Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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CNRS, LaBRI, Bordeaux

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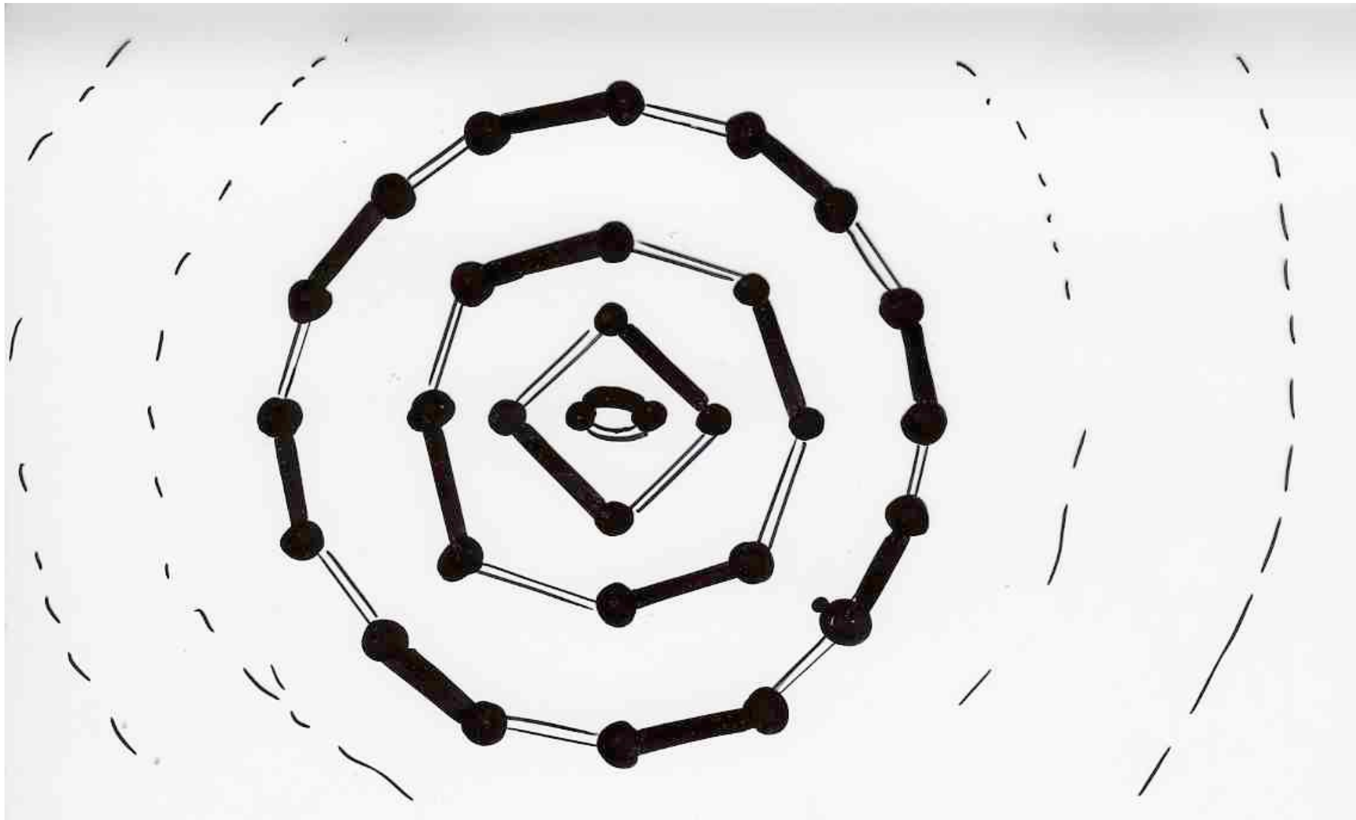
Epilogue
Kepler Towers

IMSc, Chennai
16 March 2017



System of Kepler towers

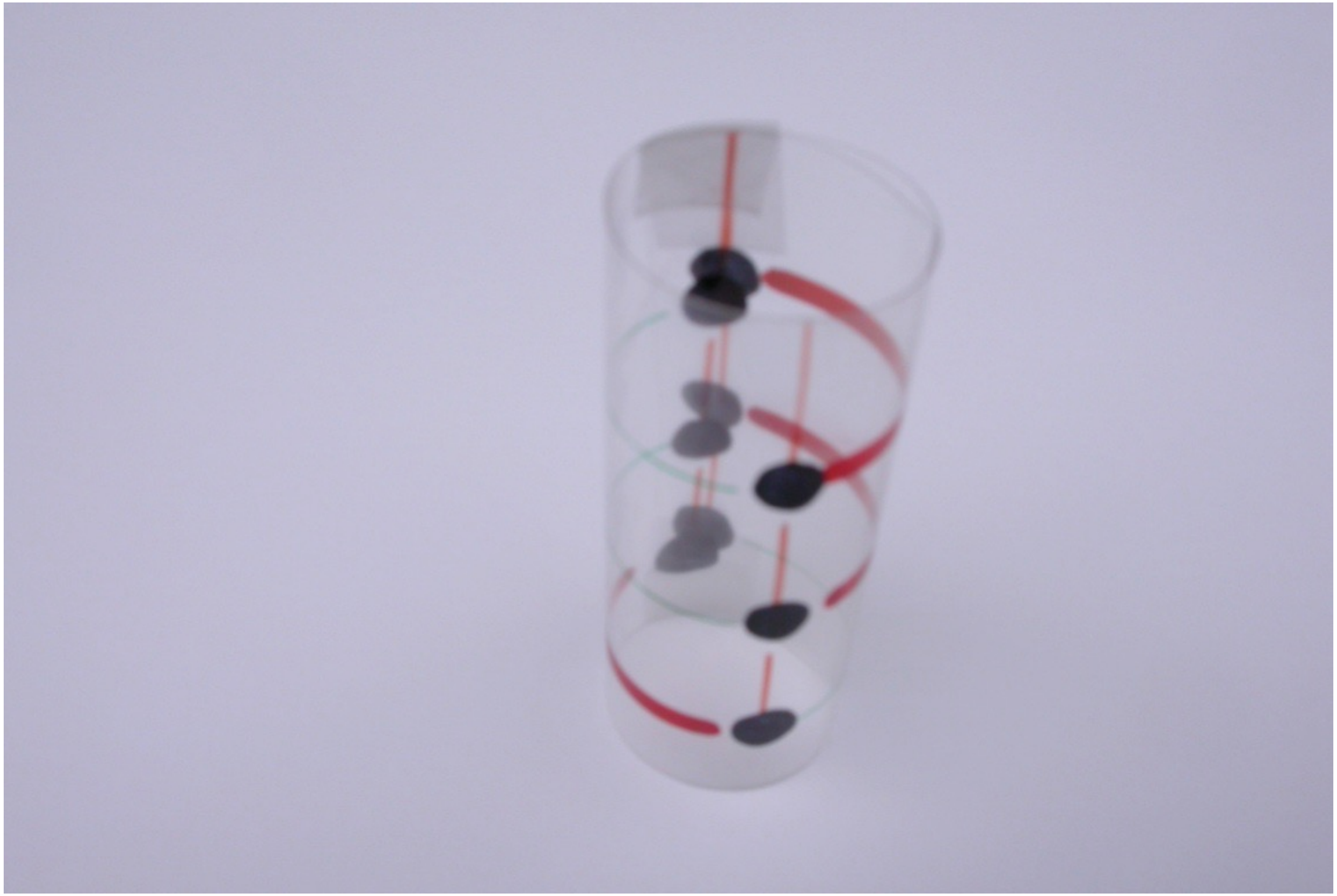
- regular polygons $P_2, P_4, P_8, \ldots$

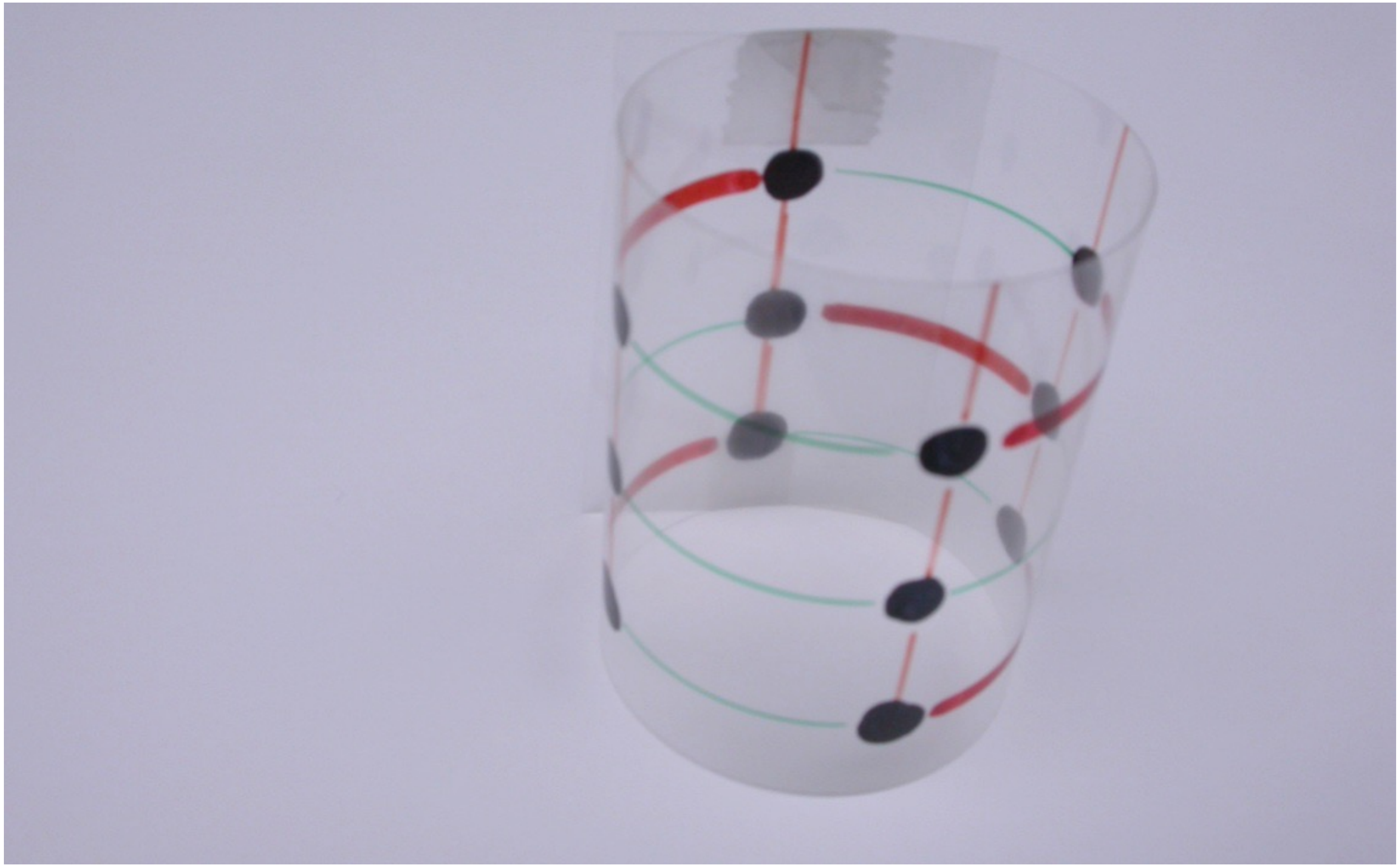
  P_i $\quad 2^i$ edges $\left\{ \right.$

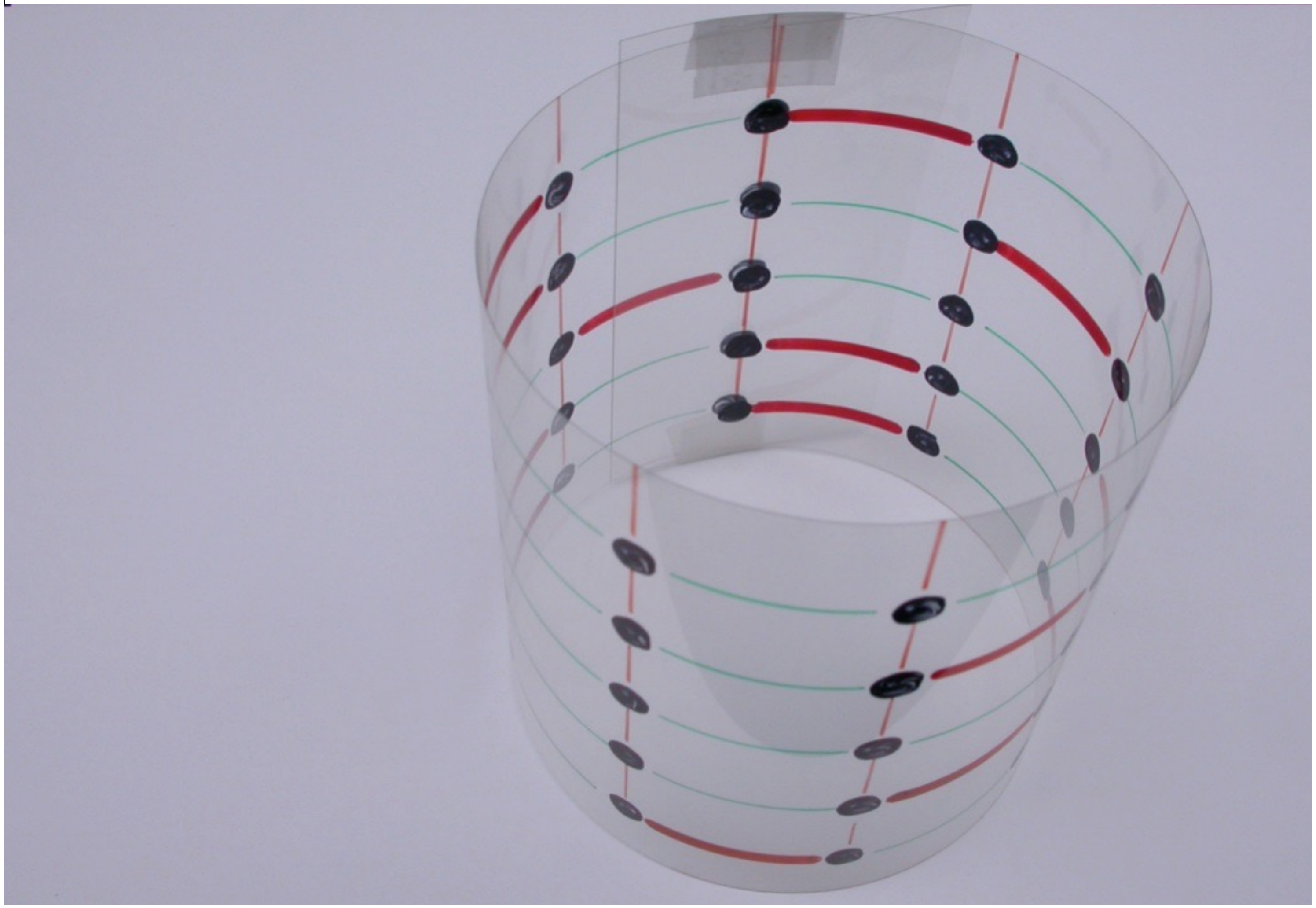
- heaps $H_1, \ldots, H_k$

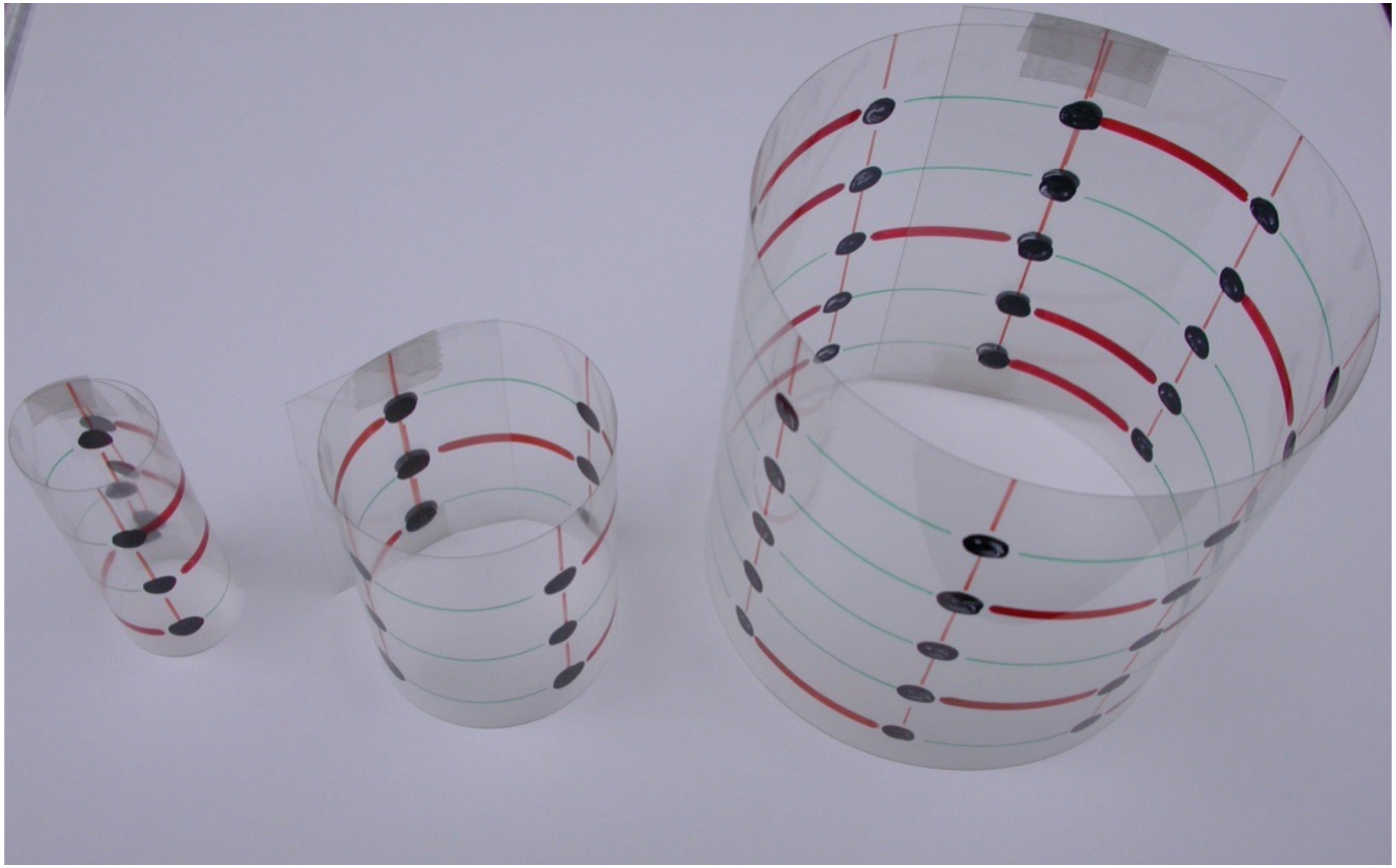
  H_i heap of dimers above P_i (= tower) $1 \le i \le k$

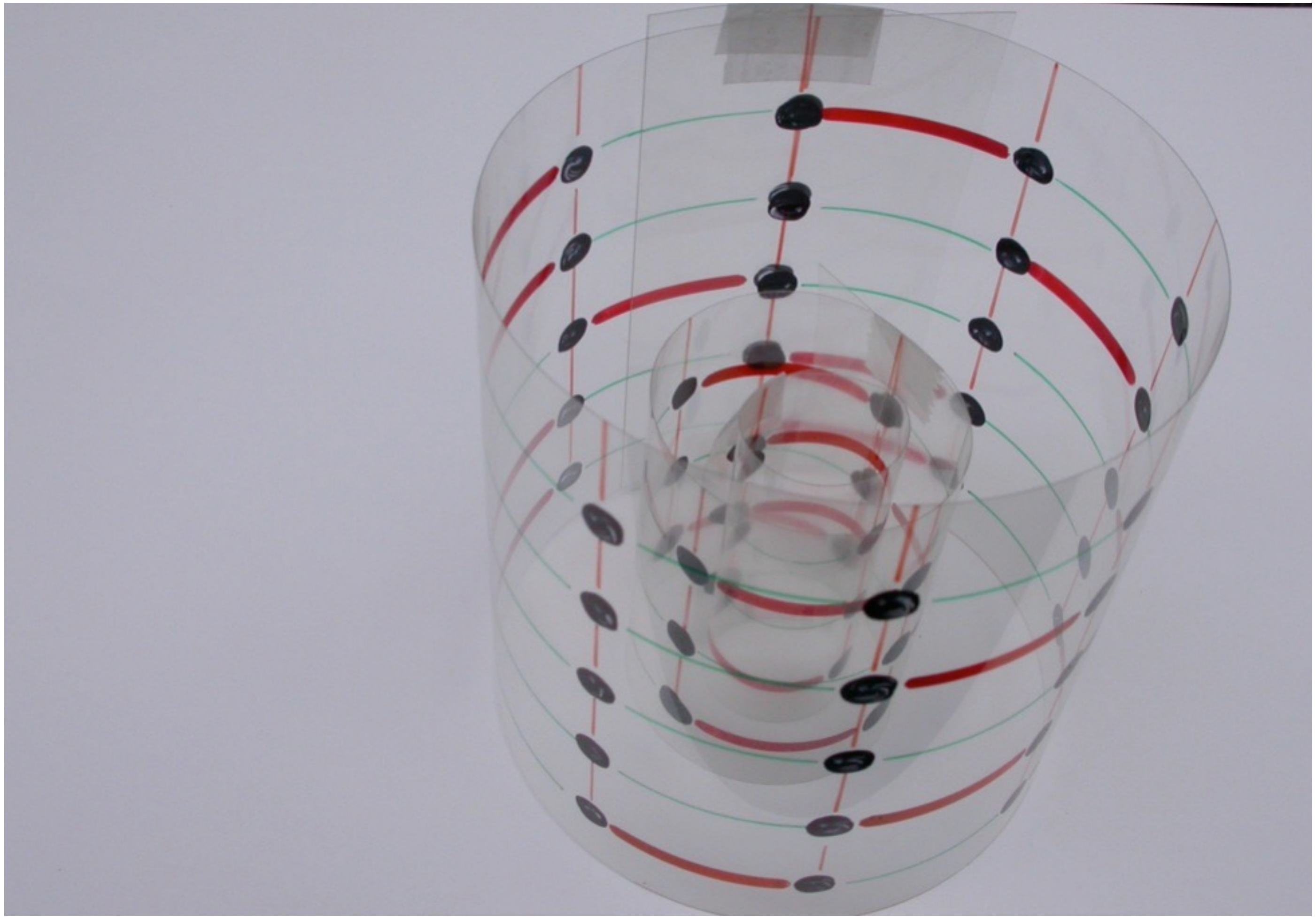
(*) at level 0, H_i contains all 2^{i-1} black edges of P_i

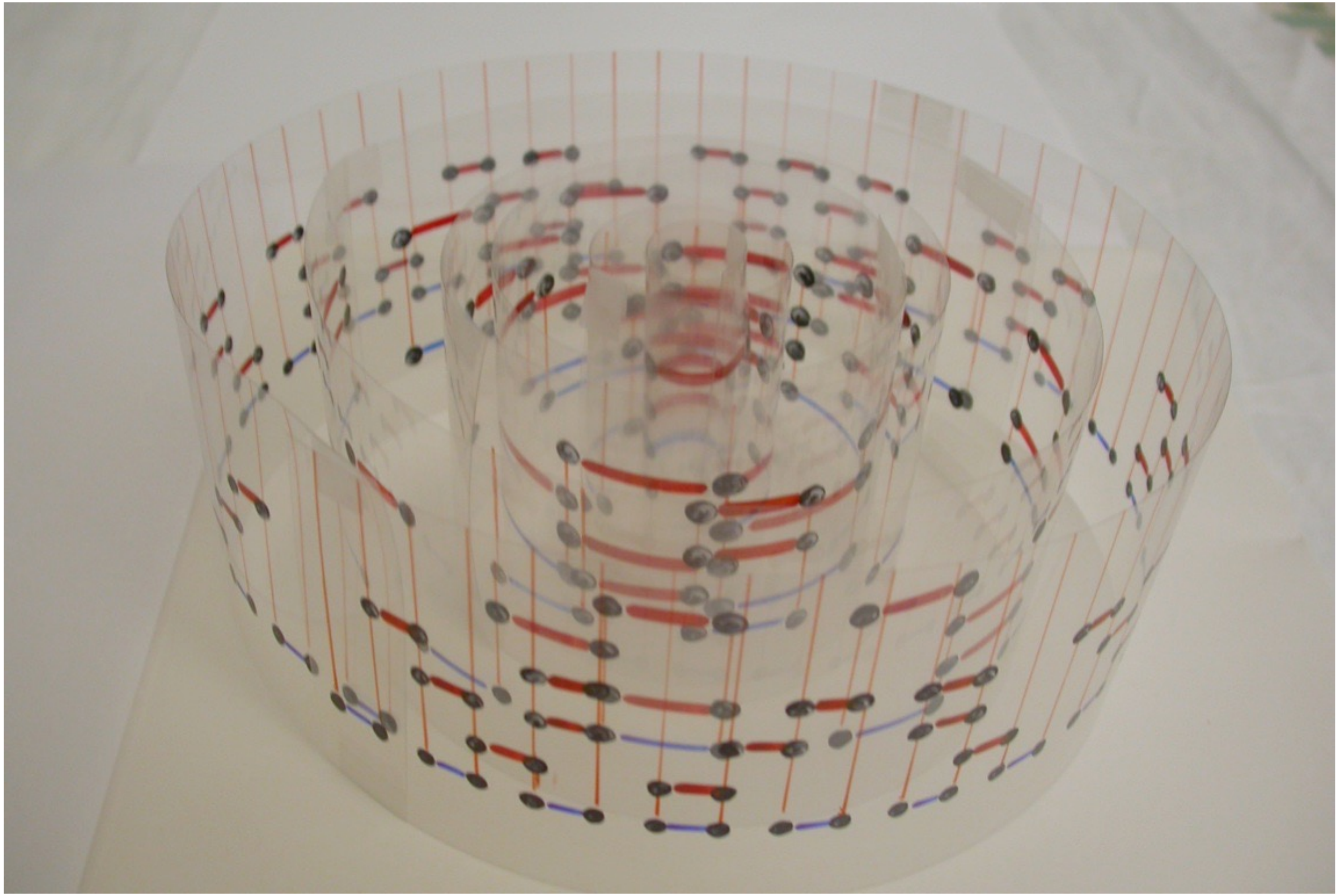






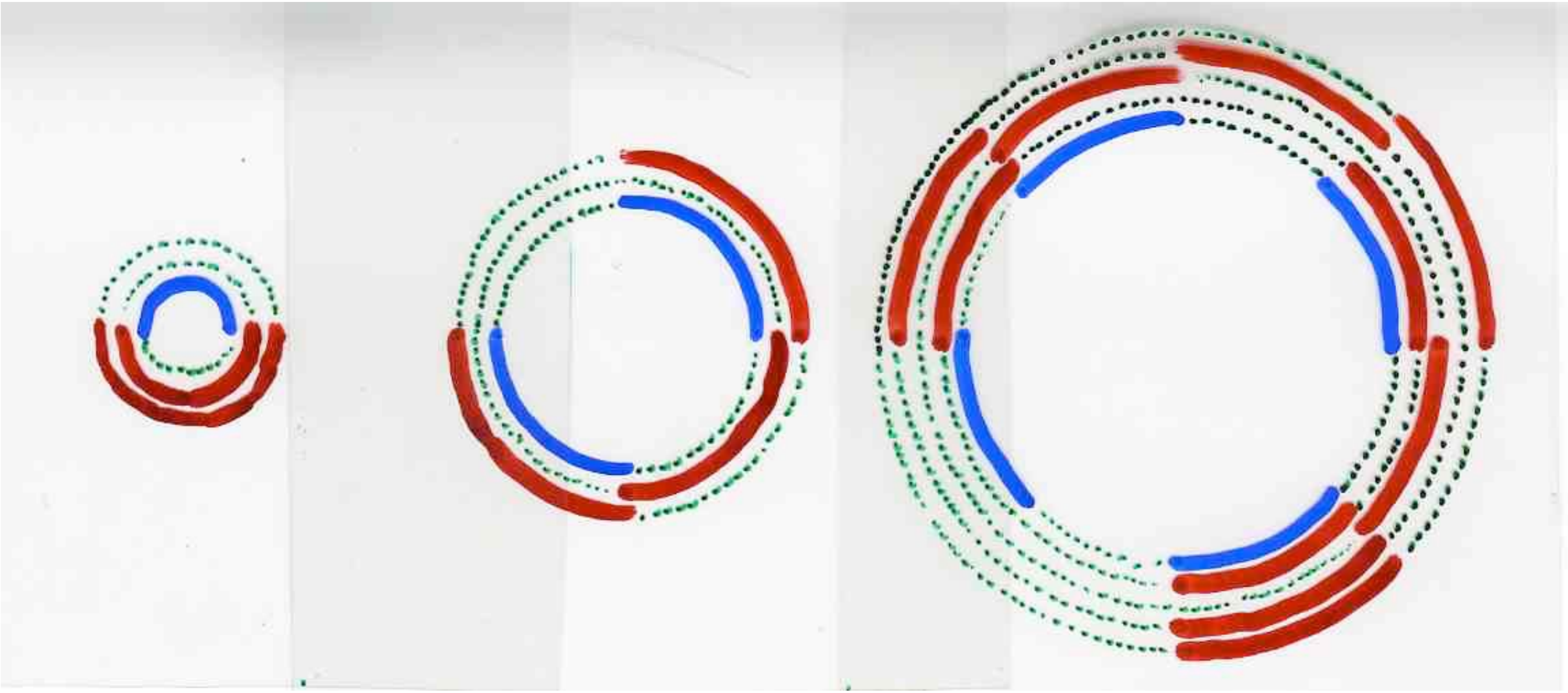


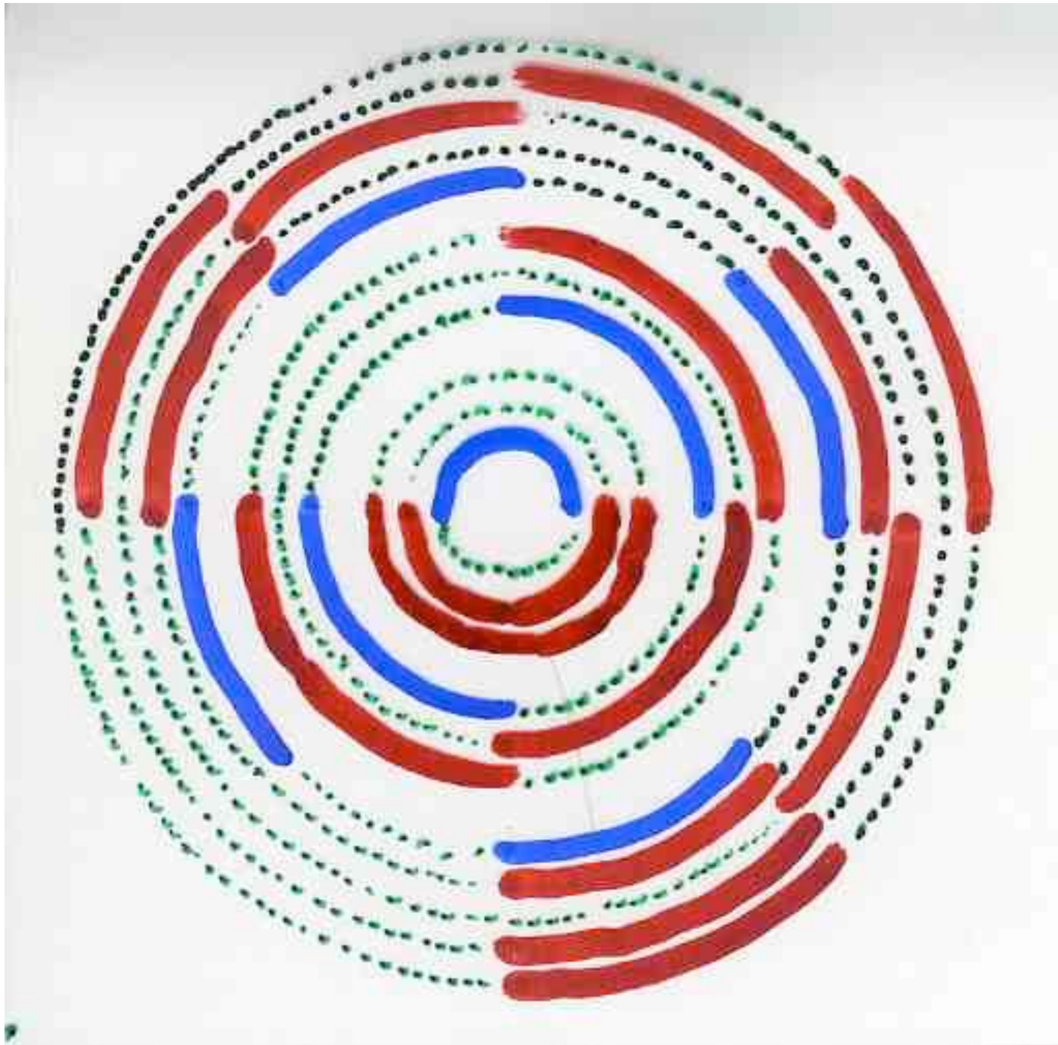


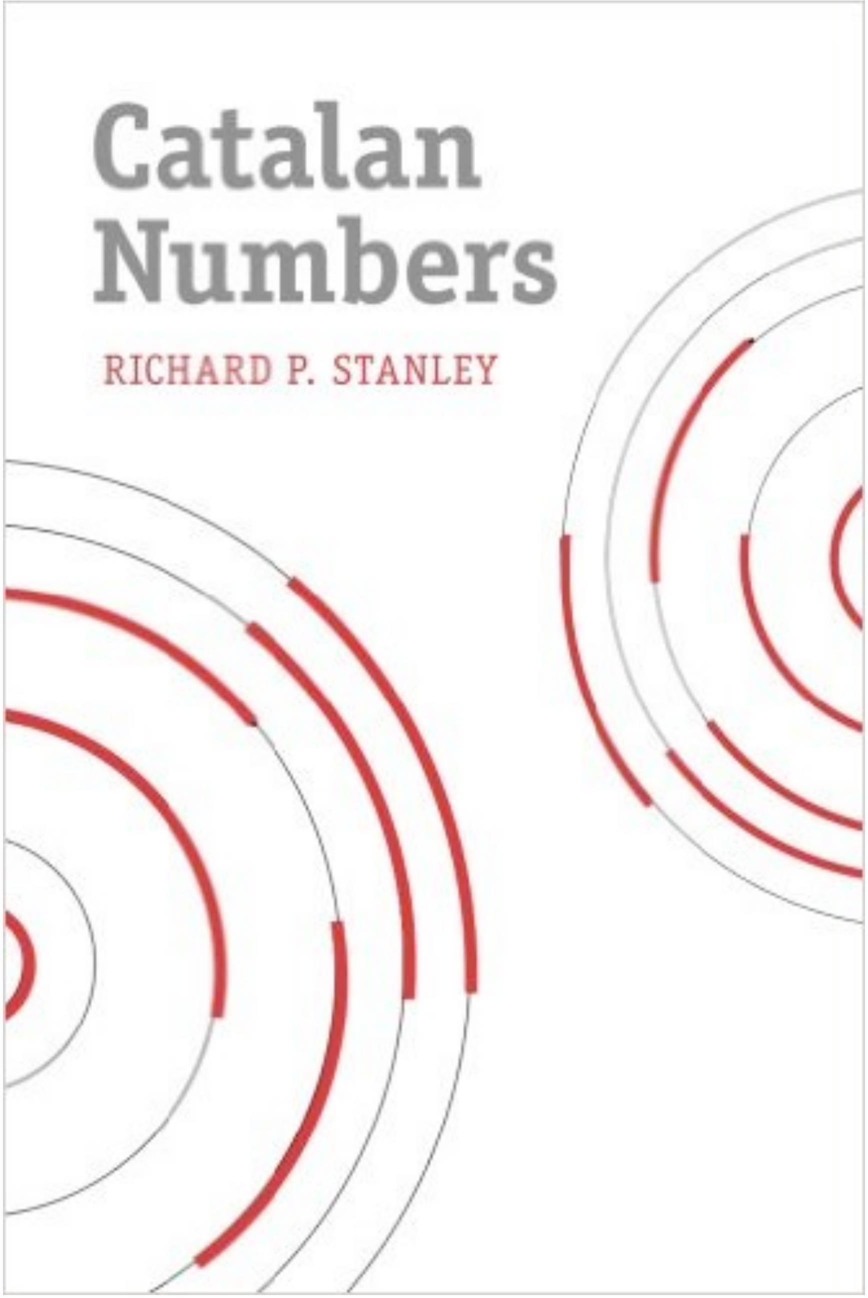


<u>Prop.</u> The number of system of Kepler towers having n dimers is

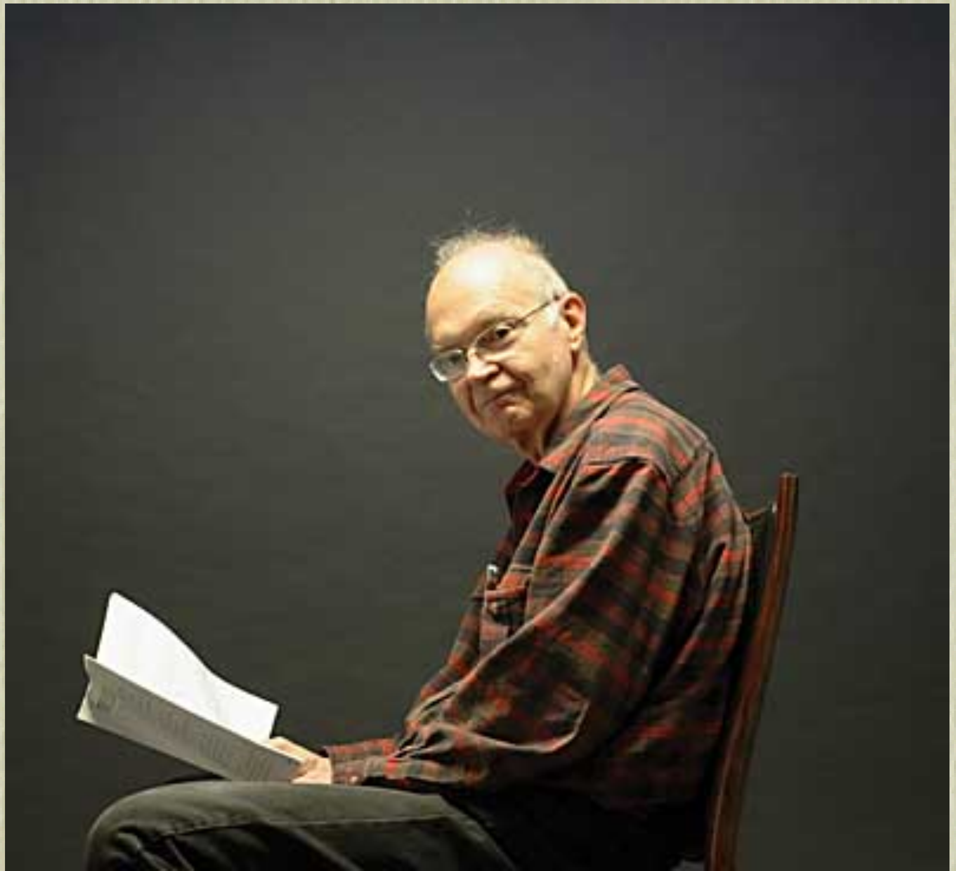
Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$






Catalan Numbers
RICHARD P. STANLEY

Why Kepler Towers ?

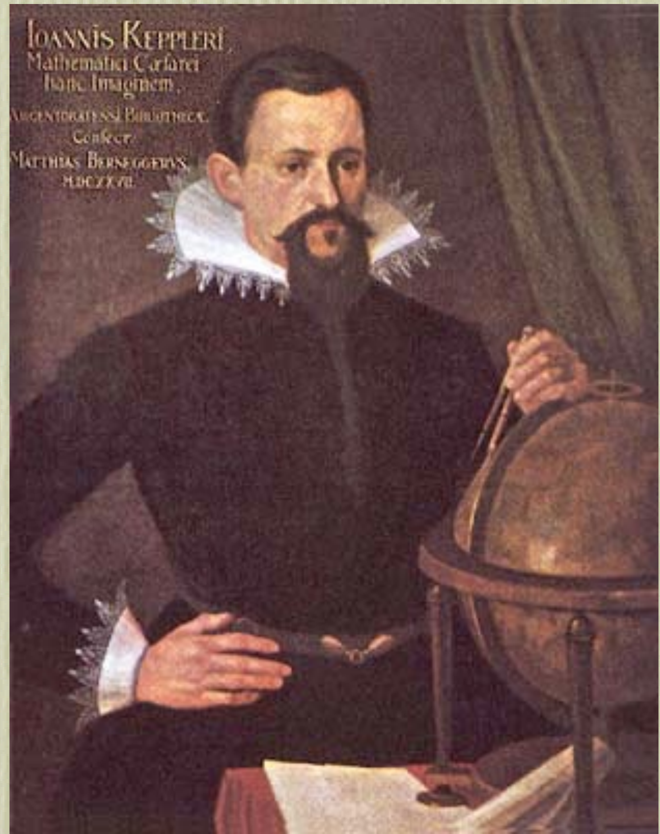


Donald Knuth

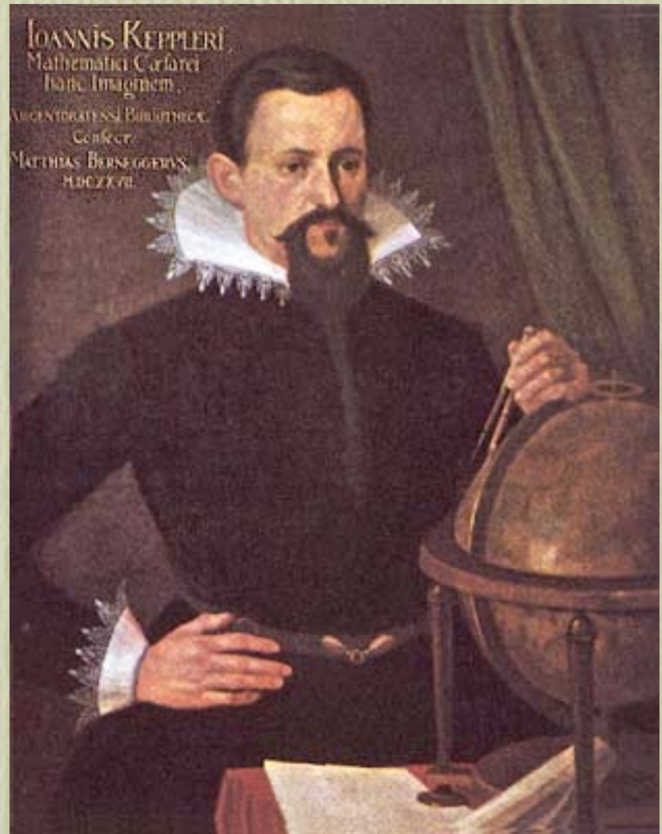
Mittag-Leffler Institute

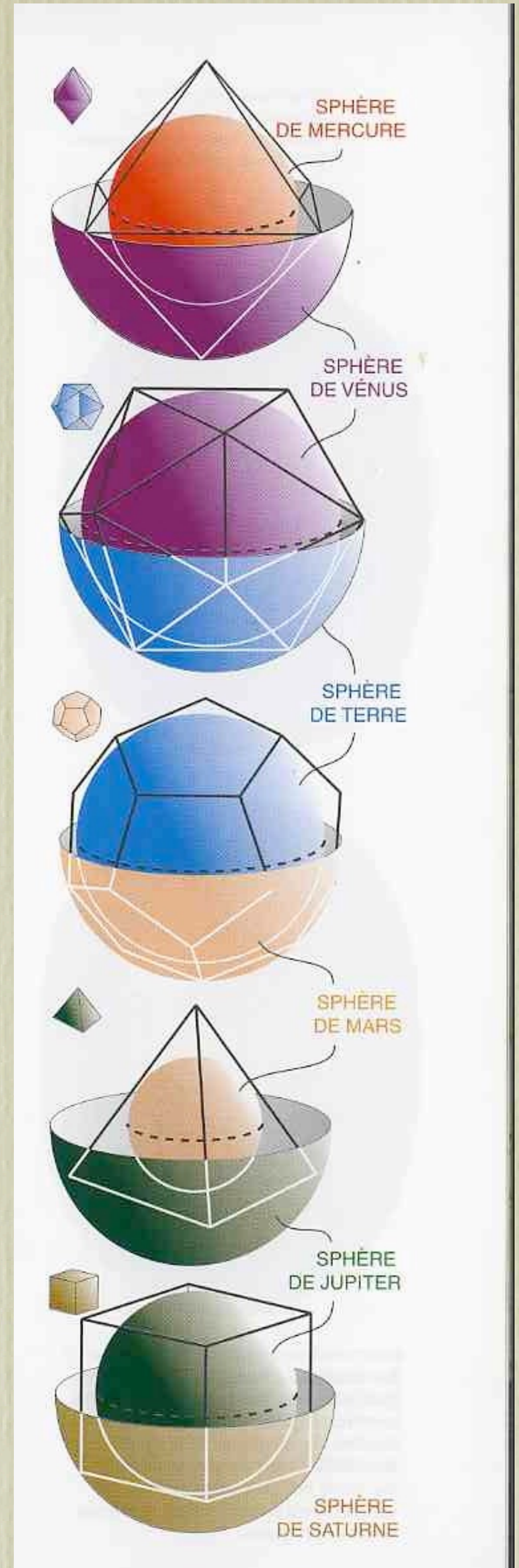

Mittag-Leffler Institute

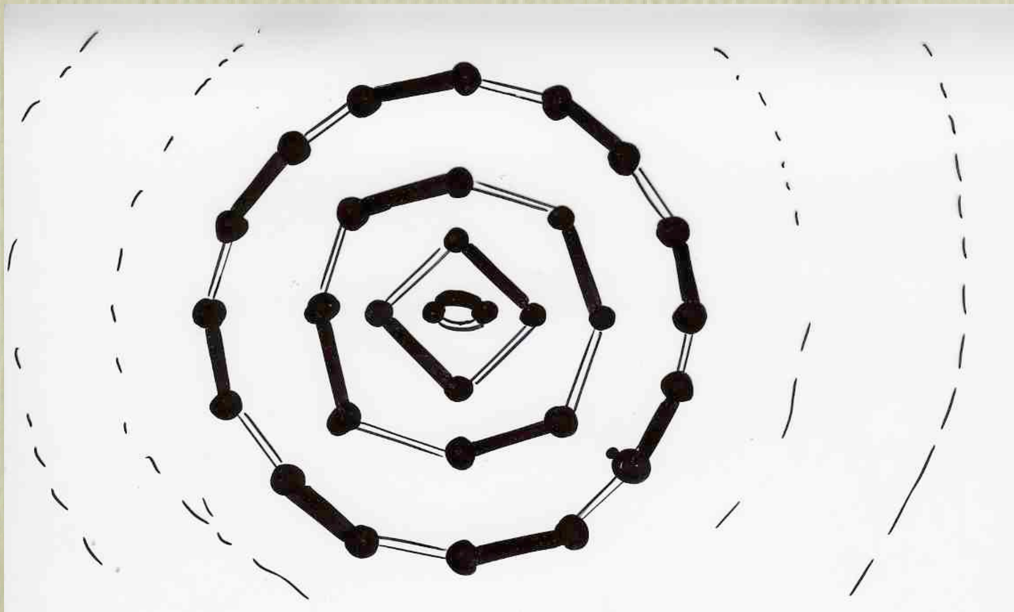

Mittag-Leffler
Institute

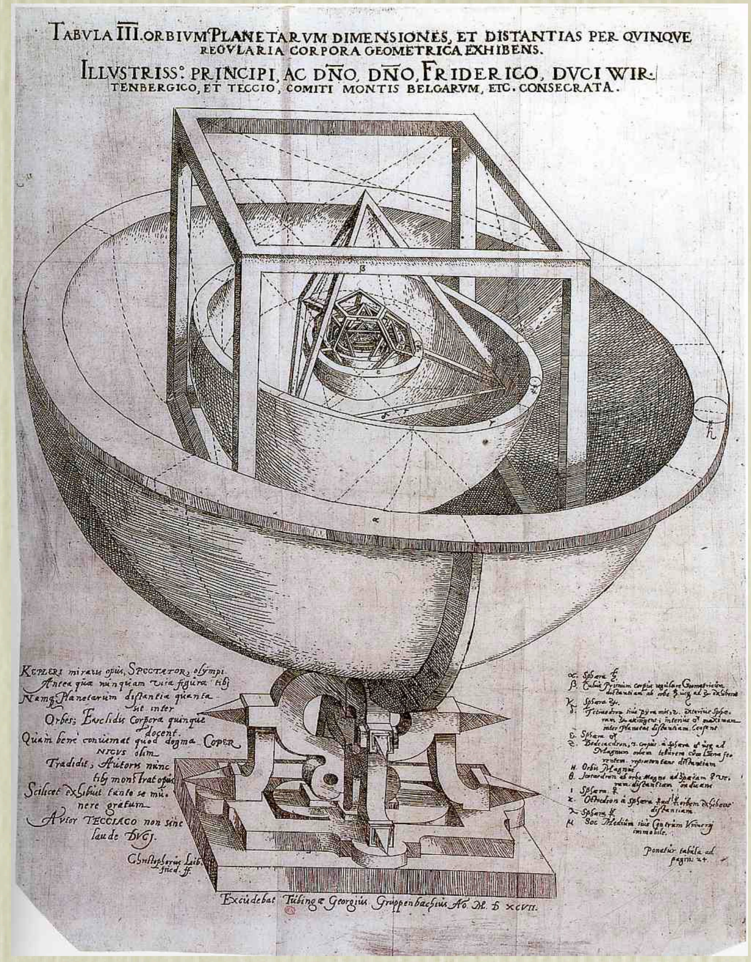


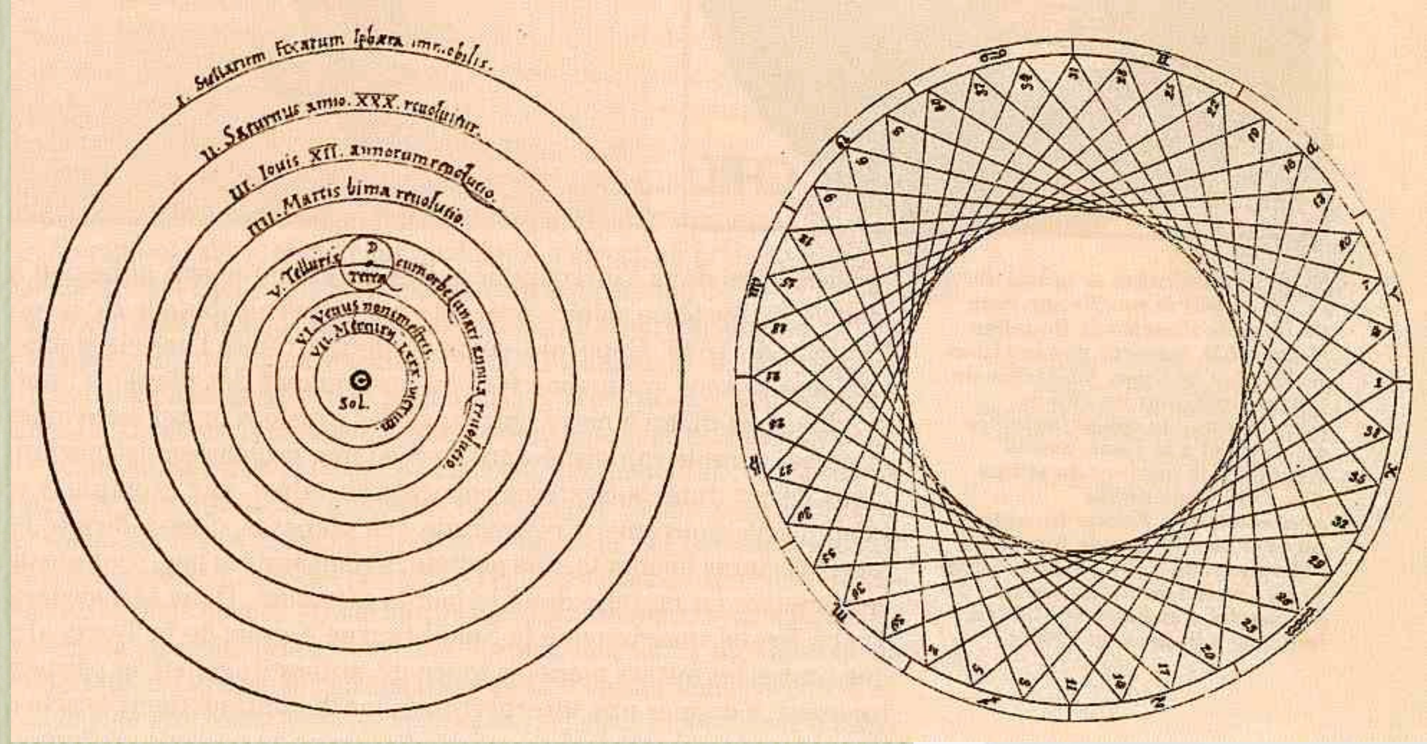
Mysterium cosmographicum (1596)

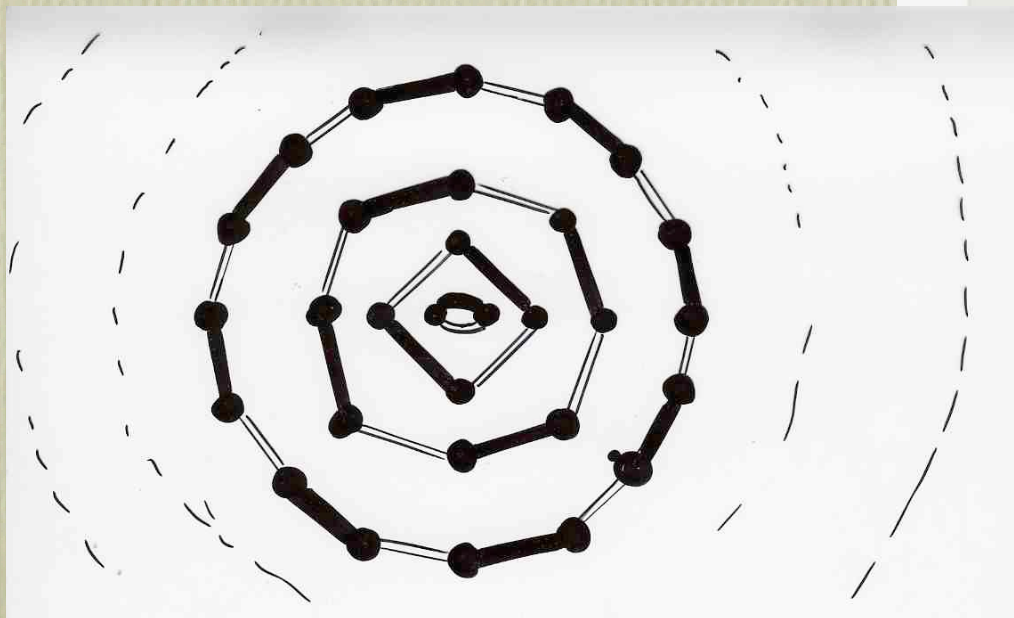

IOANNIS KEPPLERI,
Mathematici Cæsarei
hanc Imaginem,
MATTHIAS BERNEGGERUS

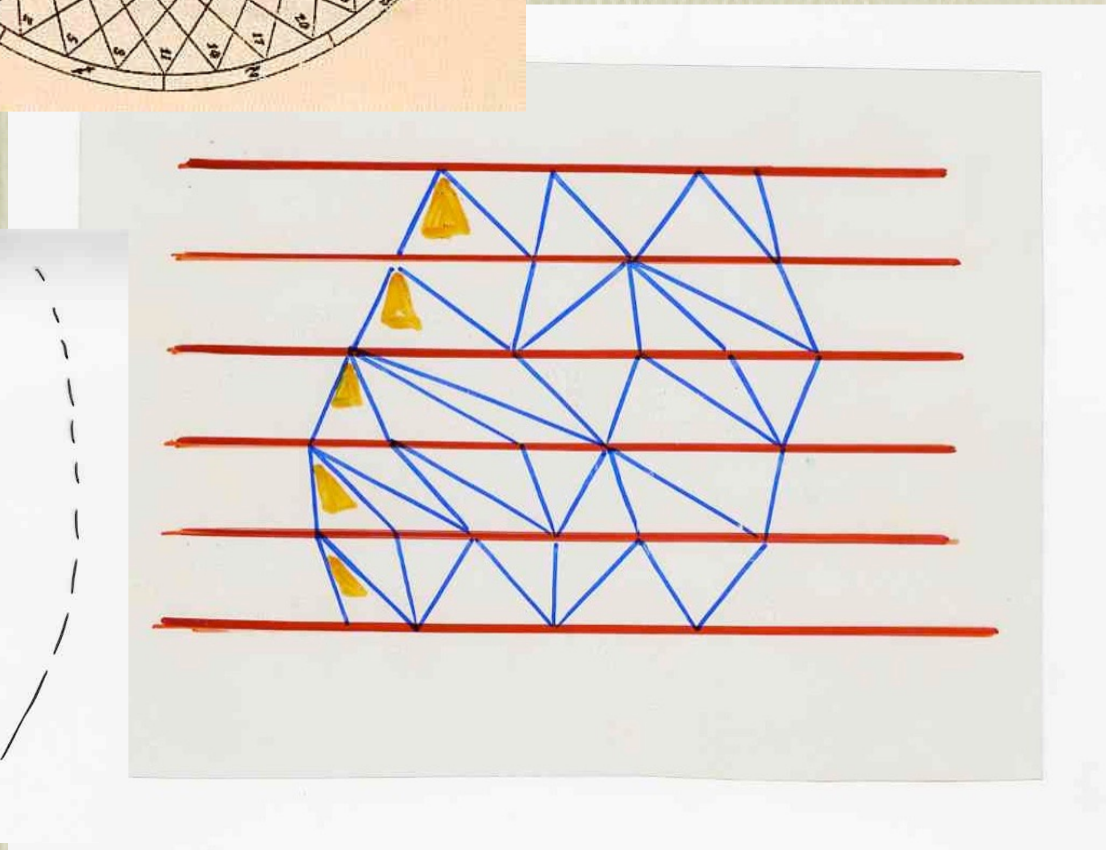

SPHÈRE DE MERCURE
SPHÈRE DE VÉNUS
SPHÈRE DE TERRE
SPHÈRE DE MARS
SPHÈRE DE JUPITER
SPHÈRE DE SATURNE




TABVLA III. ORBIVM PLANETARVM DIMENSIONES, ET DISTANTIAS PER QVINQVE
REGVLARIA CORPORA GEOMETRICA EXHIBENS.
ILLVSTRISS. PRINCIPI, AC DNO, DNO, FRIDERICO, DVCI WIR:
TENBERGICO, ET TECCIO, COMITI MONTIS BELGARVM, ETC. CONSECRATA.


I. Stellarum Fixarum sphaera immobilis.
II. Saturnus anno. XXX. reuolutitur.
III. Iouis XII. annorum reuolutio.
IIII. Martis bima reuolutio.
V. Telluris cum orbe lunari annua reuolutio
terra
VI. Venus nono mense.
VII. Mercurius. LXXX. dierum.
Sol.

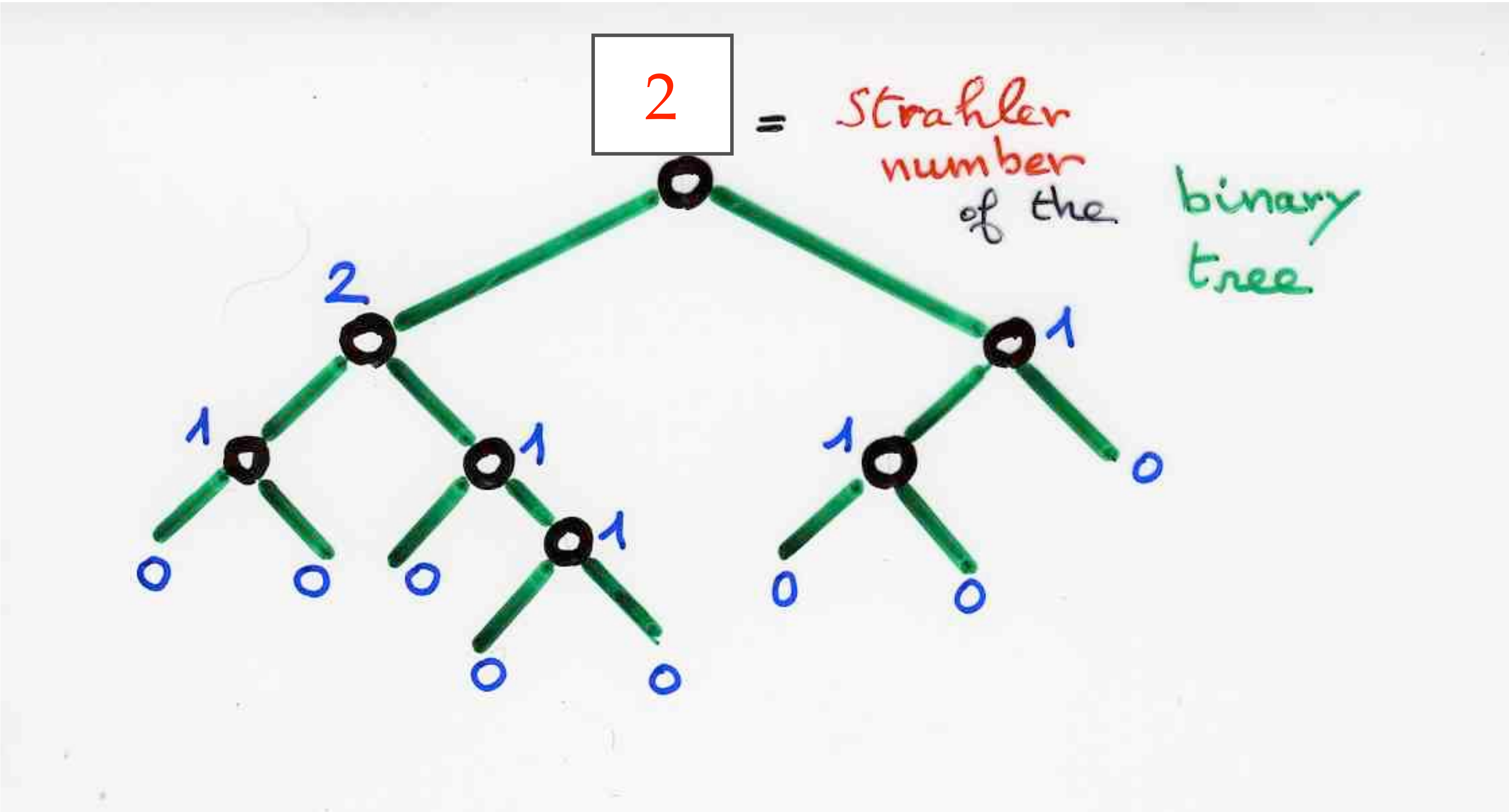


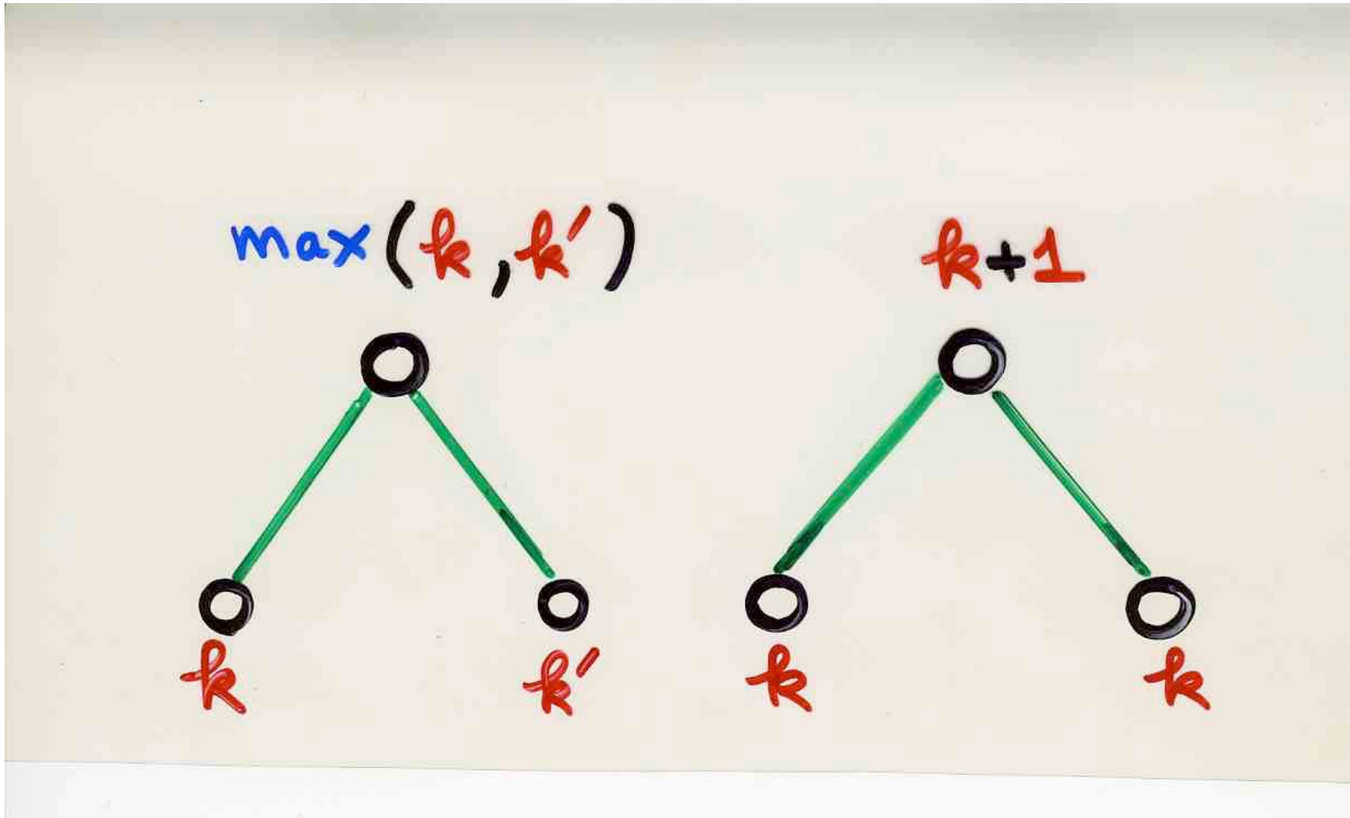


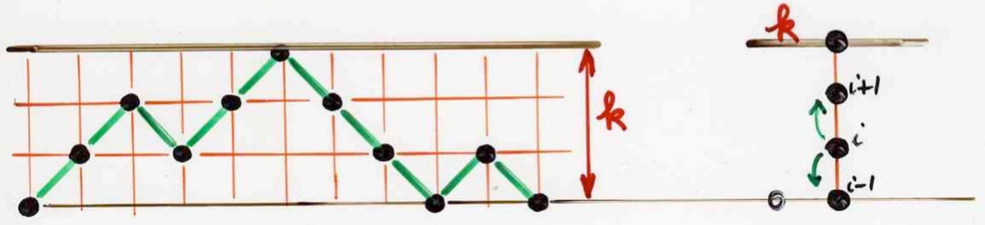
Prop. The number of system of Kepler towers having n dimers is

Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

The distribution of system of Kepler towers according to the number of towers is the Strahler distribution


2
= Strahler number of the binary tree
2
1
1
1
1
1
0
0
0
0
0
0
0
0


max(k, k')
k+1
k
k'
k
k



Dyck path w

Height $h(w)$

logarithmic height $lh(w)$

$$= \lfloor \log_2(1+h(w)) \rfloor$$

$$lh(w) = k$$

$$\Longleftrightarrow \quad 2^k - 1 \leq h(w) < 2^{k+1} - 1$$

(complete)

binary trees $\xleftrightarrow[(1984)]{\text{Françon}}$ Dyck paths

n (internal) vertices — length $2n$

Strahler nb $= k$ — log. height $\ell h(w) = k$

$\xleftrightarrow[(2005)]{\text{Knuth}}$

$$S_{\leq k}(t) = \frac{F_{2^{k+1}-2}(t)}{F_{2^{k+1}-1}(t)}$$

Fibonacci polynomial

$$S_k(t) = \frac{t^{2^k-1}}{F_{2^{k+1}-1}(t)}$$

ex:

$$S_3(t) = \frac{t^7}{F_{15}(t)}$$

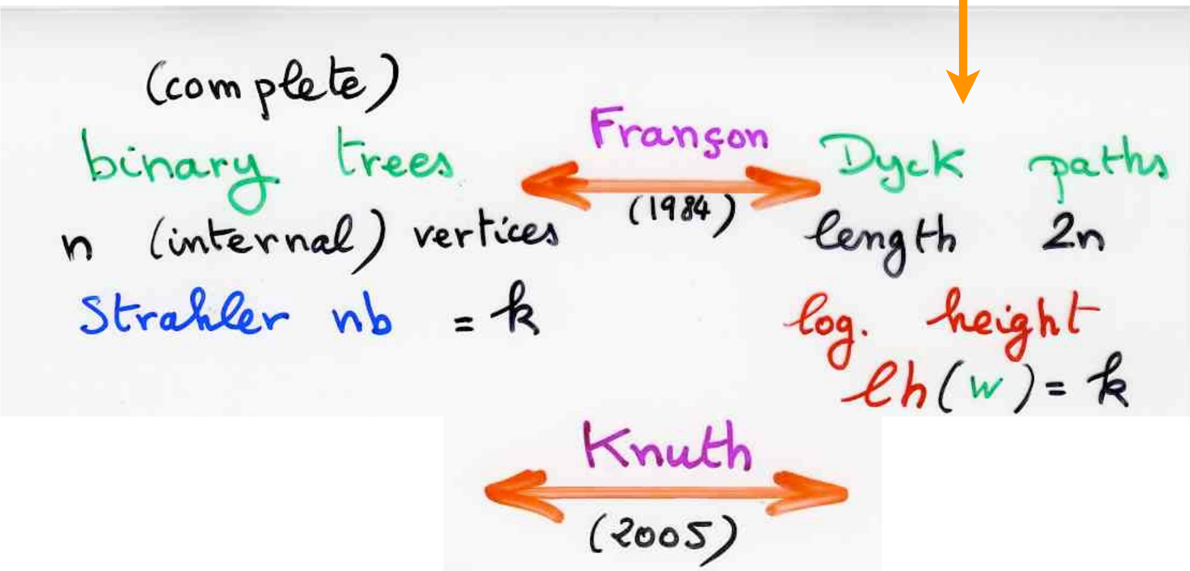
$$= \underbrace{\frac{1}{F_1}}_{1} \times \frac{t}{L_2(t)} \times \frac{t}{L_4(t)} \times \frac{t}{L_8(t)}$$

$$S_k(t) = S_{k-1}(t) \times \frac{t^{(2^{k-1})}}{L_{2^k}(t)}$$

$L_n(t)$ Lucas polynomial

system of Kepler towers

number of towers



Programs to Read

ZEILBERGER, FRANÇON, VIENNOT, an explanatory introduction, and a MetaPost source file for VIENNOT   Three Catalan bijections related to Strahler numbers, pruning orders, and Kepler towers (February 2005)

Thank you very much !

for all of you, students, professors, friends,

video technicians,

and matsciencechannel

special thanks to Amri Prasad