Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 7

Heaps in statistical mechanics (3)

q-Bessel functions in physics

IMSc, Chennai
16 March 2017

Bessel functions

Bessel functions

$$J_\alpha(x) = \sum_m \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}$$

$$\Gamma(m) = (m-1)!$$

canonical solutions

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\alpha)y = 0$$

modified Bessel functions

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$$

q-analog

$$n! \to 1(1+q)\cdots(1+q+\cdots+q^{n-1})$$

$$\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

$$J_0 = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

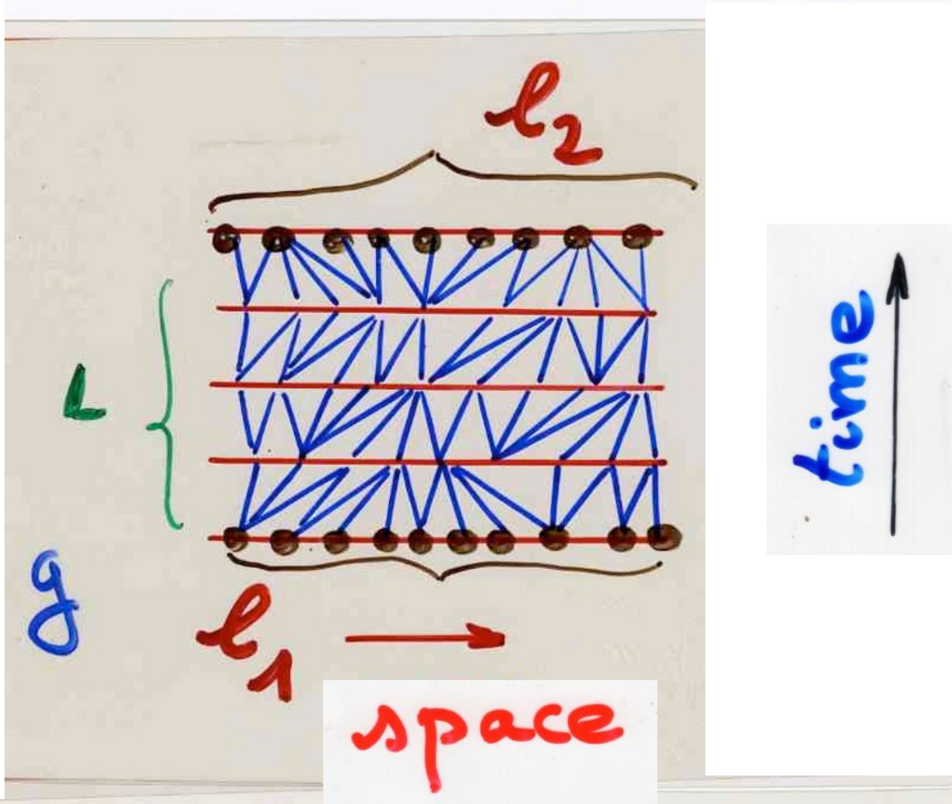
$$J_1 = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $\quad (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

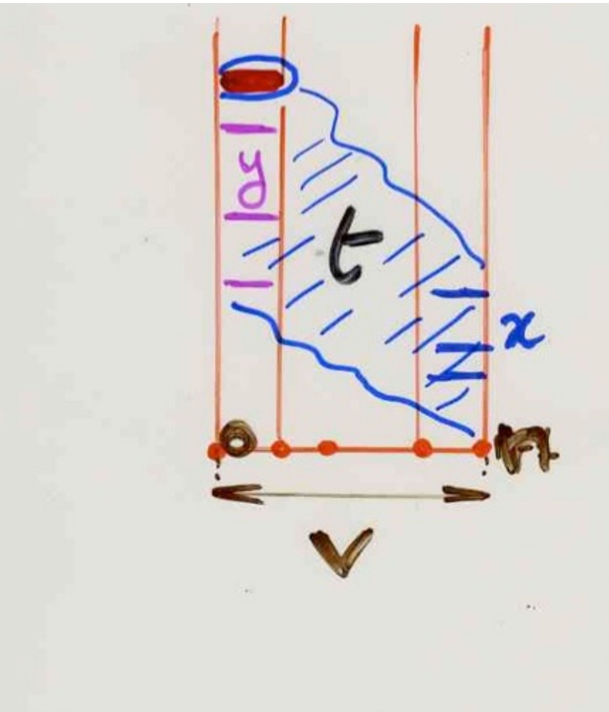
from the previous lecture

Lorentzian triangulations

in 2D qantum gravity



Path integral amplitude for the propagation from geometry ℓ_1 to ℓ_2

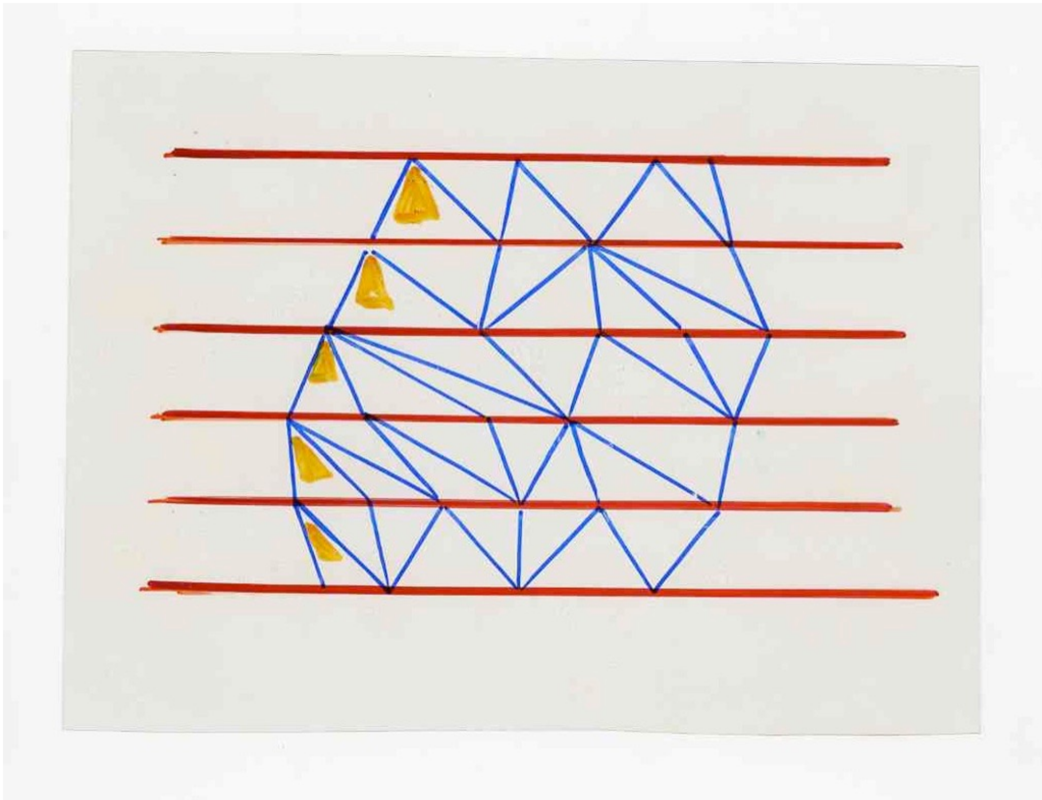


generating function
for pyramids of
dimers with 4
parameters

- t, v, y
- x number of dimers in the last column

Catalan number !

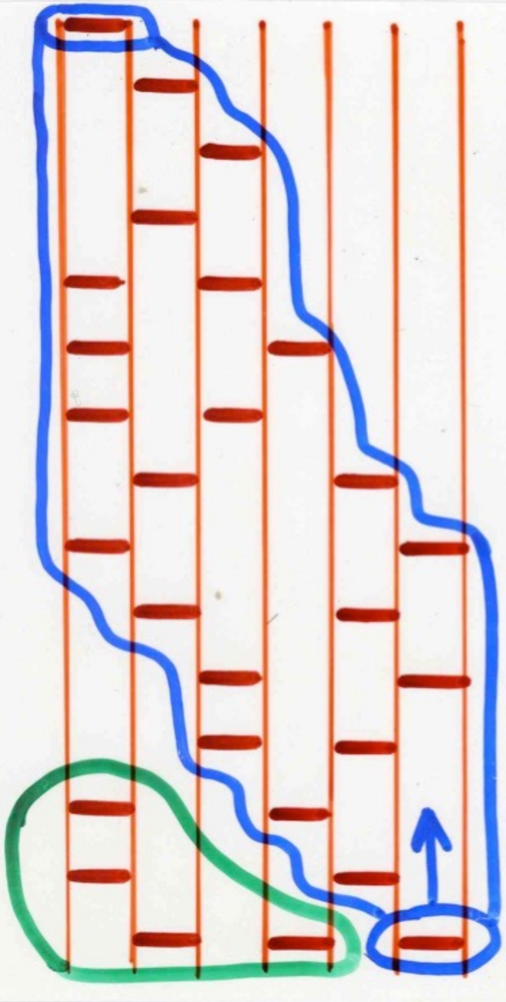
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$



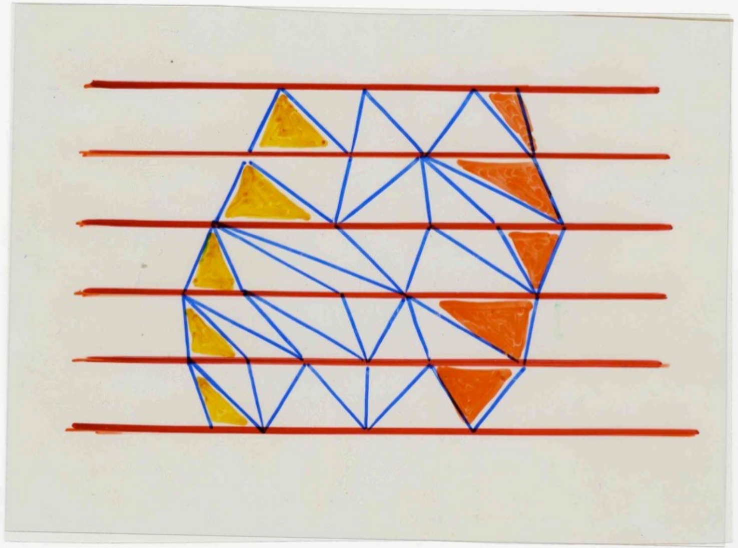
Proposition

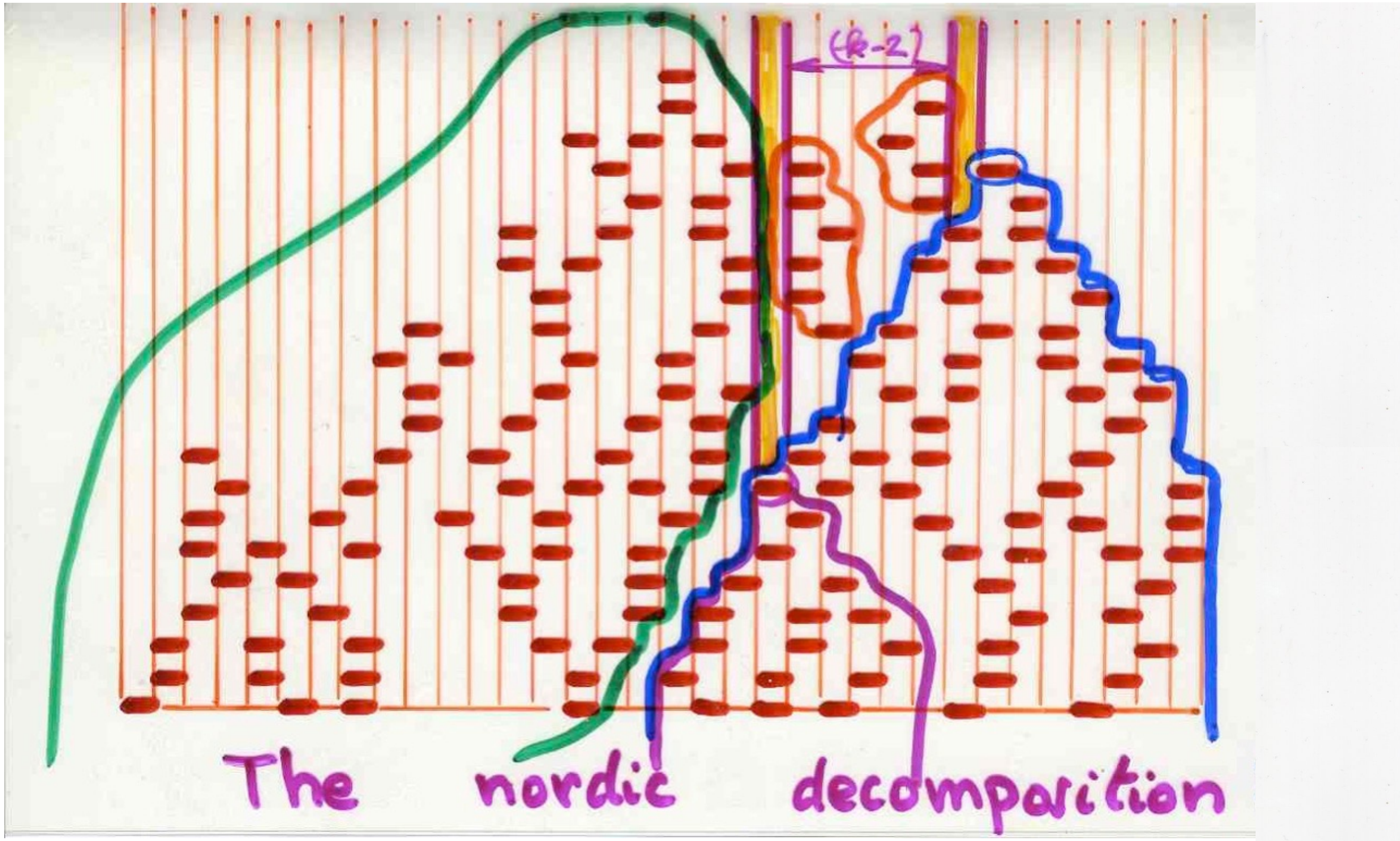
generating function for pyramids of dimers with 4 parameters

- t, v, y
- x number of dimers in the last column



$$\frac{y\,t^{n}v^{n}}{\tilde{F}_{n}(t,y,1)\,\tilde{F}_{n+1}(t,y,x)}$$





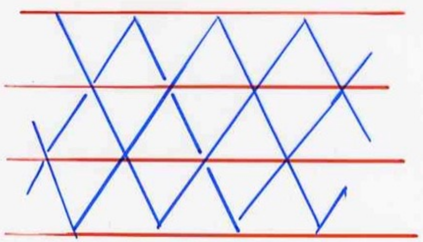
$$C = \frac{Q}{1-Q} + C\sum_{k\geq 1} \frac{Q}{1-Q}\times Q^{k}\times \frac{1}{F_{k-1}}$$

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

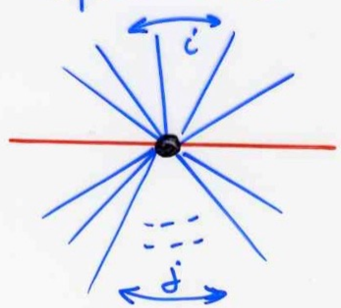
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n}\times(1+Q+\dots+Q^n)$$

curvature

of the space-time



flat



$a^{|i-3|+|j-3|}$

$$\text{total curvature} = \prod_{\text{all points}} a^{(\cdots)}$$

continuum limit

I_1 modified Bessel function

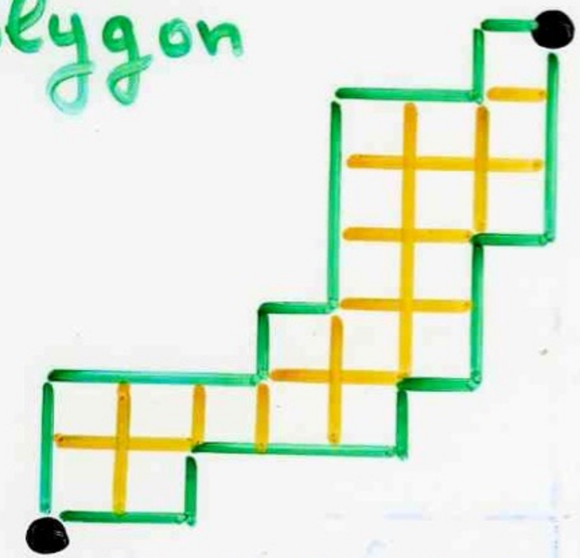
$$G_\Lambda(L_1, L_2; T) = \frac{e^{-(\coth\sqrt{\Lambda}T)\sqrt{\Lambda}(L_1+L_2)}}{\operatorname{sh}\sqrt{\Lambda}\,T}\;\frac{\sqrt{\Lambda L_1 L_2}}{L_2}\, I_1\!\left(\frac{2\sqrt{\Lambda L_1 L_2}}{\operatorname{sh}\sqrt{\Lambda}\,T}\right)$$

Parallelogram polyominoes (staircase polygons)

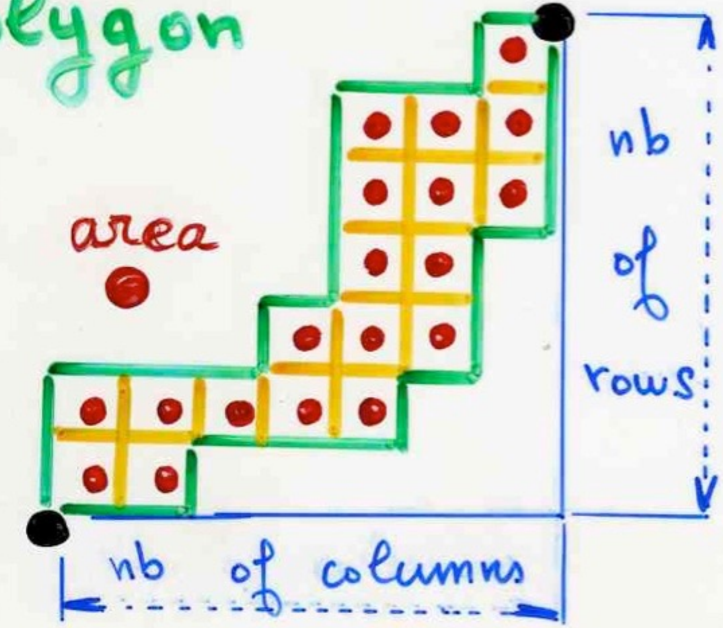
and q-Bessel functions

M.Bousquet-Mélou, X.V. (1992)°

staircase polygon



staircase polygon



$a_{m,n,p}$

generating function

$$f(x, y; q) = \sum_{m,n,p} a_{m,n,p}\, x^m y^n q^p$$

$$= \sum_{P} x^{c(P)} y^{r(P)} q^{\alpha(P)}$$

P: staircase polygons

$c(P)$: nb of columns

$r(P)$: nb of rows

$\alpha(P)$: area

parallelogram polyominoes

$$\begin{cases} x & \text{length} & (\text{nb of columns}) \\ y & \text{height} & (\text{" rows "}) \\ q & \text{area} & \end{cases}$$

Klarner, Rivest (1974)
Bender
Delest, Fedou (1989)
Brak, Guttmann (1990)
Bousquet-Mélou, X.V. (1990)

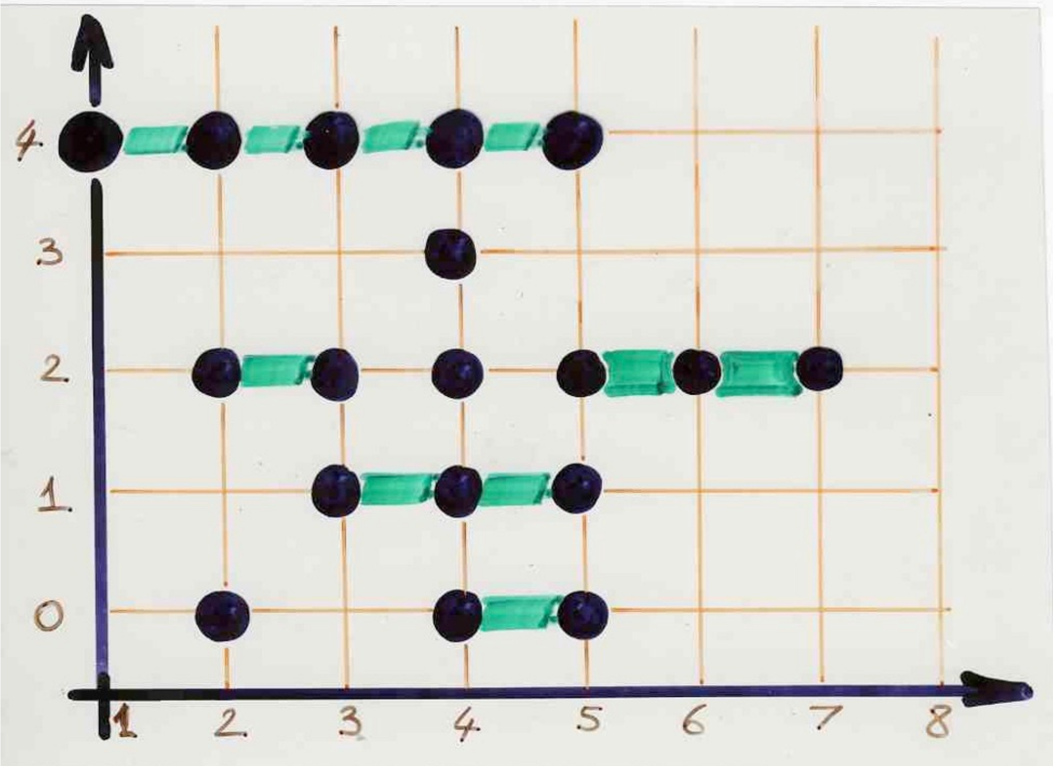
$$y \frac{J_1(x,y,q)}{J_0(x,y,q)}$$

$$J_0 = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}$$

$$J_1 = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

notation $\quad (a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$

bijetion parallelogram polyominoes
semi-pyramids of segments



poids $q^{j} t u^{(j-i)}$

bijection

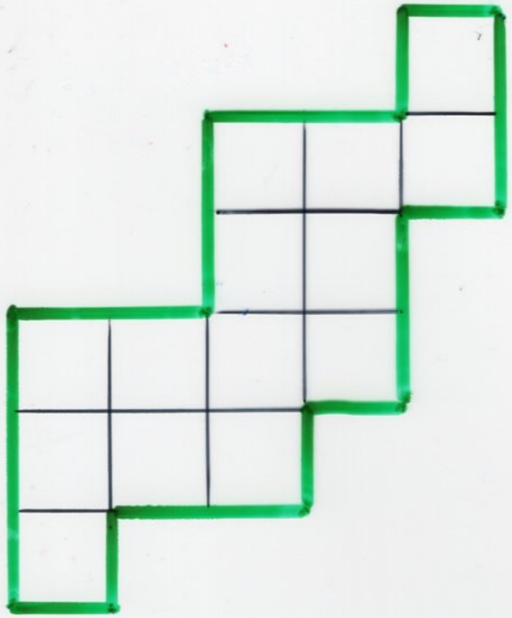
- pyramids of segments E on $\mathbb{N}^+$

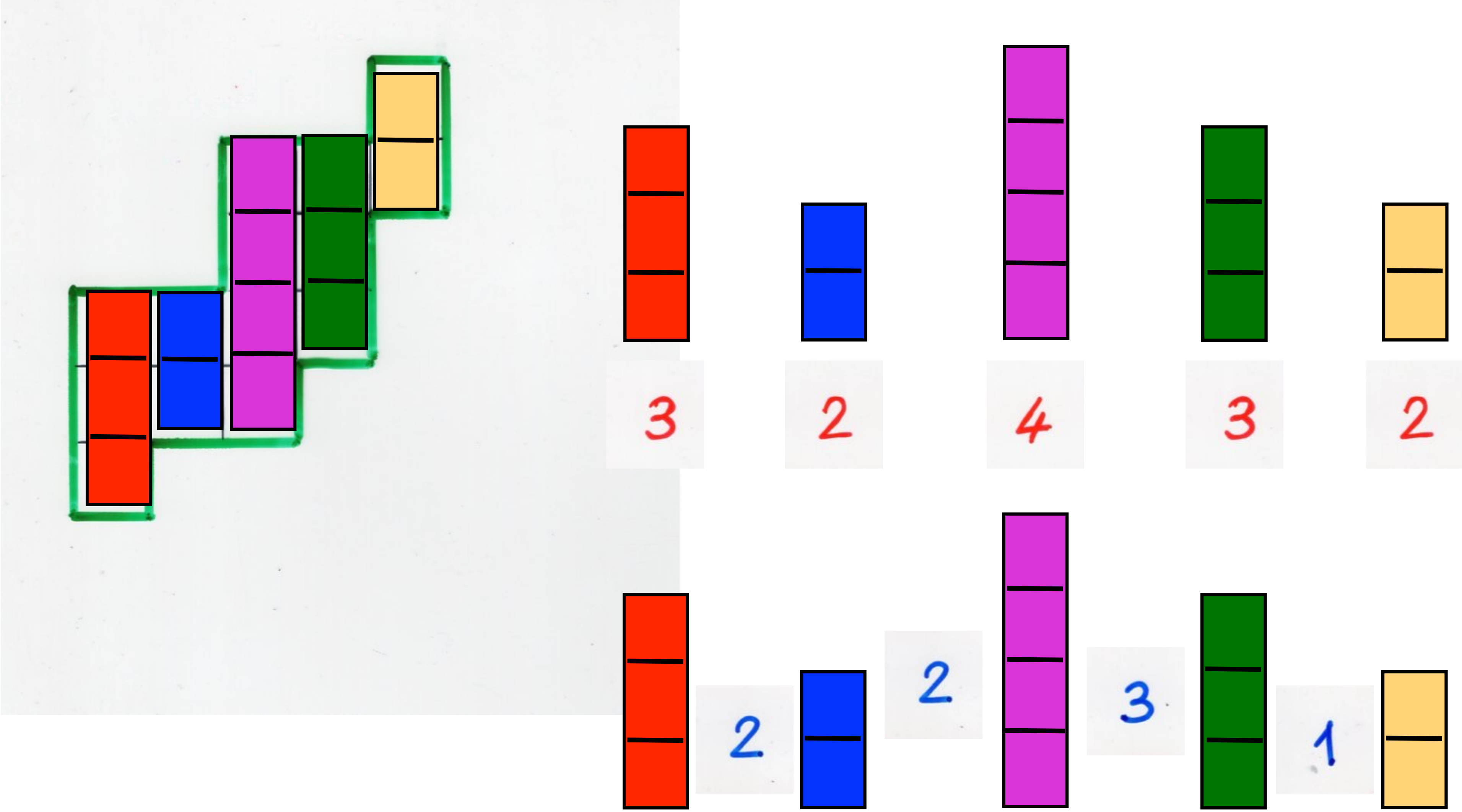
$$\pi\left(\begin{array}{c}\text{unique}\\ \text{maximal}\\ \text{piece}\end{array}\right) = [1, k] \quad , k \geq 0$$

- parallelogram polyominoes Λ

$$\underset{\text{area}}{q^{a(\Lambda)}}\ \underset{\substack{\text{number}\\ \text{of}\\ \text{columns}}}{t^{c(\Lambda)}}\ \underset{\substack{\text{number}\\ \text{of}\\ \text{rows}}}{u^{r(\Lambda)-1}} = v(E)$$

$$v(\underset{\text{segment}}{[i;j]}) = q^{j} t\, u^{(j-i)}$$

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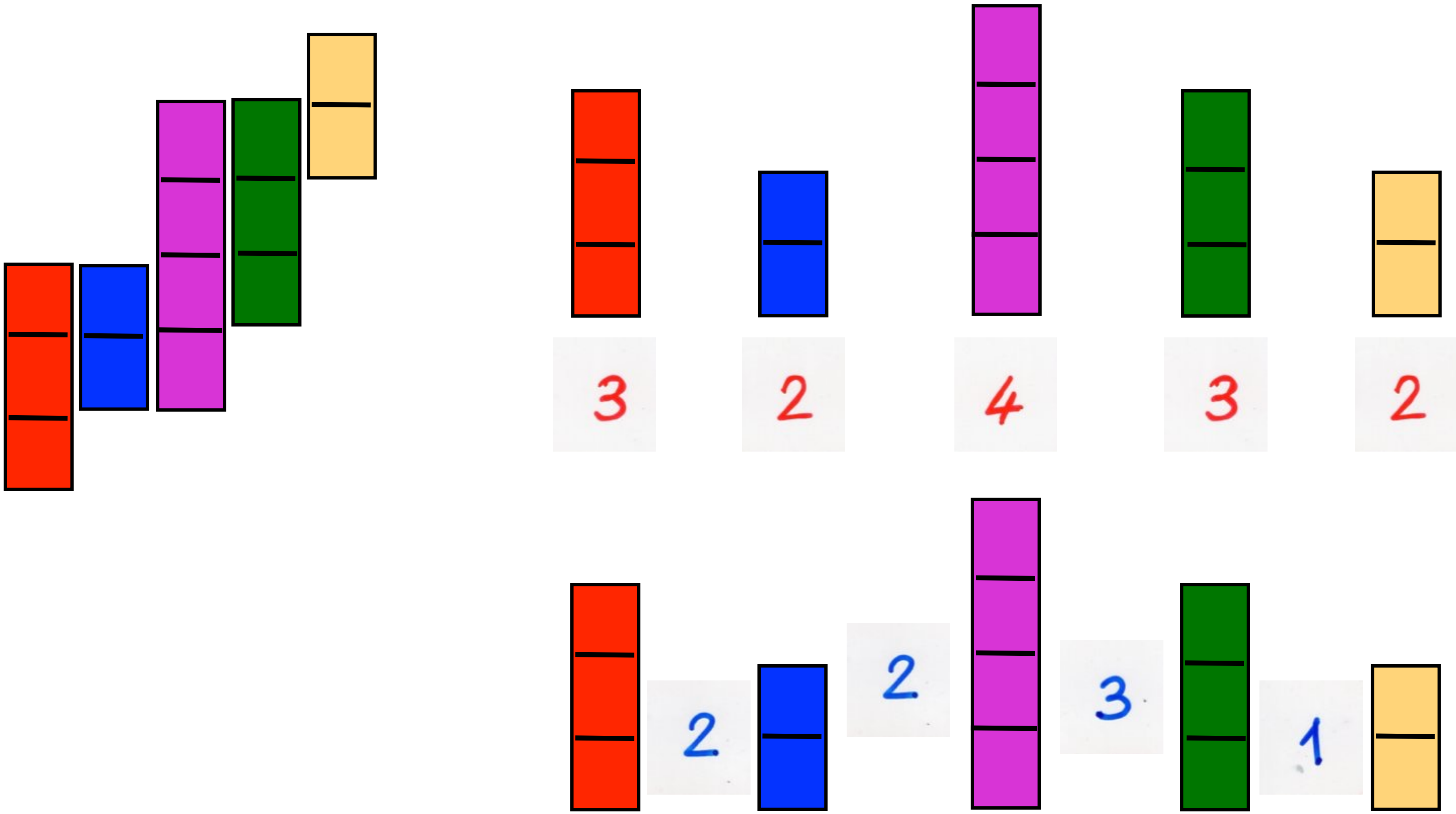
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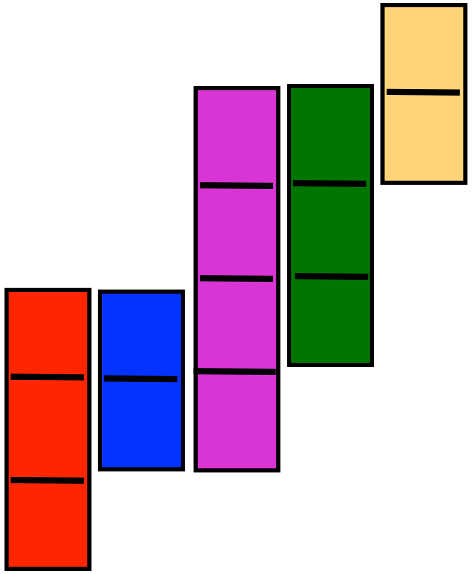
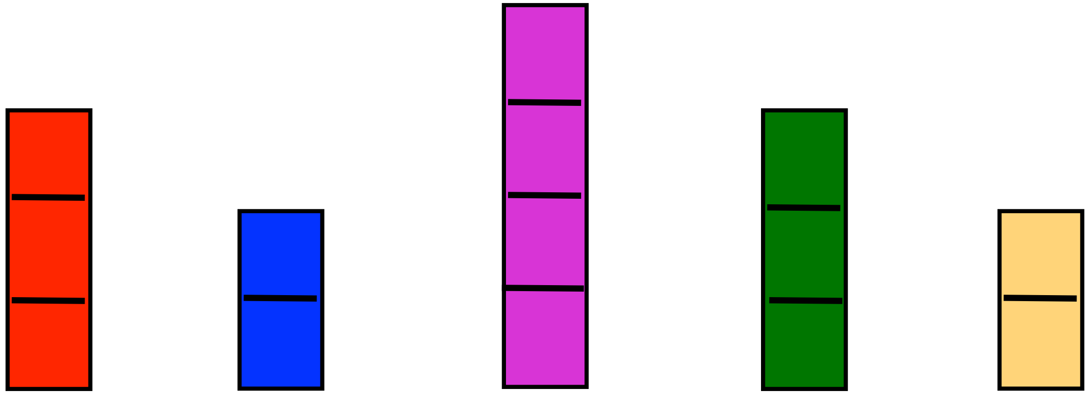
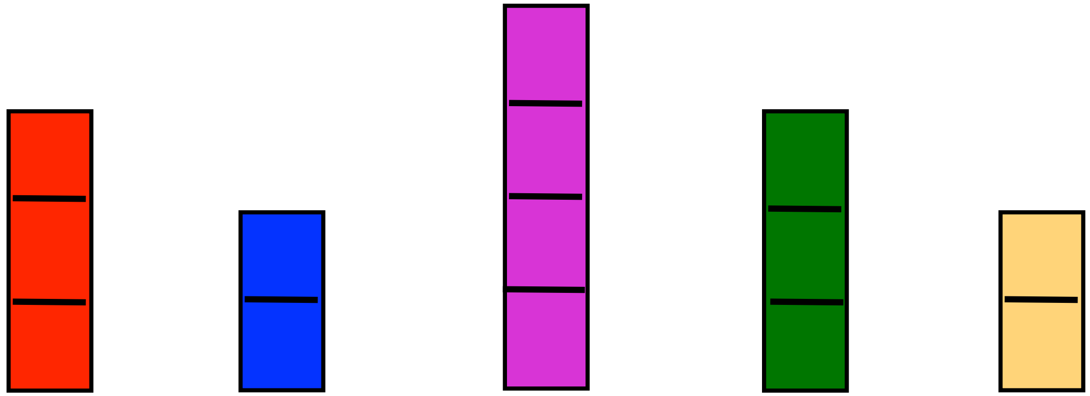
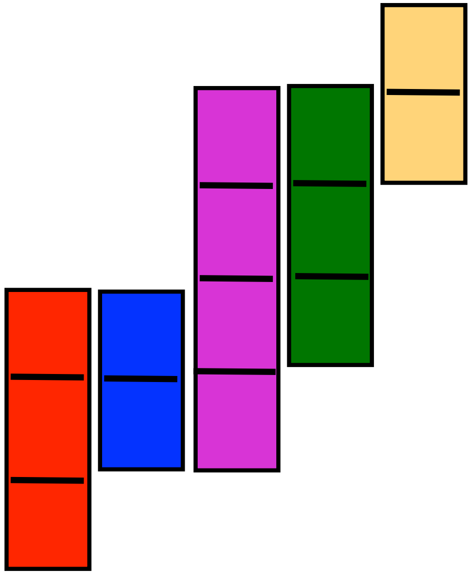
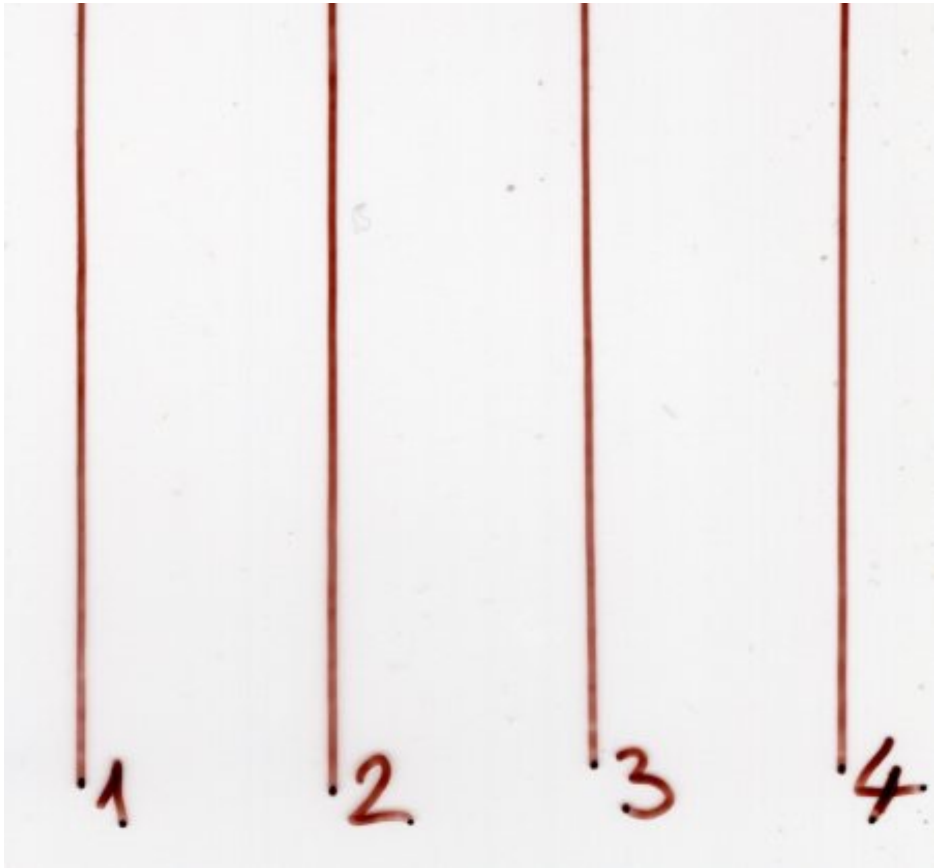






2 3 | 2 2 | 3 4 | 1 3 | 2

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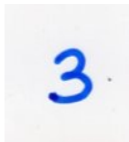
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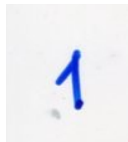
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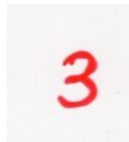
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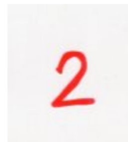
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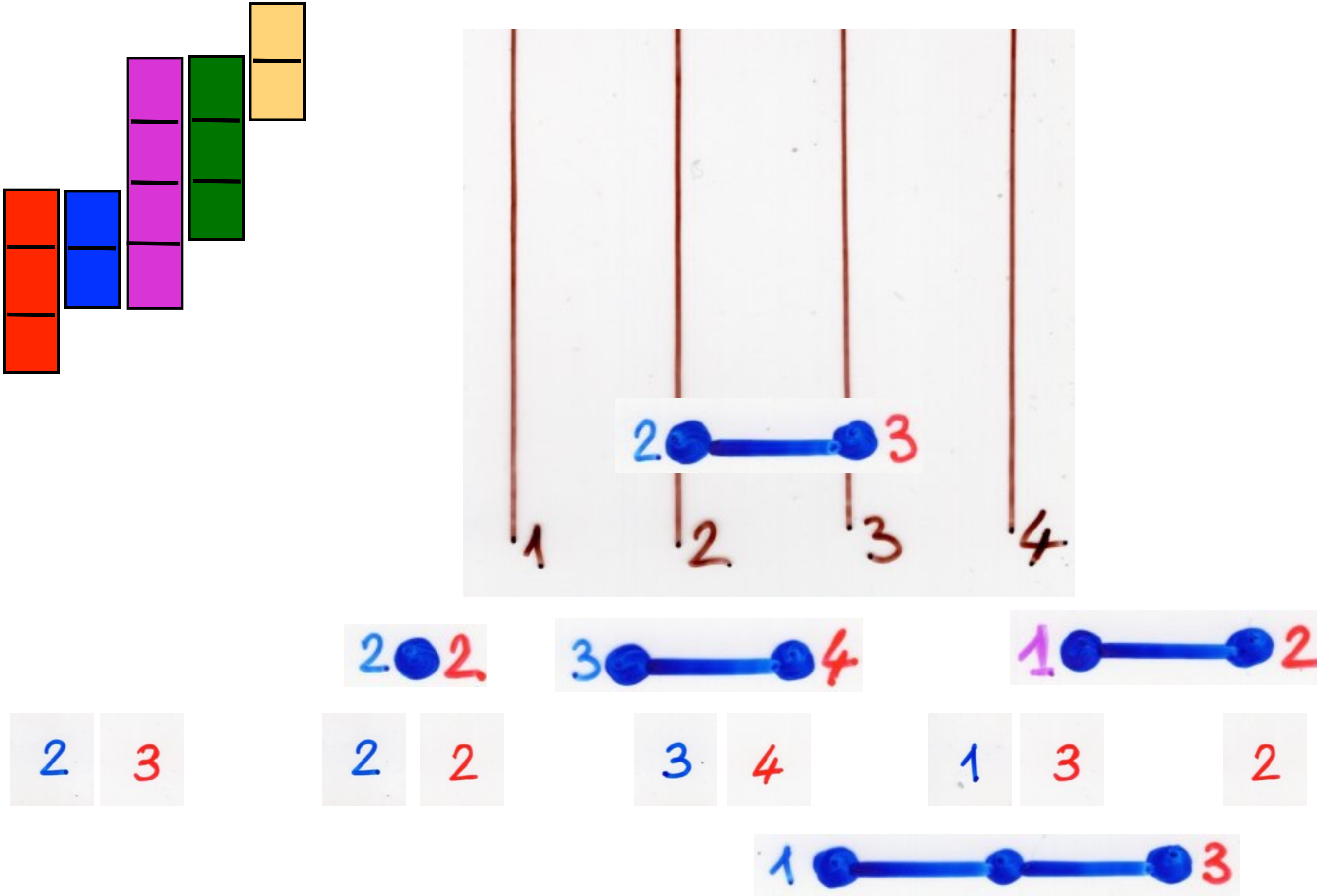
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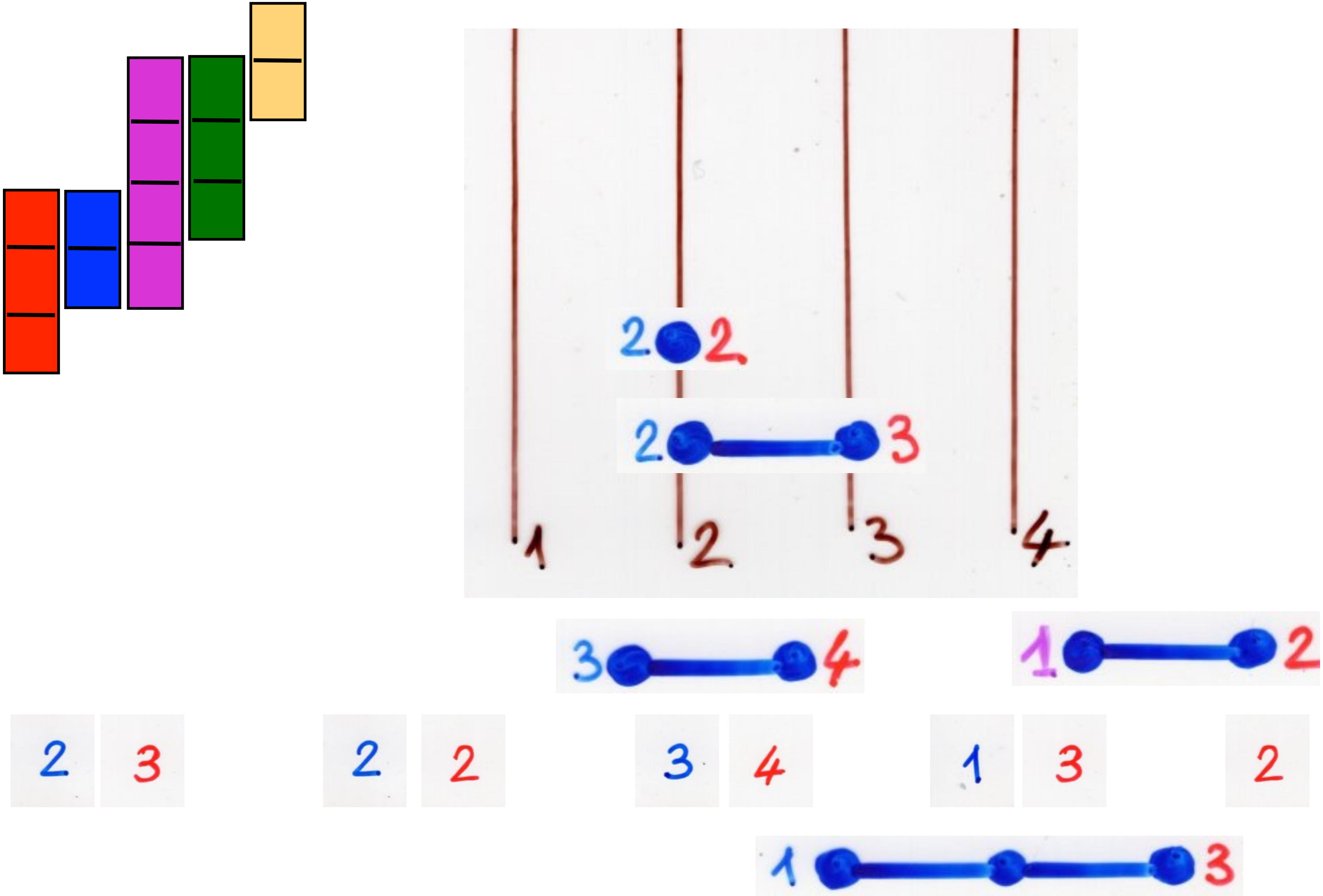
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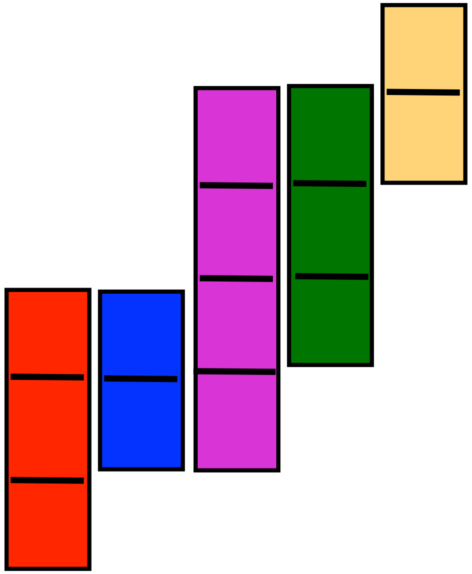
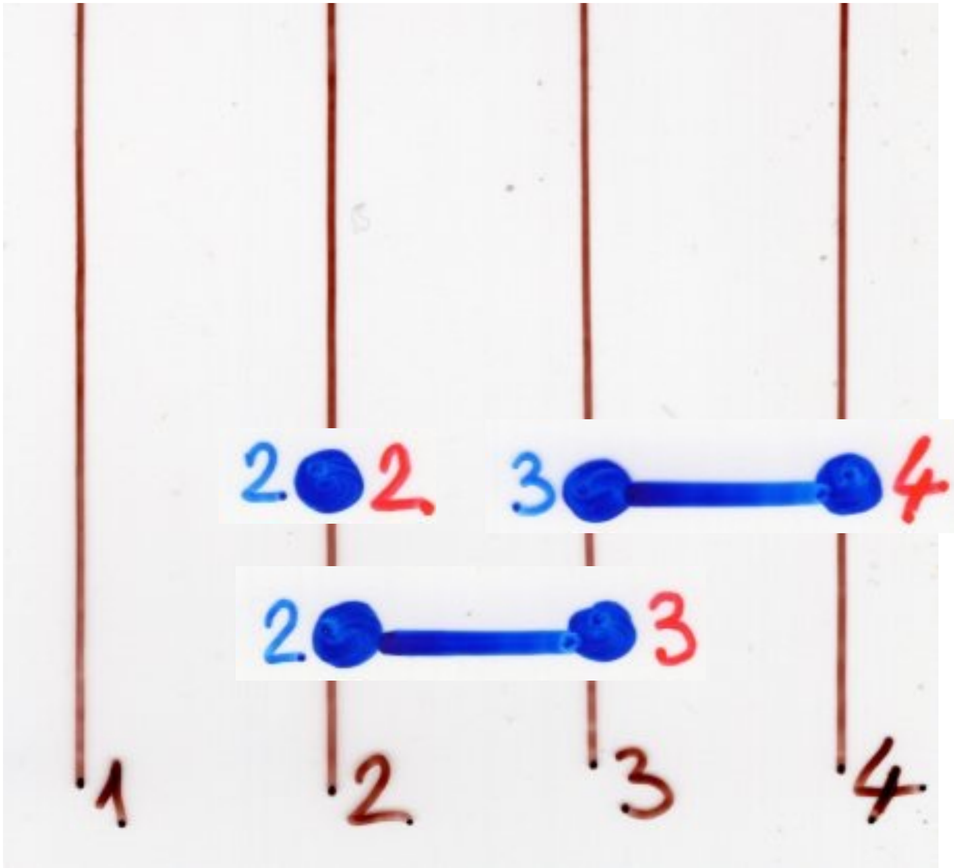
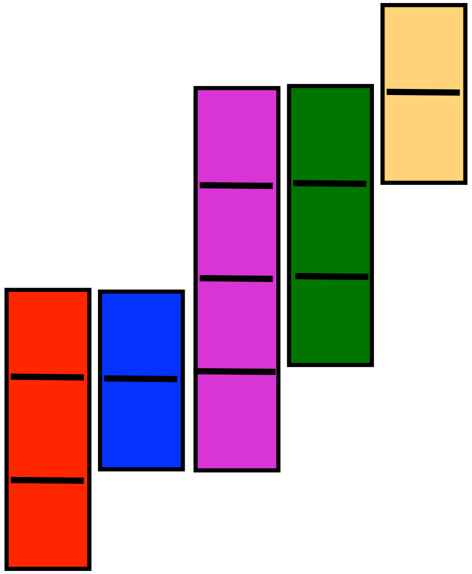
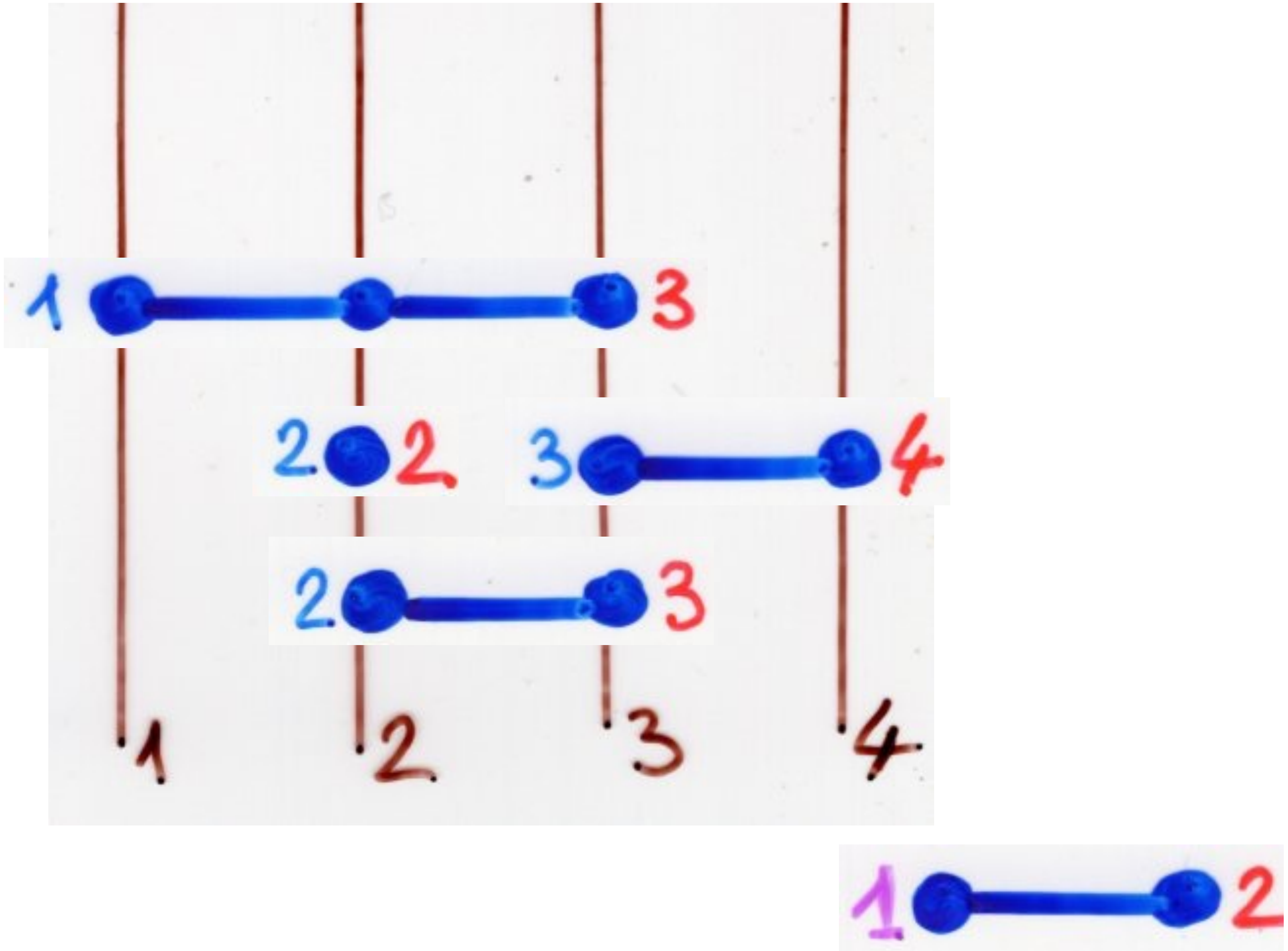




2 3 2 2 3 4 1 3 2

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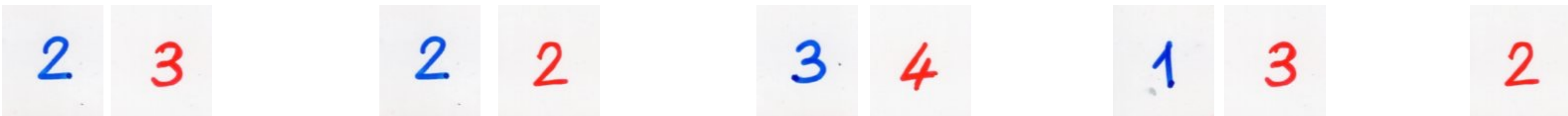
1 2


2 3
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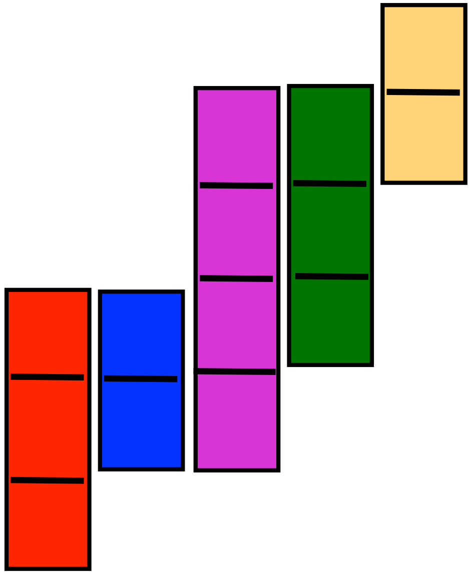

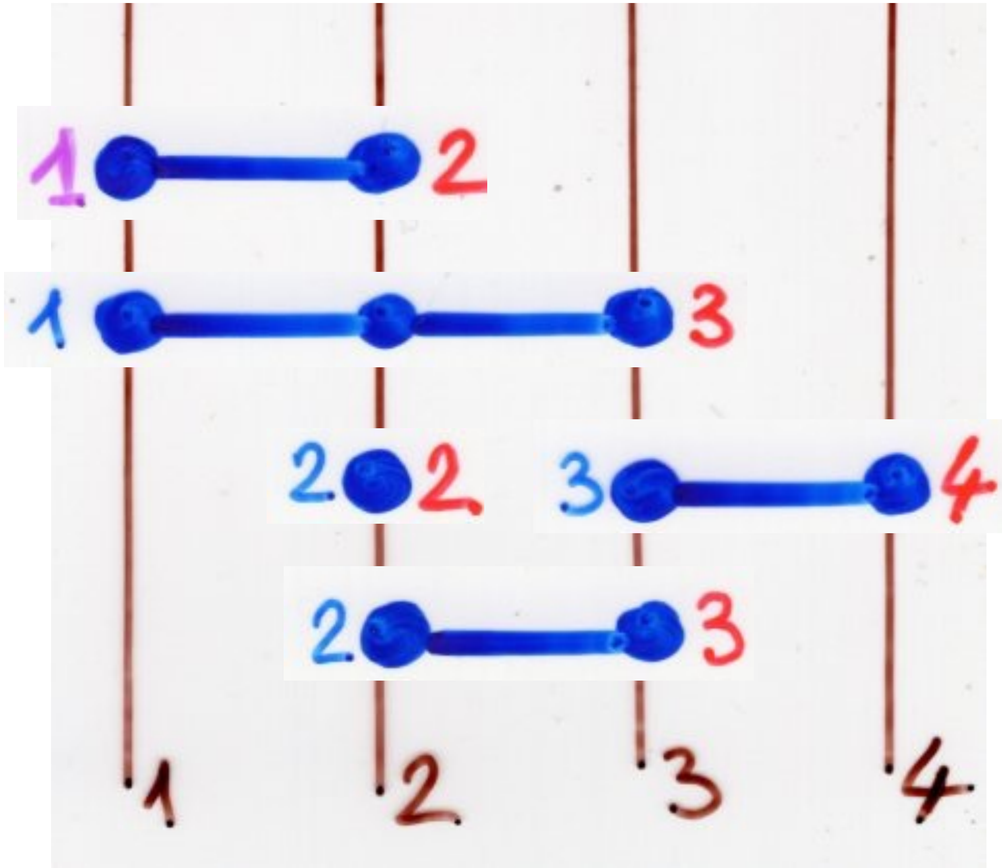

2 3

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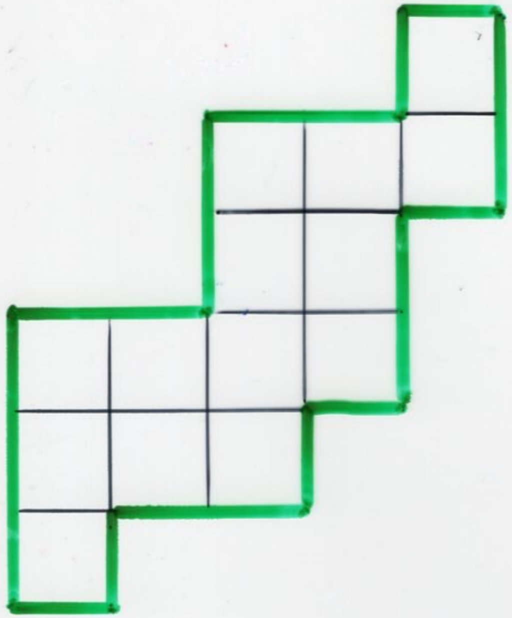

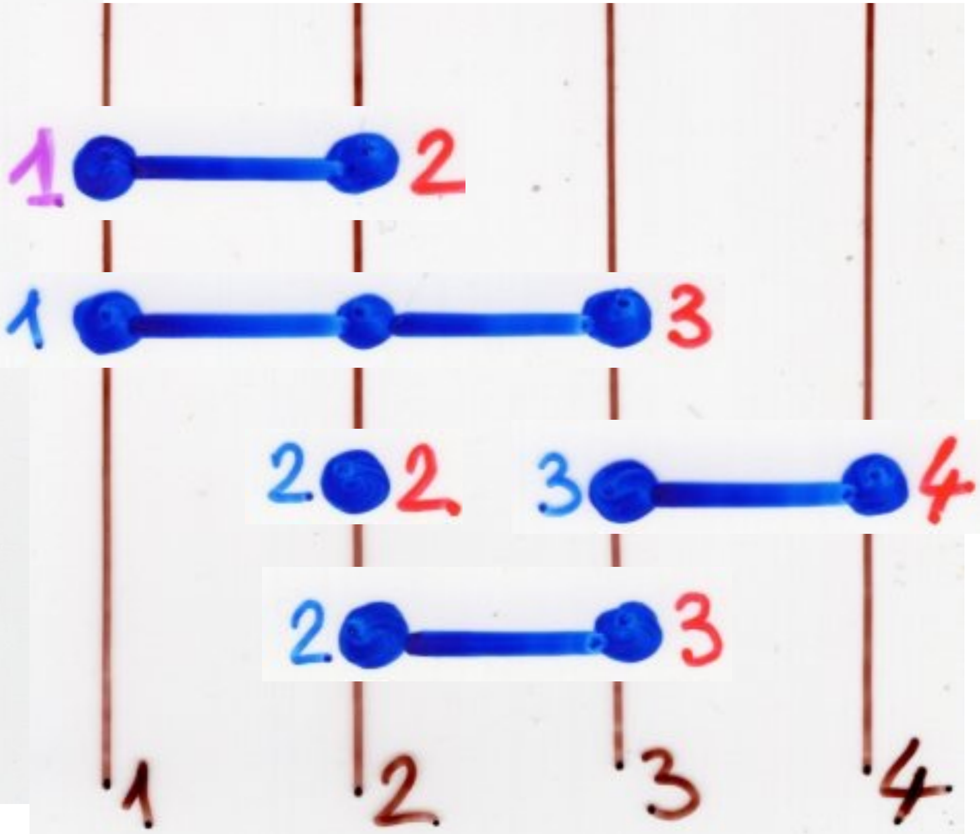

1 2
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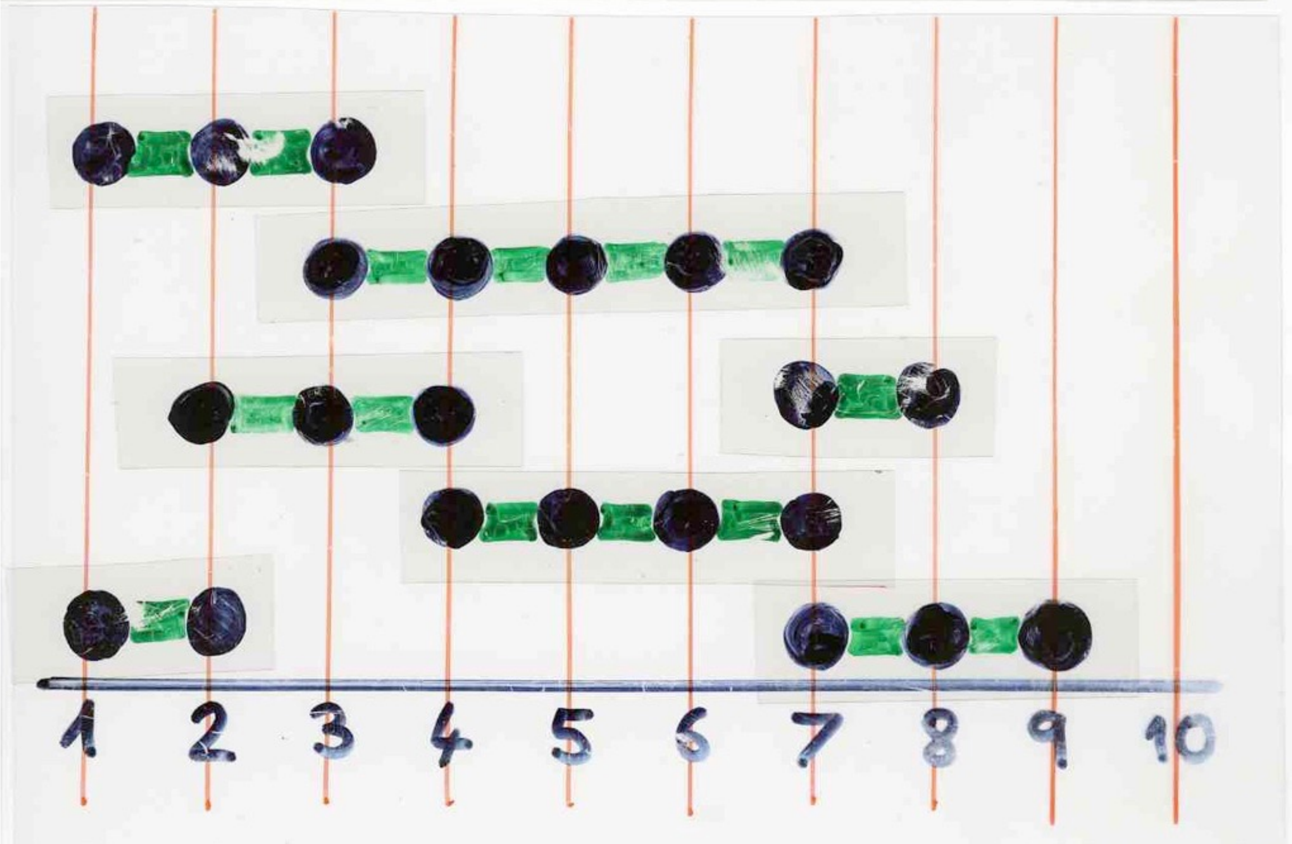
generating function

$$f(t,u;q) = \frac{N}{D}$$



extension of the inversion lemma

$M \subseteq P$

$$\sum_{E} v(E) = \frac{N}{D}$$

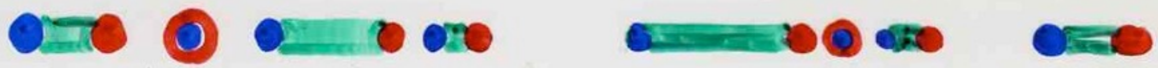
Π(maximal pieces) $\in M$

$$D = \sum_{F \text{ trivial heaps}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \text{ trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$

Segments $v([i, j]) = q^{j} t u^{(j-i)}$

$$D = \sum_{n \geq 0} \frac{(-1)^n t^n q^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$



$$D = \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} v(\mathcal{C})$$

(q-Bessel)

$\mathcal{C}$ configuration of 2 by 2 disjoint segments

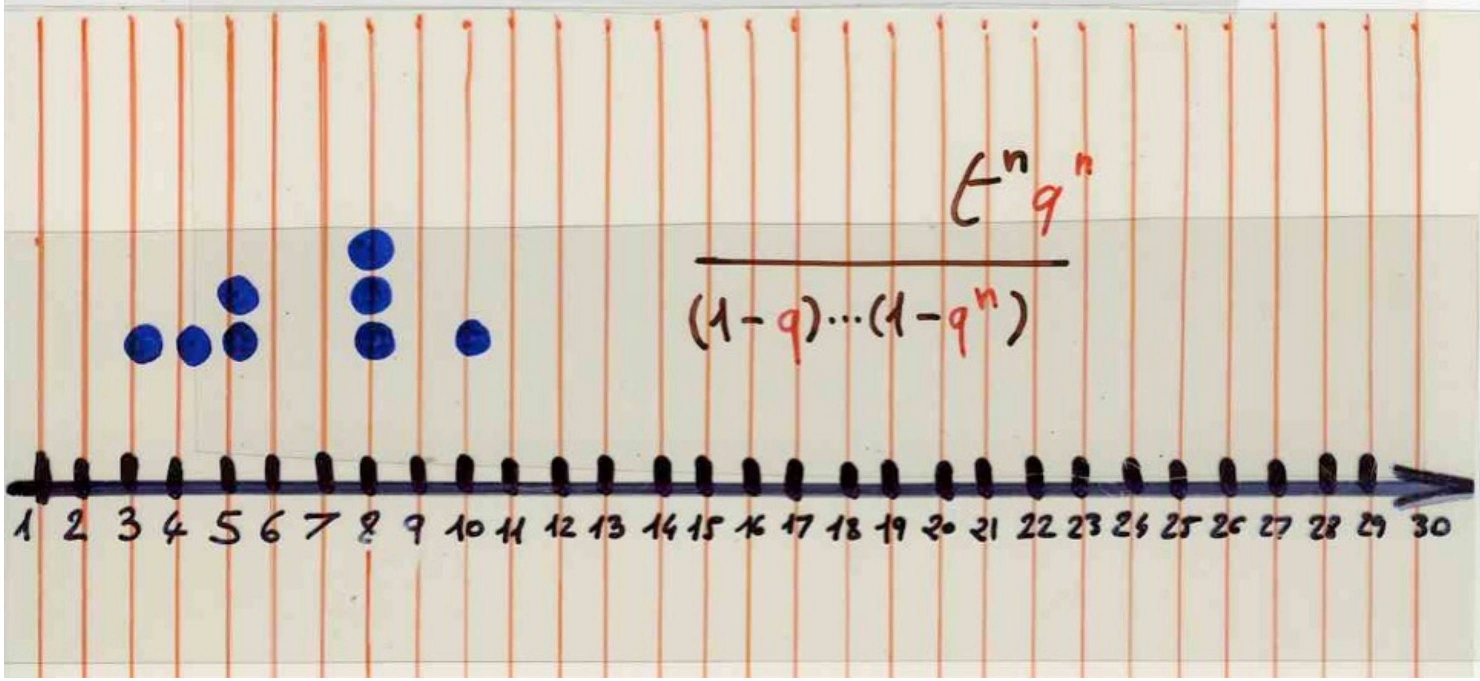
$$v(\mathcal{C}) = \prod v(\text{each segment})$$

from integers partitions

to q-Bessel functions

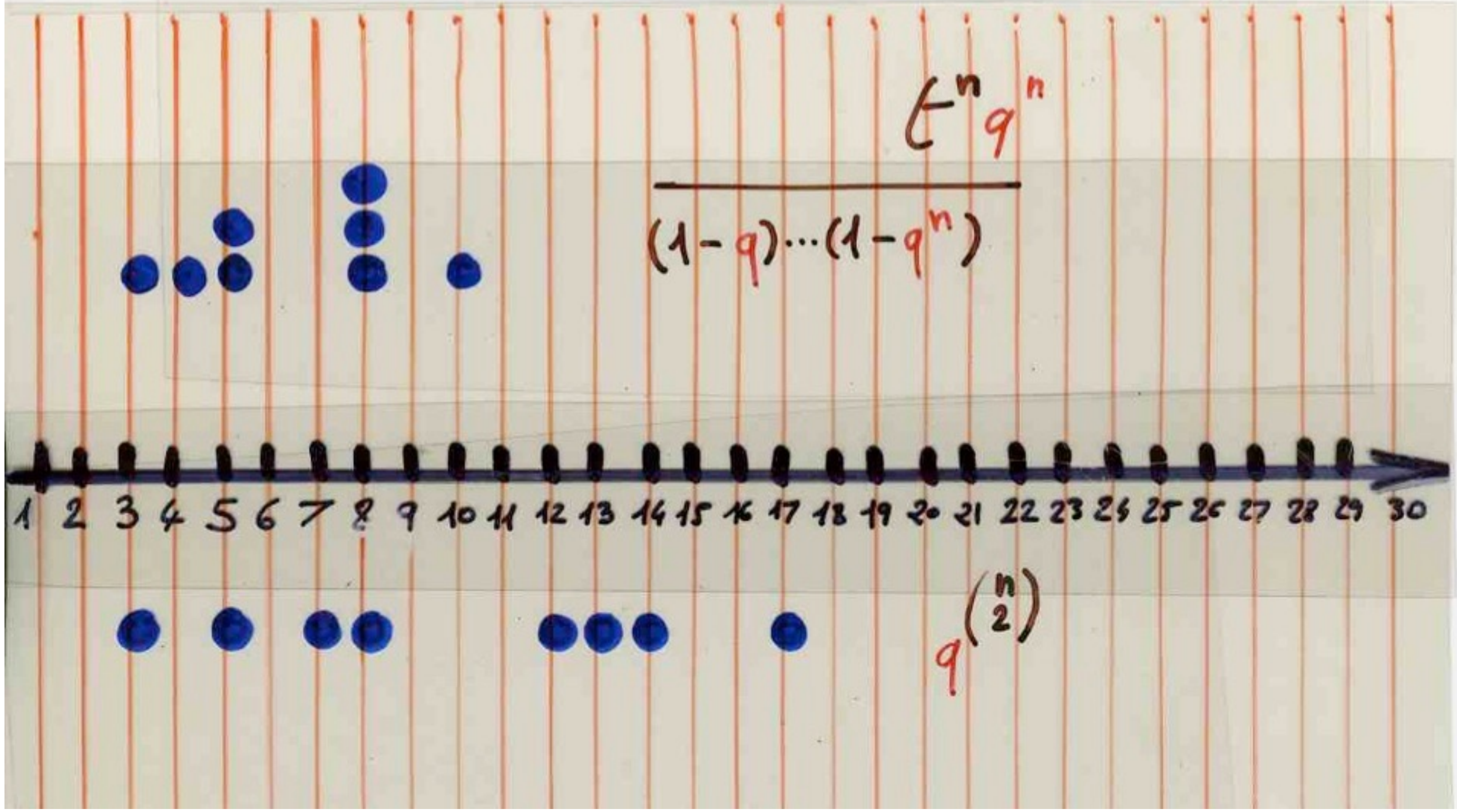
$$D = \sum_{n \geq 0} \frac{(-1)^n \, t^n q^n \, q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

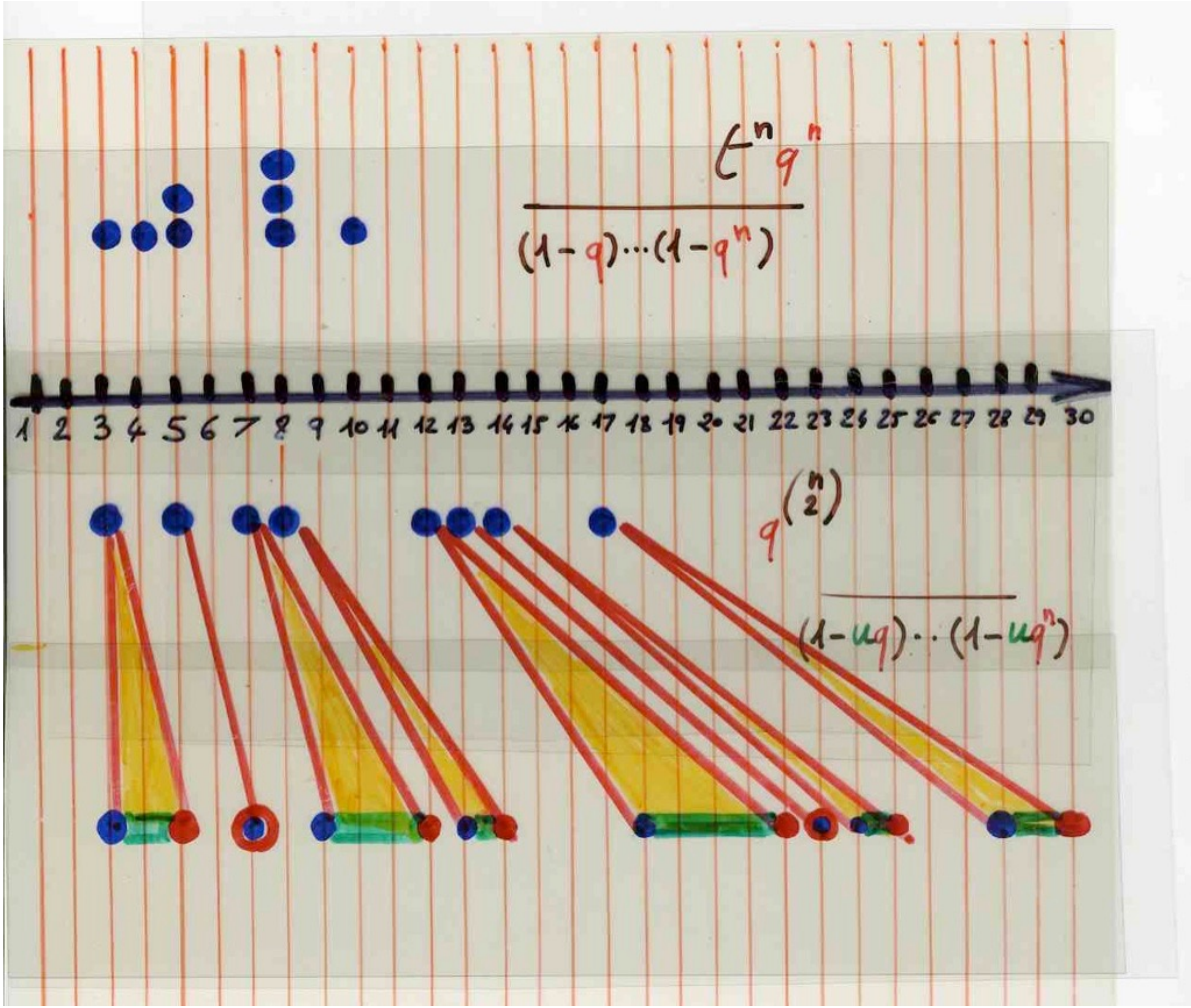
$$D = \sum_{n \geq 0} \frac{(-1)^n \, q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$$



$$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$$

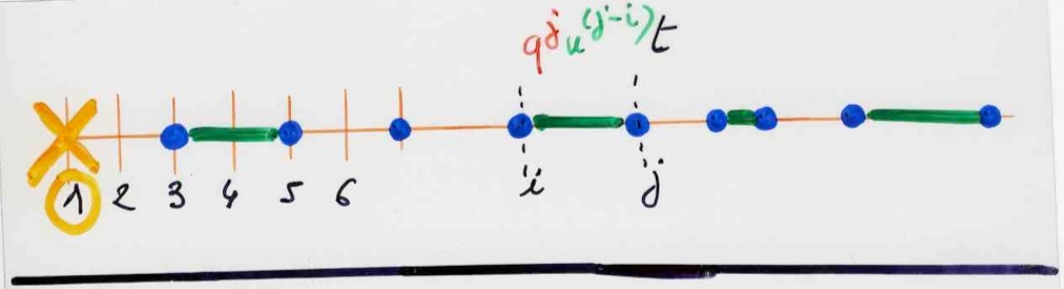
$$D = \sum_{n \geq 0} \frac{(-1)^n}{(1-uq)\cdots(1-uq^n)}$$

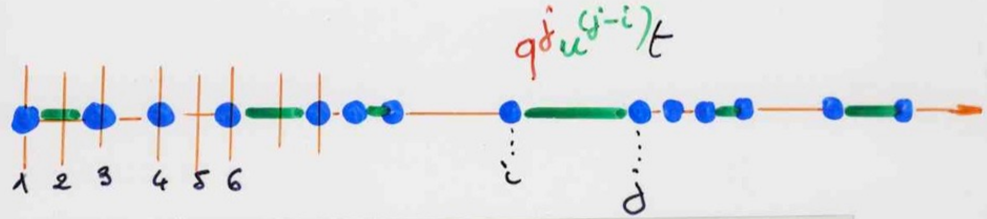




$\frac{t^n q^n}{(1-q)\cdots(1-q^n)}$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
$\frac{q^{\binom{n}{2}}}{(1-uq)\cdots(1-uq^n)}$

$$N = u \sum_{n\geq 1} \frac{(-1)^{n-1} t^n q^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^{n-1})(1-uq)\cdots(1-uq^n)}$$

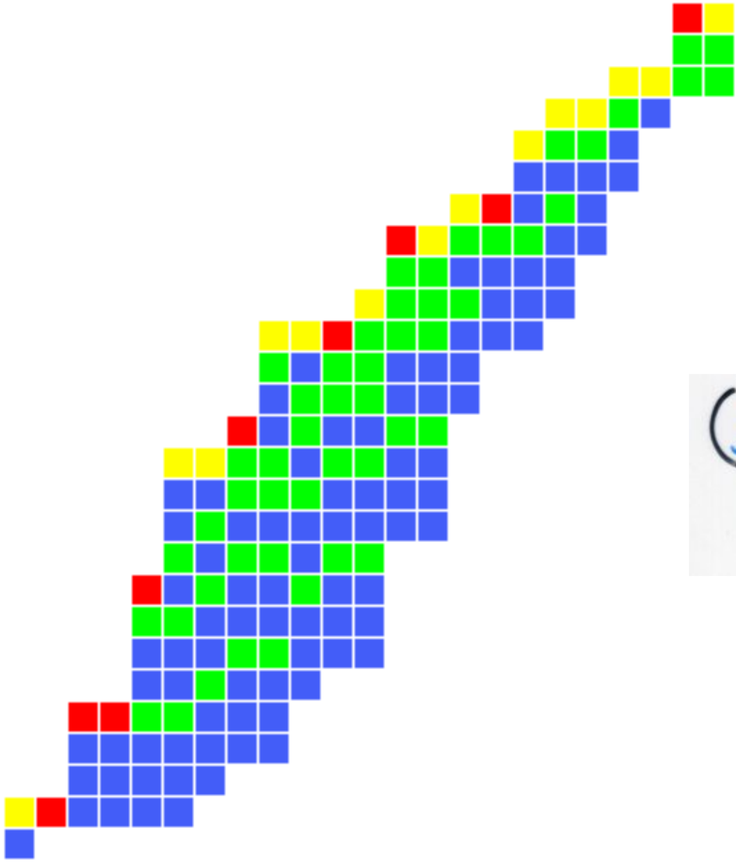




Segments $\quad v([i,j]) = q^j t\, u^{(j-i)}$

$$D = \sum_{n\geq 0} \frac{(-1)^n t^n q^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)(1-uq)\cdots(1-uq^n)}$$

random parallelogram polyominoes



random parallelogram polyominoe

(staircase polygon)

fixed perimeter

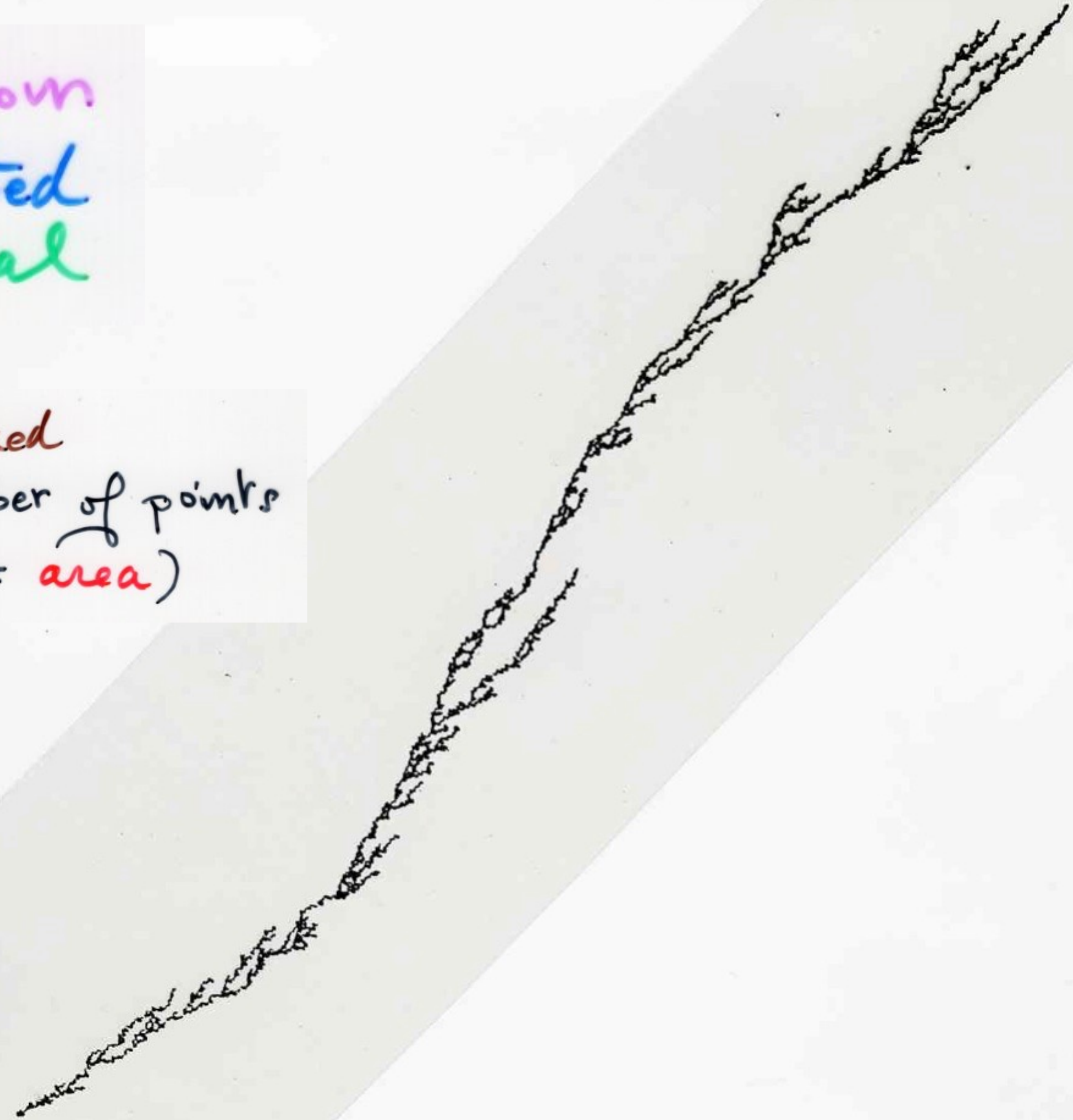
fixed area

(I. Dutour, J.M. Fedou)

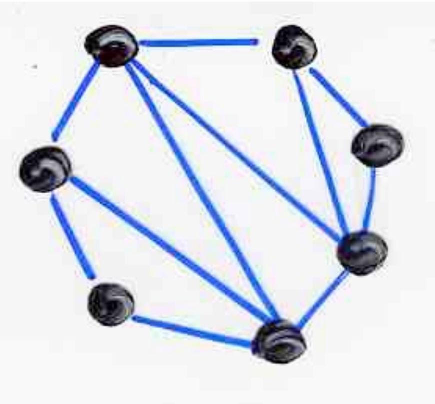
q- algebraic grammar

random
directed
animal

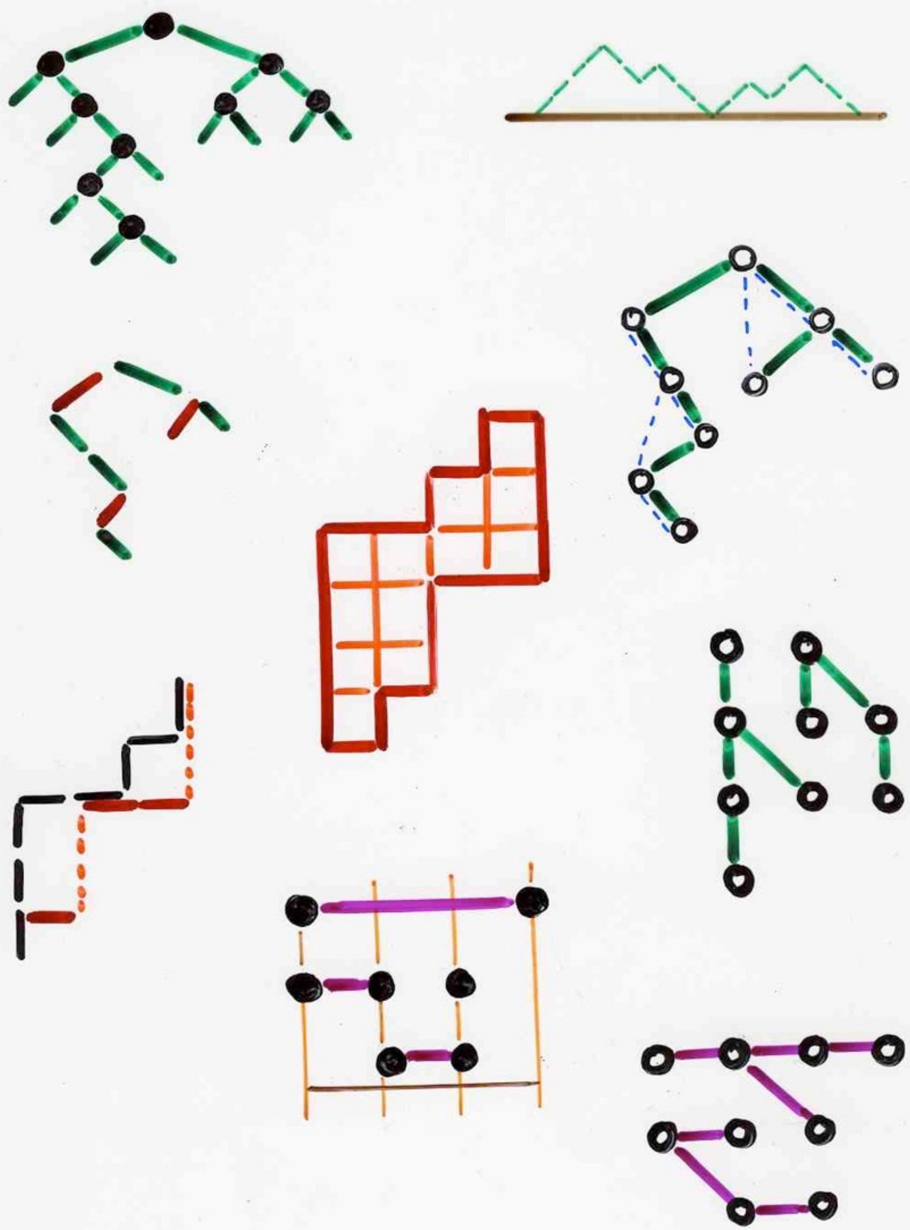
fixed
number of points
(= area)



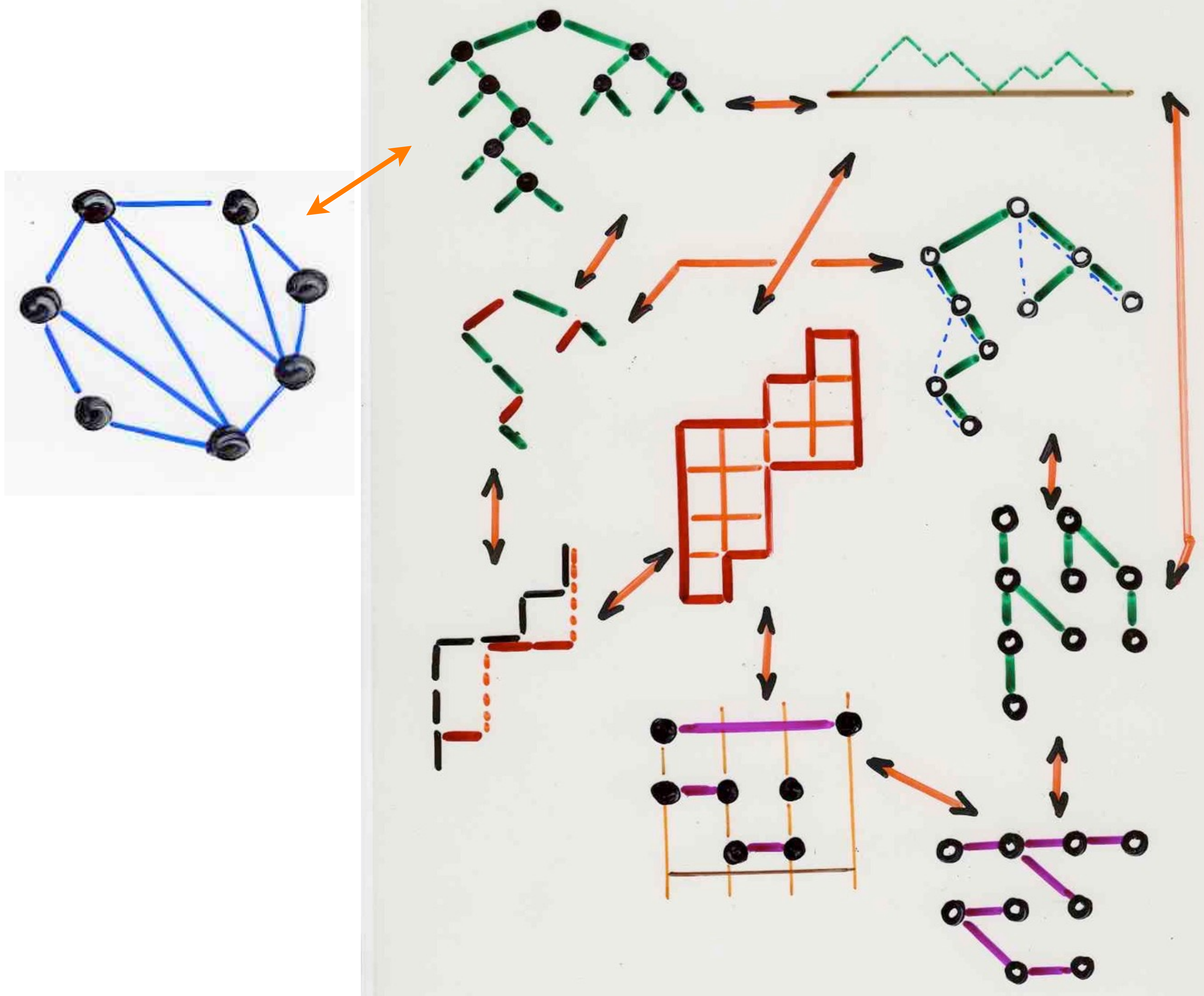
The Catalan garden



the Catalan garden



the Catalan garden



A festival of bijections ....

other description of the bijection:

1. with the stairs decomposition of a heap of dimers

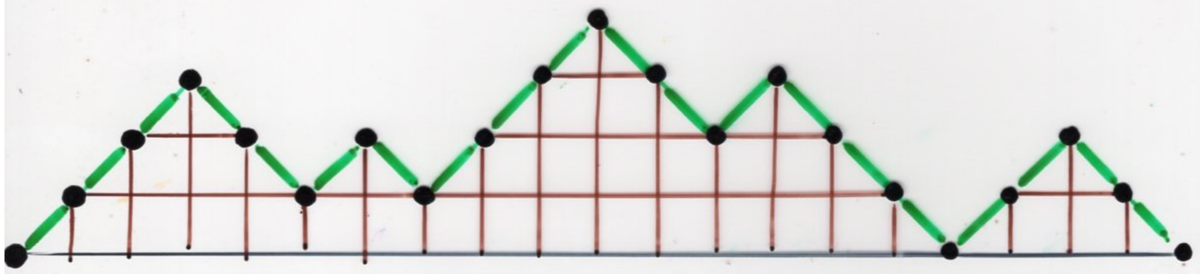
bijection

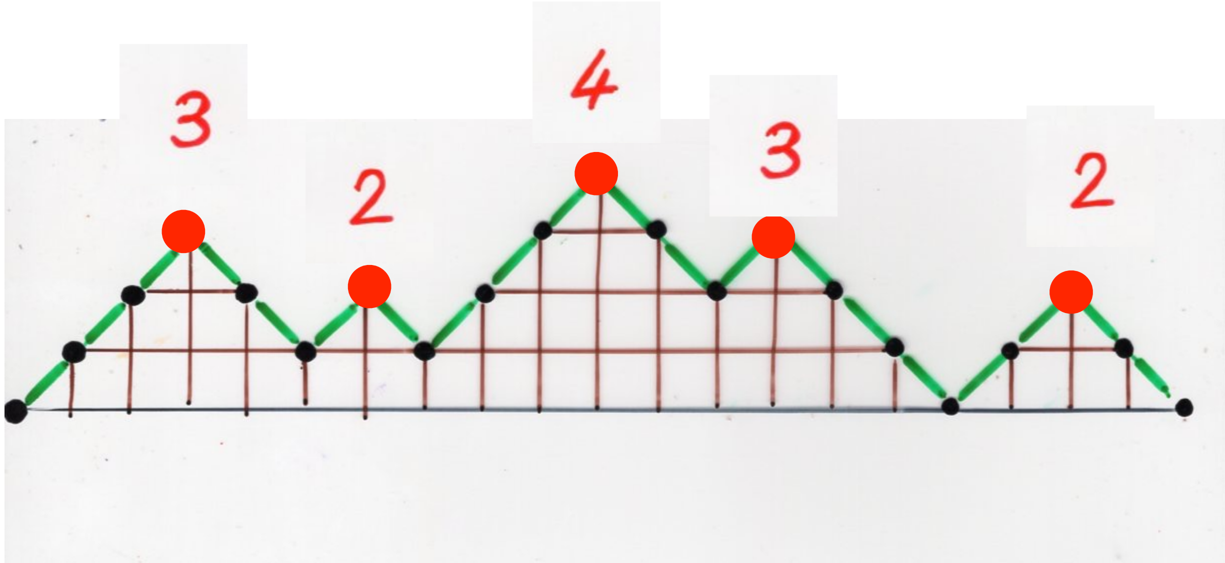
staircase polygons

Dyck paths

Ch 2a (IMSc 2016)
p 110-116

The Catalan garden

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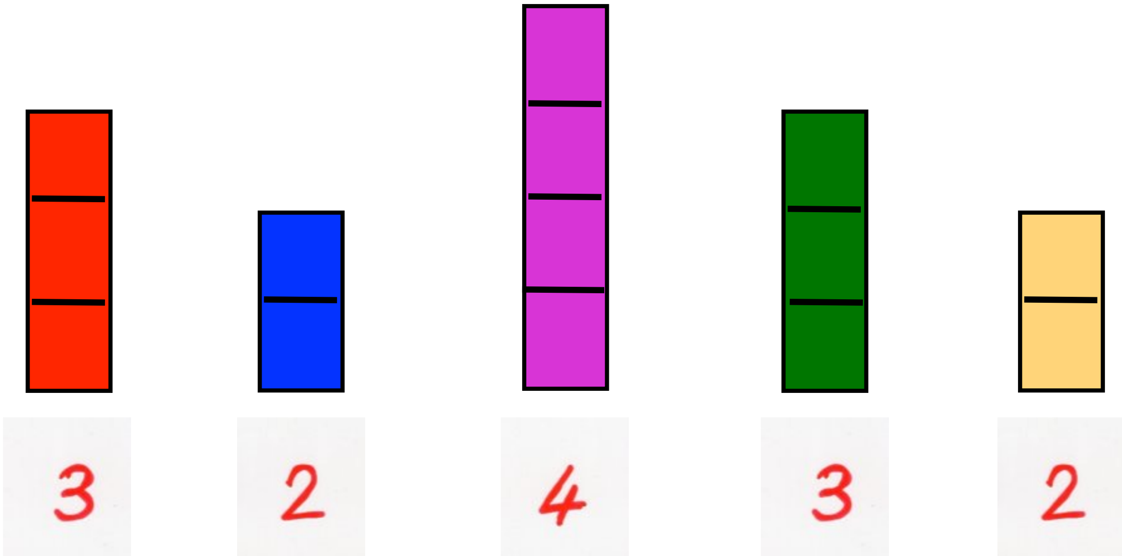
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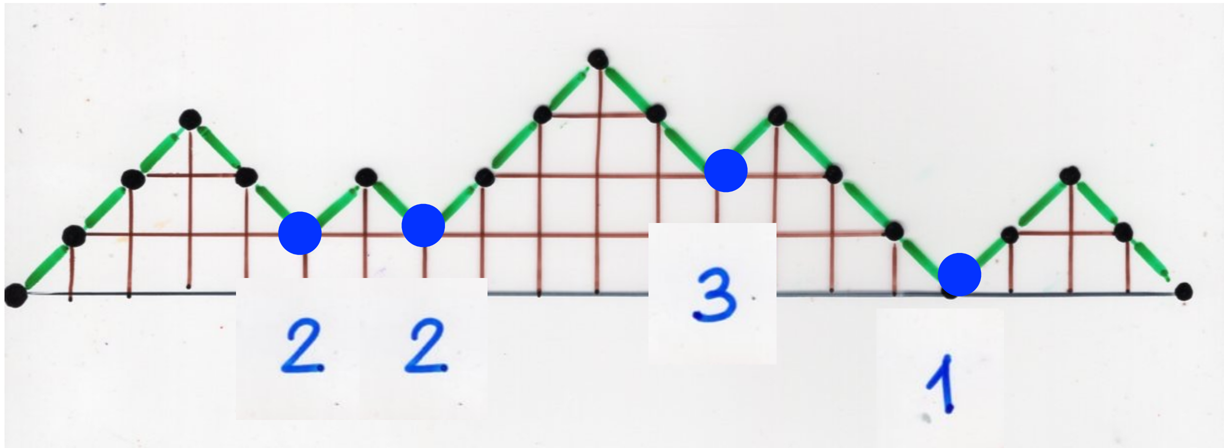
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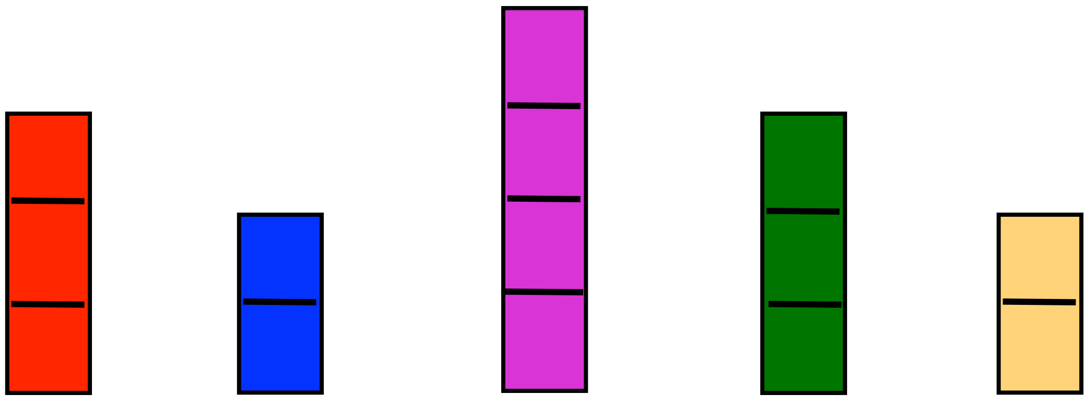

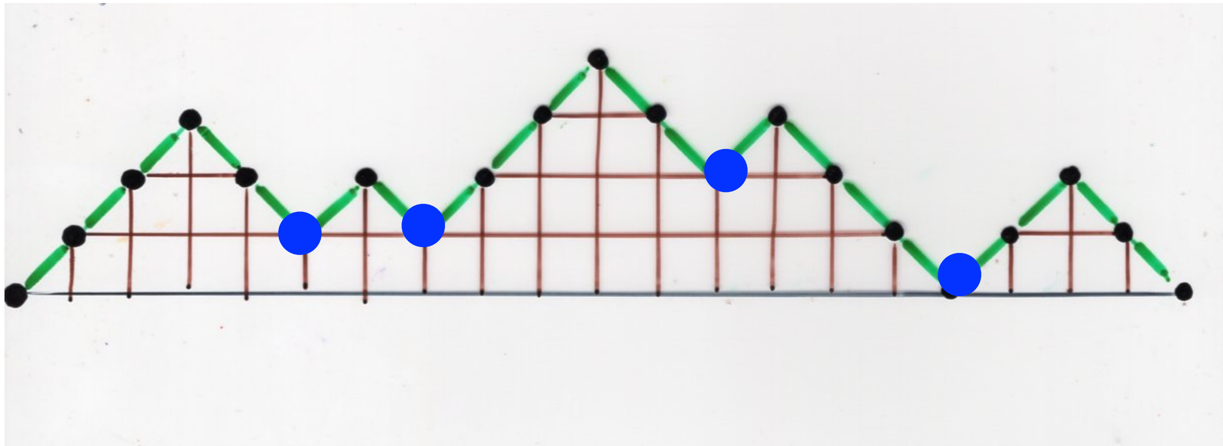

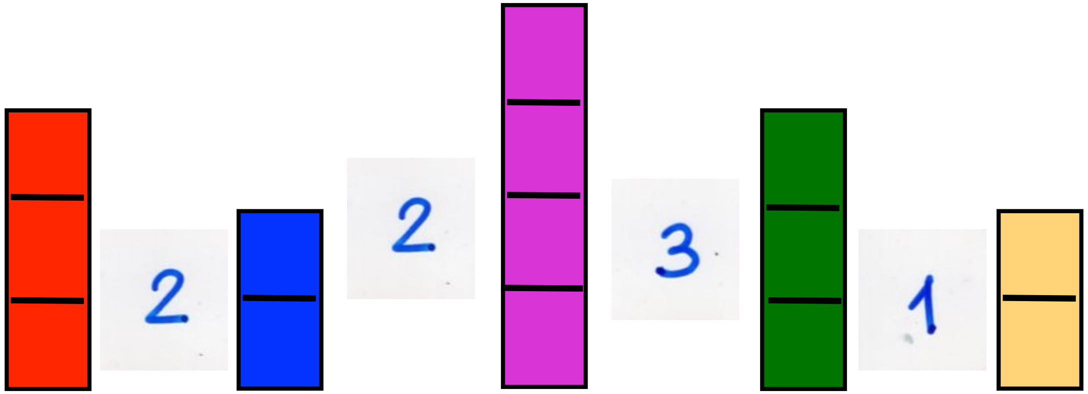
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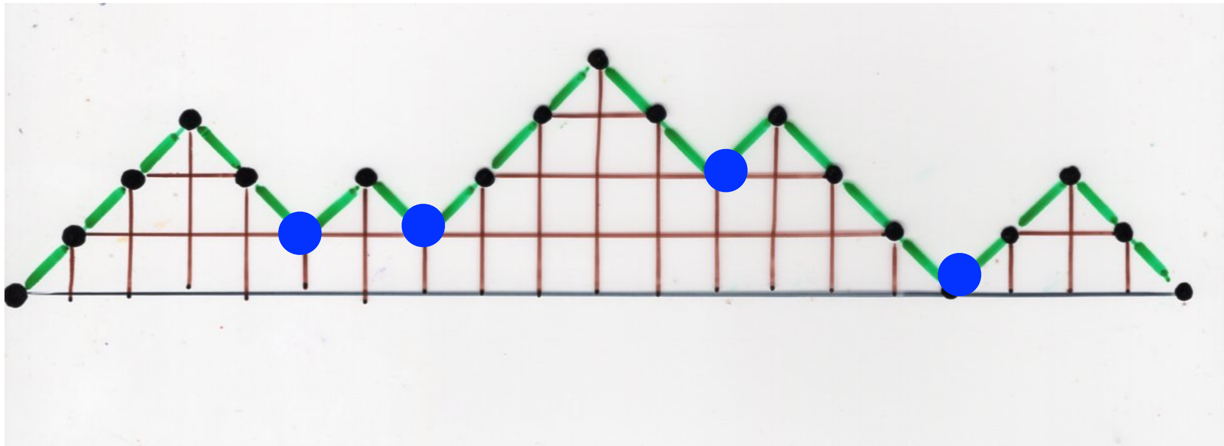
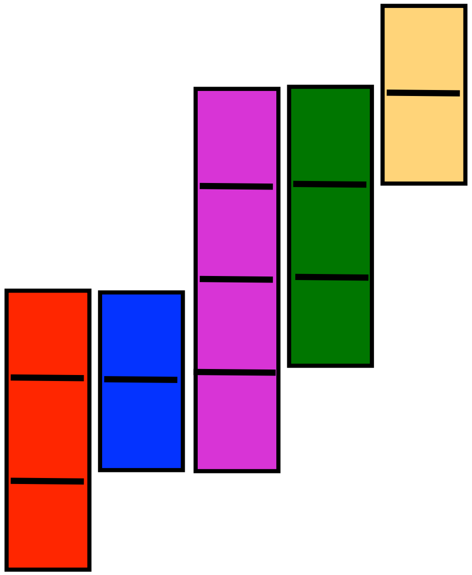
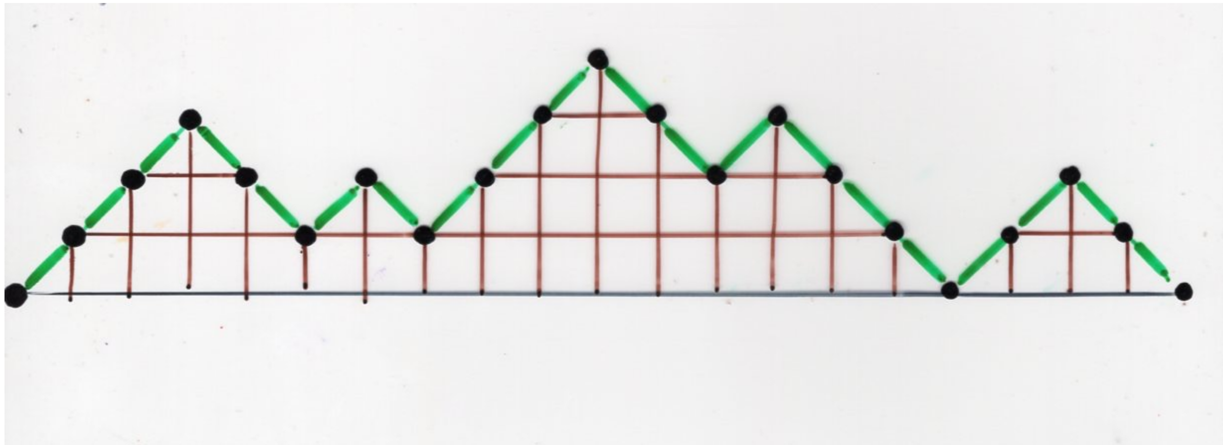
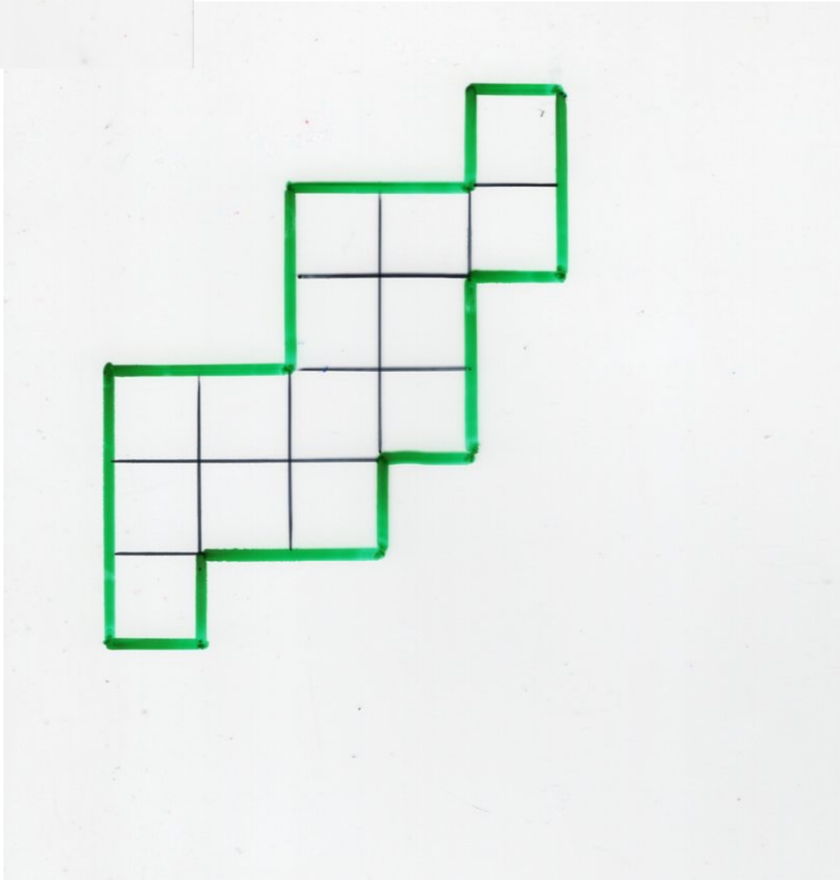








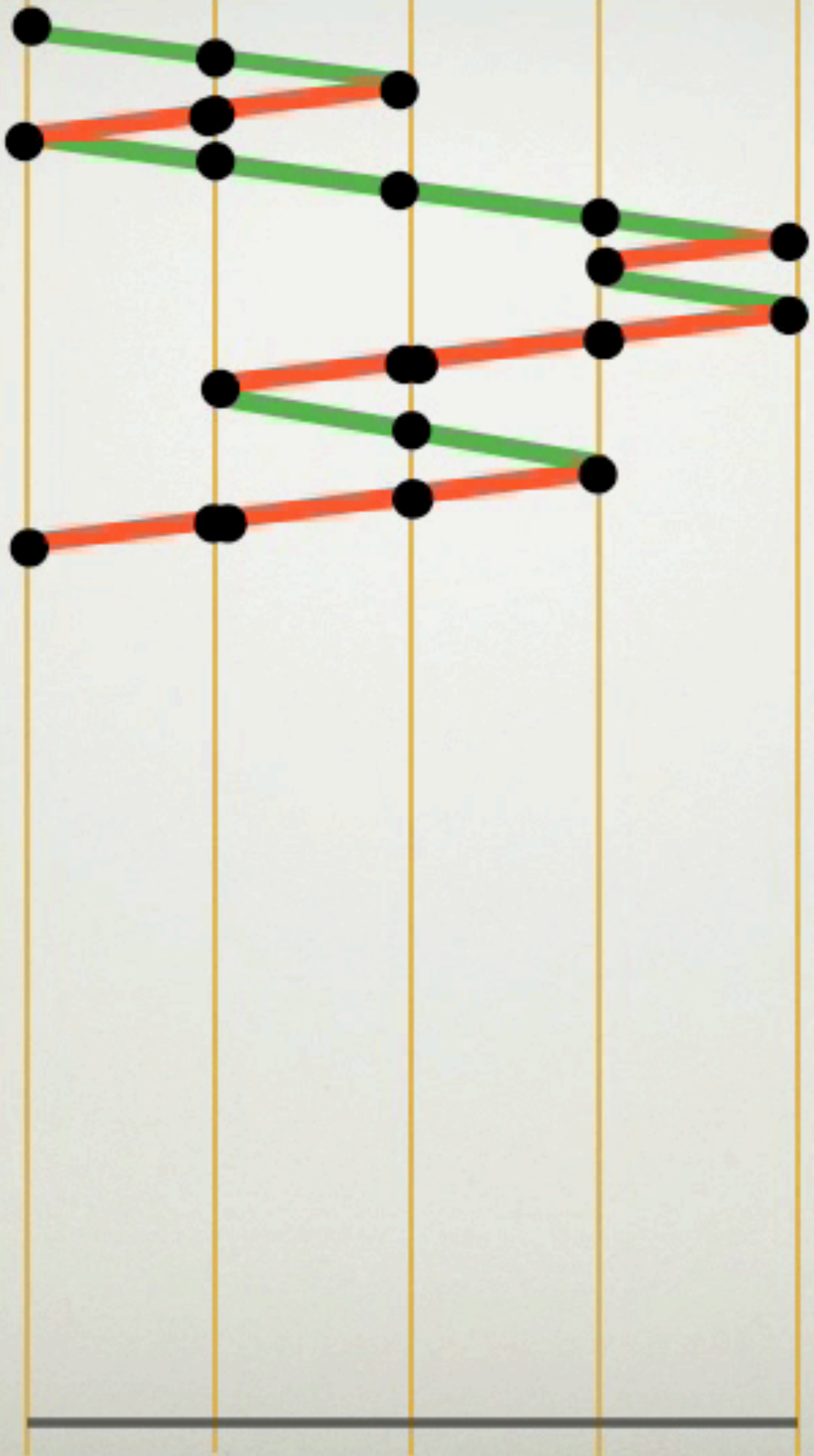
bijections

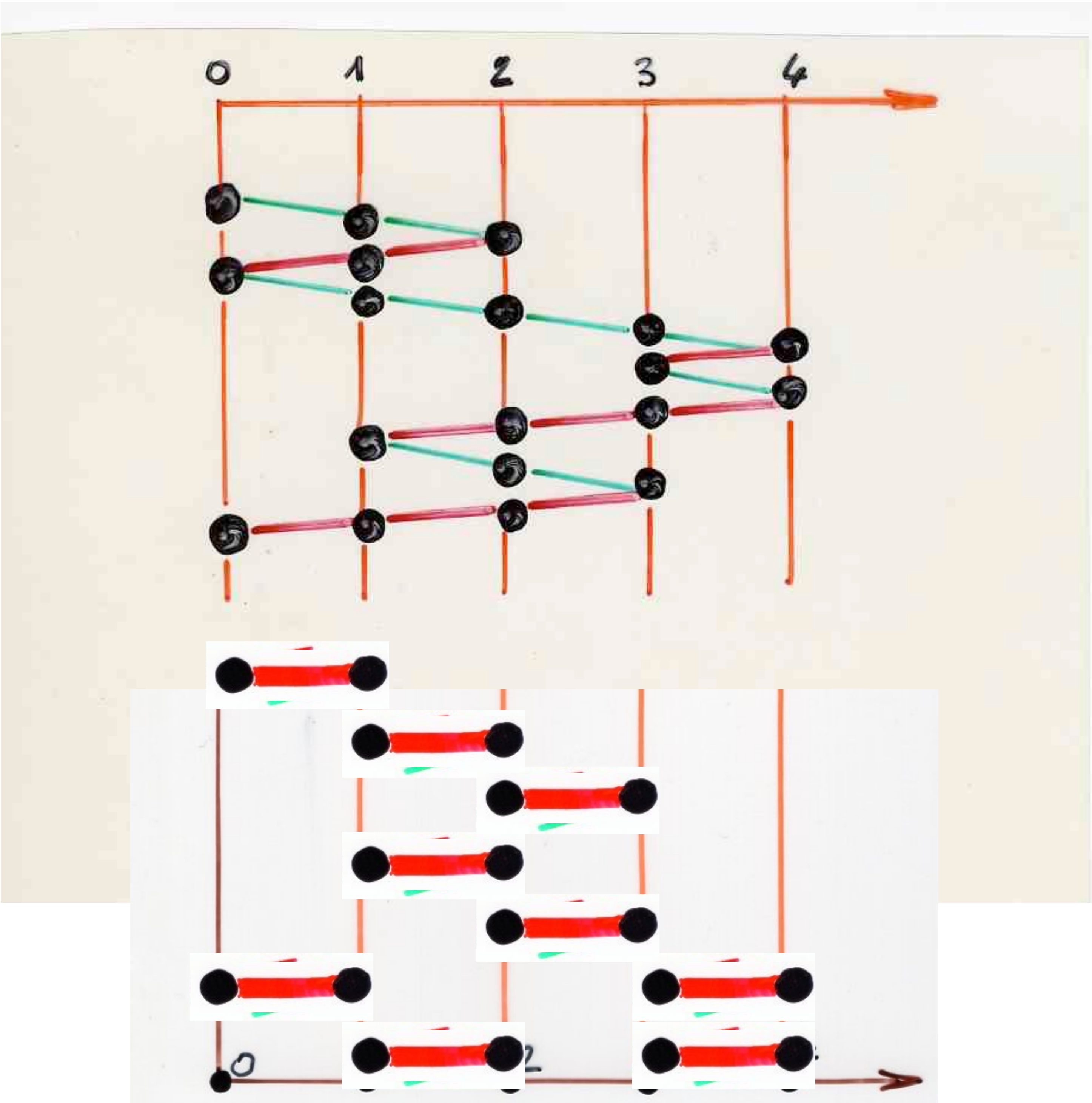
staircase polygons

Dyck paths

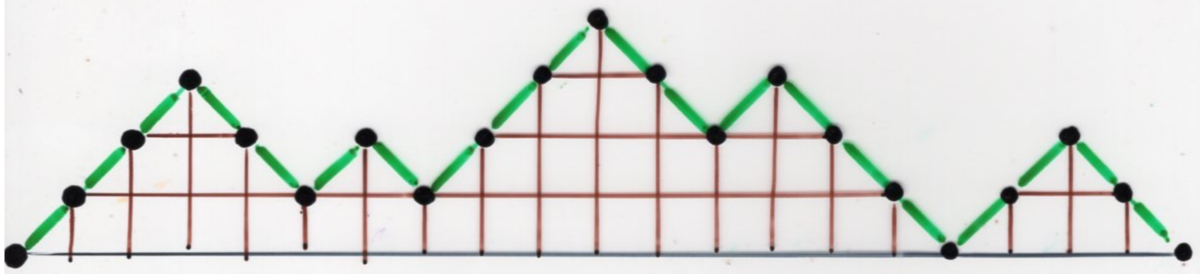
path ω on X $\overset{\chi}{\longleftrightarrow}$ (η, E)

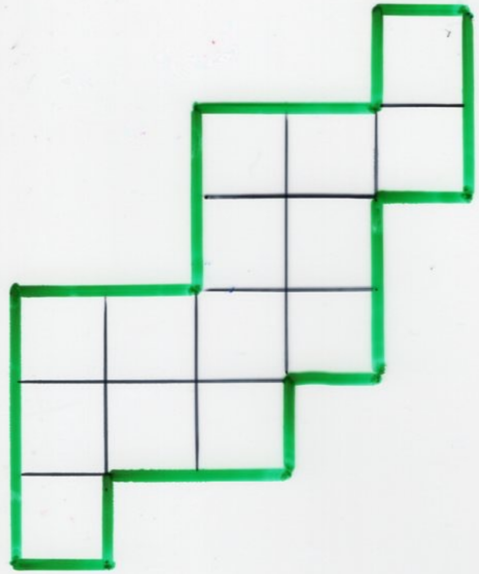
semi-pyramids of dimers

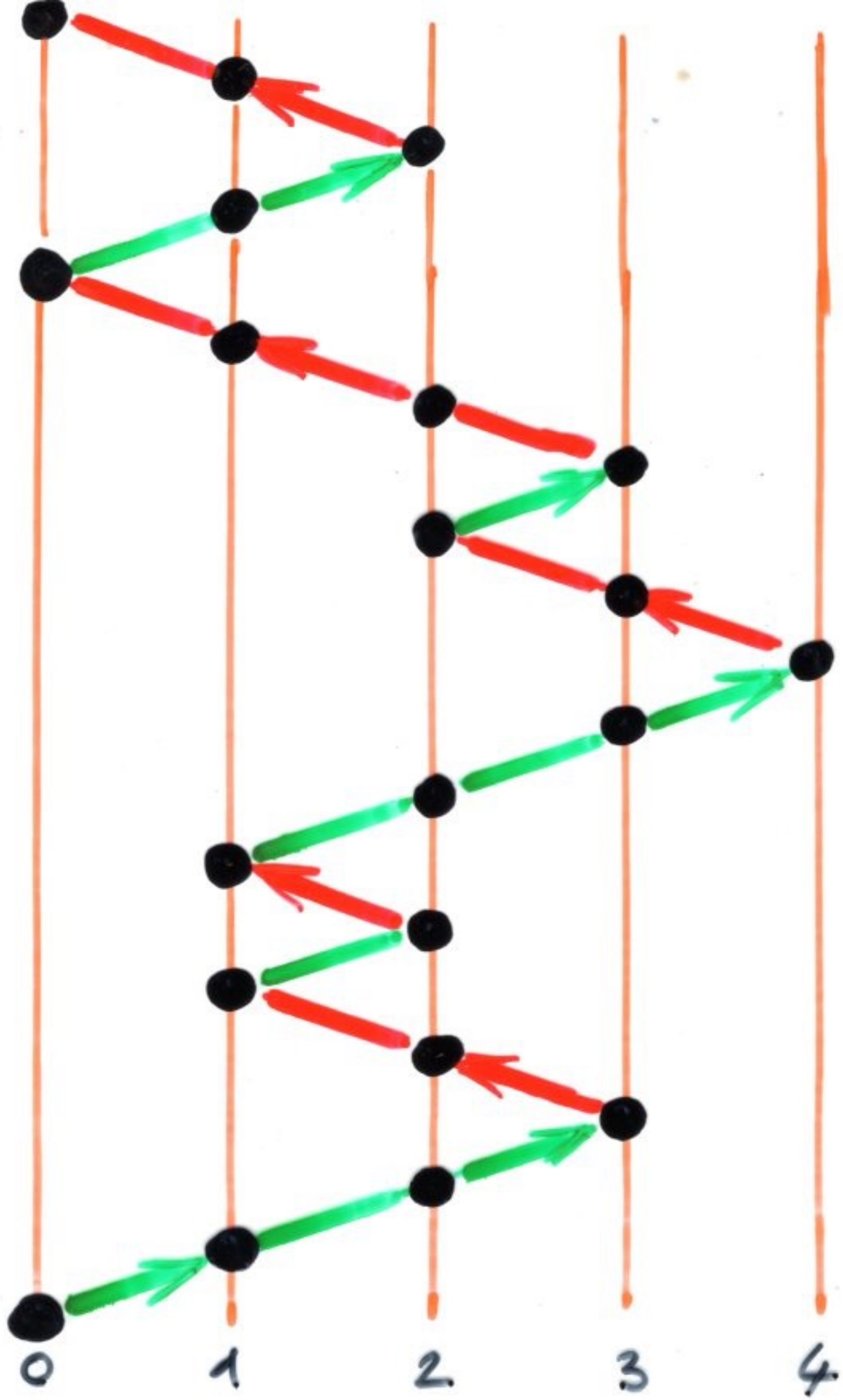




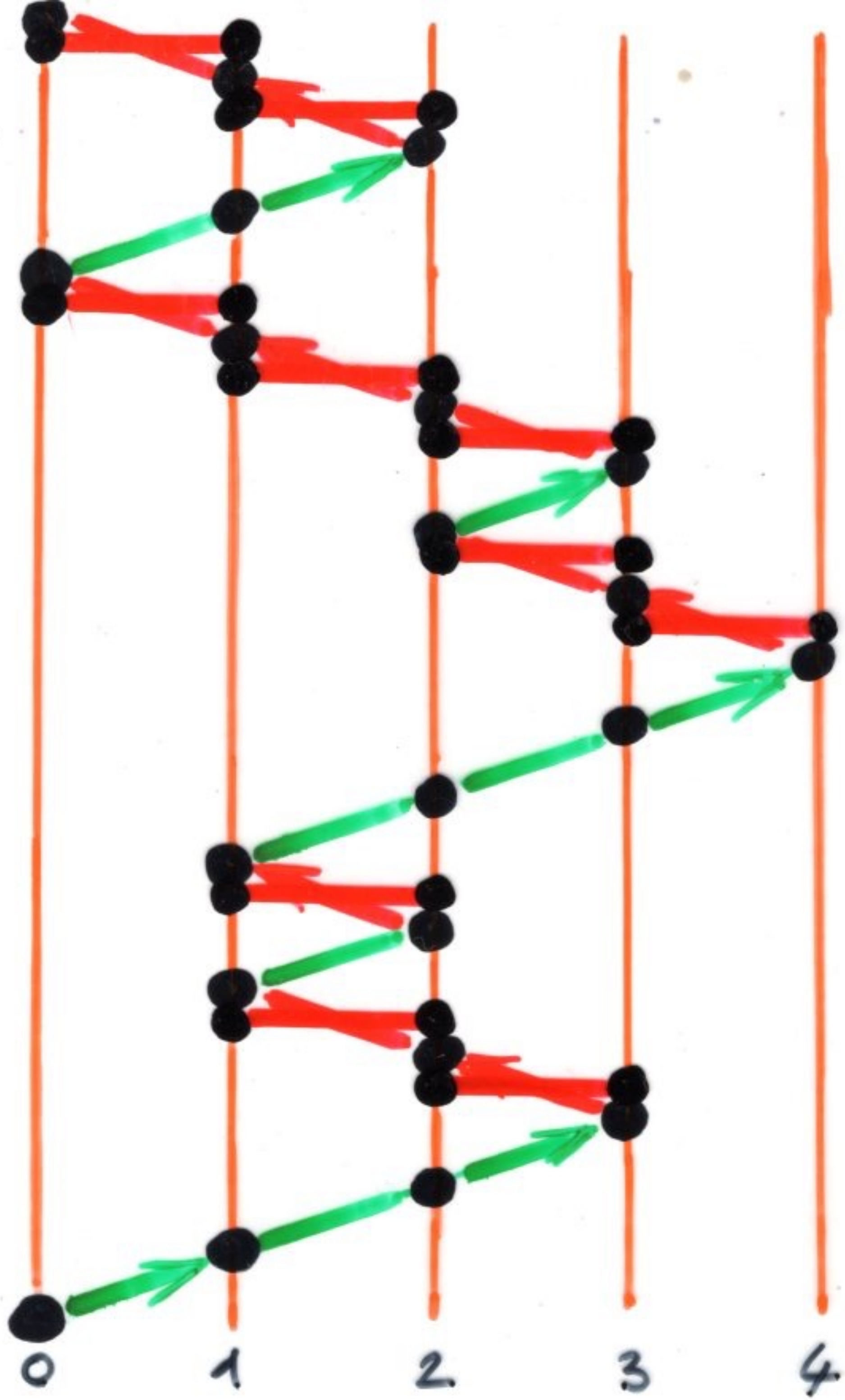
violin:
G. Duchamp



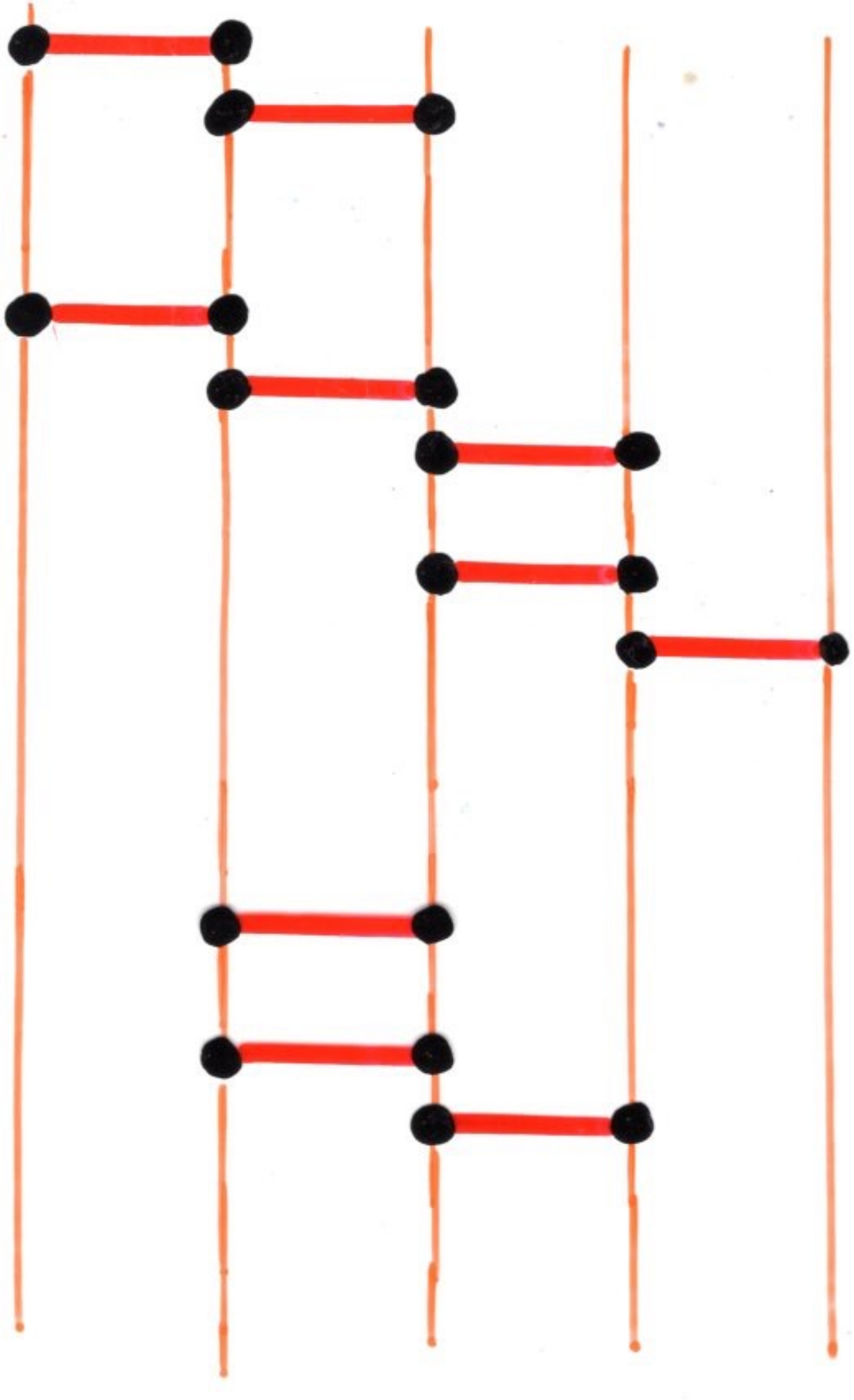




path ω on X $\overset{\chi}{\longleftrightarrow}$ (η, E)



path ω on X $\xleftrightarrow{\chi}$ (η, E)



path ω on X $\xleftrightarrow{\chi}$ (η, E)

bijections

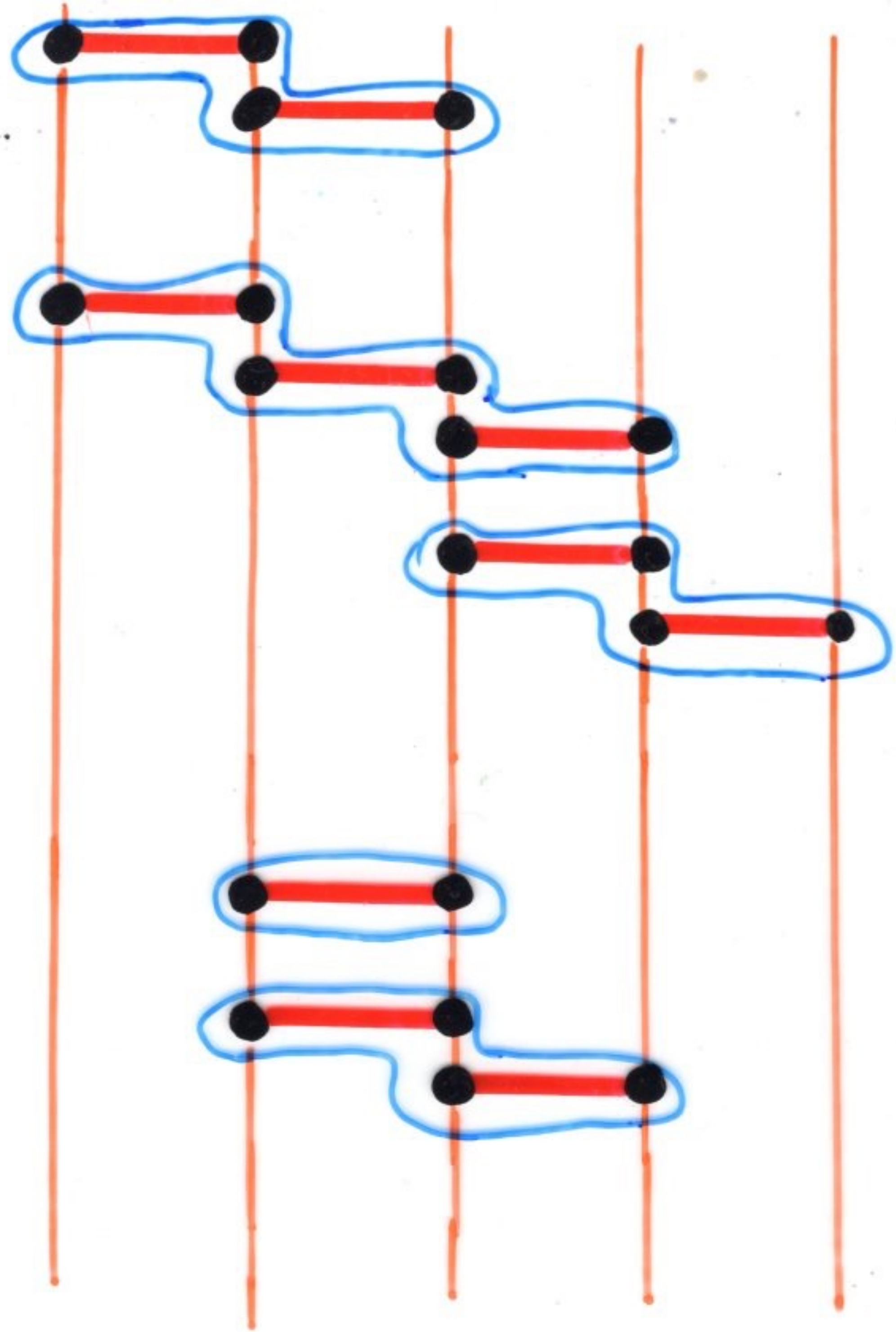
staircase polygons

Dyck paths

semi-pyramids of dimers

stair decomposition

Ch6a, p 50



bijections

staircase polygons

Dyck paths

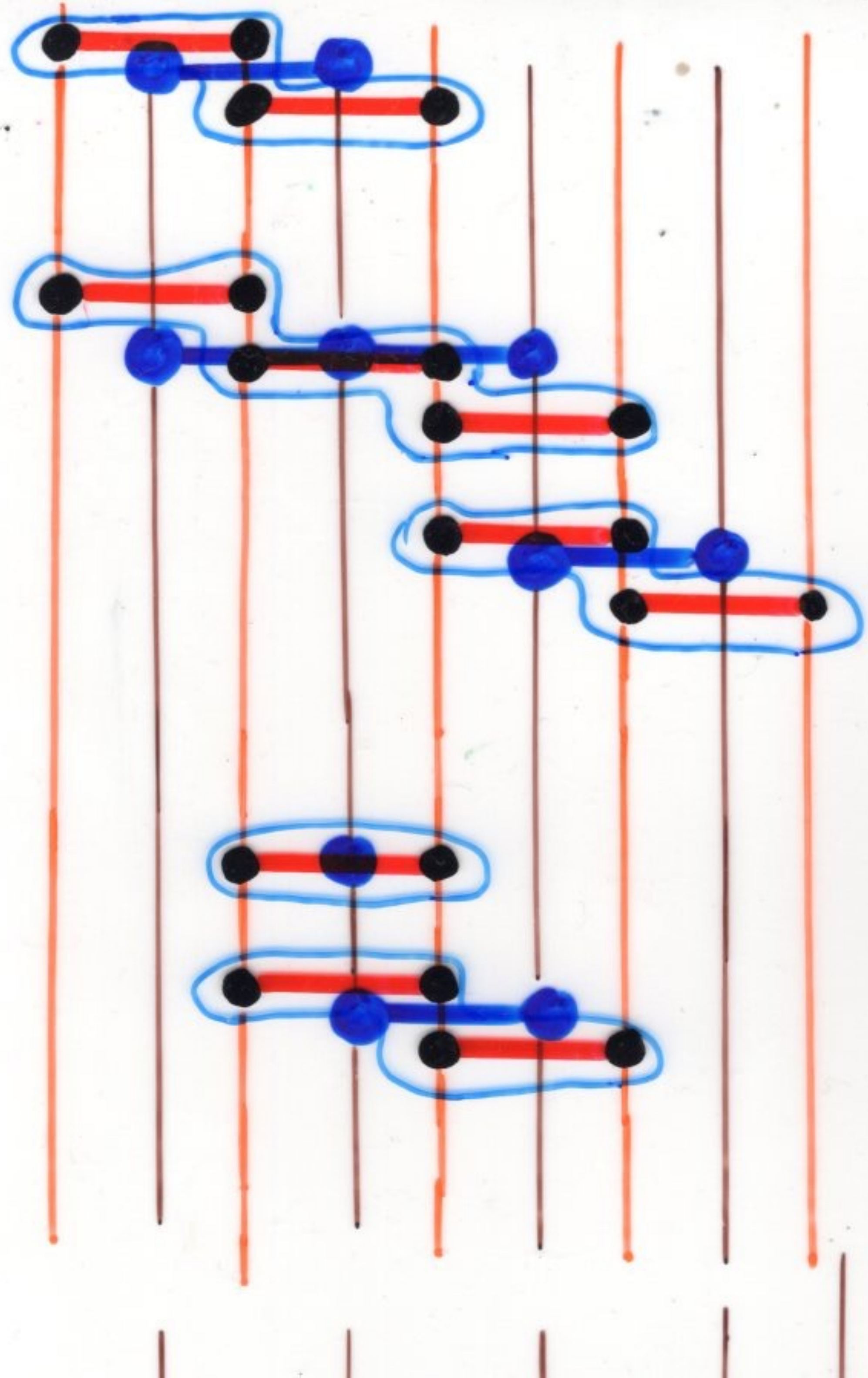
semi-pyramids of dimers

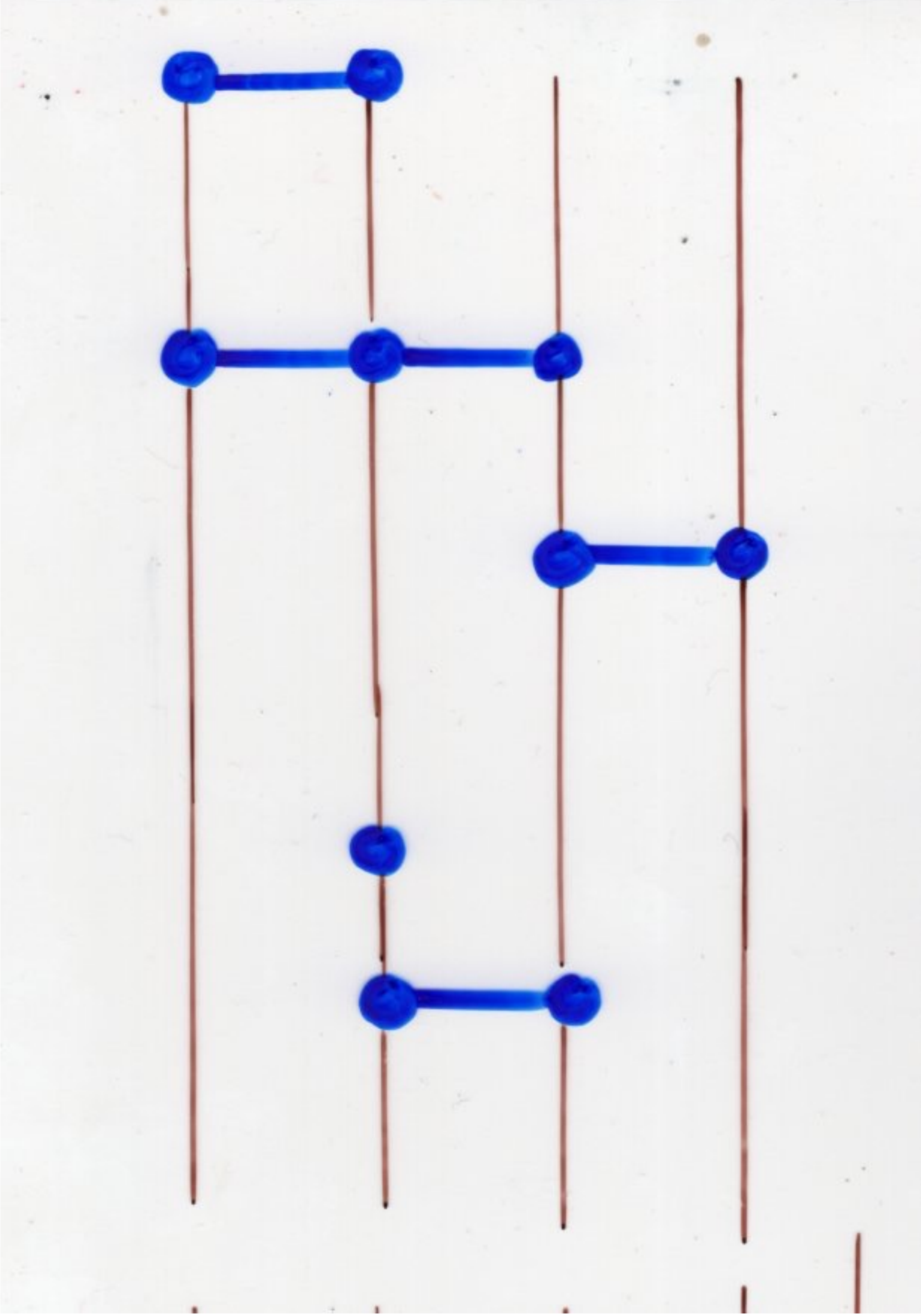
stair decomposition

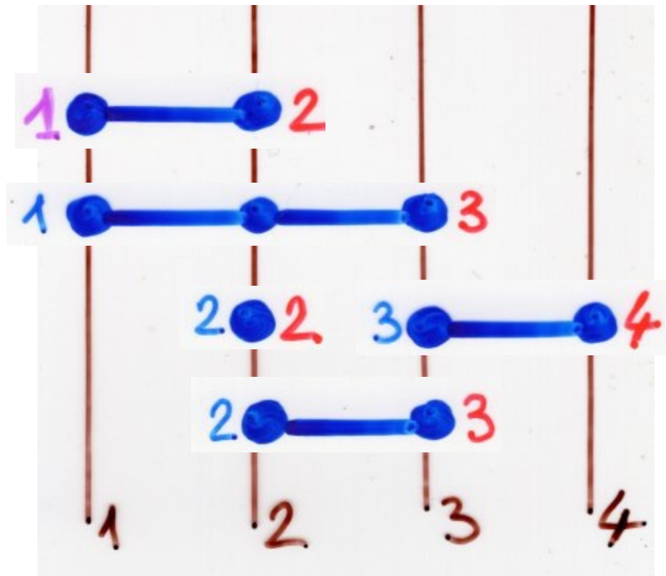
Ch6a, p 50

semi-pyramids of segments

Ch6a, p 55







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a festival of bijections

parallelogram polyominoes

(staircase polygons)

Dyck paths

semi-pyramids of dimers (on $\mathbb{N}$)

stairs decomposition

semi-pyramids of segments (on $\mathbb{N}$)

other description of the bijection:

2. with Lukasiewicz paths

Lukasiewicz path

$\omega = (s_0, \dots, s_n)$

$s_0 = (0,0)$, $s_n = (n, 0)$

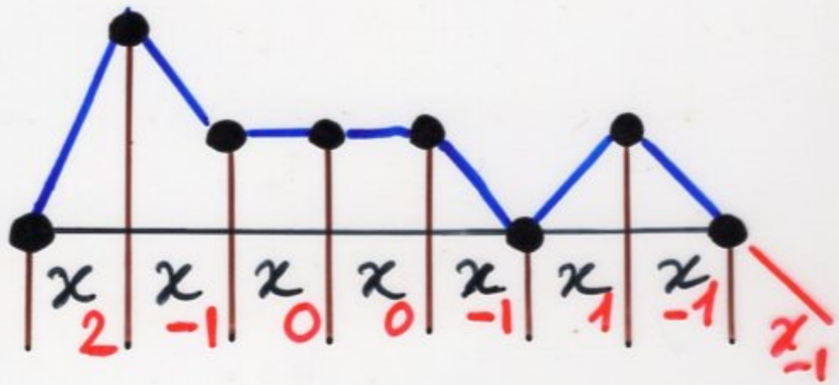
elementary step $s_i = (x_i, y_i)$ $s_{i+1} = (x_{i+1}, y_{i+1})$

with

$x_{i+1} = 1 + x_i$ $y_{i+1} \geqslant y_i - 1$

Ch 2a (IMSc 2016)

p 60



Ch2a, course 2016, p 60-63

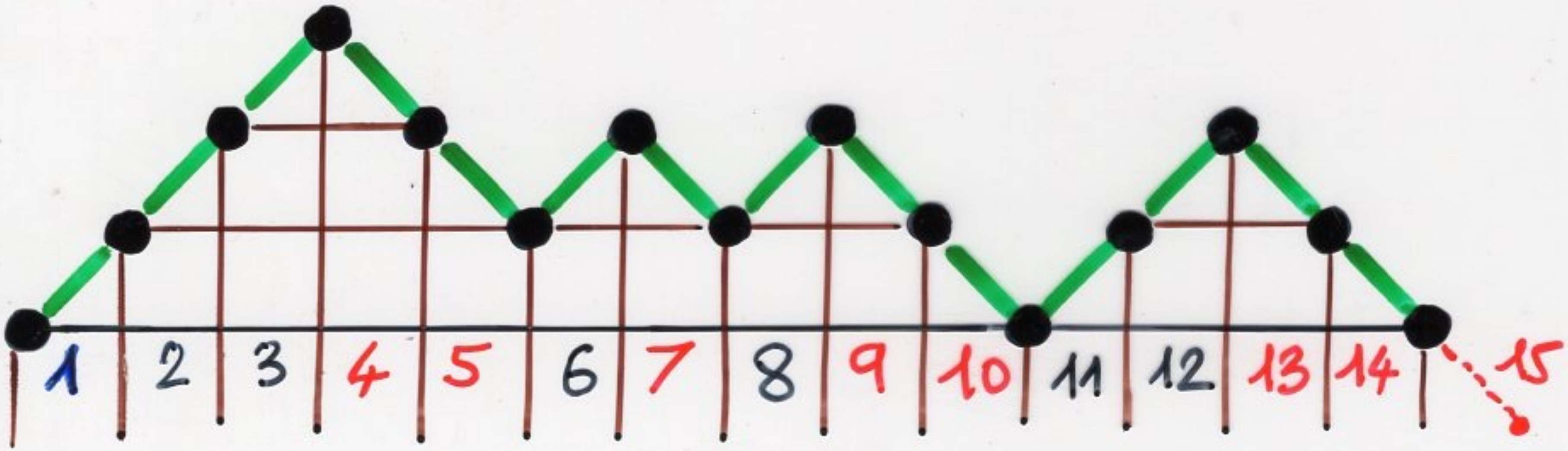
bijection

Dyck paths

Ch2a (IMSc 2016)
p60

Lukasiewicz paths

The Catalan garden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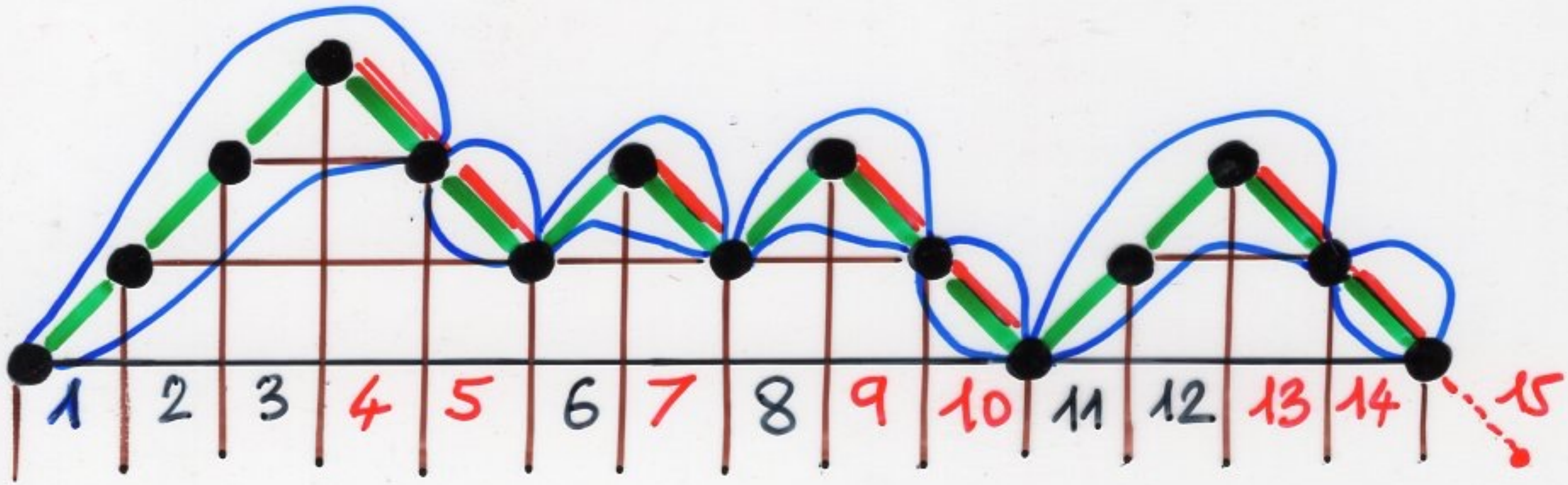
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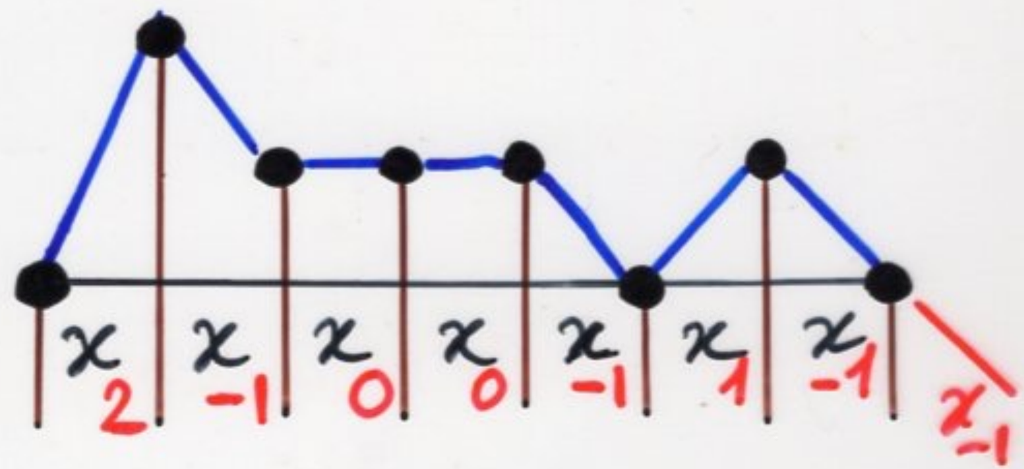
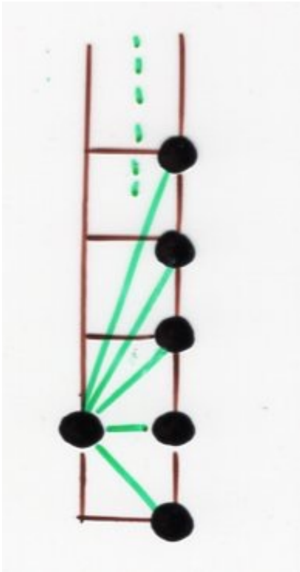
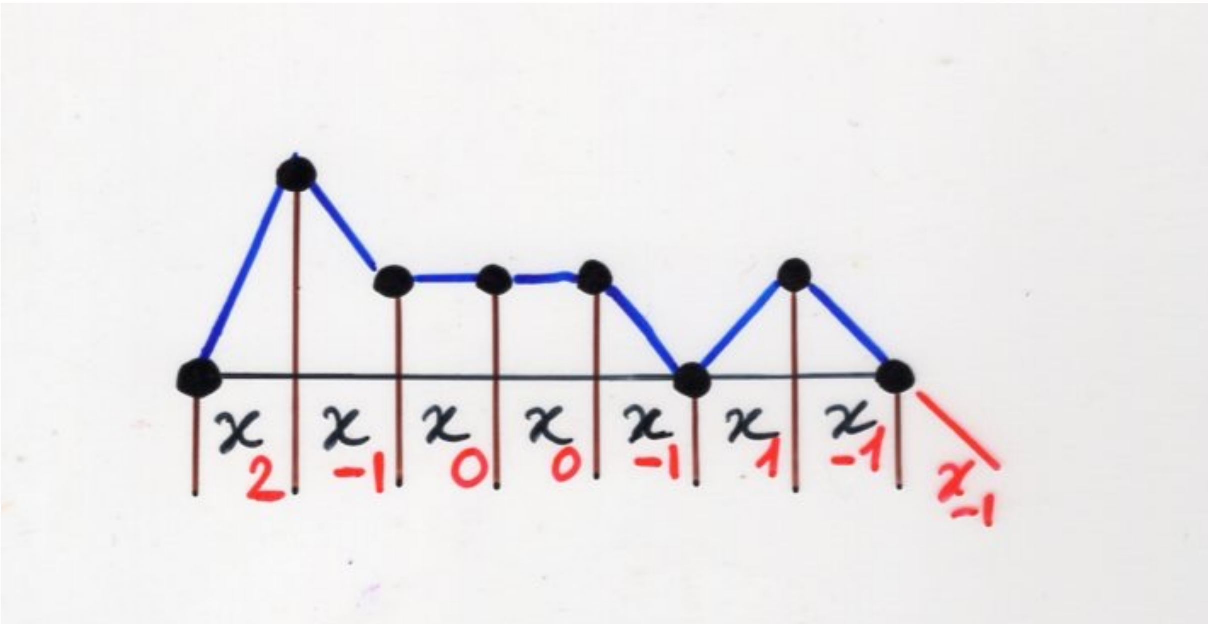
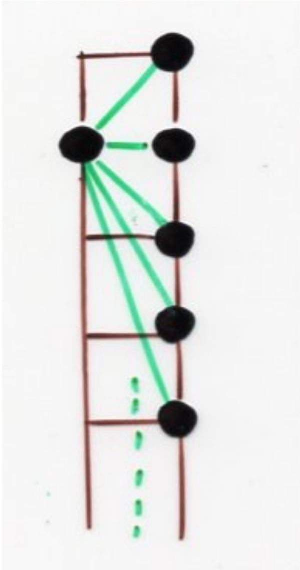


x_2
x_{-1}
x_0
x_0
x_{-1}
x_1
x_{-1}
x_{-1}

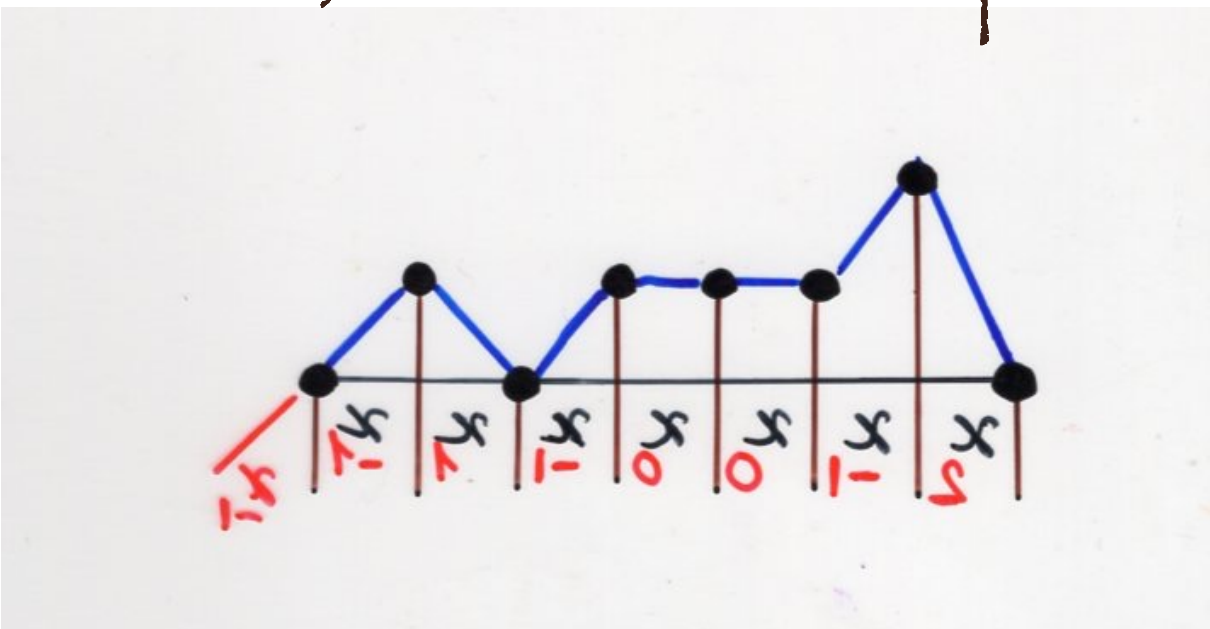




x_2 x_{-1} x_0 x_0 x_{-1} x_1 x_{-1} x_{-1}



(reverse) Lukasiewicz paths



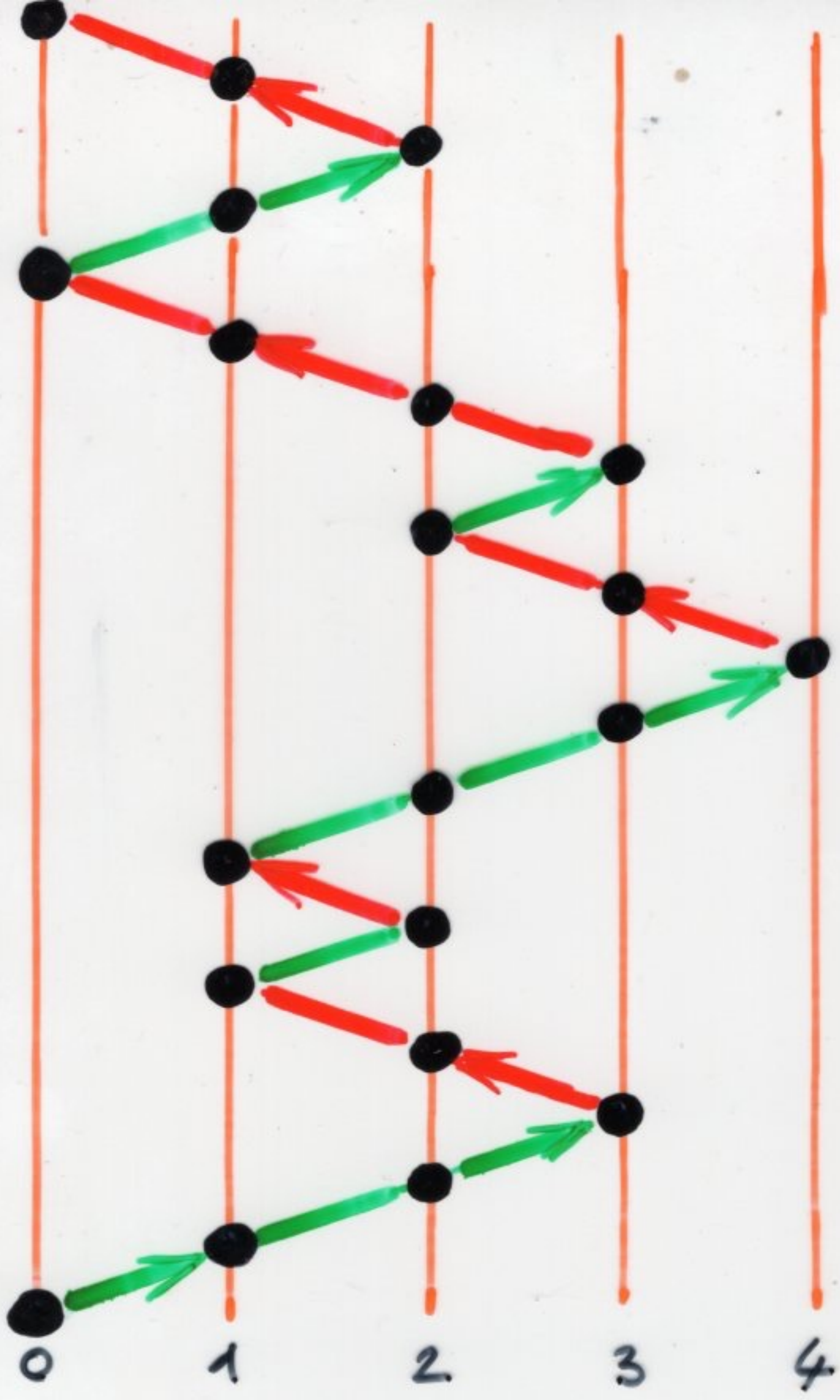
(reverse) Lukasiewicz paths

bijections

staircase polygons

Dyck paths

(reverse) Lukasiewicz paths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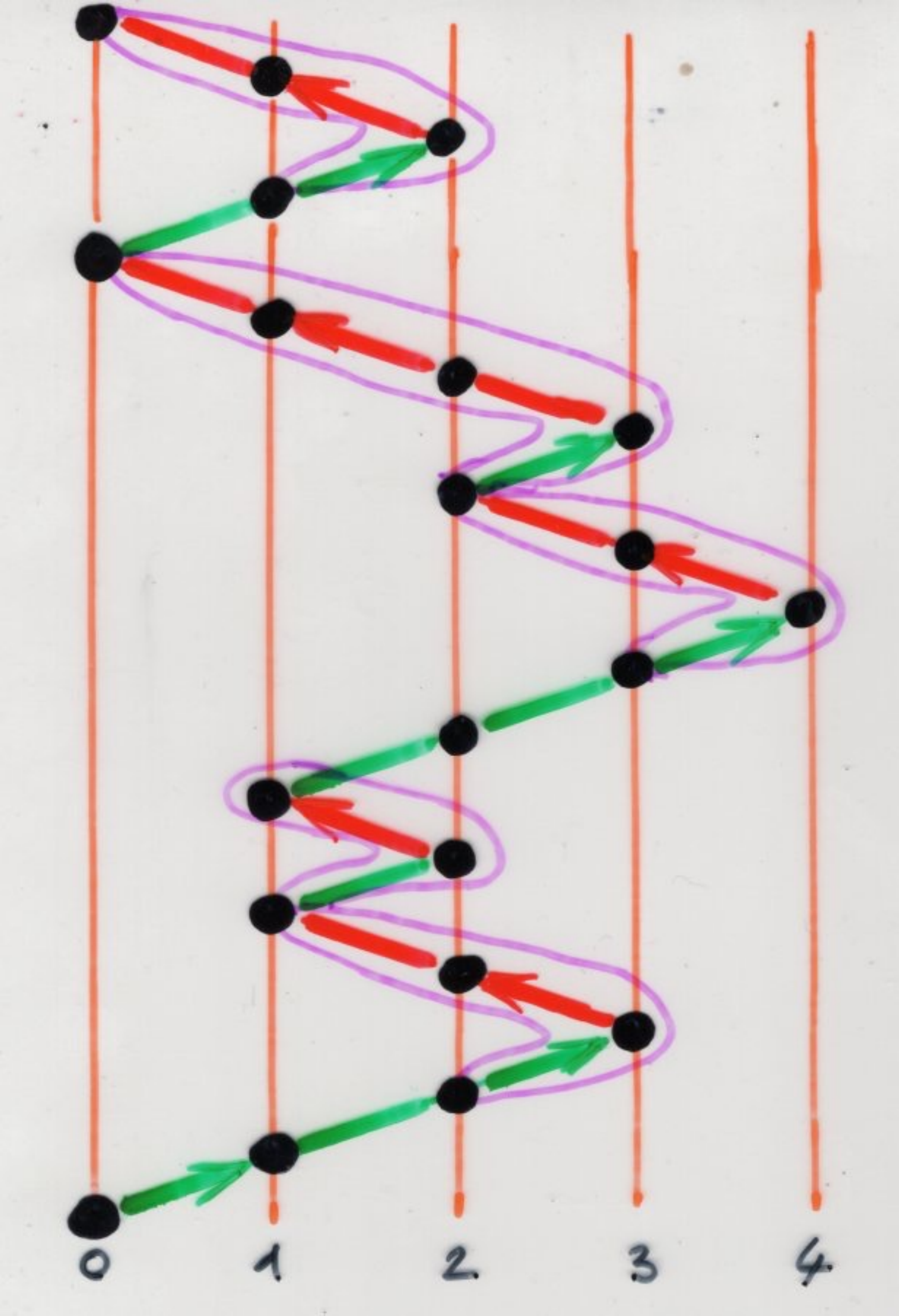
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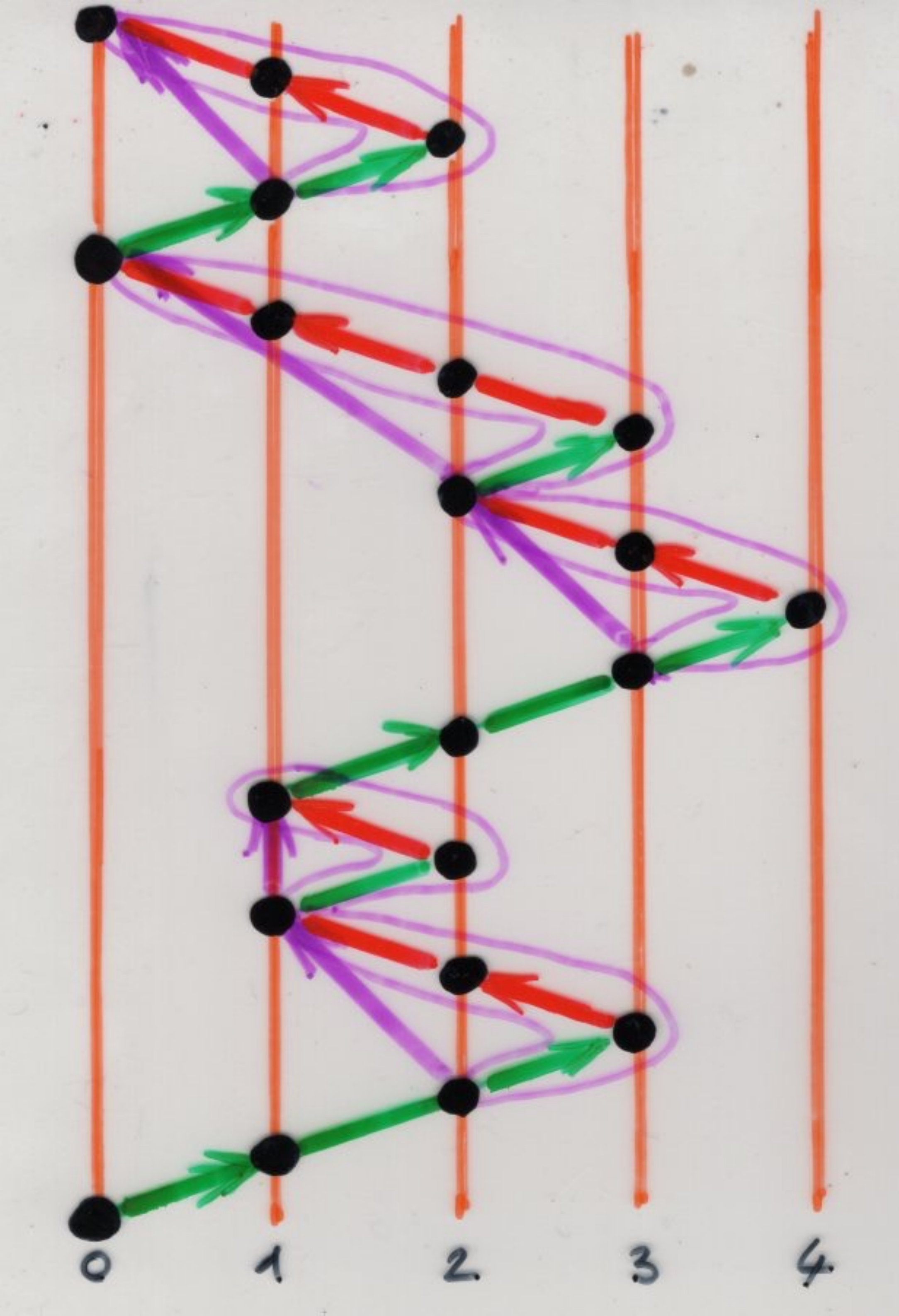

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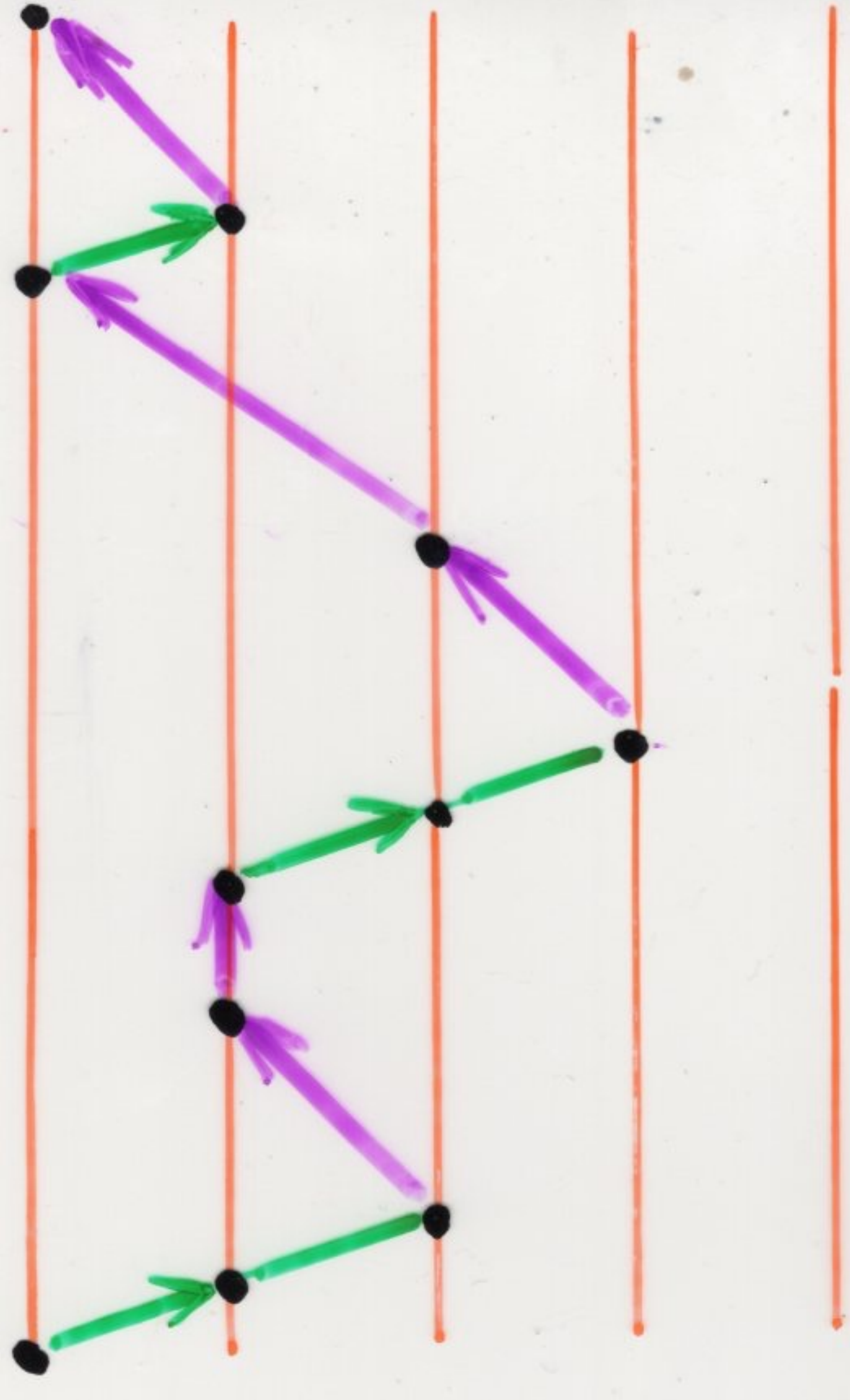


bijections

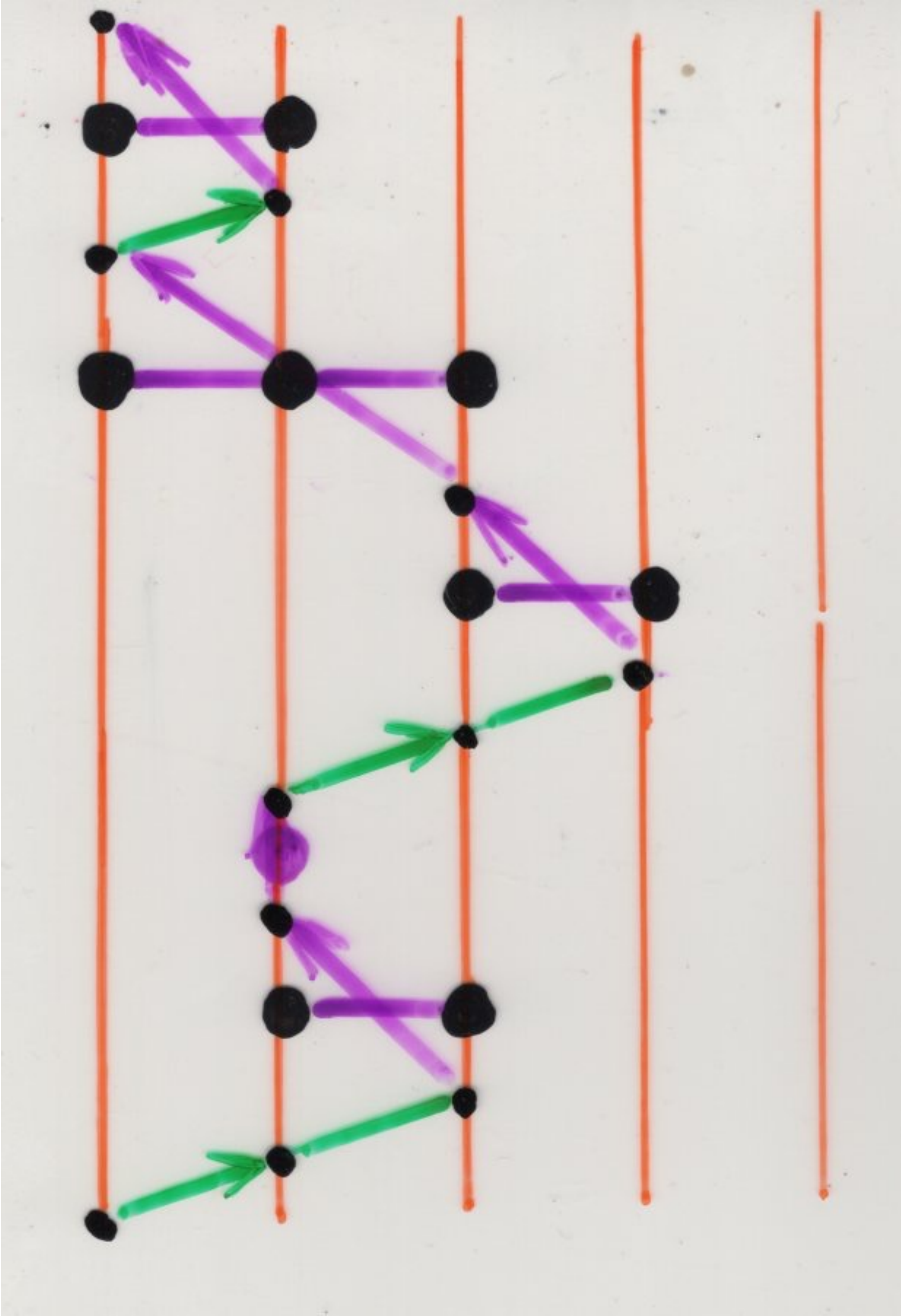
staircase polygons

Dyck paths

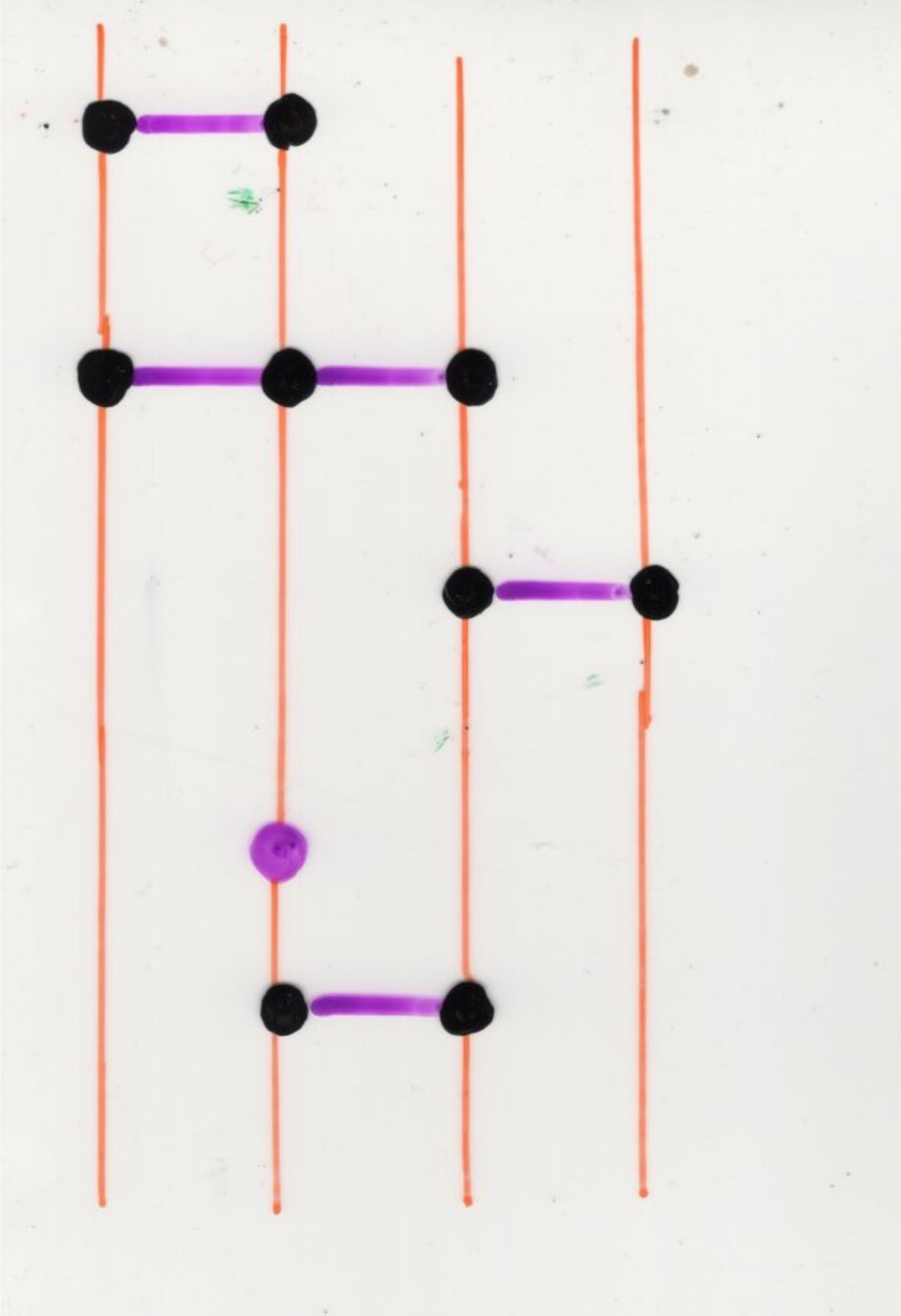
Lukasiewicz paths

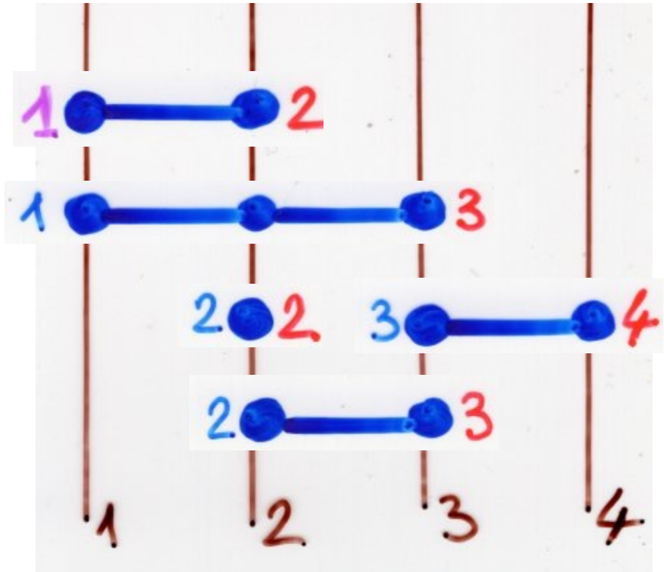
path ω on X $\xleftrightarrow{\chi}$ (η, E)

semi-pyramids of segments



path ω on X $\overset{\chi}{\longleftrightarrow}$ (η, E)





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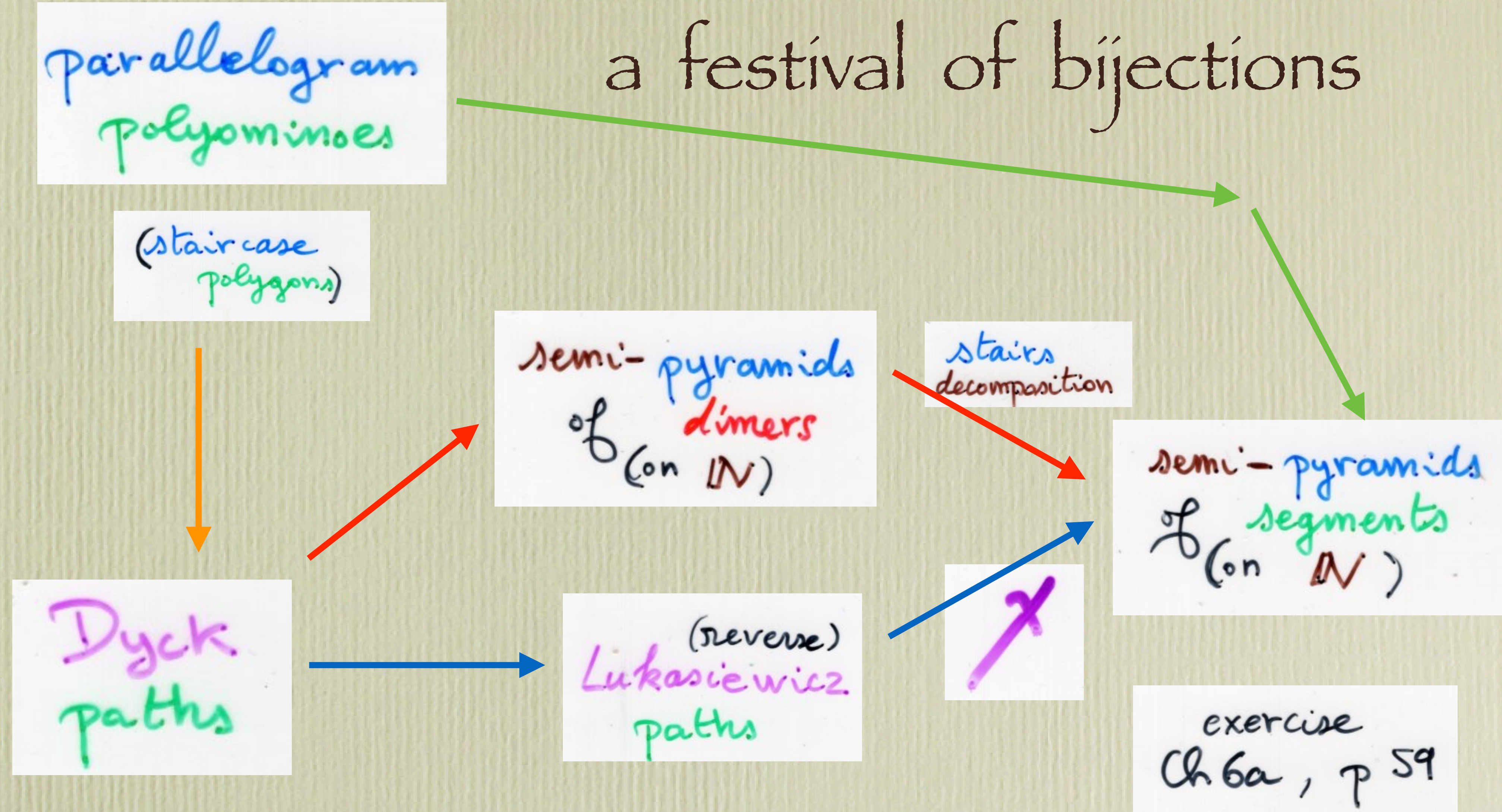
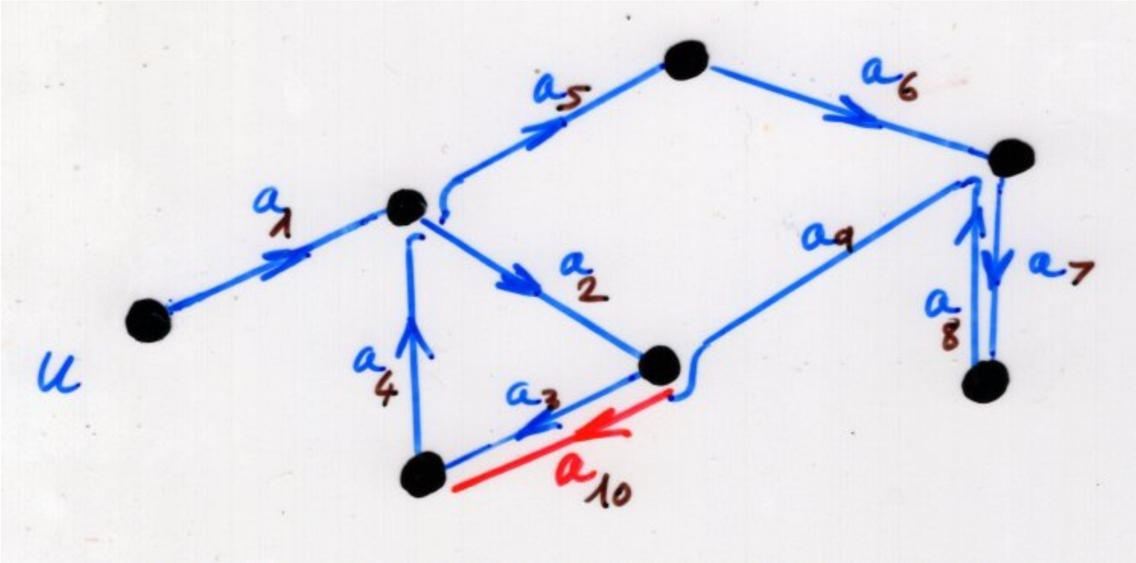
a festival of bijections



other description of the bijection:

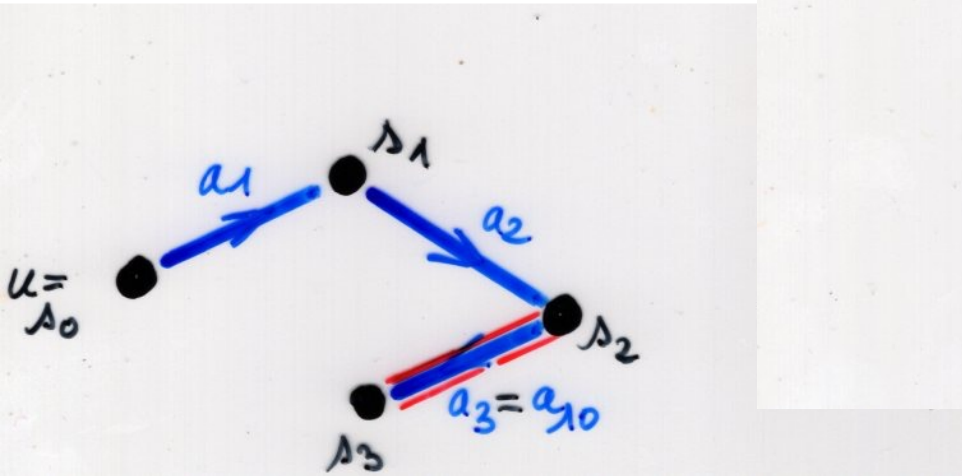
3. with the bijection
(paths — heaps of oriented loops)

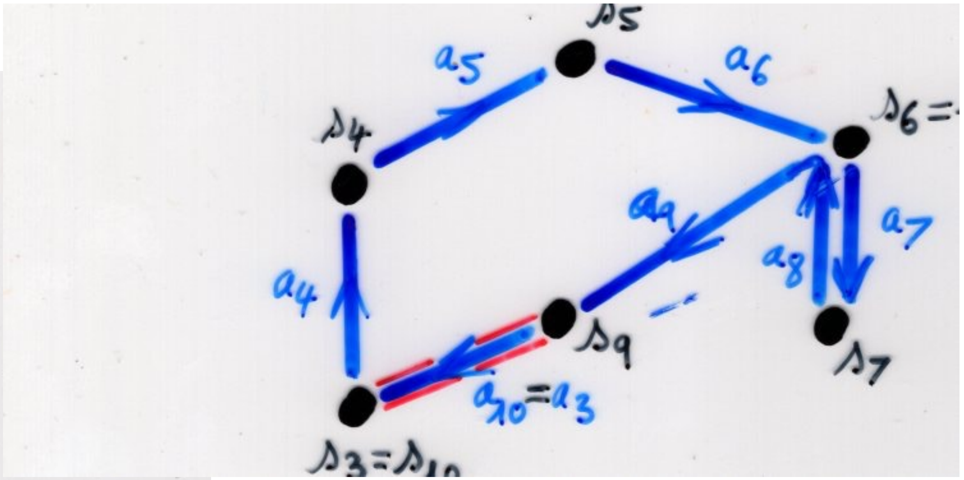
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

Ch 5b, p 21-29





bijections

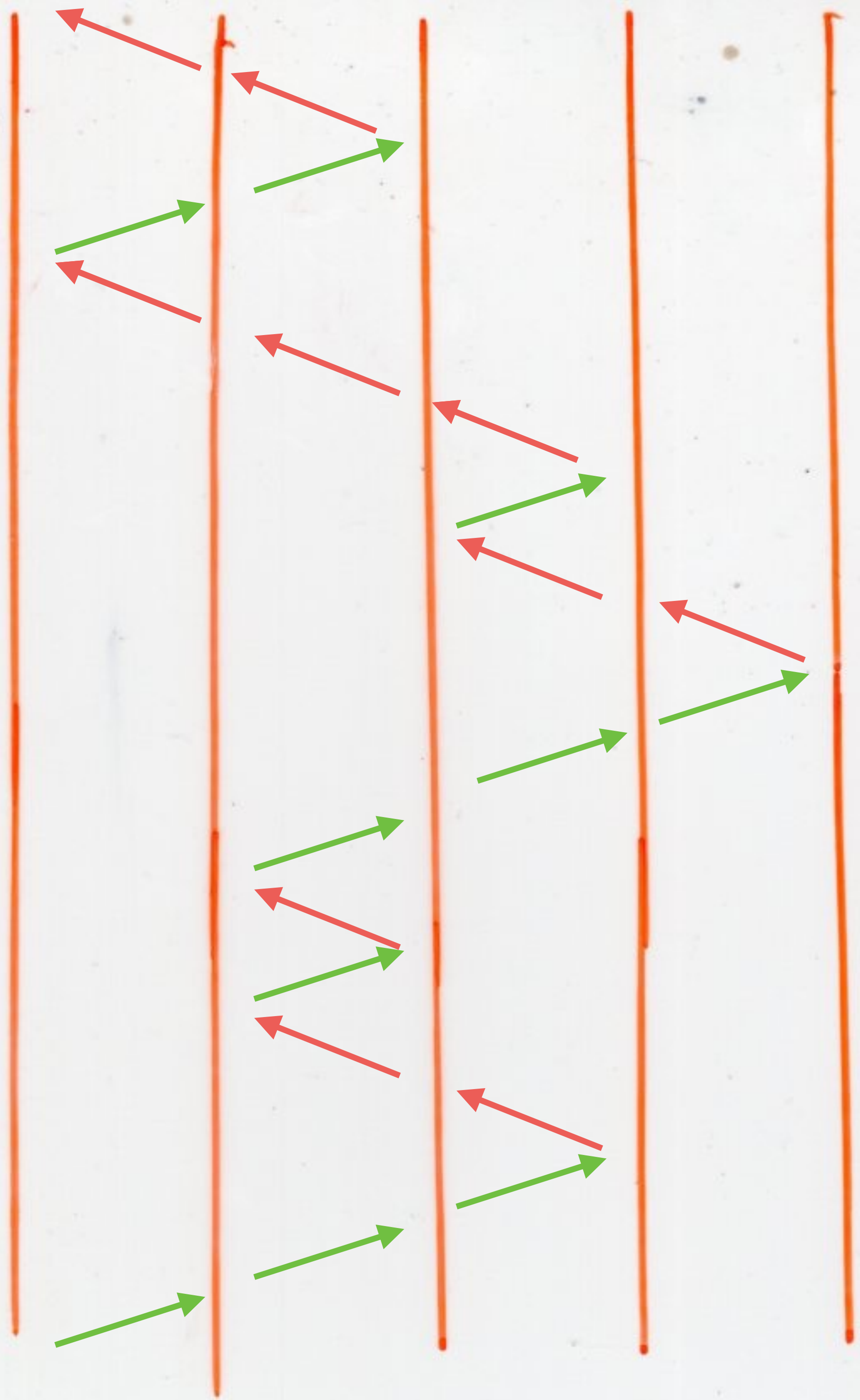
staircase polygons

Dyck paths

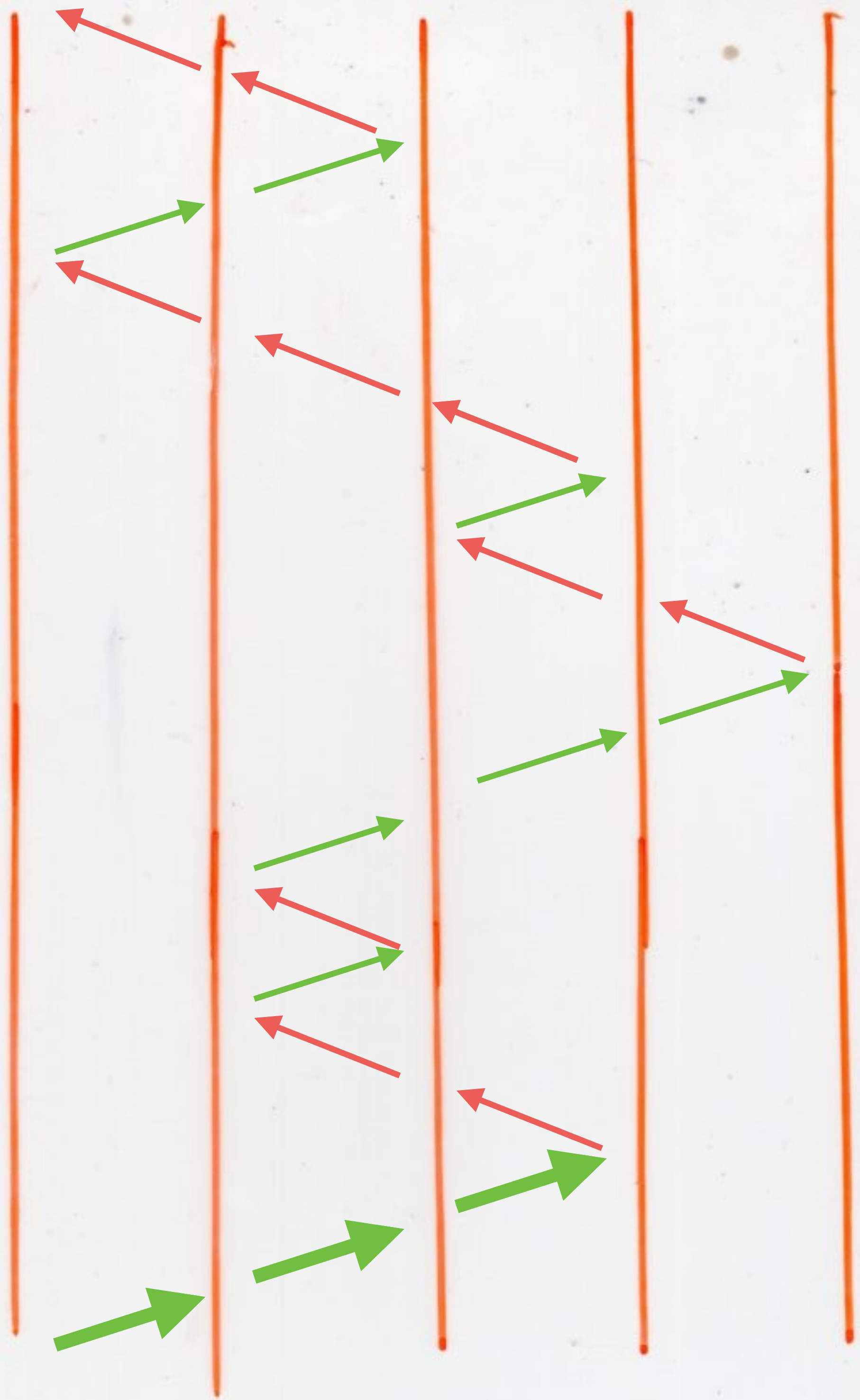
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

Ch 5b, p21-29

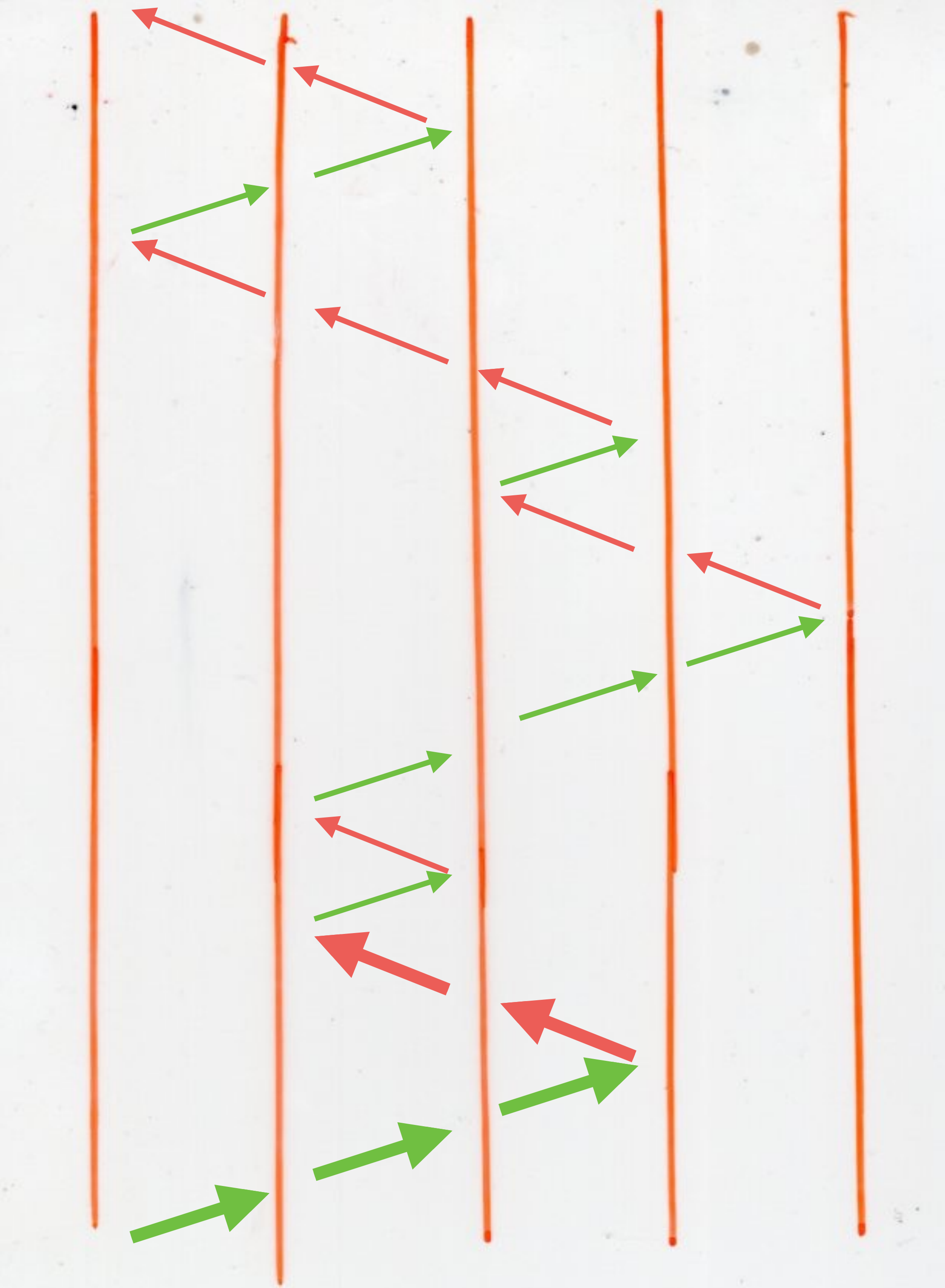
heaps of oriented loops



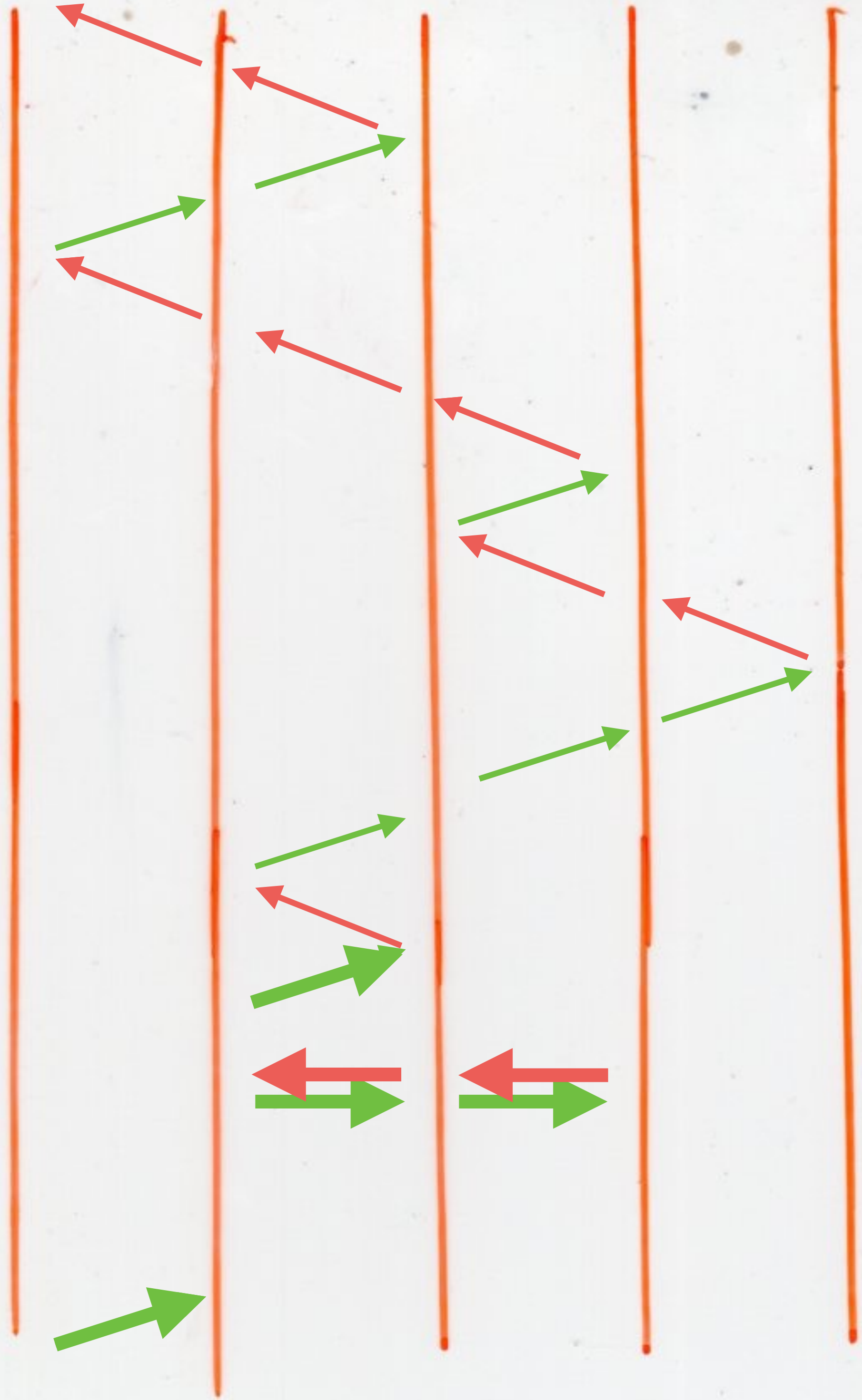
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



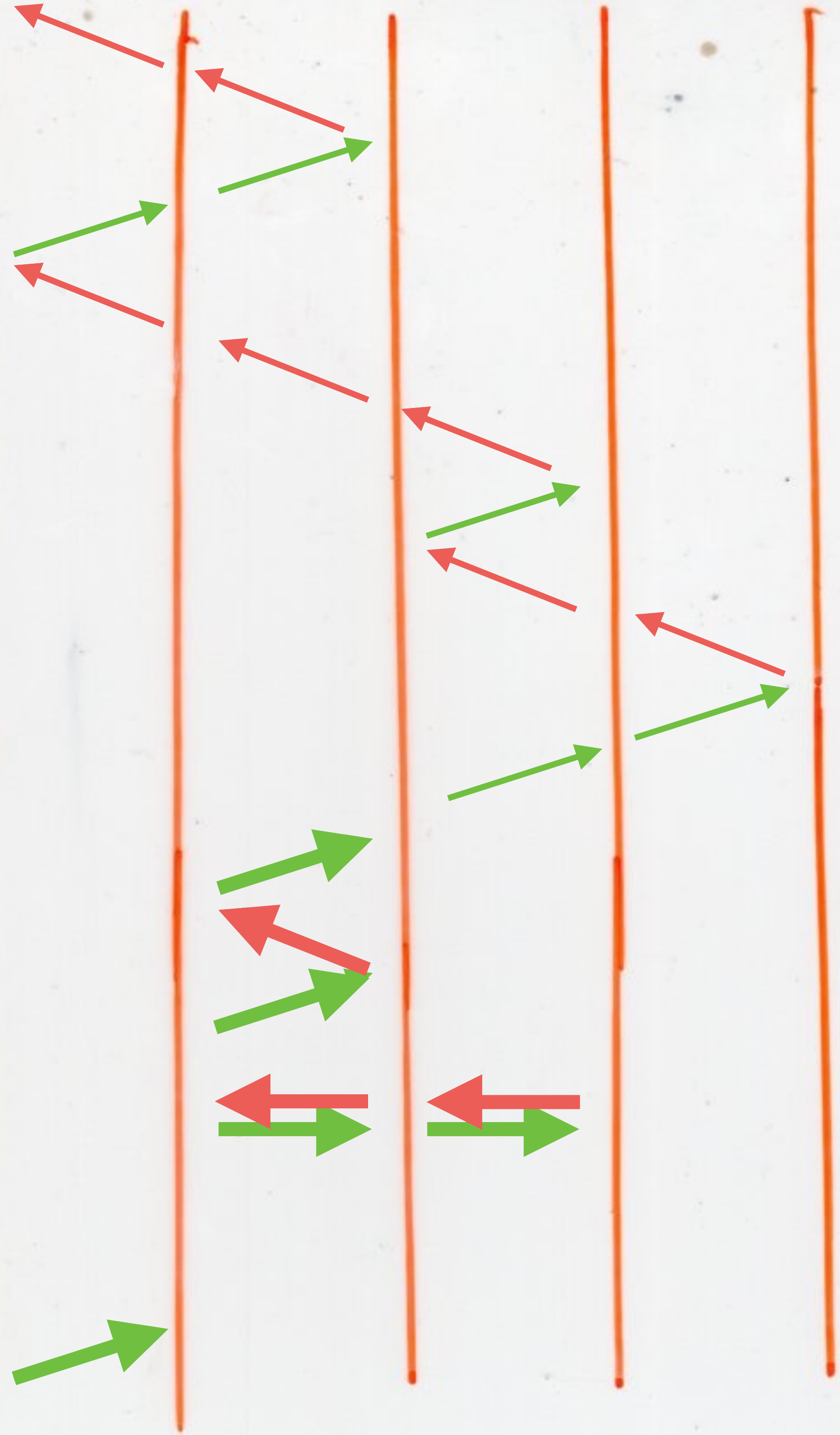
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



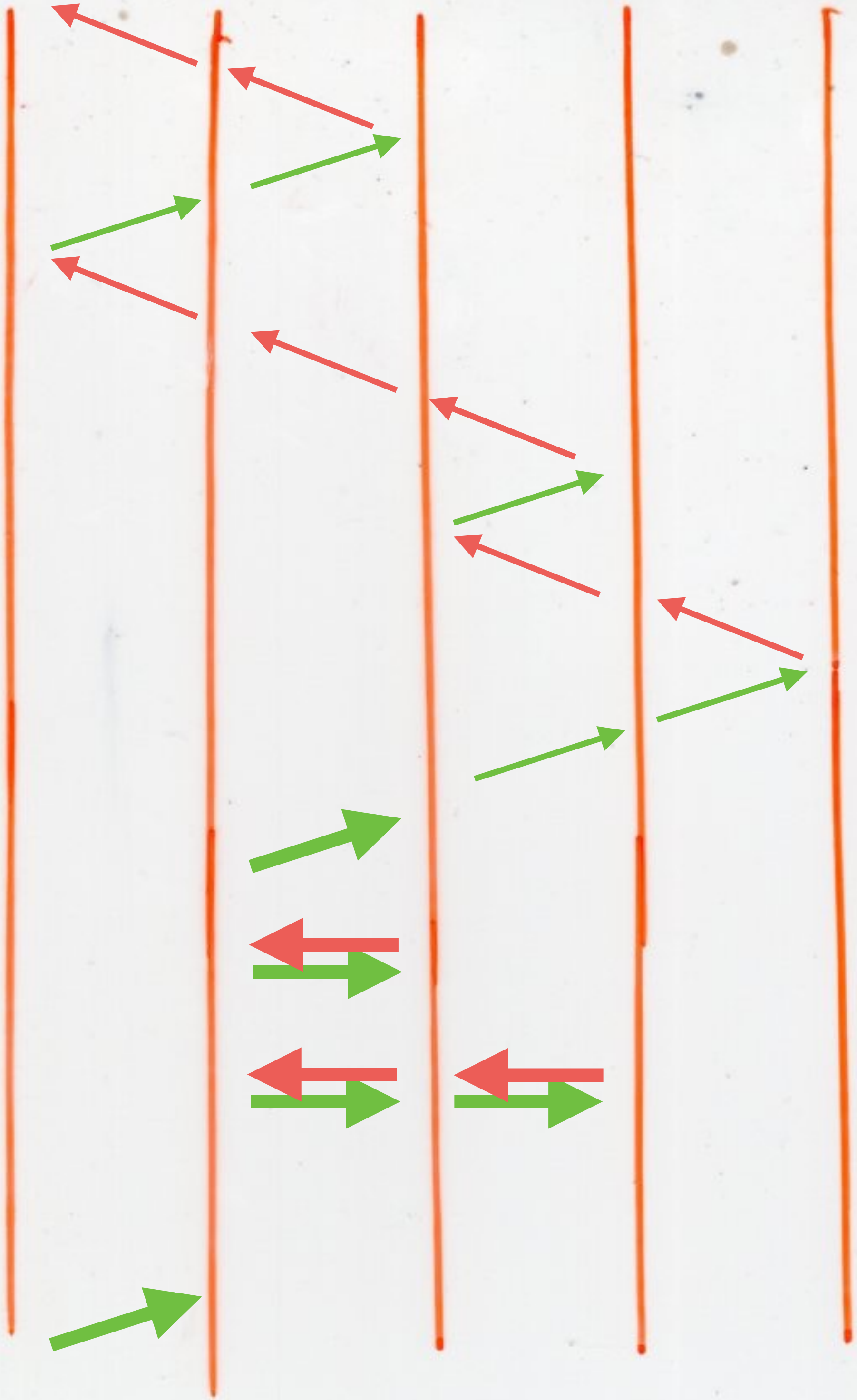
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



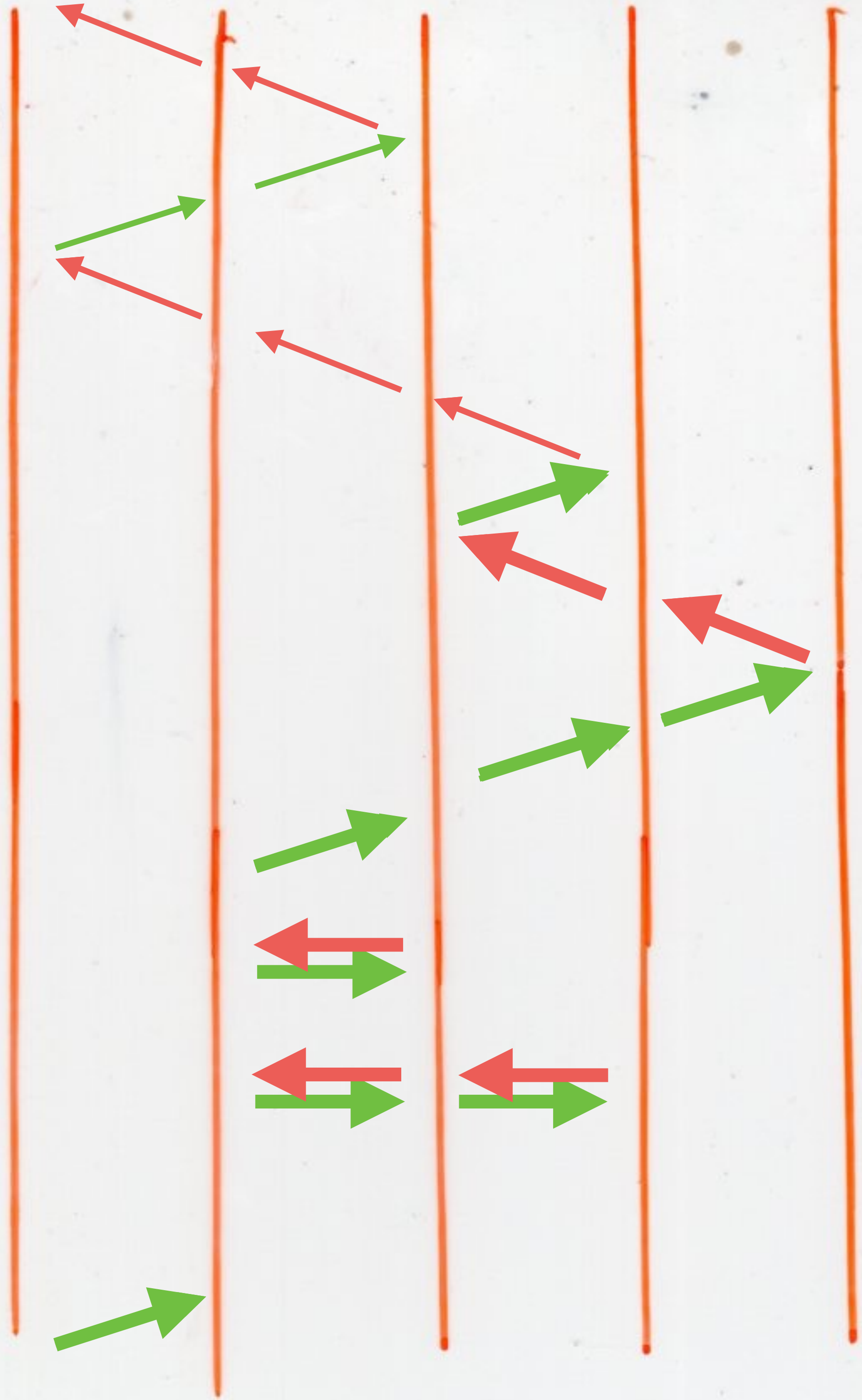
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



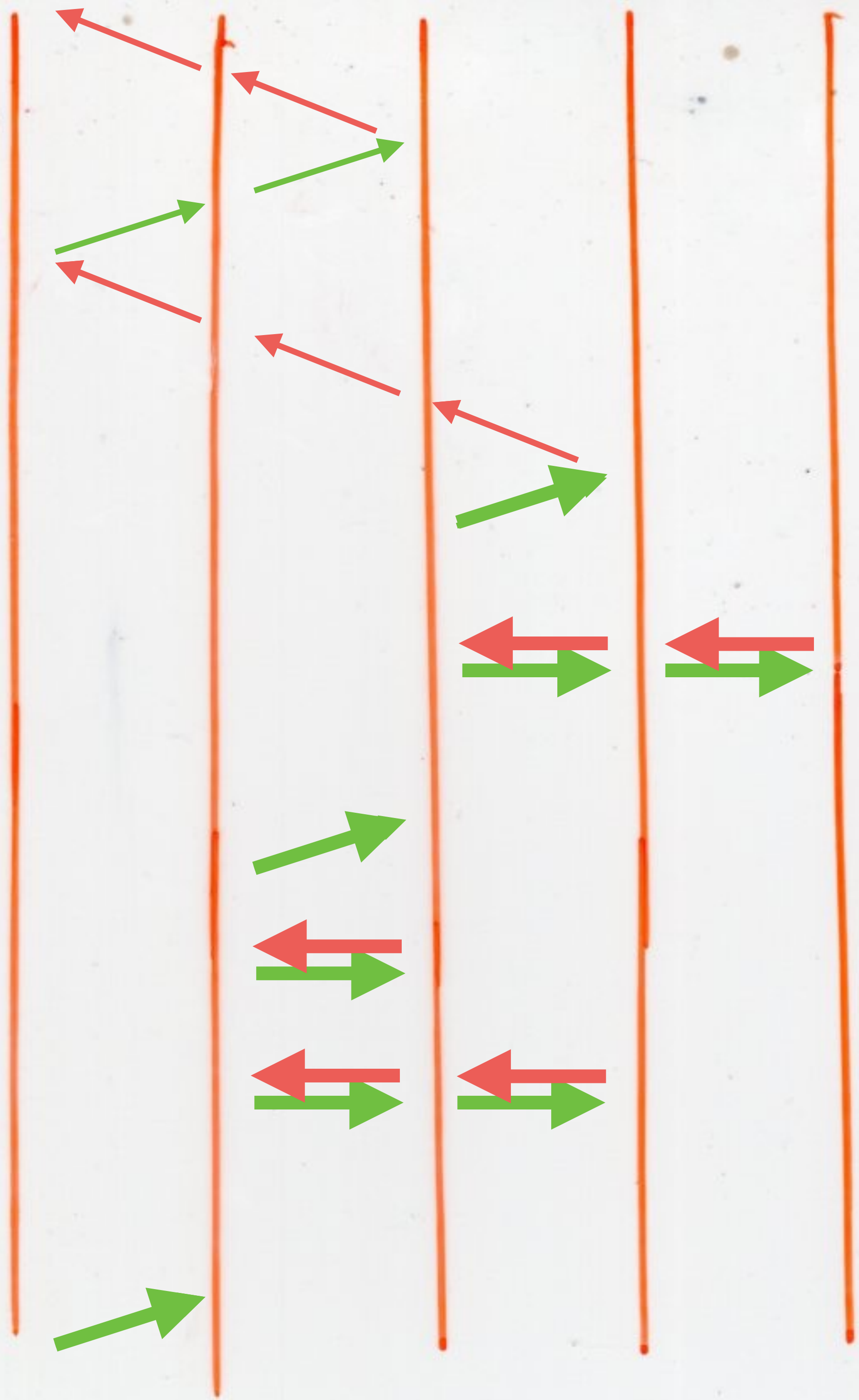
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



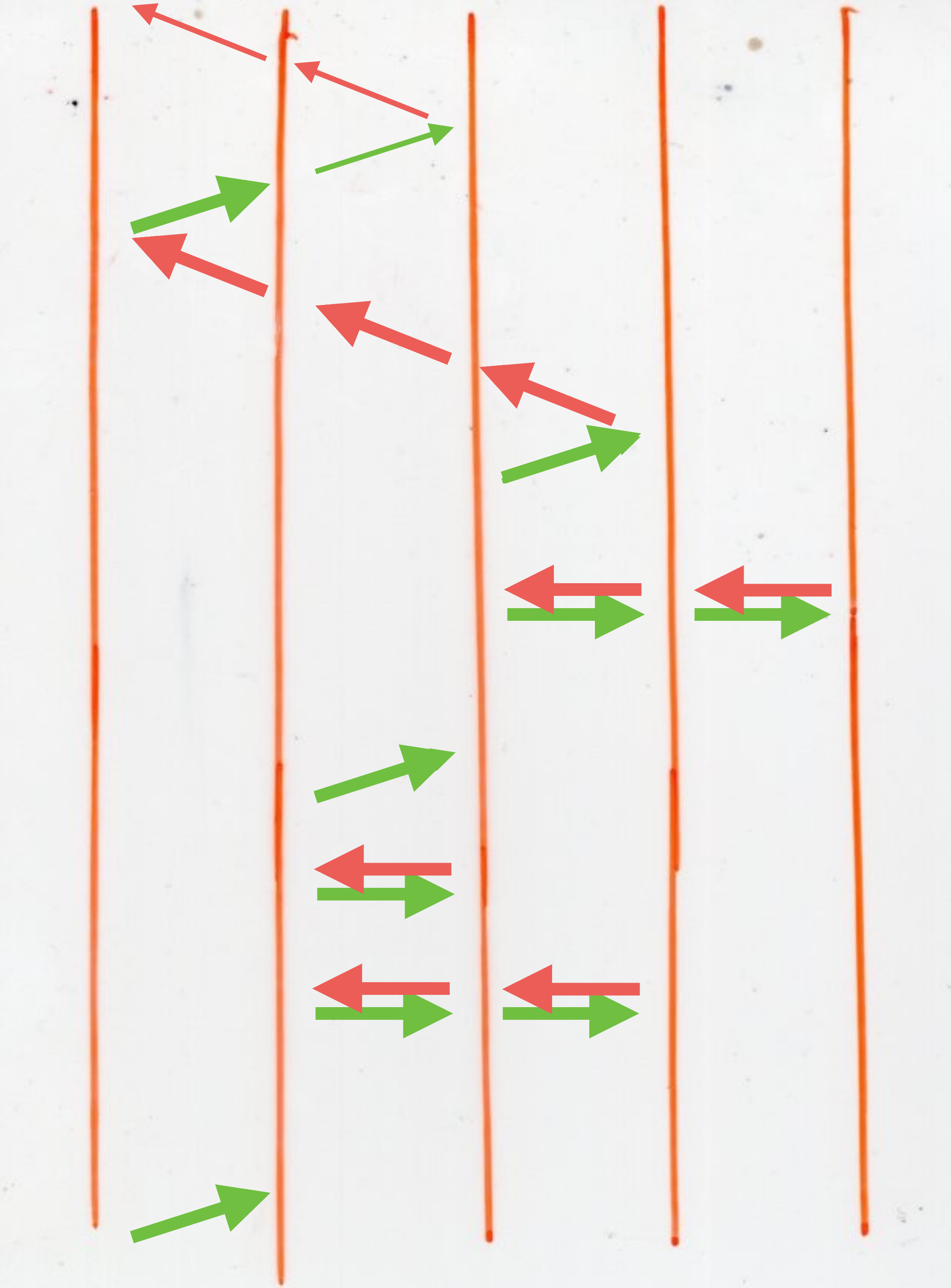
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



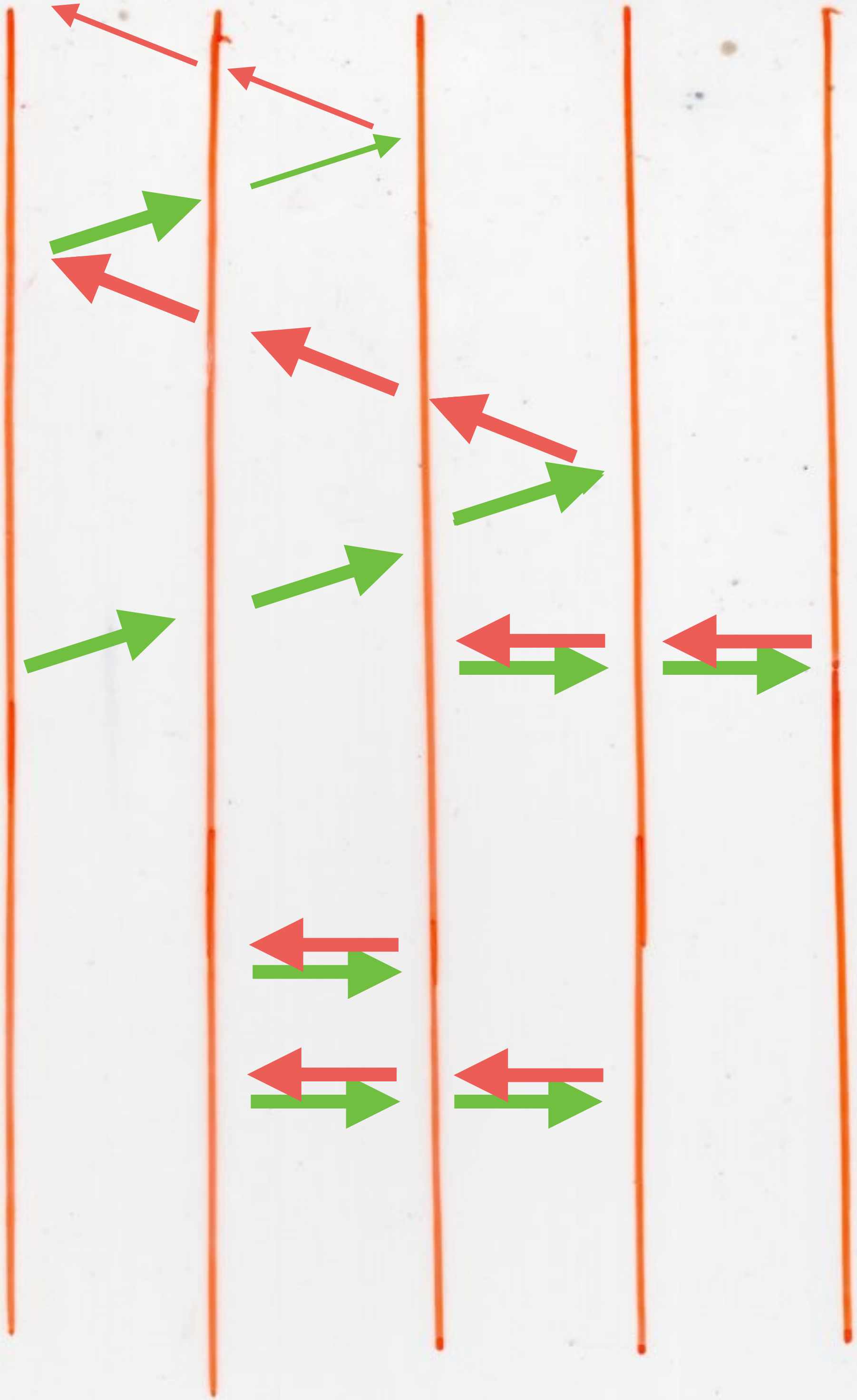
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



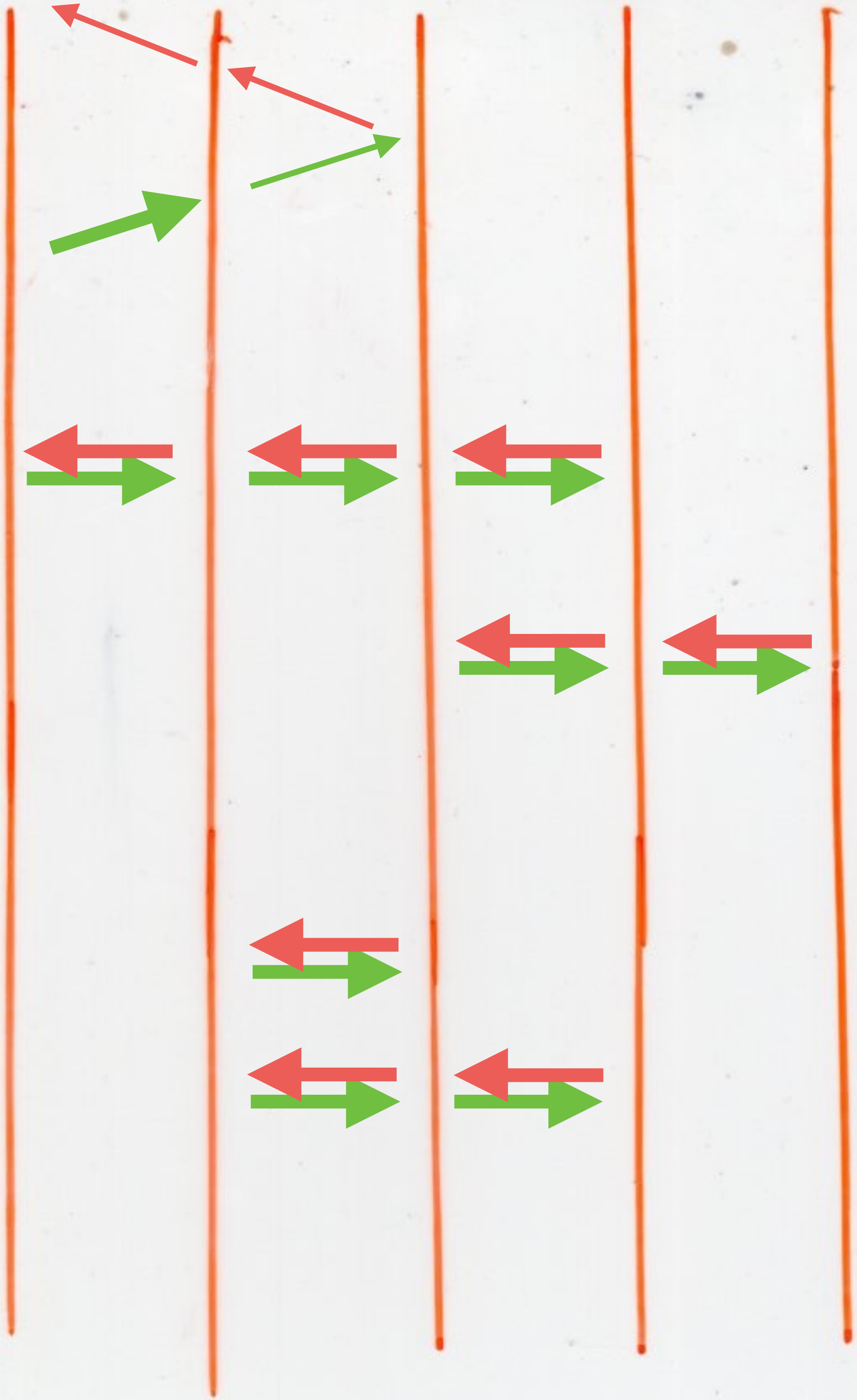
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



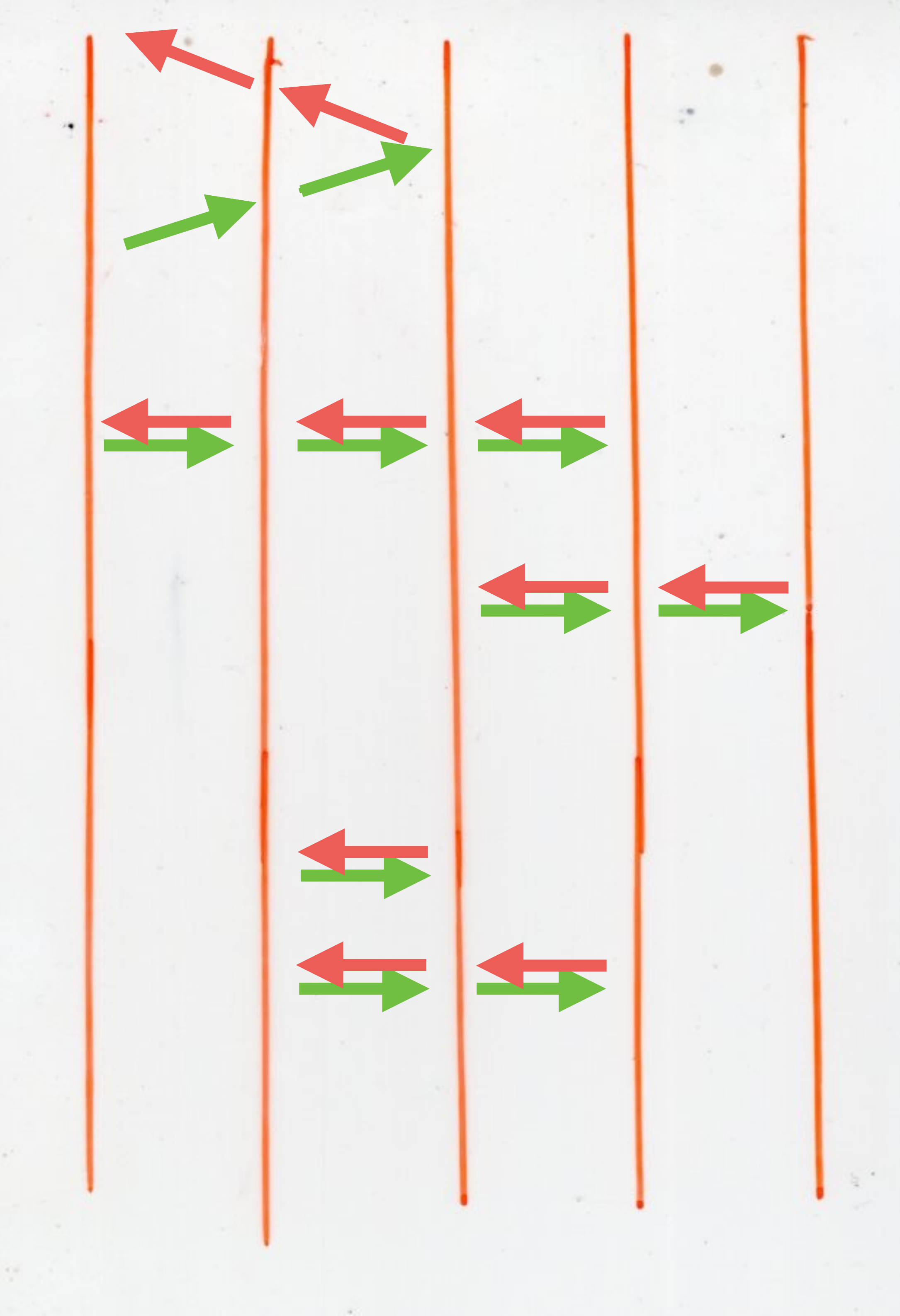
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



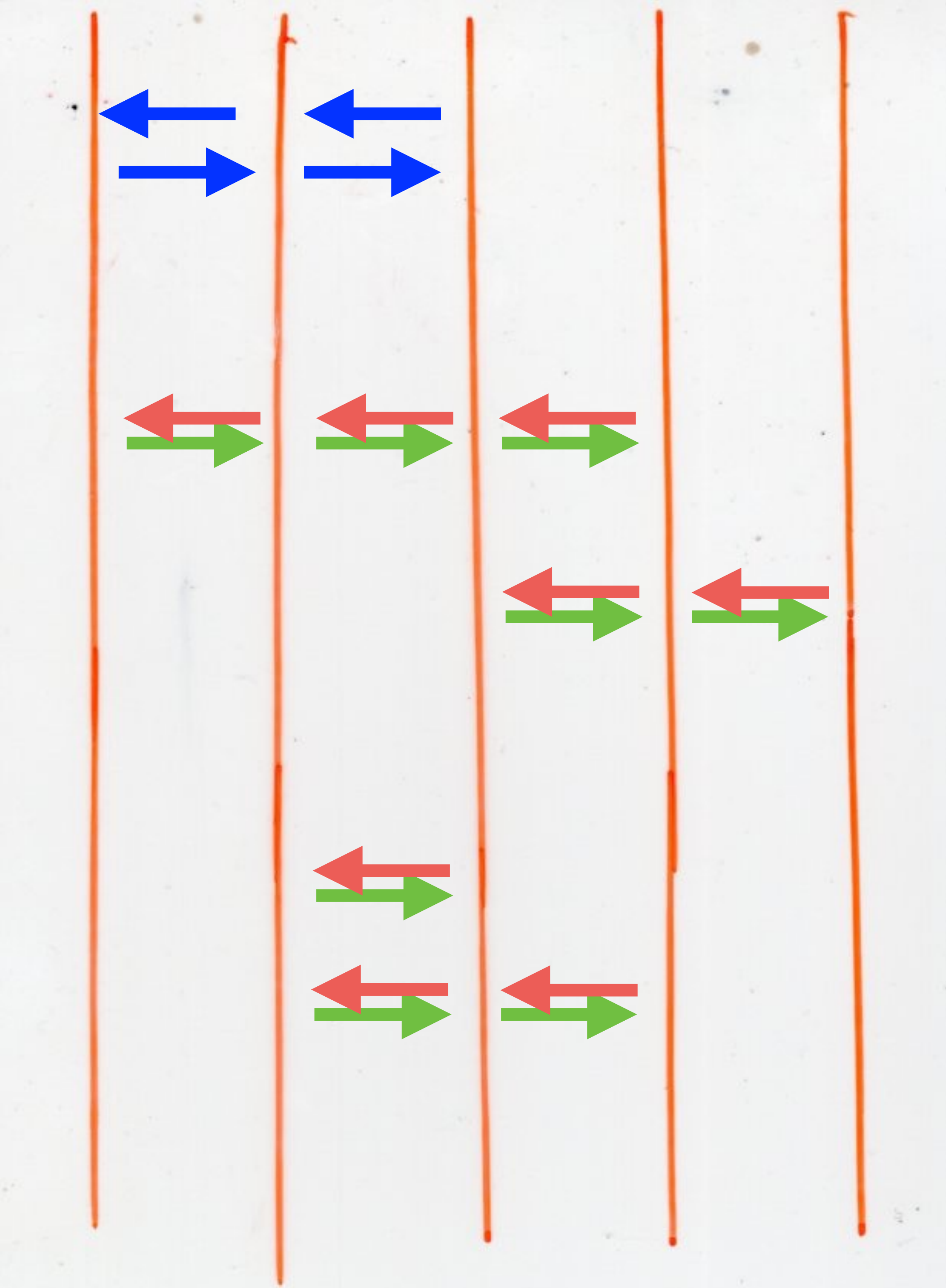
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



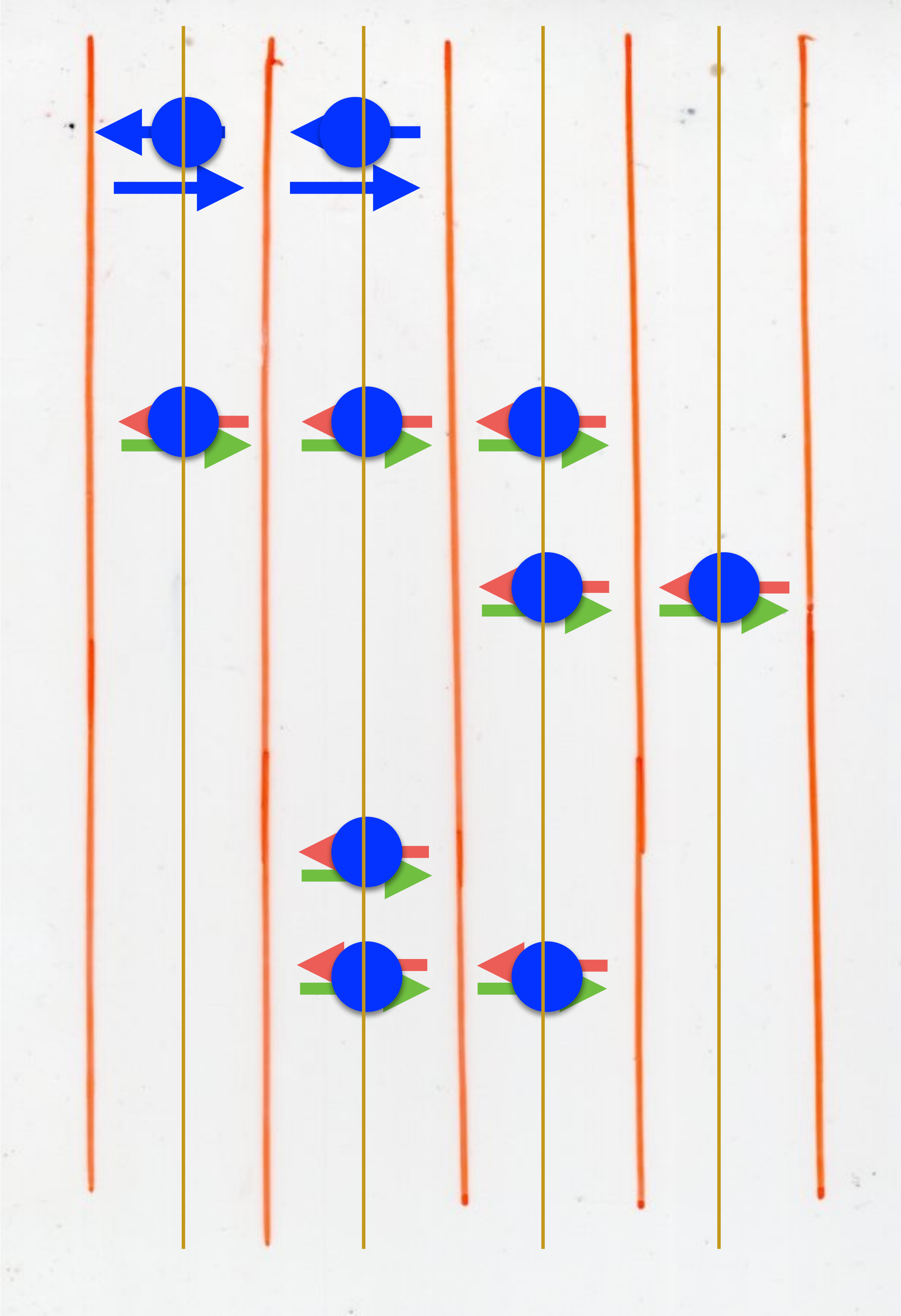
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



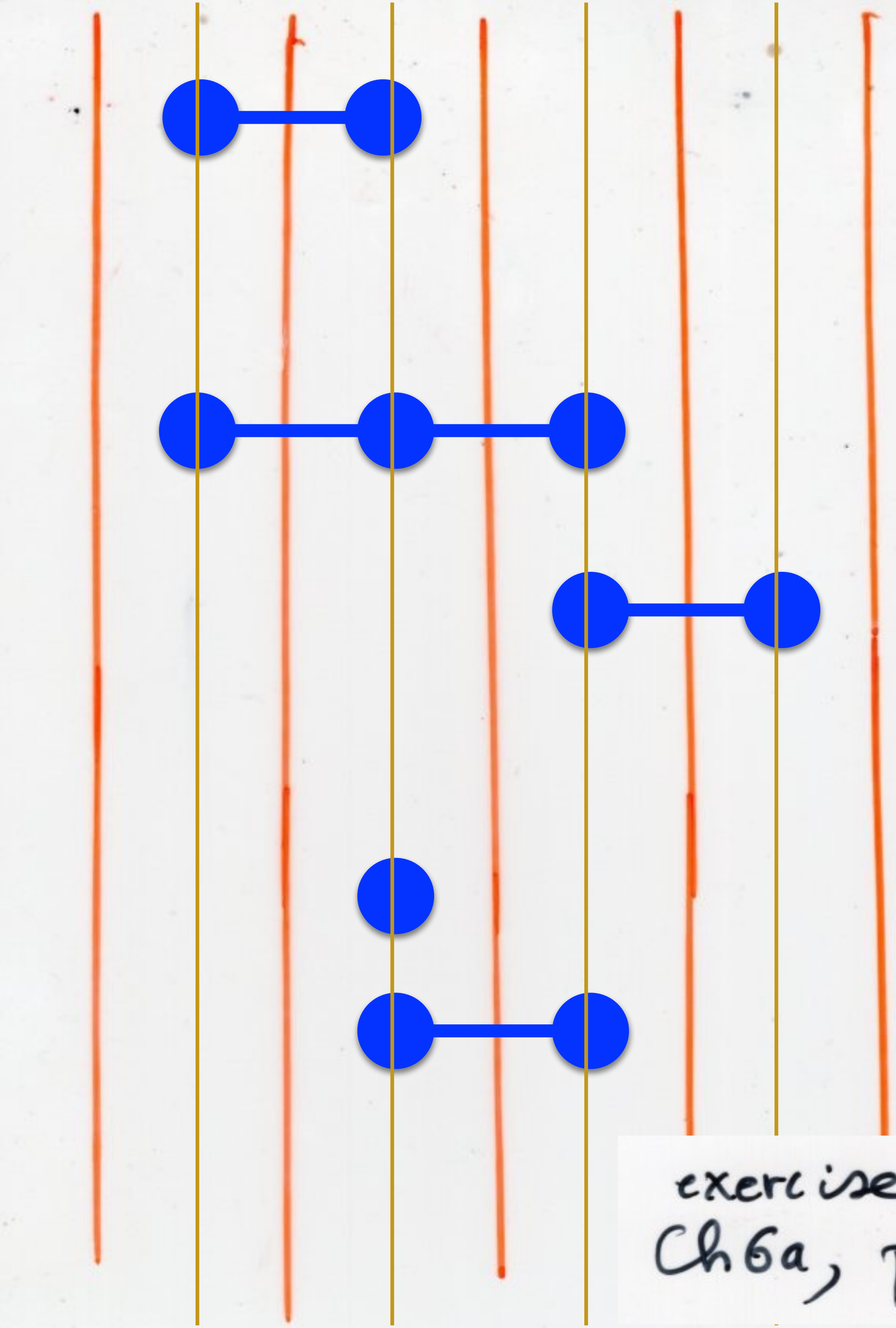
$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



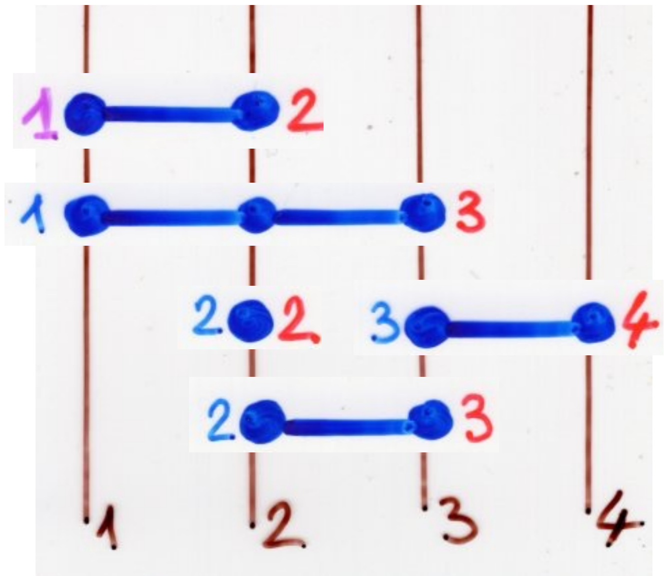
$$\underset{u\rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$



exercise
Ch6a, p65



a festival of bijections

parallelogram polyominoes

(staircase polygons)

Dyck paths

semi-pyramids of dimers (on $\mathbb{N}$)

stairs decomposition

(reverse) Lukasiewicz paths

χ

semi-pyramids of segments (on $\mathbb{N}$)

heaps of oriented loops + trail

ψ

exercise Ch6a, p65

complements

q-Bessel functions and SOS models

(Solid on solid)

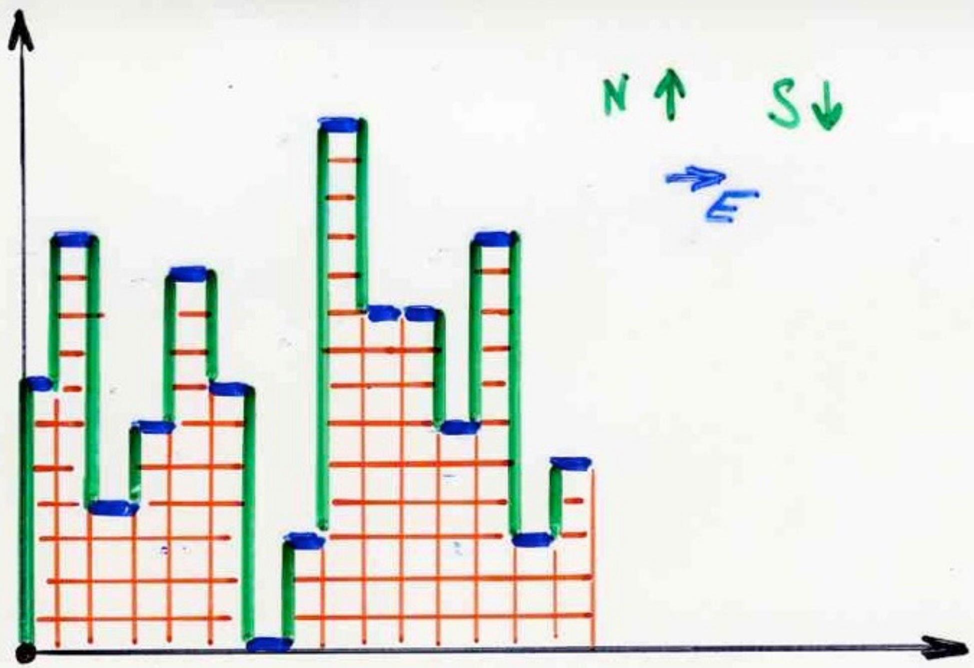
discrete (1+1)-dimensional

SOS model with

- magnetic field
- boundary potential
- surface interactions

exact solution

A. Owczarek, T. Prellberg (1993)



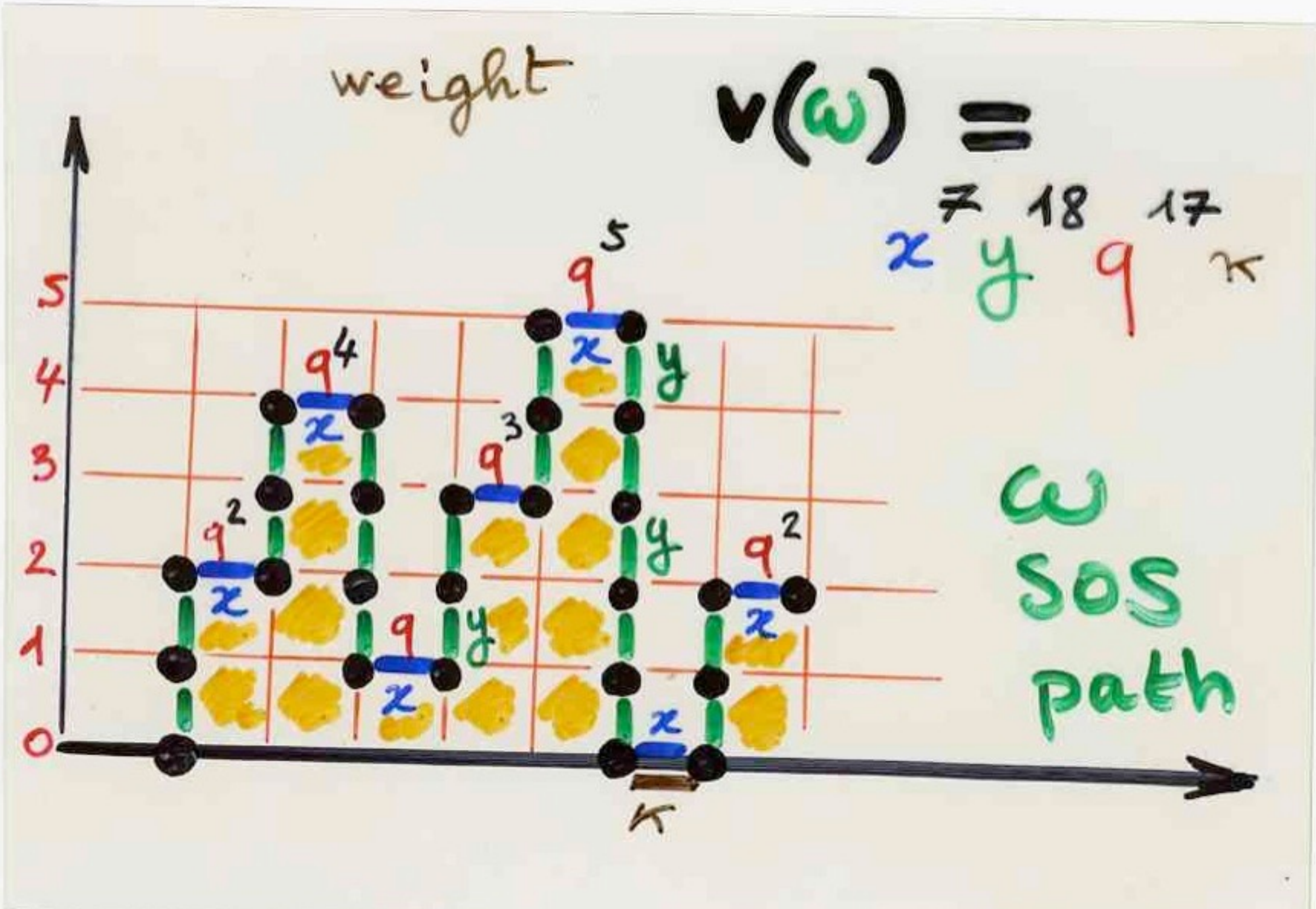
Partially directed
self-avoiding walks
(paths)

$$G(x, y, q, \kappa) = \sum_{\substack{\omega \\ \text{SOS path}}} v(\omega)$$

<u>weight</u> $\uparrow$ N $\quad \downarrow$ S $\qquad y$

$\dfrac{\text{level}}{j}$ $\bullet\!\to\!\bullet$ E $\qquad \begin{cases} x\,q^{j} & \text{if } j>0 \\ x\,\kappa & \text{if } j=0 \end{cases}$

A. Owczarek, T. Prellberg (1993)



weight
v(ω) =
x^7 y^18 q^17 κ
ω
SOS
path
κ
0
1
2
3
4
5
q^2
q^4
q
q^3
q^5
q^2
x
y

ou encore :

$$\sum_{\omega} v(\omega) = x \frac{H\left(qy^2, q, x(1-y^2)q\right)}{H\left(y^2, q, x(1-y^2)\right)}$$

chemins SOS

arrivant au niveau 0

Owczarek, Prellberg (1993)

notations : $H(u,q,t) = \displaystyle\sum_{n\geq 0} \frac{(-t)^n q^{\binom{n}{2}}}{(u,q)_n (q,q)_n}$

avec $(u,q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1})$

$(u)_n$

$$J_0 = H(uq, q, xq)$$

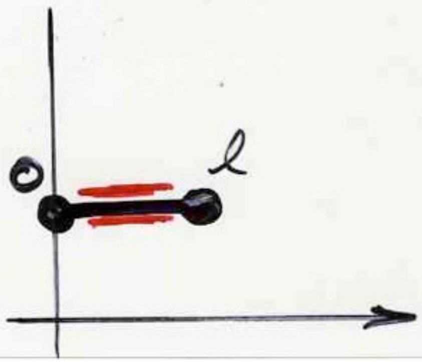
$$J_1 = H(uq, q, xq) - H(uq, q, xq^2)$$

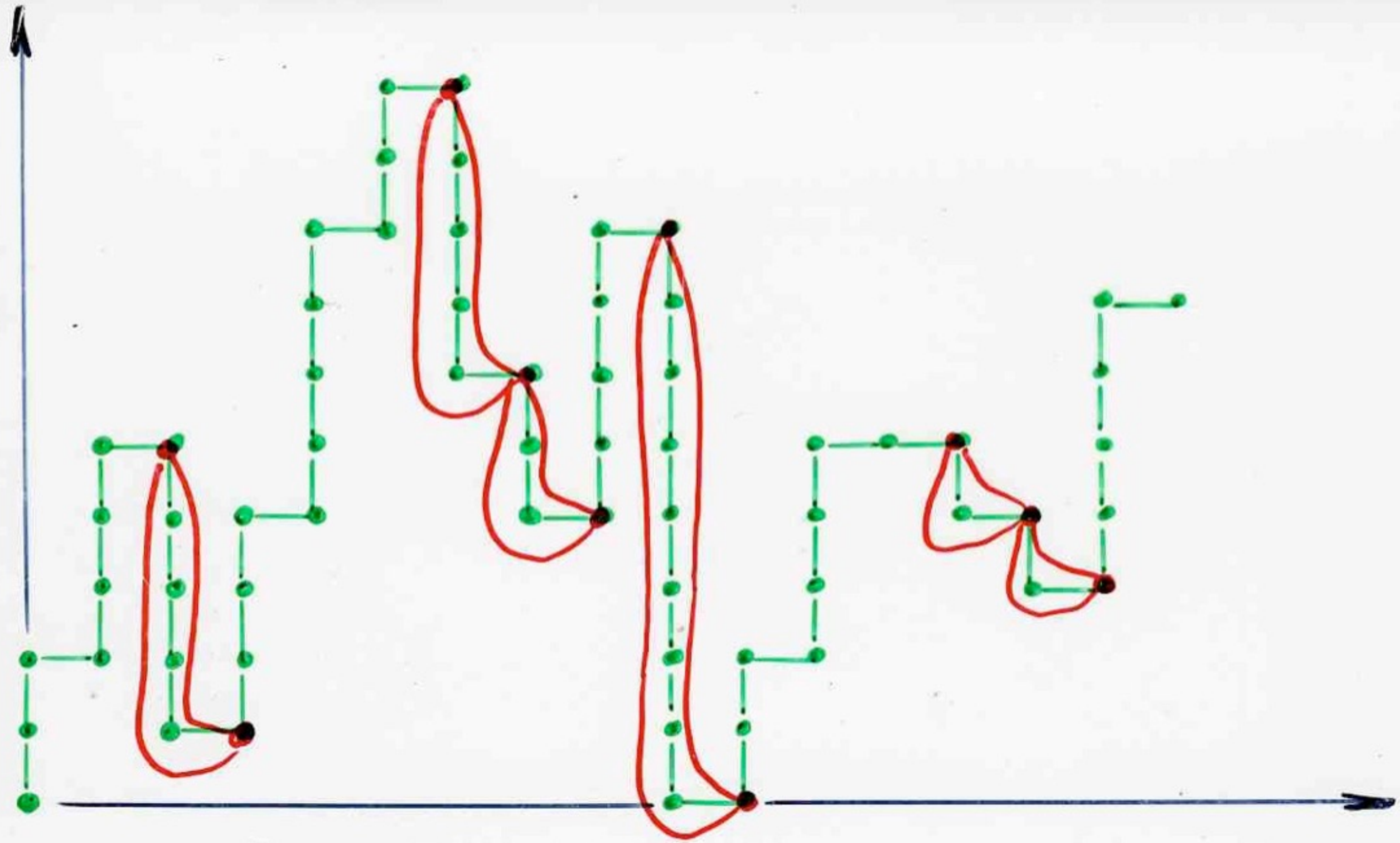
$$\frac{J_1}{J_0} \qquad \text{or} \qquad \frac{H(uq,\ q,\ xq^2)}{H(uq,\ q,\ xq)}$$

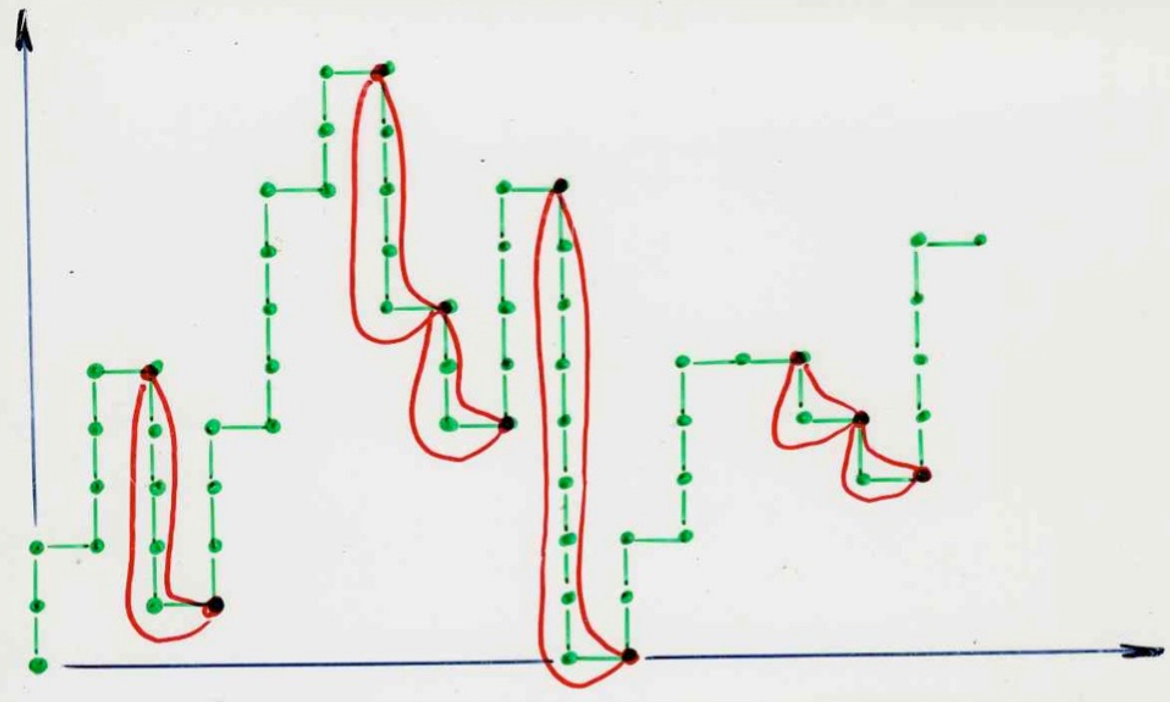
$$\frac{N}{D} = \sum_{P} v(P)$$

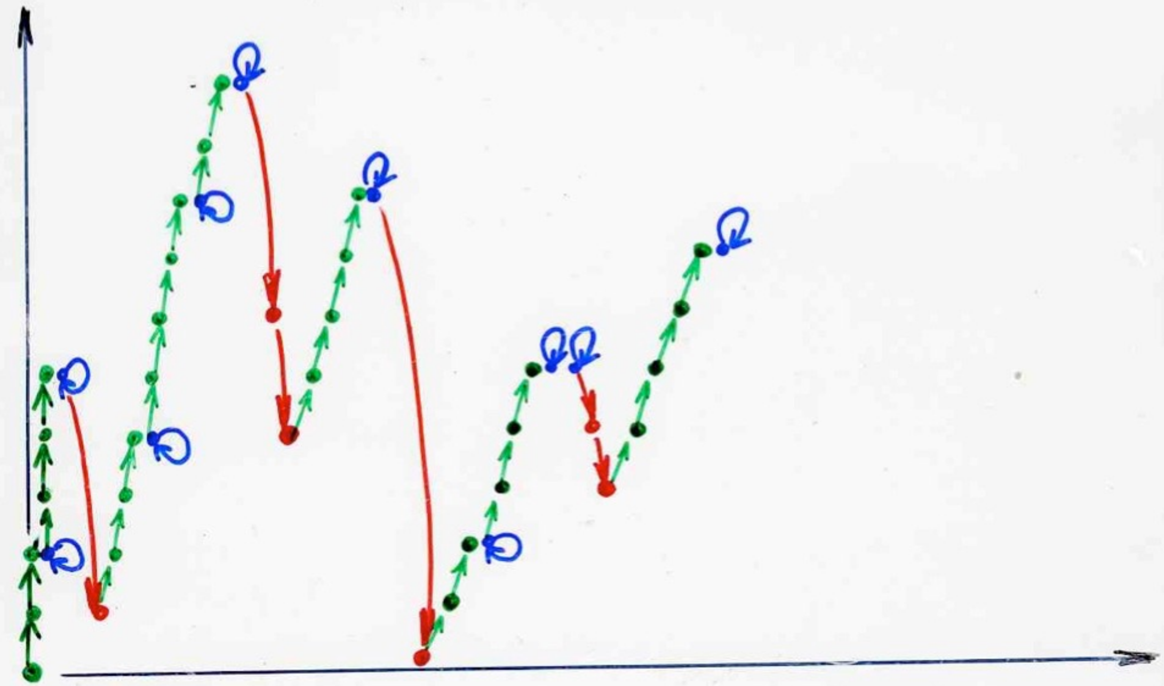
P pyramid

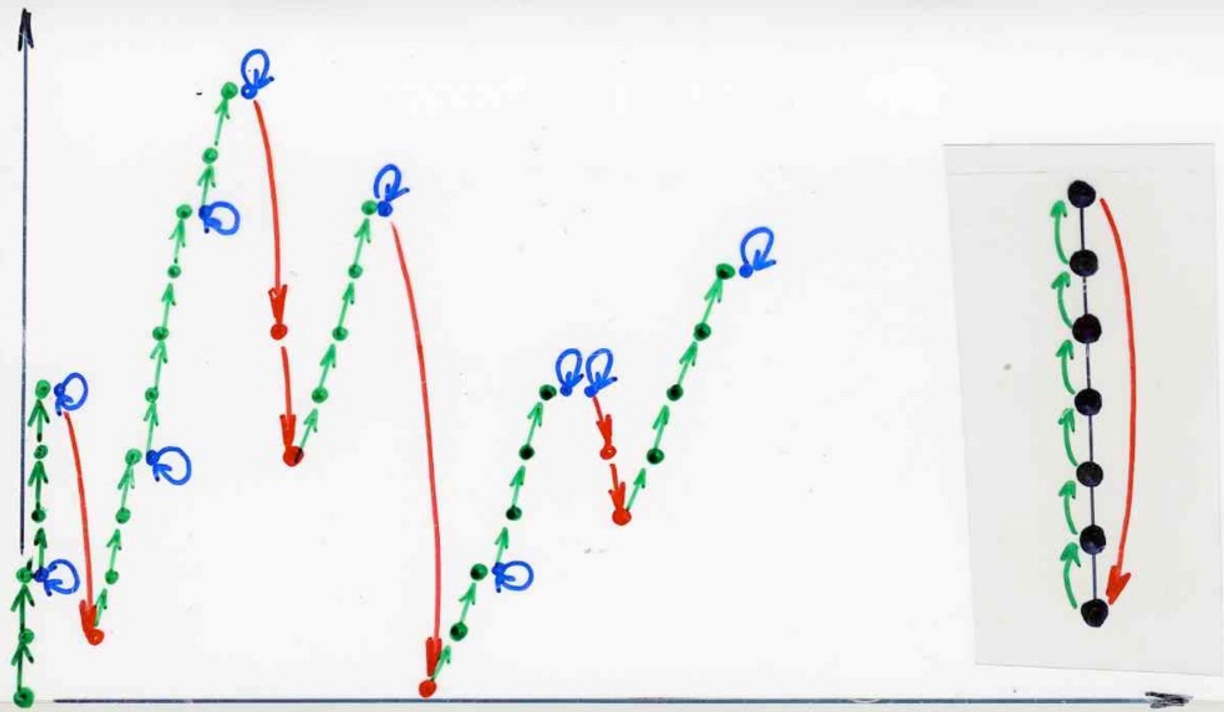
maximal piece is

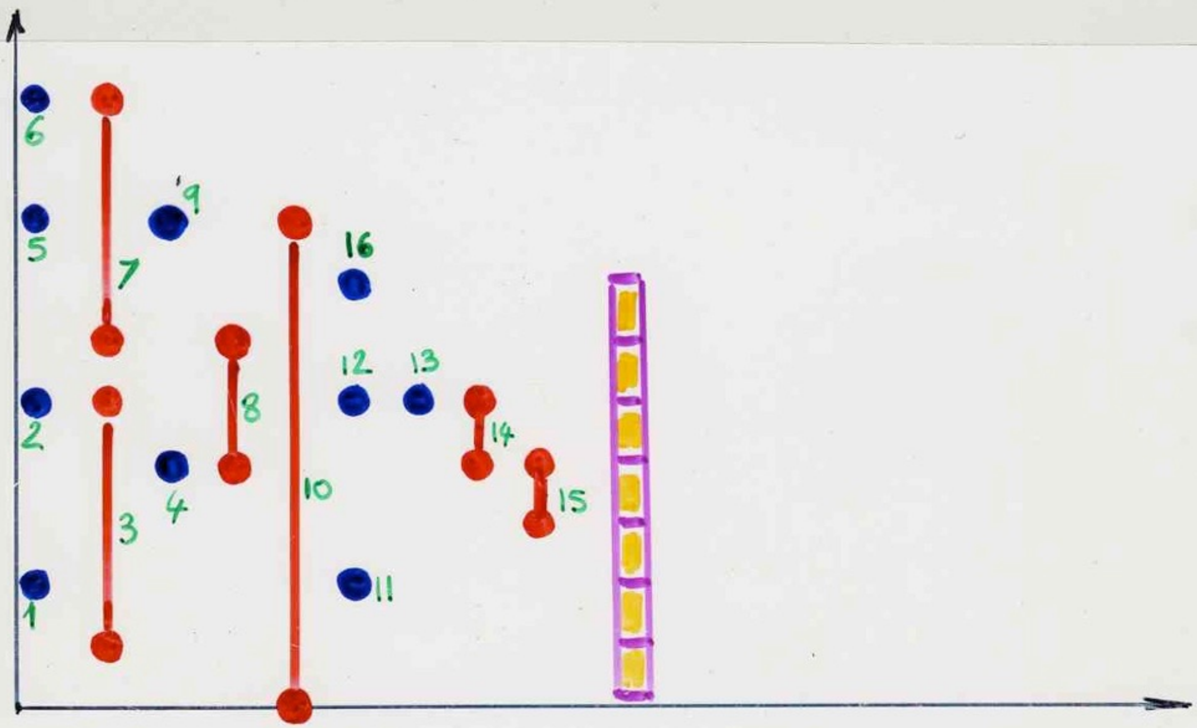








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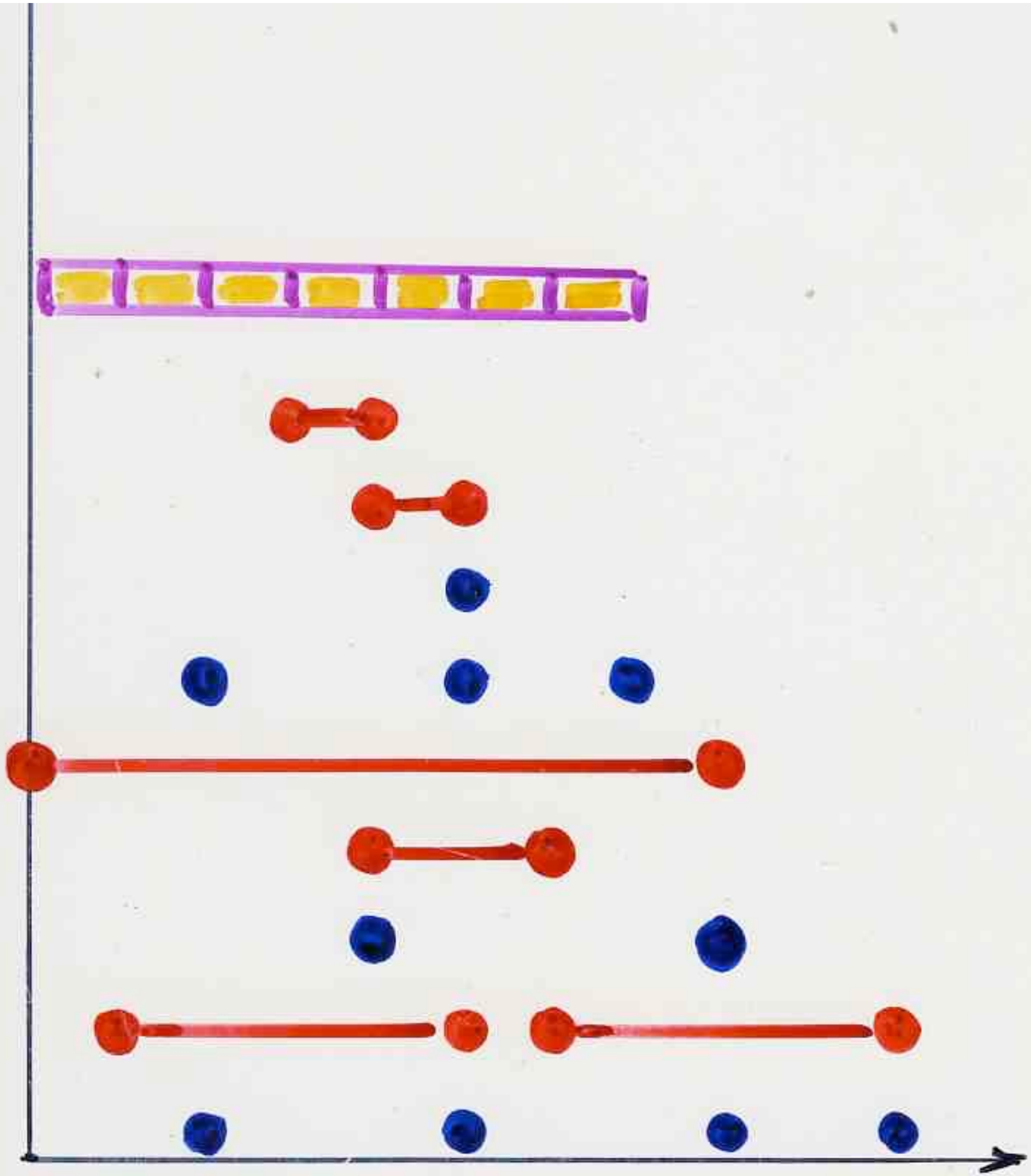
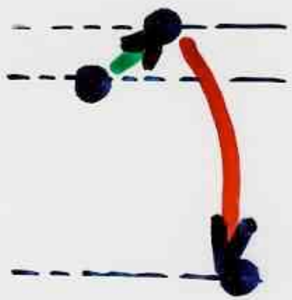


$$v_g([\![i, j]\!]) = t\, u^{(j-i)}\, q^{i}$$

$0 \leq i \leq j$

Paths with no

peaks



$$t \leftarrow x(1-y^2)$$

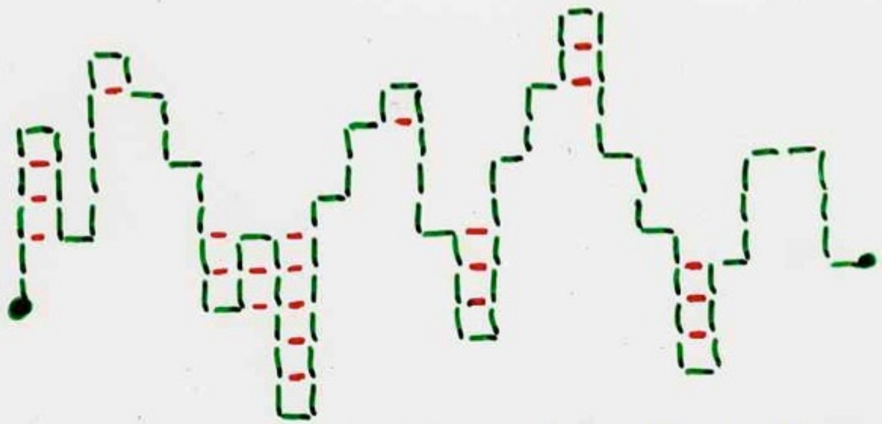
• q-Bessel

chemins partiellement dirigés
avec interaction

"effondrement" des polymères

Brak, Guttmann, Whittington 1992
Owczarek, Prellberg, Brak 1993
Zwanzig, Lauritzen 1968, 1970

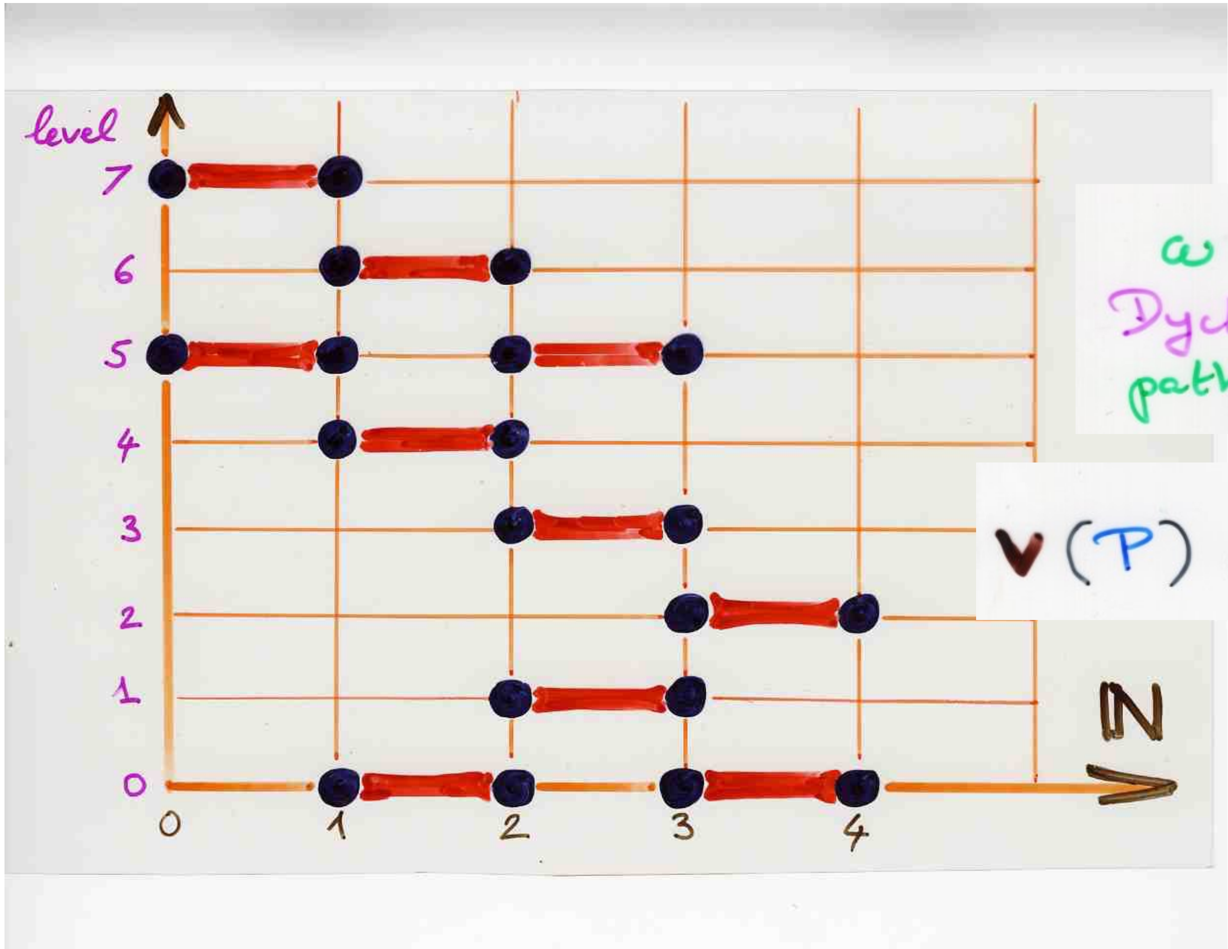
autres familles de polyominos convexes



chemin partiellement dirigé avec interactions

particular case:
heaps of dimers
and
Ramanujan contined fraction

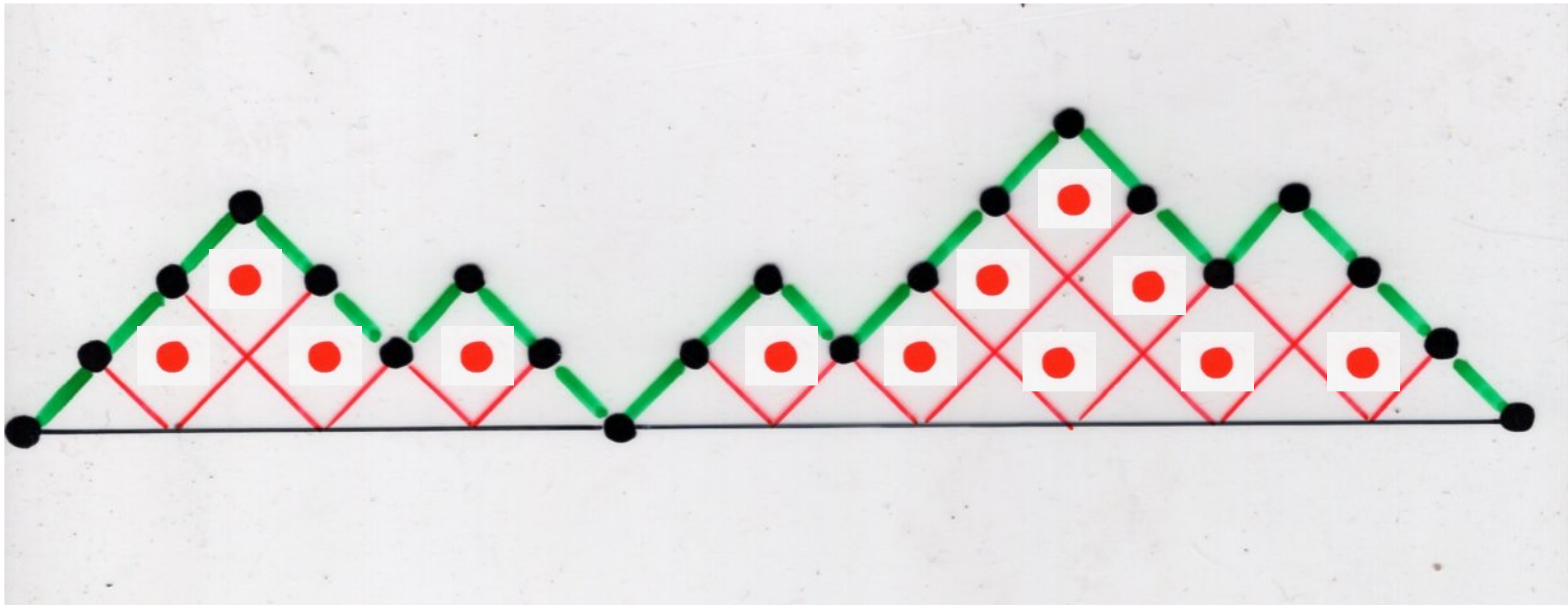
$$v([k-1, k]) = q^k t$$



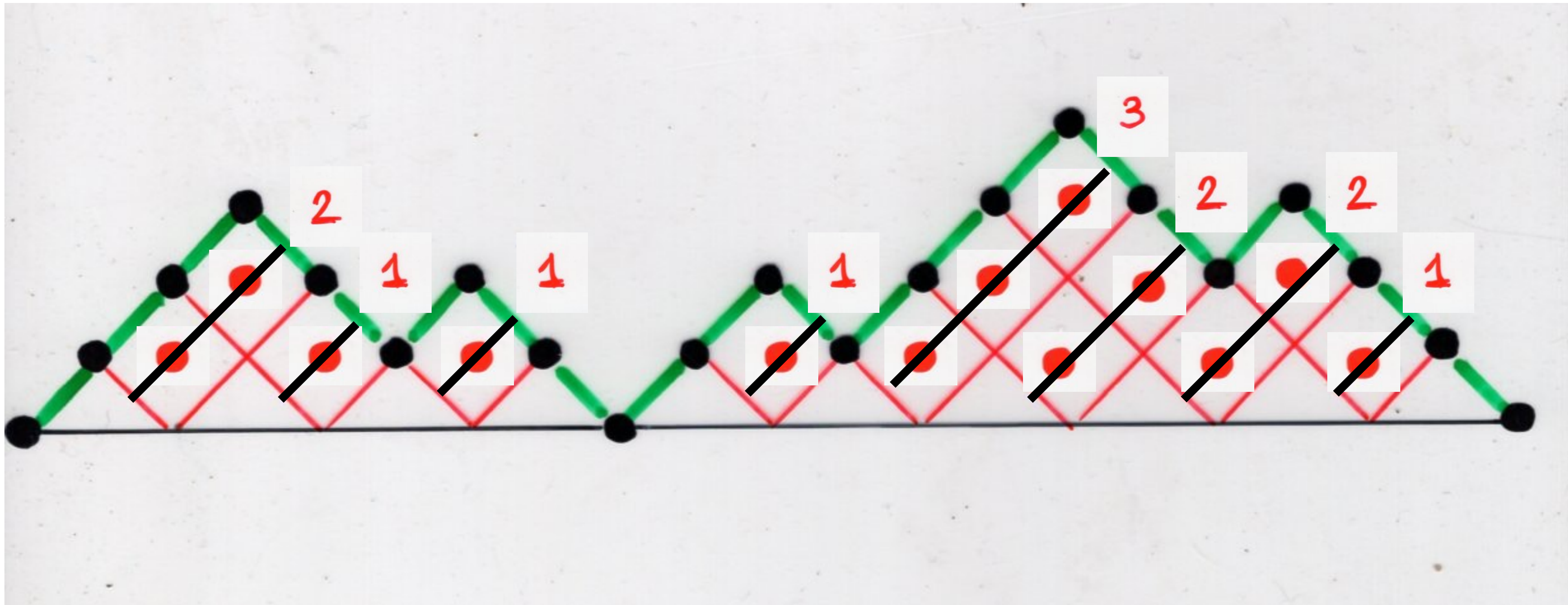
ω Dyck path $\longrightarrow$ P semi-pyramid of dimers on $\mathbb{N}$

$$v(P) = q^{|\omega|/2 + \text{area}(\omega)} \, t^{|\omega|/2}$$

area = 13



area = 13



$$v([k-1, k]) = q^k t$$

ω Dyck path $\longrightarrow$ P semi-pyramid of dimers on $\mathbb{N}$

$$v(P) = q^{|\omega|/2 + \mathrm{area}(\omega)} t^{|\omega|/2}$$

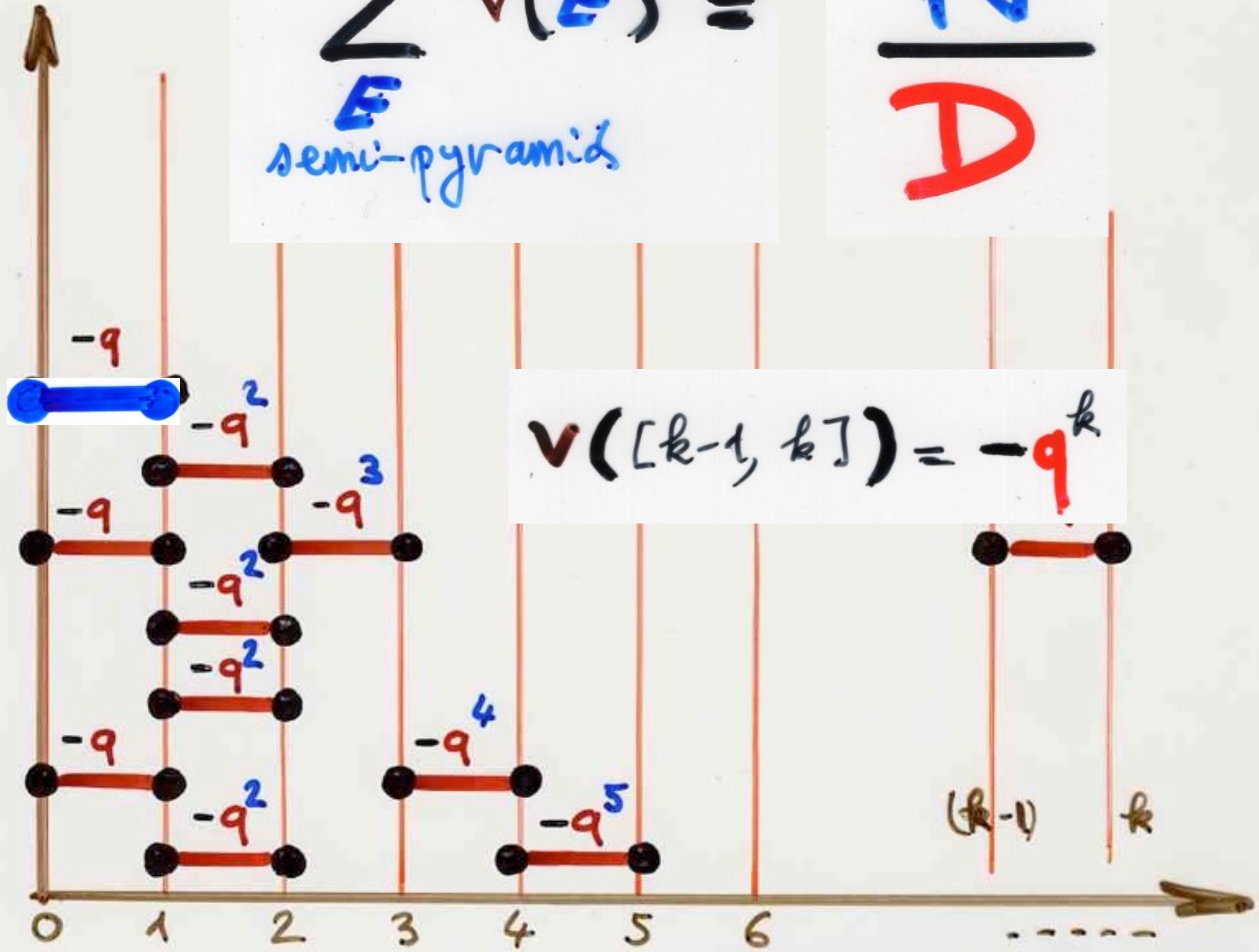
$$v([k-1, k]) = q^{k-1} t$$

$$v(P) = q^{\mathrm{area}(\omega)} t^{|\omega|/2}$$

weighted heap $v(E)$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$

$$v([k-1, k]) = -q^k$$



total weight $(-1)^{10} q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

Rogers-Ramanujan identities

Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D- partition $\quad \lambda_i - \lambda_{i+1} \geq 2$

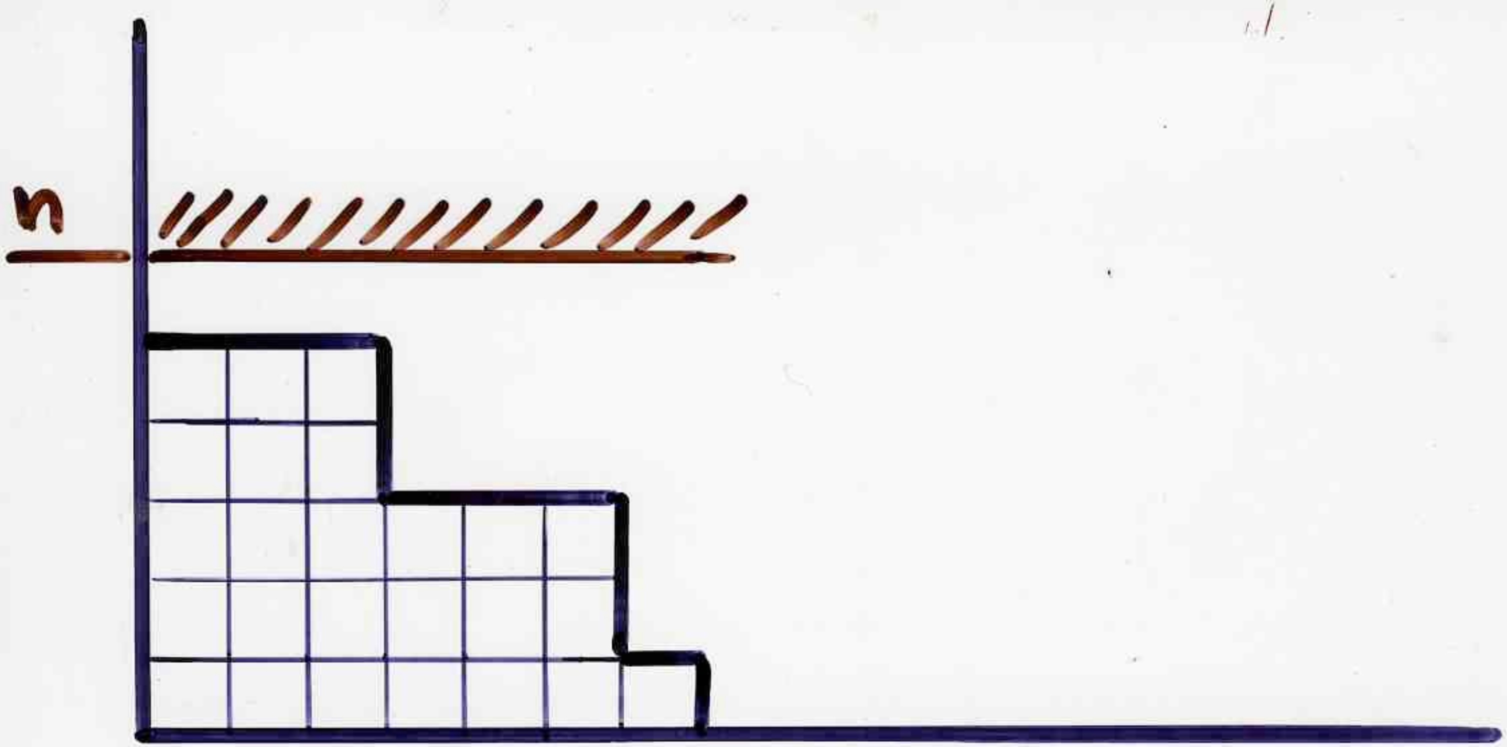
$\lambda = (\lambda_1, \dots, \lambda_k)$ $\quad (1 \leq i < k)$

generating function for D- partitions

$$\sum_{m \geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

Partition ayant au plus n parts $\longleftrightarrow$ D-partition ayant exactement n parts

$$0 \leq (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n) \qquad (1+\lambda_1, 3+\lambda_2, \ldots, (2n-1)+\lambda_n)$$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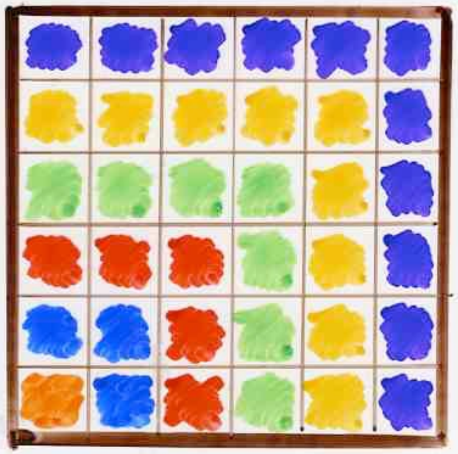
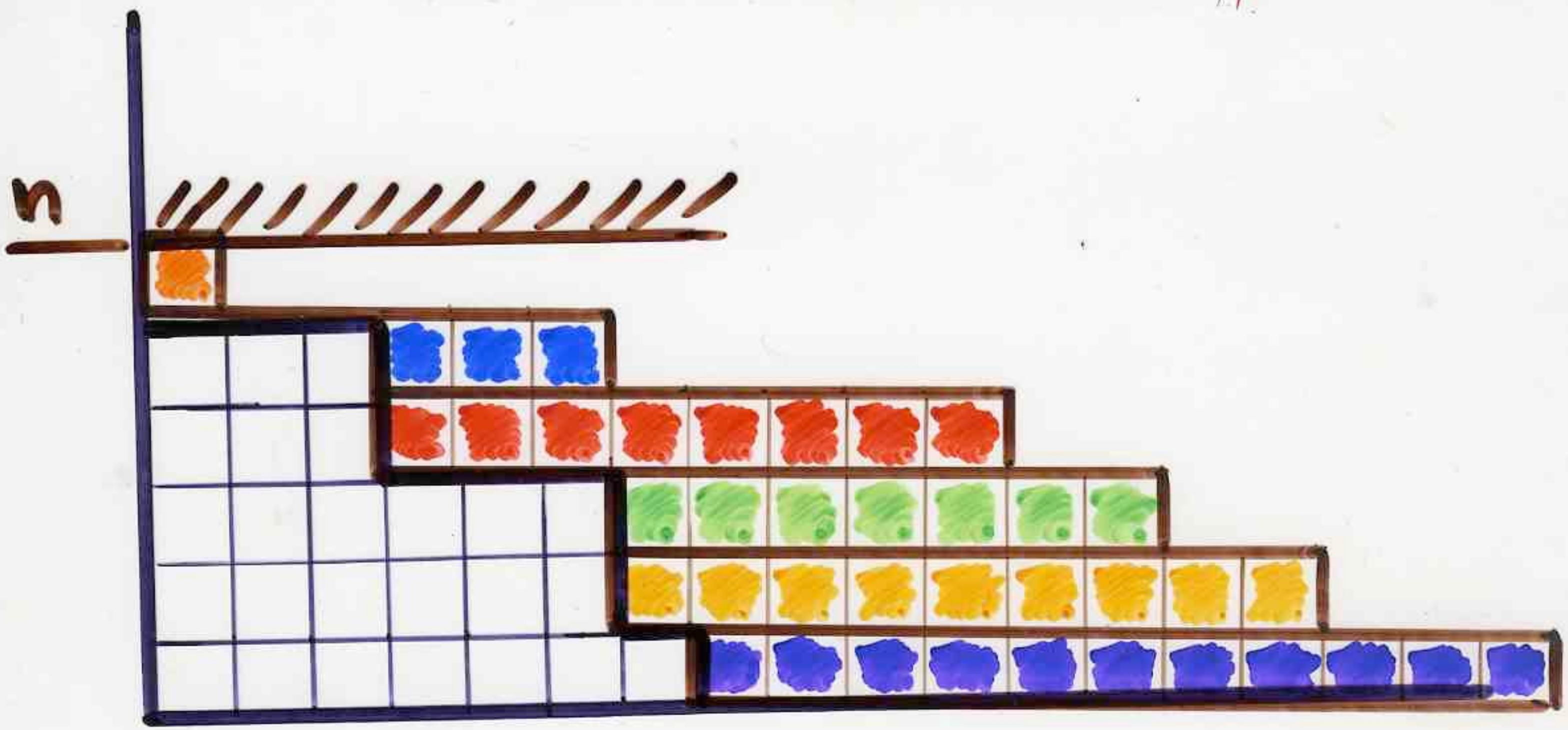


$$n^2 = 1 + 3 + \dots + (2n-1)$$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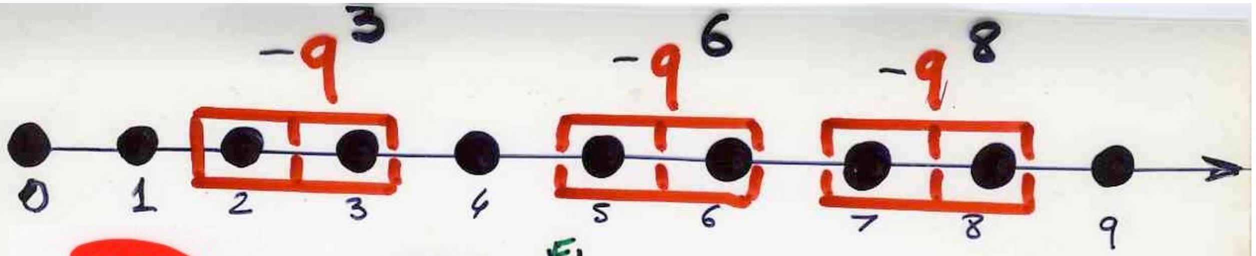
Rogers-Ramanujan 1st identity

$$D = \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$



$$D = \sum_{E} (-1)^{|E|} v(E)$$

E trivial heaps of dimers on $\mathbb{N}$

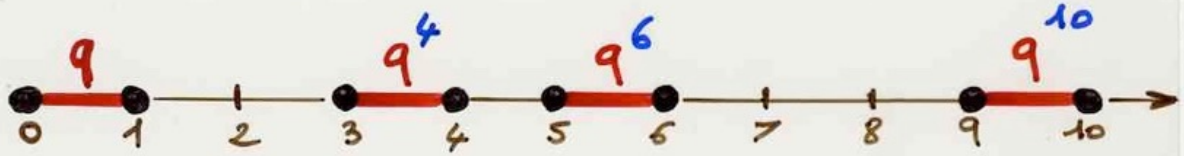
$$v([k-1, k]) = -q^k$$

Rogers-Ramanujan 1st identity

$$D = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

poids total

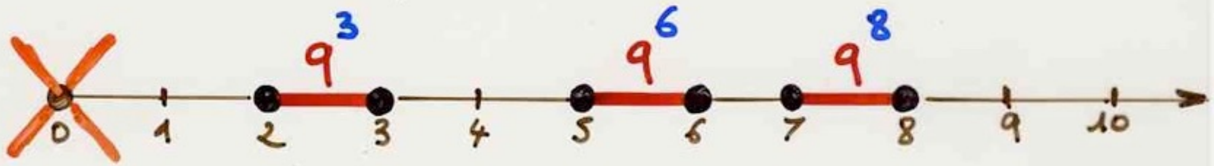
q^{21}



D-partition $\lambda = (10, 6, 4, 1)$

$21 = 10 + 6 + 4 + 1$

q^{17}

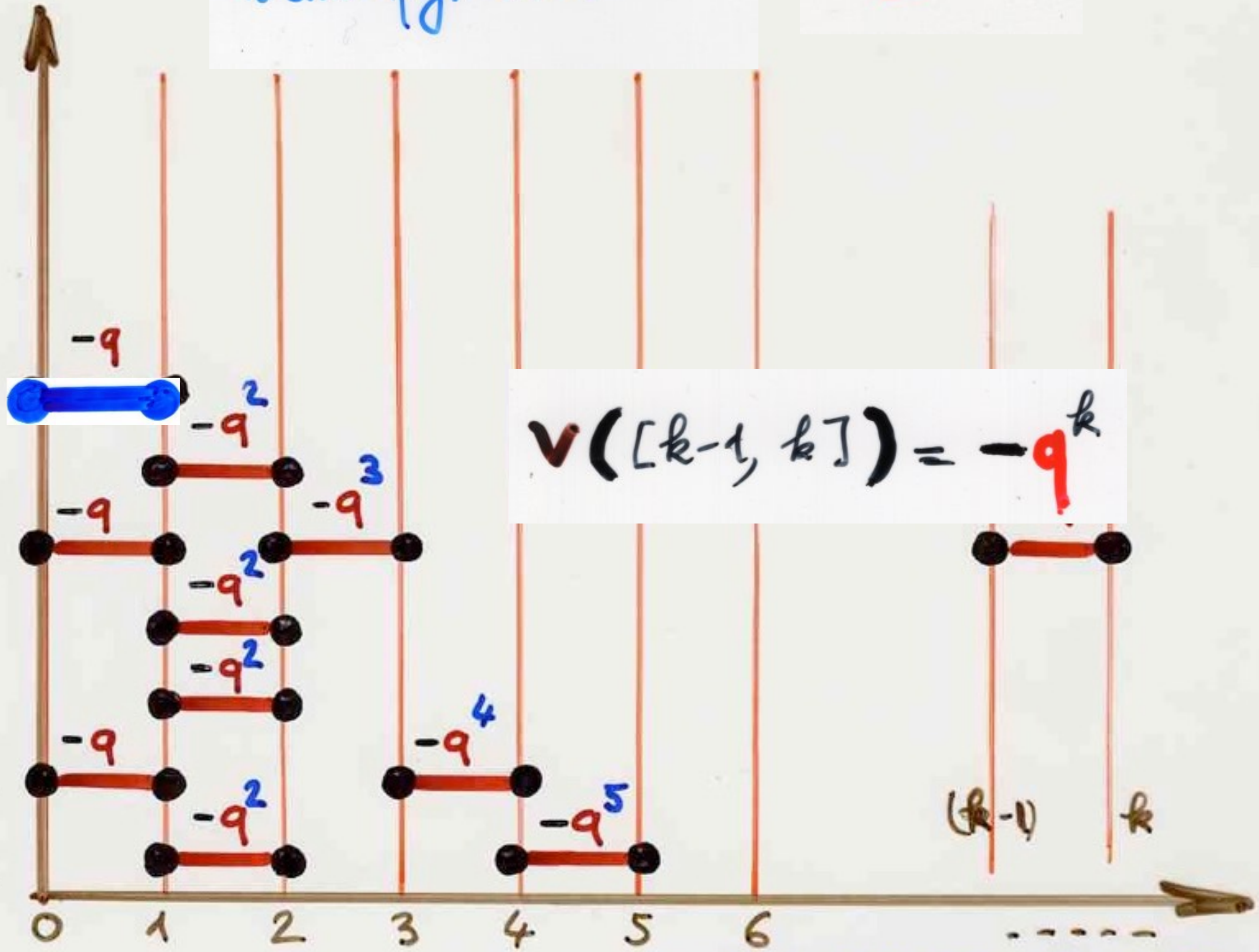


$\lambda = (8, 6, 3)$

$17 = 8 + 6 + 3$

$$N = \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \delta D$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$



$$v([k-1, k]) = -q^{k}$$

total weight $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

partitions

parts $\equiv 1,4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \quad \begin{cases} 1+\dots+1 \end{cases}$$

Rogers - Ramanujan identities

$$R_I \qquad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

D-partitions

$$\left\{\begin{array}{l} 9 \\ 8+1 \\ 7+2 \\ 6+3 \\ 5+3+1 \end{array}\right.$$

?

partitions

parts $\equiv 1,4 \bmod 5$

$$\left\{\begin{array}{l} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{array}\right. \qquad \left\{ 1+\dots+1 \right.$$

R_{II}

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

D-partitions

parts $\neq 1$

$$\left\{\begin{array}{l} 7+2 \\ 6+3 \\ 9 \end{array}\right.$$

?

Partitions

parts $\equiv 2, 3$ mod 5

$2+2+2+3$

$3+3+3$

$7+2$

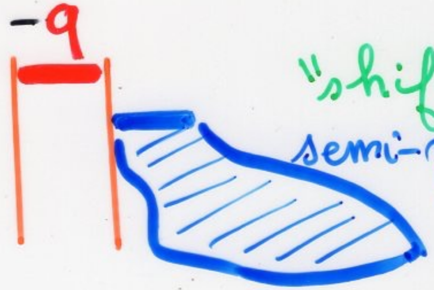
Ramanujan contined fraction

$$\sum_{\substack{E\\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$

Semi-pyramid

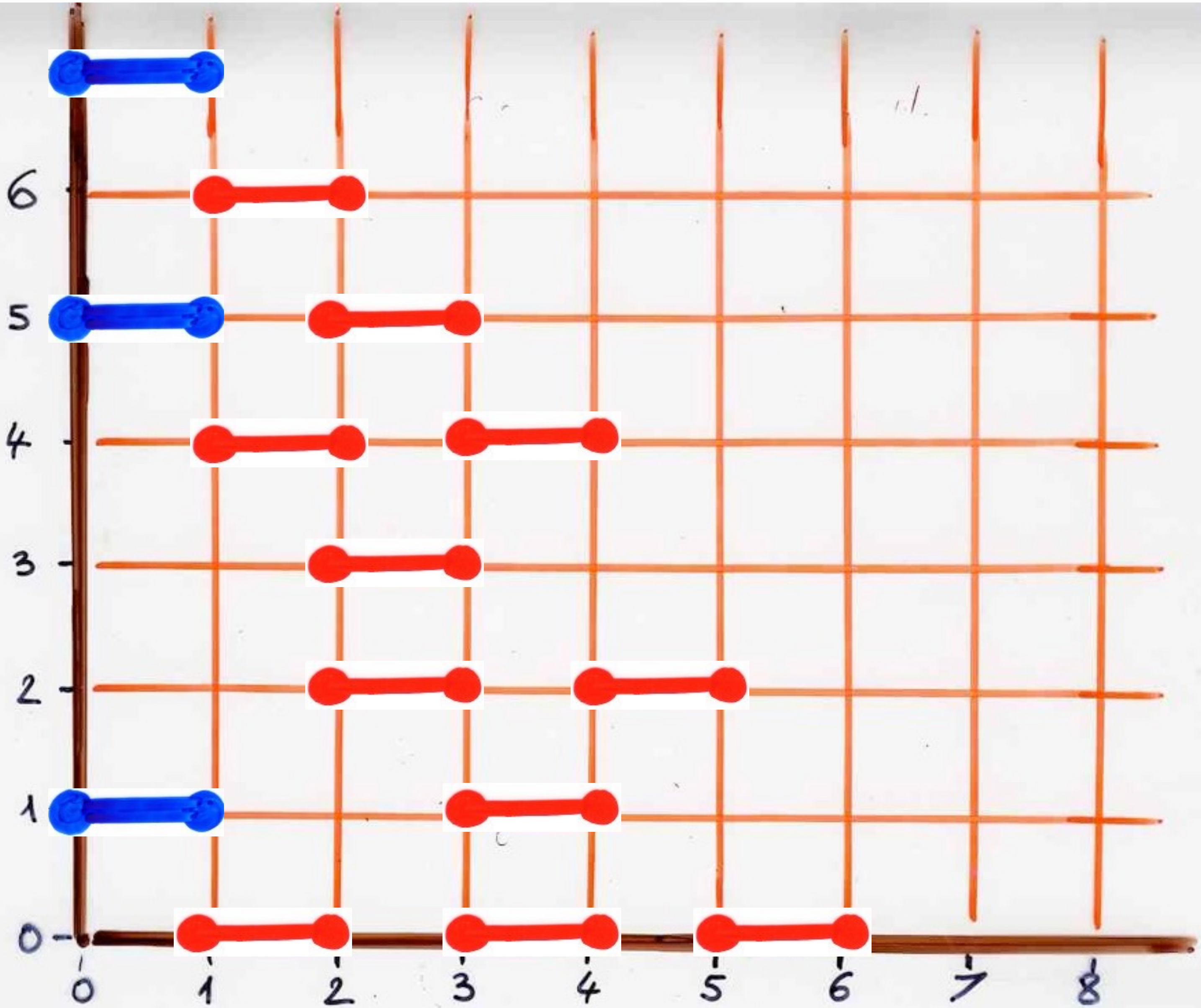
= sequence of "primitive" semi-pyramids

"Primitive" semi-pyramid =



"shifted" semi-pyramid

weight: $q^{i} \xrightarrow{\delta} q^{i+1}$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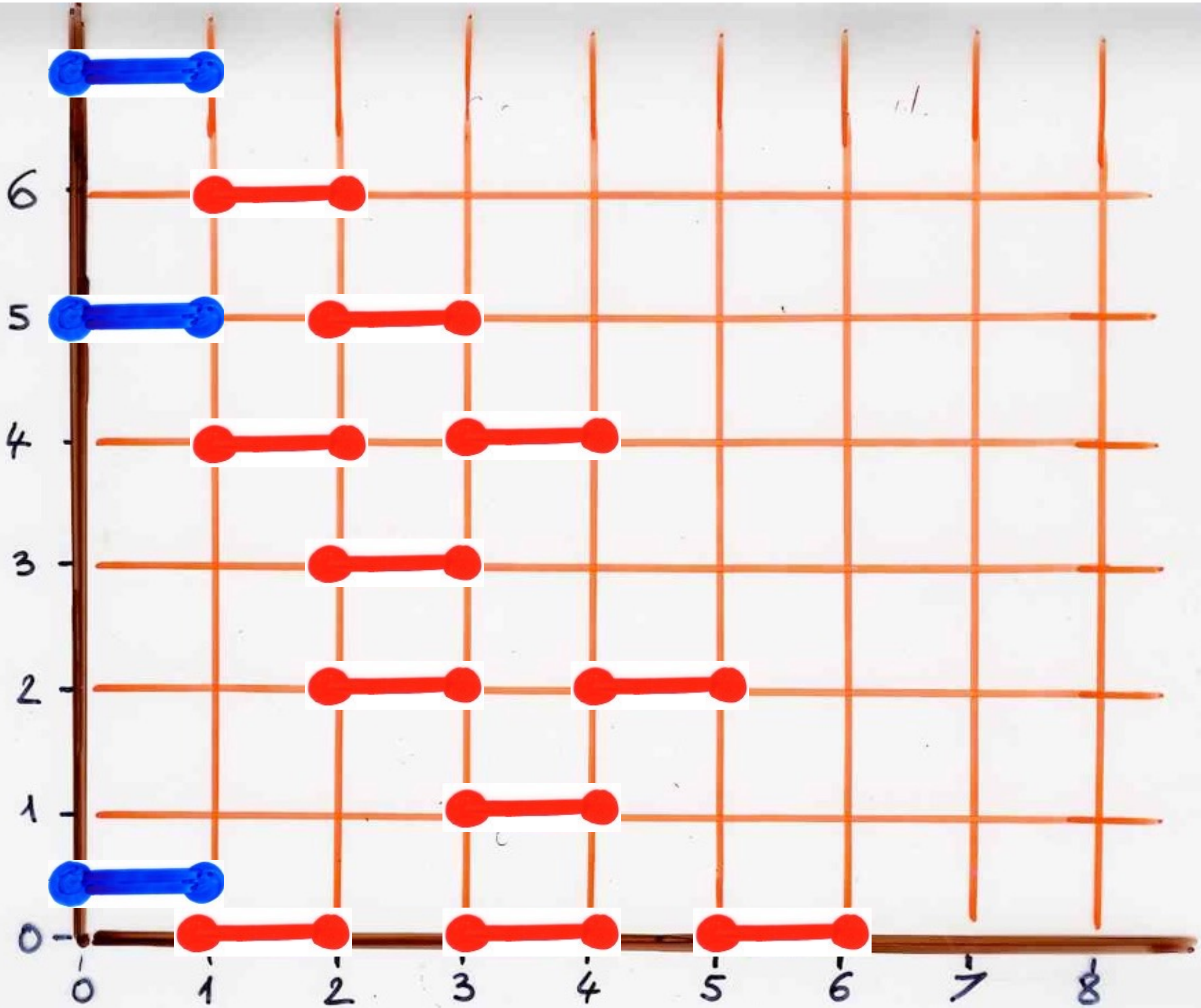

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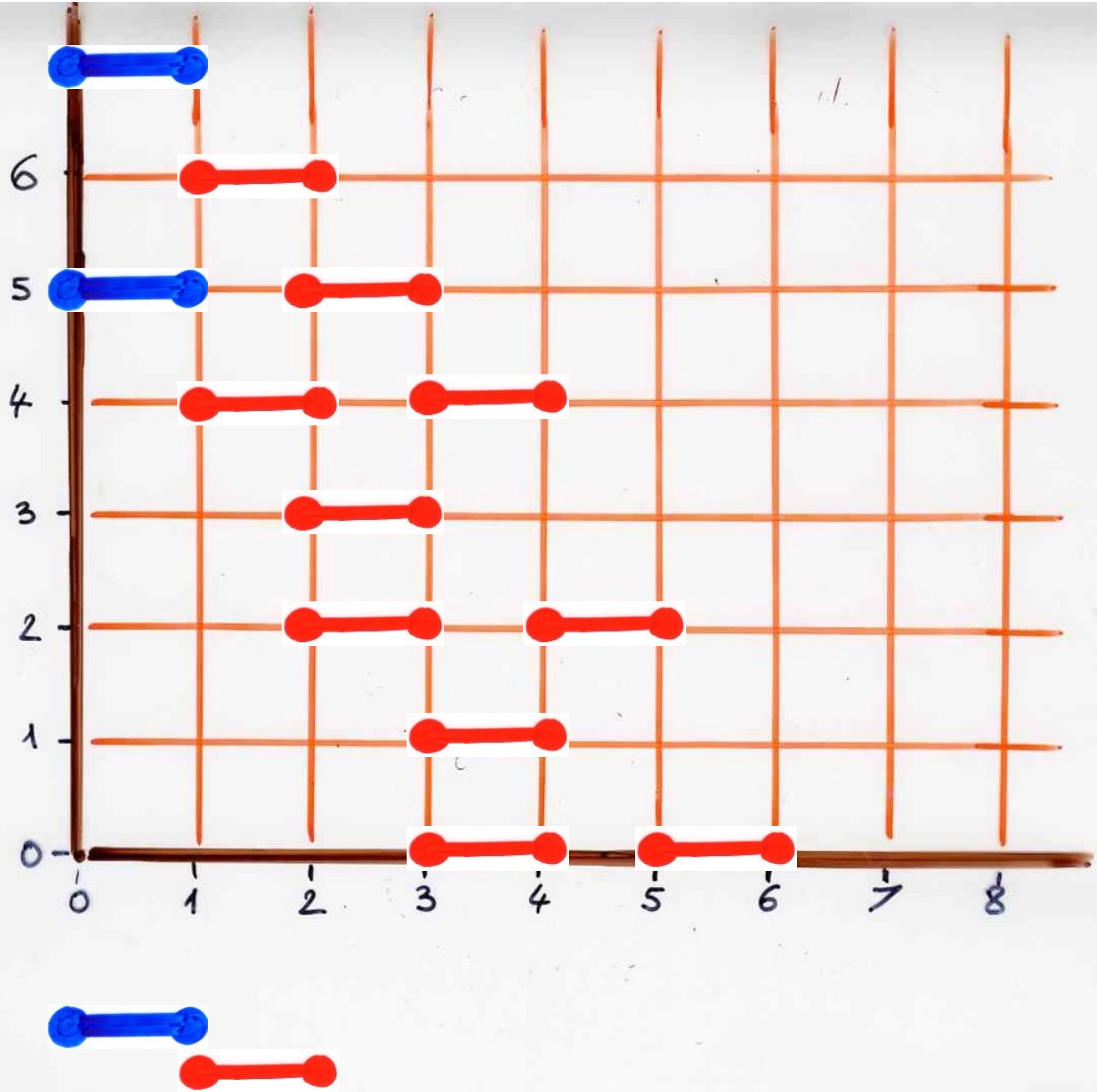

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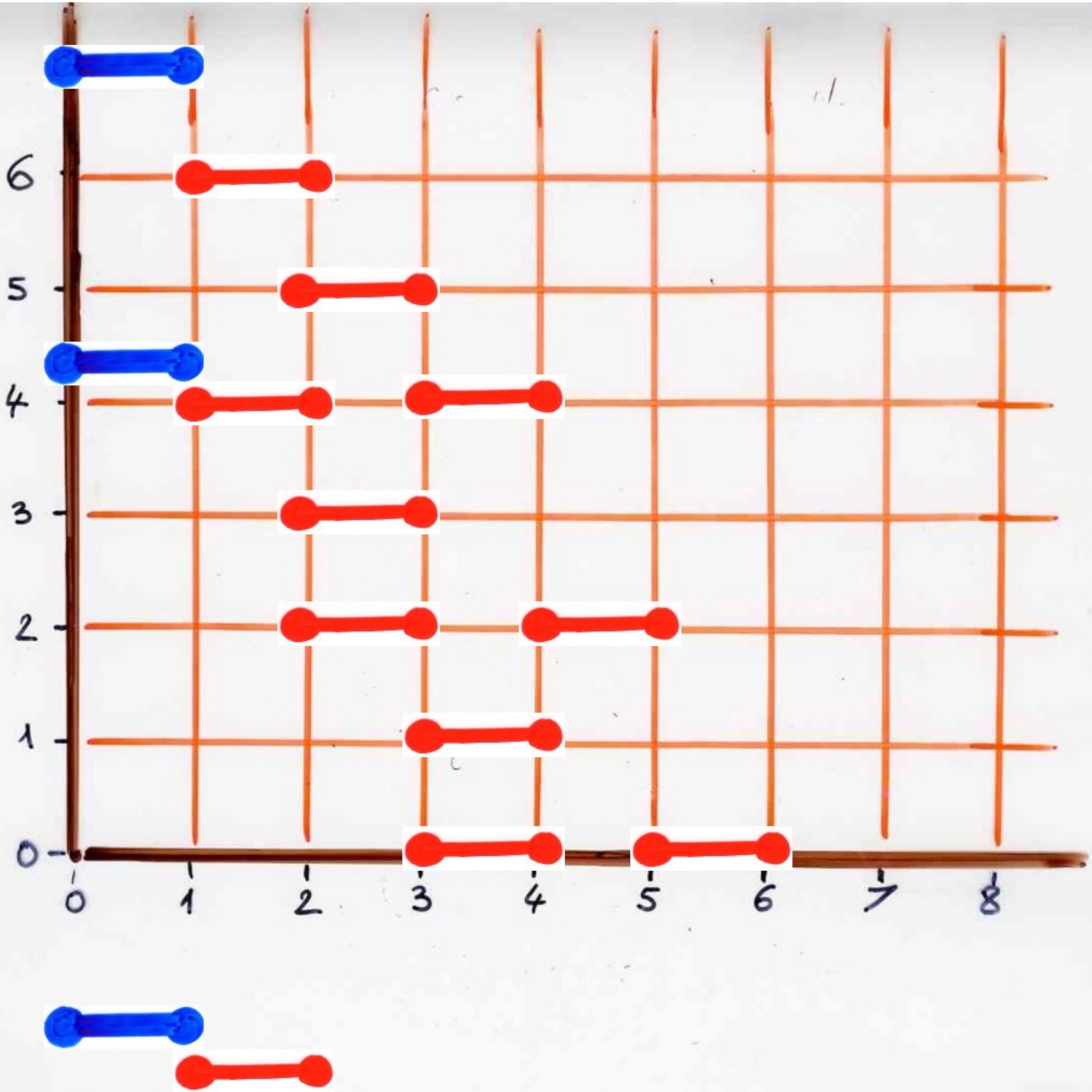
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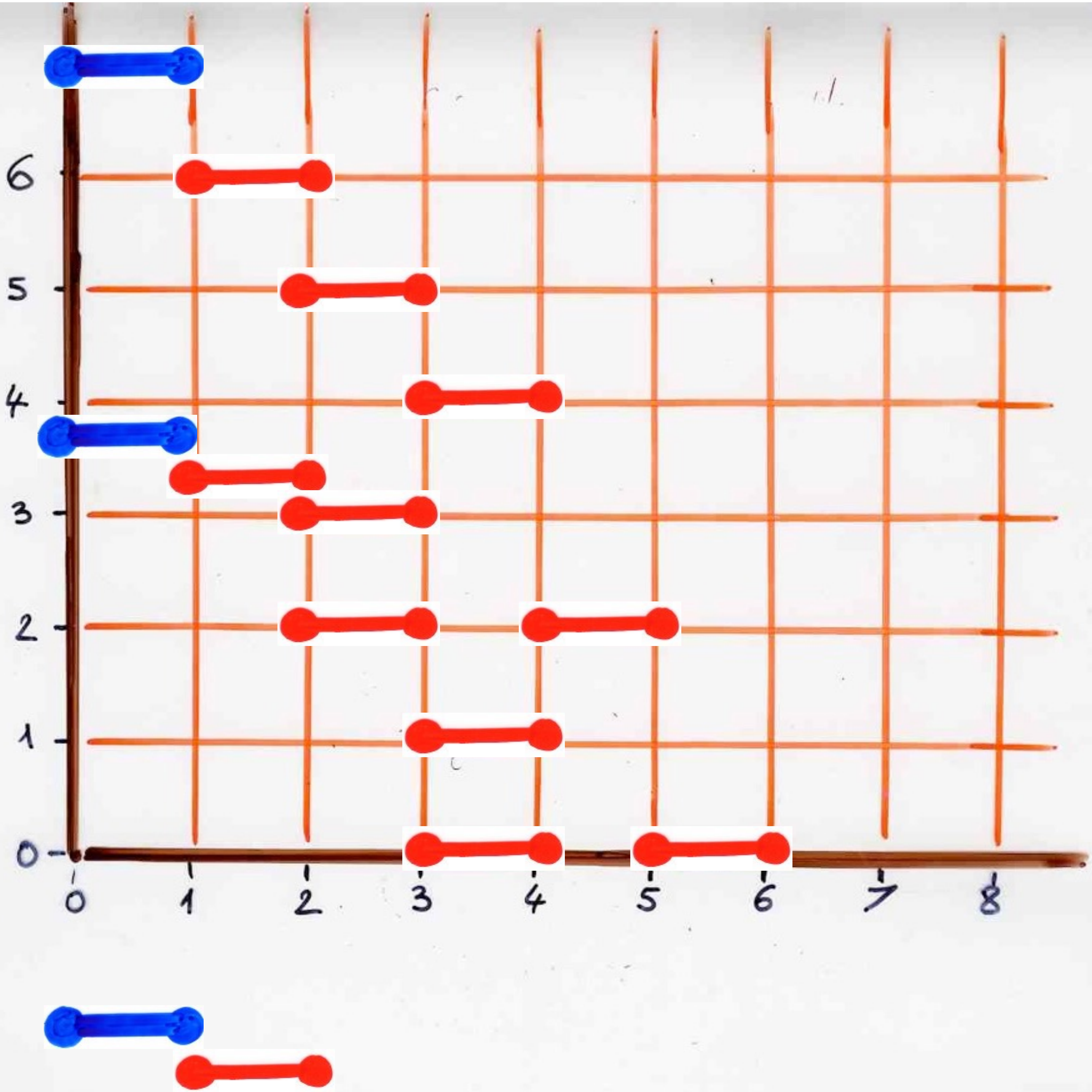

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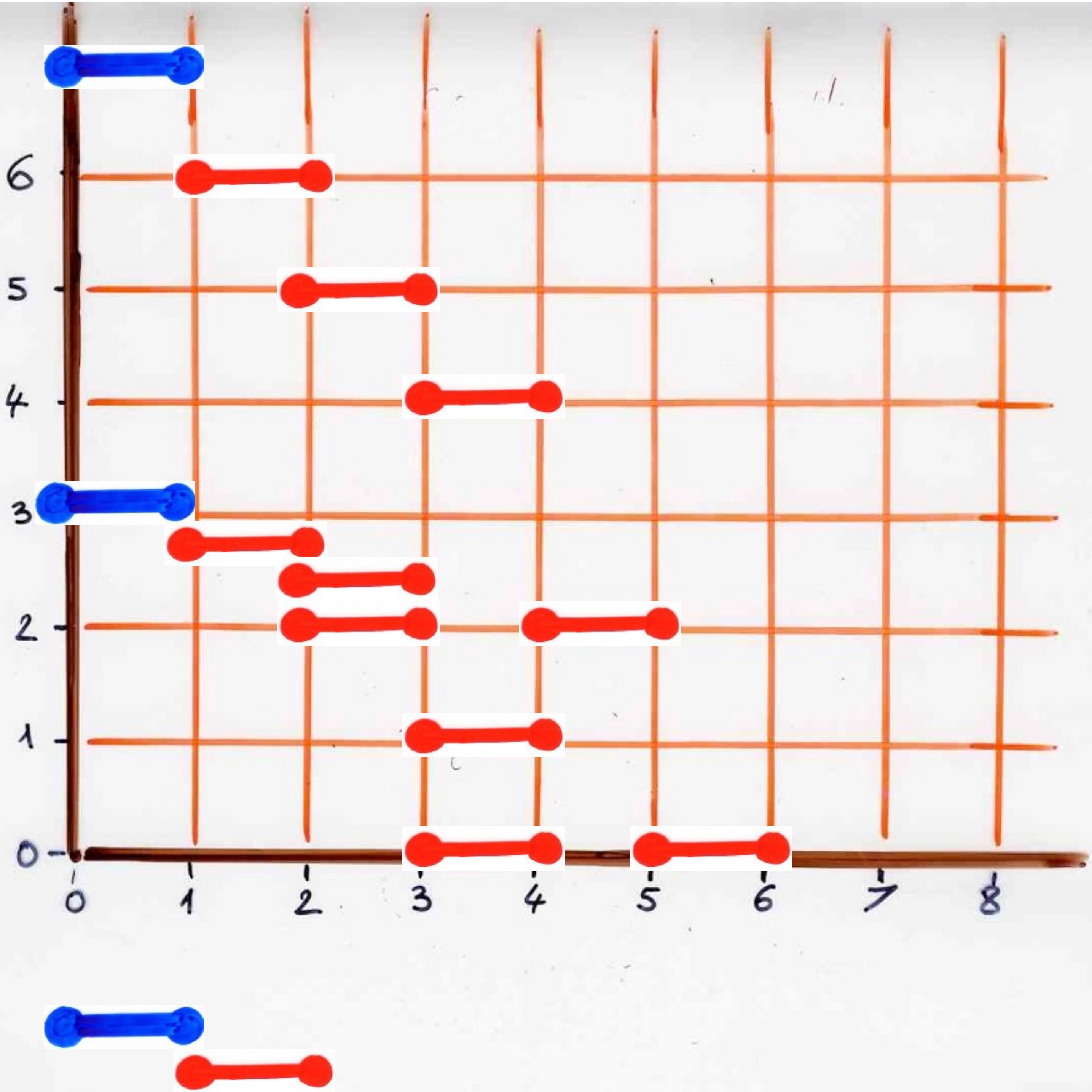

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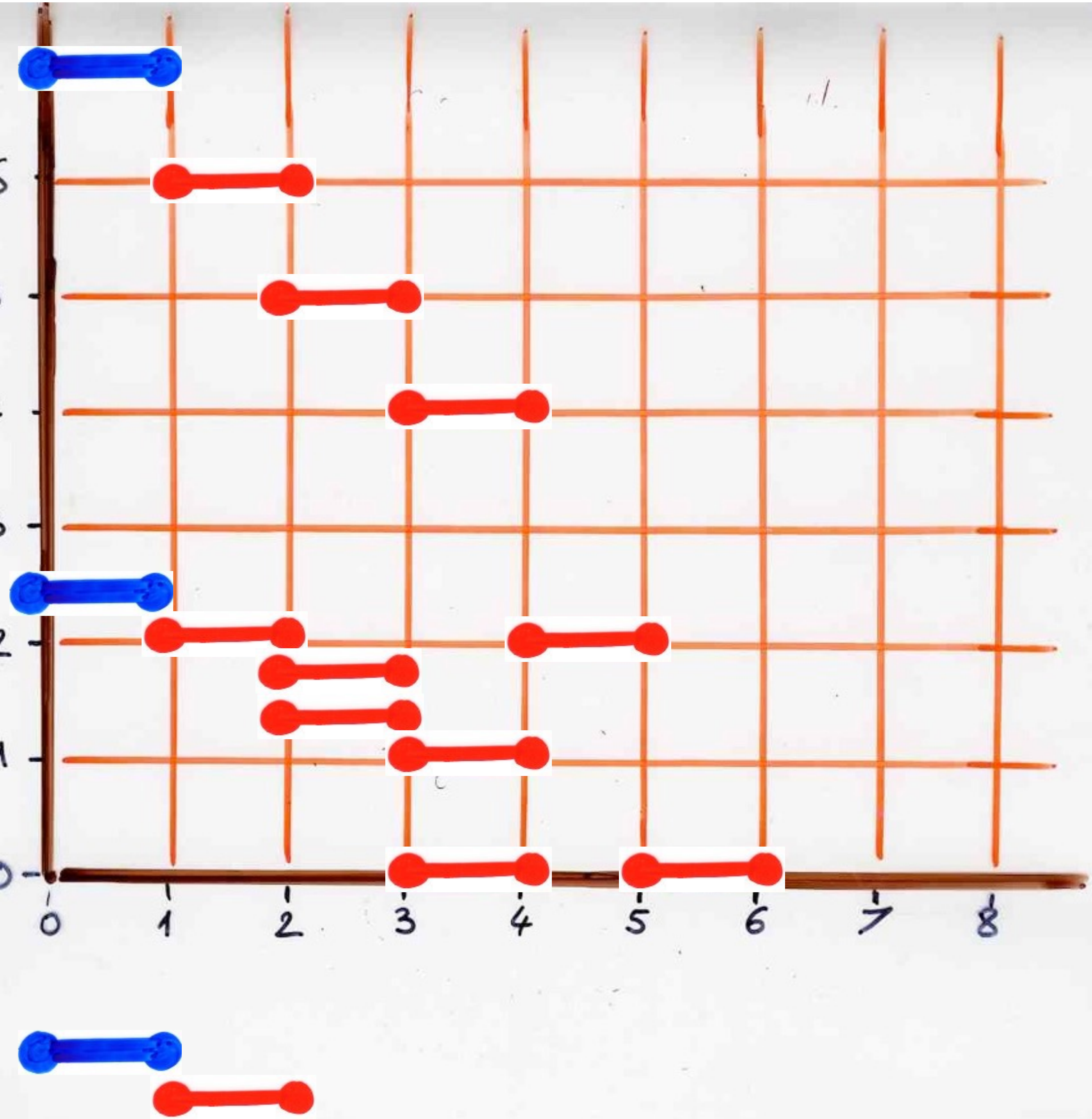

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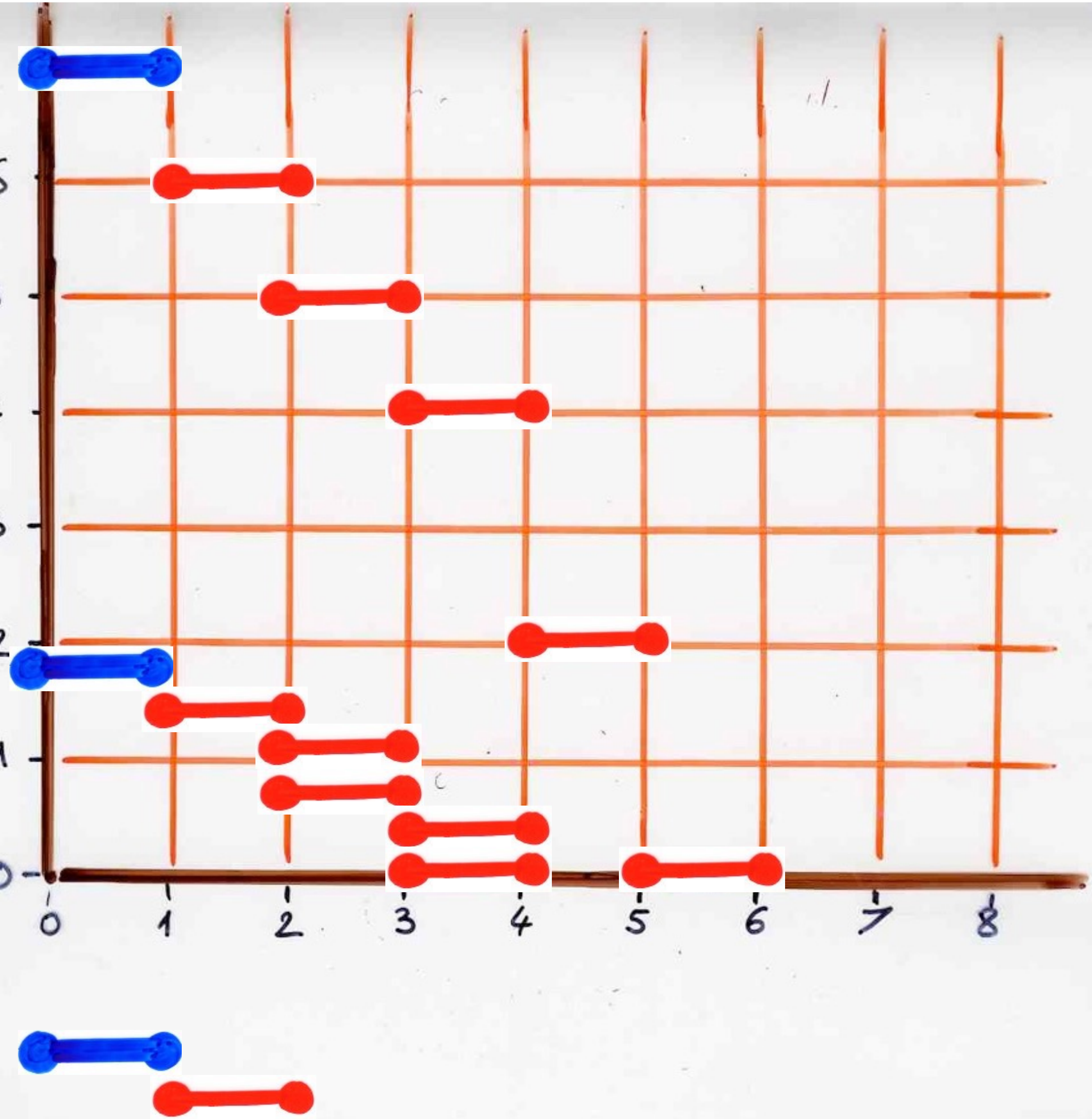

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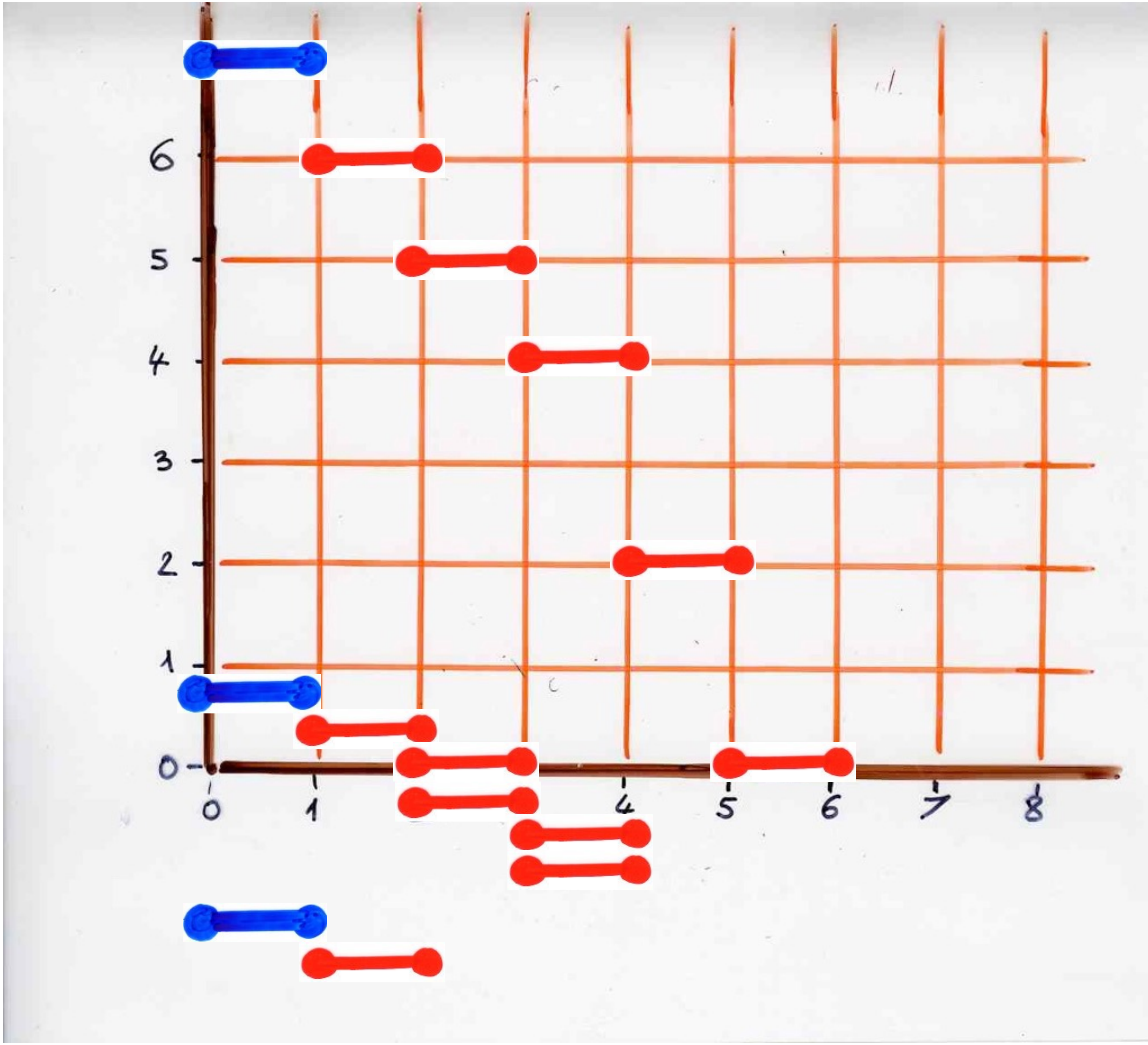

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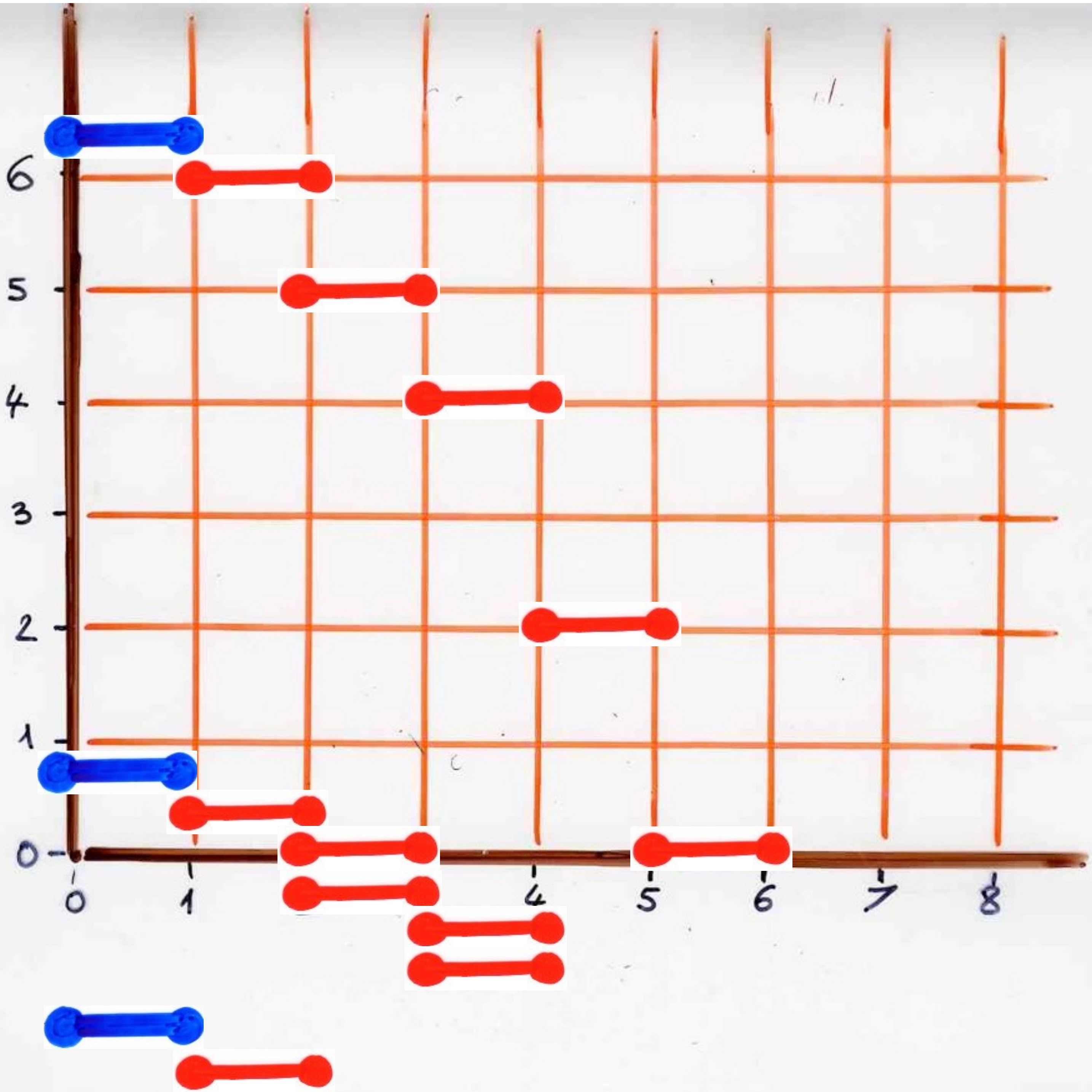

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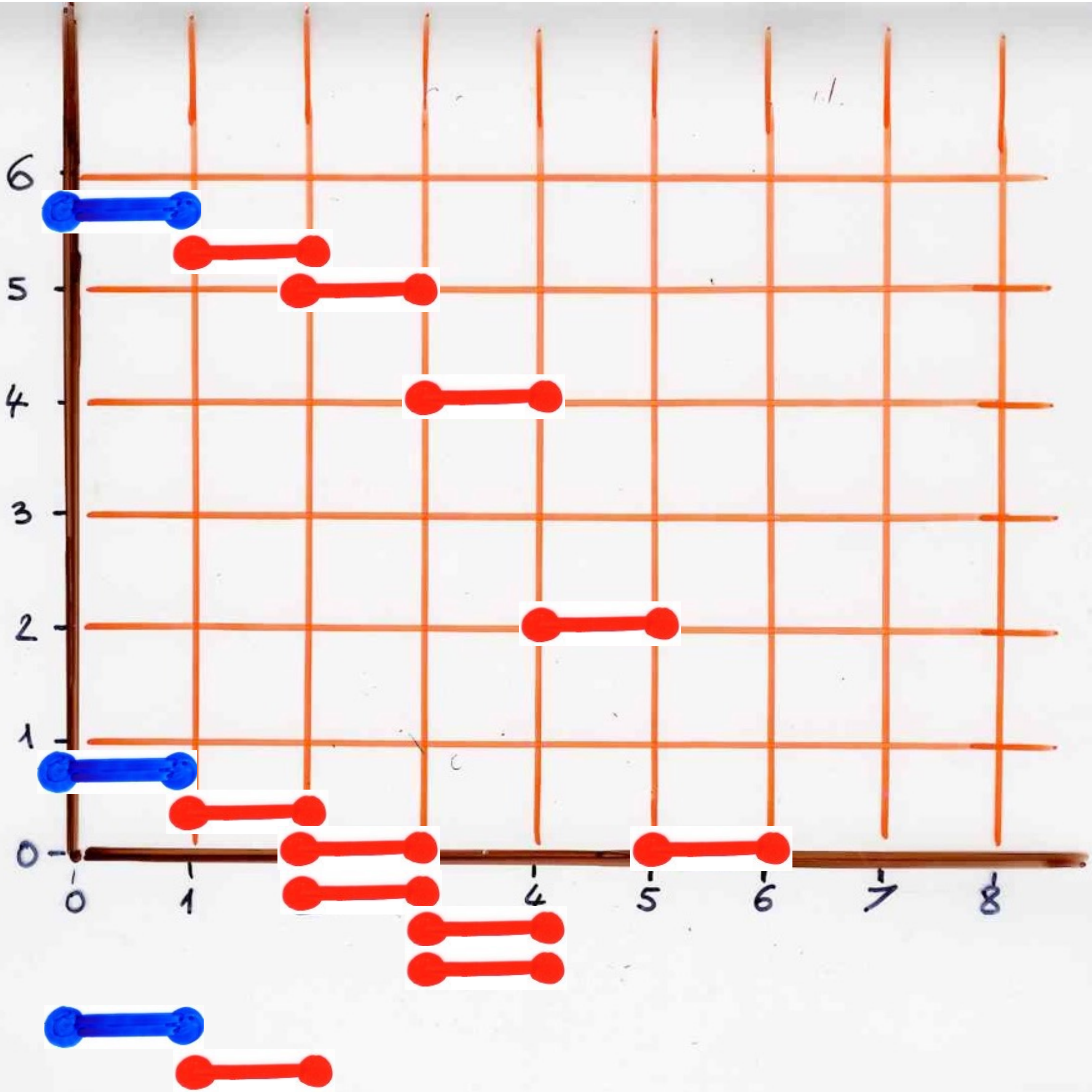

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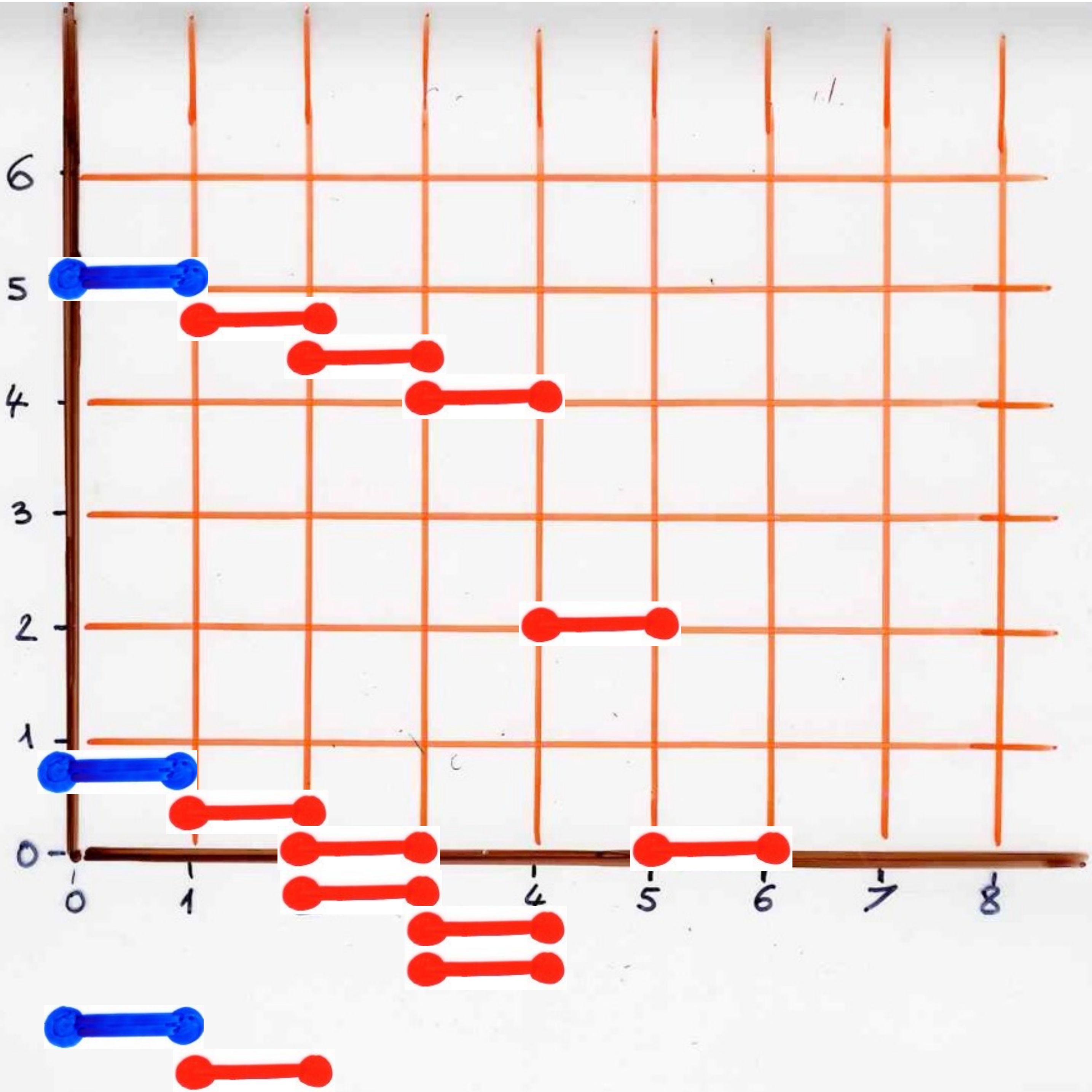

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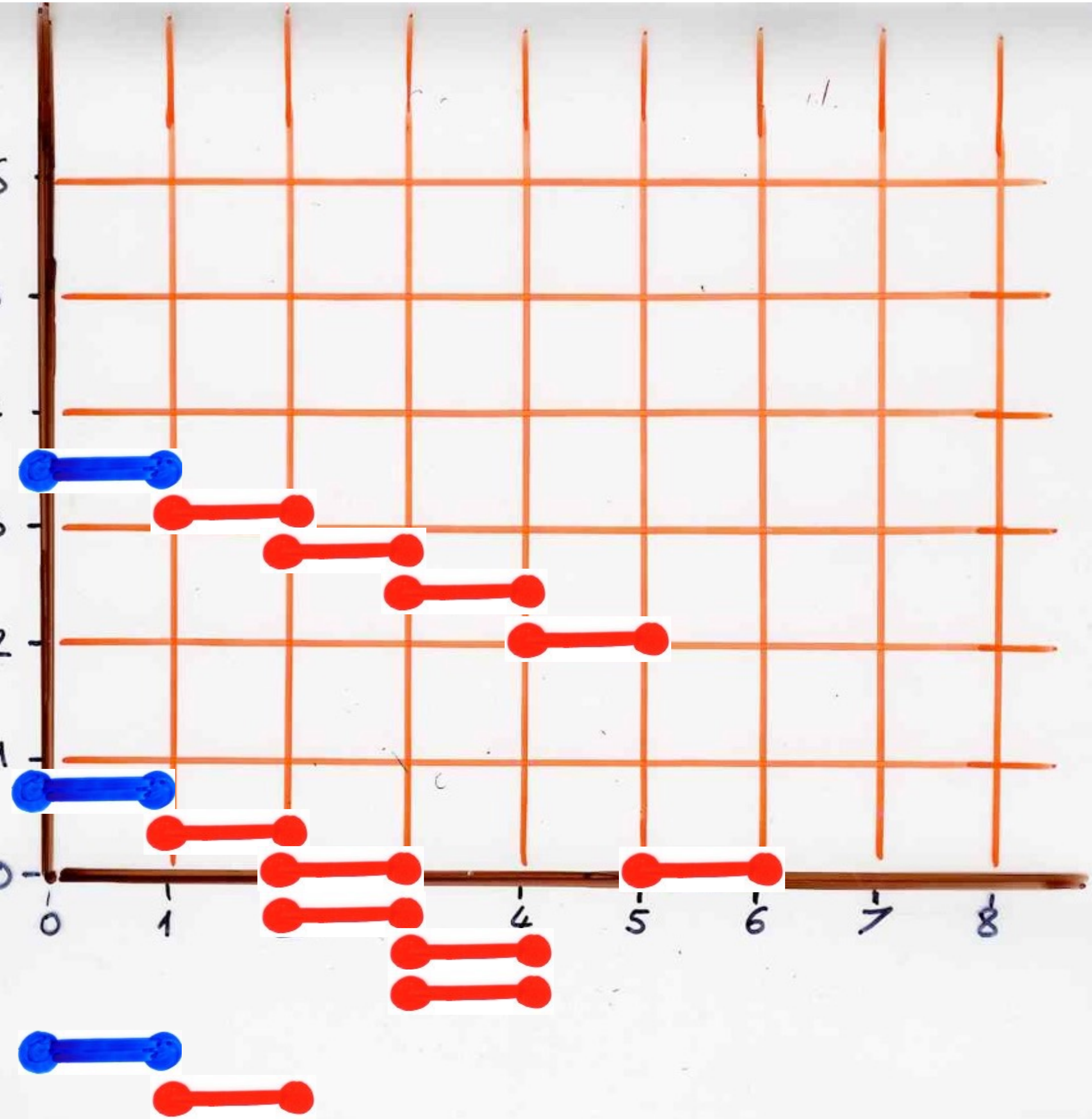
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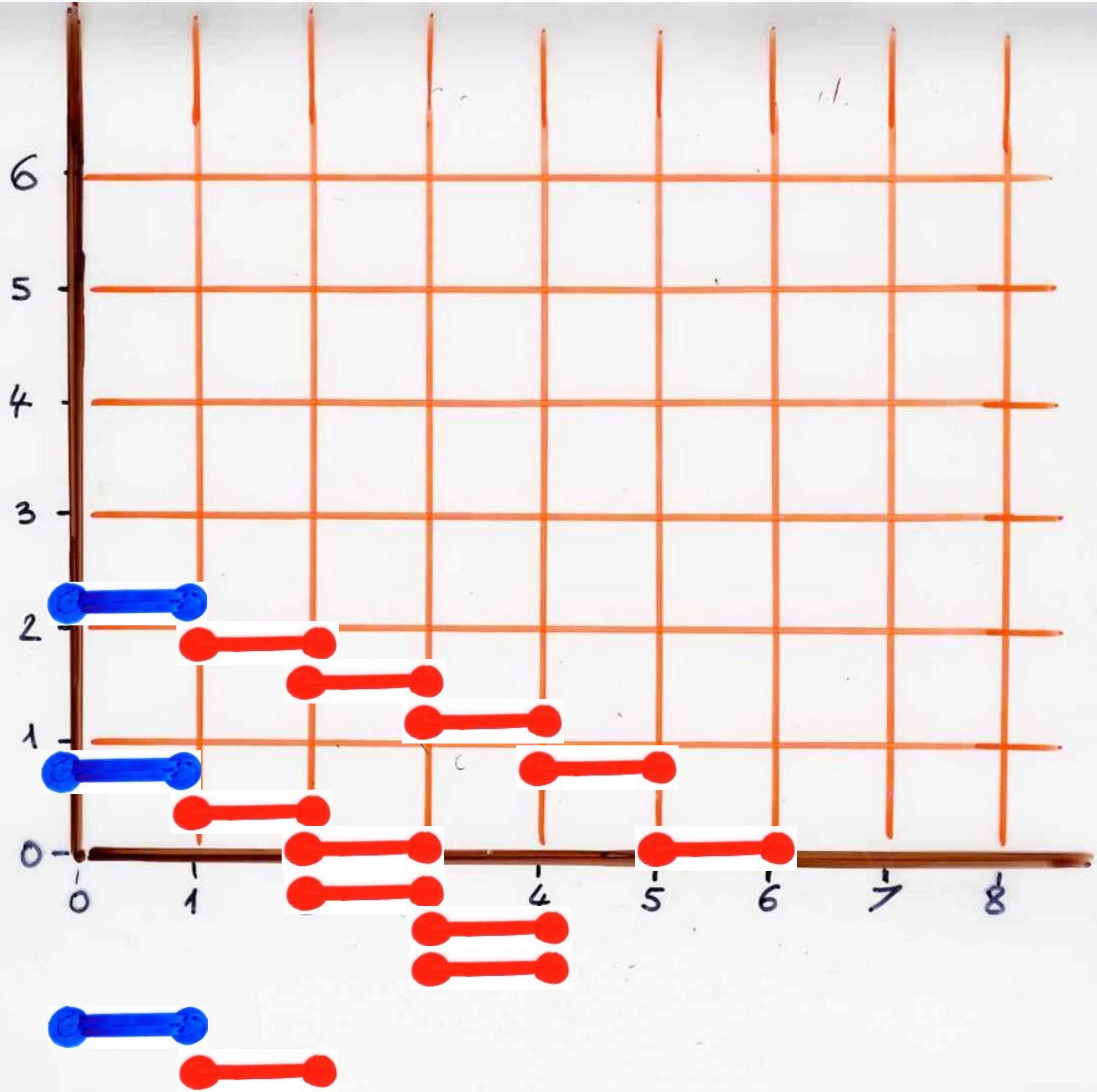

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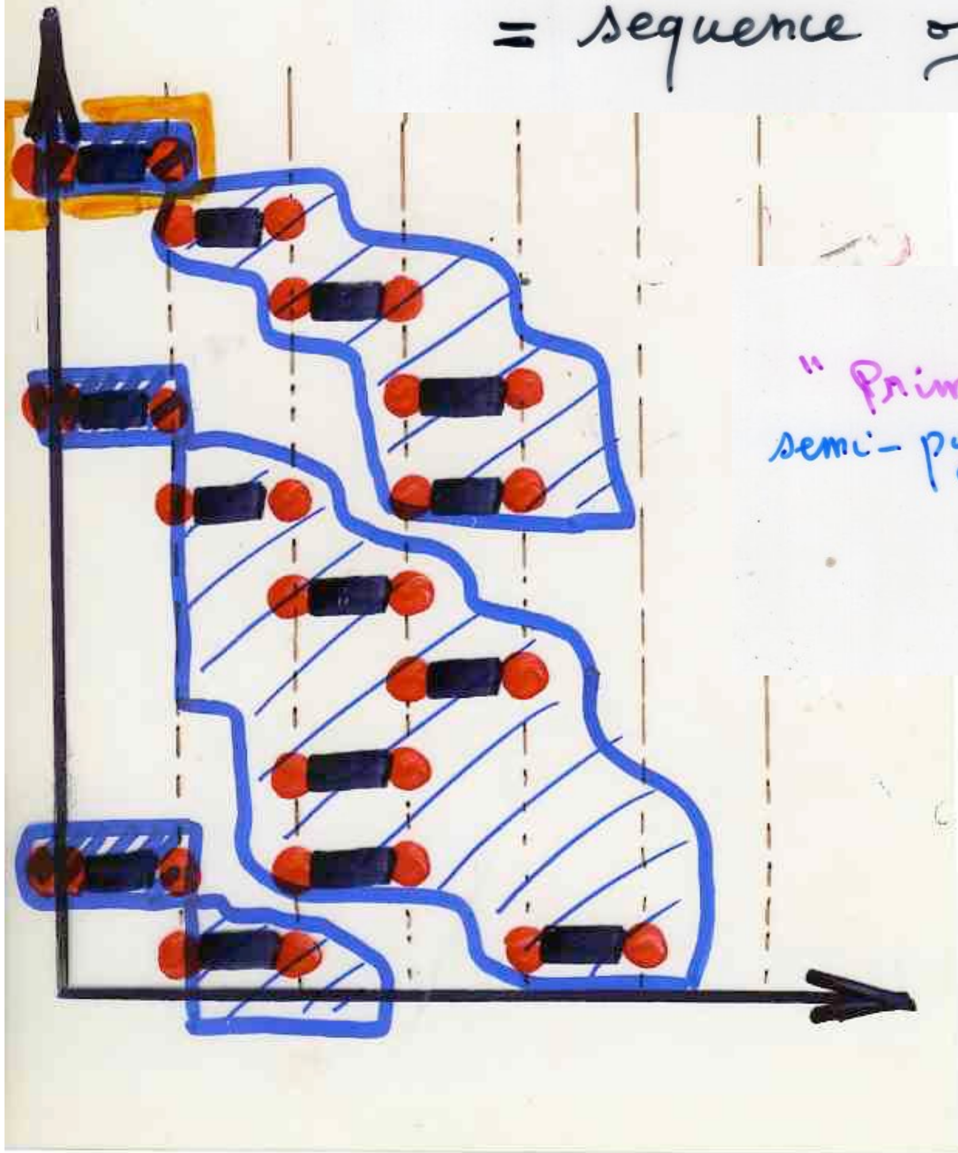
Semi-pyramid

= sequence of "primitive" semi-pyramids



"Primitive" semi-pyramid =

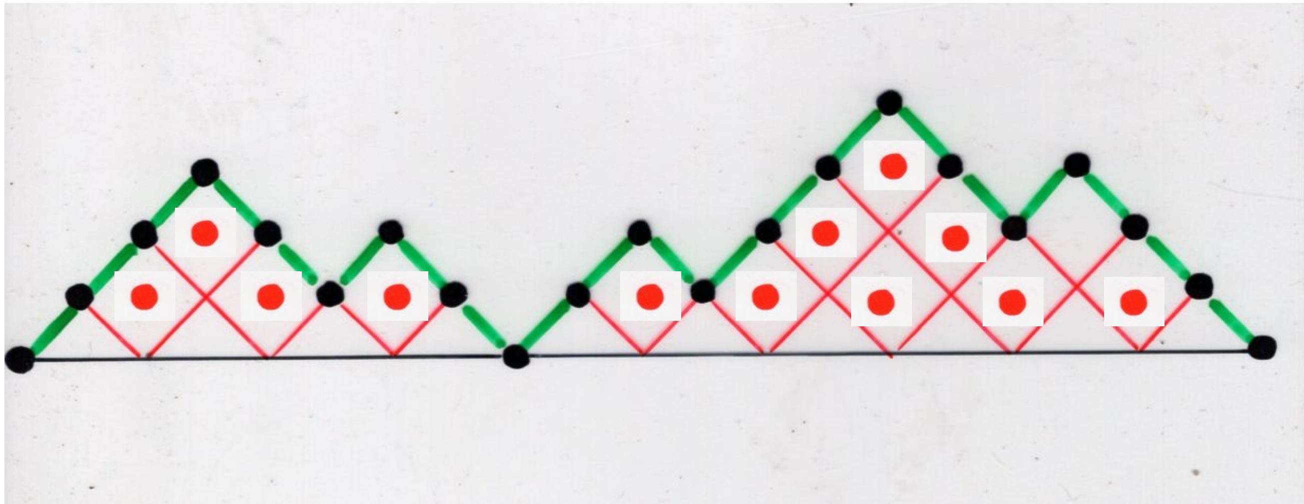
"shifted" semi-pyramid

weight : $q^{i} \xrightarrow{\delta} q^{i+1}$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{1}{1-(-q)\displaystyle\sum_{\substack{E \\ \text{semi-pyramids}}} \delta v(E)}$$

$$= \frac{1}{1+\dfrac{q}{1+q^2\displaystyle\sum_{\substack{E \\ \text{semi-pyramids}}} \delta^2 v(E)}}$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$



$$\sum_{\substack{\omega \\ \text{Dyck paths}}} q^{\mathrm{area}(\omega)}\, t^{|\omega|/2} = \cfrac{1}{1-\cfrac{t}{1-\cfrac{tq}{1-\cfrac{tq^2}{\cdots\cfrac{}{1-tq^k \cdots}}}}}$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots}{1+\cfrac{q^k}{\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

Hard Hexagons gas model

$$R(q) = \prod_{n \geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

Baxter (1980)

$$t = -q\left[R(q)\right]^5$$

$$Y(q) = \prod_{n \geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$Z(t)$

partition function

$$Z(t) = Y(q(t))$$

Andrews interpretation of the «reciprocal» of Ramanujan continued fraction

quasi-partitions of n

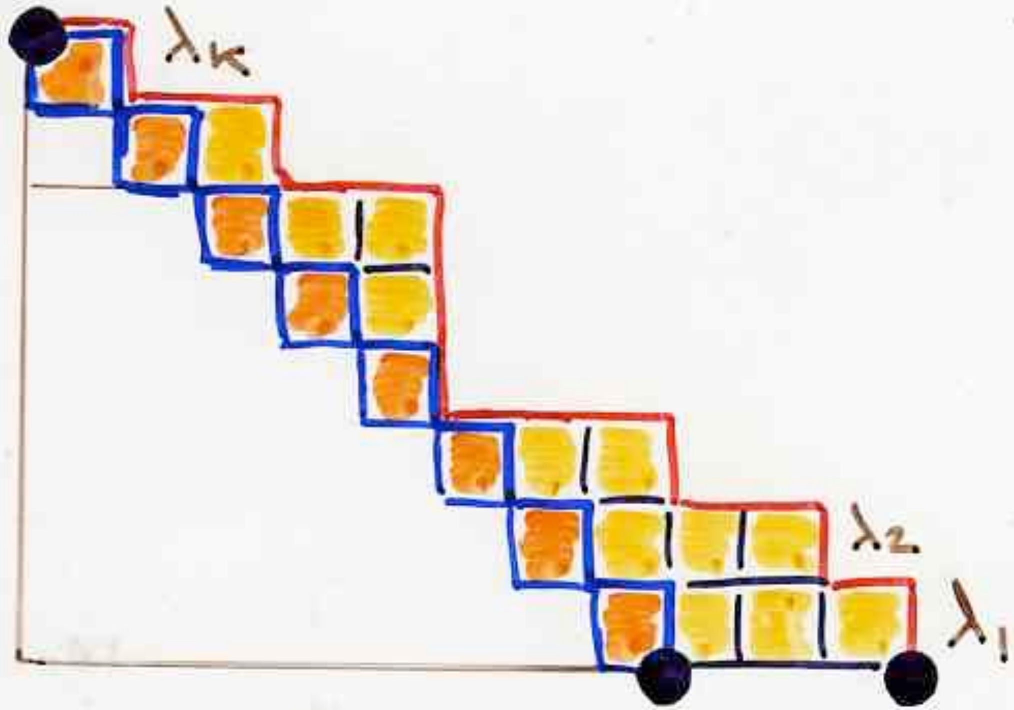
G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$1 + \lambda_i \geqslant \lambda_{i+1}$$

$$i = 1, \ldots, k-1$$

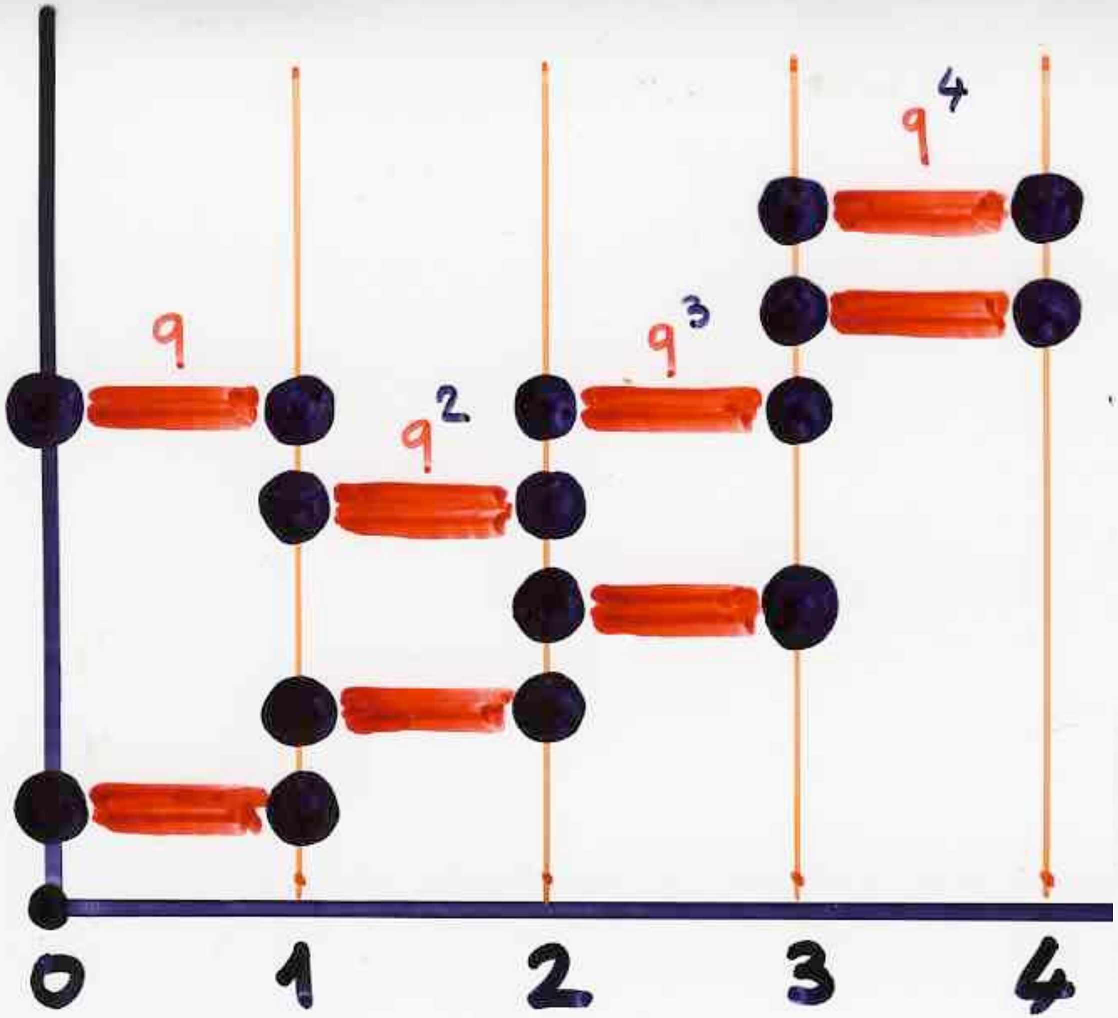
$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$



exercise
Ch1b, p71

weight-preserving
bijection

quasi-partitions
$\updownarrow$
heaps of dimers
on $\mathbb{N}$



exercise
Ch1b, p71

weight-preserving
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quasi-partitions
$\updownarrow$
heaps of dimers
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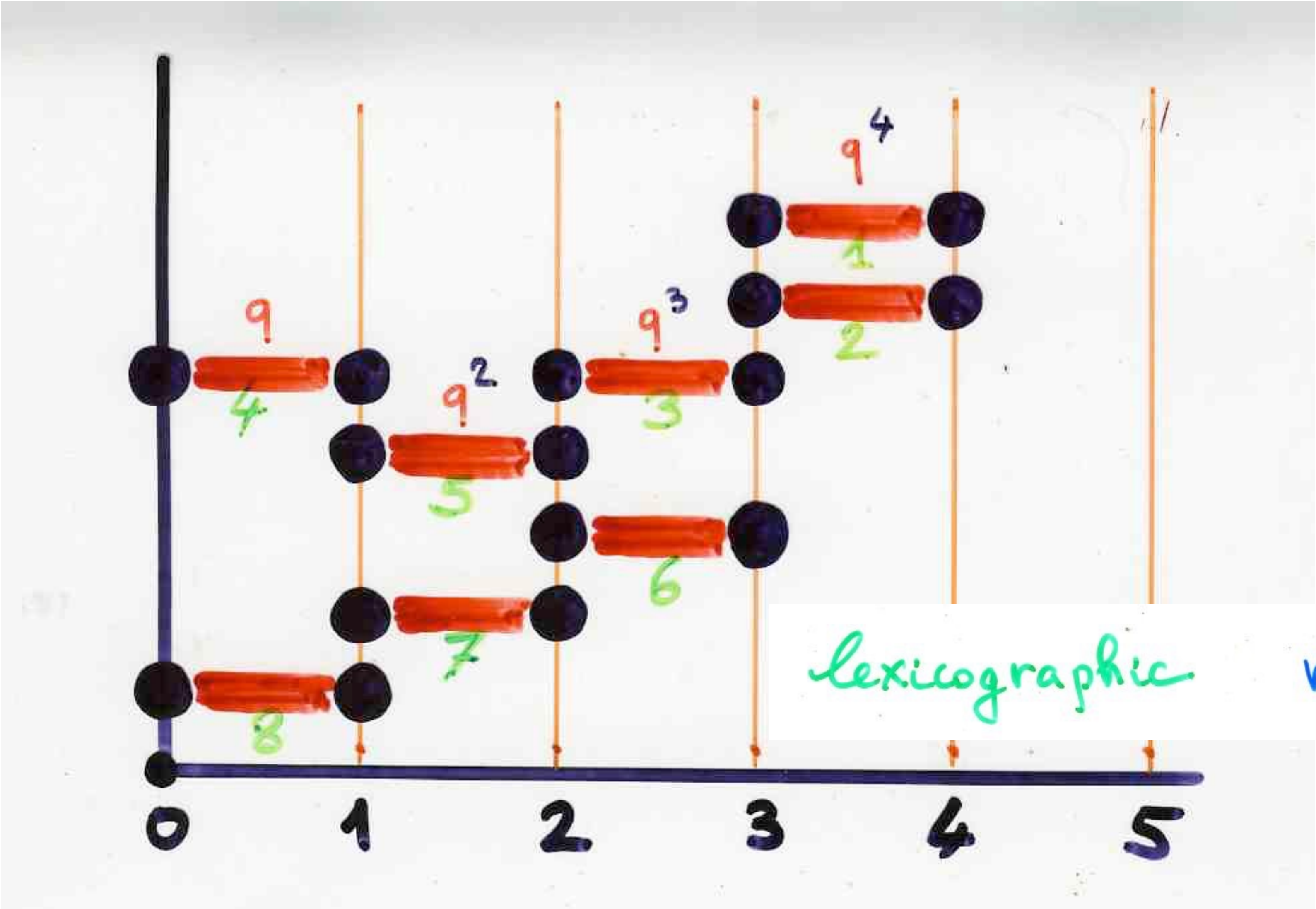


lexicographic normal form

$$H \longrightarrow \lambda = (\underset{1}{4}, \underset{2}{4}, \underset{3}{3}, \underset{4}{1}, \underset{5}{2}, \underset{6}{3}, \underset{7}{2}, \underset{8}{1})$$

quasi-partition

$$\frac{1}{D} = \sum_{\substack{E \\ \text{heaps of dimers}}} v(E)$$

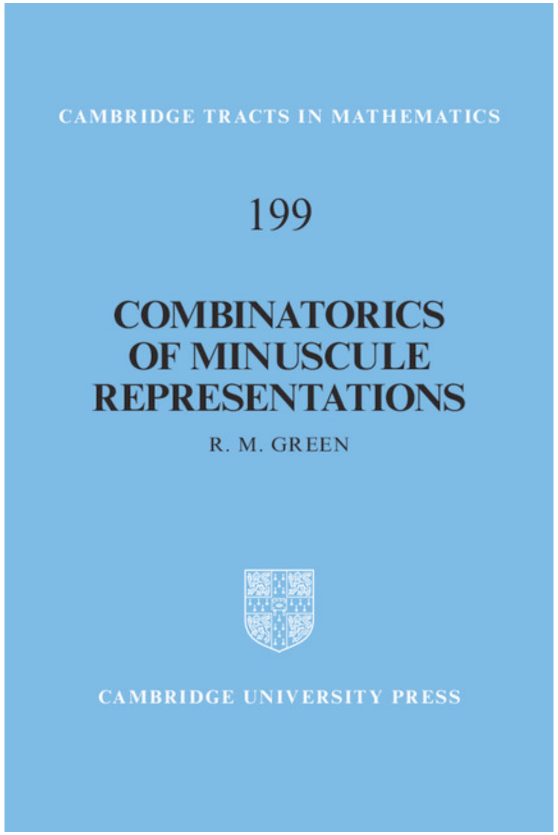
$$\frac{1}{R_I} = \sum_{\substack{\lambda \\ \text{quasi-partitions}}} (-1)^{\ell(\lambda)} q^{|\lambda|}$$

G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

other future chapters

Complementary topics

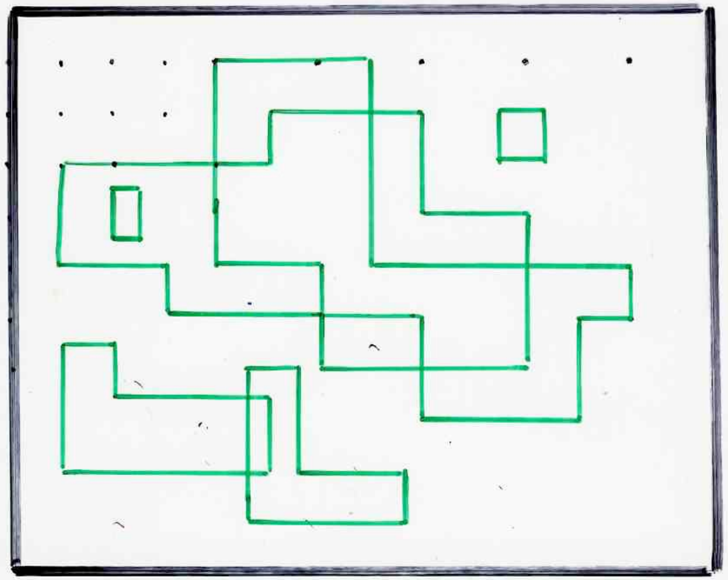
- minuscule representations of Lie algebra (R. Green and students) book
- basis of free partially commutative Lie algebra (Lalonde, Duchamp-Krob, ...)



Lyndon words

Lyndon heaps

R. Green (2013)



"closed" graph

Ising model

- statistical physics : (T. Helmuth)
  Ising model revisited
- string theory and heaps
  gauge theory, quivers
  (Ramgoolam)

Q-systems, heaps, paths and clusters positivity

Di Francesco, Kedem (2008)

- computer science:
  the SAT problem revisited with heaps
  (D. Knuth, vol 4, Fascicle 6)

- computer science:
  Petri nets, asynchronous automata,
  Zielonka theorem