Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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mirror website

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Chapter 1
RSK
The Robinson-Schensted-correspondence
(Ch1e)

IMSc, Chennai

January 25, 2018

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From Ch 1a, 1b, 1c, 1d

The Robinson-Shensted correspondence

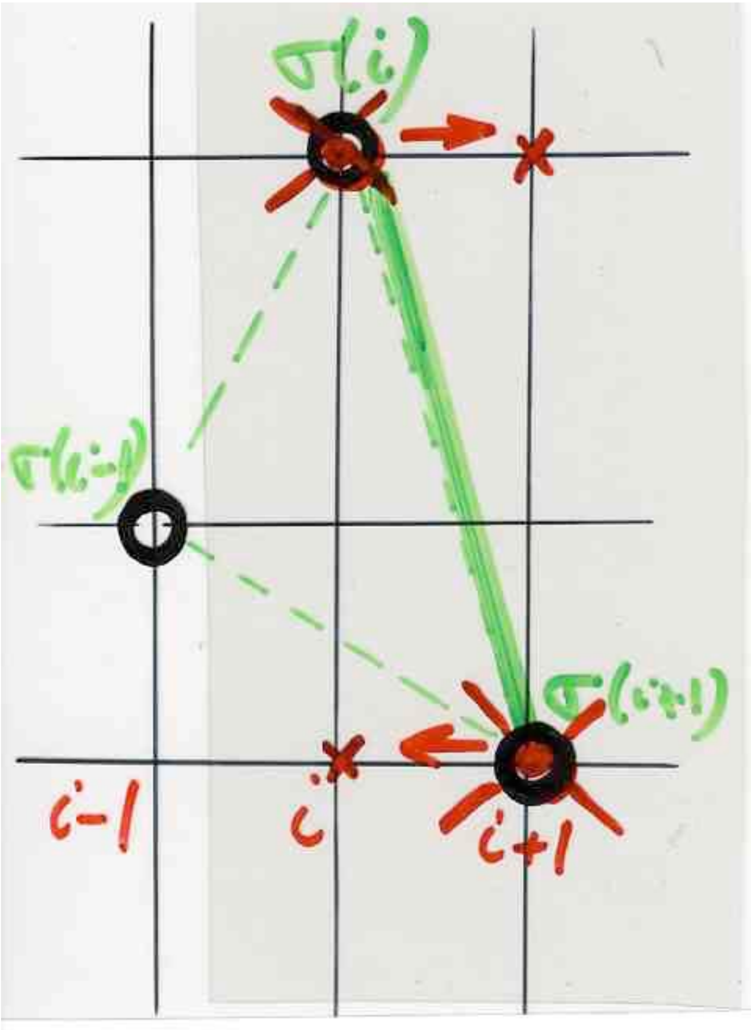
Ch 1a
- Schensted's insertions
- geometric version with "shadow lines »

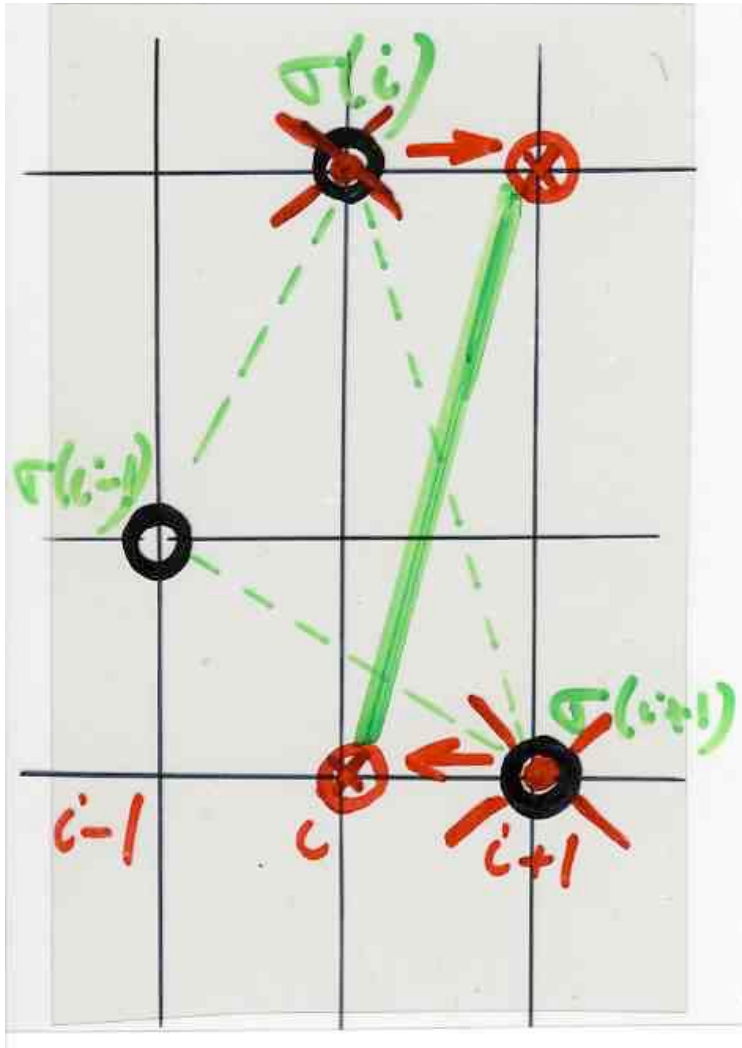
Ch 1b - Fomin "local rules" or "growth diagrams »

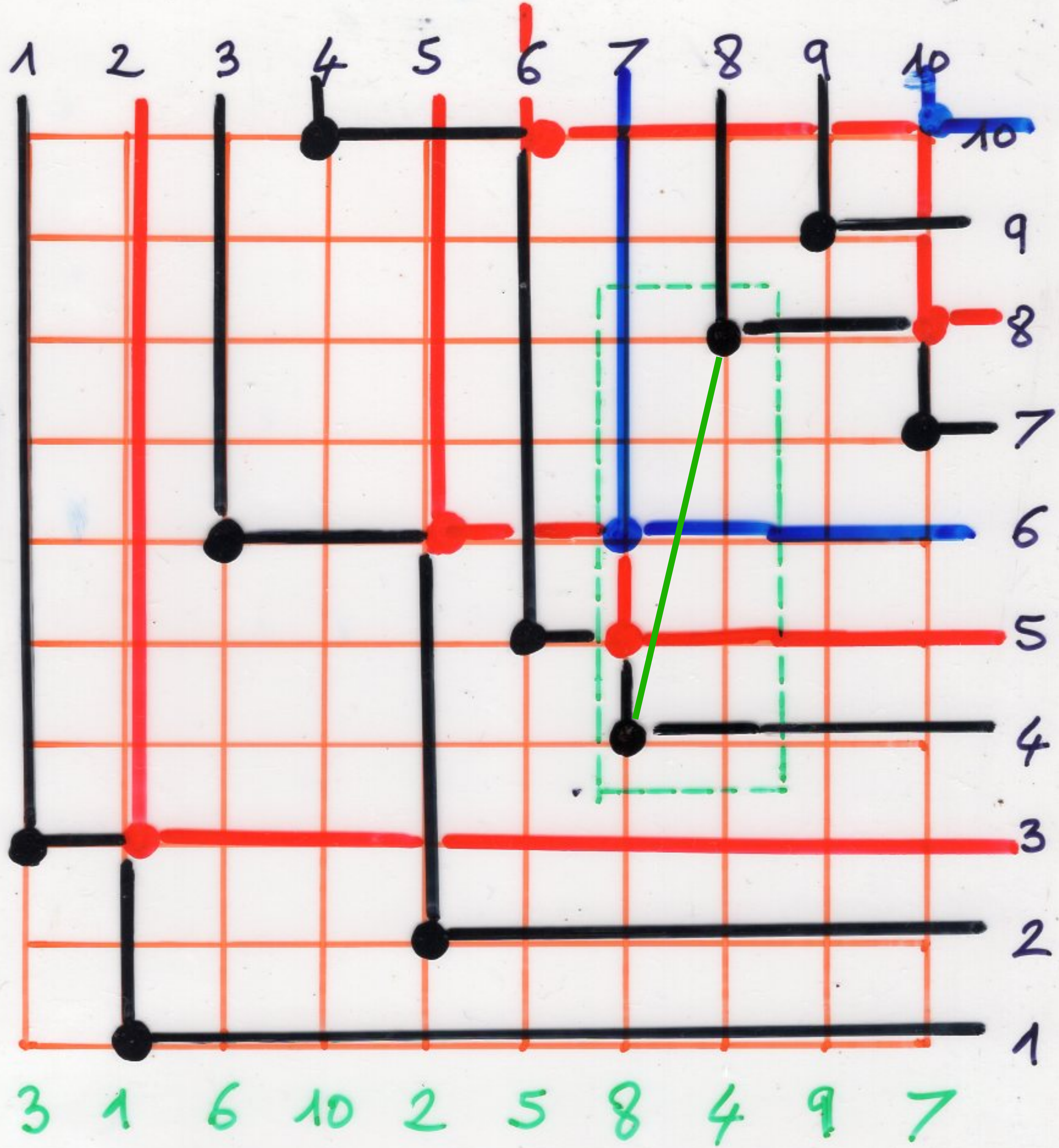
Ch 1c - from a representation of the quadratic algebra $UD=DU+I$, deduce a bijection (P,Q) — Q-tableaux

Ch 1d - Schützenberger jeu de taquin and Knuth transpositions

Knuth transposition







1 2 3 4 5 6 7 8 9 10
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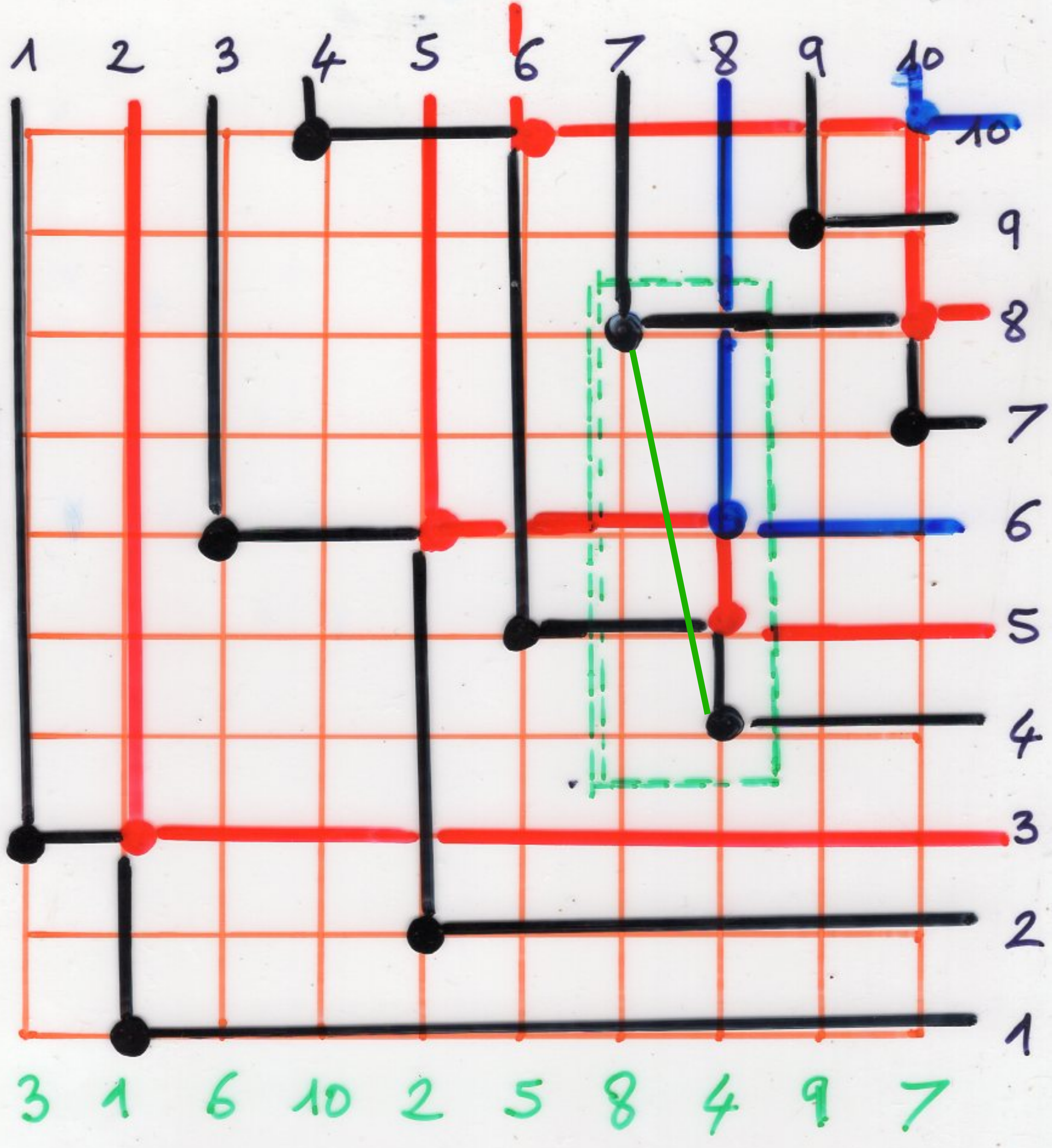
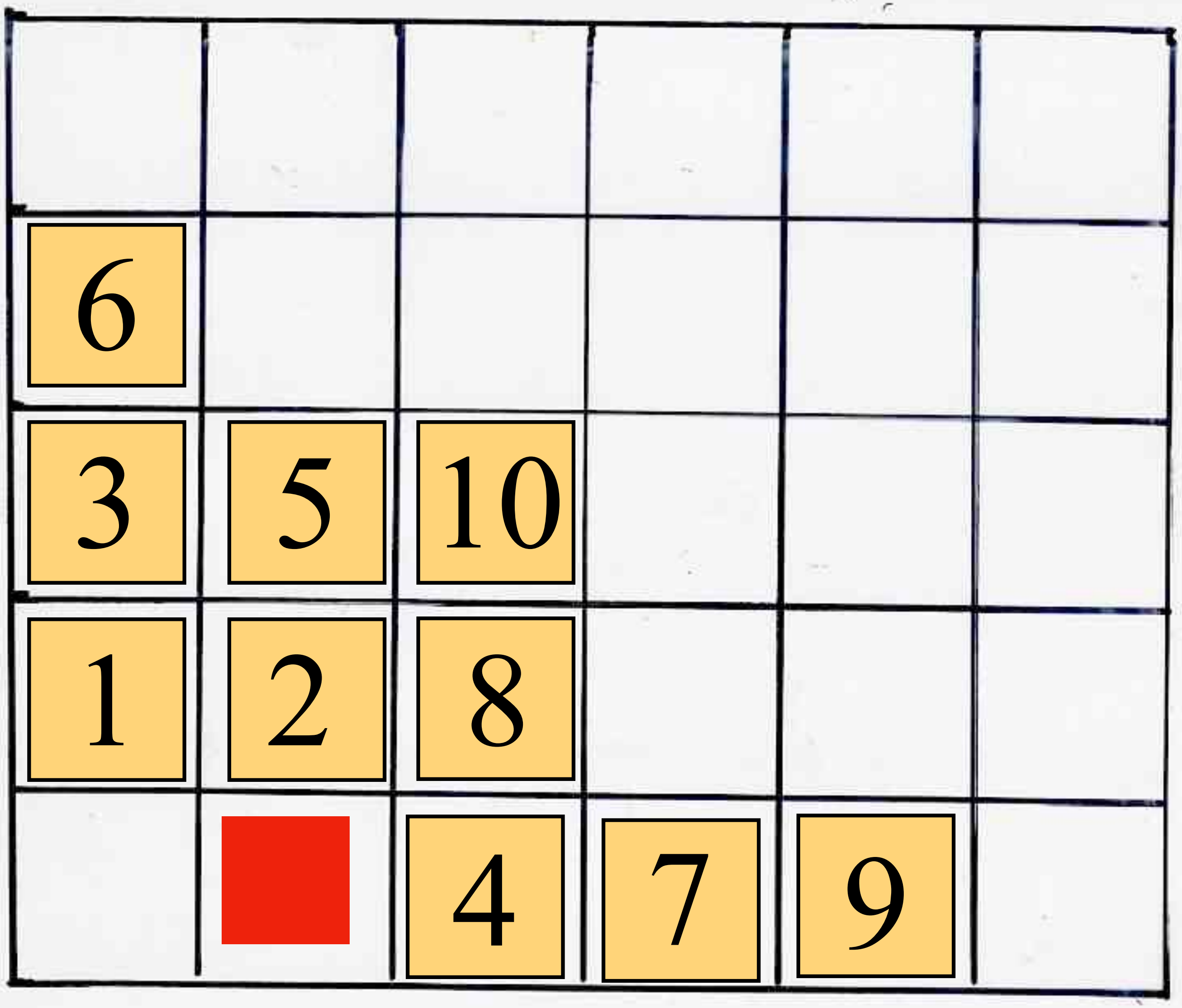
3 1 6 10 2 5 8 4 9 7


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
3 1 6 10 2 5 8 4 9 7

$T =$

6	10			
3	5	8		
1	2	4	7	9

$$\mathrm{Reading}(T) = \underbrace{6\ (10)}_{V_3}\ \underbrace{3\ 5\ 8}_{V_2}\ \underbrace{1\ 2\ 4\ 7\ 9}_{V_1}$$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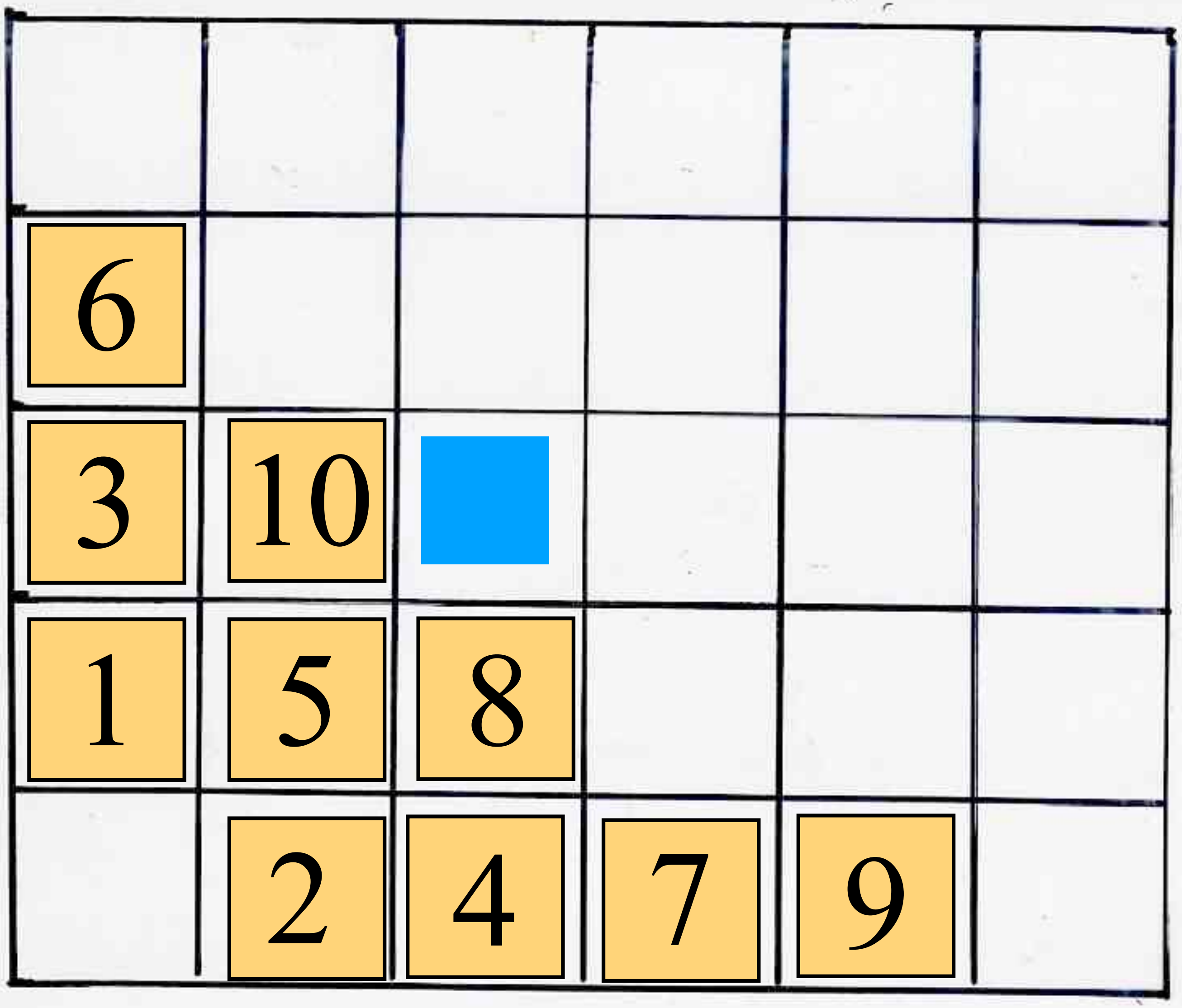
6
3
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1
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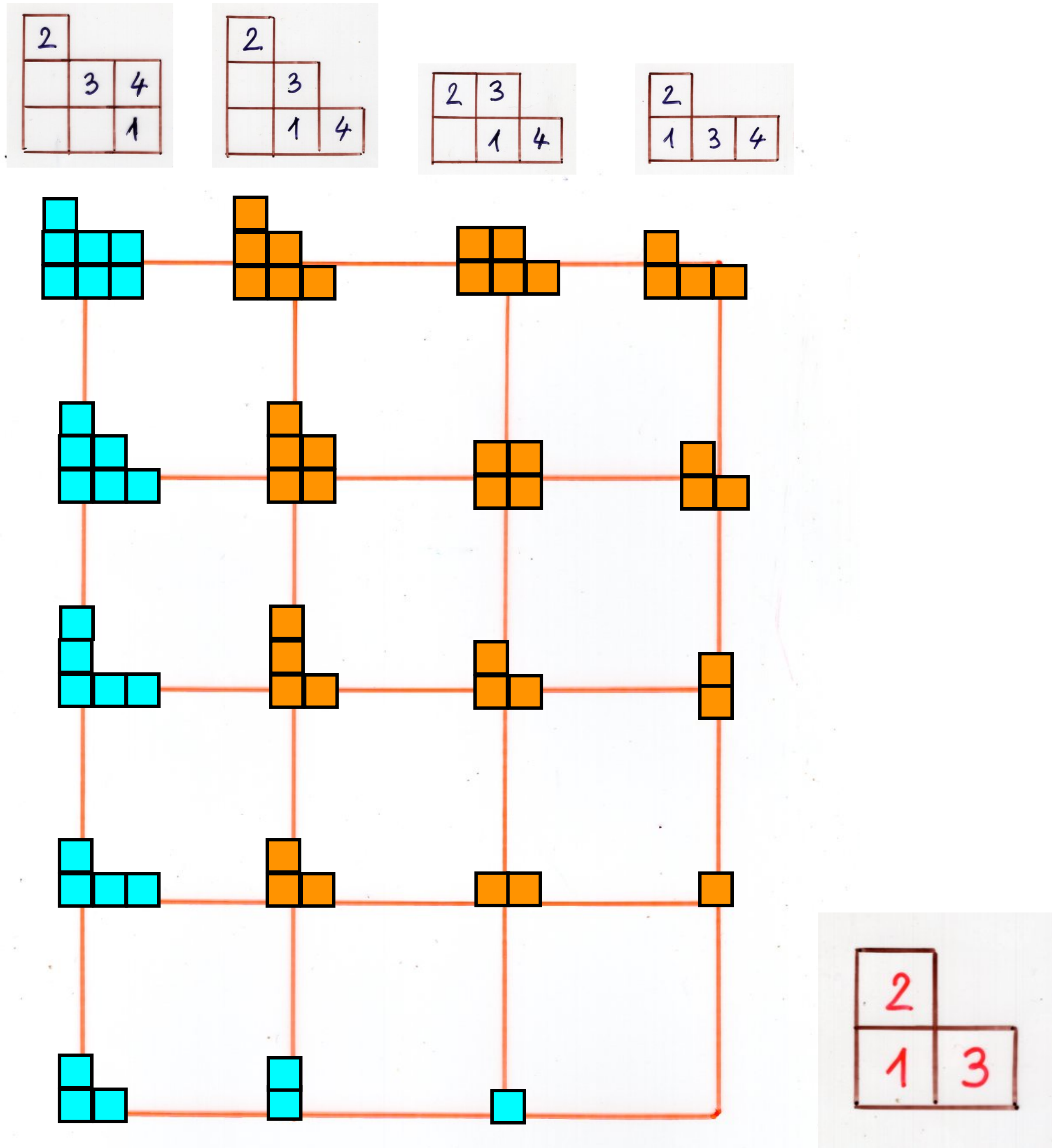
7
9

jeu de taquin
local rules

(Fomin)

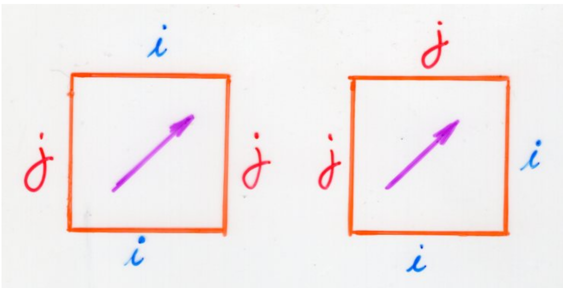


diagonal operators

$\Delta_i \quad i \in \mathbb{Z}$

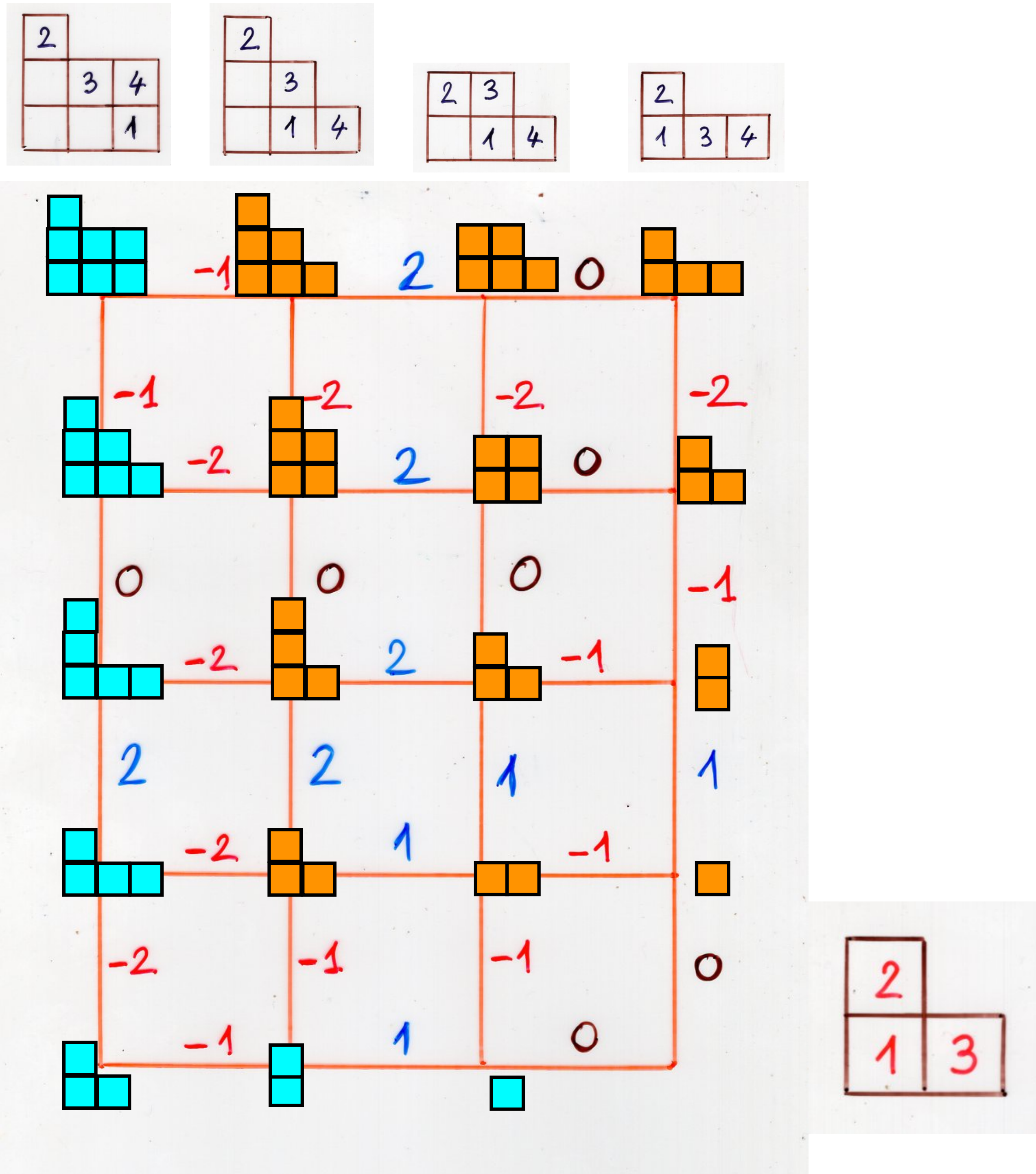
jeu de taquin

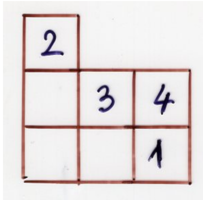
local rules on edges

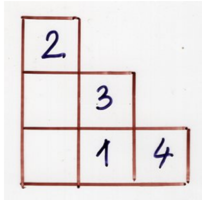


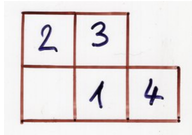
$|i-j| \geq 2$ $\quad$ $|i-j| \leq 1$

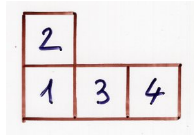
$i, j \in \mathbb{Z}$

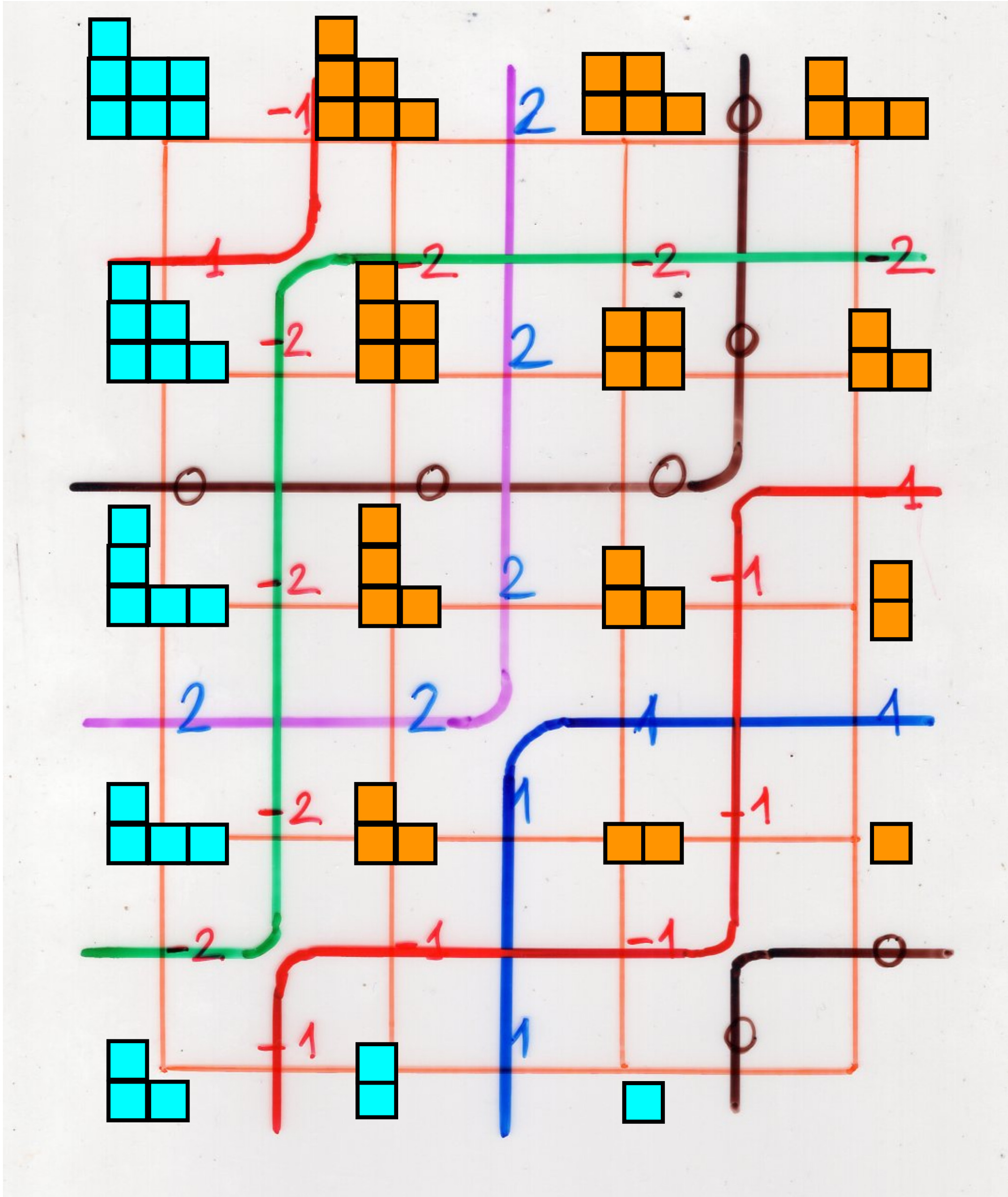


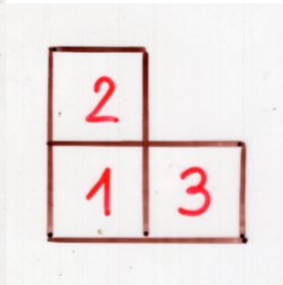

2
3 4
1


2
3
1 4

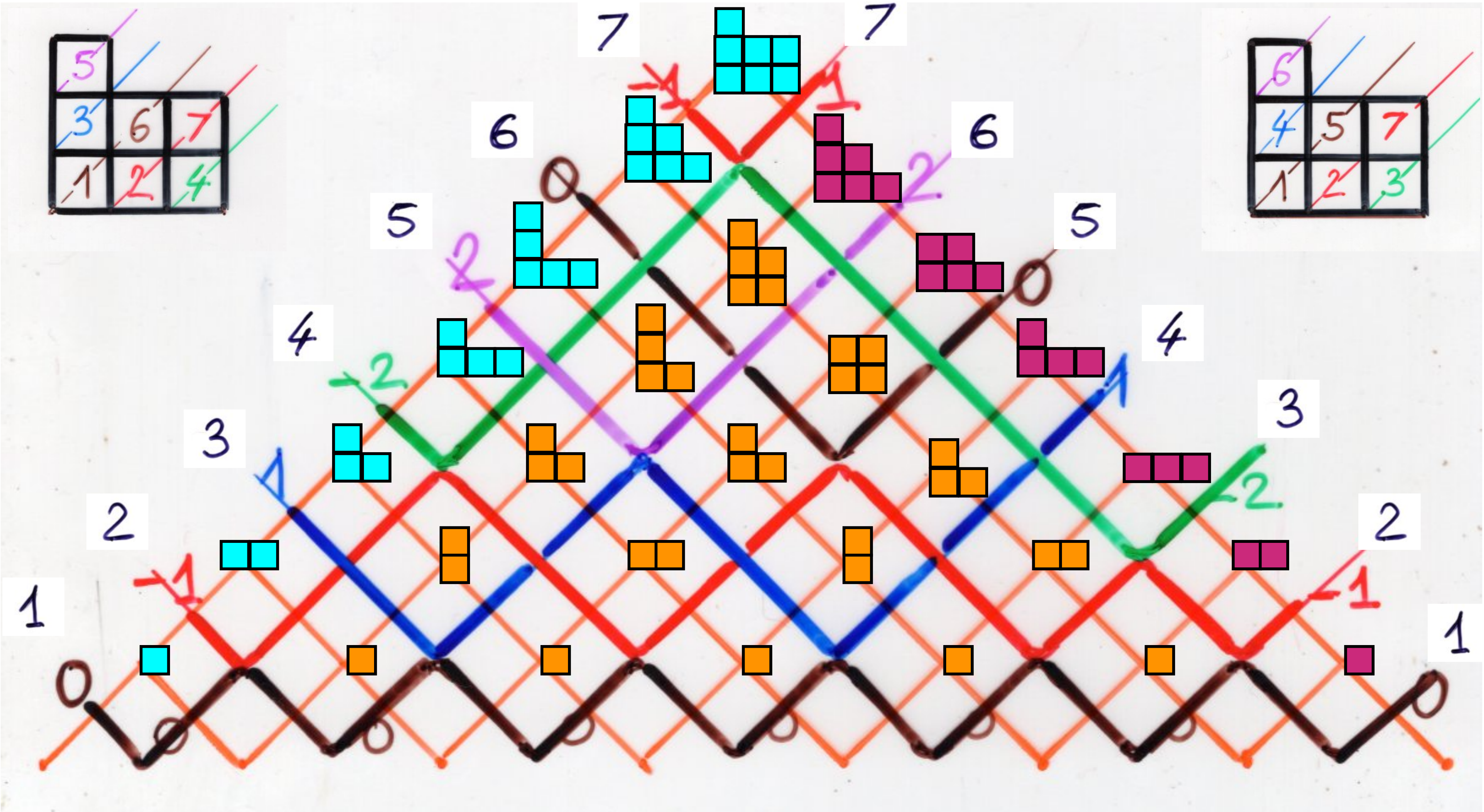

2 3
1 4


2
1 3 4




2
1 3

dual of a tableau

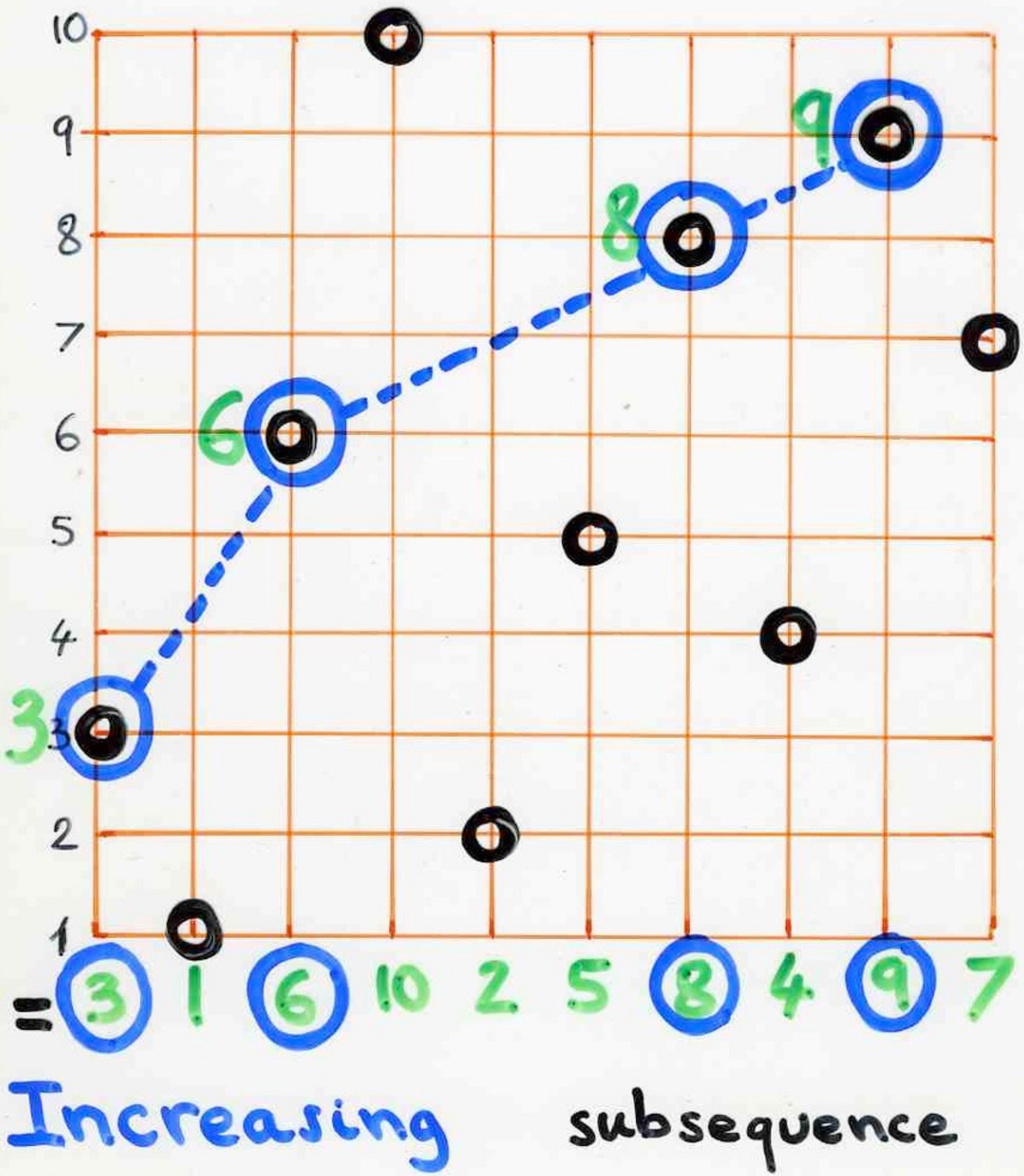


Schützenberger involution

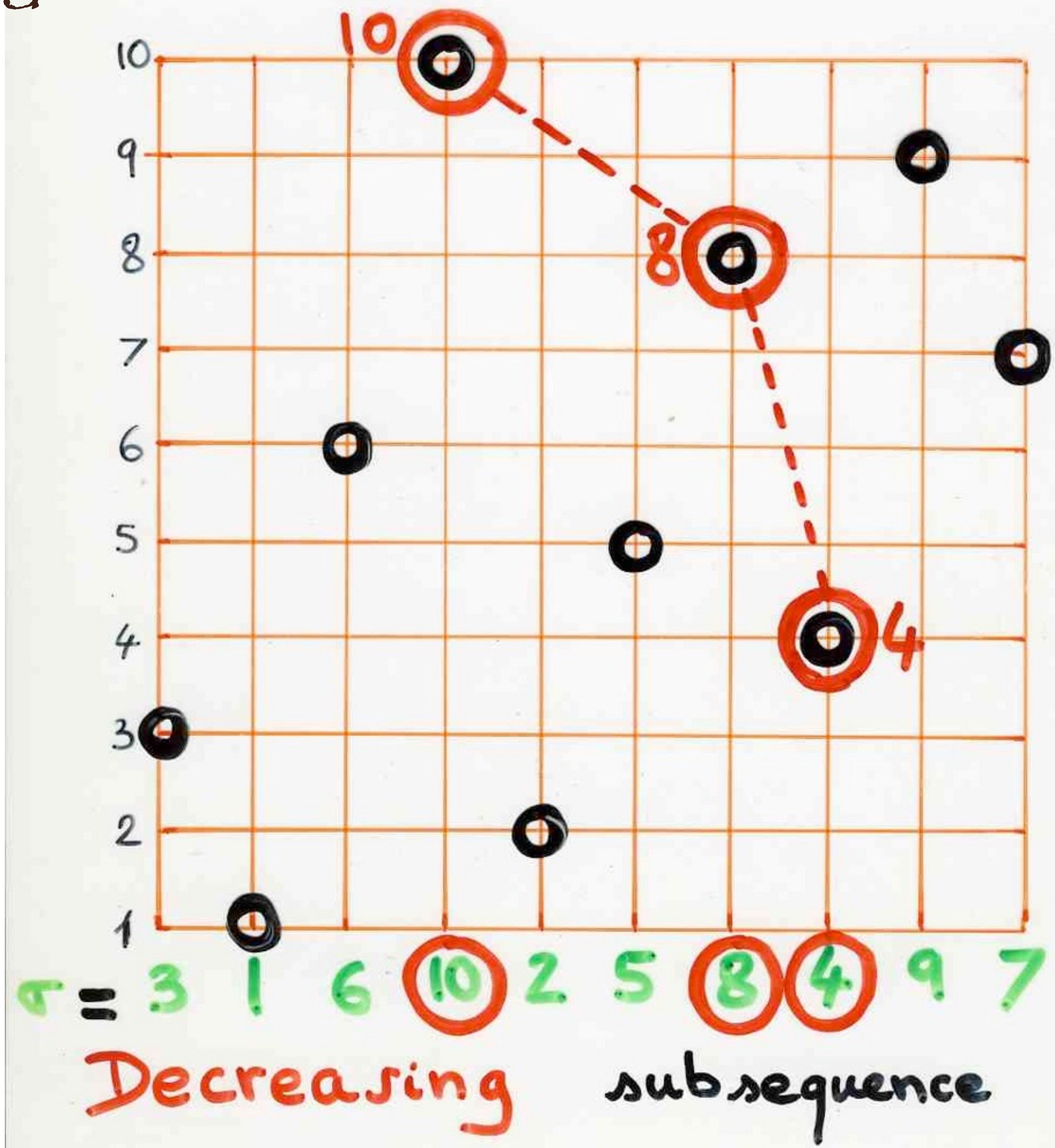
Greene theorem

C. Greene, 1974

Ch1a



Ch1a



σ permutation $\mathfrak{S}_n$

$k \in \mathbb{N}$

$I_k(\sigma)$ = maximal number of elements in a union of k increasing subsequences of σ

$D_\ell(\sigma)$ = - - - - - . . k decreasing . . .

Theorem (Greene)(1974)

$$\sigma \xrightarrow{RS} (P, Q)$$

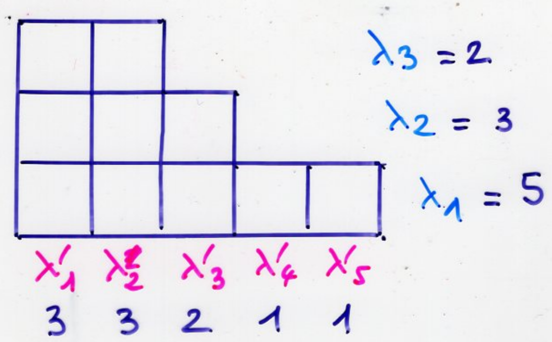
$\lambda = (\lambda_1, .., \lambda_r)$ $\quad \lambda_1 \geq \cdot\cdot \geq \lambda_r$

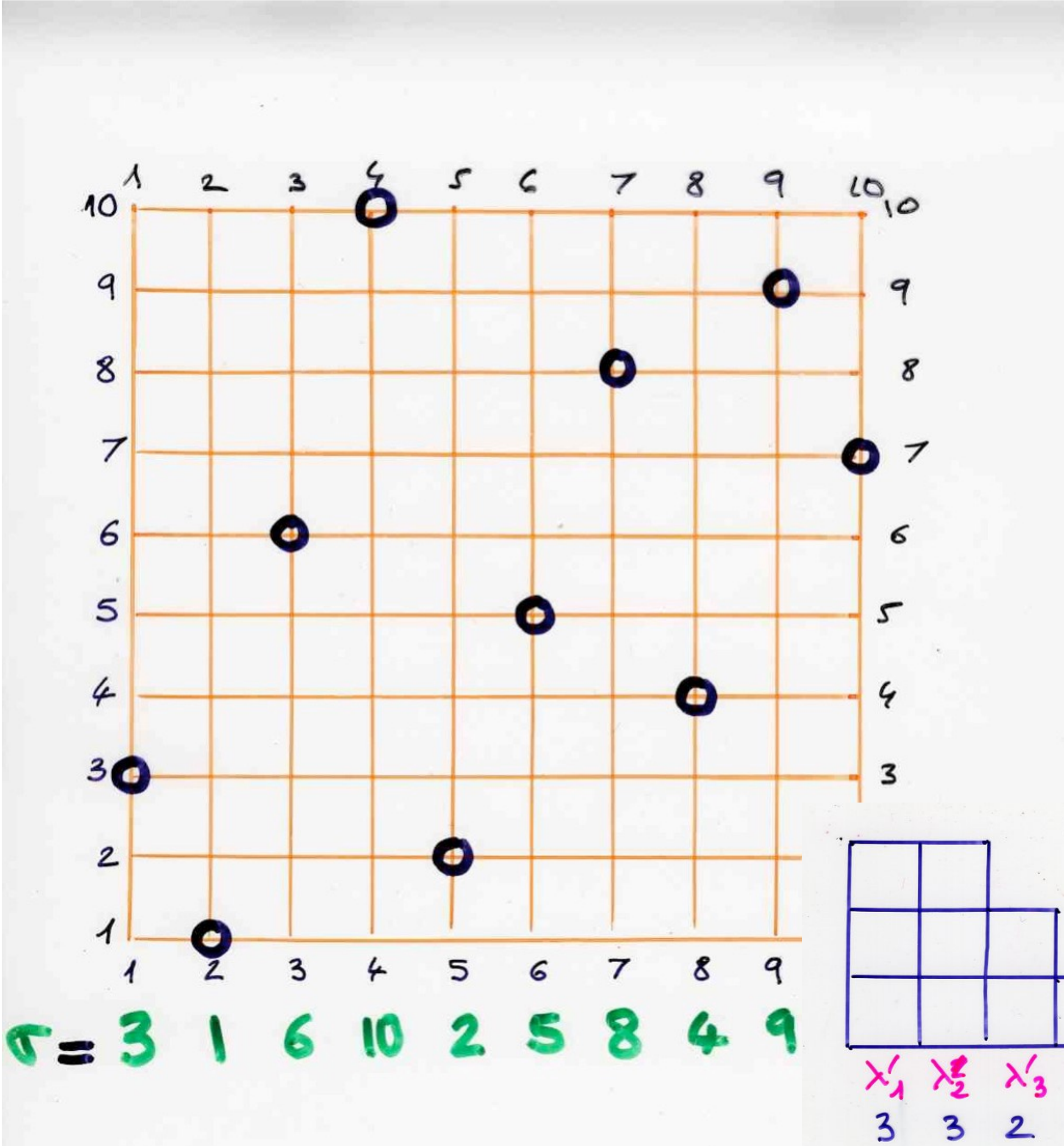
common shape of P and Q

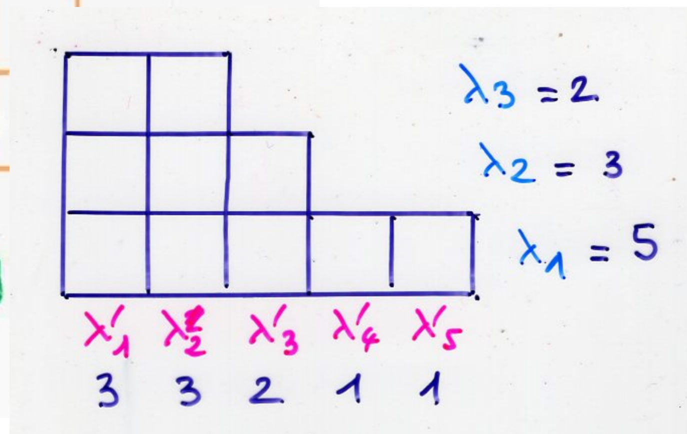
$$I_k(\sigma) = \lambda_1 + .. + \lambda_k$$

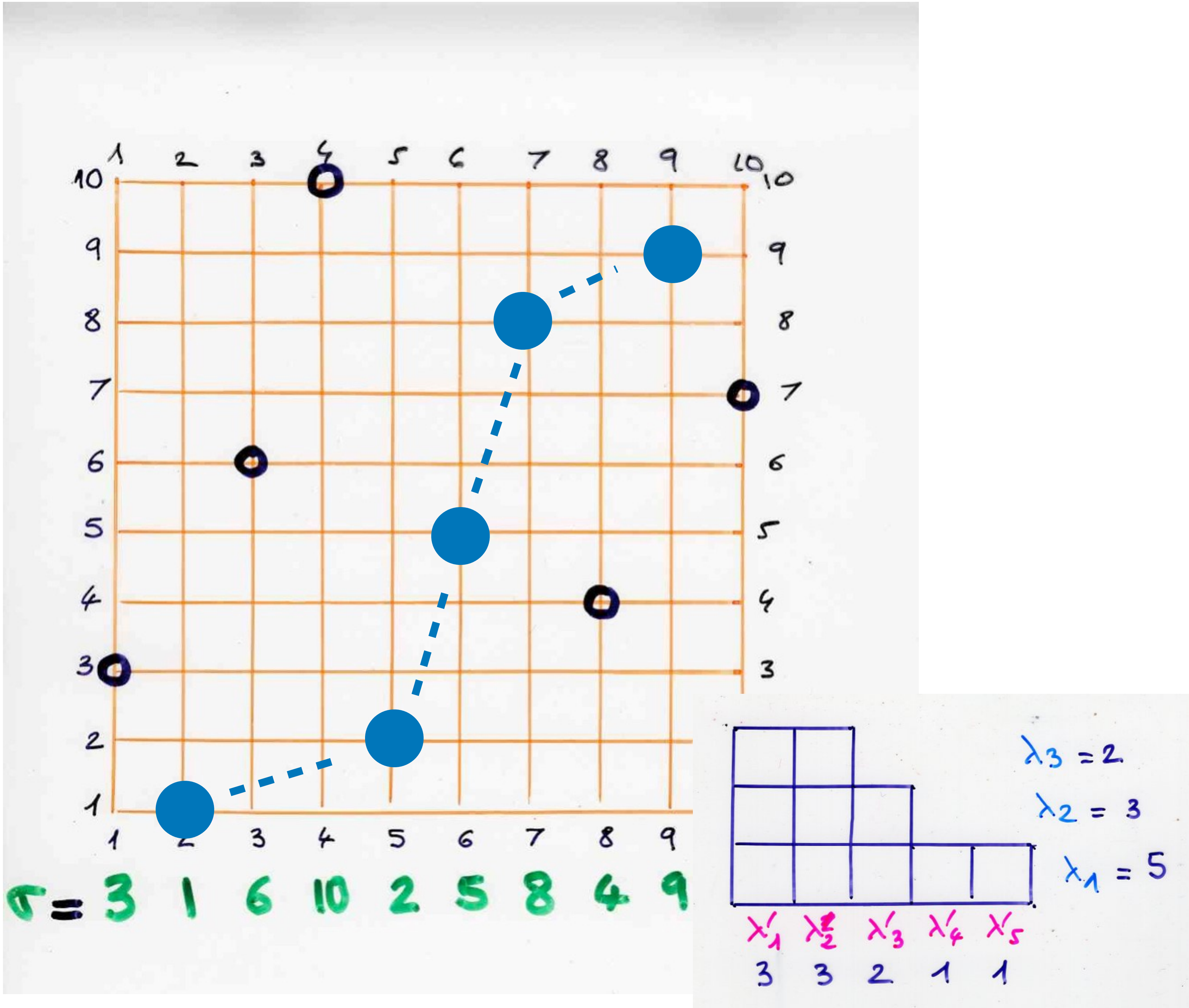
$$D_\ell(\sigma) = \lambda'_1 + .. + \lambda'_\ell$$

conjugate partition

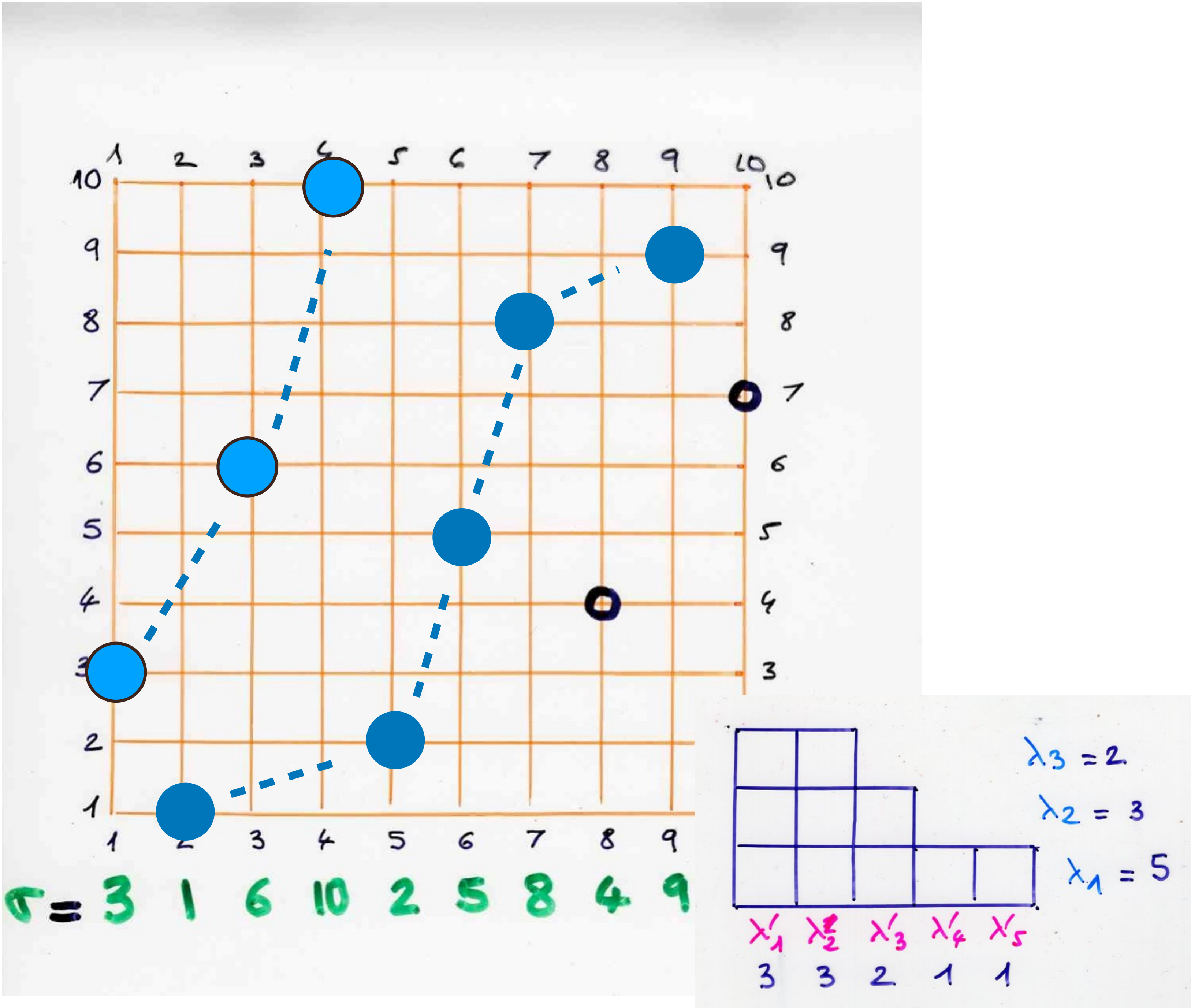


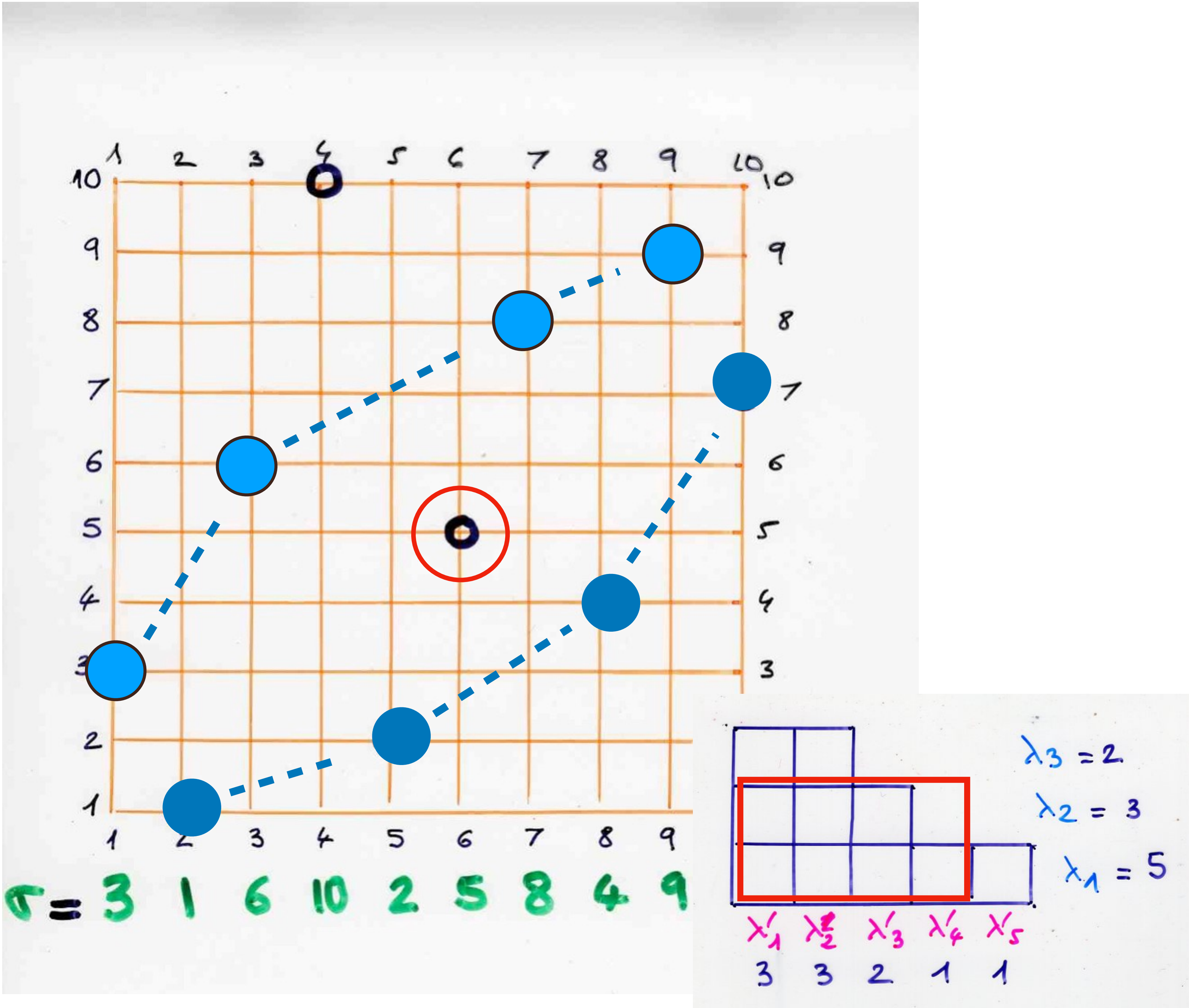

σ = 3 1 6 10 2 5 8 4 9


$\lambda_3 = 2$
$\lambda_2 = 3$
$\lambda_1 = 5$
λ'_1 λ'_2 λ'_3 λ'_4 λ'_5
3 3 2 1 1



1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9
λ3 = 2
λ2 = 3
λ1 = 5
λ'1 λ'2 λ'3 λ'4 λ'5
3 3 2 1 1


σ = 3 1 6 10 2 5 8 4 9
λ3 = 2
λ2 = 3
λ1 = 5
λ'1 λ'2 λ'3 λ'4 λ'5
3 3 2 1 1


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9
λ3 = 2
λ2 = 3
λ1 = 5
λ'1 λ'2 λ'3 λ'4 λ'5
3 3 2 1 1

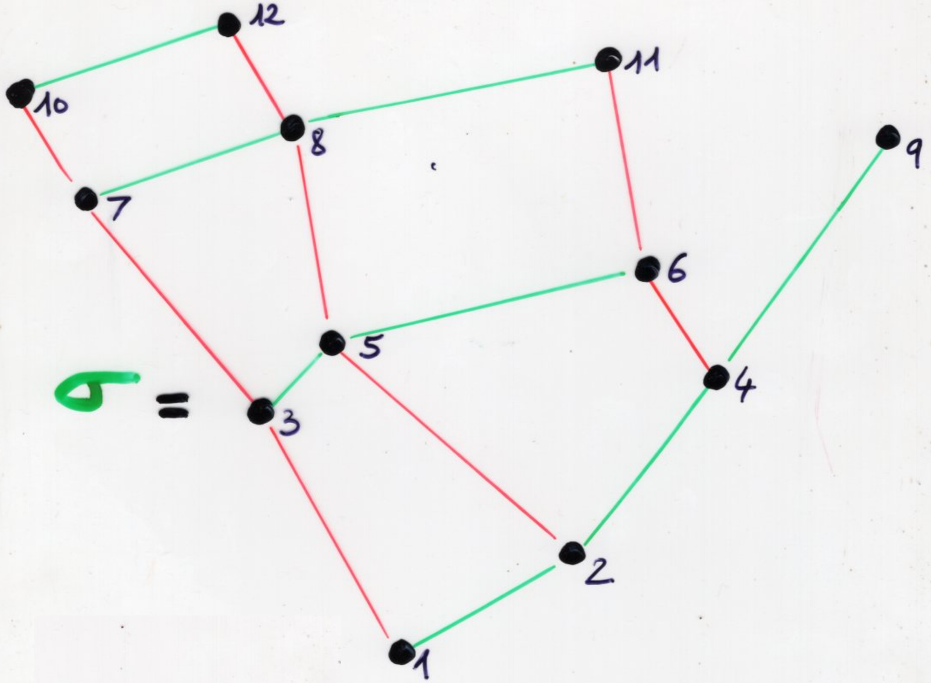
Lemma

For any k, ℓ, $I_k(\sigma)$ and $D_\ell(\sigma)$ are invariant under Knuth transpositions

$P(\sigma) =$

10	12		
7	8	11	
3	5	6	
1	2	4	9

"regular" permutation



$P(\sigma) =$

10	12		
7	8	11	
3	5	6	
1	2	4	9

example

$$\sigma = \begin{pmatrix} (10) & 7 & (12) & 3 & 8 & 5 & 1 & 2 & (11) & 6 & 4 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

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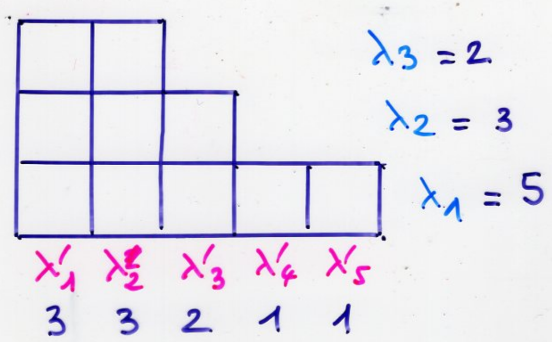
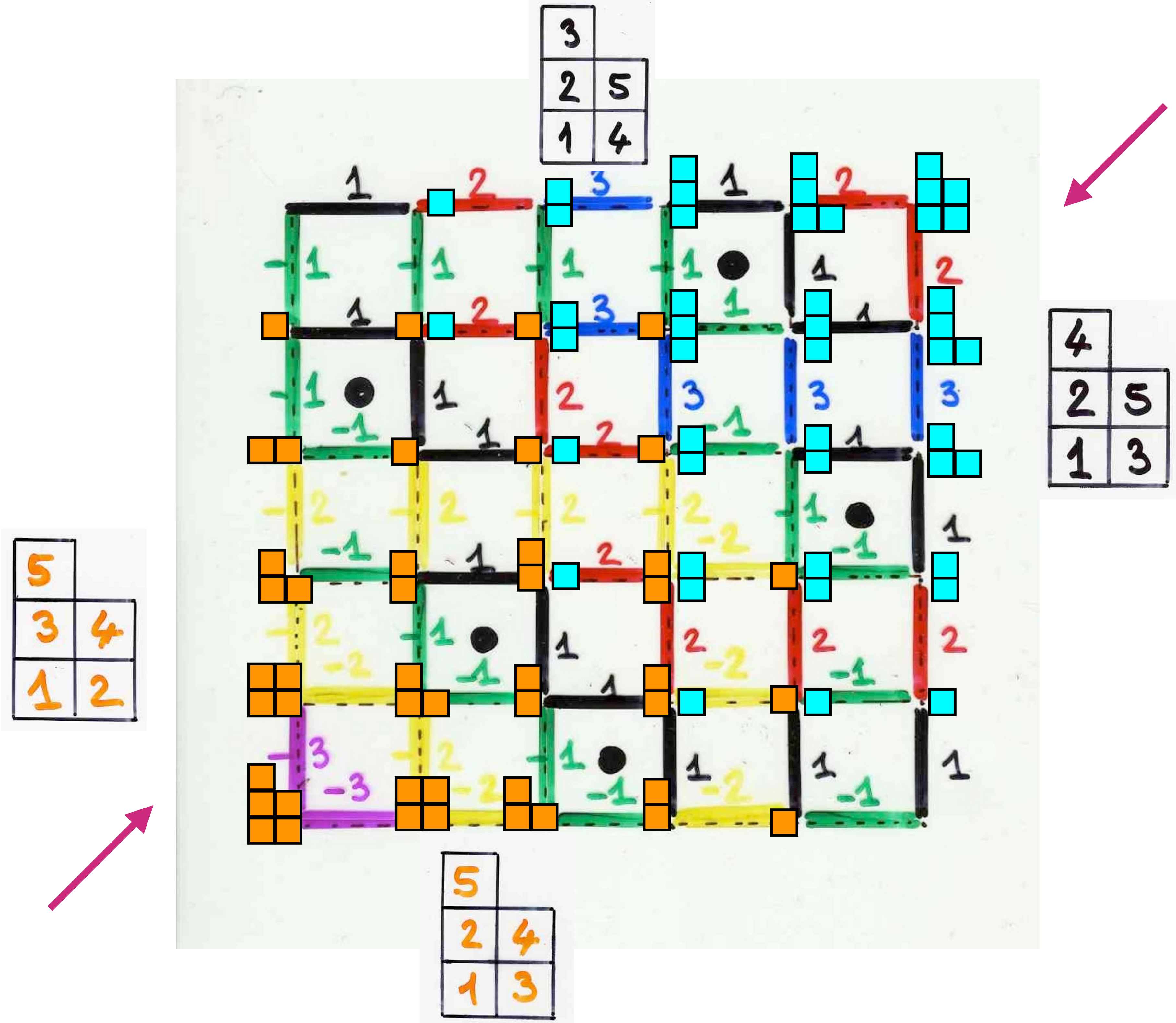


Proposition. Let $\sigma \xrightarrow{RS} (P, Q)$

permutation

then $\sigma^{\#} \xrightarrow{RS} (P^{*}, Q^{*})$


3
2 5
1 4
4
2 5
1 3
5
3 4
1 2
5
2 4
1 3

Proposition. Let σ (permutation) $\xrightarrow{RS} (P, Q)$

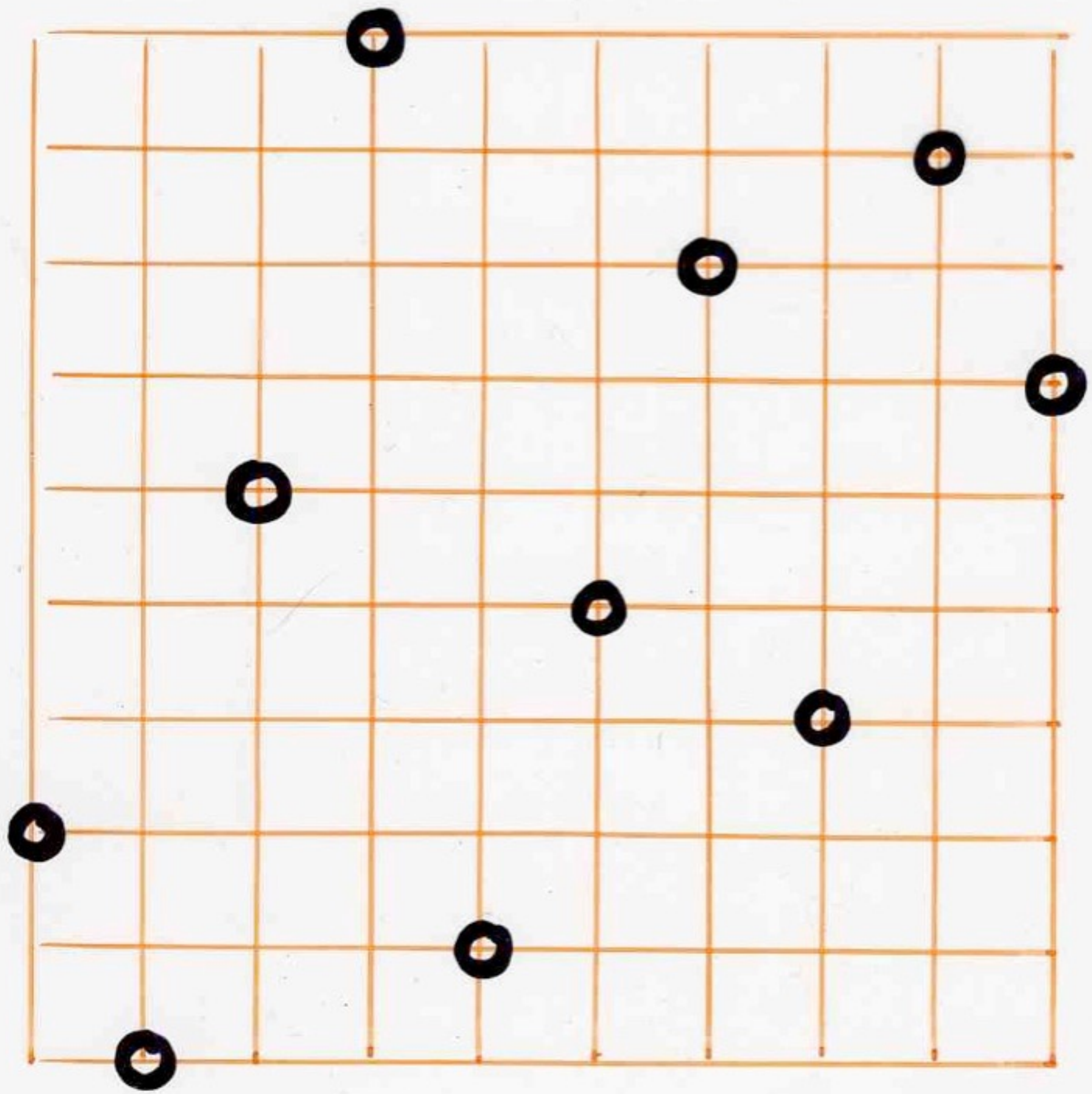
then $\sigma^{\#} \xrightarrow{RS} (P^*, Q^*)$

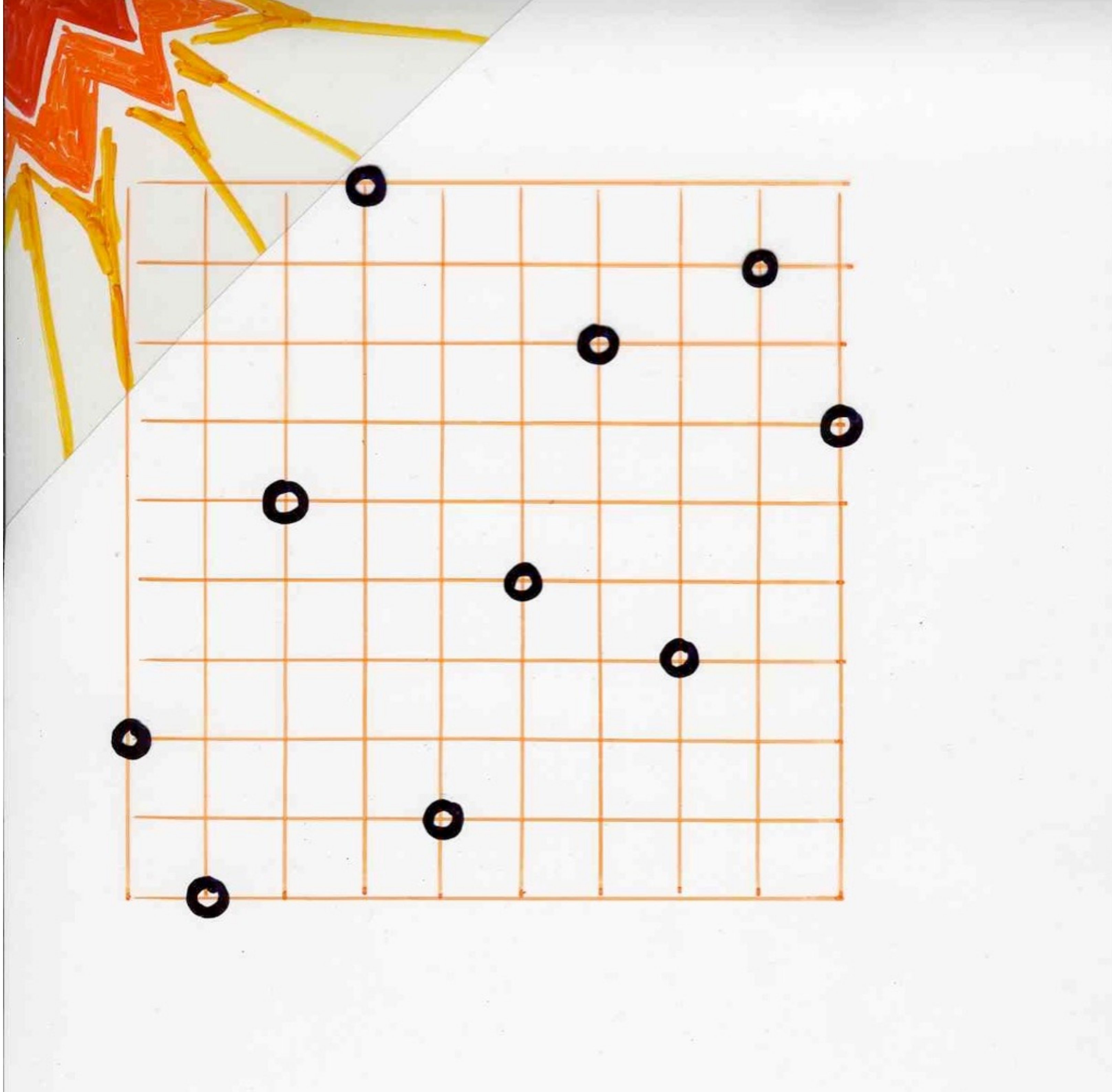
σ^{t} (transpose) $\xrightarrow{RS} (P^t, (Q^*)^t)$

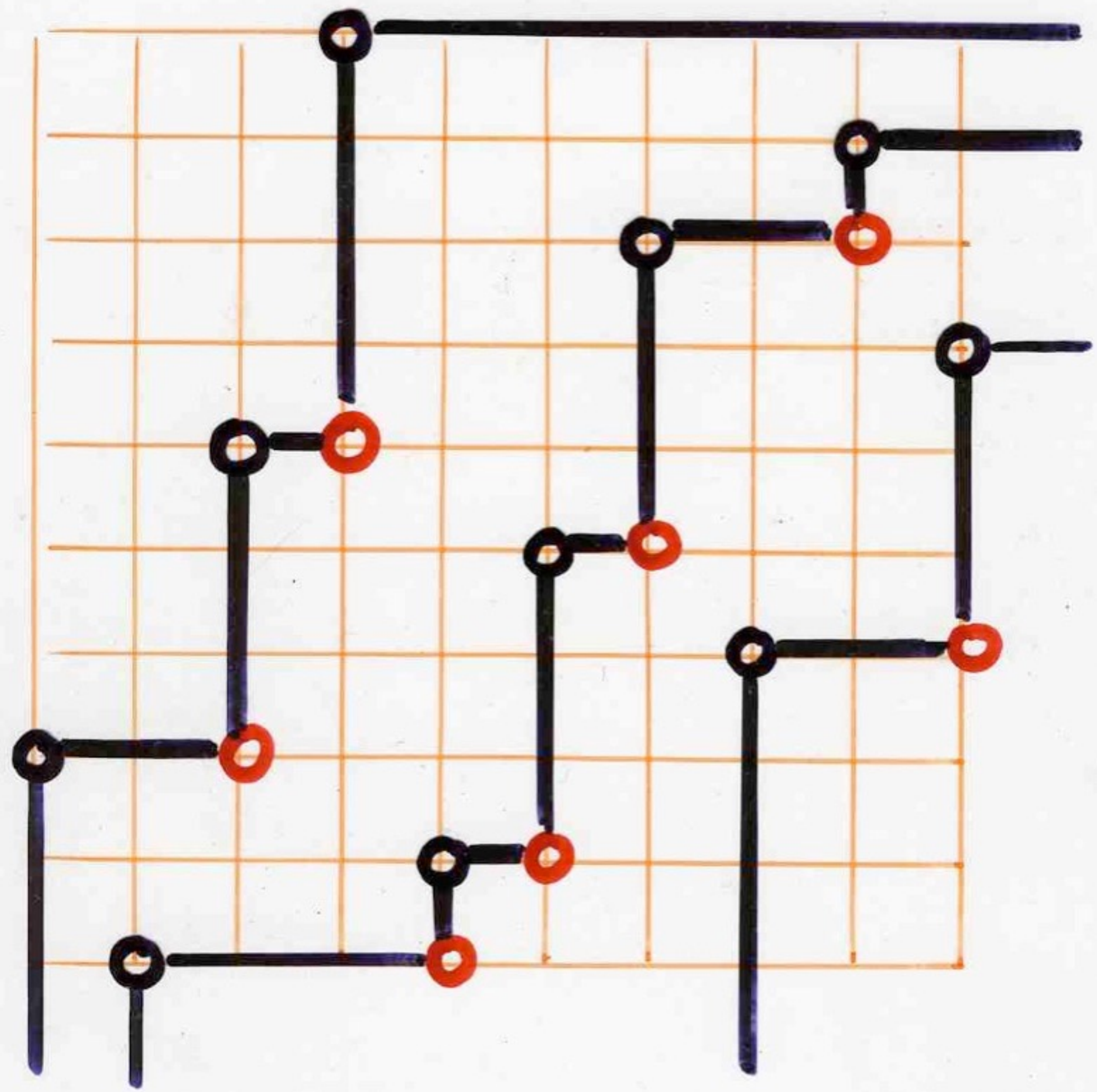
example

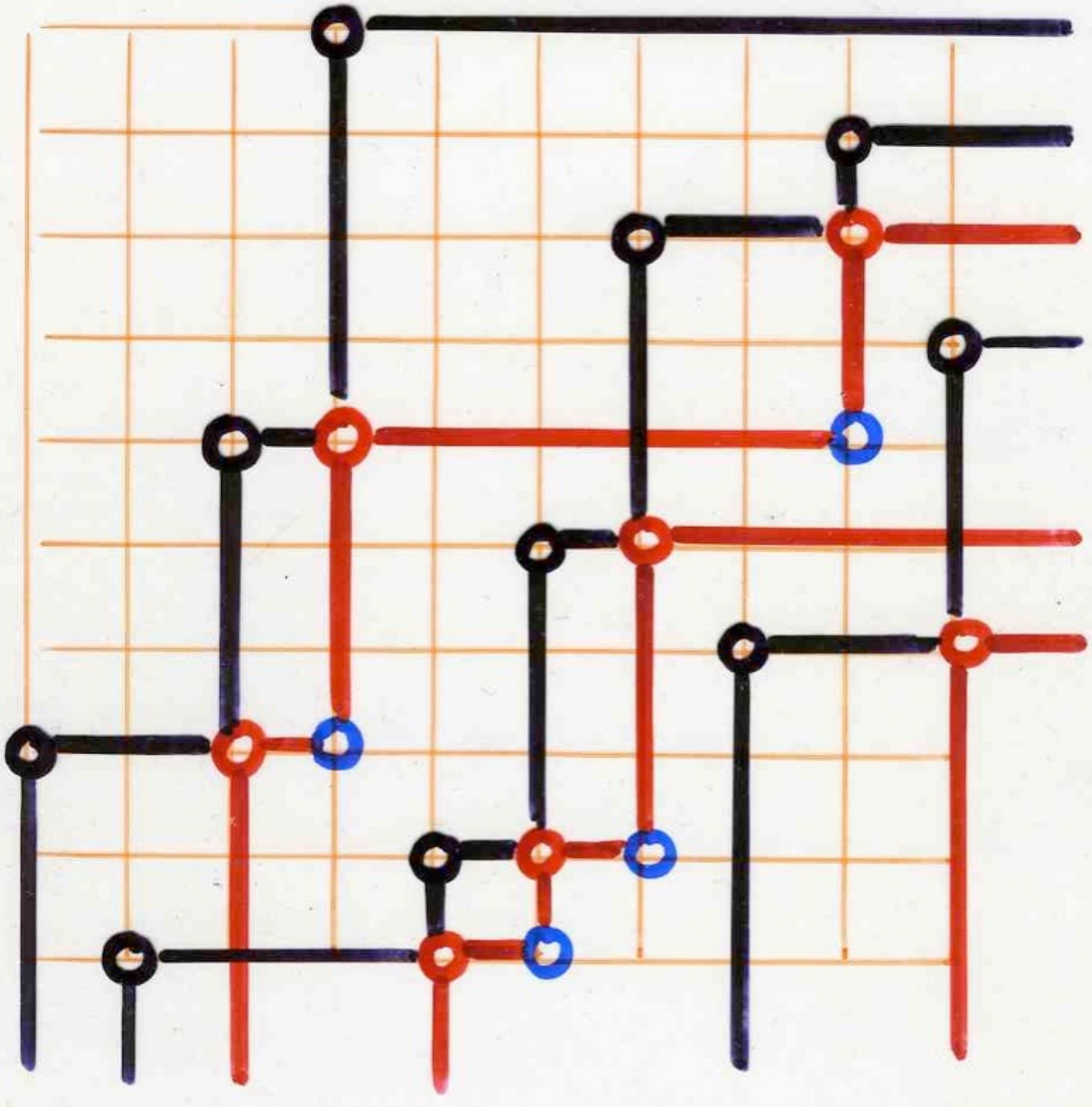
$$\sigma^t \xrightarrow{RS} (P^t, (Q^*)^t)$$

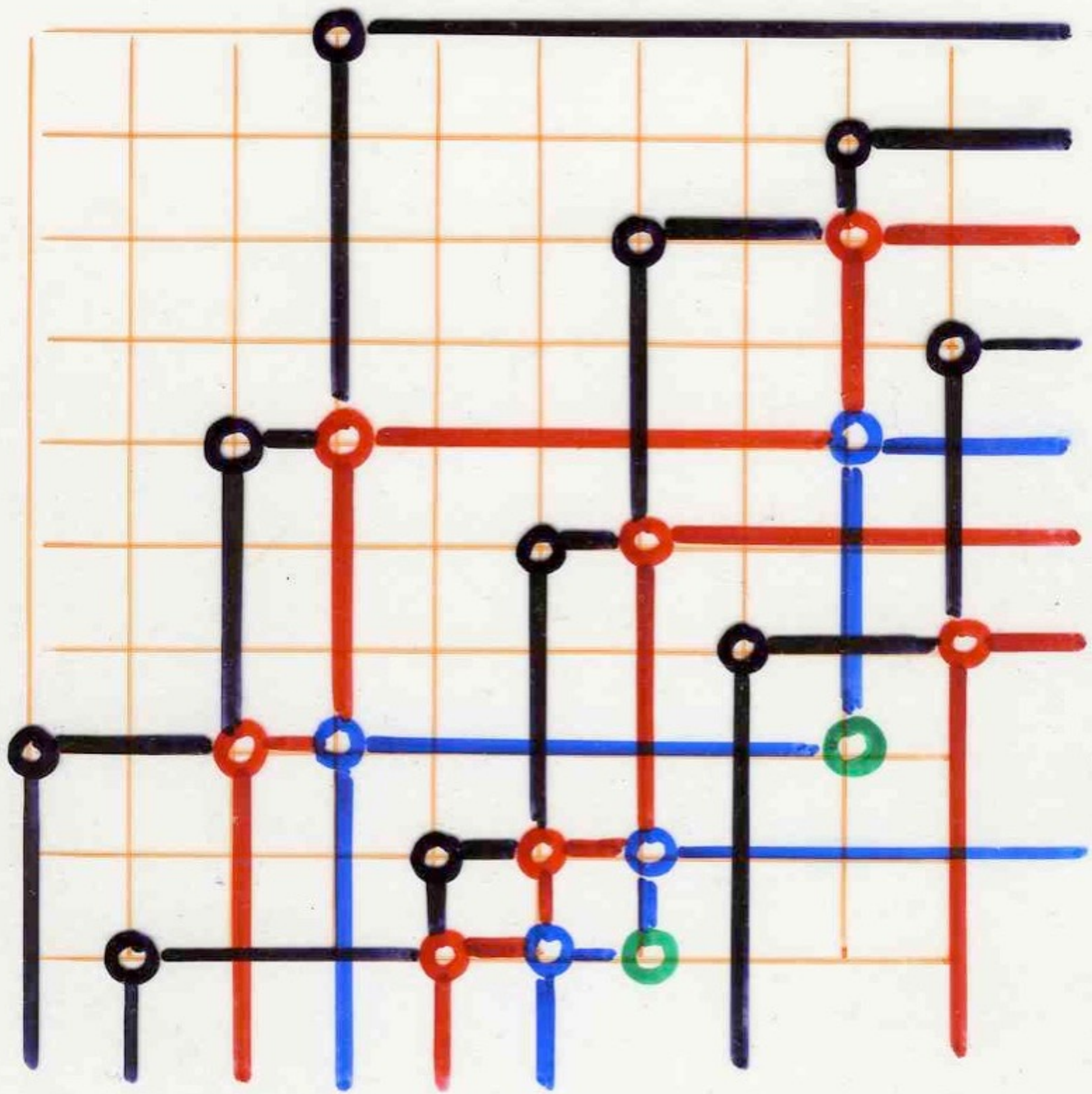
(transpose)

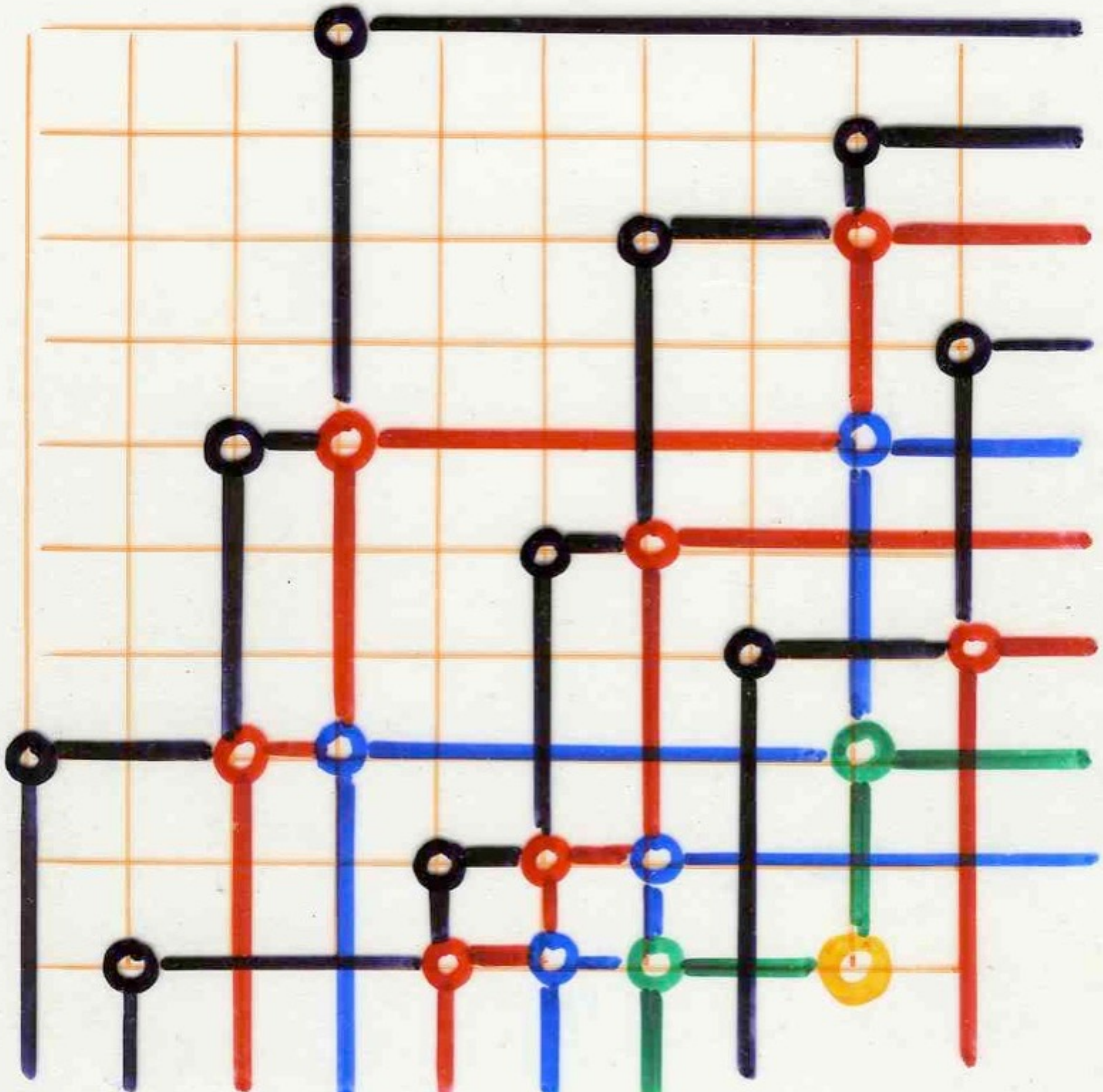


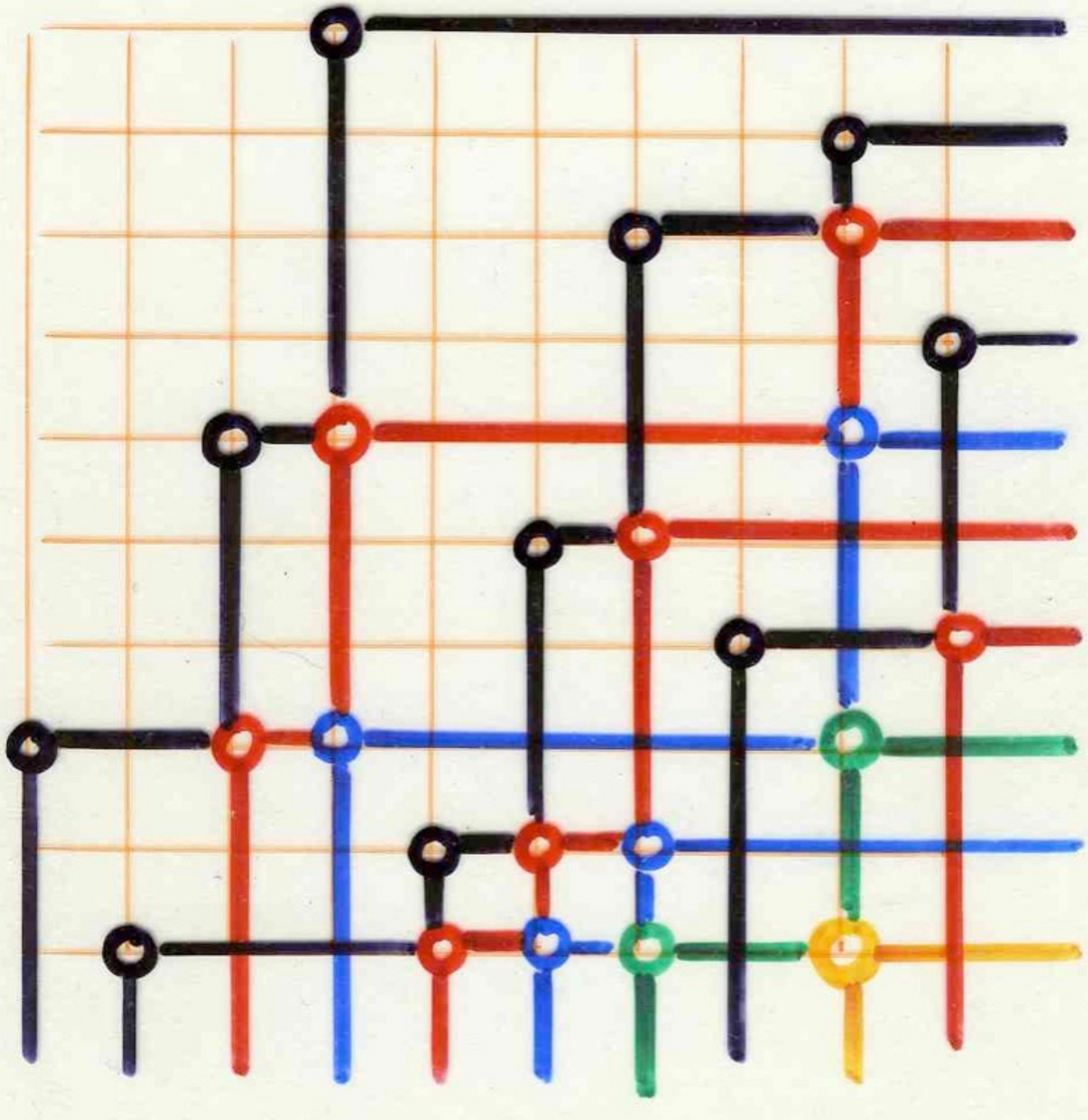


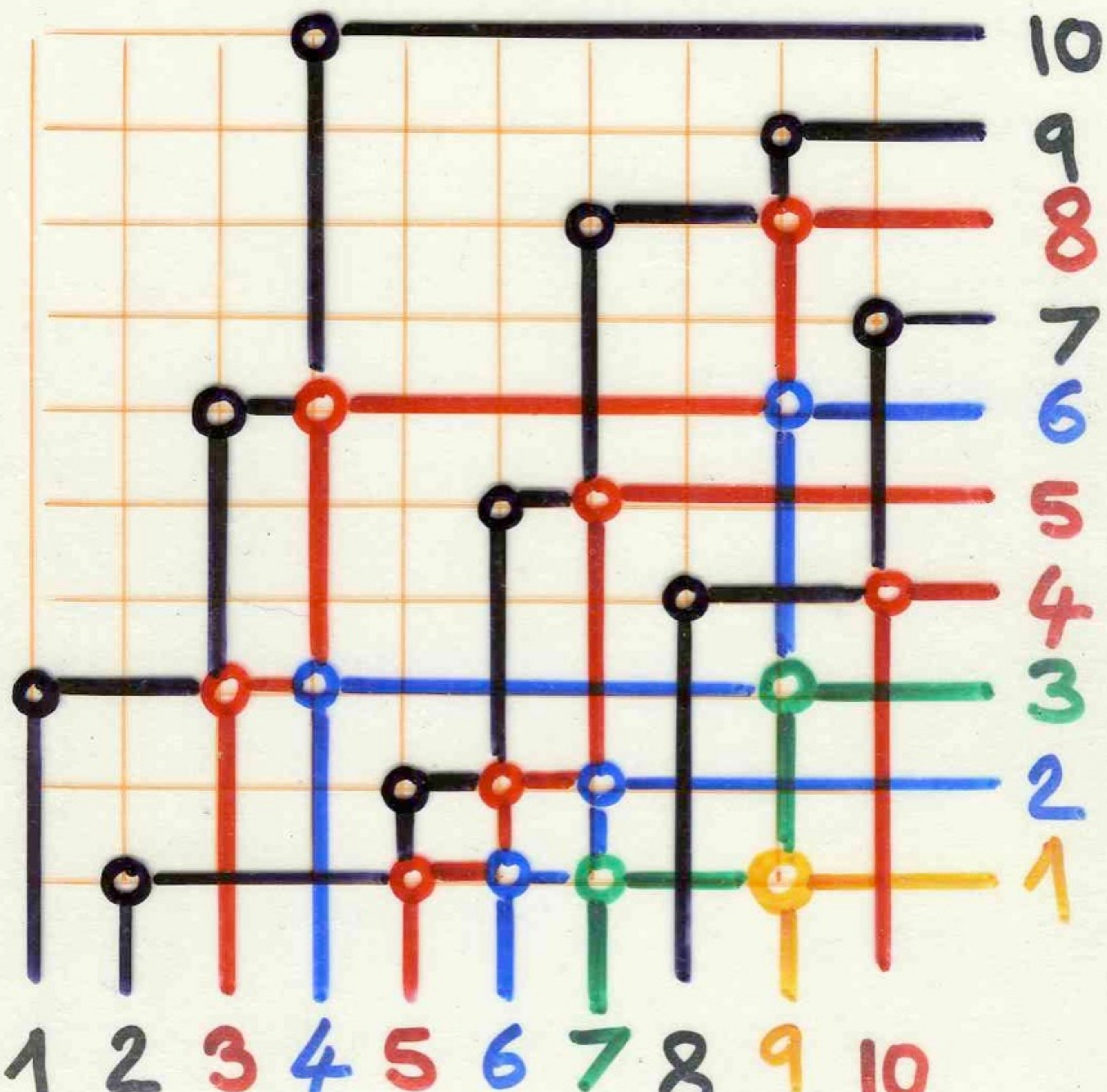











10
9
8
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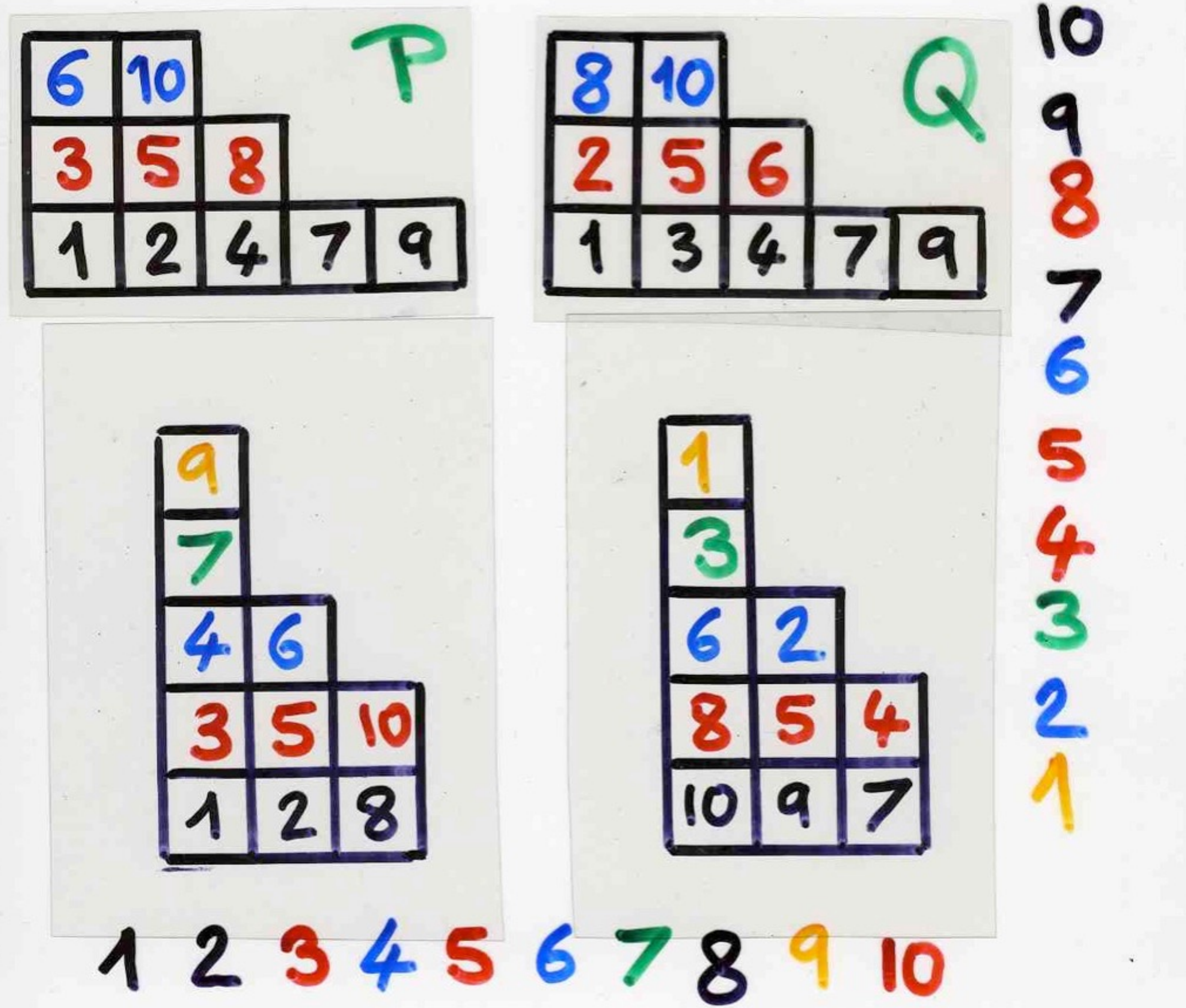
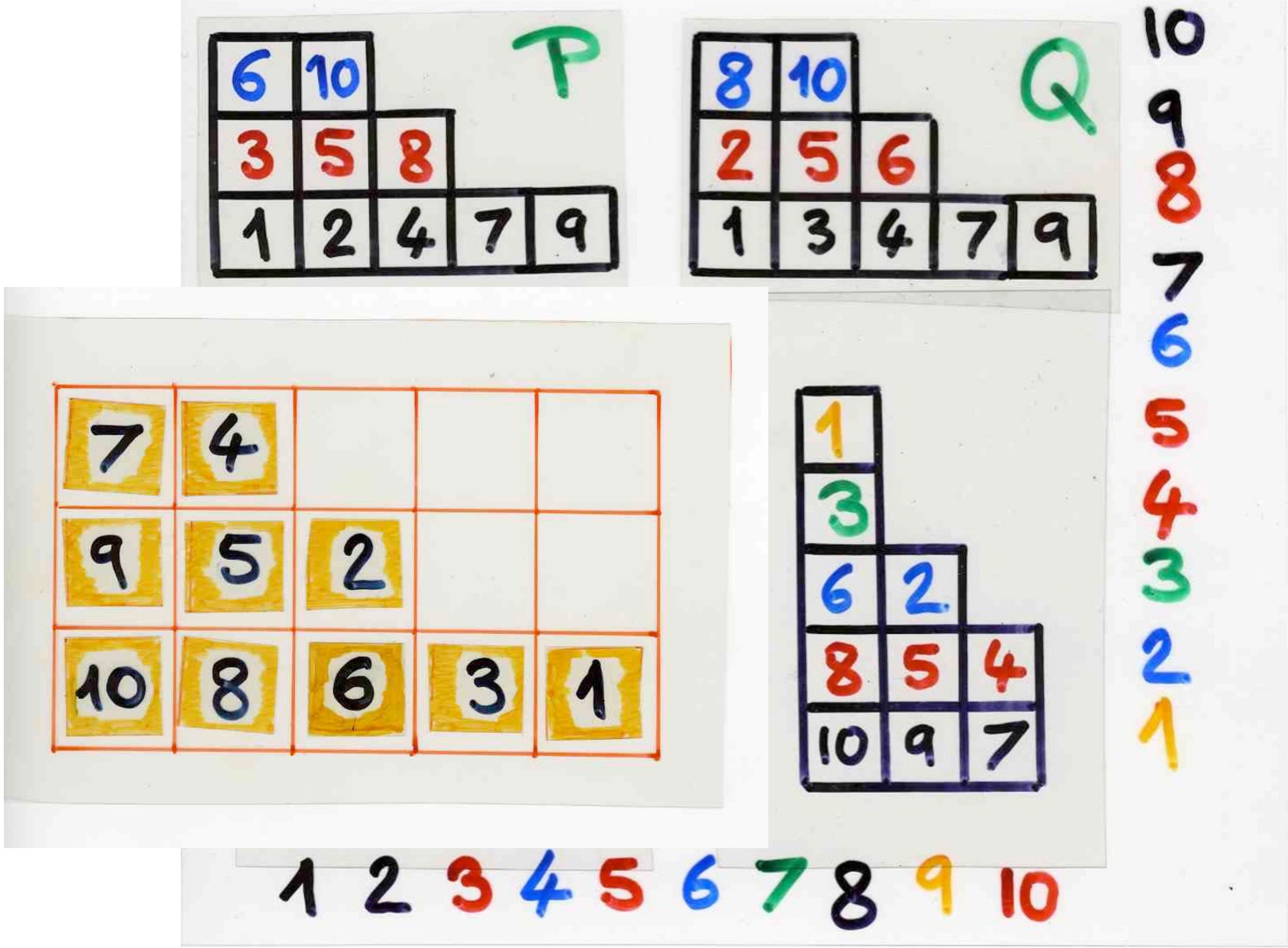
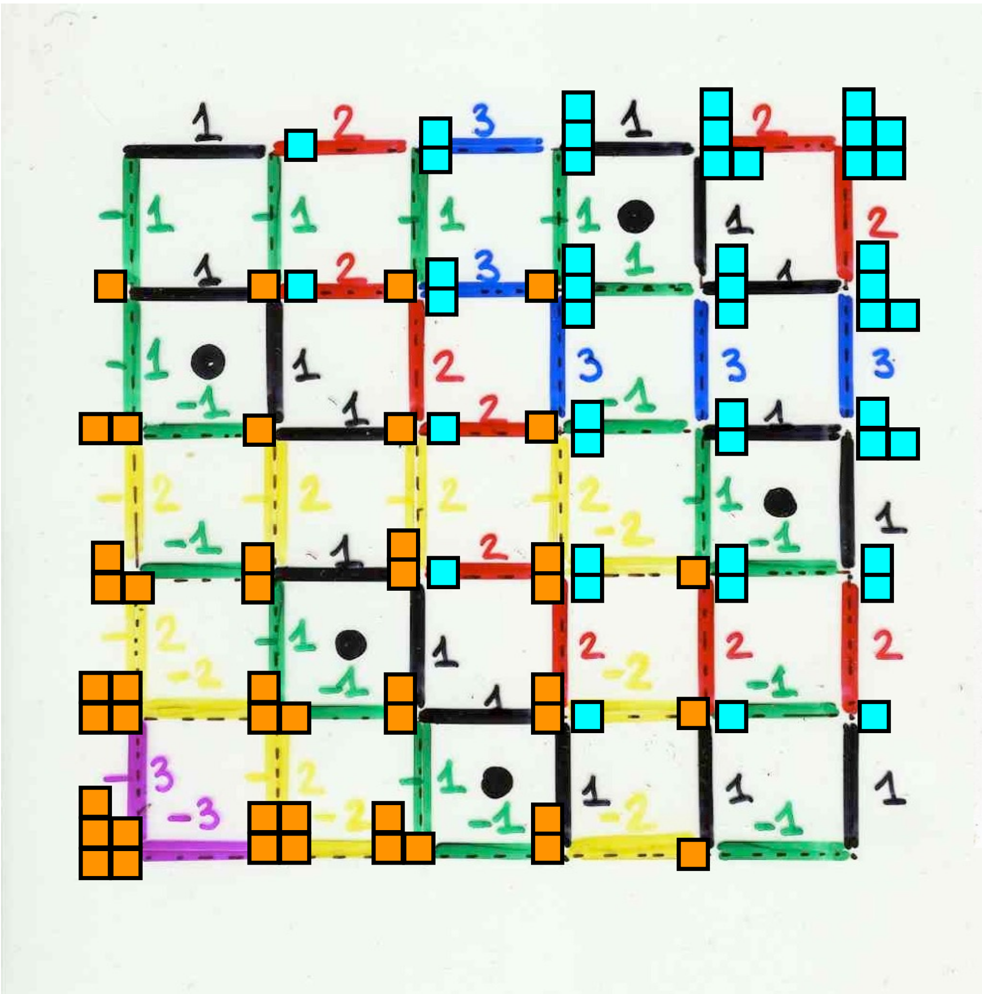
2
1
1 2 3 4 5 6 7 8 9 10

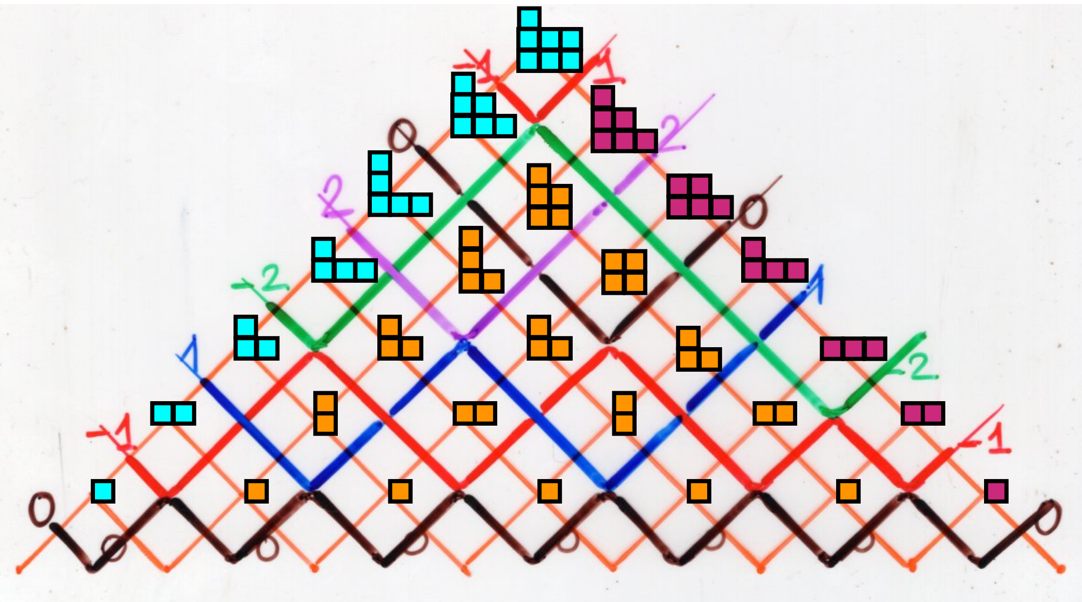

P
Q
1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1


P
6 10
3 5 8
1 2 4 7 9
Q
8 10
2 5 6
1 3 4 7 9
10 9 8 7 6 5 4 3 2 1
7 4
9 5 2
10 8 6 3 1
1
3
6 2
8 5 4
10 9 7
1 2 3 4 5 6 7 8 9 10

Research Problem

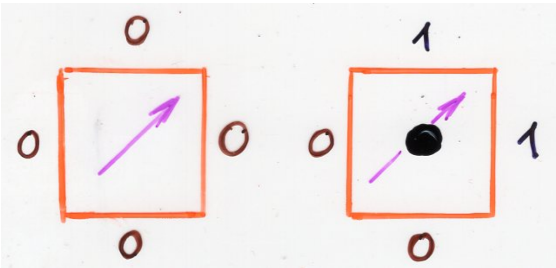
$$\sigma^{\#} \xrightarrow{RS} (P^{*}, Q^{*})$$

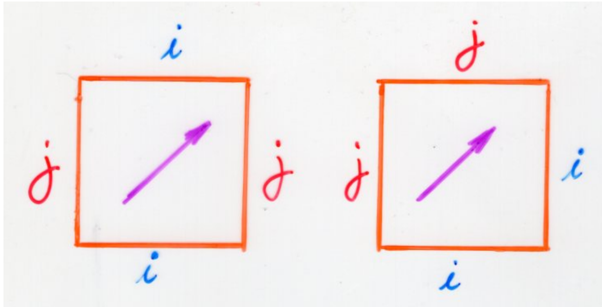




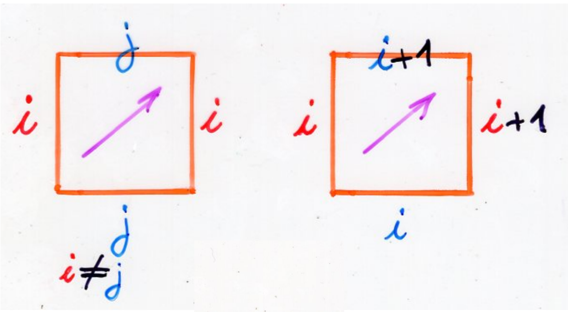
Research Problem

jeu de taquin
local rules on edges





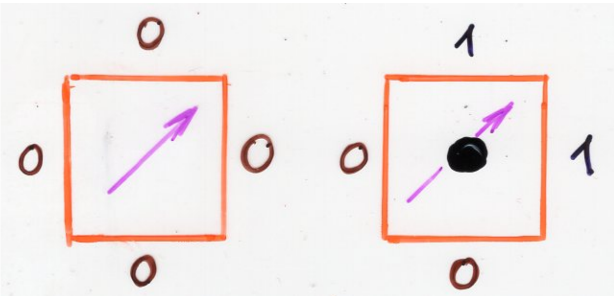
$|i-j| \geq 2$ $\quad$ $|i-j| \leq 1$



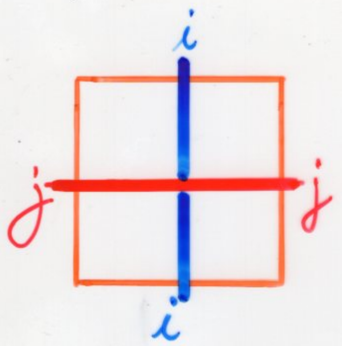
$i, j \in \mathbb{Z}$

The RSK (reverse) planar automaton

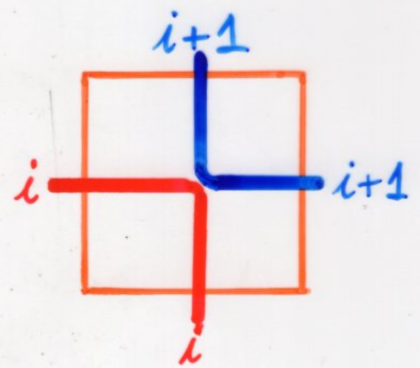
Research Problem

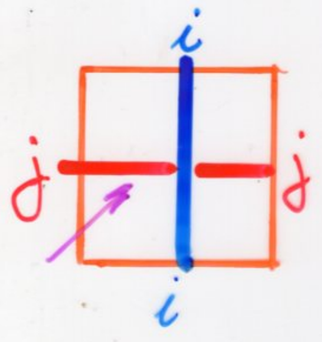


jeu de taquin
local rules on edges

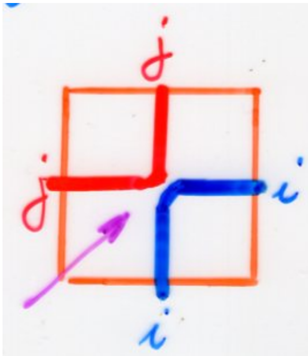


$i \neq j$





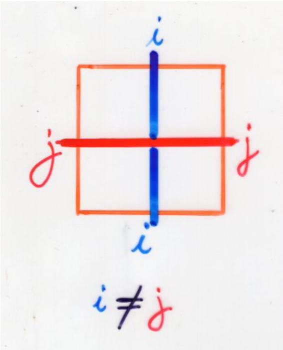
$|i-j| \geq 2$

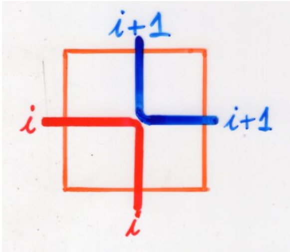


$|i-j| \leq 1$

$i, j \in \mathbb{Z}$

The RSK (reverse) planar automaton

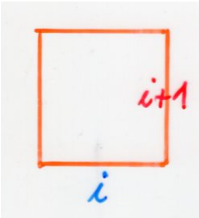


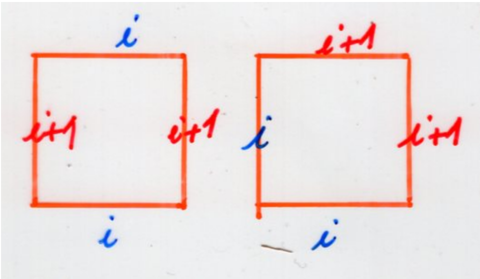


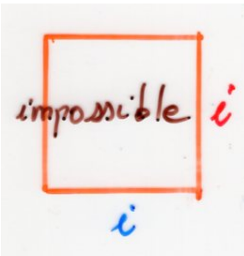
$$\sigma^t \xrightarrow{RS} (P^t, (Q^*)^t)$$

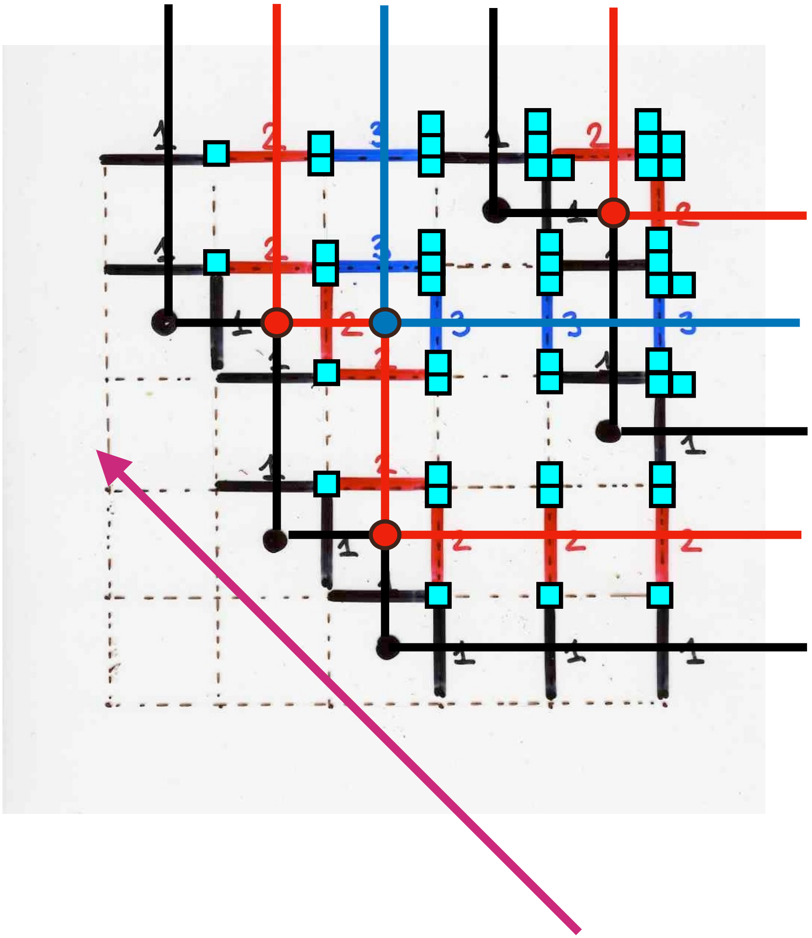
(transpose)

Direct proof ?

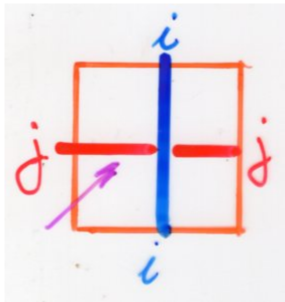


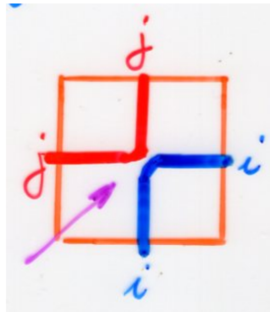






jeu de taquin
local rules on edges





$|i-j| \geqslant 2$ $\quad$ $|i-j| \leqslant 1$

$i, j \in \mathbb{Z}$

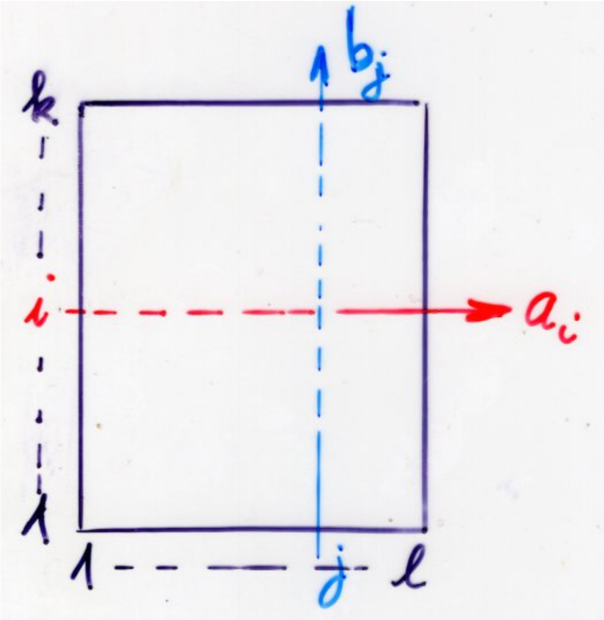
From RS to RSK

extension

to matrices

D. Knuth, 1970

$$M = (a_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}}$$

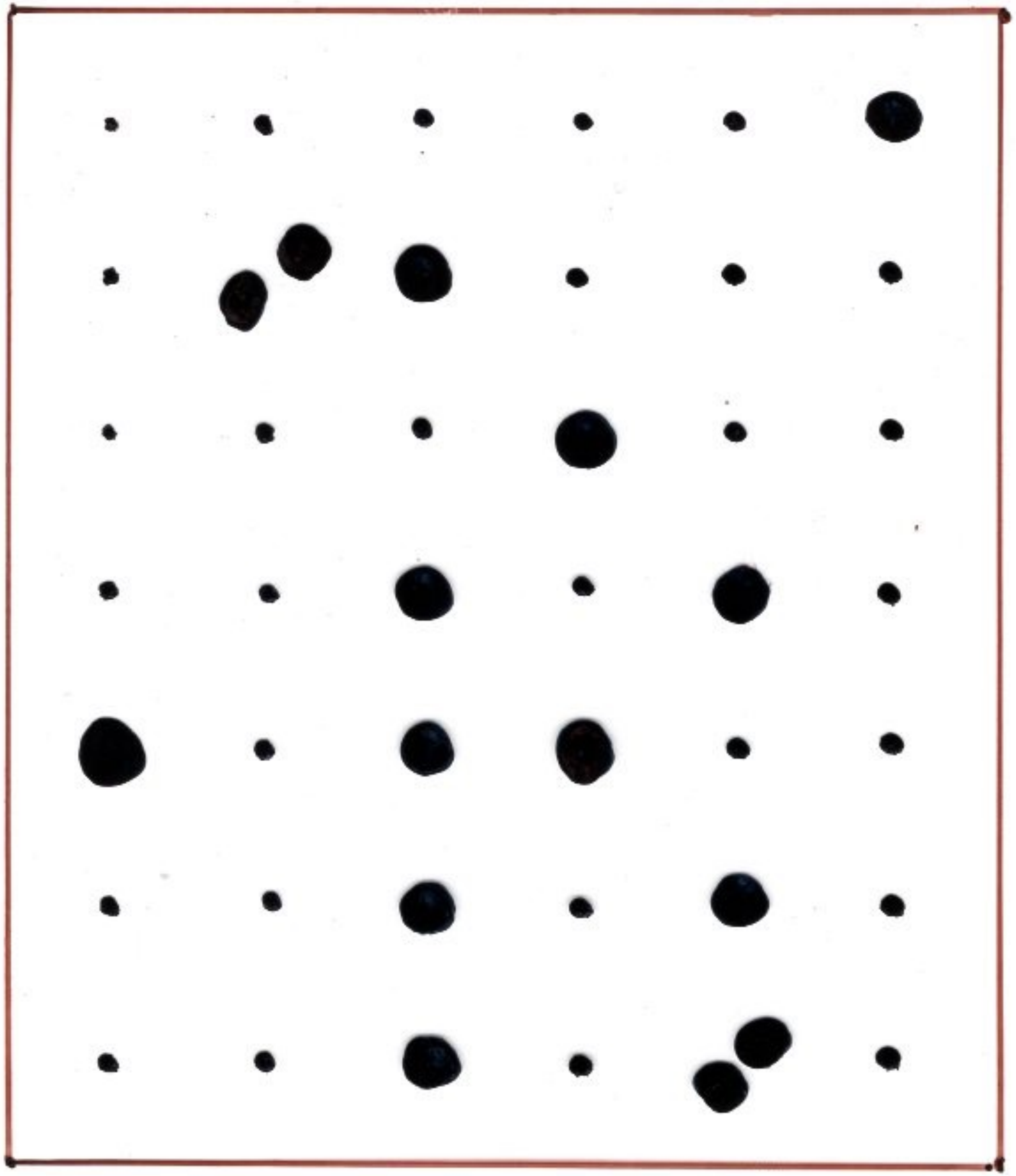


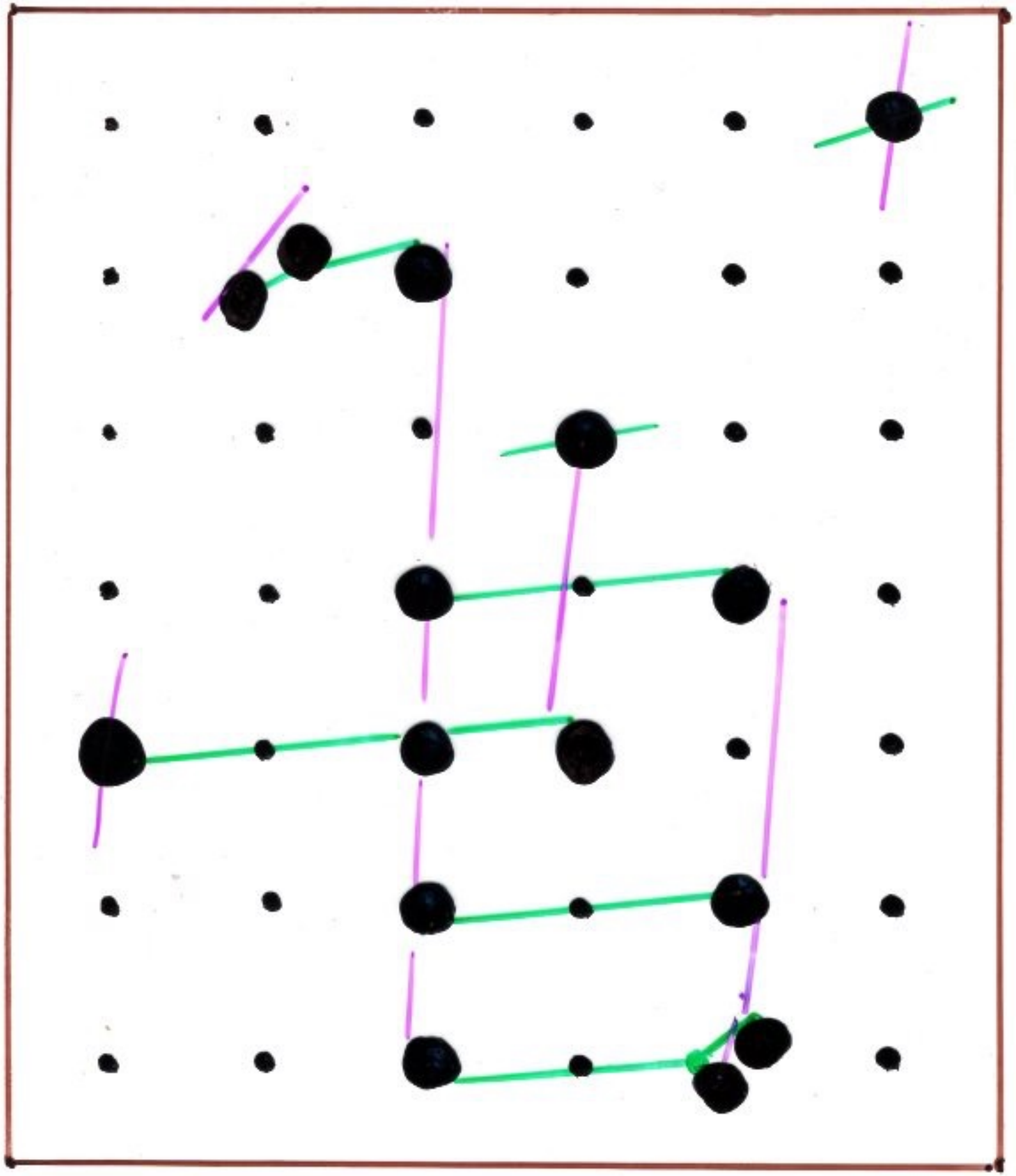
Type of M

for $i = 1, .., k$, $a_i = \sum_{1 \leq j \leq \ell} a_{ij}$ (sum of entries in row i)

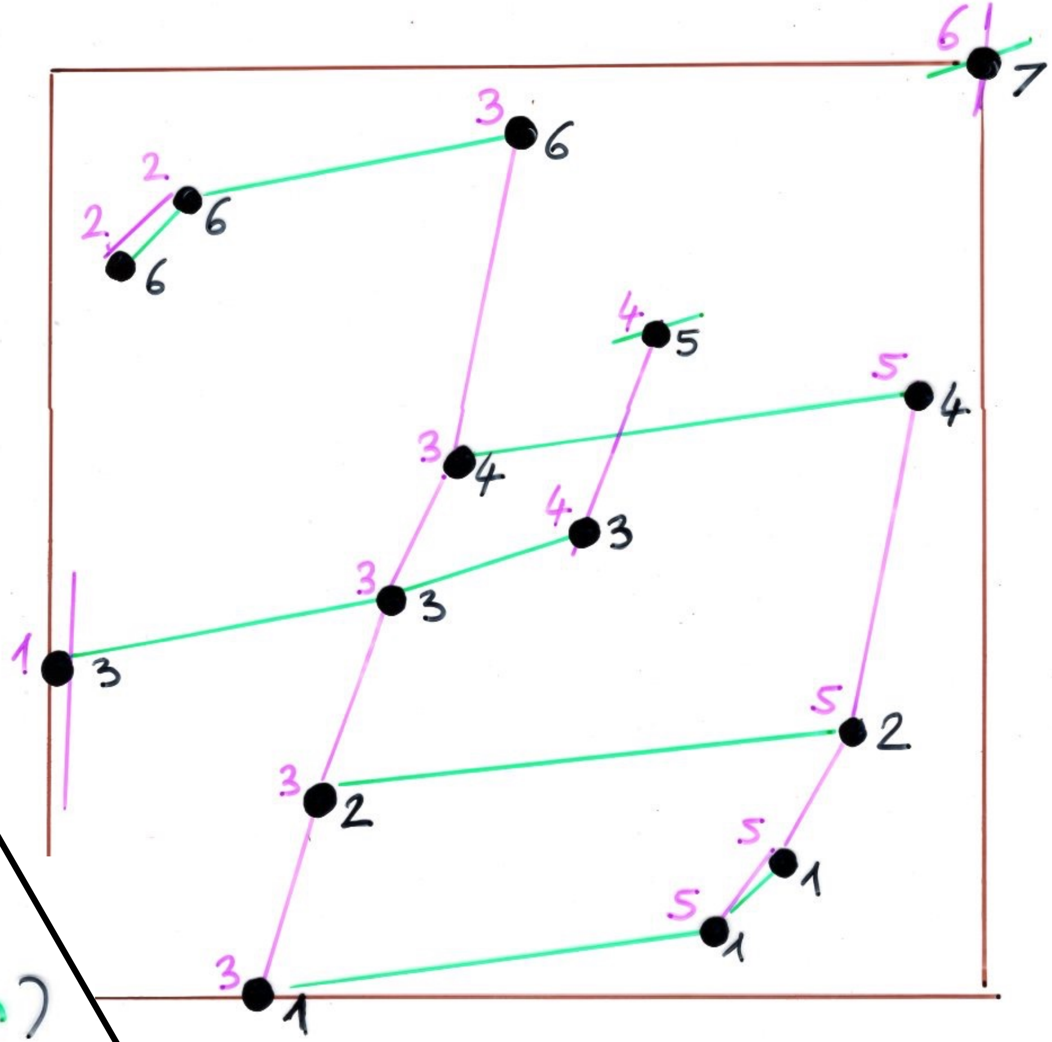
for $j = 1, .., \ell$ $b_j = \sum_{1 \leq i \leq k} a_{ij}$ (sum of entries in column j)

$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{bmatrix}$$



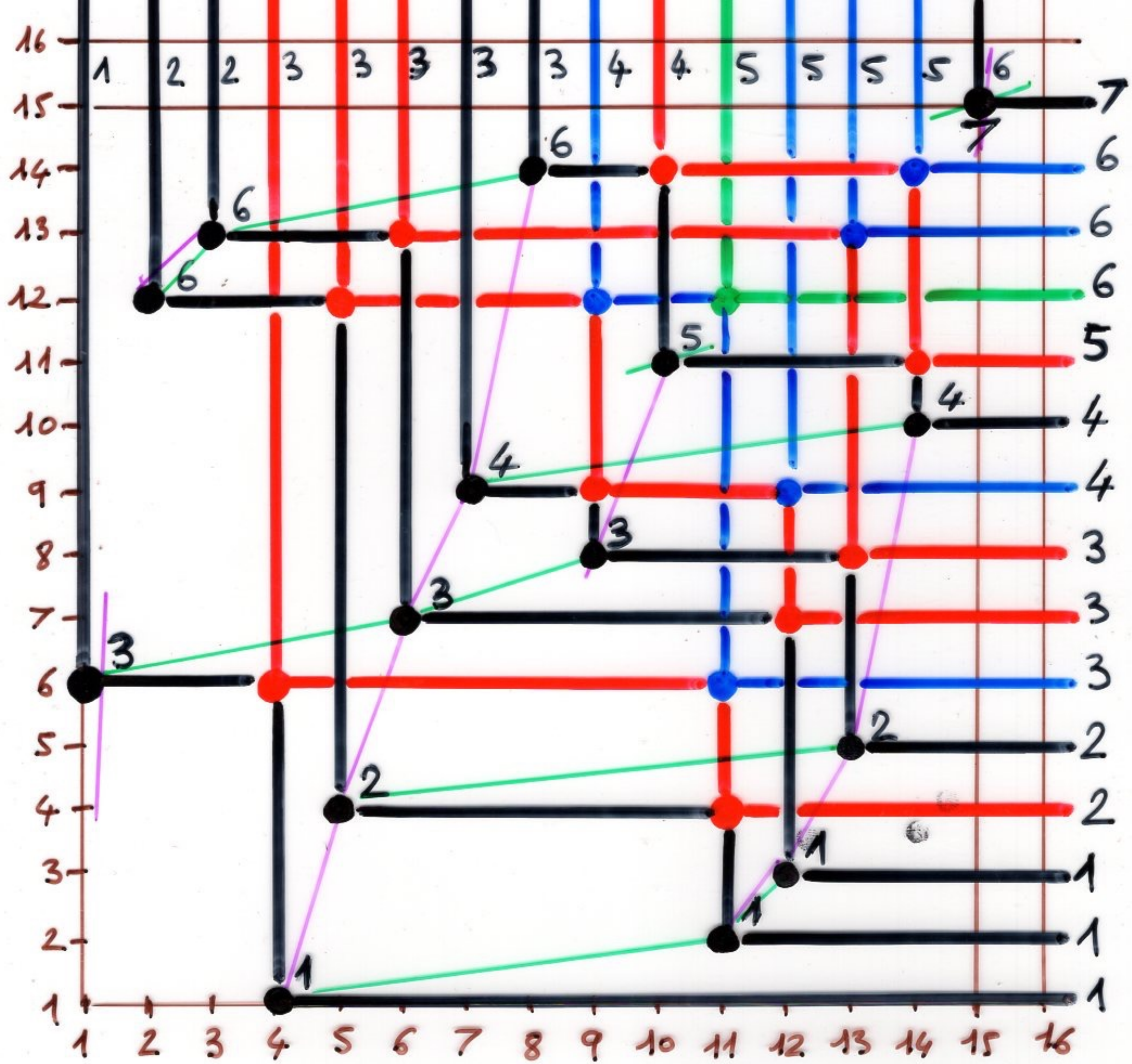


·	·	·	·	·	1
·	2	1	·	·	·
·	·	·	1	·	·
·	·	1	·	1	·
1	·	1	1	·	·
·	·	1	·	1	·
·	·	1	·	2	·

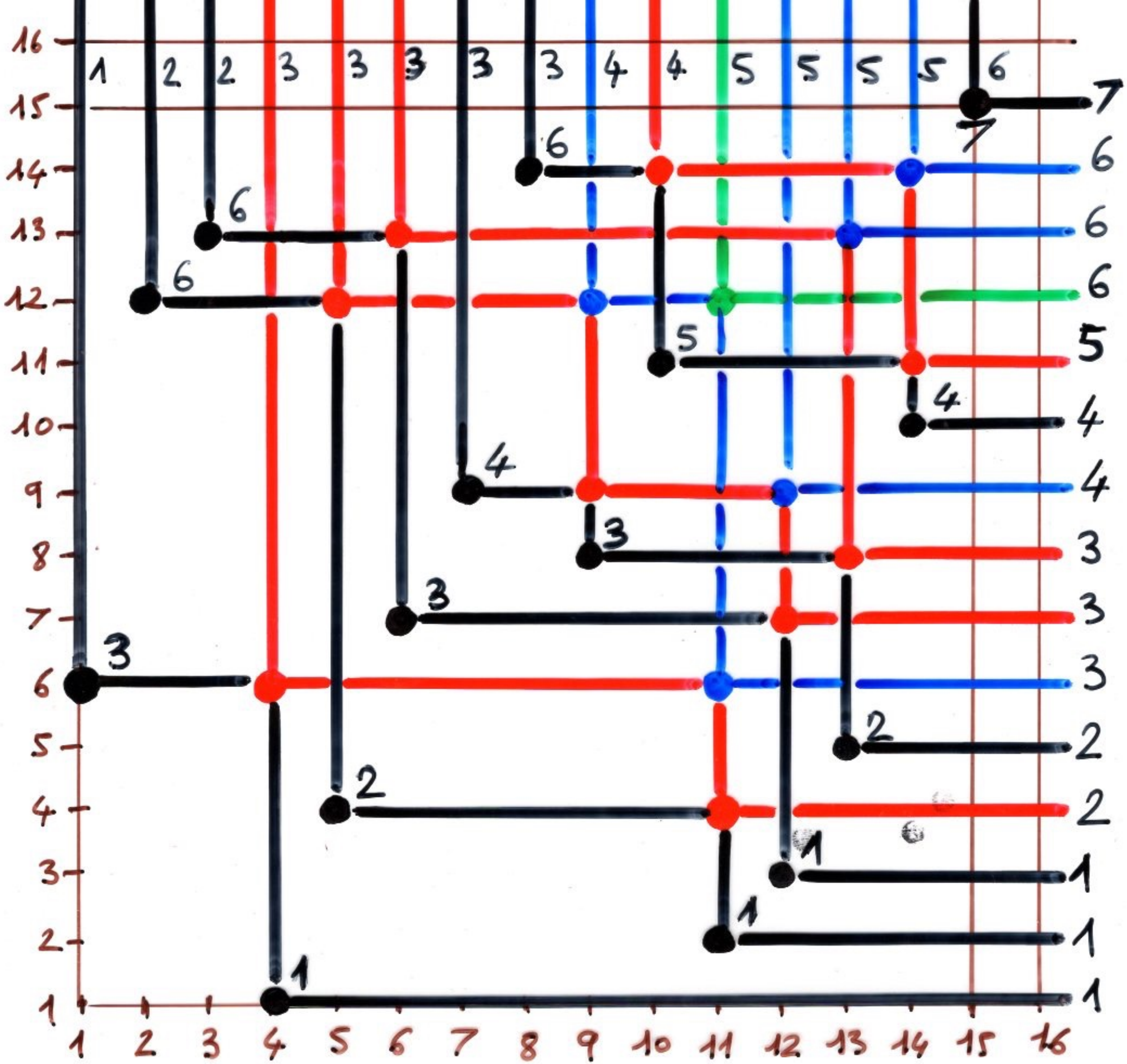


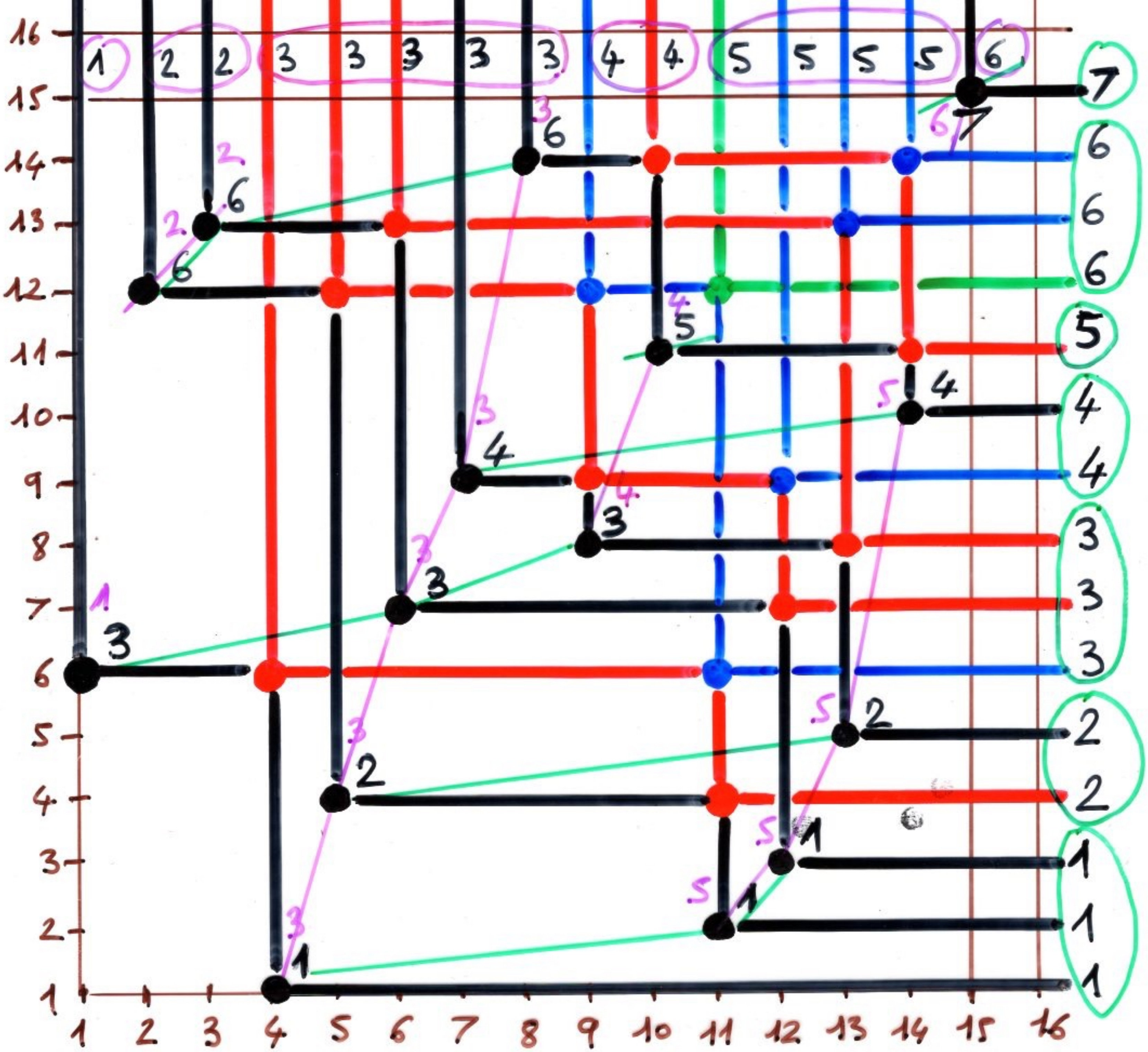
two-line array
(or generalized permutation)

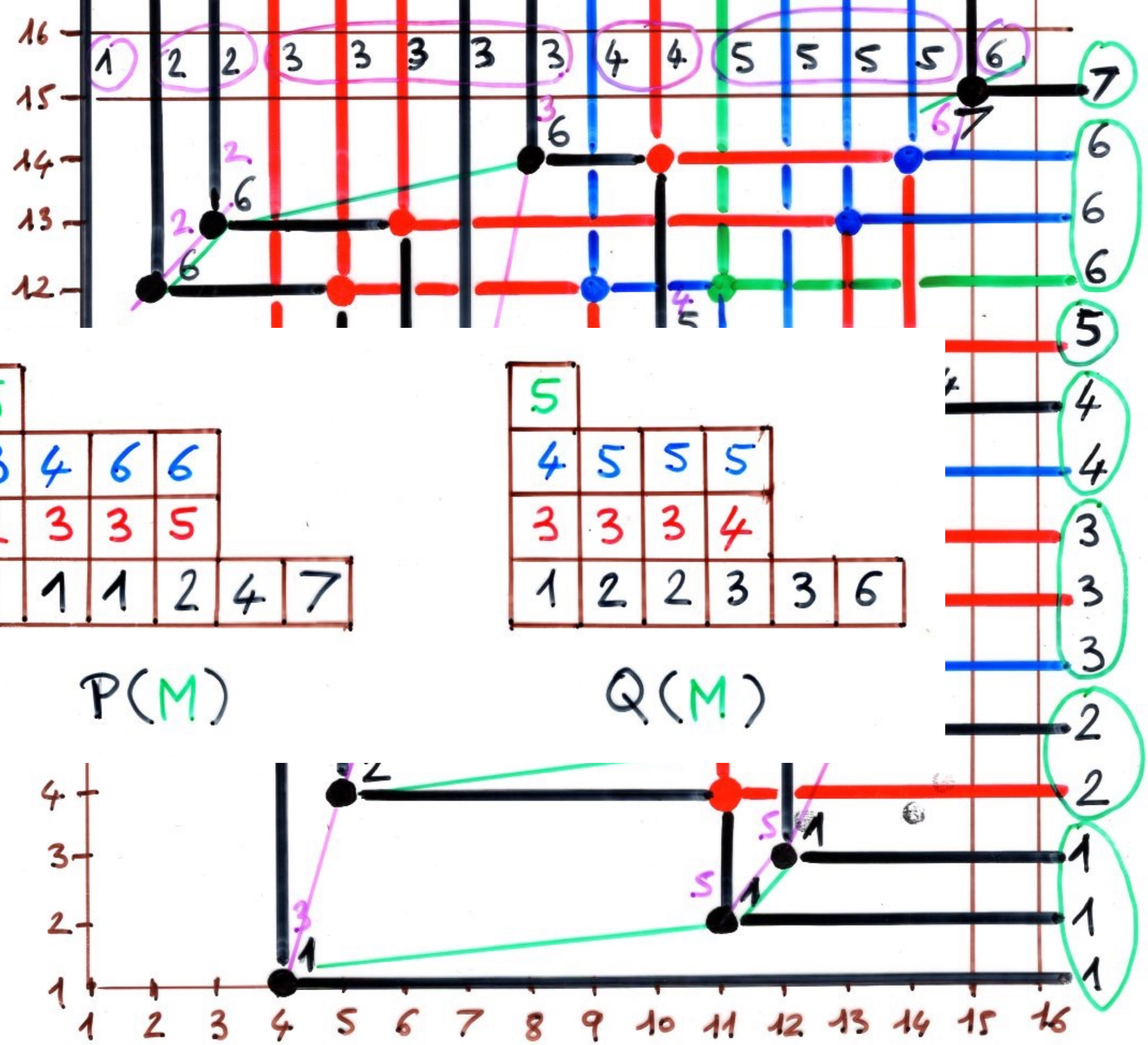
$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{c|cc|ccccc|cc|cccc|c} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{array} \right)$$



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
1 2 2 3 3 3 3 3 4 4 5 5 5 5 6
7 6 6 6 5 4 4 3 3 3 2 2 1 1 1




1 2 2 3 3 3 3 3 4 4 5 5 5 5 6
7
6
6
6
5
4
4
3
3
3
2
2
1
1
1
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

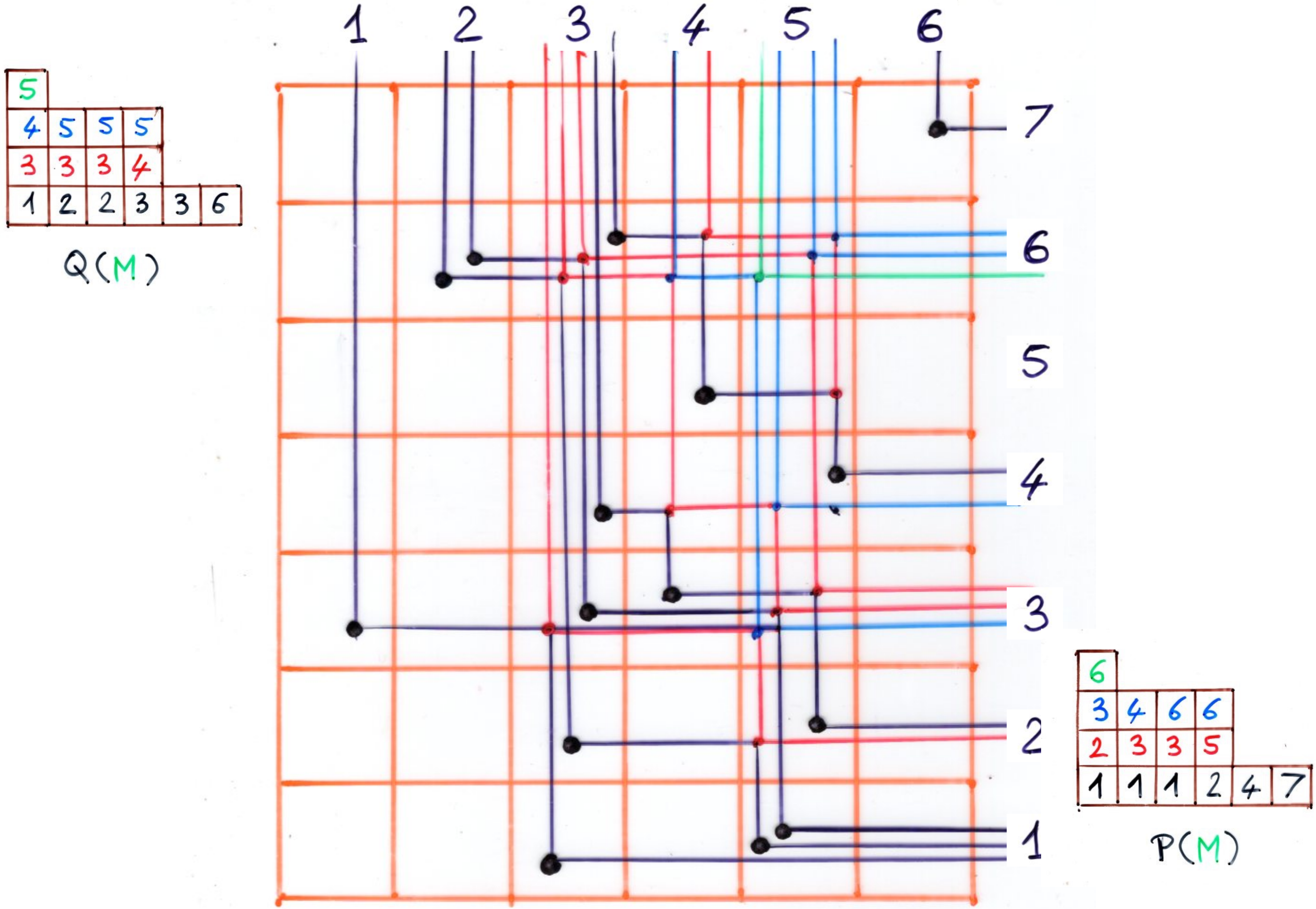


6					
3	4	6	6		
2	3	3	5		
1	1	1	2	4	7

P(M)

5					
4	5	5	5		
3	3	3	4		
1	2	2	3	3	6

Q(M)

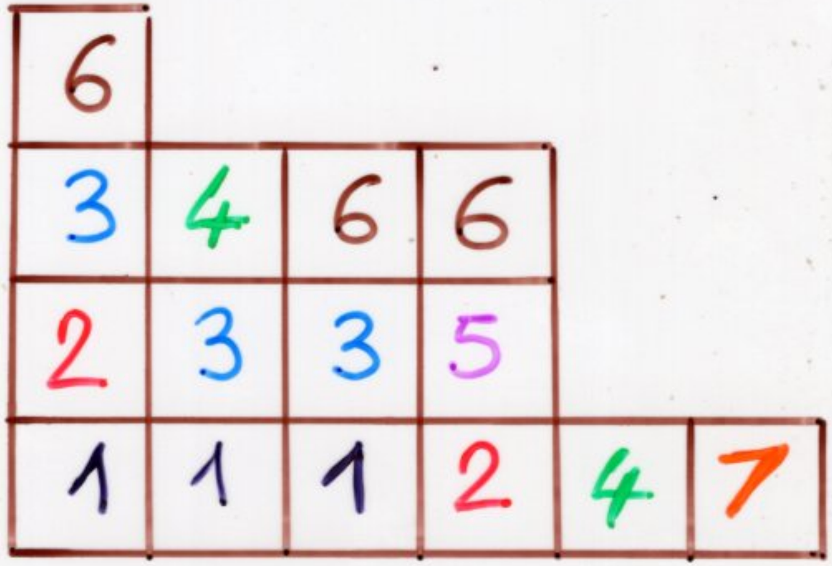

5
4 5 5 5
3 3 3 4
1 2 2 3 3 6
Q(M)
1 2 3 4 5 6
7
6
5
4
3
2
1
6
3 4 6 6
2 3 3 5
1 1 1 2 4 7
P(M)

Definition

semi-standard Young tableau (SSYT)
with shape λ (λ partition of n)

is a filling of Ferrers diagram $F(\lambda)$
with integers ≥ 1 such that

- they go increasing weakly in rows (from left to right)
- they go strictly increasing in columns (from bottom to top)



RSK Robinson-Schensted-Knuth correspondence

<u>Proposition</u> The map $M \to (P, Q)$ is a bijection between $k \times \ell$ matrices with integers entries ≥ 0 and pair (P, Q) of semi-standard Young tableaux having the same shape λ.

The "type" of P is $(a_1, \dots, a_k)$
Q is $(b_1, \dots, b_k)$

$$M = (a_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}}$$

$k \times \ell$

<u>Type of M</u>

for $i = 1, \dots, k$, $a_i = \sum_{1 \leq j \leq \ell} a_{ij}$ (sum of entries in row i)

for $j = 1, \dots, \ell$ $b_j = \sum_{1 \leq i \leq k} a_{ij}$ (sum of entries in column j)

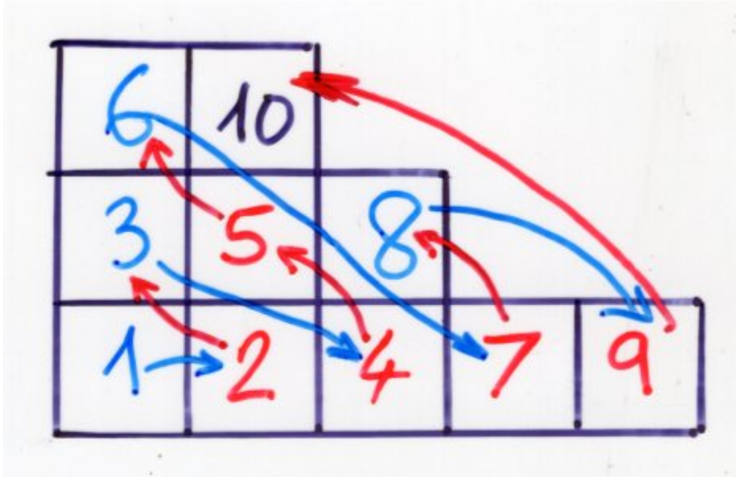
Proof:

x advance in a permutation σ

iff $x = \sigma(i)$, $x+1 = \sigma(j)$ with $i < j$

Lemma $\sigma \xrightarrow{RS} (P, Q)$

- x is an advance of σ iff in the tableau P $(x+1)$ is located at the South-East of x



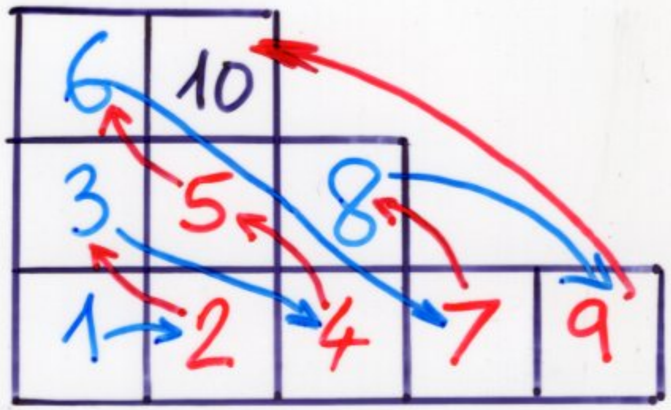
$\sigma = 3\ 1\ 6\ (10)\ 2\ 5\ 8\ 4\ 9\ 7$

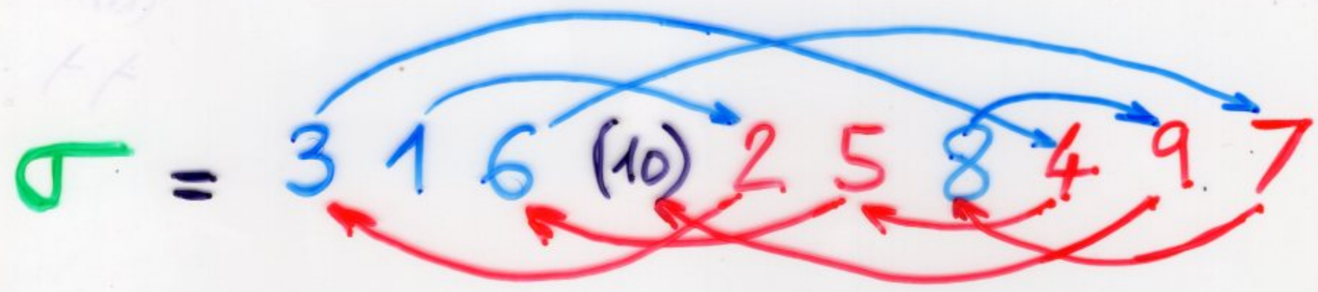
"ligne de route"

x advance in a permutation σ
iff $x = \sigma(i)$, $x+1 = \sigma(j)$
with $i < j$

Lemma $\sigma \xrightarrow{RS} (P, Q)$

- x is an advance of σ iff in the tableau P $(x+1)$ is located at the South-East of x





"ligne de route"

i is a **descent** of σ iff

$$\sigma(i) > \sigma(i+1)$$

rise $\sigma(i) < \sigma(i+1)$

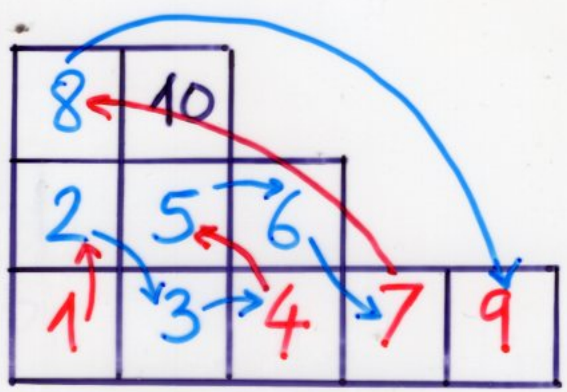
$$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$$

1	2	3	4	5	6	7	8	9

Lemma $\sigma \xrightarrow{RS} (P, Q)$

- there is a rise at the index i of σ iff $(i+1)$ is located at the South-East of i in the tableau Q

$Q(\sigma)$



RSK Robinson-Schensted-Knuth correspondence

<u>Proposition</u> The map $M \to (P, Q)$ is a bijection between $k \times \ell$ matrices with integers entries ≥ 0 and pair (P, Q) or semi-standard Young tableaux having the same shape λ.

Fulton "matrix balls" construction

Amri Prasad "VRSK algorithm"

$$M = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{pmatrix}$$

6					
3	4	6	6		
2	3	3	5		
1	1	1	2	4	7

$P(M)$

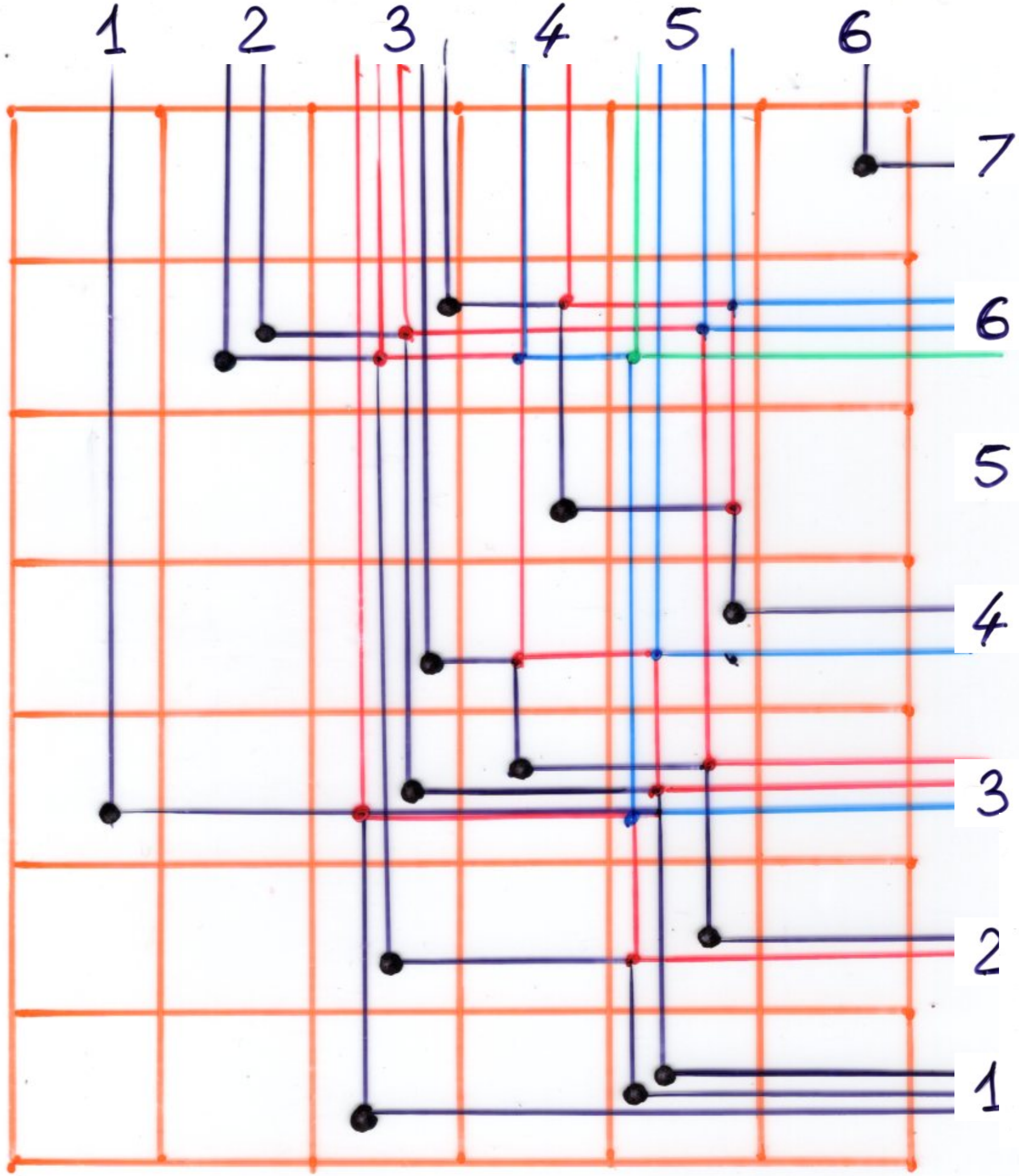
5					
4	5	5	5		
3	3	3	4		
1	2	2	3	3	6

$Q(M)$

Local rules ?

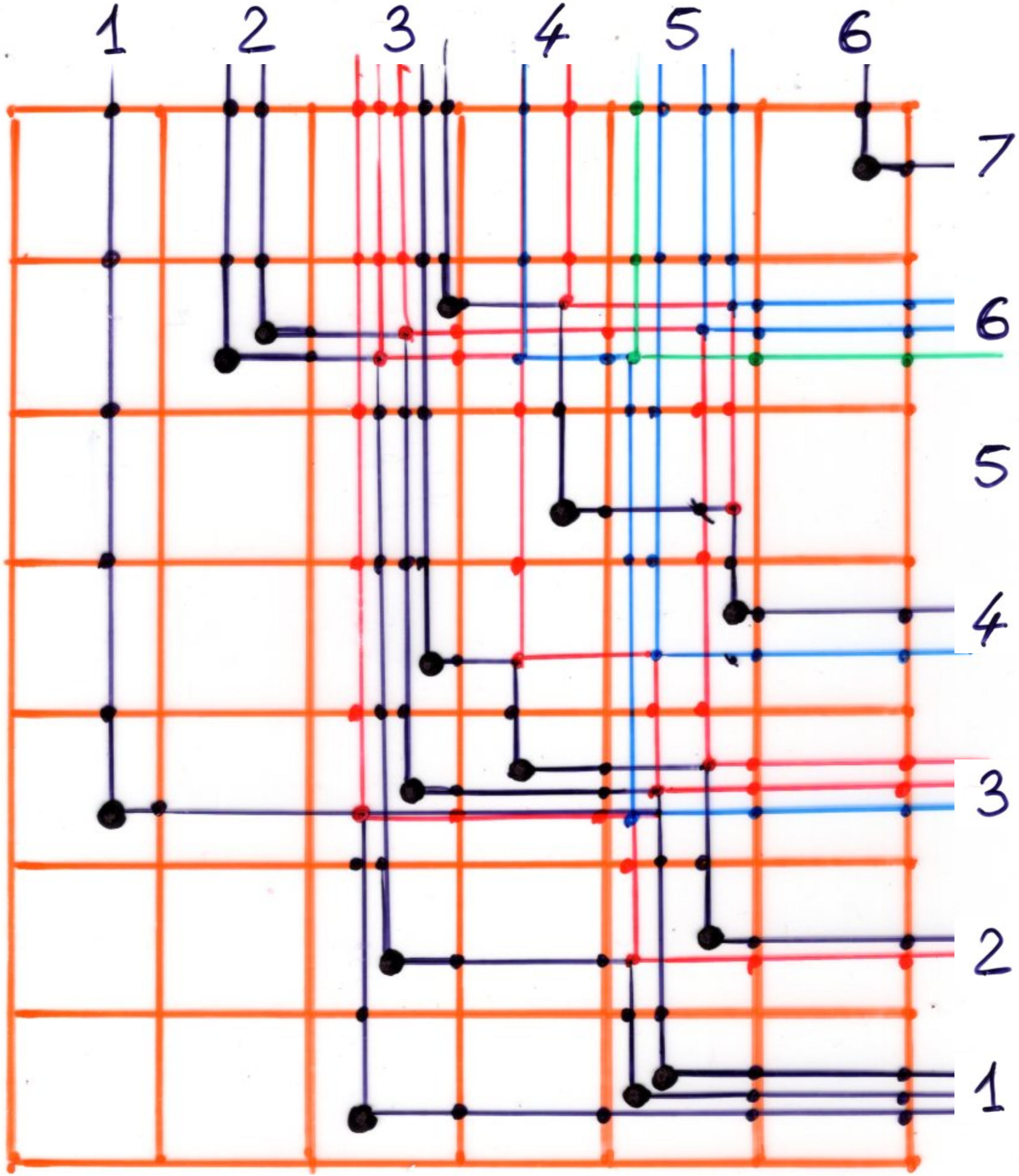
5					
4	5	5	5		
3	3	3	4		
1	2	2	3	3	6

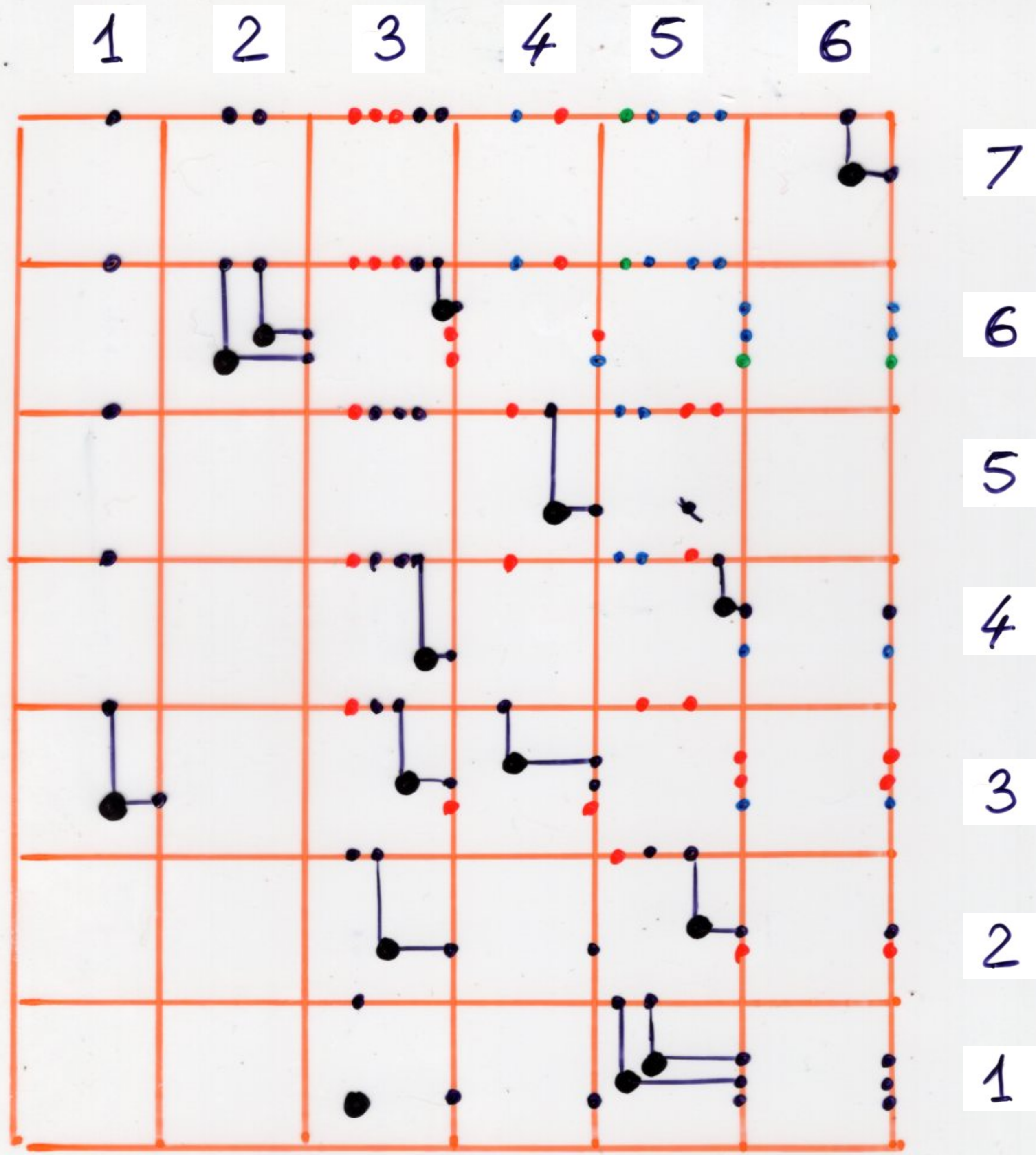
Q(M)



6					
3	4	6	6		
2	3	3	5		
1	1	1	2	4	7

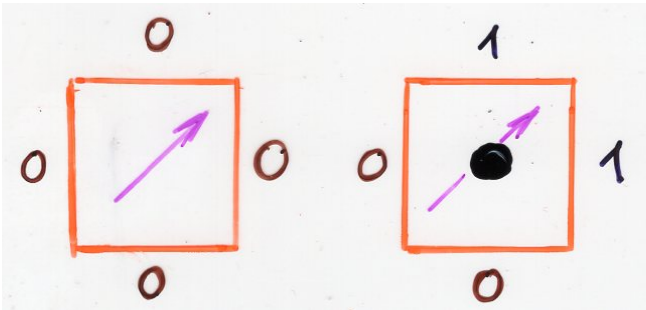
P(M)

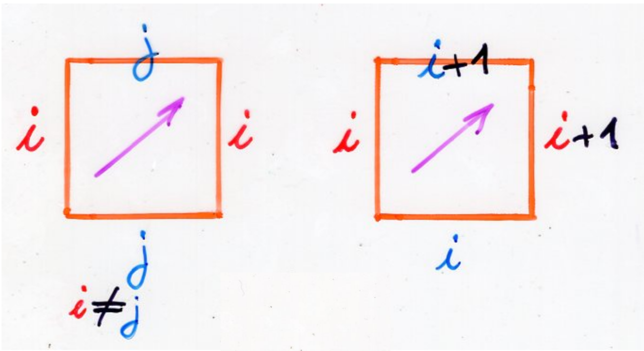

1
2
3
4
5
6
7
6
5
4
3
2
1

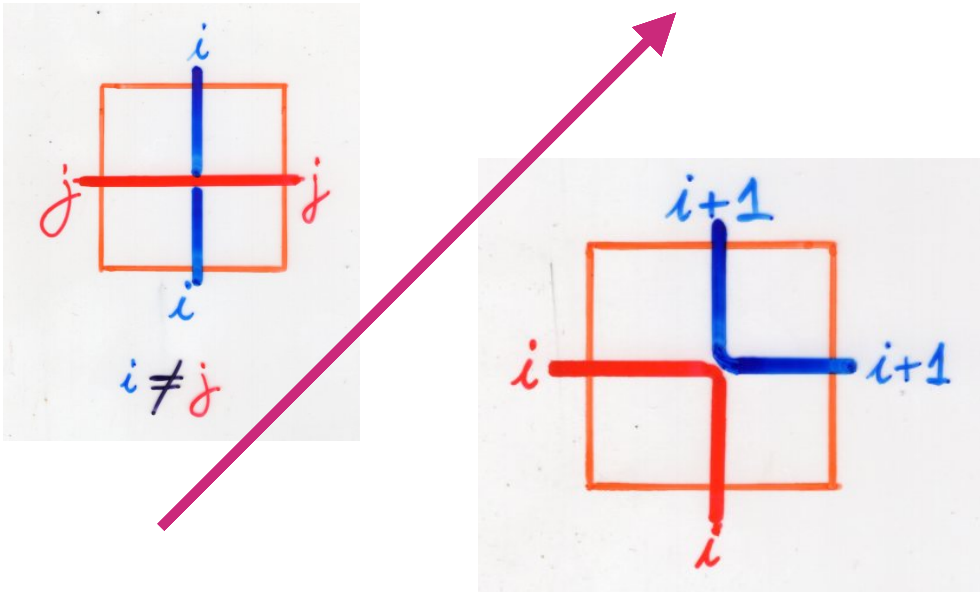

1
2
3
4
5
6
7
6
5
4
3
2
1

The RSK (reverse) planar automaton

Ch1b, p91



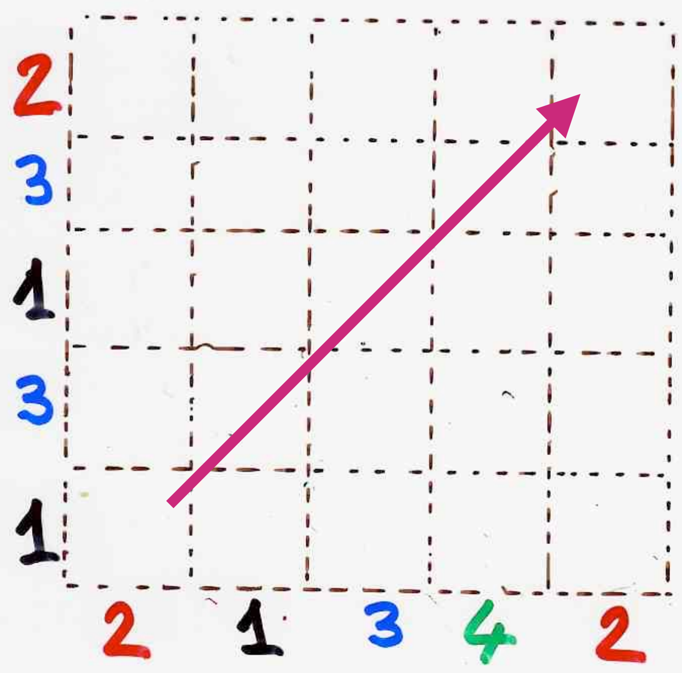


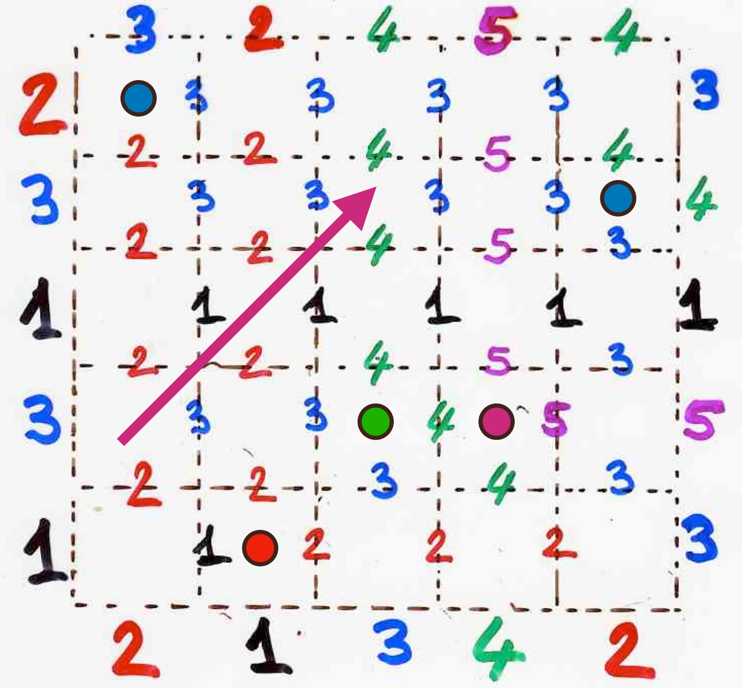


"local rules"
on the edges

$(\alpha, \beta) \rightarrow (\alpha', \beta')$

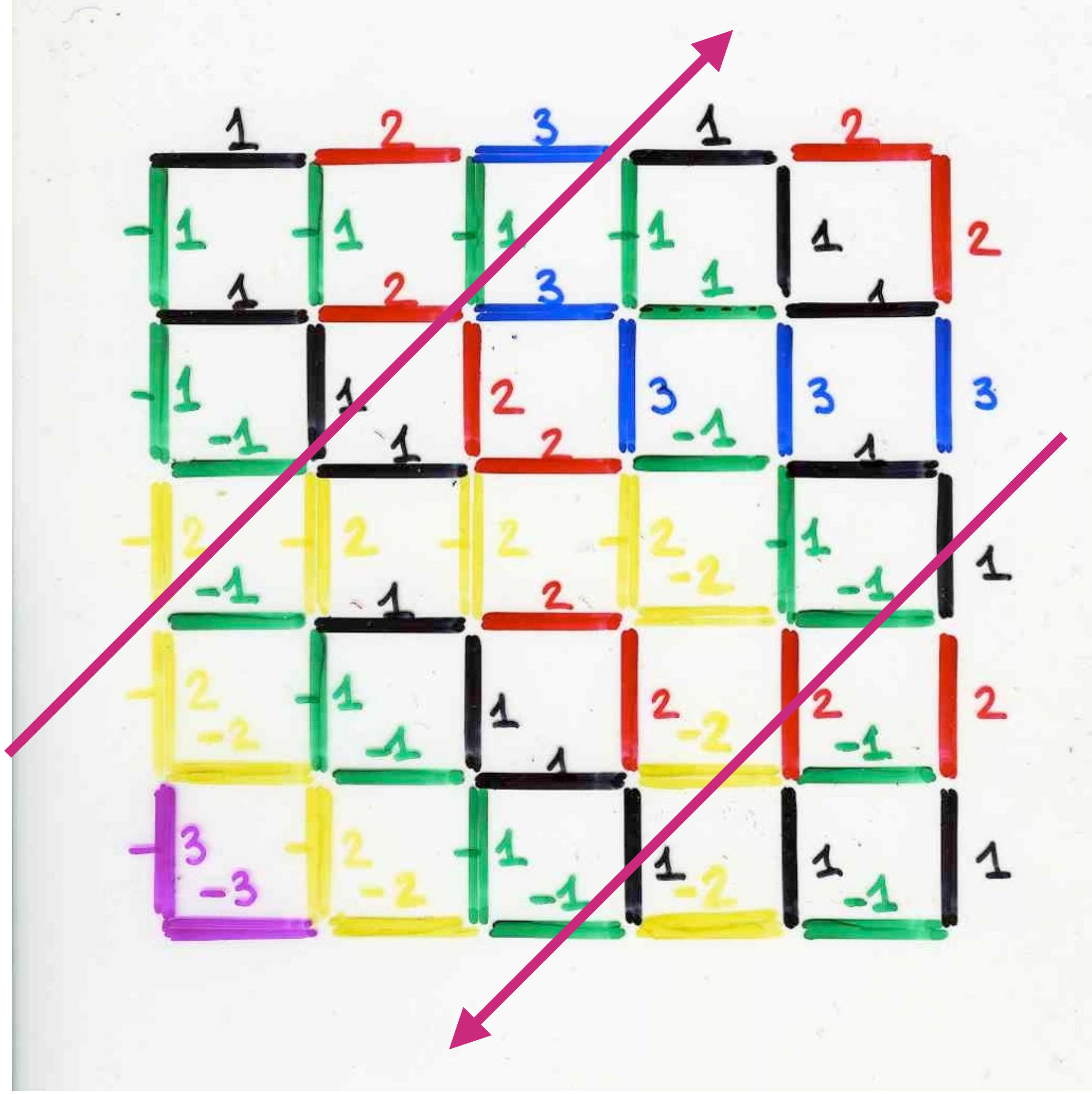
RSK product of two words (Ch1b, p111)

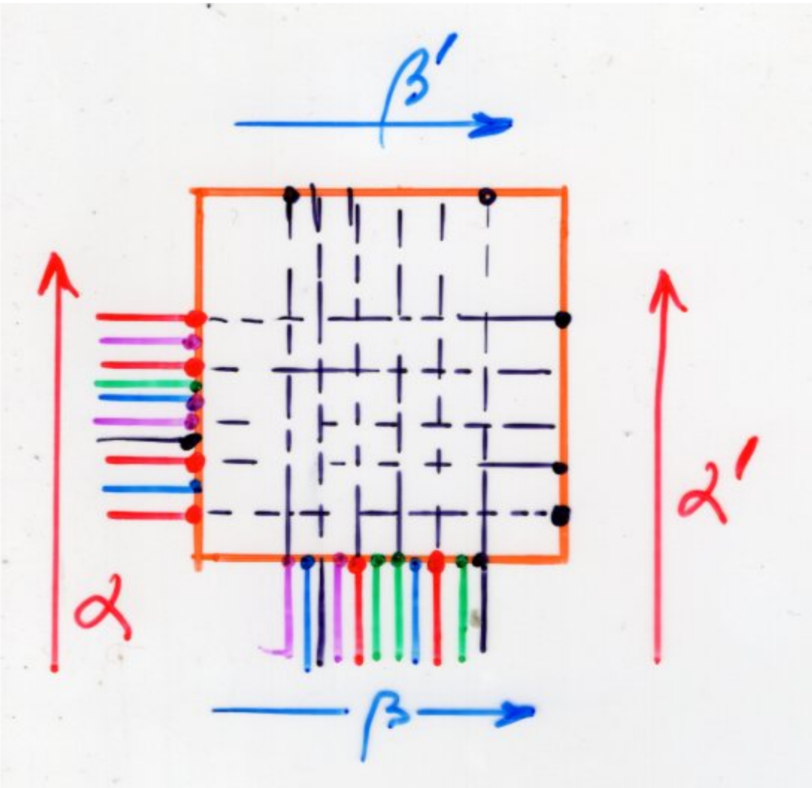




bilateral

planar automaton RSK



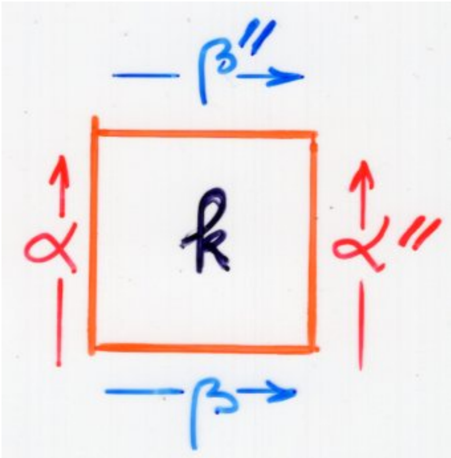


$(\alpha, \beta) \longrightarrow (\alpha', \beta')$

RSK product of two words (Ch1b, p111)

$$(\alpha', \beta') = RS(\alpha, \beta)$$

local rules on edges for RSK



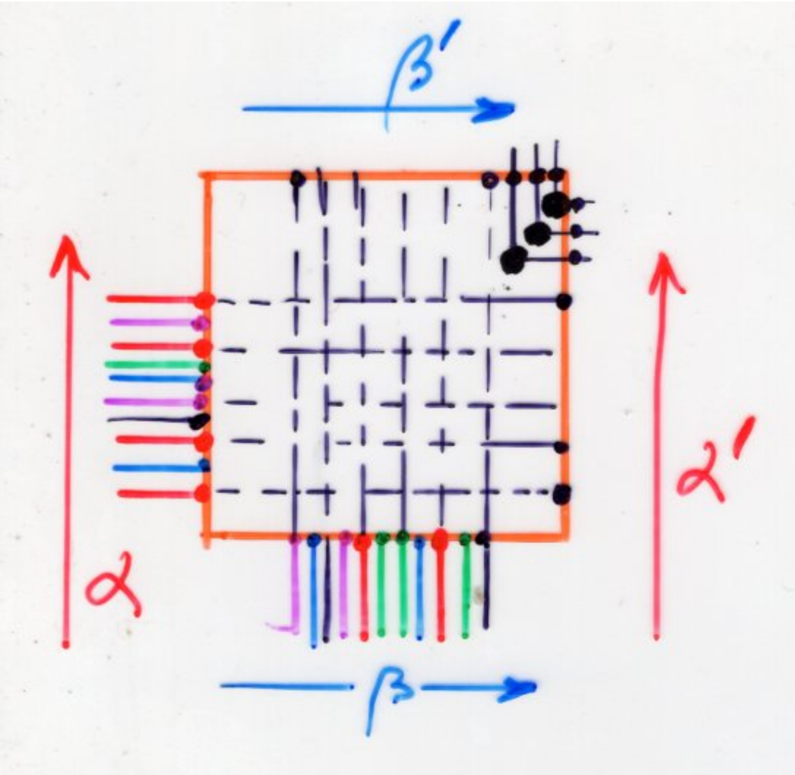
$\alpha, \alpha'', \beta, \beta''$ words $\in \{1, 2, 3, \dots\}^*$

$k \geq 0$ integer

$$(\alpha', \beta') = RS(\alpha, \beta)$$

$$\alpha'' = \alpha' \cdot 11\;1$$

$$\beta'' = \beta' \cdot \underbrace{11\;1}_{k \text{ times } 1}$$

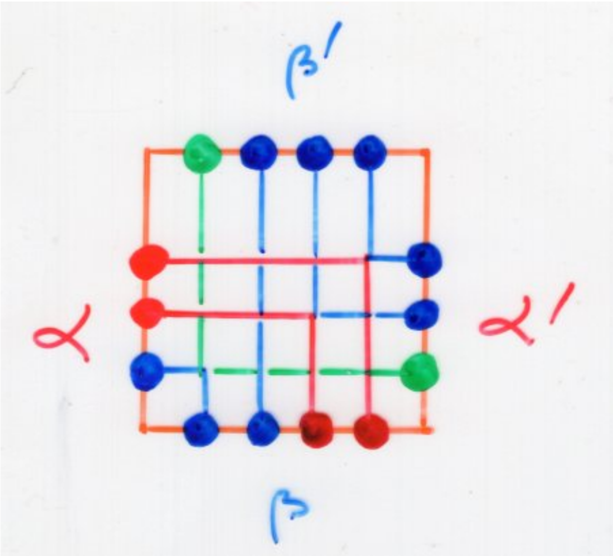


$$(\alpha', \beta') = RS(\alpha, \beta)$$

$$\alpha'' = \alpha' \cdot 11\ 1$$

$$\beta'' = \beta' \cdot \underbrace{11\ 1}_{k \text{ times } 1}$$

Example



$\alpha = 3\ 2\ 2$

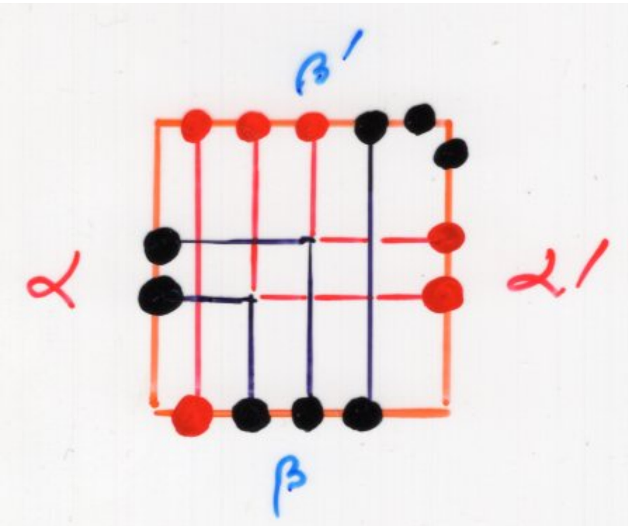
$\alpha' = 4\ 3\ 3$

$\beta = 3\ 3\ 2\ 2$

$\beta' = 4\ 3\ 3\ 3$

$\alpha'' = \alpha'$

$\beta'' = \beta'$



$\alpha = 1\ 1$

$\alpha' = 2\ 2$

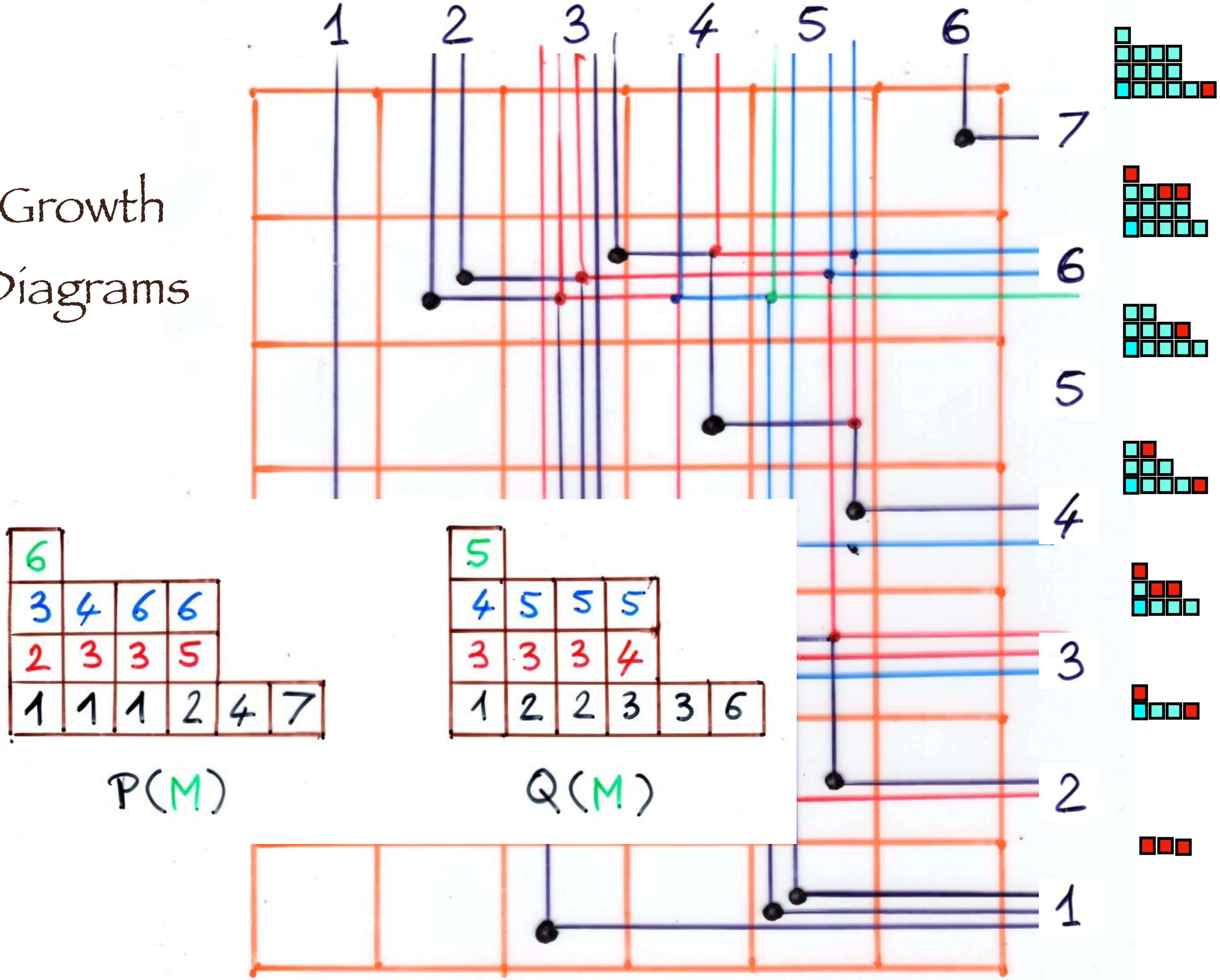
$\beta = 2\ 1\ 1\ 1$

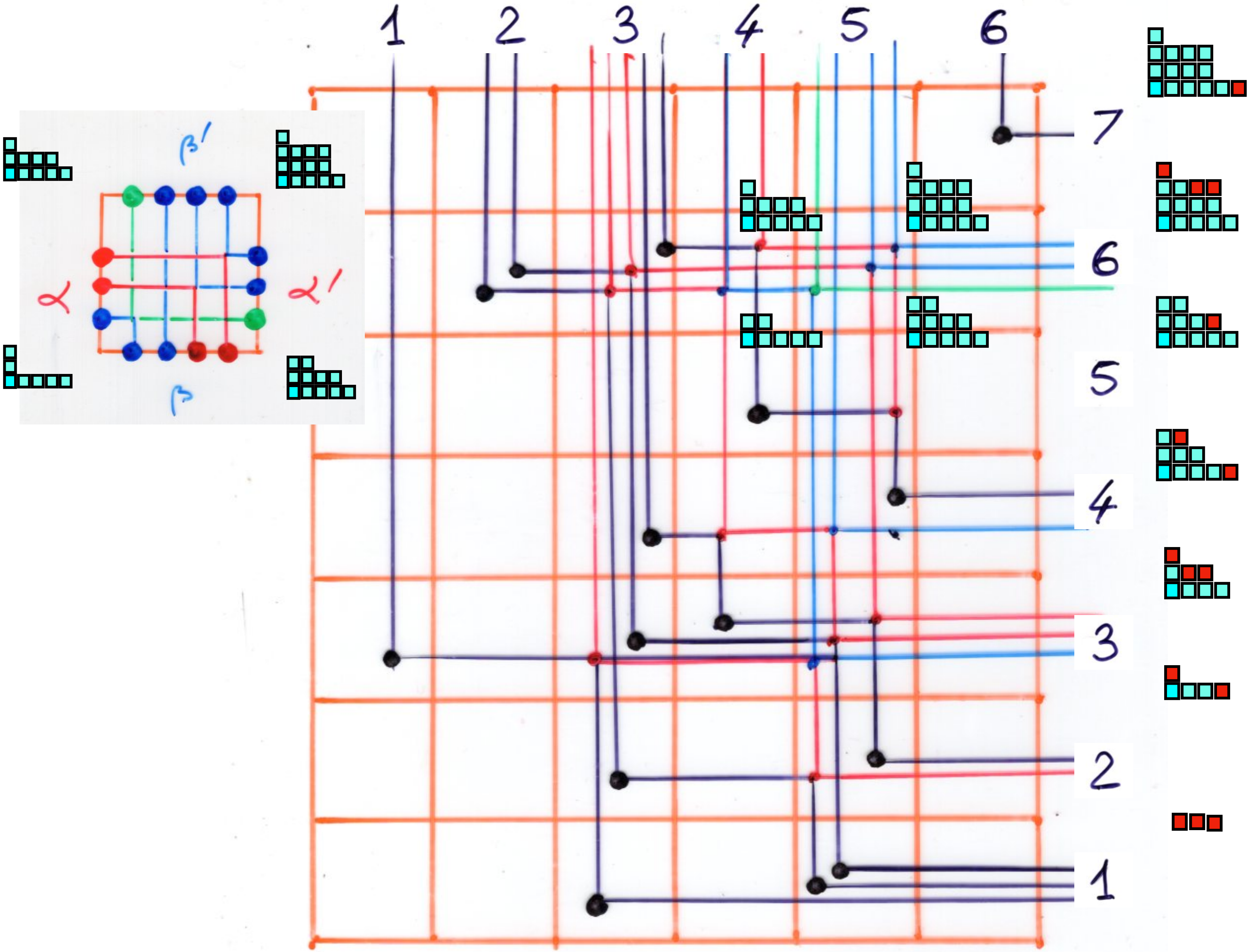
$\beta' = 2\ 2\ 2\ 1$

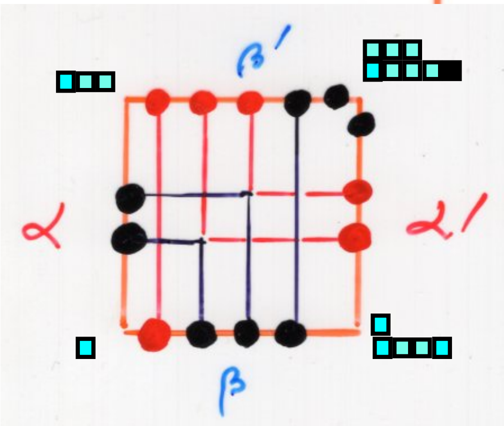
$\alpha'' = \alpha' \cdot 1$

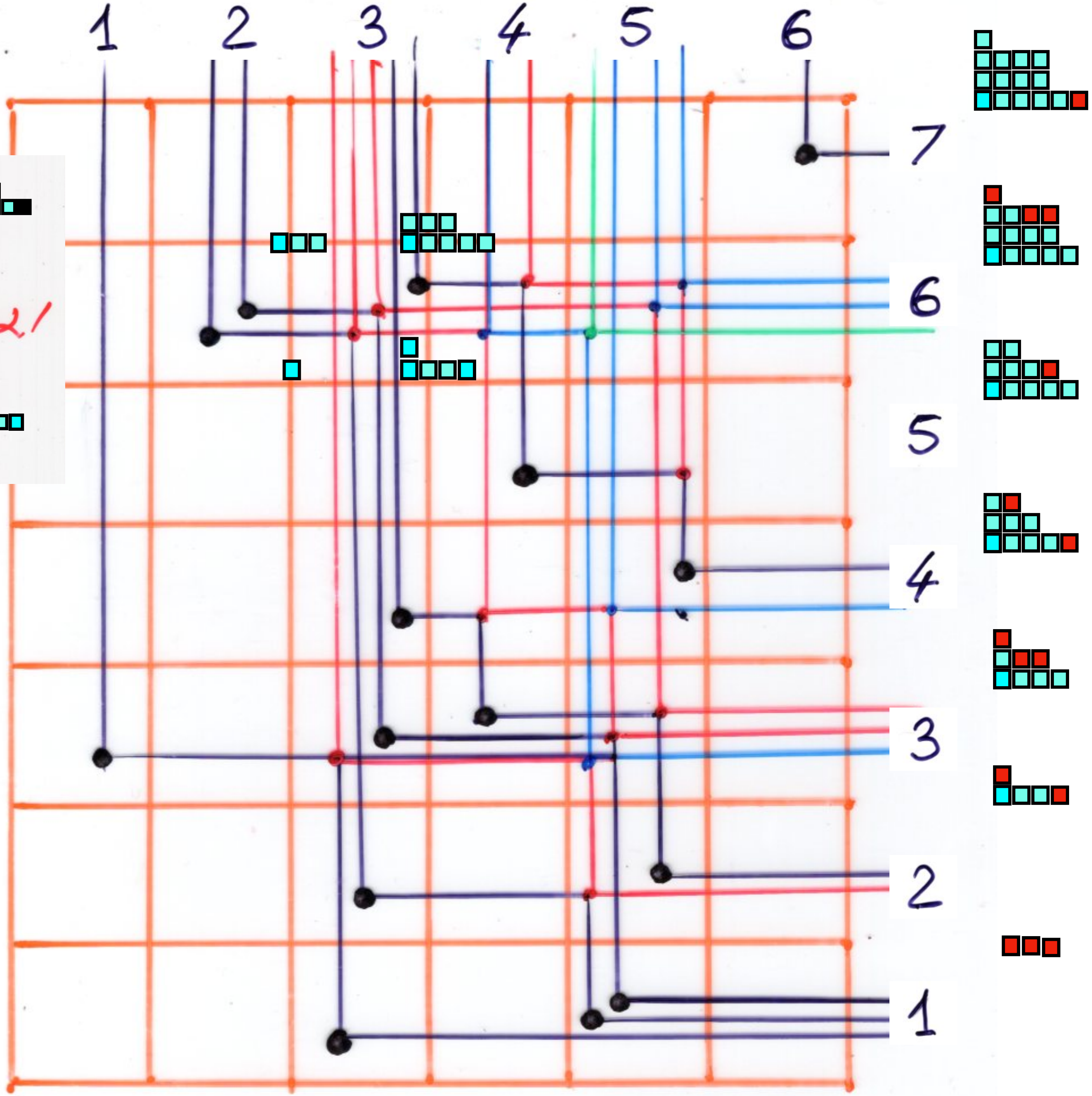
$\beta'' = \beta' \cdot 1$

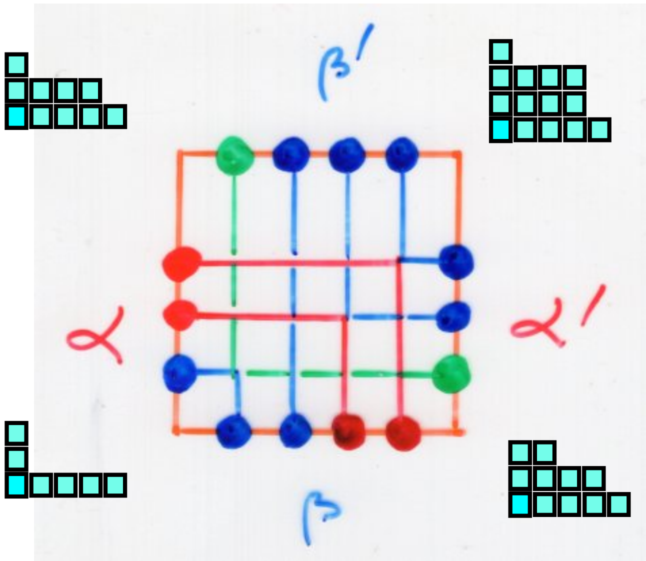
Growth Diagrams




β'
α
α'
β
1
2
3
4
5
6
7
6
5
4
3
2
1


β'
α
α'
β


1
2
3
4
5
6
7
6
5
4
3
2
1

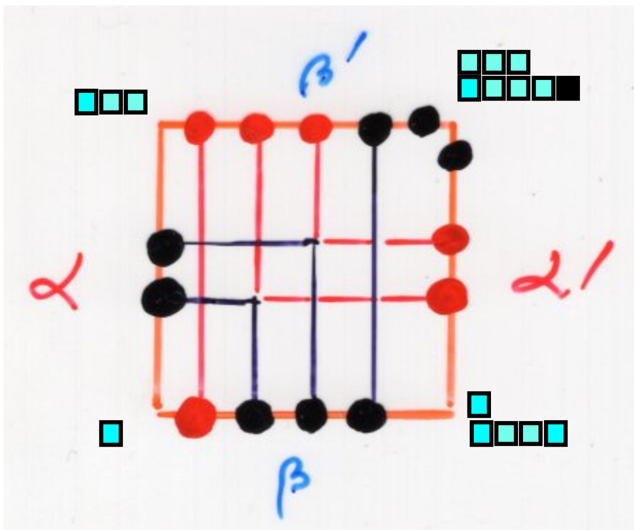


$\alpha = 3\ 2\ 2$

$\alpha' = 4\ 3\ 3$

$\beta = 3\ 3\ 2\ 2$

$\beta' = 4\ 3\ 3\ 3$



$\alpha = 1\ 1$

$\alpha' = 2\ 2$

$\beta = 2\ 1\ 1\ 1$

$\beta' = 2\ 2\ 2\ 1$

$\alpha'' = \alpha' \cdot 1$

$\beta'' = \beta' \cdot 1$

Dual RSK

dual-RSK

$M = (a_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}}$

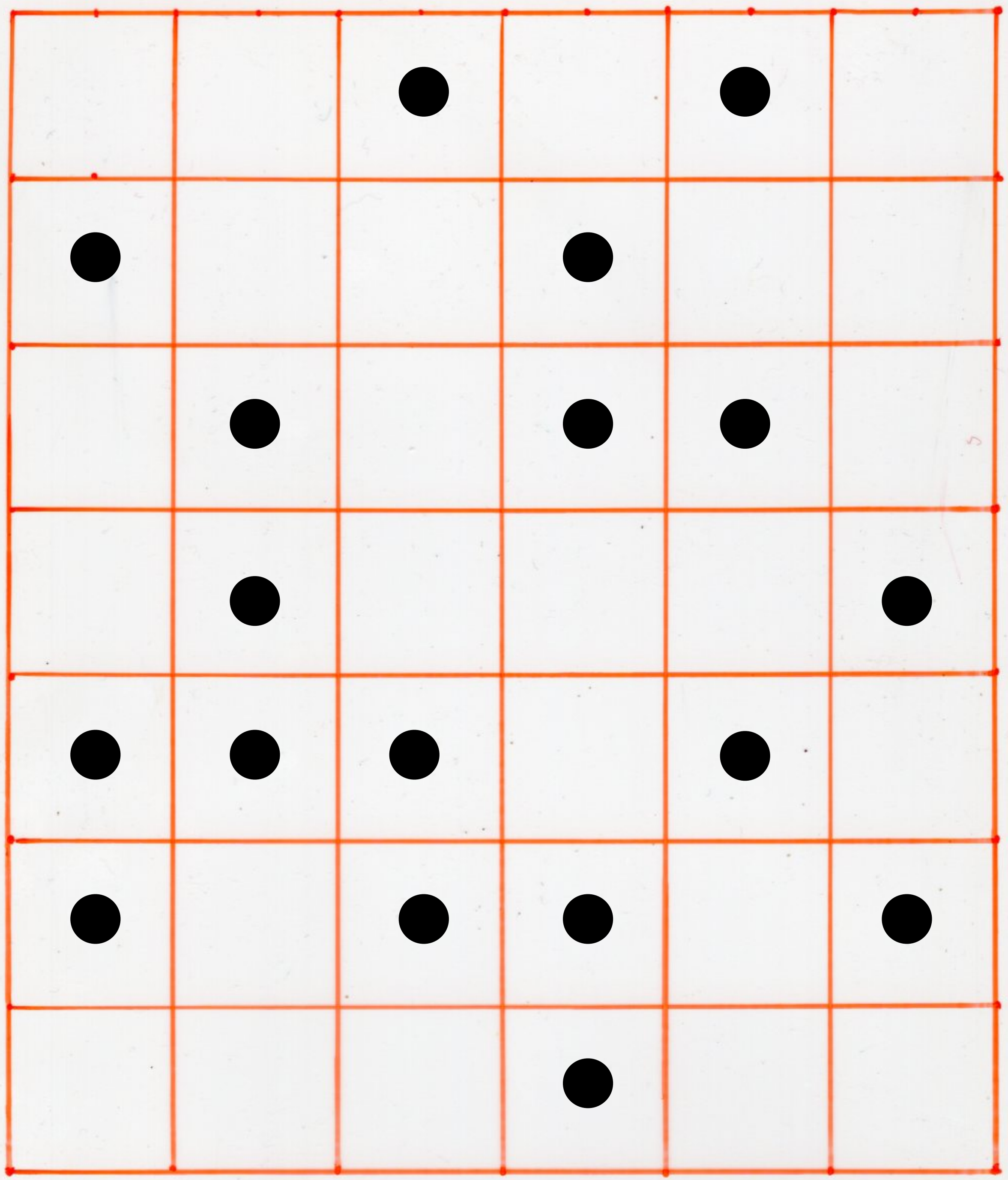
$a_{ij} = 0$ or 1

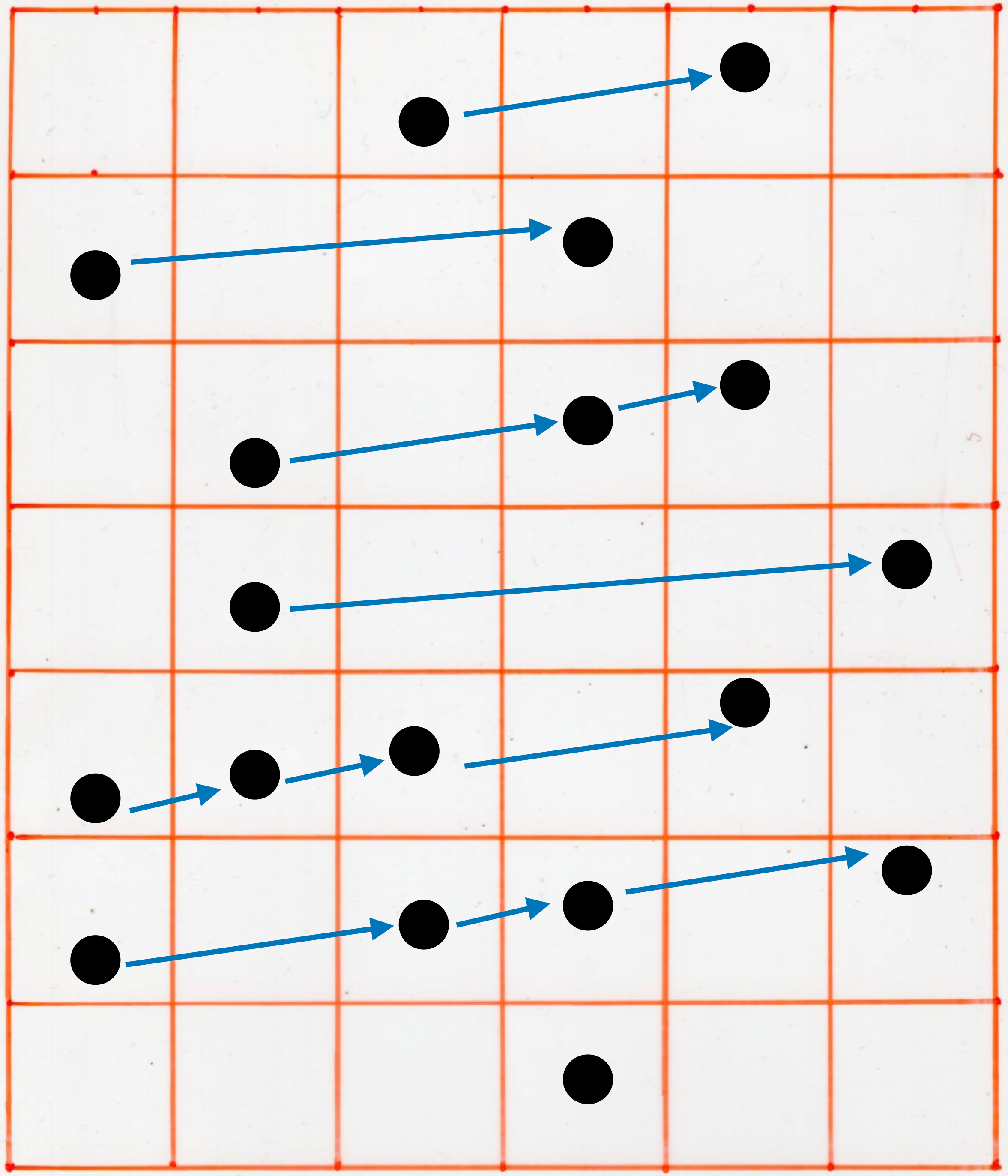
$M \xrightarrow{\text{dual RSK}} (P, Q)$

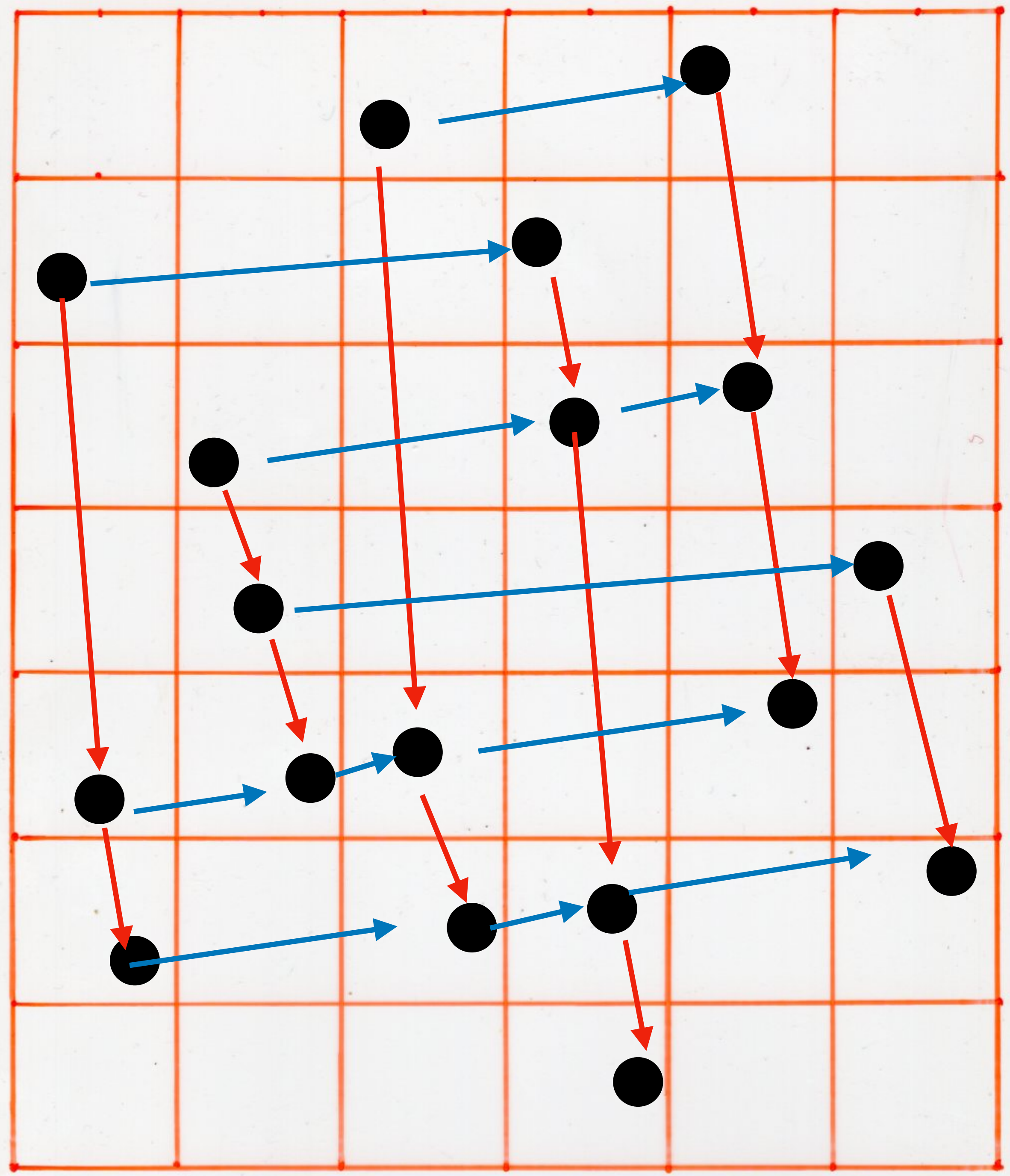
P shape λ

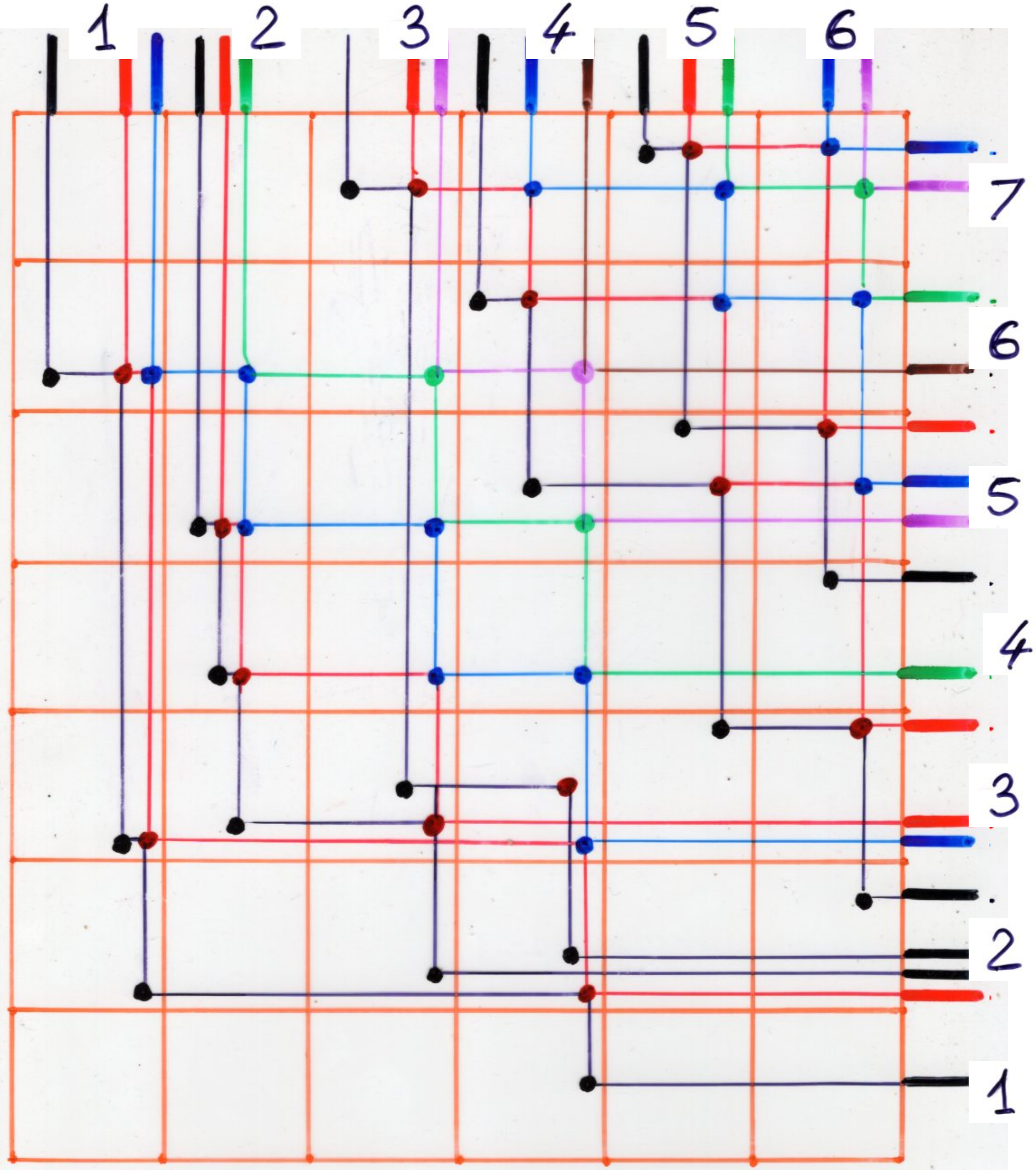
Q shape λ'

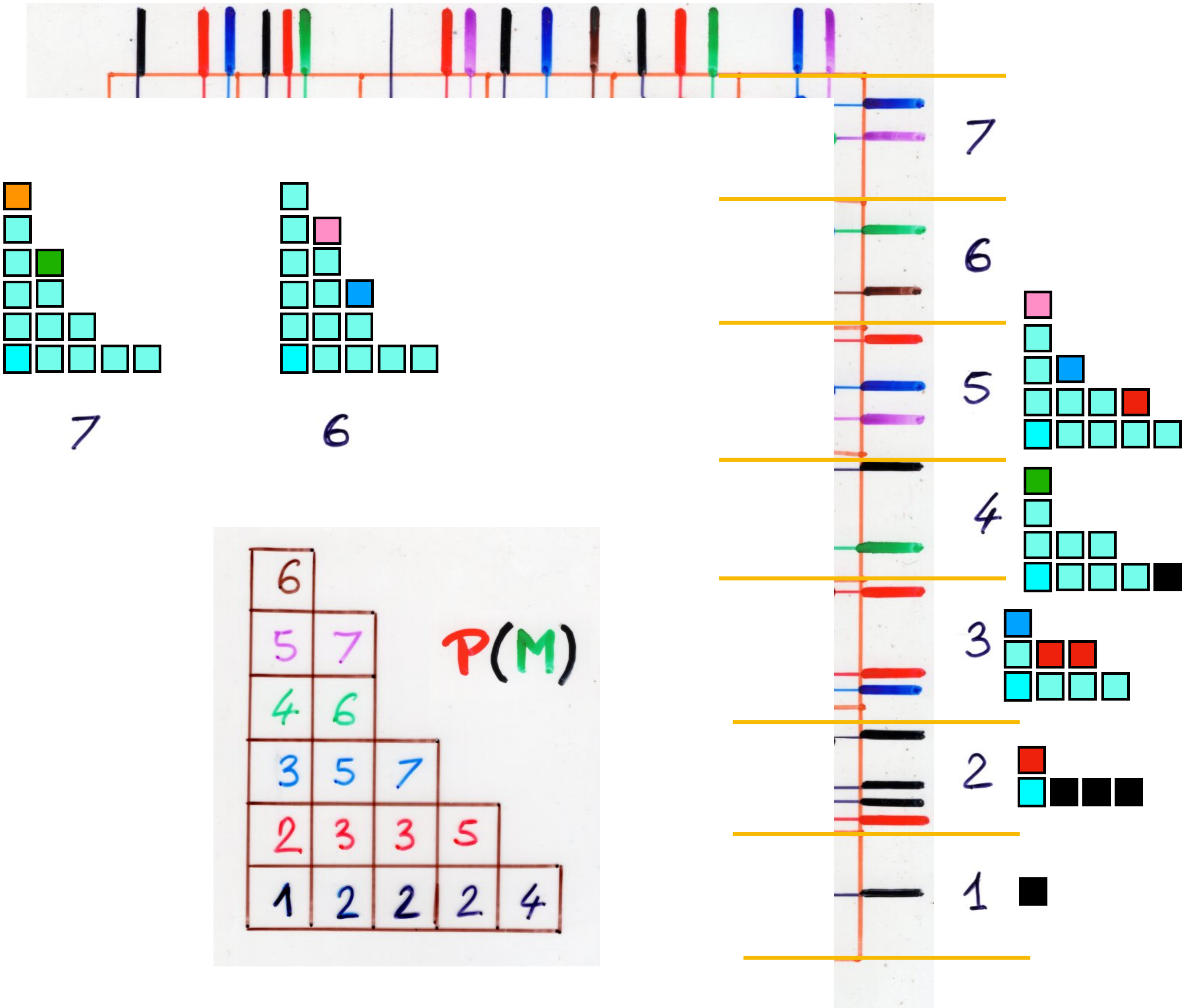
(conjugate)



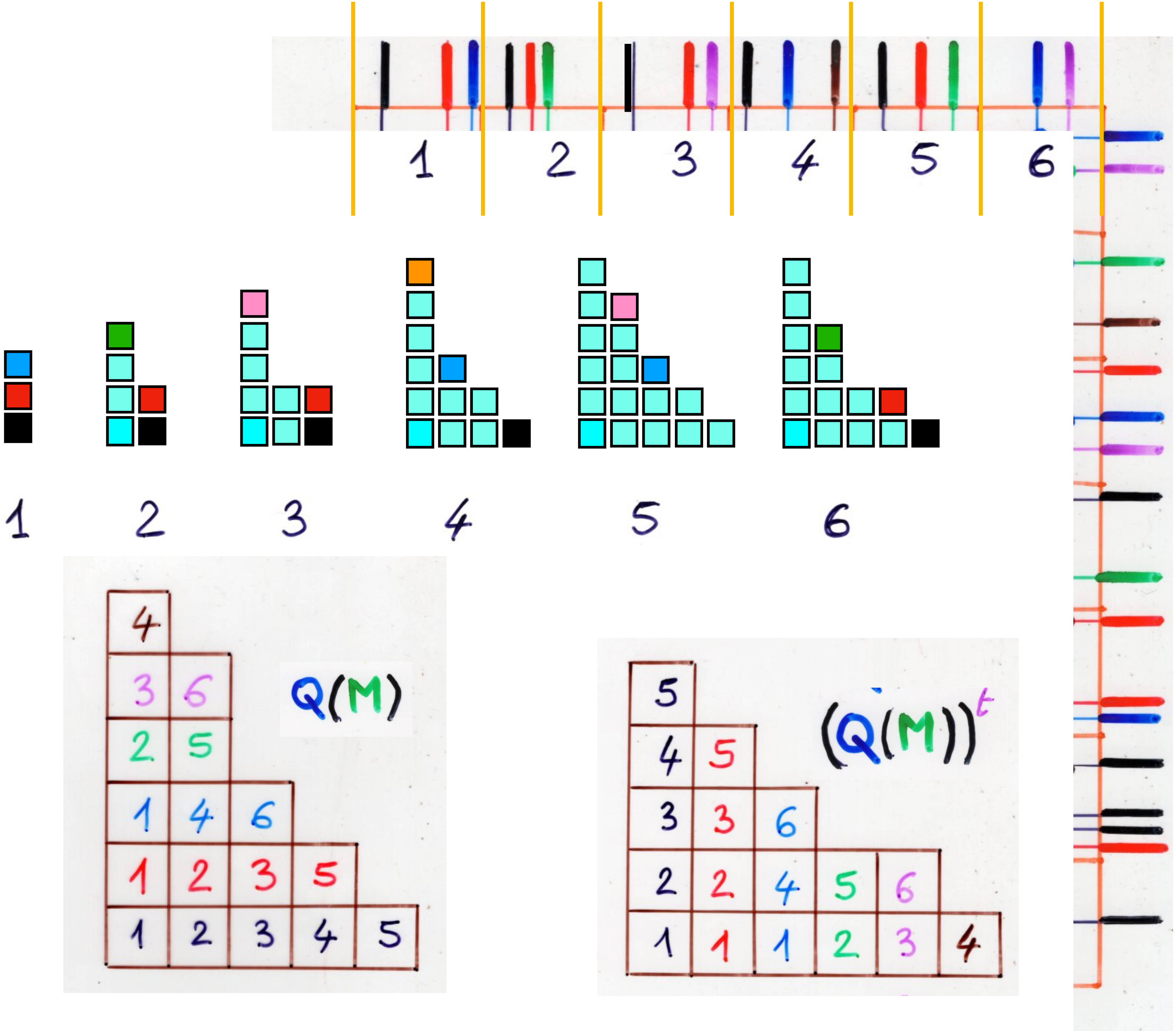




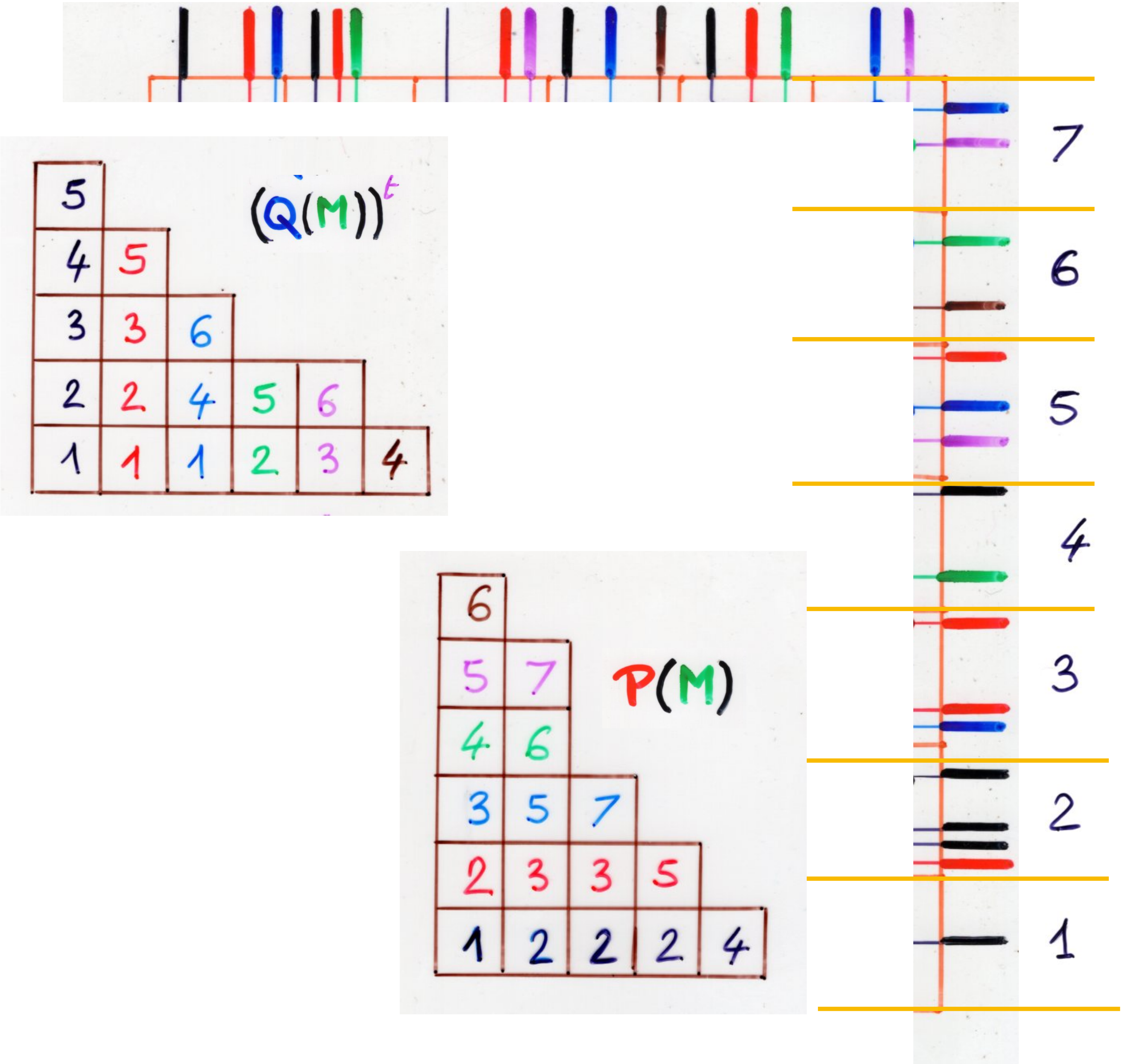

1
2
3
4
5
6
7
6
5
4
3
2
1



7
6
5
4
3
2
1
7
6
P(M)
6
5 7
4 6
3 5 7
2 3 3 5
1 2 2 2 4



1 2 3 4 5 6
1 2 3 4 5 6
Q(M)
(Q(M))^t

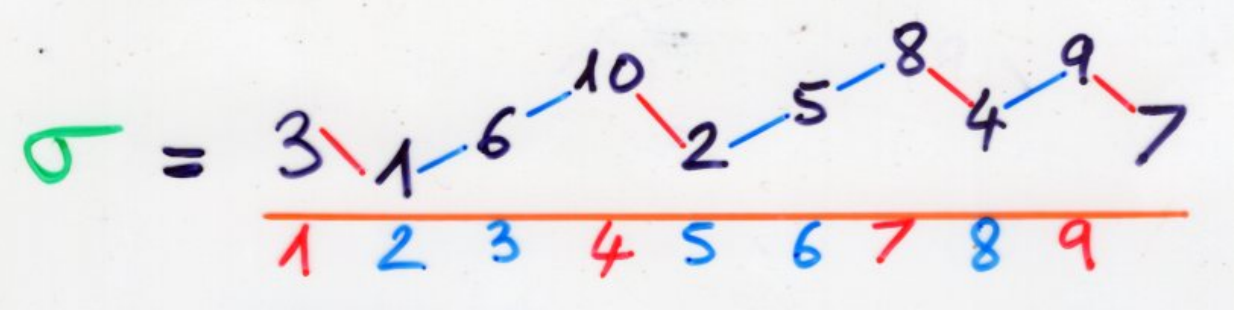


(Q(M))^t
5
4 5
3 3 6
2 2 4 5 6
1 1 1 2 3 4
P(M)
6
5 7
4 6
3 5 7
2 3 3 5
1 2 2 2 4
7
6
5
4
3
2
1

Proof:

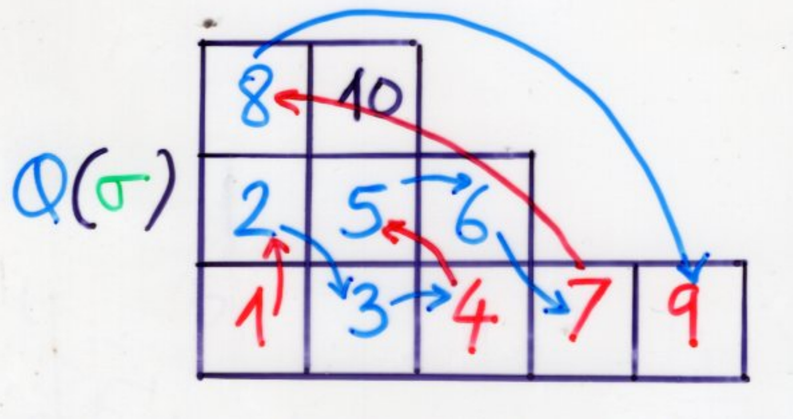
i is a descent of σ iff $\sigma(i) > \sigma(i+1)$

rise $\sigma(i) < \sigma(i+1)$



Lemma $\sigma \xrightarrow{RS} (P, Q)$

- there is a rise at the index i of σ iff $(i+1)$ is located at the South-East of i in the tableau Q



dual-RSK

$M = (a_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}}$

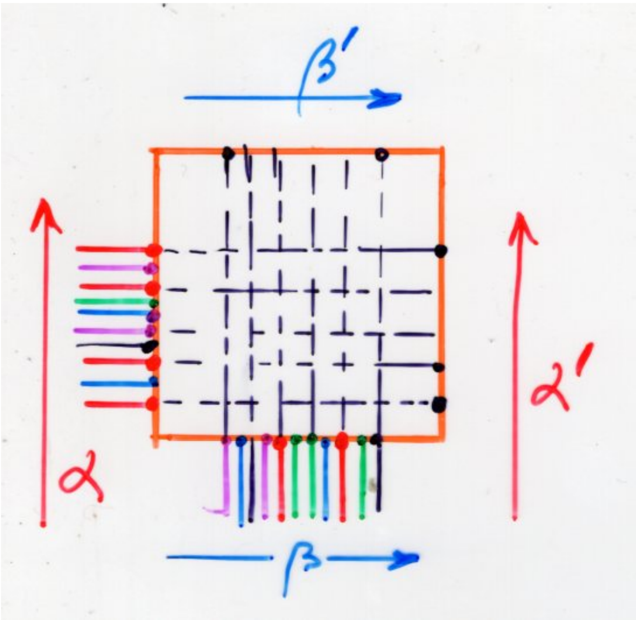
$a_{ij} = 0 \text{ or } 1$

$$M \xrightarrow{\text{dual RSK}} (P, Q)$$

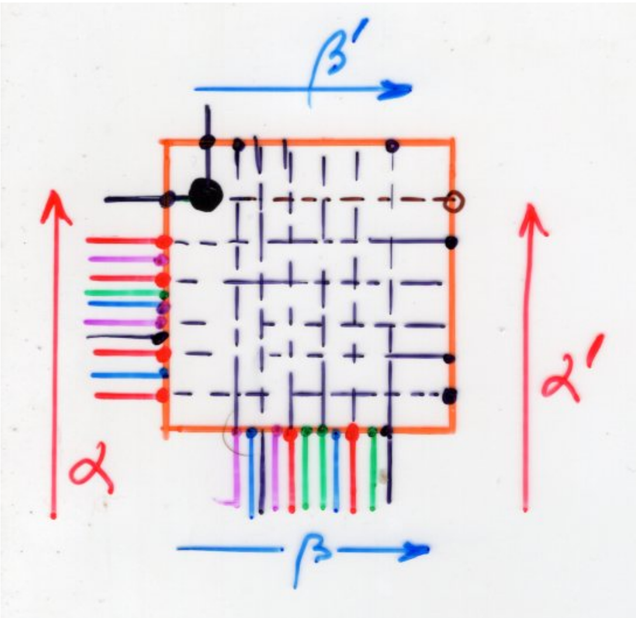
P shape λ

Q shape λ' (conjugate)

Dual local rules



$$\alpha'' = \alpha'$$
$$\beta'' = \beta'$$



$$(\alpha', \beta') = RS(\alpha \cdot 1, \beta)$$

$$\alpha'' = \alpha'$$
$$\beta'' = 1 \cdot \beta'$$

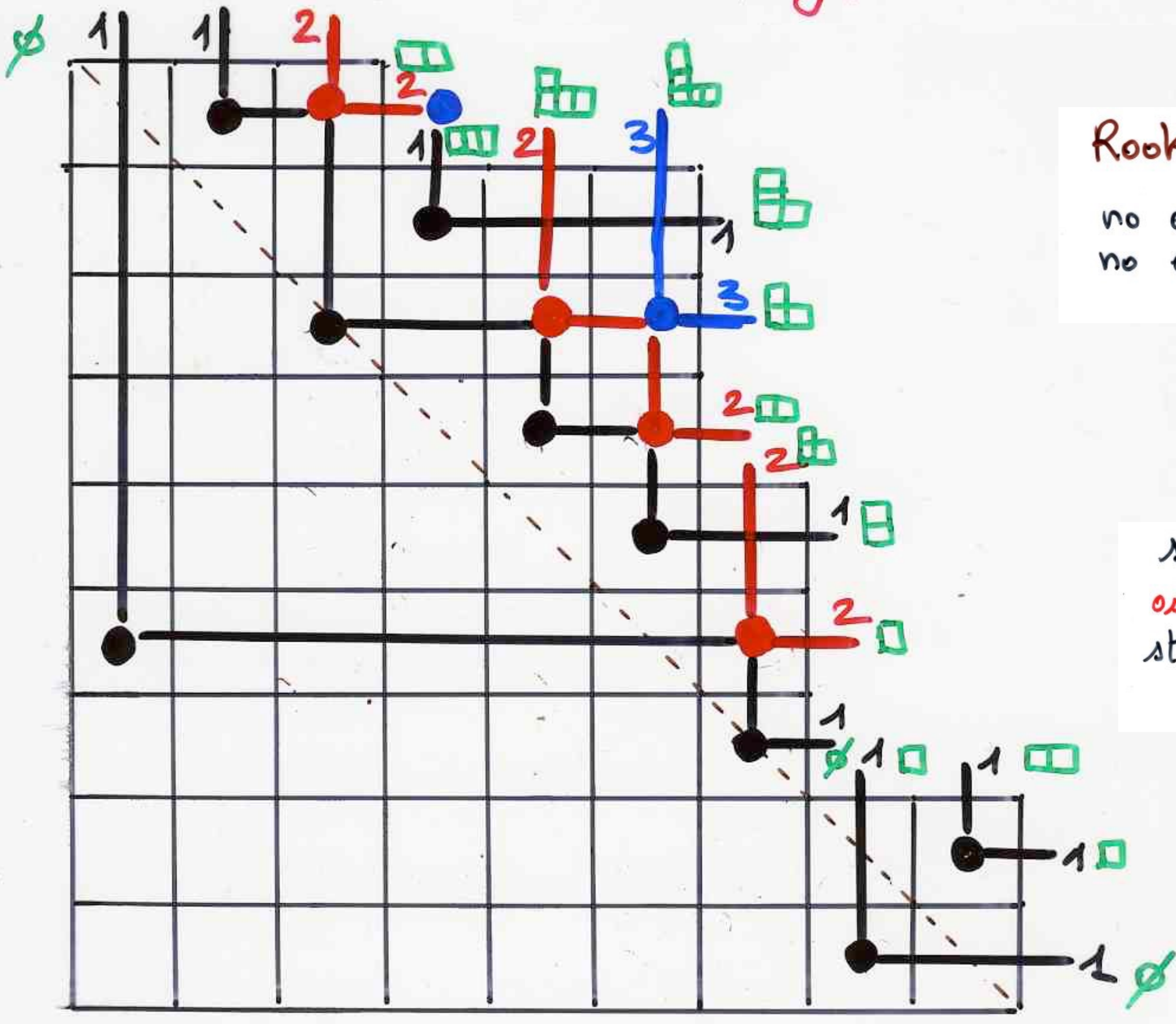
Lauren Kelly Williams

App for iPad

available on the AppStore

bijections
for
rook placements

oscillating tableaux

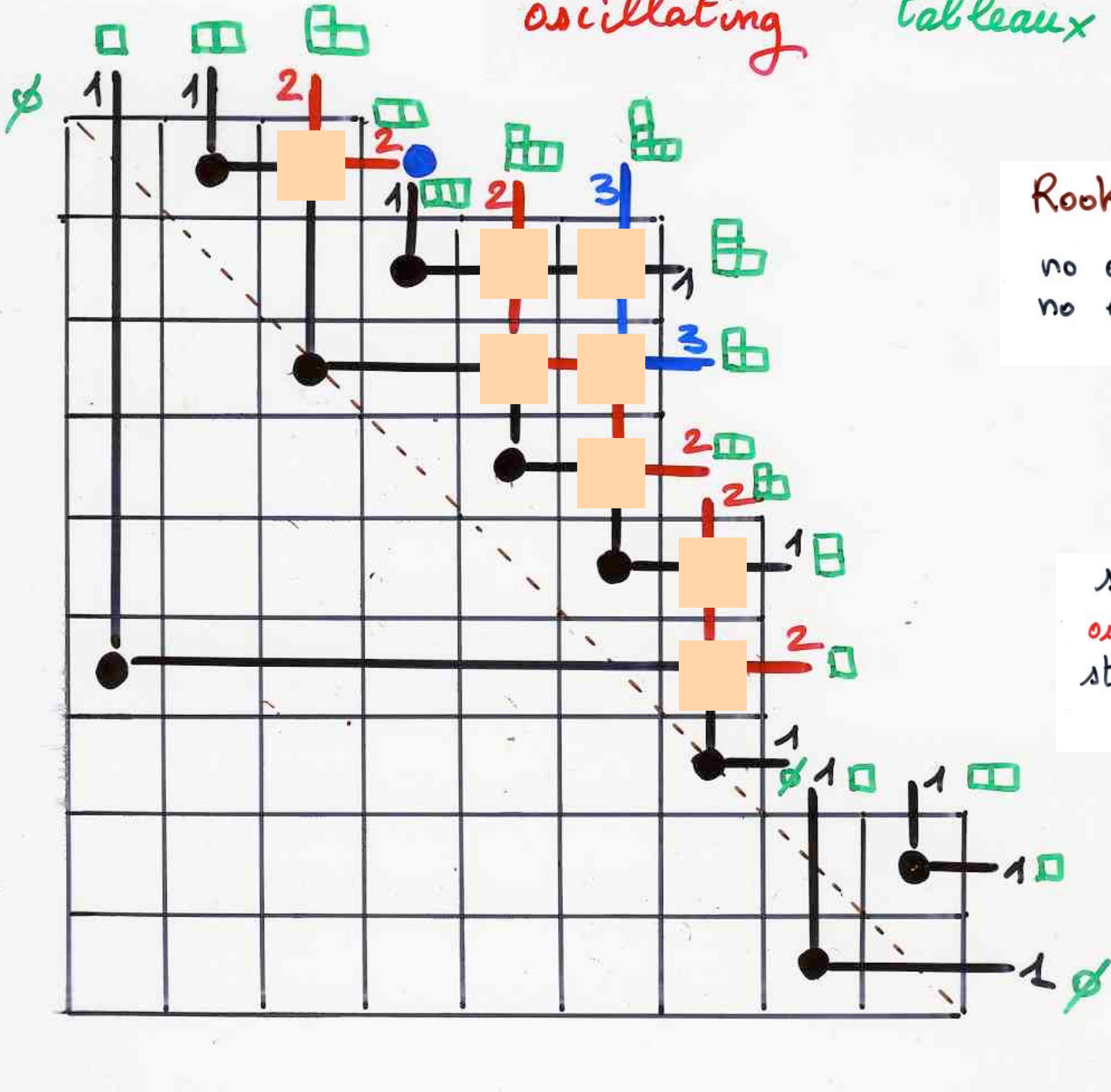


Rook placements
with
no empty row
no empty column

↕

sequences of
oscillating tableaux
starting and ending
at ∅

oscillating tableaux

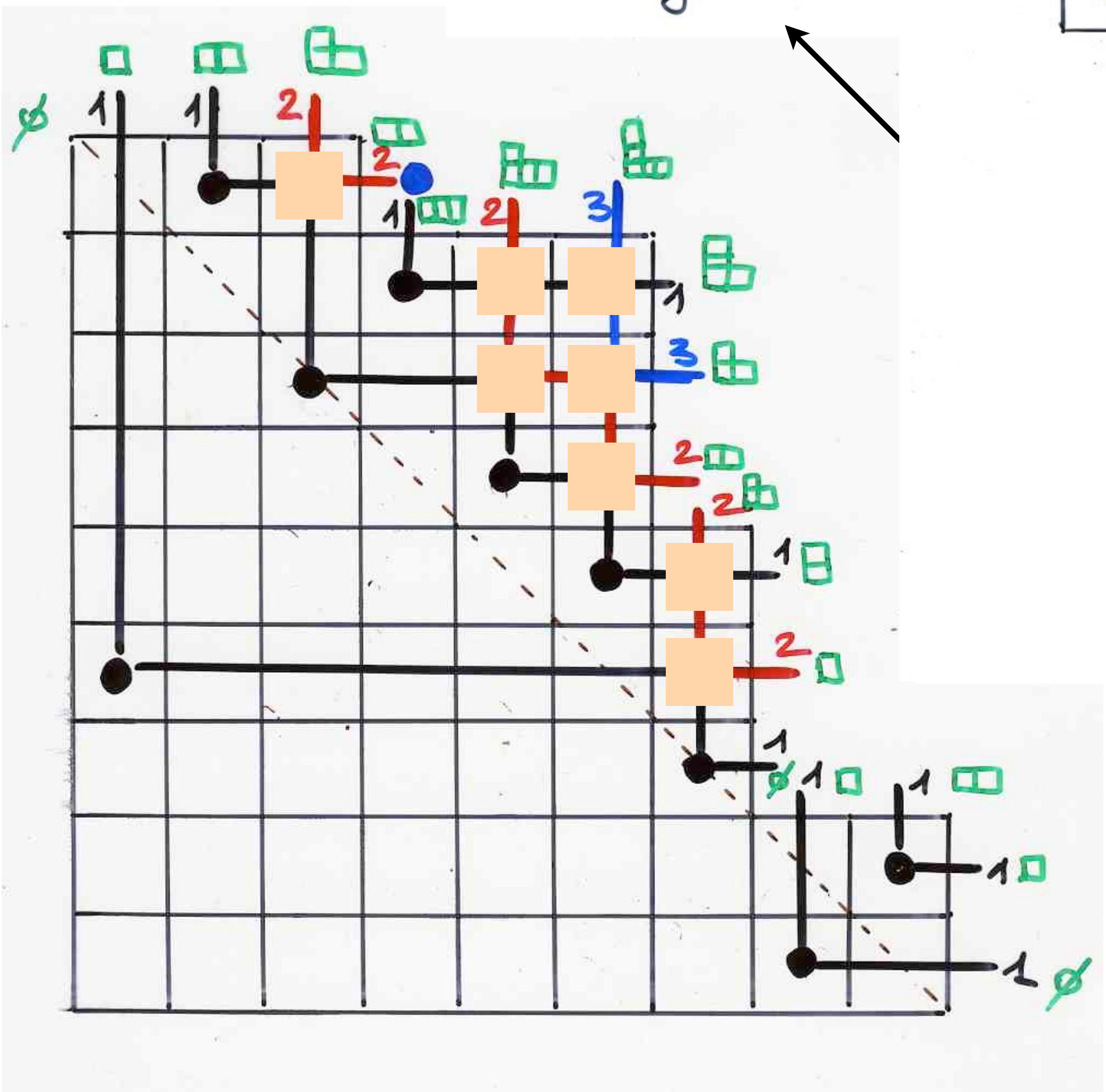


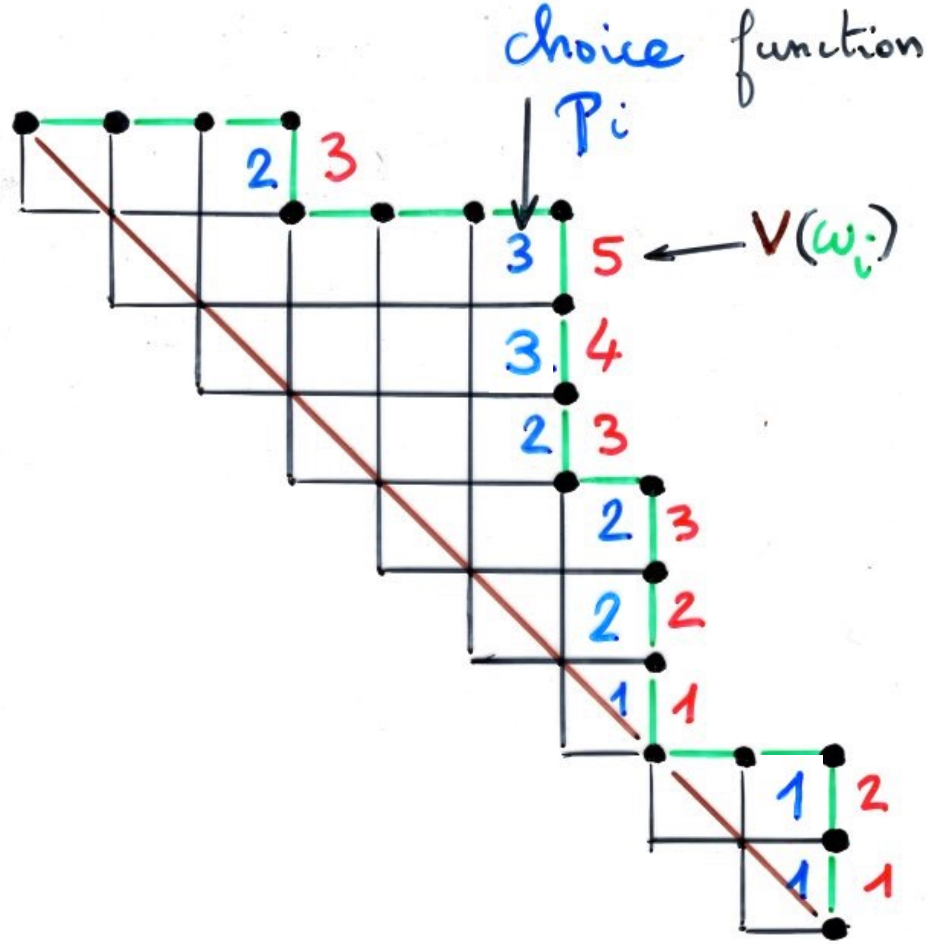
Rook placements
with
no empty row
no empty column

↕

sequences of oscillating tableaux
starting and ending
at ∅

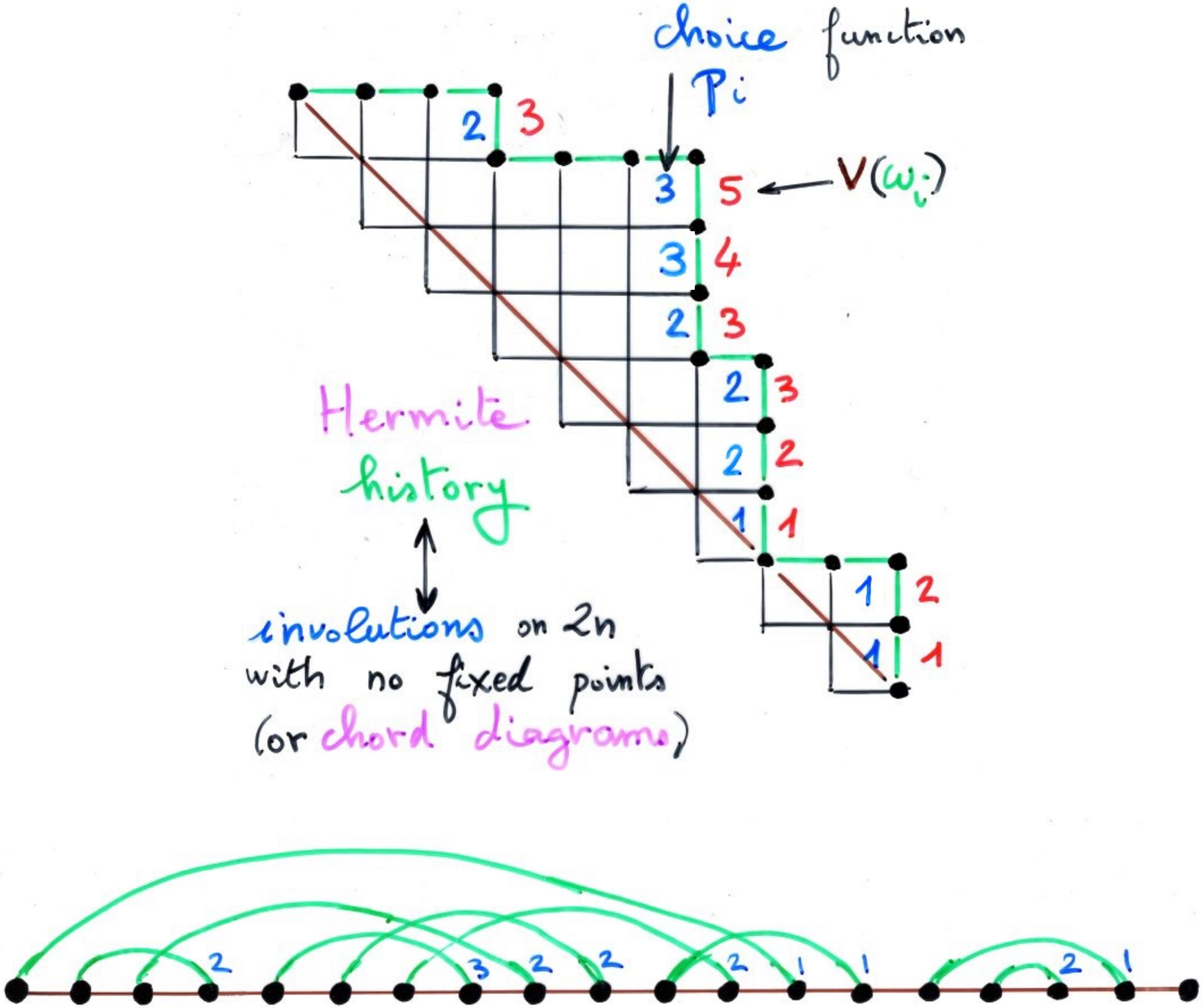
Rook placements
with
no empty row
no empty column



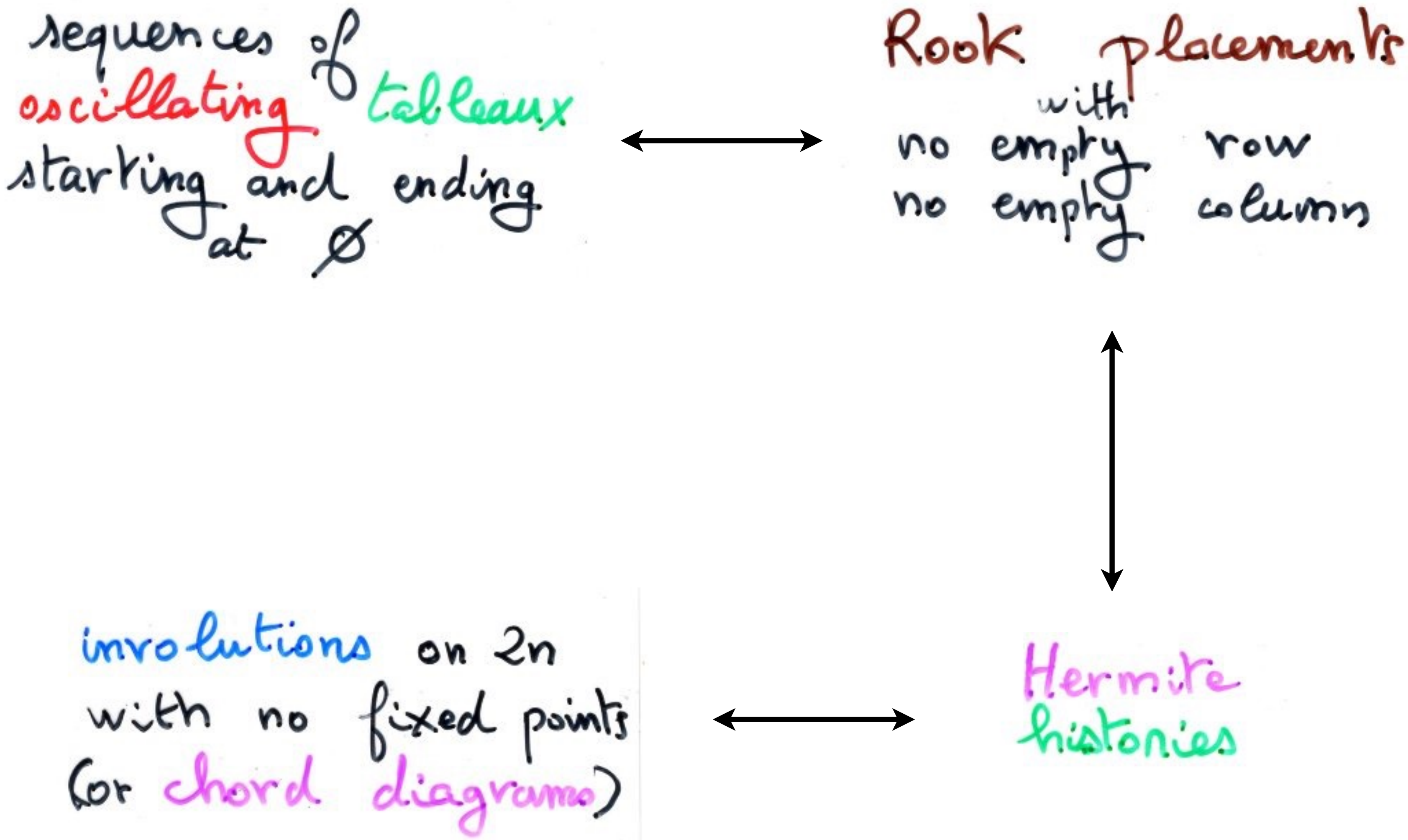


see Ch4b, Part I

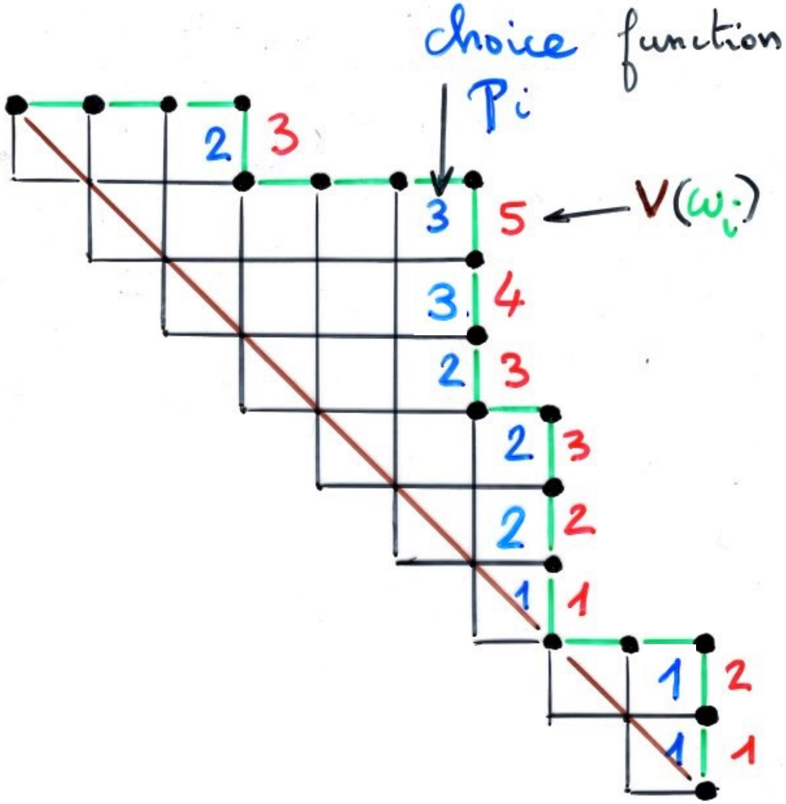
course BJC 2006



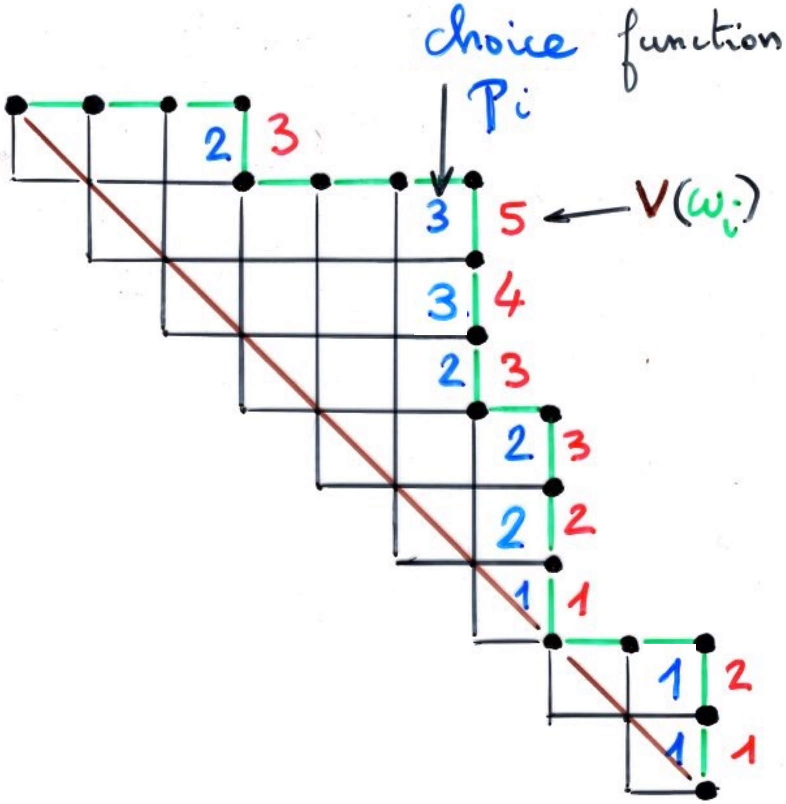
choice function
p_i
$V(\omega_i)$
Hermite history
involutions on 2n
with no fixed points
(or chord diagrams)



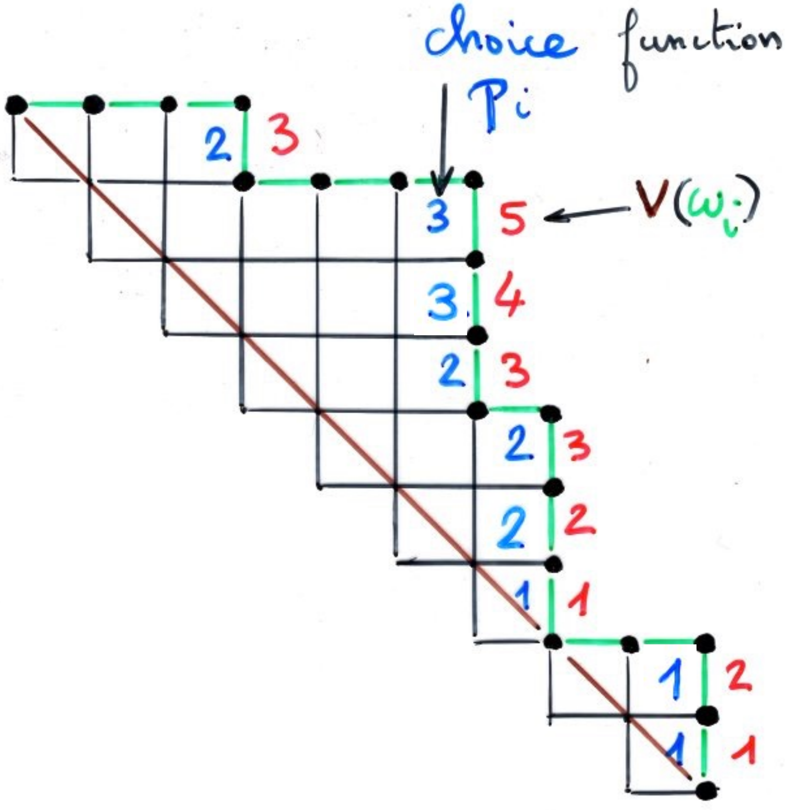
sequences of oscillating tableaux starting and ending at ∅
Rook placements with no empty row no empty column
involutions on 2n with no fixed points (or chord diagrams)
Hermite histories


choice function
p_i
$V(\omega_i)$
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1

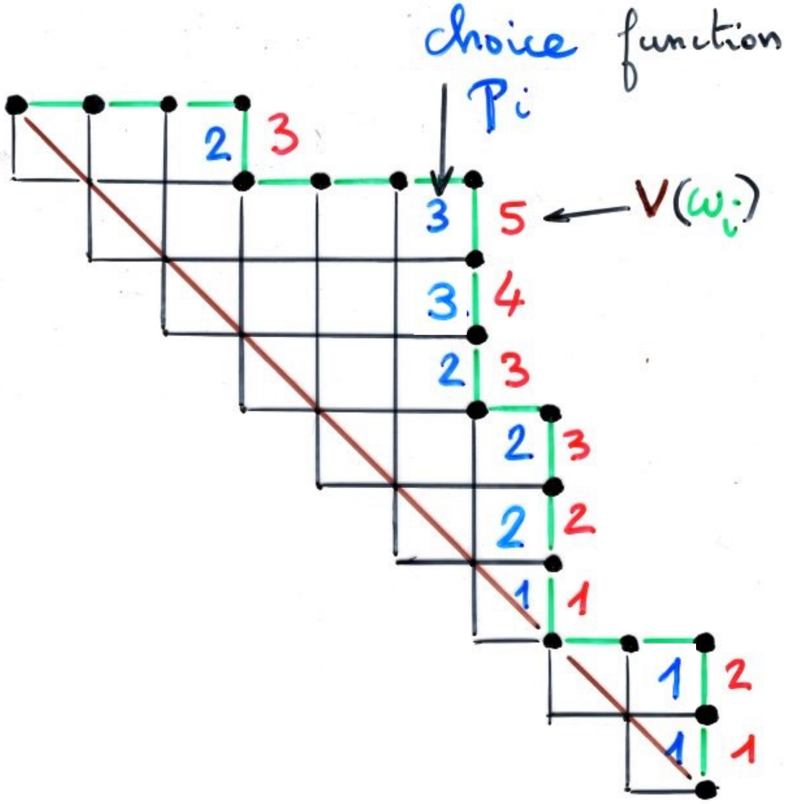

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18


choice function
Pi
V(ωi)
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18


choice function
p_i
$V(\omega_i)$
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18


choice function
P_i
$V(\omega_i)$
2 3
3 5
3 4
2 3
2 3
2 2
1 1
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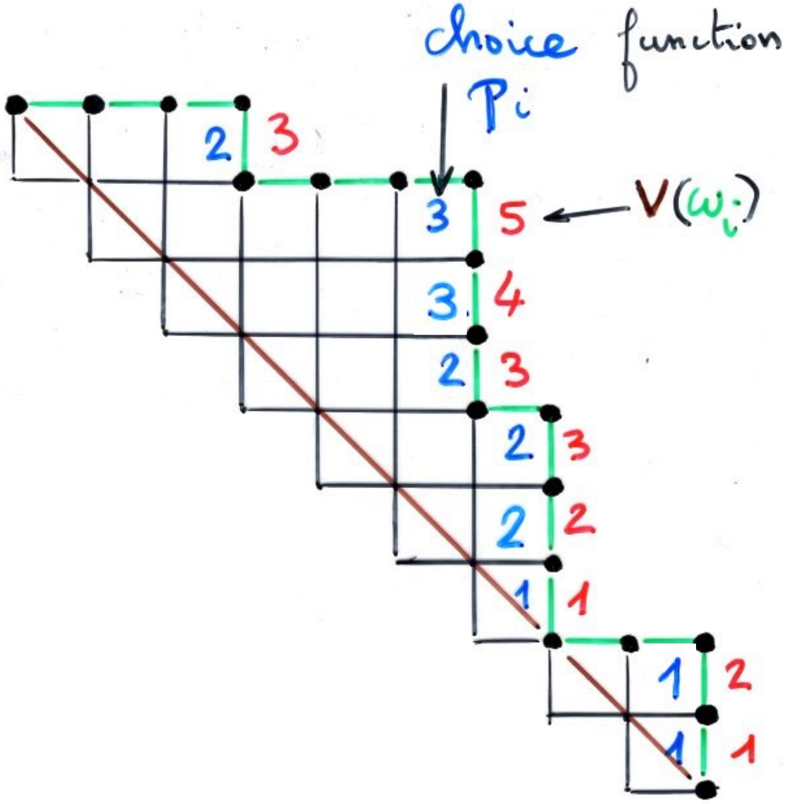
1 1


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18


choice function
p_i
$V(\omega_i)$
2 3
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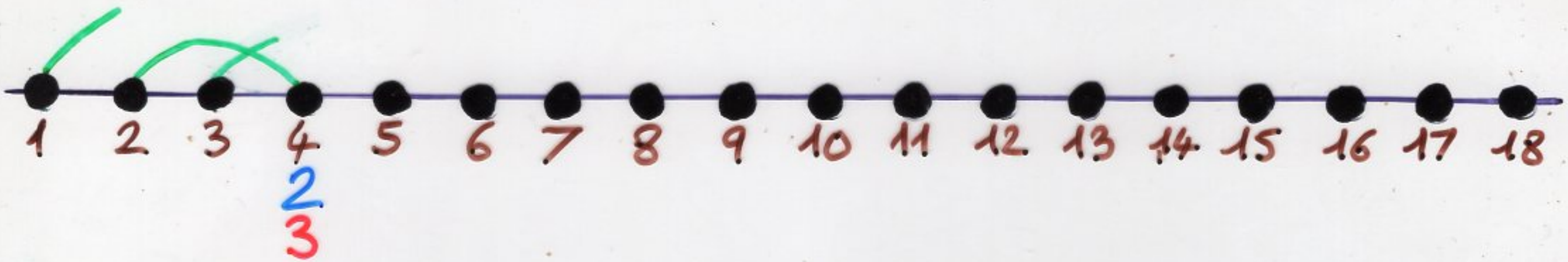
3 4
2 3
2 3
2 2
1 1
1 2
1 1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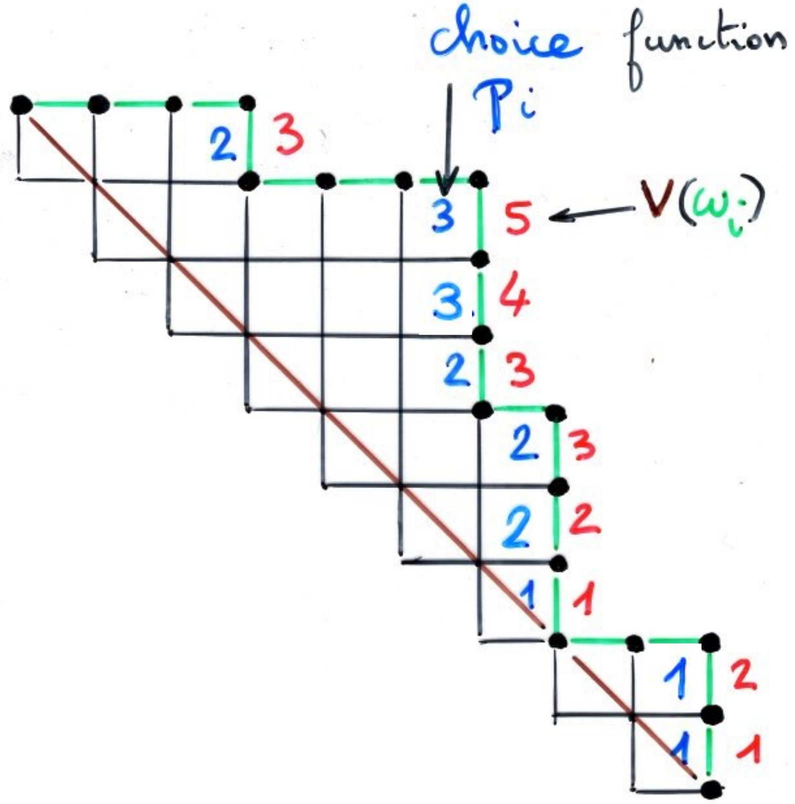
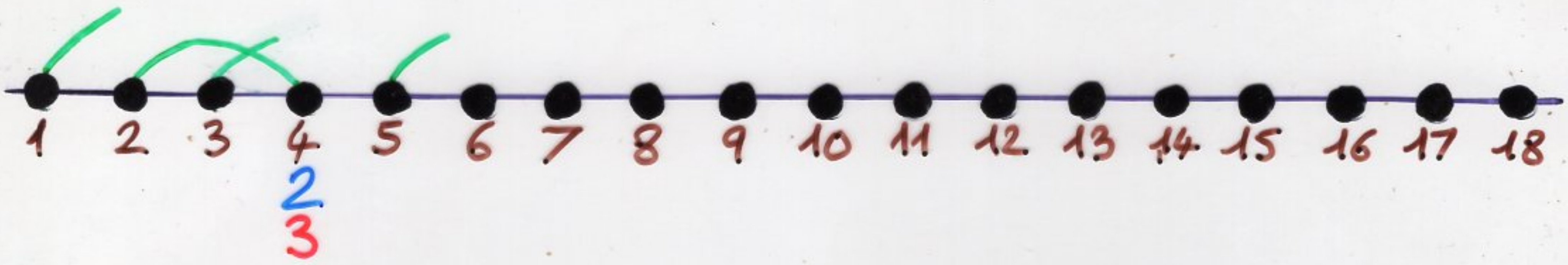
2
3


choice function
P_i
V(ω_i)
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1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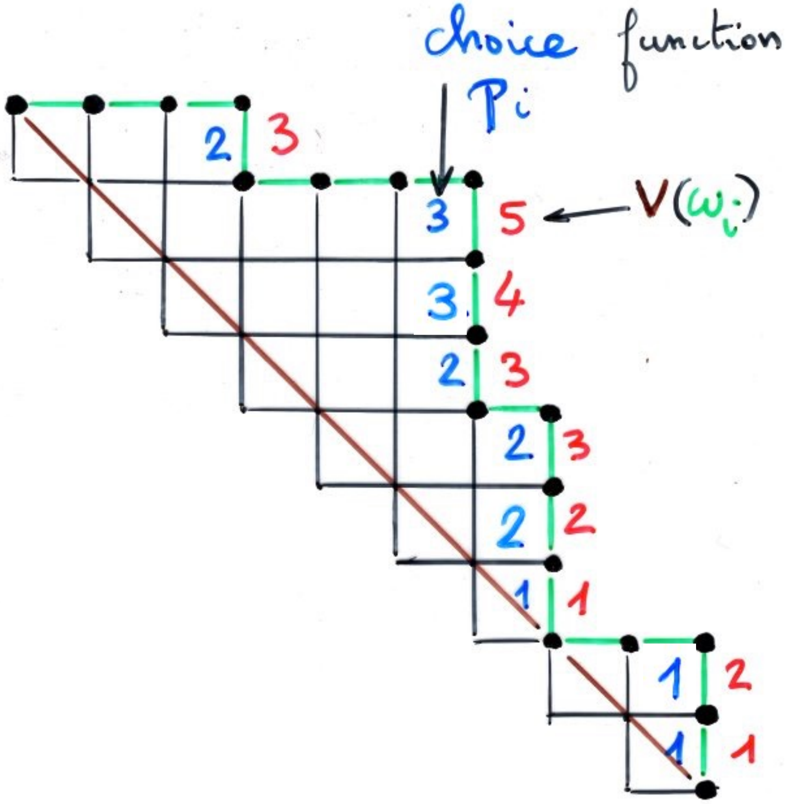
choice function
p_i
$V(\omega_i)$
2 3
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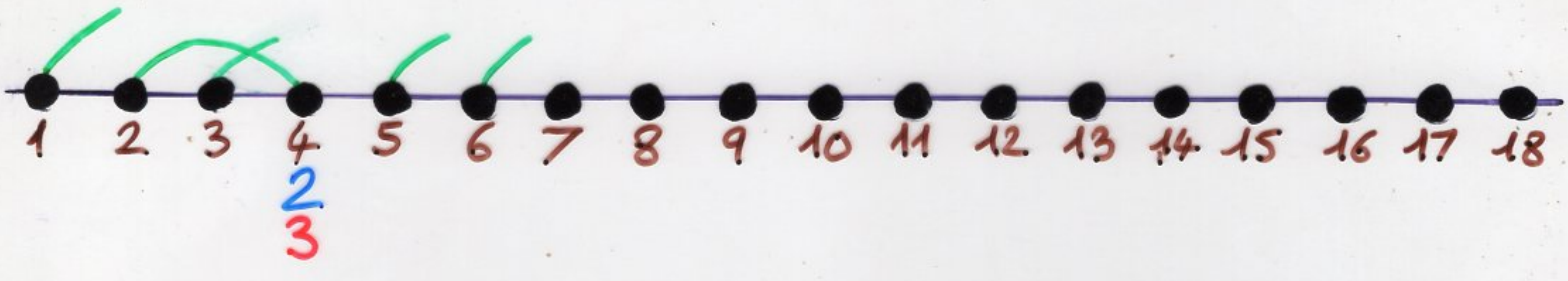
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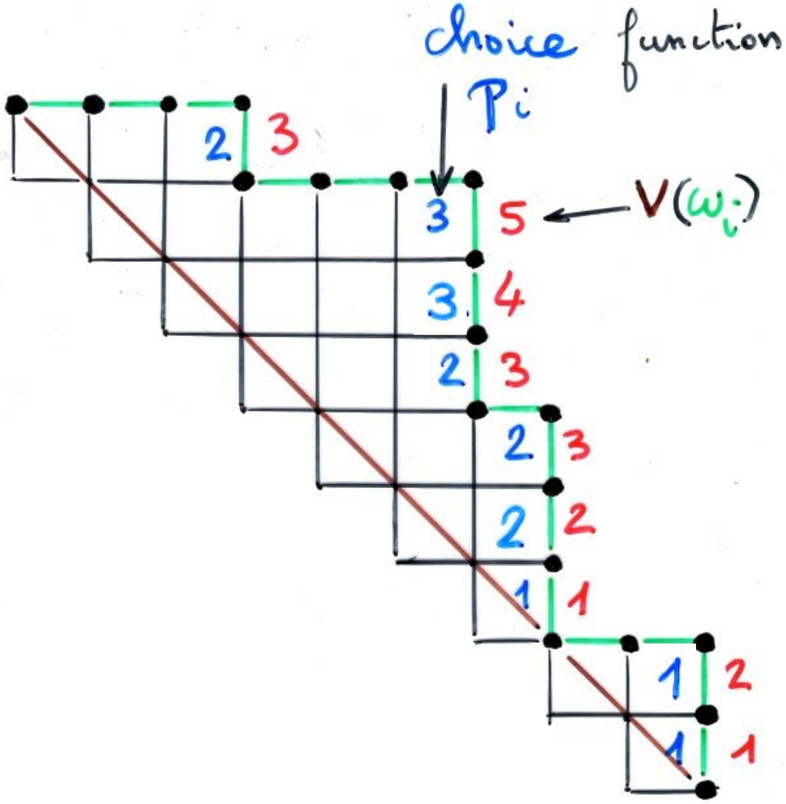
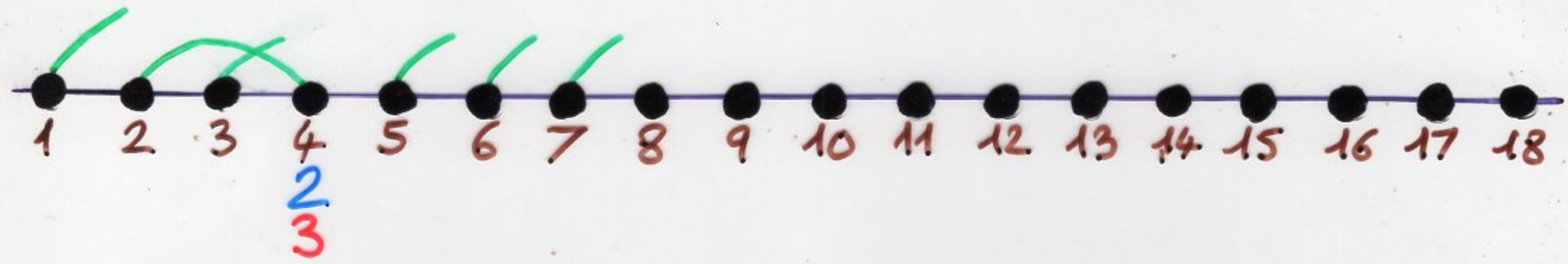

choice function
V(ω)
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1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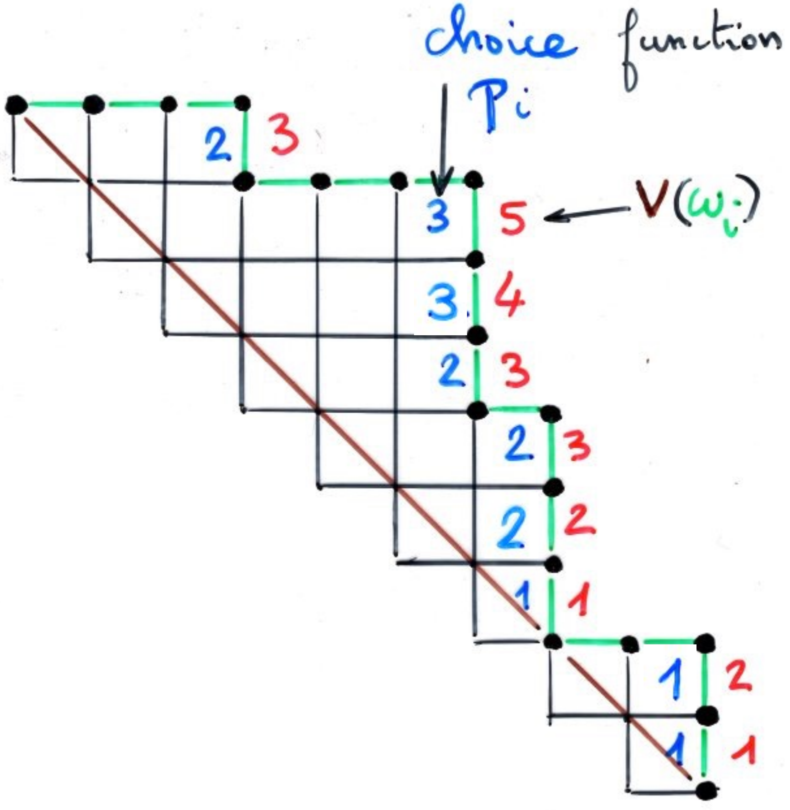
choice function
p_i
$V(\omega_i)$
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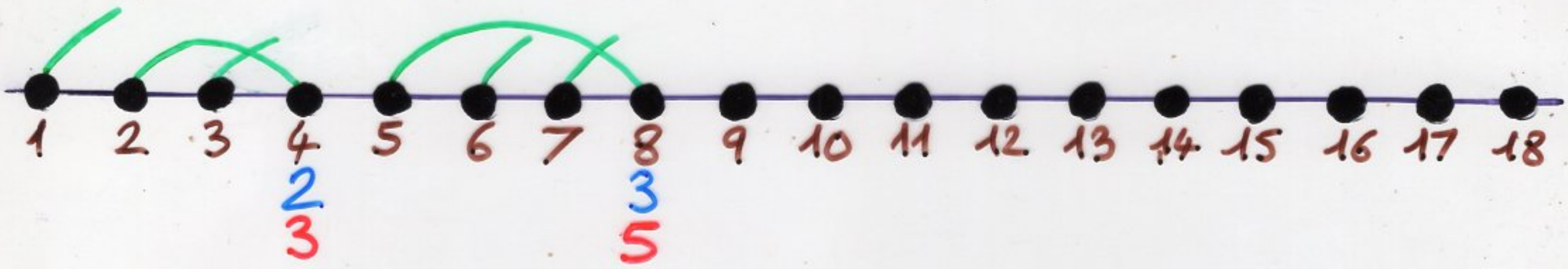
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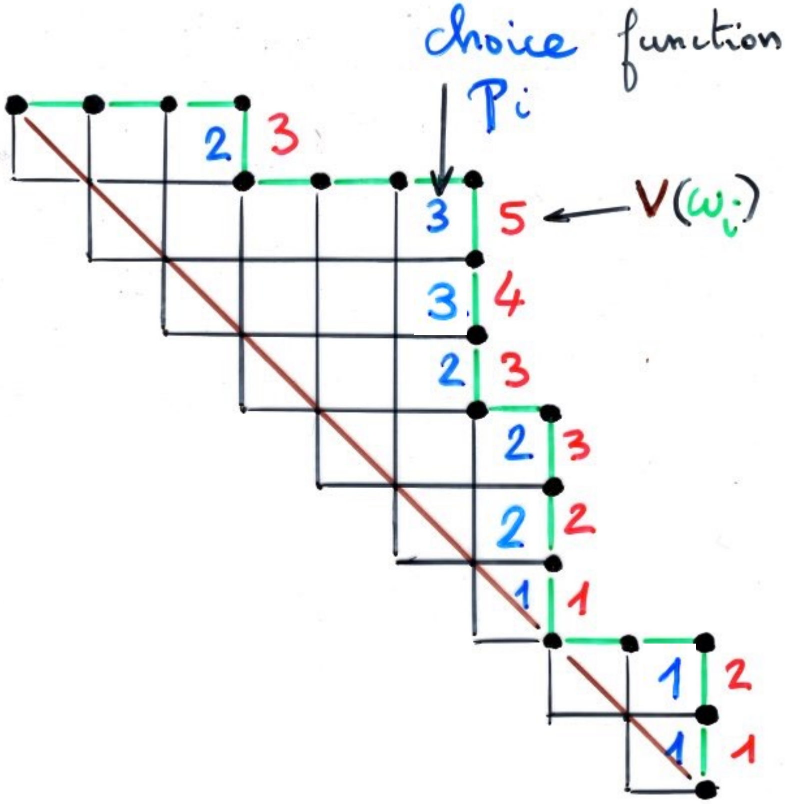

choice function
p_i
$V(\omega_i)$
2 3
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2 3
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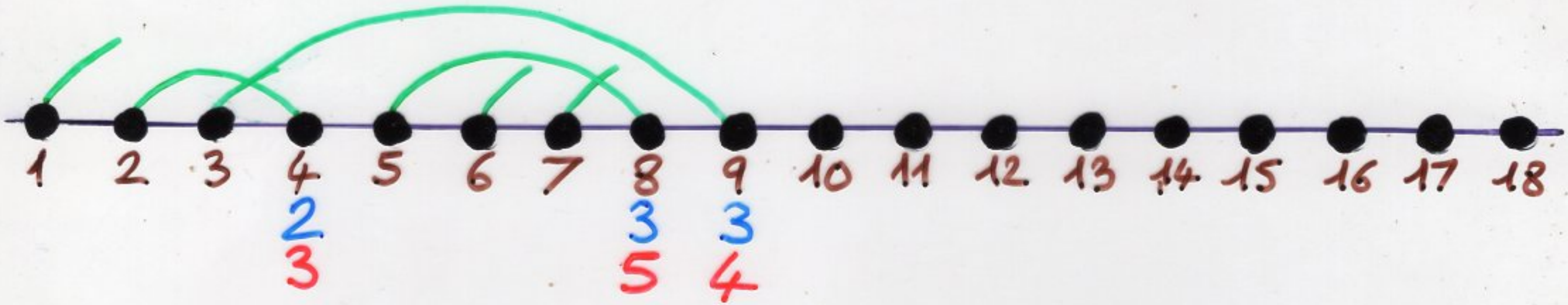

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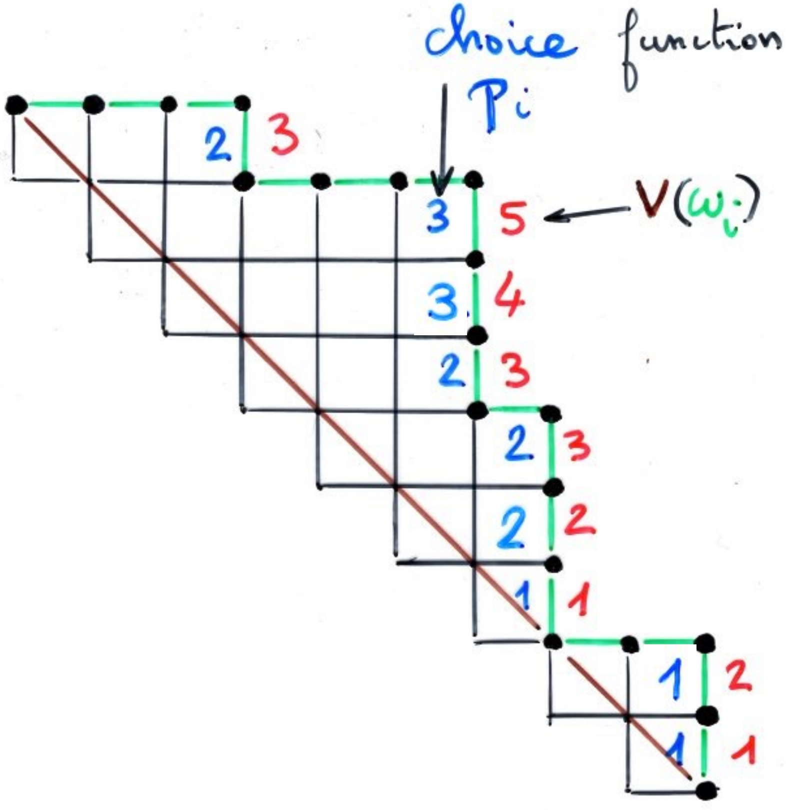
choice function
p_i
$V(\omega_i)$
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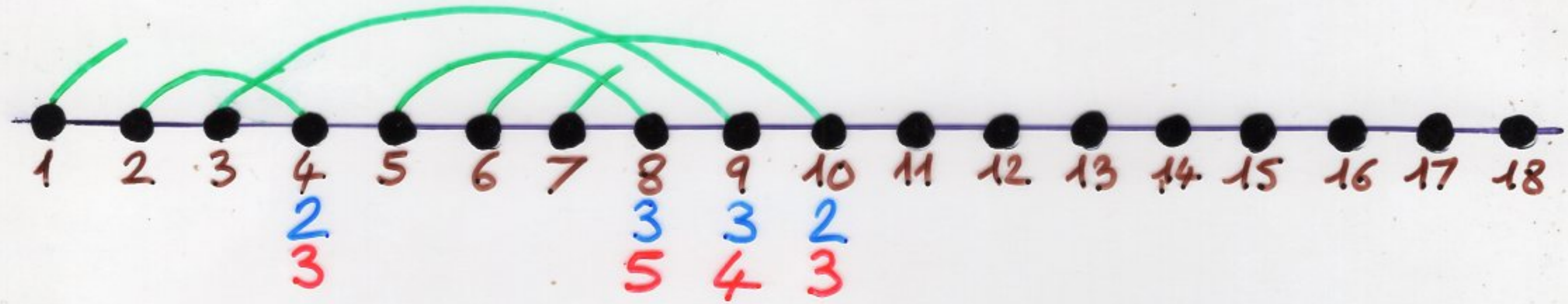
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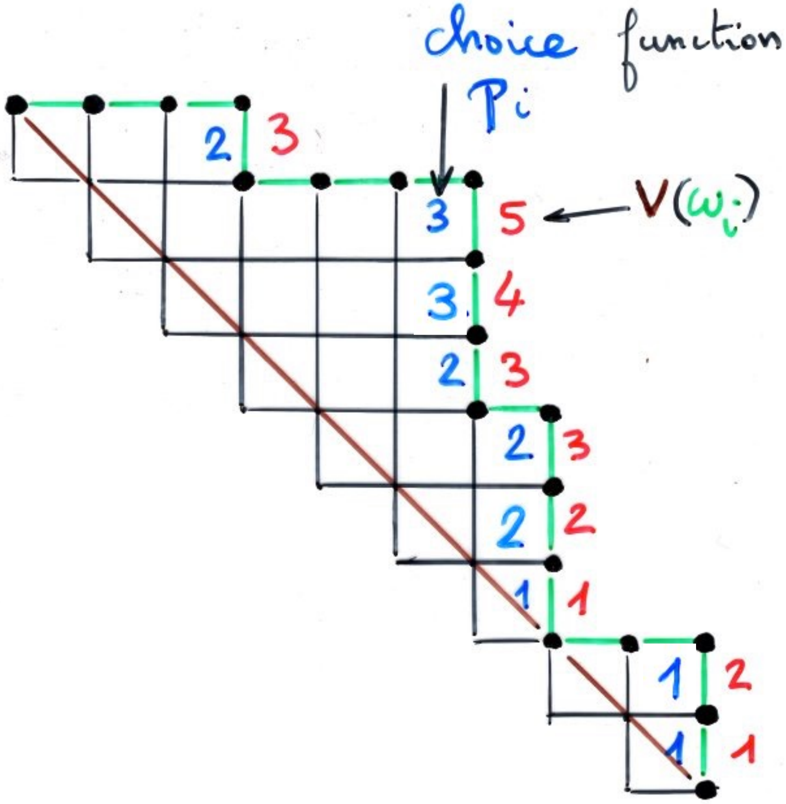
2 3 3 2
3 5 4 3


choice function
p_i
$V(\omega_i)$
2 3
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2 3
2 3
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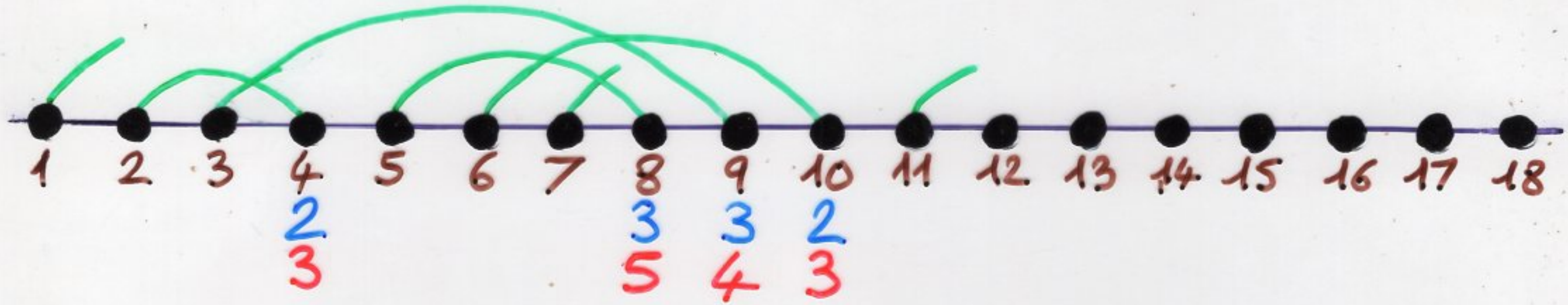
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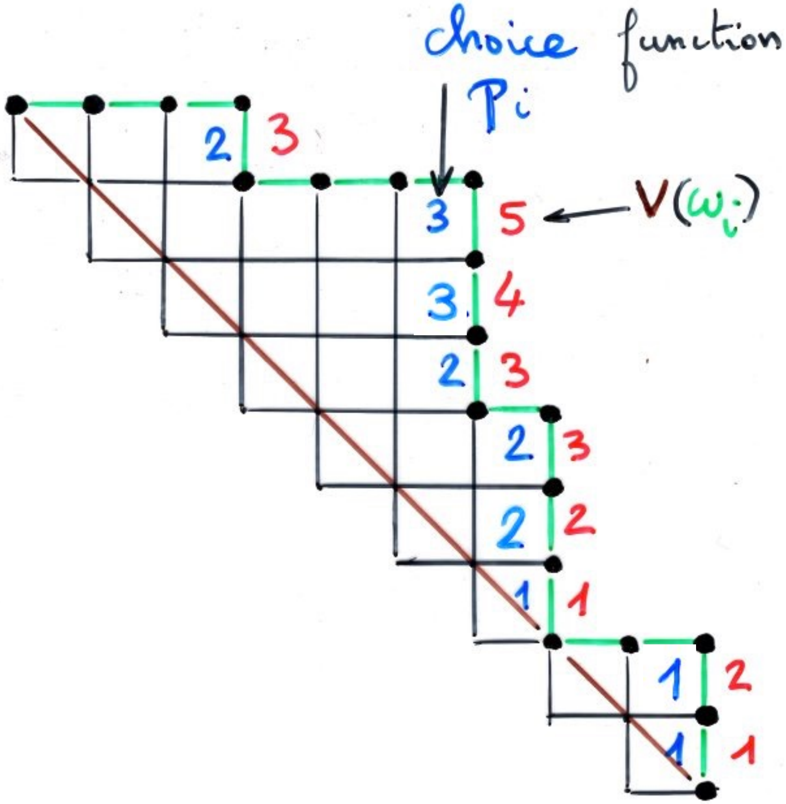
2 3 3 2
3 5 4 3


choice function
p_i
$V(\omega_i)$
2 3
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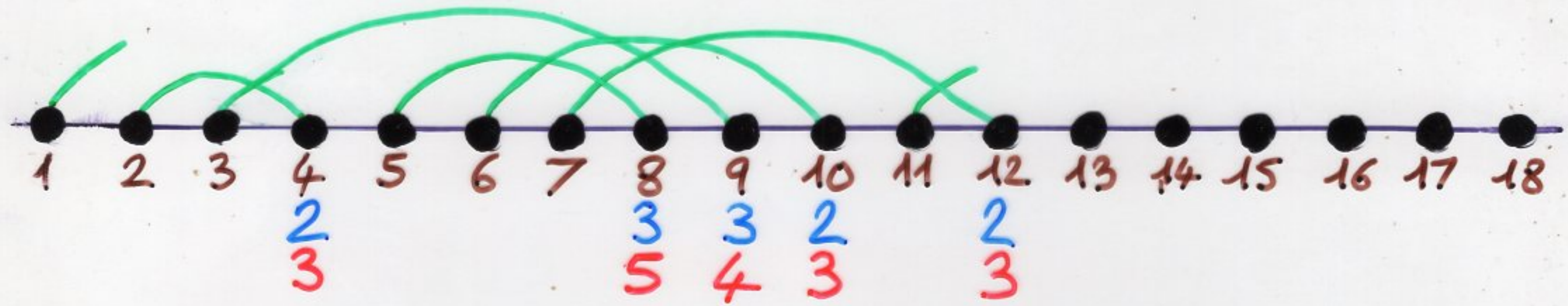
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2 3 3 2 2
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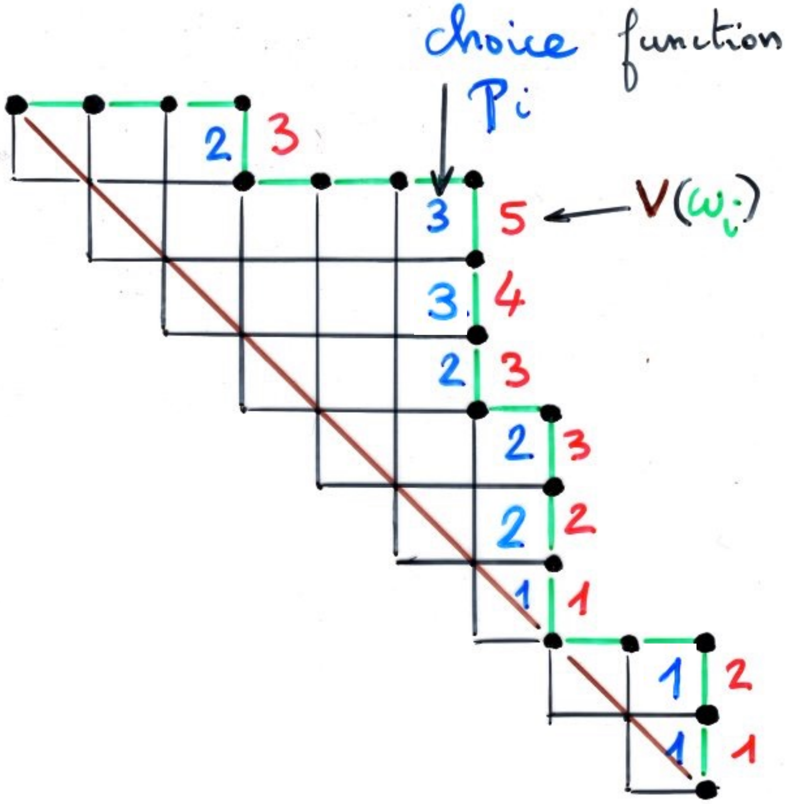
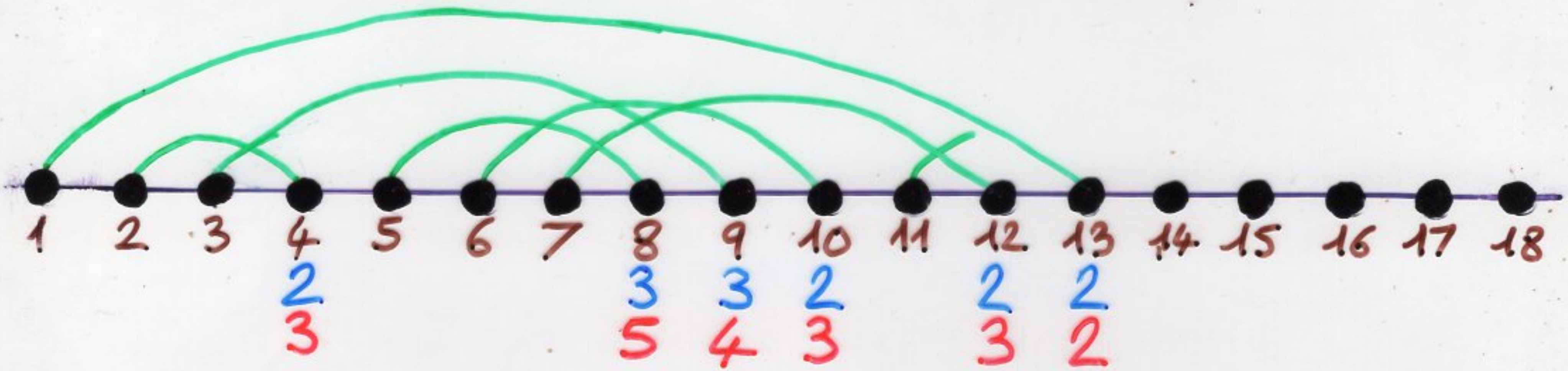


choice function
p_i
$V(\omega_i)$



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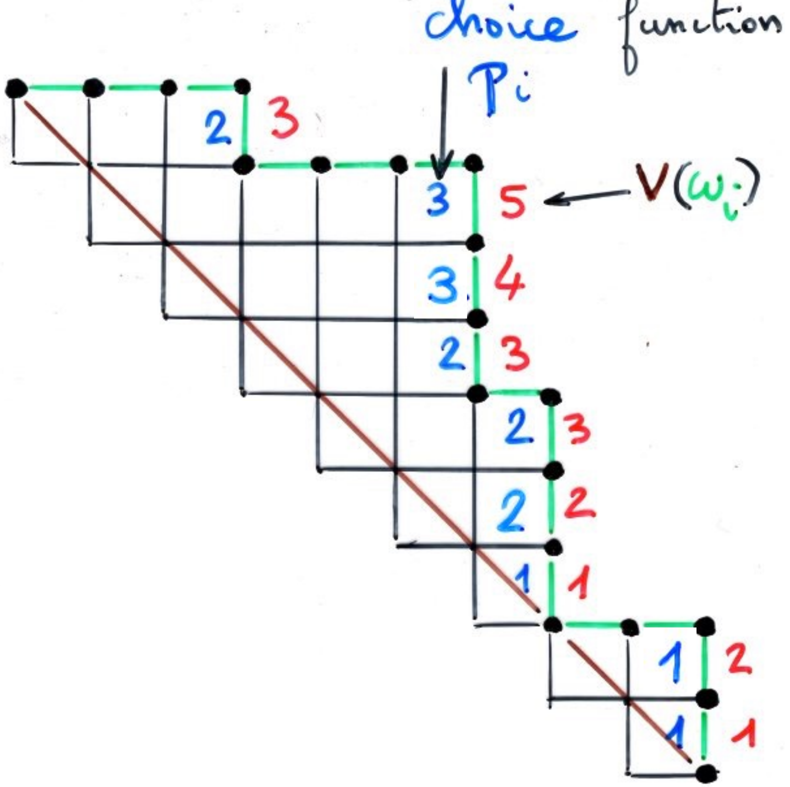
2 3 3 2 2 2
3 5 4 3 3 2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2 3
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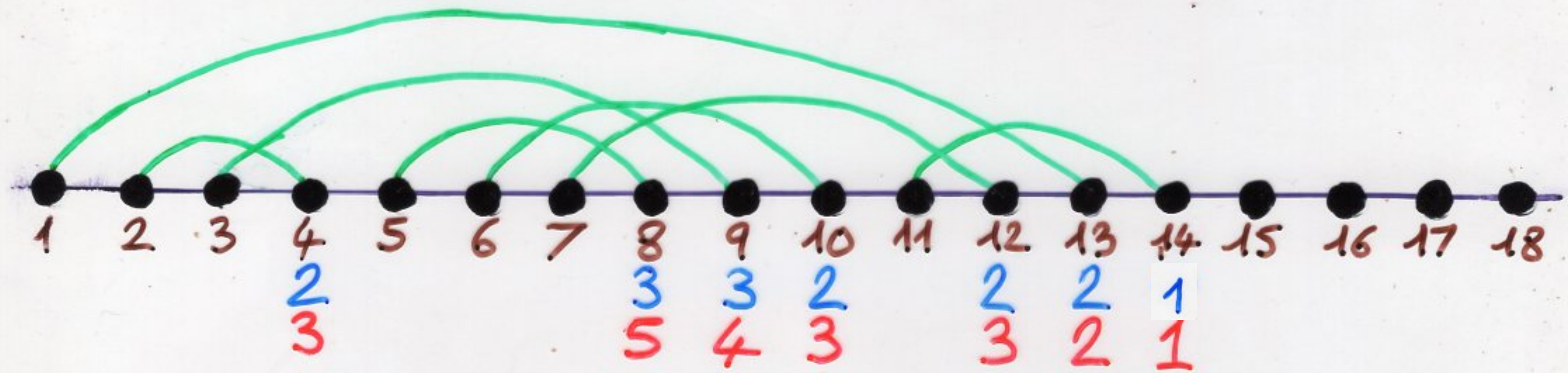
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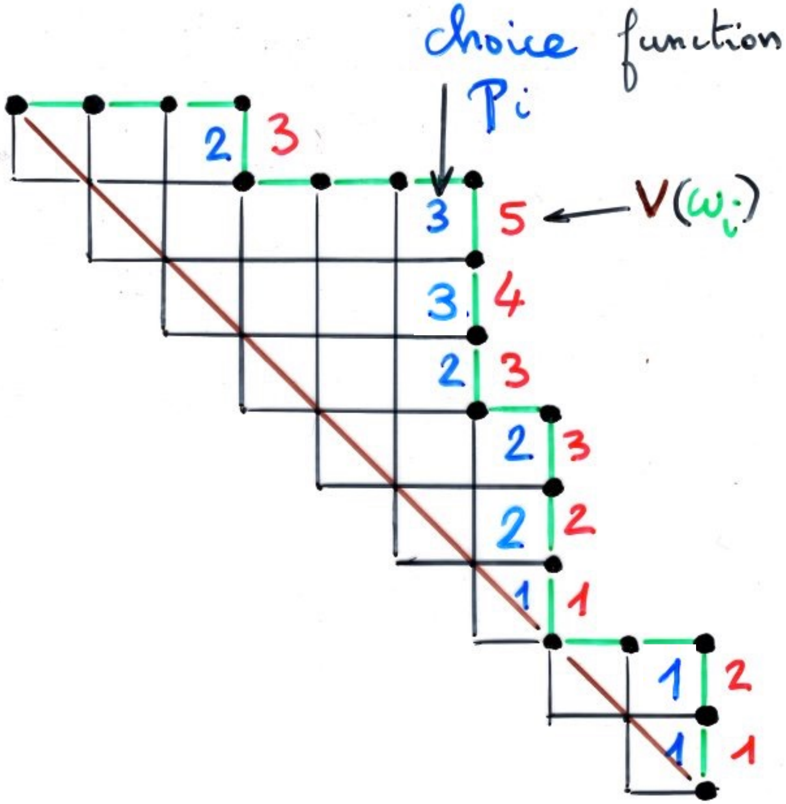
2 3 3 2 2 2 1
3 5 4 3 3 2 1


choice function
p_i
V(ω_i)
2 3
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3 4
2 3
2 3
2 2
1 1
1 2
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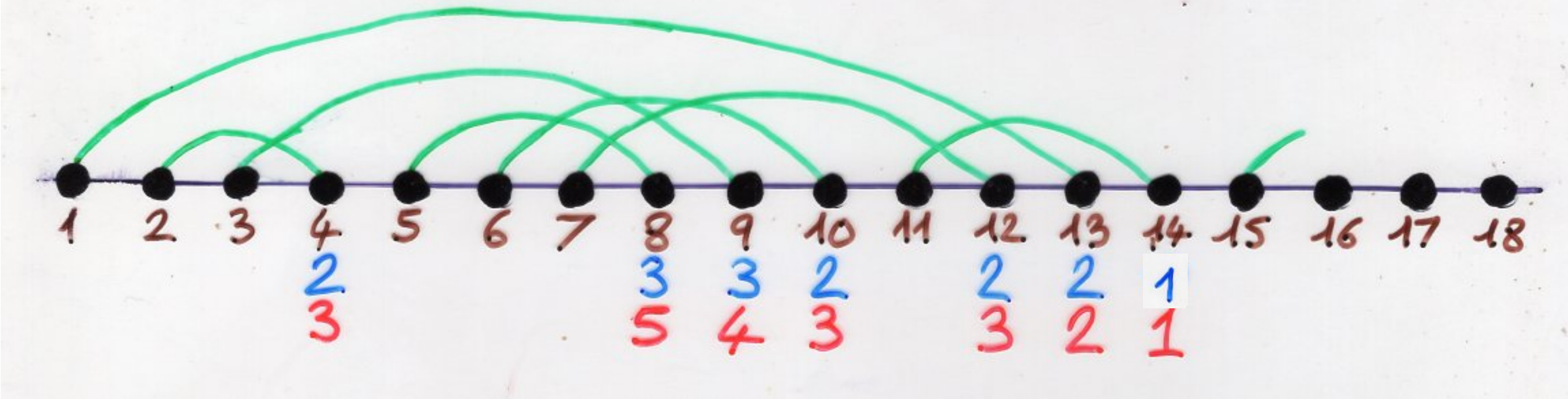
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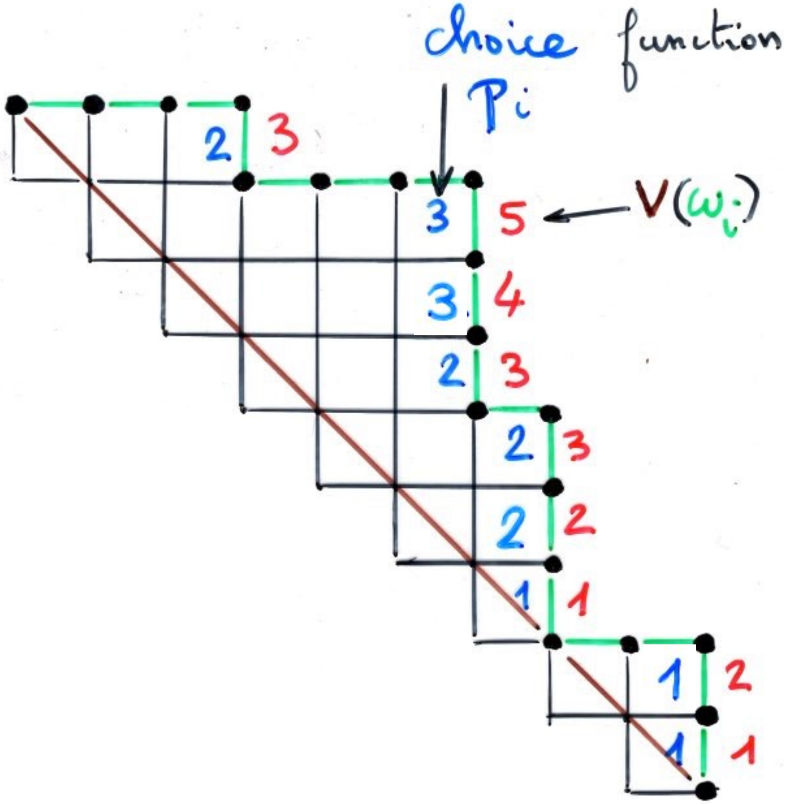

choice function
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$V(\omega_i)$
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2 3
2 3
2 2
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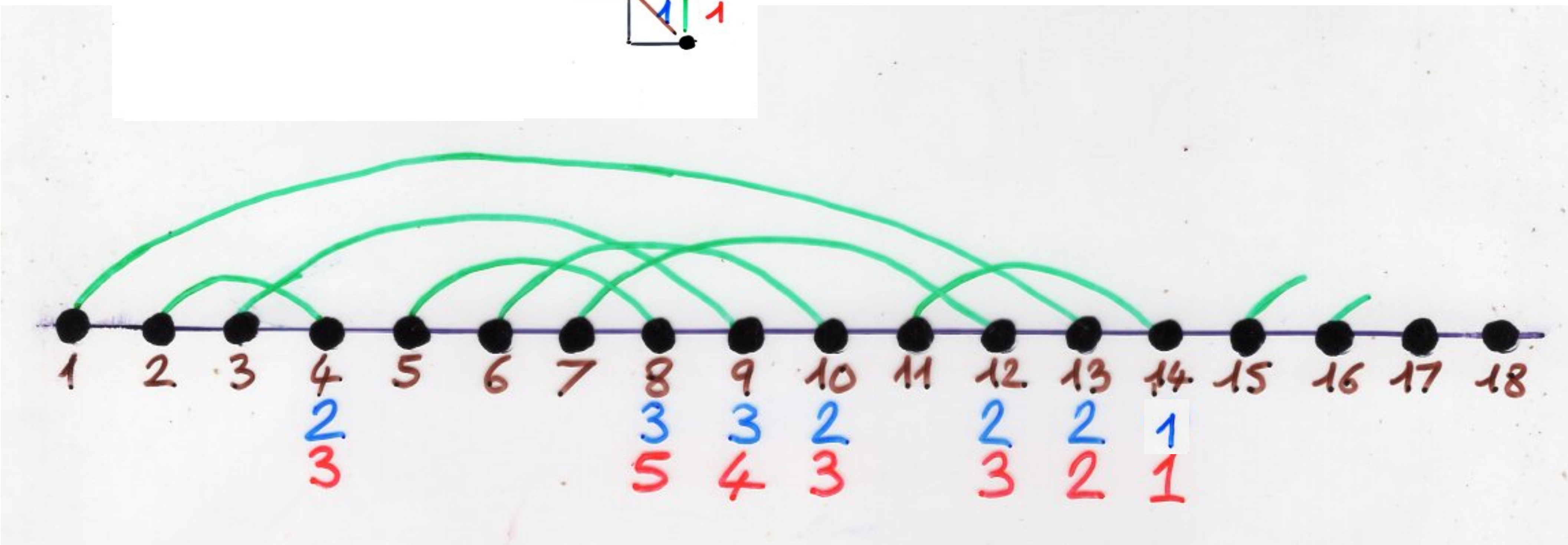
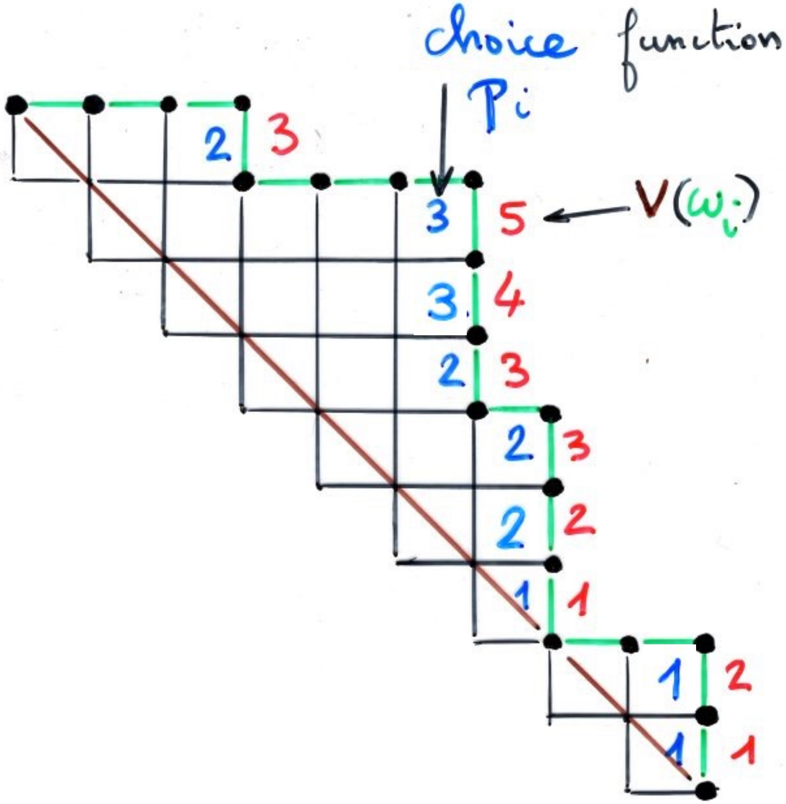
1 2
1 1



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
2 3 3 2 2 2 1
3 5 4 3 3 2 1


choice function
p_i
$V(\omega_i)$
2 3
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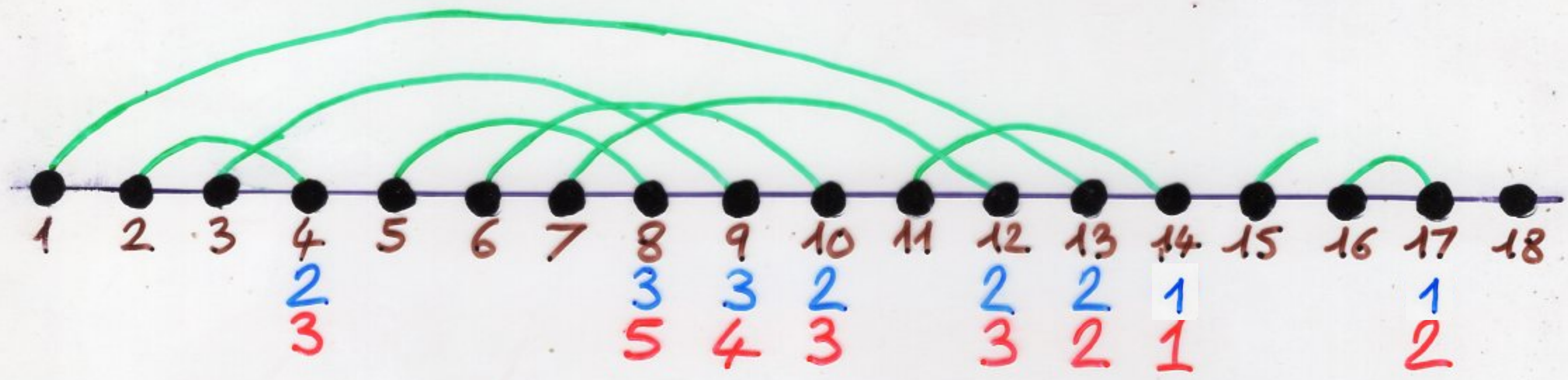
3 4
2 3
2 3
2 2
1 1
1 2
1 1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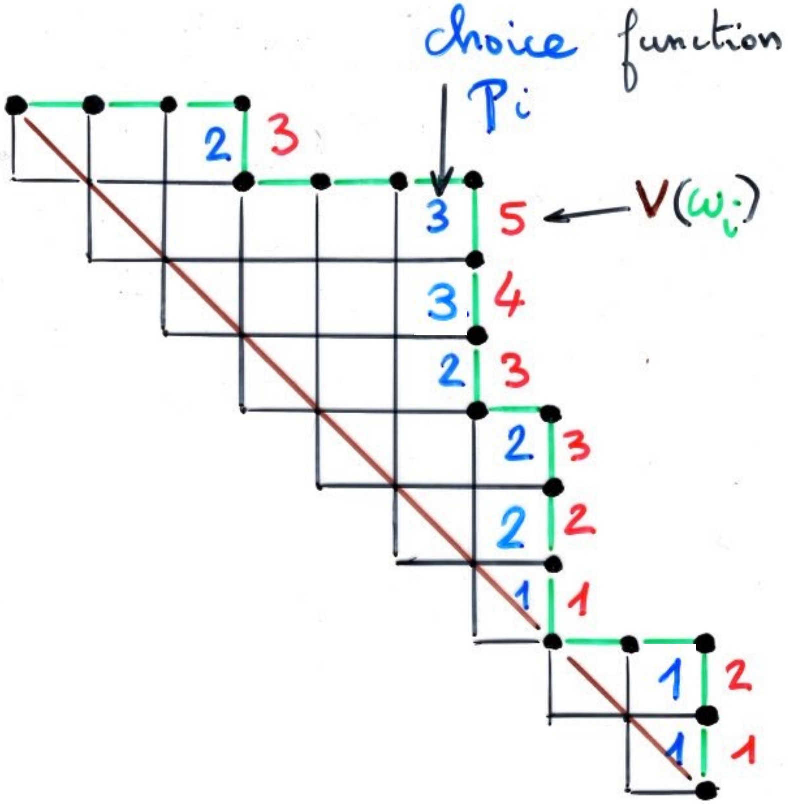
3 5 4 3 3 2 1 2


choice function
p_i
$V(\omega_i)$
2 3
3 5
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2 3
2 3
2 2
1 1
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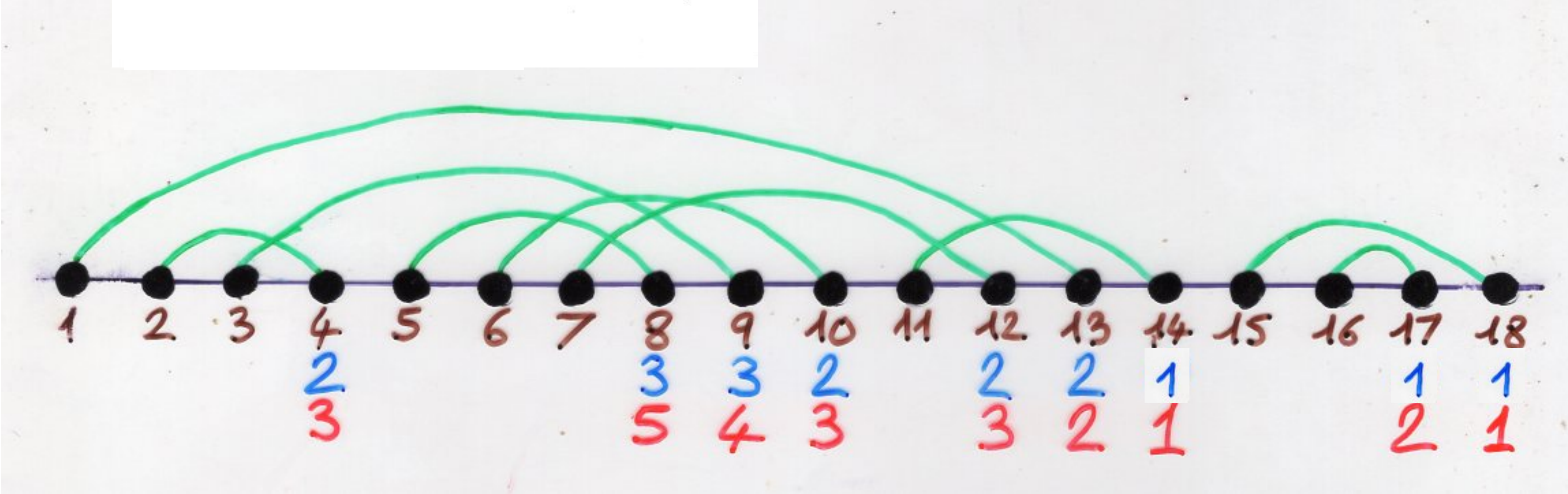
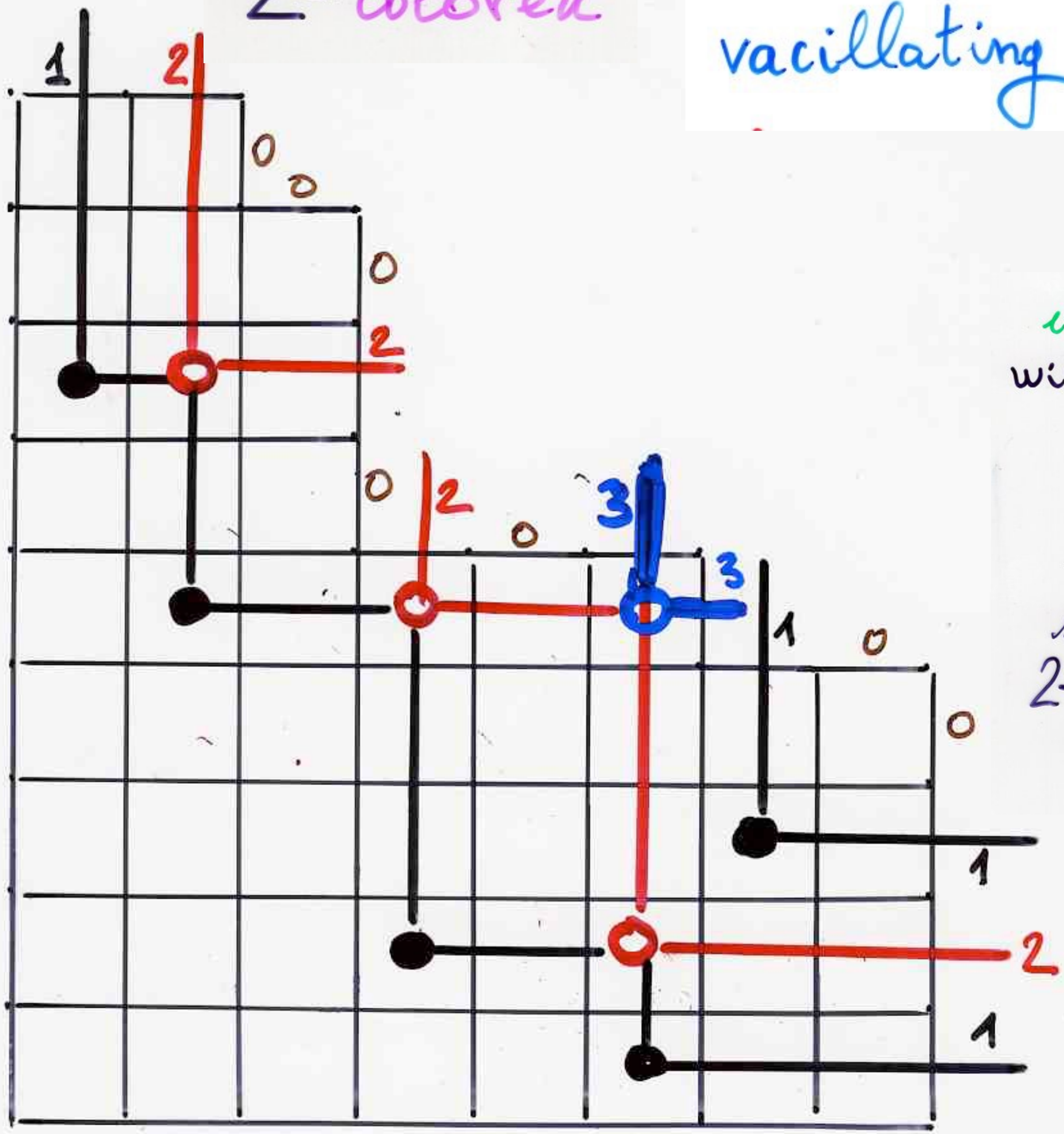

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
2 3 3 2 2 2 1 1 1
3 5 4 3 3 2 1 2 1

oscillating tableaux

vacillating tableaux

2- colored



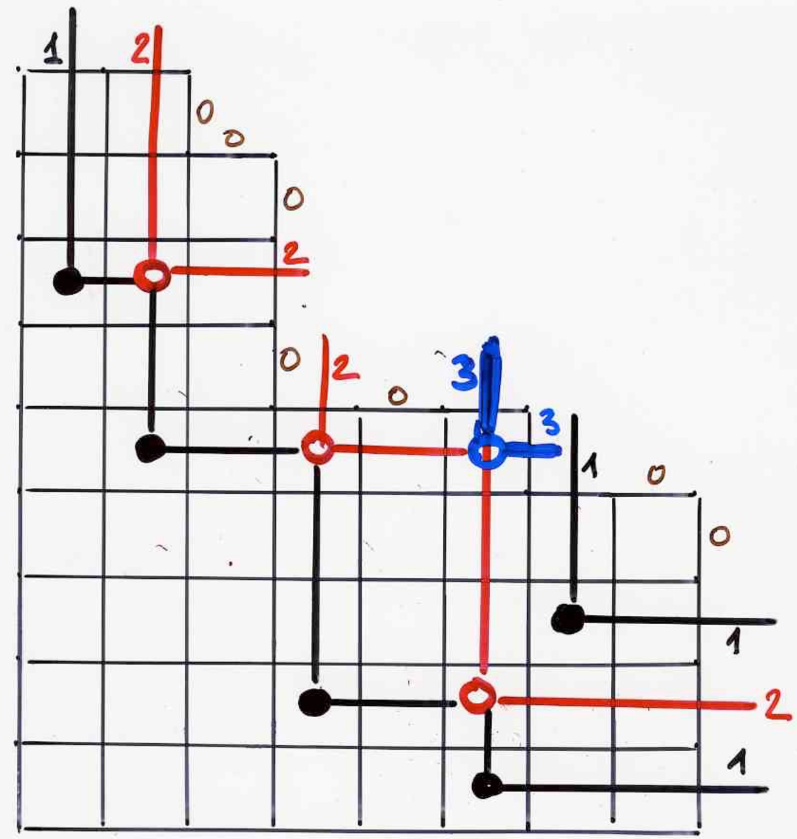
involutions on $2n$
with 2-colored fixed points

↕

Rook placements

↕

sequence of $2n$
2-colored vacillating tableaux
starting and ending at $\emptyset$



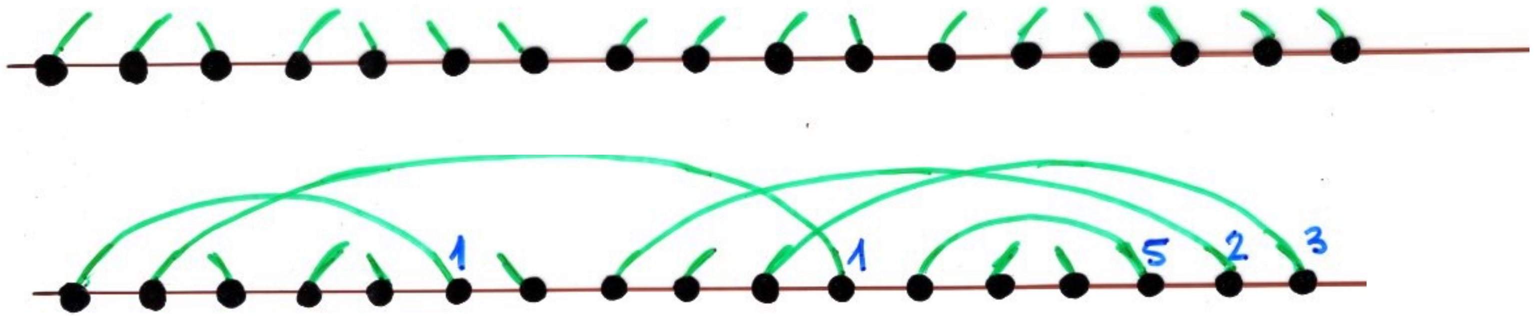
involutions on $2n$
with 2-colored fixed points

↕

Rook placements

↕

sequence of $2n$
2-colored vacillating tableaux
starting and ending at $\emptyset$



oscillating tableaux

vacillating tableaux

hesitating tableaux

Chen, Deng, Du, Stanley, Yan (2005)

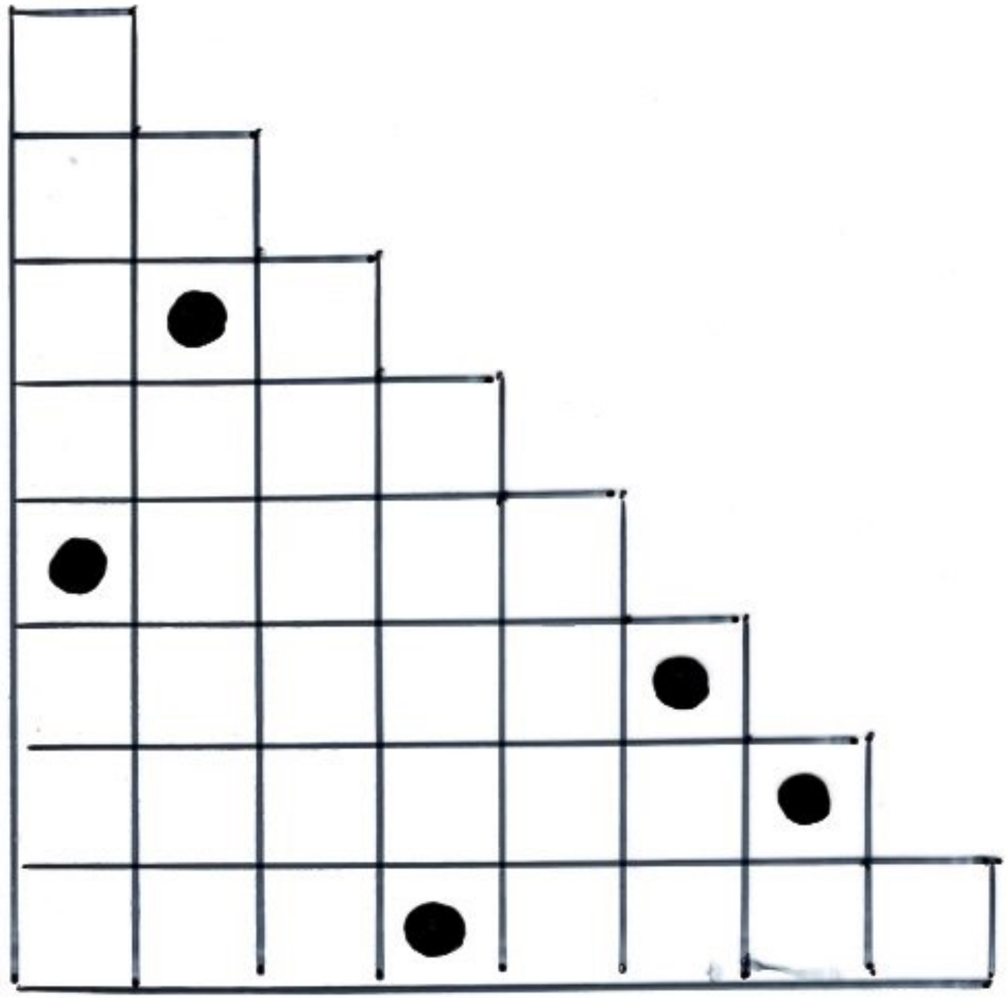
stammering tableaux

Josuat-Vergès

Josuat-Vergès (2012)

Blasiak, Horzela, Penson
Solomon, Duchamp (2007)...

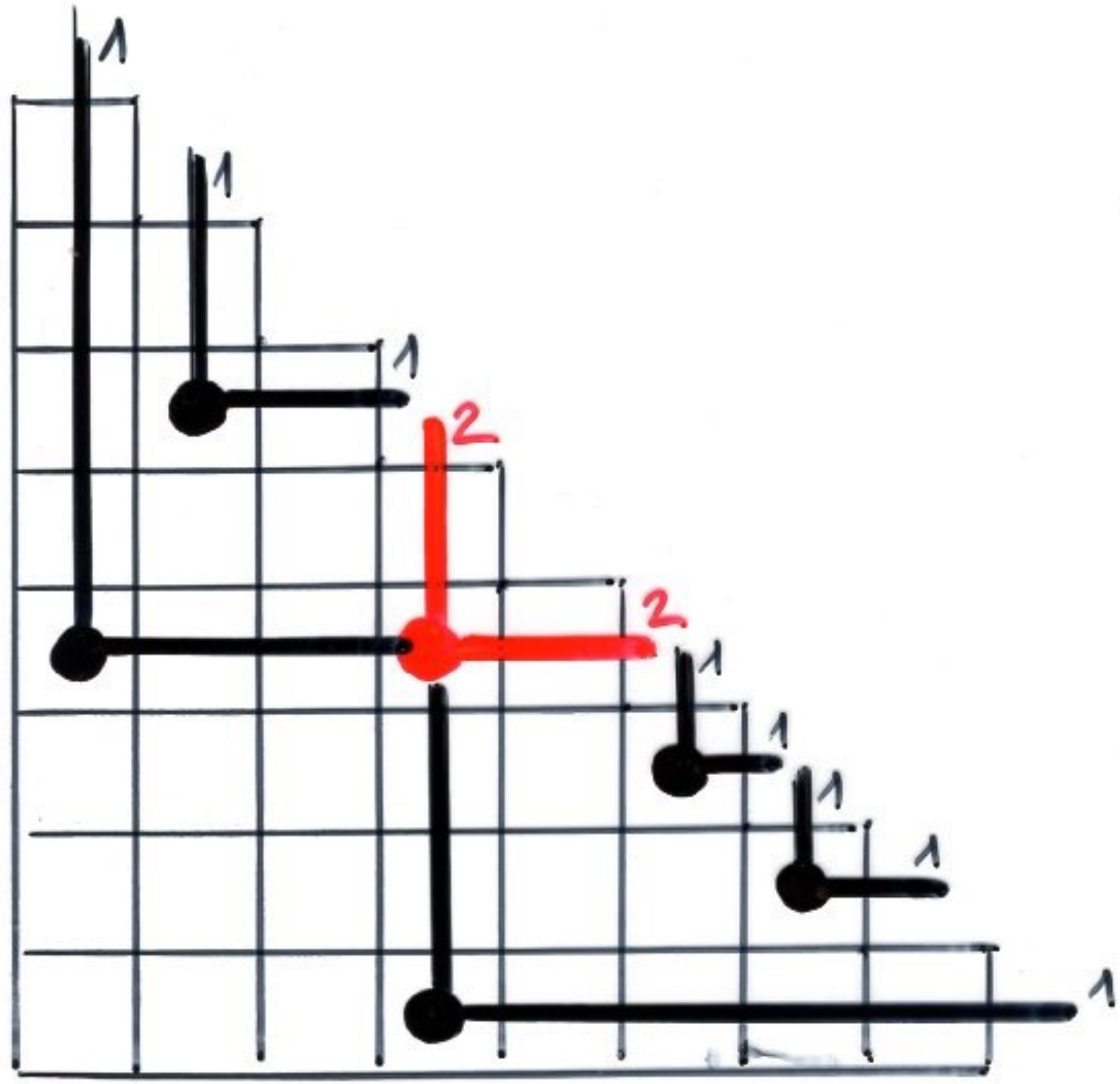
rooks placements
and
set partitions



<u>exercise</u> Find a bijection
between Rook placements in a
staircase shape
and partitions (of sets)

shape with n rows and columns
k rooks
$\updownarrow$
partition on $(n+1)$ elements
with $(n+1-k)$ blocks

Blasiak, Horzela, Penson
Solomon, Duchamp (2007)...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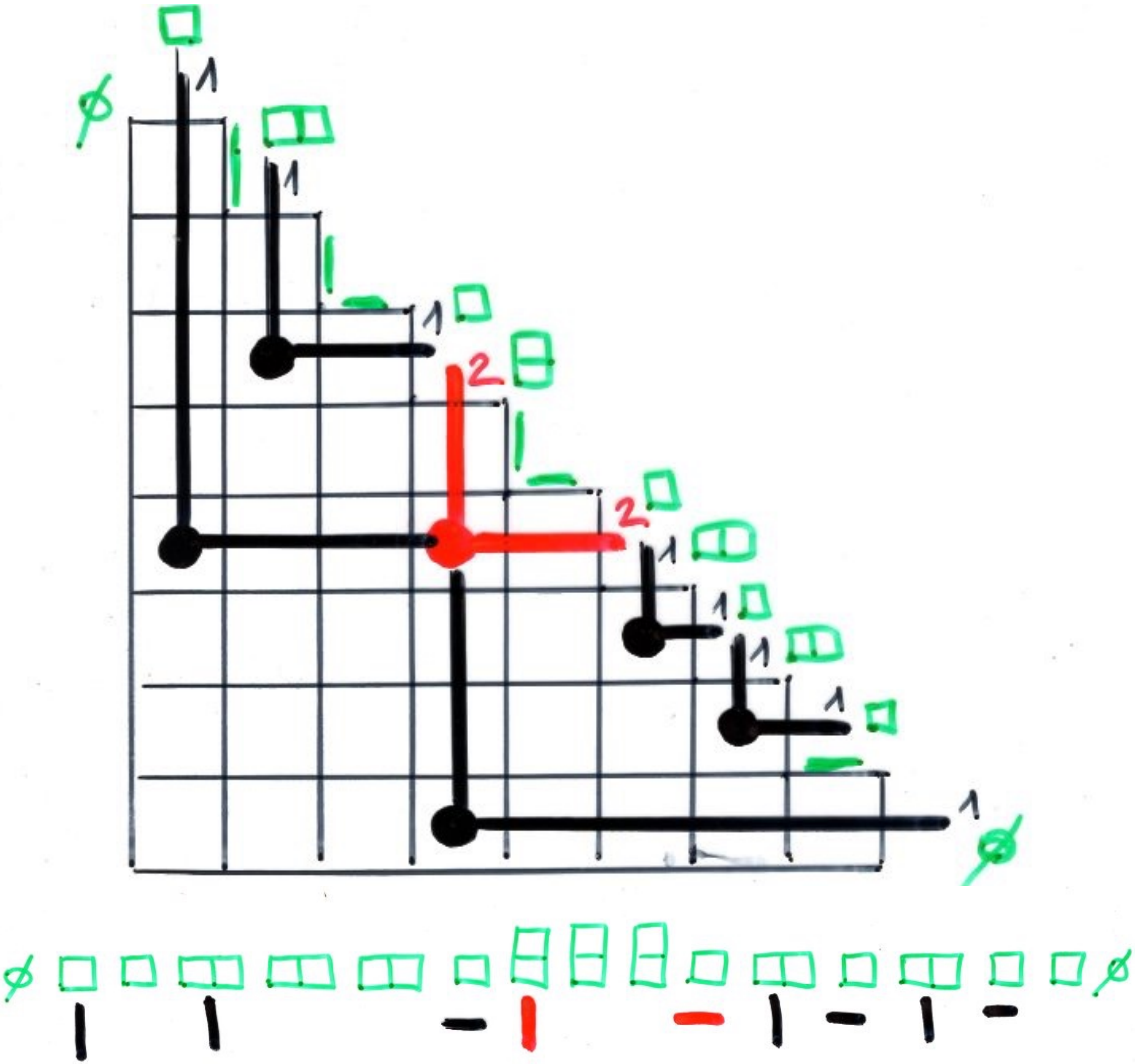
1



1
1
1
2
2
1
1
1
1
1

exercise Read (part of) the paper

W.Chen, E.Deng, R. Du, R. Stanley, C. Yan
arXiv:math.CO/0501230. Trans.A.M.S. (2005)

and reprove the fact that
rook placements in a staircase shape
are in bijection with sequences of
vacillating (resp. hesitating) tableaux
[and thus with set partitions]

stammering tableaux

Josuat-Vergès arXiv:1601.02212 [math.CO]

Blasiak, Horzela, Penson
Solomon, Duchamp (2007)...

Wick's theorem

In

quantum mechanics

quantum
mechanics

a annihilation D

$a^{\dagger}$ creation U

$[a, a^{\dagger}] = 1$

←

n number of particles

$\mathcal{H}$ Hilbert space

$\langle m|n\rangle = \delta_{m,n}$

$\{n\}$ basis of $\mathcal{H}$

Fock space

bosons

$a|n\rangle = \sqrt{n}\,|(n-1)\rangle$

$a^{\dagger}|n\rangle = \sqrt{n+1}\,|(n+1)\rangle$

$N|n\rangle = n|n\rangle$

$N = a^{\dagger}a$

←

double dot operation $:w:$

<u>Wick's theorem</u> w word $\in \{U, D\}^*$

the polynomial

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

is obtained by applying the double dot operation to the sum of all possible expressions obtained by removing pairs ..U....D.. in the word w

$U D U U D U$

$D^2 U^4 + 4 D U^3 + 2 U^2$

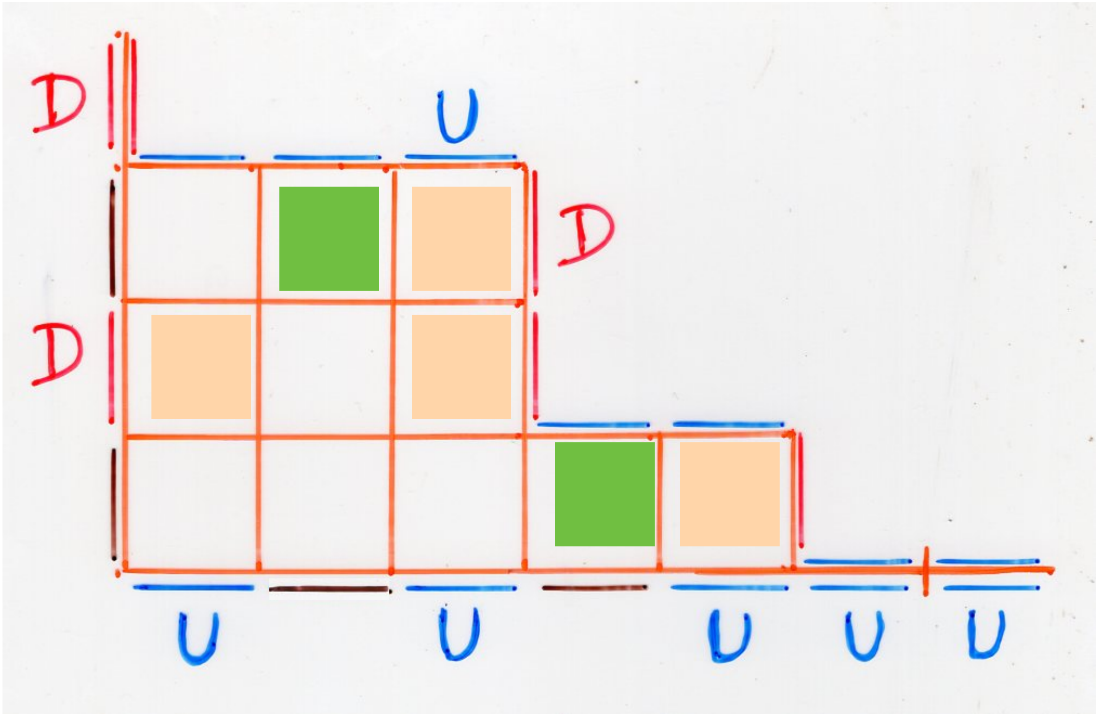
Blasiak, Horzela, Penson, Solomon, Duchamp (2007)...

$w = D\,U^3 D^2 U^2 D\,U^2$

$w \longrightarrow F = F(w)$

F Ferrers diagram

Rooks placement



$\mathbb{K}[x]$

polynomials

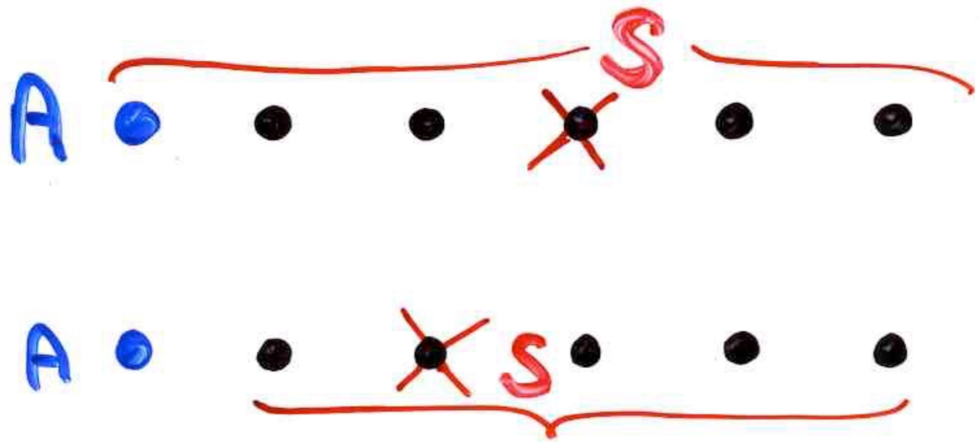
A $\longrightarrow$ product by x $\quad x \cdot P$

S $\longrightarrow$ $\dfrac{d}{dx}(P)$

$$x \cdot x^k = x^{k+1}$$

$$\frac{d}{dx}(x^k) = k\, x^{k-1}$$

Polya urn



bosons

fermions

$q = -1$

$UD + DU = I$

bosons $[a_i, a_j^{+}] = \delta_{ij}$

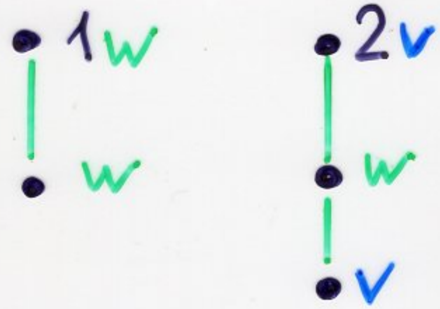
fermions $\{f_i, f_j^{+}\} = \delta_{ij}$

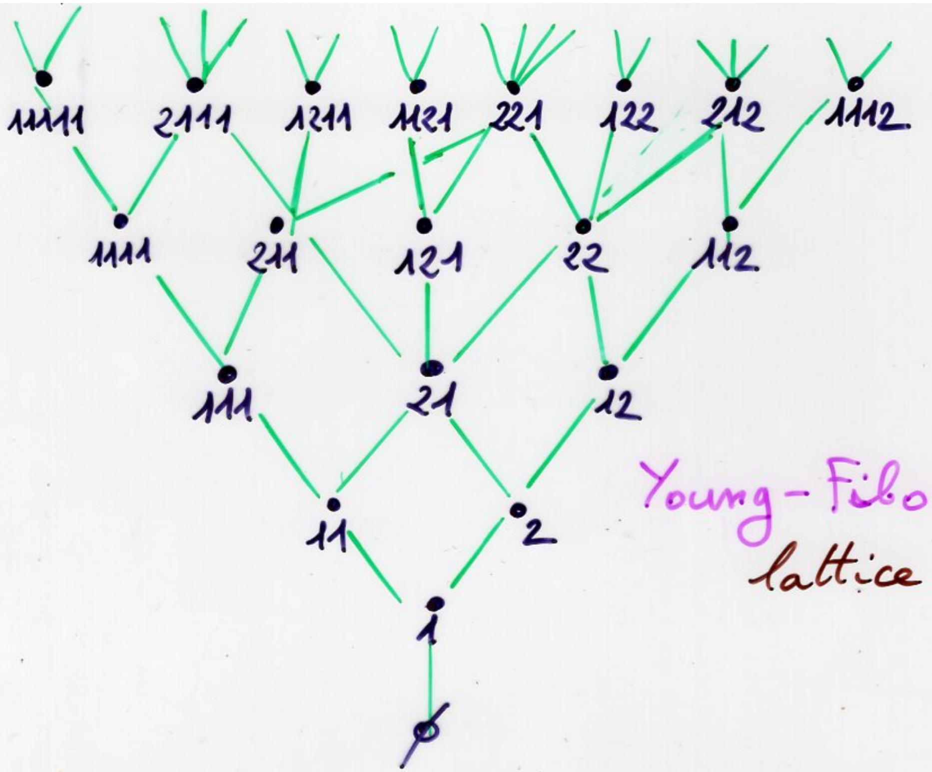
$[a_i, f_j^{+}] = 0$

Differential posets

Definition Young-Fibonacci poset YF

- word $w \in \{1,2\}^*$ are the vertices
- w' covers w iff $\begin{cases} w' = 1w & \text{(concatenation)} \\ \text{or} \\ w' = 2v & \text{where } w \text{ covers } v \end{cases}$





Young-Fibonacci lattice

exercise

- YF is a graded poset
  rank function $r(x)$ = sum of its "digits"

- number of elements $r(x) = n$
  is the Fibonacci number F_n

- find a bijection
  involutions on $[n]$ $\longleftrightarrow$ maximal chains $\emptyset \to x$, $r(x)=n$

  permutations $\longleftrightarrow$ pairs of maximal chains $\emptyset \to x$, $r(x)=n$

Roby (1991)

- YF is a differential poset

Definition P differential poset

P is a graded poset with a minimum $\hat{0}$

- finite number of elements for each rank

$$r(x) = k$$

- for any $x \in P$, $C^+(x)$ and $C^-(x)$ are finite

with $C^+(x) = \{y \in P,\ y \text{ covers } x\}$

$C^-(x) = \{y \in P,\ x \text{ covers } y\}$

(i) If $x \neq y$ in P, there are exactly k elements of P which are covered by both x and y, then they are exactly k elements of P which cover both x and y

(ii) $$|C^+(x)| = |C^-(x)| + 1$$

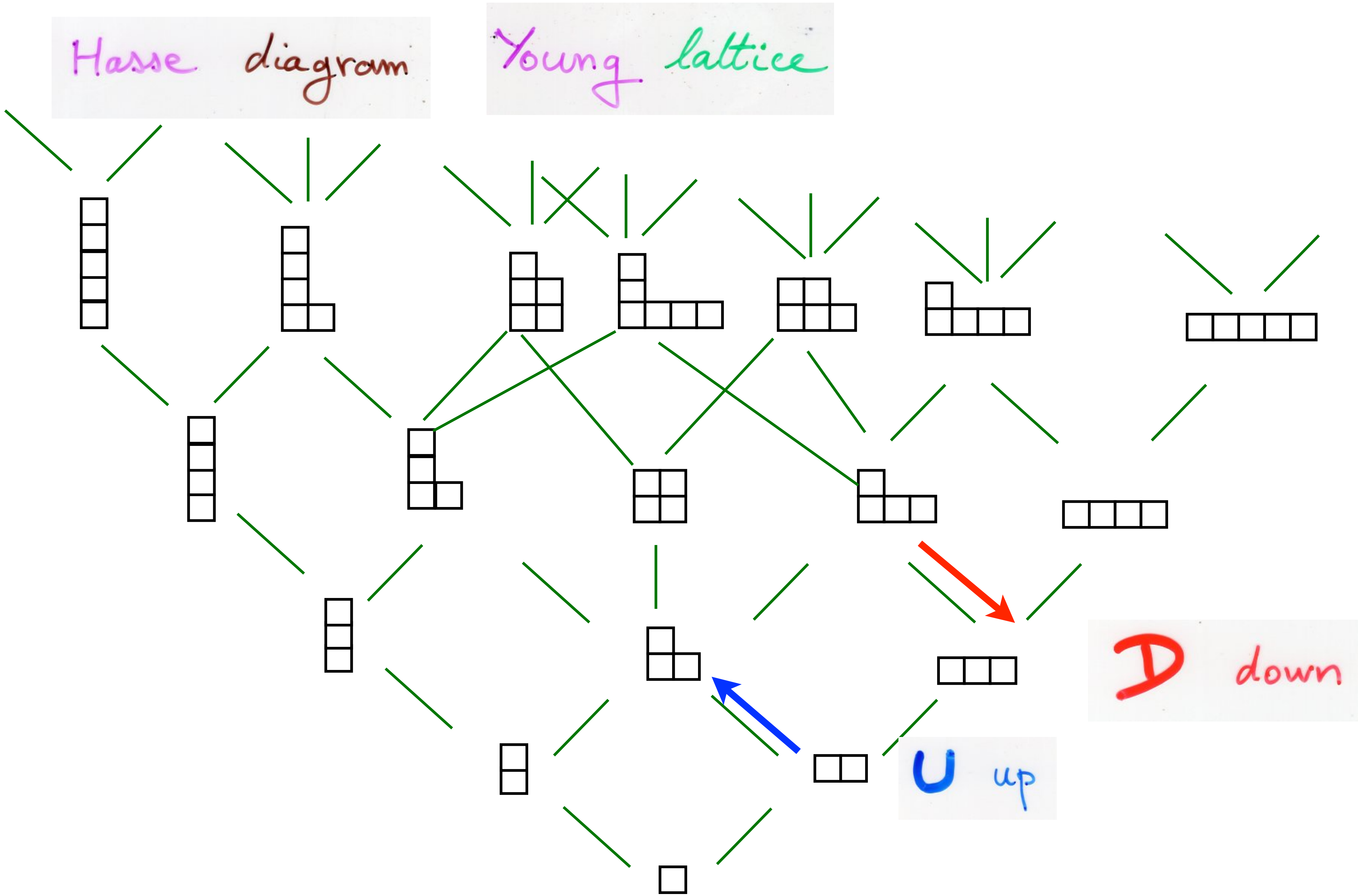
differential poset

Fomin (1992, 1995)

Stanley (1988, 1990)

Roby (1991)

(i) $\Longrightarrow$ $k = 0$ or 1


Hasse diagram
Young lattice
D down
U up

$$U(x) = \sum_{y \in C^{+}(x)} y$$

$$D(x) = \sum_{y \in C^{-}(x)} y$$

<u>Proposition</u>

P differential poset $\iff UD - DU = I$

Proposition P differential poset

- number of maximal chains $\emptyset \to x$, $r(x)=n$
  = number of involutions
  exponential generating function $\exp\left(t+\frac{t^2}{2}\right)$

- $$\sum_{\substack{x \\ r(x)=n}} \left(\text{number of maximal chains } \emptyset \to x\right)^2 = n!$$