Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 3

Tableaux for the PASEP quadratic algebra

Ch3b

IMSc, Chennai
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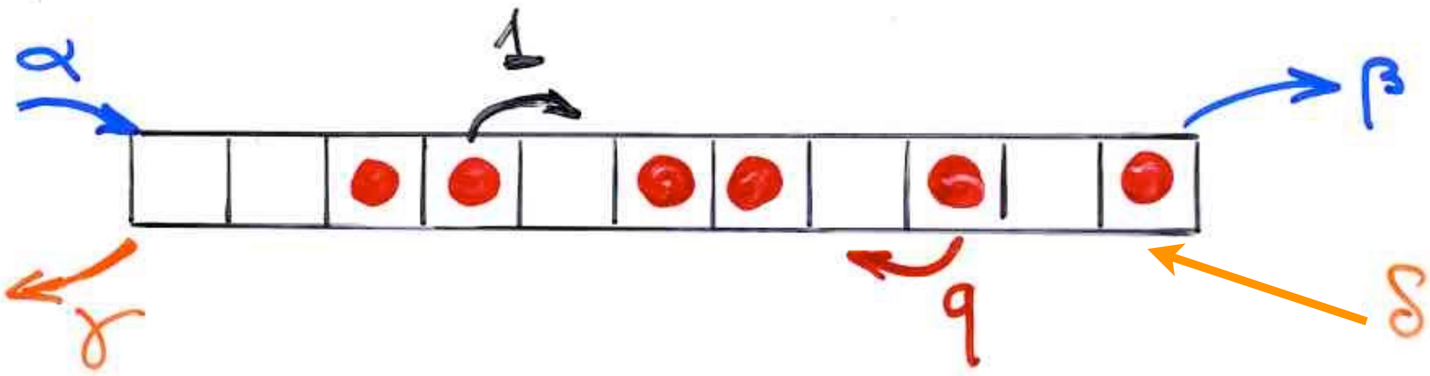
From Ch 3a

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

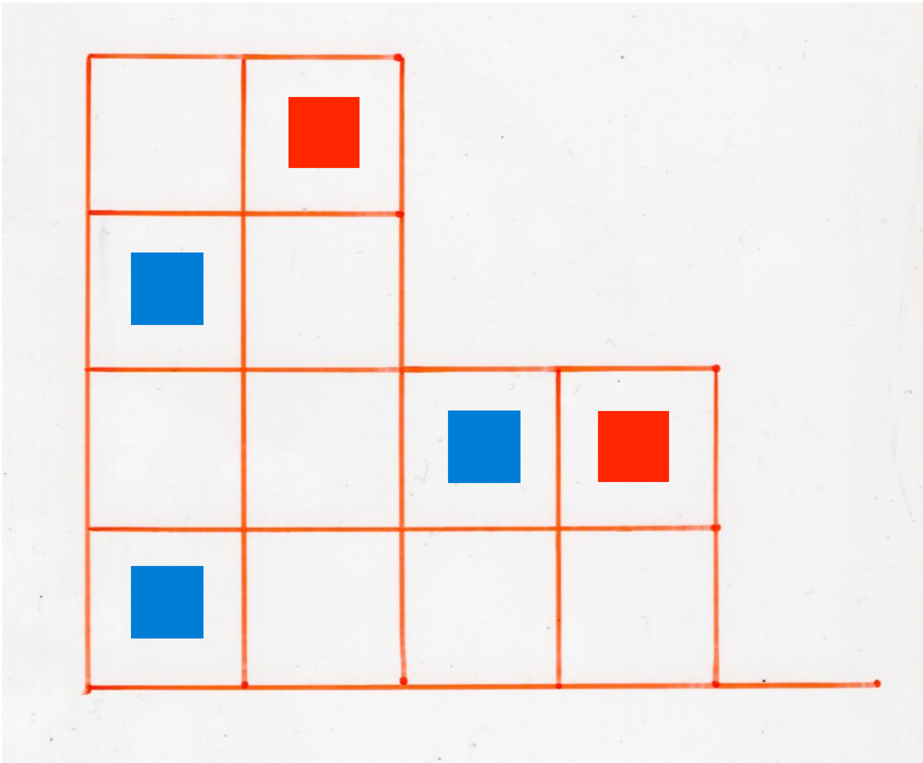
D, E matrices (may be ∞)

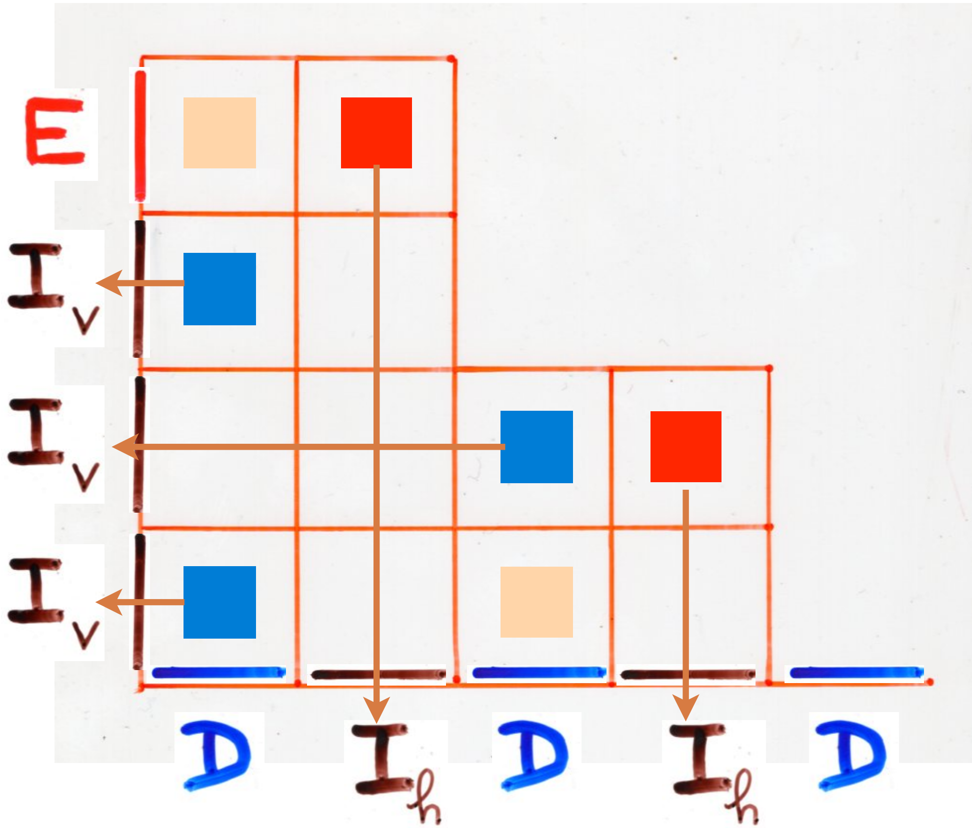
colum vector V

row vector W

$$\begin{cases} DE = qED + E + D \\ \langle W|(\alpha E - \gamma D) = \langle W| \\ (\beta D - \delta E)|V\rangle = |V\rangle \end{cases}$$

Then




E
Iv
Iv
Iv
D
Ih
D
Ih
D

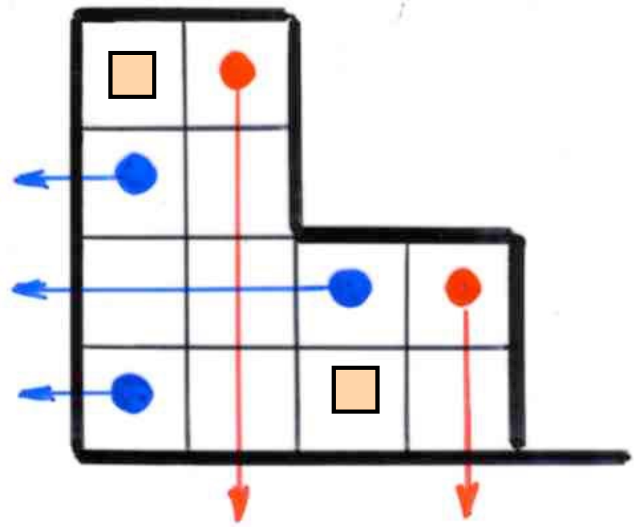
$$DE = qED + E + D$$

In the **PASEP** algebra

any word $w(E,D)$ can be uniquely written

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T: alternative tableaux, profile w



$k(T)$ = nb of cells ☐

$i(T)$ = nb of rows without ● (blue)

$j(T)$ = nb of columns without ● (red)

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \ldots, \tau_n)$ is

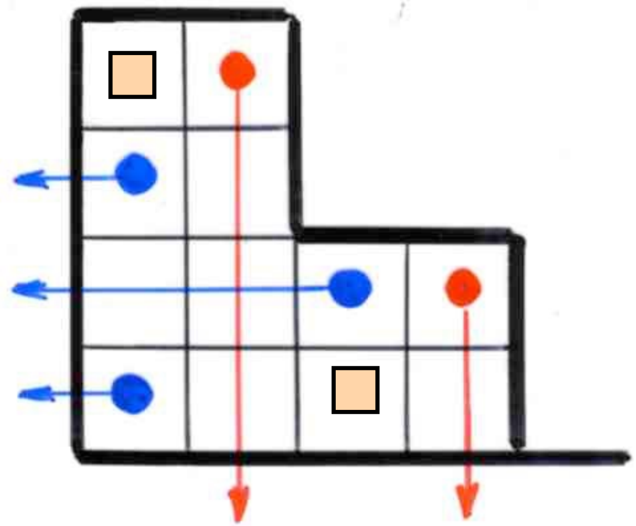
$$\text{proba}_{\tau}(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

T alternative tableaux profile τ

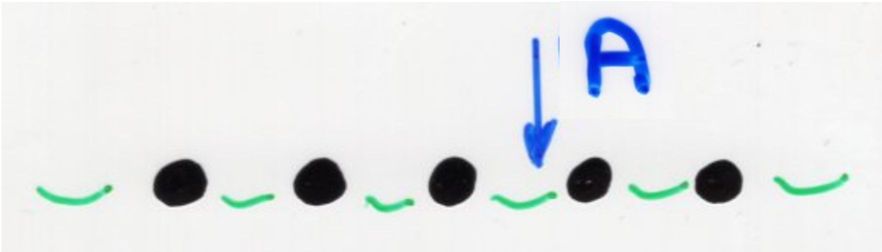
$k(T)$ = nb of cells

$i(T)$ = nb of rows without

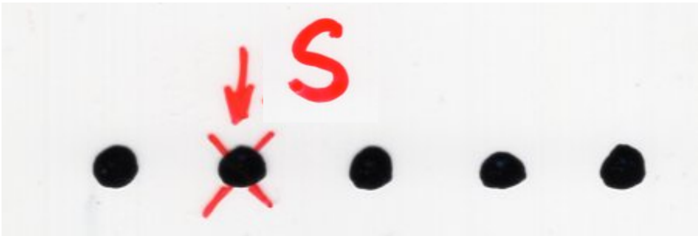
$j(T)$ = nb of columns without



$A\,|k\rangle = (k+1)\,|(k+1)\rangle$



$S\,|k\rangle = |(k-1)\rangle$



$$\begin{cases} D = A + K \\ E = S + J \end{cases}$$

$\langle k | O$

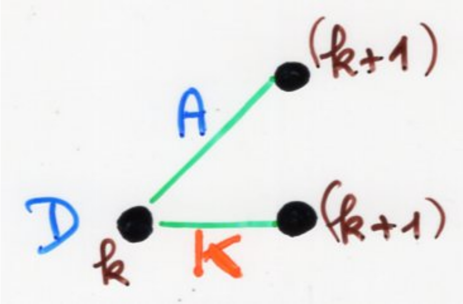
$J\,|k\rangle = k\,|k\rangle$

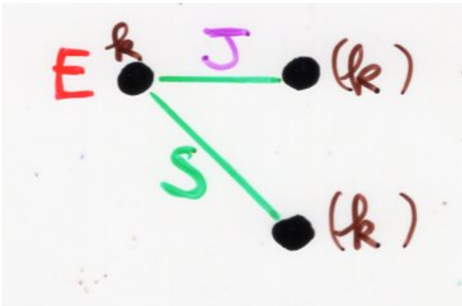
$K\,|k\rangle = (k+1)\,|k\rangle$

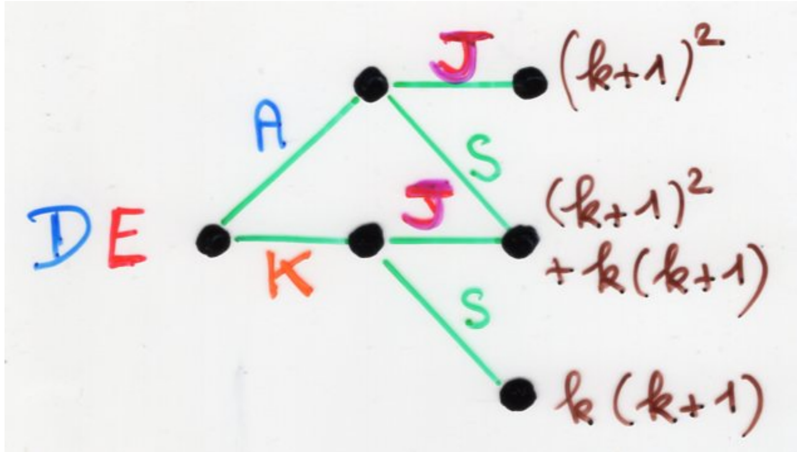
$$DE = ED + E + D$$

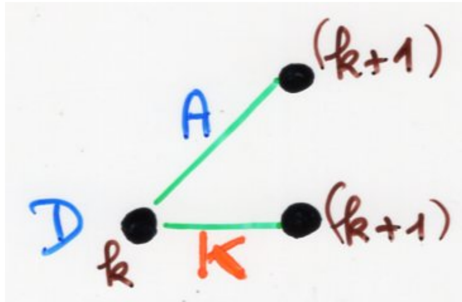
$O = O_1 \cdots O_2$ (word)

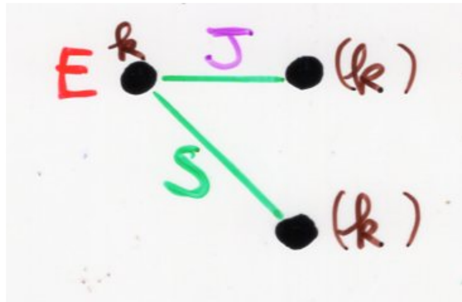
$\longrightarrow$ product from left to right

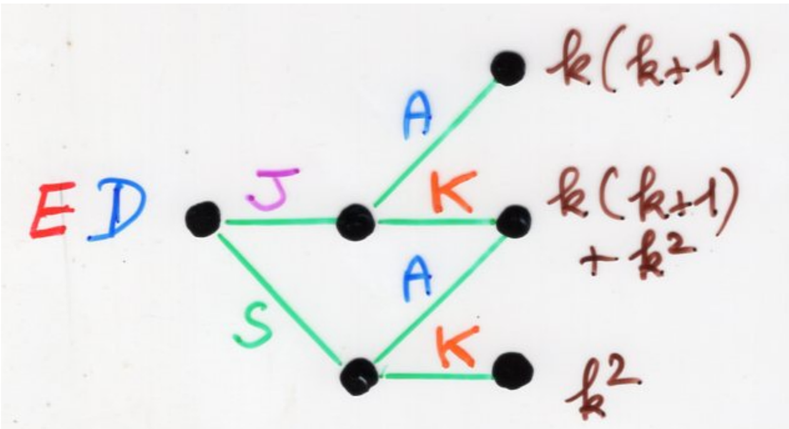

D
k
A
K
(k+1)
(k+1)


E
k
J
S
(k)
(k)

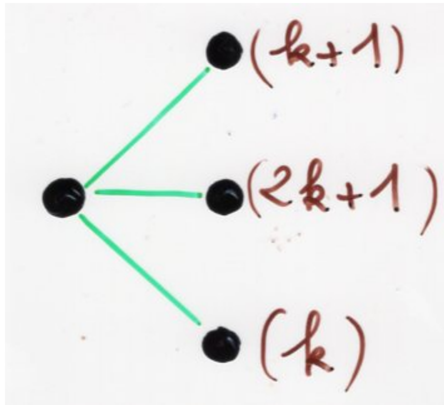

DE
A
K
J
S
J
S
(k+1)²
(k+1)²
+ k(k+1)
k(k+1)


D
k
A
K
(k+1)
(k+1)


E
k
J
S
(k)
(k)


ED
J
S
A
K
A
K
k(k+1)
k(k+1)
+ k²
k²

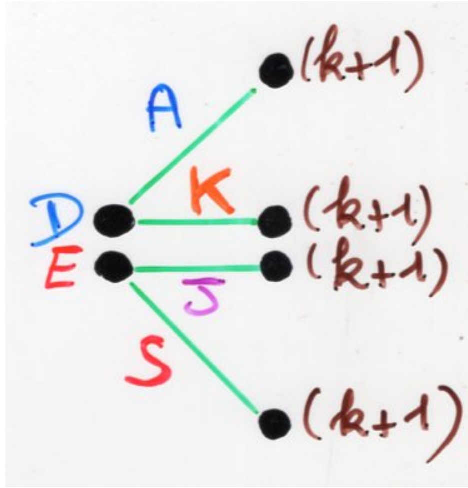
DE - ED


(k+1)
(2k+1)
(k)

$$\langle k|\, A = (k+1)\, \langle (k+1)|$$
$$\langle k|\, K = (k+1)\, \langle k|$$
$$\langle k|\, J = (k+1)\, \langle k|$$
$$\langle k|\, S = (k+1)\, \langle (k-1)|$$

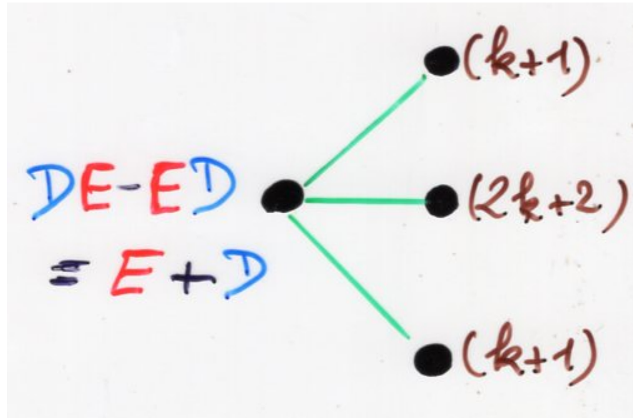
D, E "large"

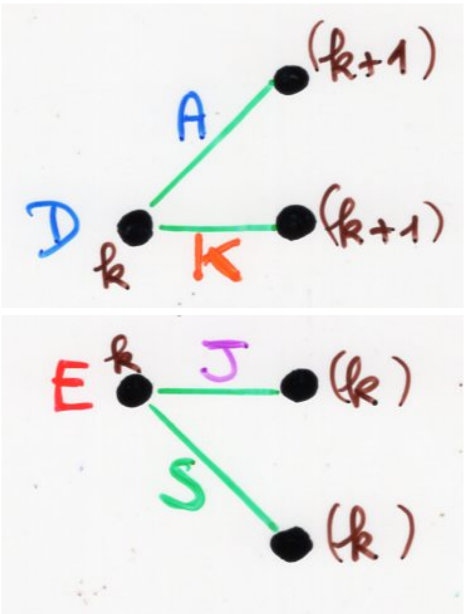
$$DE = ED + E + D$$



Corrections
after the video

$$D = A + K$$
$$E = S + J$$





D, E "restricted"

$$DE = ED + E + D$$

V vector space

notations – A operator $V \longrightarrow V$ (linear map)

$v \in V$

$$\langle v | A = A(v)$$

B basis of V, $v_0 \in B$

$$v | v_0 \rangle = \text{coeff. of } v \text{ on } v_0 \in B$$

$$\langle v_0 | A | v_0 \rangle$$

combinatorial representation of the operators E and D

PASEP algebra

$$DE = qED + E + D$$

V vector space generated by $\mathcal{B}$ basis

$\mathcal{B}$ alternating words two letters $\{○, ●\}$
(no occurences of ○○ or ●●)

4 operators A, S, J, K

4 operators A, S, J, K, $u \in B$

$\langle u | A = \sum_{\text{letter } ○ \text{ of } u} v$, v obtained by : ○ → ○●○

$\langle u | S = \sum_{○ \text{ of } u} v$ v obtained by : ○ → ● (and ●● → ● ●●● → ●)

$\langle u | J = \sum_{○ \text{ of } u} v$ v, ○ → ●○ (and ●● → ●)

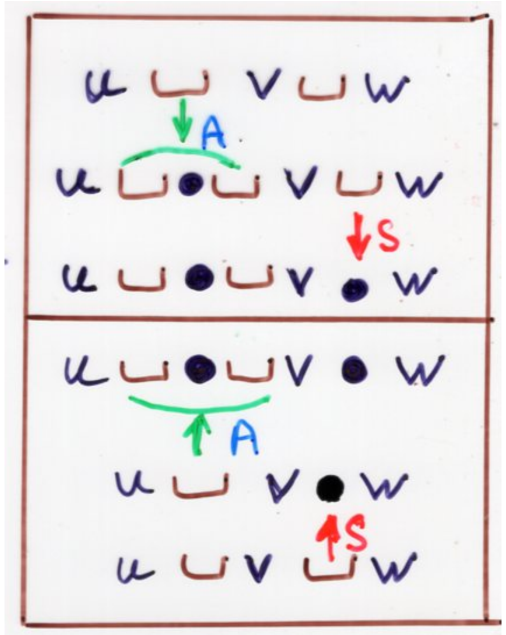
$\langle u | K = \sum_{○ \text{ of } u} v$ v, ○ → ○● (and ●● → ●)

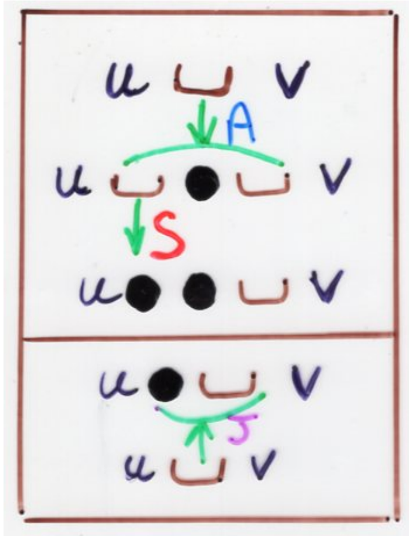
○●○●| S = ●○● + ○●

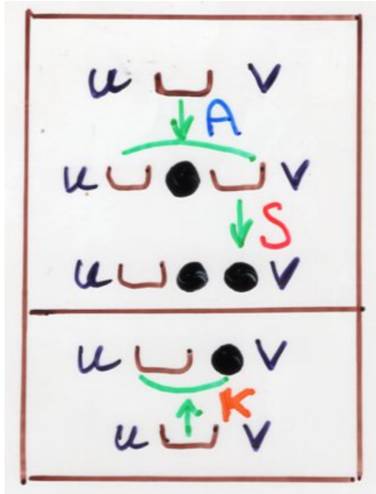
$$D = A + J$$
$$E = S + K$$

claim:

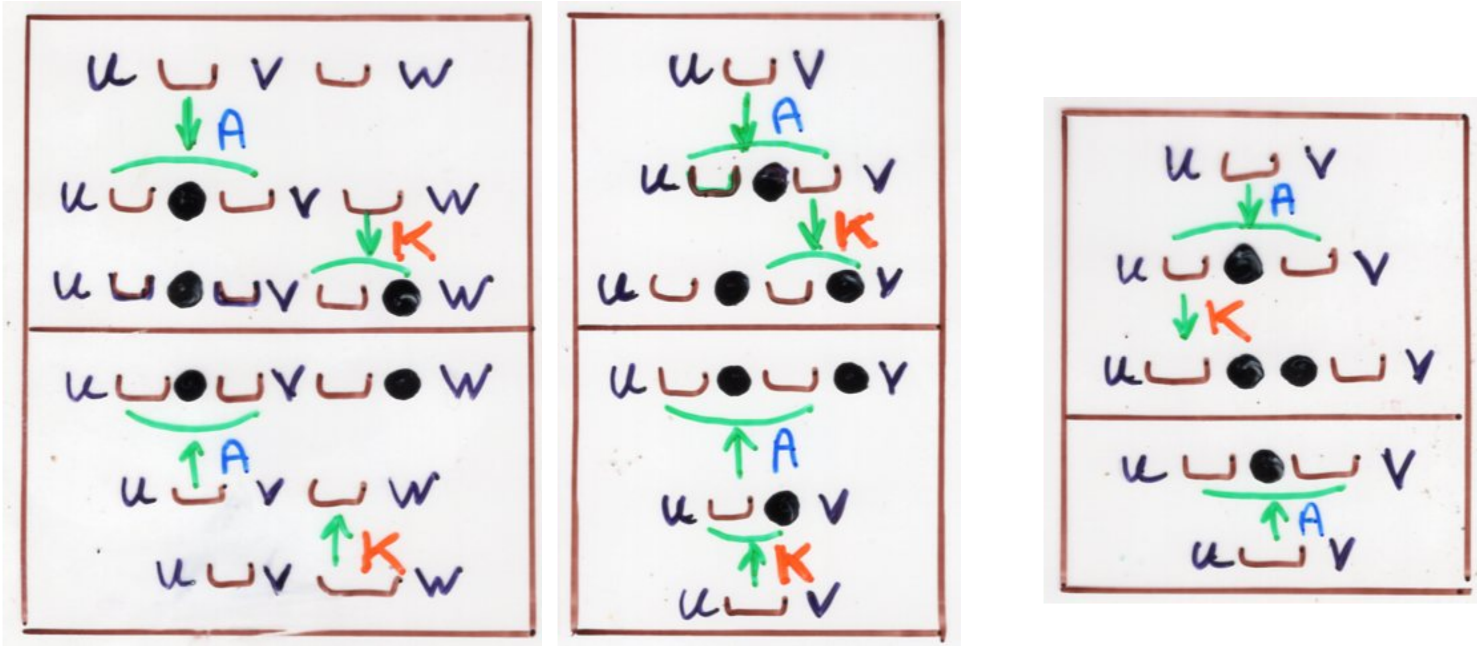
$$DE = ED + E + D$$



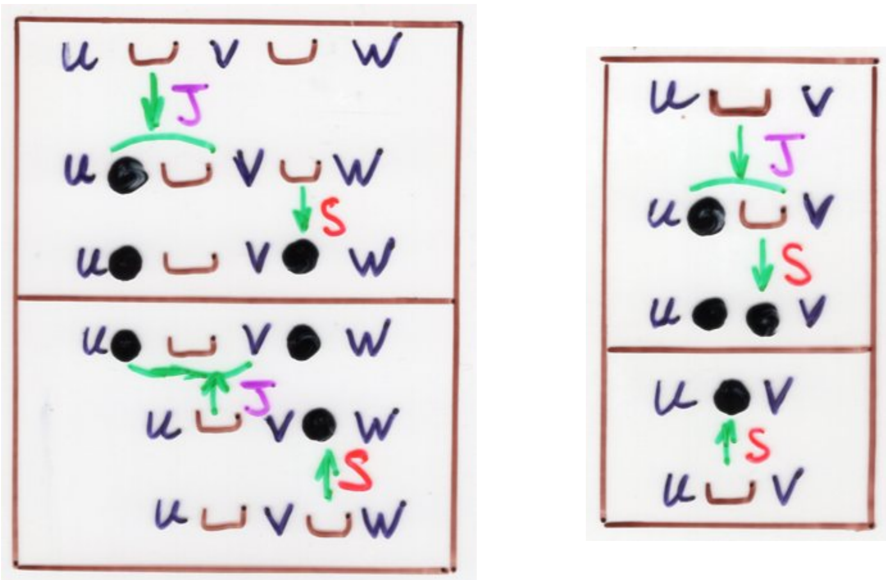


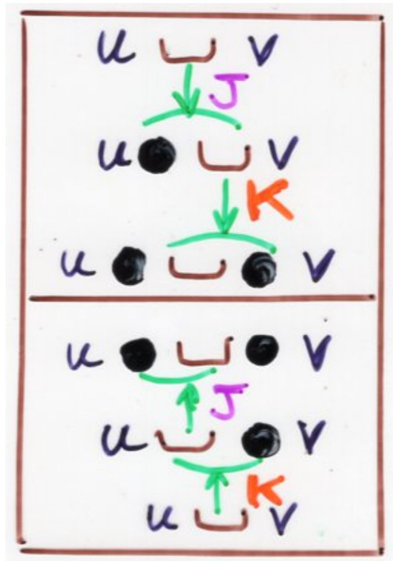


$$AS = SA + J + K$$



AK = KA + A





JS = SJ + S

JK = KJ

AS = SA + J + K

AK = KA + A

JS = SJ + S

JK = KJ

D = A + J

E = S + K

$$D = A + J$$

$$E = S + K$$

$$\begin{aligned} DE &= (A+J)(S+K) \\ &= AS + AK + JS + JK \\ &= \underbrace{\underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)}}_{ED} + \underbrace{J + K + A + S}_{E + D} \end{aligned}$$

$$DE = ED + E + D$$

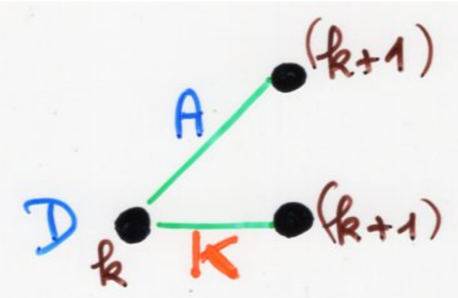
claim:

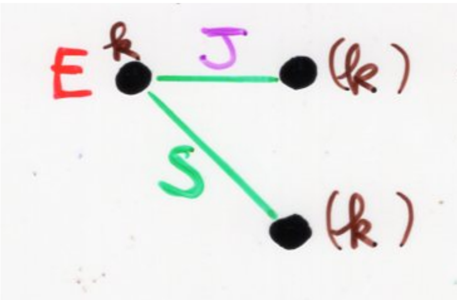
D, E "large"

$\langle k | O = \langle k+1 | (k+1)$ $O = A, S, J, K$

$$\langle 0 | (D+E)^n | 0 \rangle = (n+1)!$$

D, E "restricted"





$$\langle 0 | (D+E)^n | 0 \rangle = n!$$

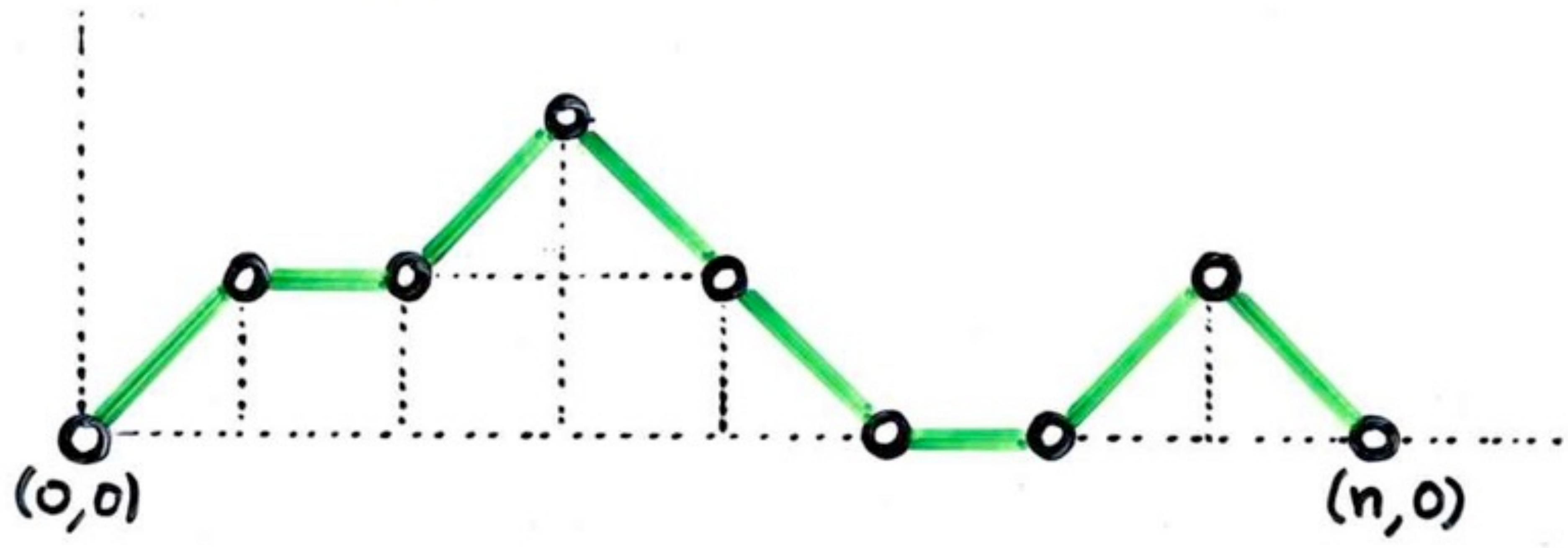
Laguerre histories

definition

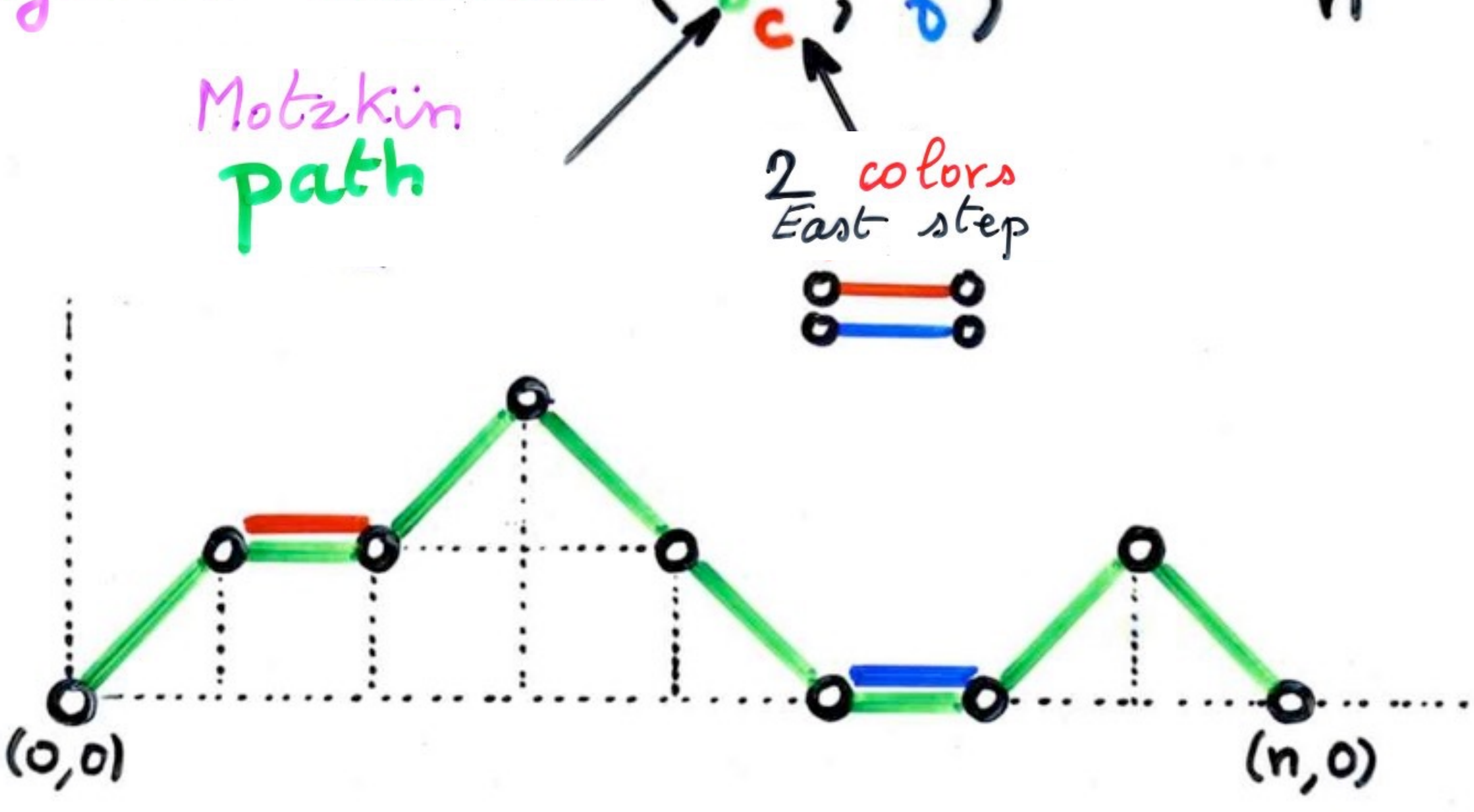
Laguerre histories (γ_c, f) n

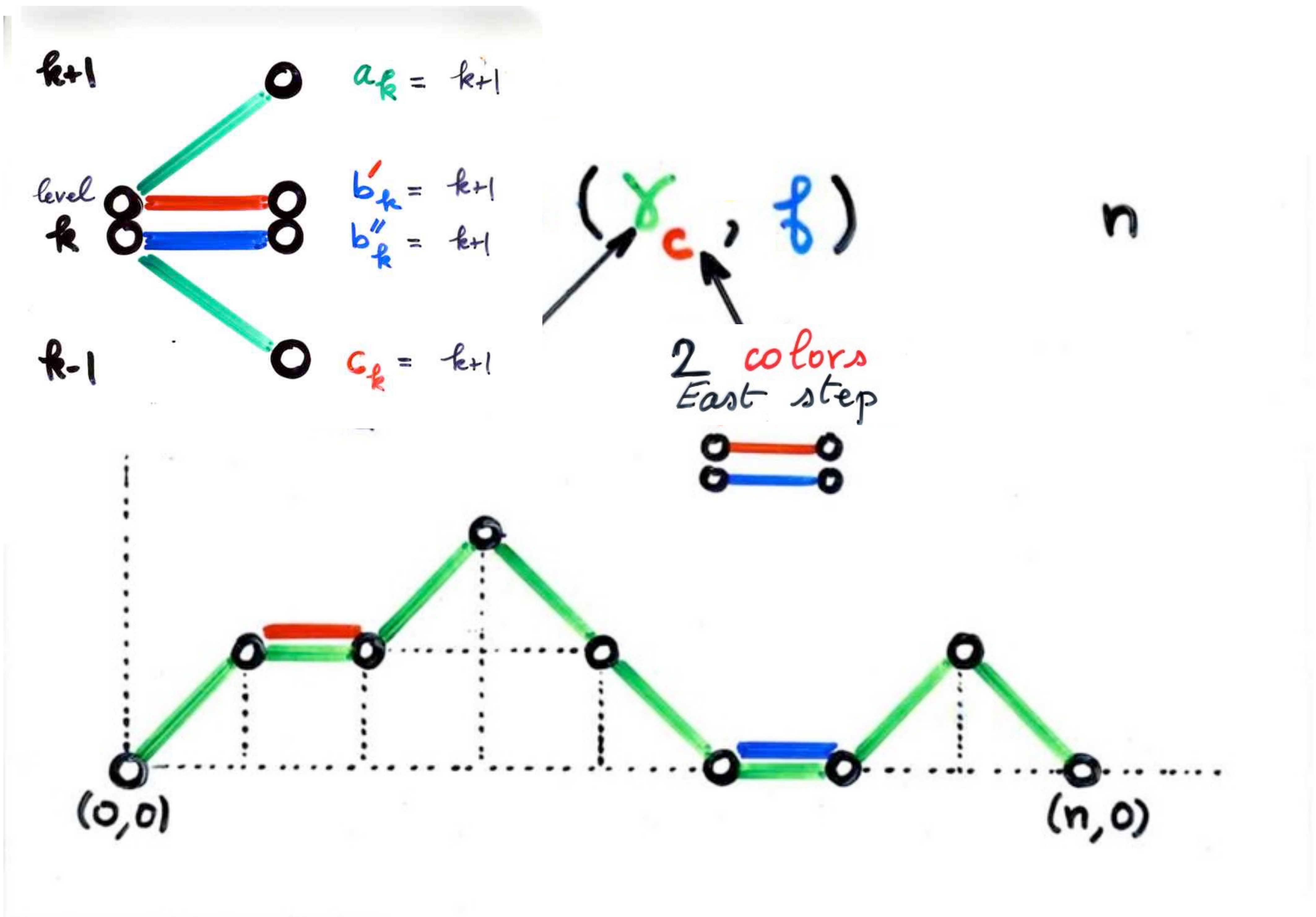
Laguerre histories (γ_c, f) n

Motzkin path

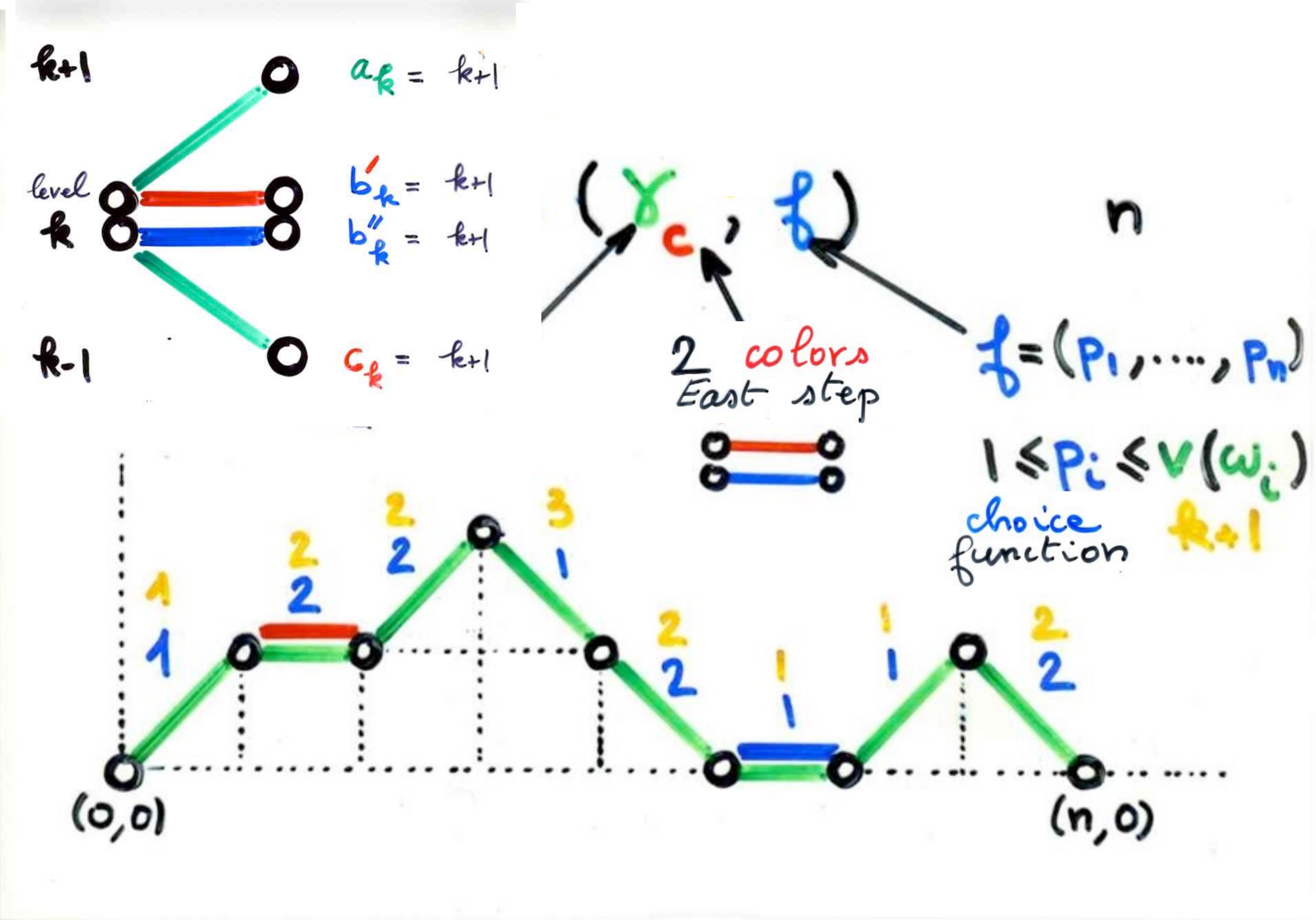


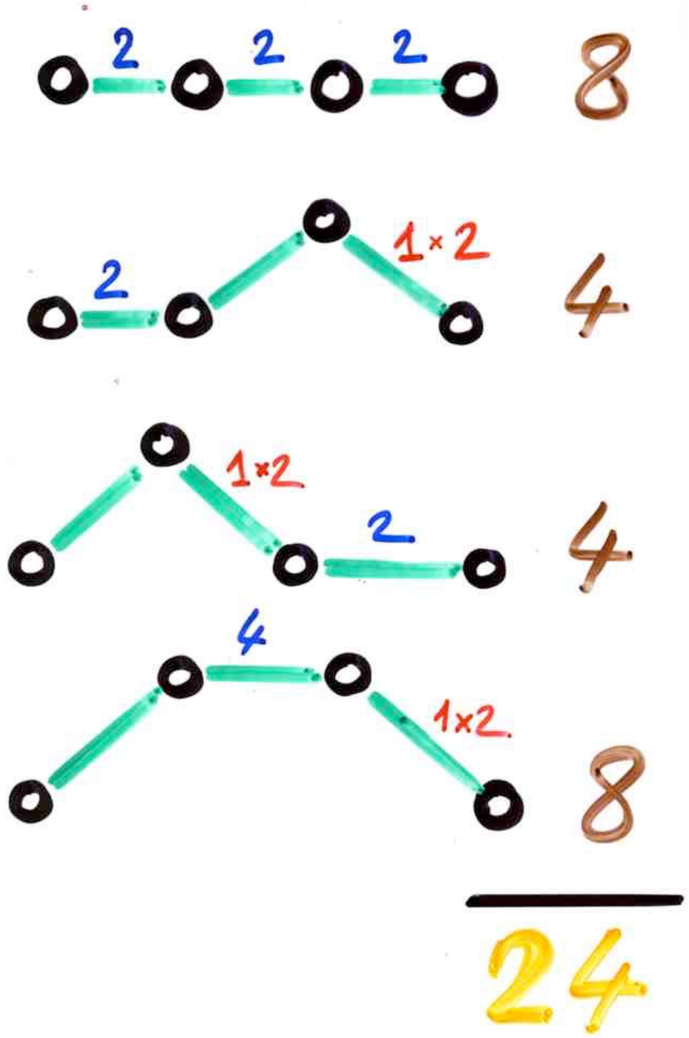
Laguerre histories (γ_c, f) n





k+1
a_k = k+1
level
k
b'_k = k+1
b''_k = k+1
k-1
c_k = k+1
(γ_c , f)
n
2 colors
East step
(0,0)
(n,0)


k+1
a_k = k+1
level
k
b'_k = k+1
b''_k = k+1
k-1
c_k = k+1
(γ_c , f)
n
2 colors
East step
f=(p_1,...,p_n)
1 ≤ p_i ≤ v(ω_i)
choice
function
k+1
(0,0)
(n,0)



$$(n+1)! = \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin}}} v^{*}(\omega)$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

Bijection

Laguerre histories $(\omega ; (p_1, \ldots, p_n)) \longleftrightarrow$ permutations

$(n+1)!$

bijection

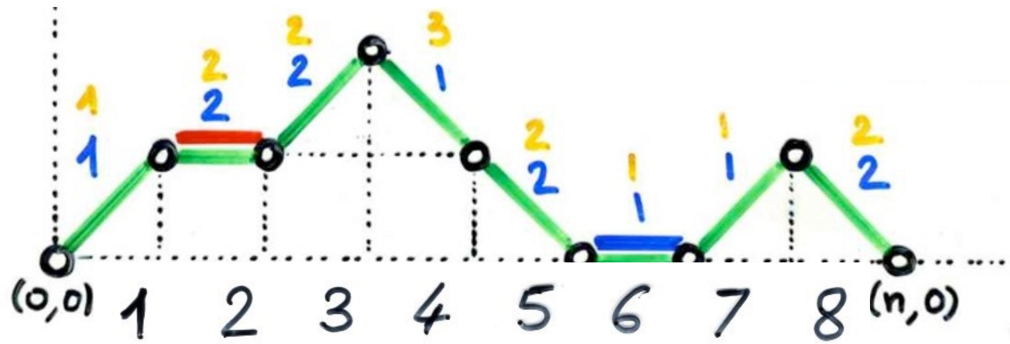
Laguerre histories ⟶ permutations

description with words

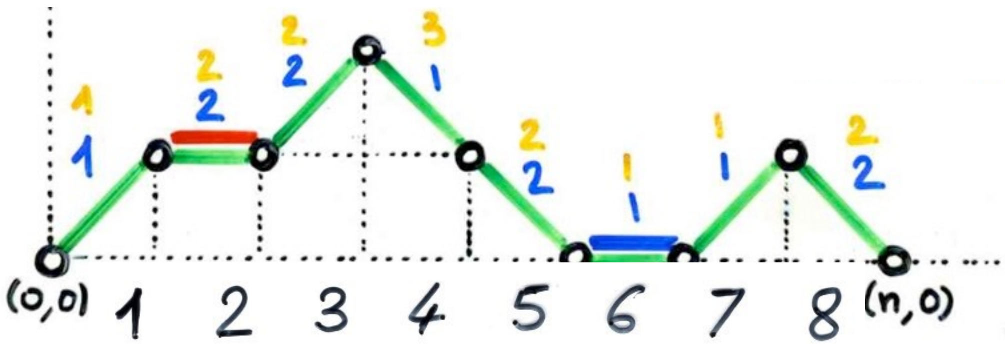
The FV bijection

(Françon-XV 1978)

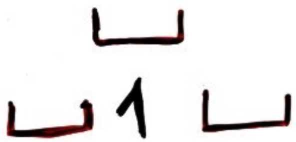
$h = (\omega_c ; (p_1, \ldots, p_n))$

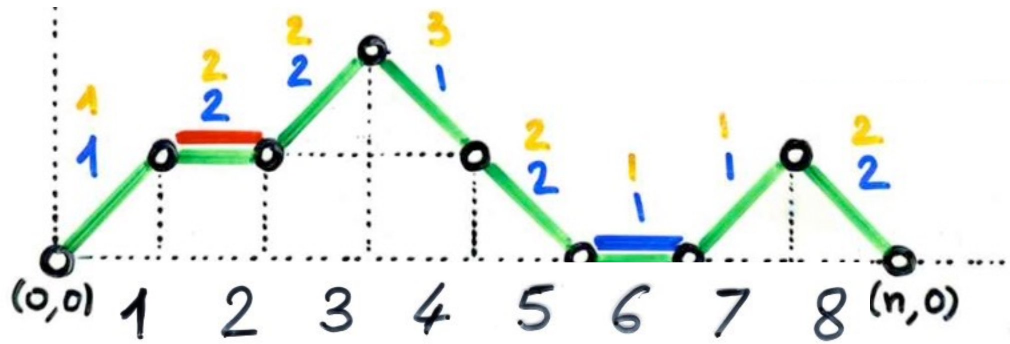


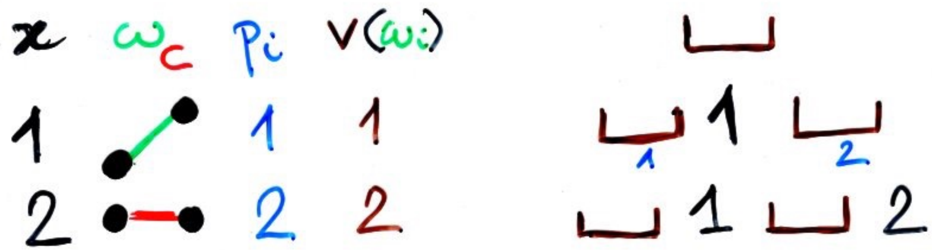
x ω_c p_i $v(\omega_i)$ $\sqcup$

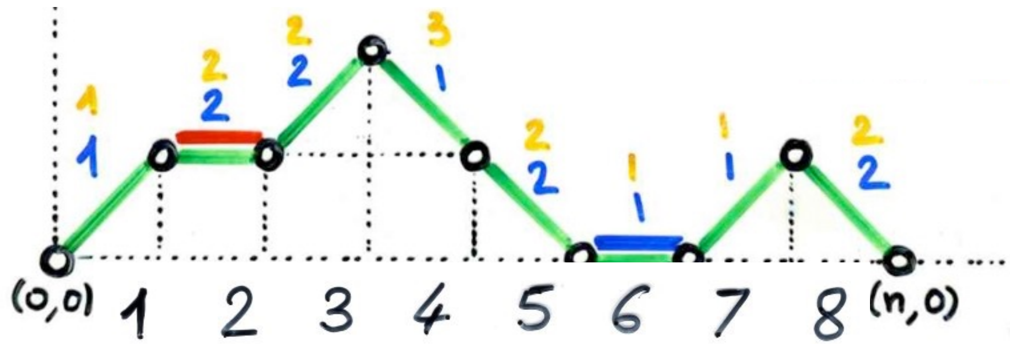


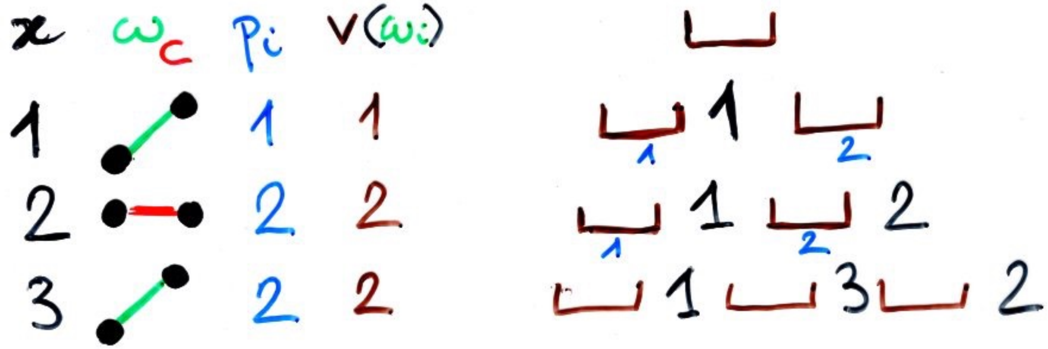
x	ω_c	p_i	$v(\omega_i)$
1		1	1

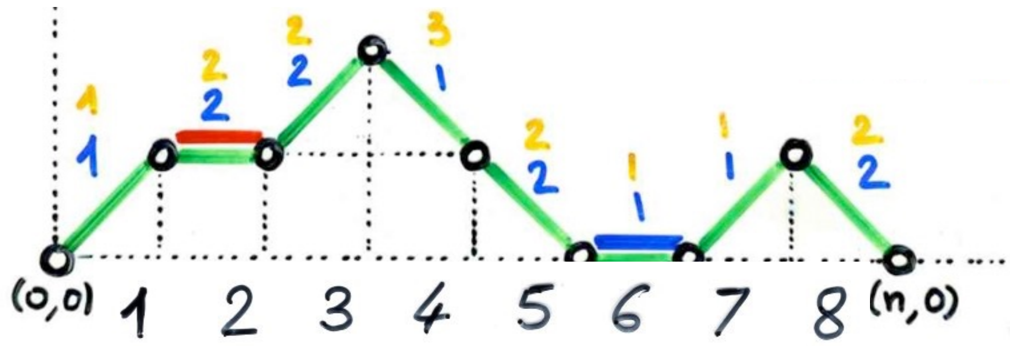


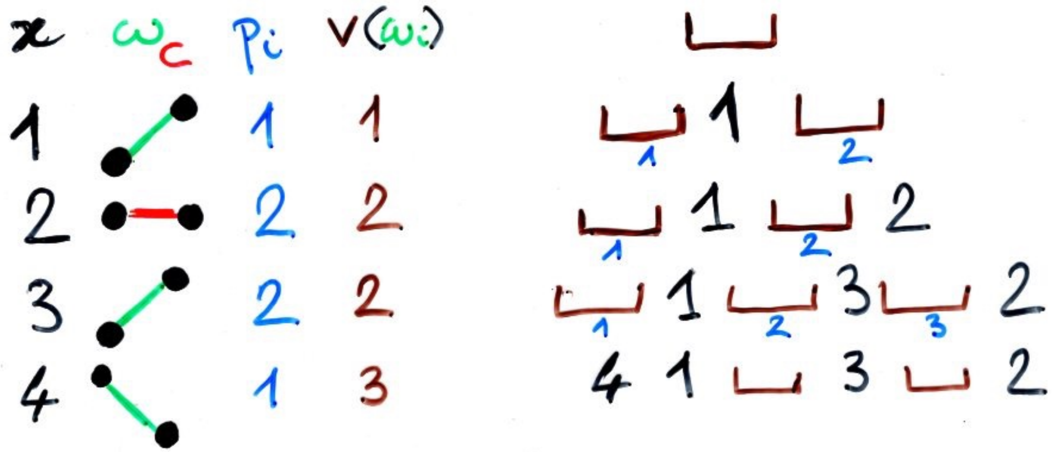

(0,0)
1
2
3
4
5
6
7
8
(n,0)

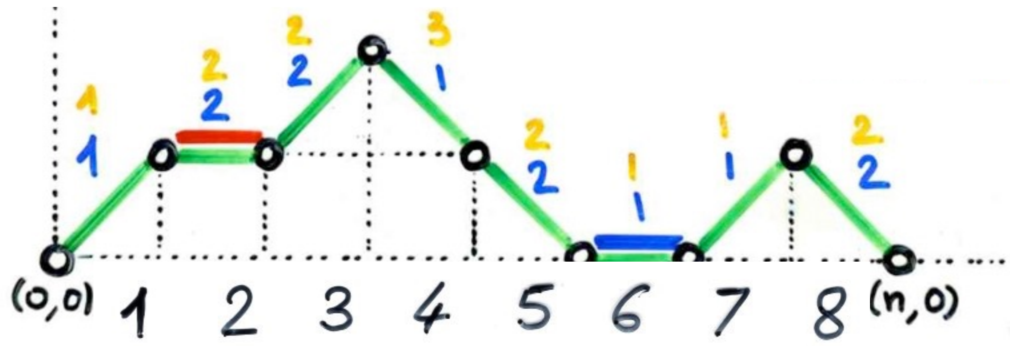

x
ω c
p i
v(ω i)
1
1
1
2
2
2


(0,0)
1 2 3 4 5 6 7 8
(n,0)
1 2 2 3 2 1 1 2
1 2 2 1 2 1 1 2

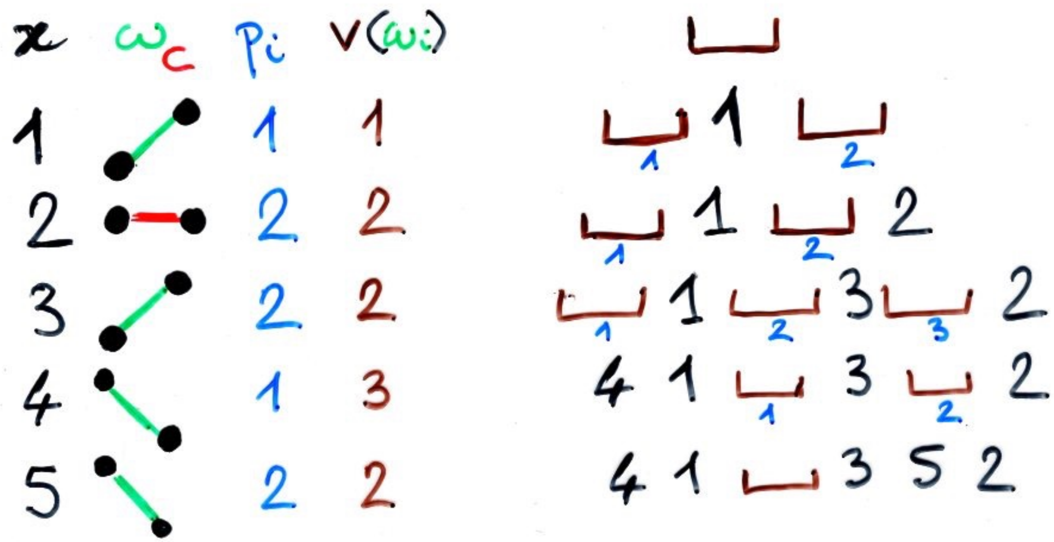

x
ω c
pi
v(ωi)
1 1 1
2 2 2
3 2 2
1
1 2
1 2
1 2
1 3 2


(0,0)
1 2 3 4 5 6 7 8
(n,0)
1 1
2 2
2 2
3 1
2 2
1 1
1 1
2 2

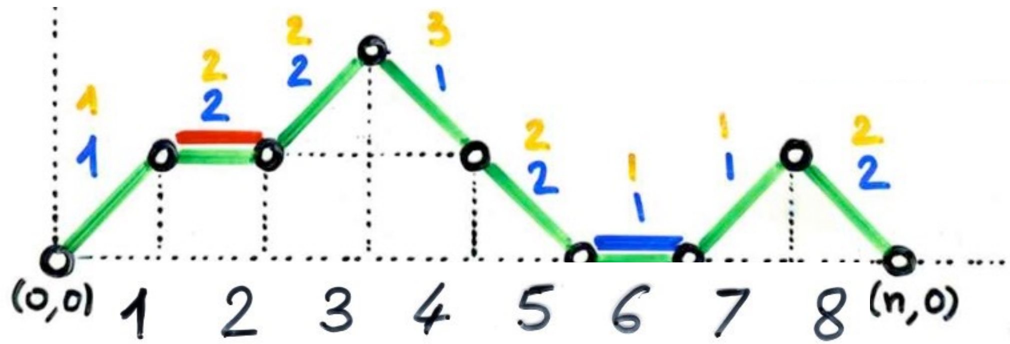

x
ω c
pi
v(ωi)
1 1 1
2 2 2
3 2 2
4 1 3
1
1 2
1 2
1 2
1 2 3
1 3 2
4 1 3 2

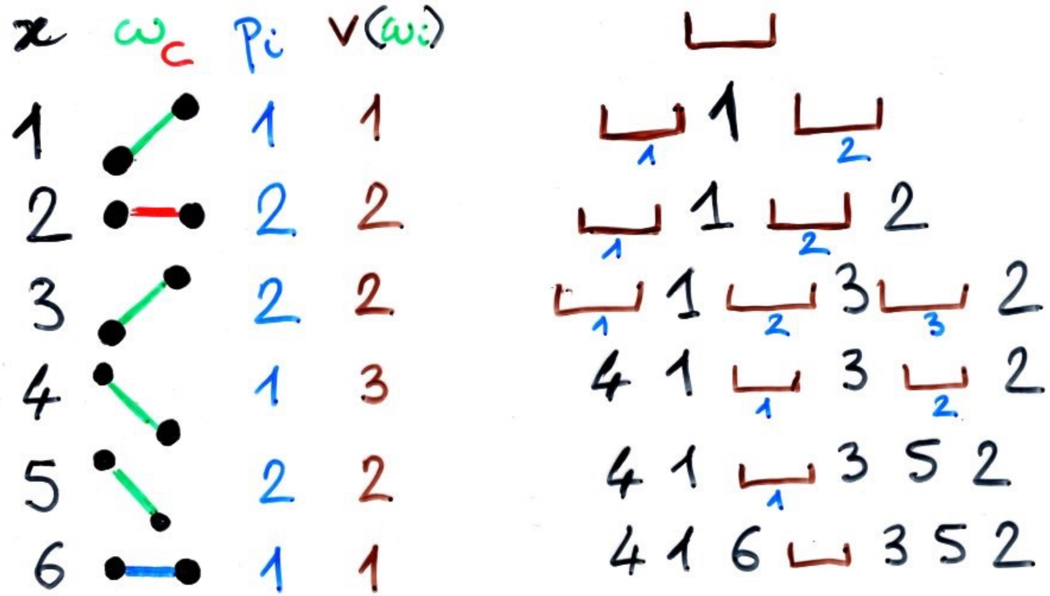


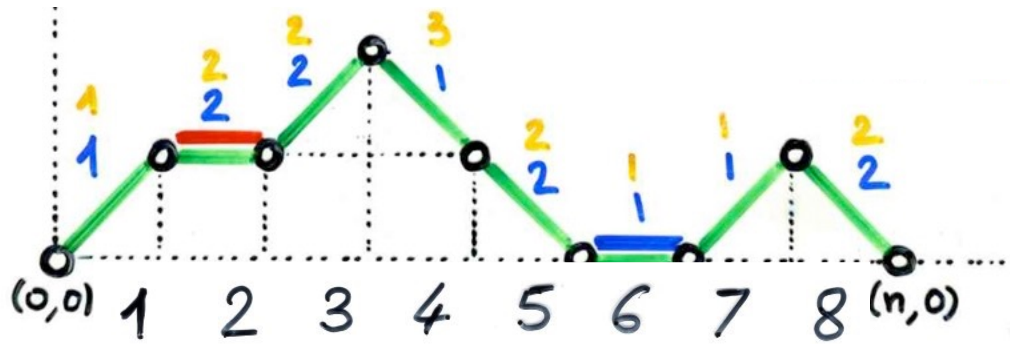
(0,0)
1 2 3 4 5 6 7 8
(n,0)

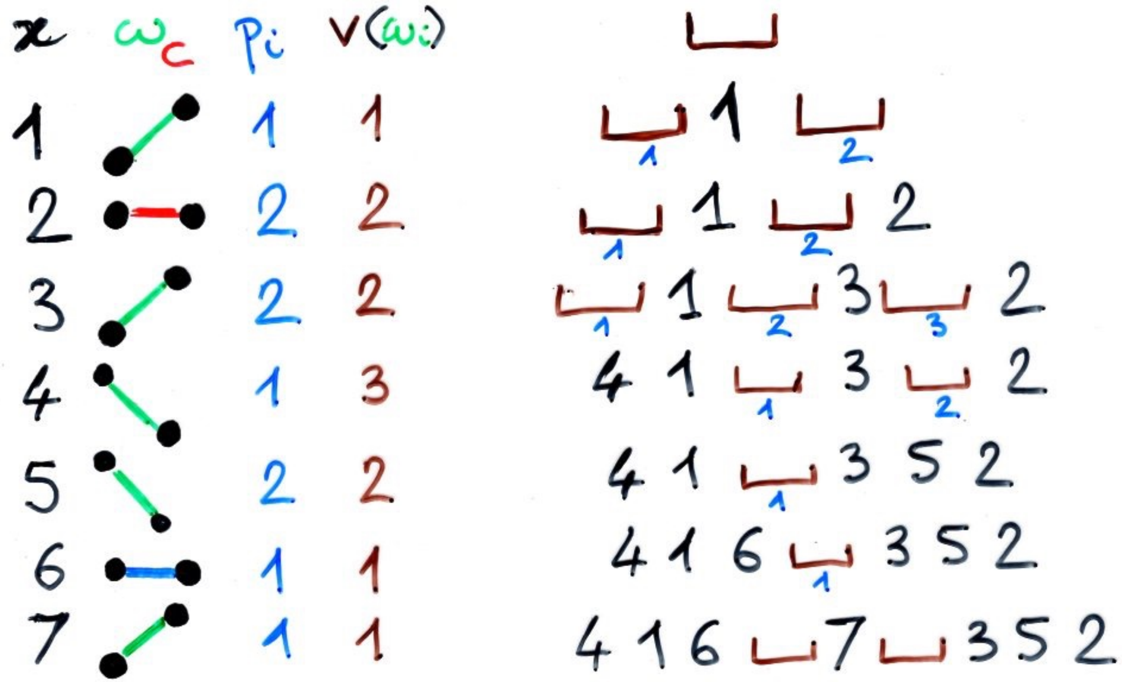


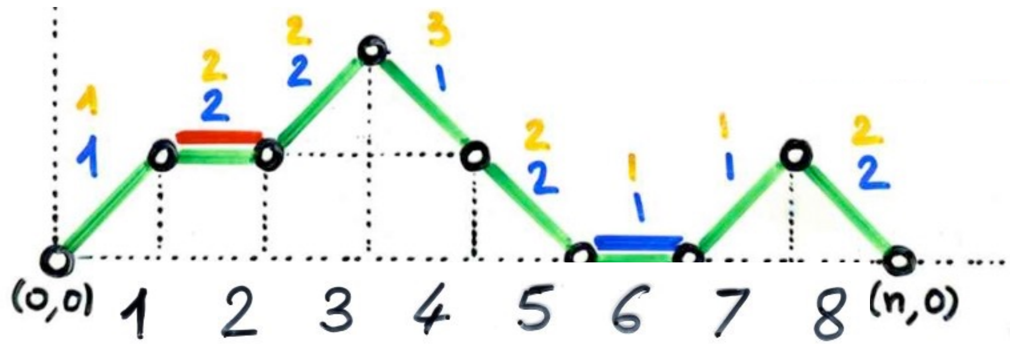
x
ω_c
pi
v(ωi)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2

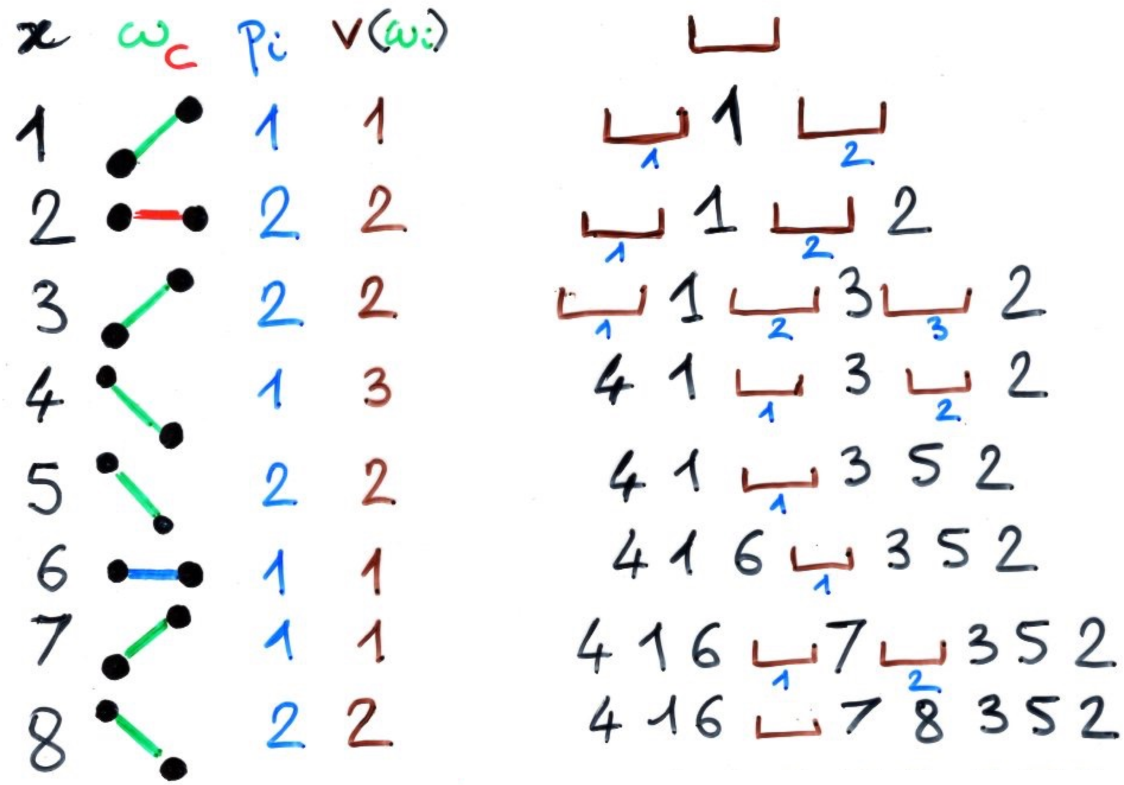

(0,0)
1 2 3 4 5 6 7 8
(n,0)
1 2 2 3 2 1 1 2
1 2 2 1 2 1 1 2

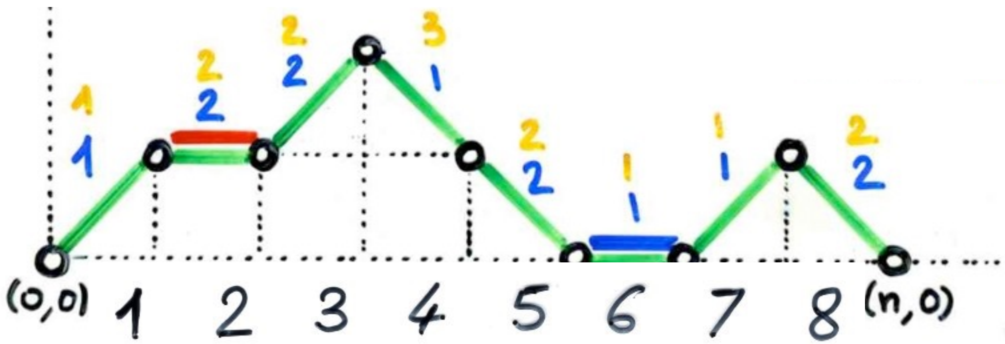

x ω_c p_i v(ω_i)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
6 1 1
1
1 2
1 2
1 2 3
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2


(0,0)
1 2 3 4 5 6 7 8
(n,0)

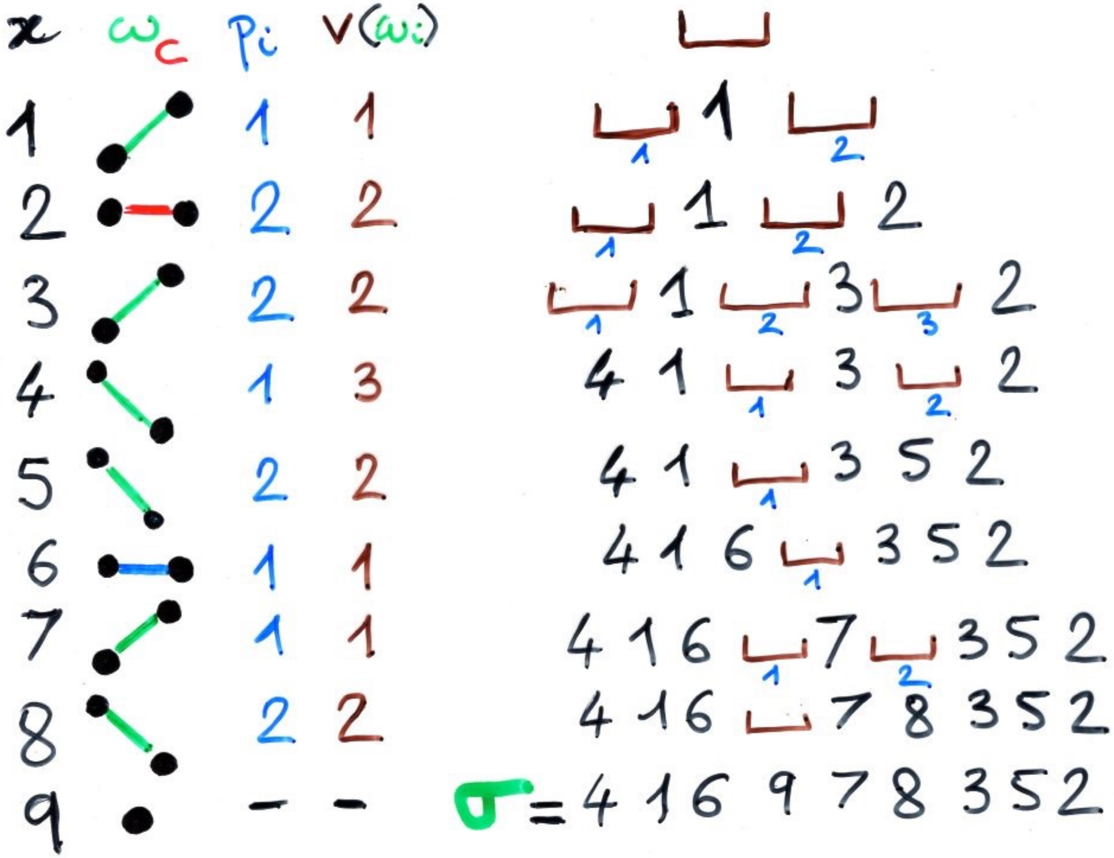

x
ω_c
p_i
v(ω_i)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
6 1 1
7 1 1
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2


(0,0)
1 2 3 4 5 6 7 8
(n,0)


x
ωc
pi
v(ωi)
4 1 6 7 8 3 5 2



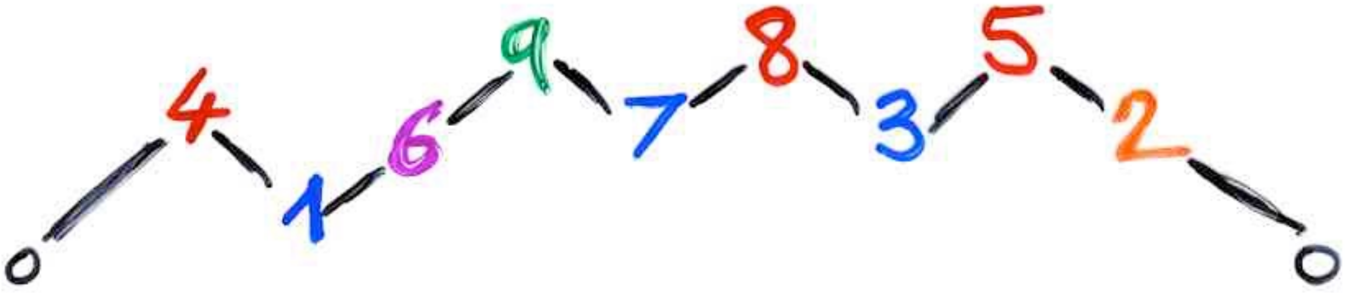
(0,0)
1 2 3 4 5 6 7 8
(n,0)



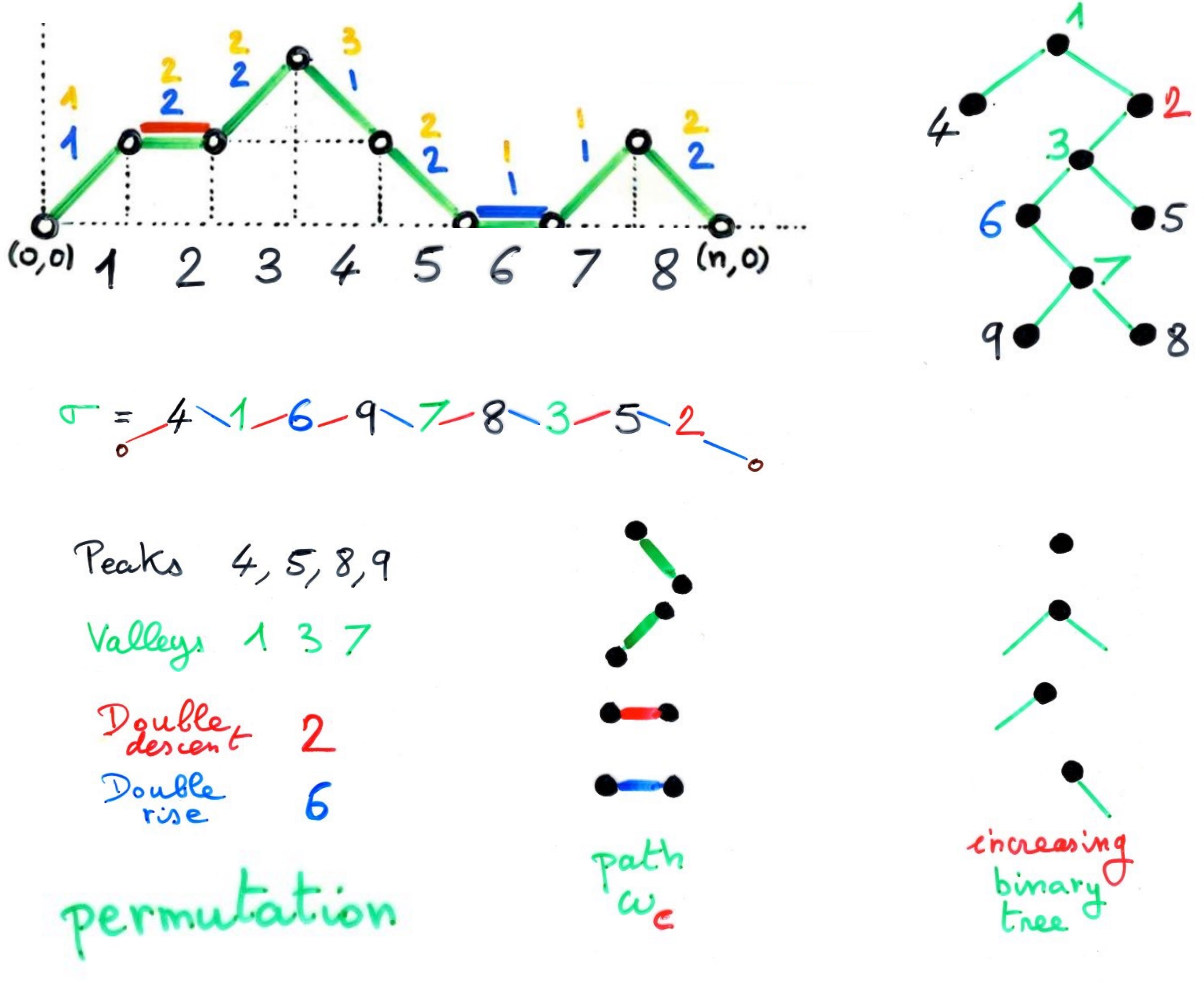
x
ω_c
p_i
v(ω_i)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
6 1 1
7 1 1
8 2 2
9 – –
σ = 4 1 6 9 7 8 3 5 2

reciprocal bijection

permutations ⟶ Laguerre histories



$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$



(0,0)
1 2 3 4 5 6 7 8
(n,0)
σ = 4 1 6 9 7 8 3 5 2
Peaks 4, 5, 8, 9
Valleys 1 3 7
Double descent 2
Double rise 6
permutation
path ω c
increasing binary tree

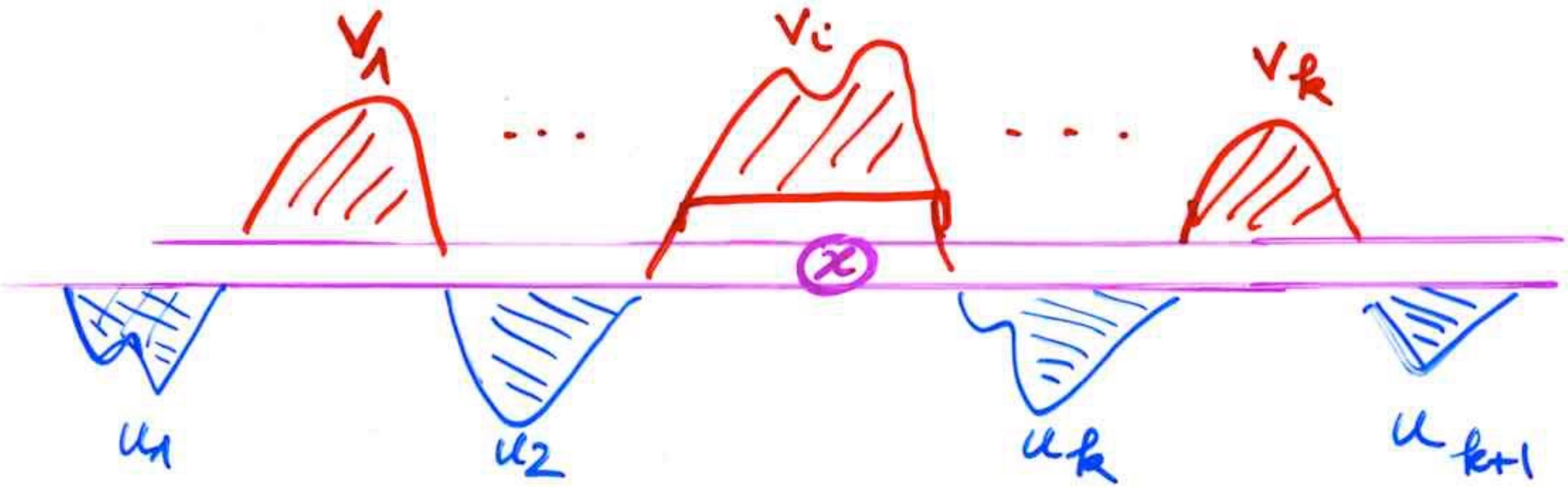
Def - $\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

x- decomposition

- $\sigma = u_1 v_1 \cdots u_k v_k u_{k+1}$
- letters $(u_i) < x$
- letters $(v_j) \geq x$
- words $v_1, u_2, \ldots, u_k, v_k$ non empty

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$

$\underbrace{\ }_{u_1}\ \overbrace{4}^{v_1}\ \underbrace{1}_{u_2}\ \overbrace{6\,9\,7\,8\,③\,5}^{v_2}\ \underbrace{2}_{u_3}$



v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}

reciprocal bijection

$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow \left(\omega_c \,;\, (p_1, \dots, p_n)\right)$$

$$\omega_c = \omega_1 \dots \omega_n$$

- ω_i (i^{th} step) is

iff i is a valley / peak / double rise / double descent

- $p_i = j$ iff i is a letter of v_j in the i-decomposition of σ

$$\sigma = u_1 v_1 \dots v_j \dots u_k v_k u_{k+1}$$

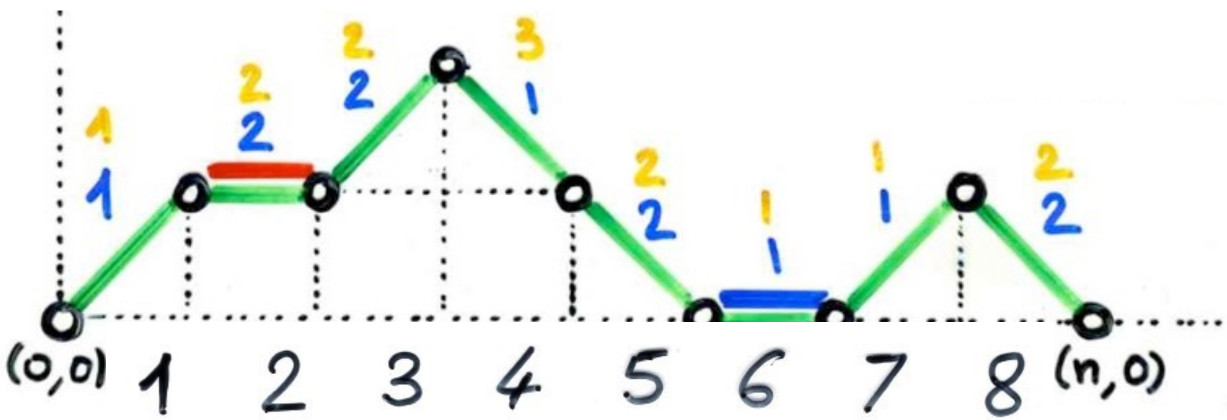
example

- $p_i = j$ iff i is a letter of v_j in the i-decomposition of σ

$$\sigma = u_1 v_1 \dots v_j \dots u_k v_k u_{k+1}$$

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$

$$\underbrace{\ }_{u_1}\ \overbrace{4}^{v_1}\ \underbrace{1}_{u_2}\ \overbrace{6\,9\,7\,8\,③\,5}^{v_2}\ \underbrace{2}_{u_3}$$

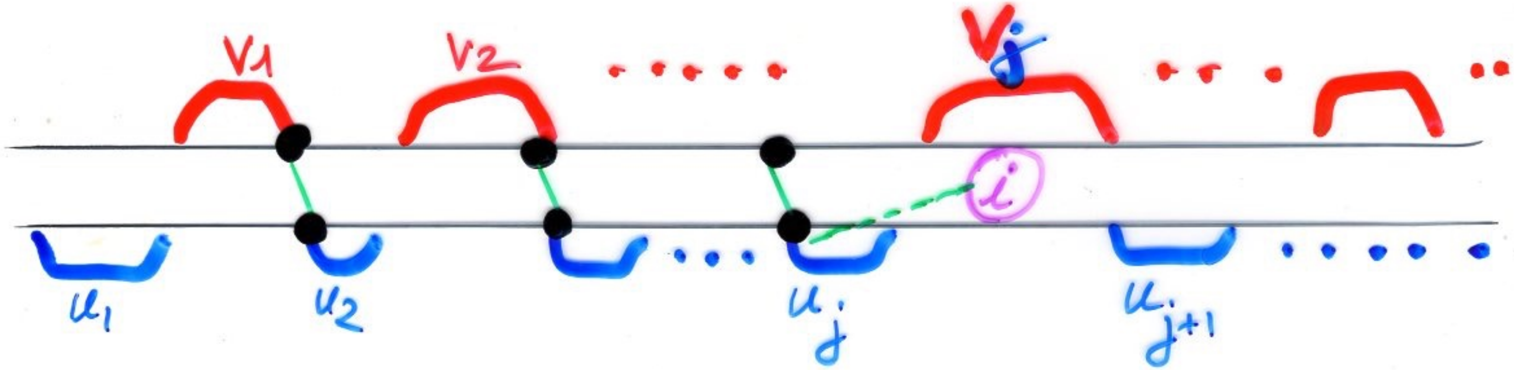


Lemma $p_i = j$ is also defined by:

$j = 1 +$ number of triples (a, b, i) having the pattern $(31-2)$, that is

$a = \sigma(k)$, $b = \sigma(k+1)$, $i = \sigma(\ell)$

with $k < k+1 < \ell$ and $b < i < a$

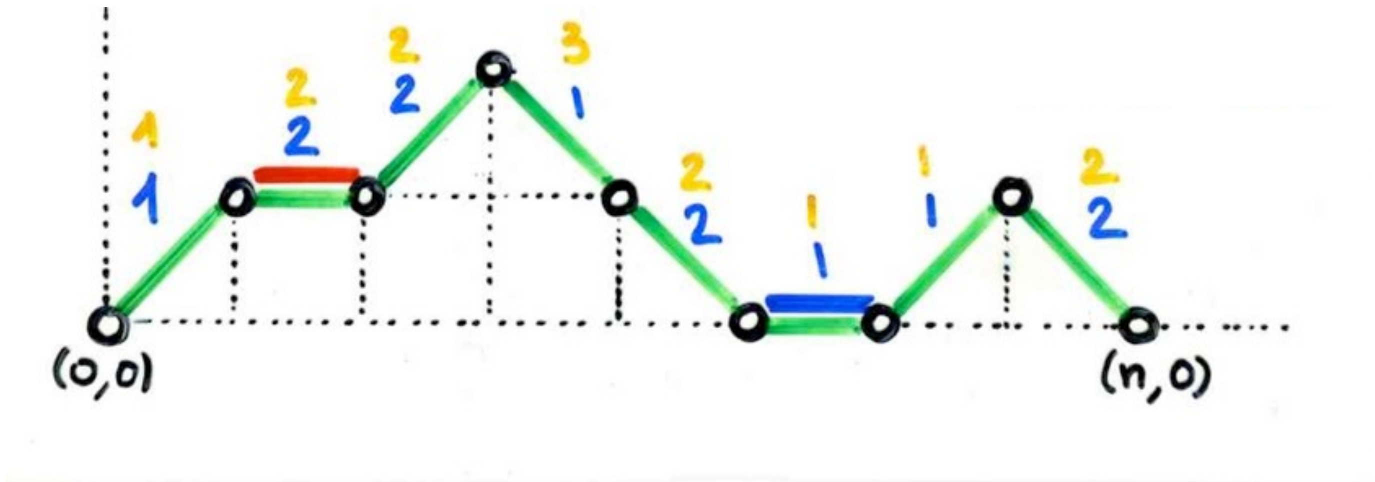


Laguerre histories
and
orthogonal polynomials

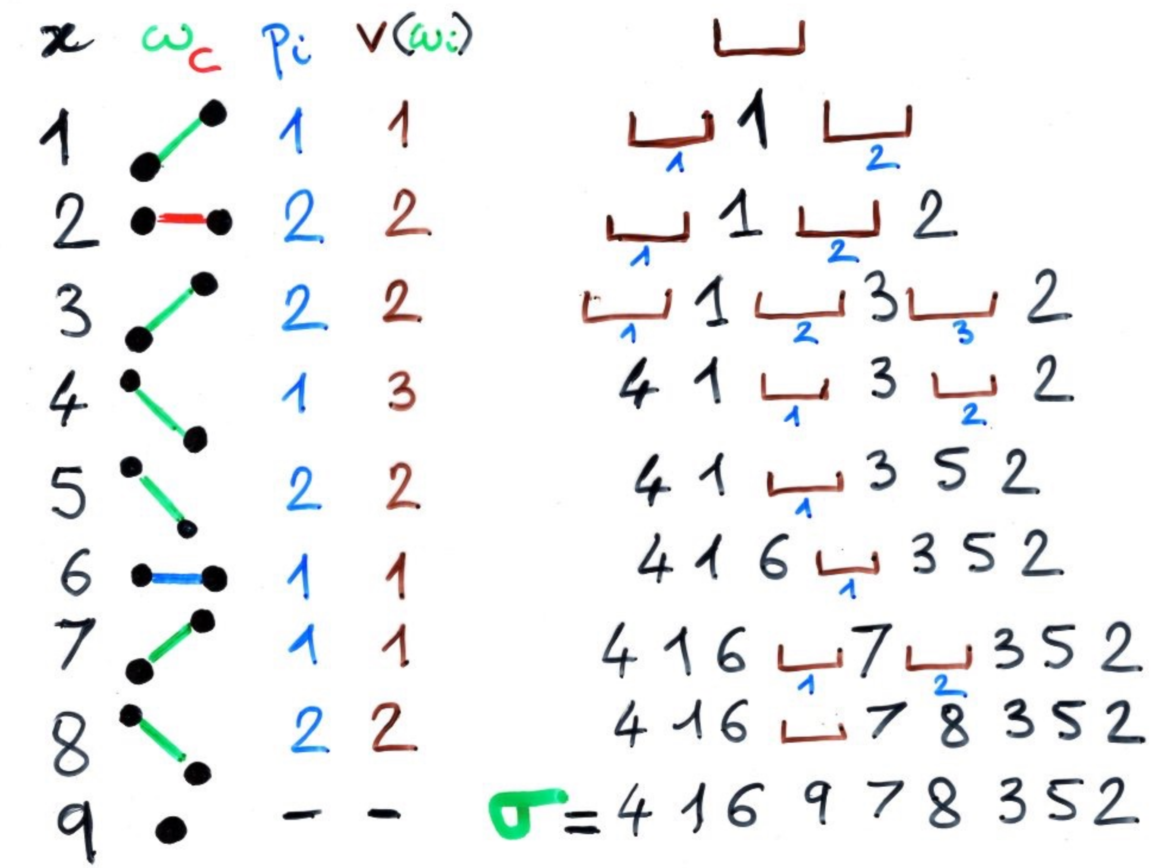
("abstract") Data structures	Possibility functions a_k	q_k	s_k	Number of Histories. h_n	
Dictionary	$k+1$	$2k+1$	k	$n!$ Permutations	
Linear list	$k+1$	0	k	E_{2n} alternating permutations	
Priority Queue	$k+1$	0	1	$1.3 \ldots (2n-1)$ involutions with no fixed pts.	
Symbol table	$k+1$	k	1	$B_n^{(2)}$ Partitions	
Stack	1	0	1	$C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan nb.	

("abstract") Data structures	Possibility functions			Number of Histories. Moments	Orthogonal Polynomials
	a_k	q_k	s_k	h_n	
Dictionary	$k+1$	$2k+1$	k	$n!$ Permutations	Laguerre
Linear list	$k+1$	0	k	E_{2n} alternating permutations	Meixner
Priority Queue	$k+1$	0	1	$1.3\ldots(2n-1)$ involutions with no fixed pts.	Hermite
Symbol table	$k+1$	k	1	$B_n^{(2)}$ Partitions	Charlier
Stack	1	0	1	$C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan nb.	Tchebycheff

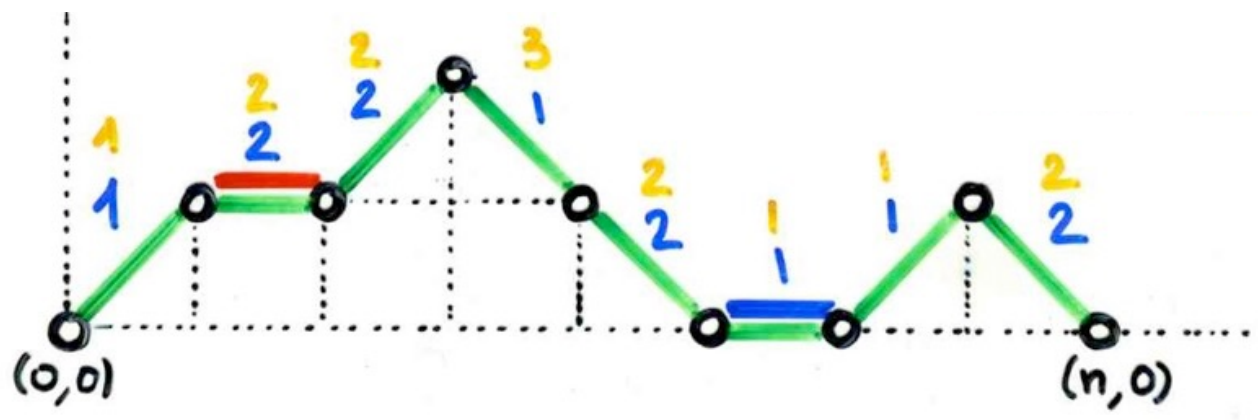
The FV-bijection with operators

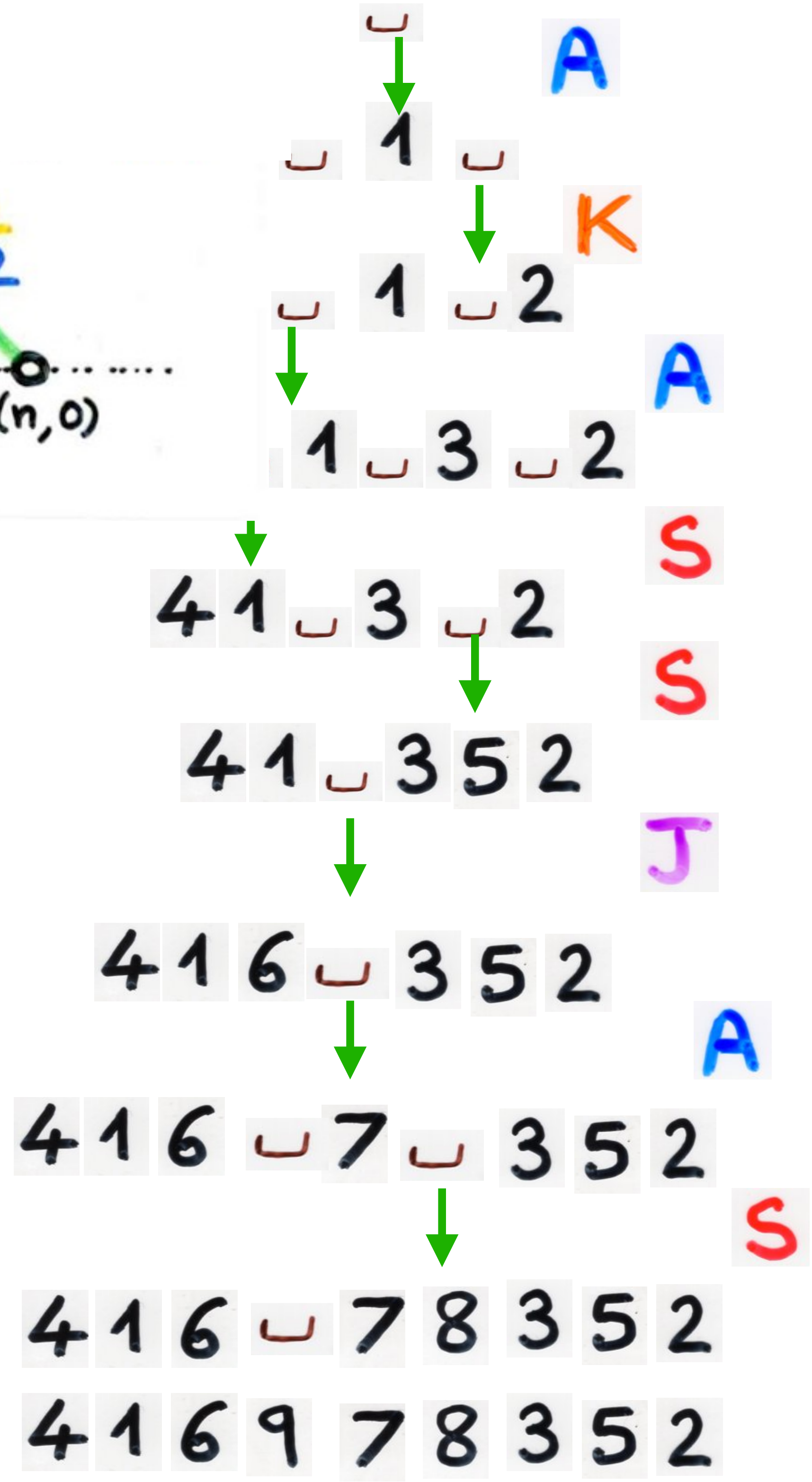


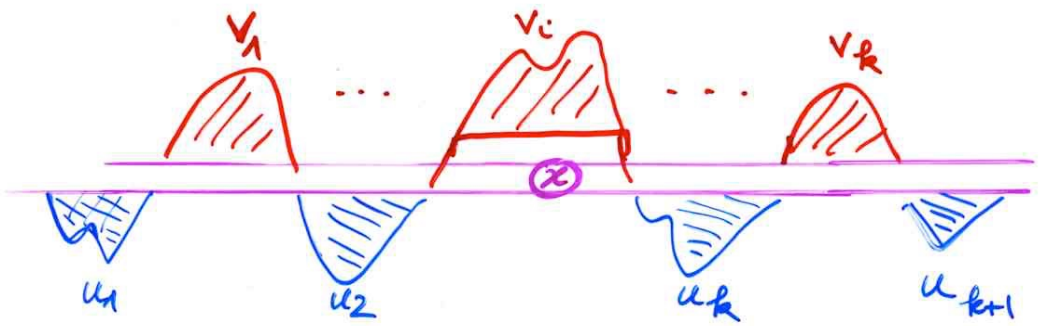
(0,0)
(n,0)
1 1
2 2
2 2
3 1
2 2
1 1
1 1
2 2



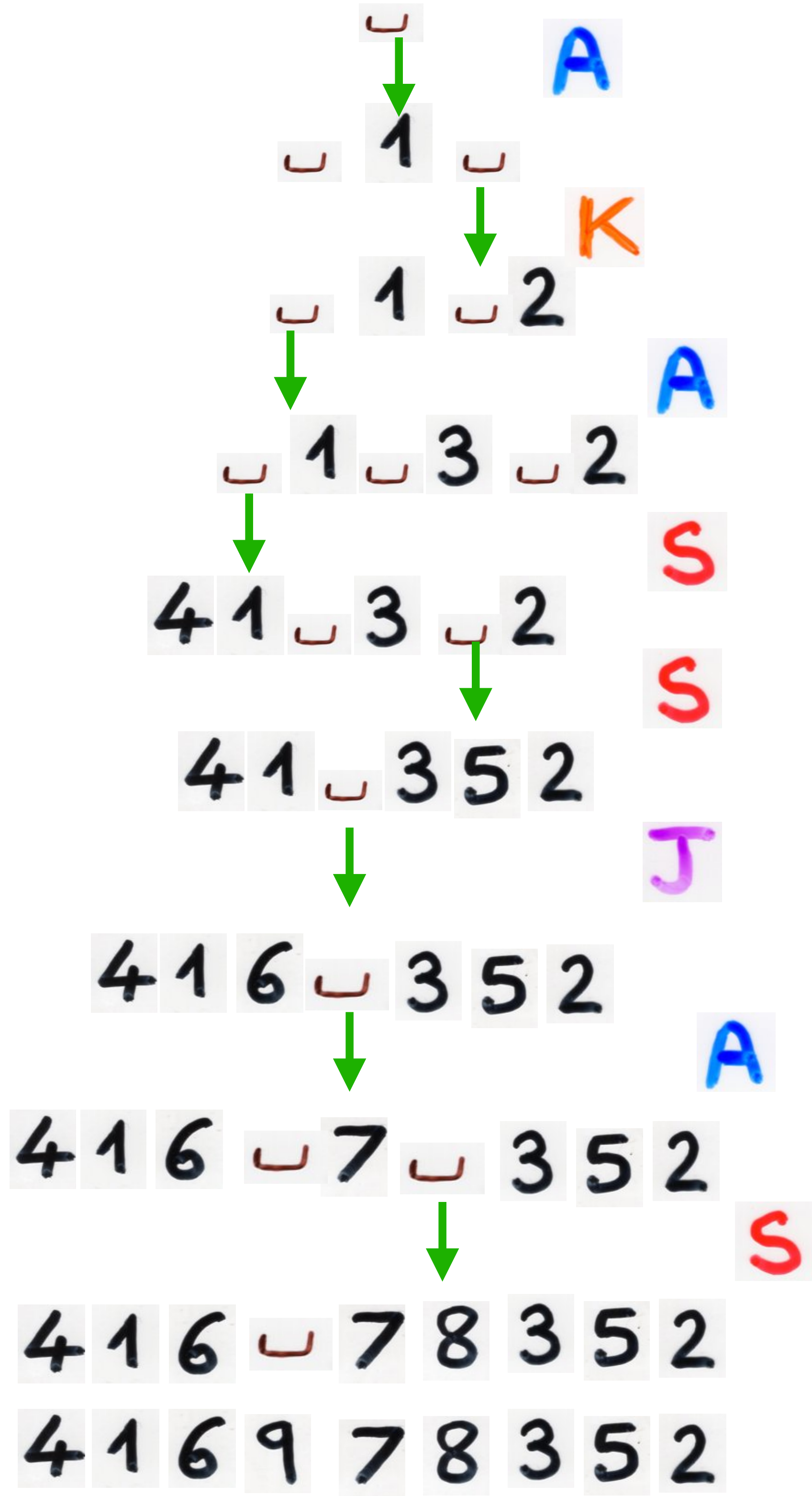
x ω_c p_i v(ω_i)
1 1 1 ⊔ 1 ⊔ 2
2 2 2 ⊔ 1 ⊔ 2
3 2 2 ⊔ 1 ⊔ 3 ⊔ 2
4 1 3 4 1 ⊔ 3 ⊔ 2
5 2 2 4 1 ⊔ 3 5 2
6 1 1 4 1 6 ⊔ 3 5 2
7 1 1 4 1 6 ⊔ 7 ⊔ 3 5 2
8 2 2 4 1 6 ⊔ 7 8 3 5 2
9 – – σ = 4 1 6 9 7 8 3 5 2


(0,0)
(n,0)
1 1
2 2
2 2
3 1
2 2
1 1
1 1
2 2


A
K
A
S
S
J
A
S
1
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2



v1
vi
vk
x
u1
u2
uk
uk+1



␣
A
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

direct proof for the number
of alternating tableaux of size n
= $(n+1)!$

$$\langle \circ | (D+E)^n | \circ \bullet \rangle$$
$$+ \langle \circ | (D+E)^n | \bullet \circ \rangle$$
$$+ \langle \circ | (D+E)^n | \bullet \circ \bullet \rangle = (n+1)!$$

from the bijection Laguerre histories -- permutations

$$\langle \bullet \circ \bullet | (D+E)^n | \bullet \circ \bullet \rangle = (n+1)!$$

Lemma - • V vector space
B base, $v_0 \in B$

• E, D operators $V \to V$ such that

$$DE = ED + E + D$$

$$\forall i,j, \quad \langle v_0 | E^i D^j | v_0 \rangle = 1$$

Then the number of alternating tableaux of size n is equal to

$$\sum_{\substack{w \in \{E,D\}^* \\ |w| = n}} \langle v_0 | w | v_0 \rangle$$

$$D = A + J$$

$$E = S + K$$

$$v_0 = \bullet \circ \bullet$$

satisfies condition of lemma

(i) $DE = ED + E + D$

(ii) $\forall i, j \geqslant 0 \quad \langle v_0 | E^i D^j | v_0 \rangle = 1$

proof:

$$\begin{cases} E = S + K \\ D = A + J \end{cases}$$

$$\langle \bullet 0 \bullet | E^i D^j | \bullet 0 \bullet \rangle$$

$$= \langle \bullet 0 \bullet | K^i D^j | \bullet 0 \bullet \rangle$$

$$= \langle \bullet 0 \bullet | D^j | \bullet 0 \bullet \rangle$$

$$= \langle \bullet 0 \bullet | J^j | \bullet 0 \bullet \rangle$$

$$= \langle \bullet 0 \bullet | \bullet 0 \bullet \rangle = 1$$

(i) $DE = ED + E + D$

(ii) $\forall i,j \geqslant 0 \quad \langle V_0 | E^i D^j | V_0 \rangle = 1$

(iii)
$$\sum_{\substack{w \in \{E,D\}^* \\ |w| = n}} \langle V_0 | w | V_0 \rangle =$$

Number of alternative tableaux of size n

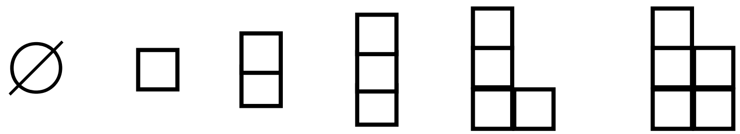
$$\langle \bullet \circ \bullet | (D+E)^n | \bullet \circ \bullet \rangle = (n+1)!$$

$$\langle \circ | (D+E)^n | \circ \bullet \rangle$$
$$+ \langle \circ | (D+E)^n | \bullet \circ \rangle$$
$$+ \langle \circ | (D+E)^n | \bullet \circ \bullet \rangle = (n+1)!$$

analogy with direct proofs
using operators U,D
representing the Weyl-algebra
(see Ch 2)

$$UD = DU + I$$

$$\langle \emptyset | U^n D^n | \emptyset \rangle$$

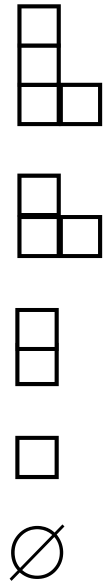


$\emptyset$ empty Ferrers diagram

$$= \sum_{\lambda} \left(f_\lambda\right)^2$$

λ partition of n

$$= \sum_{i \geq 0} c_{n,i} \langle \emptyset | D^i U^i | \emptyset \rangle$$



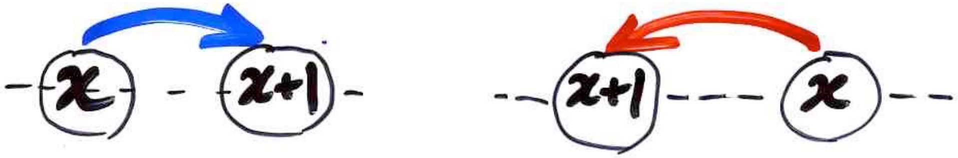
$$= c_{n,0} \qquad = n!$$

The “exchange-fusion” algorithm

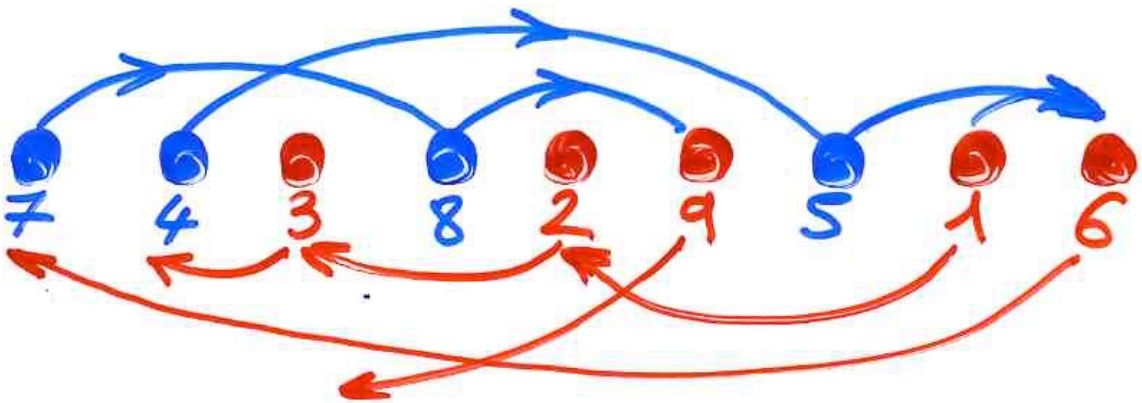
x advance in a permutation σ

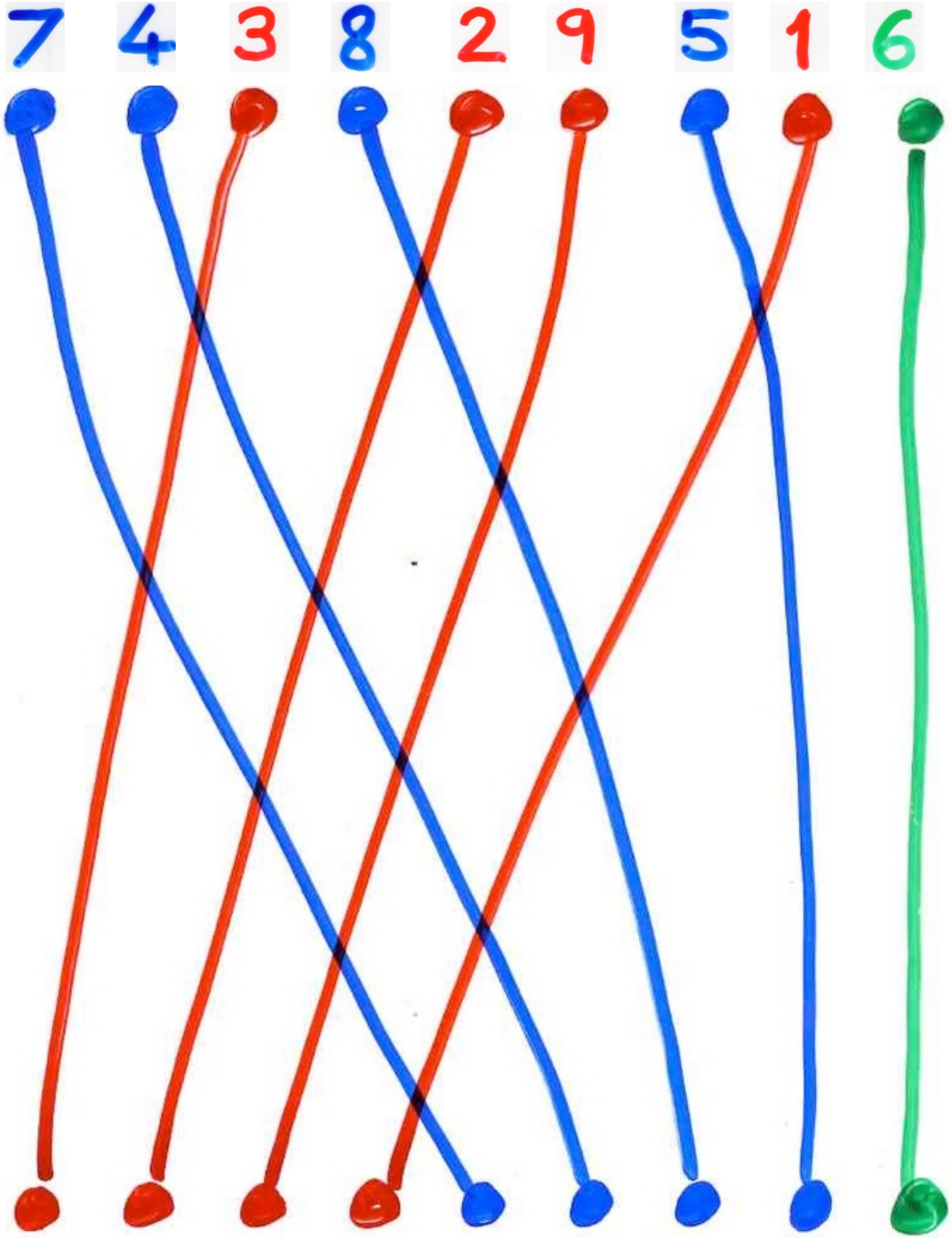
iff $x = \sigma(i)$, $x+1 = \sigma(j)$

with $i < j$



σ = 7 4 3 8 2 9 5 1 6




7
4
3
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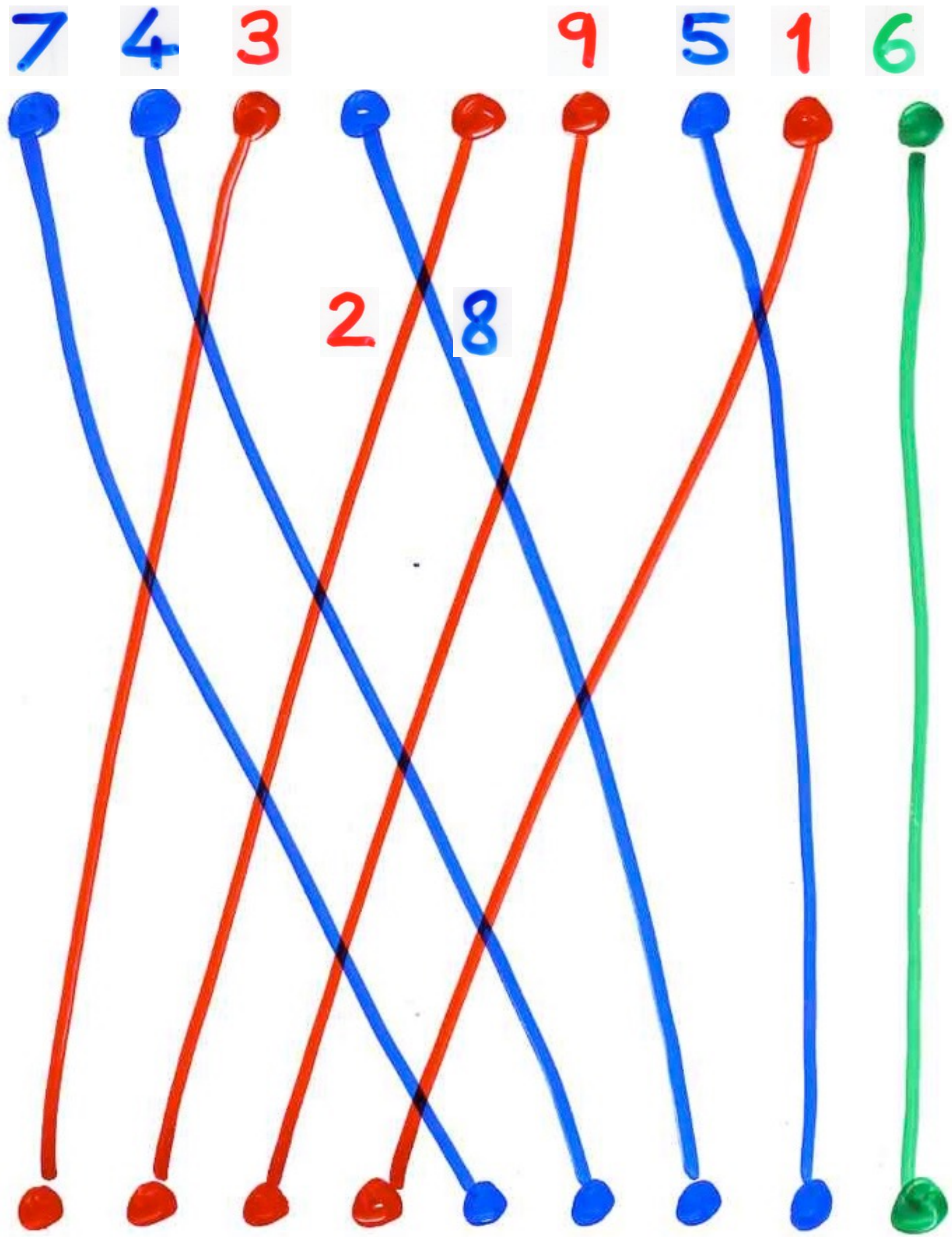
2
9
5
1
6



7
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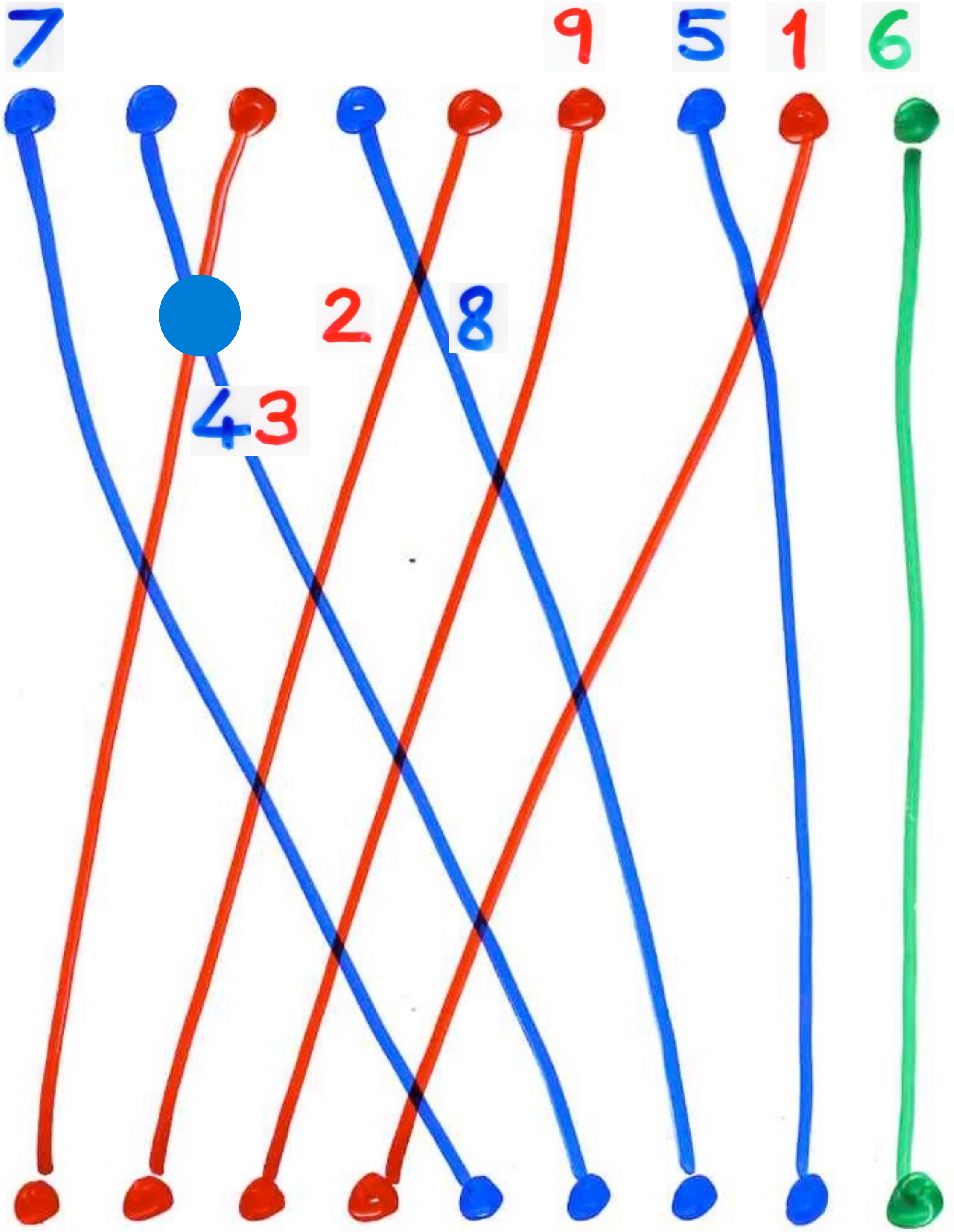
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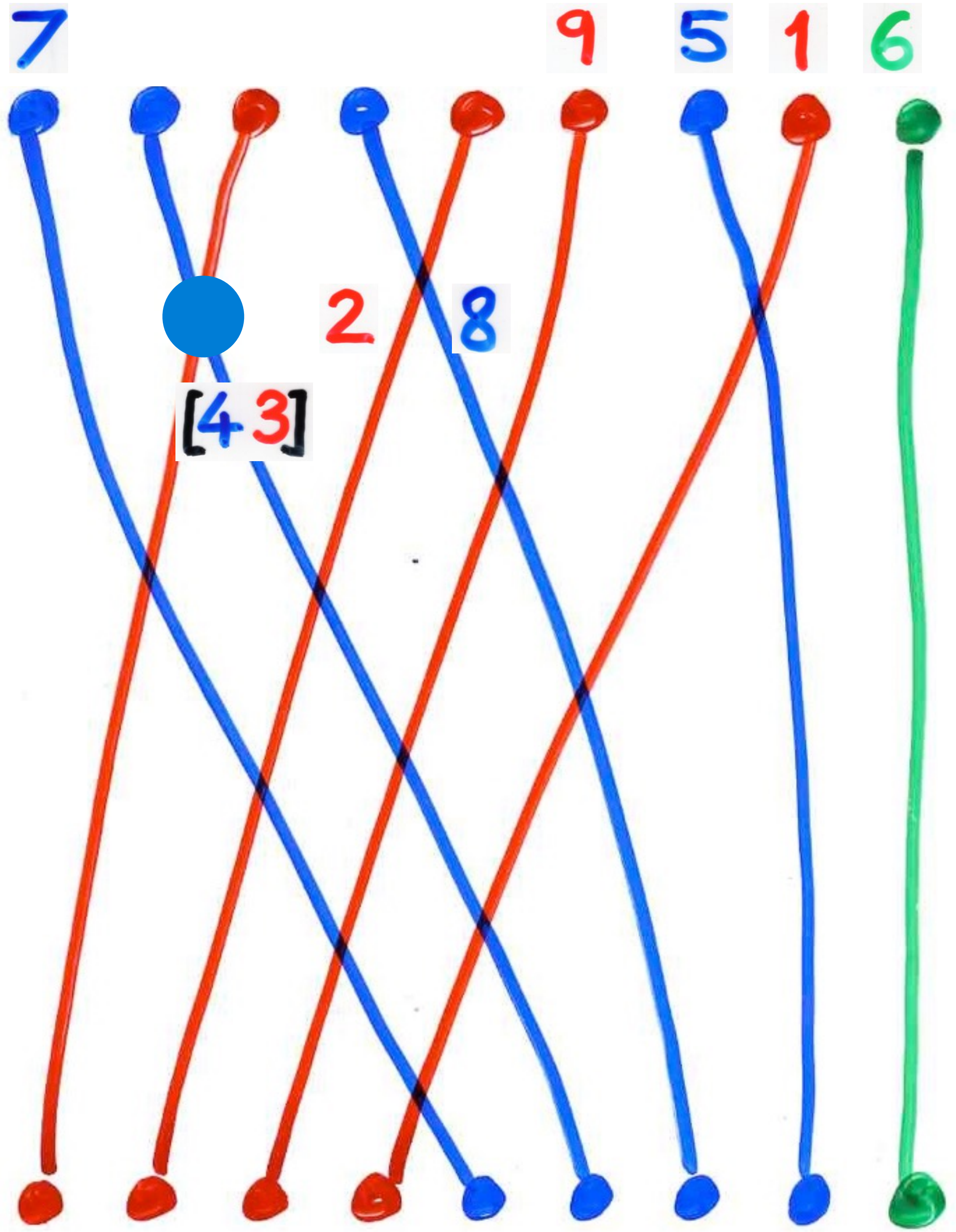
7
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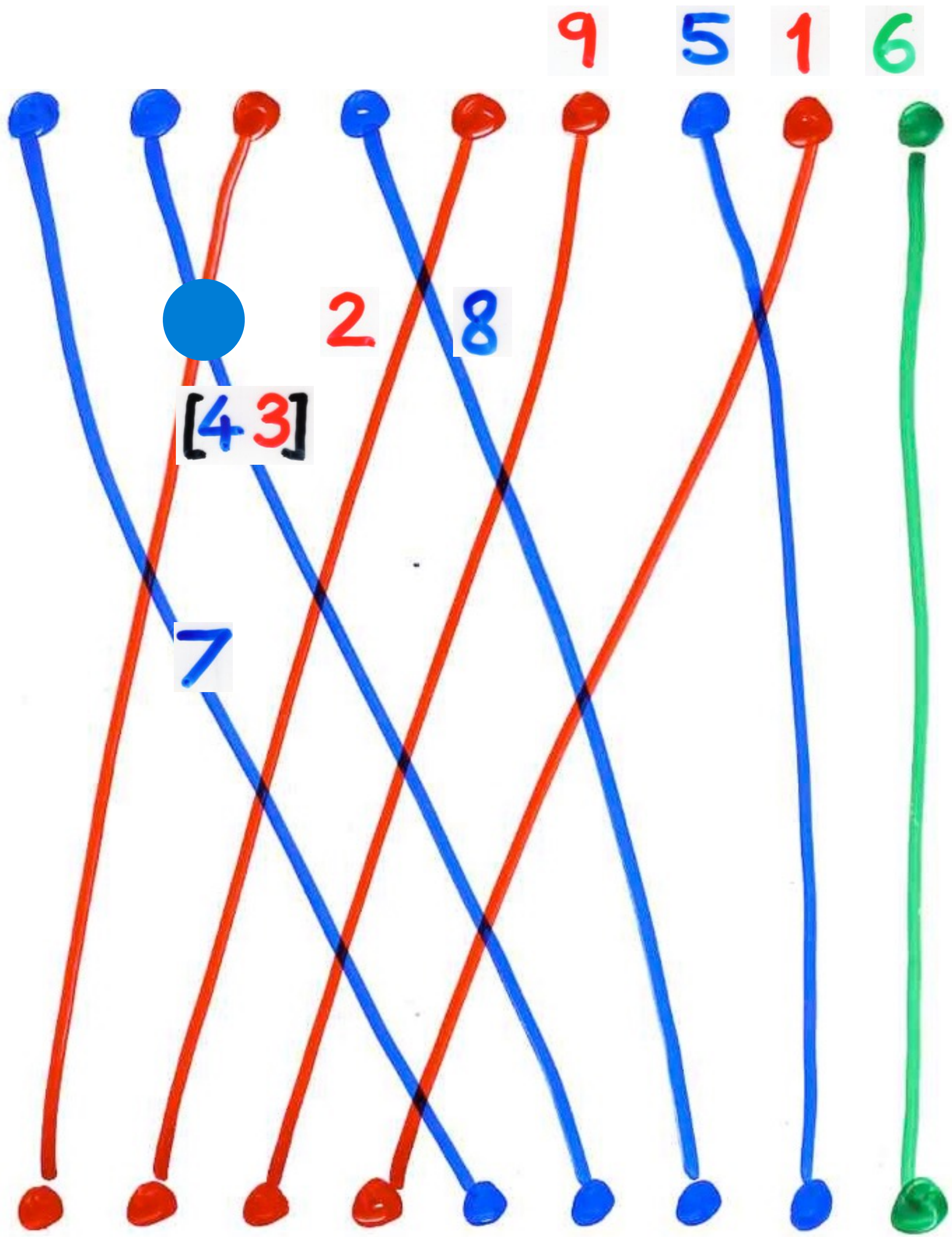
[4 3]


9
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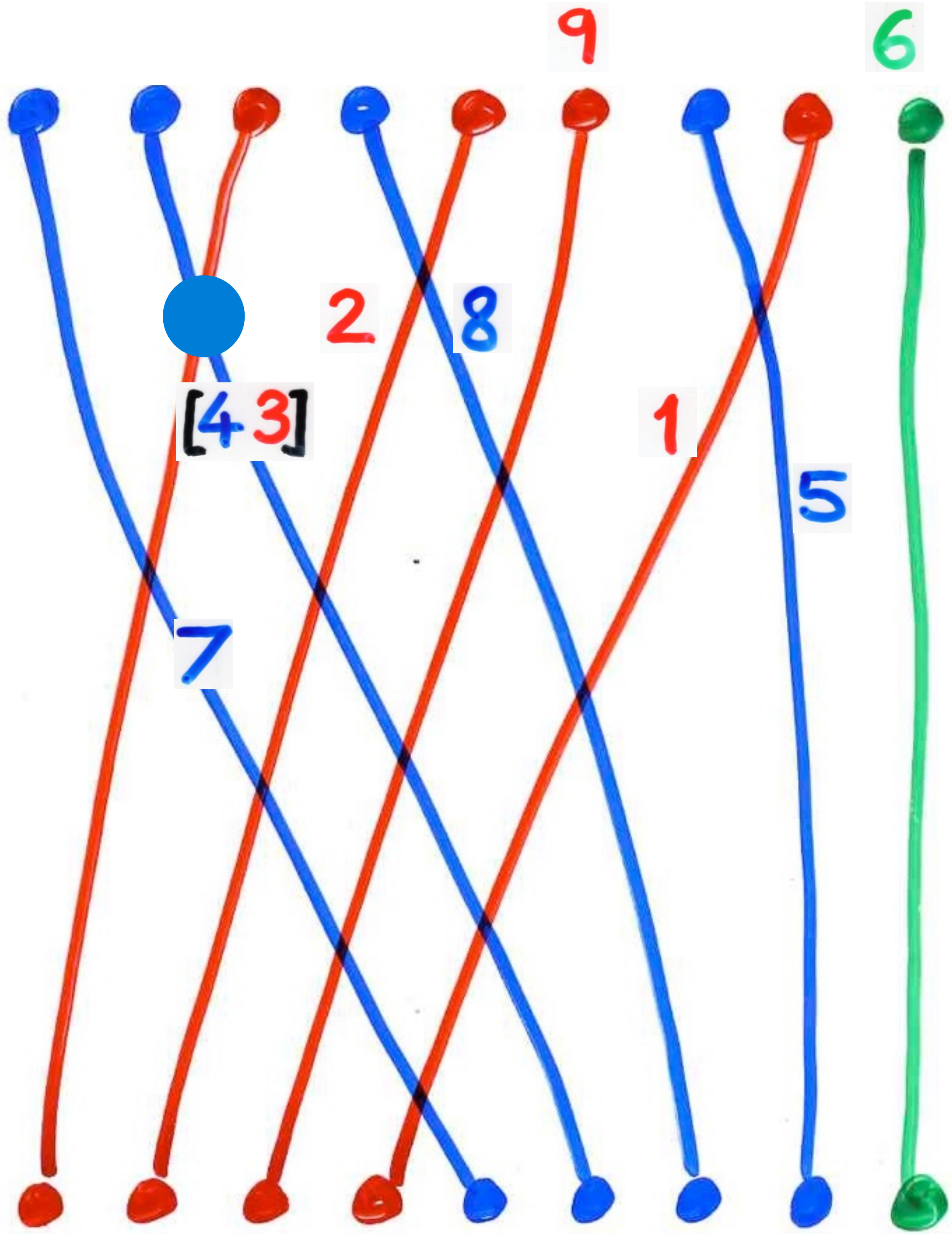
[43]
7



9
6
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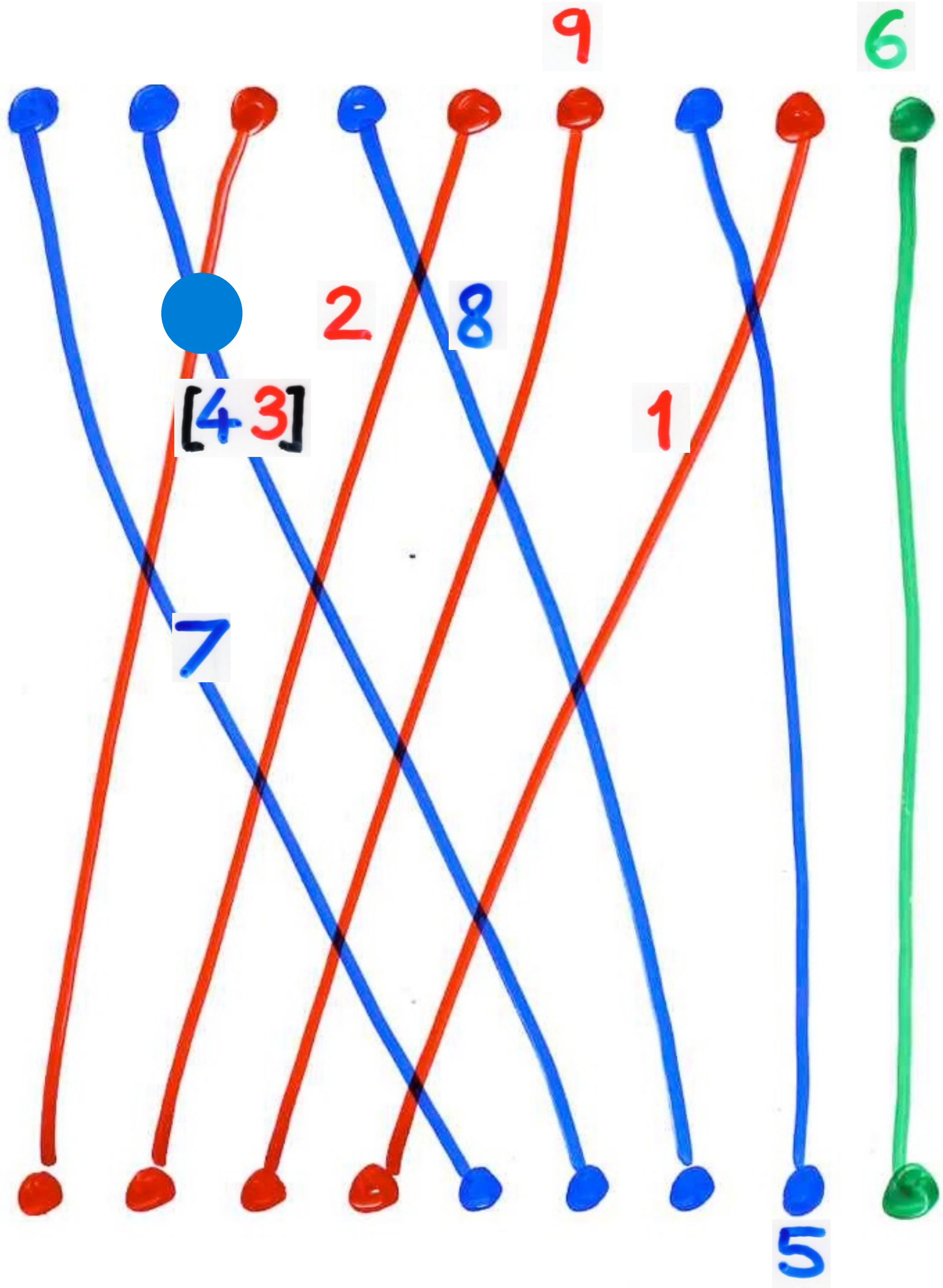
[43]
1
5
7


9
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[4 3]
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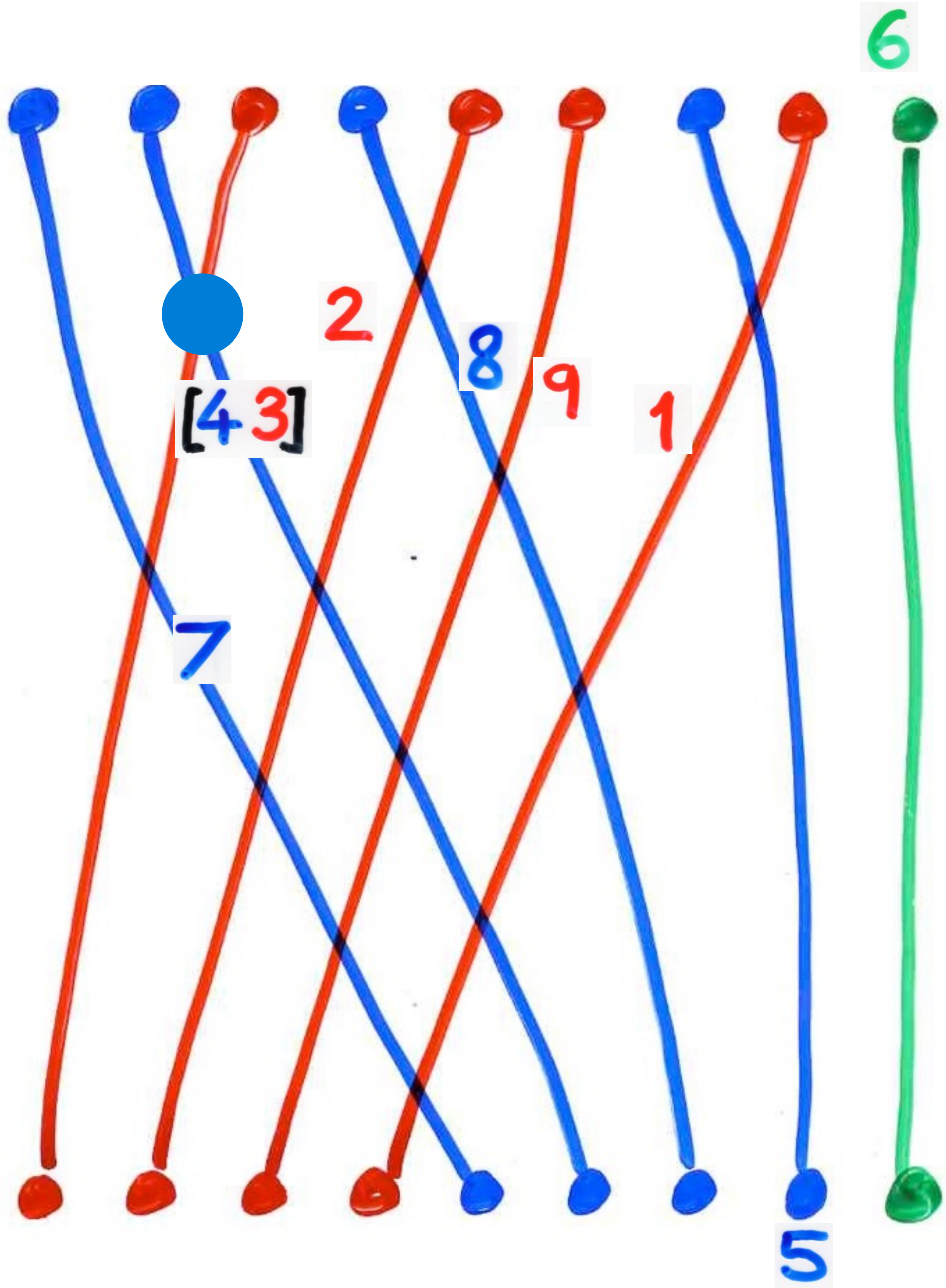
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[43]
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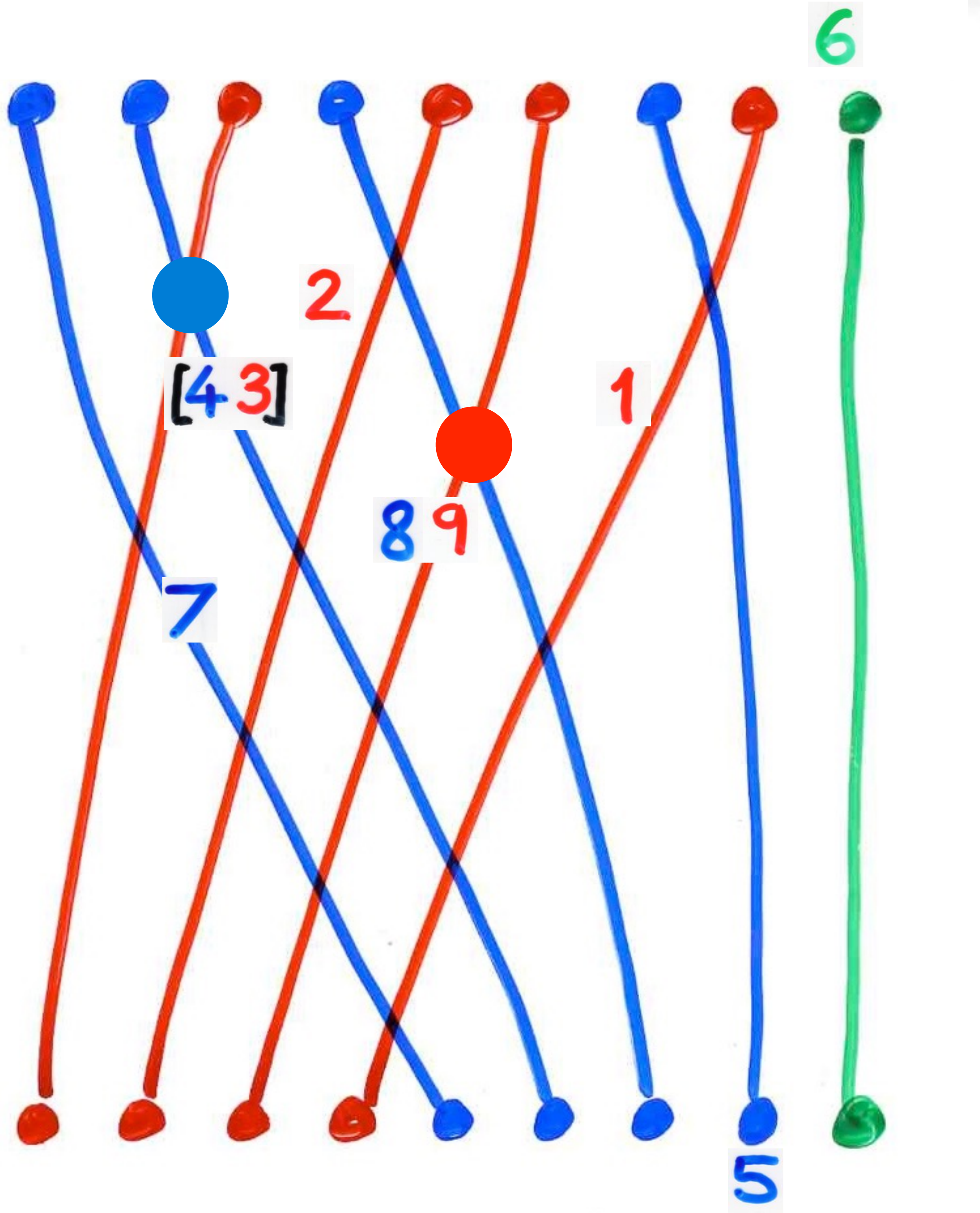
6
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[4 3]
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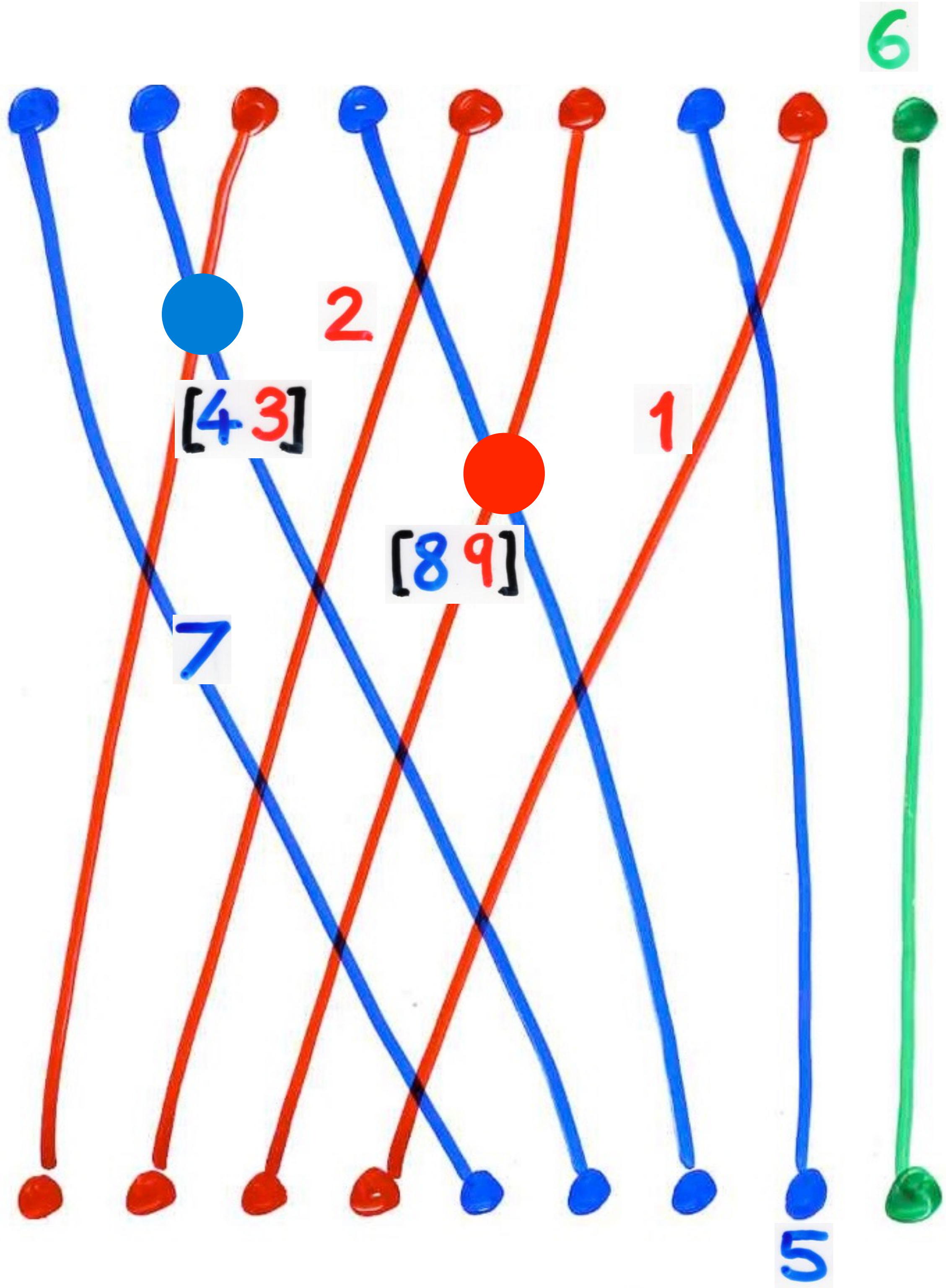
6
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[4 3]
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[8 9]
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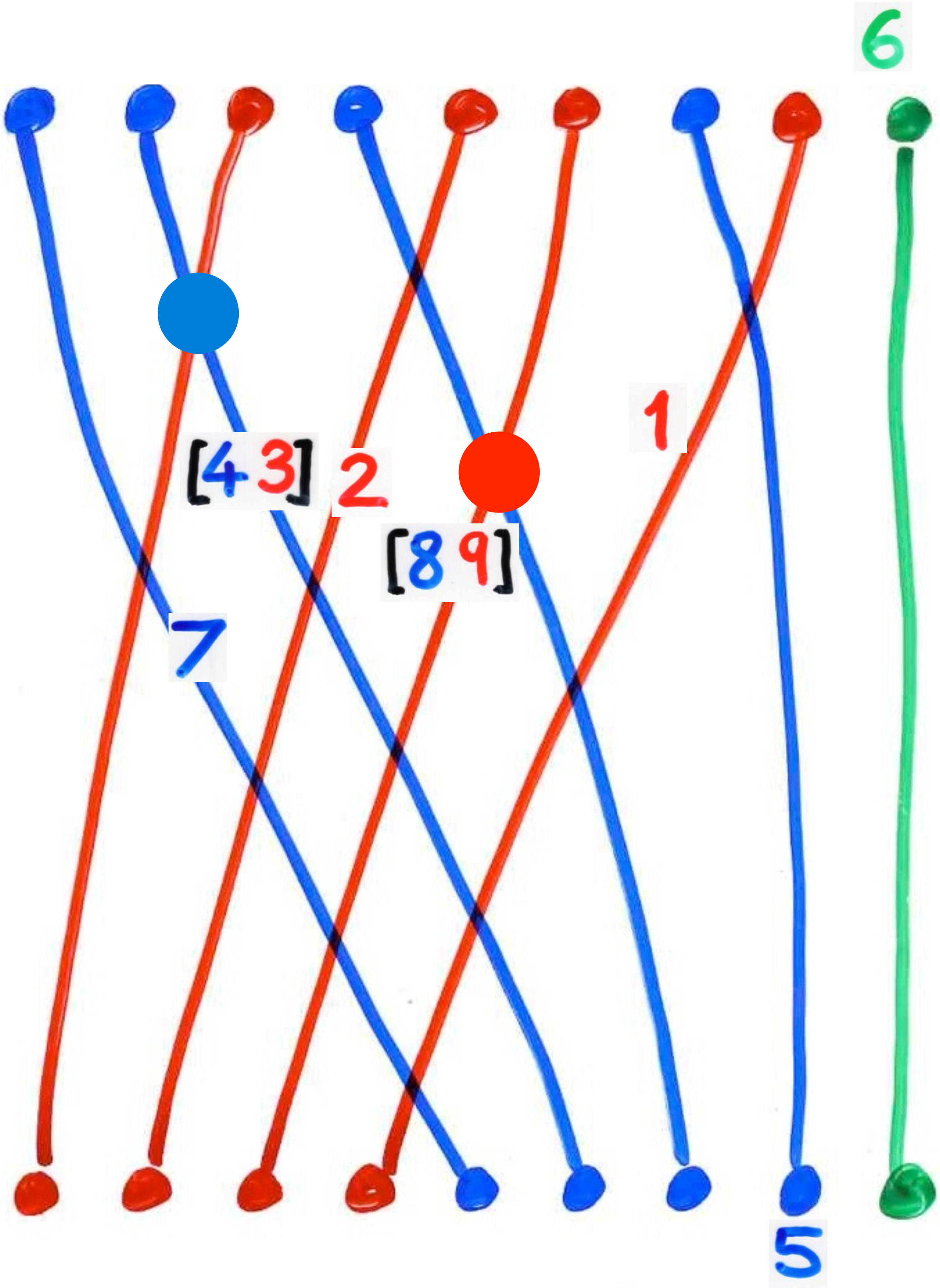
6
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[4 3]
2
[8 9]
7
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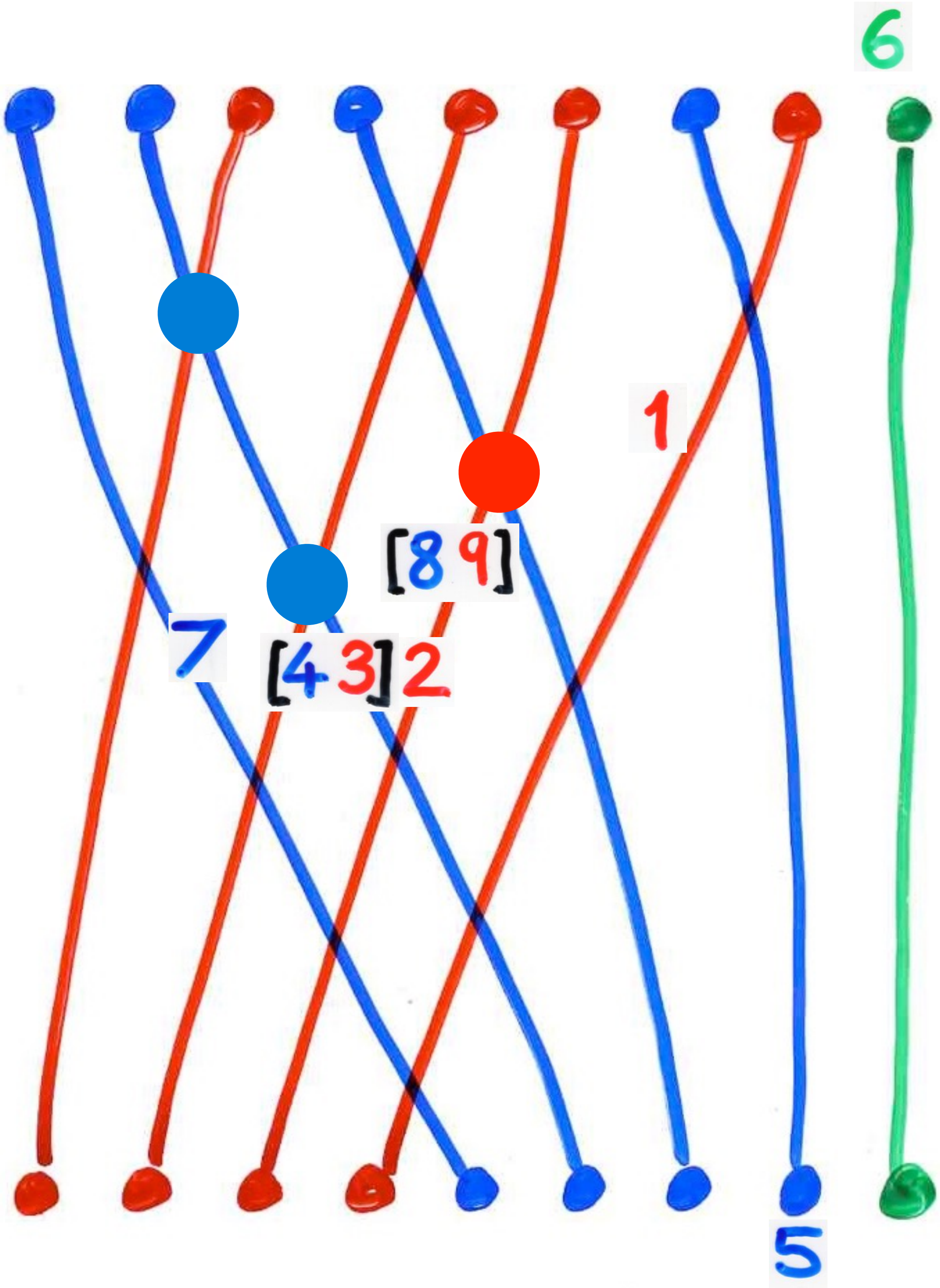
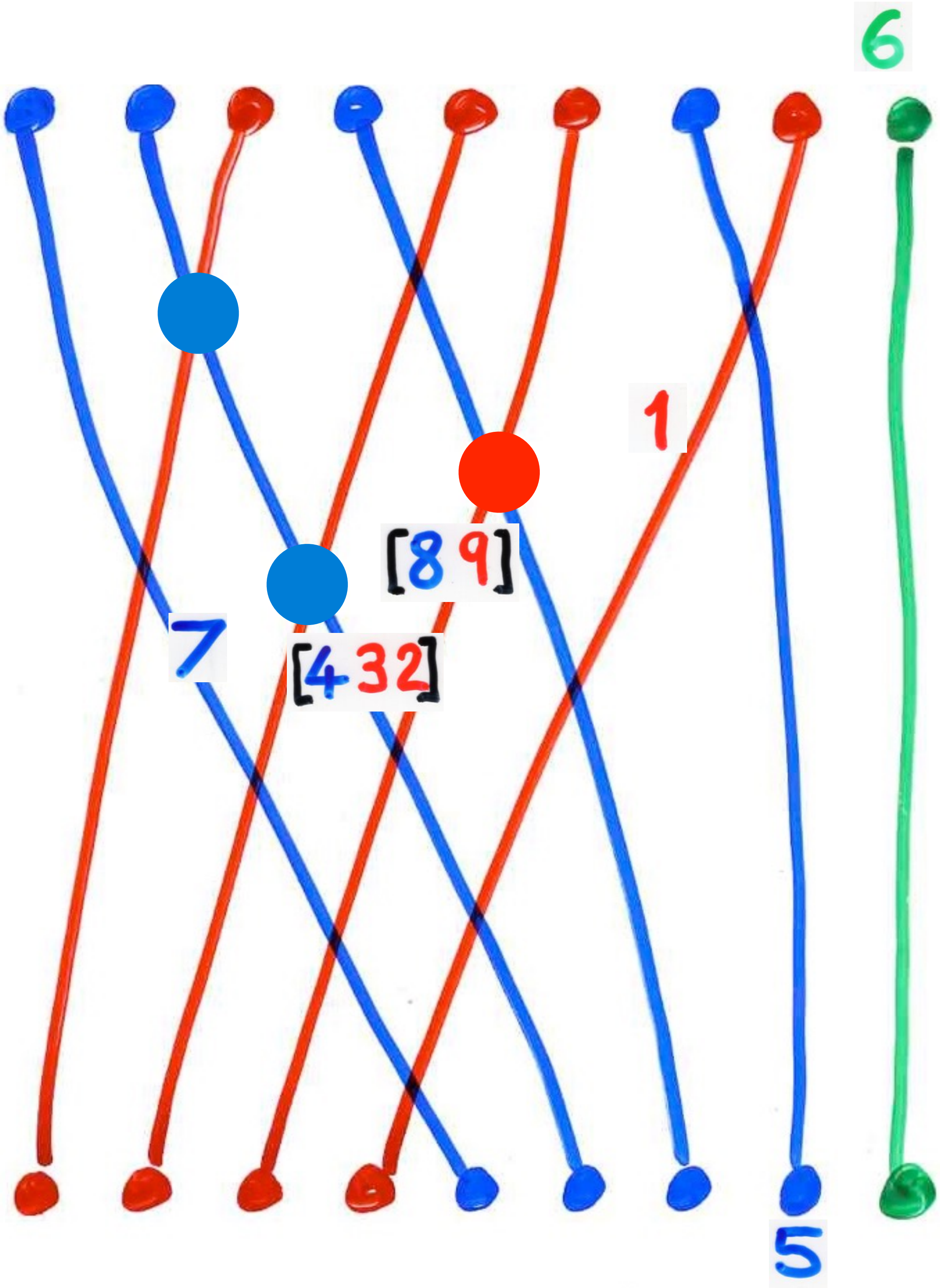

6
1
[8 9]
7
[4 3] 2
5


6
1
[8 9]
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[4 3 2]
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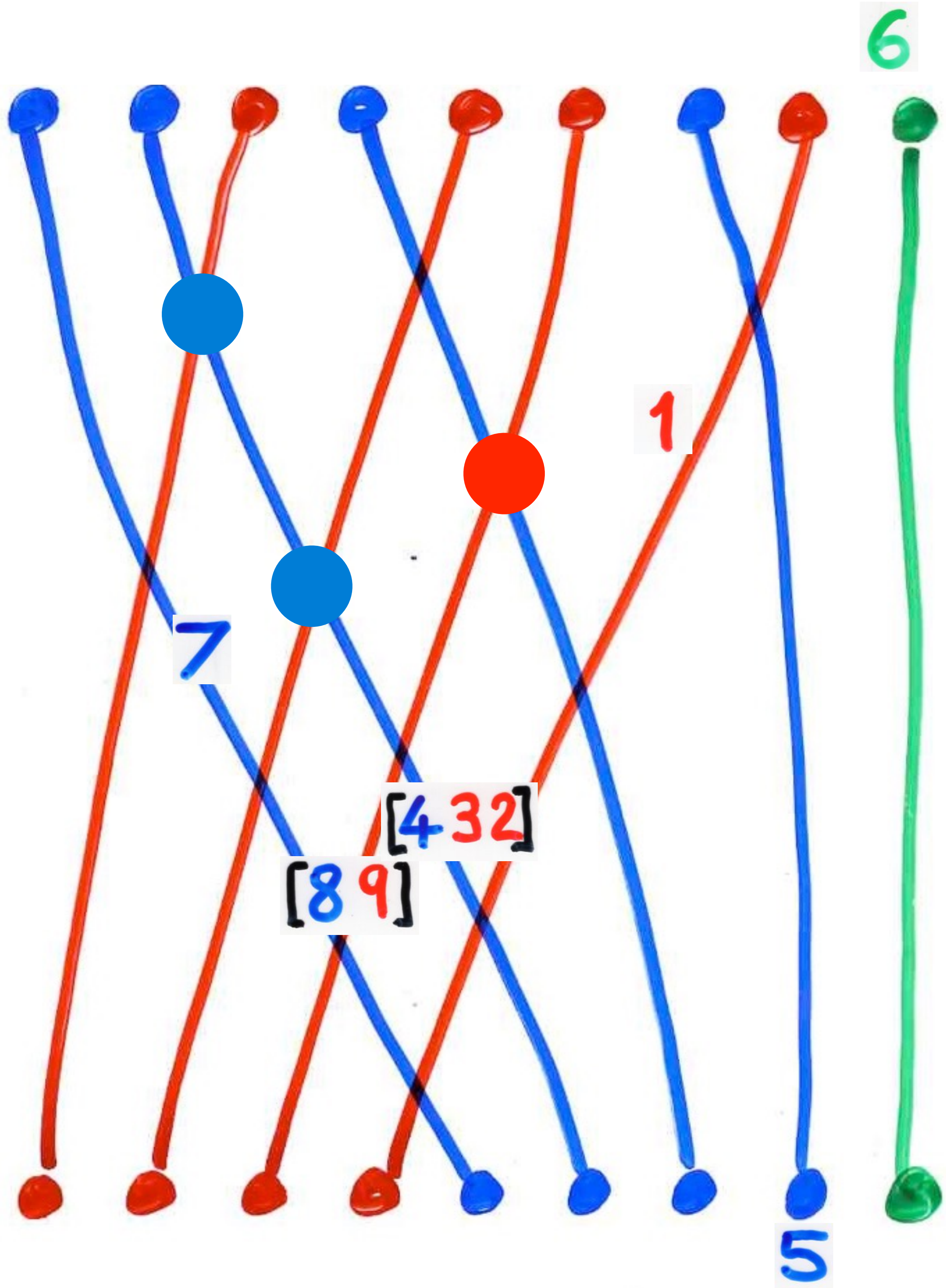
6
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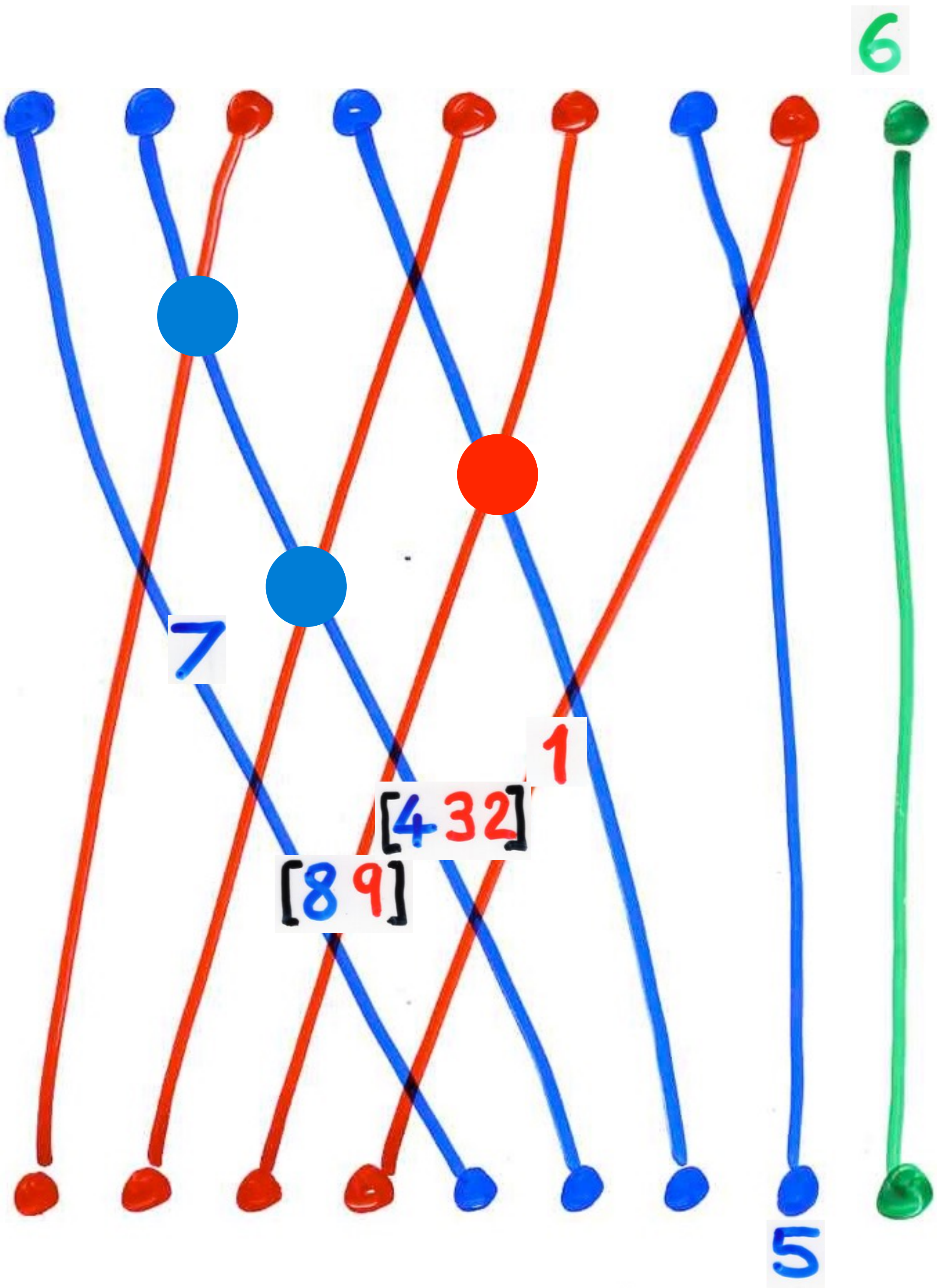
[4 3 2]
[8 9]
5


6
7
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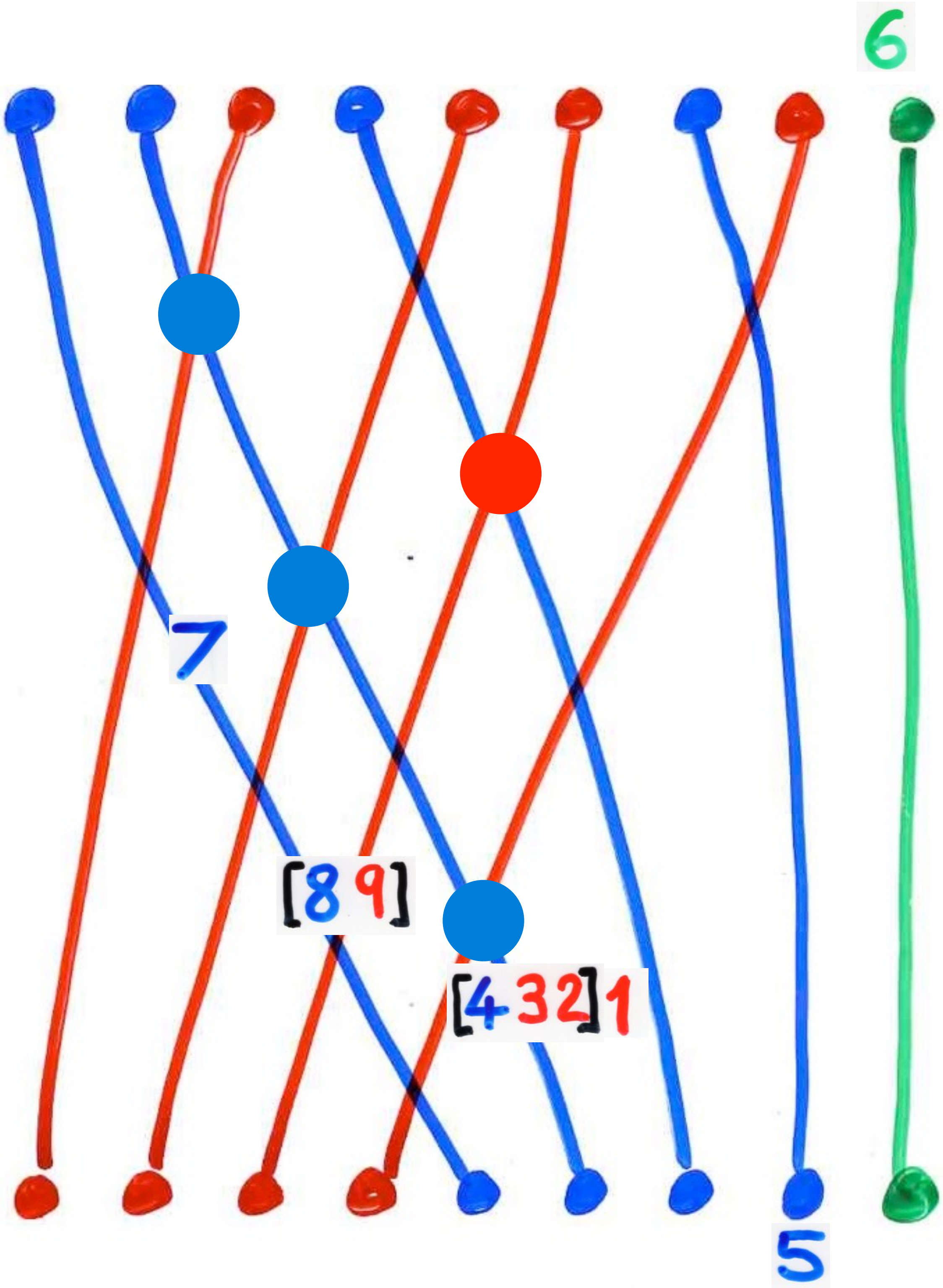
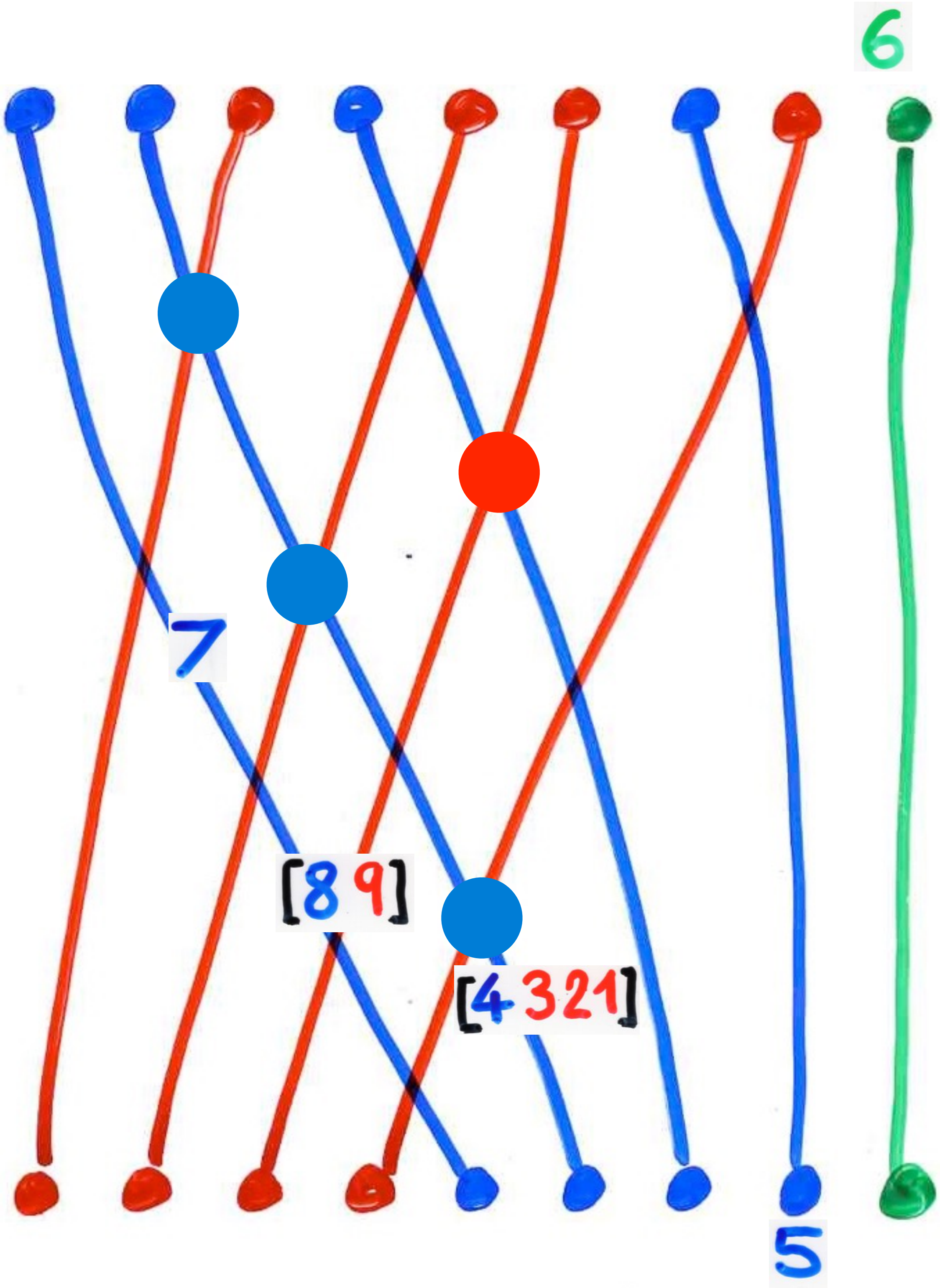
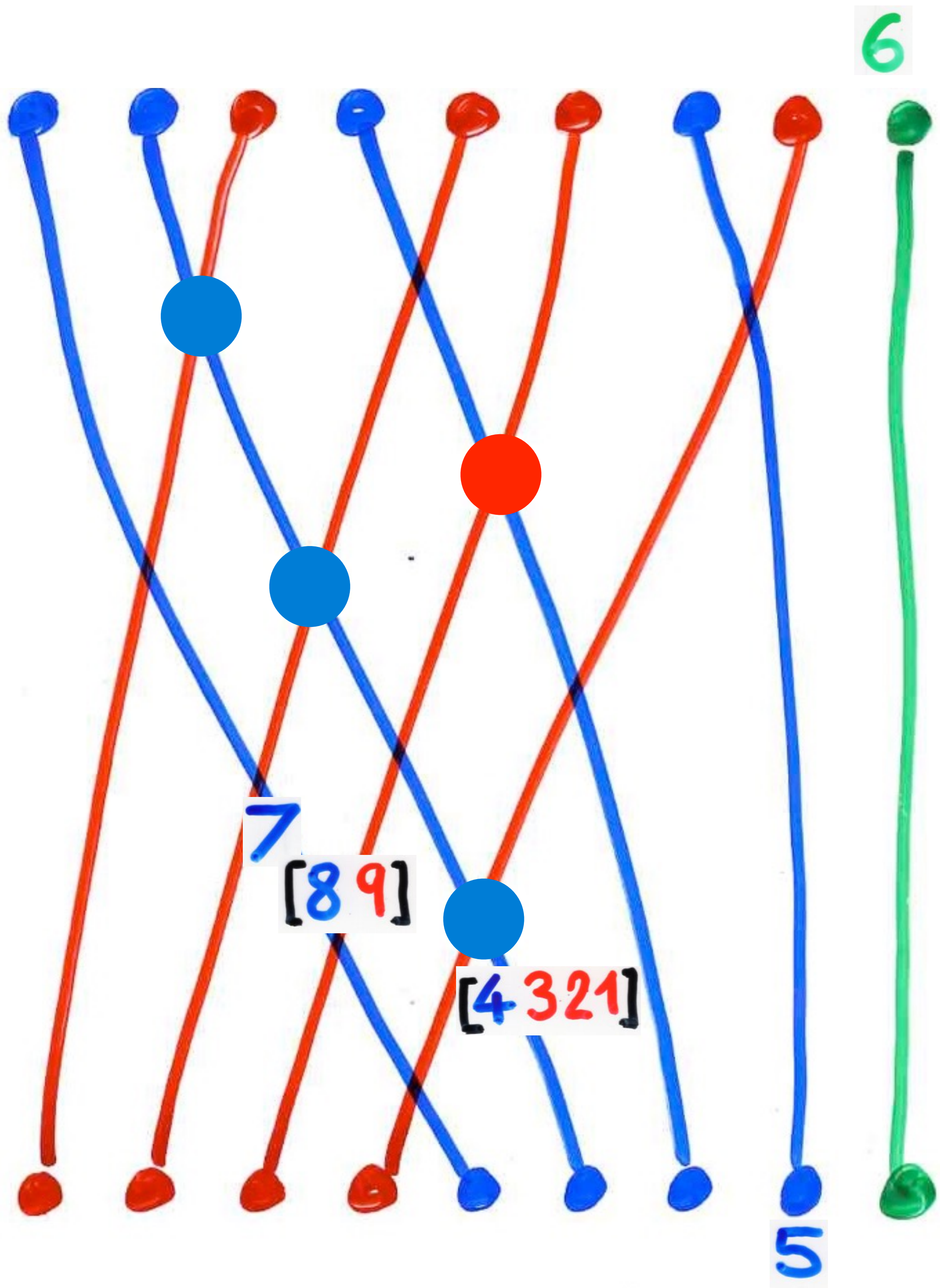
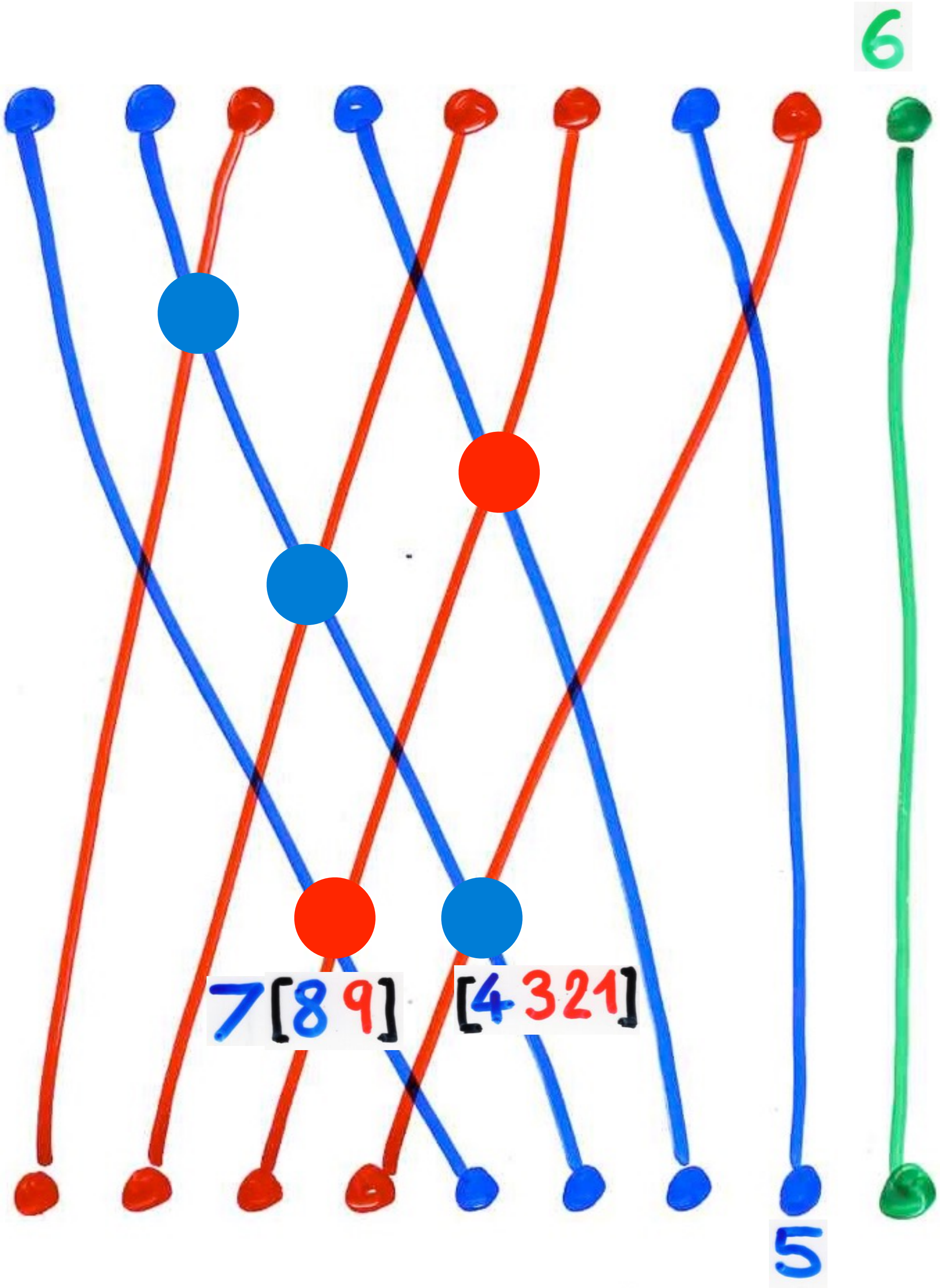
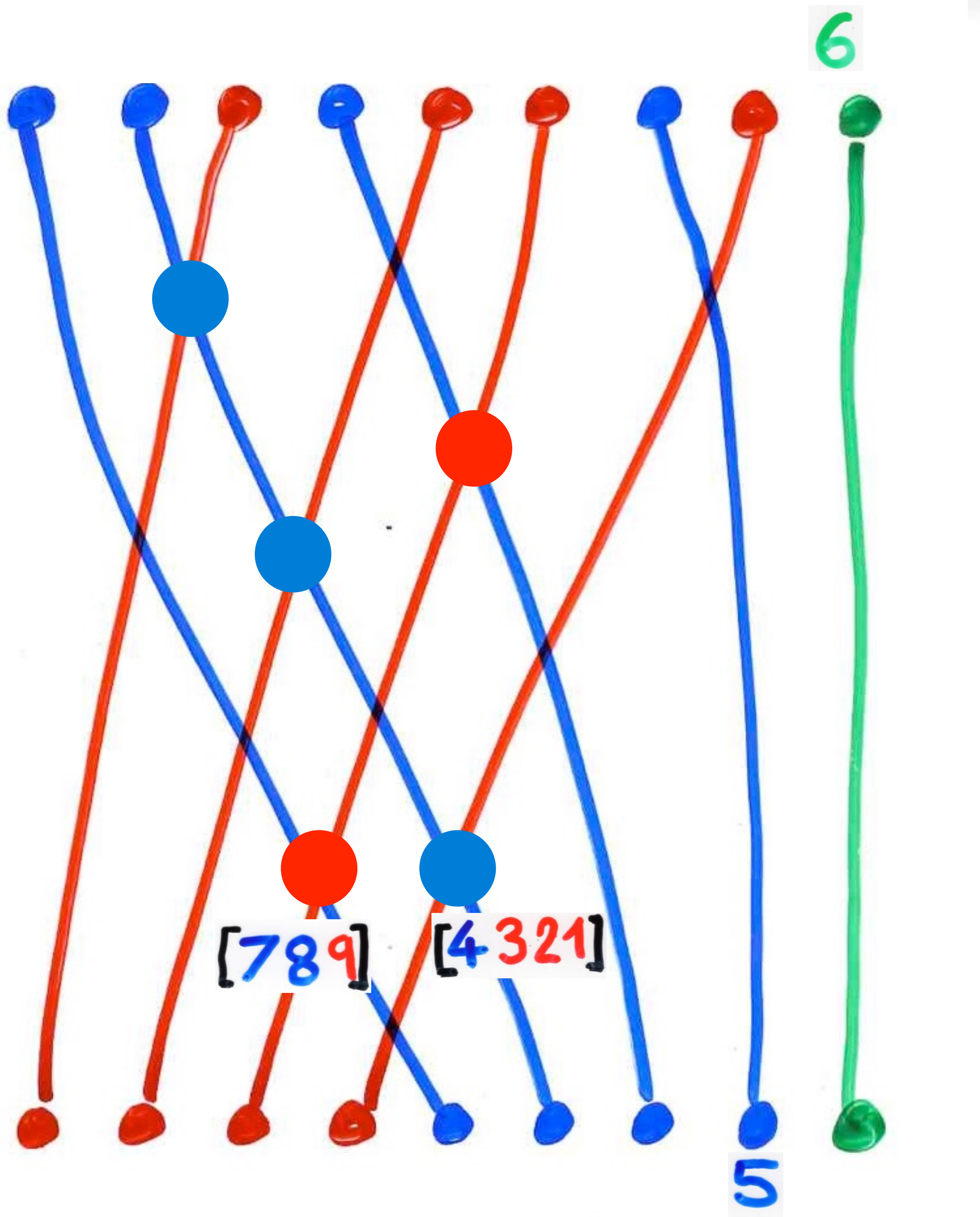
[4 3 2]
[8 9]
5

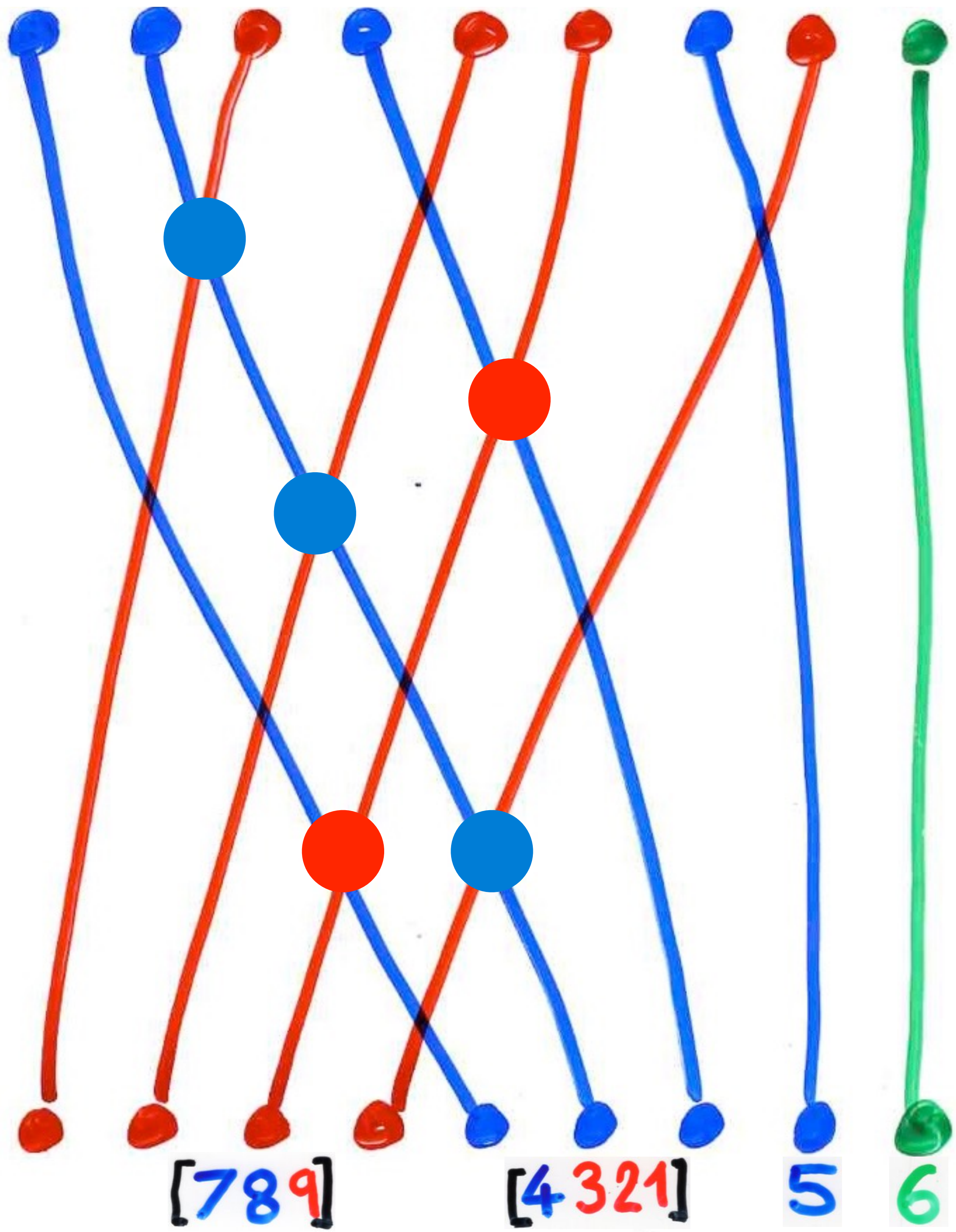

6
7
[8 9]
[4 3 2] 1
5


6
7
[8 9]
[4 3 2 1]
5

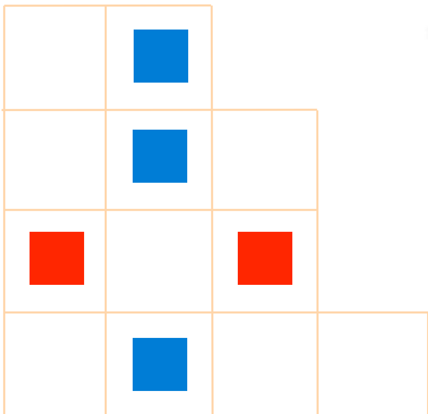

6
7
[8 9]
[4 3 2 1]
5

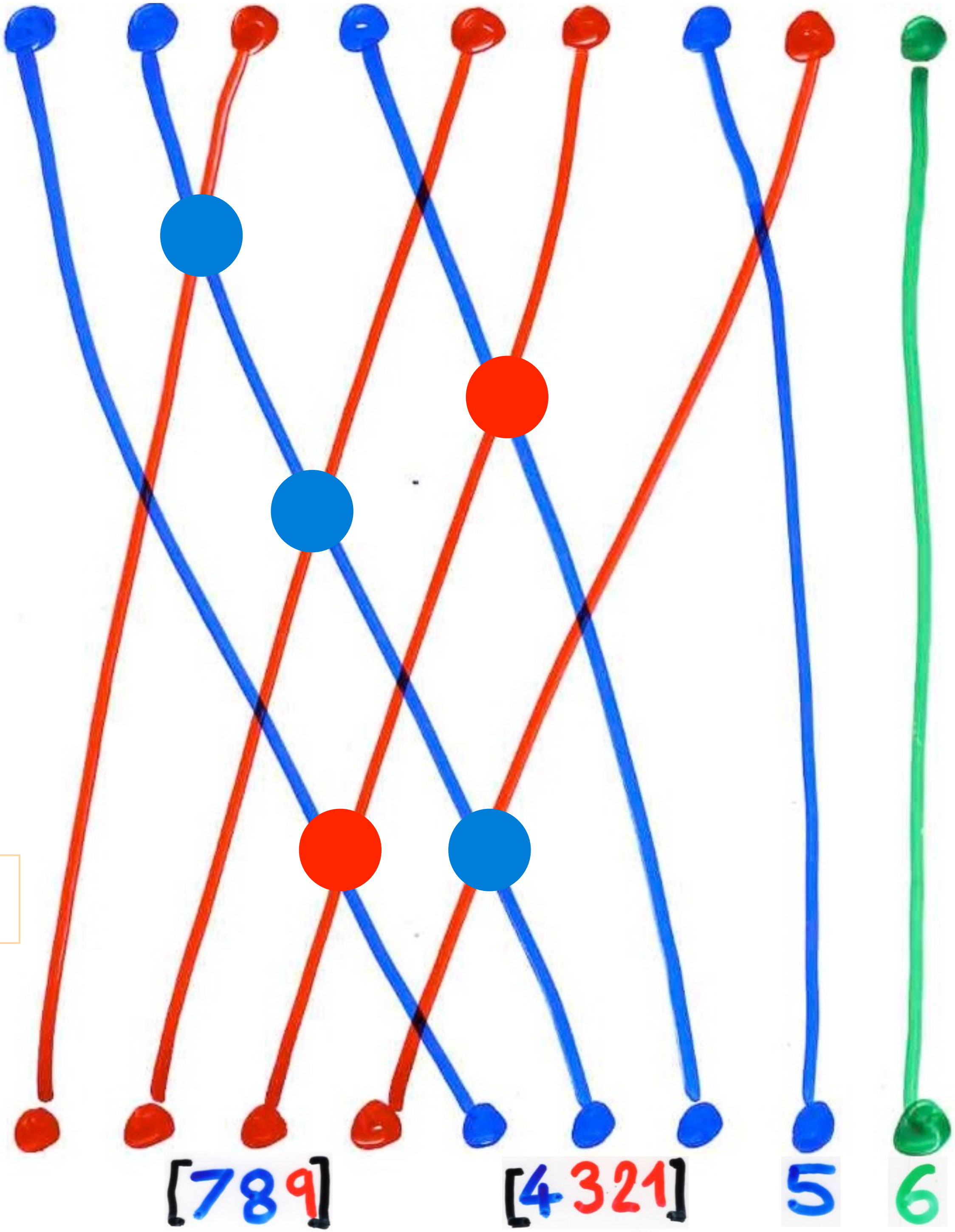

6
7[89]
[4321]
5


6
[789]
[4321]
5

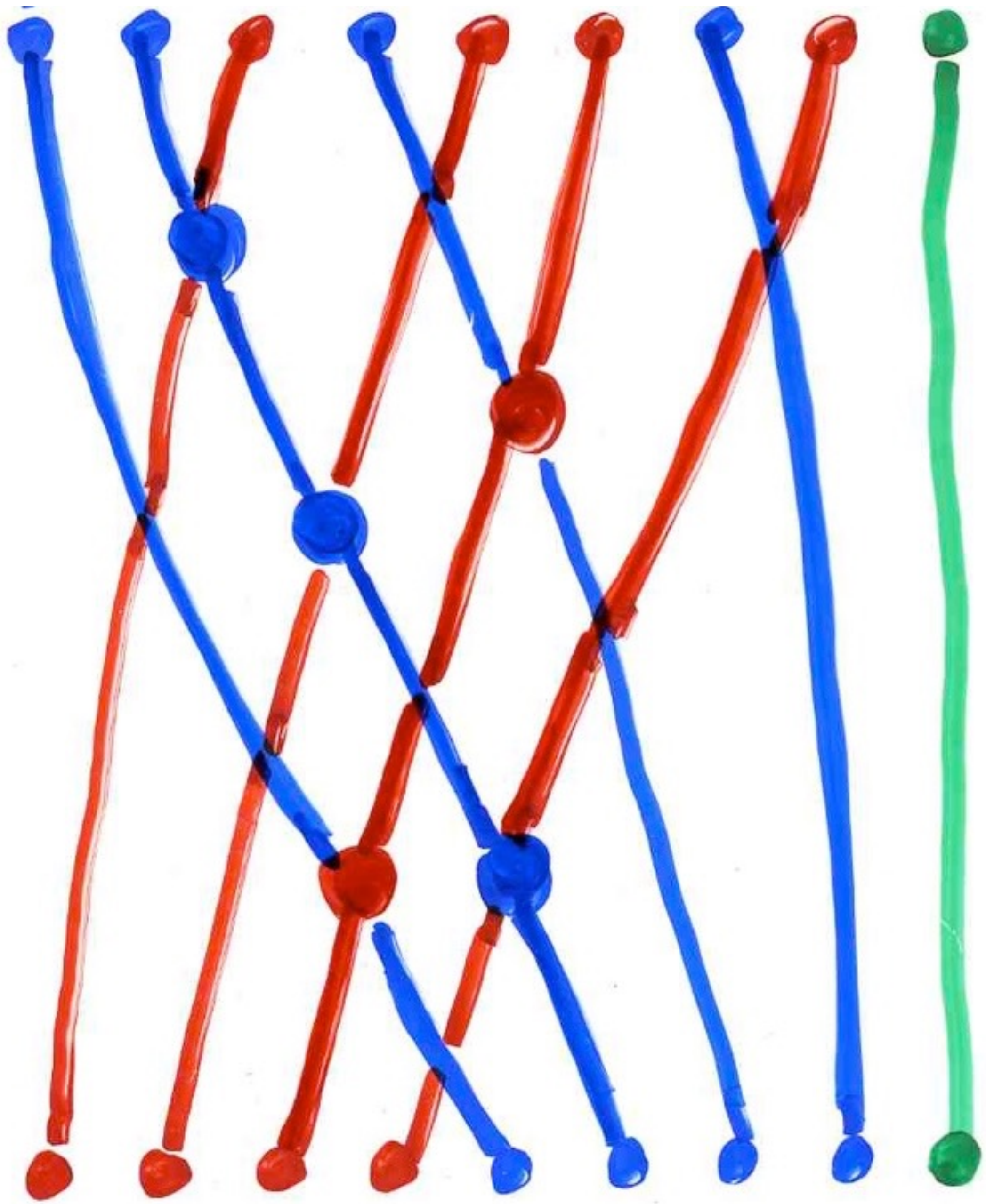

[789]
[4321]
5
6

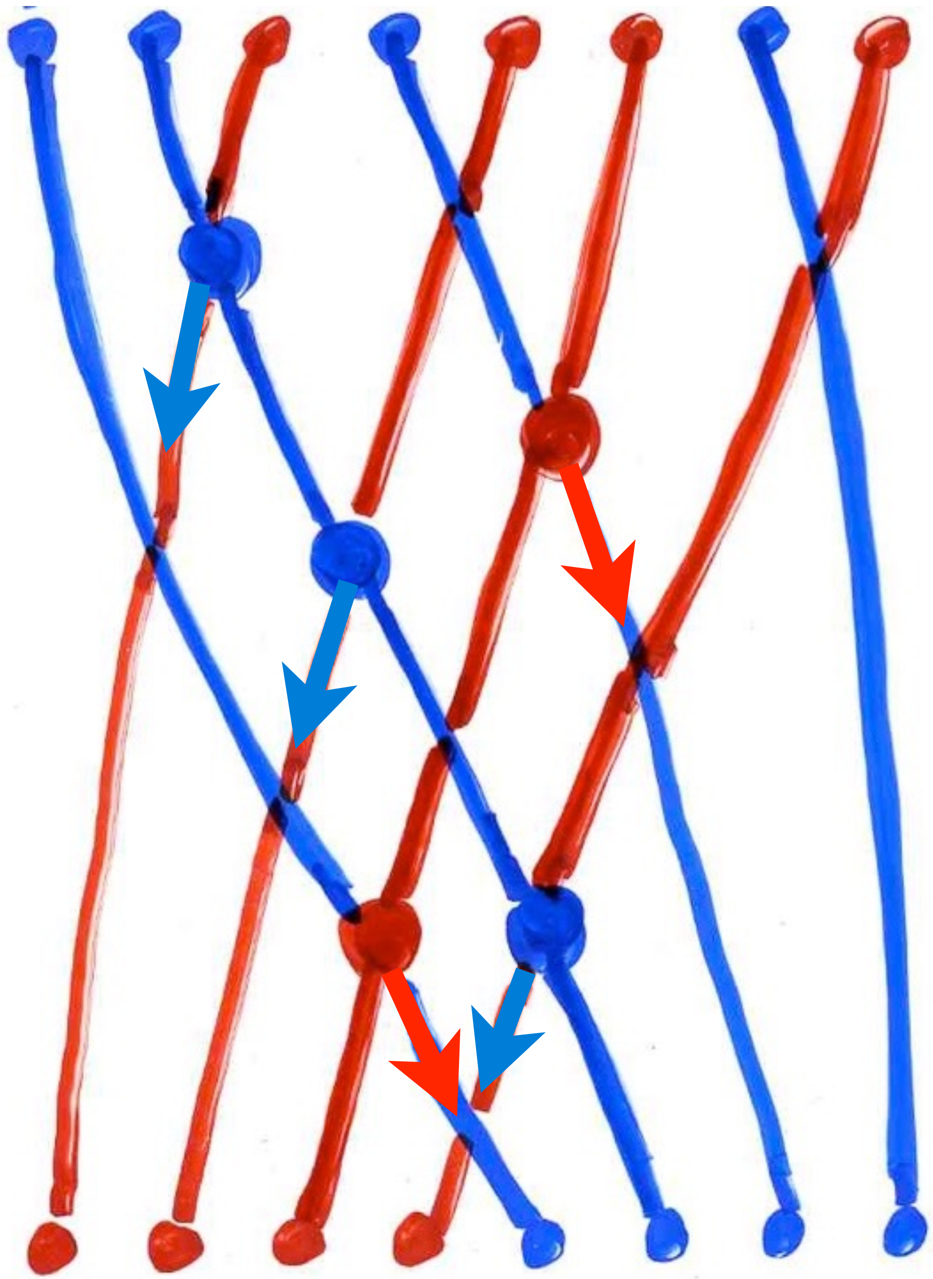
“exchange-fusion” algorithm

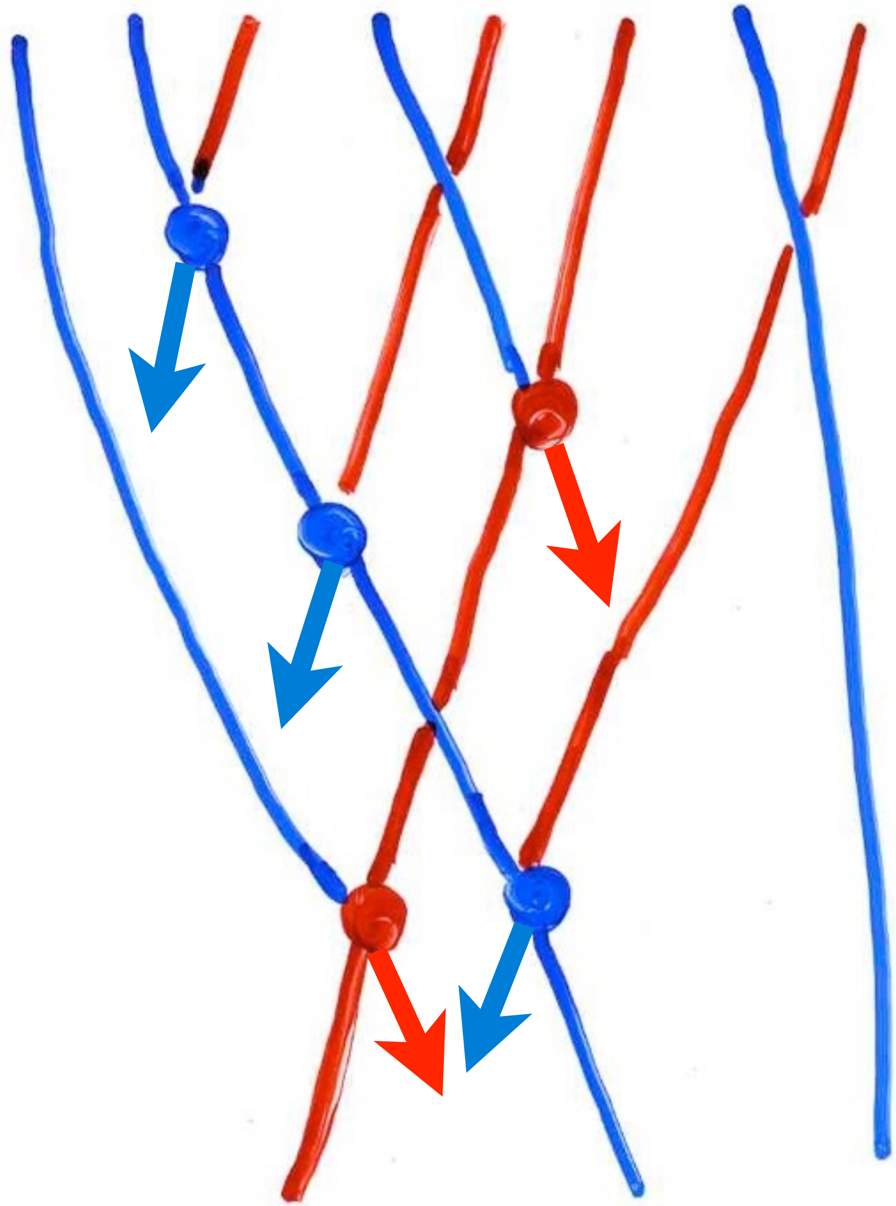


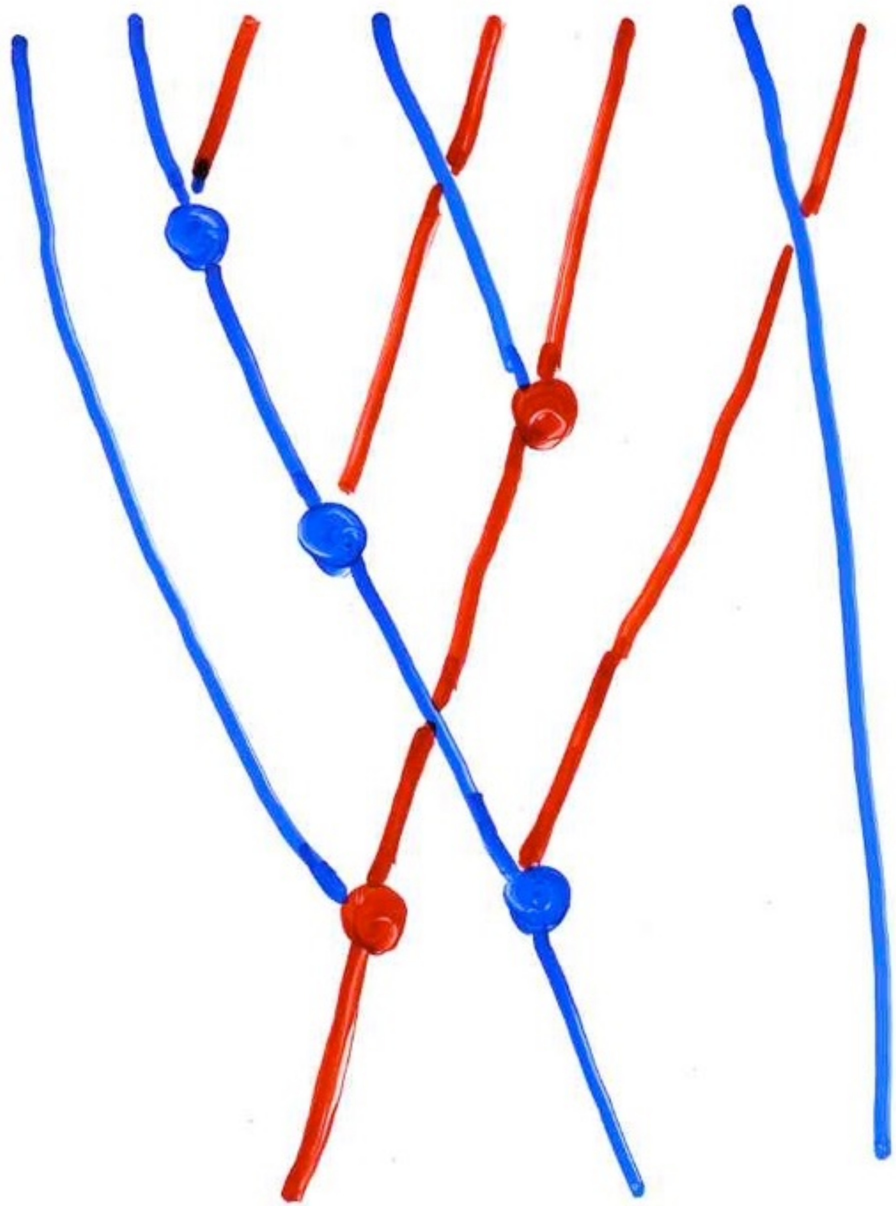


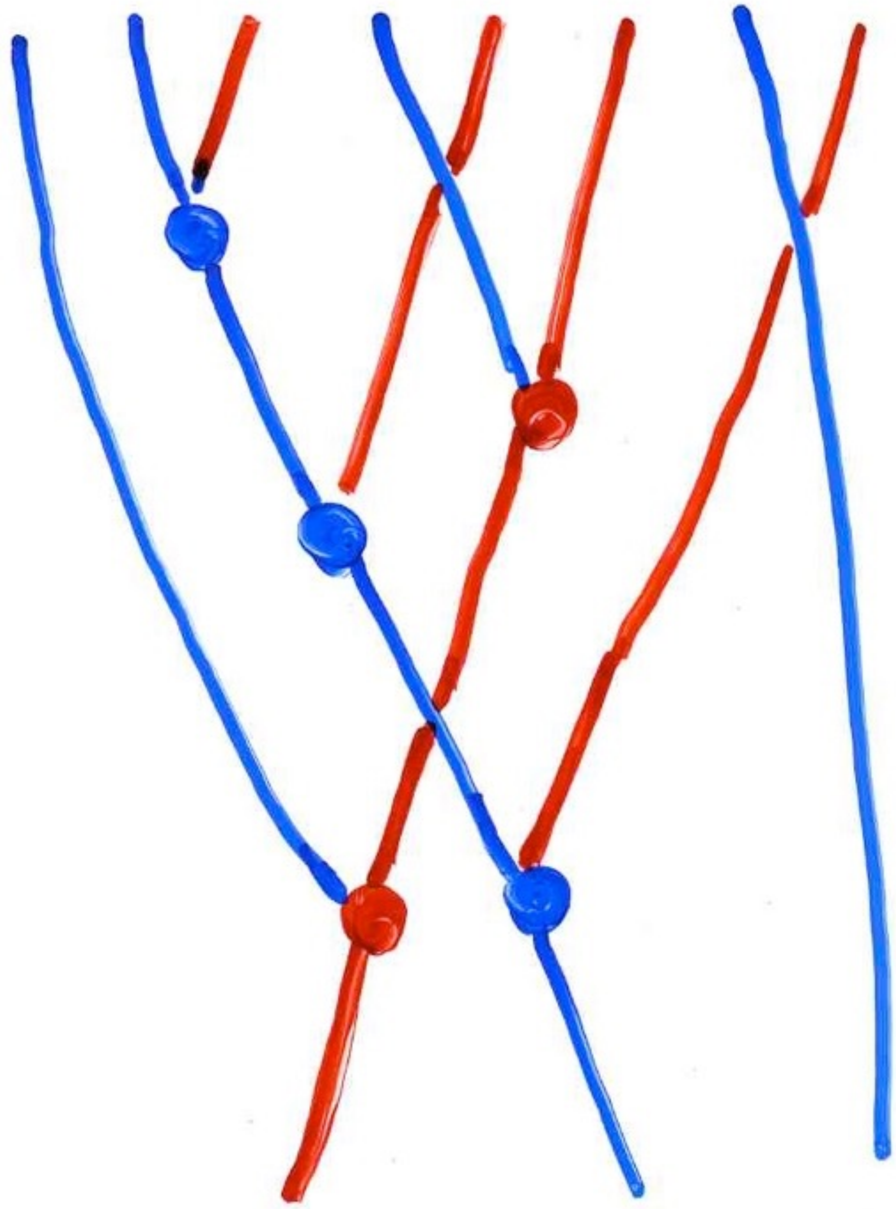
The inverse "exchange-fusion" algorithm

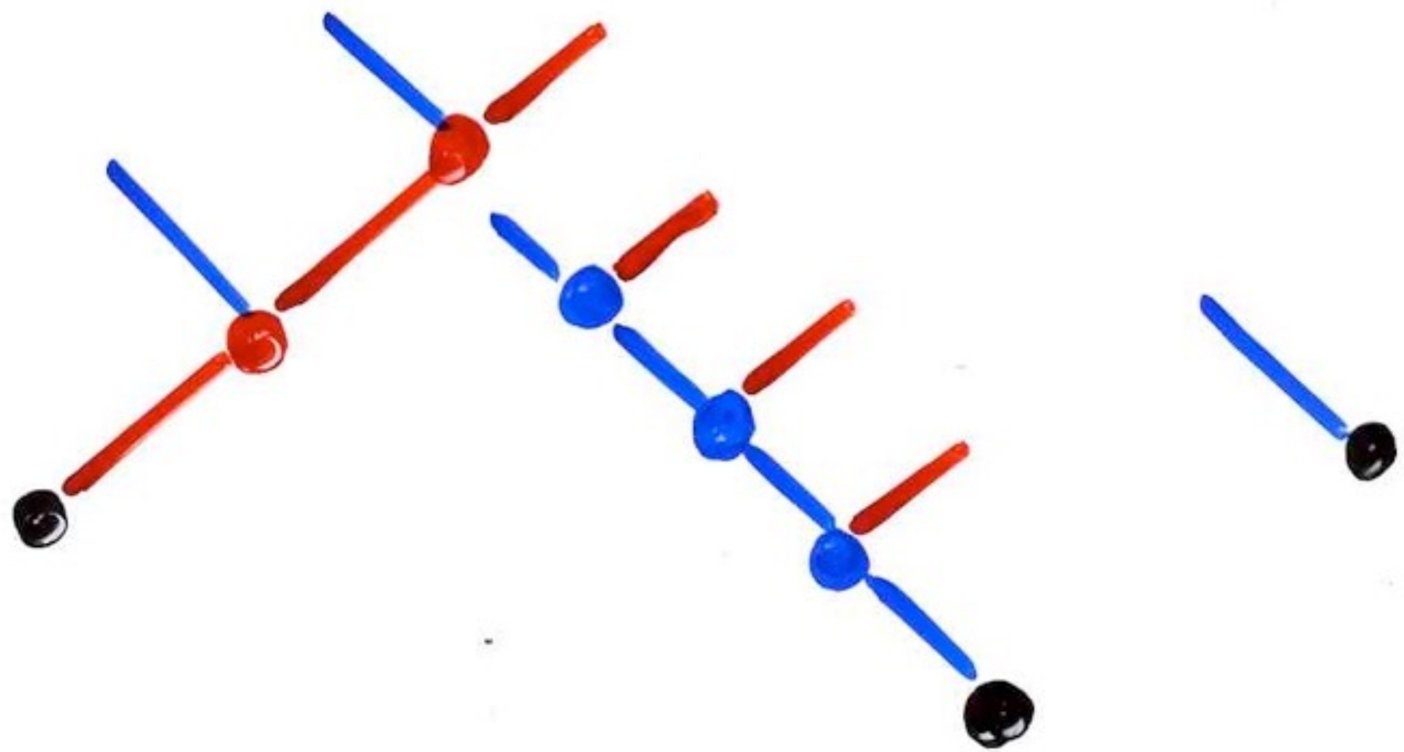


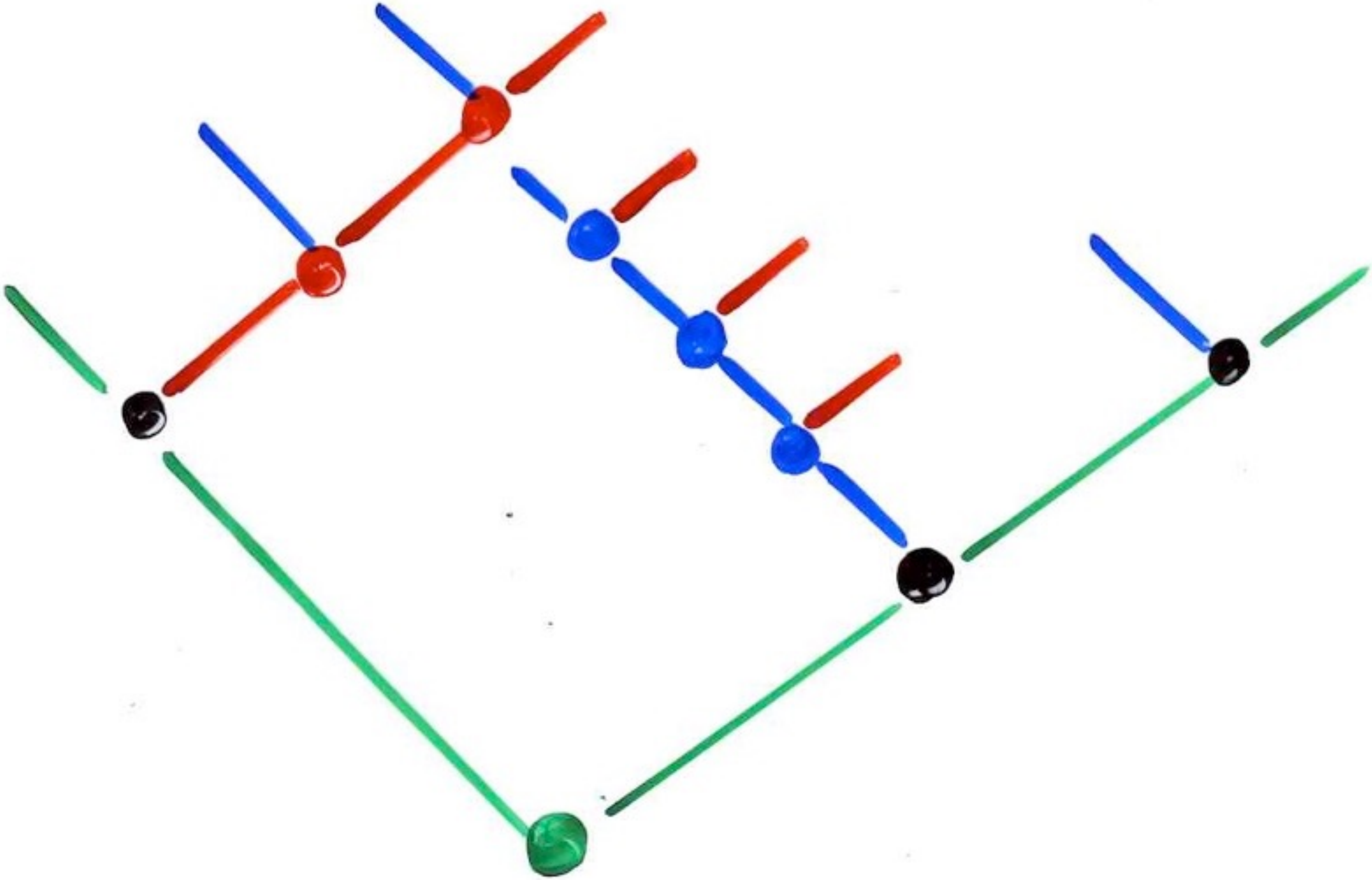


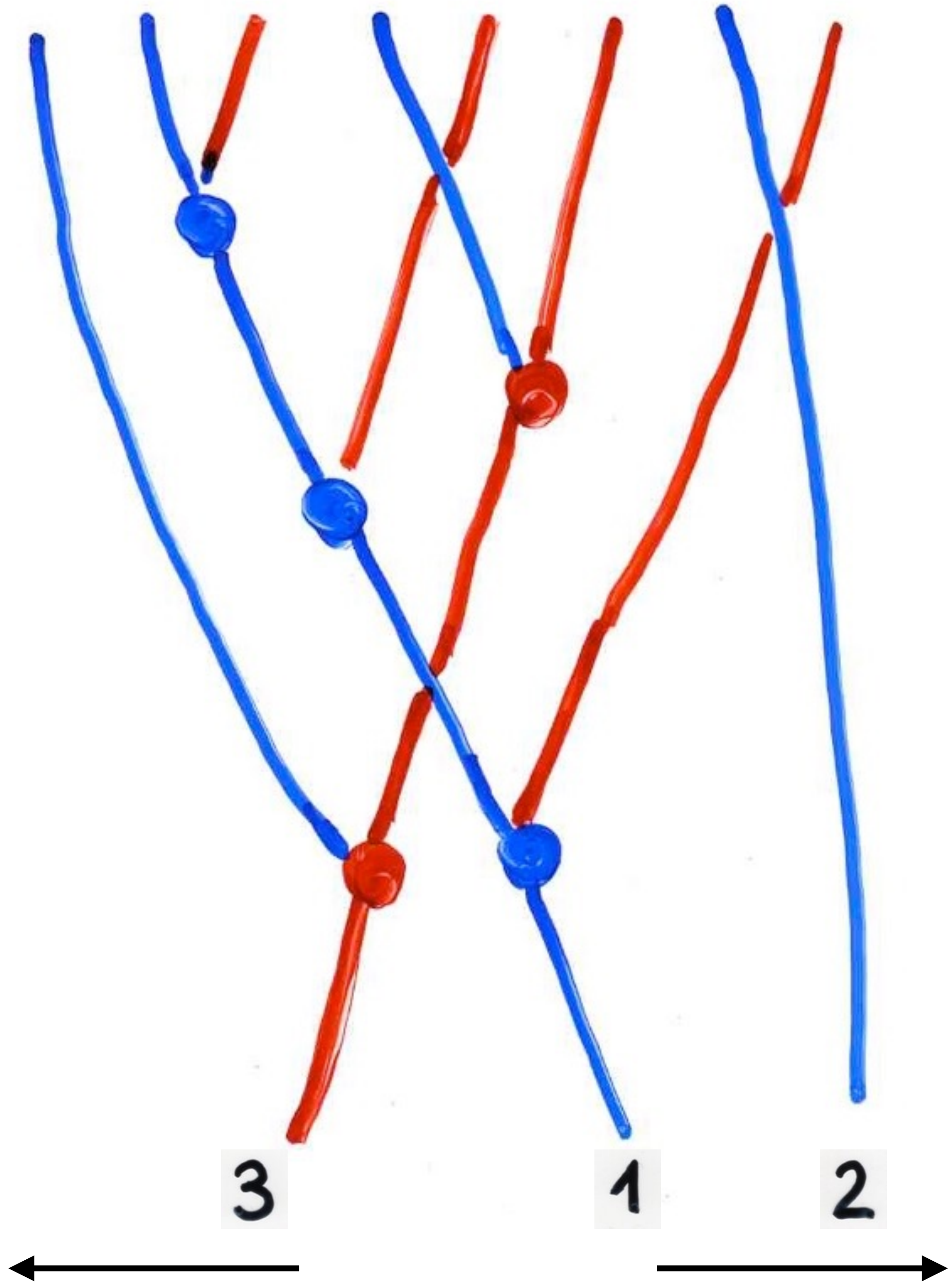


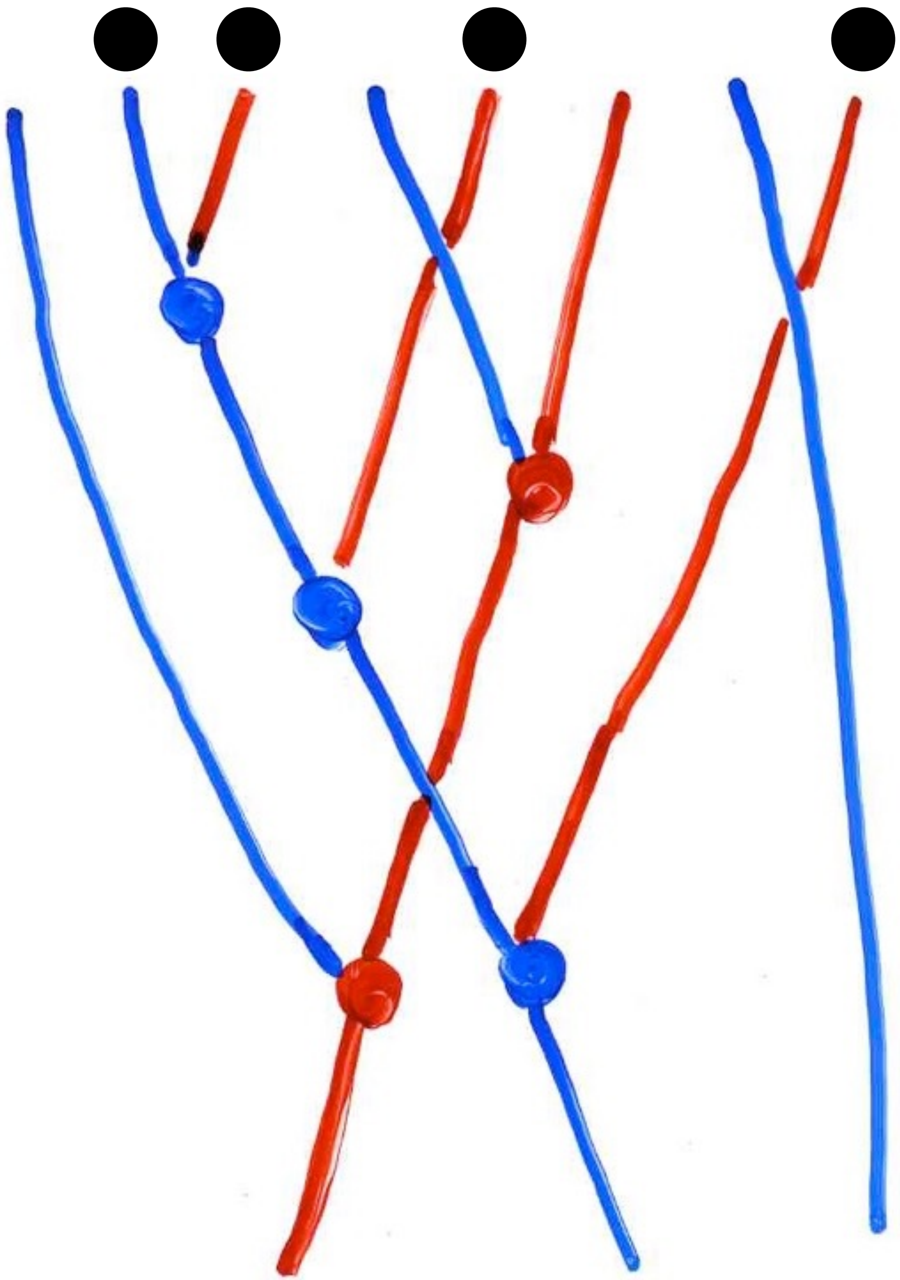




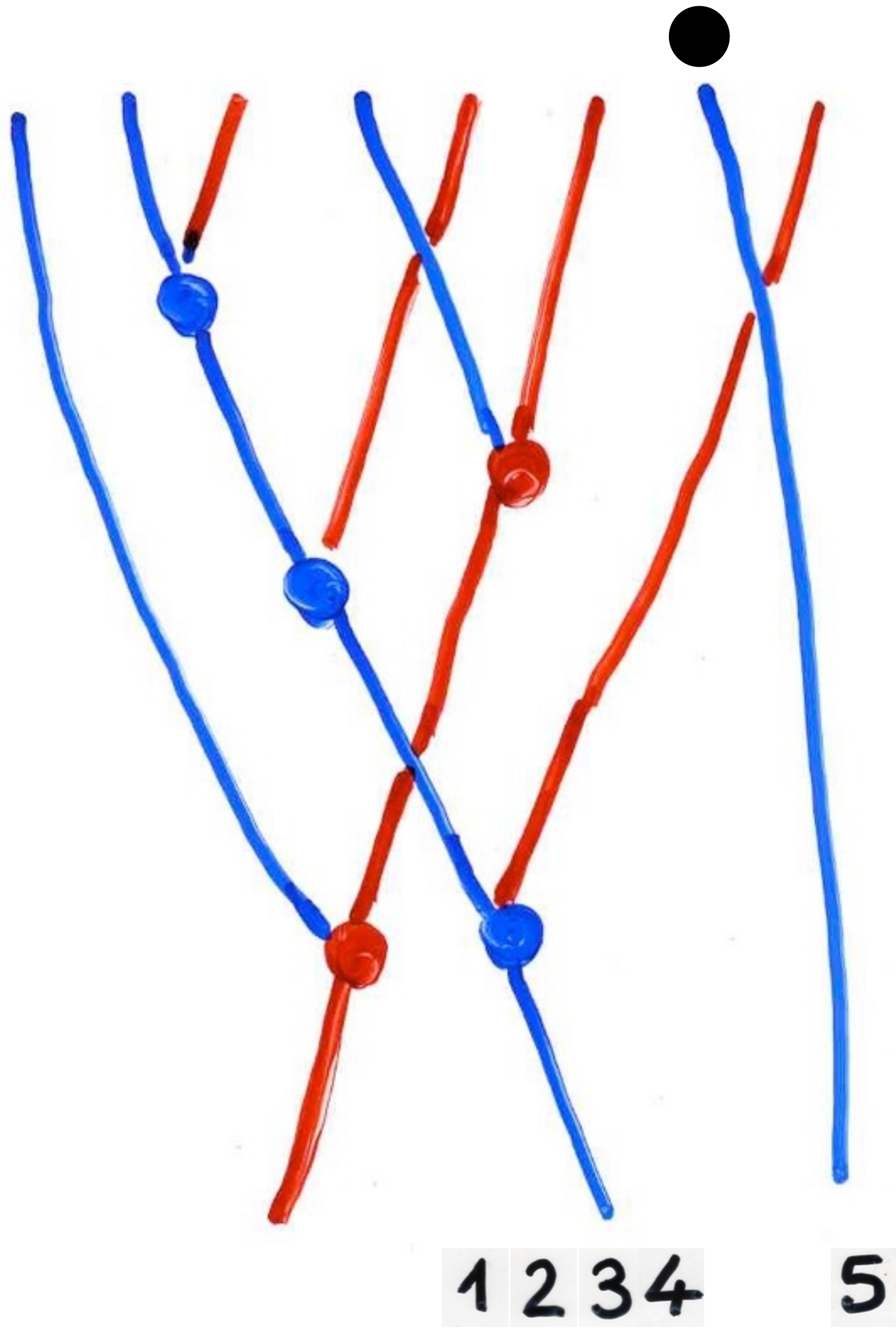




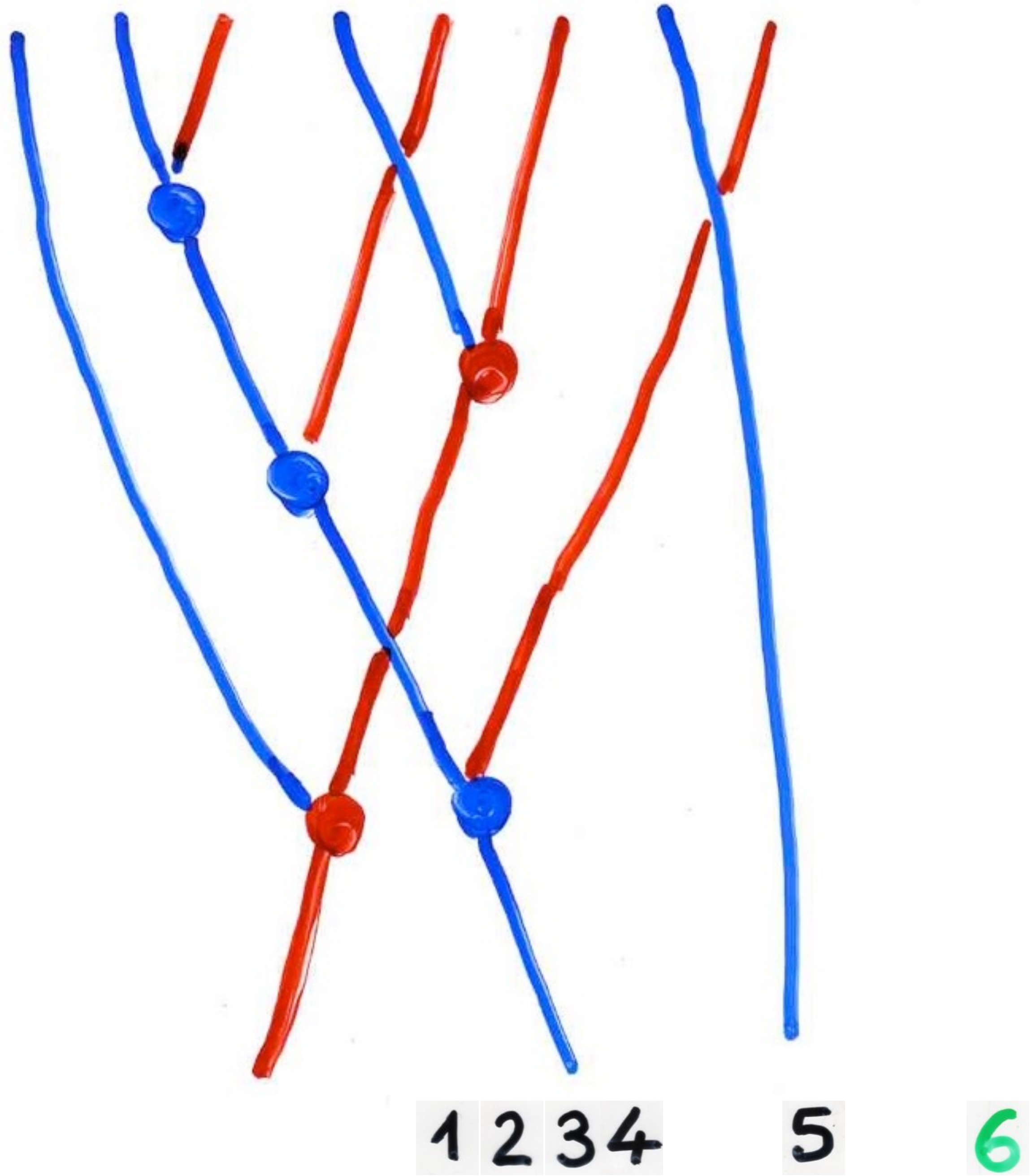

3
1
2



1 2 3 4



1 2 3 4
5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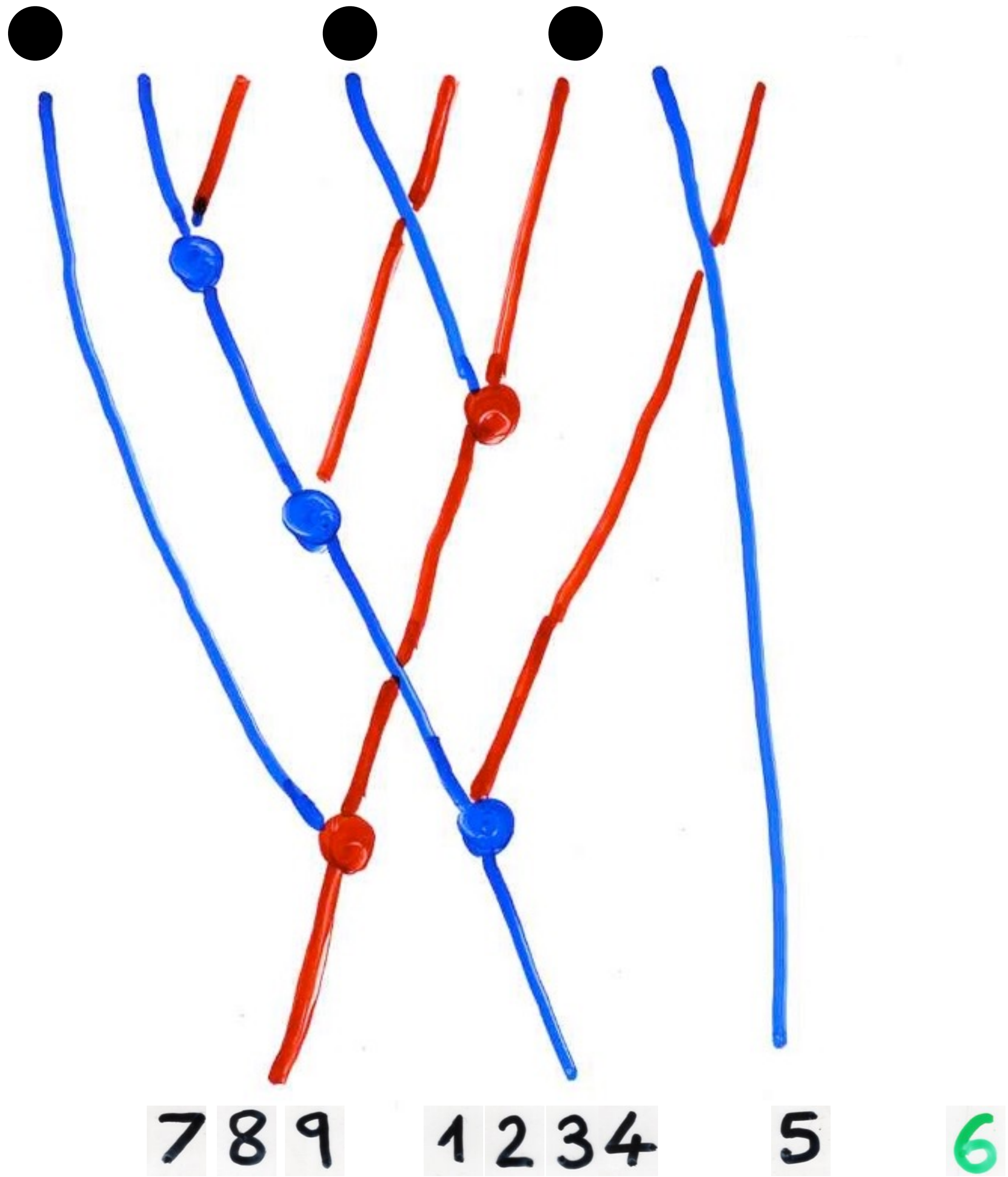
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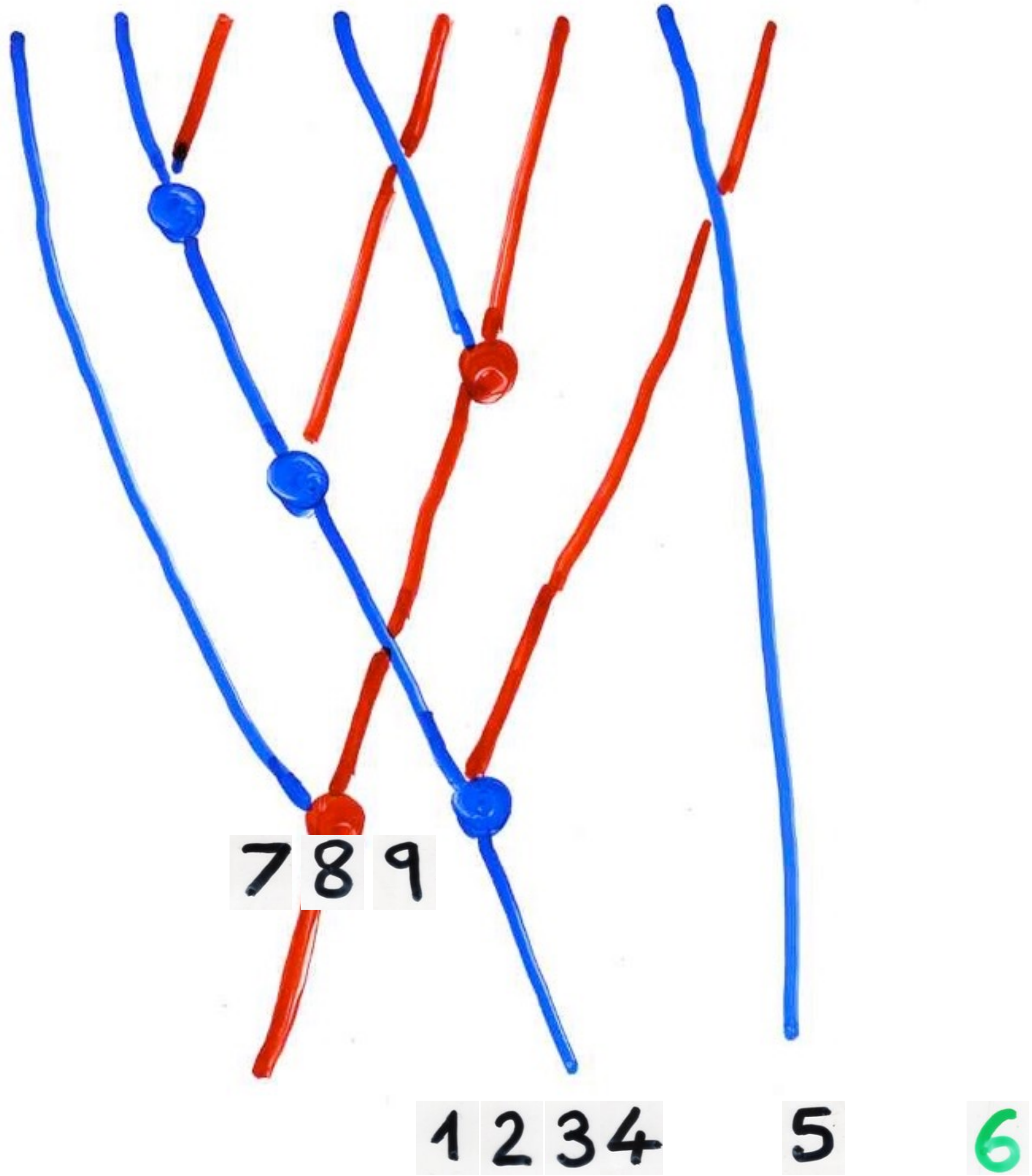

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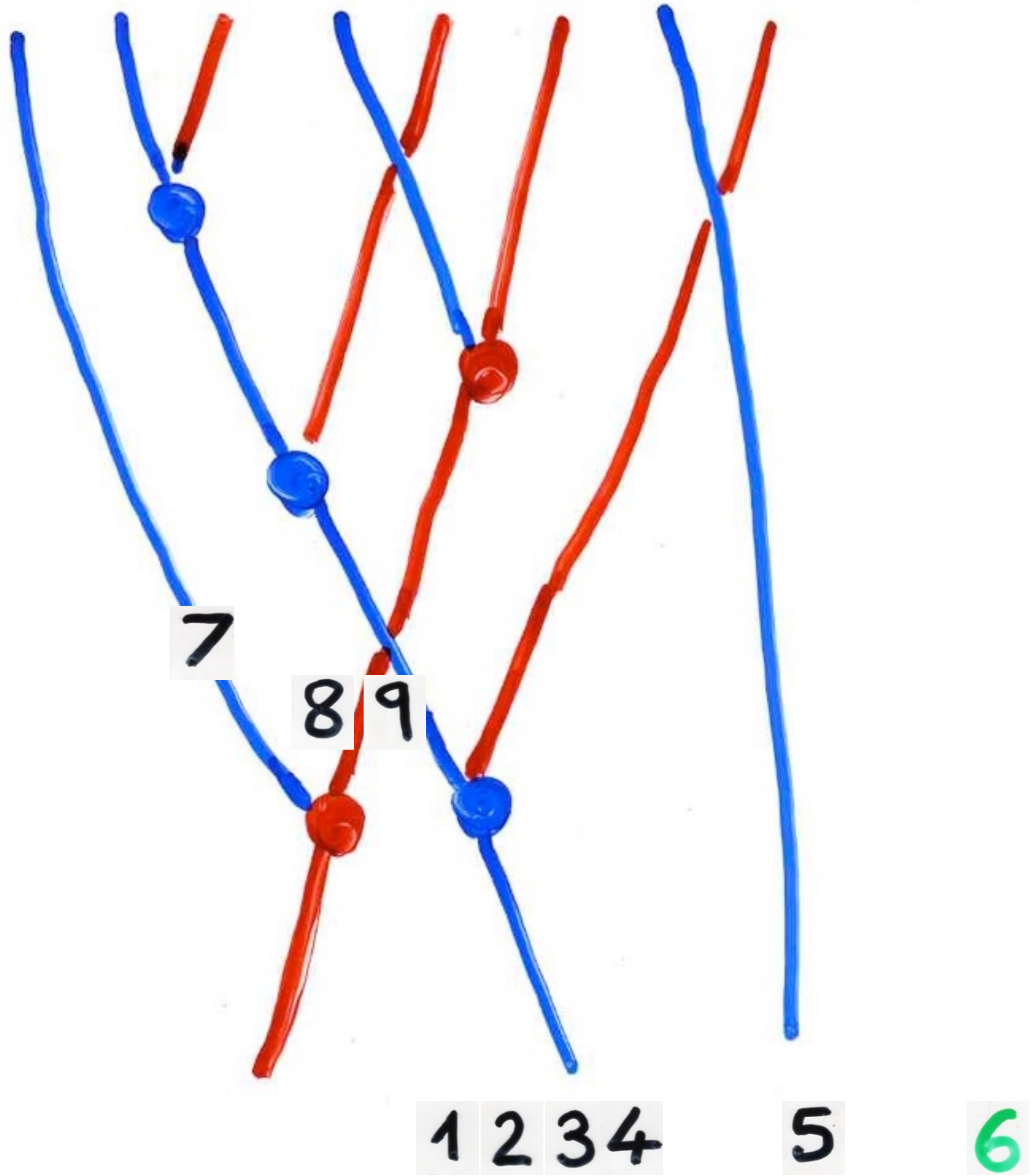
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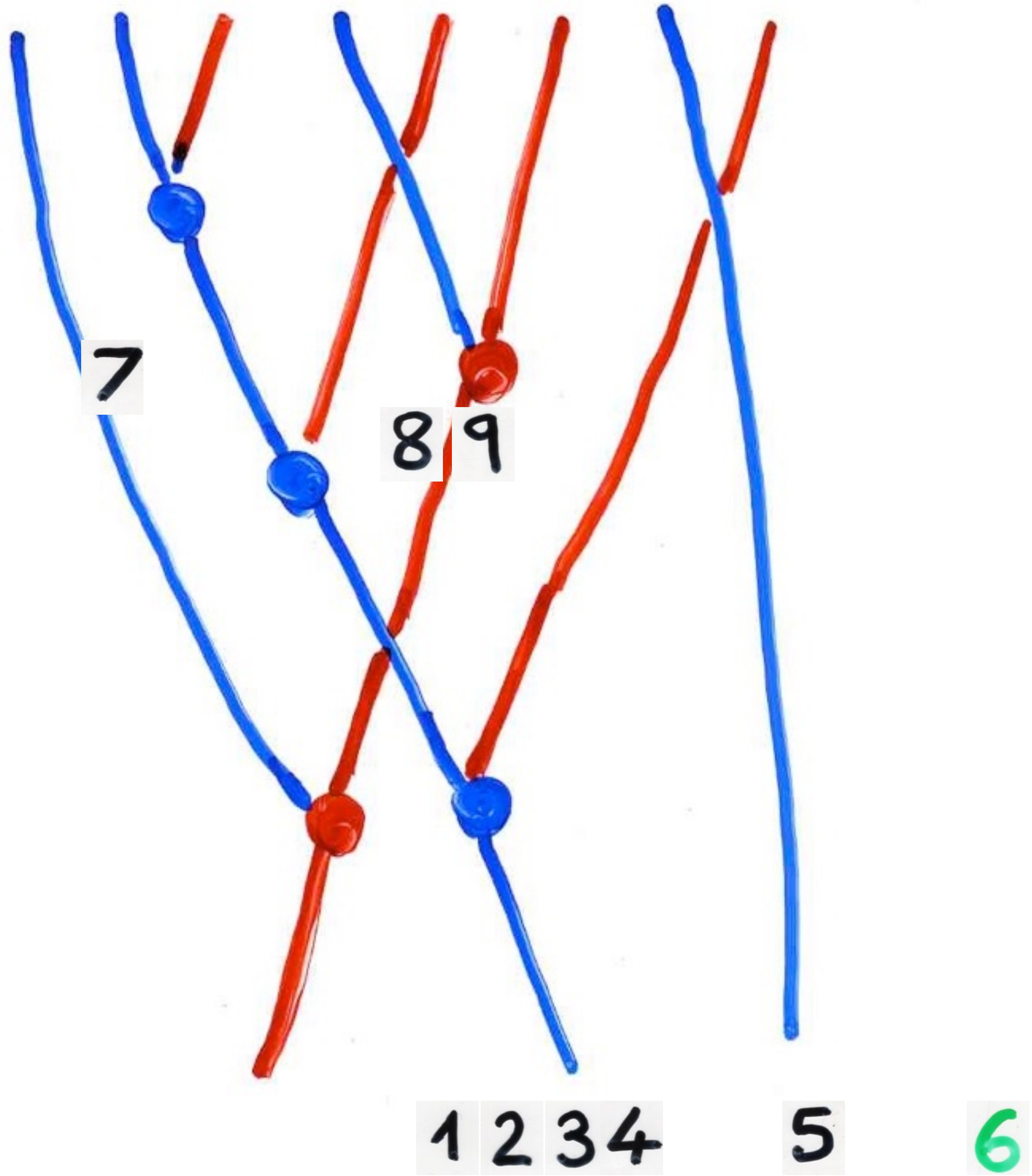
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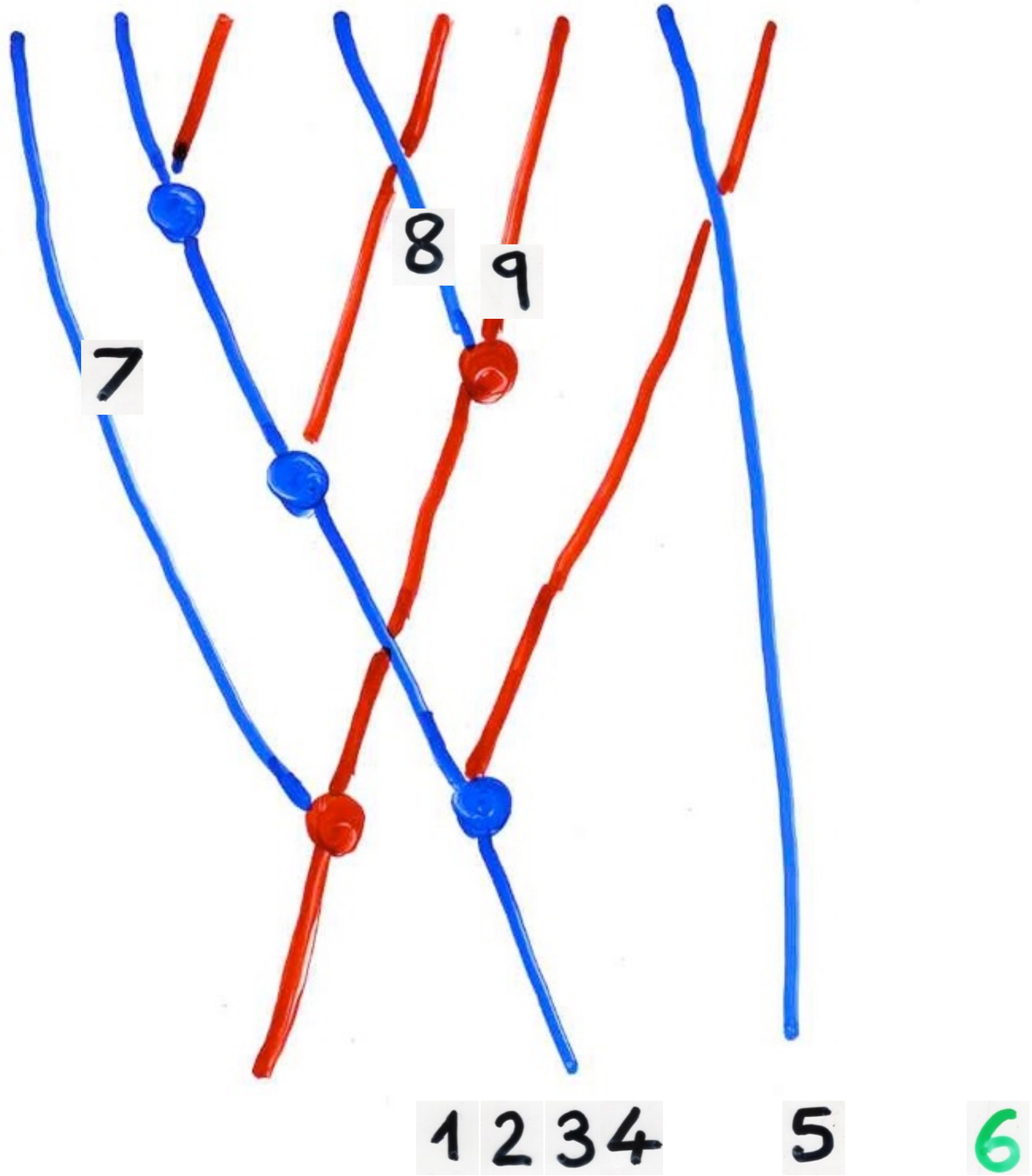
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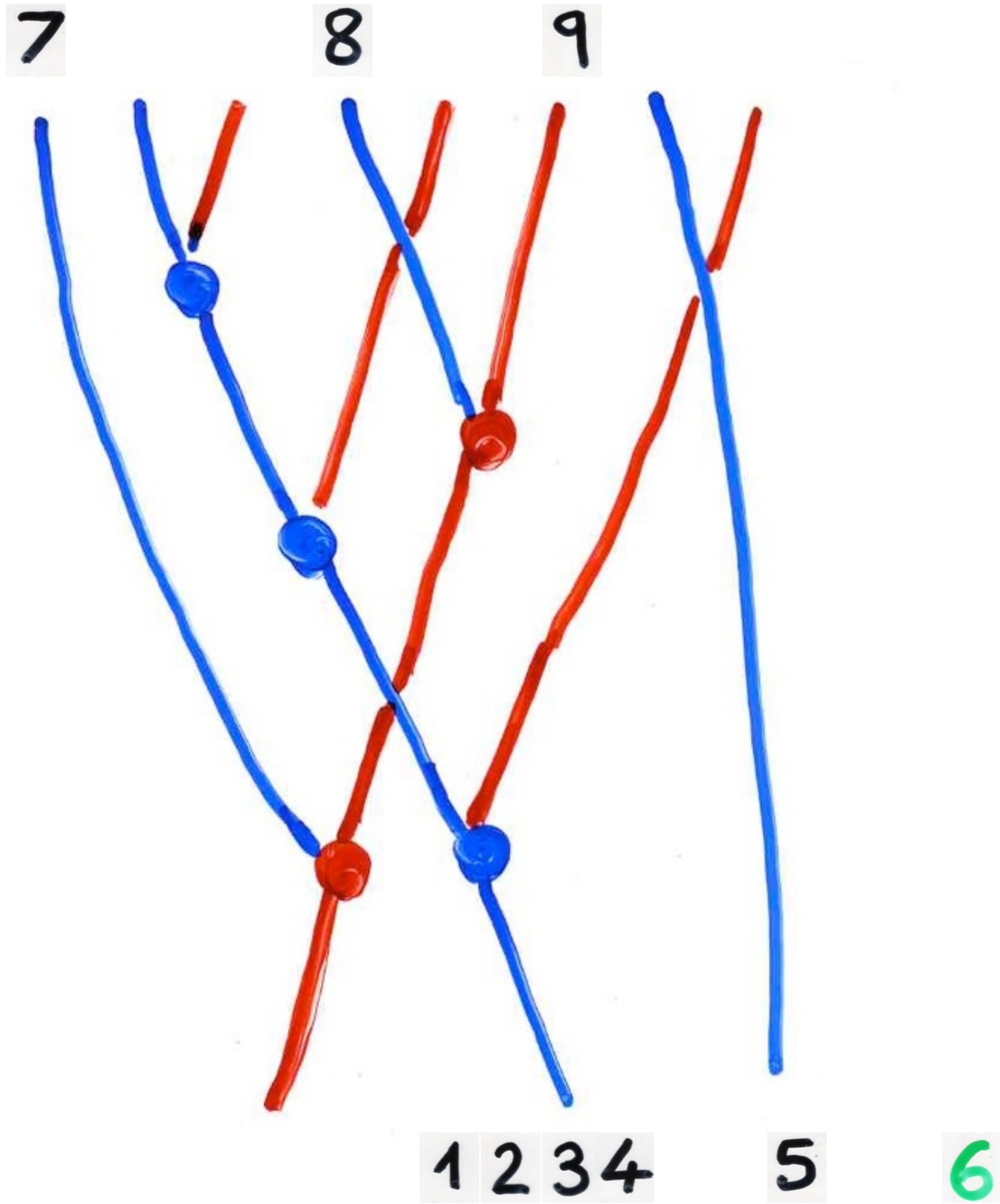

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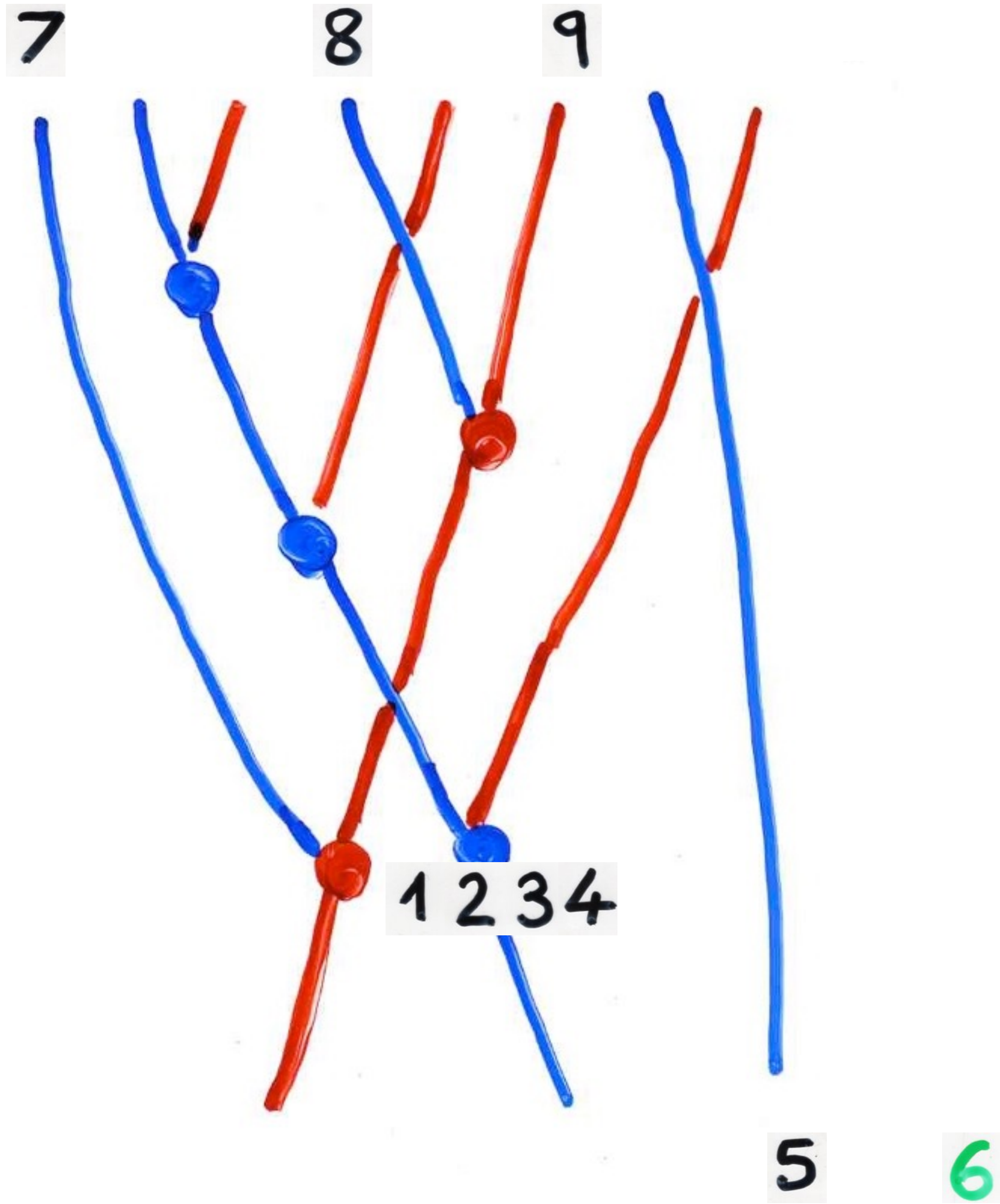
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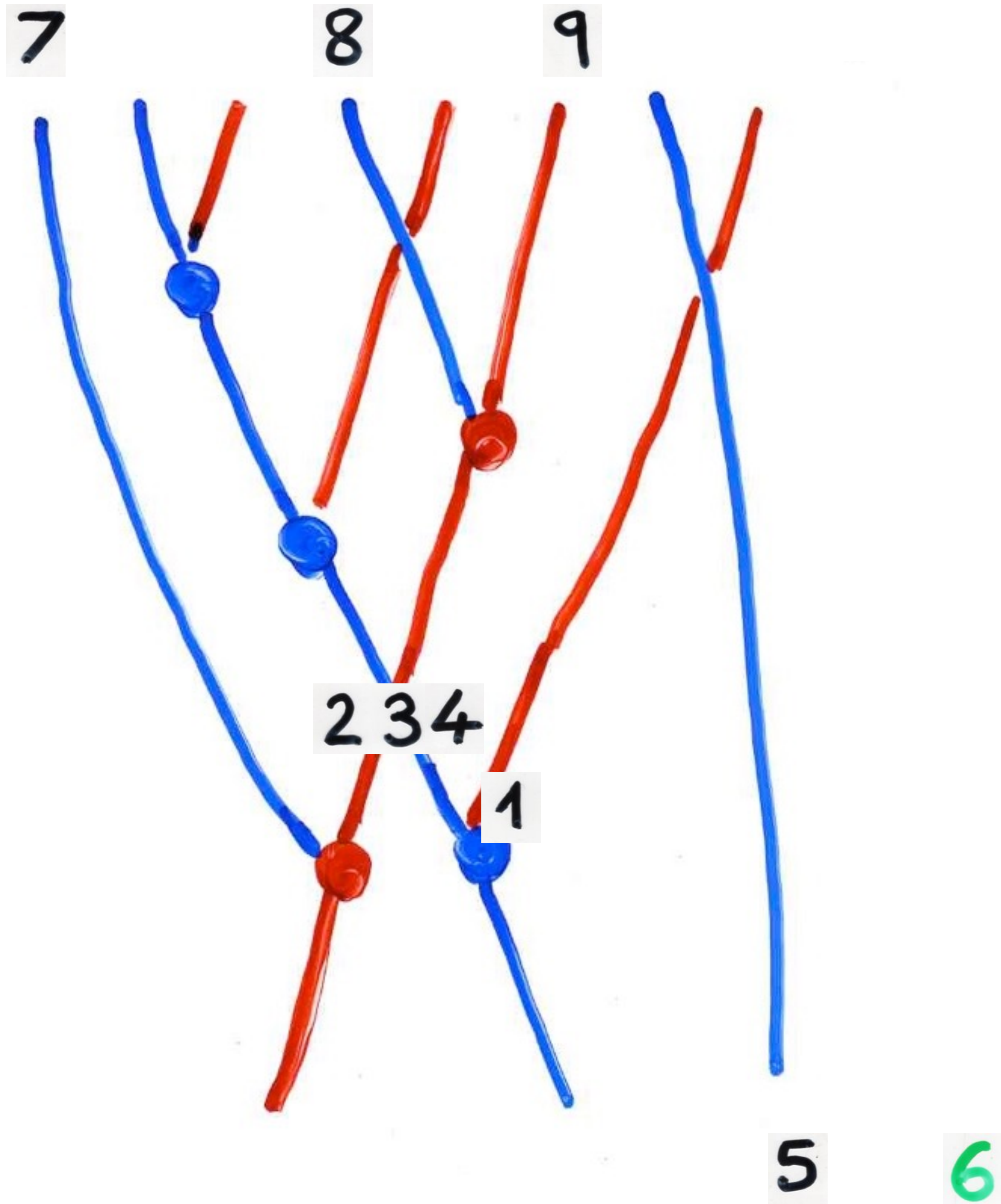
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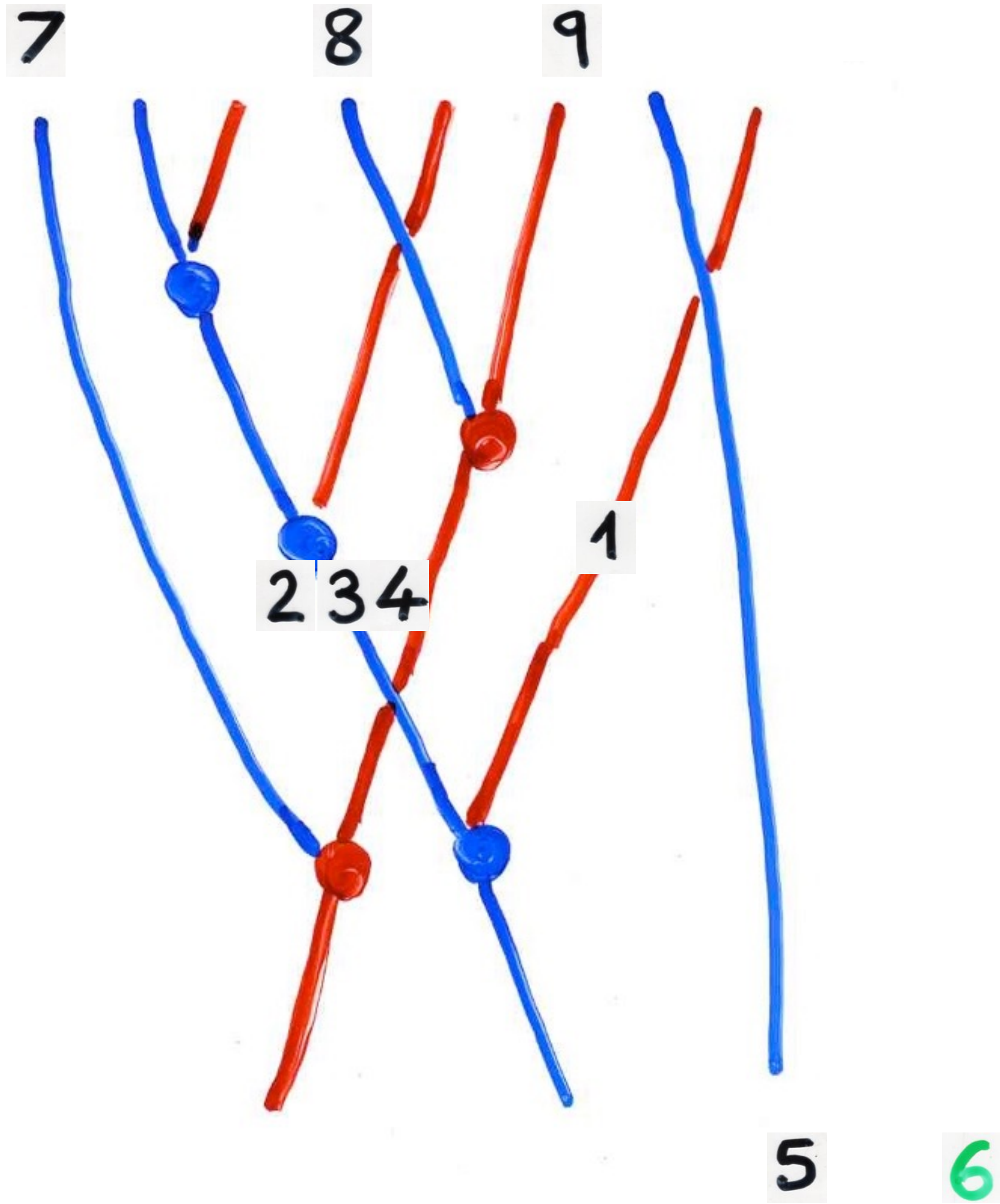
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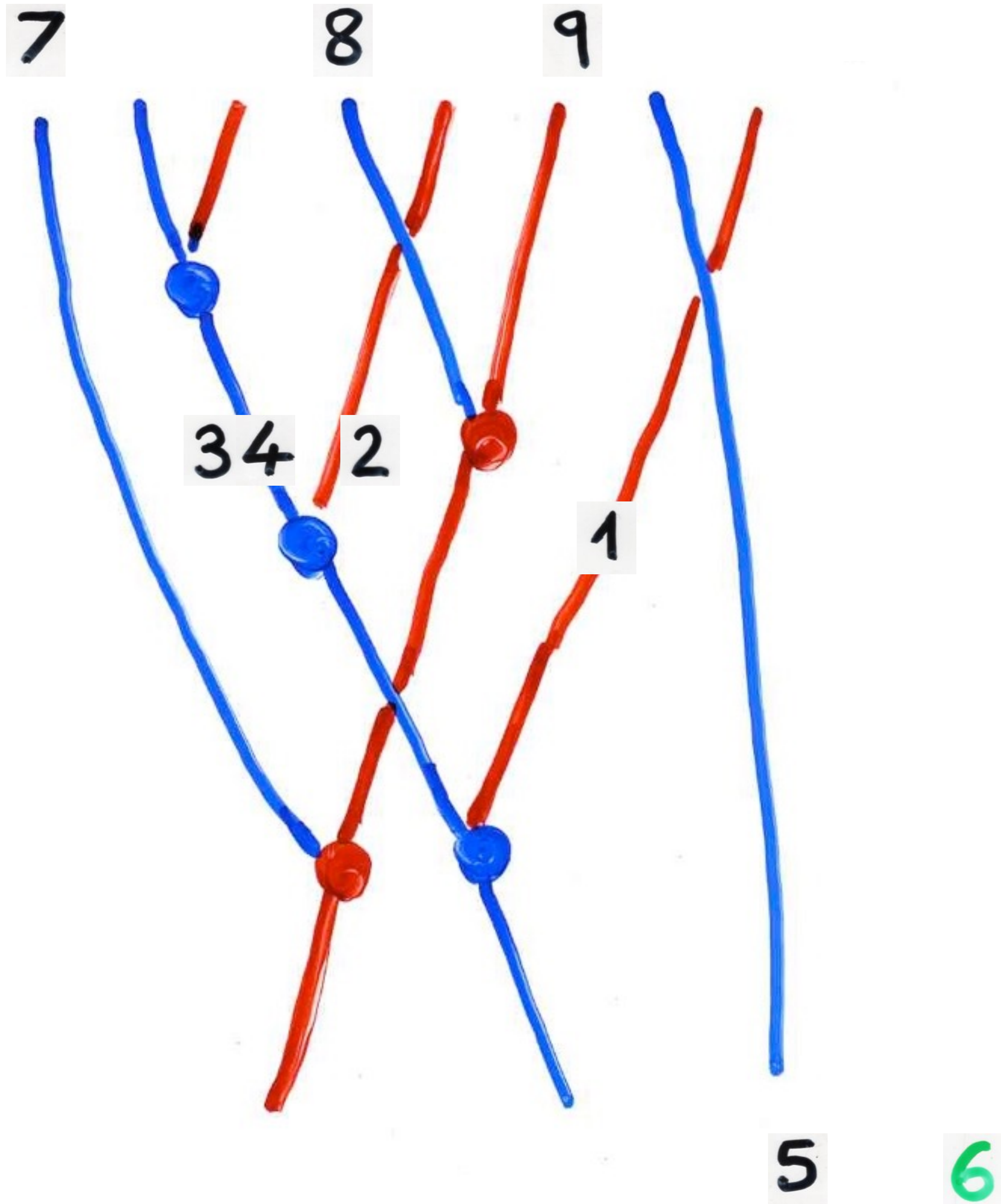
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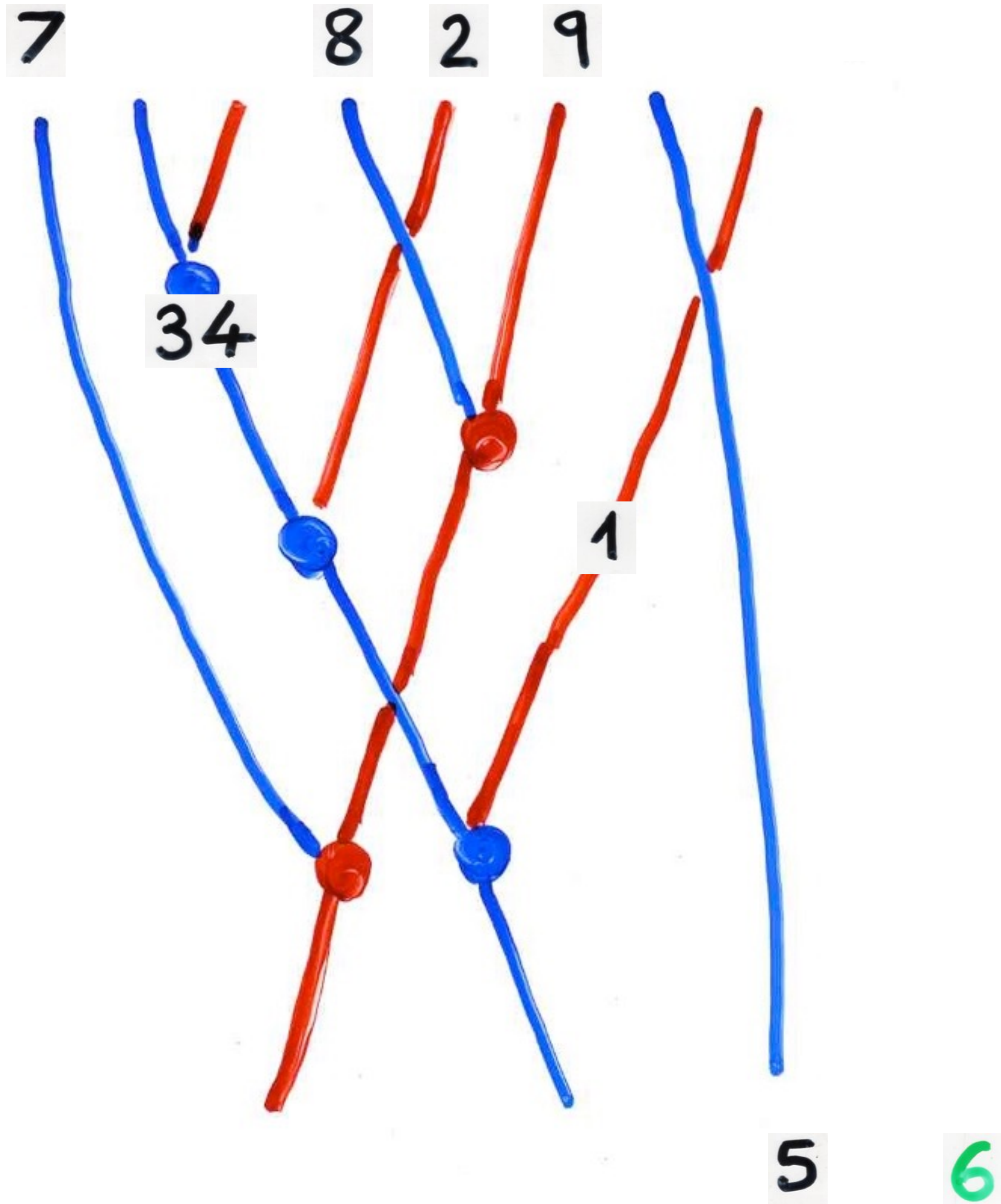
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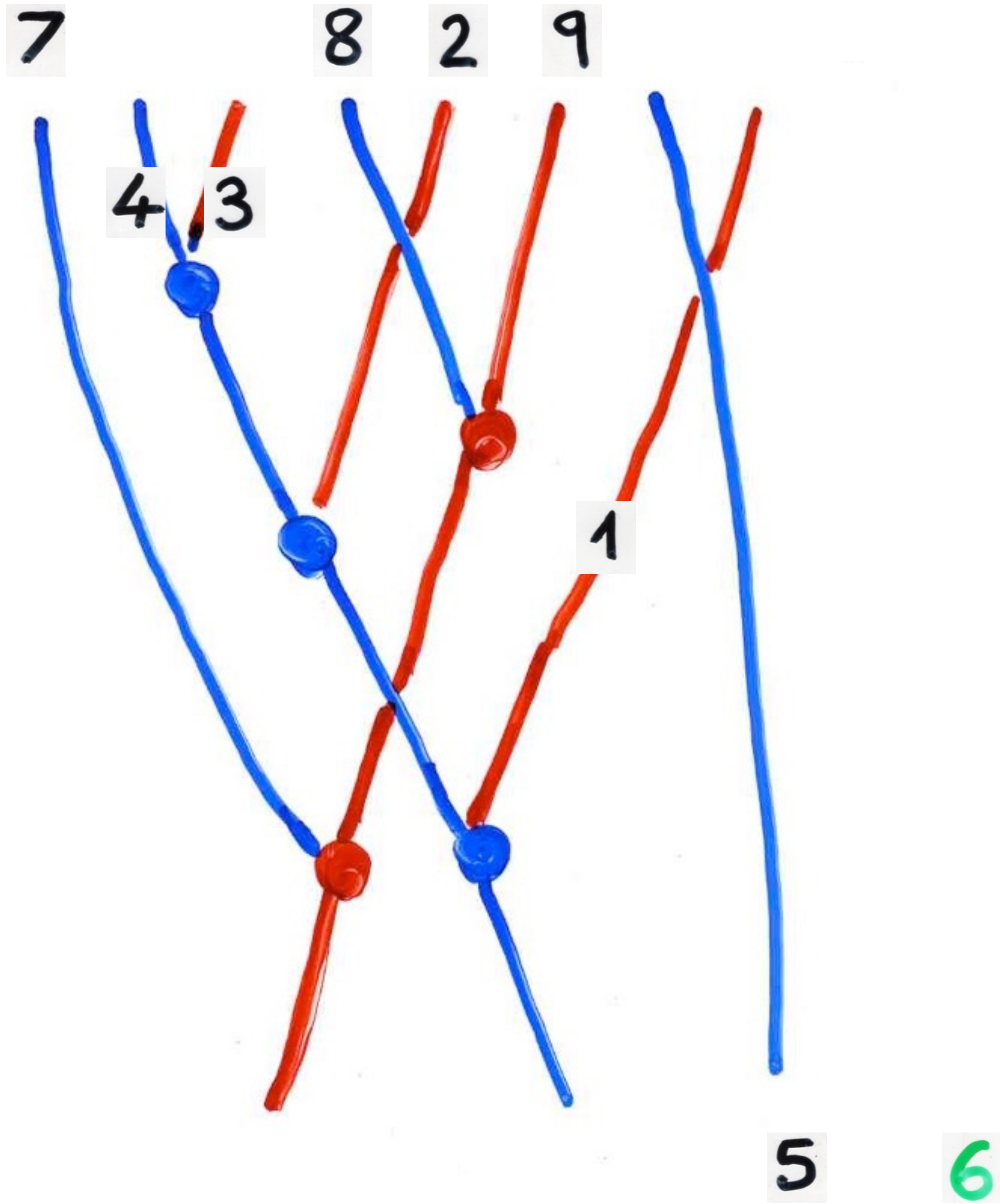

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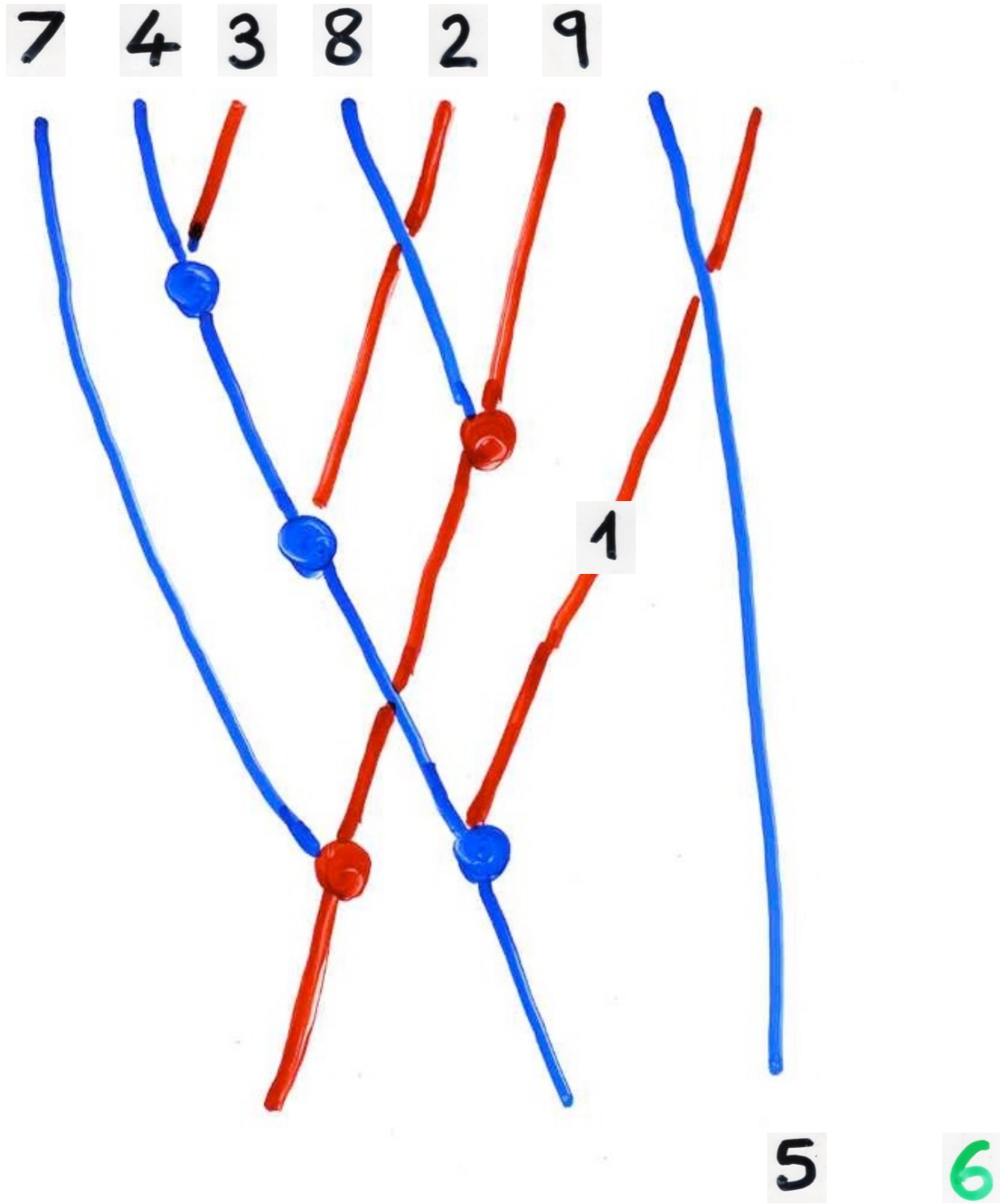
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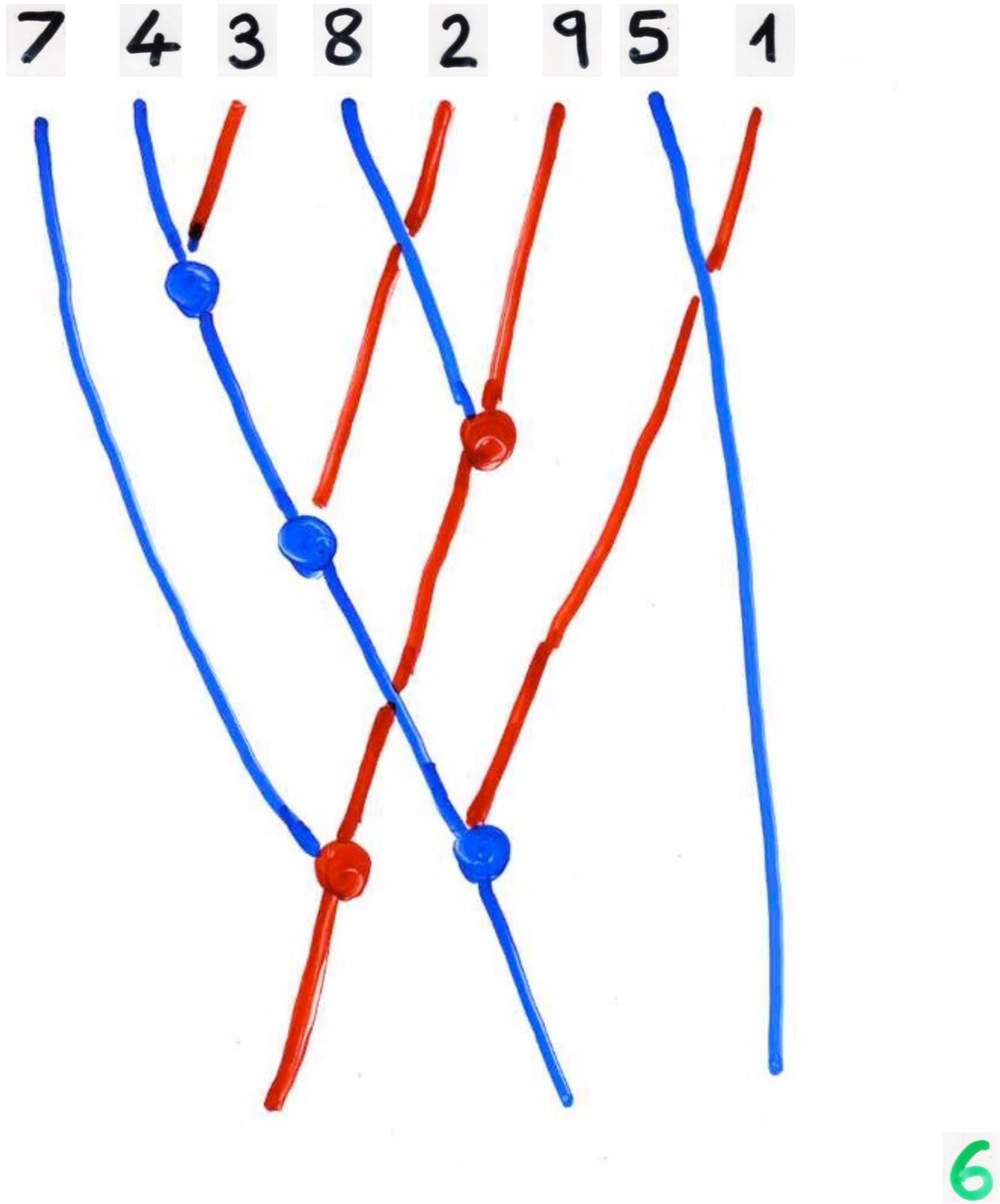
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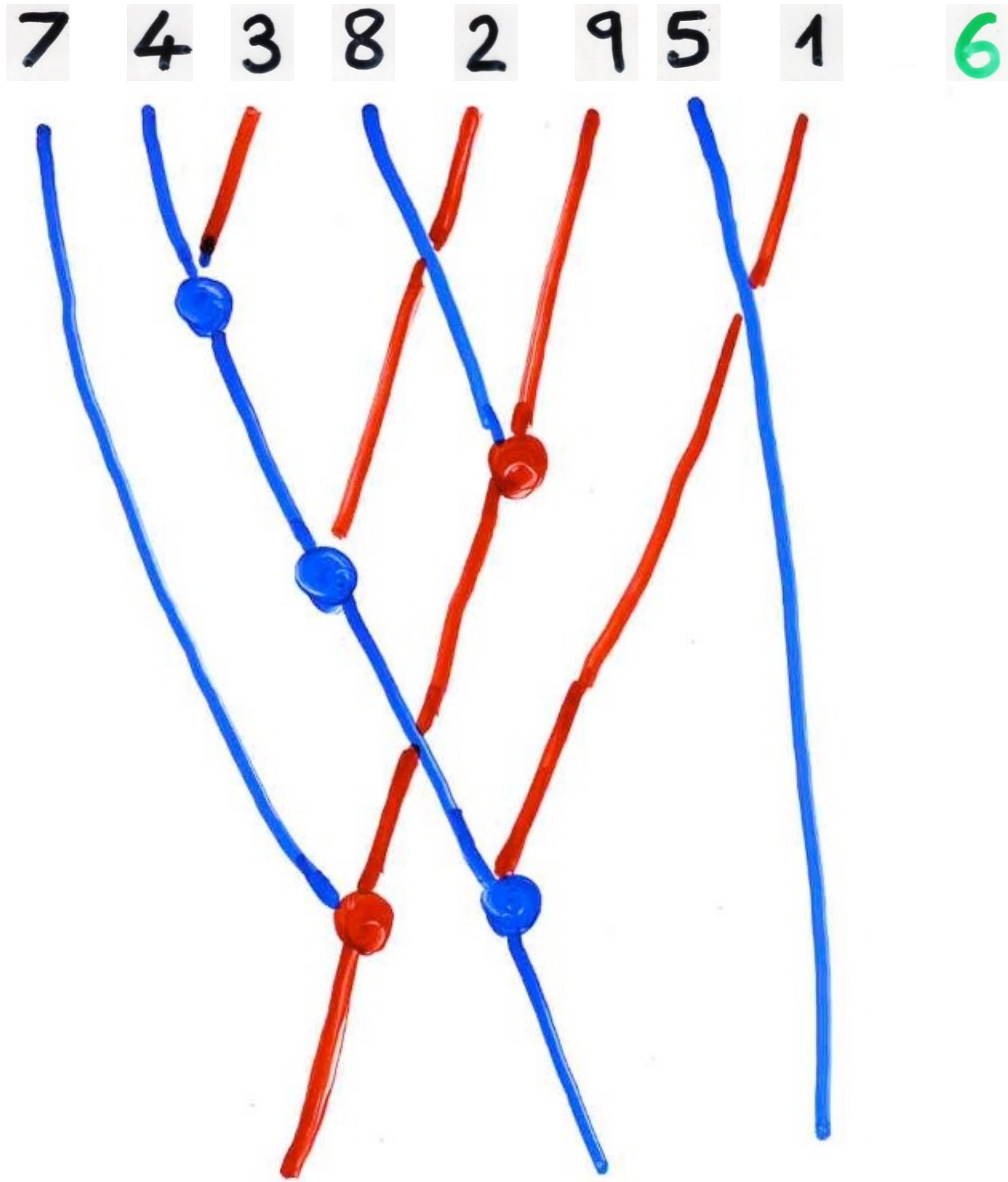

7 4 3 8 2 9 5 1 6

A variation of
the “exchange-fusion” algorithm:

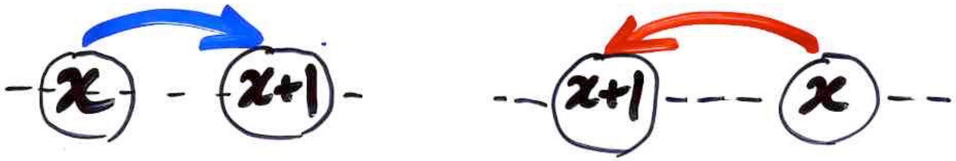
The “exchange-delete” algorithm

Def- Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

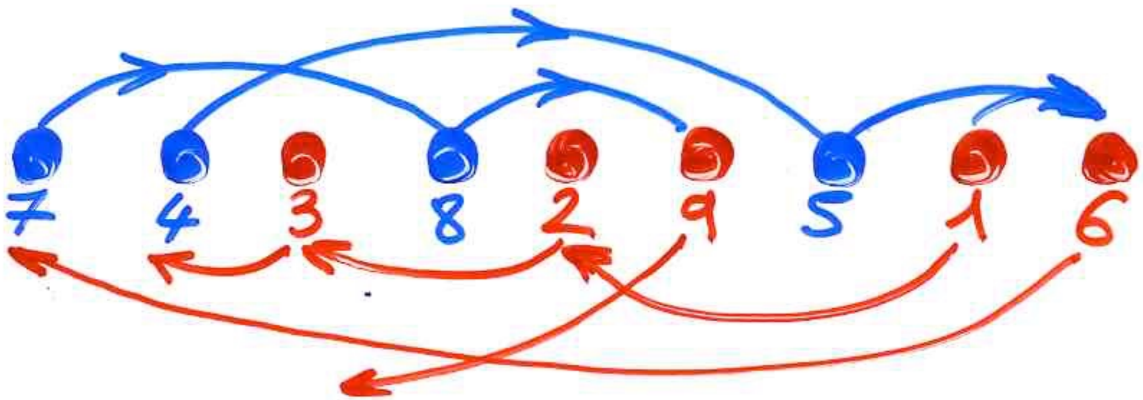
$x = \sigma(i)$, $1 \leq x < n$

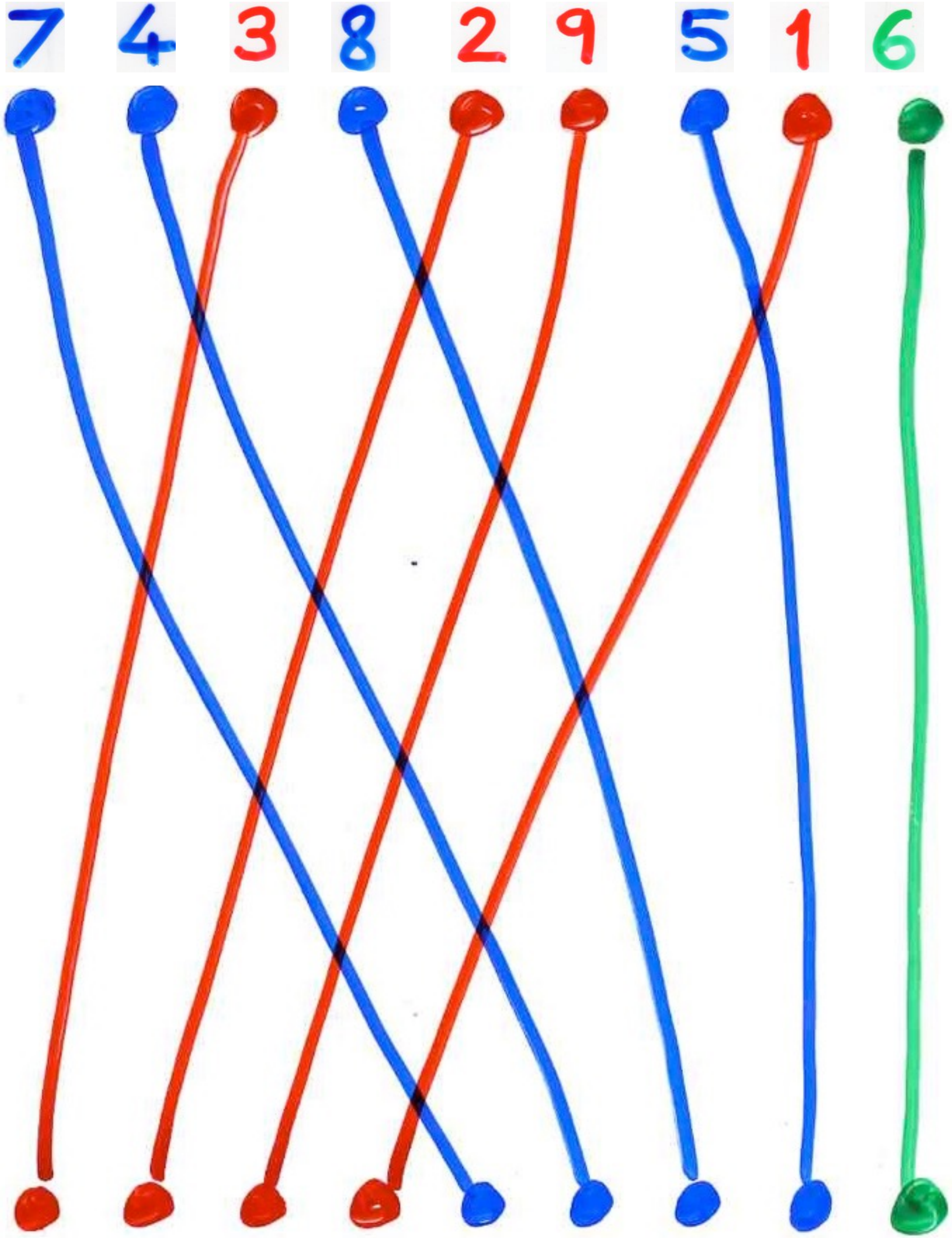
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i<j \\ j<i \end{cases}$

- convention $x=n$ est un recul



$\sigma =$ 7 4 3 8 2 9 5 1 6

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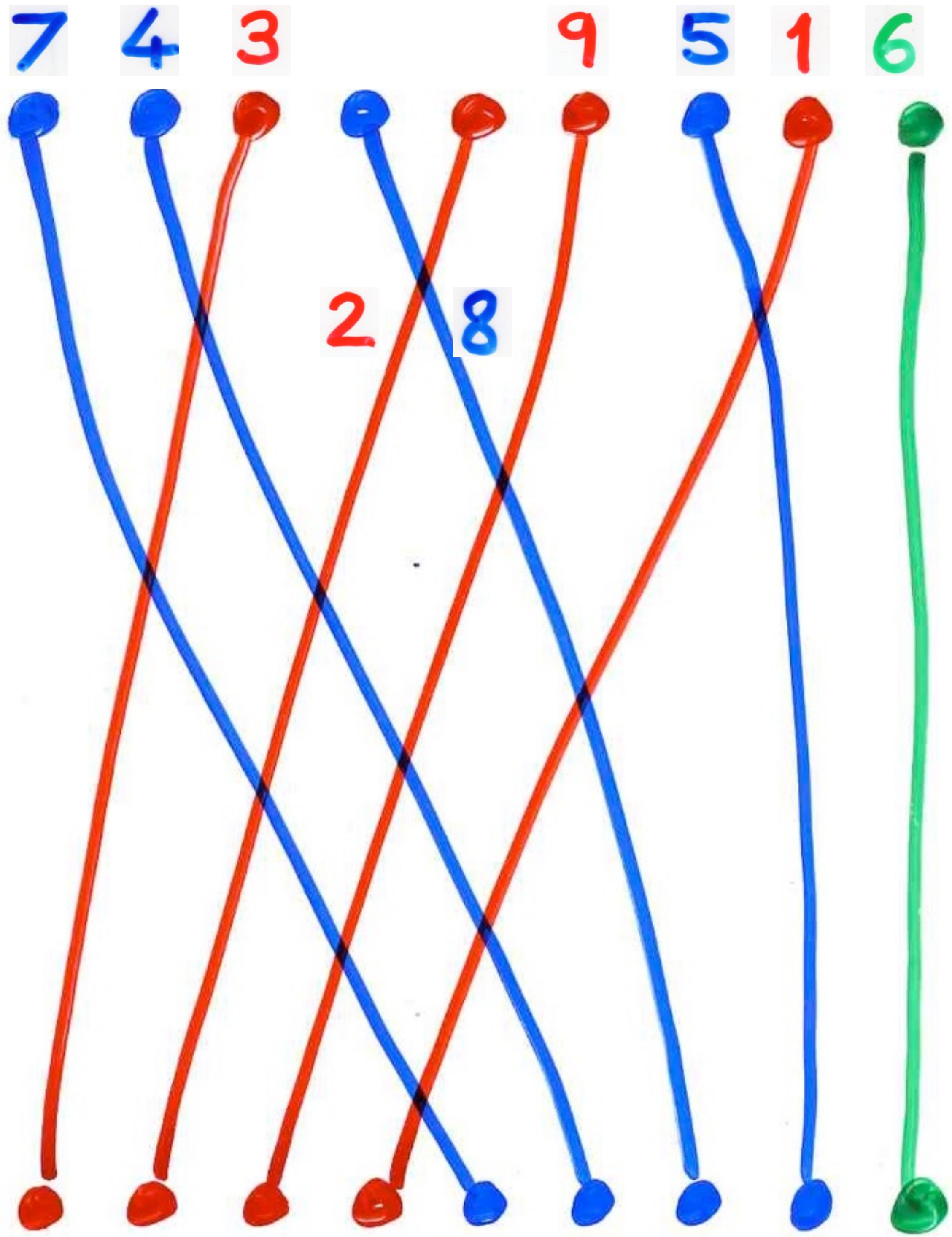
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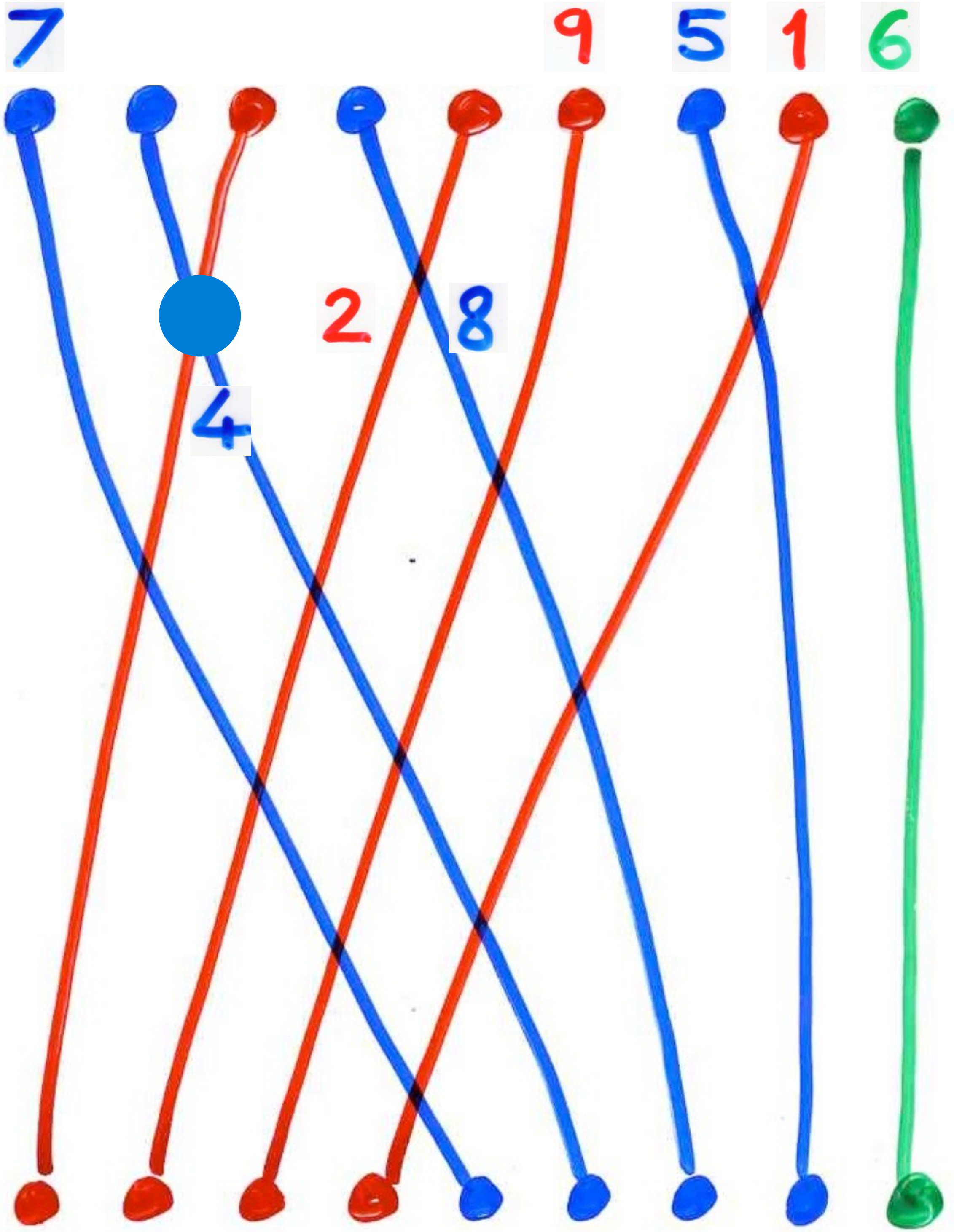
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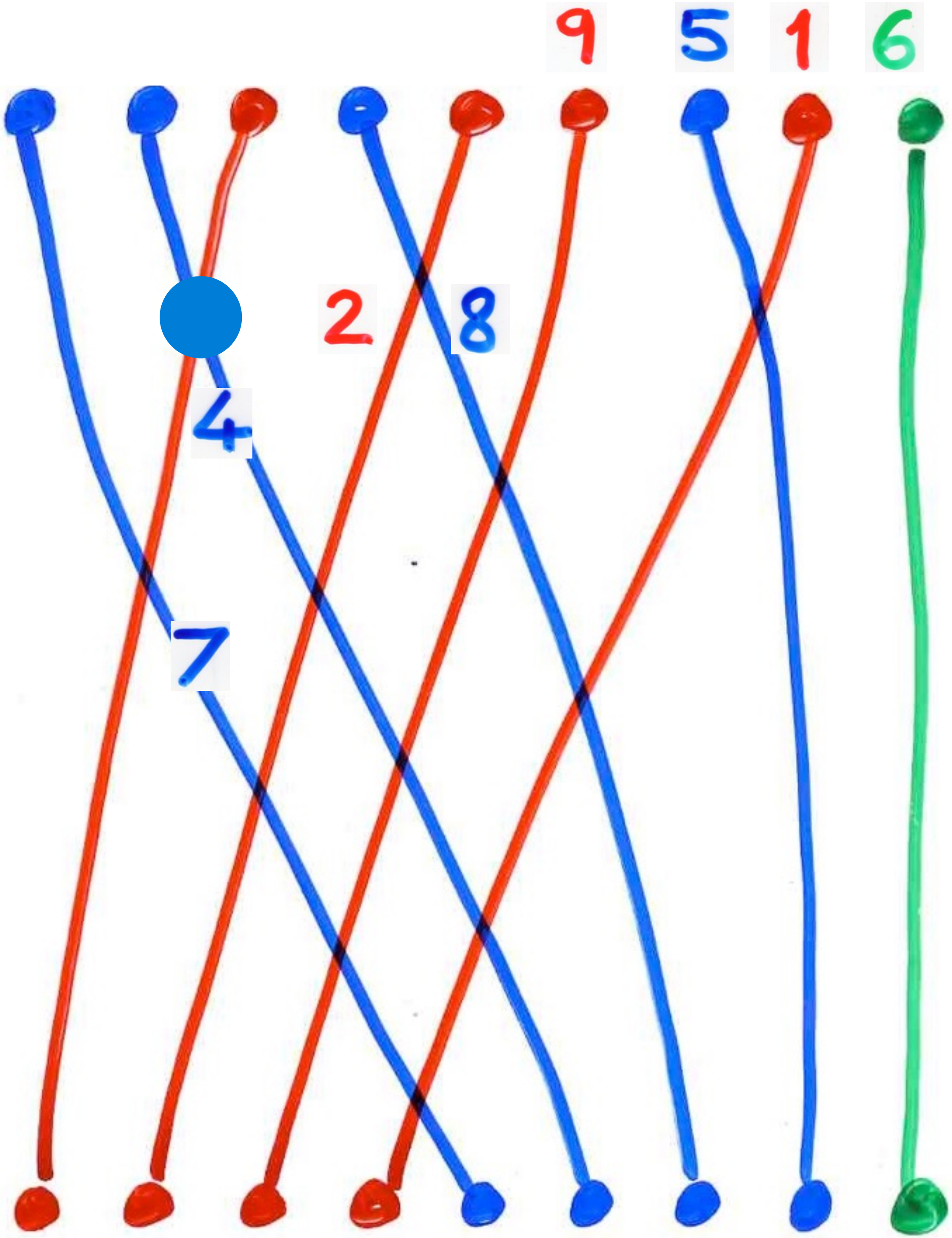

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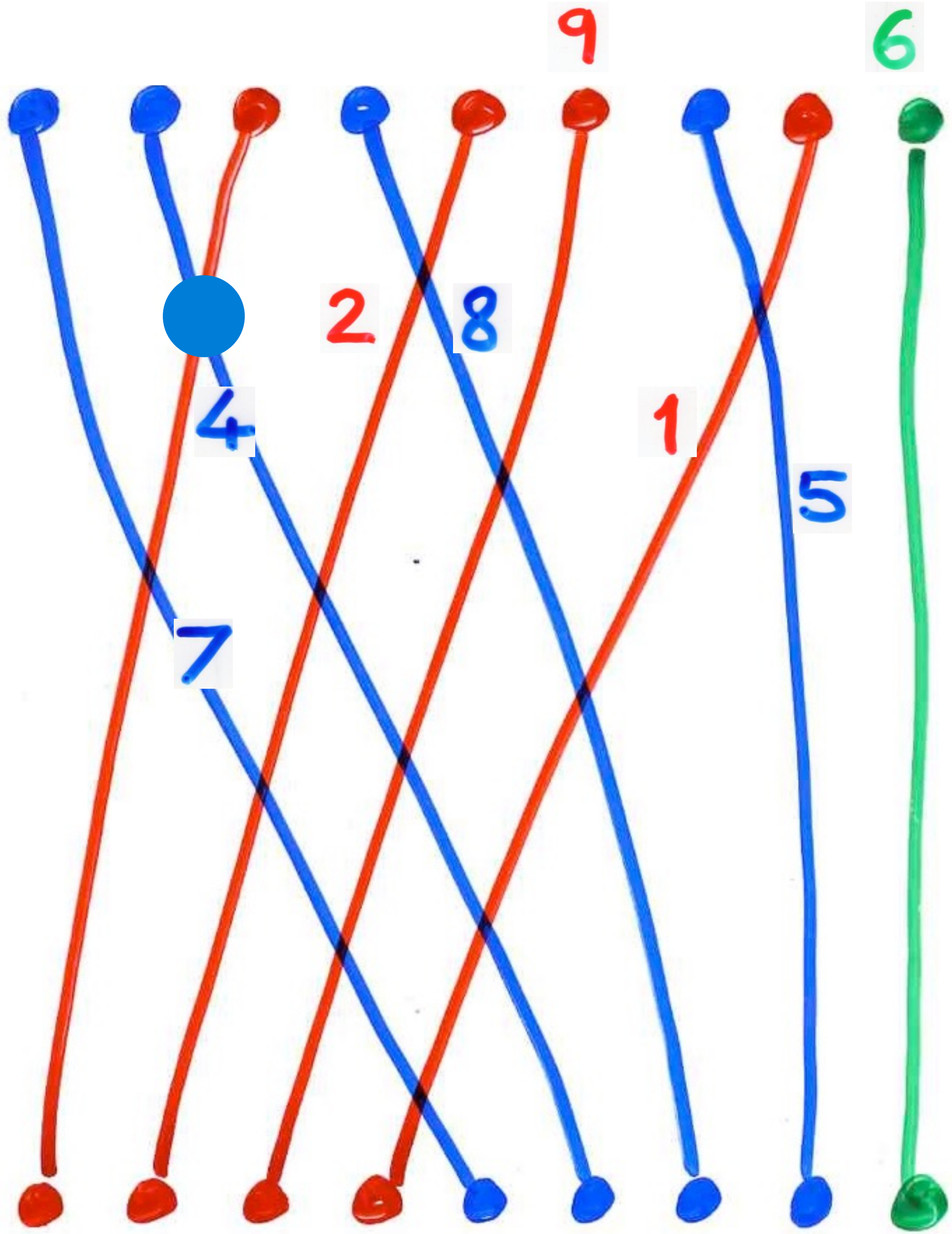

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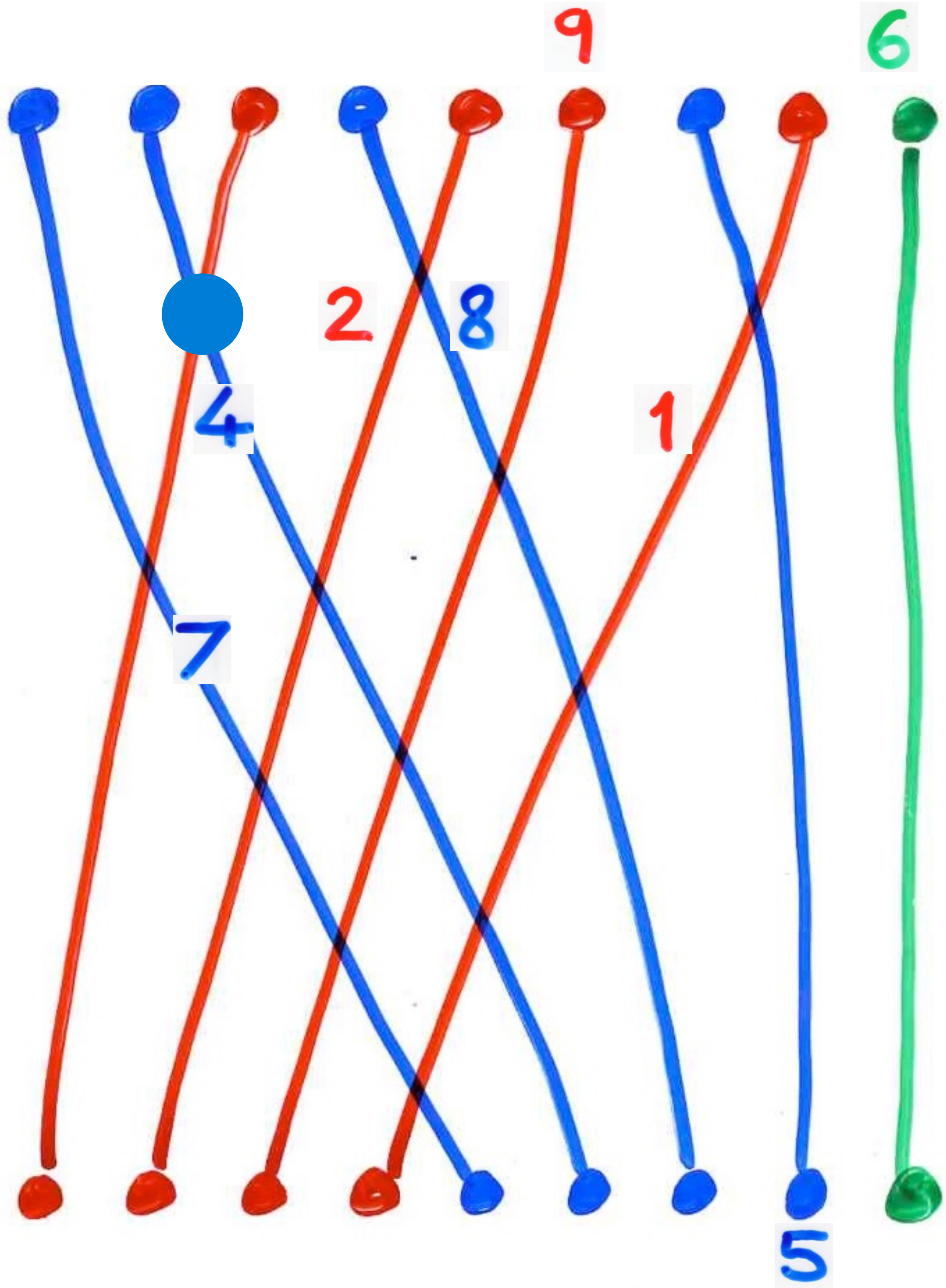
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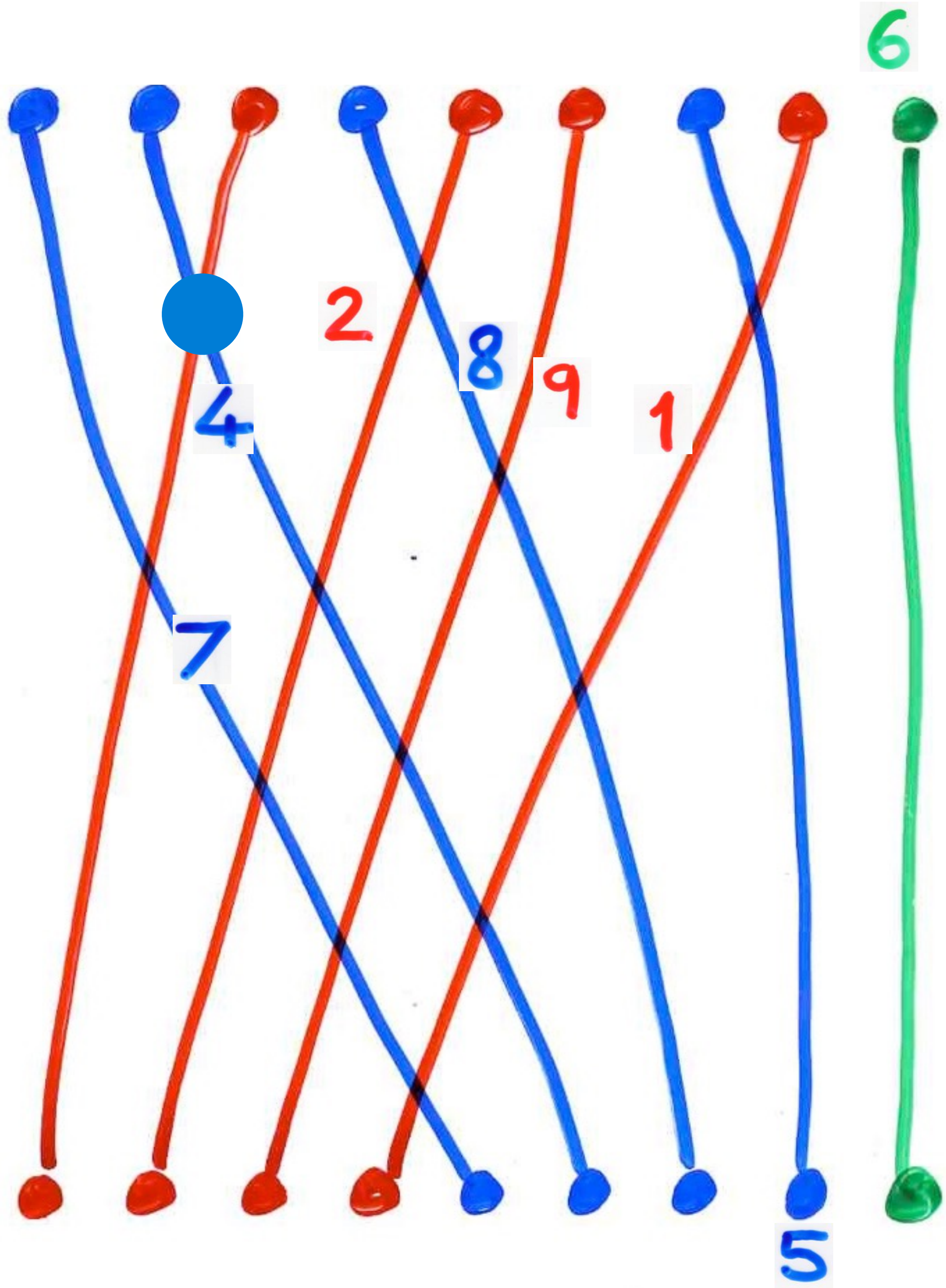
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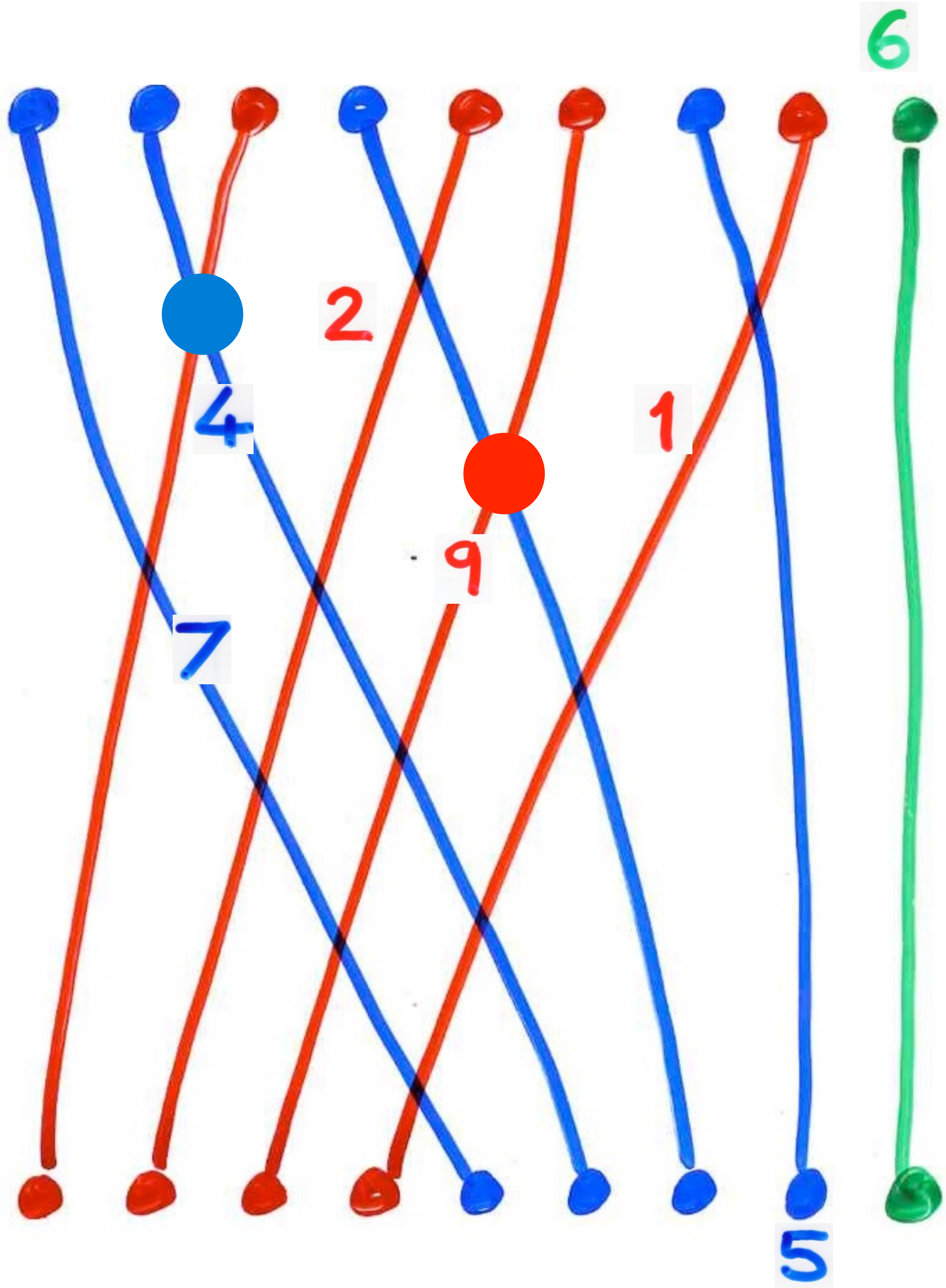
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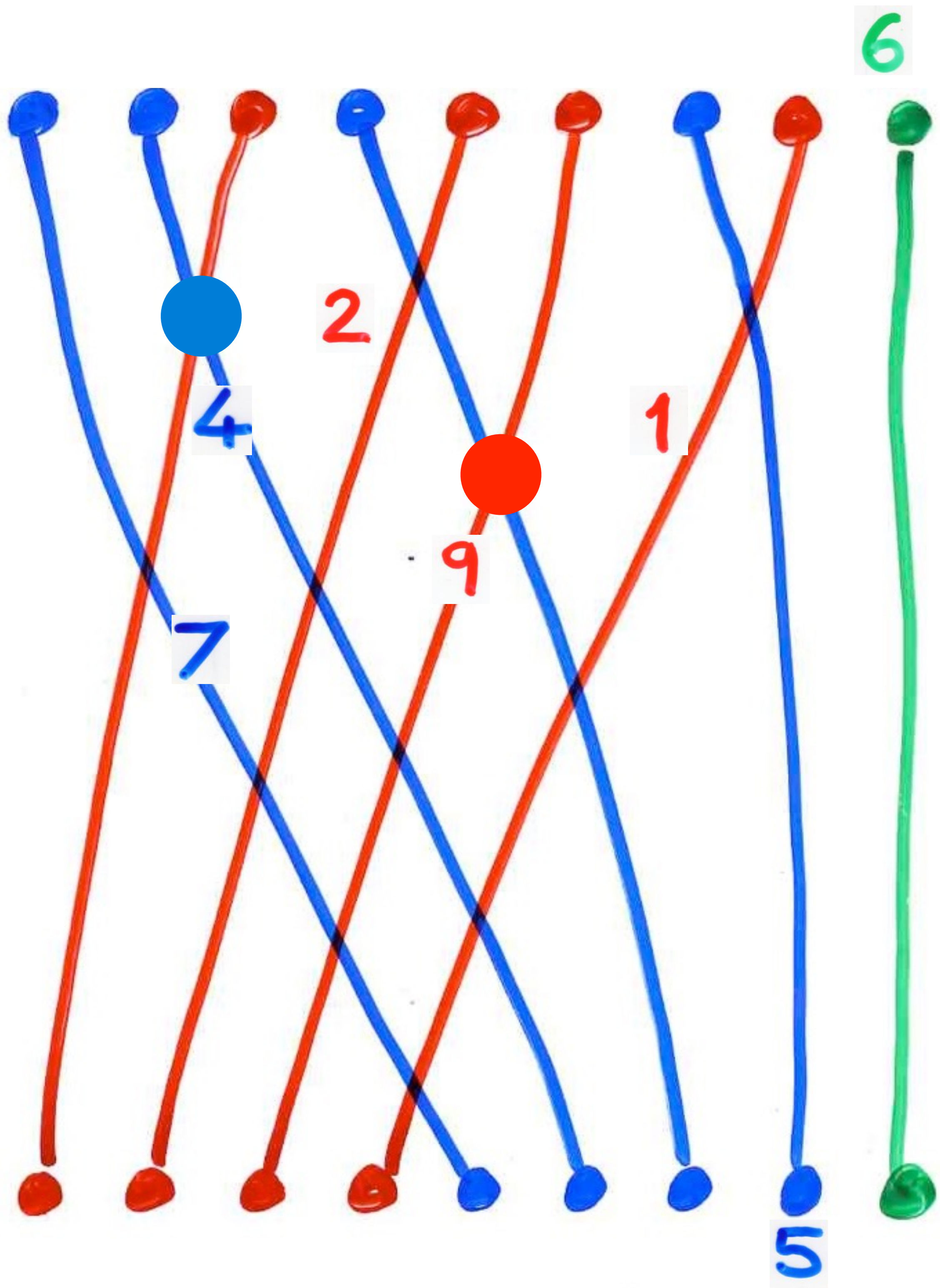
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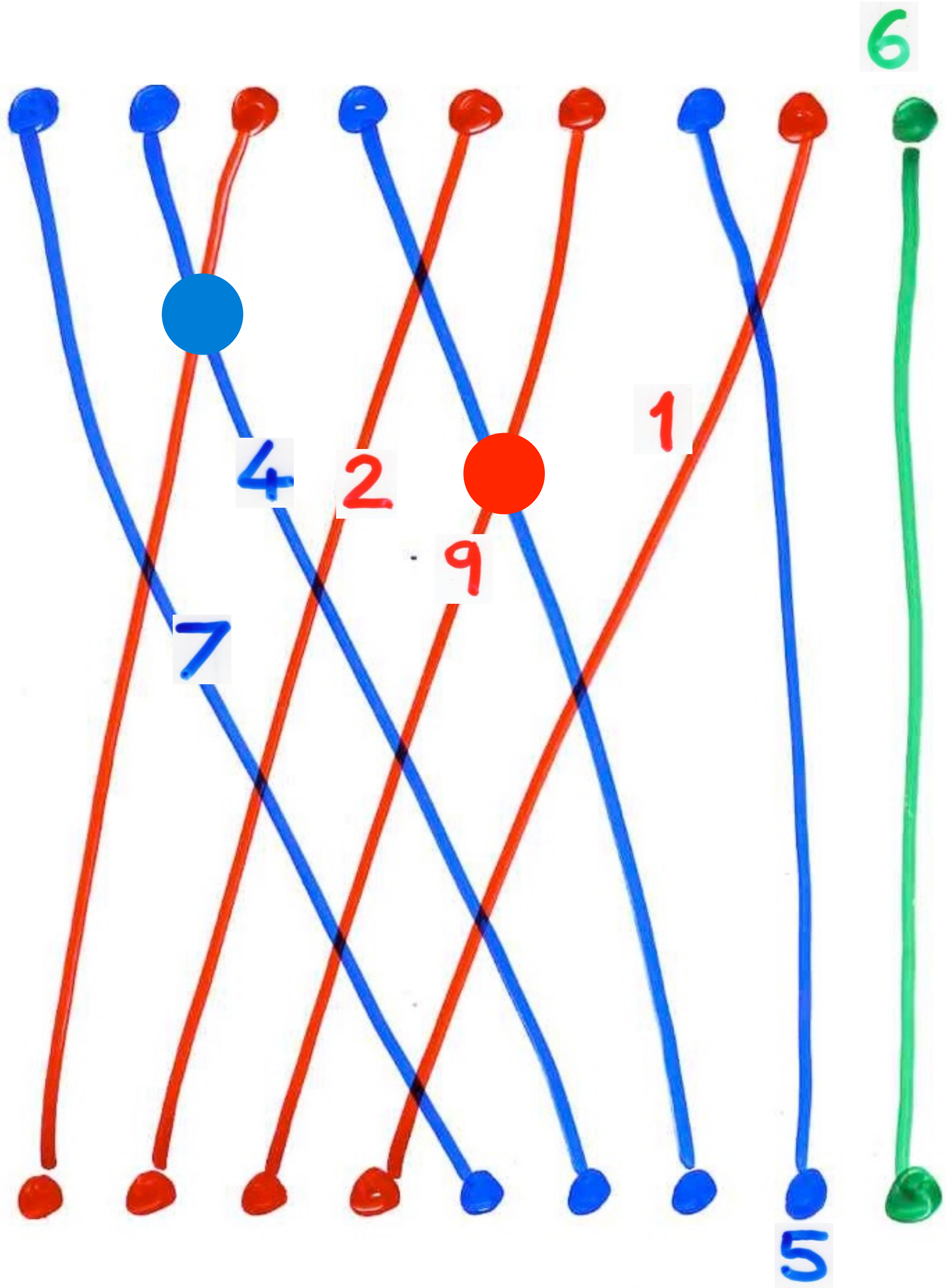

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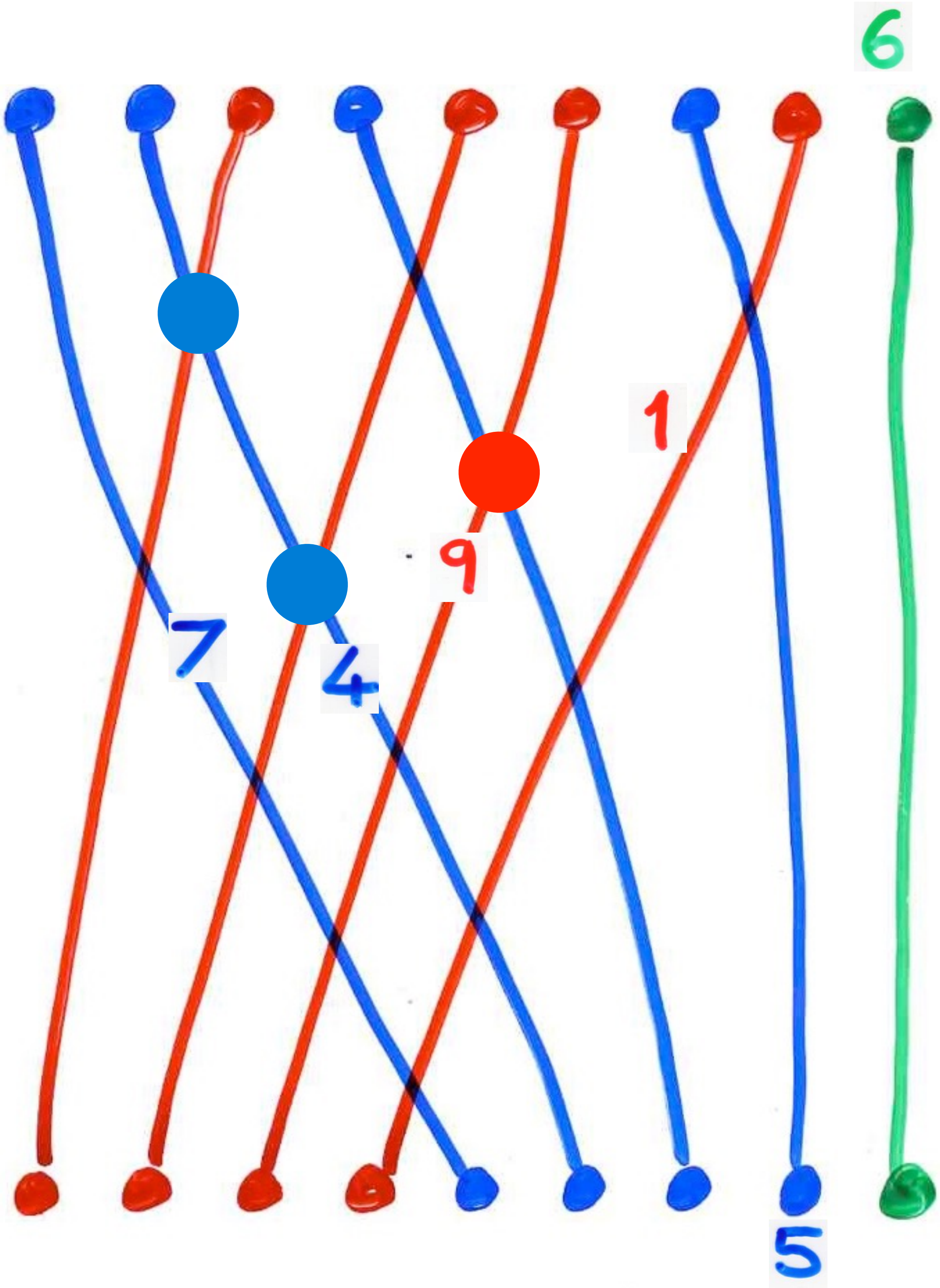
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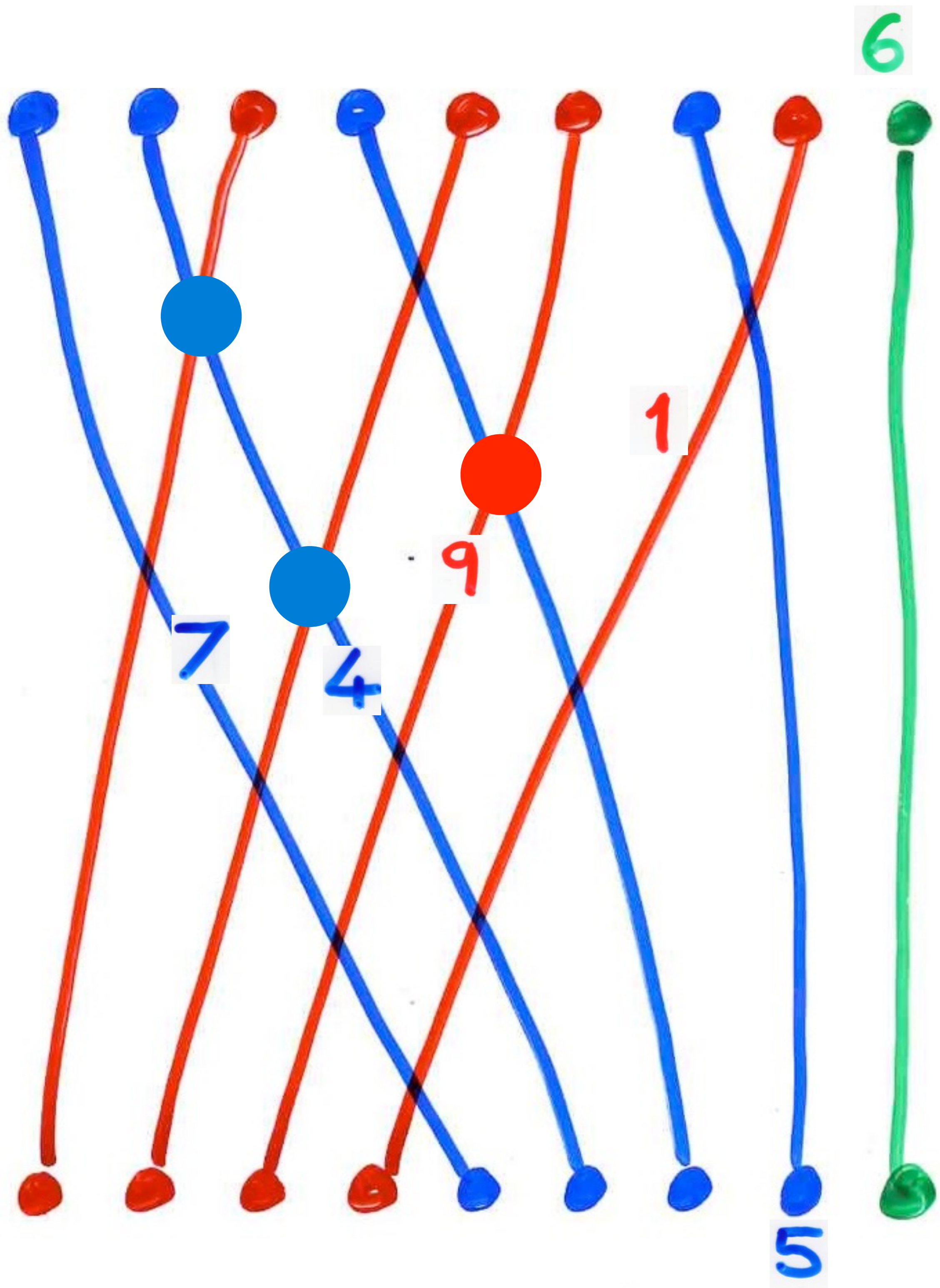
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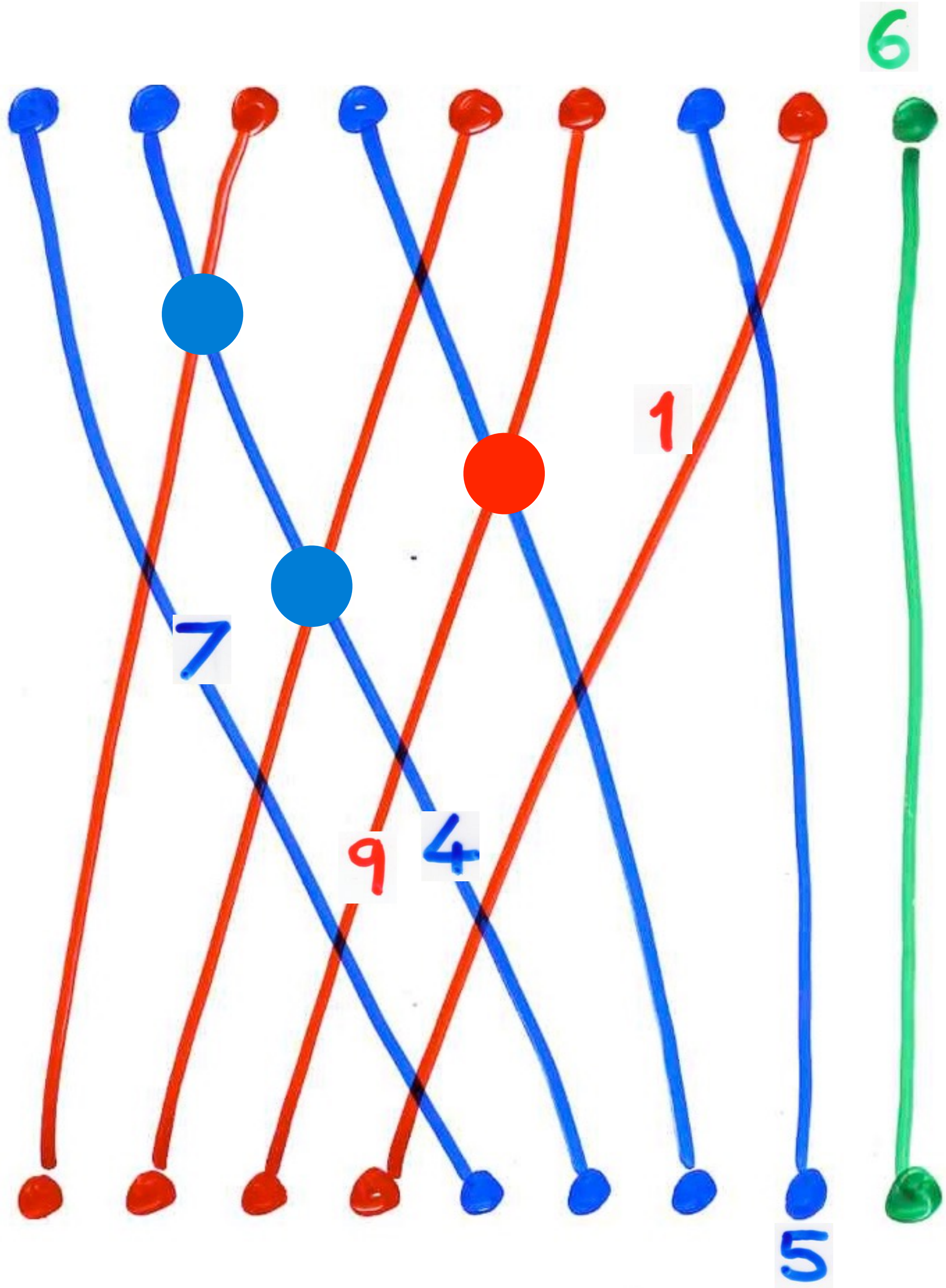

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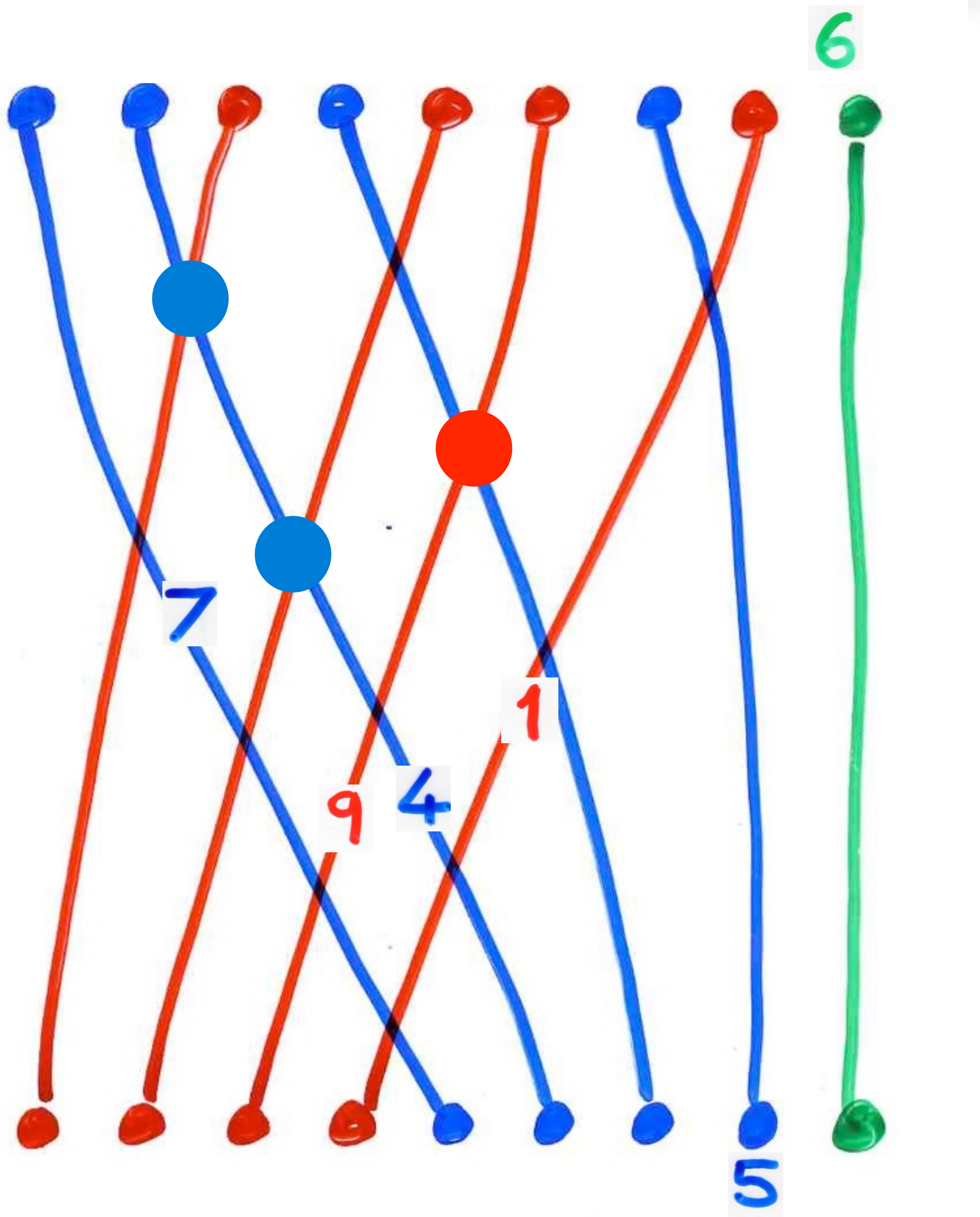
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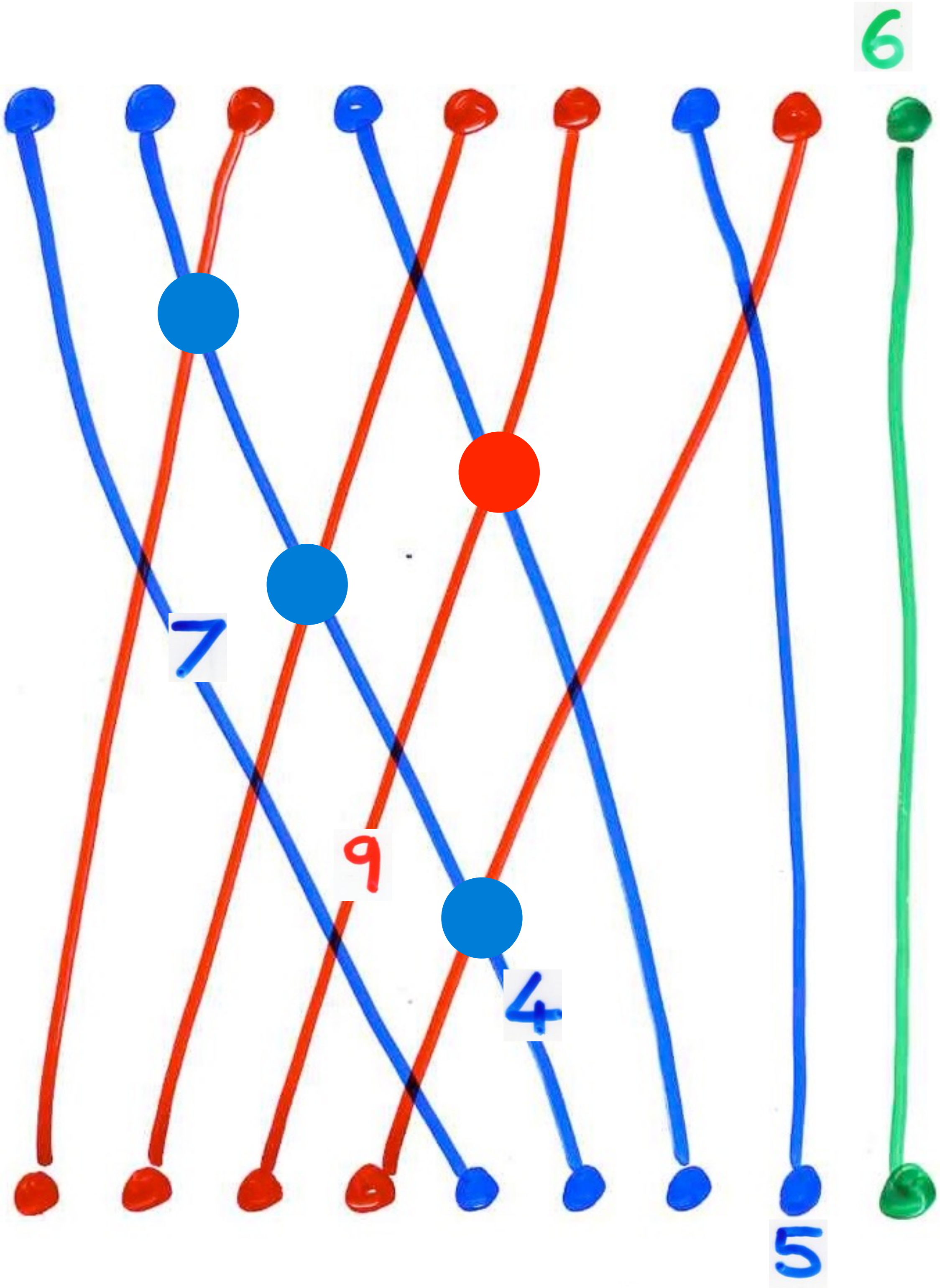
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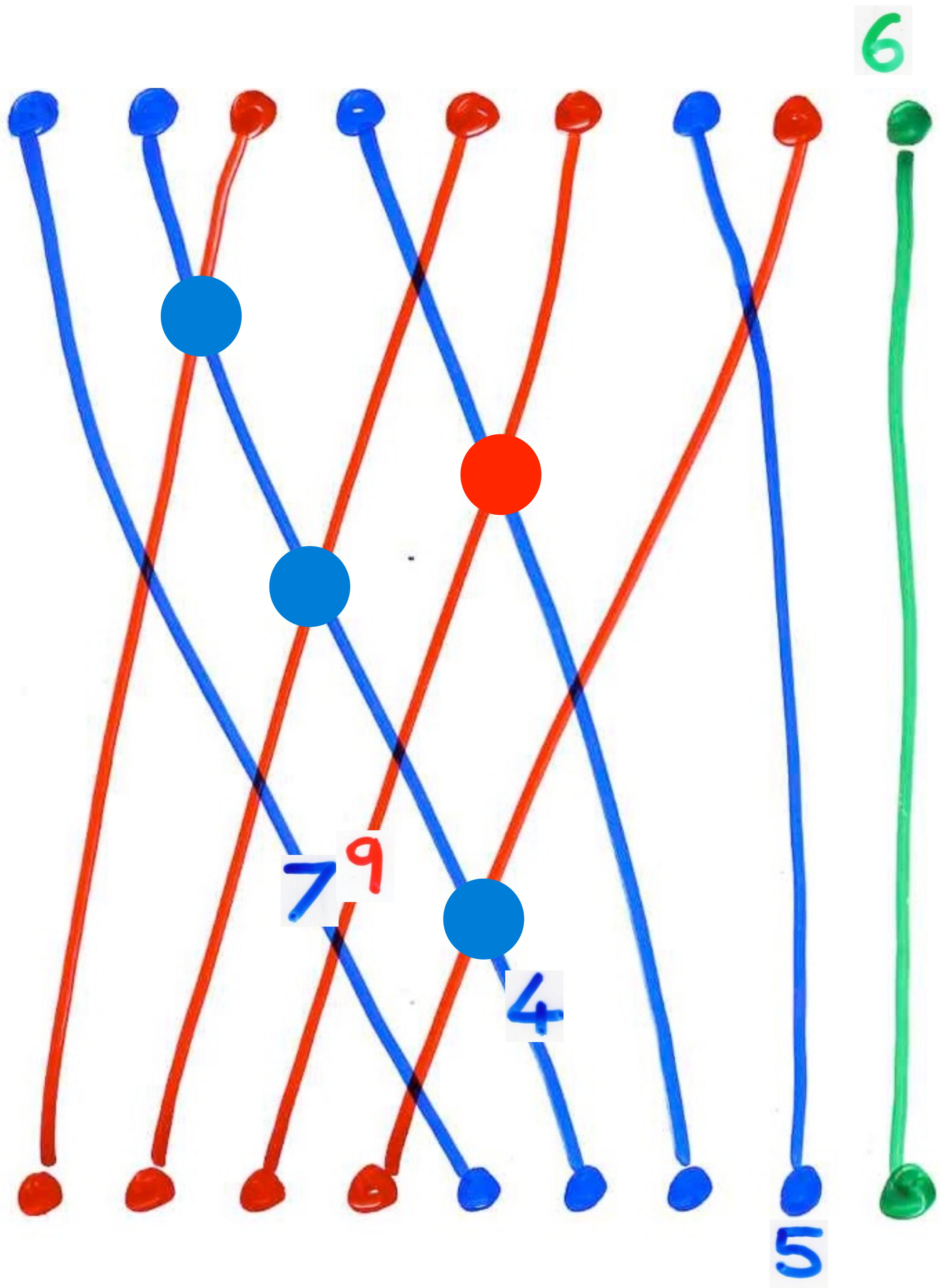
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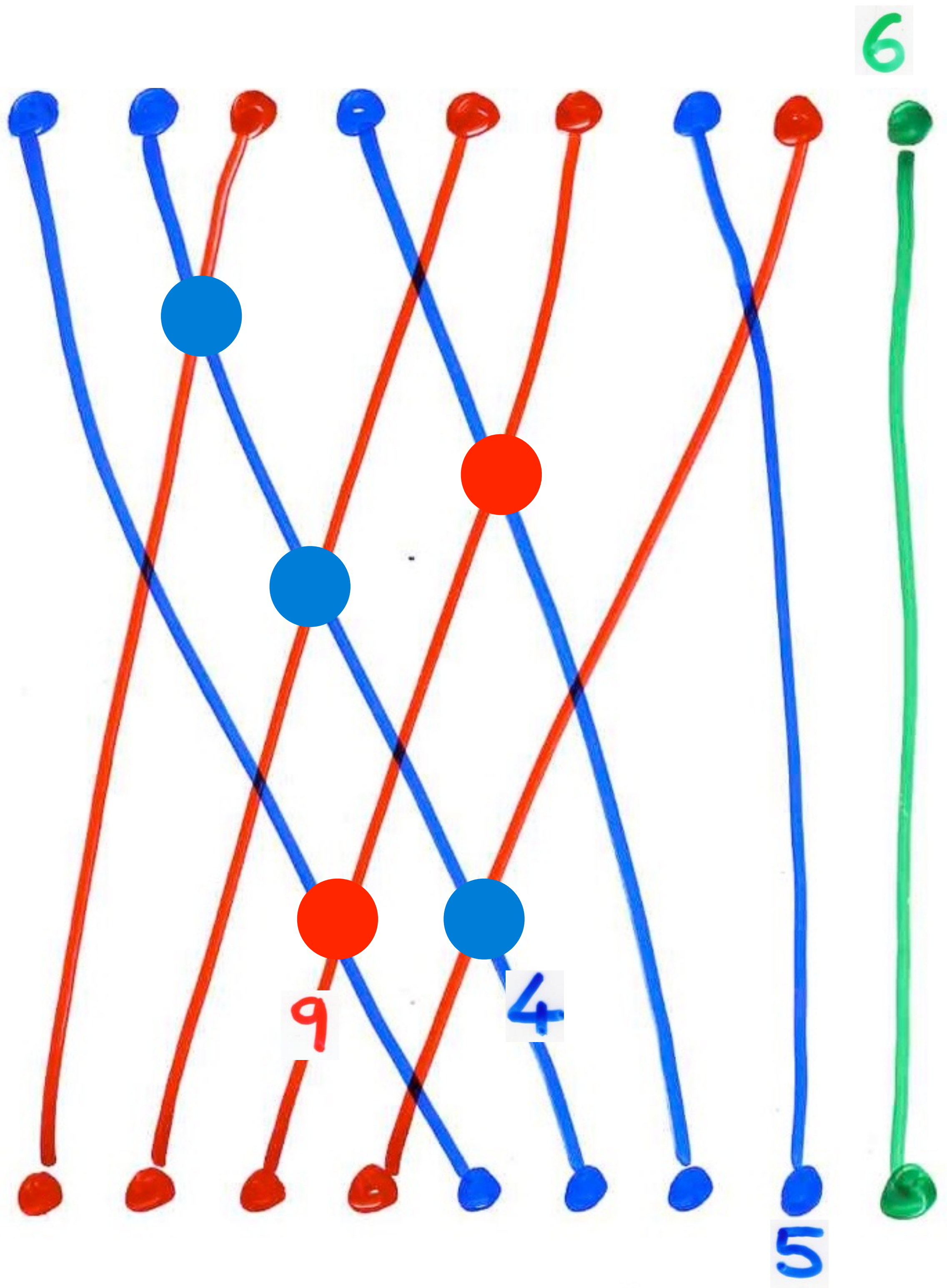
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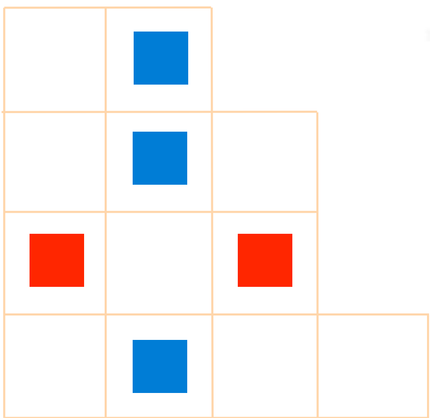
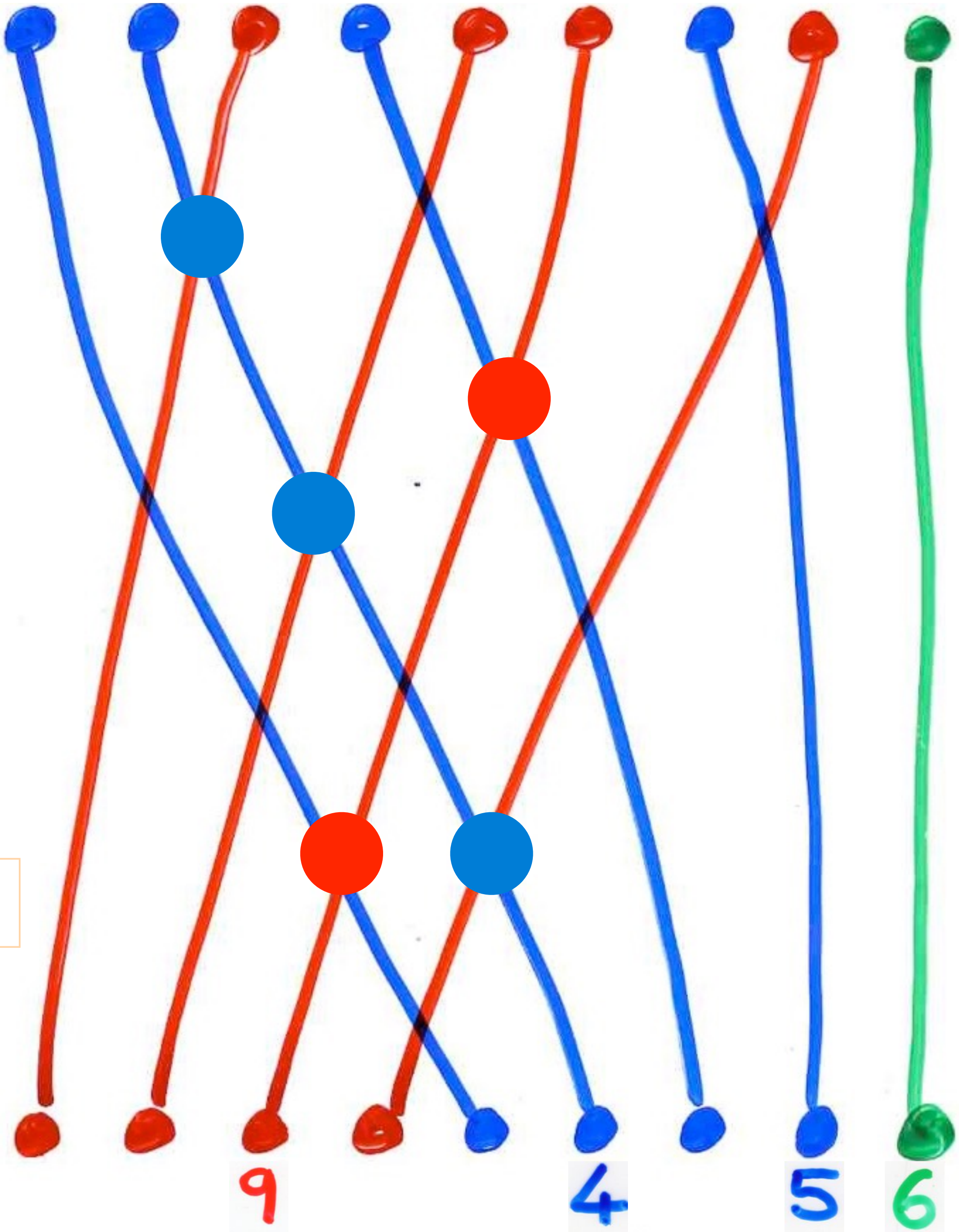
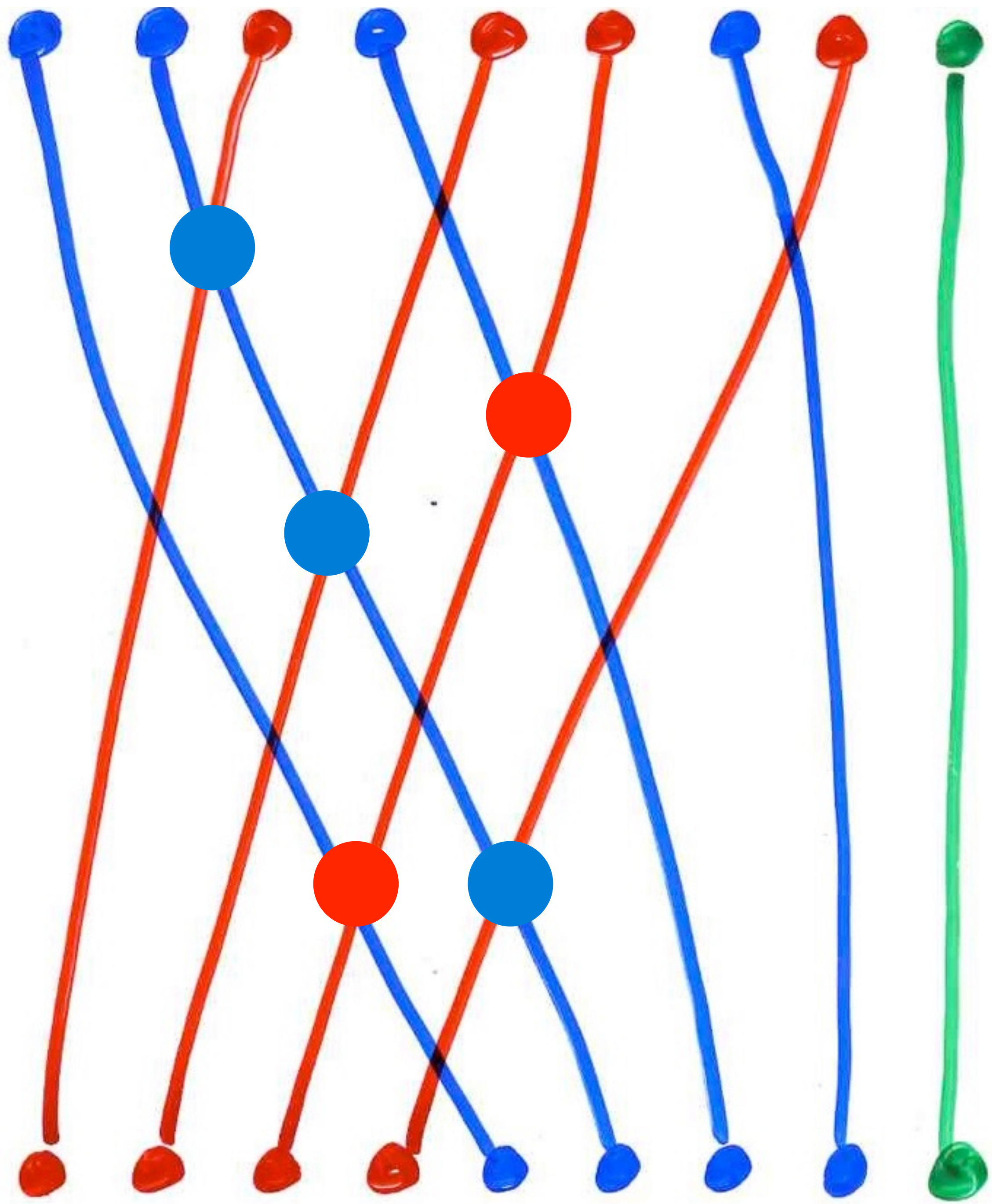
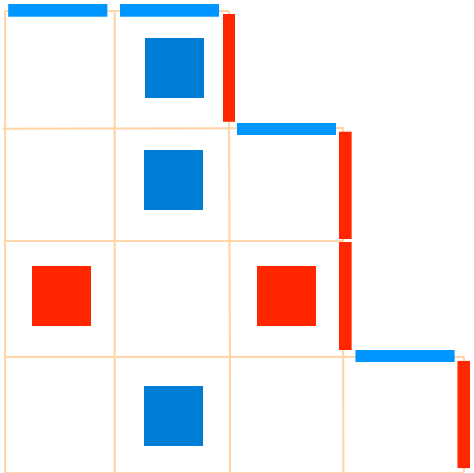
"exchange-
deletion"
algorithm

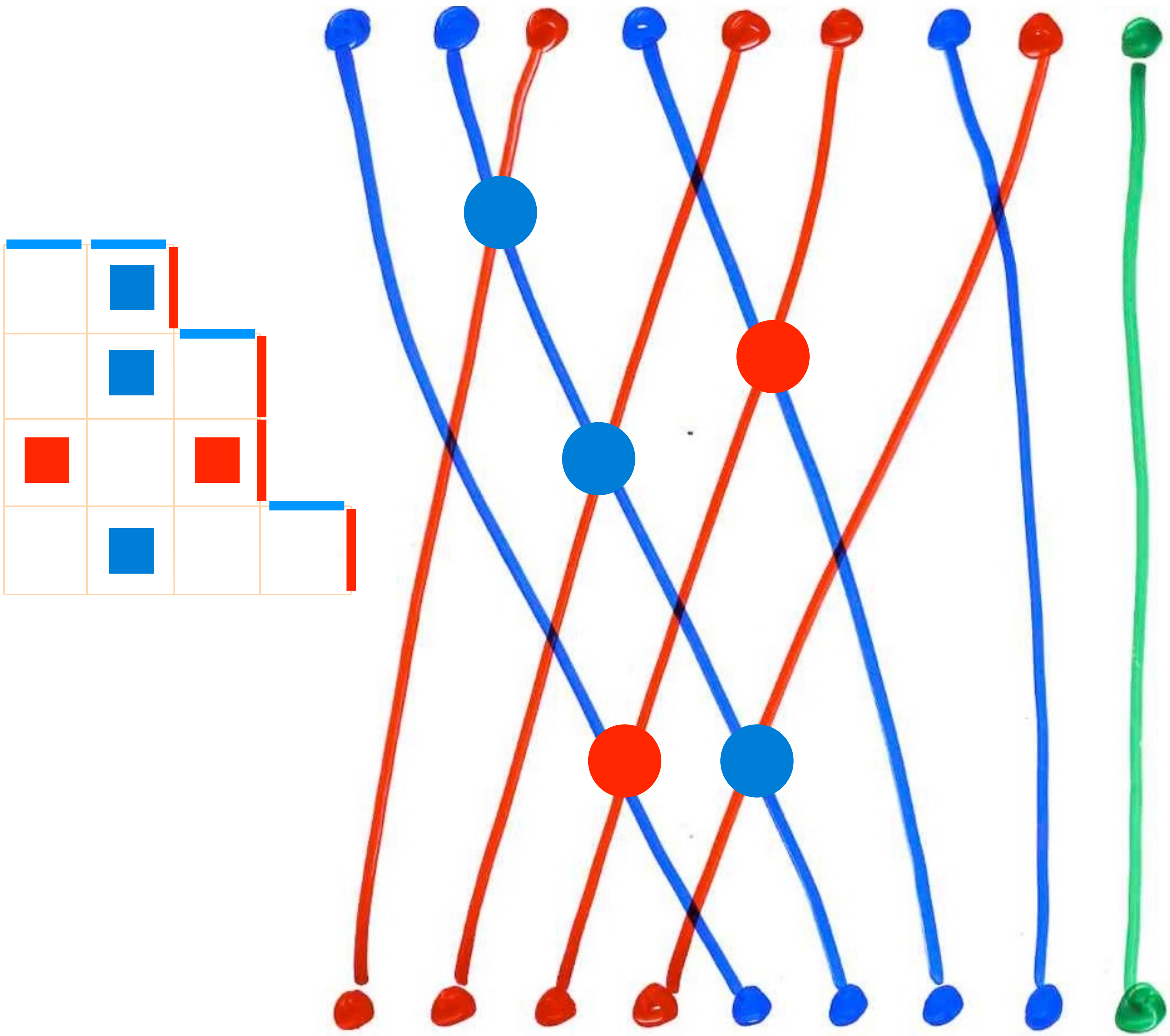


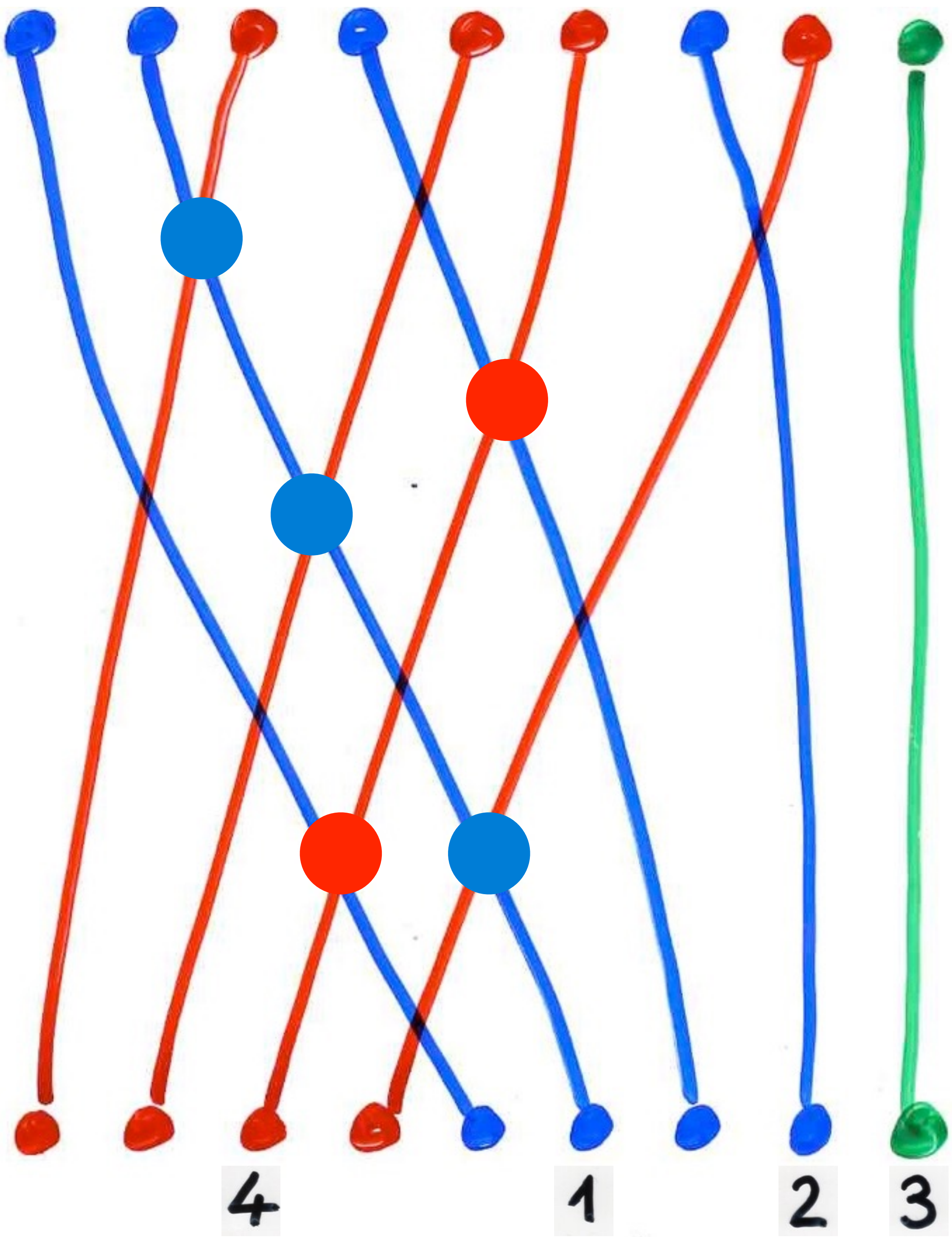




The inverse “exchange-delete” algorithm



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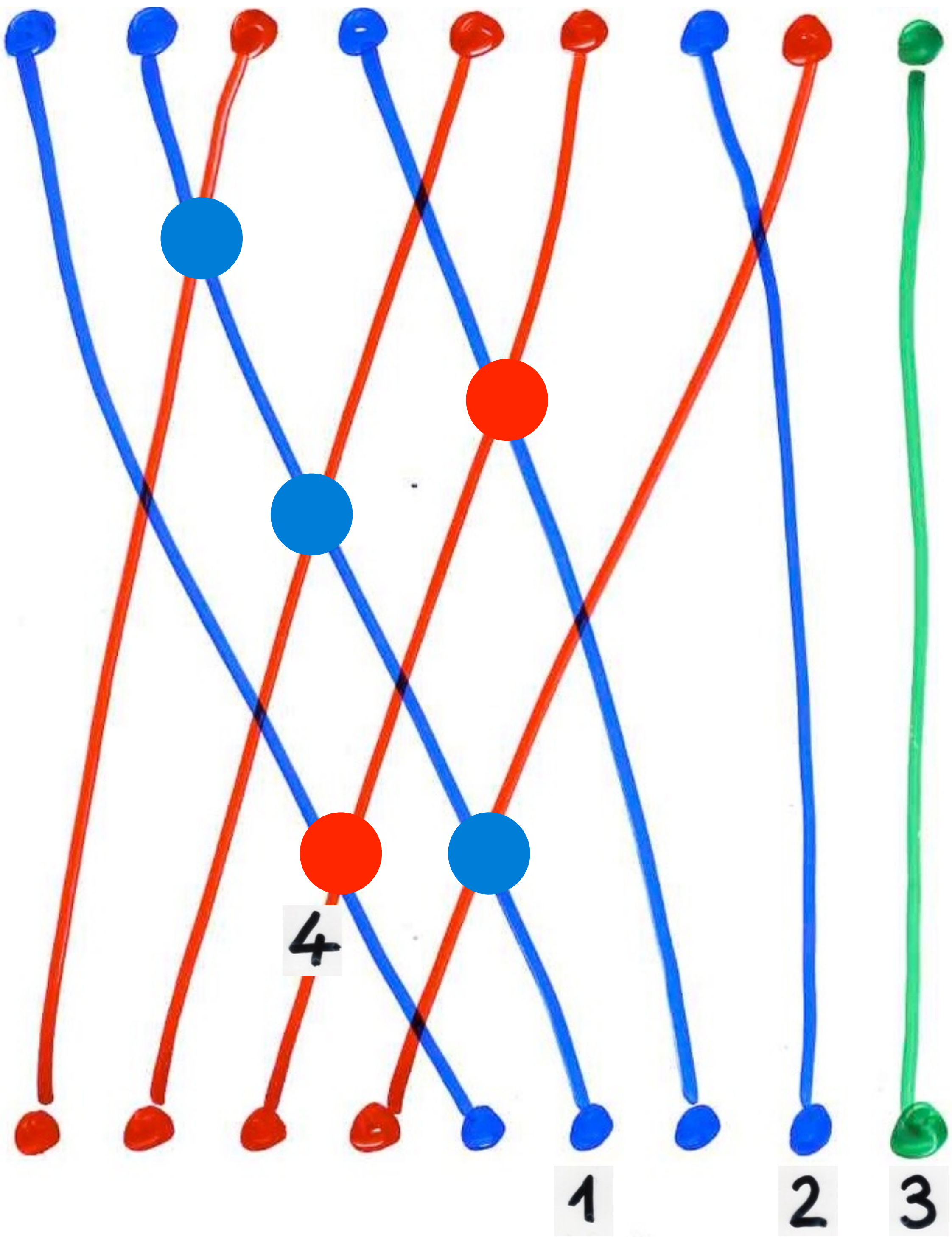
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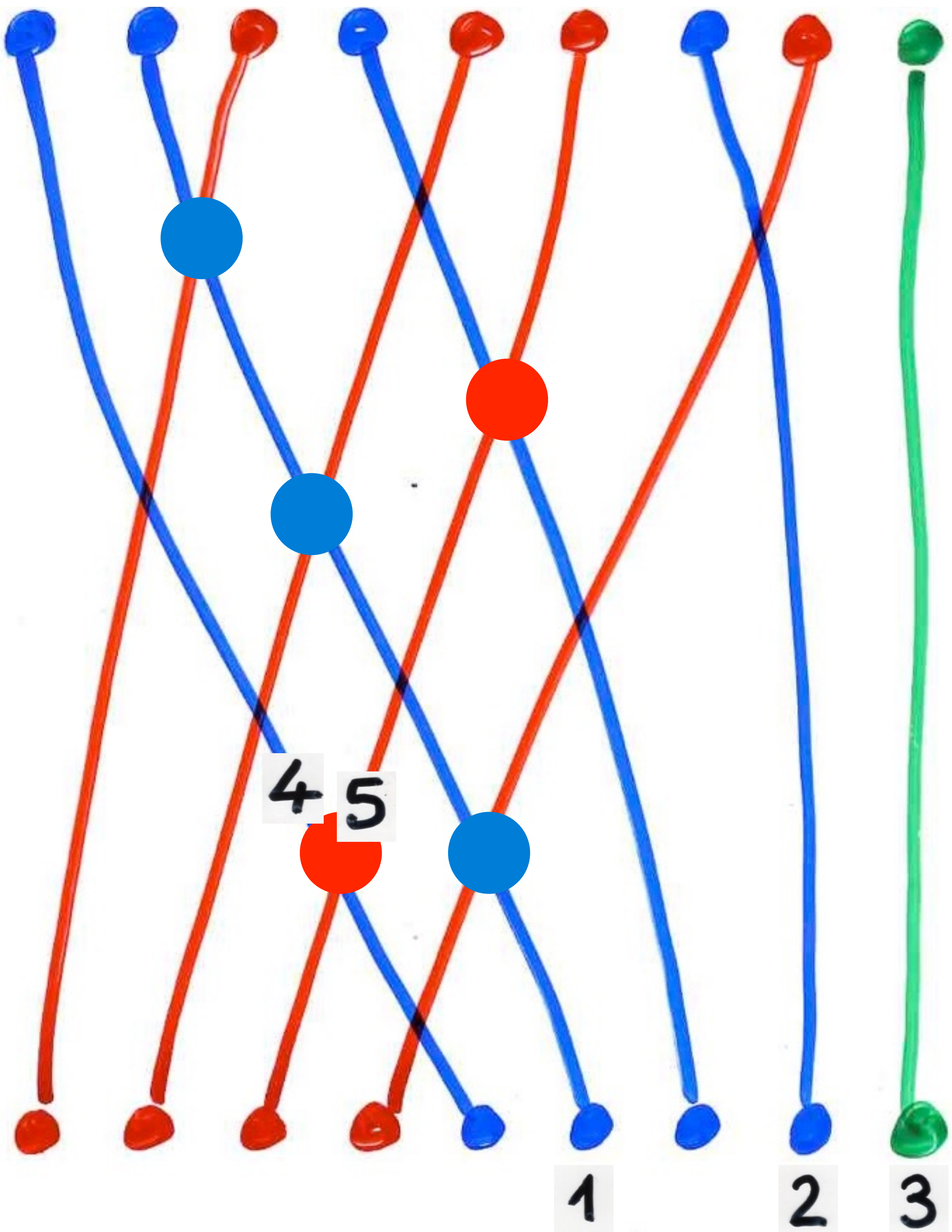
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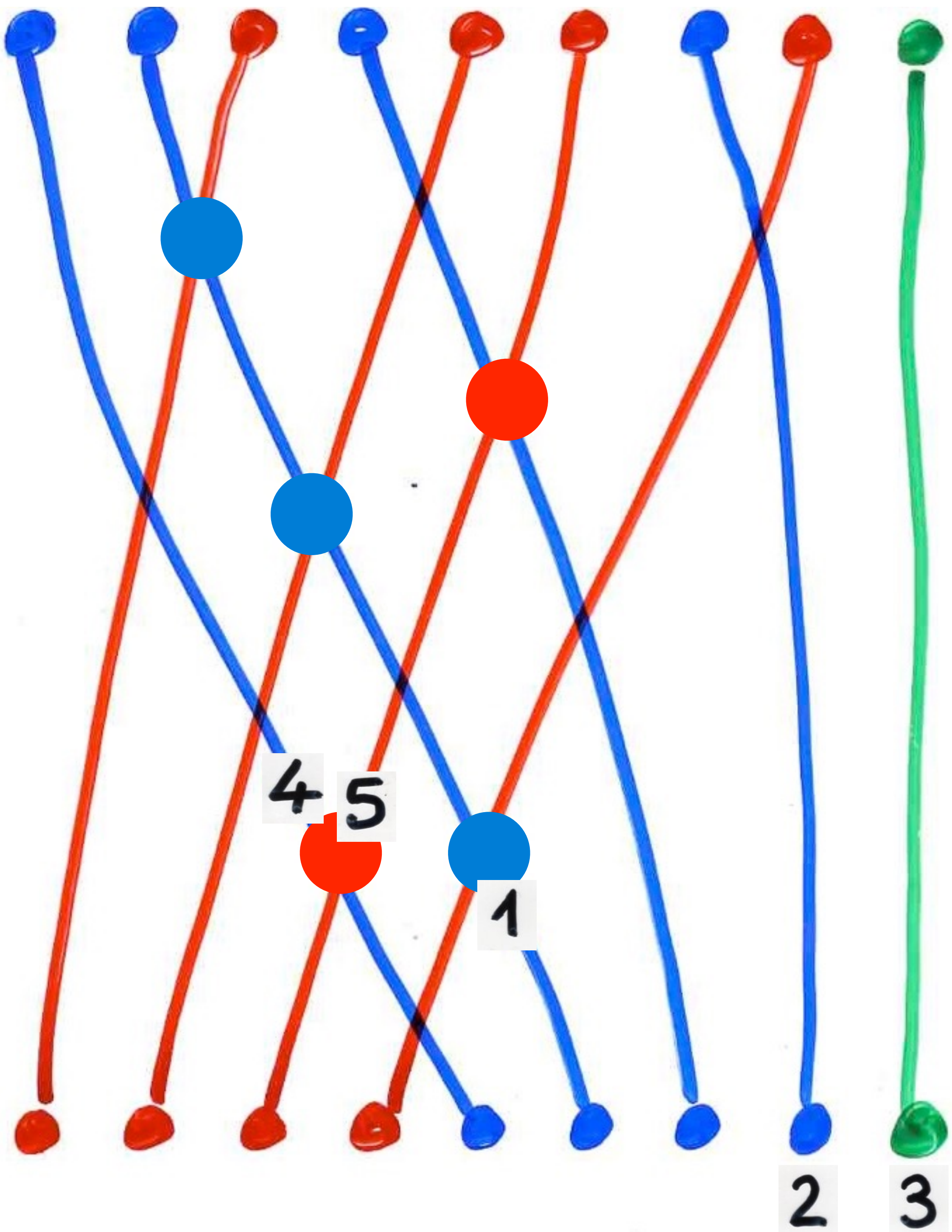
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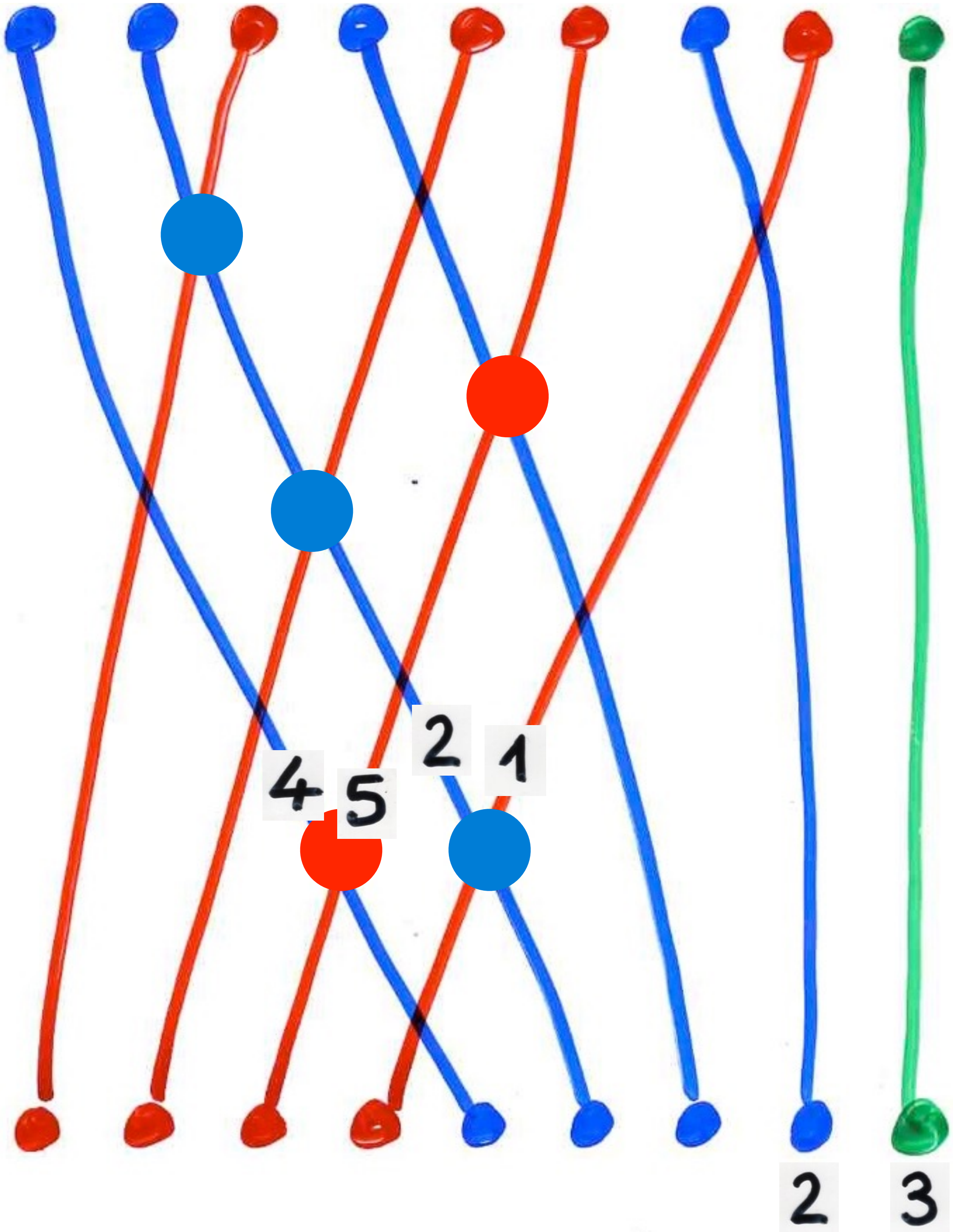
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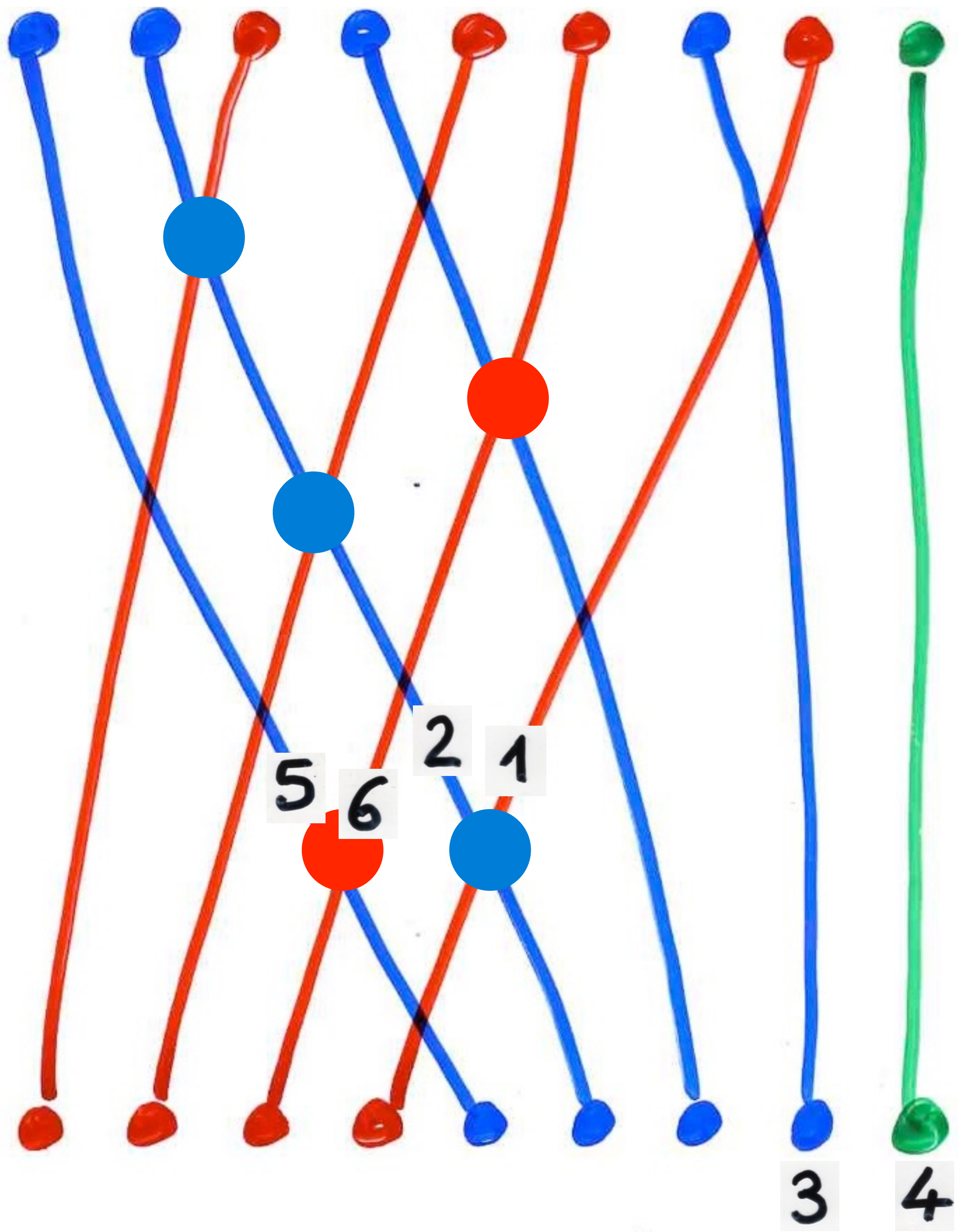

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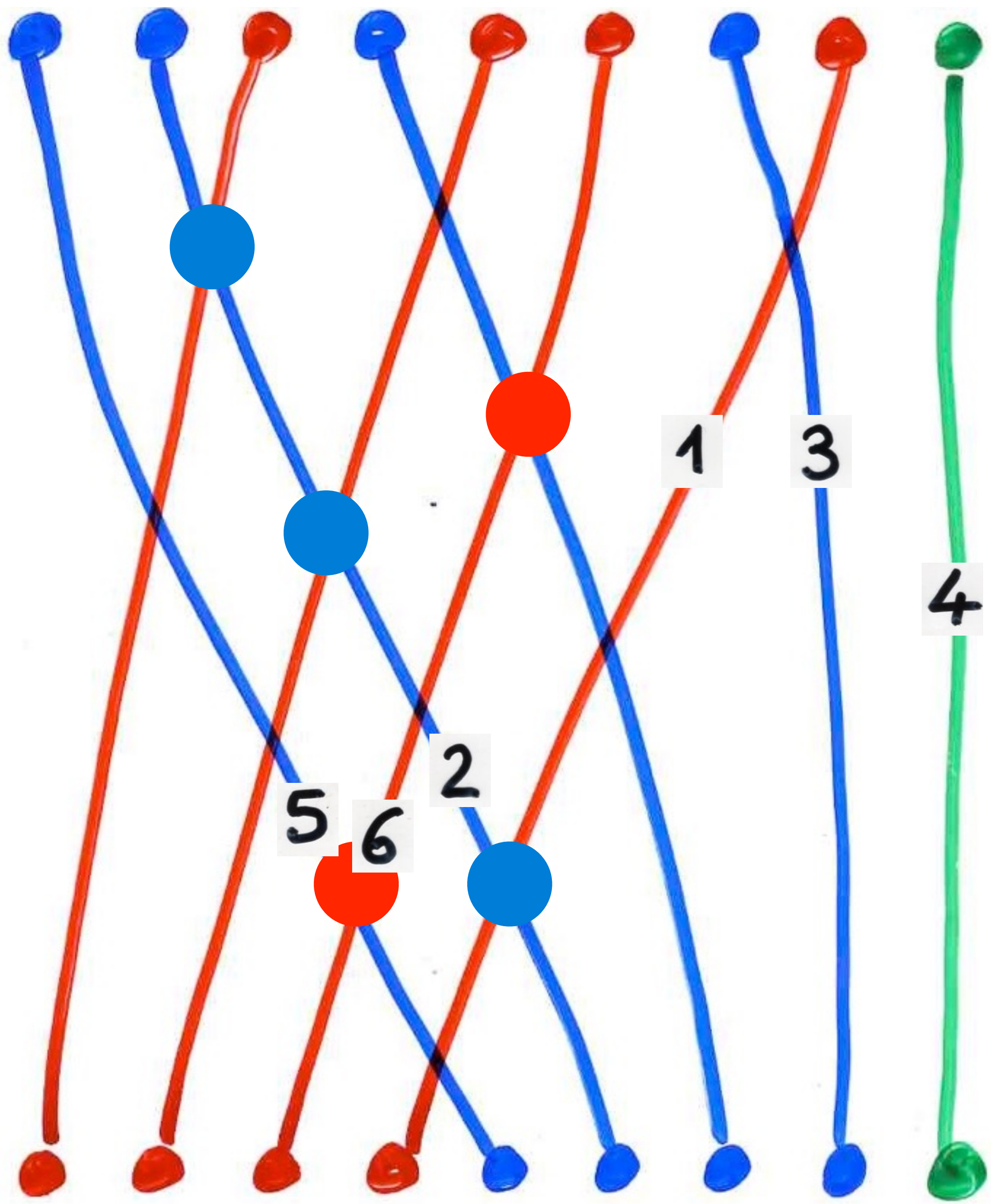
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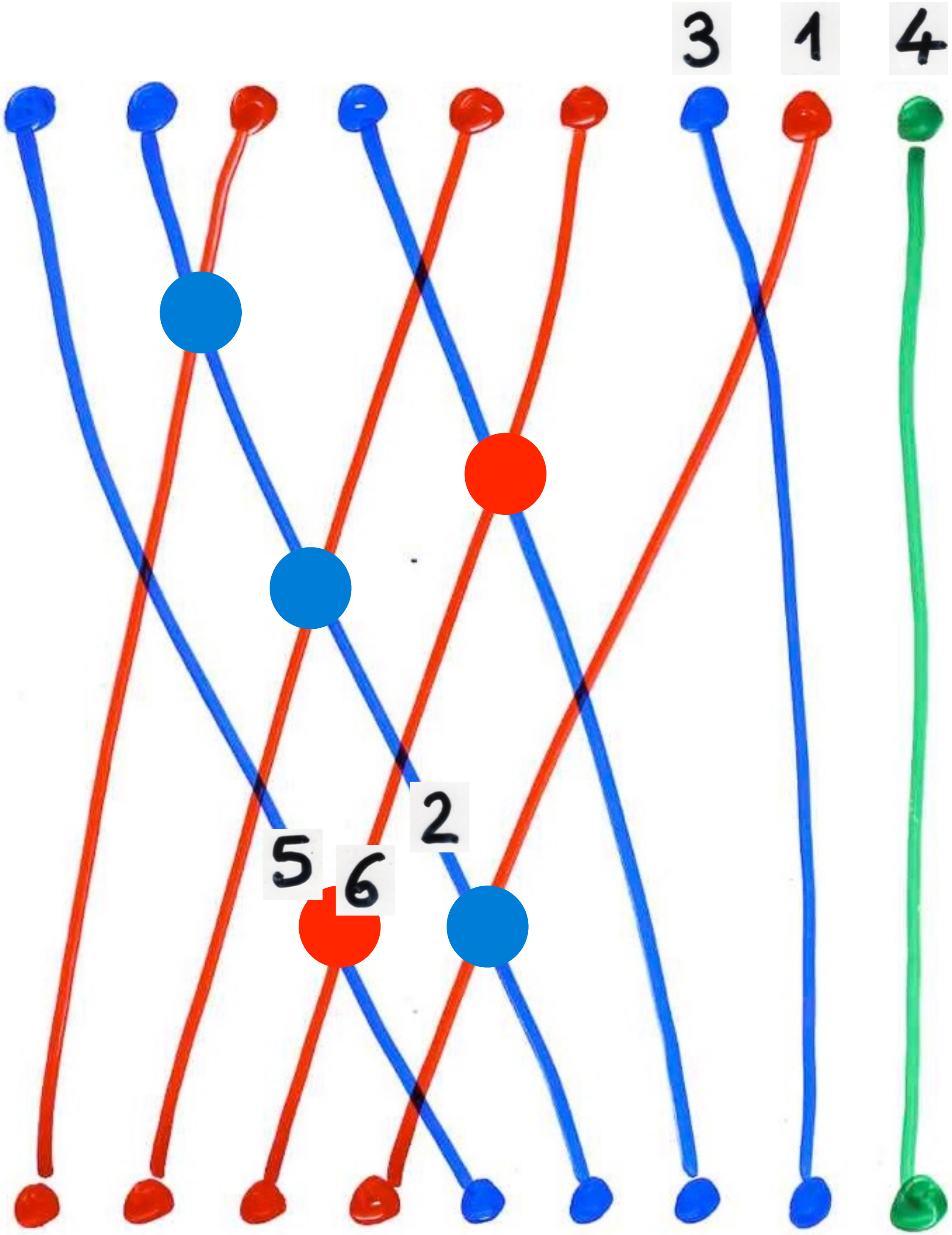
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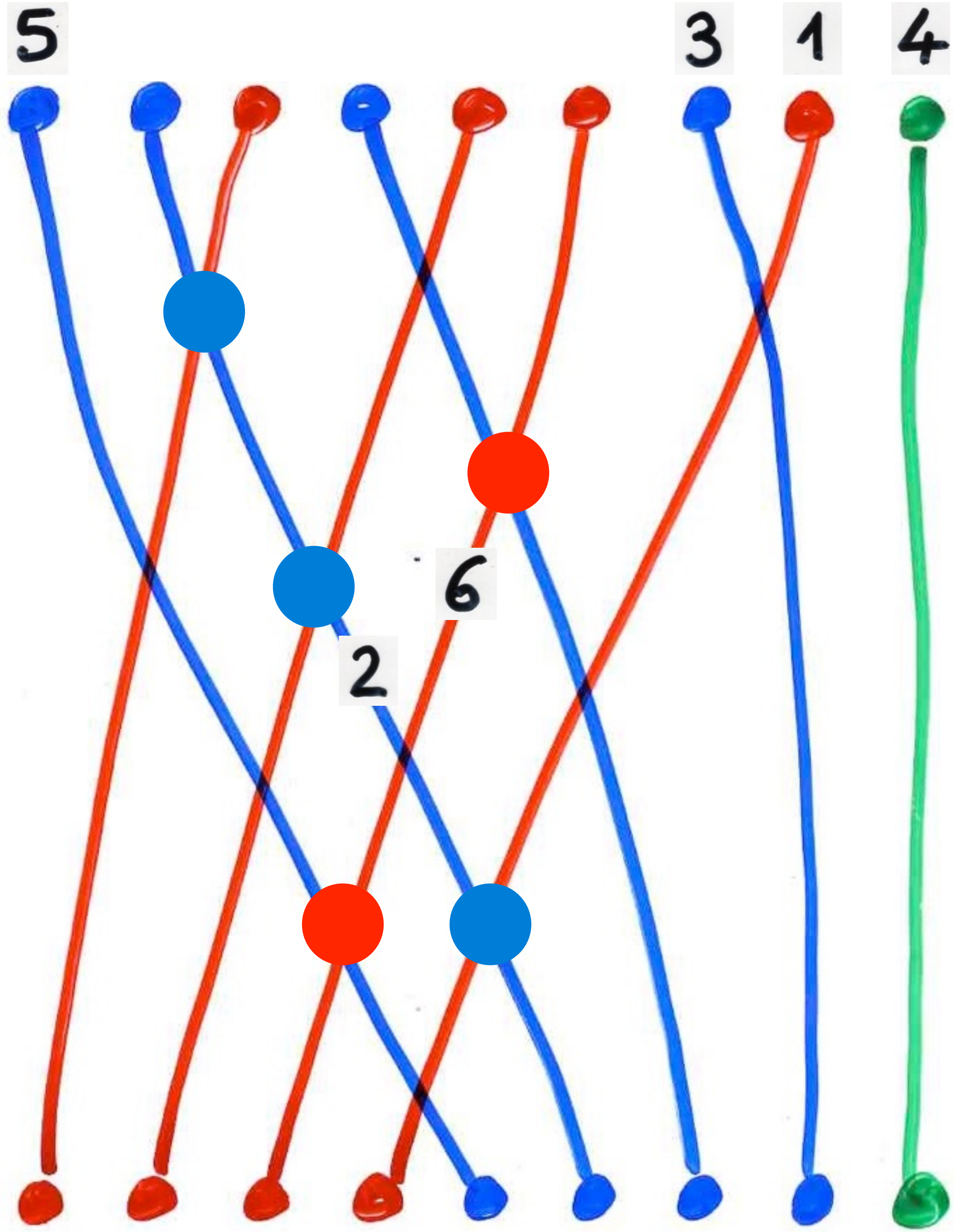
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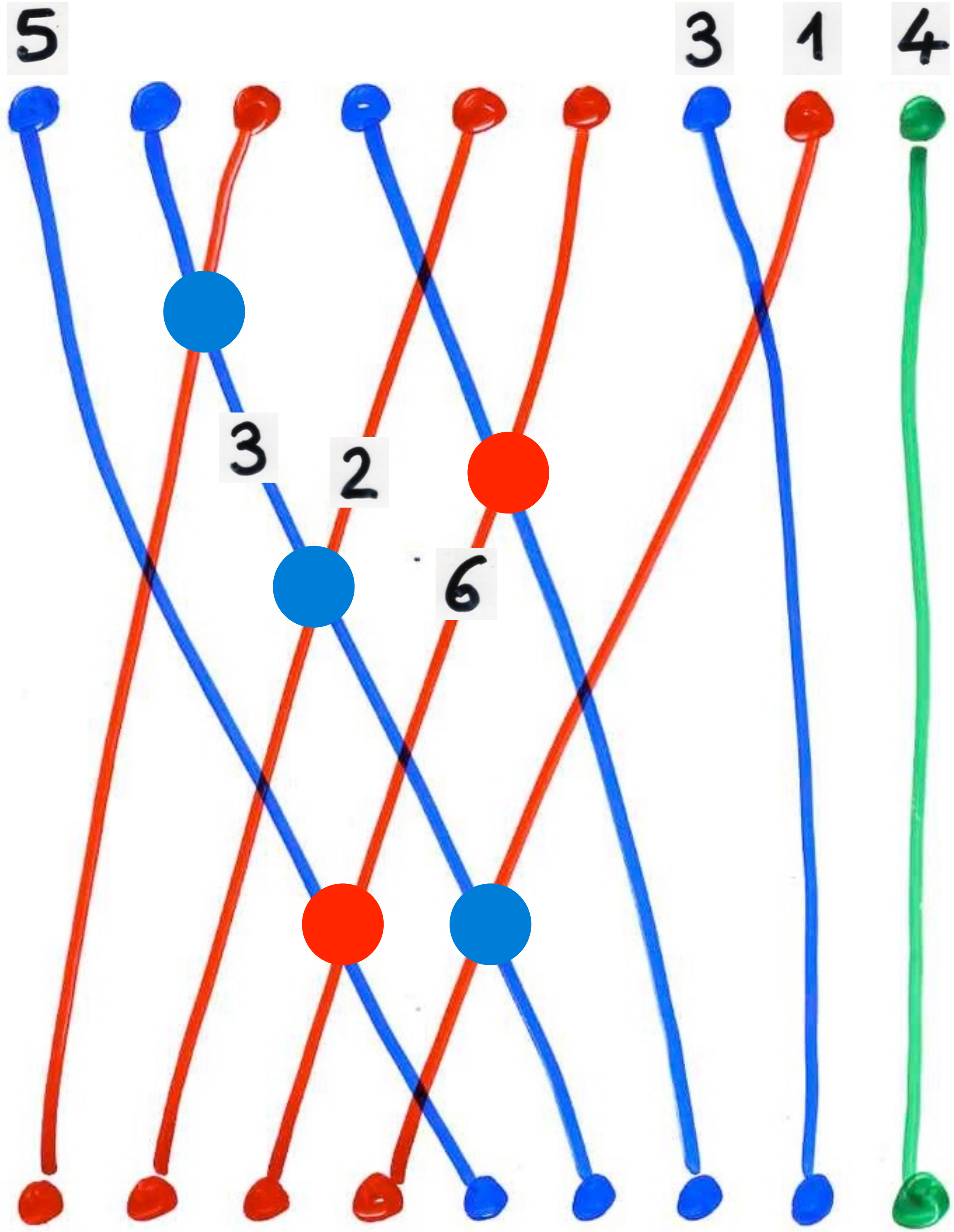

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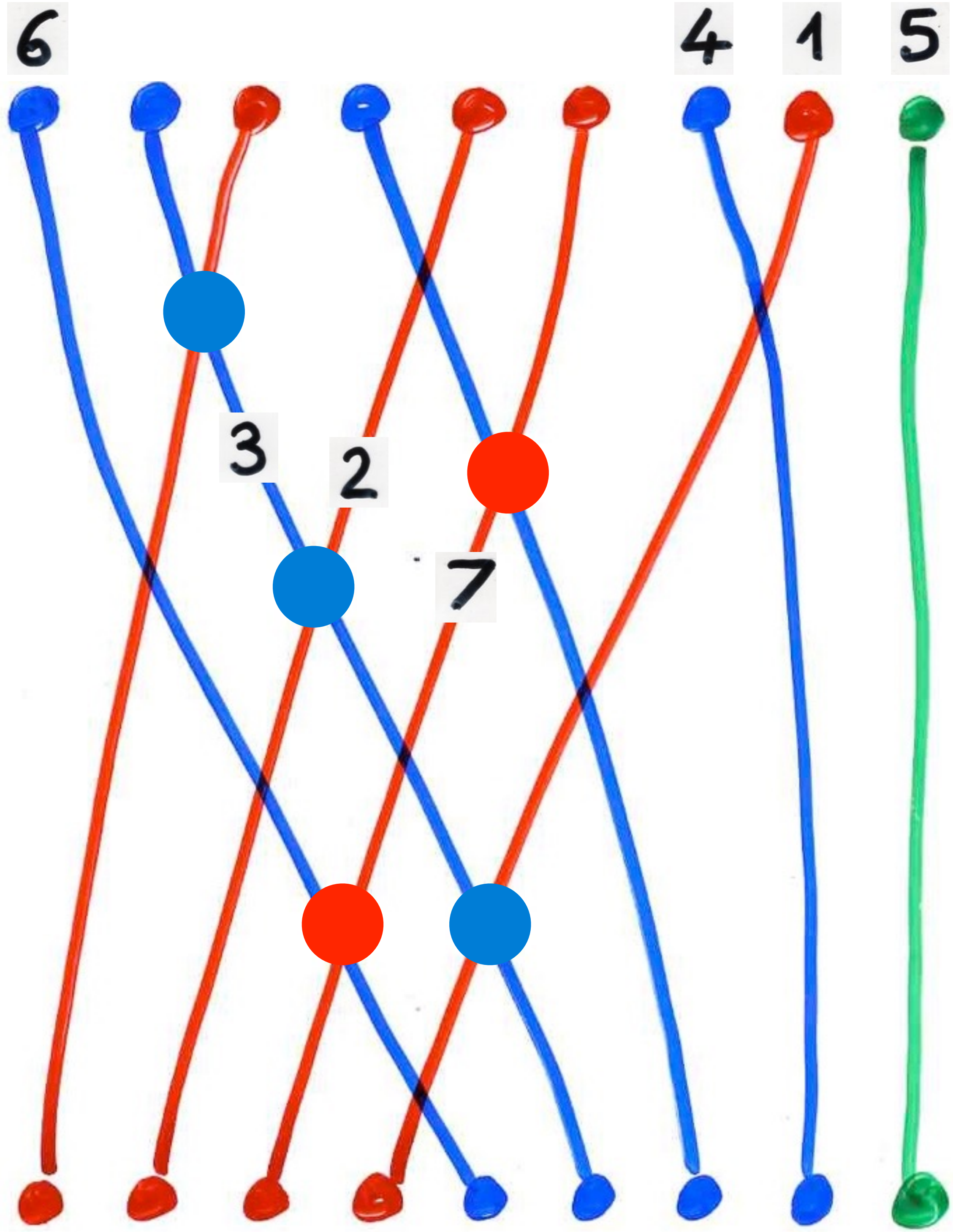
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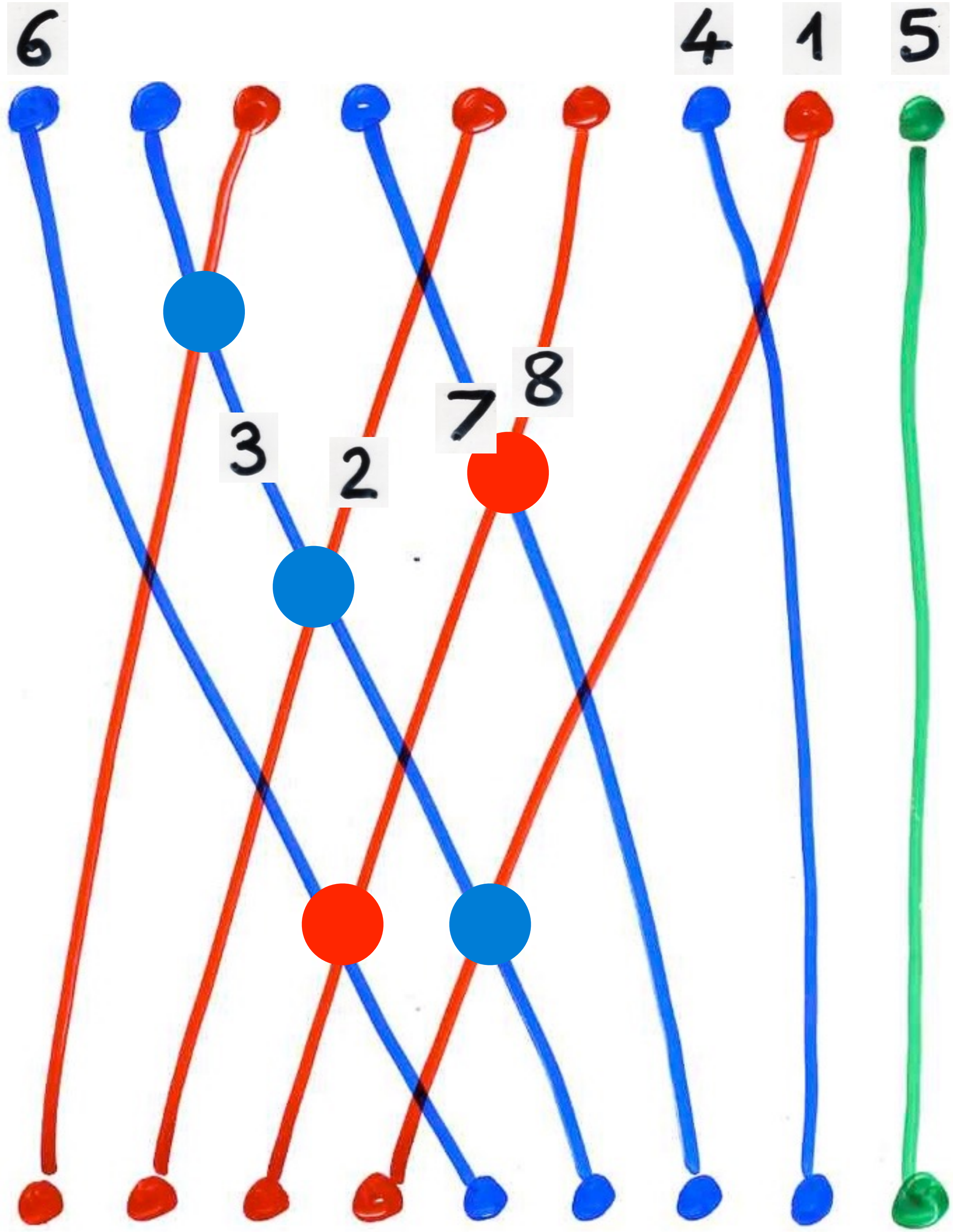

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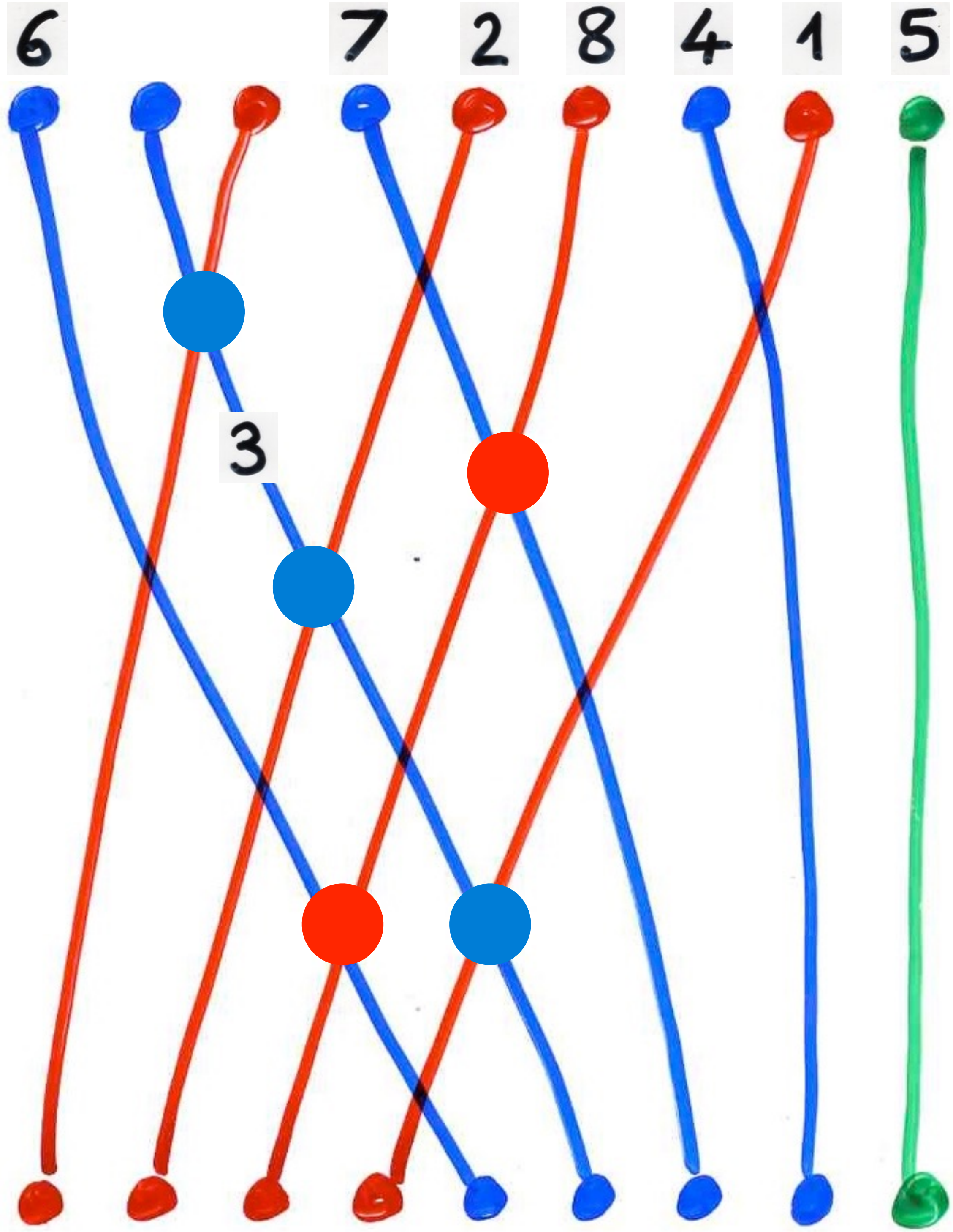
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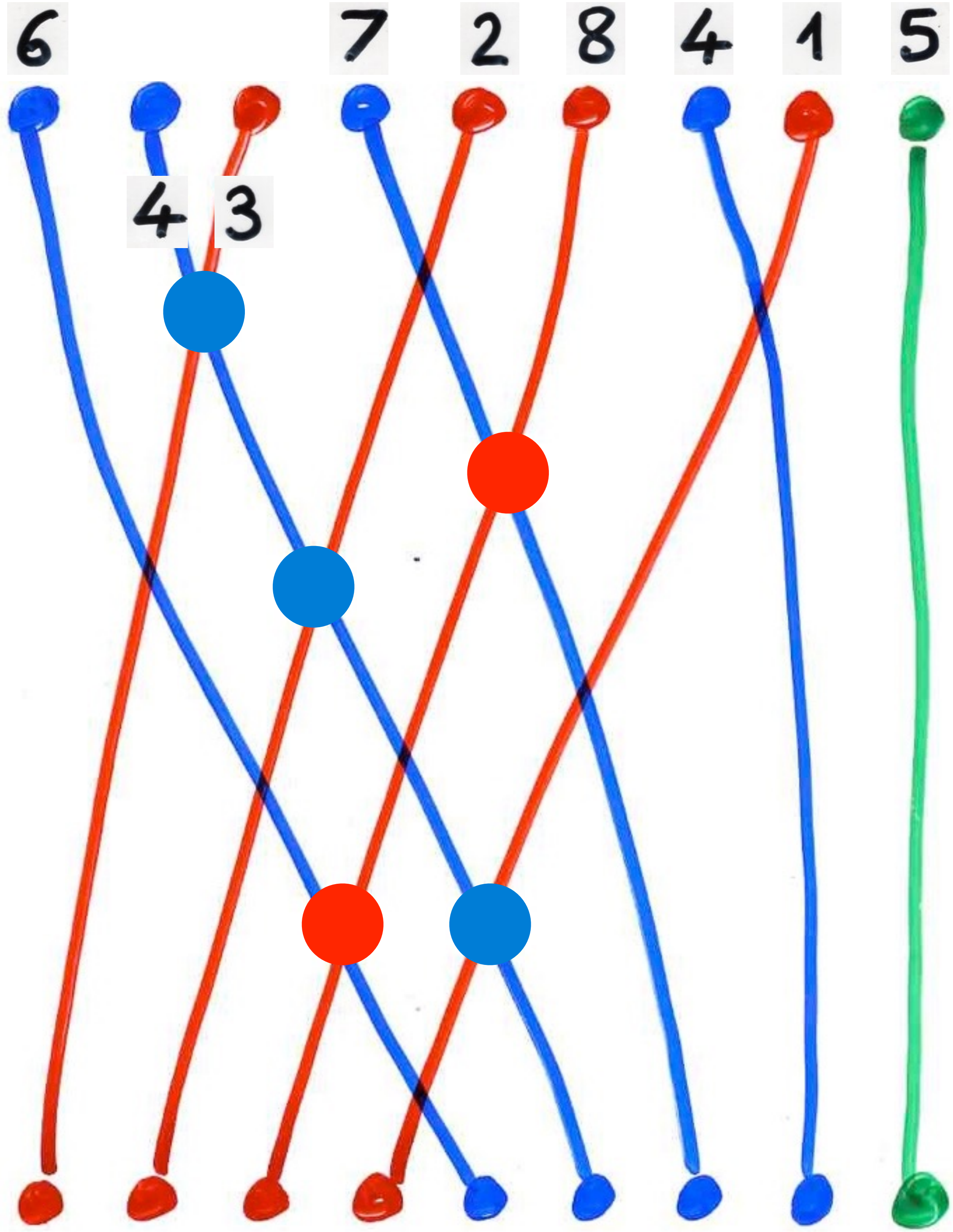
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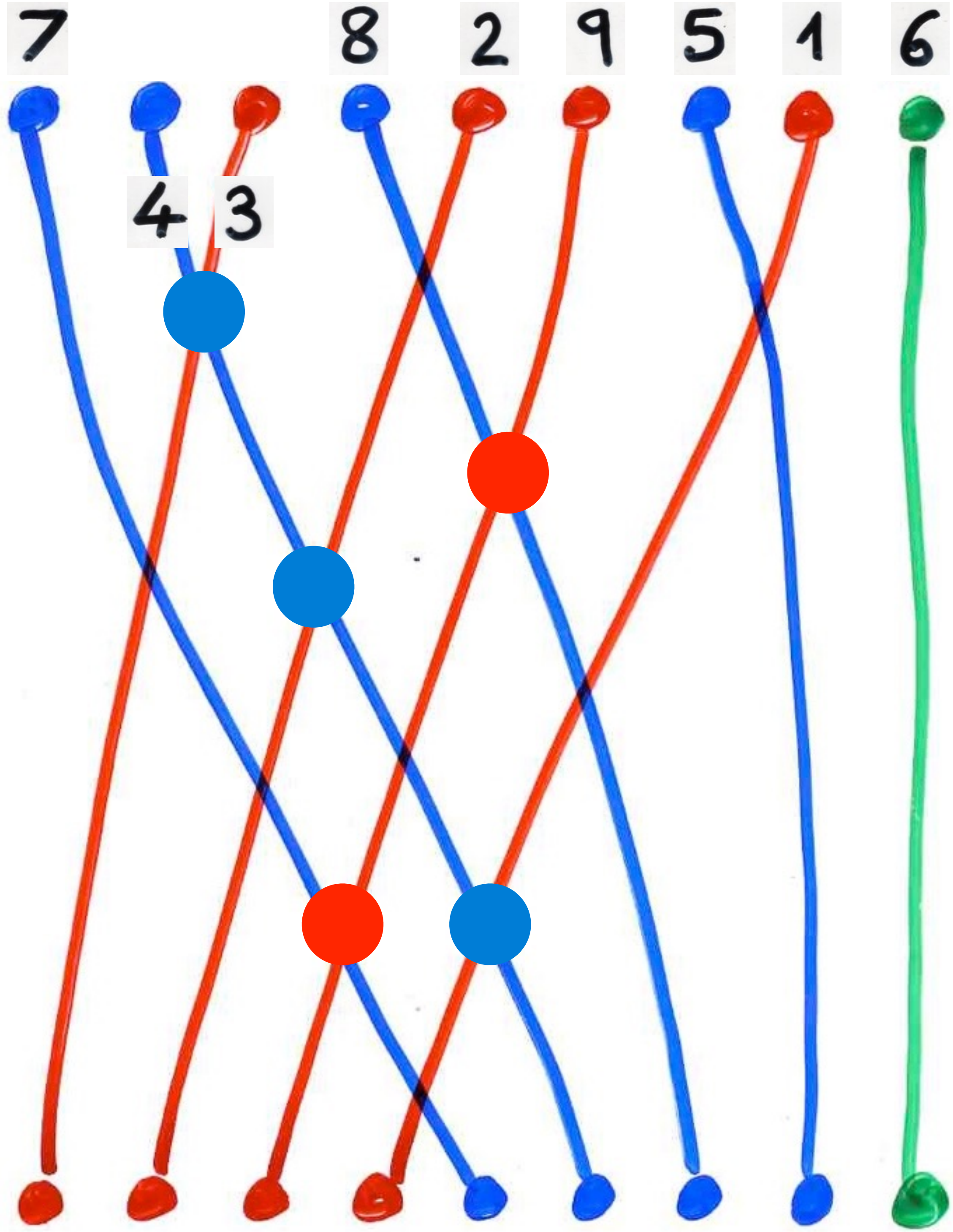
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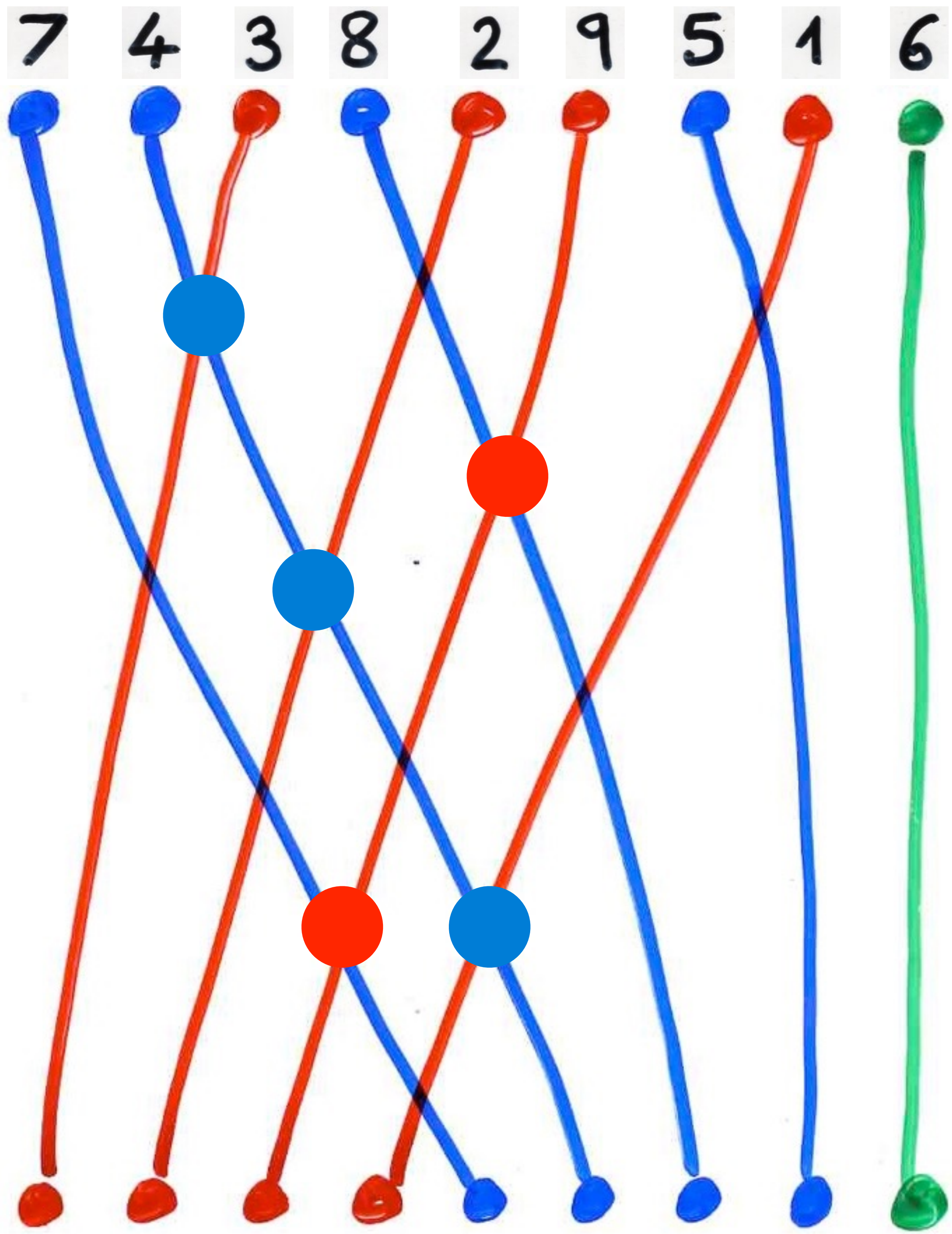

7 4 3 8 2 9 5 1 6

commutation diagrams

$$AS = SA + I_v J + K I_h$$

$$AK = KA + I_v A$$

$$JS = SJ + S I_h$$

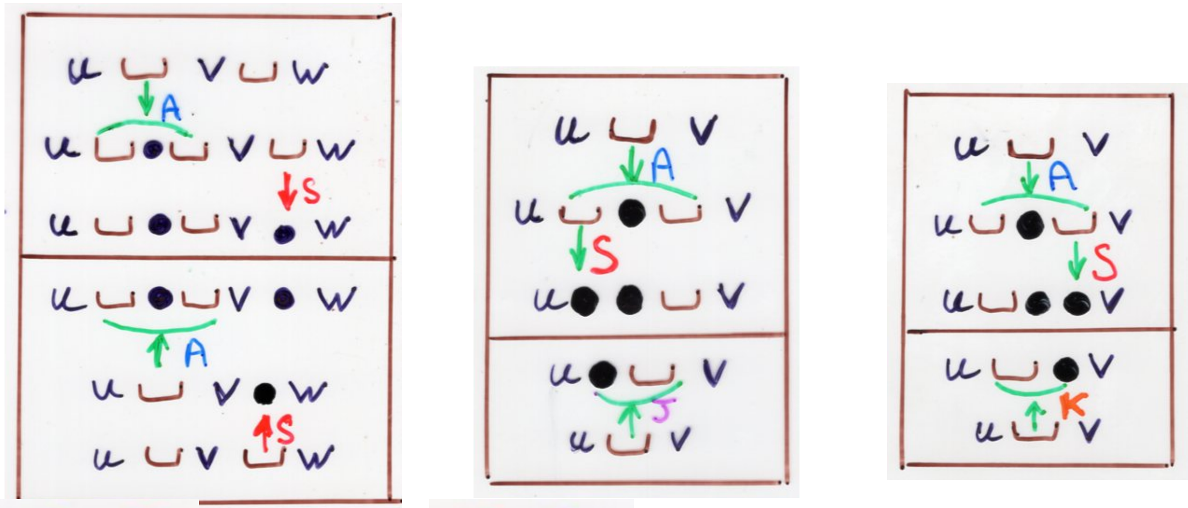
$$JK = KJ$$

$$A I_v = I_v A$$

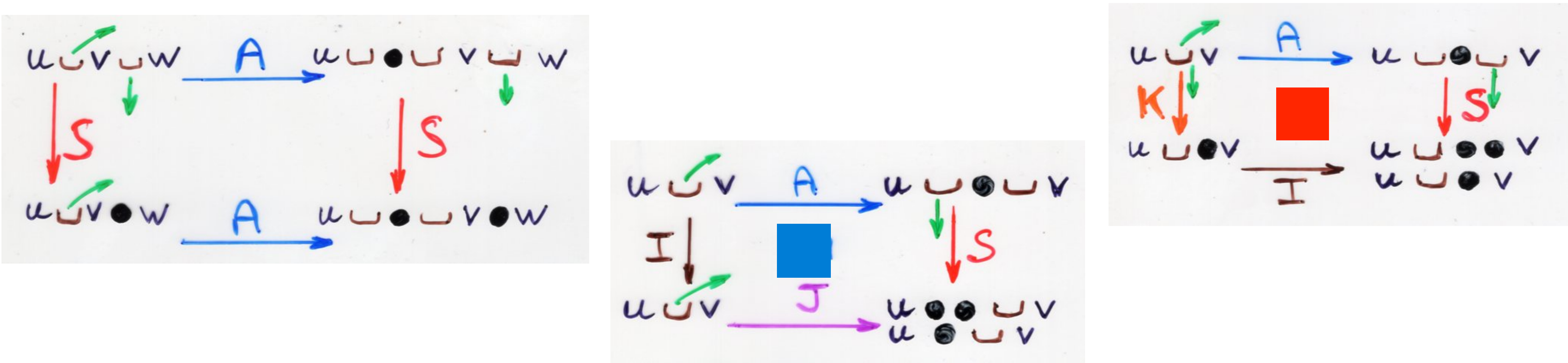
$$J I_v = I_v J$$

$$I_h S = S I_h$$

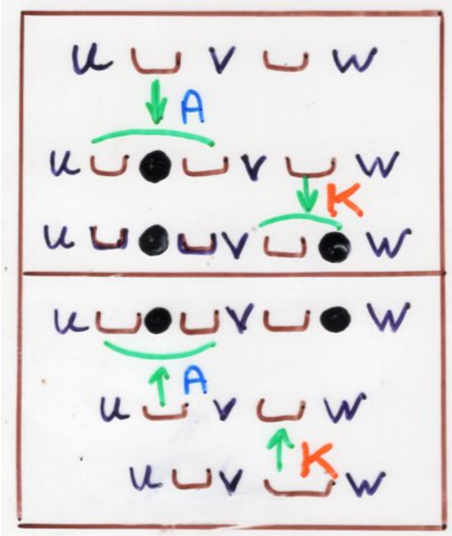
$$I_h K = K I_h$$

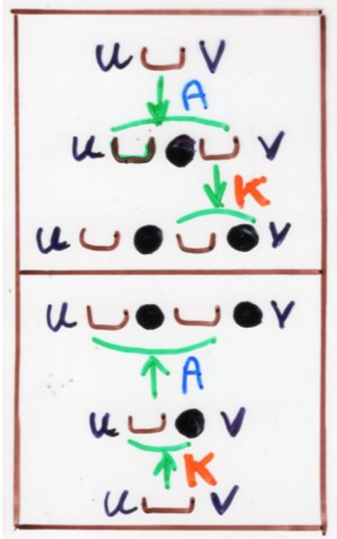


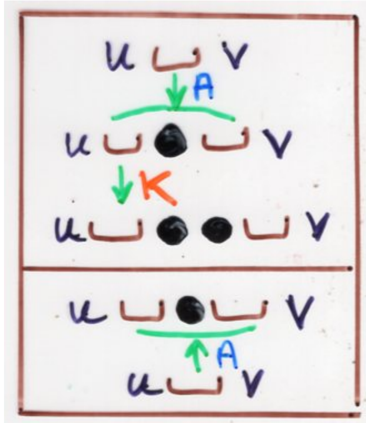
$$AS = SA + J + K$$



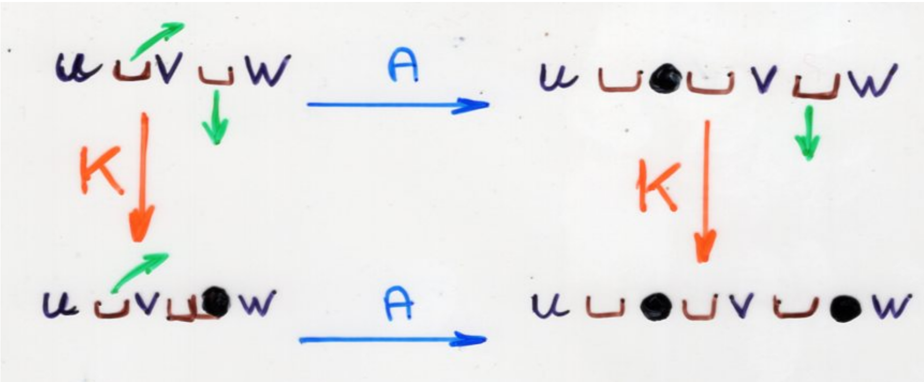
$$AS = SA + I_v J + K I_h$$

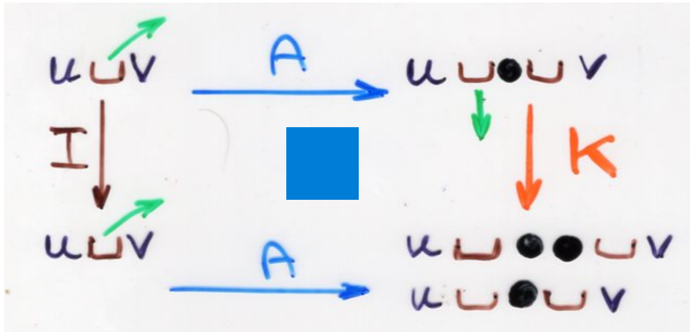


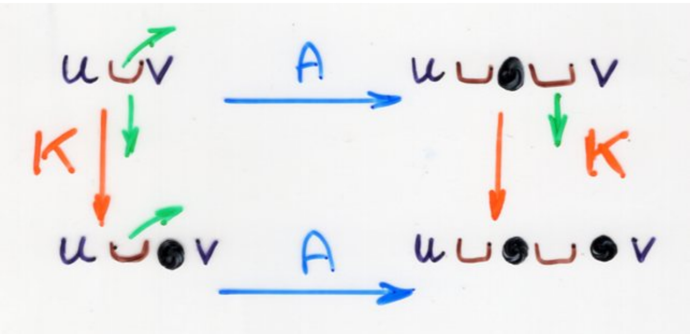




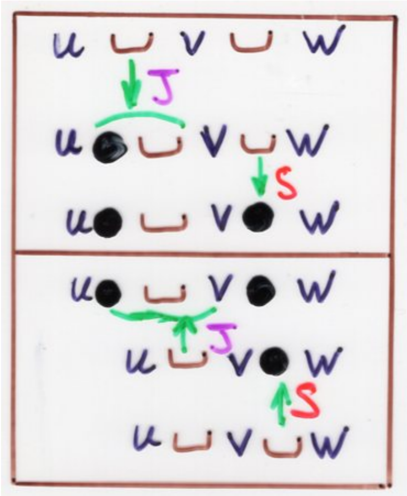
$$AK = KA + A$$

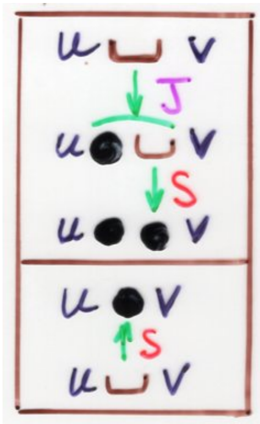




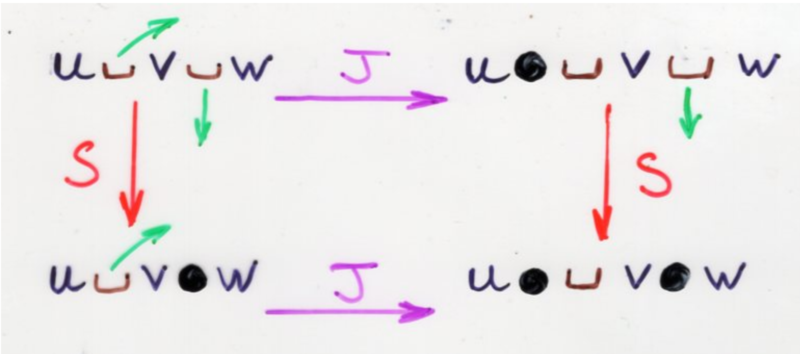


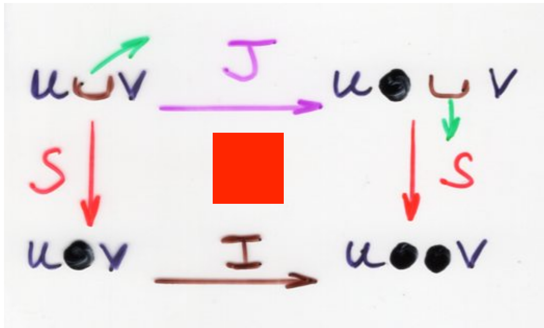
$$AK = KA + I_v A$$



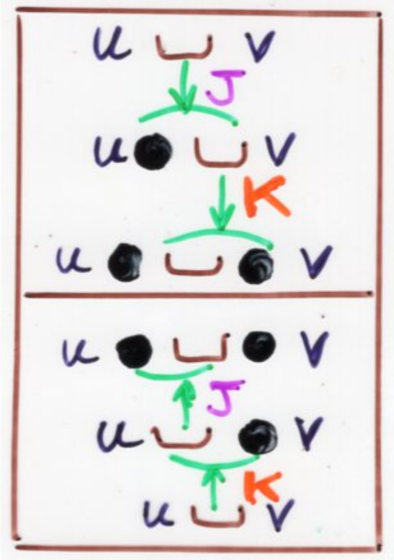


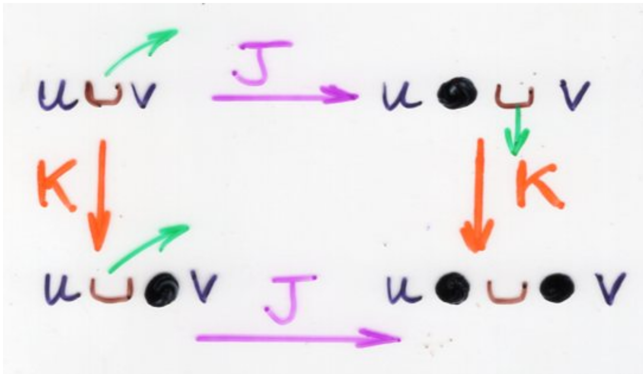
$$JS = SJ + S$$





$$JS = SJ + SI_h$$





$$JK = KJ$$

$$JK = KJ$$

$$A\,I_v = I_v\,A$$

$$J\,I_v = I_v\,J$$

$$I_h\,S = S\,I_h$$

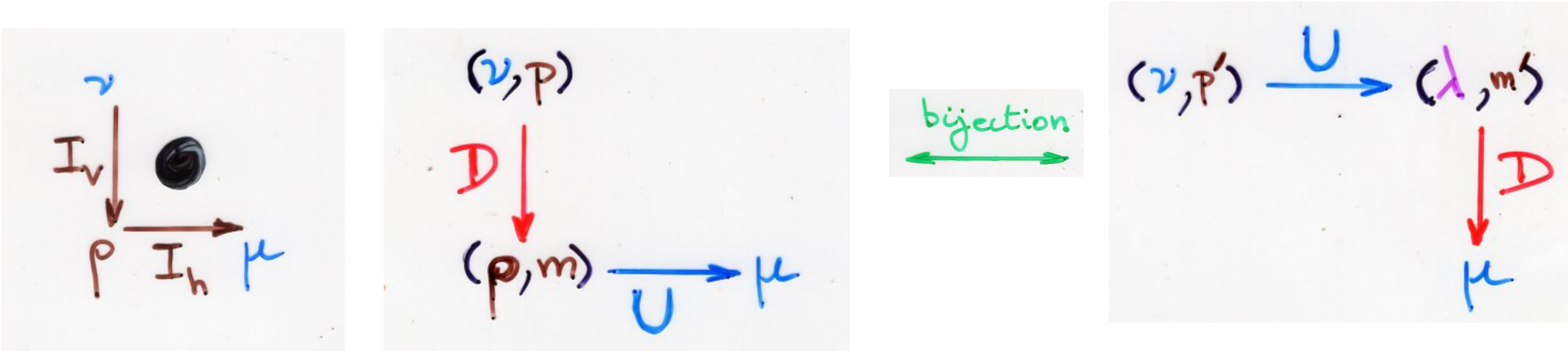
$$I_h\,K = K\,I_h$$

commutation diagrams bijections

analogy with commutation diagrams bijection
for the representation of the Weyl-Heisenberg algebra
(Ch2)

$$UD = DU + I_v I_h$$

"commutation diagrams"



p, m, p', m' are "positions"

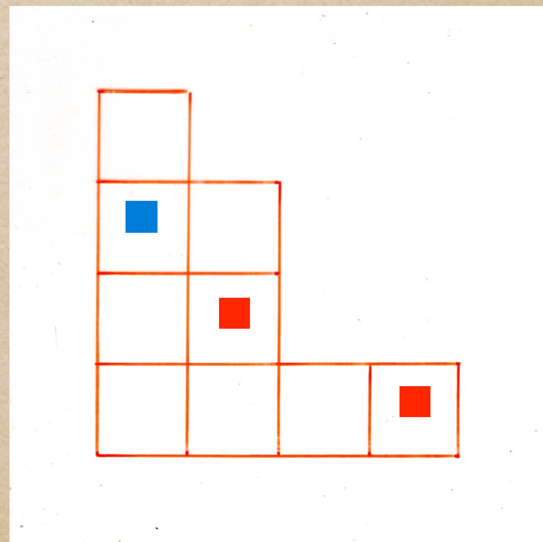
in ν, ρ, ν, λ respectively

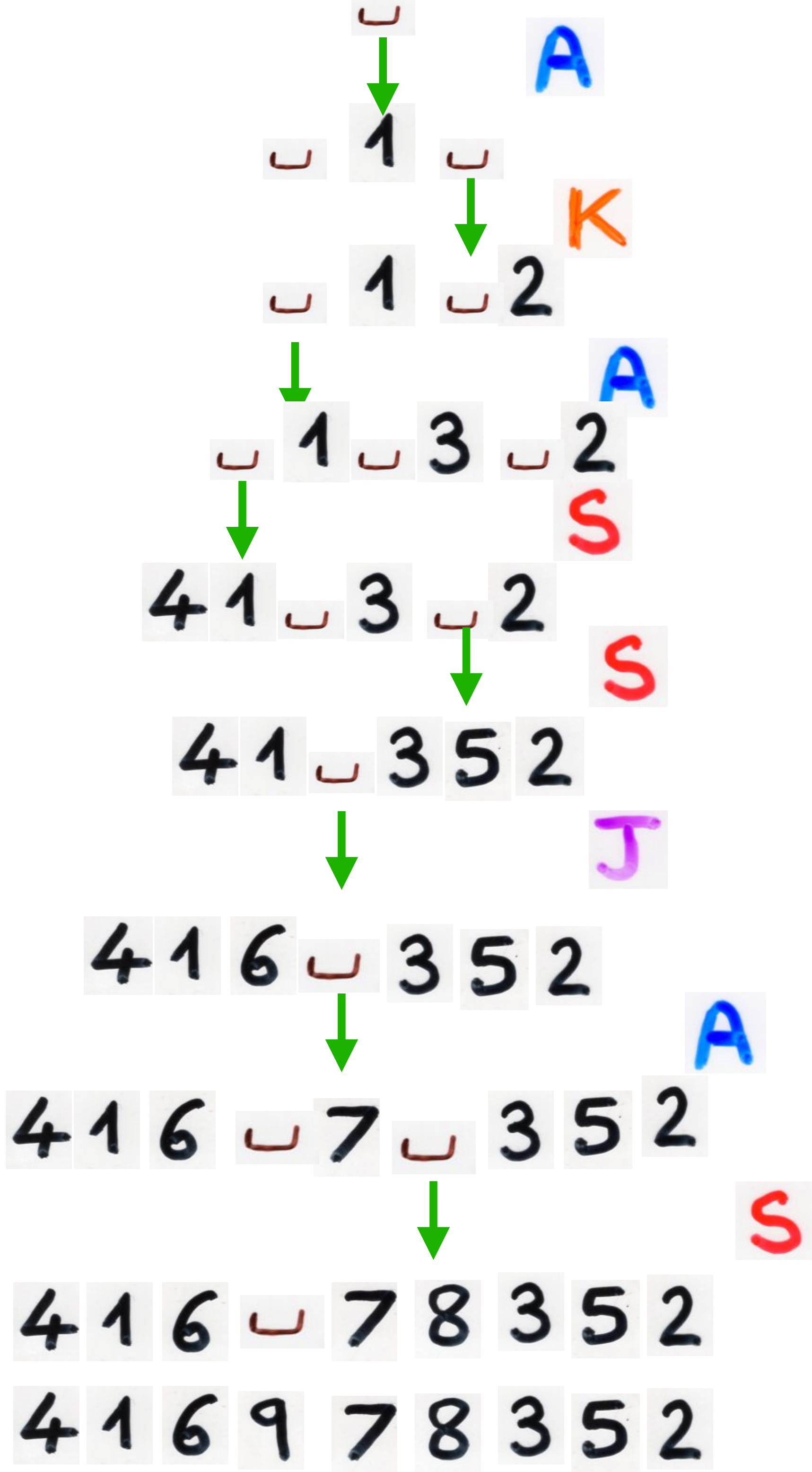
the bijection

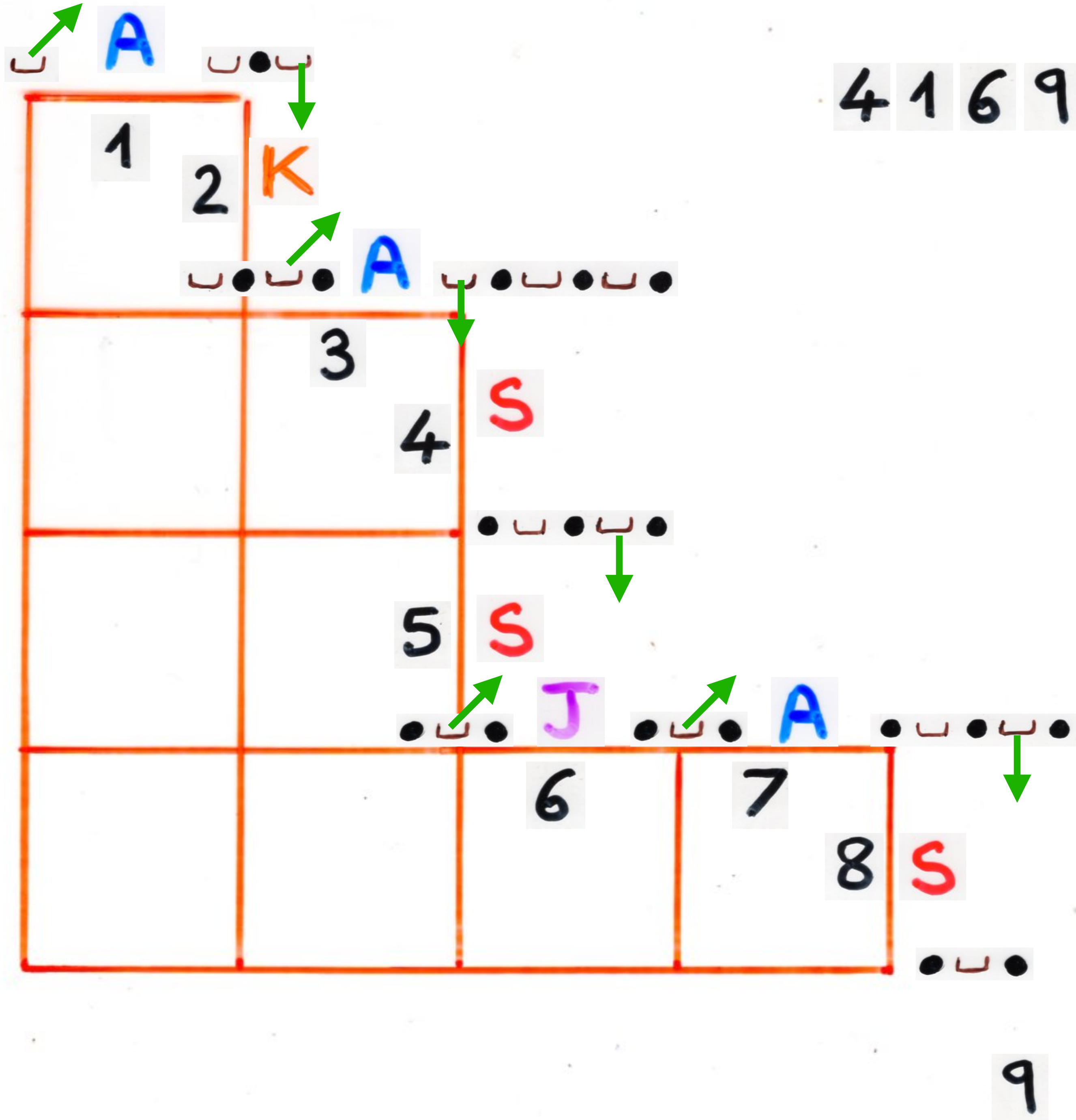
permutations — alternative tableaux
(Laguerre histories)

with local rules
(commutation diagrams)

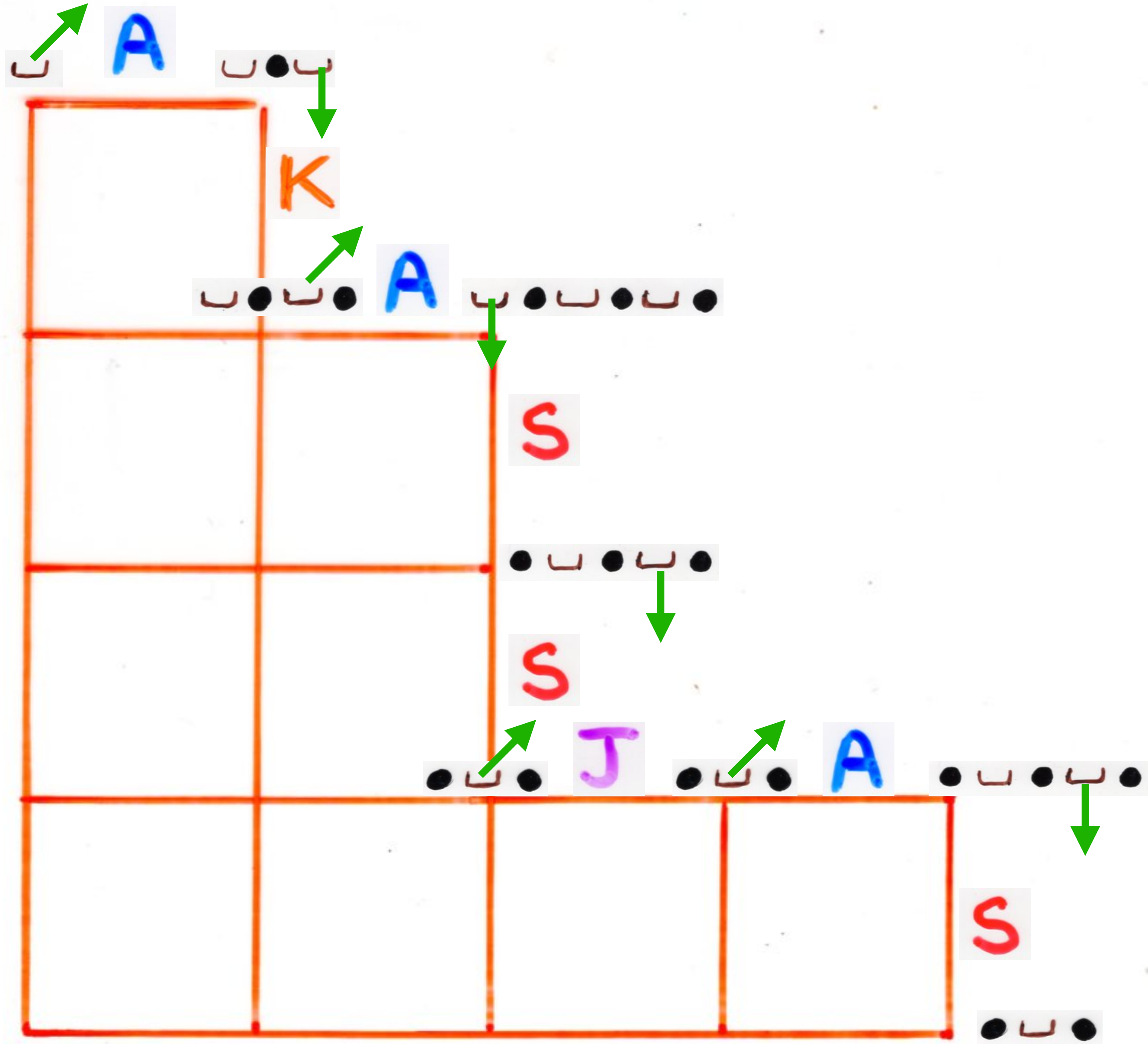
4 1 6 9 7 8 3 5 2 →




␣
A
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2



4 1 6 9 7 8 3 5 2


A
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J
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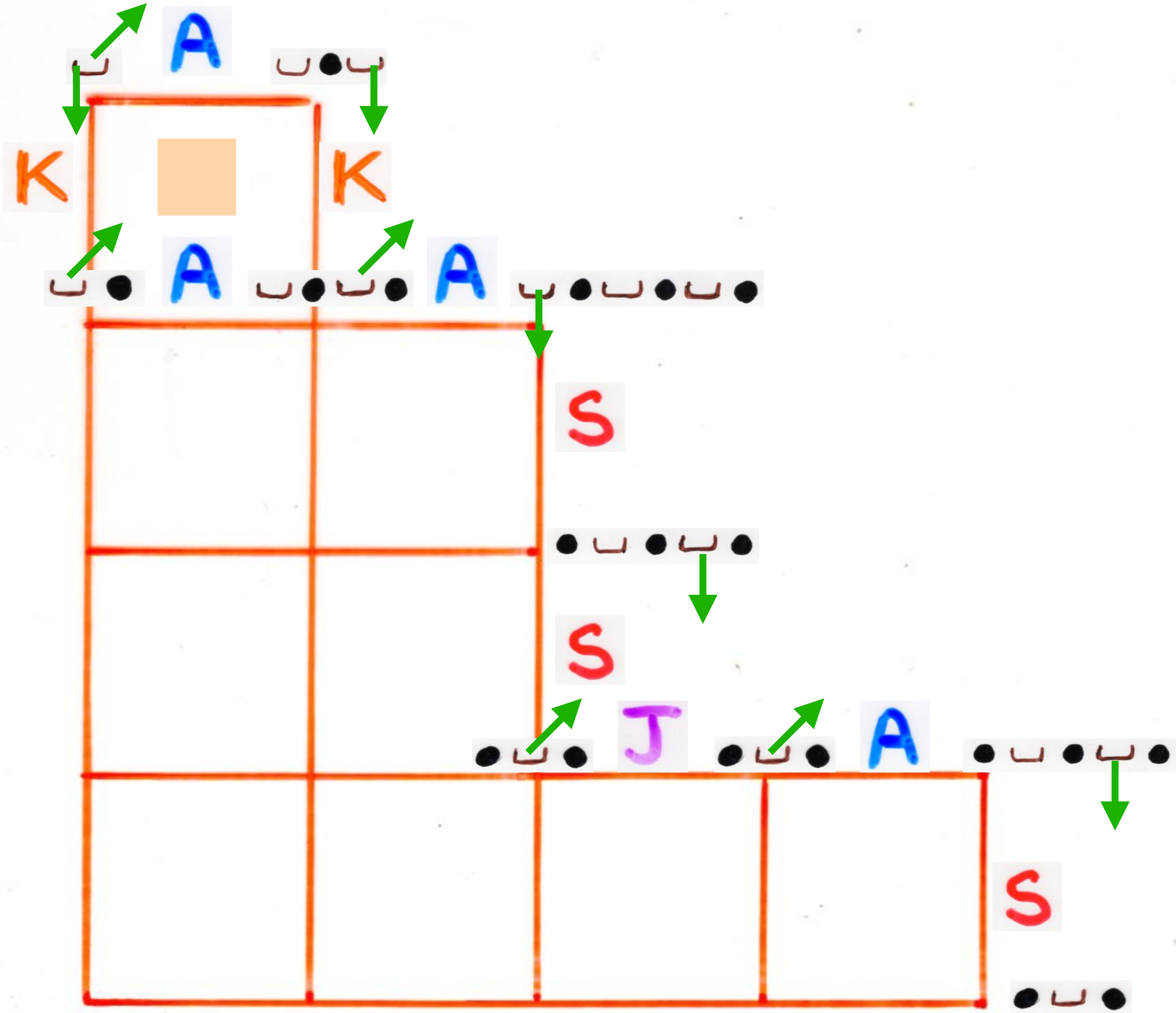
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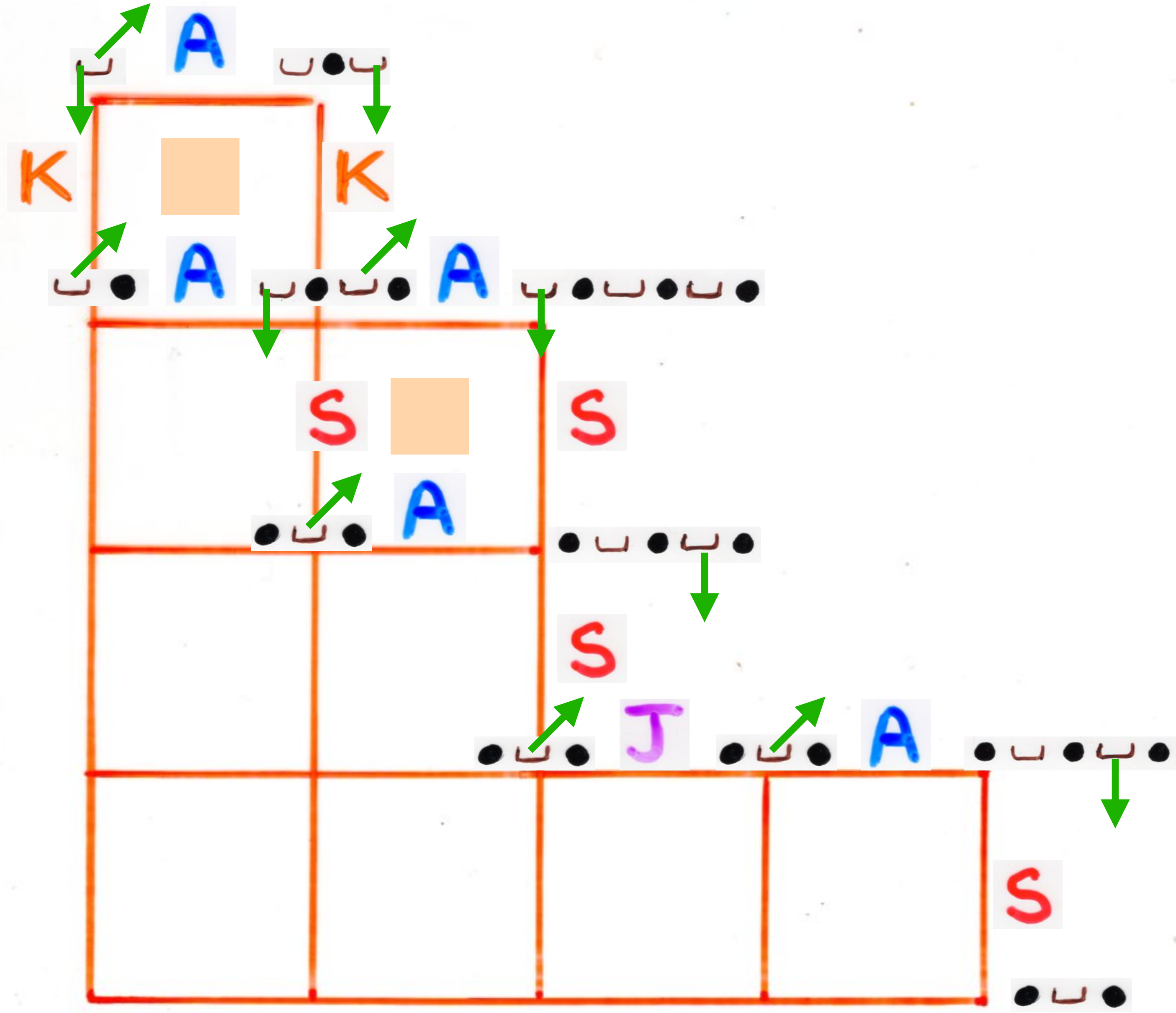
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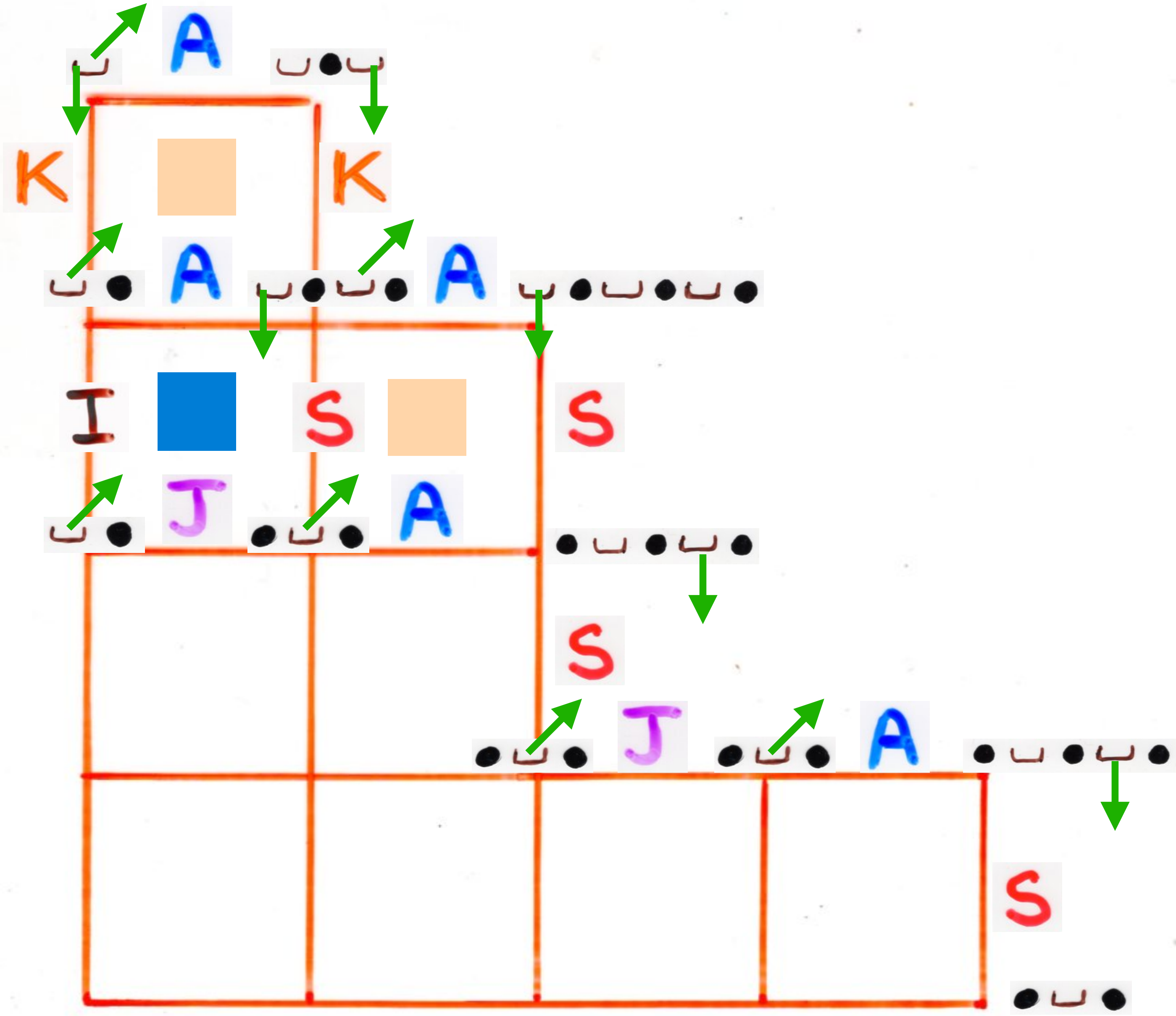
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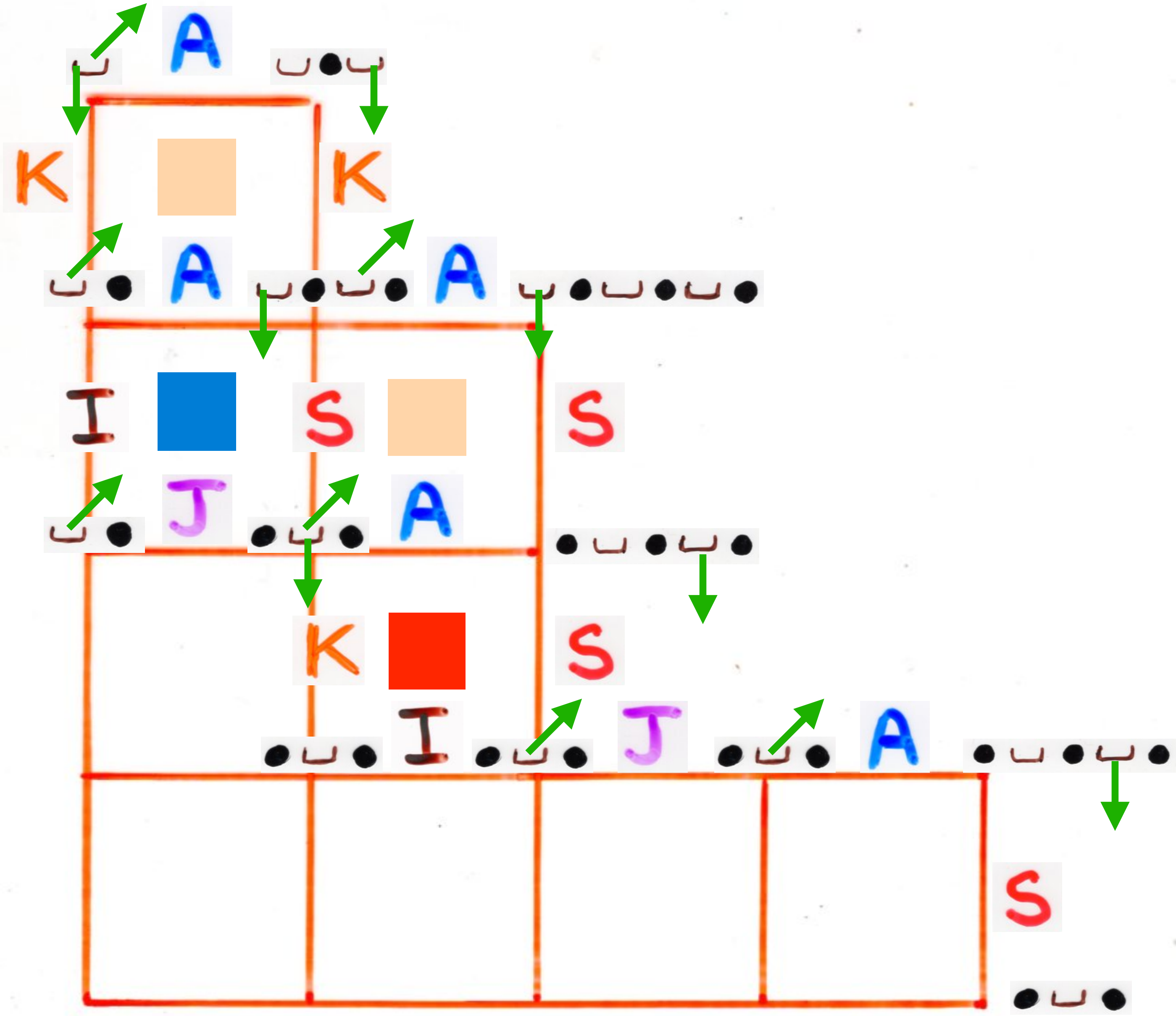
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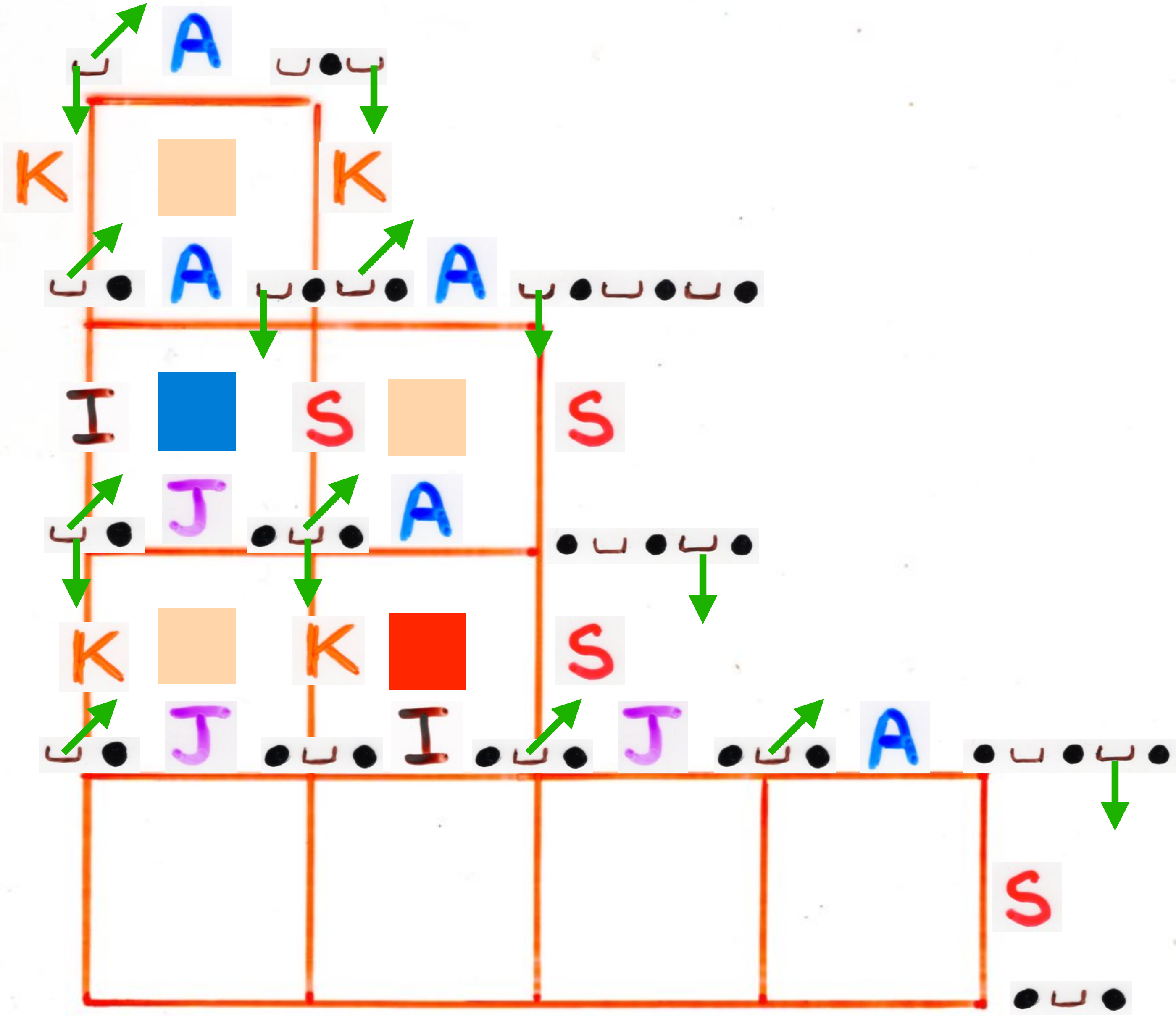
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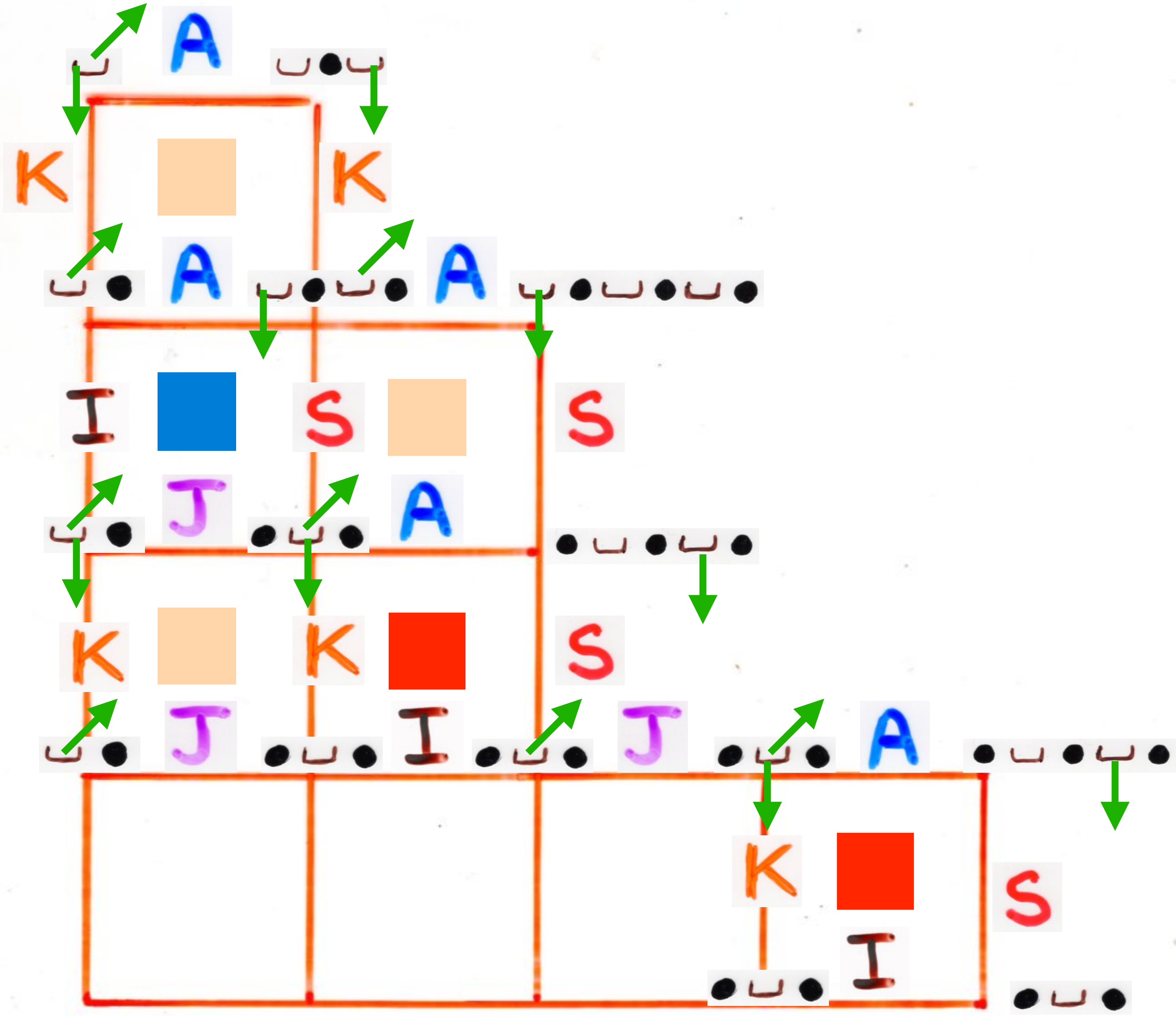
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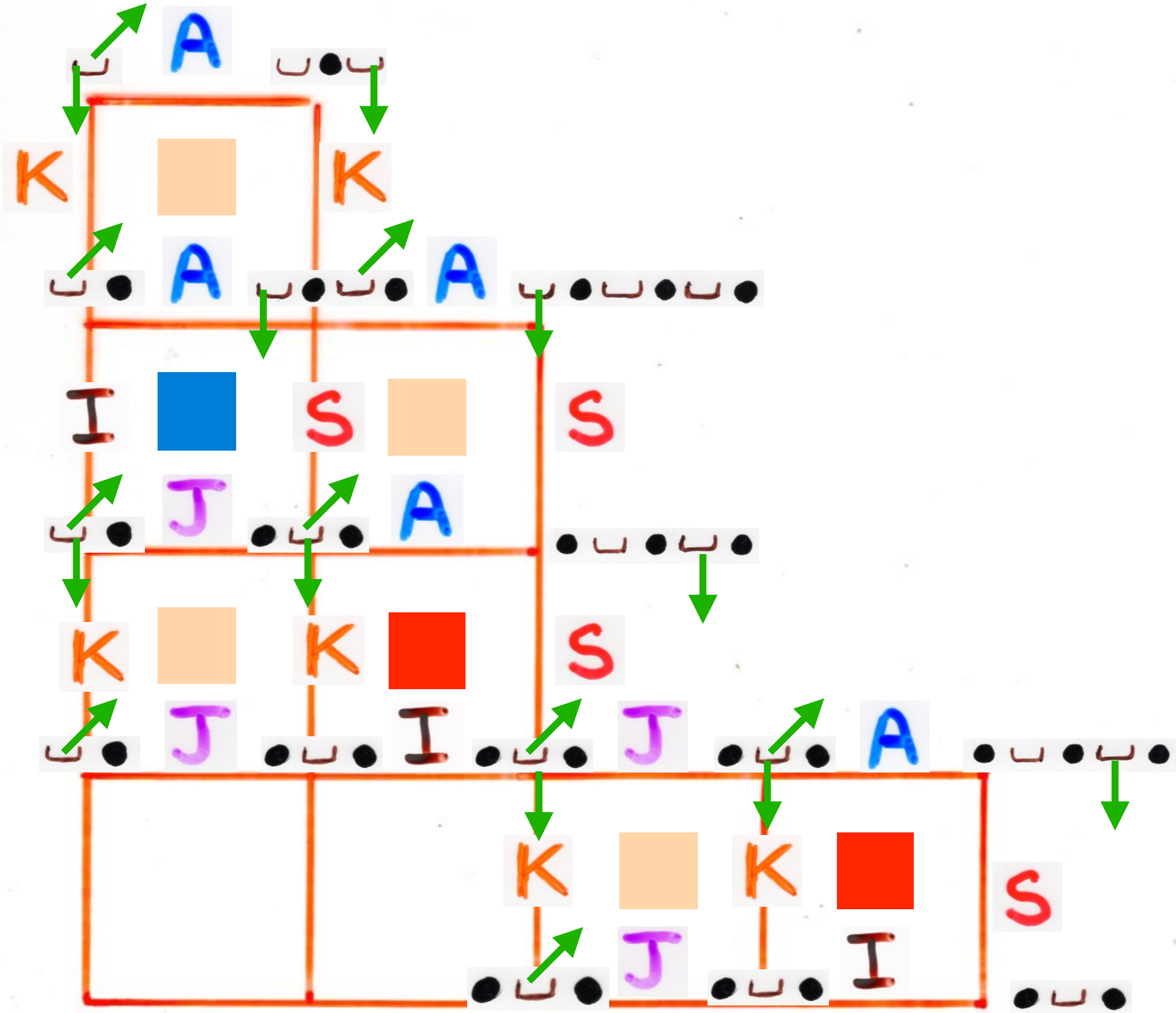
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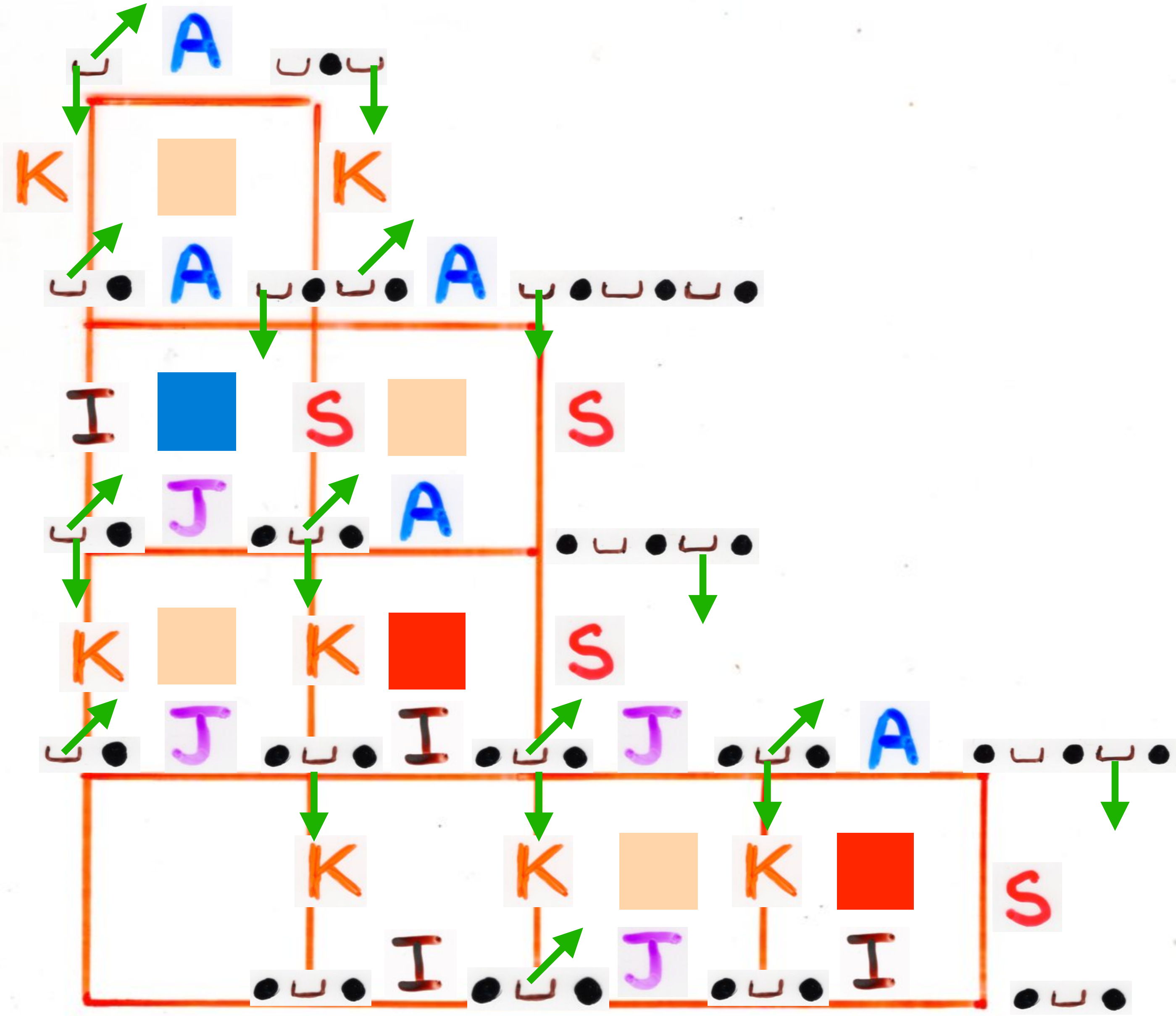
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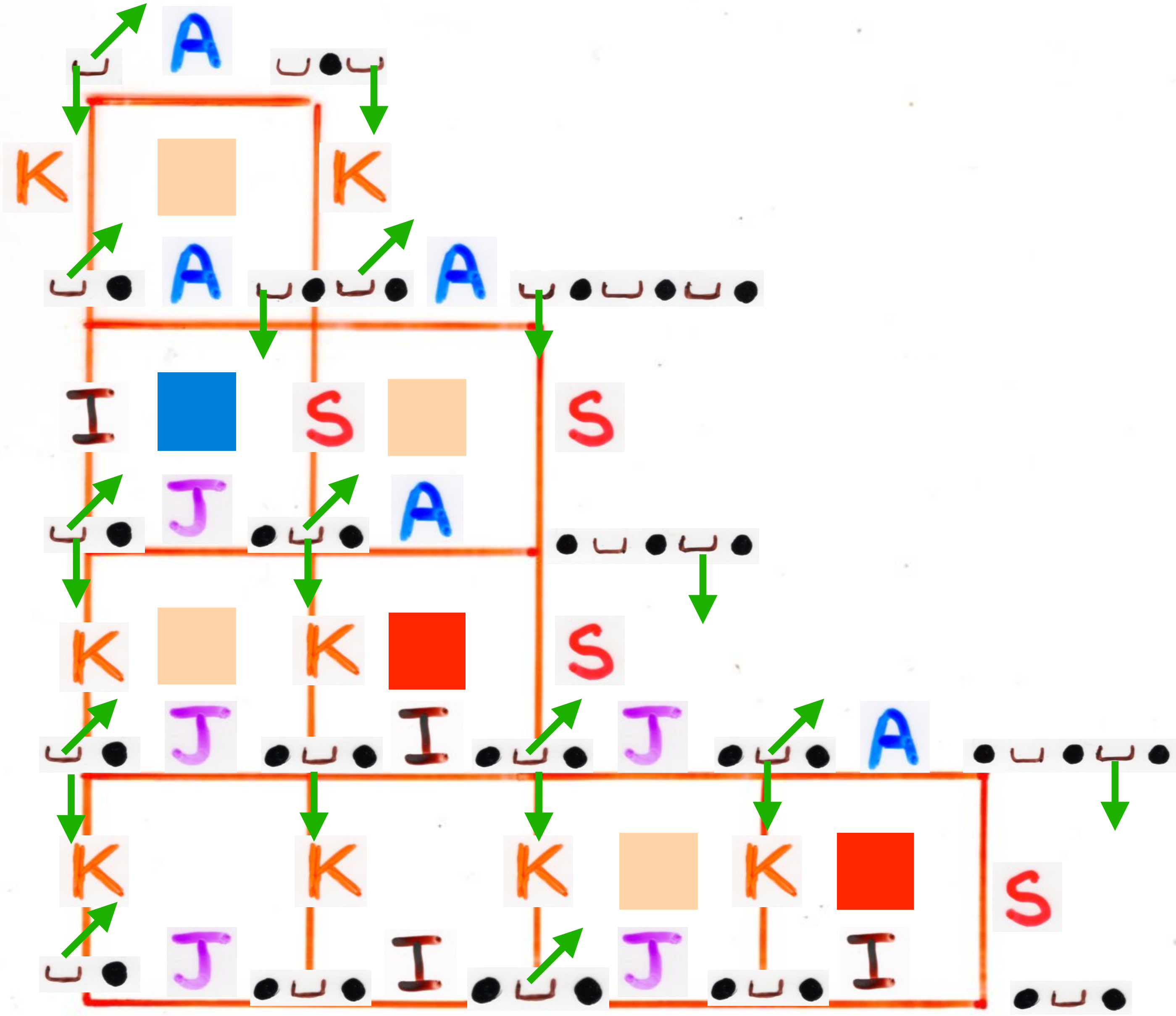
K K K S
I J I



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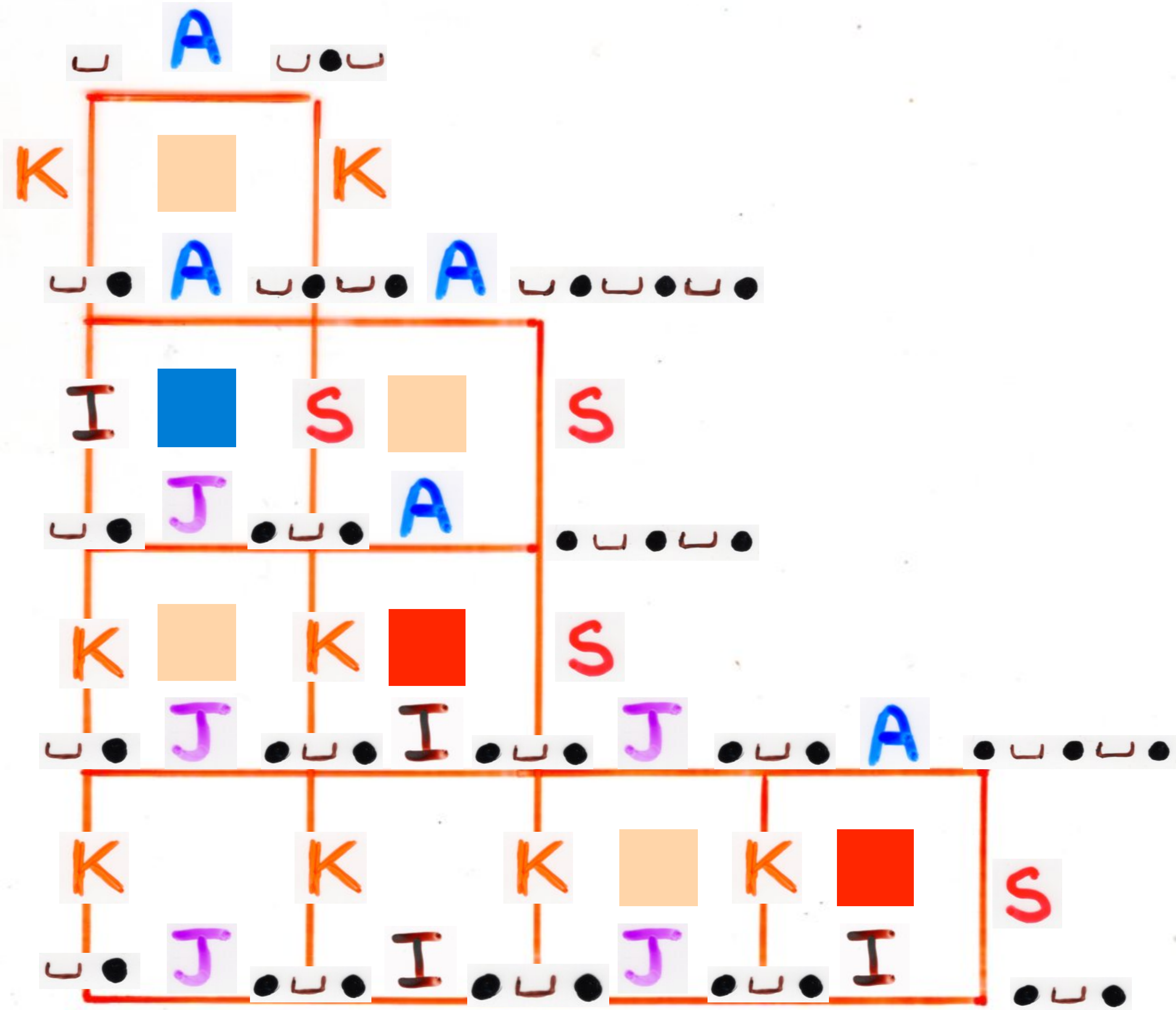

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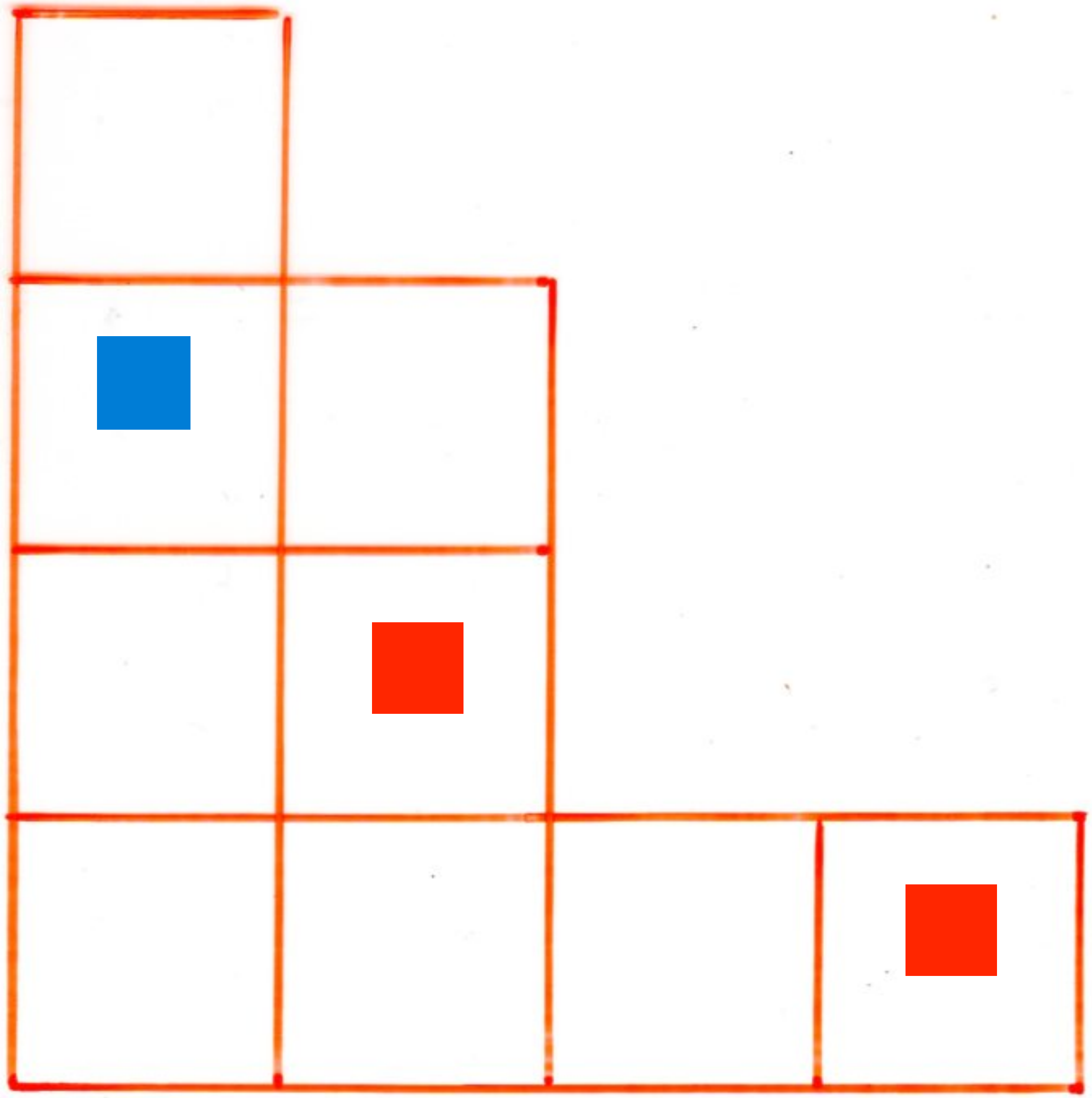


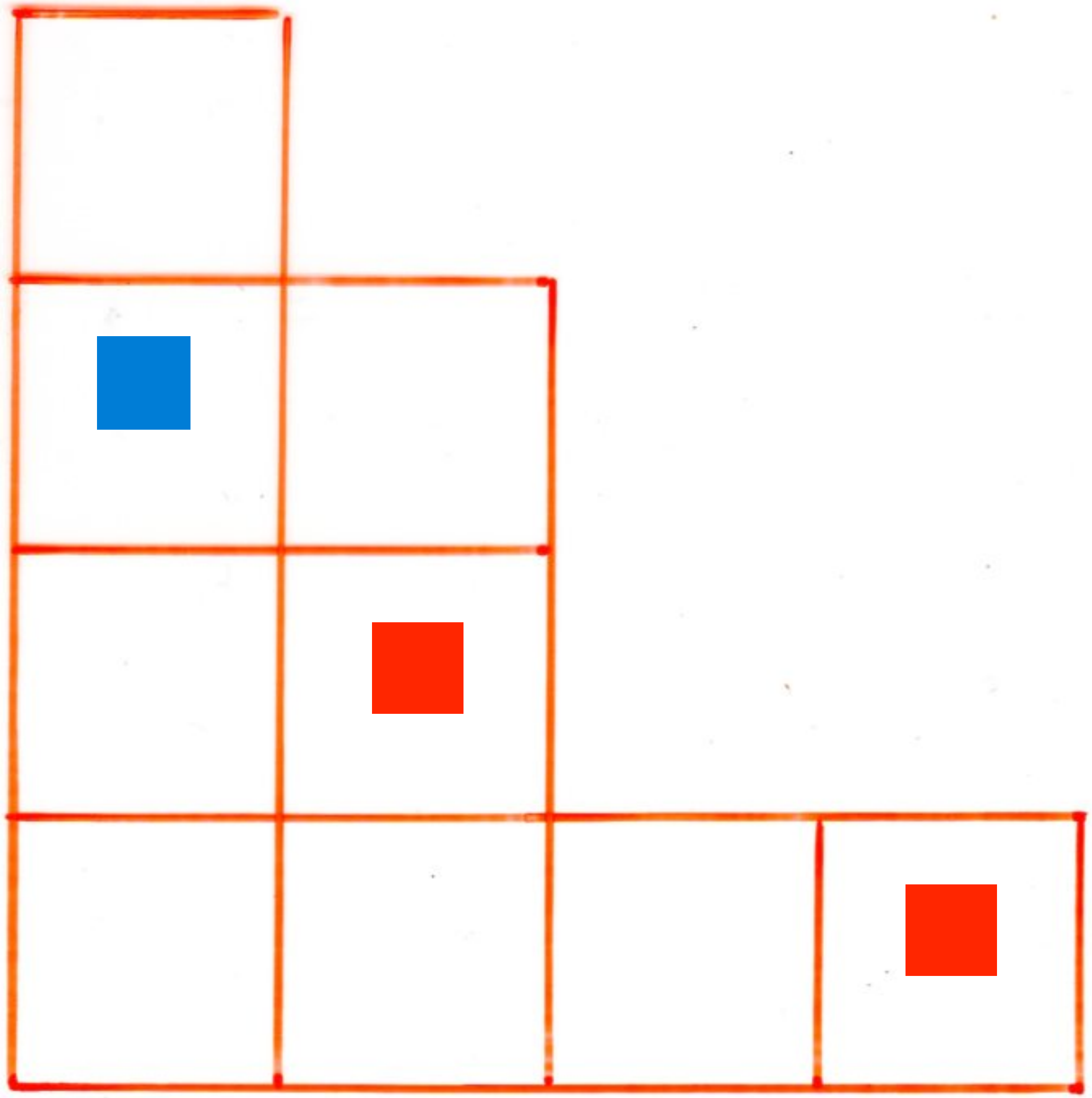
the reverse bijection

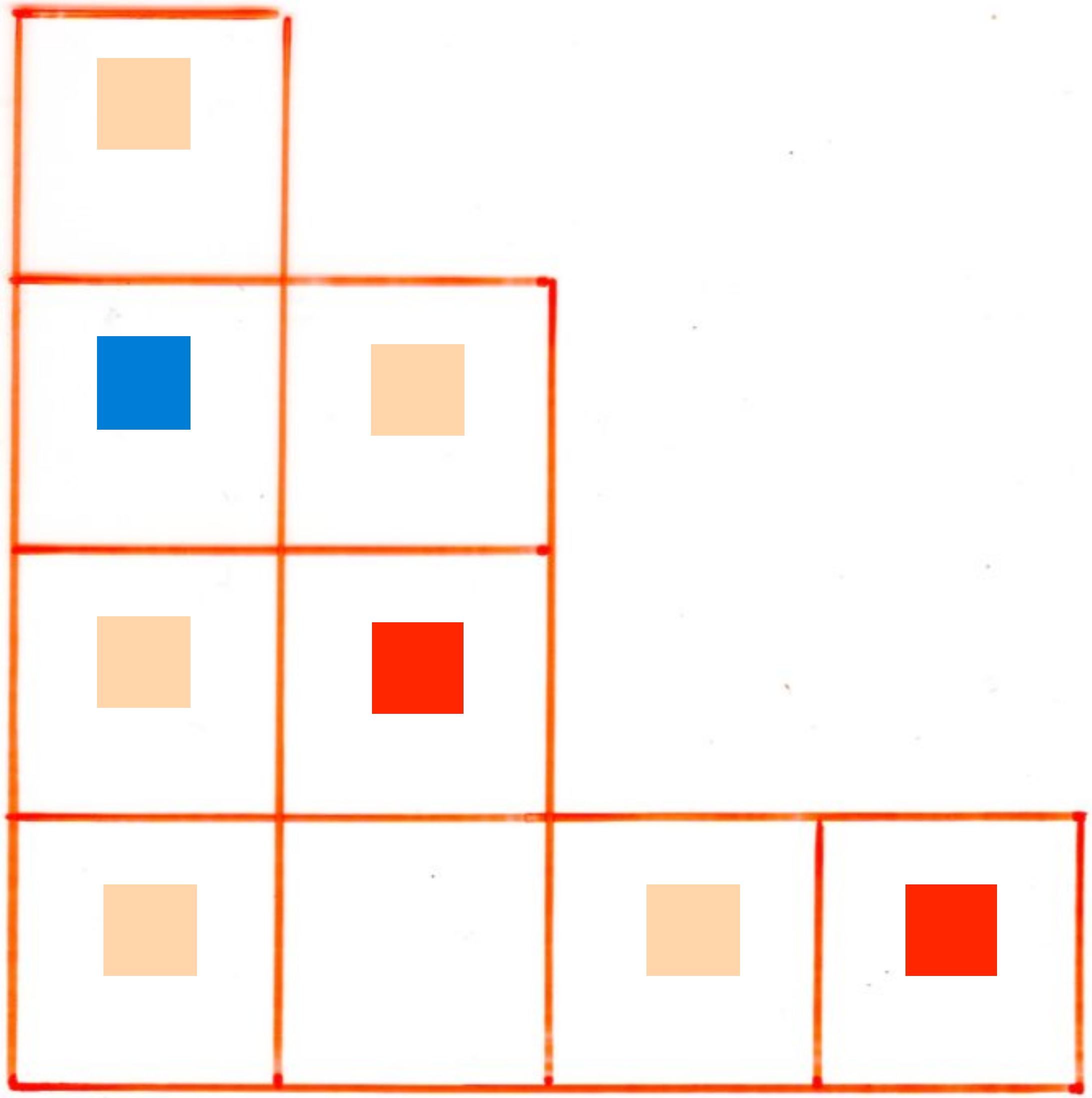
permutations — alternative tableaux

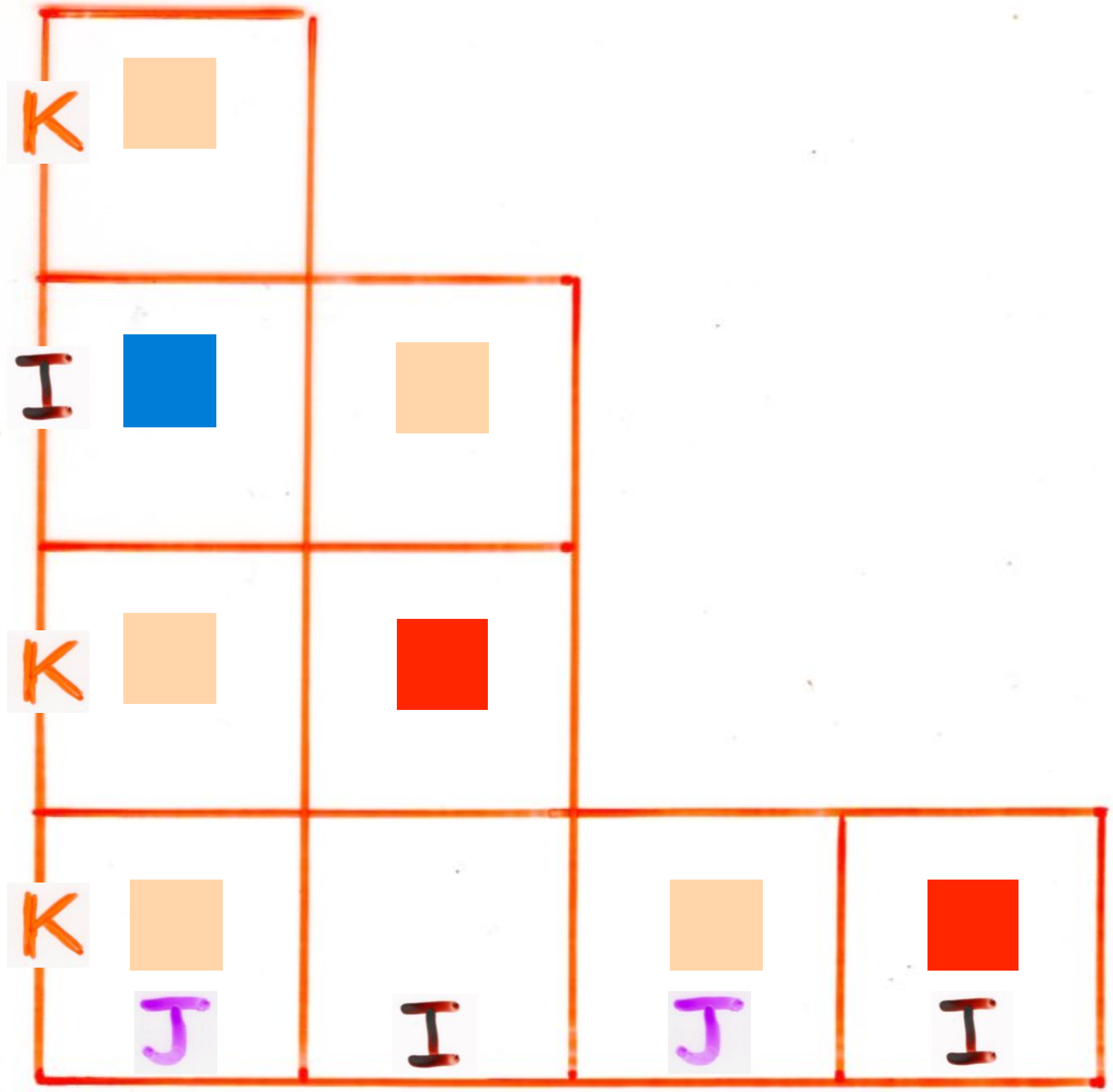
(Laguerre histories)

local rules

(commutation diagrams)






K
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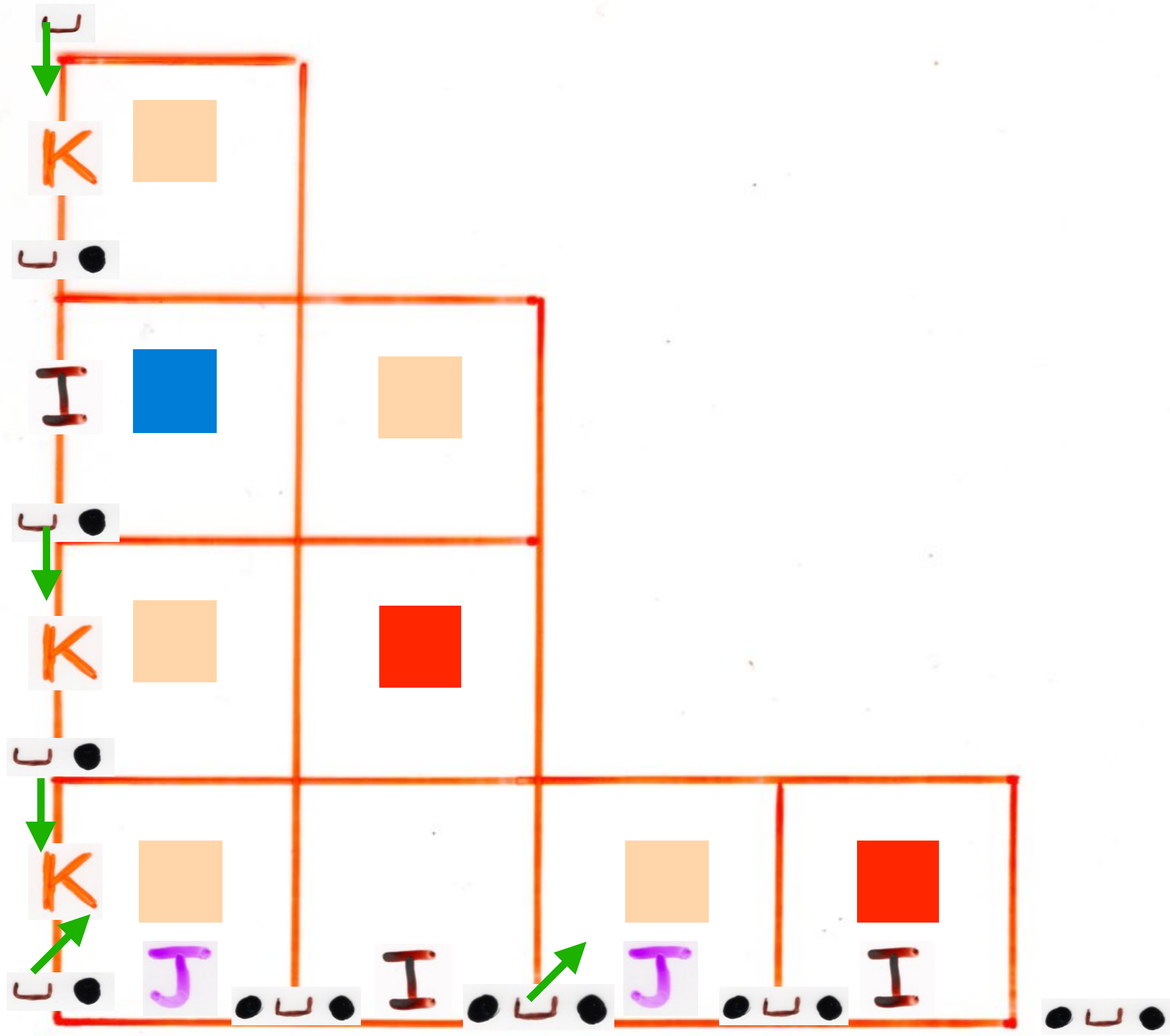
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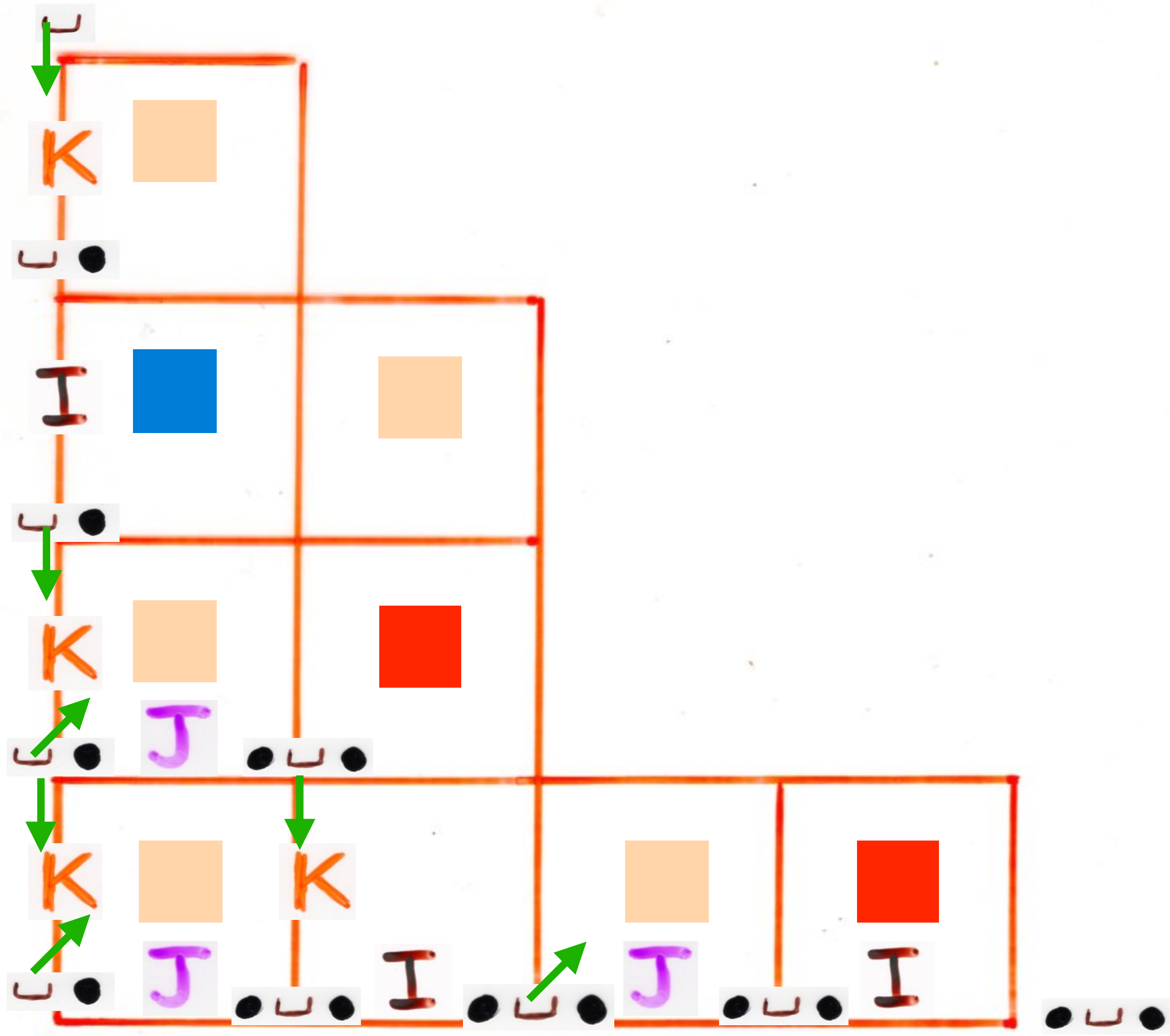
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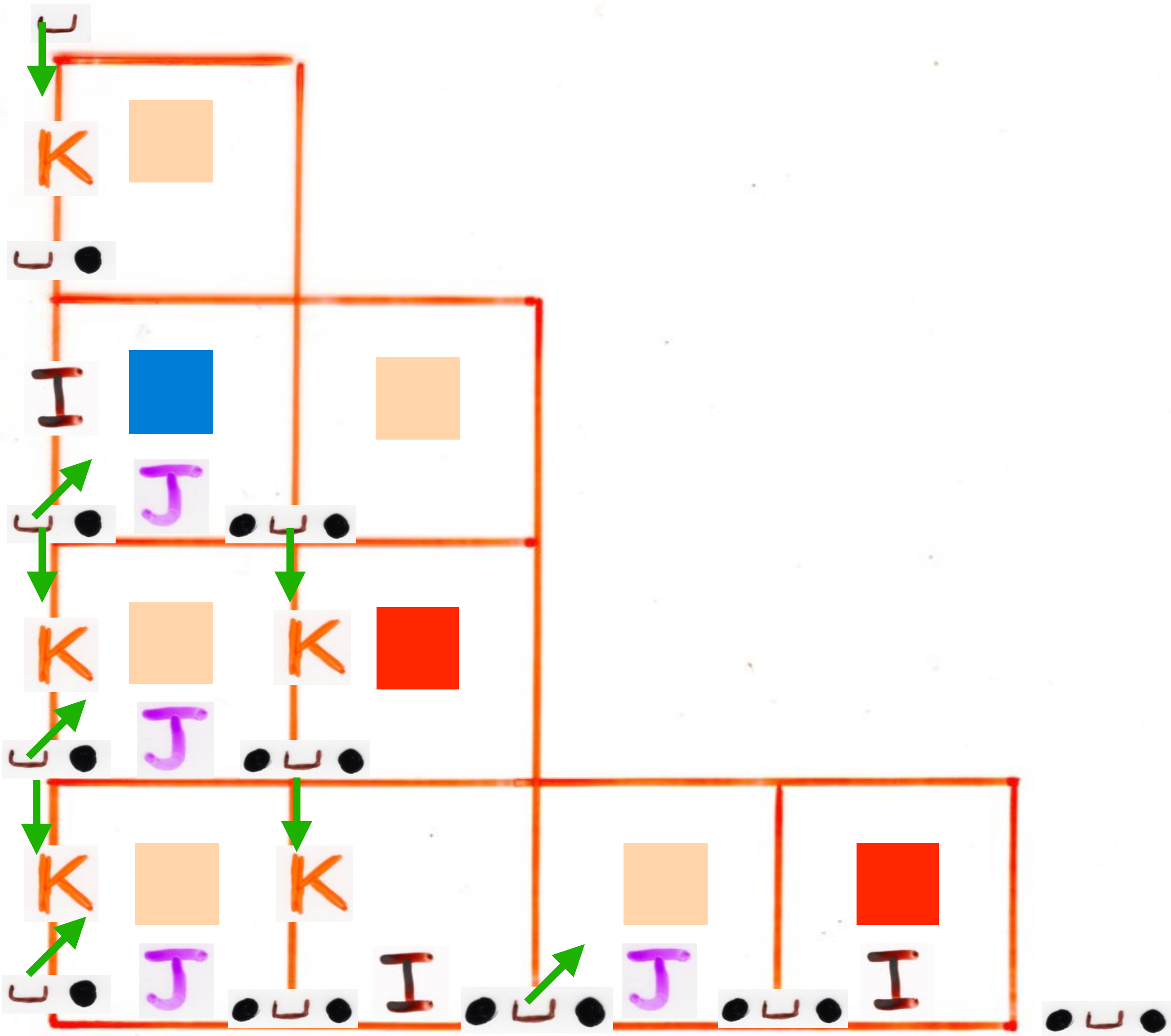
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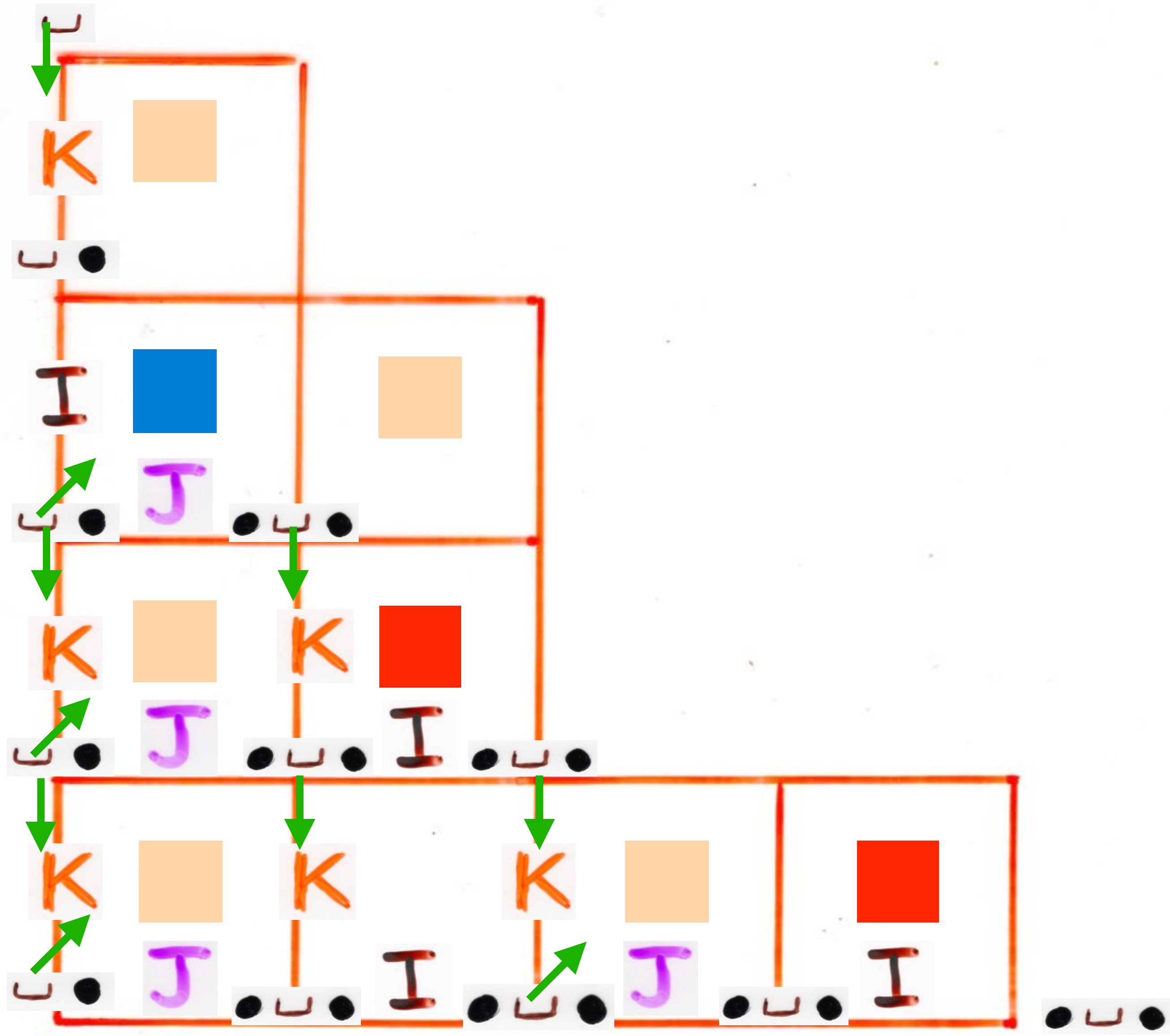

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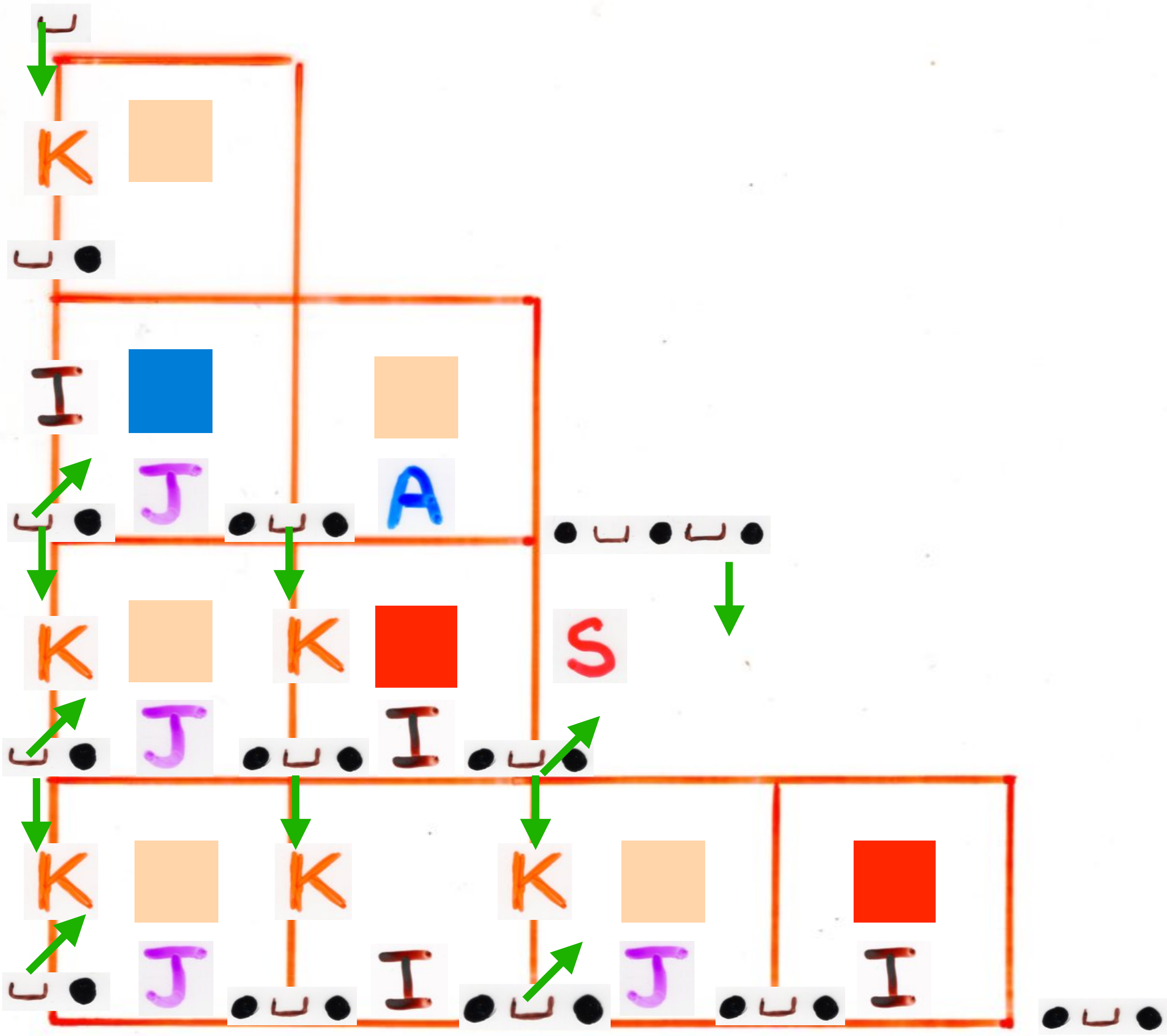
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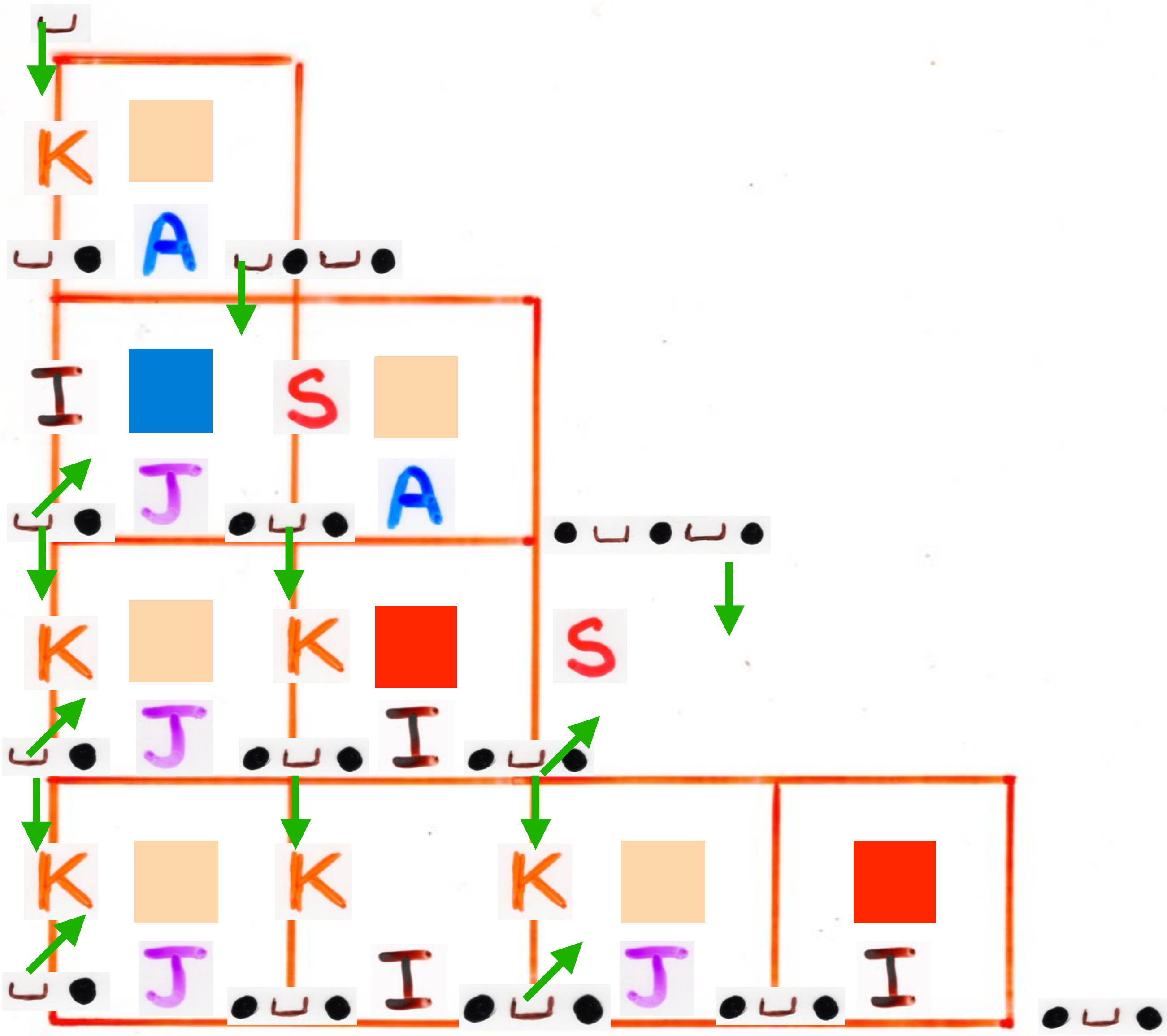
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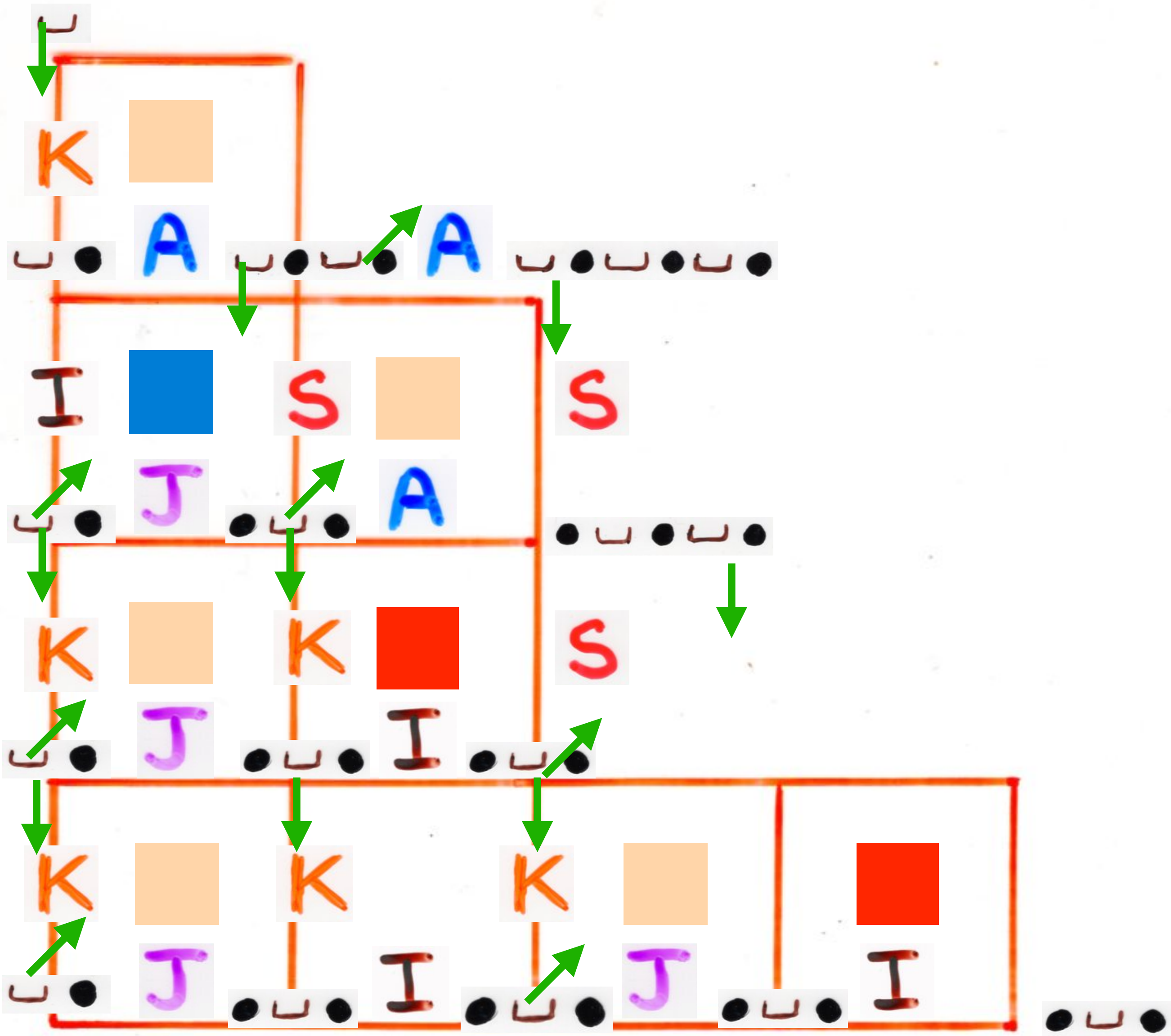

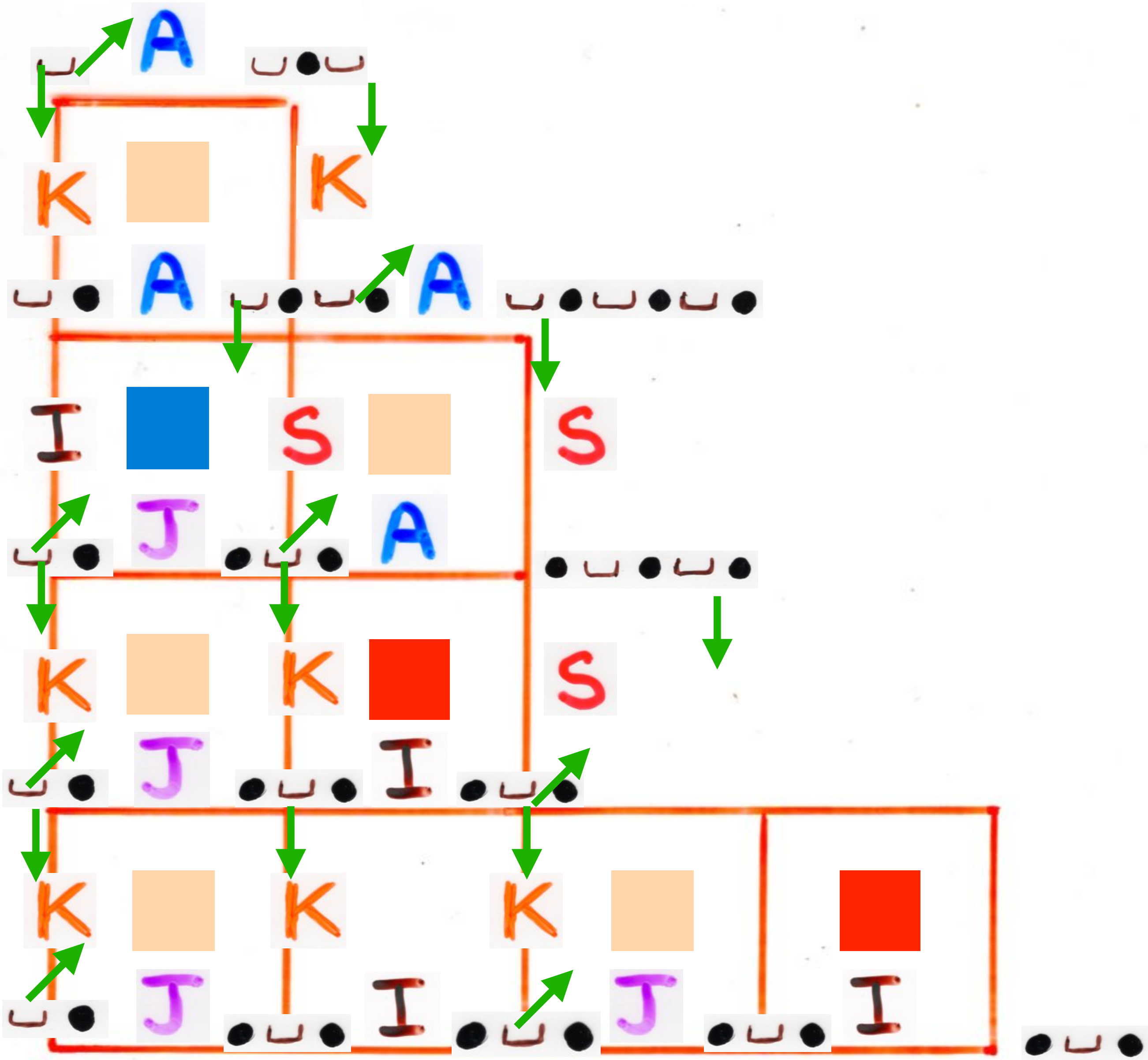
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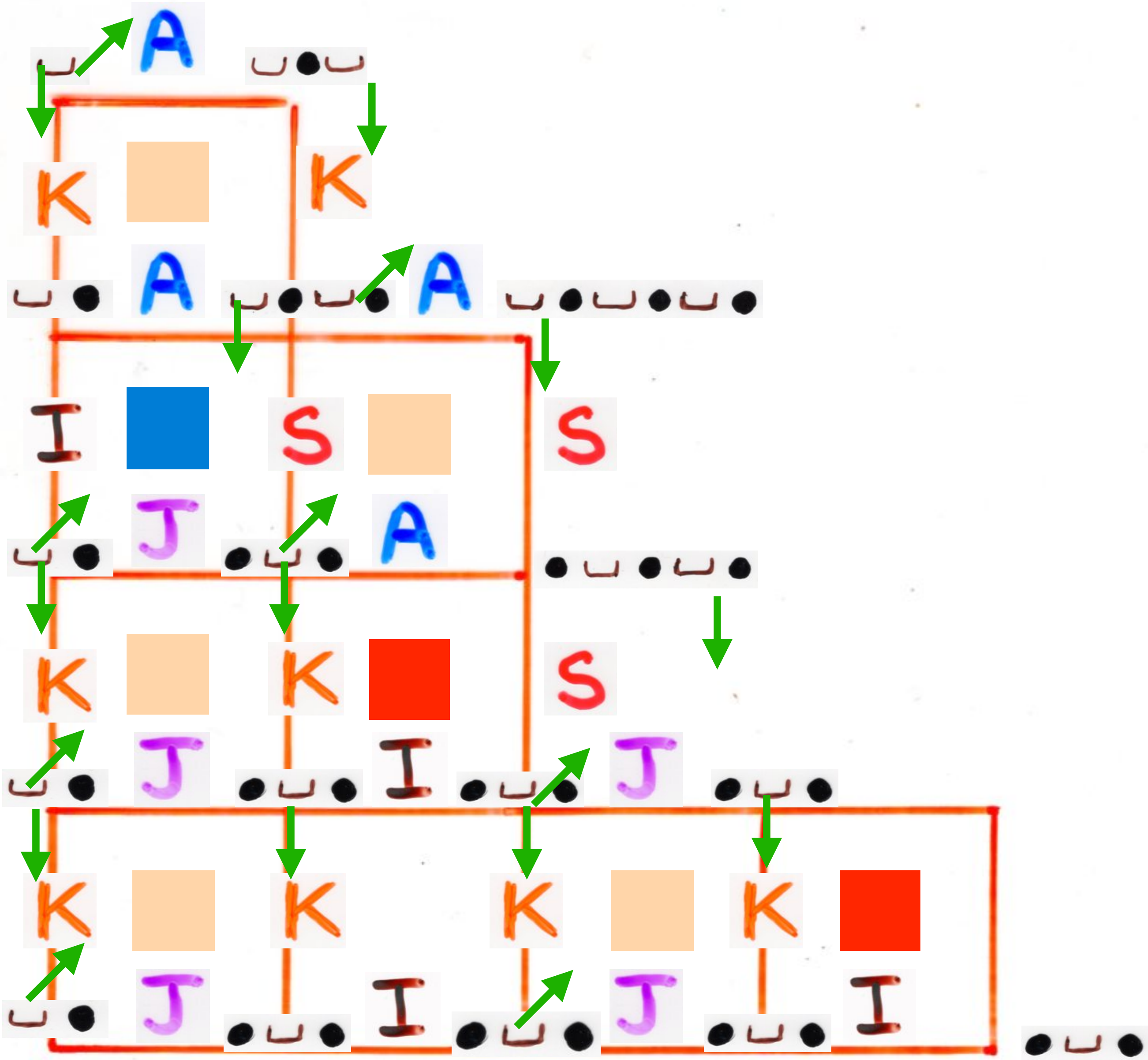
I S S
J A
K K S
J I
K K K
J I J I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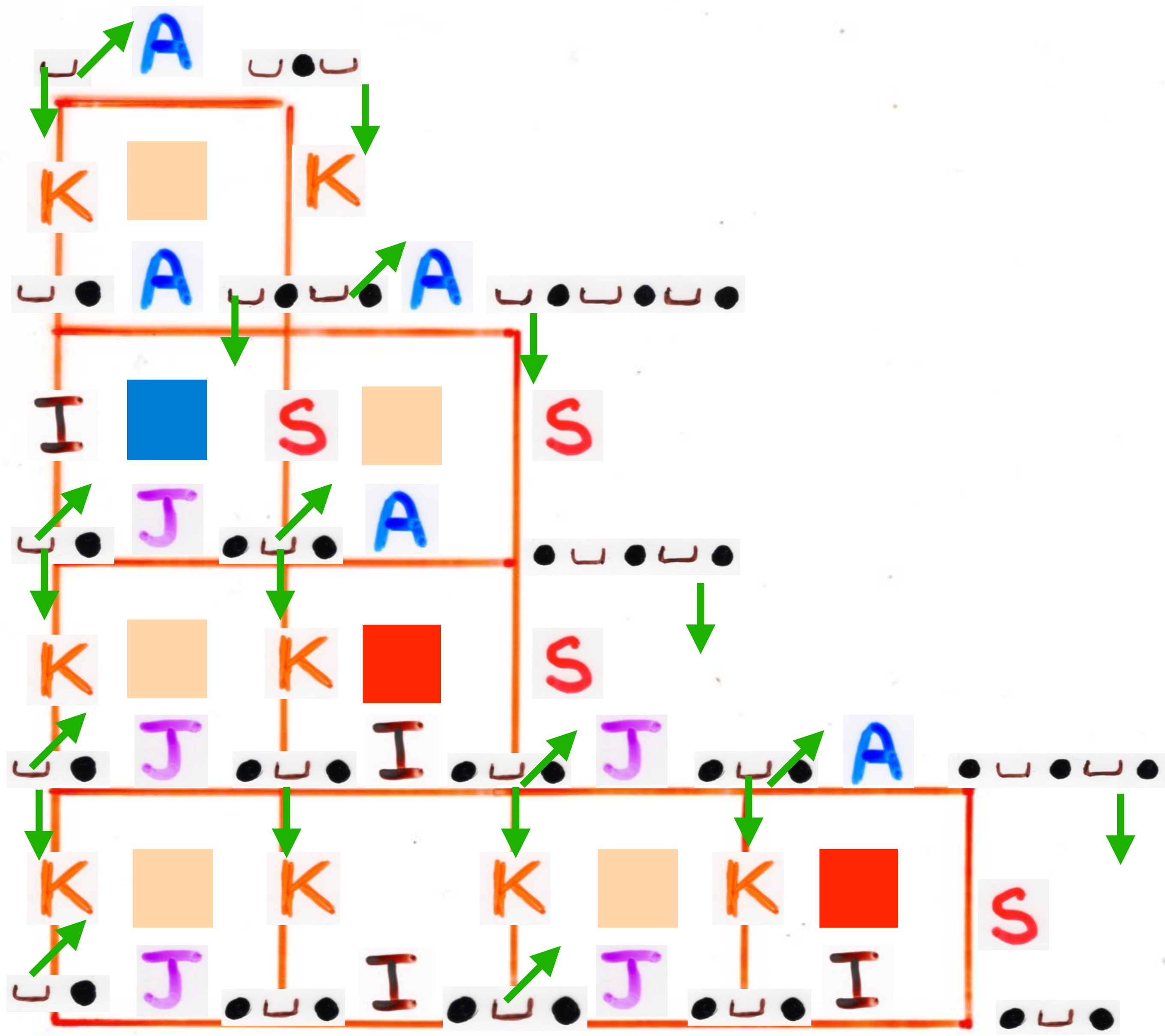

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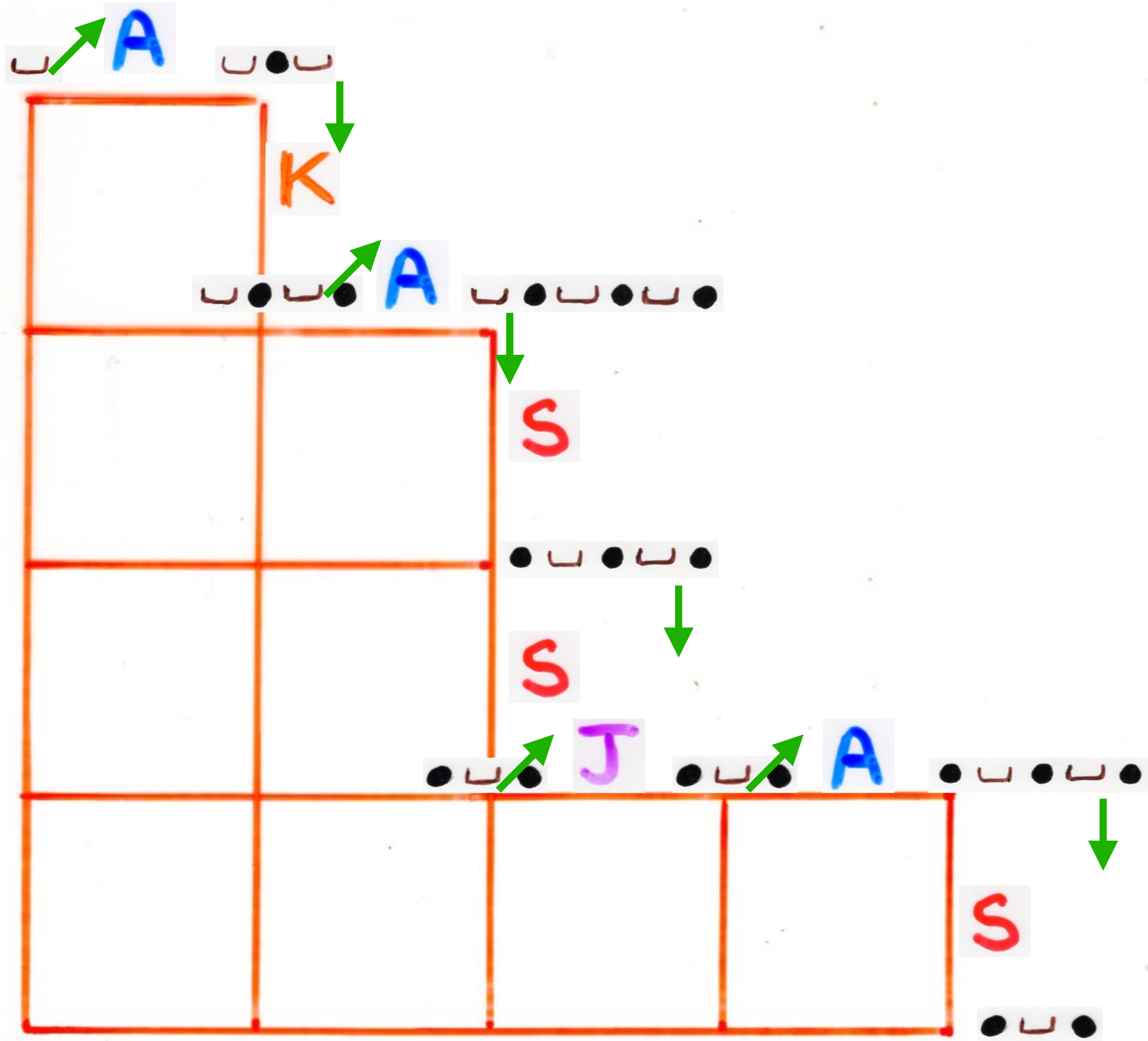
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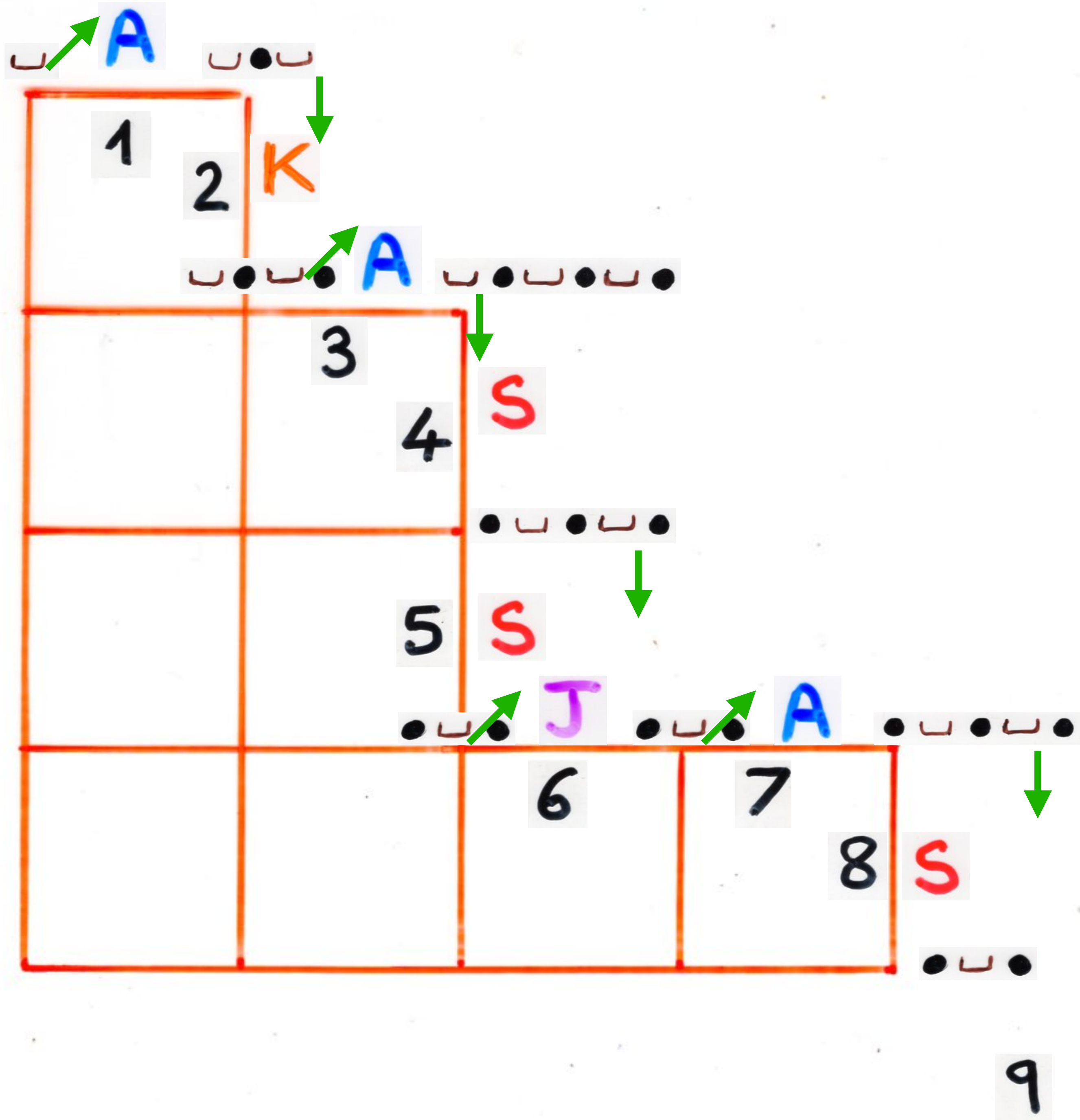

A
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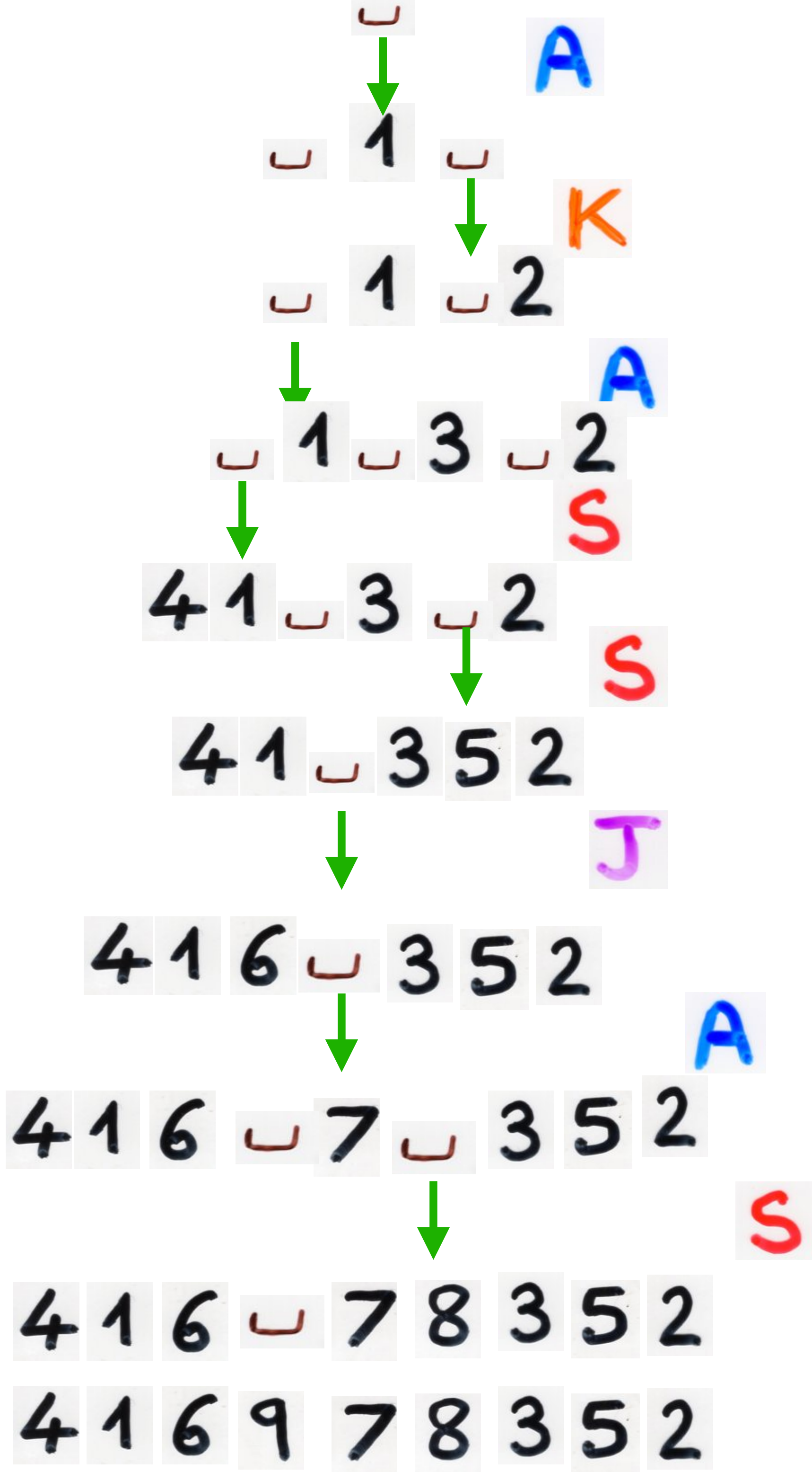
A
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1 3 2
S
4 1 3 2
S
4 1 3 5 2
J
4 1 6 3 5 2
A
4 1 6 7 3 5 2
S
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

two bijections
one theorem

Prop.

T alternative tableau → σ "exchange-fusion" inverse algorithm

T alternative tableau → τ "local" algorithm from $DE = ED + E + D$

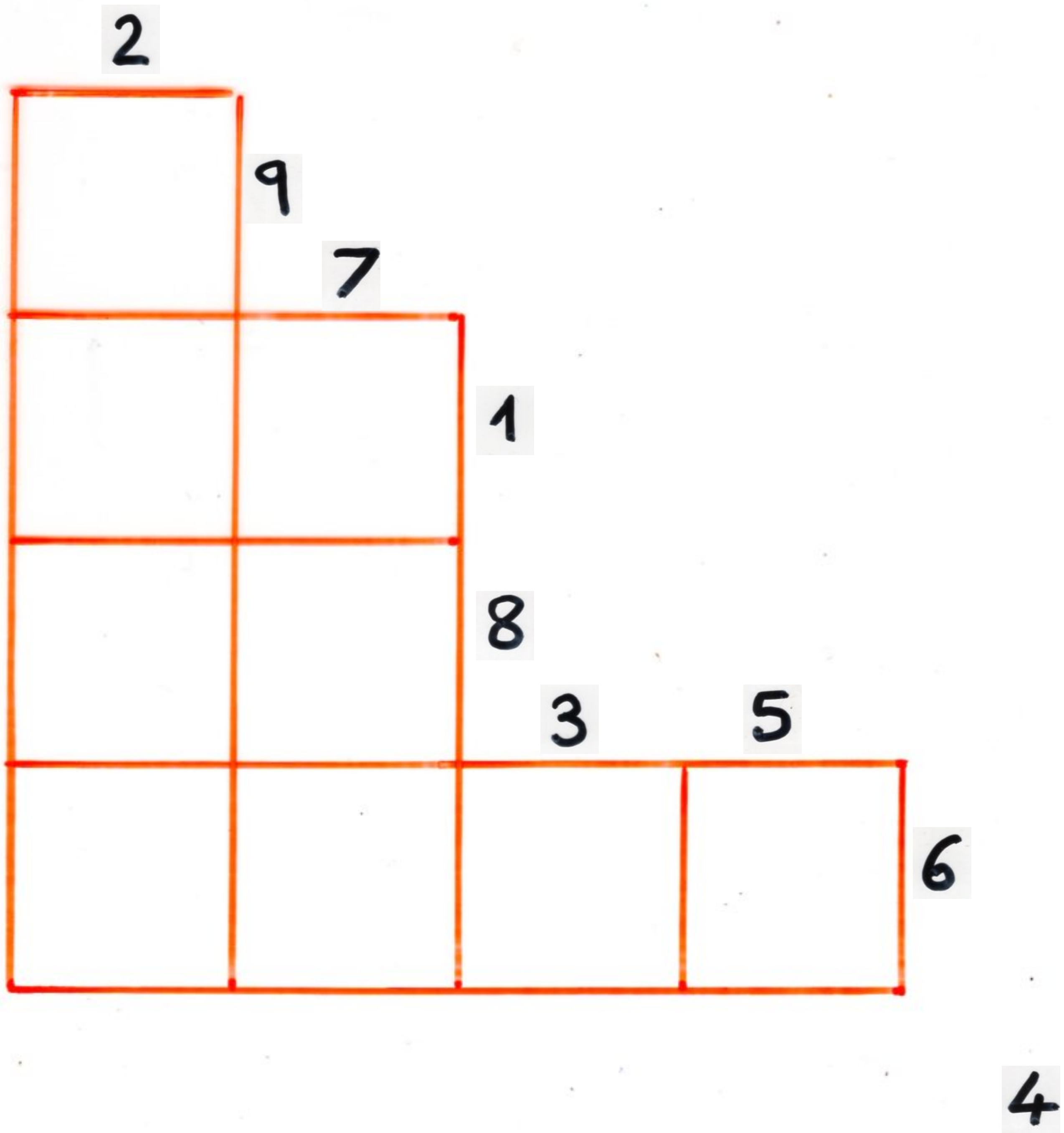
$$\sigma = \tau^{-1}$$

exchange-delelete algorithm

with the inverse permutation

4 1 6 9 7 8 3 5 2 → 2 9 7 1 8 3 5 6 4

(with a variant: keep the min instead of the max)


2
9
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3
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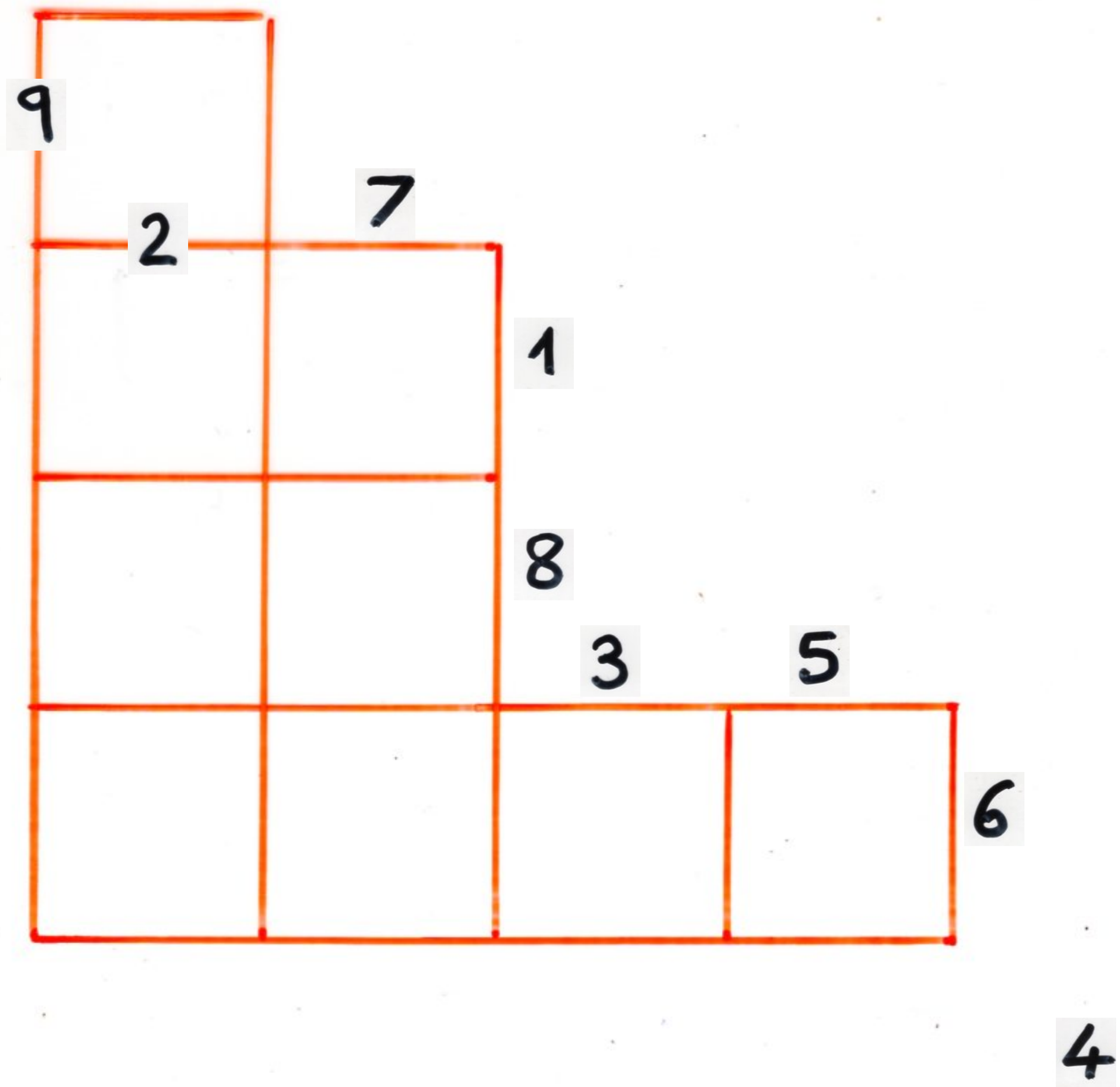
6
4


9
2
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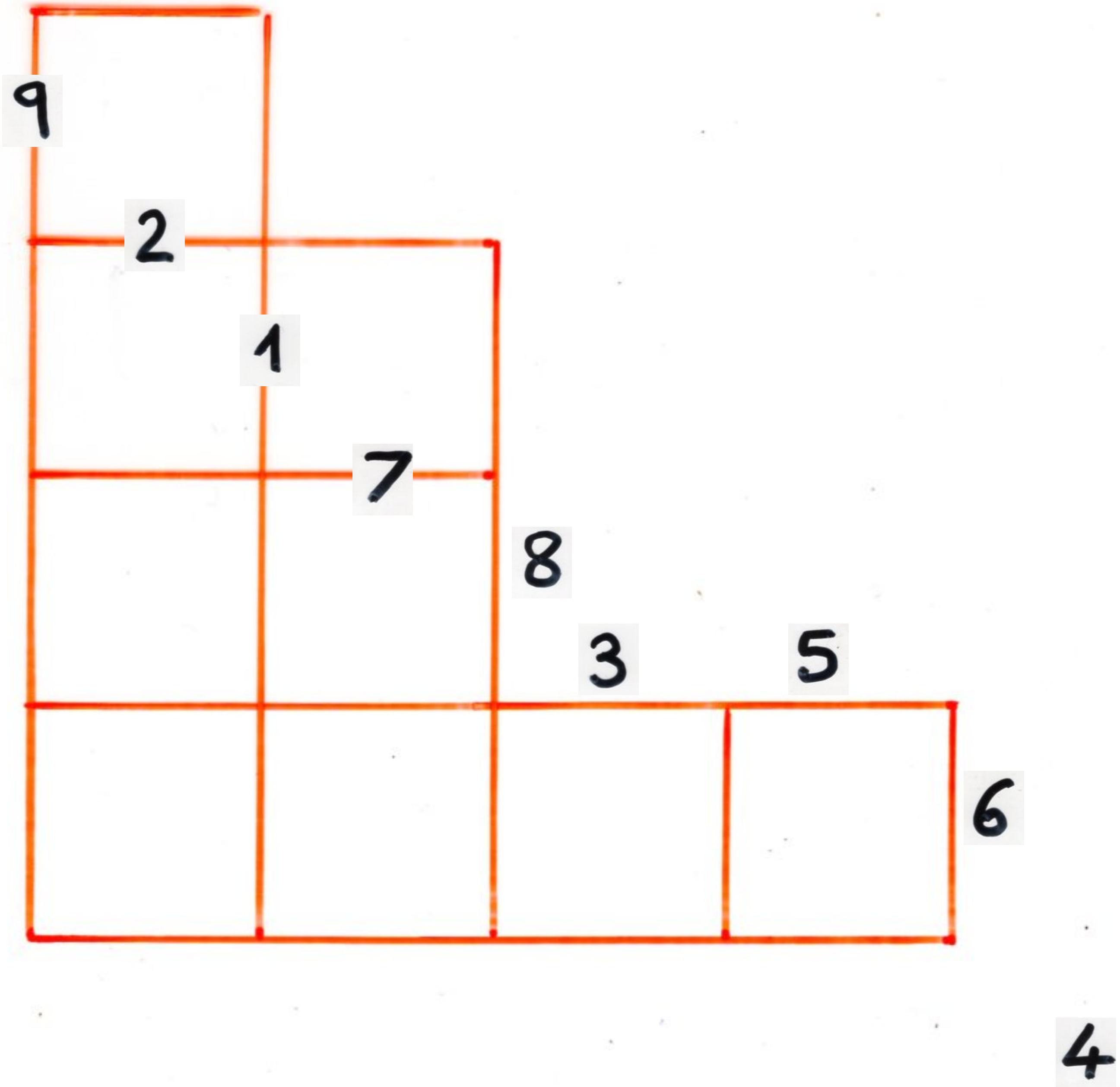
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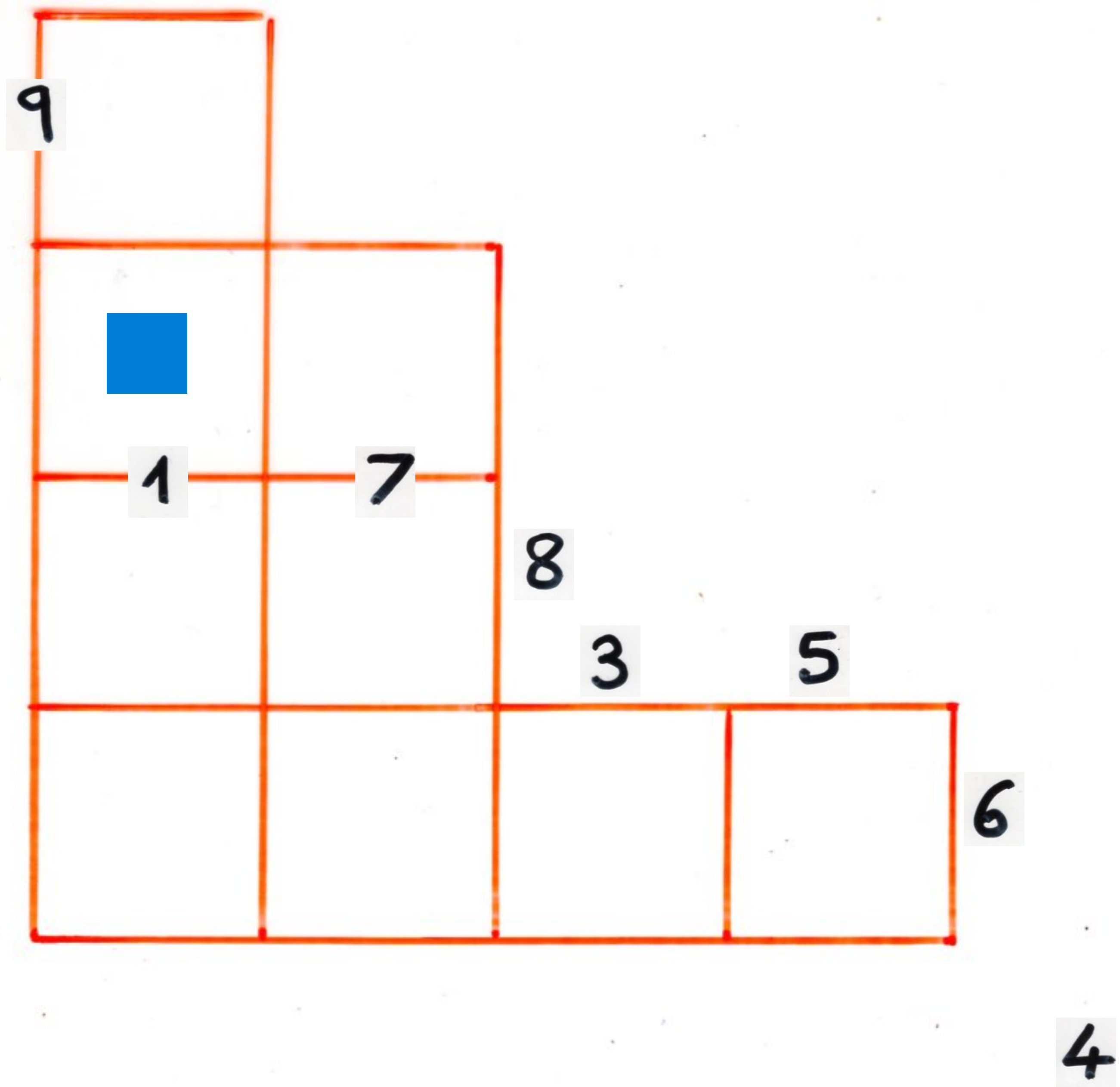
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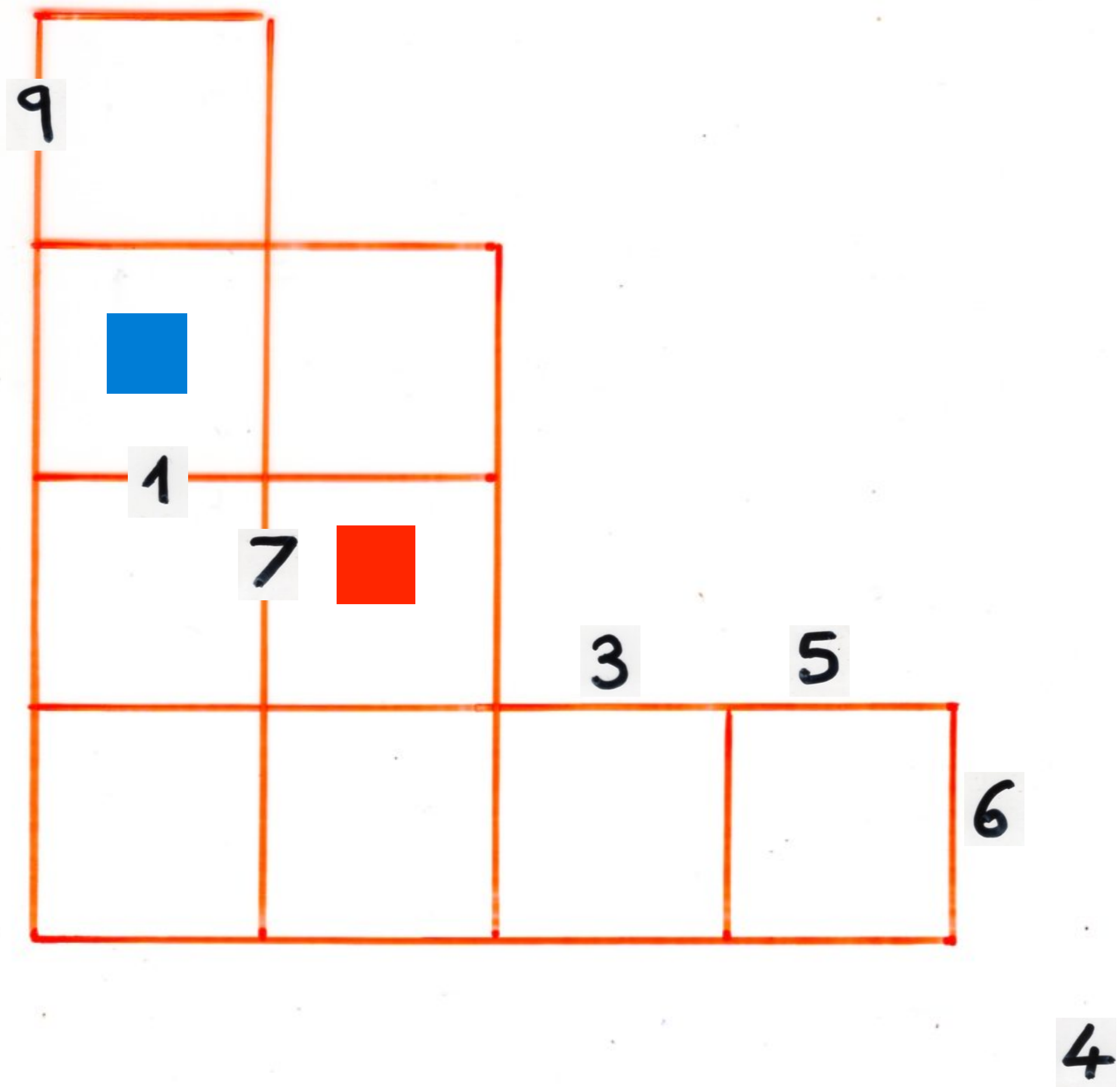
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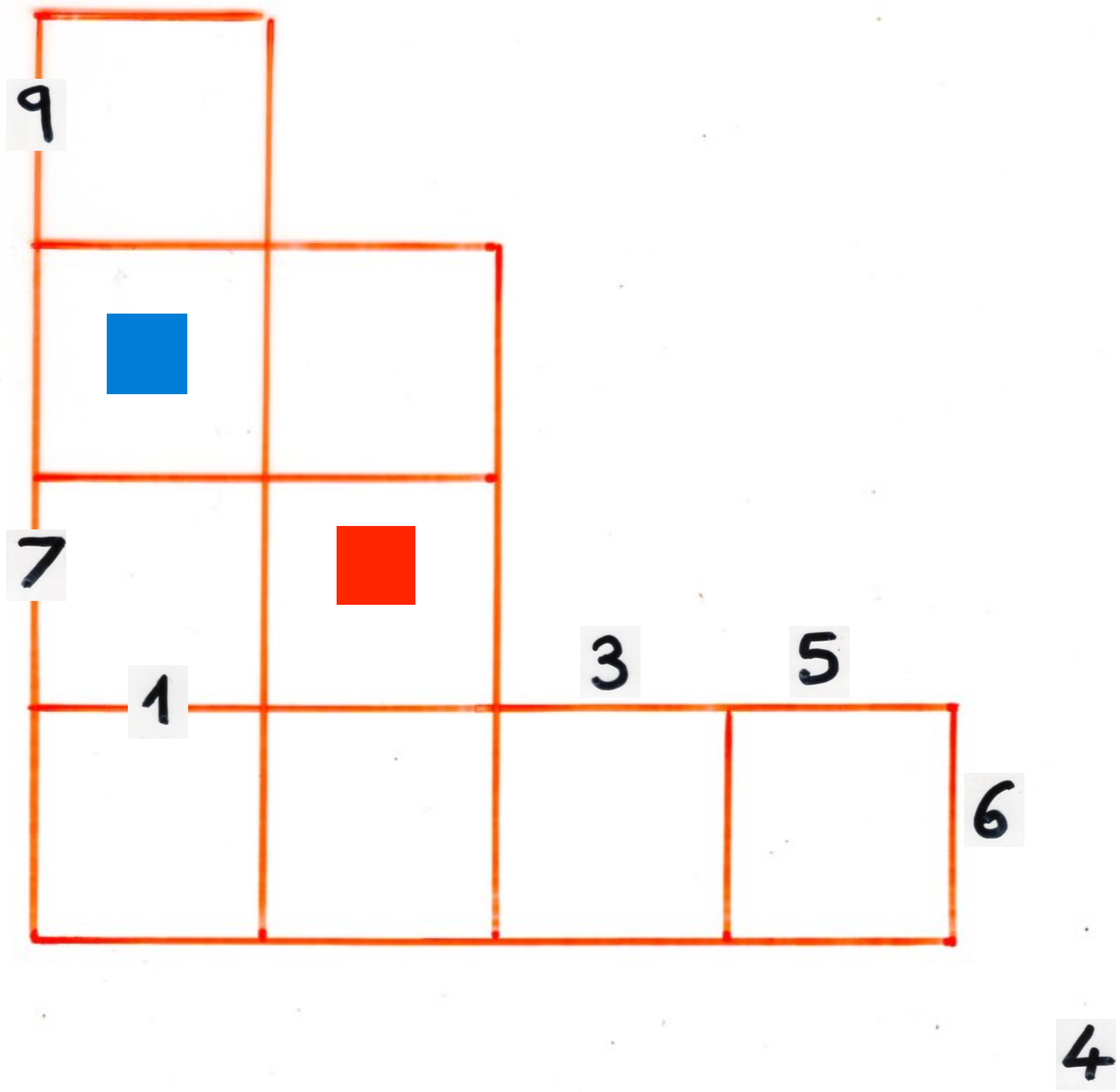
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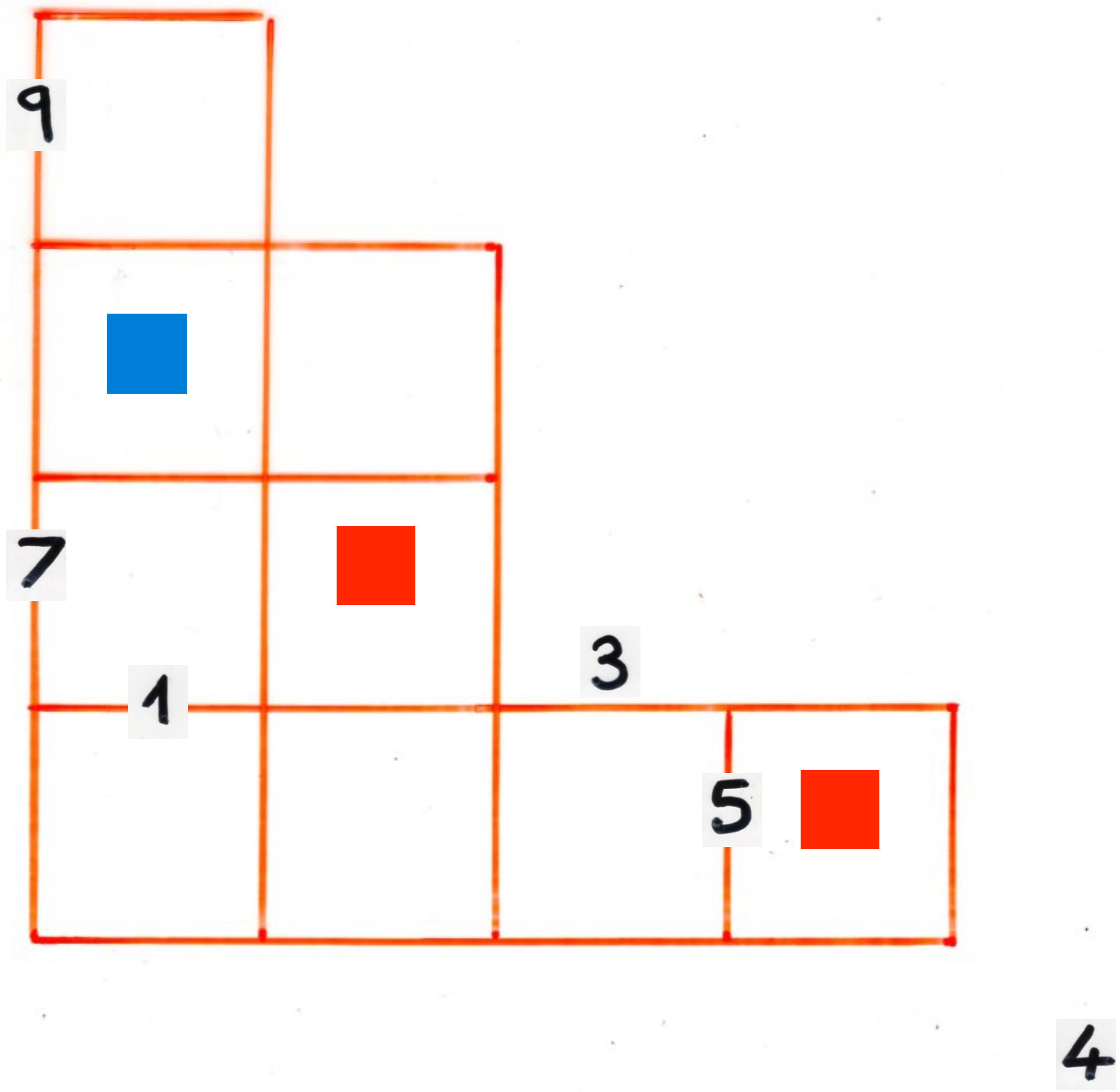
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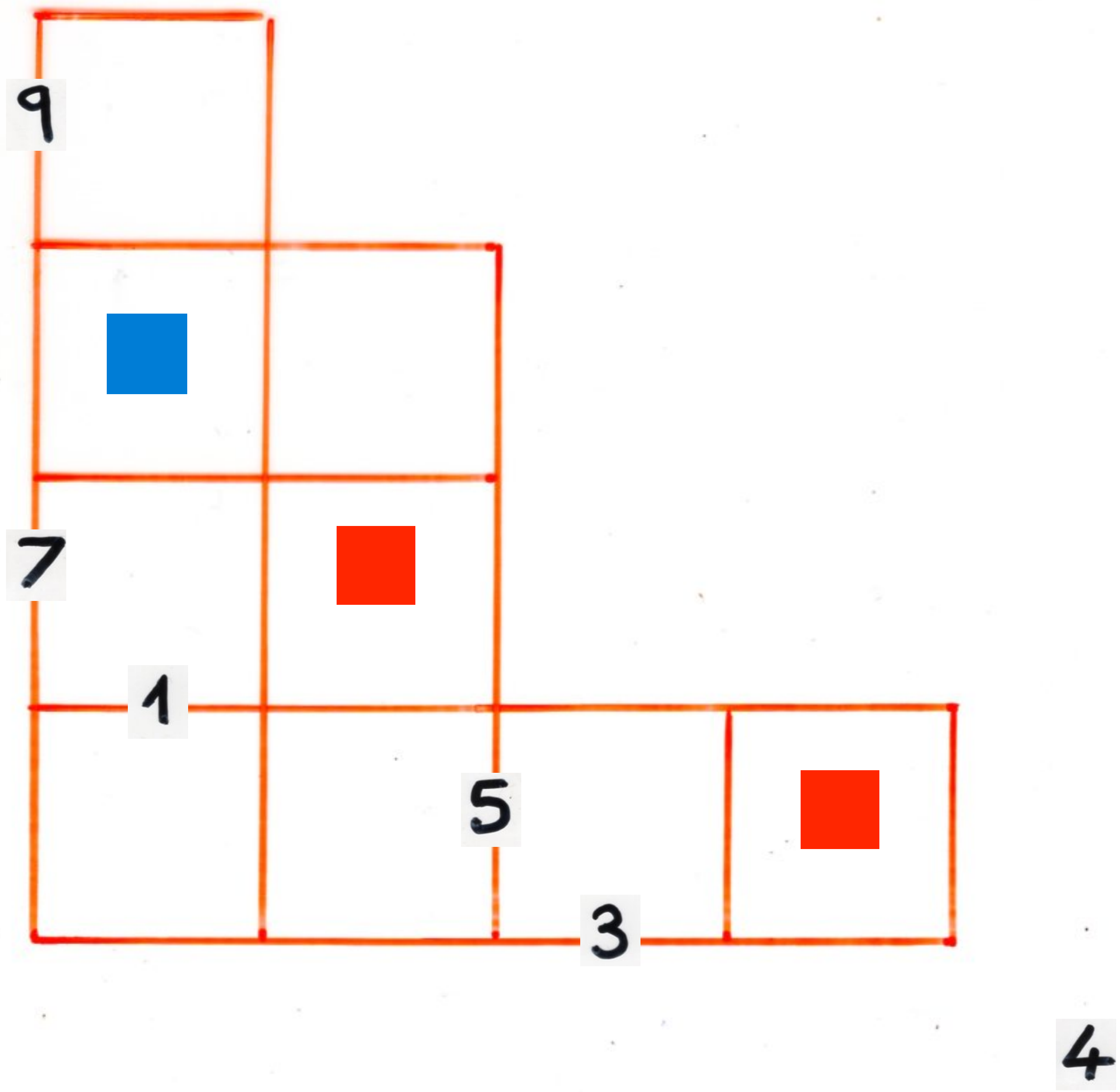
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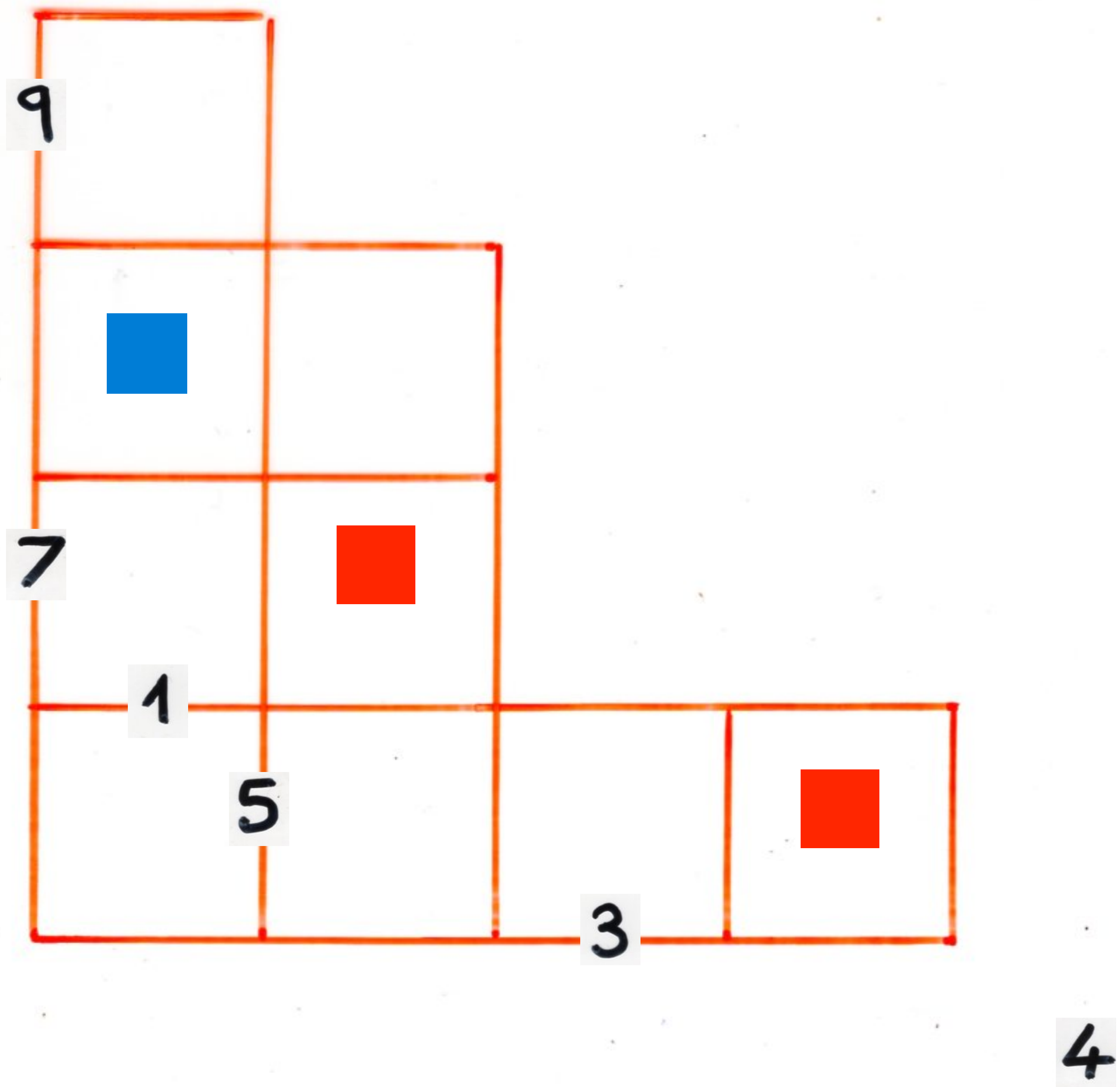
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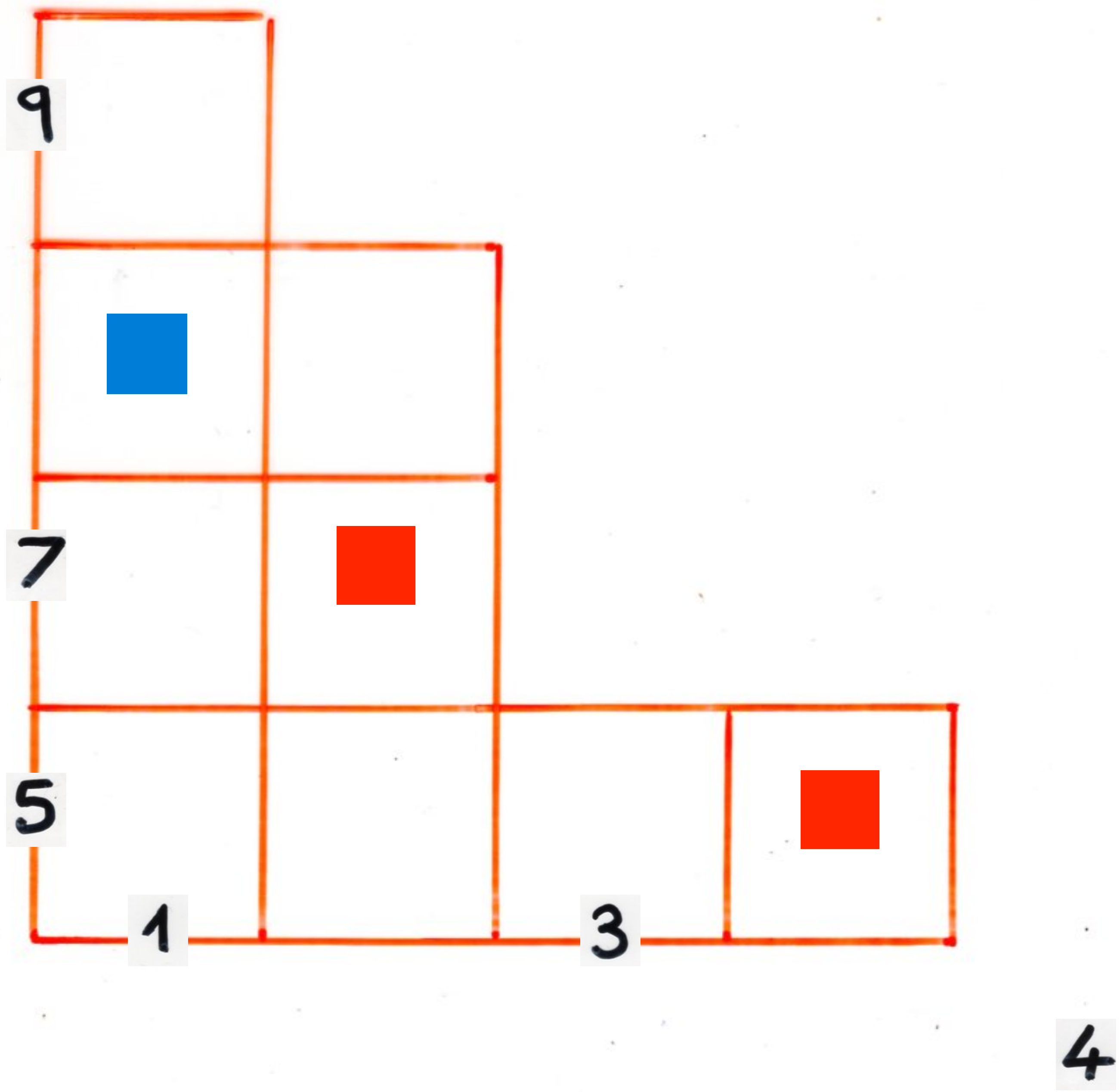
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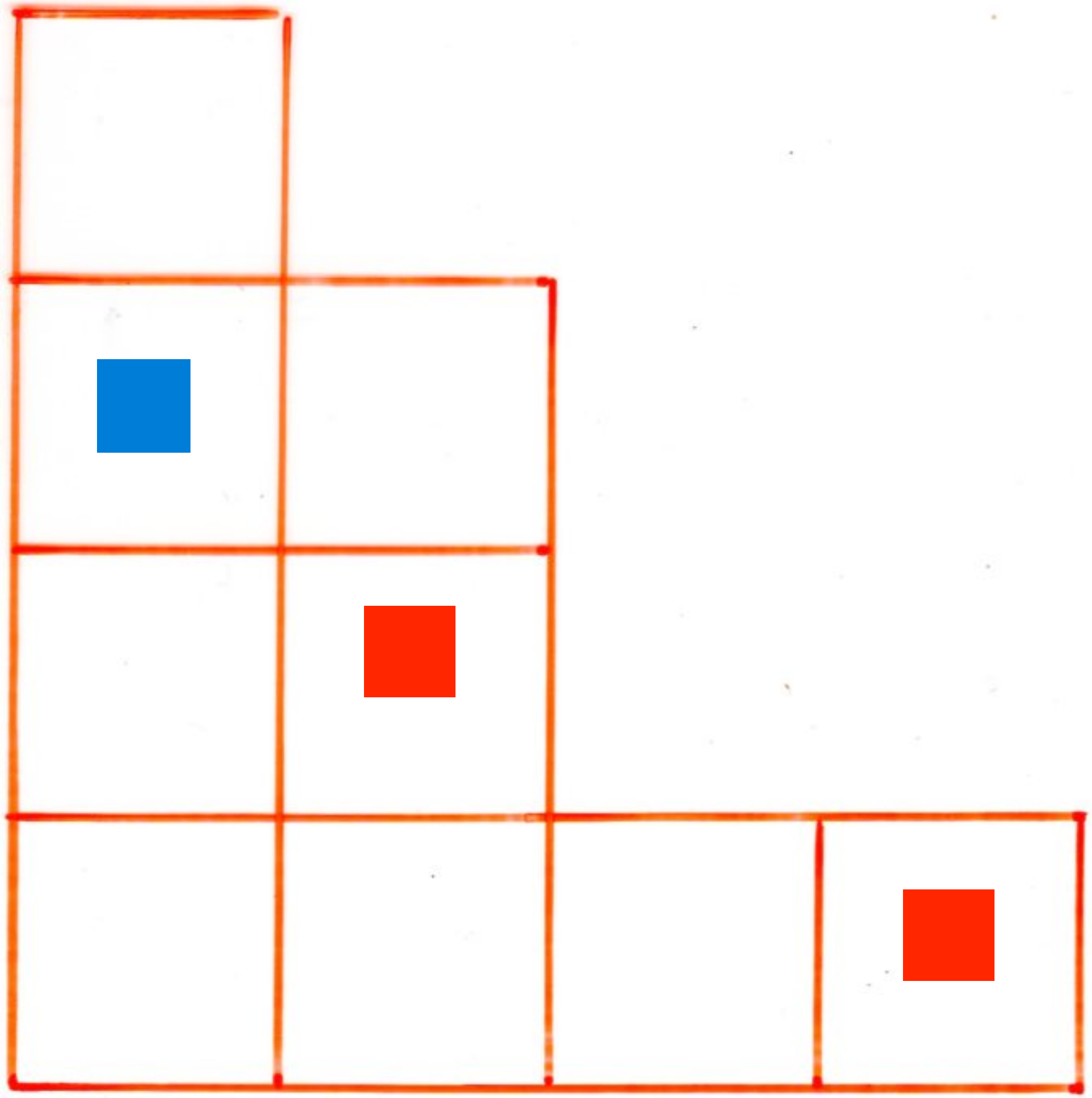
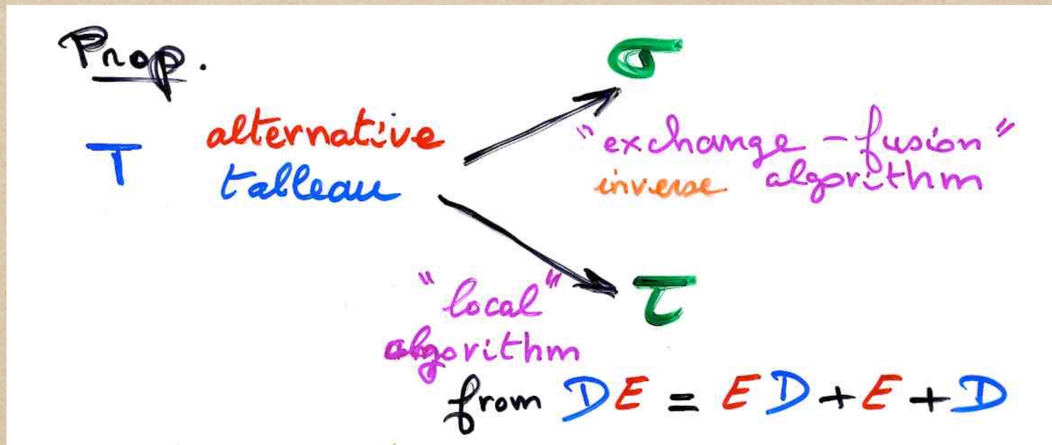


proof of the main theorem



Proof of the equivalence

local rules
(commutation diagrams)
and Laguerre histories

exchange-fusion
(or exchange-delete)
algorithm

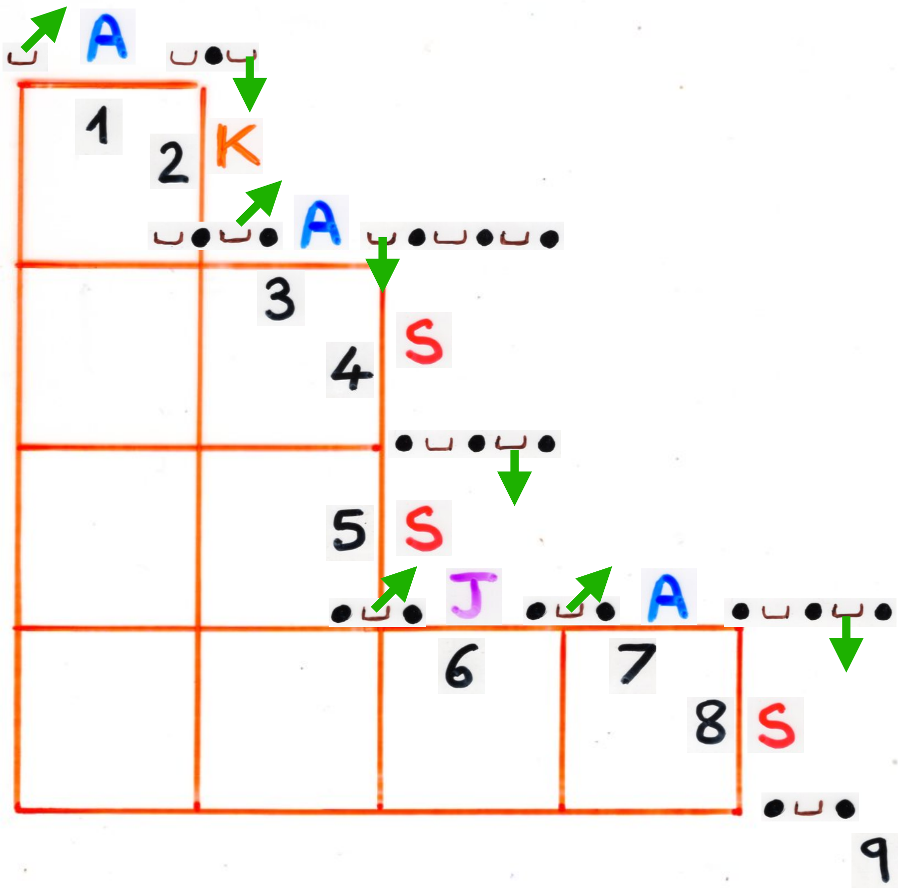
4 1 6 9 7 8 3 5 2

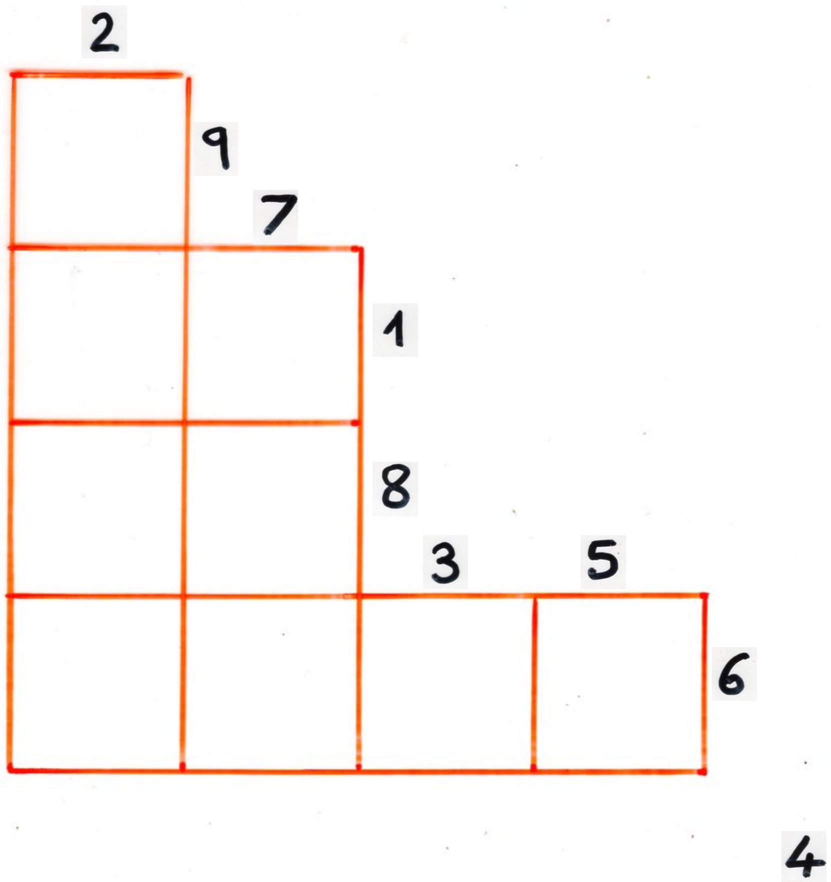
2 9 7 1 8 3 5 6 4

$$\sigma = \tau^{-1}$$

4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4



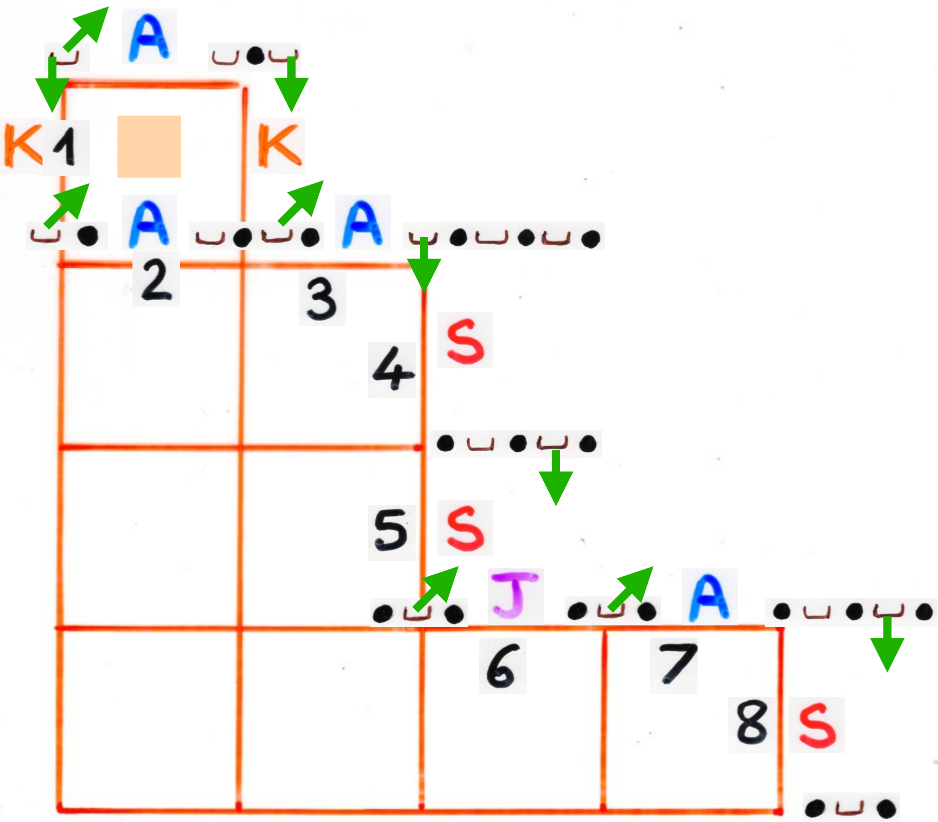


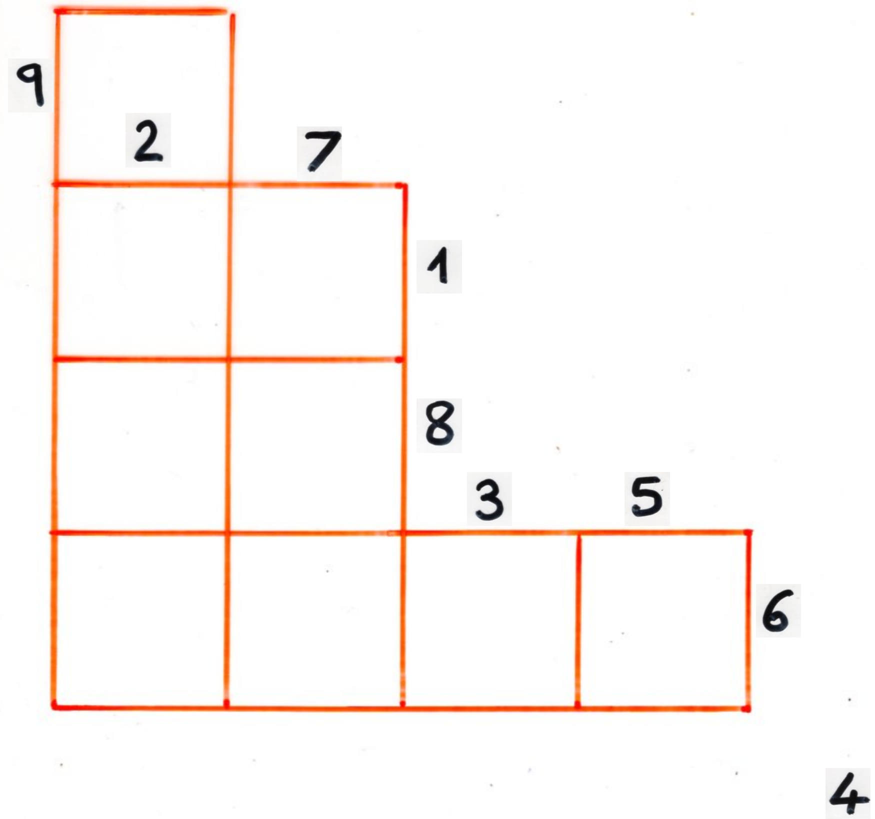
4 1 6 9 7 8 3 5 2

4 2 6 9 7 8 3 5 1

2 9 7 1 8 3 5 6 4

9 2 7 1 8 3 5 6 4



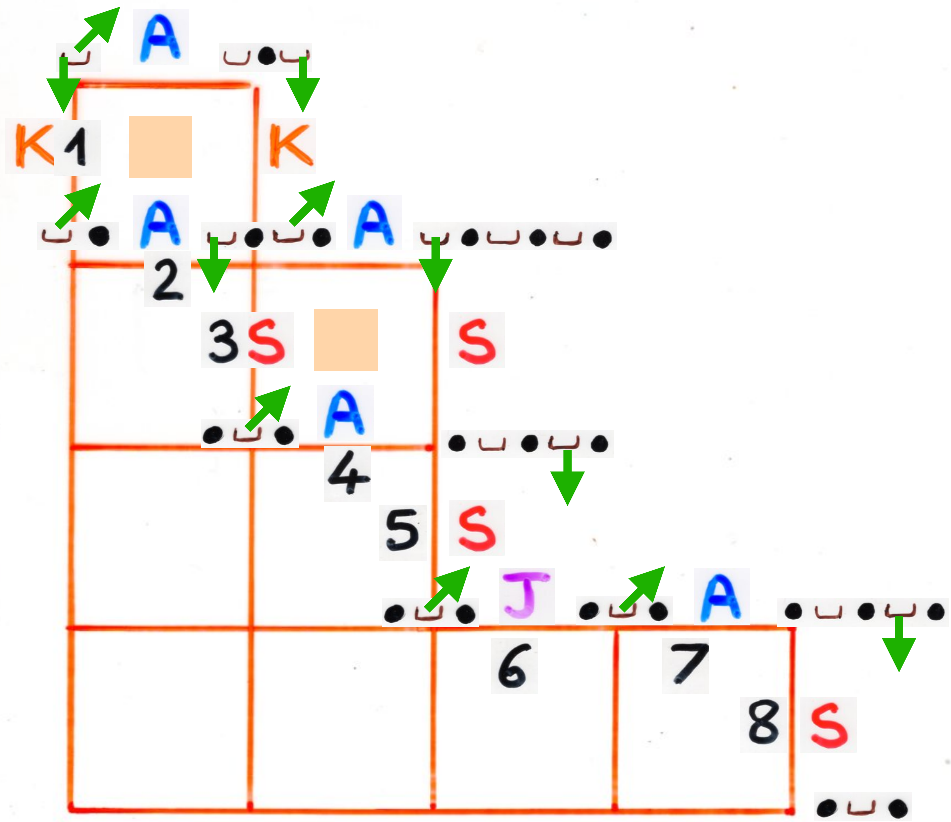


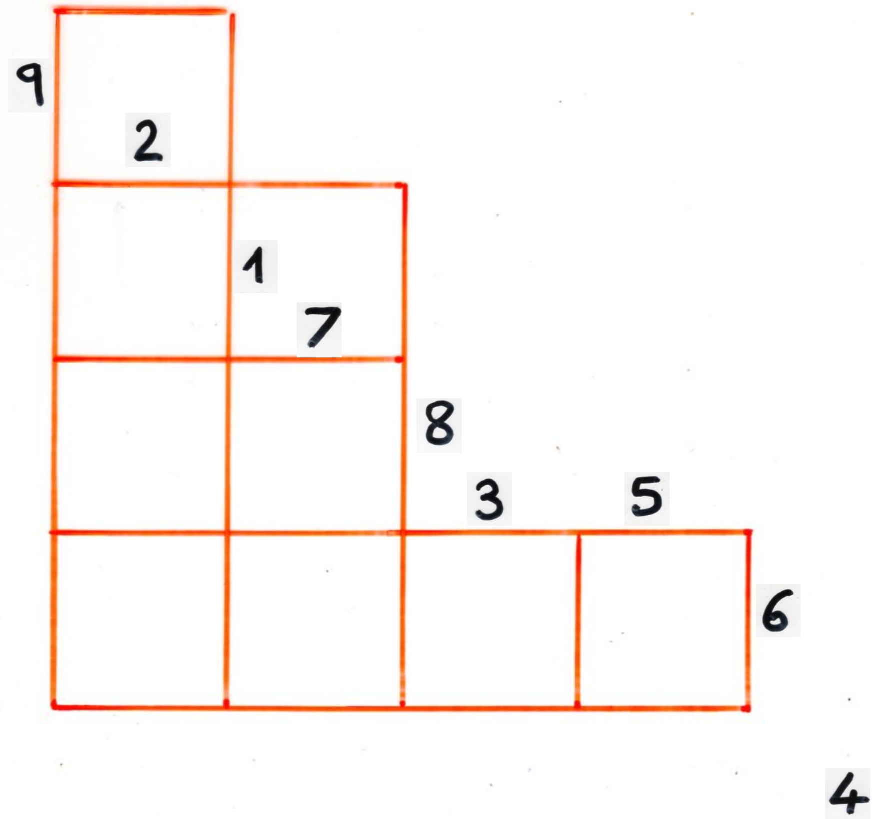
4 2 6 9 7 8 3 5 1

3 2 6 9 7 8 4 5 1

9 2 7 1 8 3 5 6 4

9 2 1 7 8 3 5 6 4



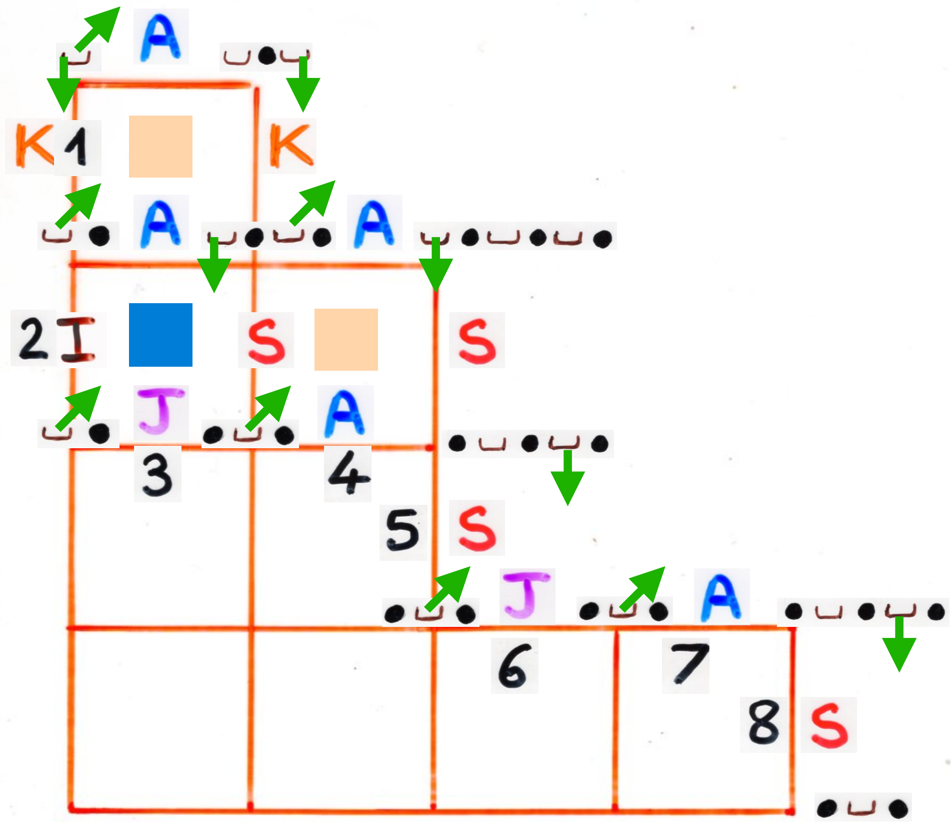


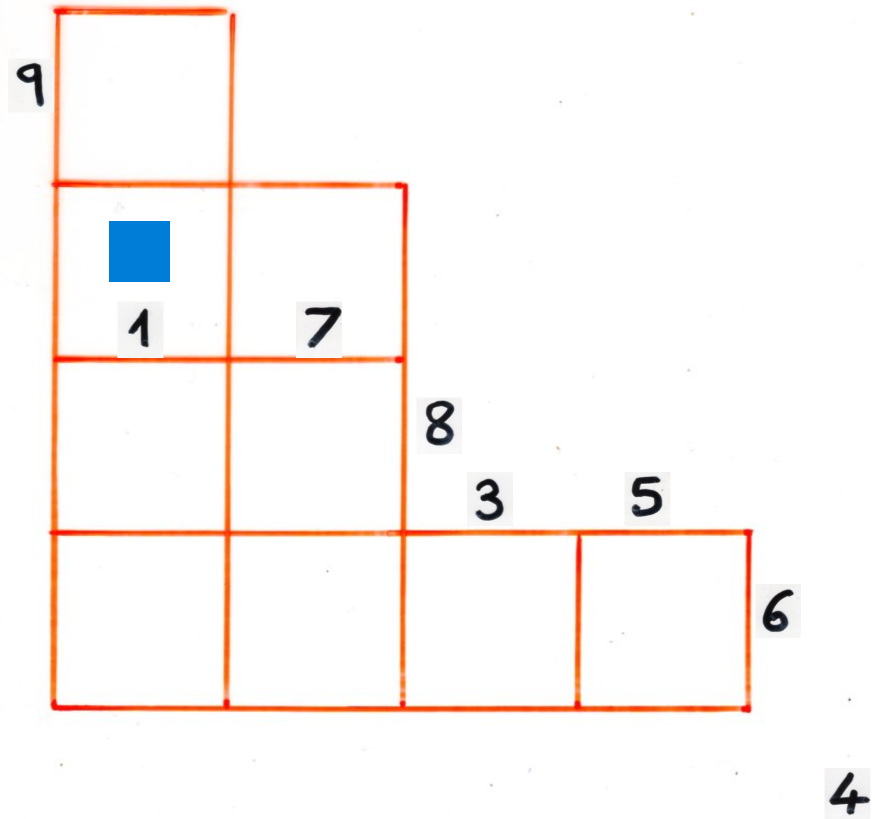
3 2 6 9 7 8 4 5 1

3 6 9 7 8 4 5 1

9 2 1 7 8 3 5 6 4

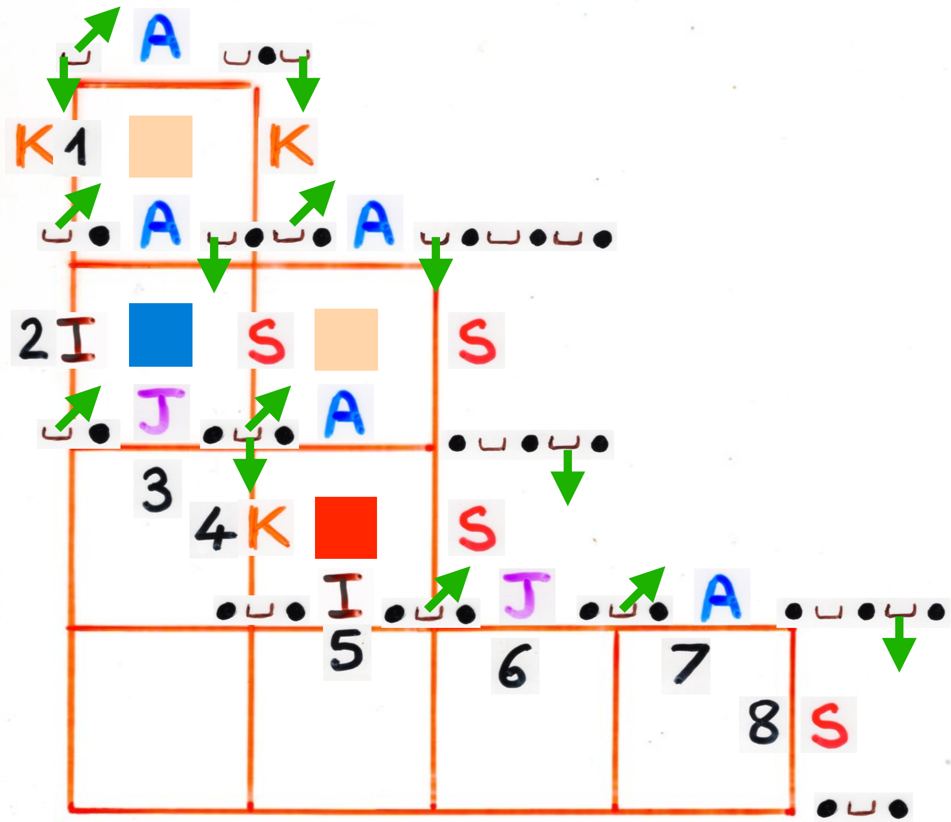
9 1 7 8 3 5 6 4

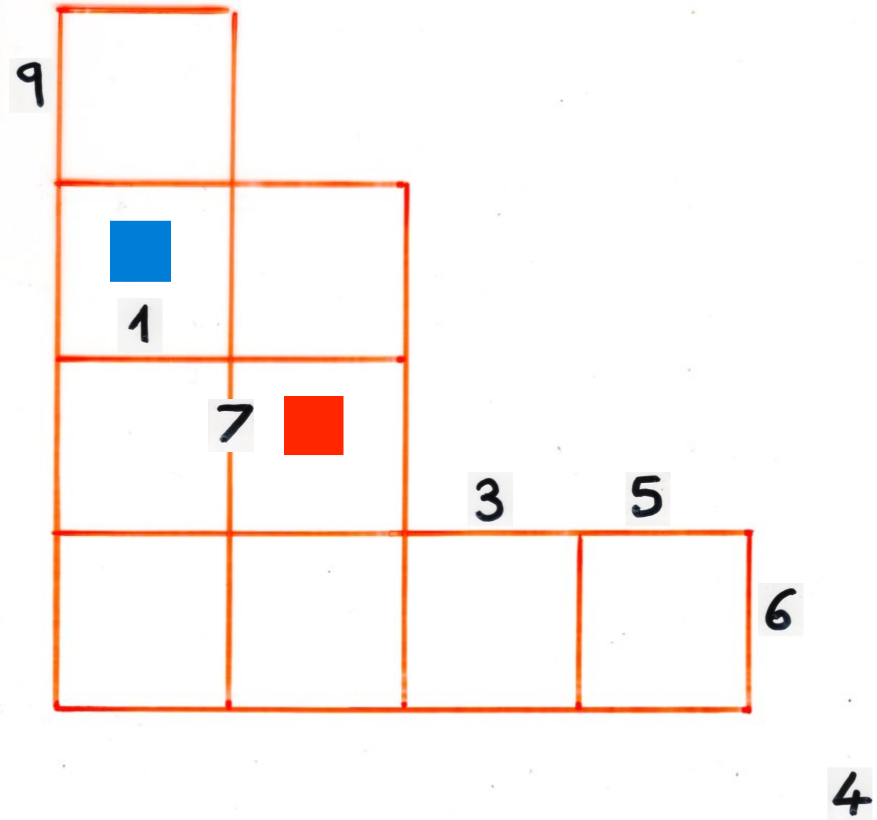




3		6	9	7	8	4	5	1
3		6	9	7	8	4		1

9		1	7	8	3	5	6	4
9		1	7		3	5	6	4





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9		1	7	8	3	5	6	4
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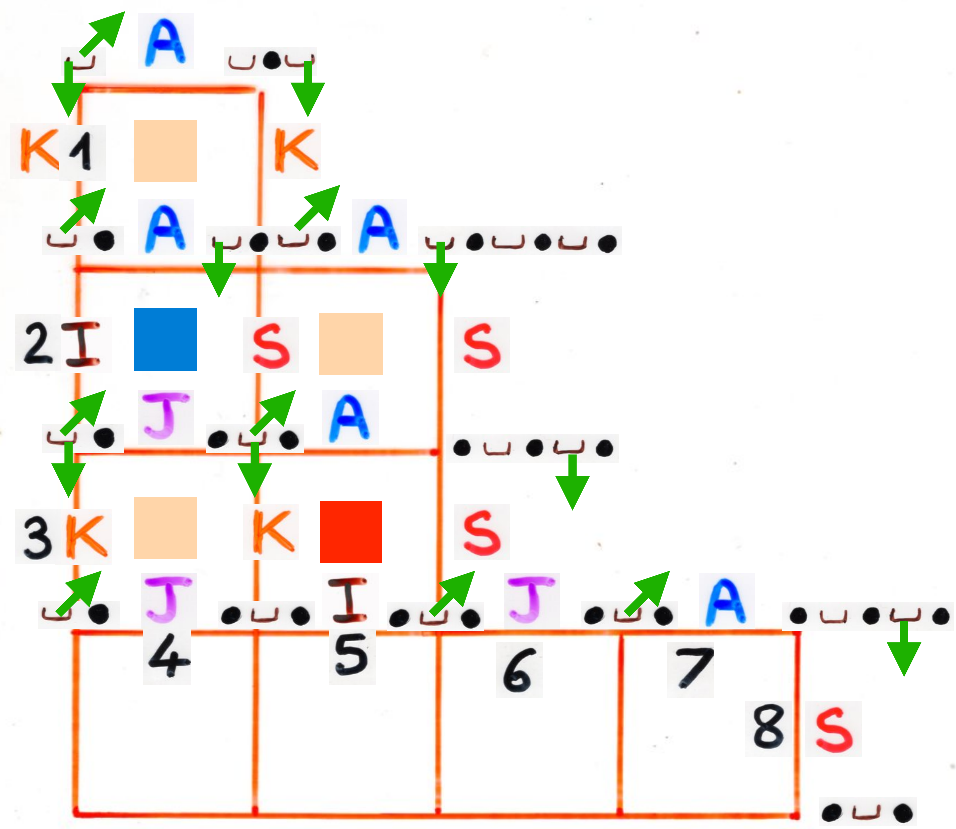

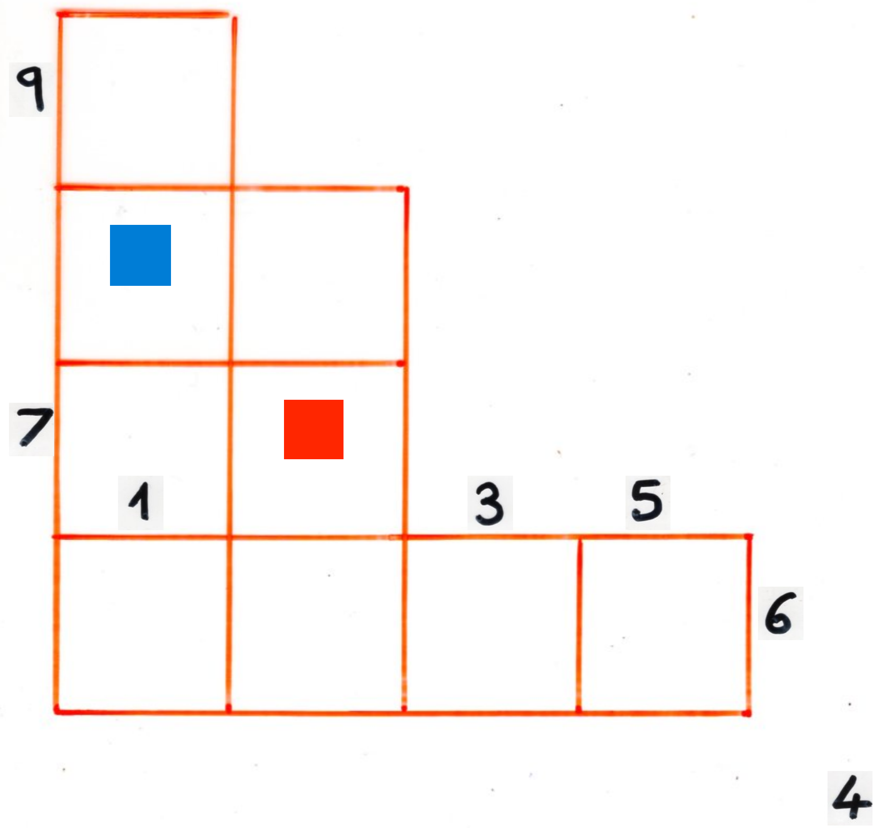

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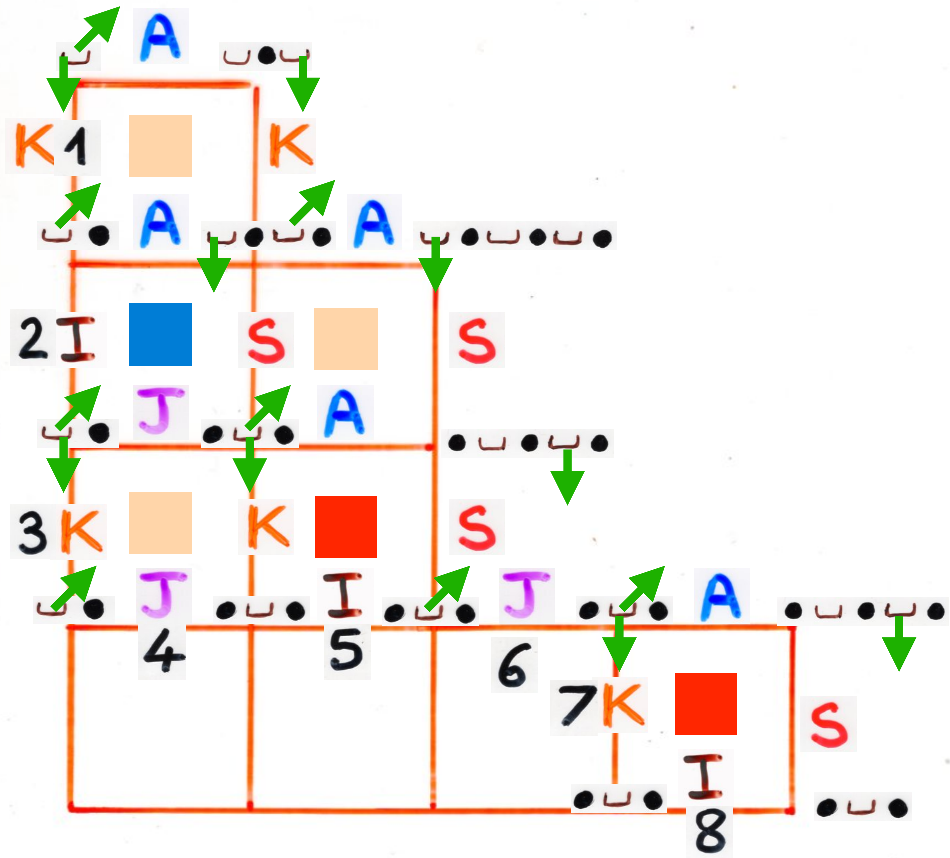
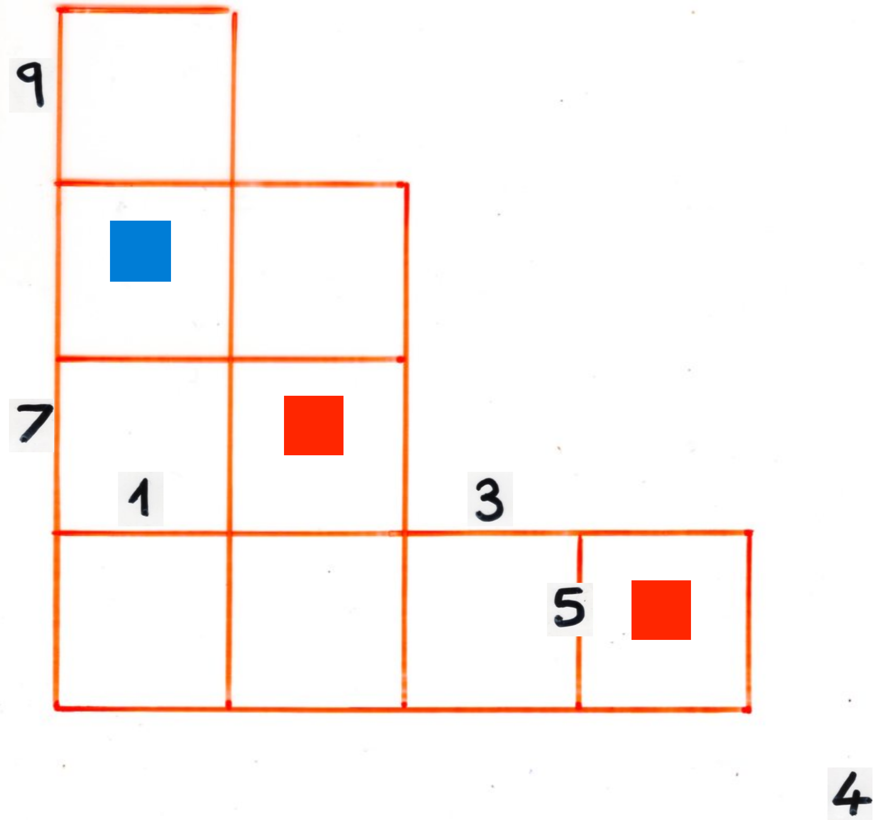


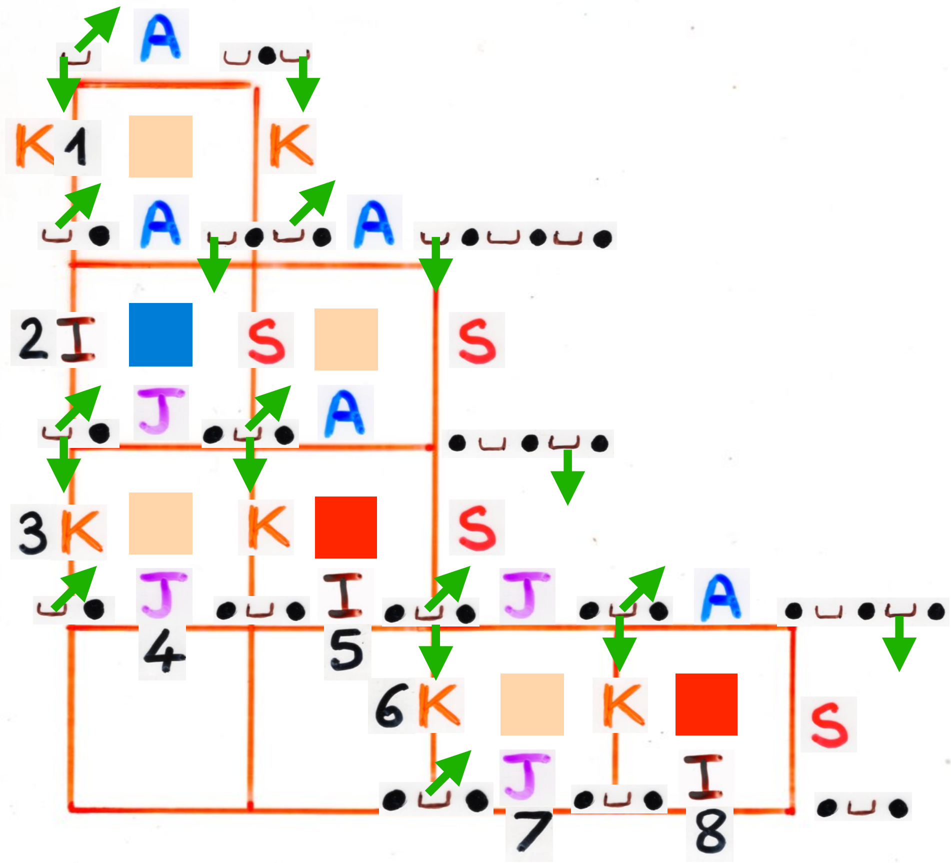


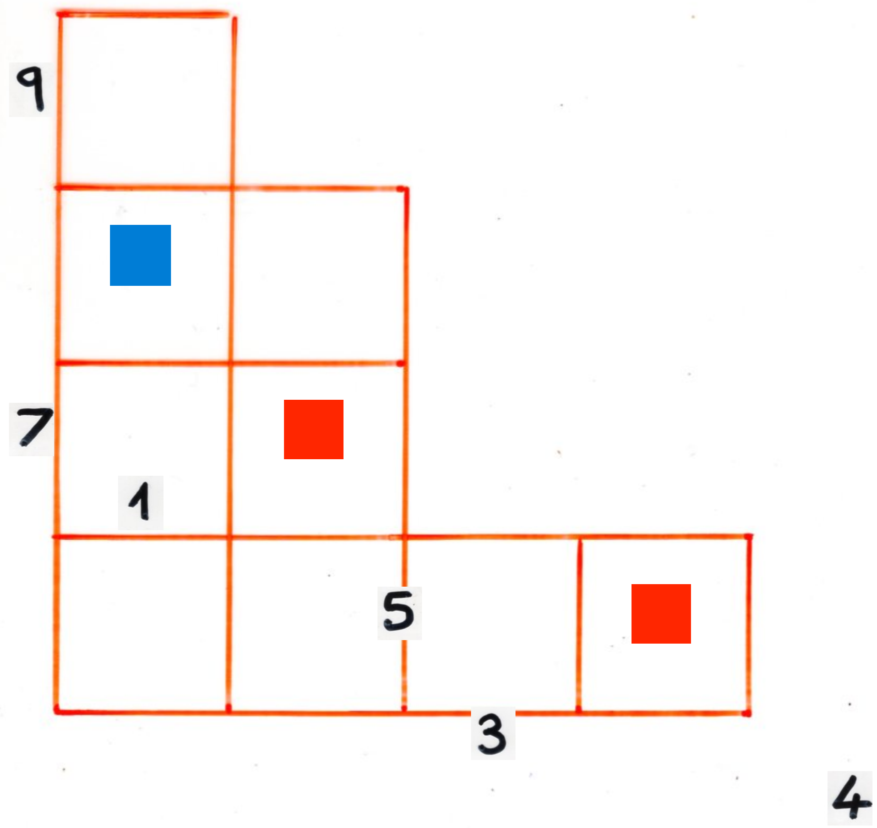
4 6 9 7 3 1

4 7 9 6 3 1

9 7 1 3 5 4

9 7 1 5 3 4





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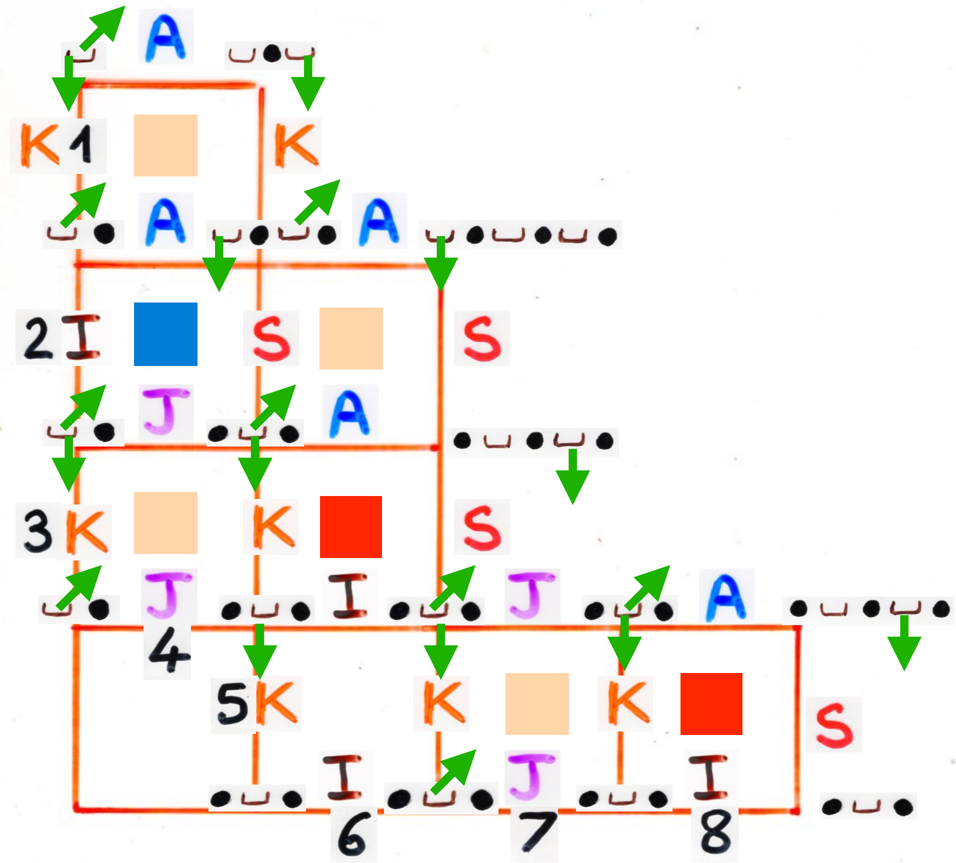
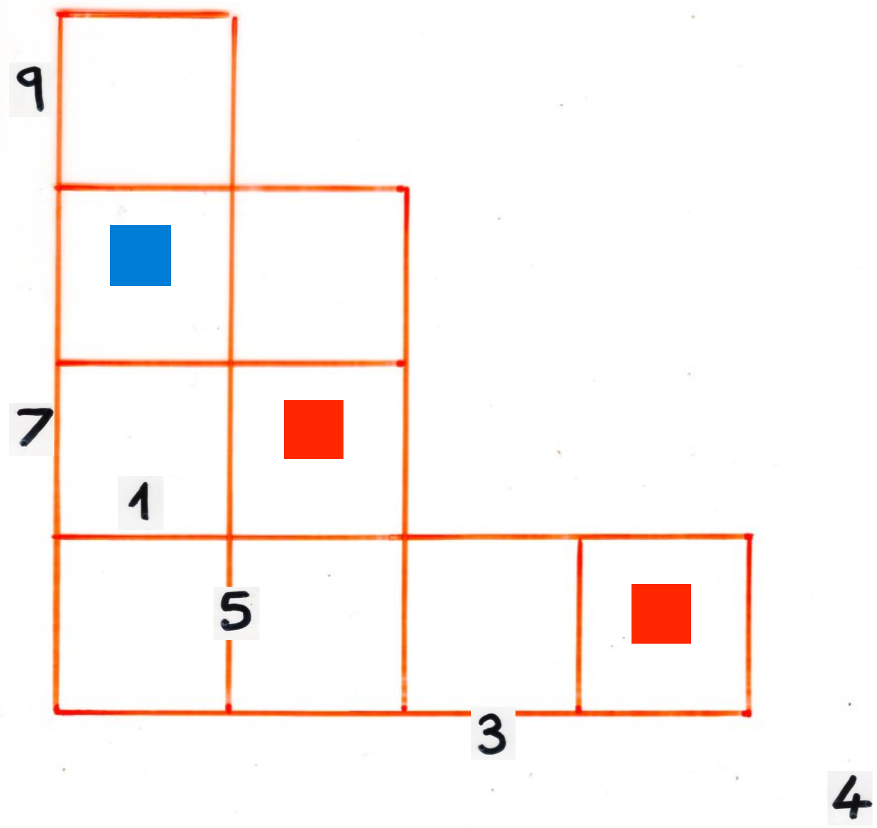
4 795 3 1   9 715 3 4





4 795 3 1

5 794 3 1

9 715 3 4

9 751 3 4