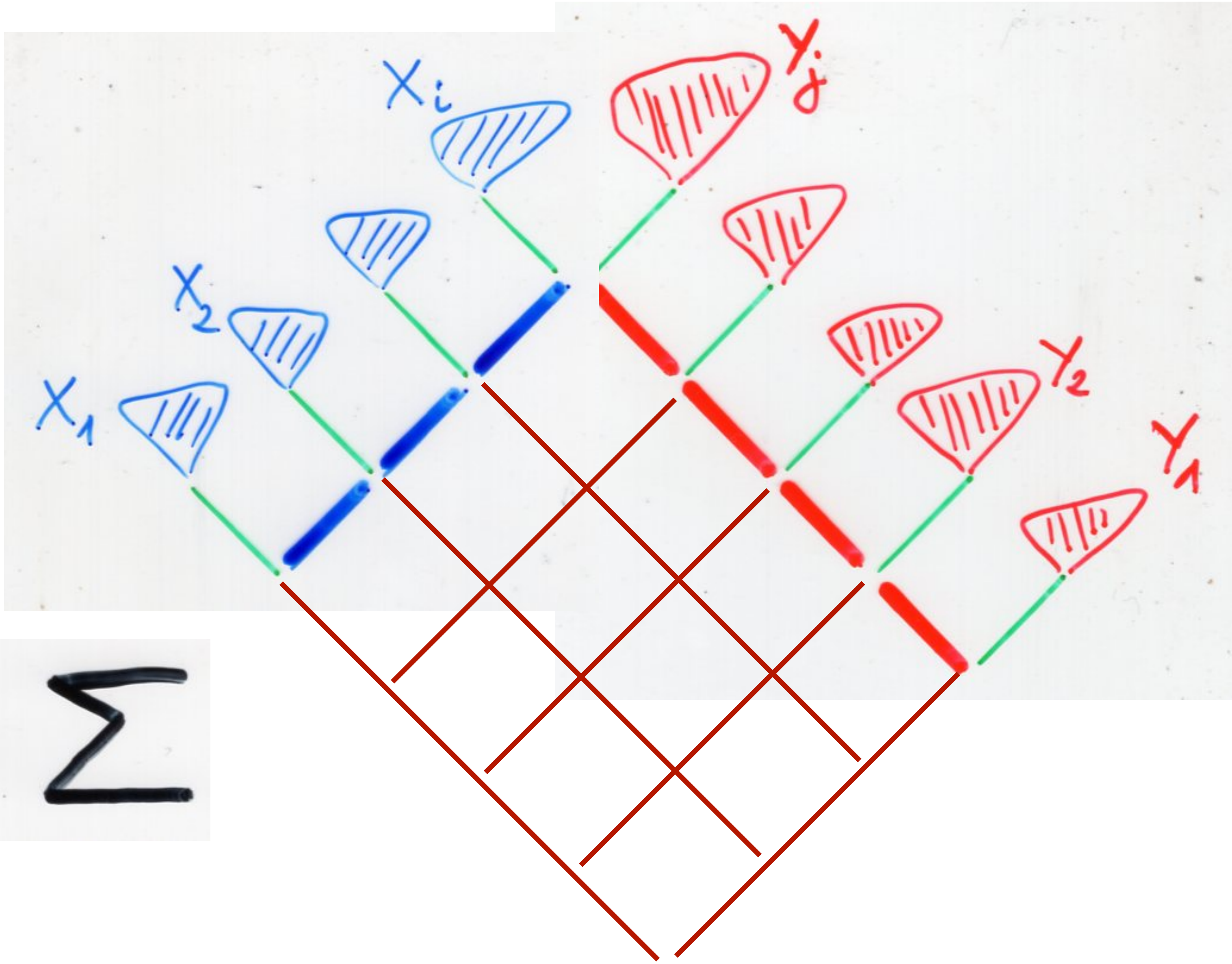






#-product

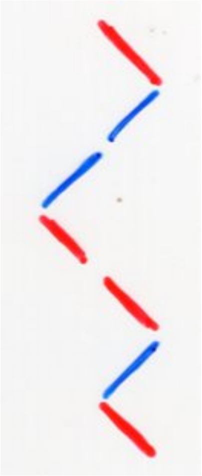
#-product

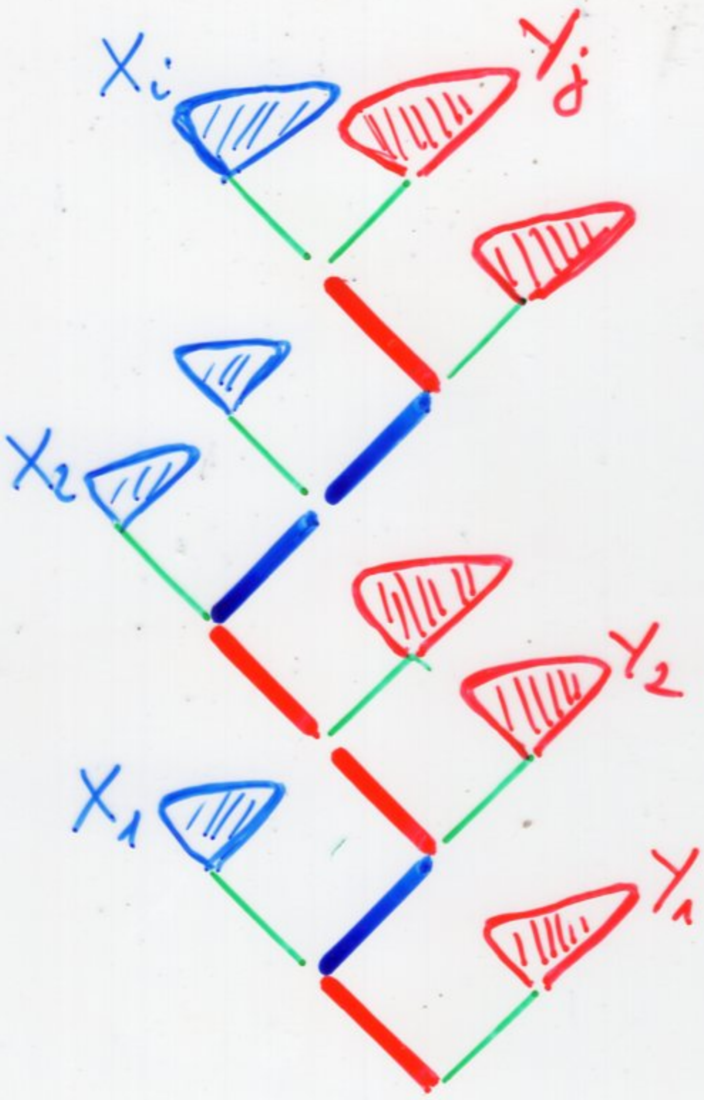


$= \sum$

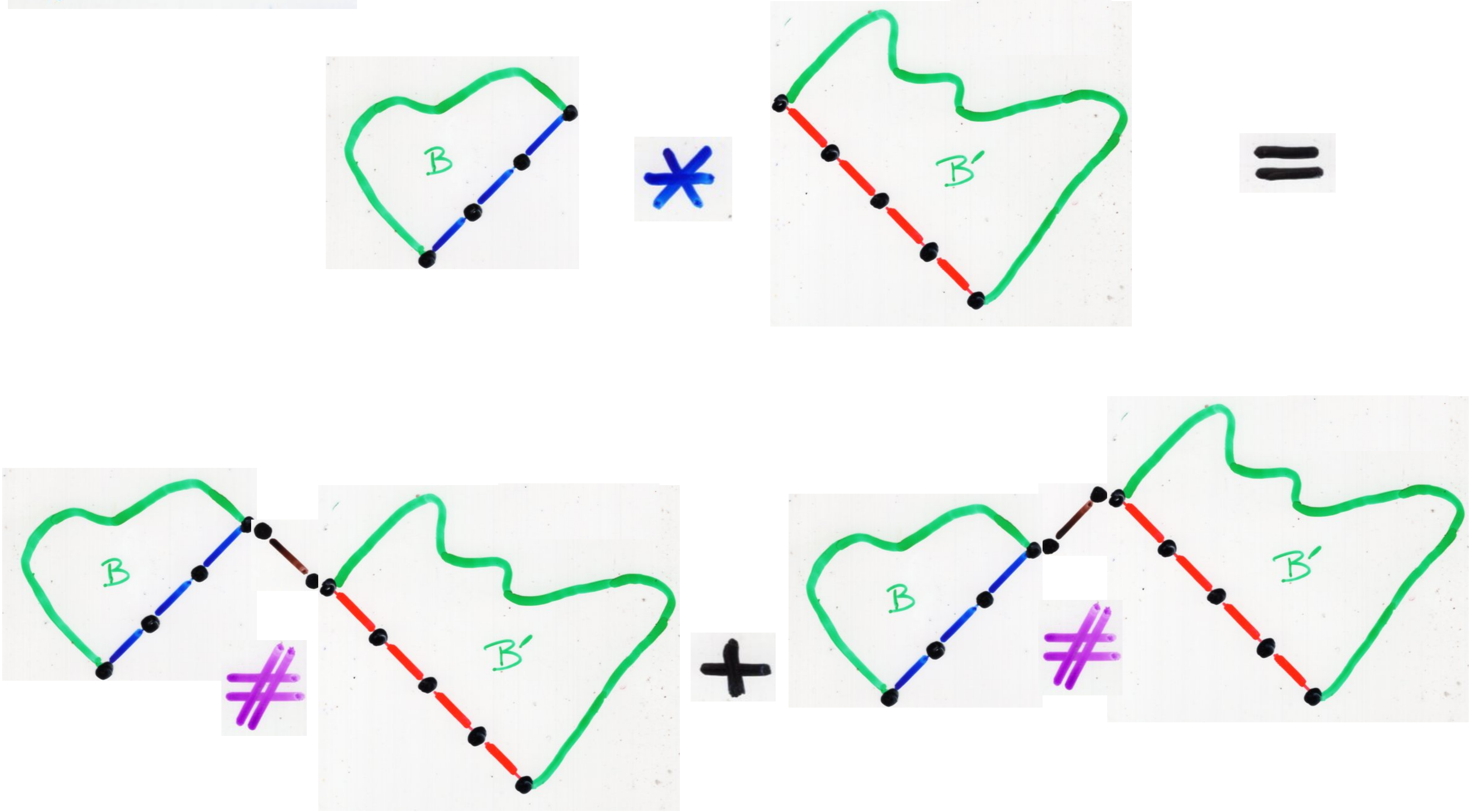
#-product

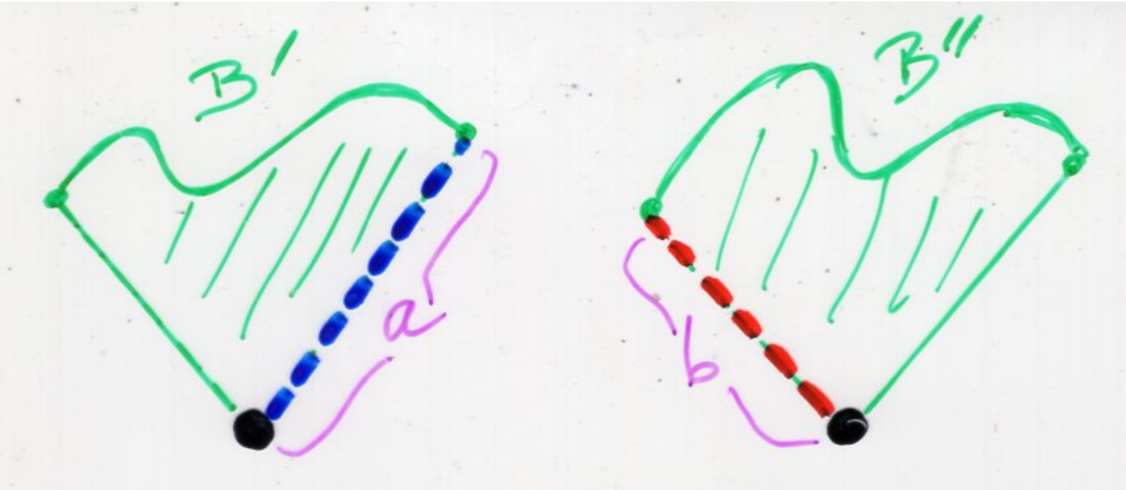
$$= \sum$$





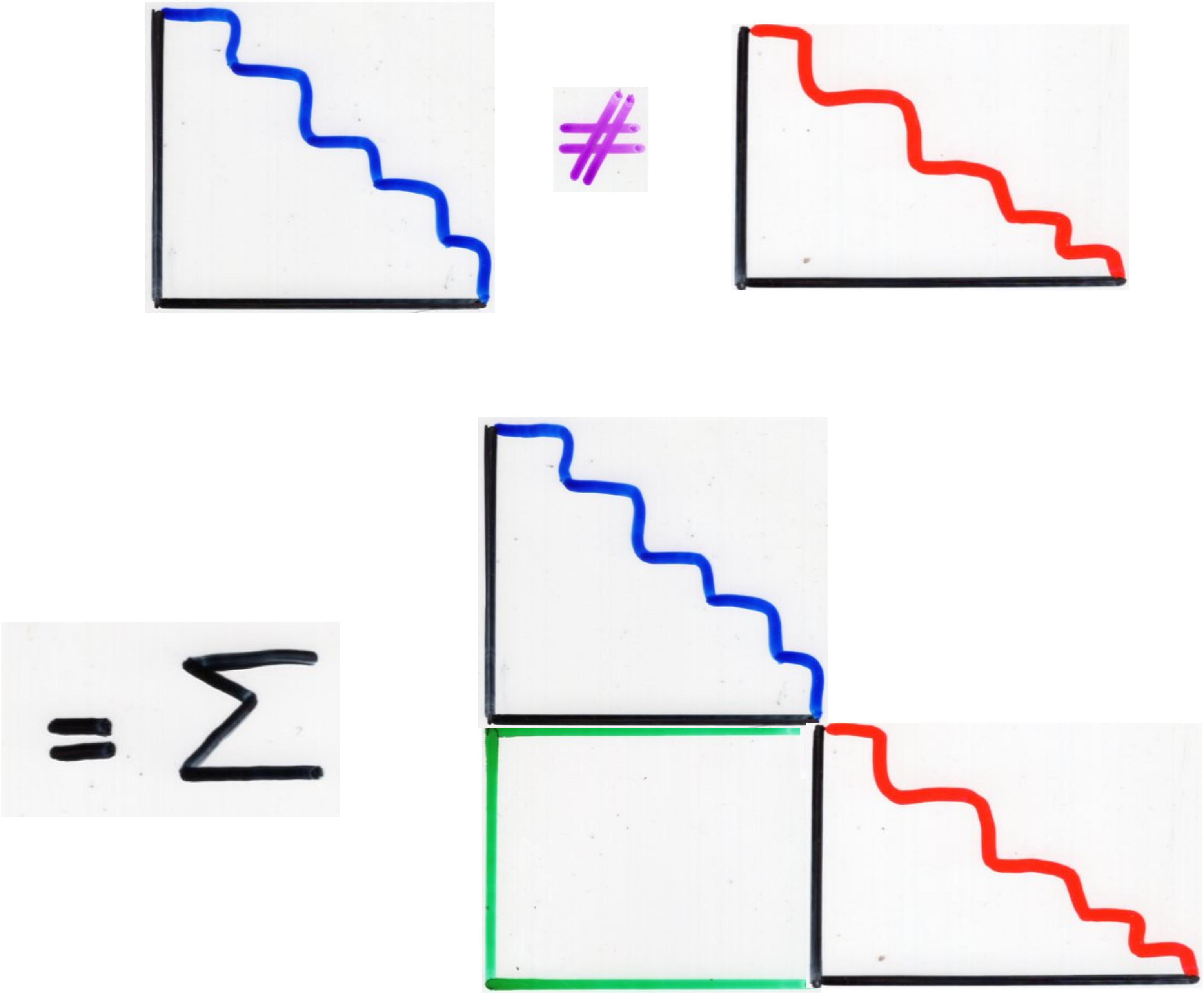
Loday-Ronco product *





The number of binary trees B in the product $B' * B''$ is

$$\binom{a+b+1}{a} + \binom{a+b+1}{a+1} = \binom{a+b+2}{a+1}$$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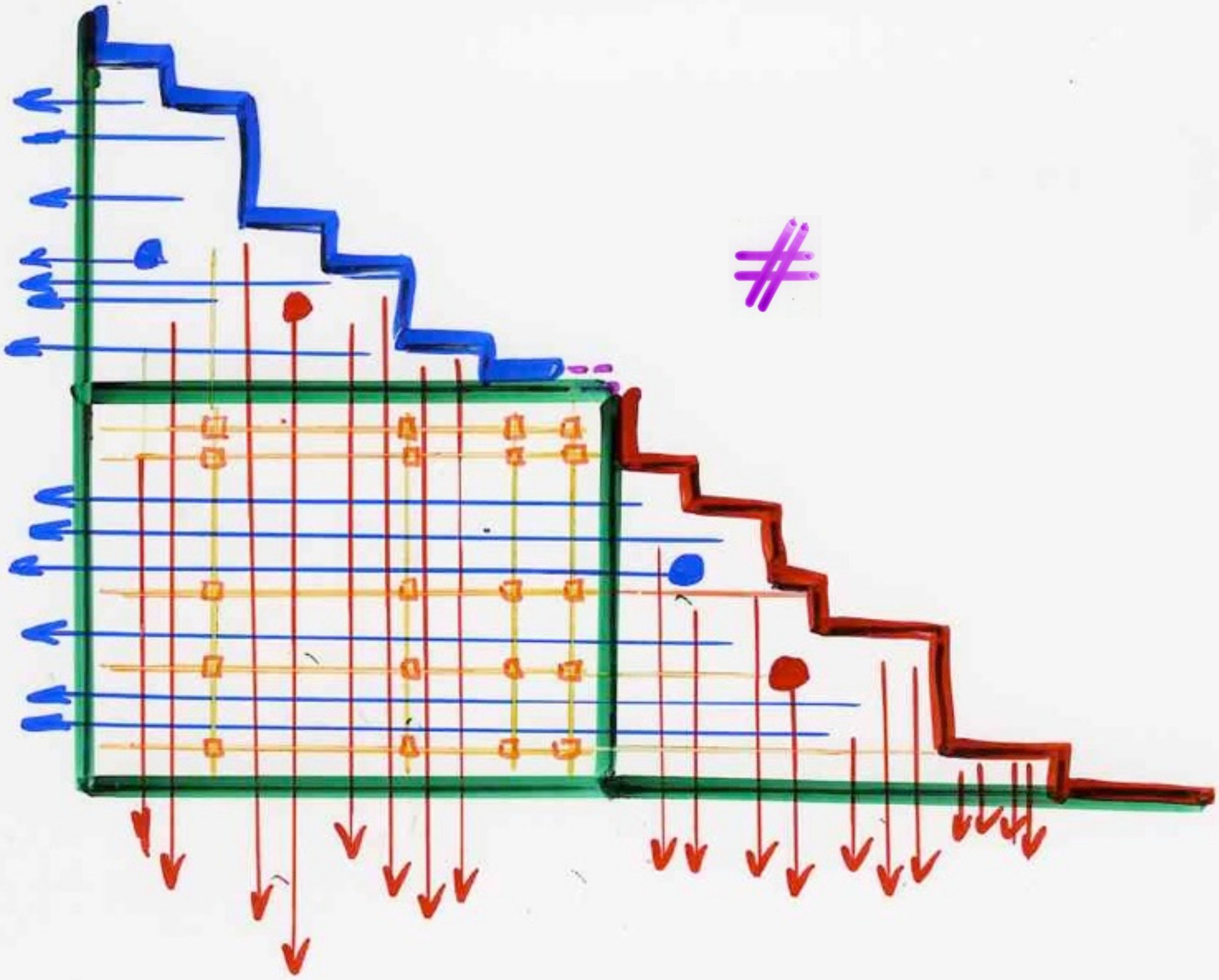
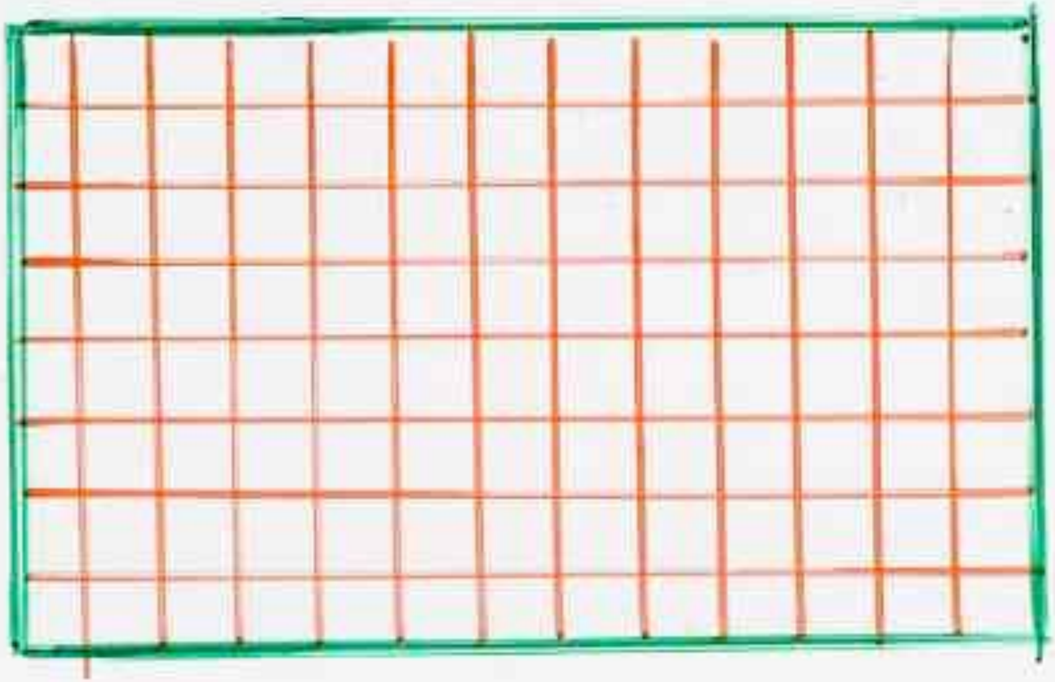
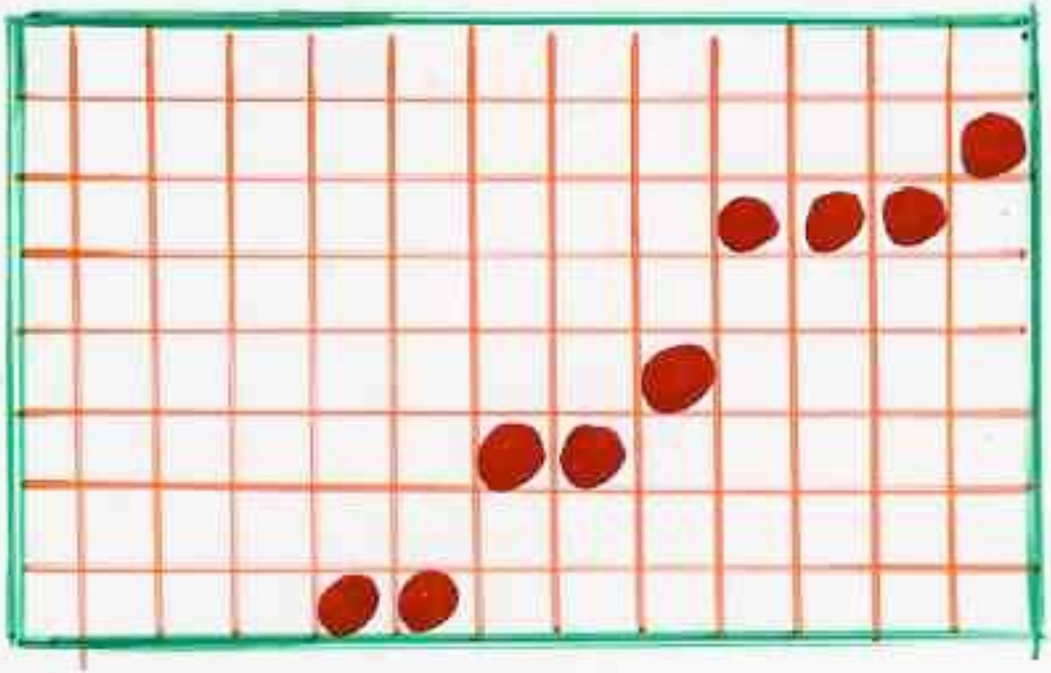
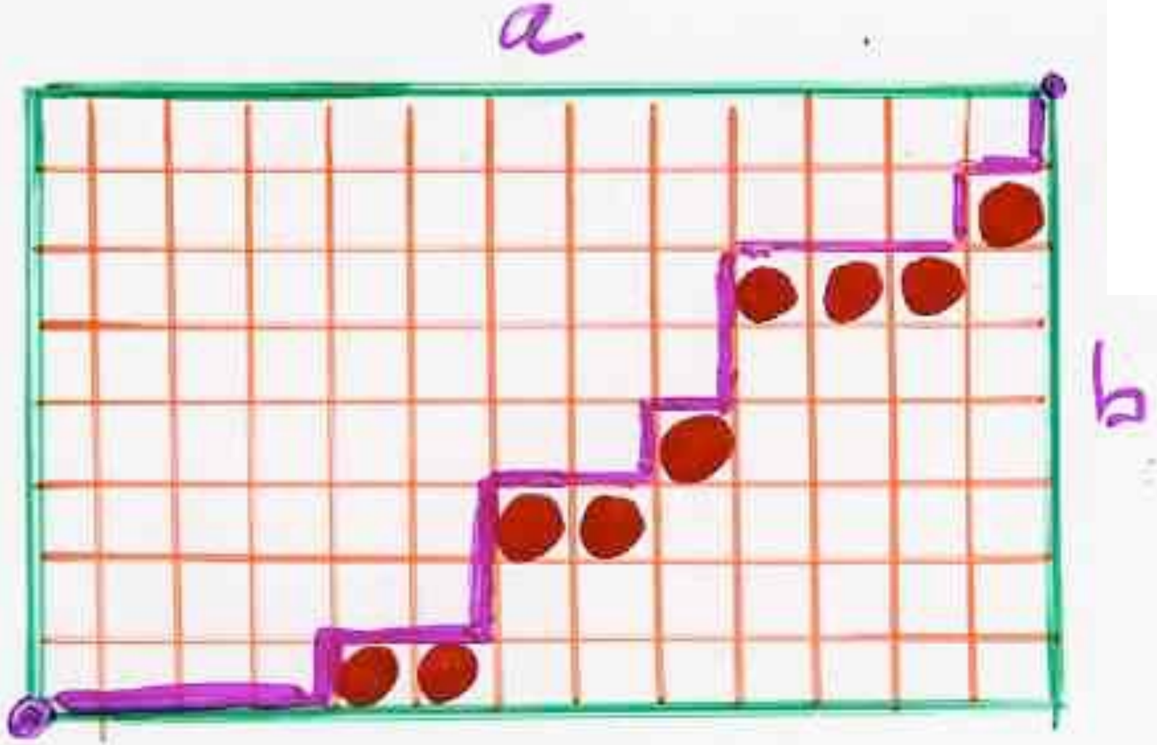
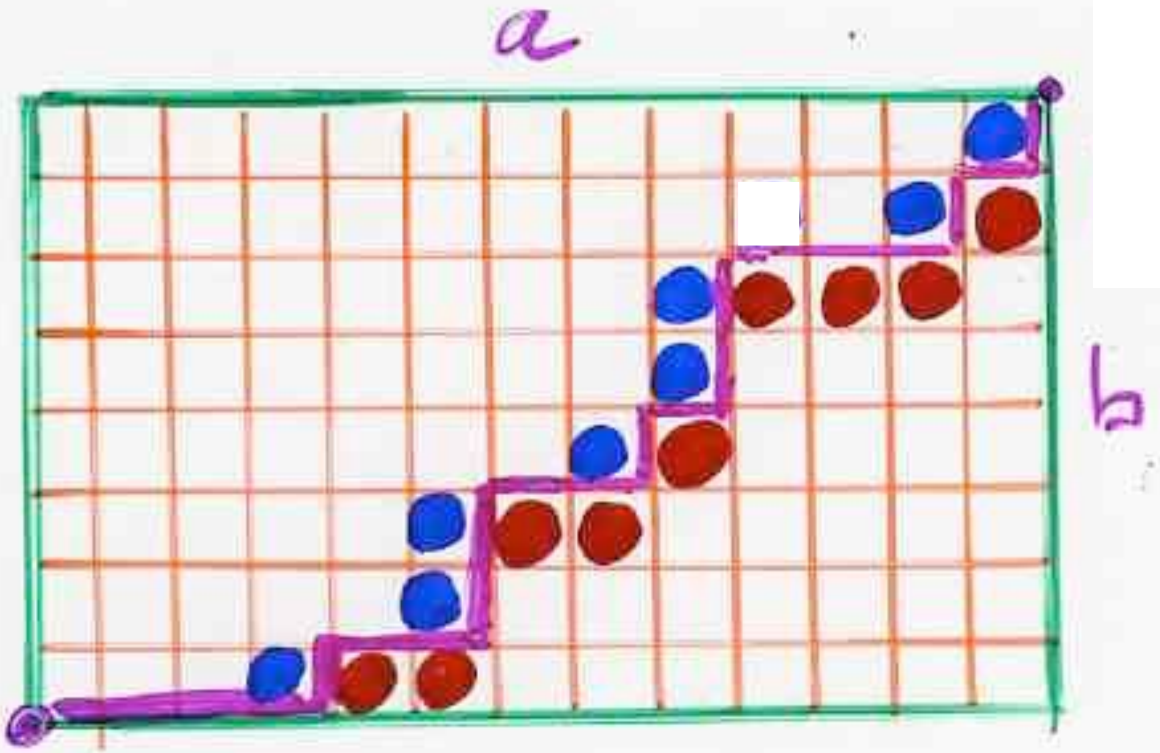




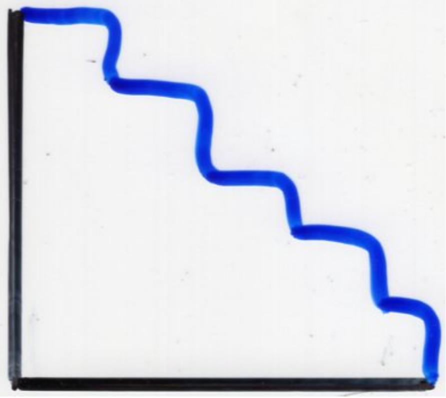




$$\binom{a+b}{a}$$


a
b

Loday-Ronco
product *



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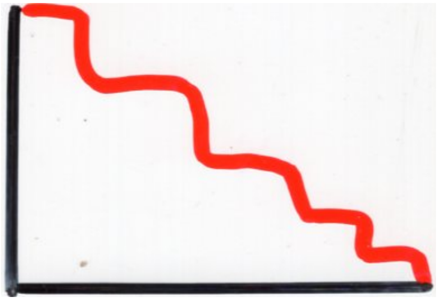

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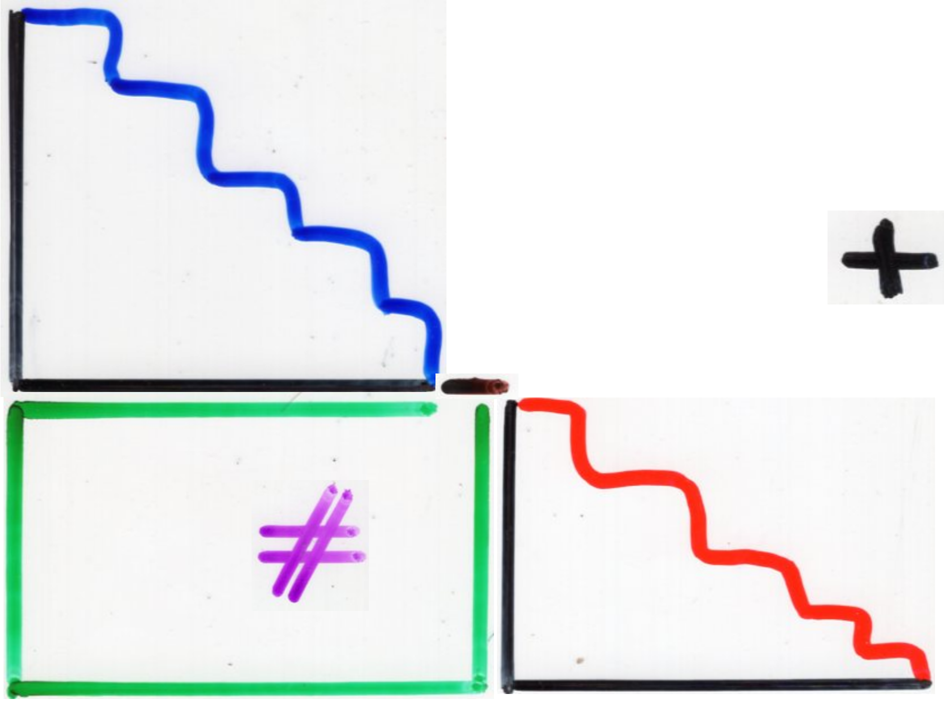

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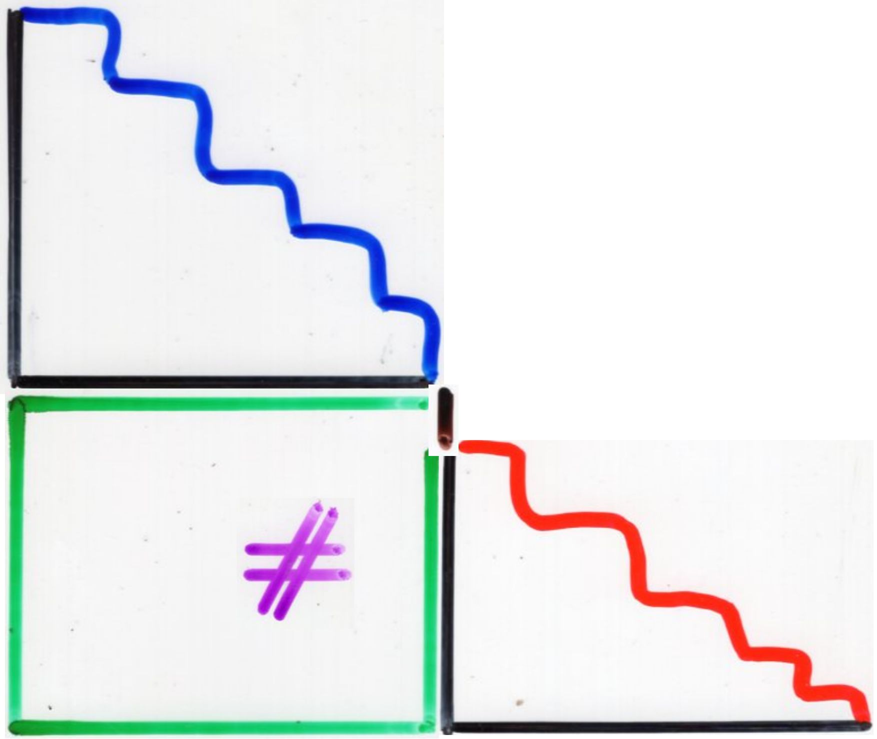

Malvenuto, Reutenauer (1995)

Loday, Ronco (1998)

Aval, X.V. (2010)

Aval, Novelli, Thibon (2011)
The # product in combinatorial Hopf algebra

algebraic structures

Hopf algebra

	descent algebra	Loday-Ronco algebra	Reutenauer-Malvenuto algebra
dim	2^{n-1}	C_n Catalan	$n!$

combinatorial structures

hypercube — associahedron — permutahedron

Boolean lattice inclusion ↔ Tamari order ↔ weak Bruhat order

dim 2^{n-1} — C_n — $n!$

Catalan