Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.xavierviennot.org

Chapter 7

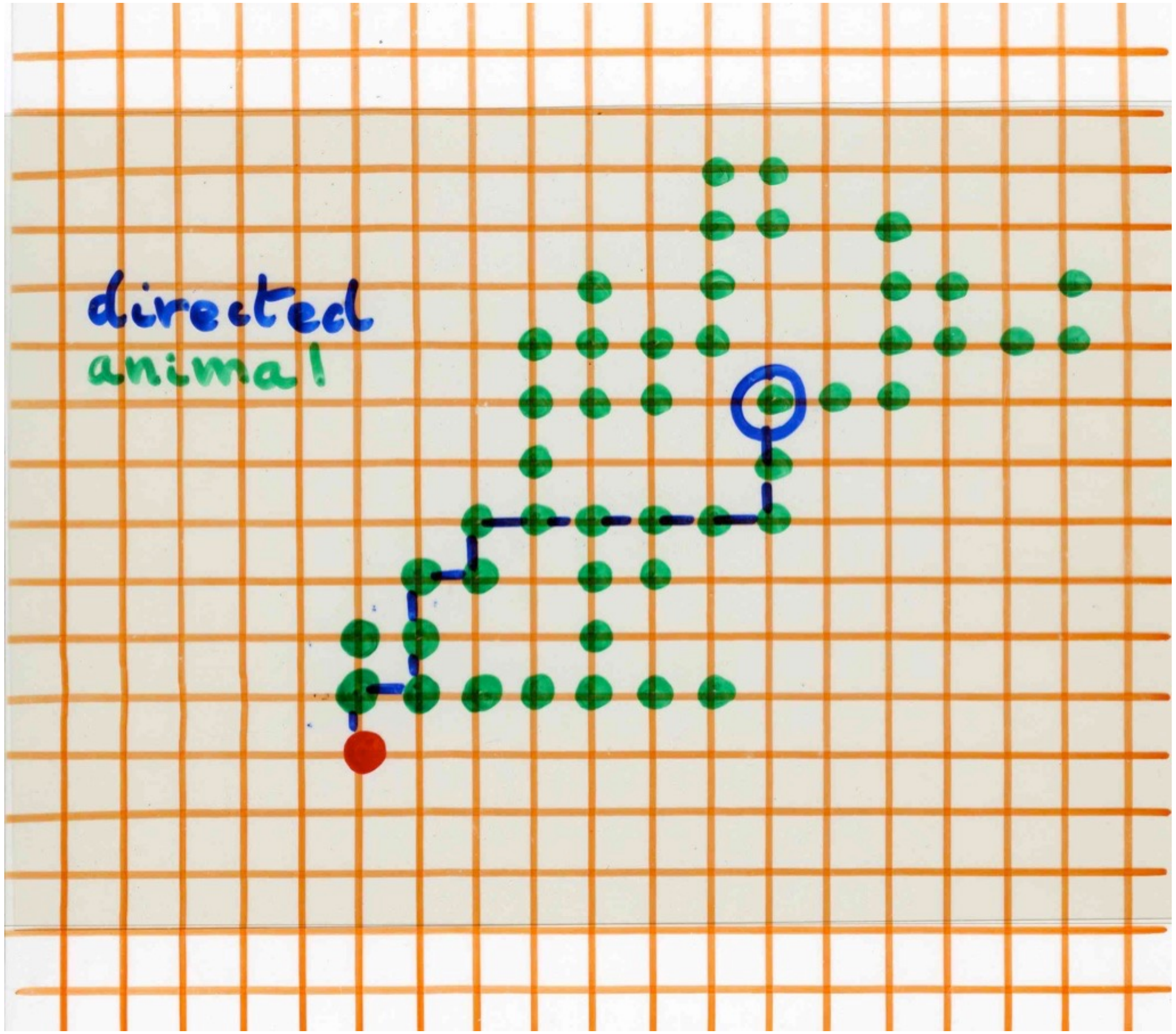
Heaps in statistical mechanics

(2)

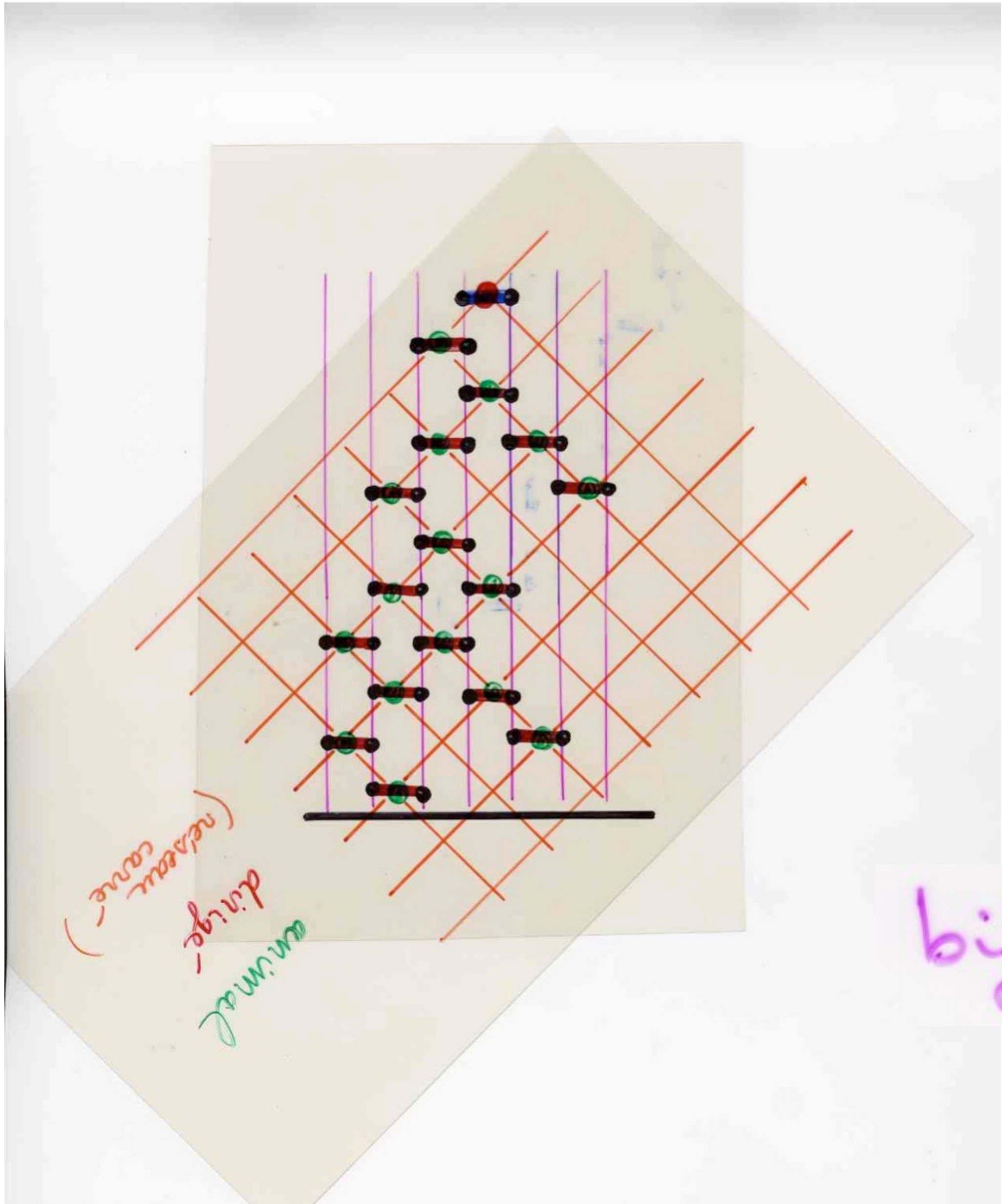
IMSc, Chennai

13 March 2017

from the previous lecture



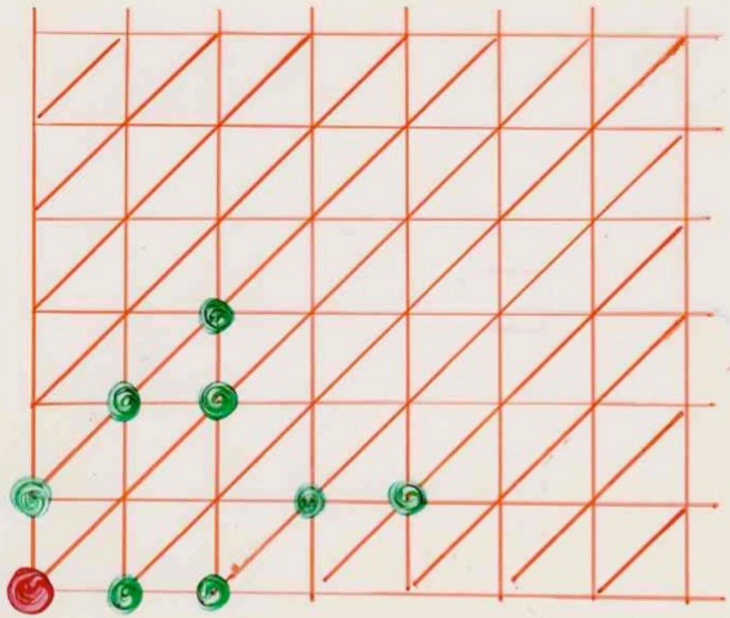
directed
animal
(square
lattice)



bijection

directed
animal
(square
lattice)

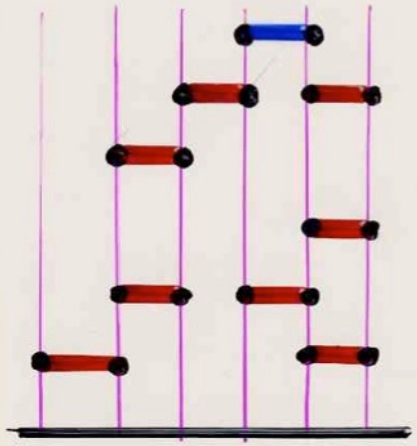
strict
pyramids



animal dirigé

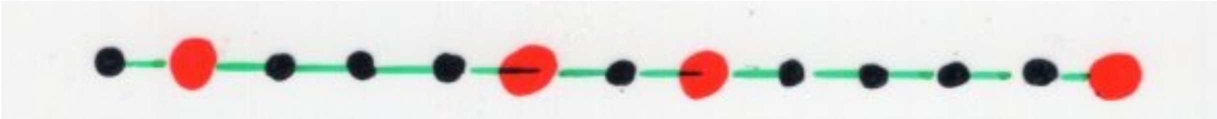
(réseau triangulaire)

bijection



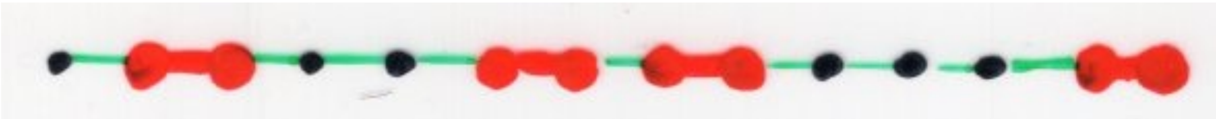
pyramids

1D Gas model



1 n

Partition function
$Z_n(t)$



$$Z_n(t) = F_n(-t)$$

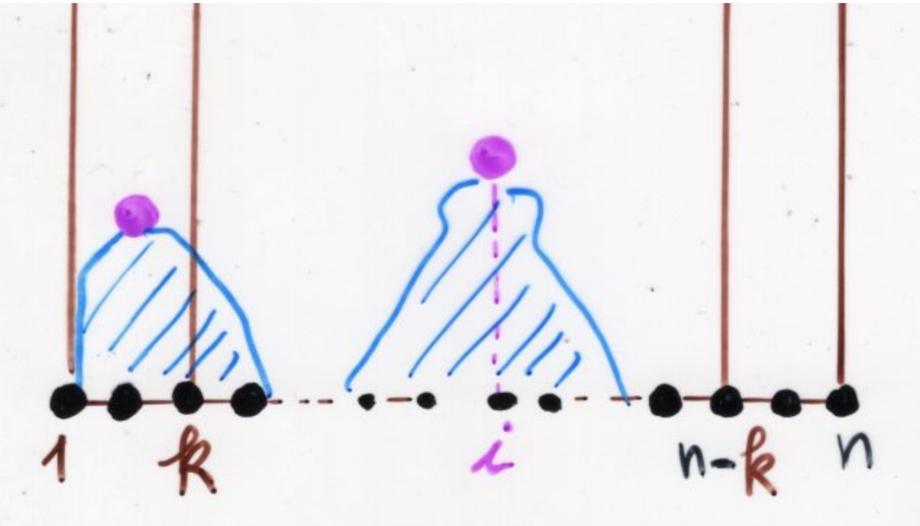
Fibonacci polynomials

$$\lim_{n\to\infty} \left(Z_n(t)\right)^{1/n}$$

thermodynamic limit

logarithmic lemma

$$-t\,\frac{d}{dt}\log \mathbf{Z}_n^{1/n}(-t) = \underbrace{\frac{1}{n}\,t\,\frac{d}{dt}\log\frac{1}{\mathbf{Z}_n(-t)}}_{\frac{1}{n}\,\mathbf{Pyr}_n(t)}$$



density of the gas

t activity

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

$-\rho(-t)$ is the generating function of pyramids on $\mathbb{Z}$ (up to translation)

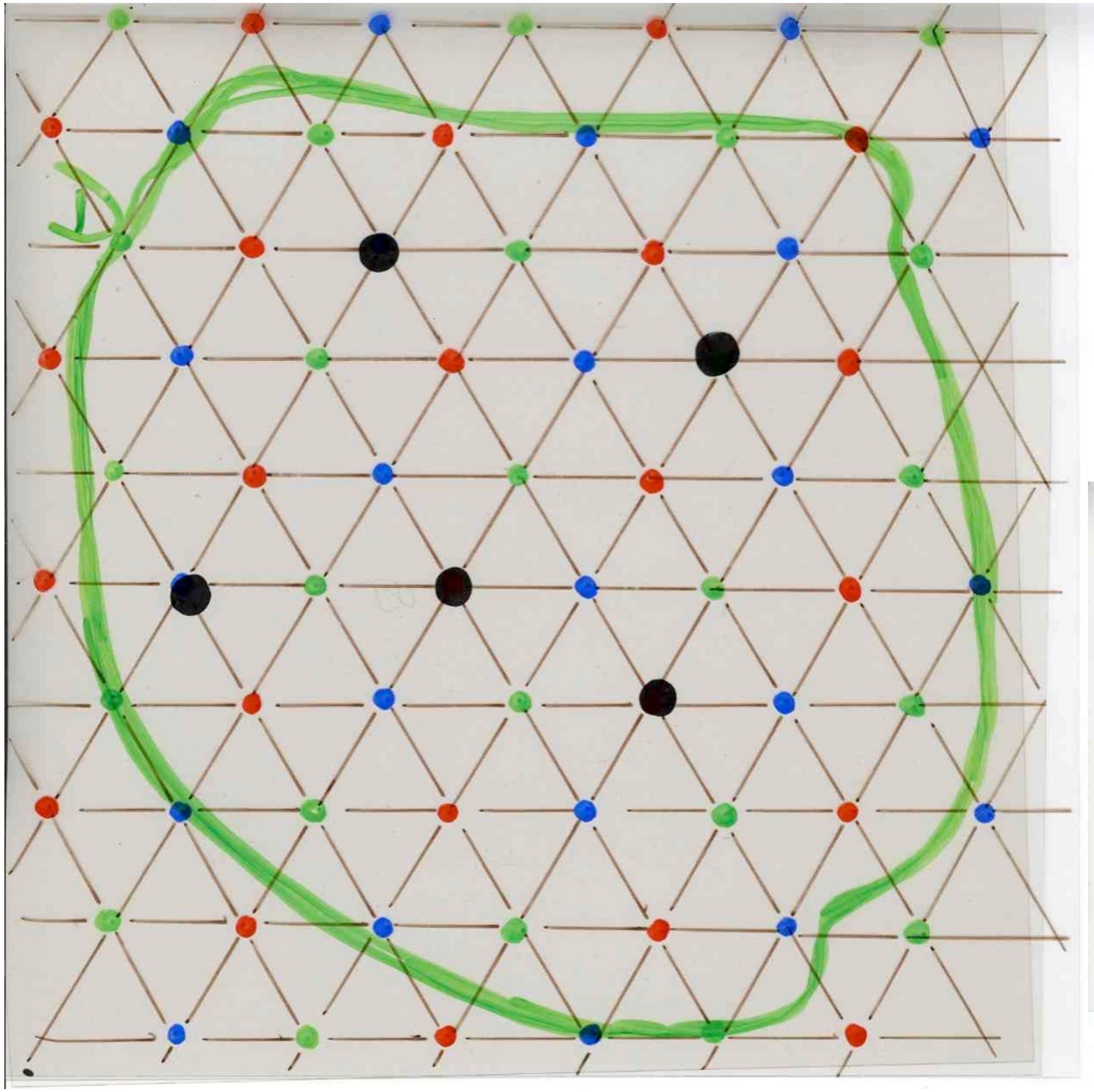
$$\frac{1}{2}\binom{2n}{n}$$

$$\frac{1}{2}\frac{1}{\sqrt{1-4t}}$$

directed animals

on the triangular lattice

lattice



partition function

$$Z_D(t) = \sum_{n\geqslant 0} a_{n,D}\, t^n$$

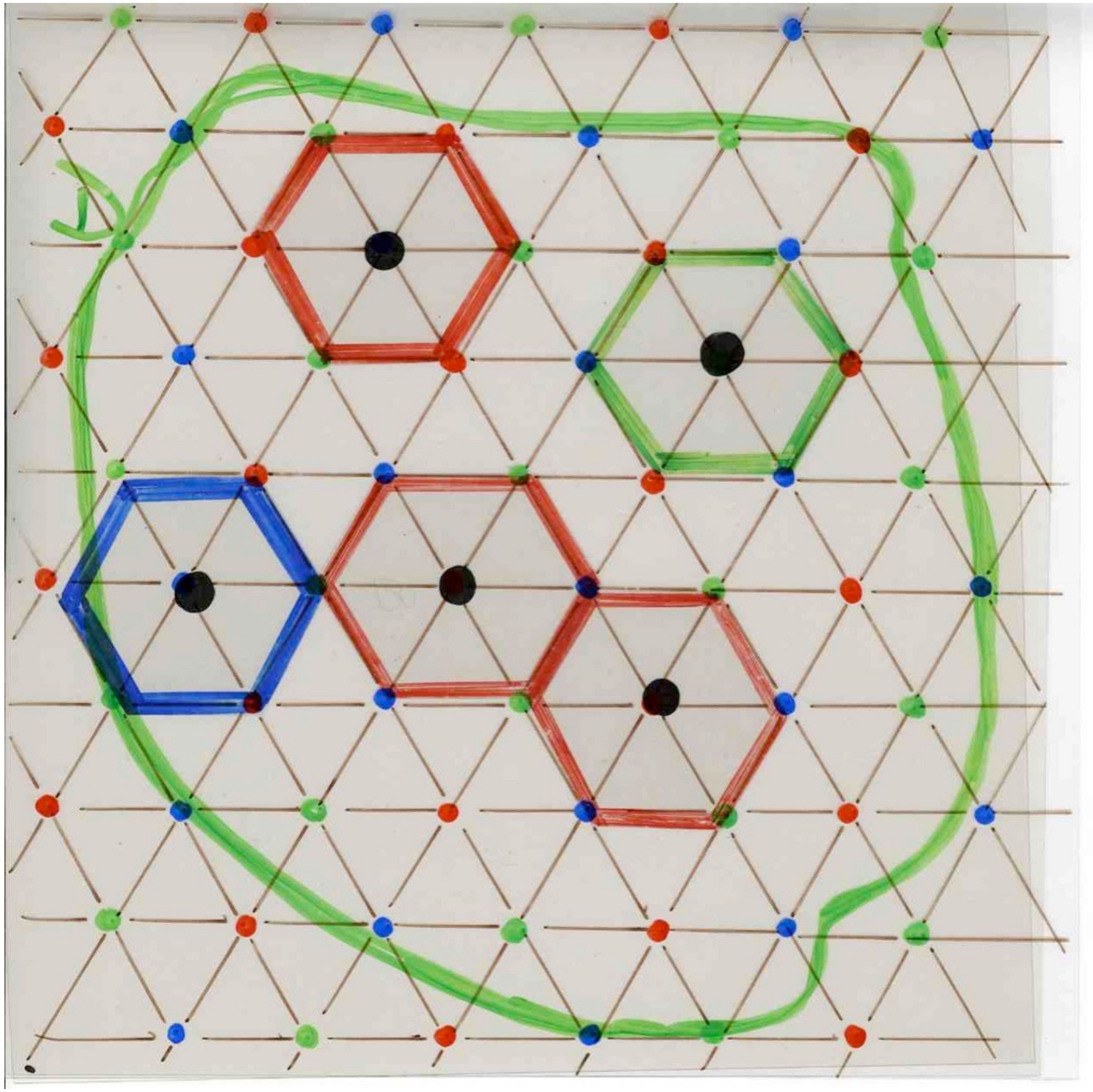
$$Z(t) = \lim_{\text{“}D\to\infty\text{”}} \left(Z_D(t)\right)^{1/|D|}$$

thermodynamic limit

density of the gas

t activity

$$\rho(t) = t\,\frac{d}{dt}\log Z(t)$$



partition function

$$Z_D(t) = \sum_{n \geq 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{“}D \to \infty\text{”}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

density of the gas

t activity

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

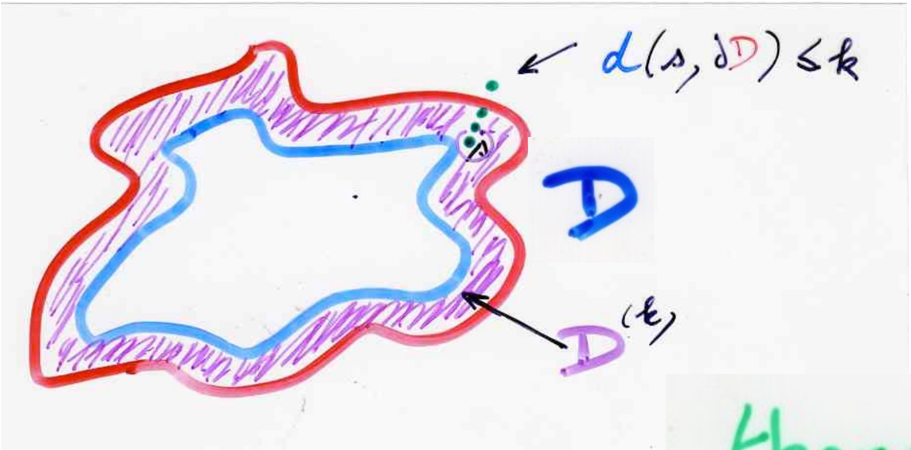
$$\log Z_D(t)^{1/|D|} = \frac{1}{|D|} \log Z_D(t)$$

logarithmic lemma

$$\frac{1}{|D|} P_D(t)$$

$$P_D(t) = (-t)\frac{d}{dt} \log Z_D^{-1}(-t)$$

generating function
for pyramids of
hexagons on D



thermodynamic limit

$$\frac{1}{|D|} P_D(t)$$

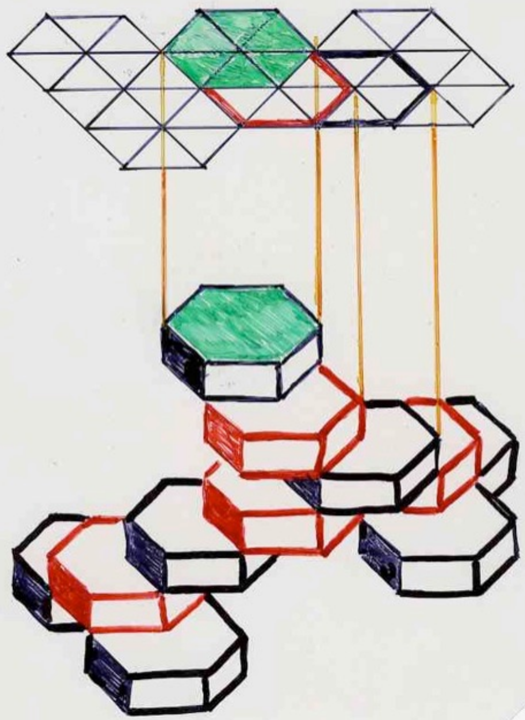
generating function for pyramids of hexagons on D

Proposition

$$-\rho(-t) = \sum_{n \geq 1} a_n t^n$$

number of pyramids on the triangular lattice with n hexagons (up to translation)

$$-p(-t) = y$$



Hard hexagons
algebricity ?

Research project
5^{+++}
(scale: 1, 2, .., 4, 5)

Prove (directly) that the generating function $-\rho(-t)$ for the number of pyramids of hexagons (up to translation) on the triangular lattice satisfies the following algebraic system of equations:

["directly" means: without using Baxter resolution of the hard hexagonal model]

Research project

5^{+++}

(scale: 1, 2, .., 4, 5)

S

$$y = 1 + t y^2 + t y^3$$

$$f = t^2 y^5$$

$$g = 1 + 3 f g^2 - f^2 g^3 (g - 1)$$

$$h = t g y^3$$

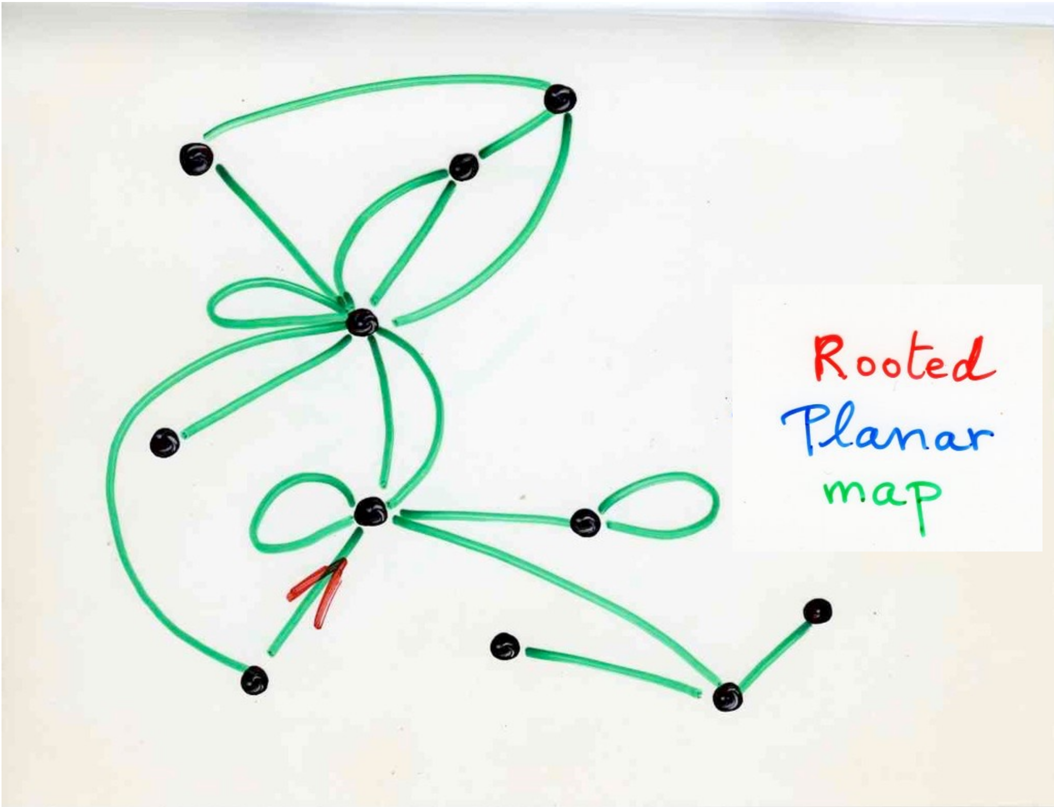
$$-p(-t) = \frac{h}{1 - h}$$

primitive pyramid

hexagons pyramid

similiraty

with the algebraic
system of equations for



Tutte (1960)

a_n number of rooted planar maps with n edges

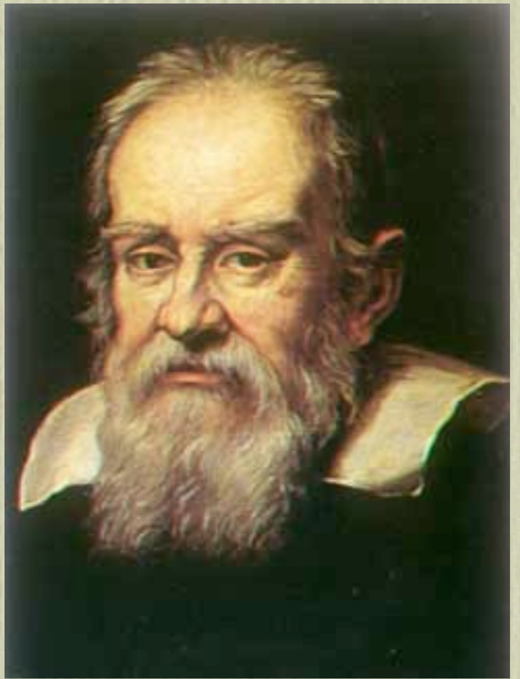
$$y = \sum_{n \geq 0} a_n t^n$$

$$\begin{cases} h = 1 + 3t h^2 \\ y = h - t h^3 \end{cases}$$

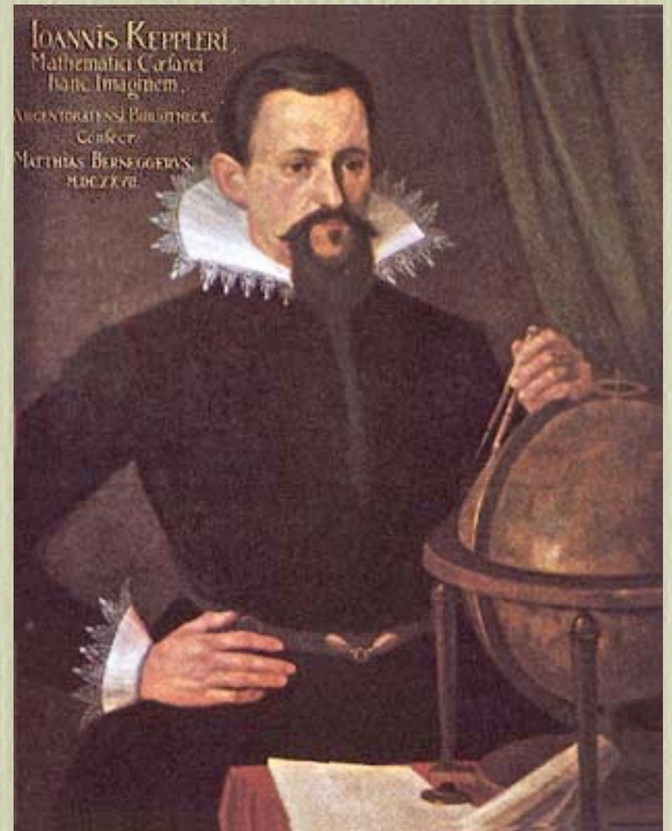
Cori, Vauquelin (1970)

Lorentzian triangulations in 2D qantum gravity

a (very) brief introduction
to
quatum gravity



Galileo Galilei
1564-1642



Johannes Kepler
1571 - 1630

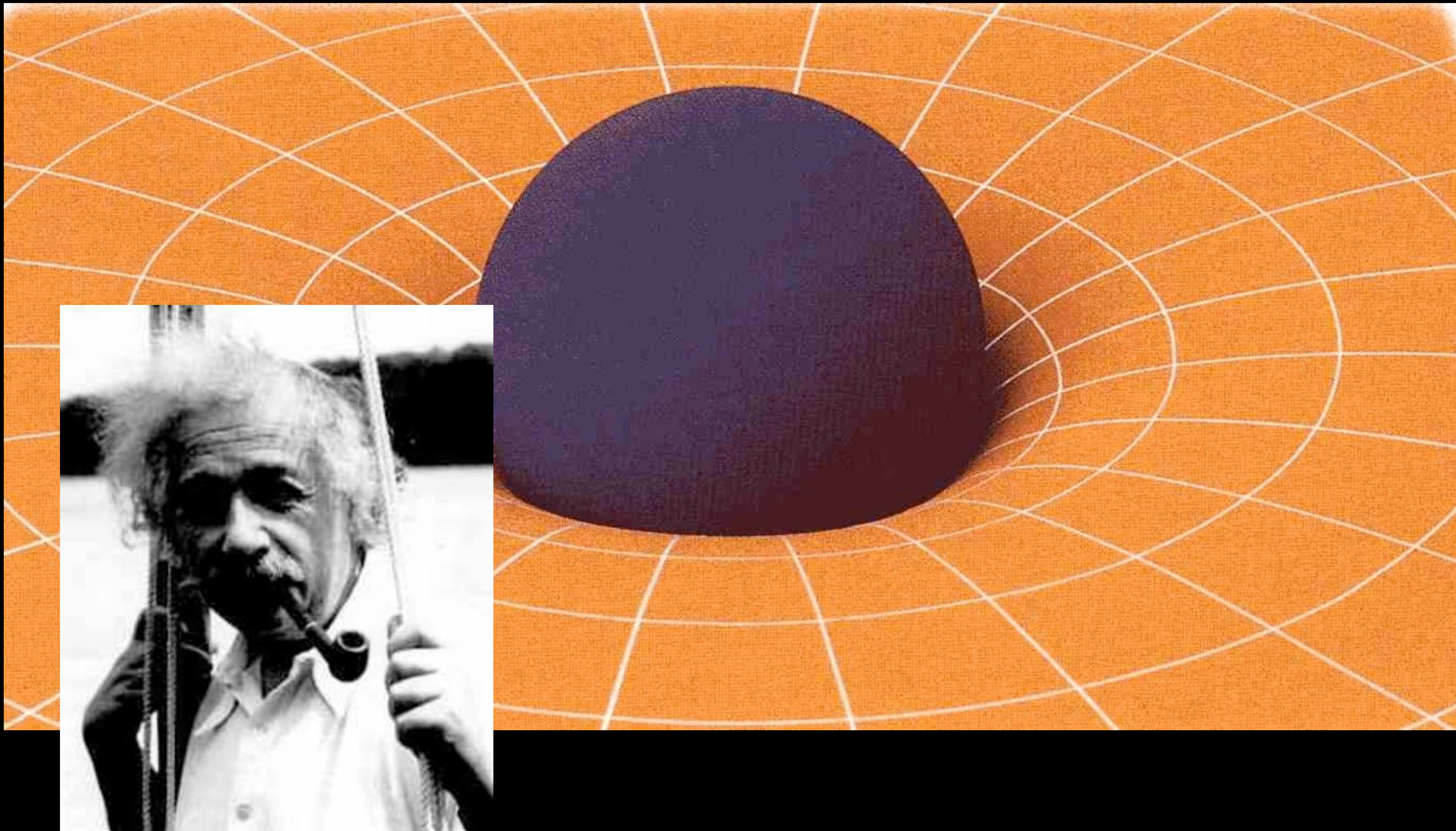
Isaac Newton
1643-1727

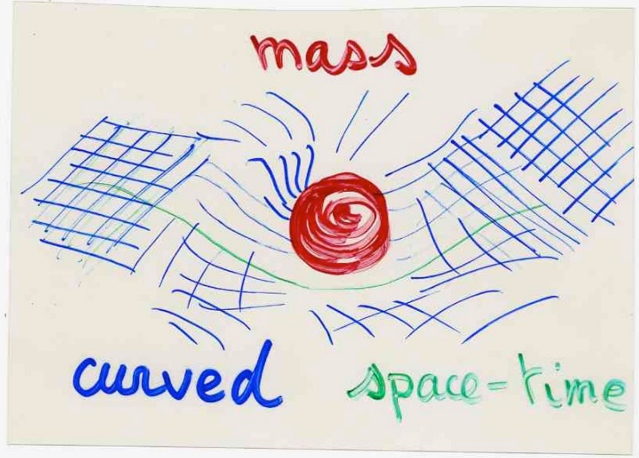
classical geometry
Euclidian geometry

classical
mechanics

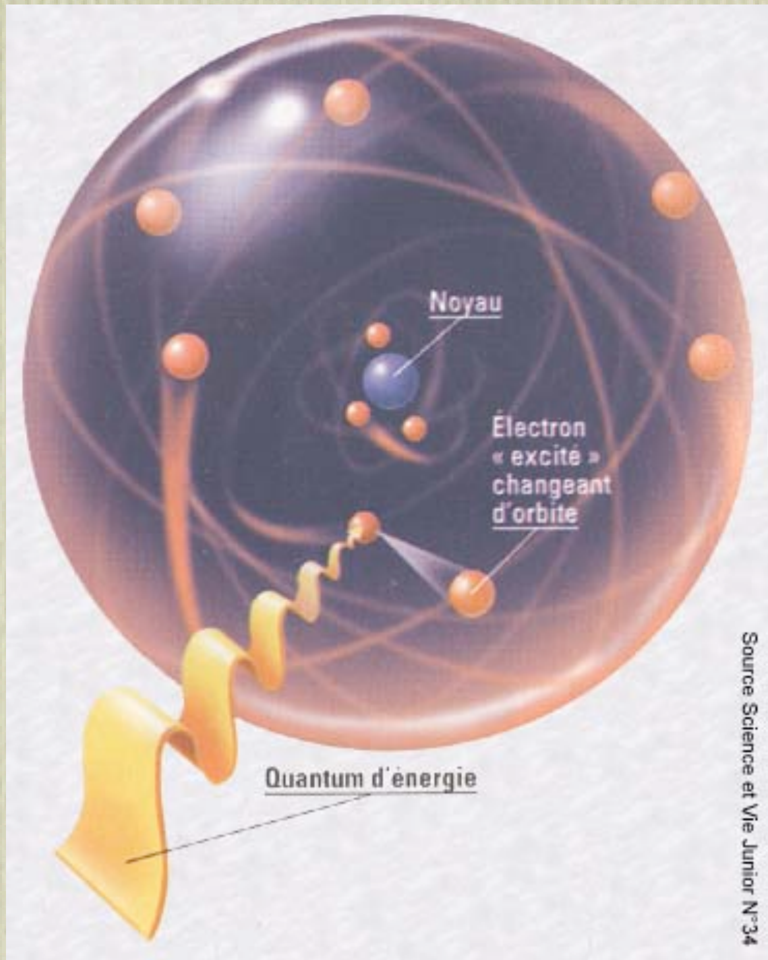
Euclidian geometry

general relativity



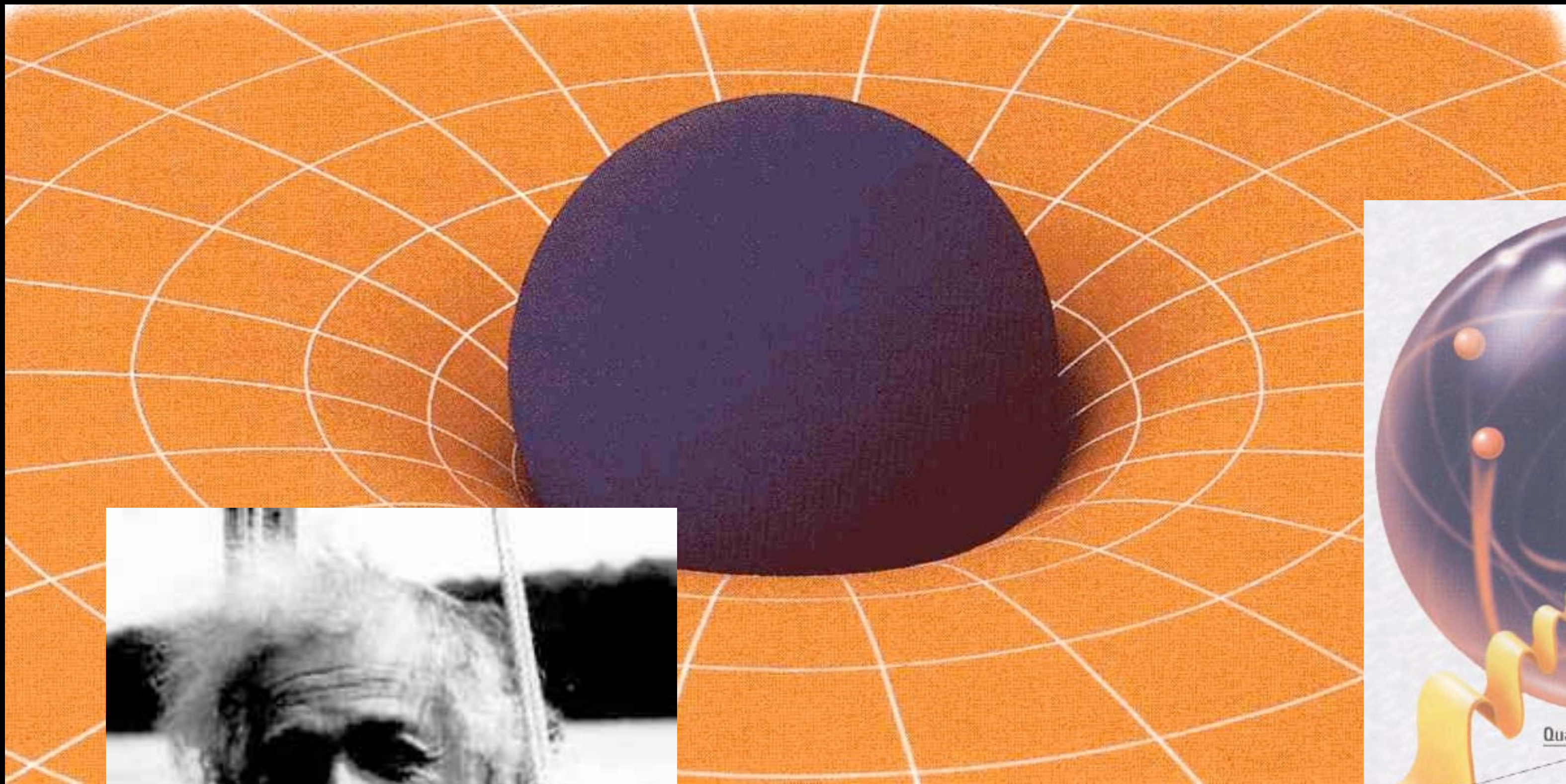

mass
curved
space-time

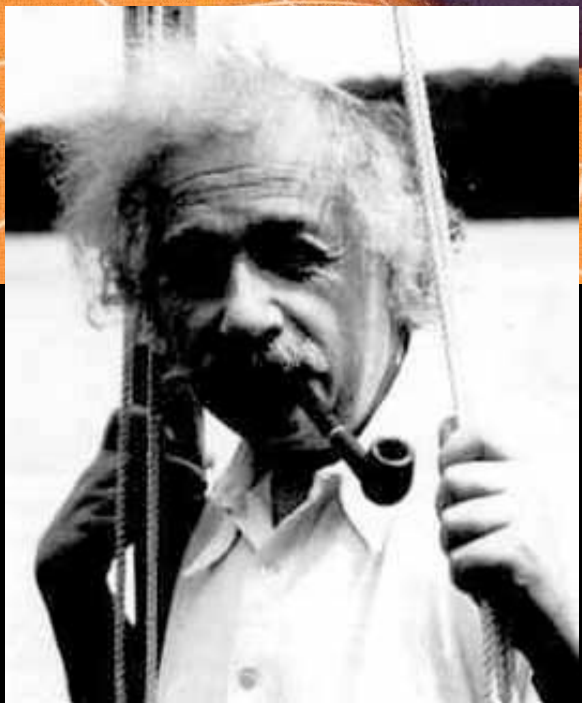
the quantum world

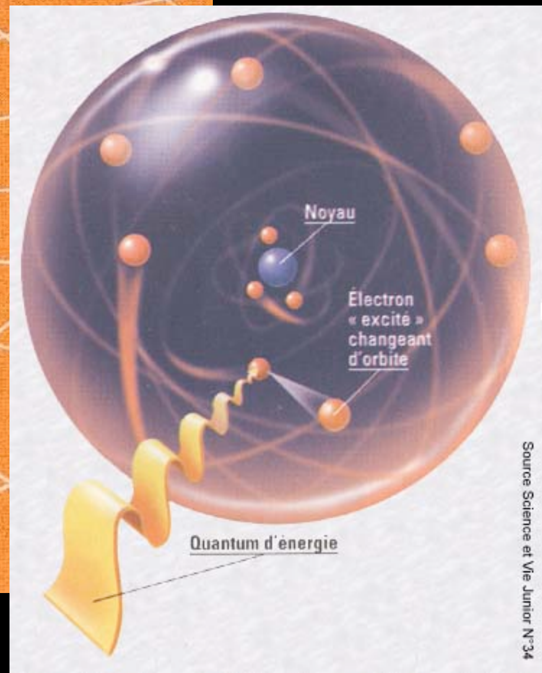


general relativity

quantum mechanics







quantum gravity

quantum gravity

- quantum theory
- general relativity

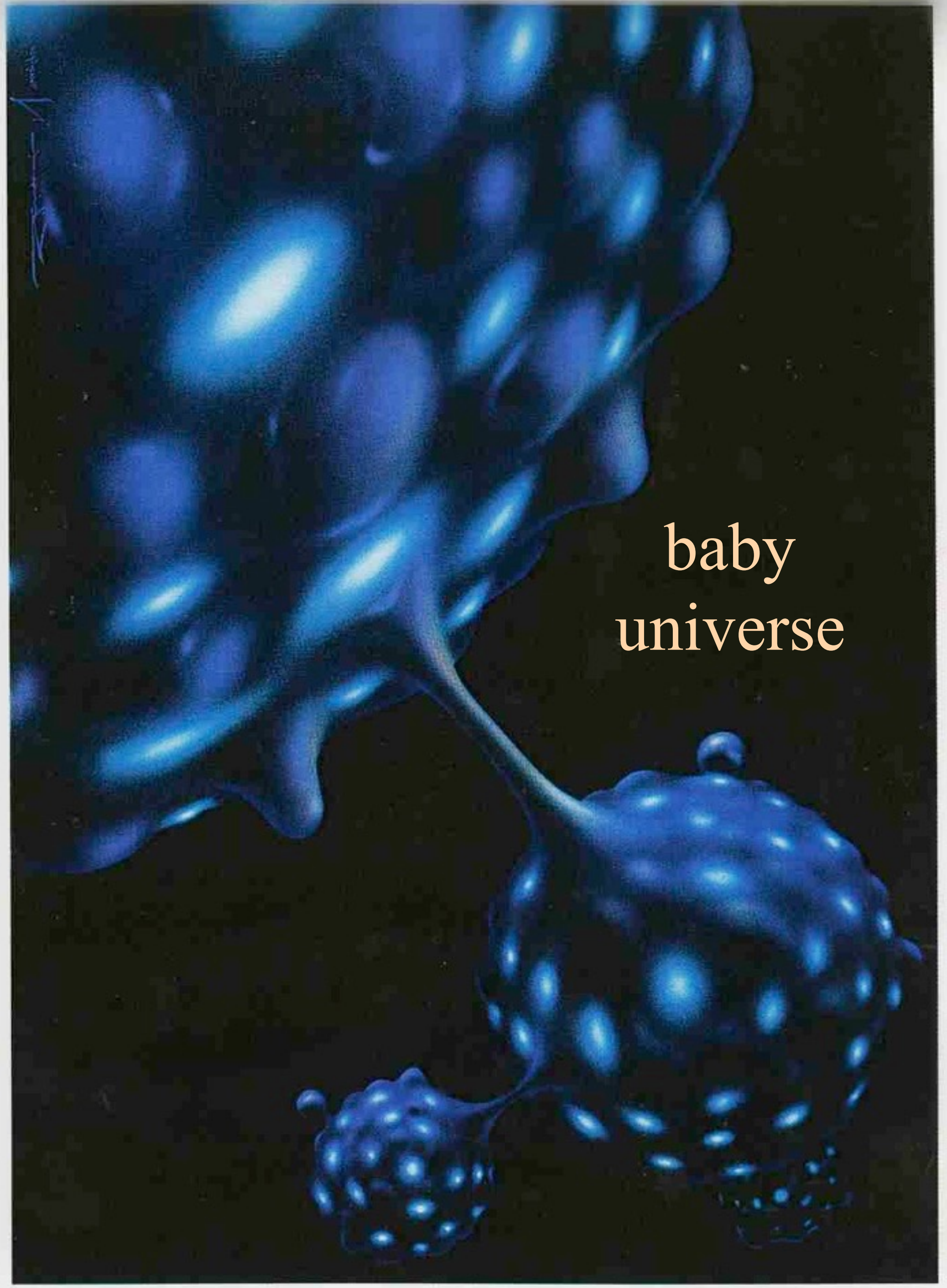
unification ?
4 forces

gravity
electromagnetism
weak interaction
strong interaction

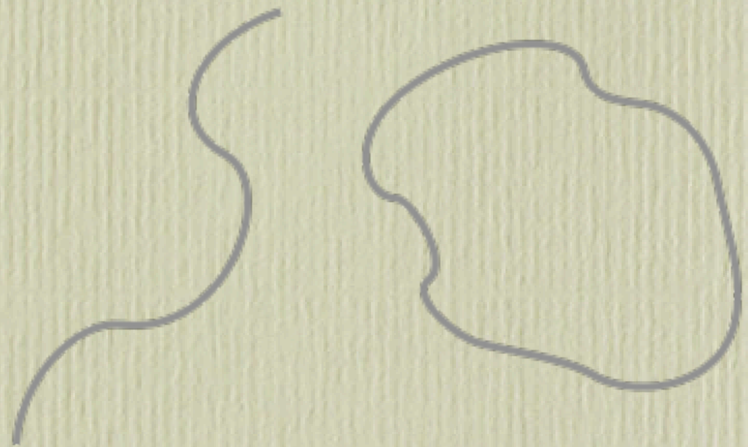

foam of
space-time

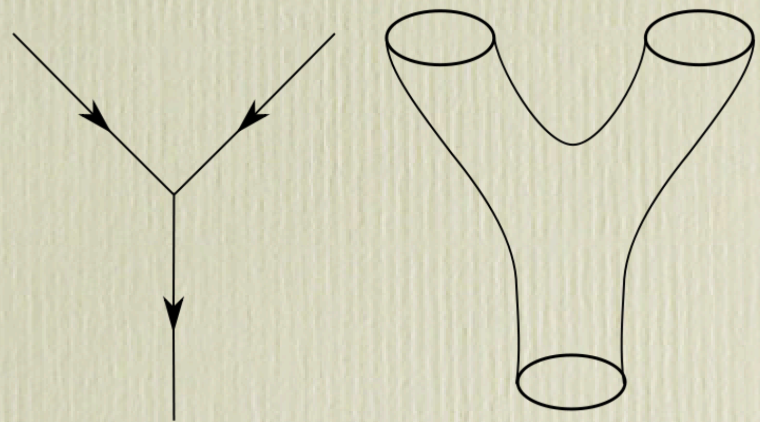
Dessin
S. Numazawa

Ciel & Espace

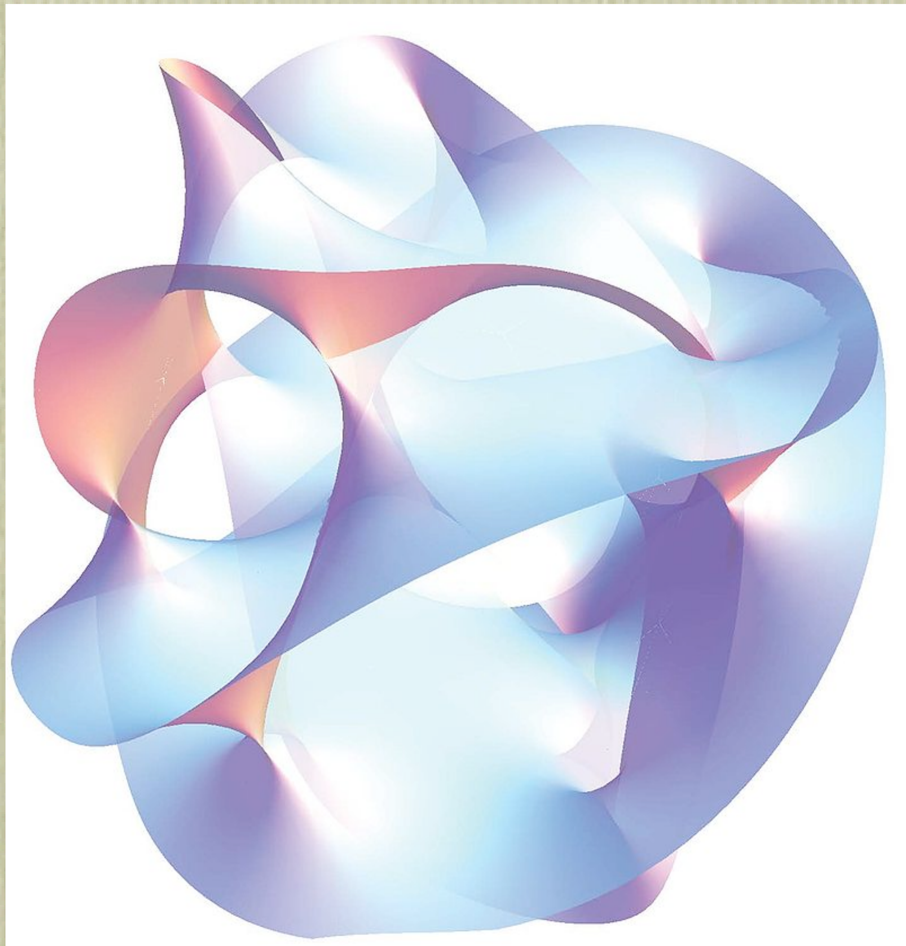


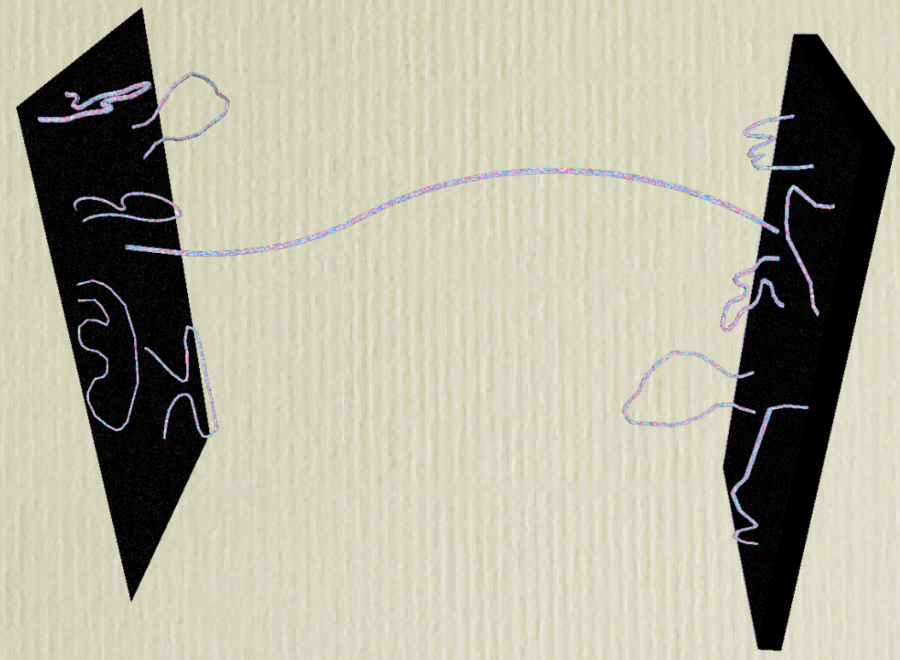
string theory



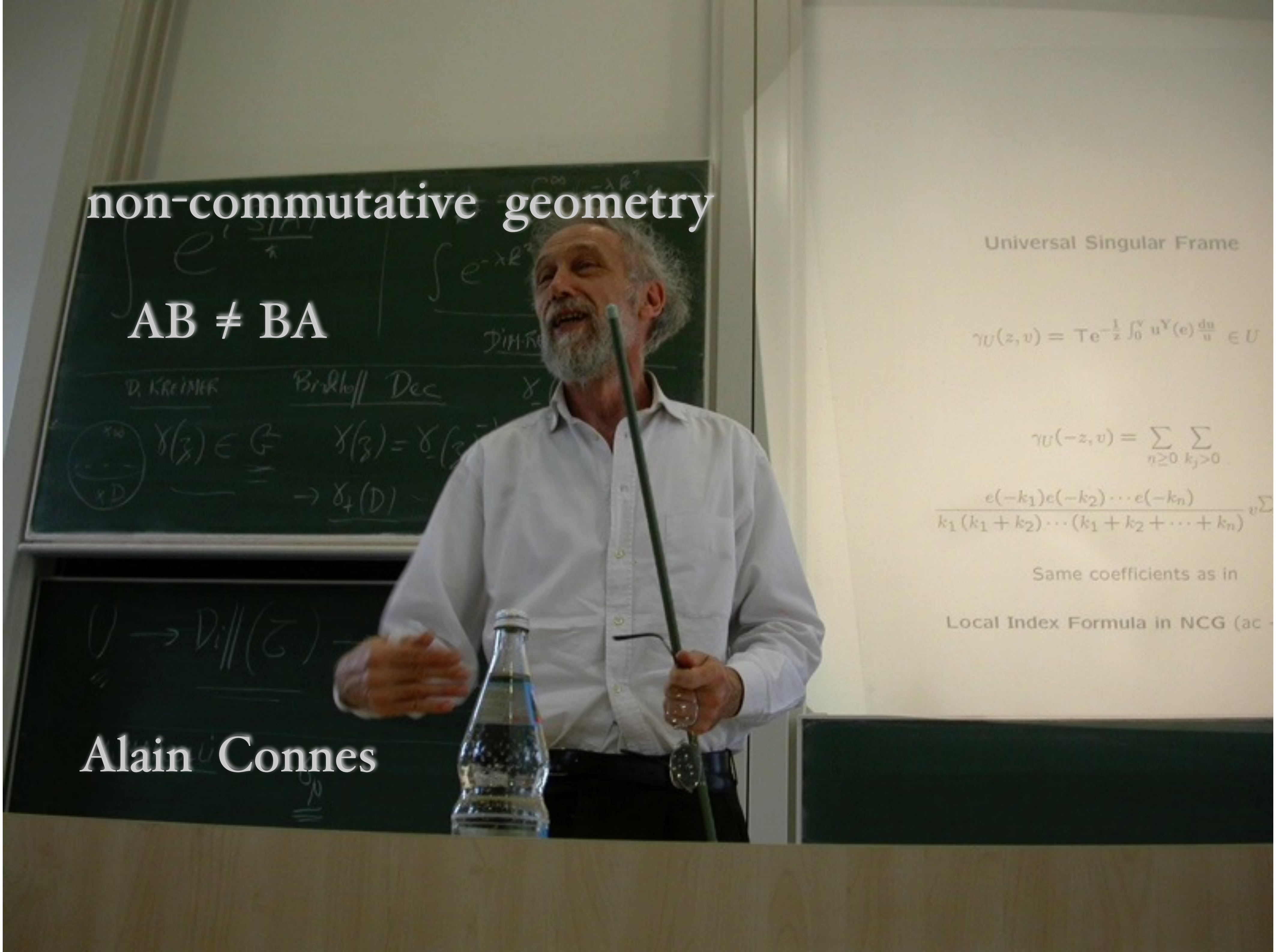


A cross section of a quintic Calabi–Yau manifold



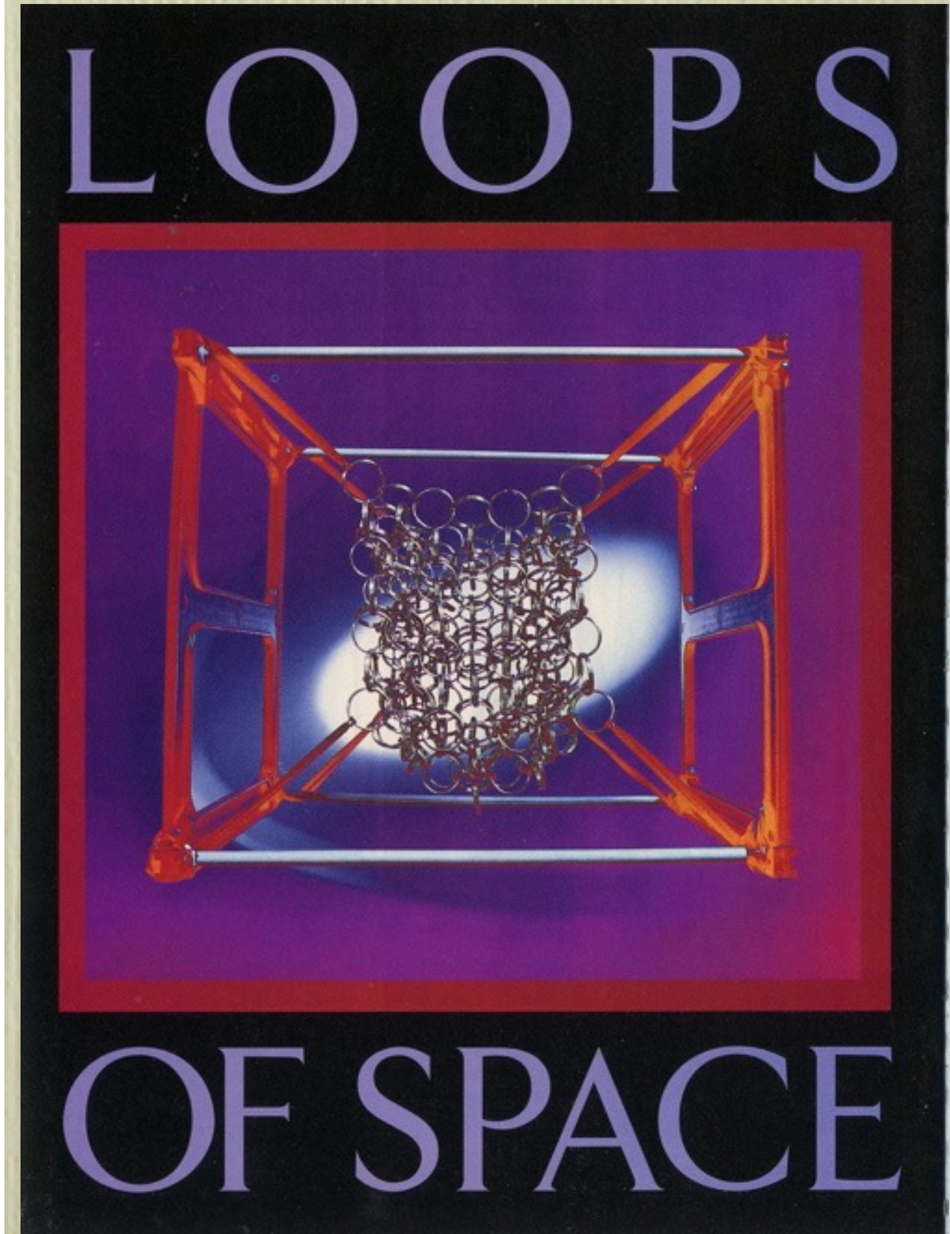


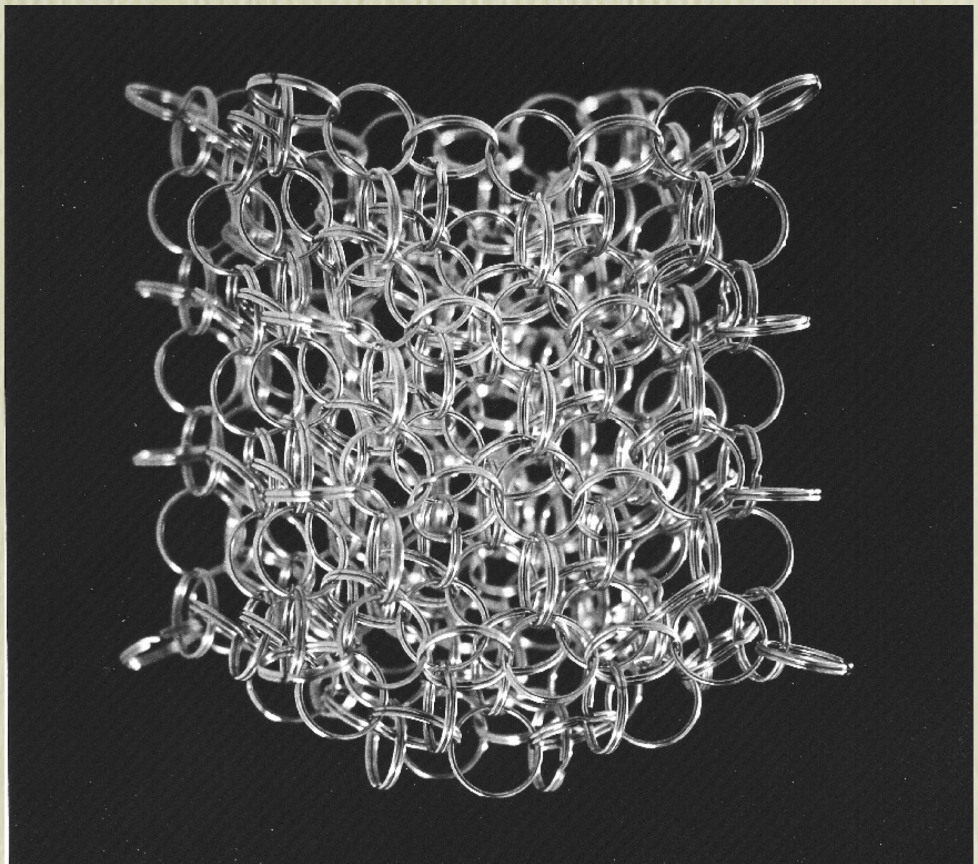
Open strings attached to a pair of D-branes

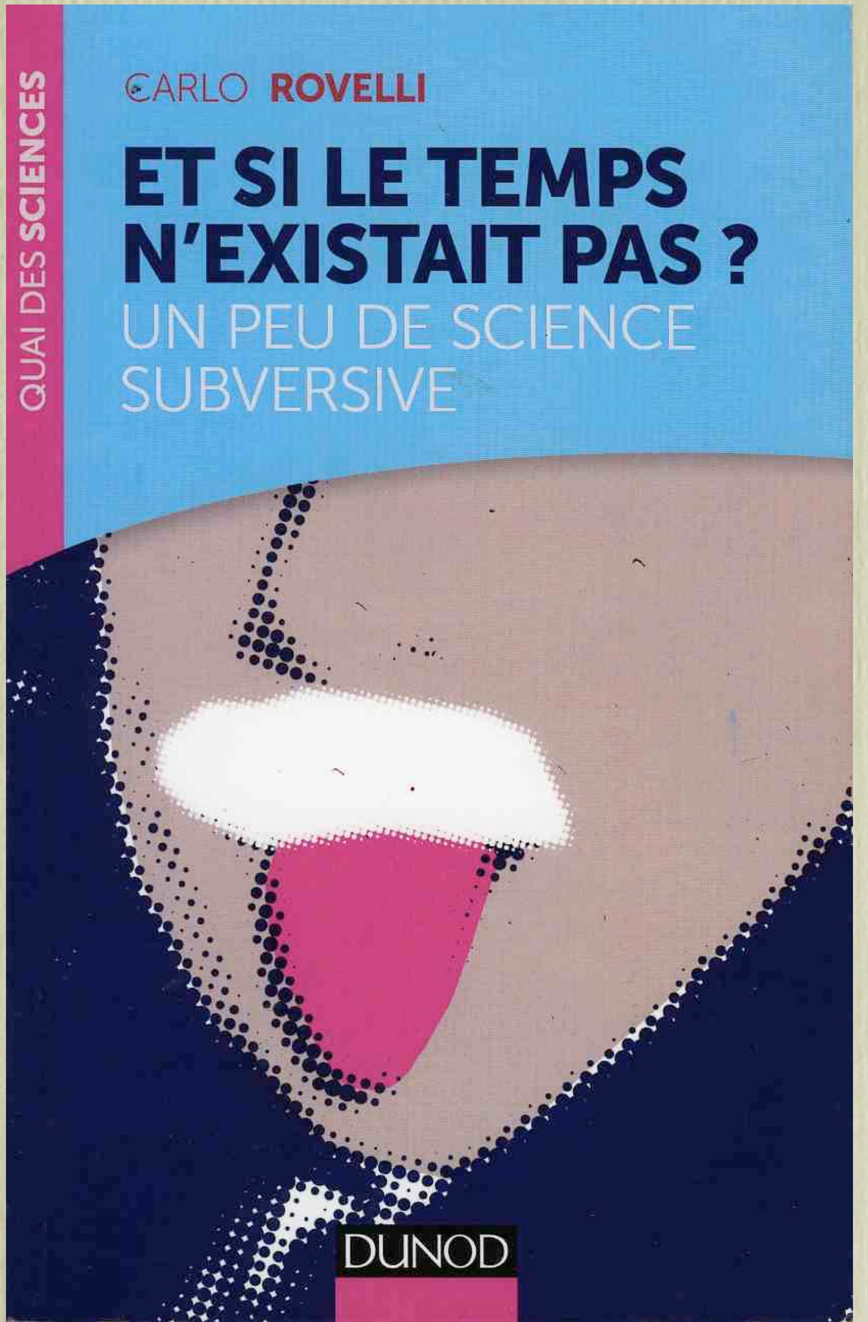

non-commutative geometry
$AB \neq BA$
Alain Connes
Universal Singular Frame
Same coefficients as in
D. KREIMER
Birkhoff Dec

loop quantum gravity

Carlo Rovelli

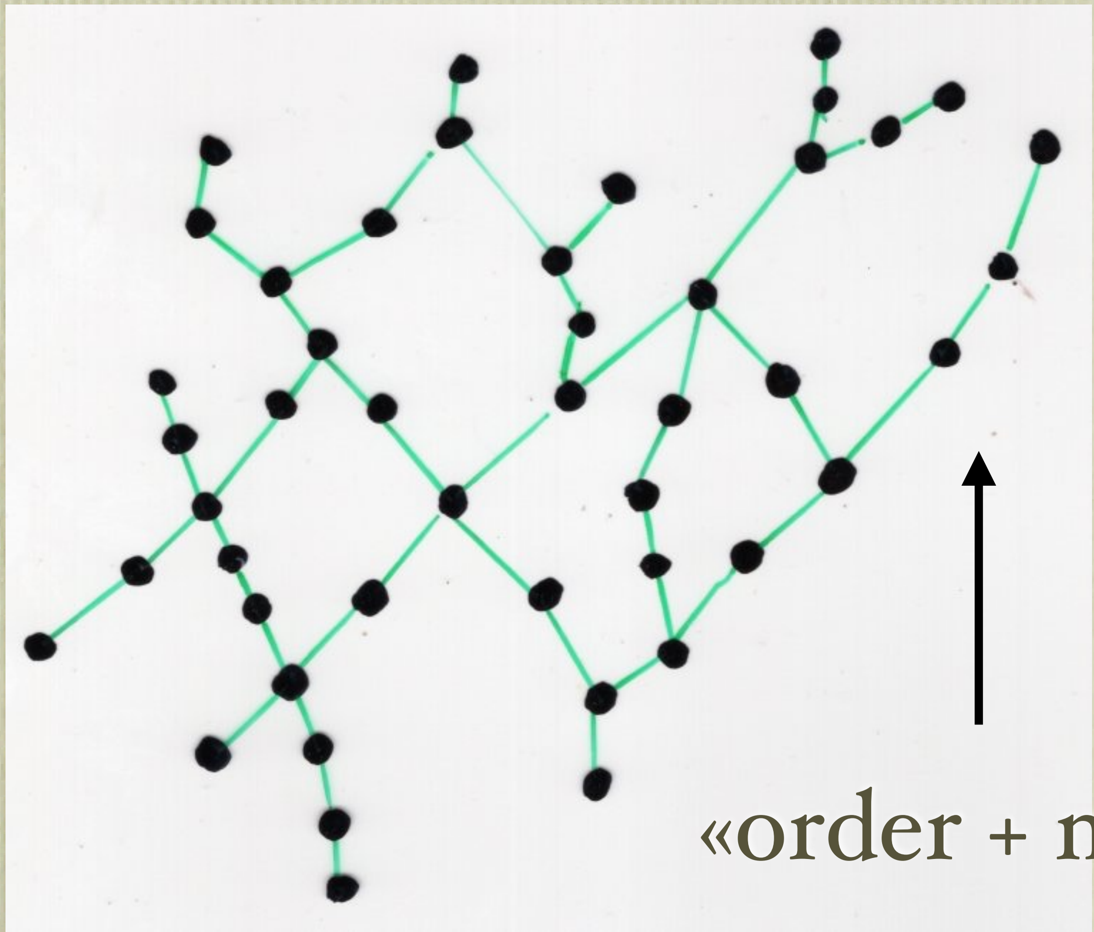

LOOPS
OF SPACE




QUAI DES SCIENCES
CARLO ROVELLI
ET SI LE TEMPS N'EXISTAIT PAS ?
UN PEU DE SCIENCE SUBVERSIVE
DUNOD

causal sets

Rafael Sorkin



poset of
spacetime events

causal relation

«order + number = geometry»

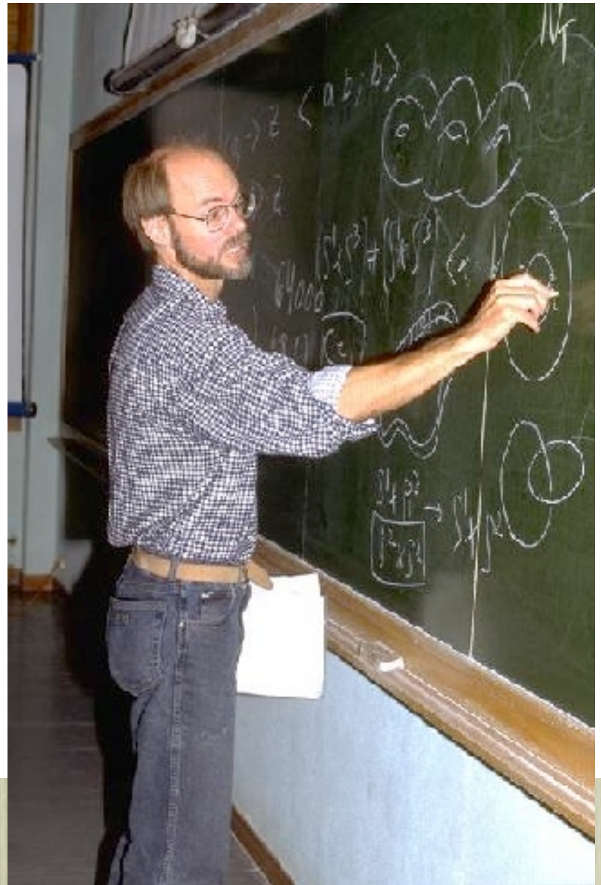
causal dynamical triangulations

(CDT)

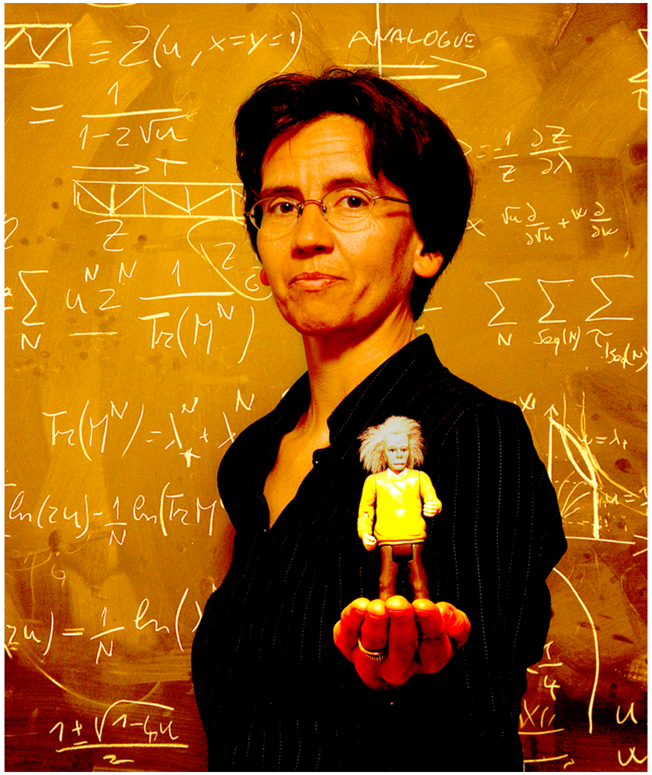
Deepak Dhar
TIFR Bombay

Xavier, you should have
a look at these papers:

- J. Ambjørn, R. Loll, "Non-perturbative Lorentzian quantum gravity and topology change", Nucl. Phys. B536 (1998) 407-436
arXiv: hep-th/9805108

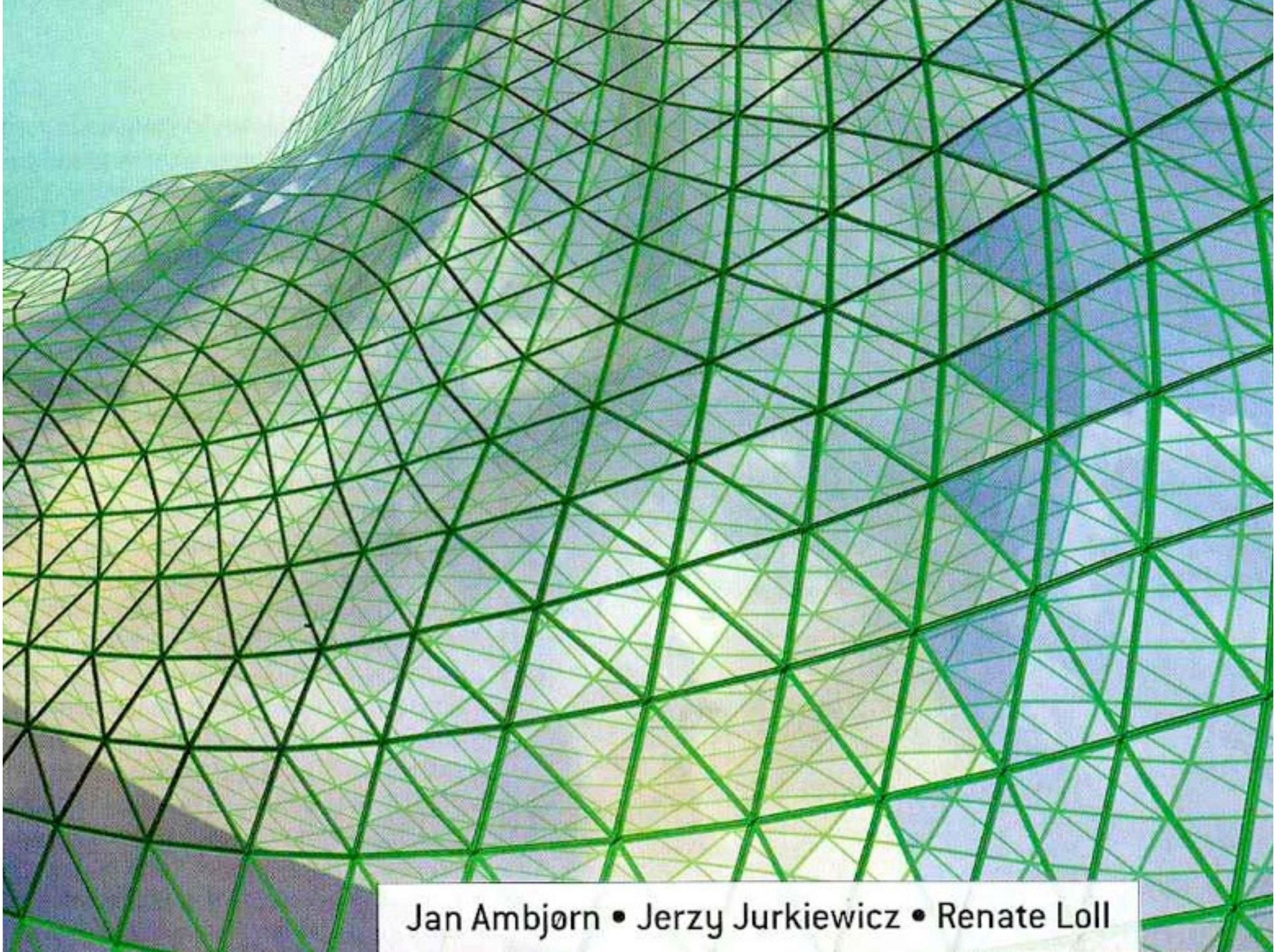


J.Ambjørn

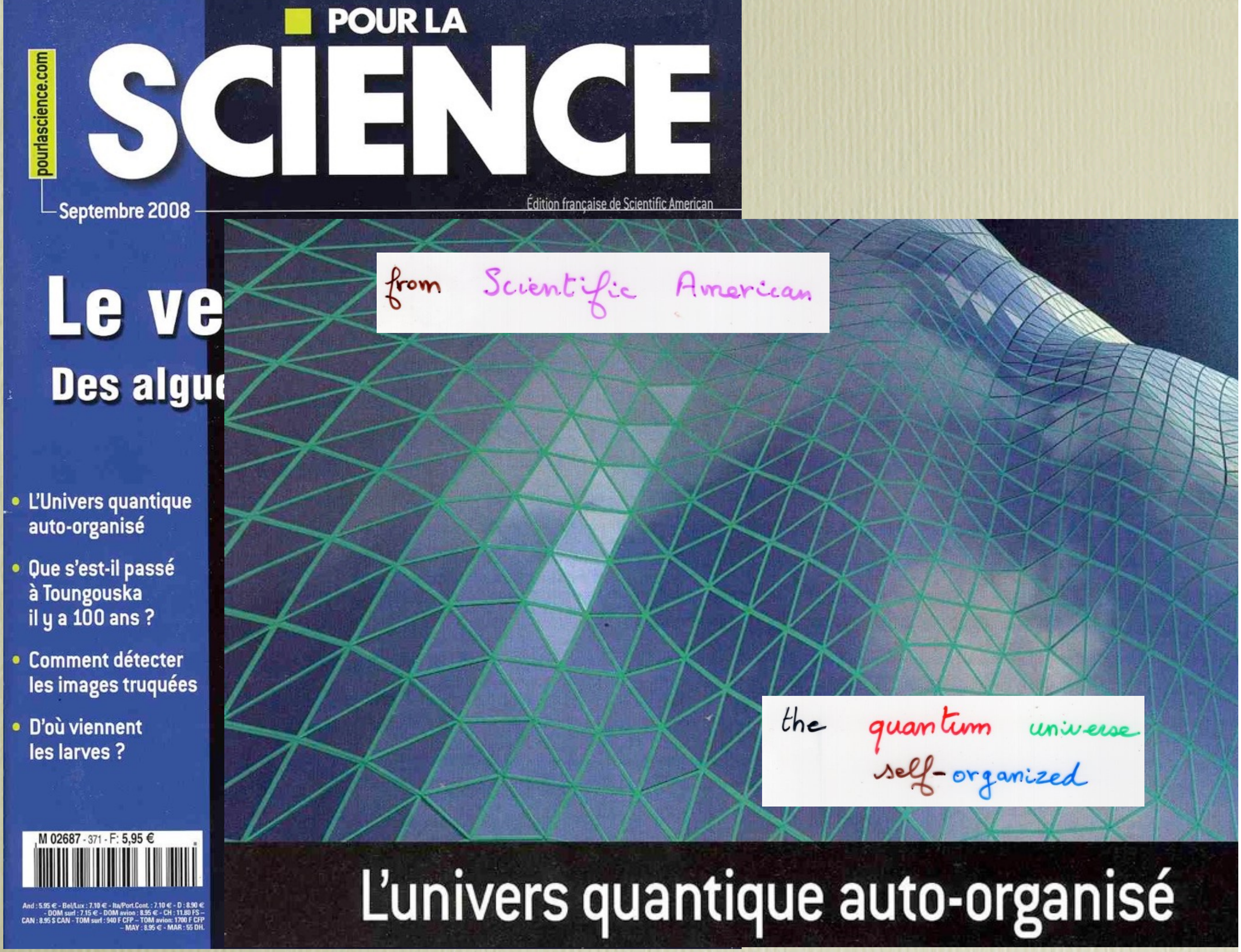


R. Loll

2D Lorentzian quantum gravity



Jan Ambjørn • Jerzy Jurkiewicz • Renate Loll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SCIENCE
pourlascience.com
Septembre 2008
Édition française de Scientific American
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Des algue
• L'Univers quantique auto-organisé
• Que s'est-il passé à Toungouska il y a 100 ans ?
• Comment détecter les images truquées
• D'où viennent les larves ?
M 02687 - 371 - F: 5,95 €
L'univers quantique auto-organisé
from Scientific American
the quantum universe
self-organized

(p5)

.... In quantum gravity we are instructed to sum over all geometries connecting, say, two spatial boundaries of length ℓ_1 and ℓ_2, with the weight of each geometry g given by

$$(5) \qquad e^{iS[g]} \qquad\qquad S[g] = \Lambda \int \sqrt{-g} \, , \quad (\text{in } 2d)$$

where Λ is the cosmological constant.

(p7)

(21) $$F_t(x) = F\,\frac{1 - xF + F^{2t-1}(x - F)}{1 - xF + F^{2t-1}(x - F)}$$

$$F = \frac{1 - \sqrt{1 - 4g^2}}{2g}$$

Catalan number !

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

— P. Di Francesco, E. Guitter, C. Kristjansen, "Integrable 2D Lorentzian gravity and random walks", Nucl. Phys. B 567 (2000) 515-553
arXiv: hep-th/9907084

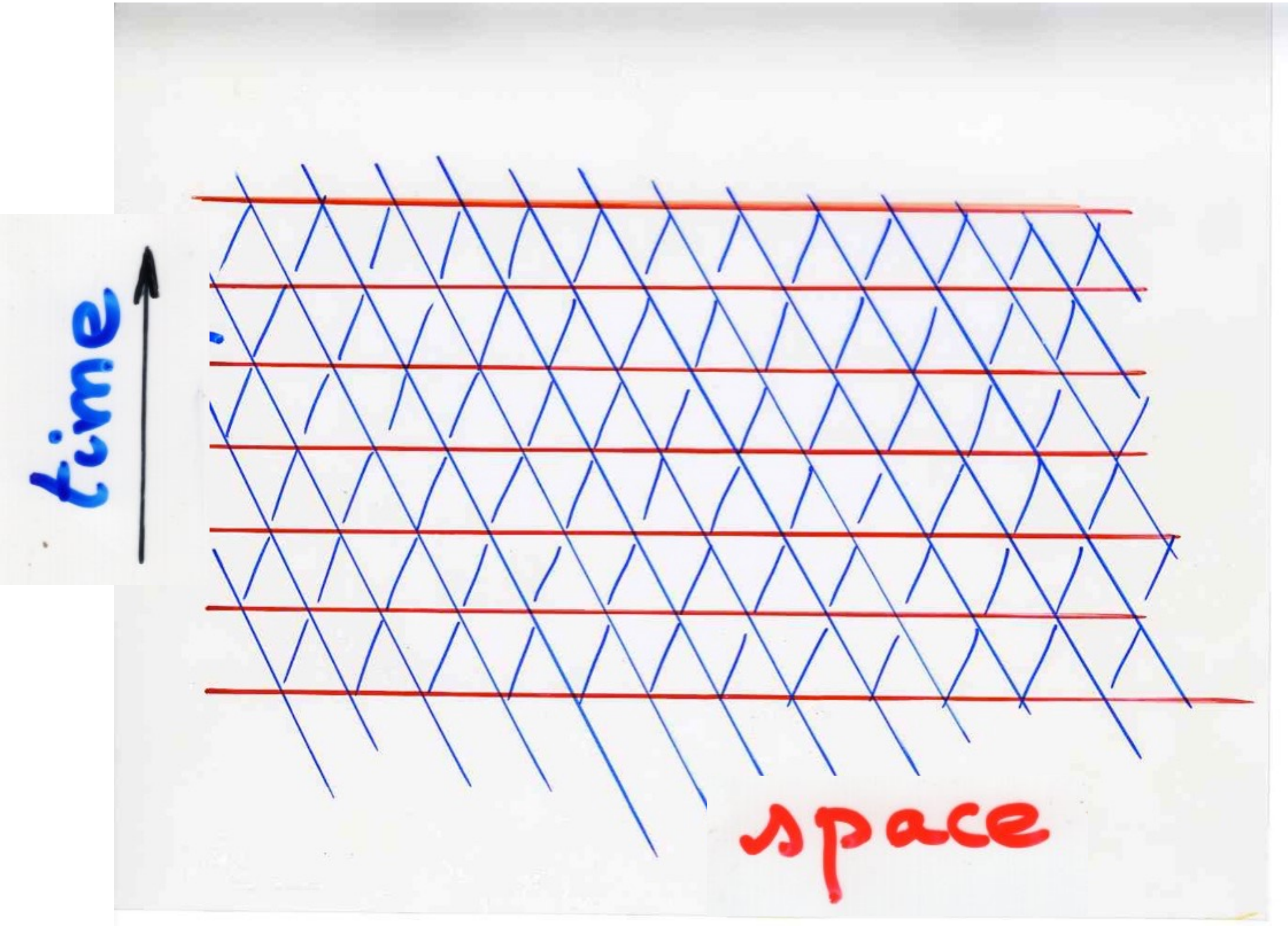
P. Di Francesco

E.Guitter

C. Kristjansen

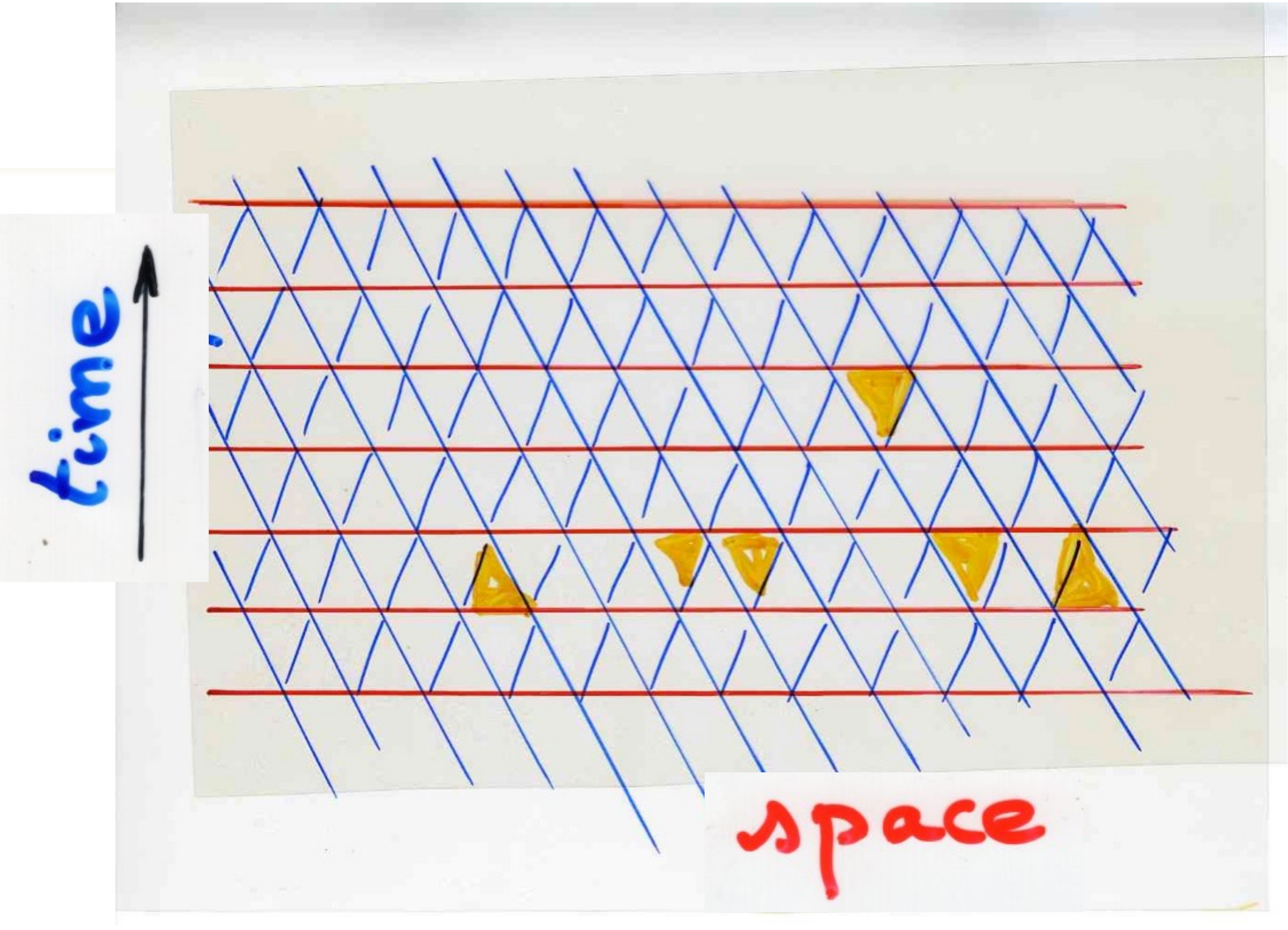
2D Lorentzian quantum gravity

2D Lorentzian triangulation

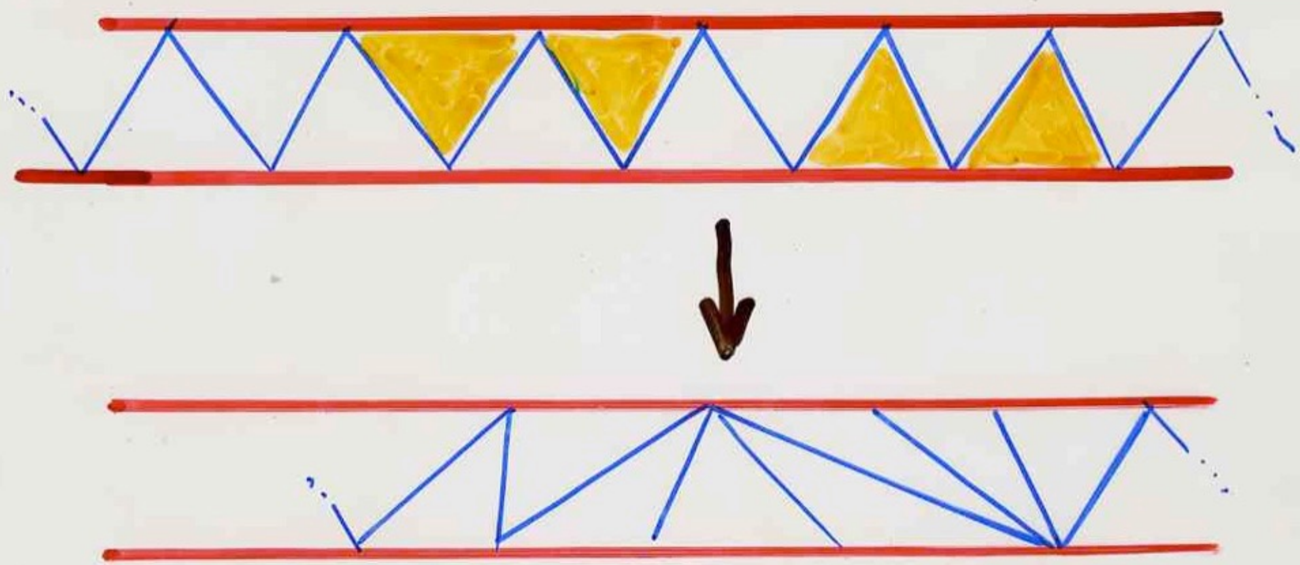

time
space

- "quantum fluctuations" of the space-time

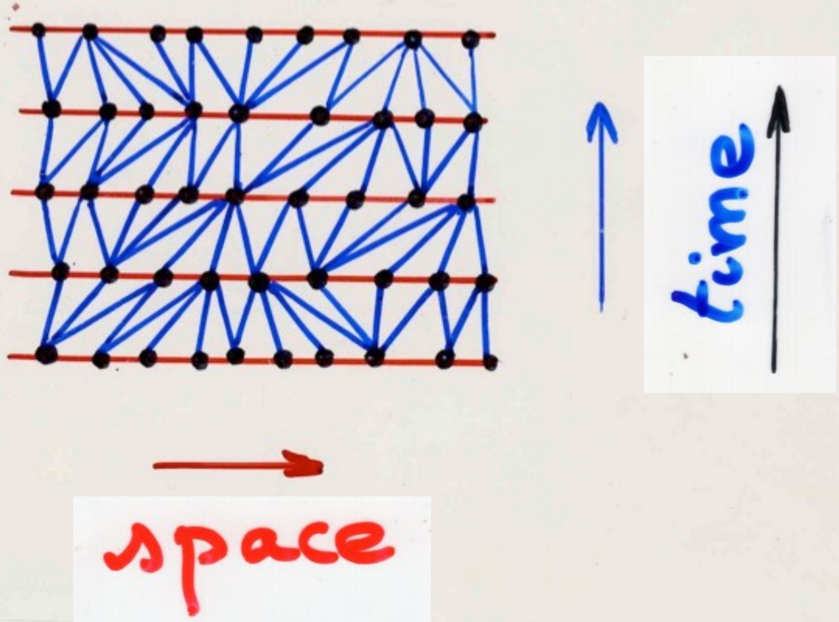
"quantum geometry"



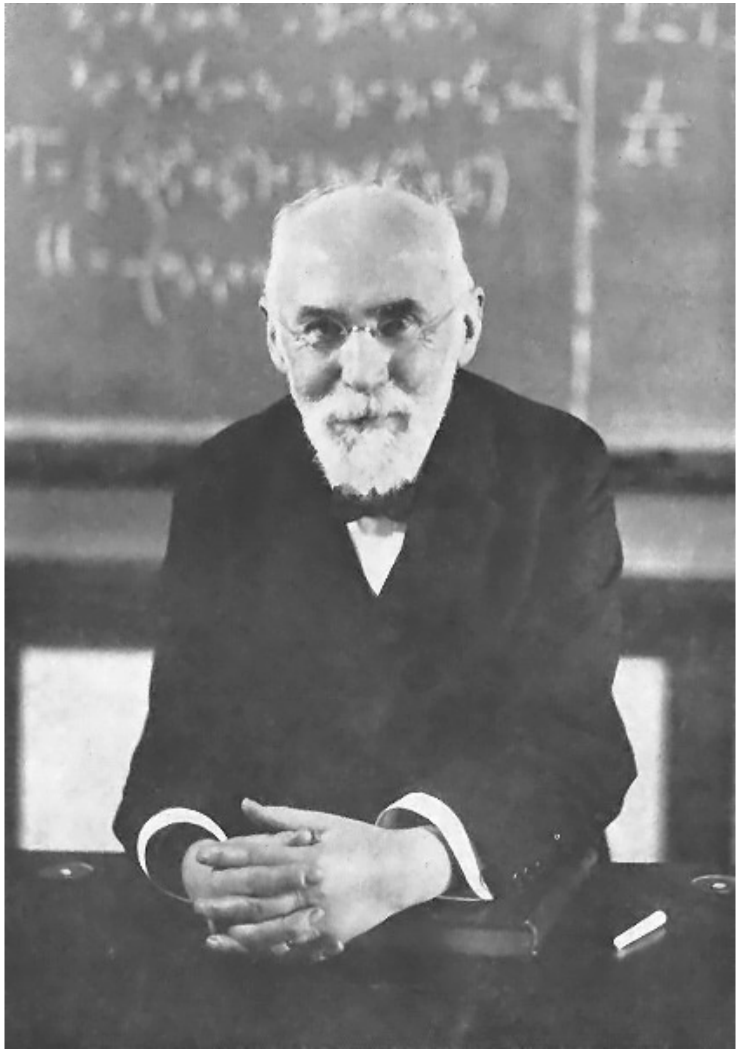
• "quantum fluctuations"
of the space-time
"quantum geometry"



Lorentzian triangulation



2D Lorentzian quantum gravity



Lorentzian
quantum
gravity

$\underbrace{+++}_{\text{space}}\underbrace{-}_{\text{time}}$

2D

$\underbrace{1}_{\text{space}} + \underbrace{1}_{\text{time}}$

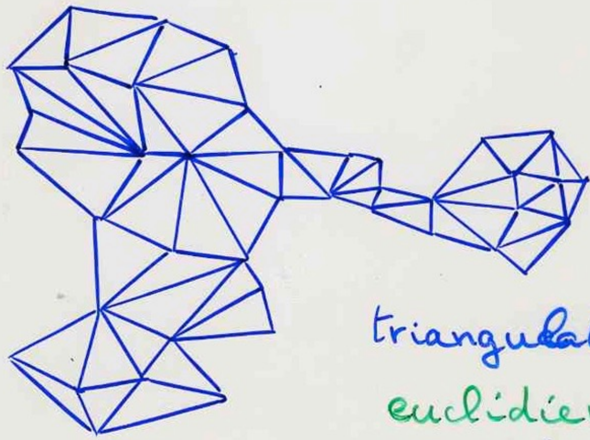
euclidian

$+ + + \quad +$

space time

$\underbrace{+ + +}_{\text{space}} \quad \underbrace{-}_{\text{time}}$

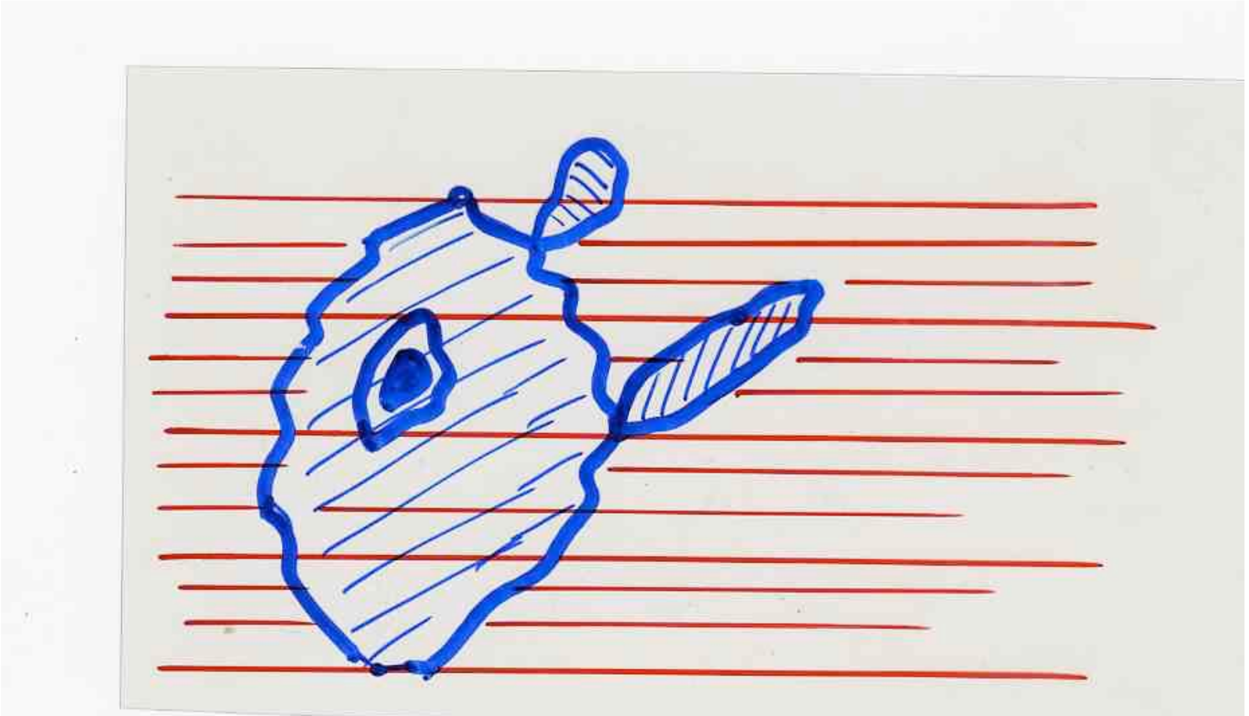
Lorentzian
quantum
gravity



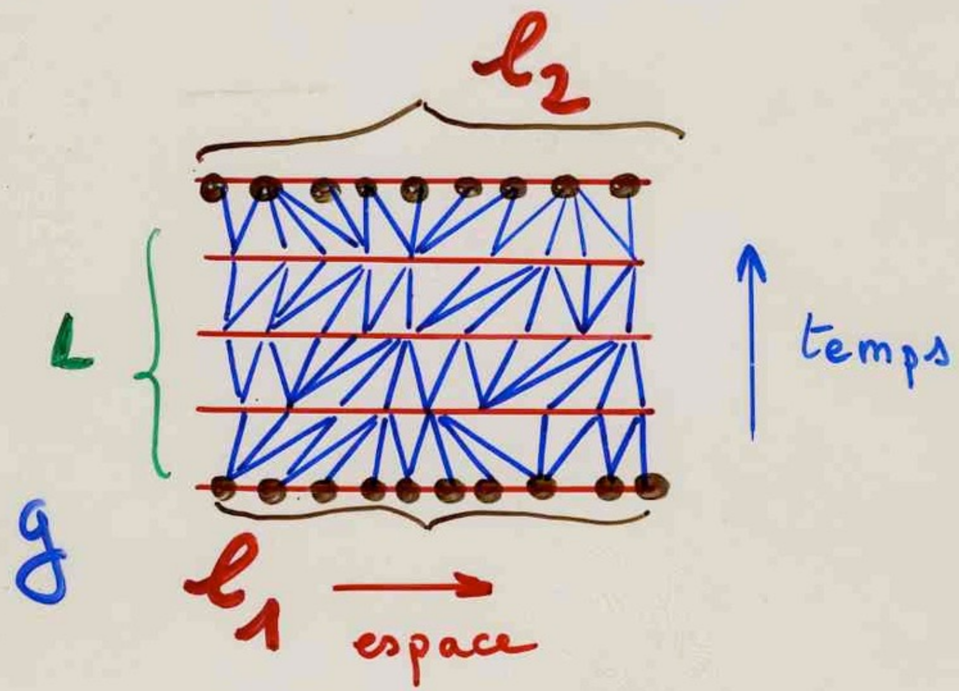
triangulation
euclidienne
$+ +$

2D $\quad 1 + 1$

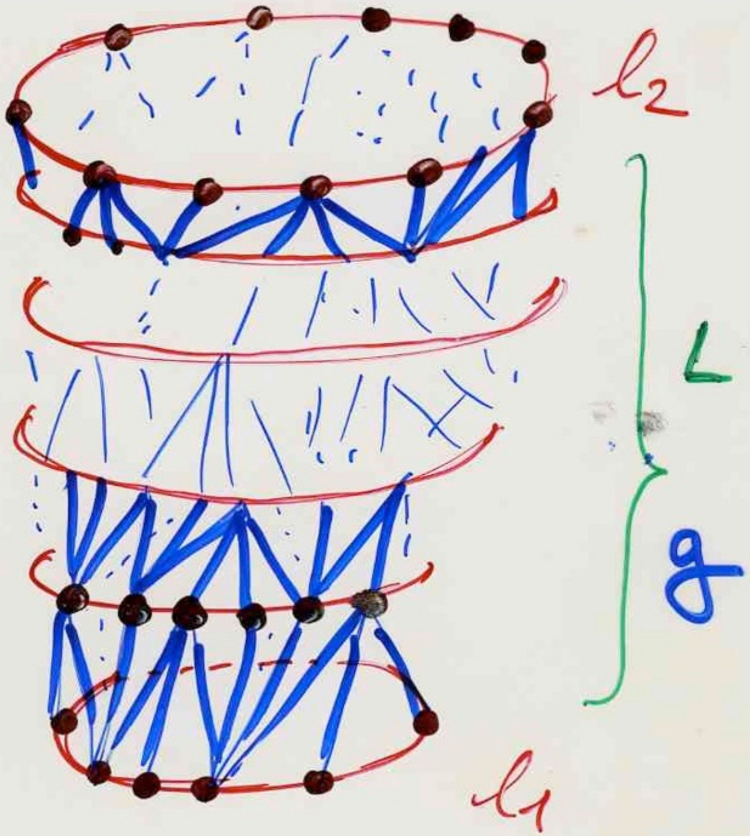
space time



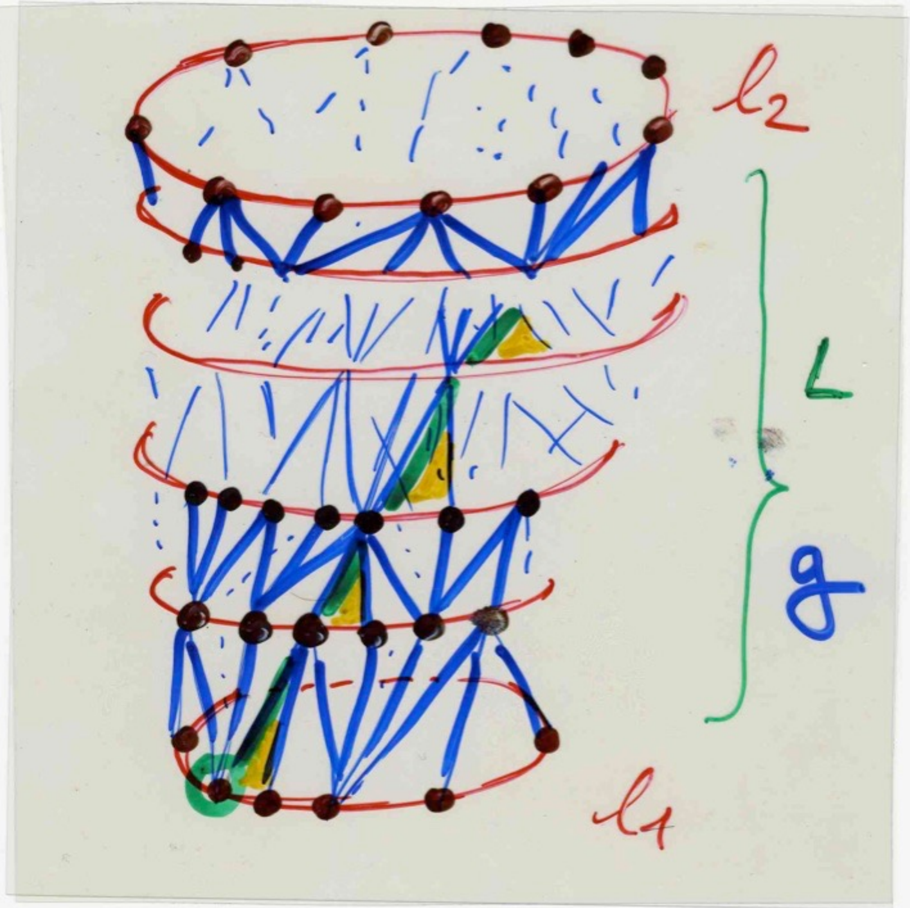
no baby
universe

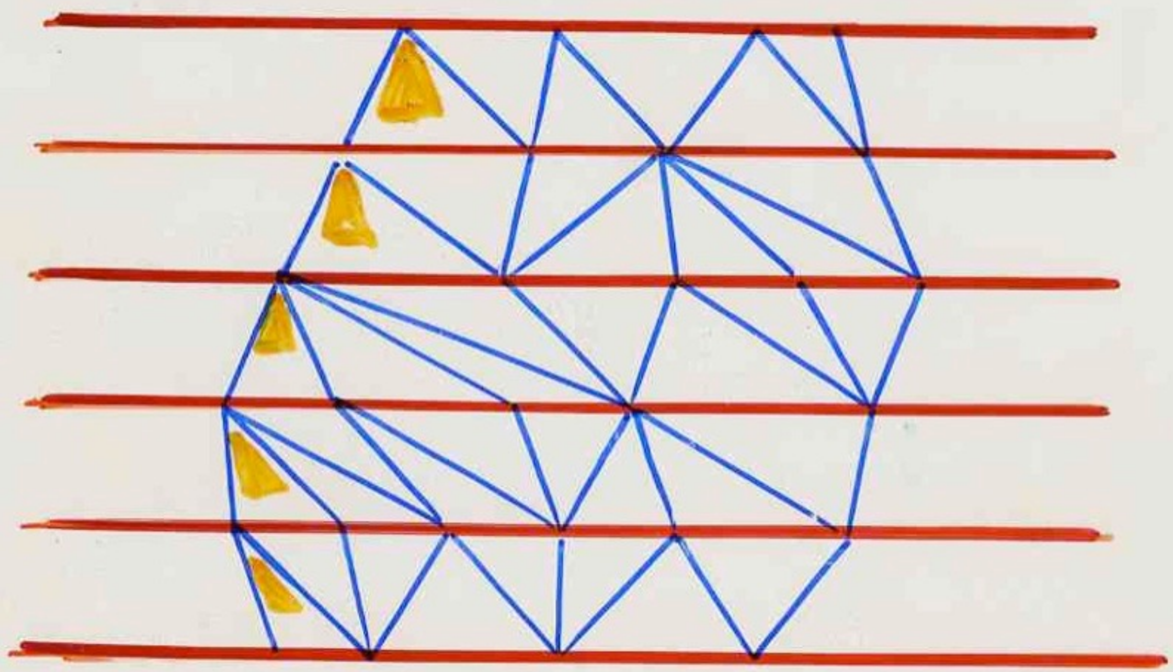


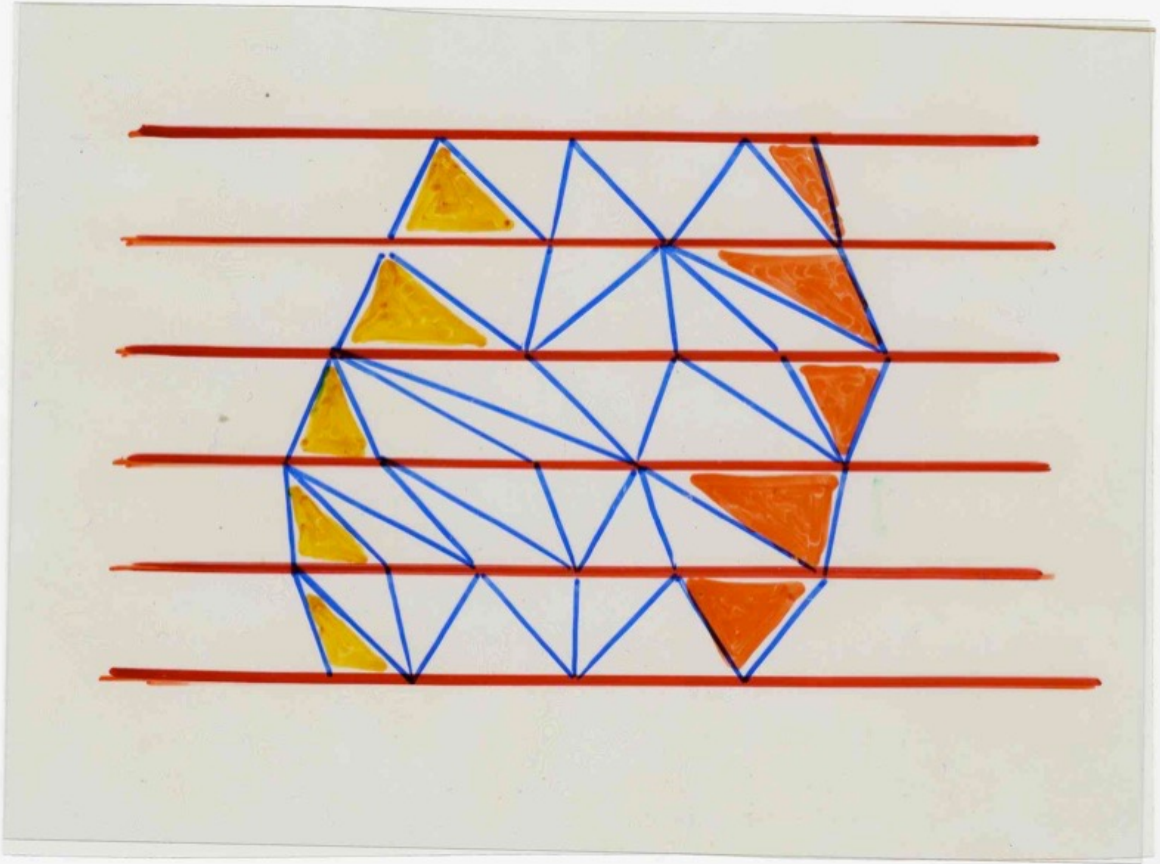
Path integral amplitude
for the propagation from
geometry ℓ_1 to ℓ_2



ℓ_2
L
g
ℓ_1

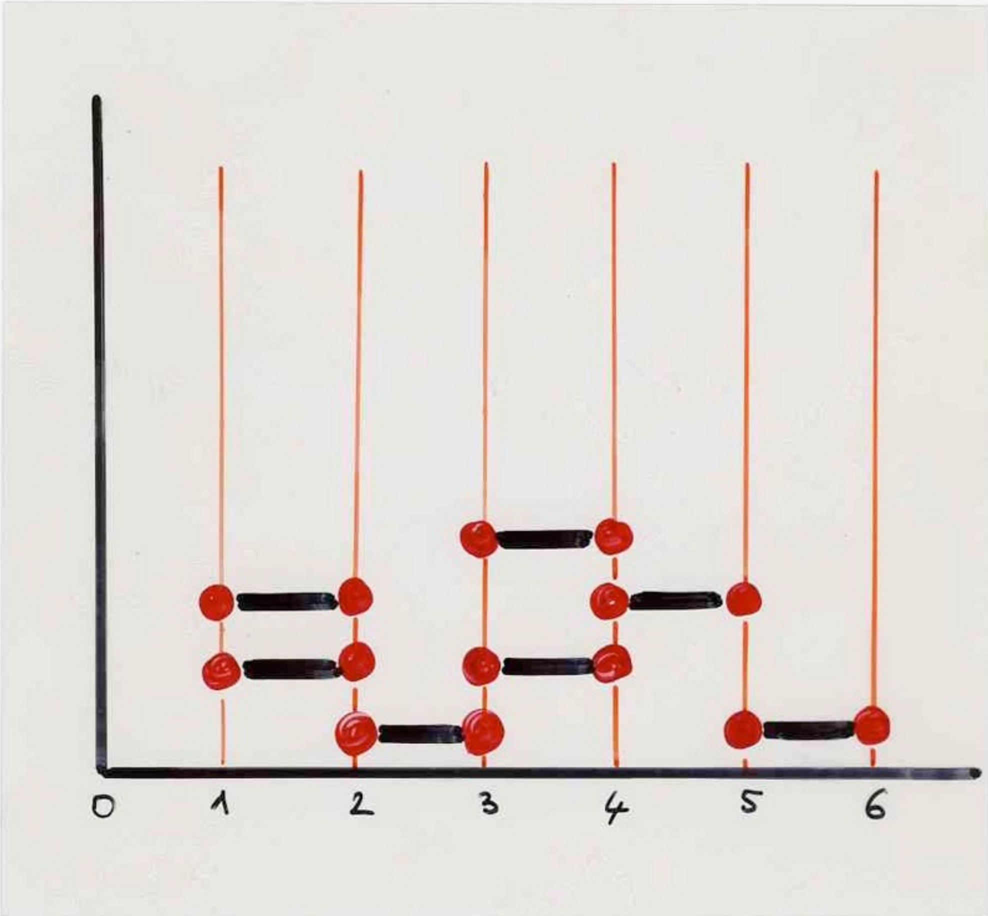

l_2
L
g
l_1

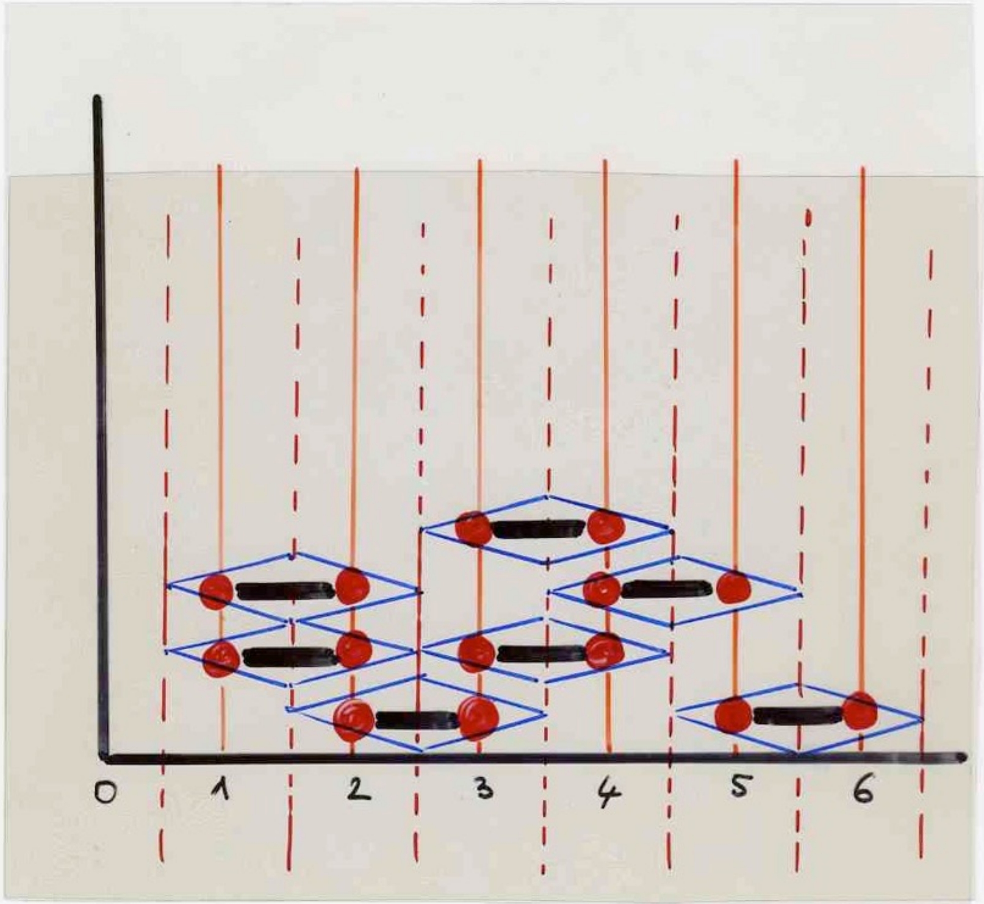




Heaps of dimers

Lorentzian triangulations


0
1
2
3
4
5
6


0
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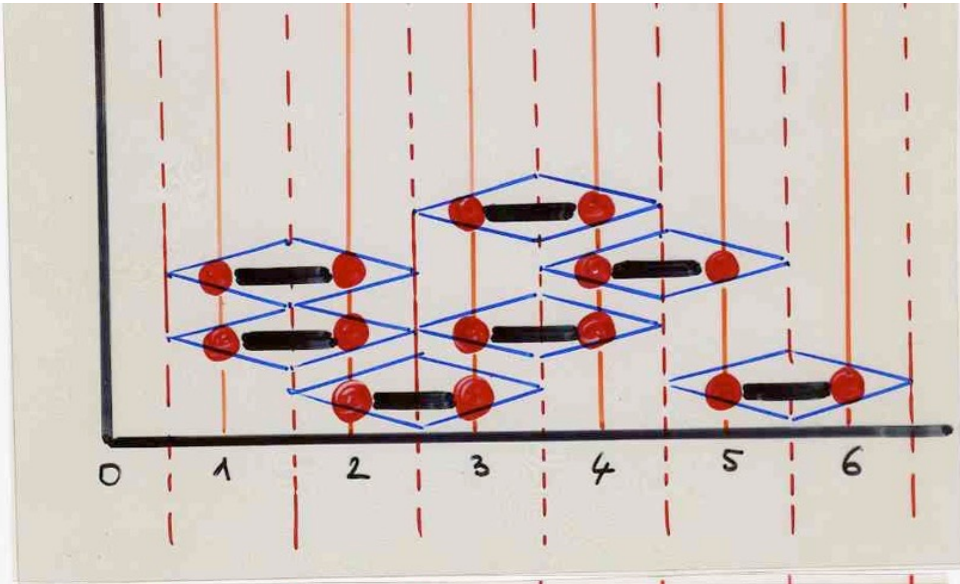
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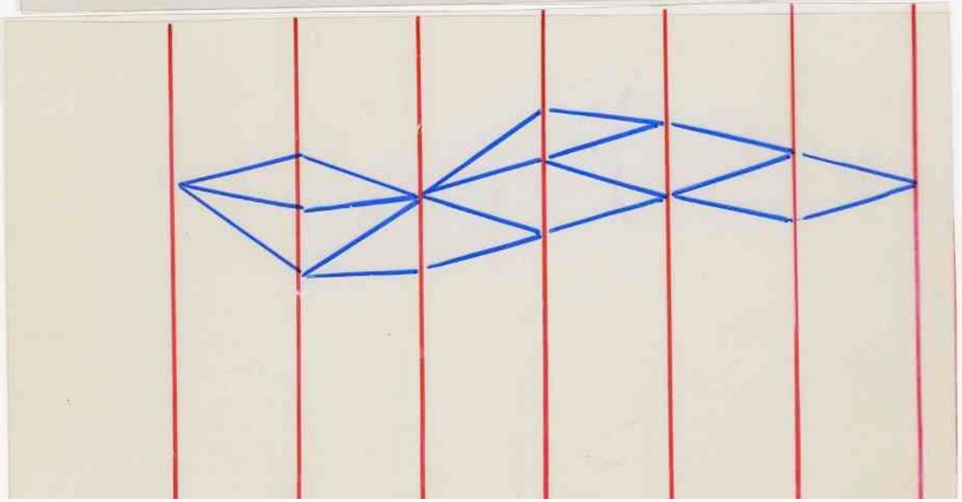

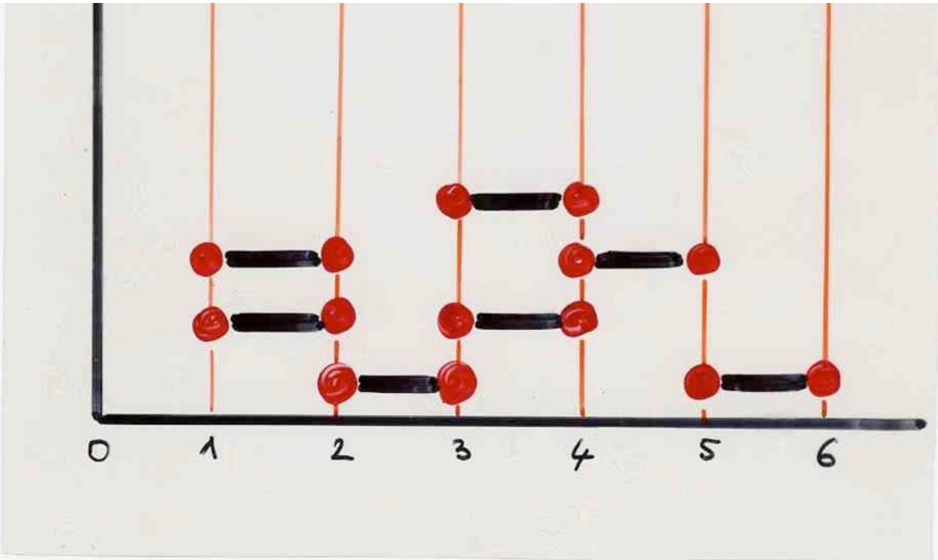
0
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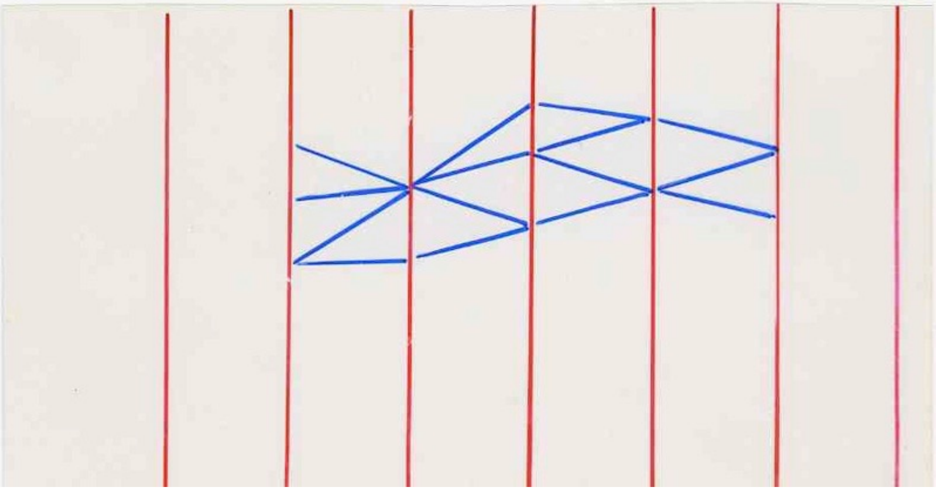
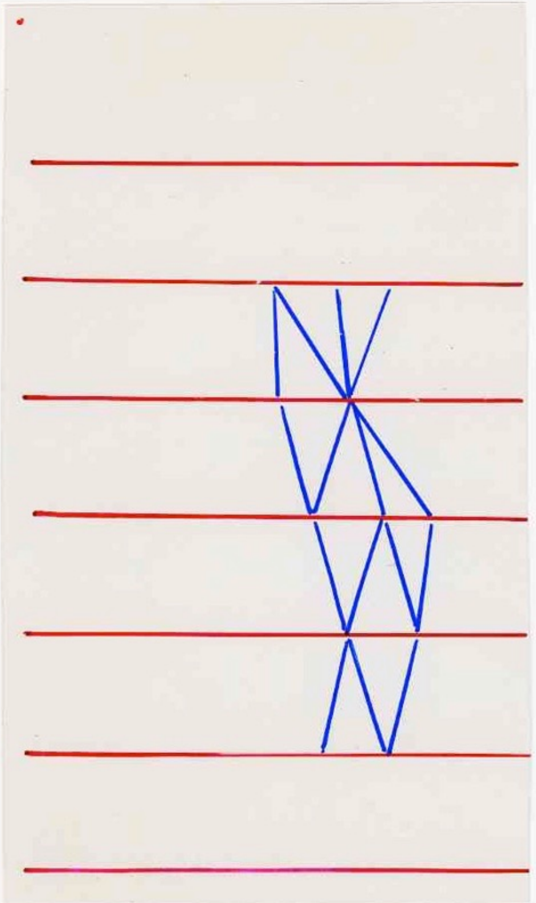
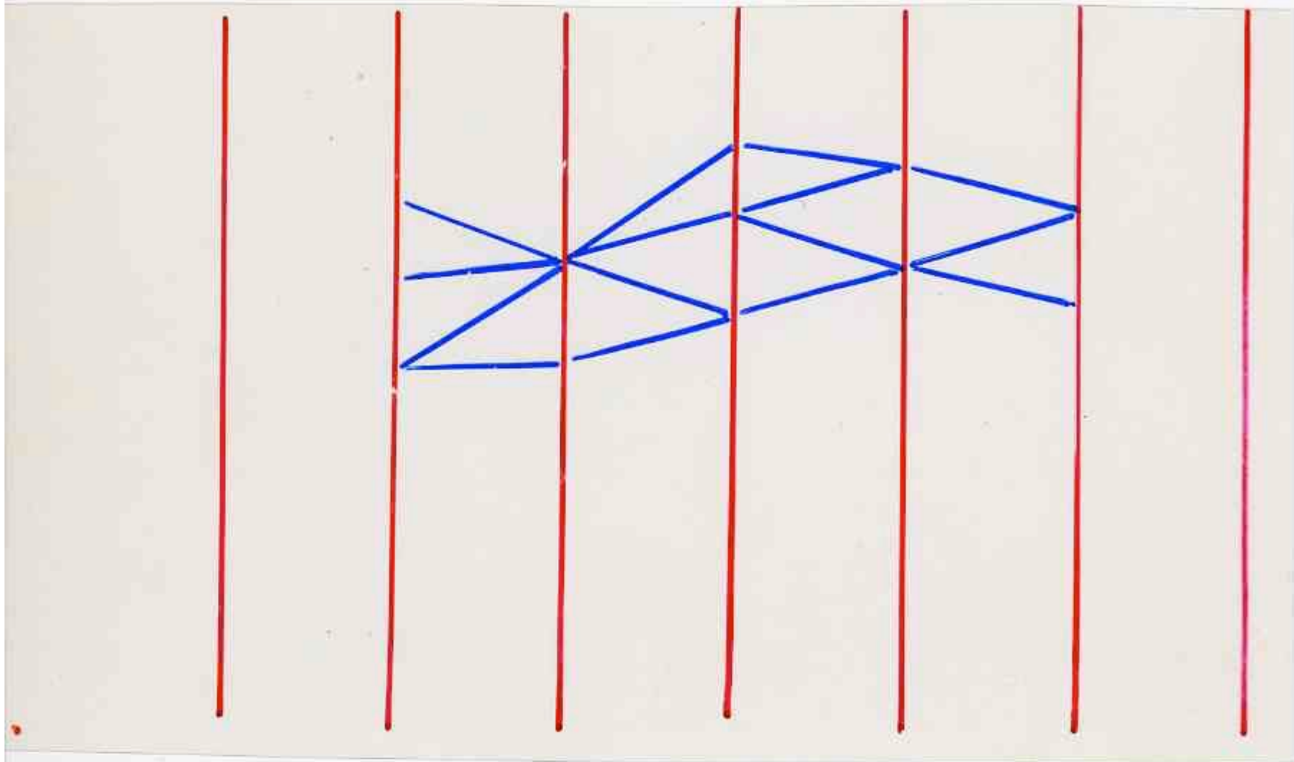
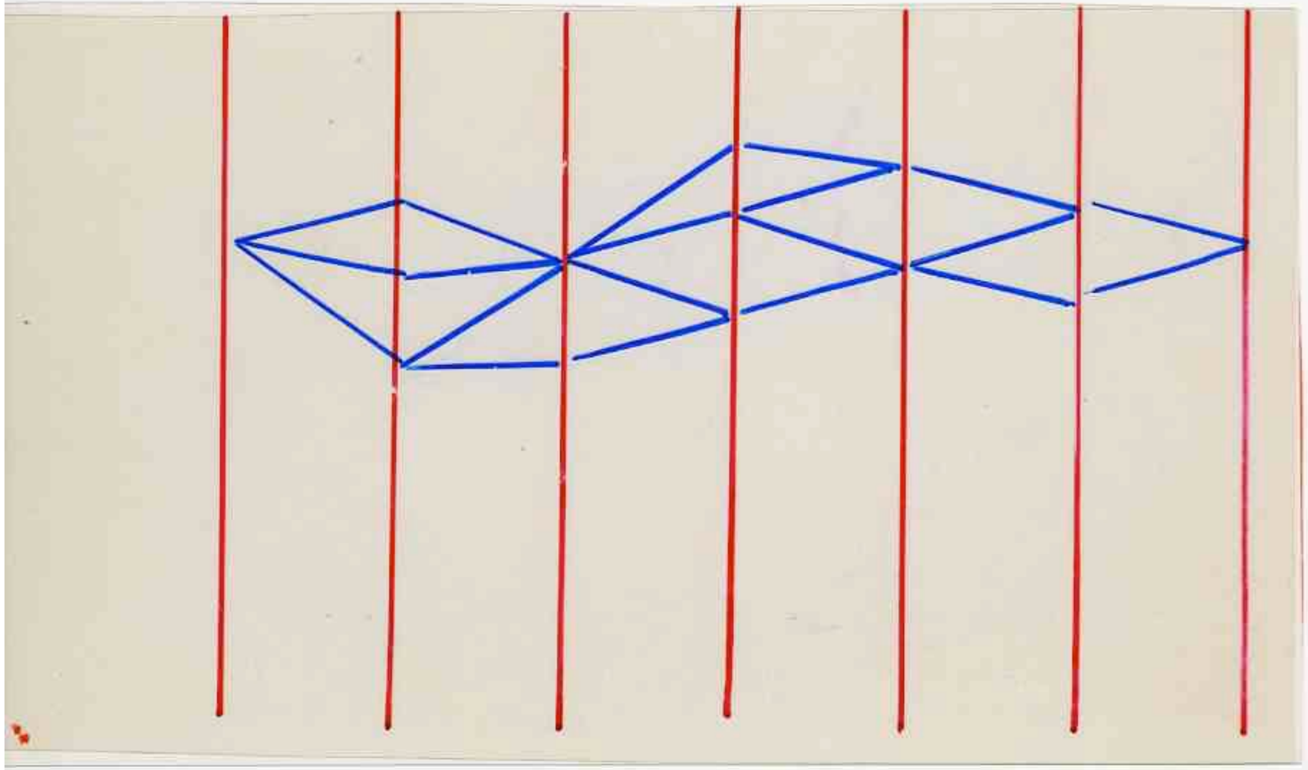
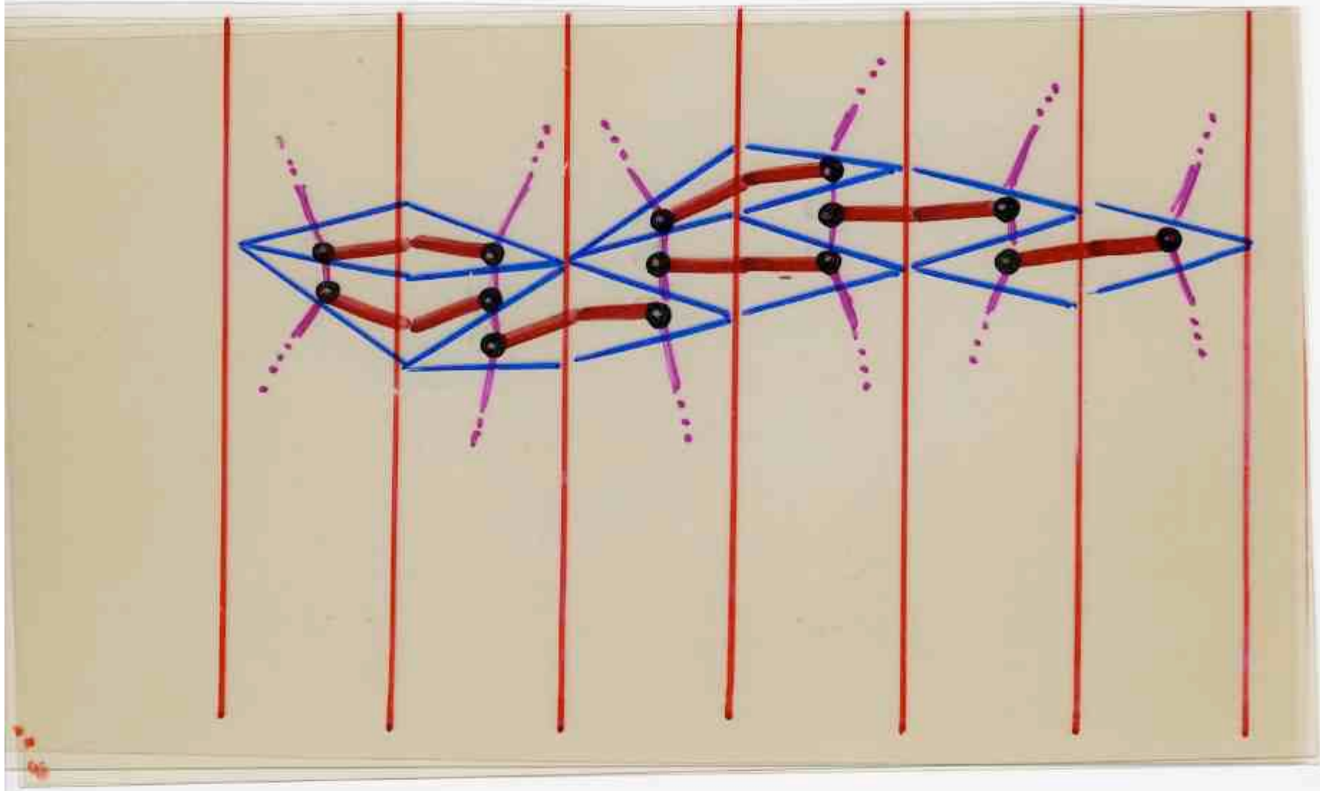
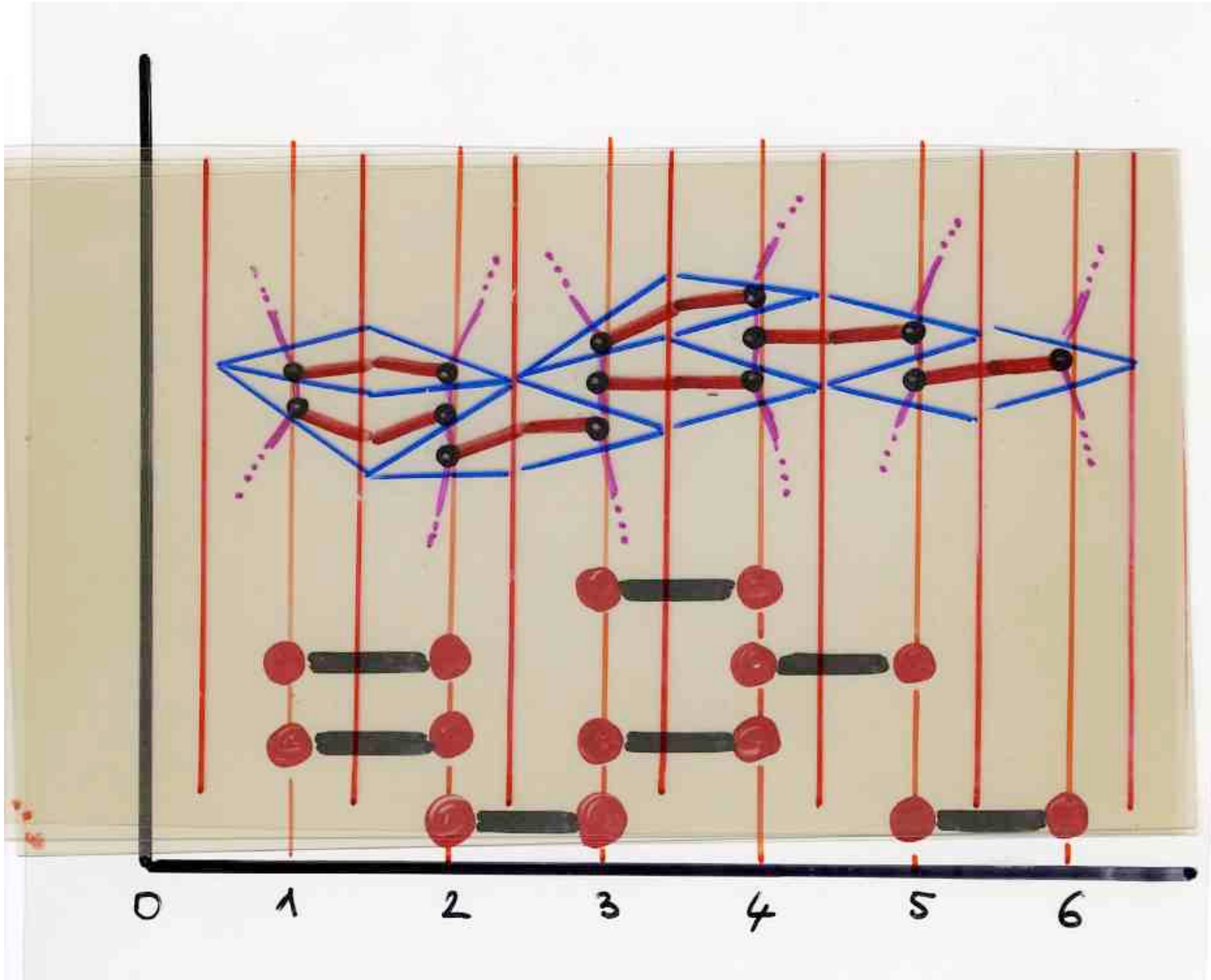






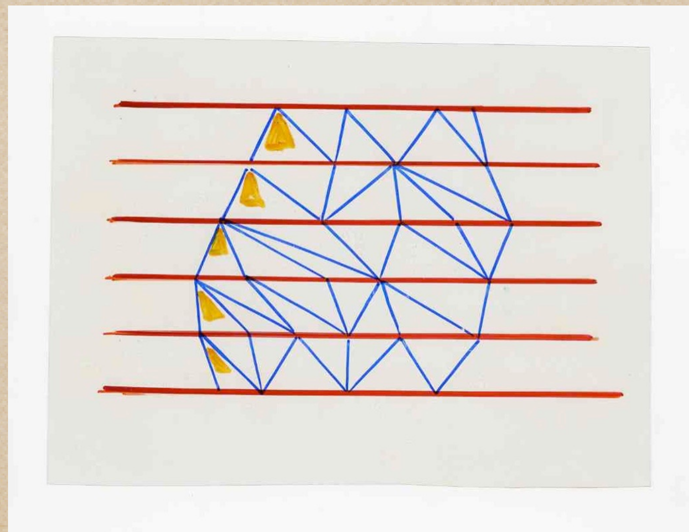


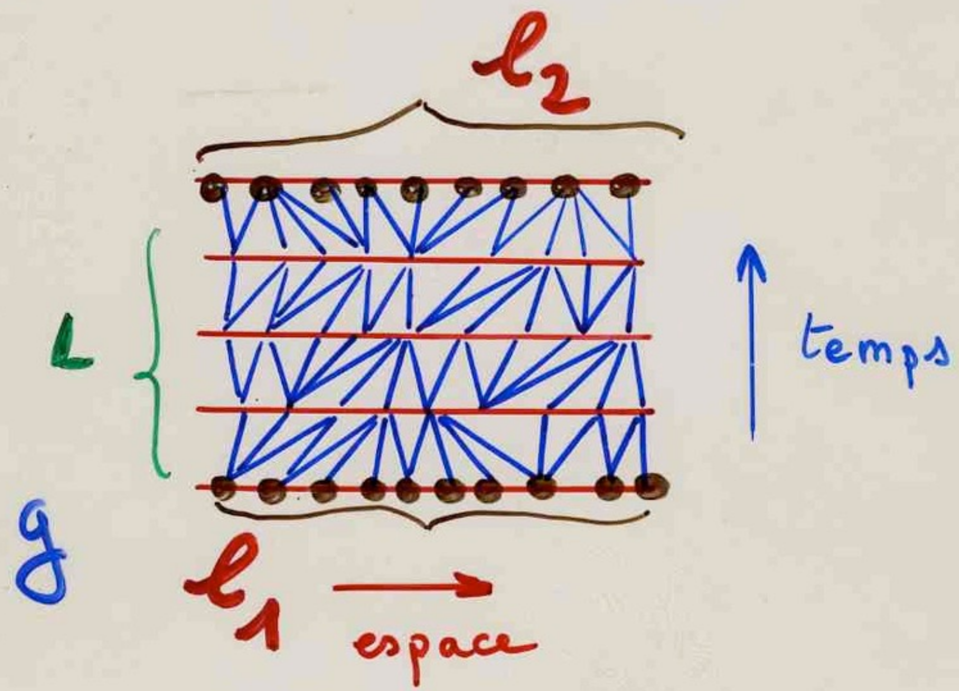



0
1
2
3
4
5
6

4 parameters

generating functions





Path integral amplitude
for the propagation from
geometry ℓ_1 to ℓ_2

Dyck paths

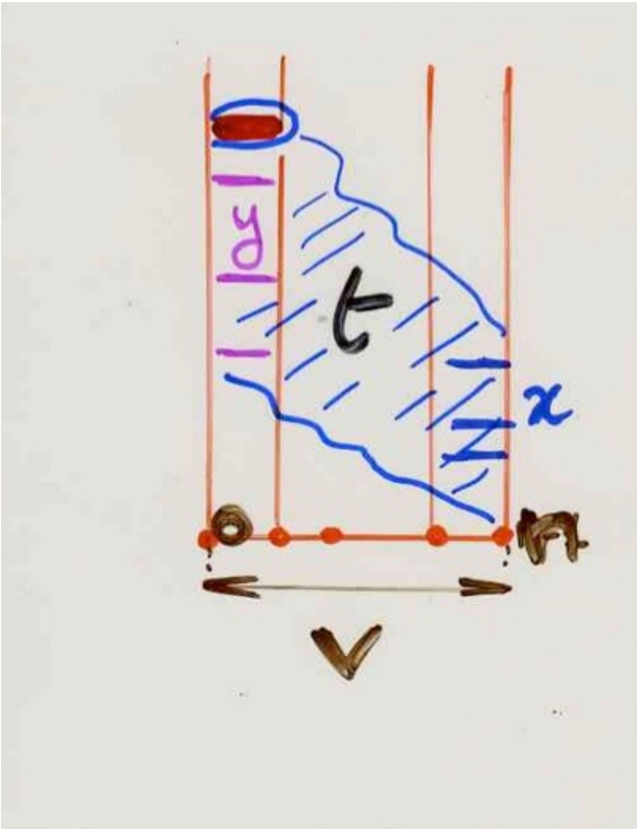
Heaps of dimers

(Pyramids)

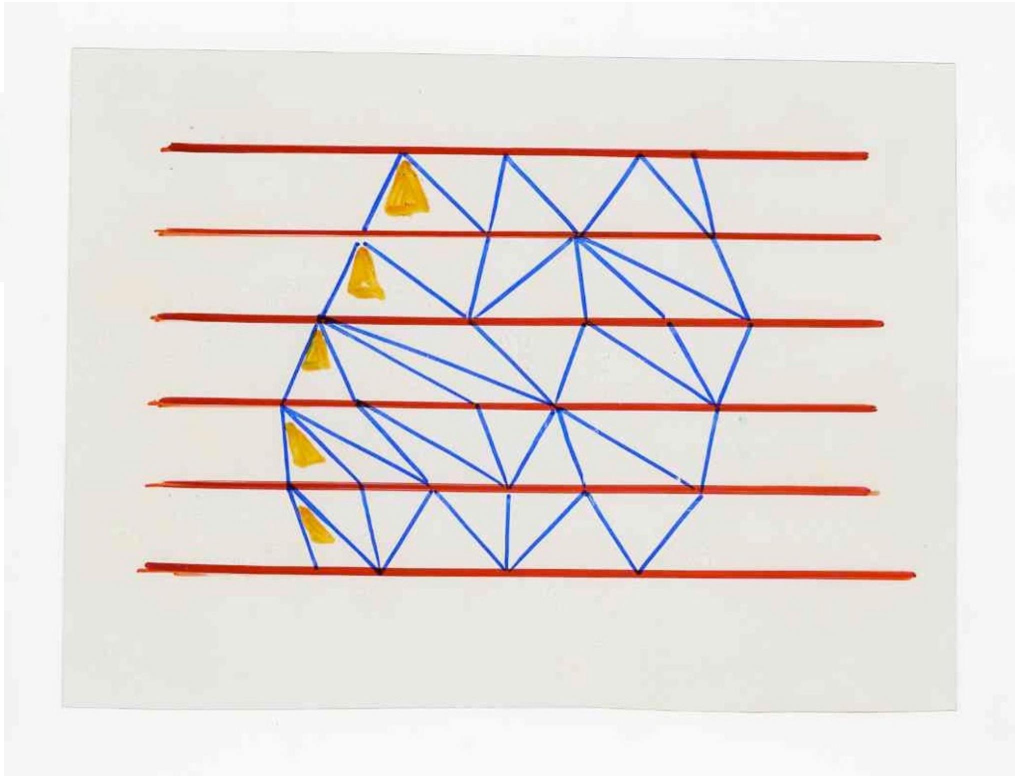
Lorentzian triangulations

(*) border condition

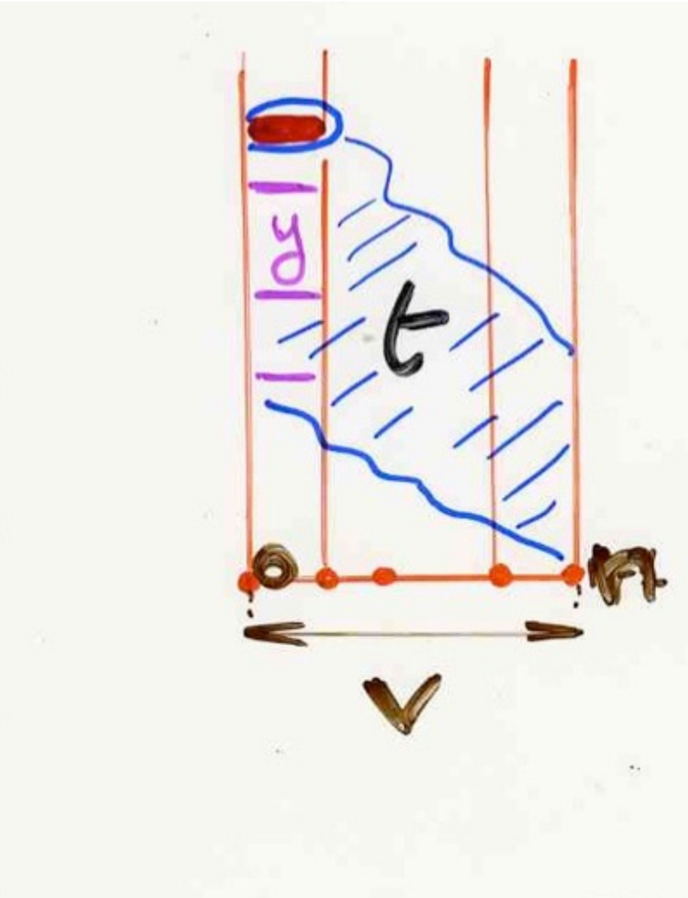
semi-pyramid



bijection



exercise

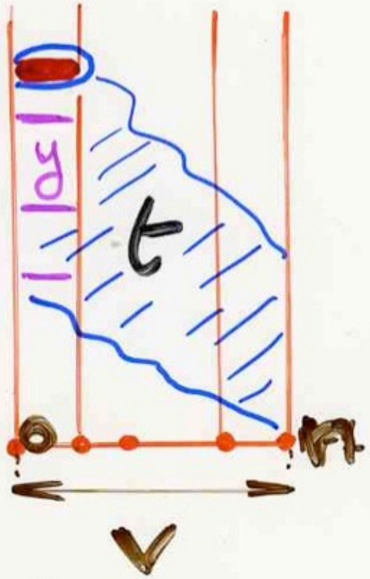


prove that the generating function $Q(t, y, v)$ for pyramids with 3 parameters is given by:

- t number of dimers
- v width
- y number of dimers in the first column

$$Q(t,y,v) = \sum_{n\geq 1} \frac{y\, t^n v^n}{\tilde{F}_n \, \tilde{F}_{n+1}}$$

(Bousquet-Mélou, Rechnitzer)



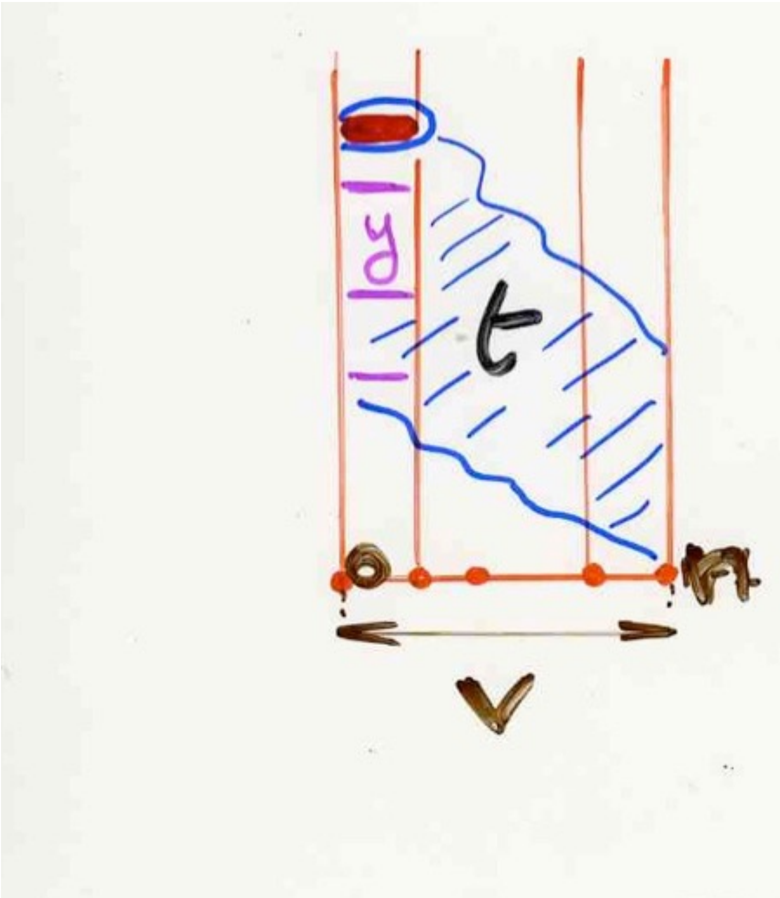
n width

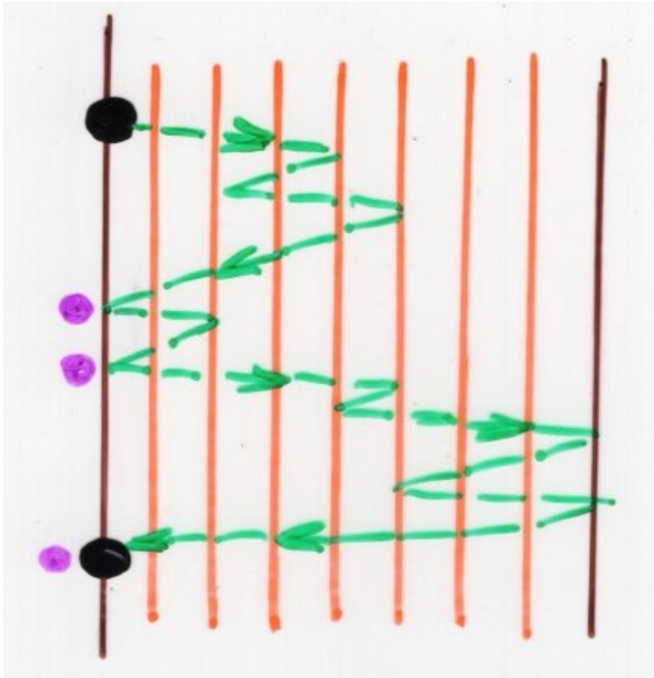
$\tilde{F}_n(t,y)$ defined by :

$$\tilde{F}_n(t,y) = F_{n-1}(t) - y\, F_{n-2}(t)$$

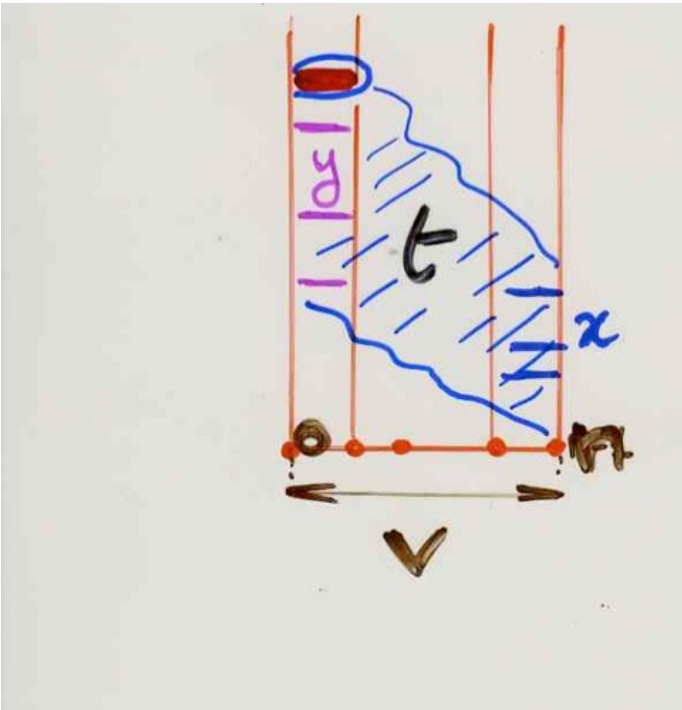
$F_n(t)$ Fibonacci polynomial

hint: use the bijection
with Dyck paths.





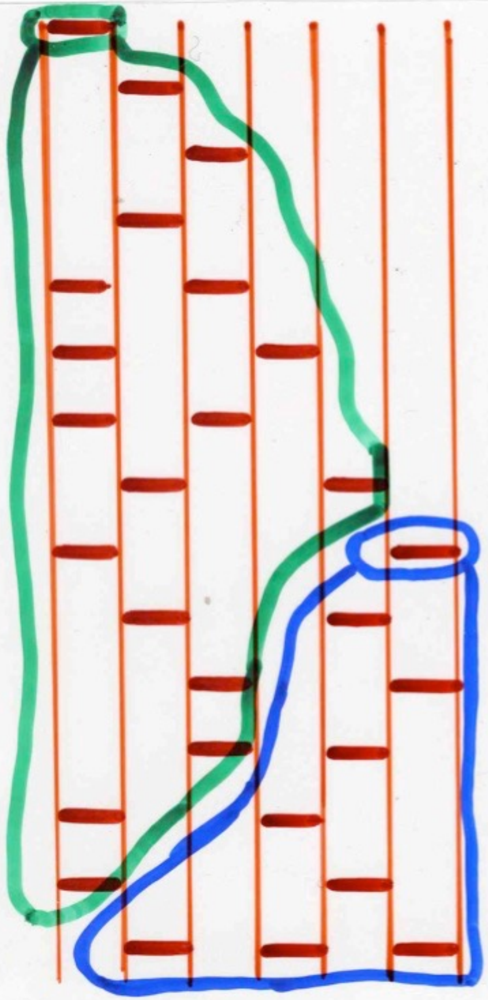
Proposition



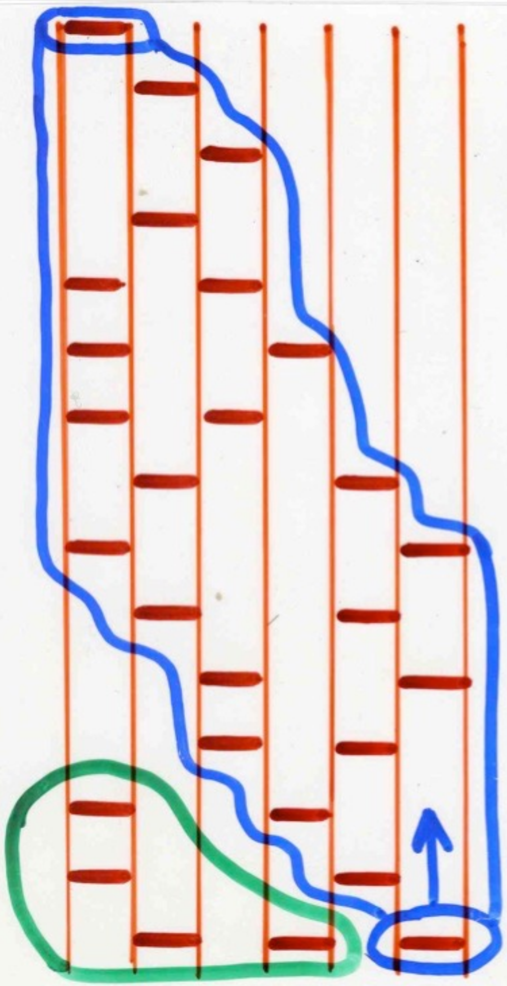
generating function for pyramids of dimers with 4 parameters

- t, v, y
- x number of dimers in the last column

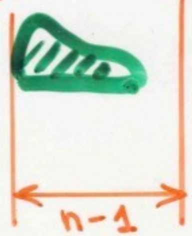
Proposition

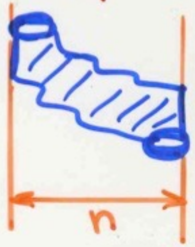


$$\frac{{}_y t^{n} v^{n}_{x}}{\tilde{F}_n(t,y,1)\,\tilde{F}_{n+1}(t,y,x)} = \frac{{}_y t^{n-1} v^{n-1}}{\tilde{F}_{n-1}(t,y,1)\,\tilde{F}_n(t,y,1)} \times \frac{x\,t\,v\,\tilde{F}_{n-1}(t,y,1)}{\tilde{F}_{n+1}(t,y,x)}$$

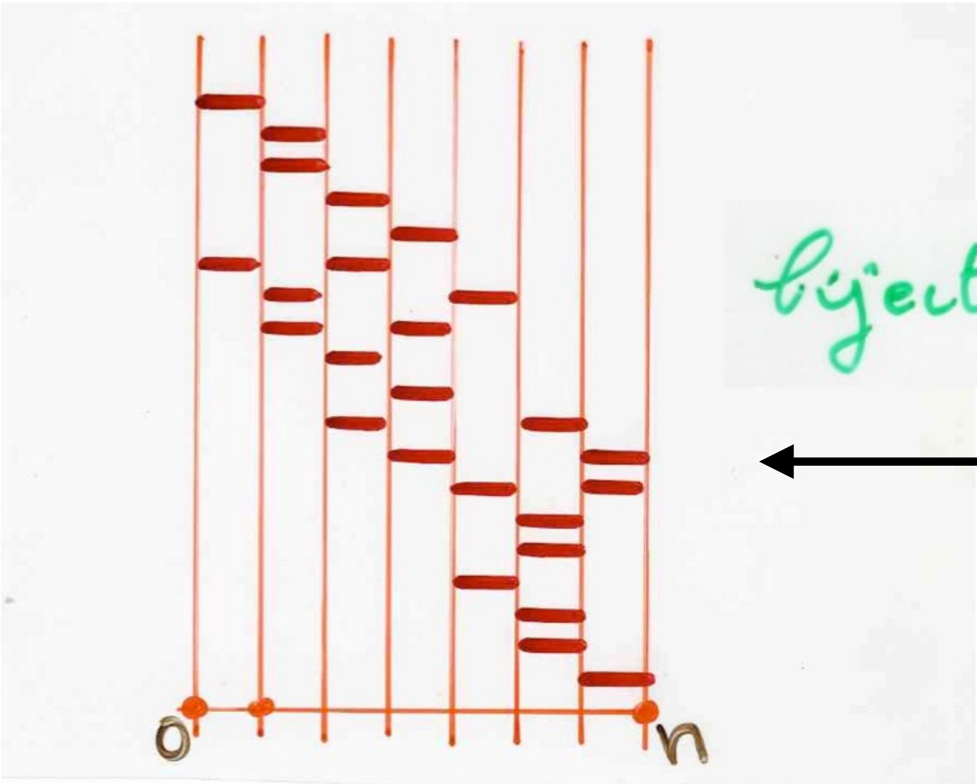


$$\frac{y\,t^{n} v^{n}}{\tilde{F}_{n}(t,y,1)\,\tilde{F}_{n+1}(t,y,x)} = \frac{1}{\tilde{F}_{n}(t,y,1)} \times \frac{y\,t^{n} v^{n}}{\tilde{F}_{n+1}(t,y,x)}$$

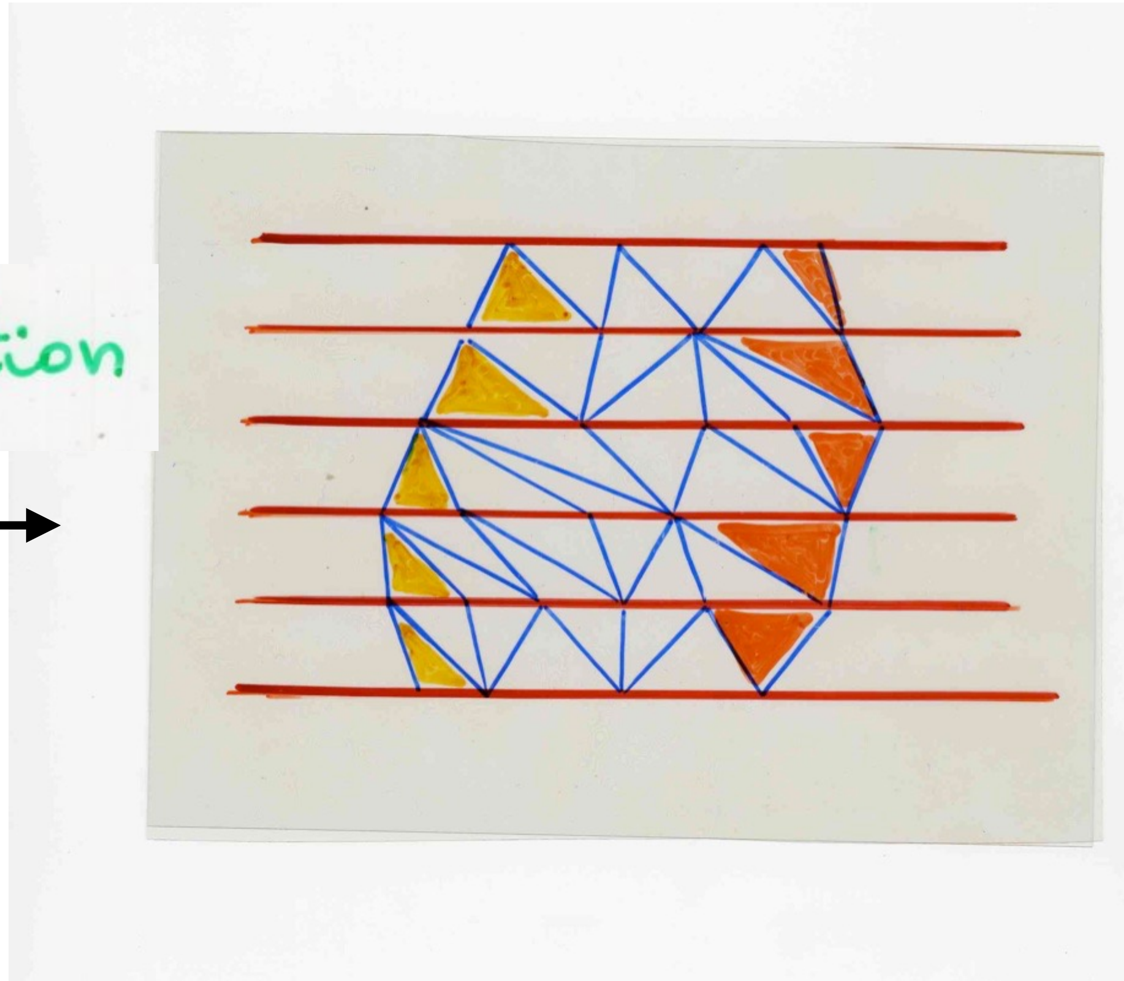




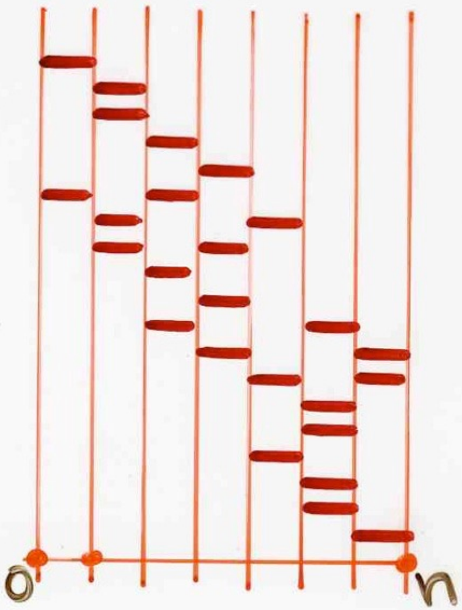
semi- double pyramid



bijection



exercise



semi- double pyramid

in bijection with (general) heap on $[0,n]$

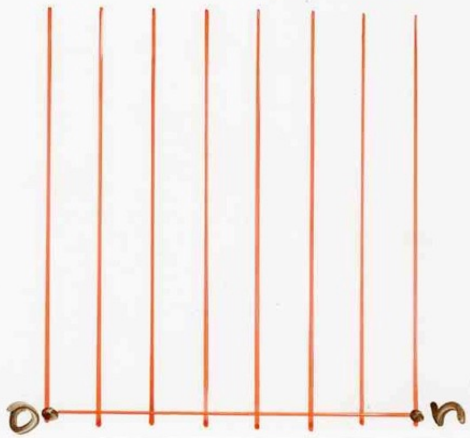
P_m t^m

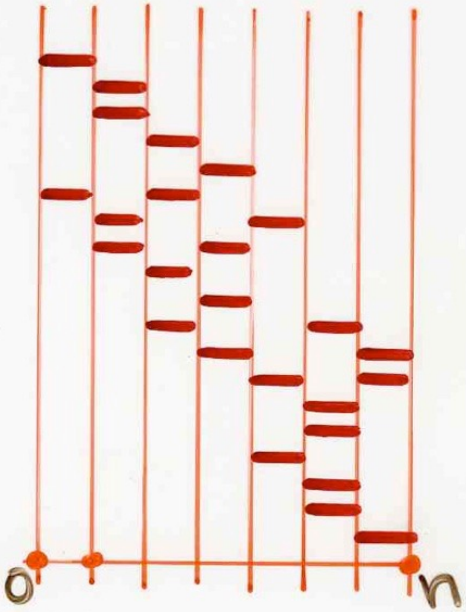
m dimers

H_{m-n} t^{m-n}

$m-n$ dimers

preserve x^{i-1} y^{j-1}





semi-double pyramid

in bijection with (general) heap on $[0,n]$

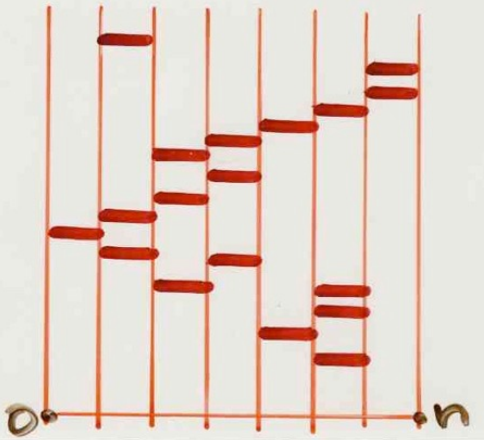
$P_m \quad t^m$

m dimers

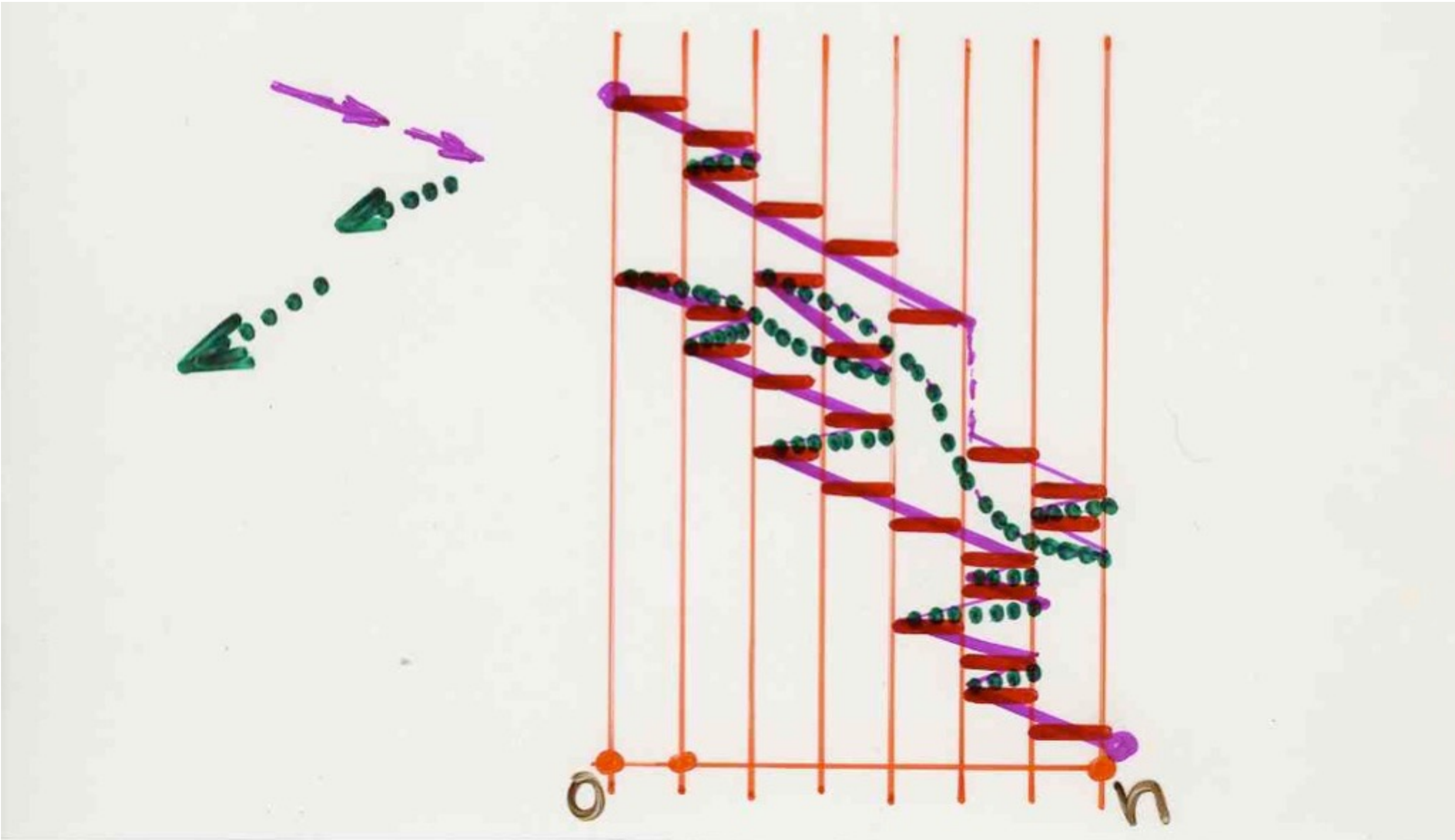
$H_{m-n} \quad t^{m-n}$

$m-n$ dimers

preserve $x^{i-1} \quad y^{j-1}$



double
semi- pyramid



in bijection

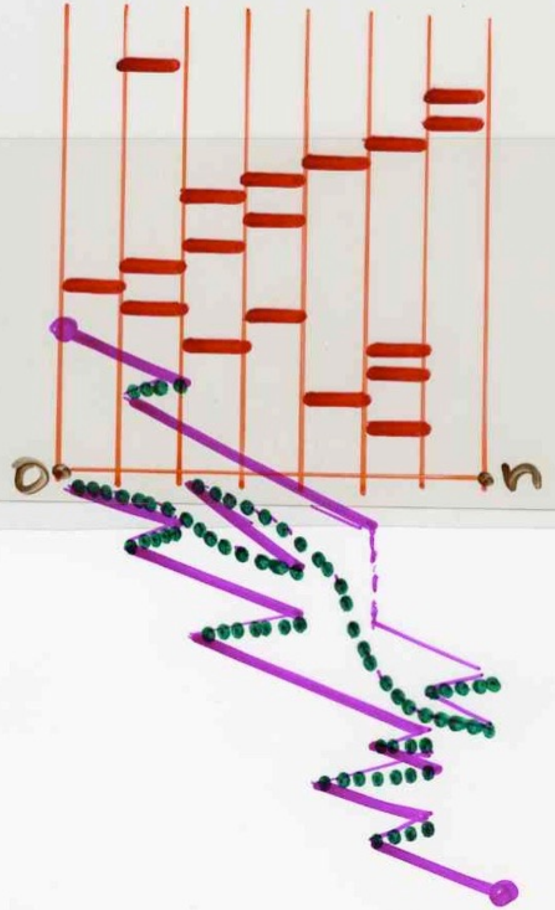
P_m t^m with (general) **heap** on $[0,n]$

m dimers

H_{m-n} t^{m-n}

$m-n$ dimers

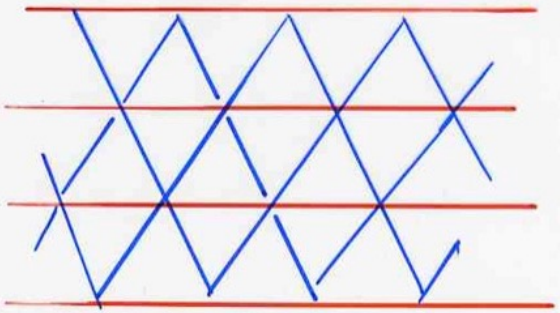
preserve x^{i-1} y^{j-1}



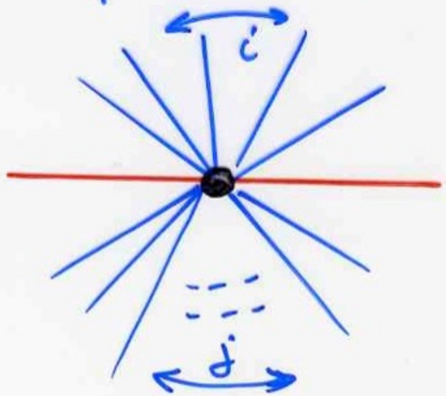
curvature

curvature

of the space-time



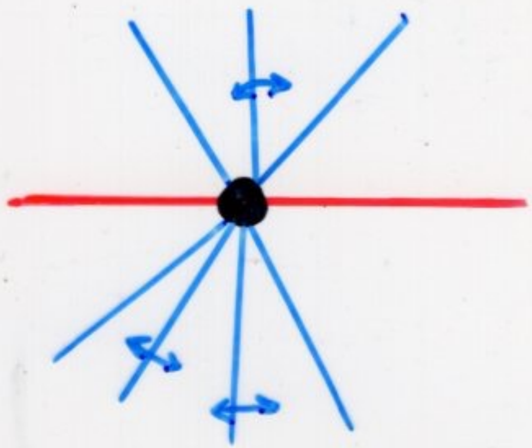
flat

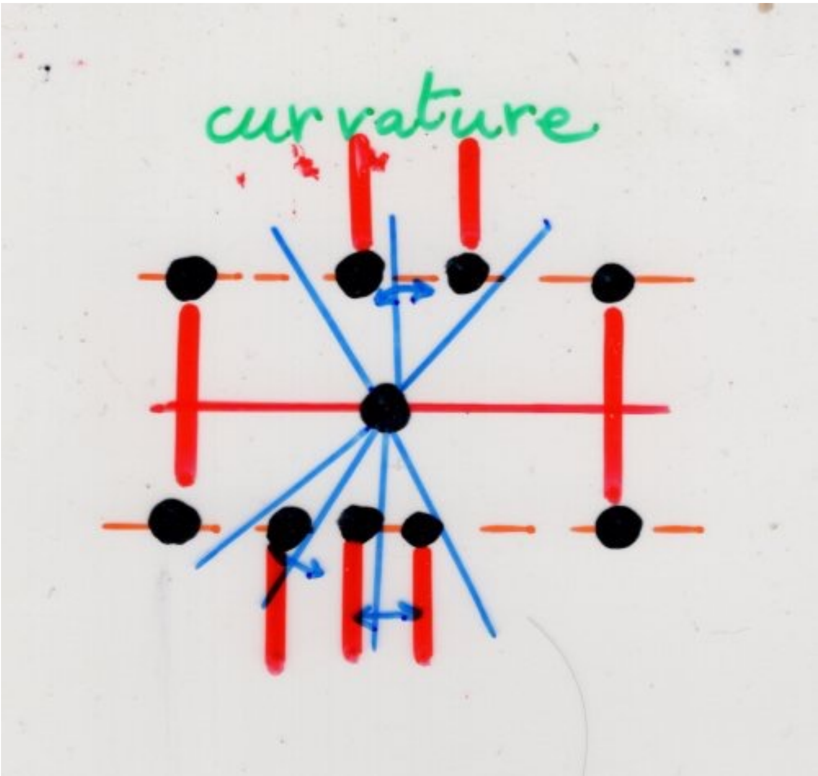


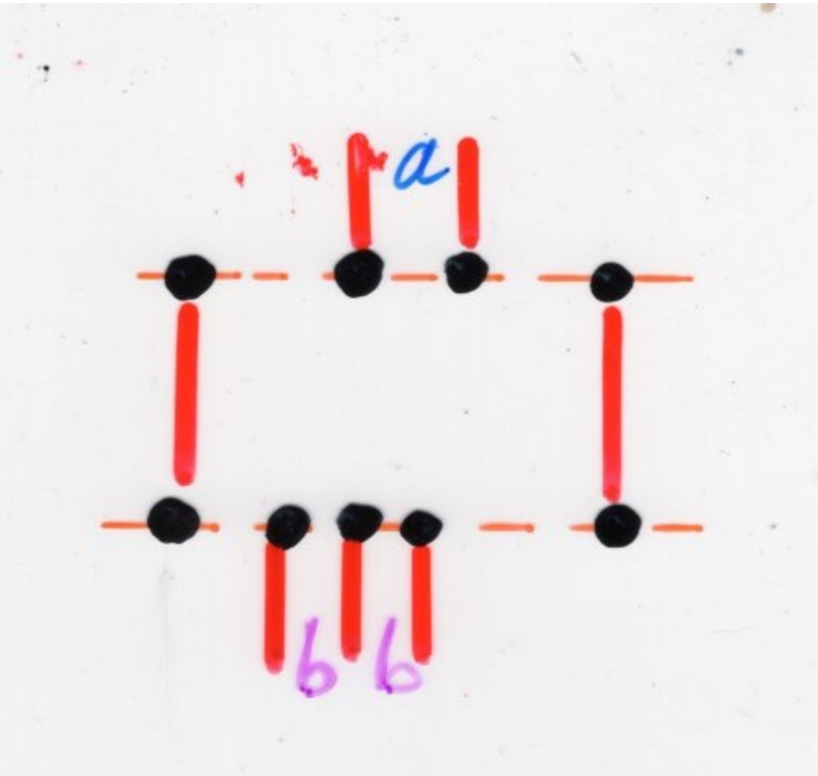
$$a^{|i-3|+|j-3|}$$

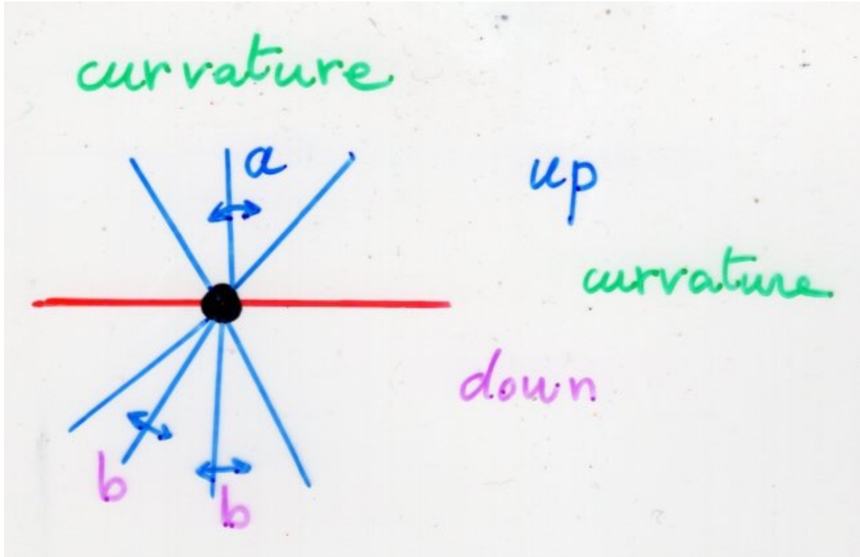
$$\text{total curvature} = \prod_{\text{all points}} a^{(\cdots)}$$

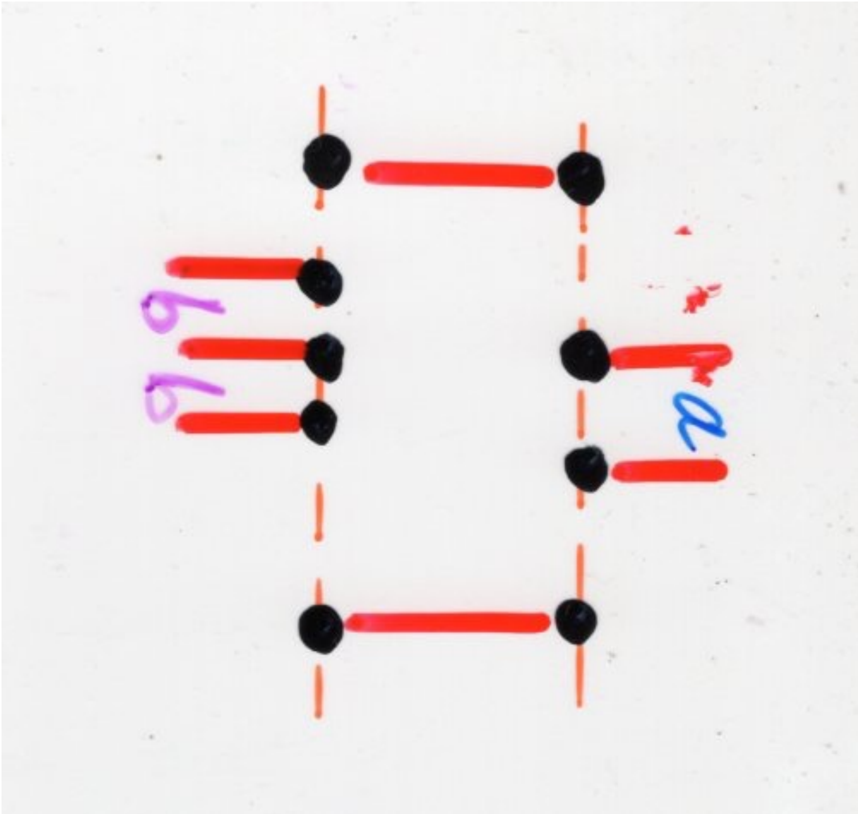
curvature

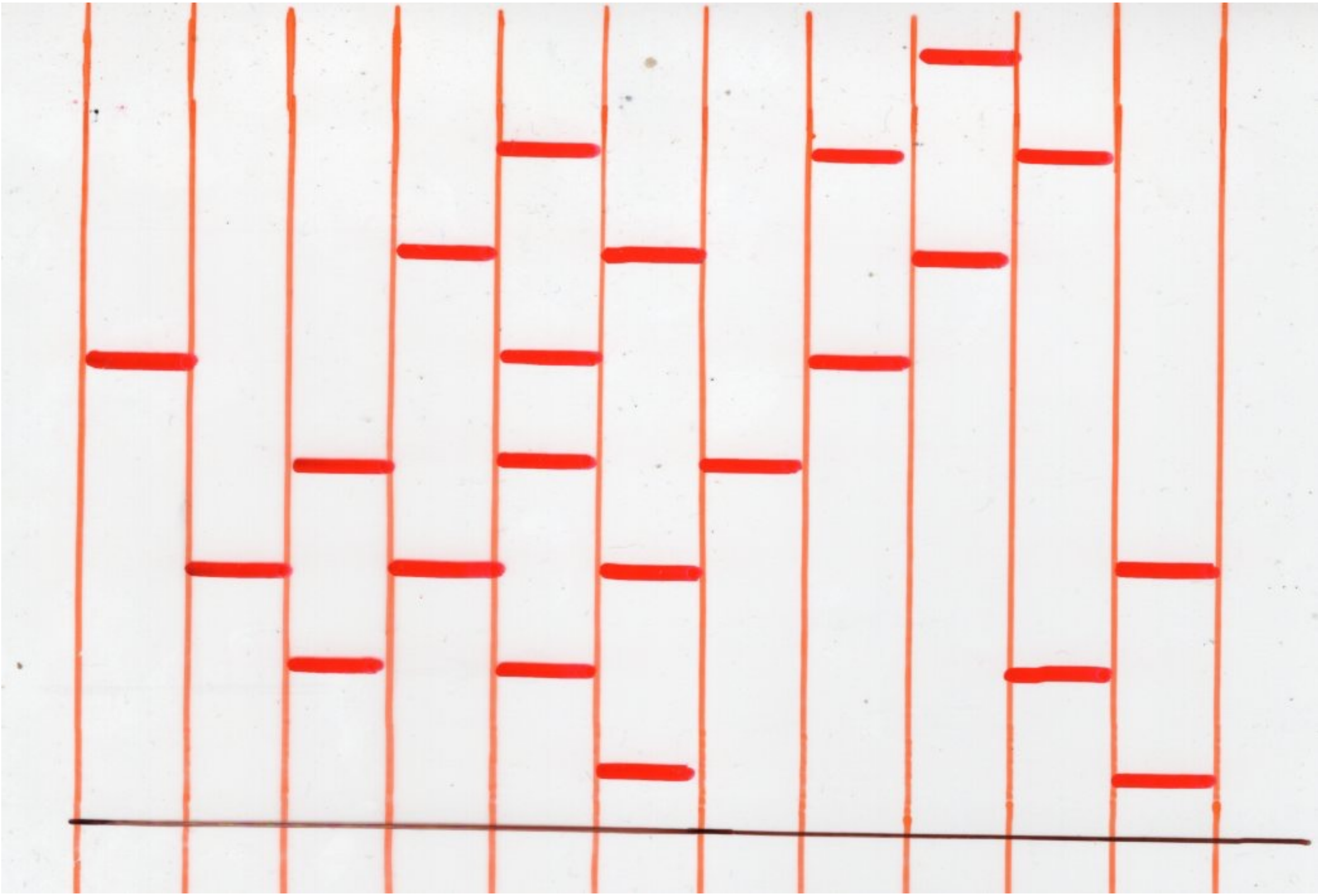


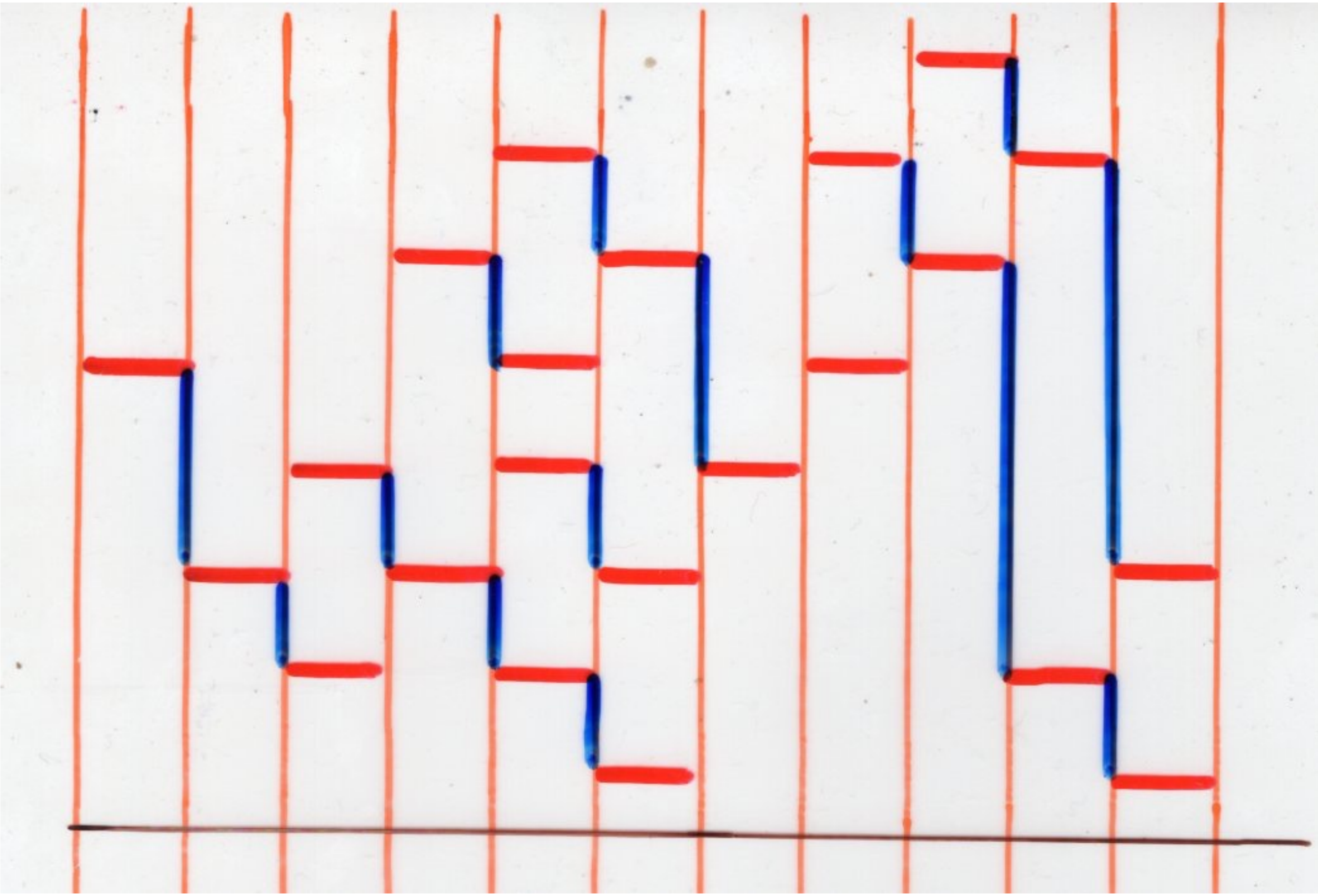

curvature

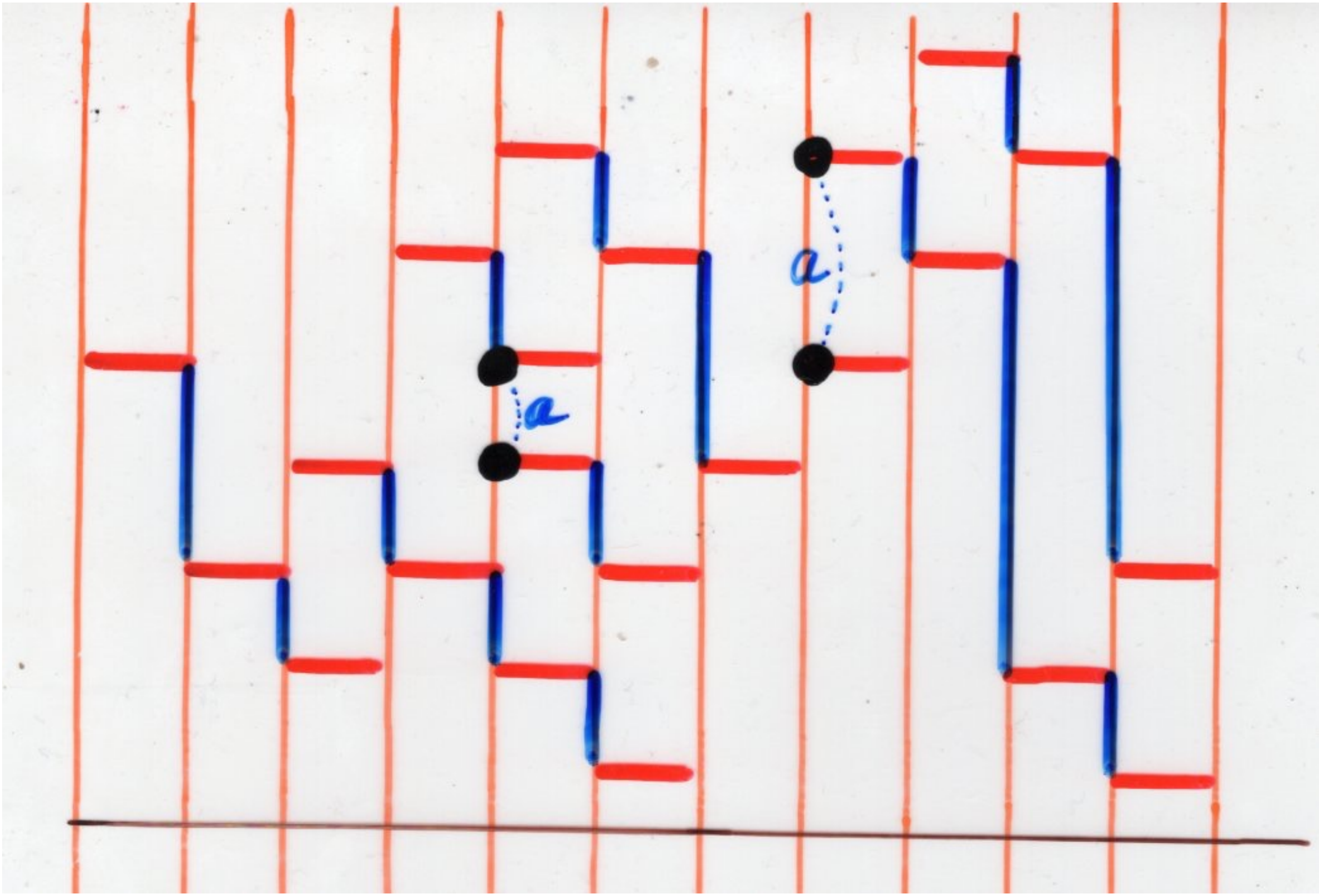

a
b
b

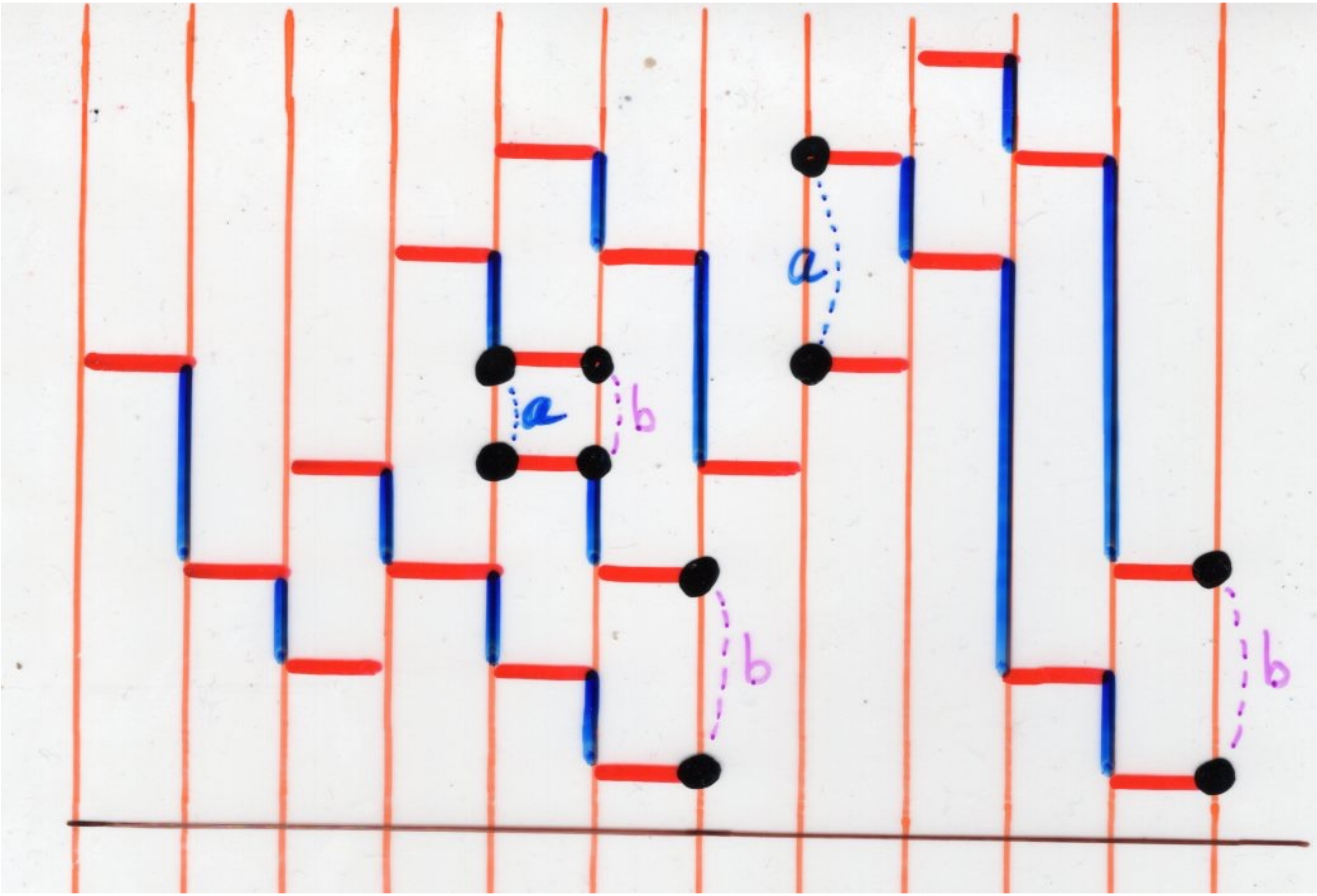

curvature
a
up
curvature
down
b
b

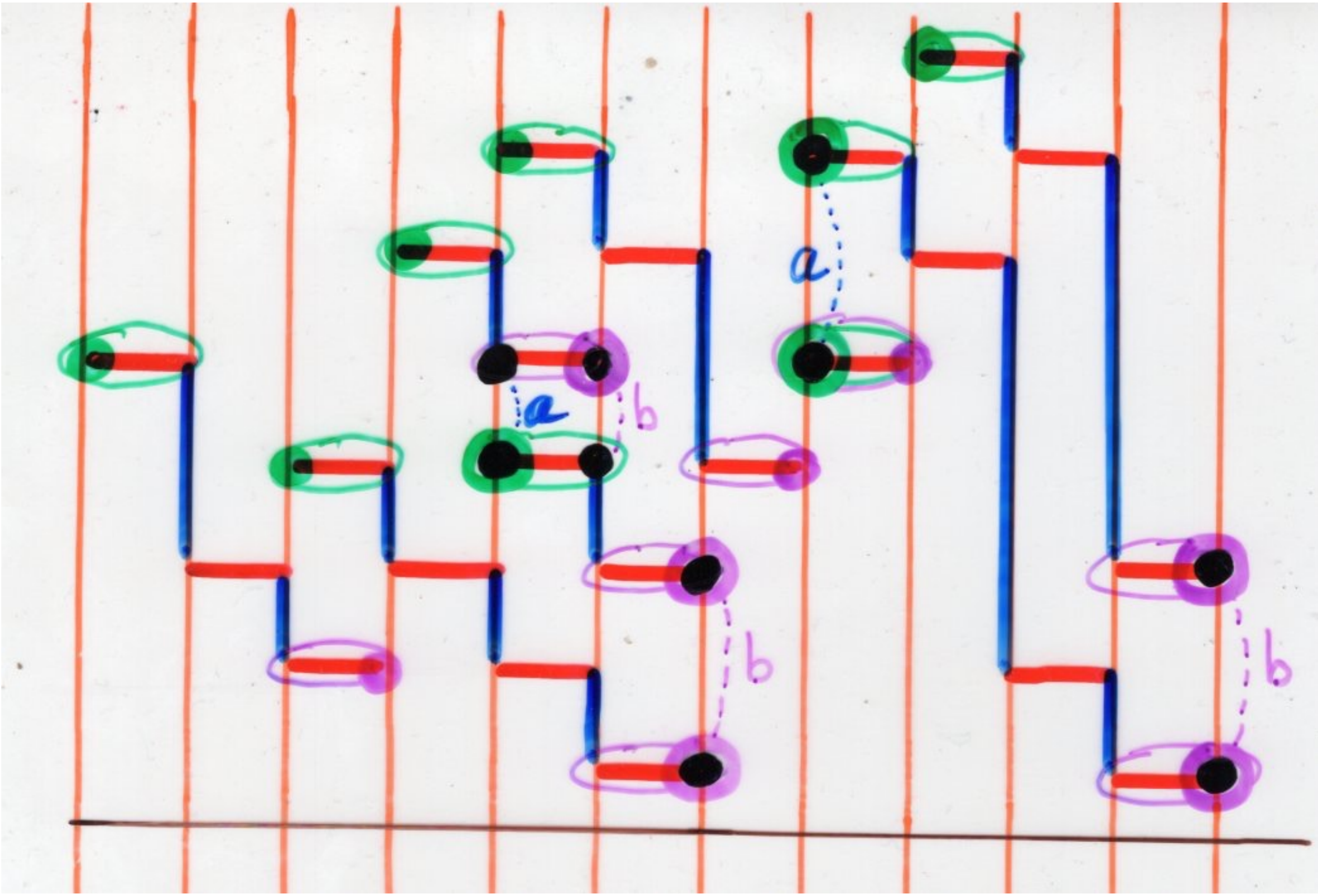


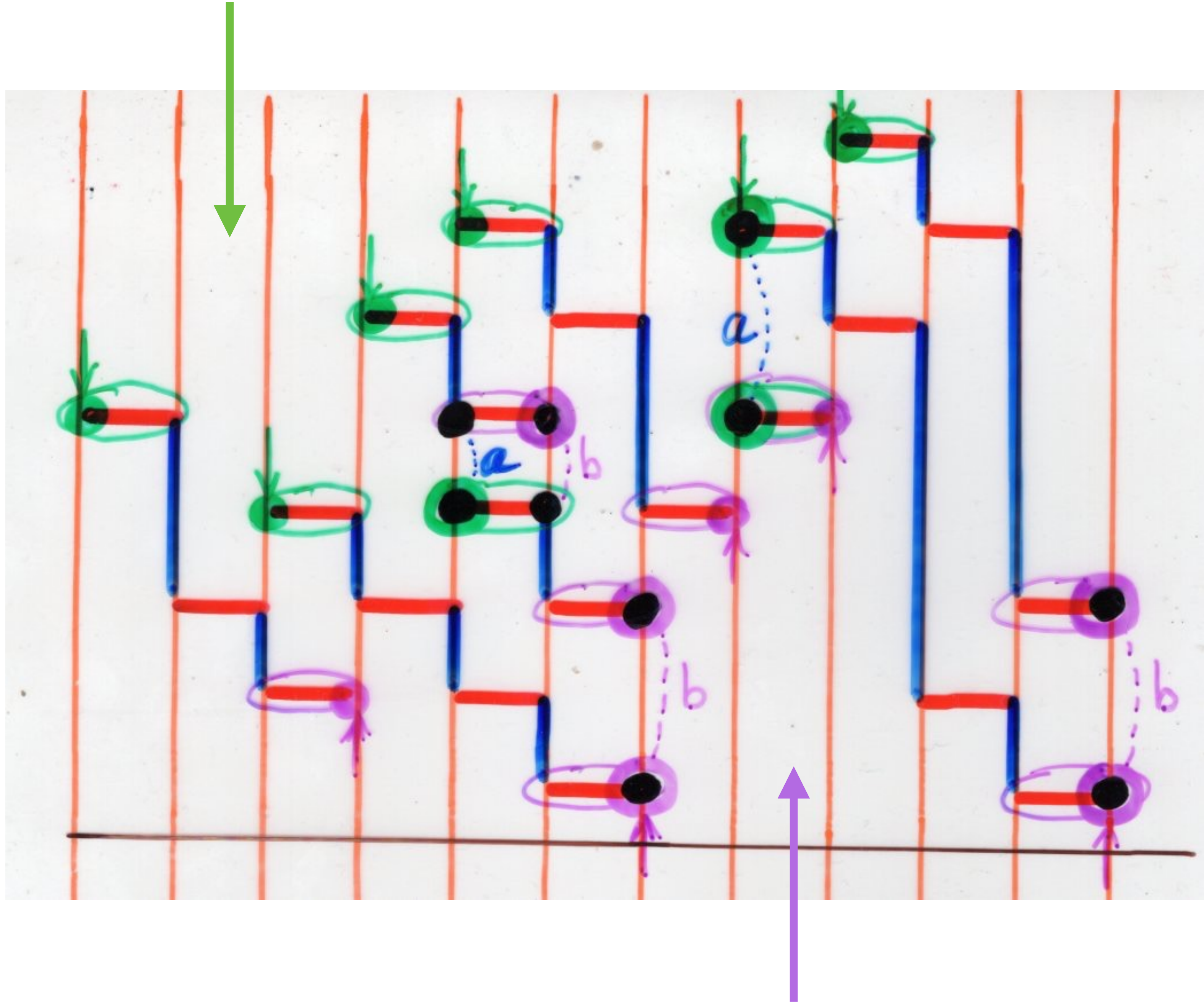




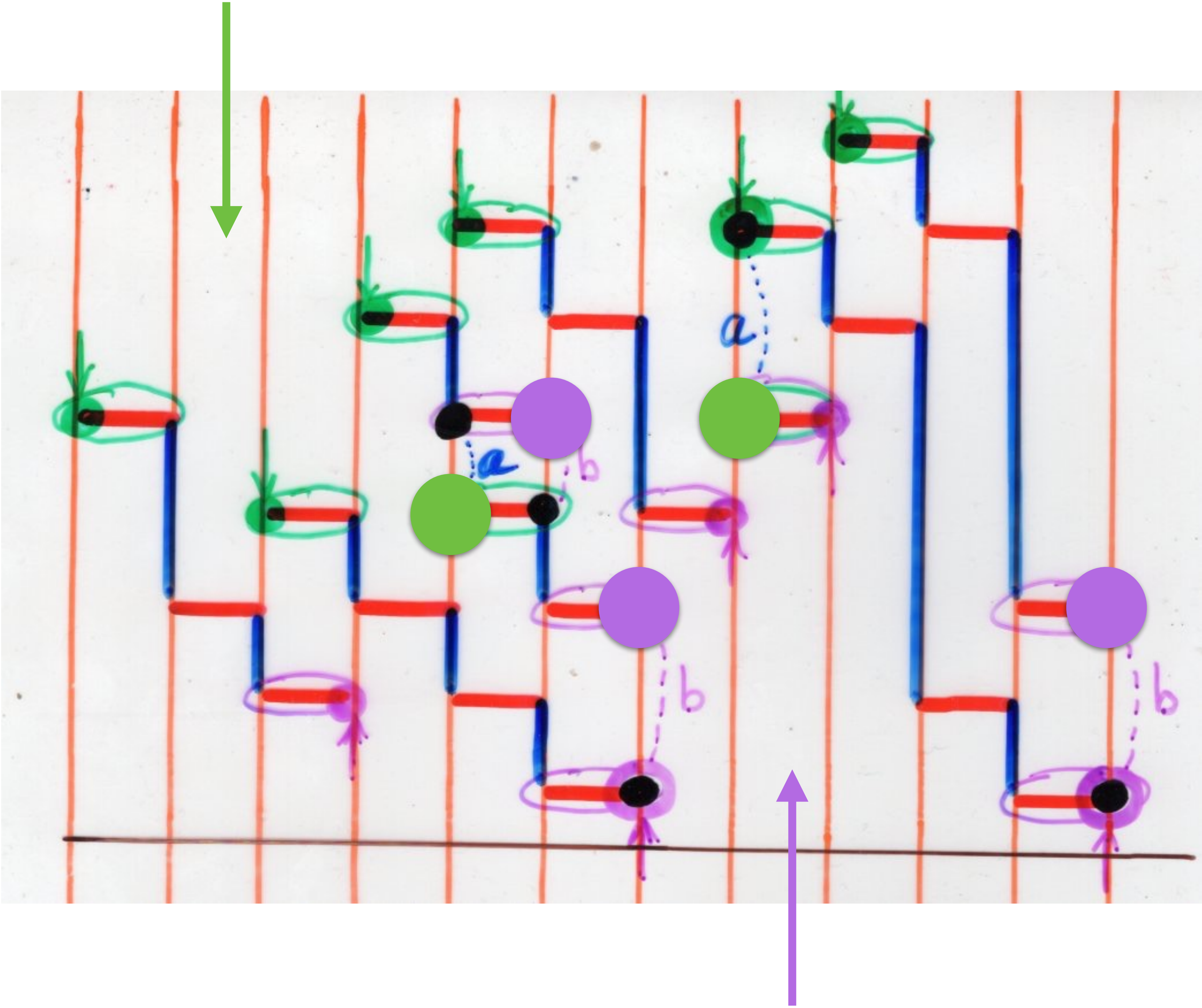

a
a

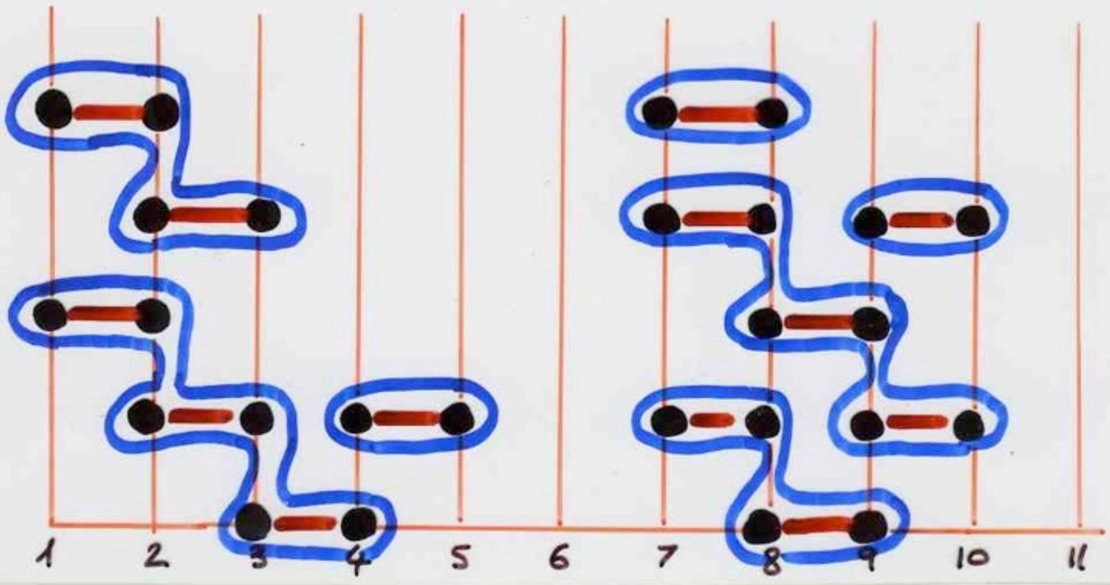

a
b
b
a
b


a
a
b
b
b

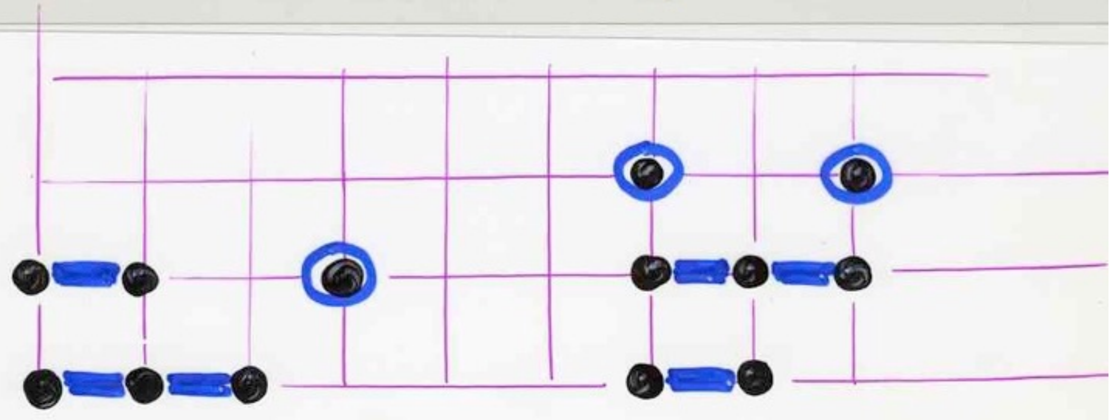


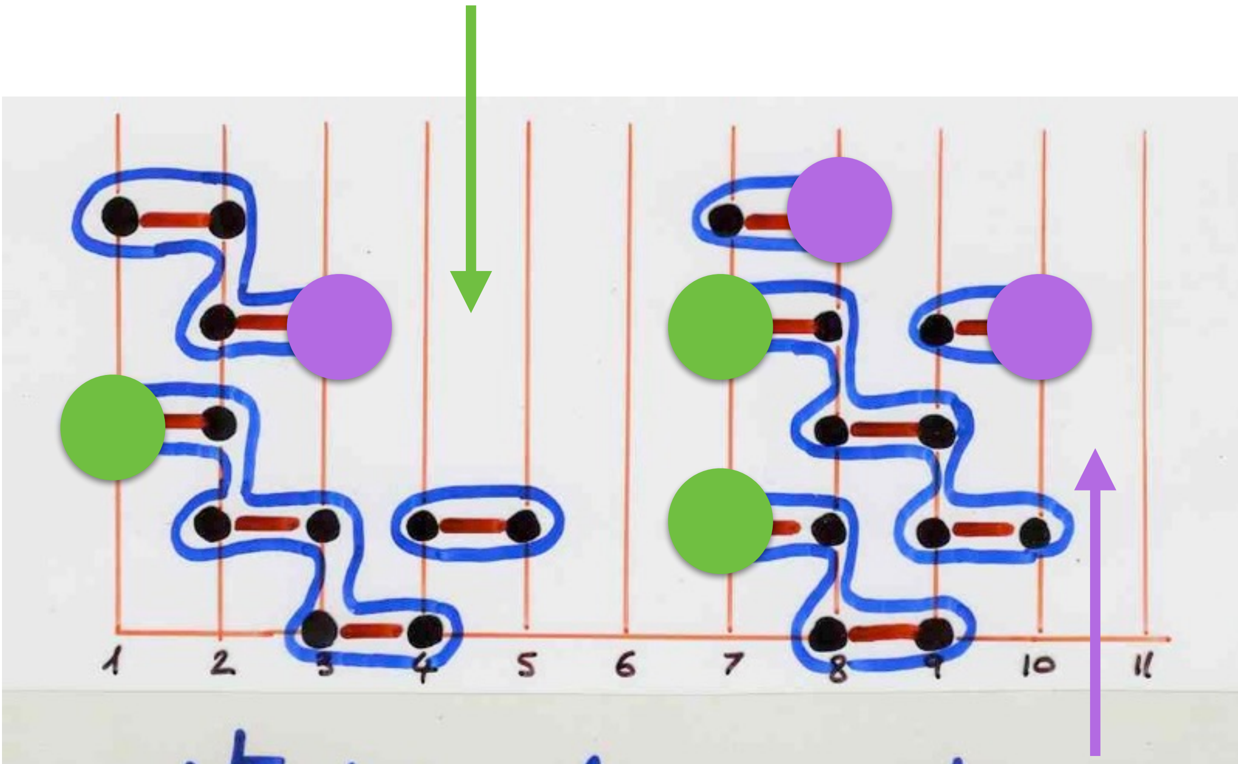
a
b
a
b
b


a
a
b
b
b

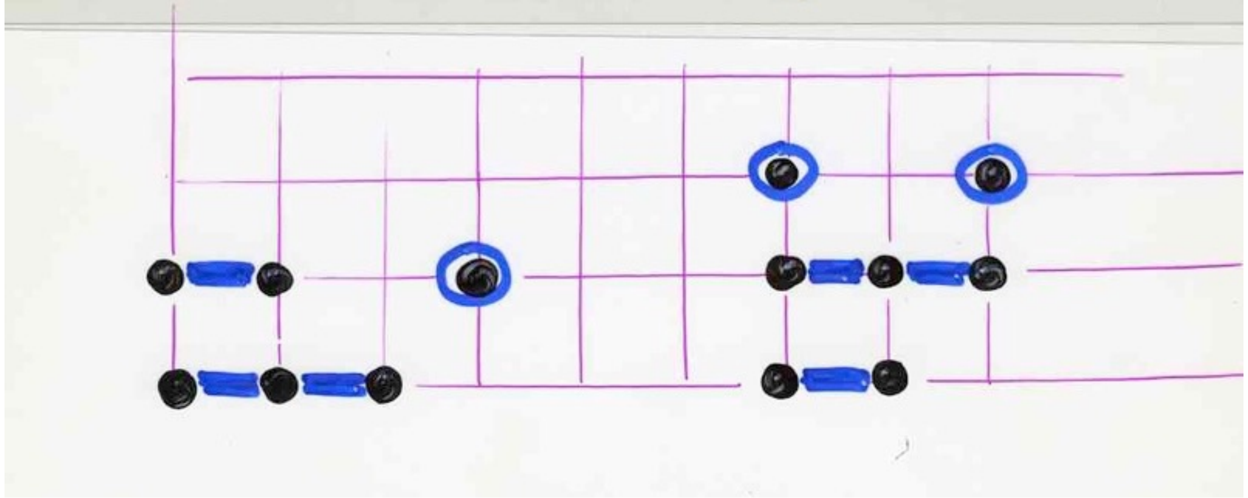


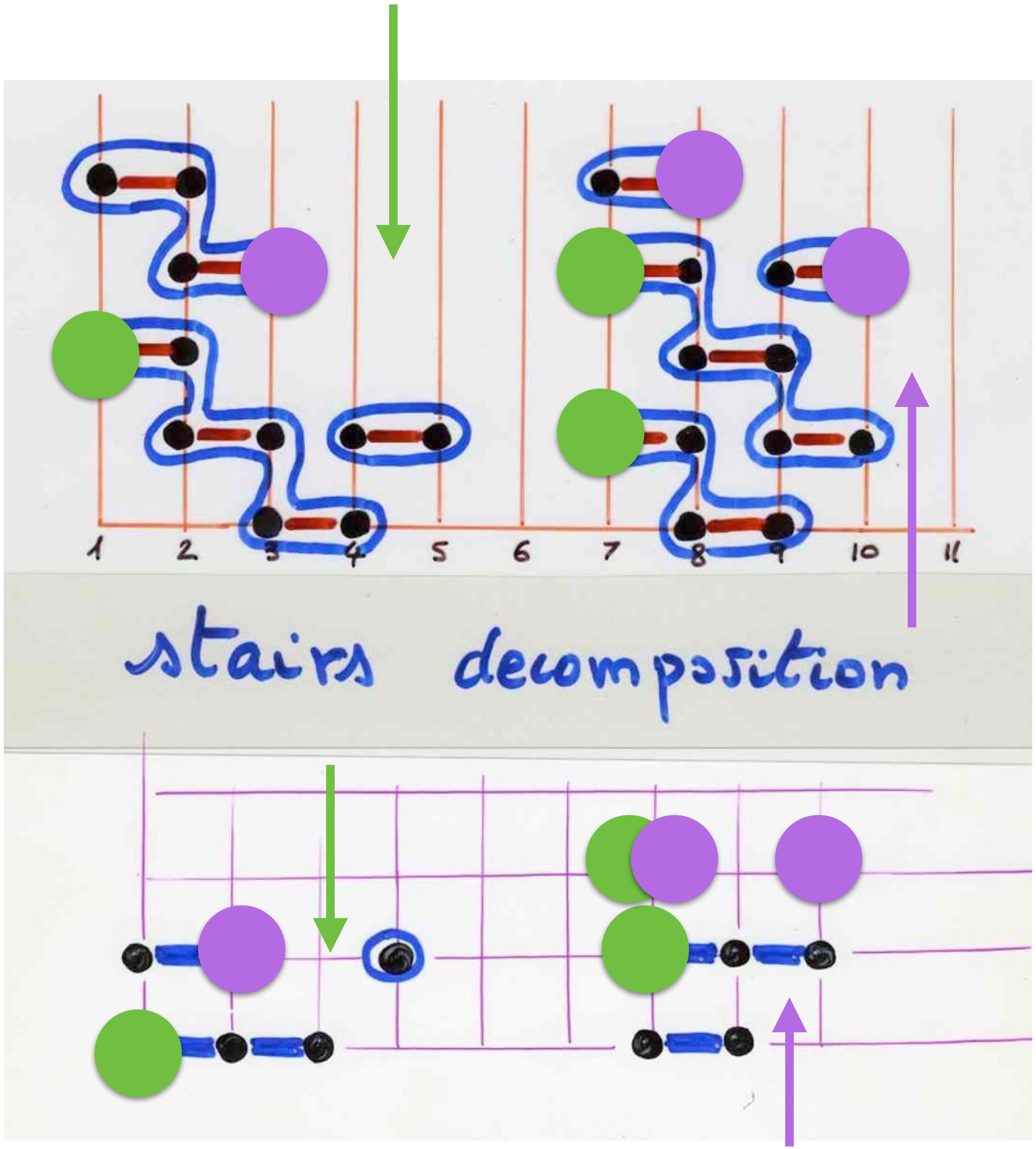
stairs decomposition





stairs decomposition




1 2 3 4 5 6 7 8 9 10 11
stairs decomposition

heaps of dimers

on $[0,n]$ with

total curvature $= 0$

up- curvature $= 0$

(or down-)

?

heaps of dimers

on $[0, n]$ with

total curvature $= 0$

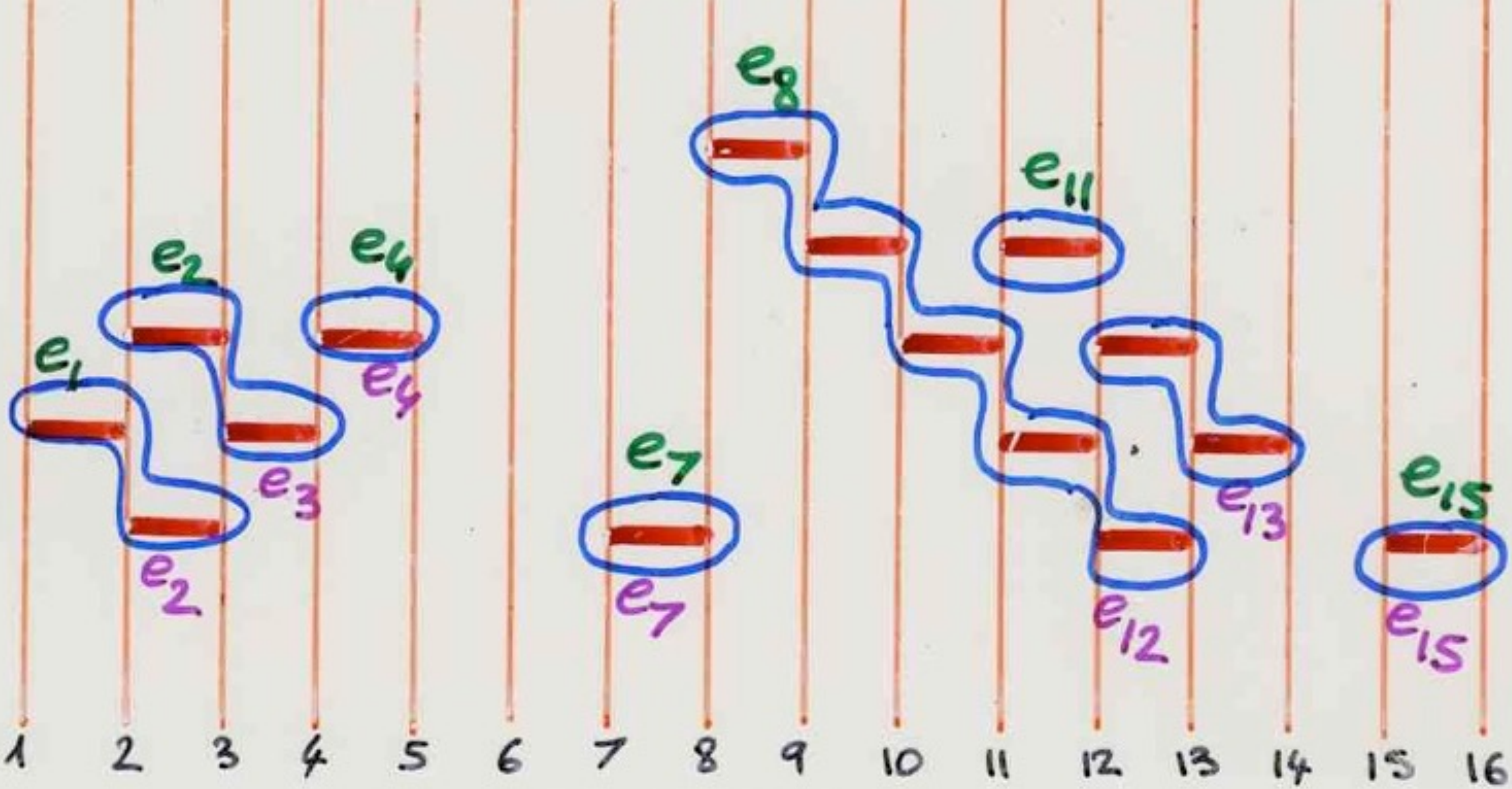
up- curvature $= 0$

(or down-)

number

C_n Catalan number

$n!$



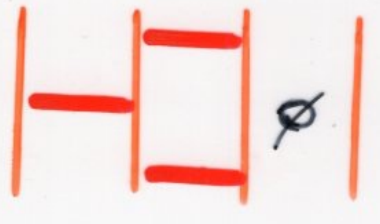
from Chapter 6a

$$
\begin{array}{cccccccccccccccc}
 & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \\
1 \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}
$$

exercise

from Chapter 6a, p86

The number of strict heaps satisfying the condition :



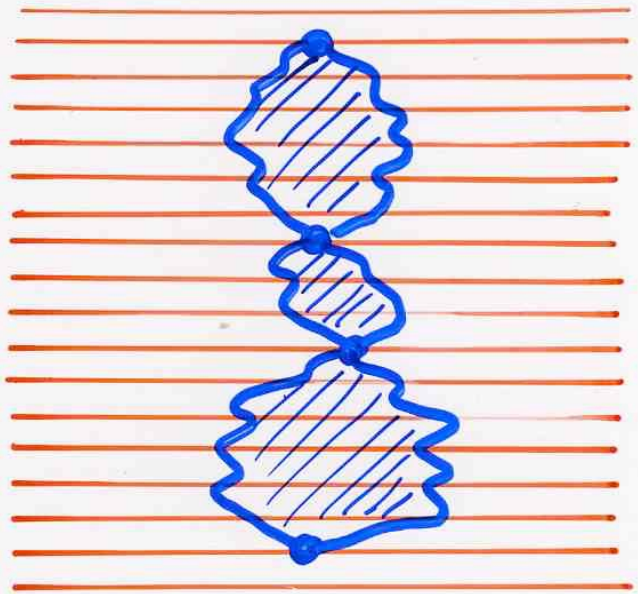
$\Longleftrightarrow$ $\min(S_1) < \cdots < \min(S_k)$

is $n!$

$\Longleftrightarrow$ $\max(S_1) < \cdots < \max(S_k)$

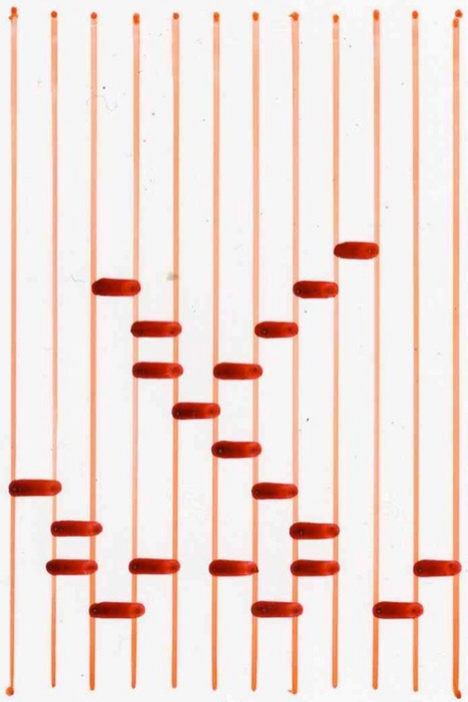
Lorentzian triangulations
in 2D qantum gravity

the nordic decomposition
of a heap of dimers

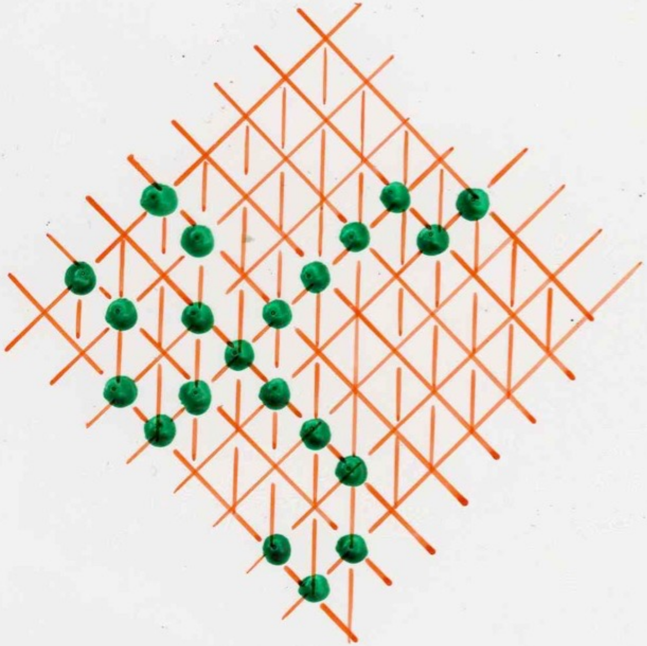


(general)
Lorentzian
triangulation

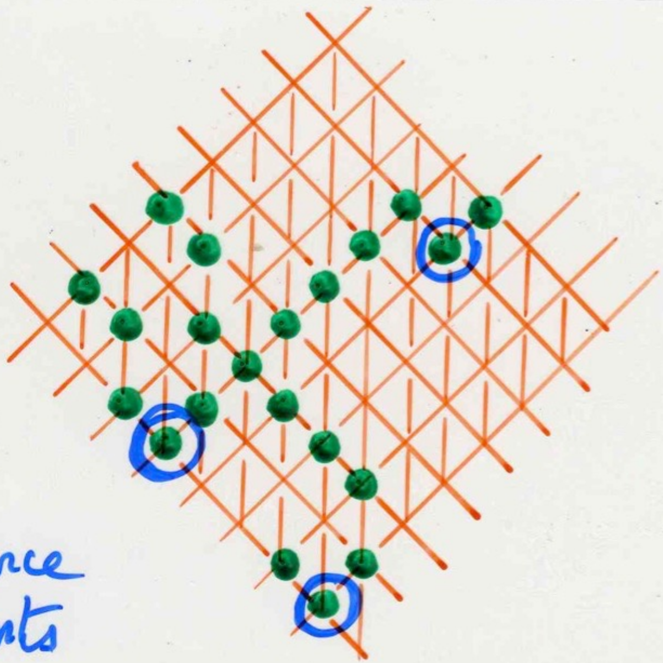
Lorentzian triangulation with no articulation points $\Longleftrightarrow$ connected **heap** of dimers



connected
heap
of
dimers



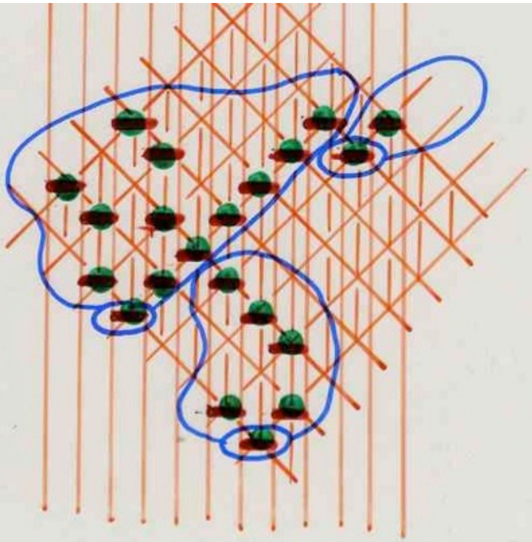
multidirected
animal
(triangular
lattice)
(Bousquet-Mélou,
Rechnitzer, 2002)



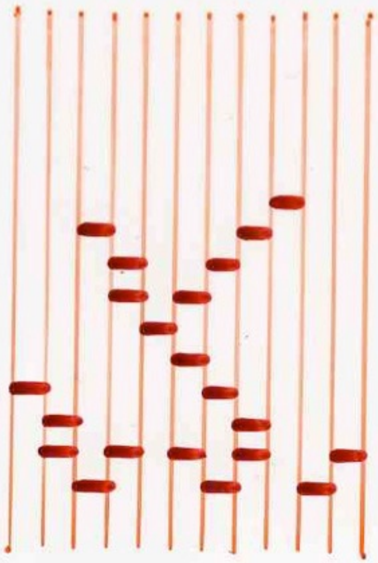
multidirected animal

(triangular lattice)

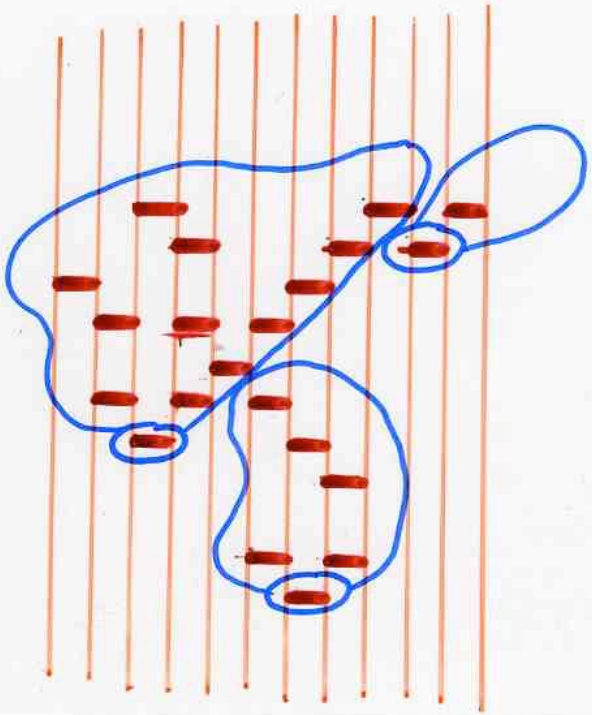
(Bousquet-Mélou, Rechnitzer, 2002)



multidirected
animal
(triangular
lattice)
(Bousquet-Mélou,
Rechnitzer, 2002)



connected
heap
of
dimers



$$Q(t) = \frac{1 - 2t - \sqrt{1-4t}}{2t}$$

generating function for

half-pyramid $(\neq \emptyset)$

$$= \sum_{n \geq 1} C_n \, t^n$$

Catalan

$C(t)$ g.f. connected heap

Bousquet-Mélou, Rechnitzer (2002)

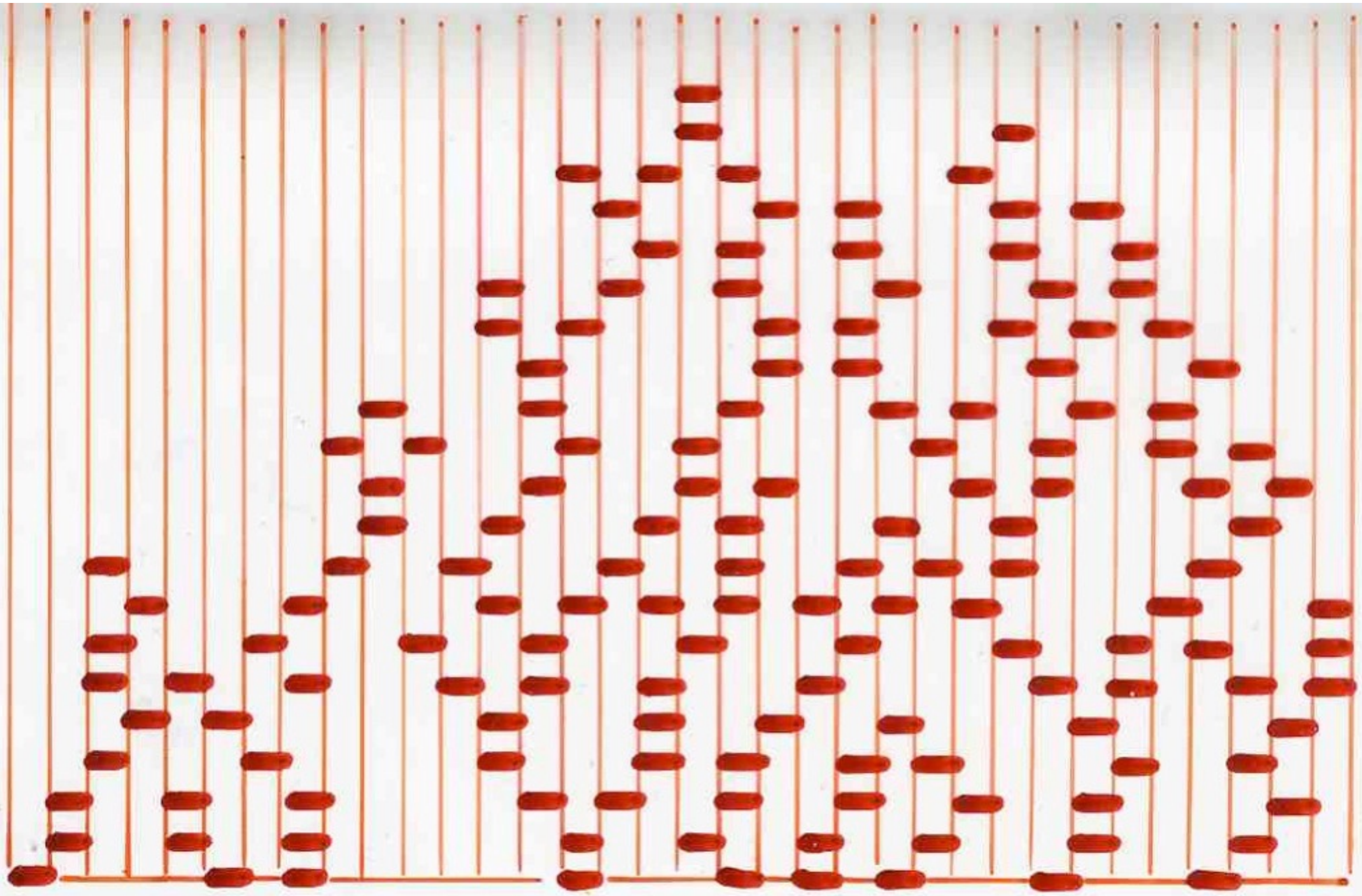
$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1-\boxed{\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{k-1}}{1-Q^k}}\right]}$$

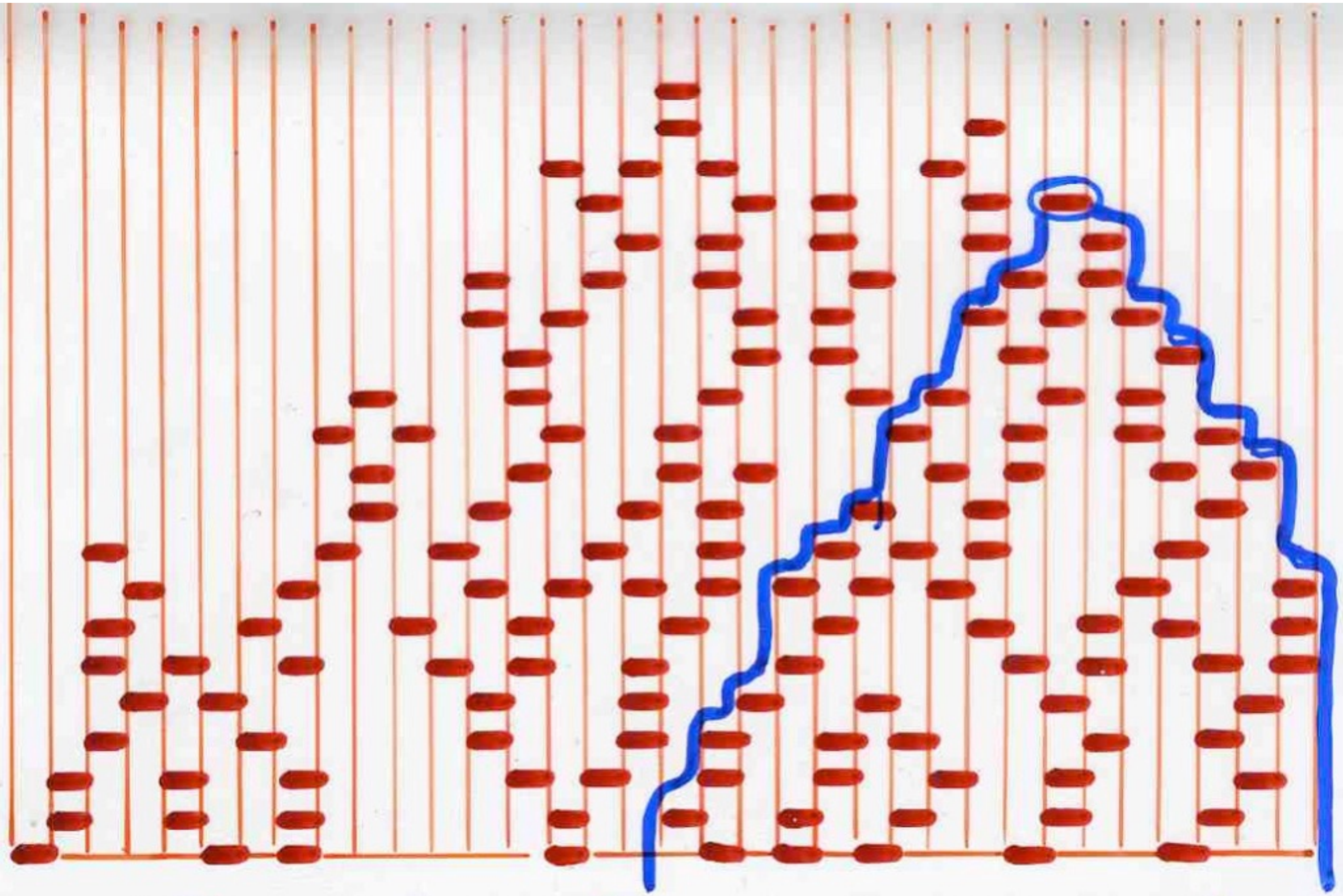
$$= \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}$$

bijective proof

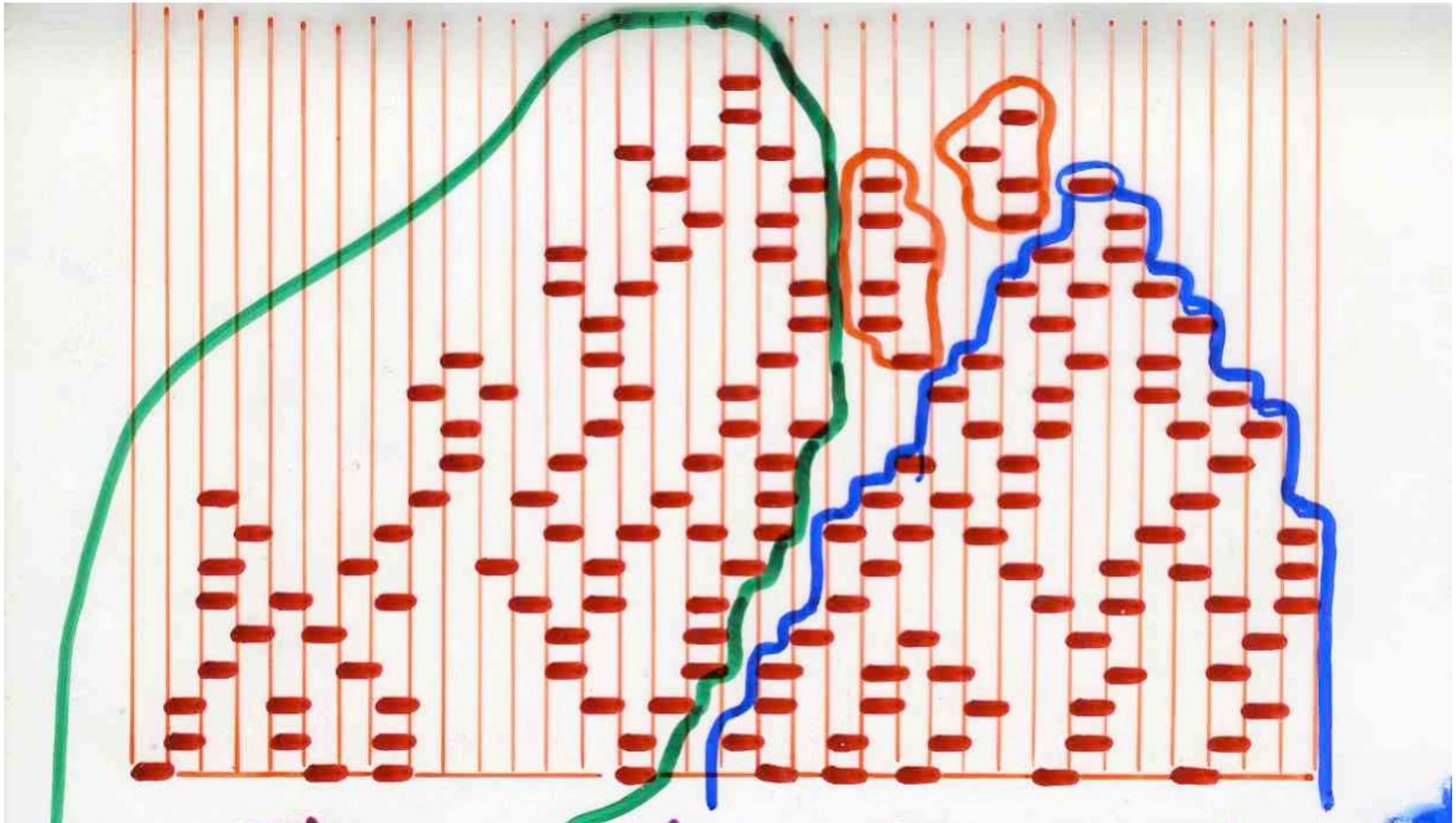
X.V. (2005)



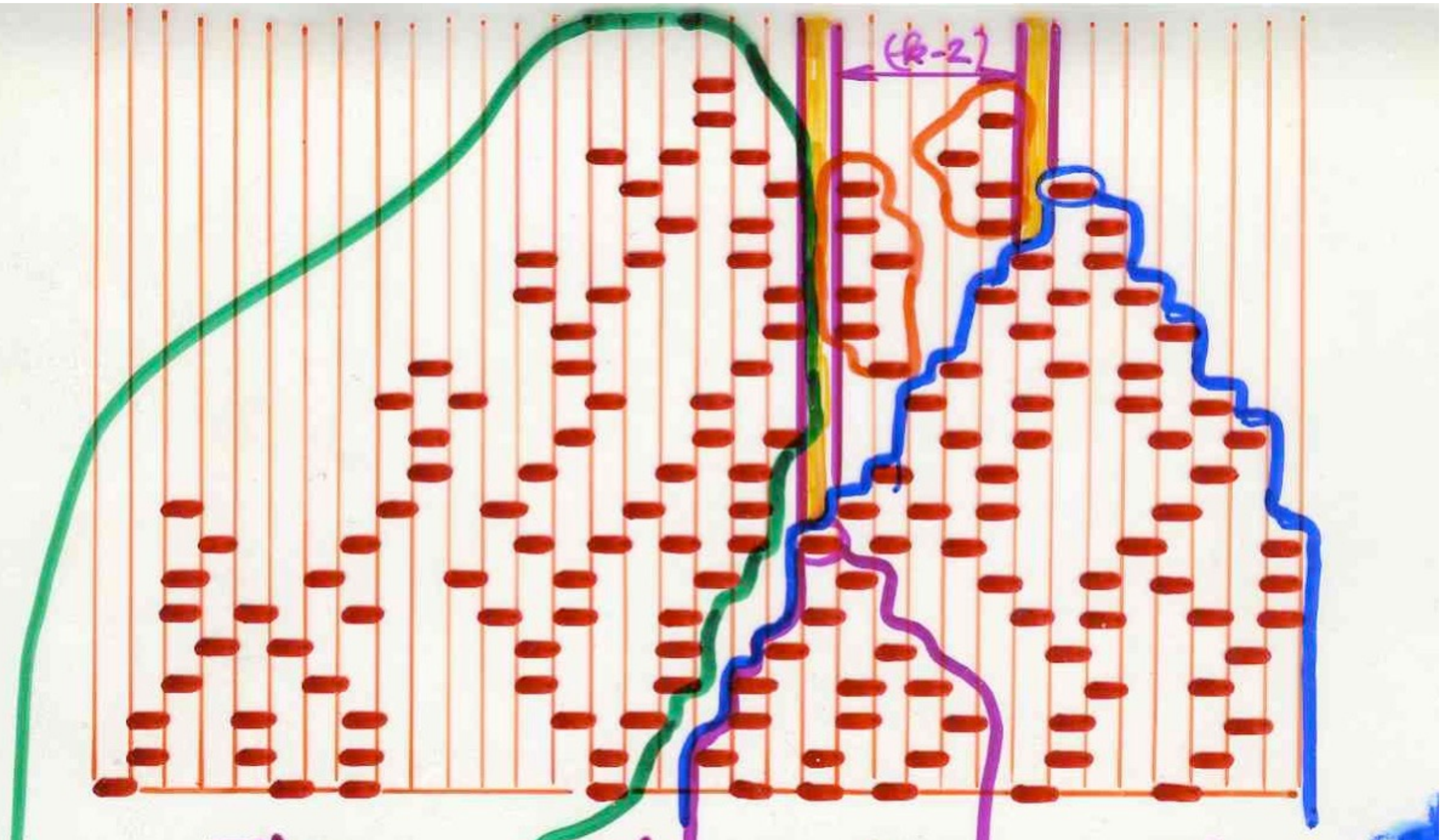
The nordic decomposition



The nordic decomposition

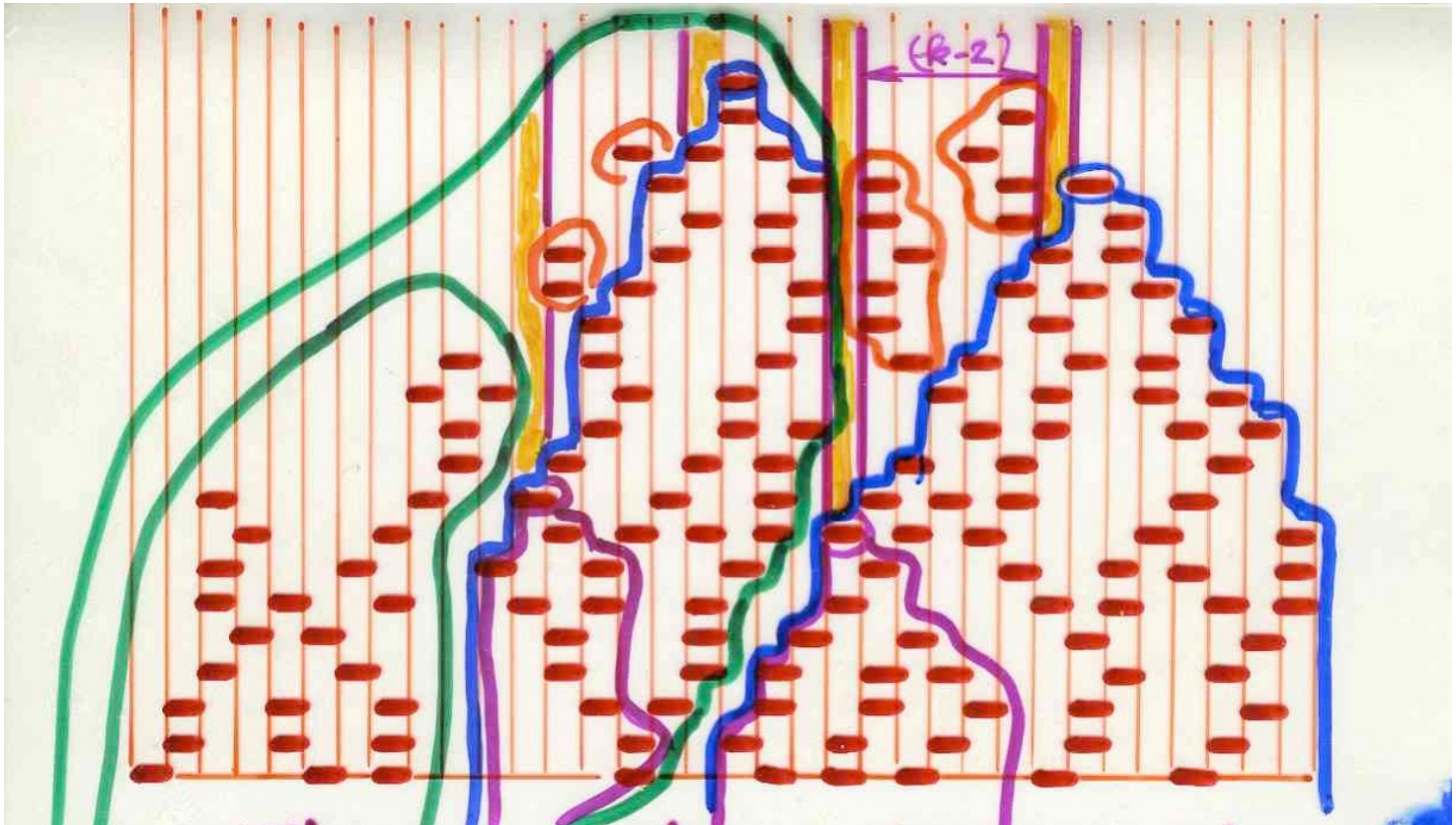


The nordic decomposition

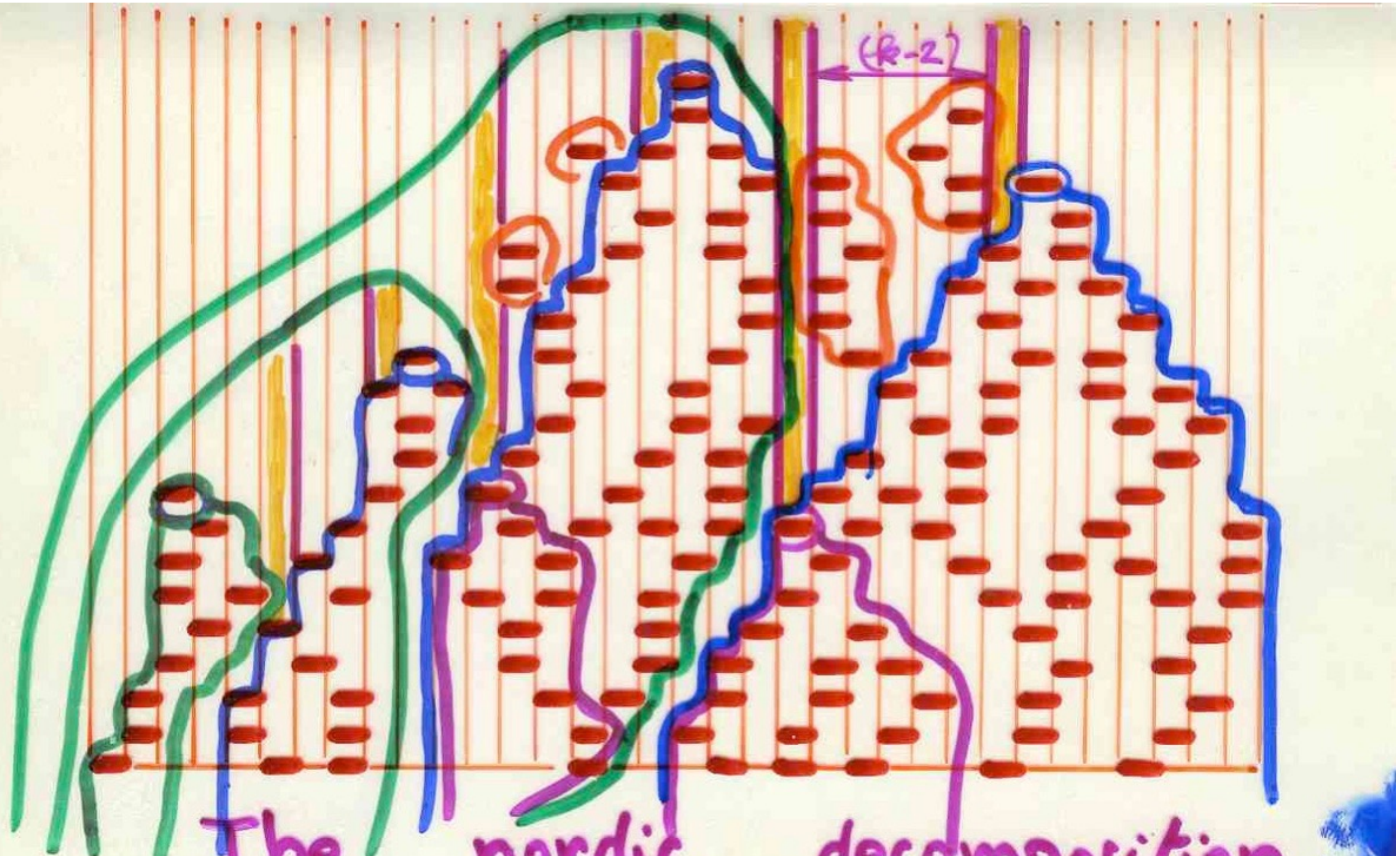


The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$



The nordic decomposition



The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$

$$C = \frac{Q}{1-Q} \times \frac{1}{\left[1-\left(\sum_{k\geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}\right)\right]}$$

connected heap

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$C = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1 - \boxed{\sum_{k\geqslant 1} \frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}}\;\right]}$$

$$= \sum_{k\geqslant 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

solution exercise Ch2b, p103

Fibonacci polynomials
and
generating function of Catalan numbers

notations $D = 1 + Q$

generating function of Catalan numbers

$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for half-pyramid $(\neq \emptyset)$

$$= \sum_{n \geq 1} C_n t^n$$

Catalan

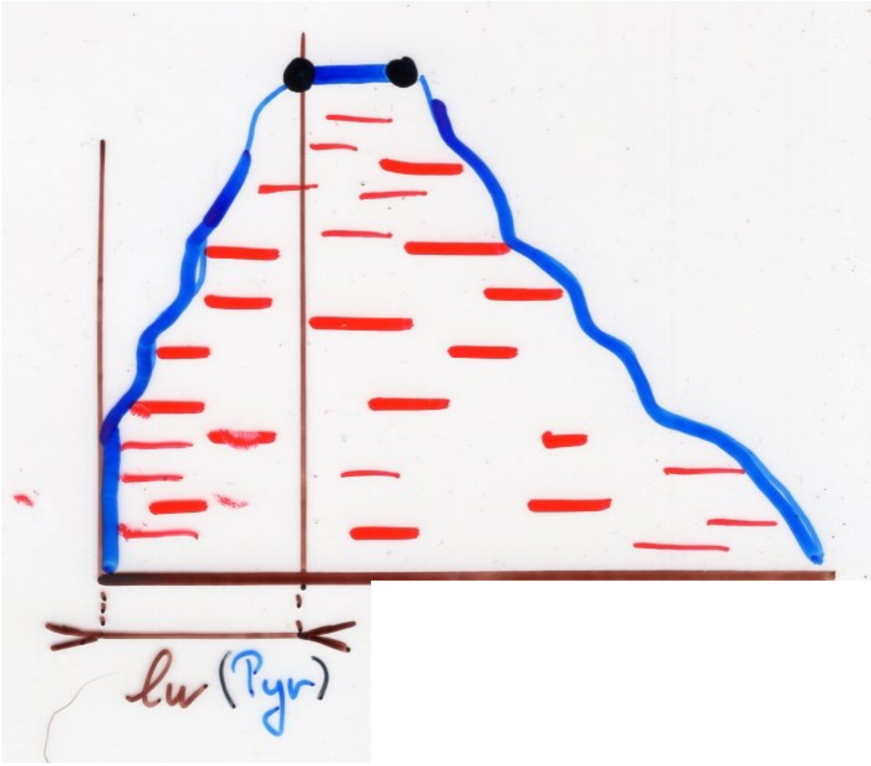
$F_n(t)$ n^{th} Fibonacci polynomial

we want to prove the following identity.

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\Longleftrightarrow$$

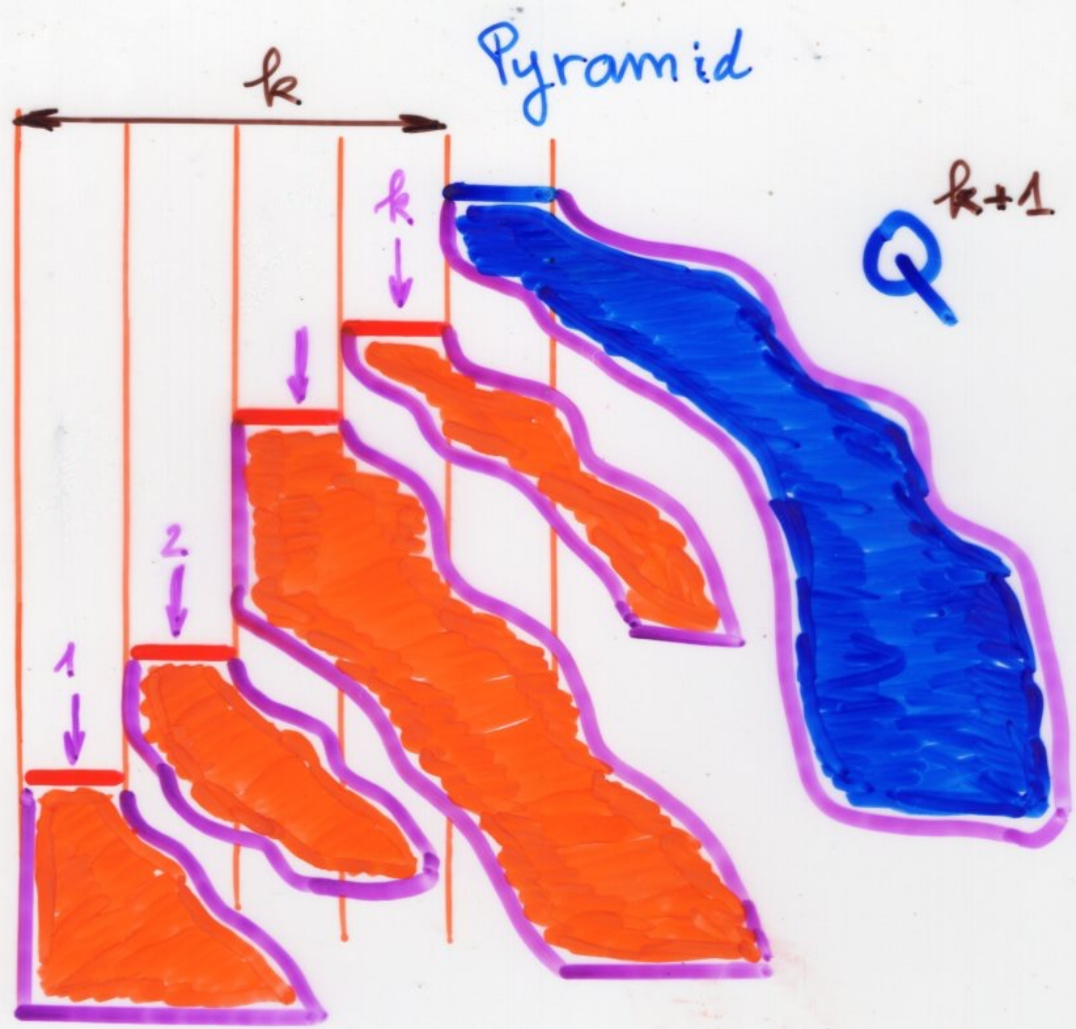
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\ldots+Q^n)$$



semi-pyramid :
lw (Pyr) = 0

left-width of a pyramid of dimers lw (Pyr)

a) Prove that the generating function of (non-empty) pyramids of dimers Pyr with left-width lw (Pyr) = k, is equal to Q^{k+1}


Pyramid
k
k
2
1
Q^{k+1}

b)

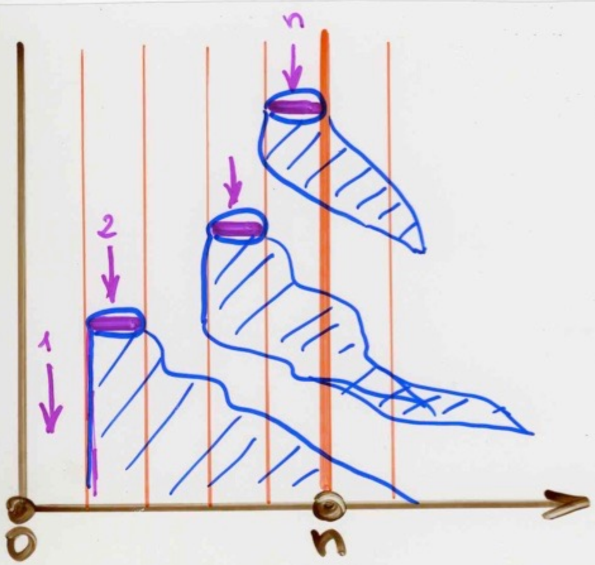
Prove that both sides of the identity are the generating function of:

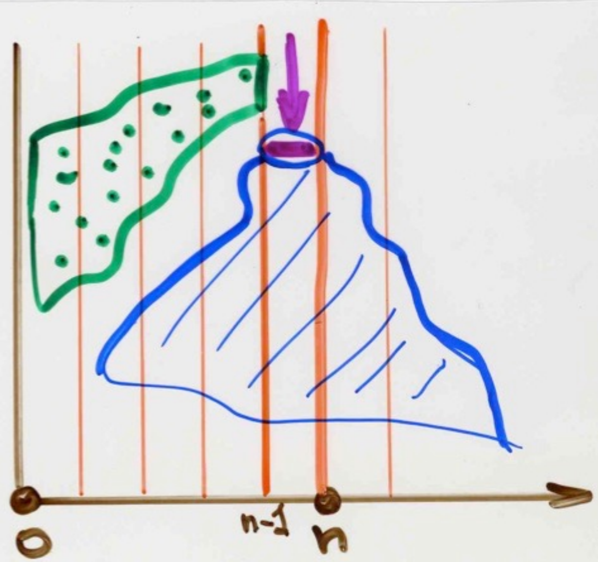
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$

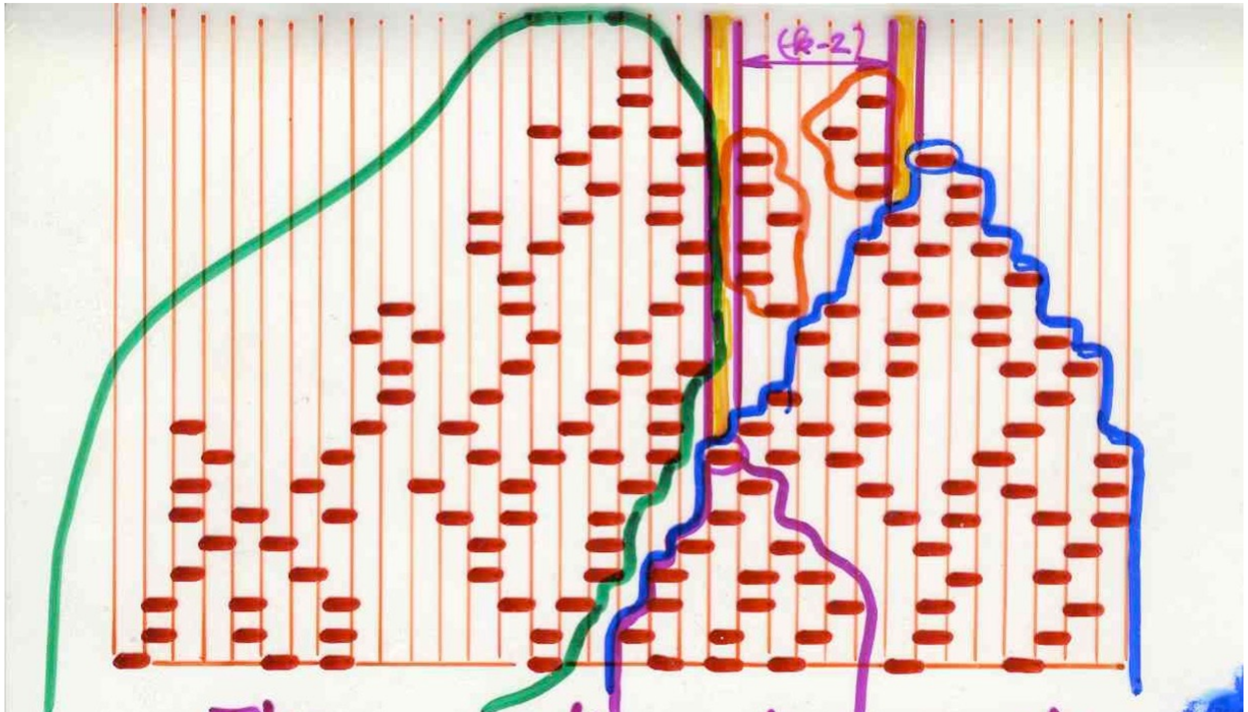
heaps of dimers on $[0, \infty[$

maximal pieces, projection $\subseteq [0, n]$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$



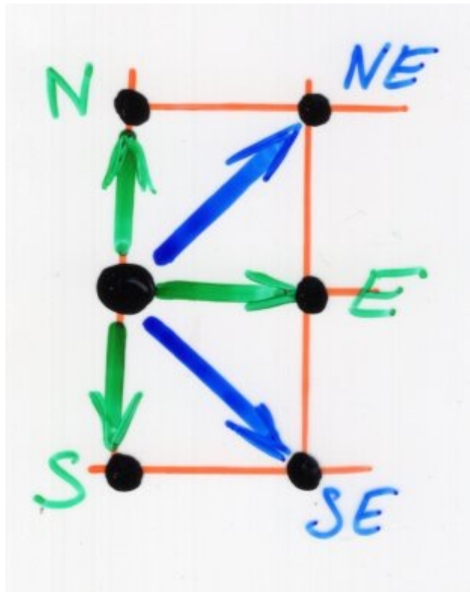




The nordic decomposition

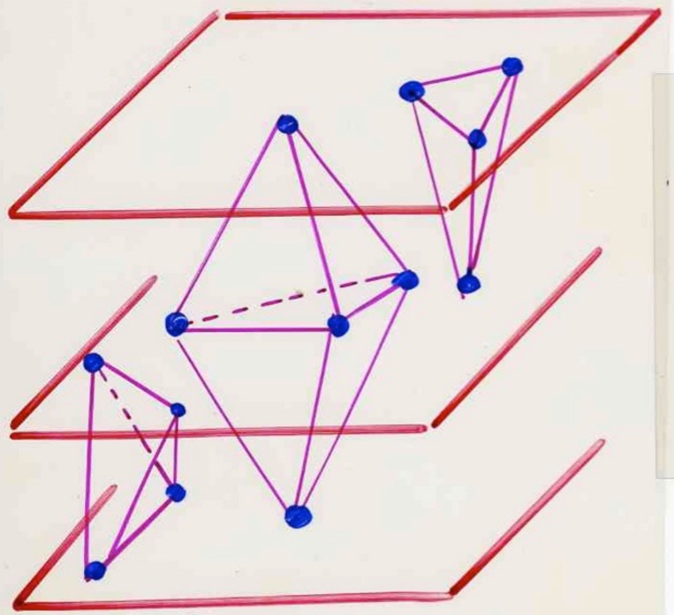
A. Bacher (2016)

partially directed animals



Lorentzian quantum gravity

(1+1) +1 dimension



3d

4d

?

Benedetti, Loll, Zamponi (2007)
arXiv:0704.3214

Benedetti, thesis (2007)
arXiv: 0707.3070

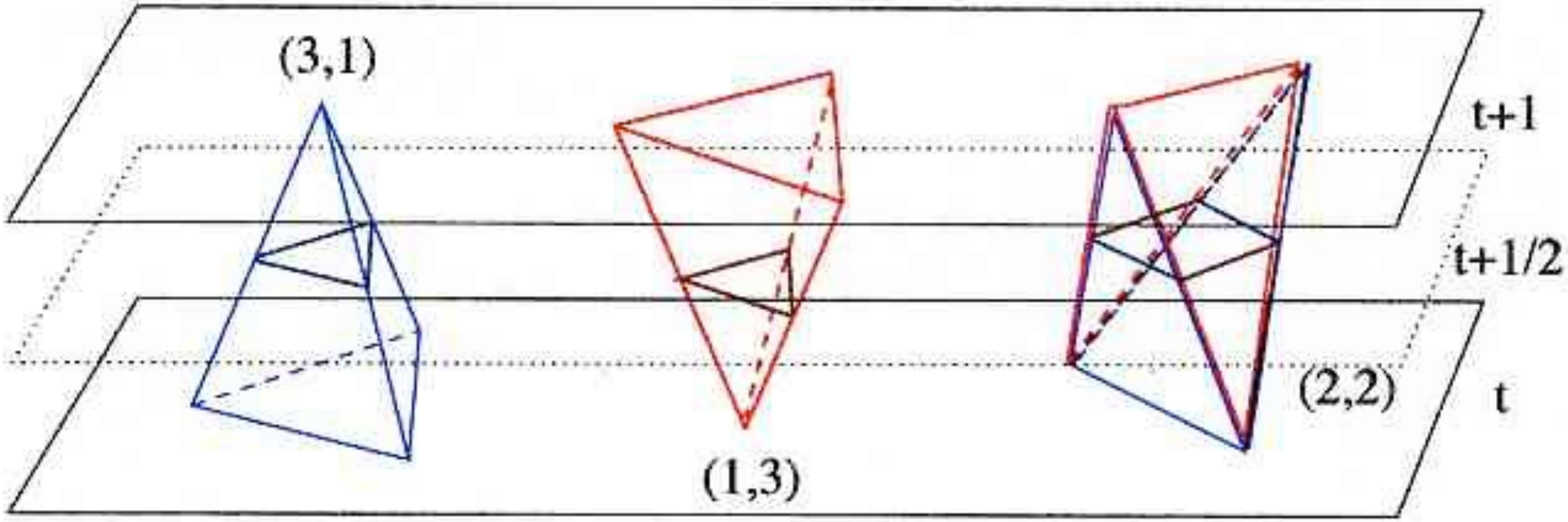
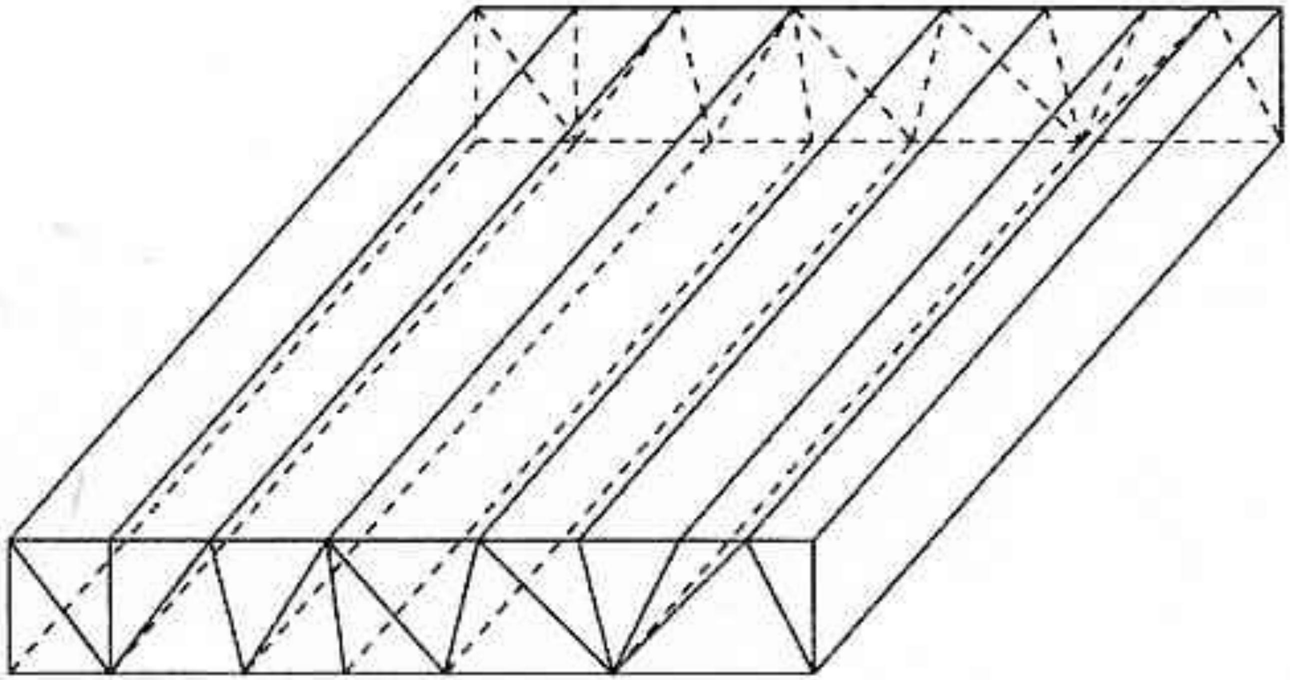


Figure 4: The three types of tetrahedral building blocks and their intersections at time $t + 1/2$.


t+1
t

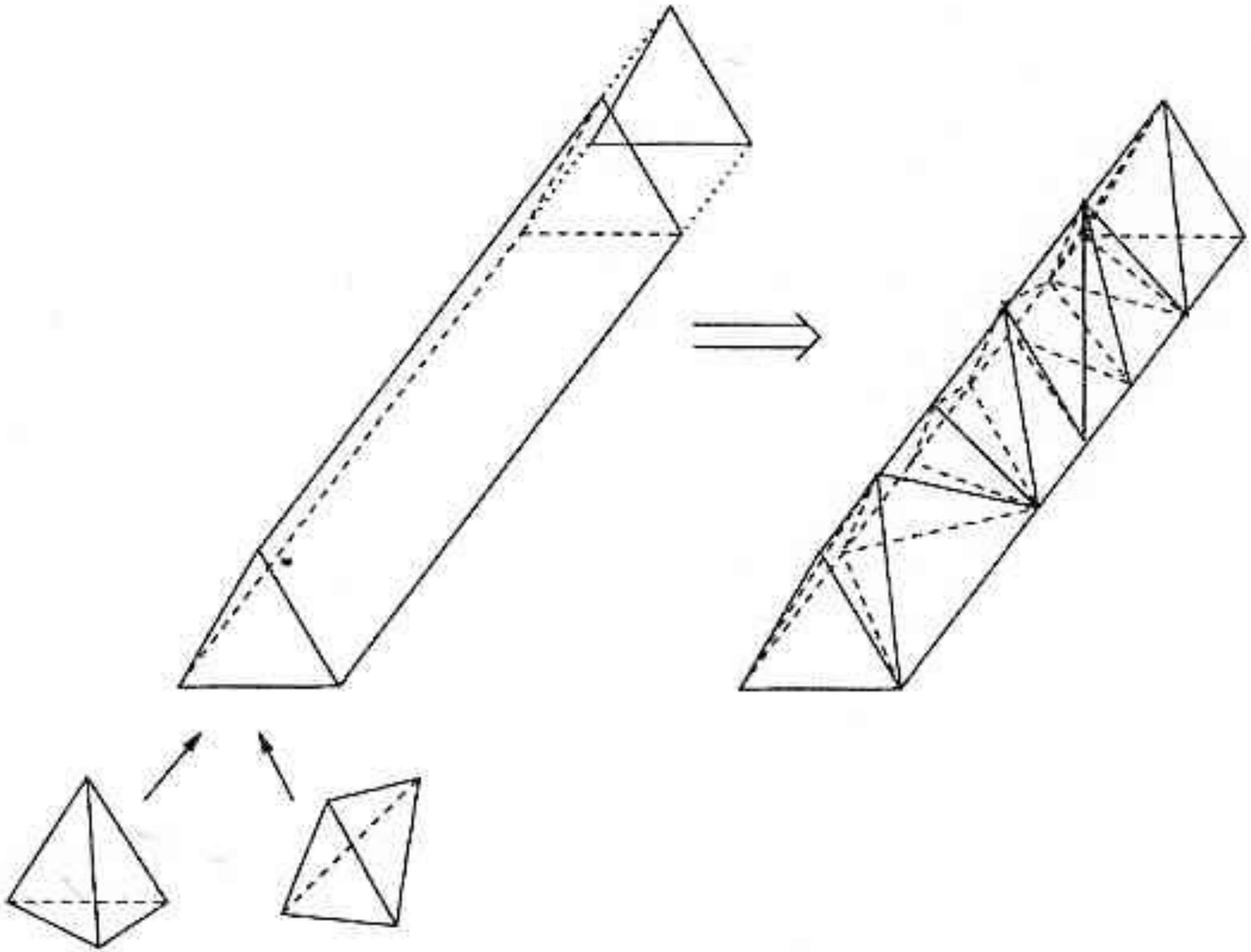
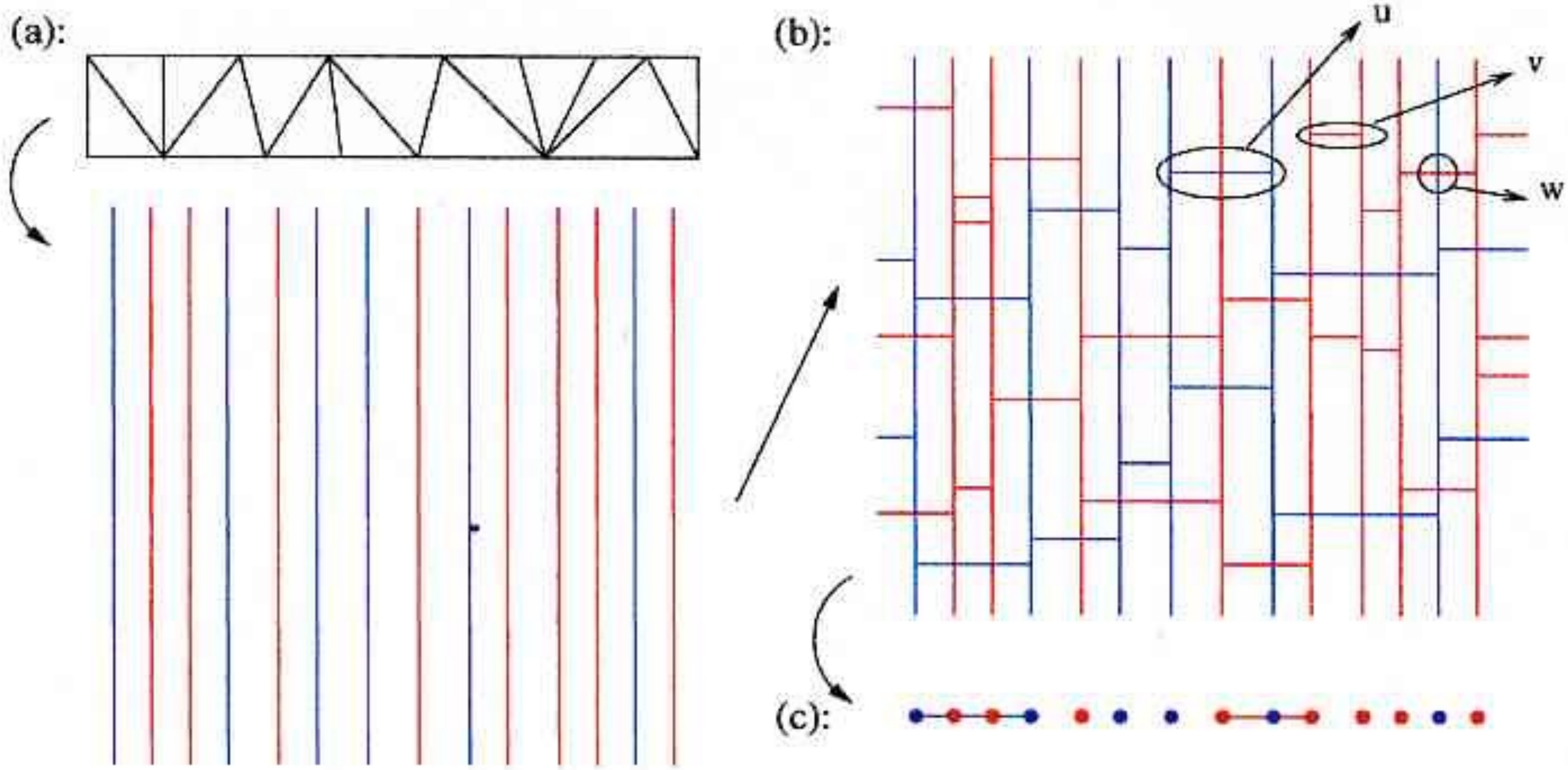


Figure 2: A triangulated prism constructed as a tower over a two-dimensional triangle.


(a):
(b):
u
v
w
(c):