An introduction to

enumerative
algebraic   combinatorics
bijective

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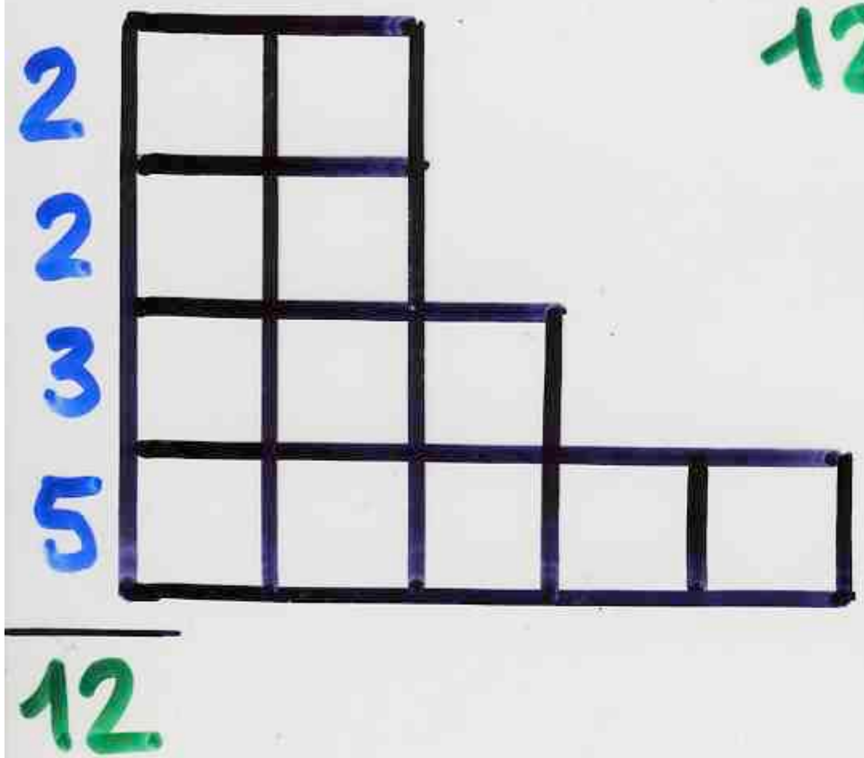
Chapter 4

The n! garden
(3)

IMSc
23 February 2016

Young tableaux



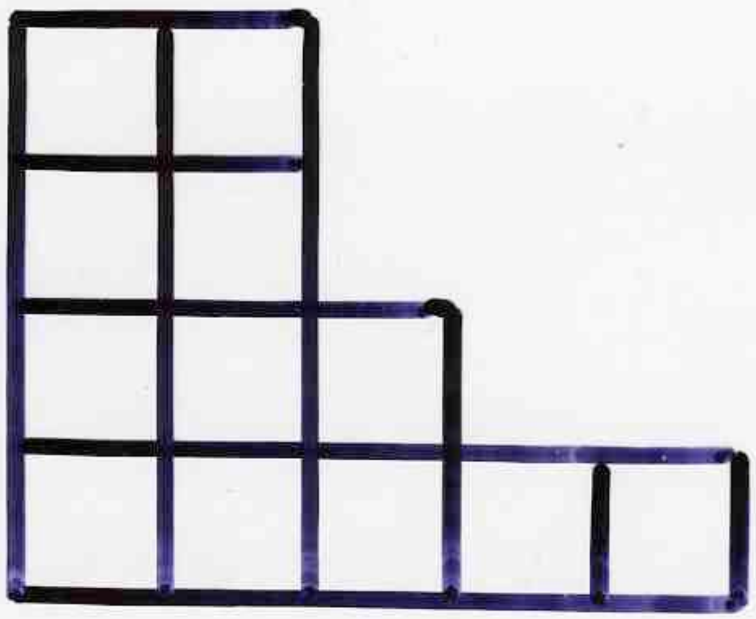
$12 = n = 5+3+2+2$

Ferrers

diagram

Partition of n

λ



7	12			
6	10			
3	5	9		
1	2	4	8	11

Young tableau

shape λ

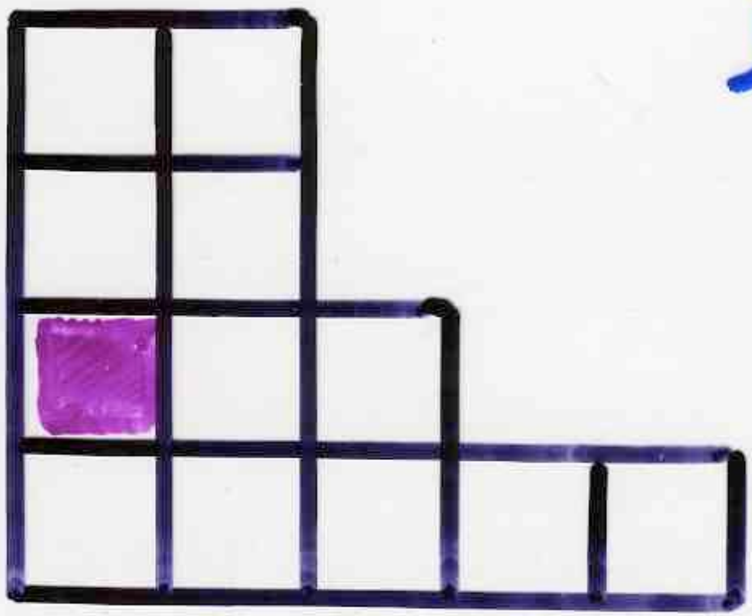
f_λ = nb of

Young tableaux

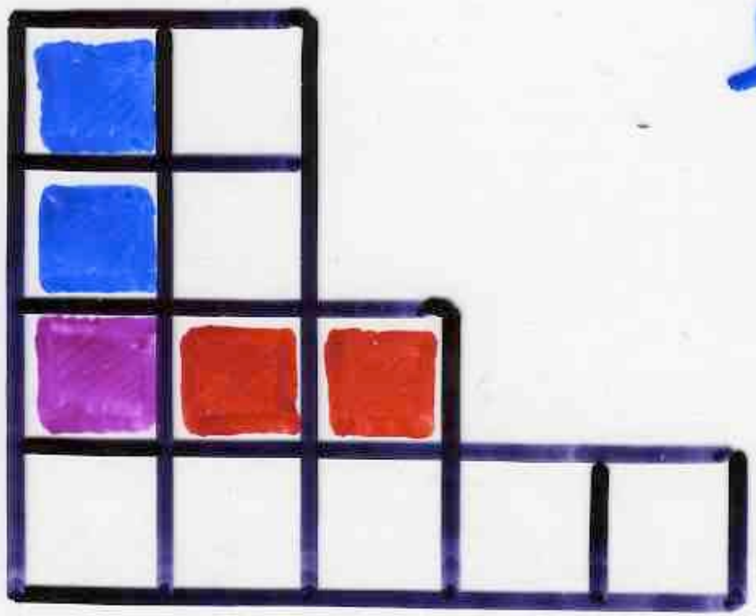
shape λ

Hook length formula

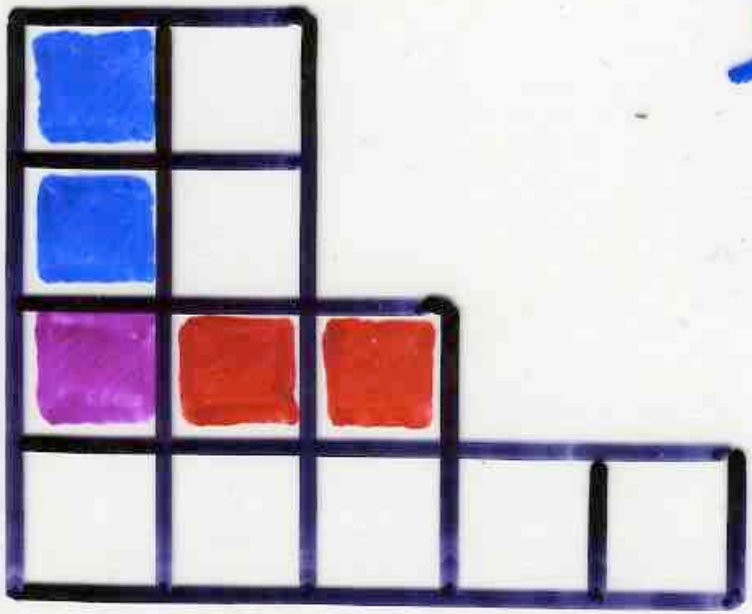
J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954



hook



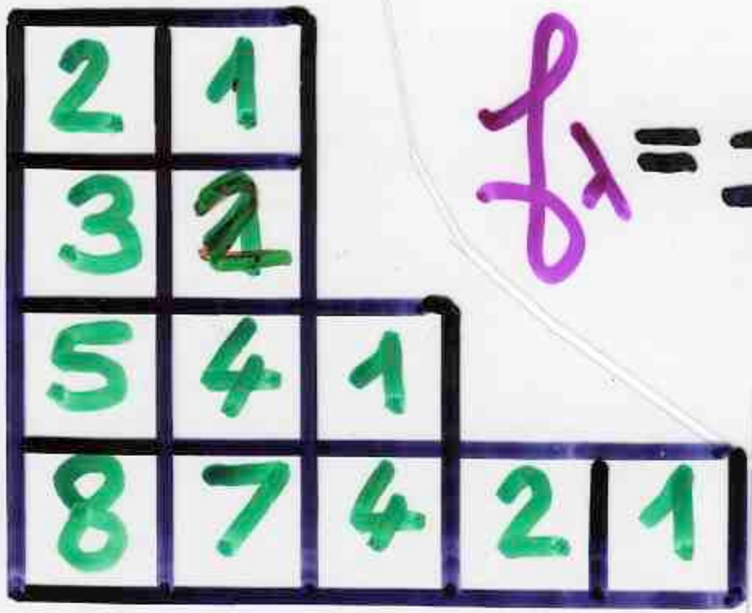
hook



hook  length

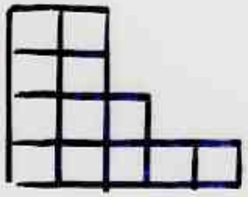
5

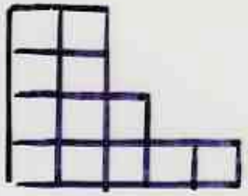
2	1			
3	2			
5	4	1		
8	7	4	2	1

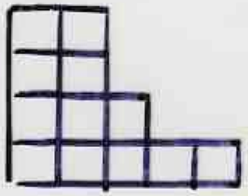


$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula

f  $=$

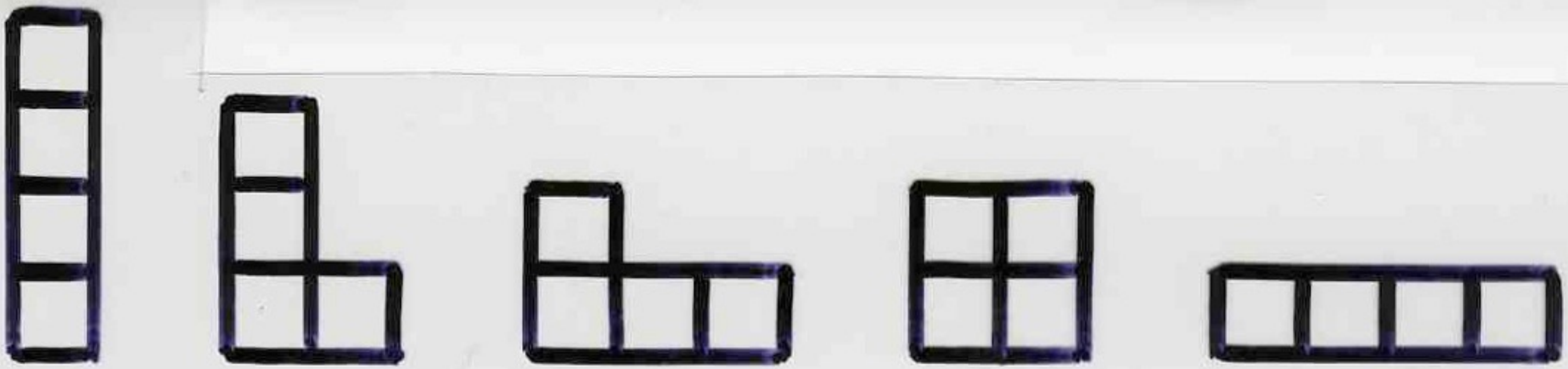
f  $= \dfrac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$

f  $= \dfrac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$

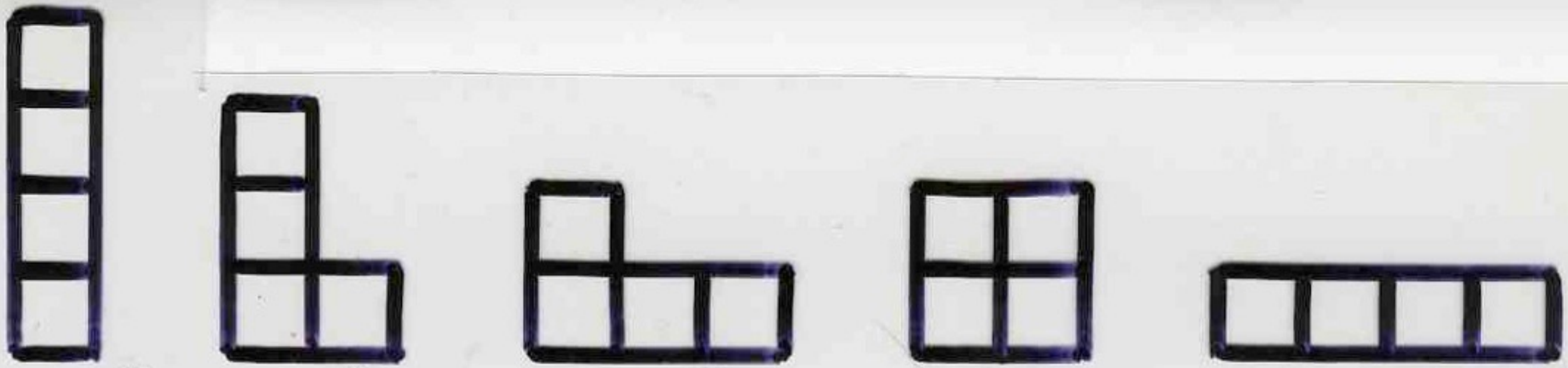
$= 3^4 \times 5 \times 11 = 4455$

An introduction to RSK

G. de B. Robinson, 1938
C. Schensted, 1961
D. Knuth, 1970



1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

number of permutations on $\{1, 2, \ldots, n\}$ $= 1 \times 2 \times 3 \times \ldots \times n$

$= n!$

$$n! = \sum_{\substack{\lambda \\ \text{partitions} \\ \text{of } n}} (f_\lambda)^2$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\begin{array}{|c|c|c|c|c|}
\hline
6 & 10 \\
\hline
3 & 5 & 8 \\
\hline
1 & 2 & 4 & 7 & 9 \\
\hline
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
8 & 10 \\
\hline
2 & 5 & 6 \\
\hline
1 & 3 & 4 & 7 & 9 \\
\hline
\end{array}$$

$$P \qquad\qquad\qquad\qquad Q$$

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1					

3					
1					

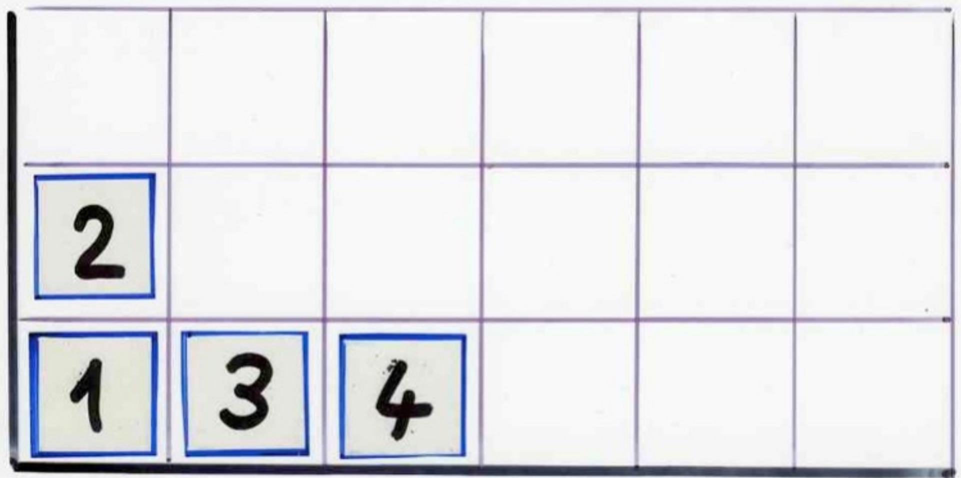
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

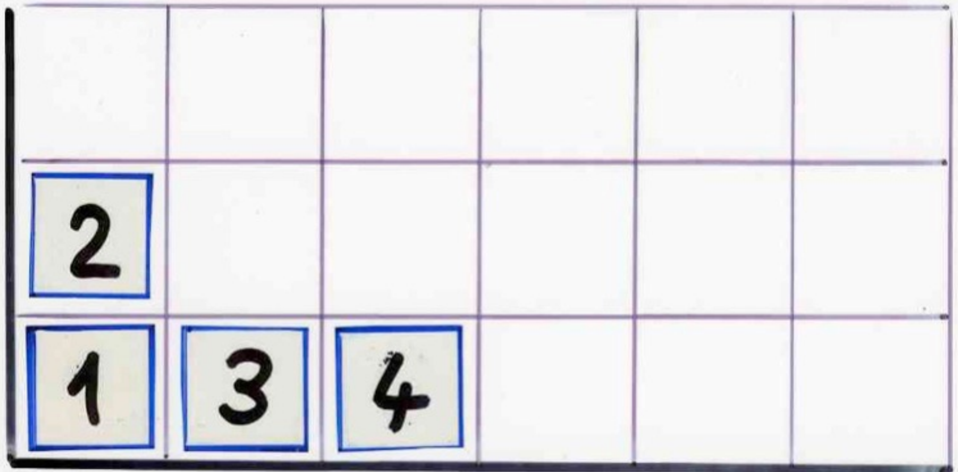
2					
1	3	4			

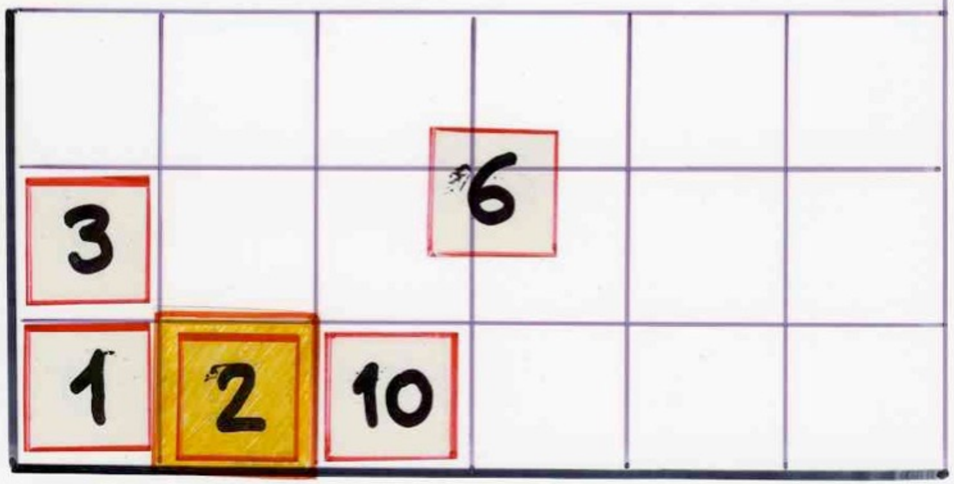
3					
1	6	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



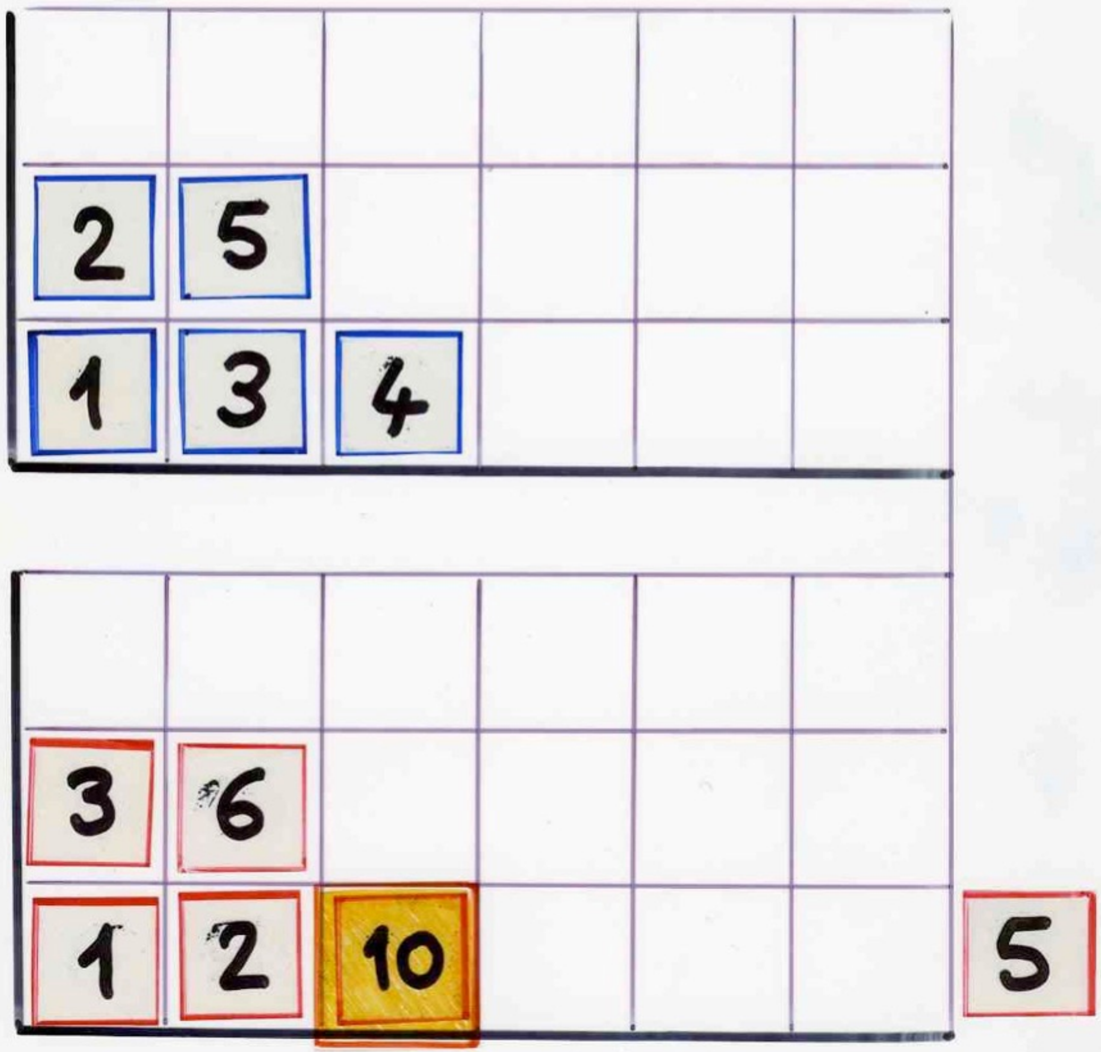


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

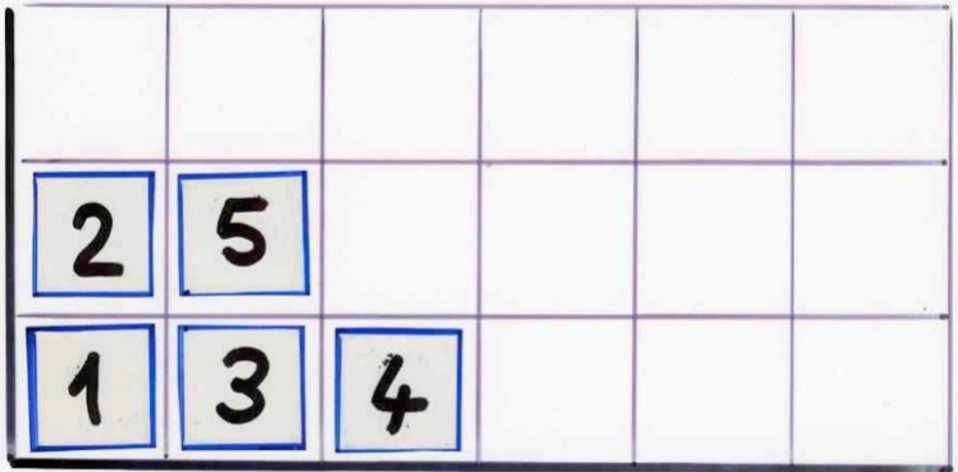
2	5				
1	3	4			

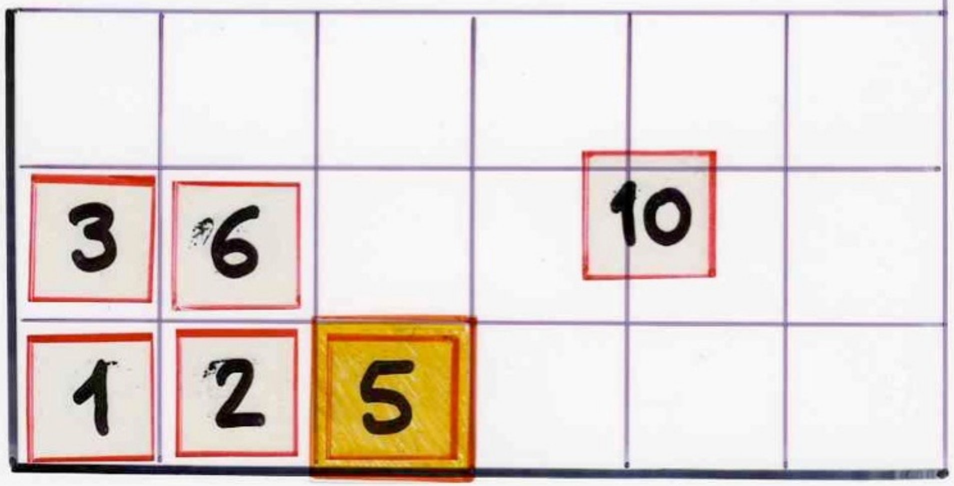
3	6				
1	2	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

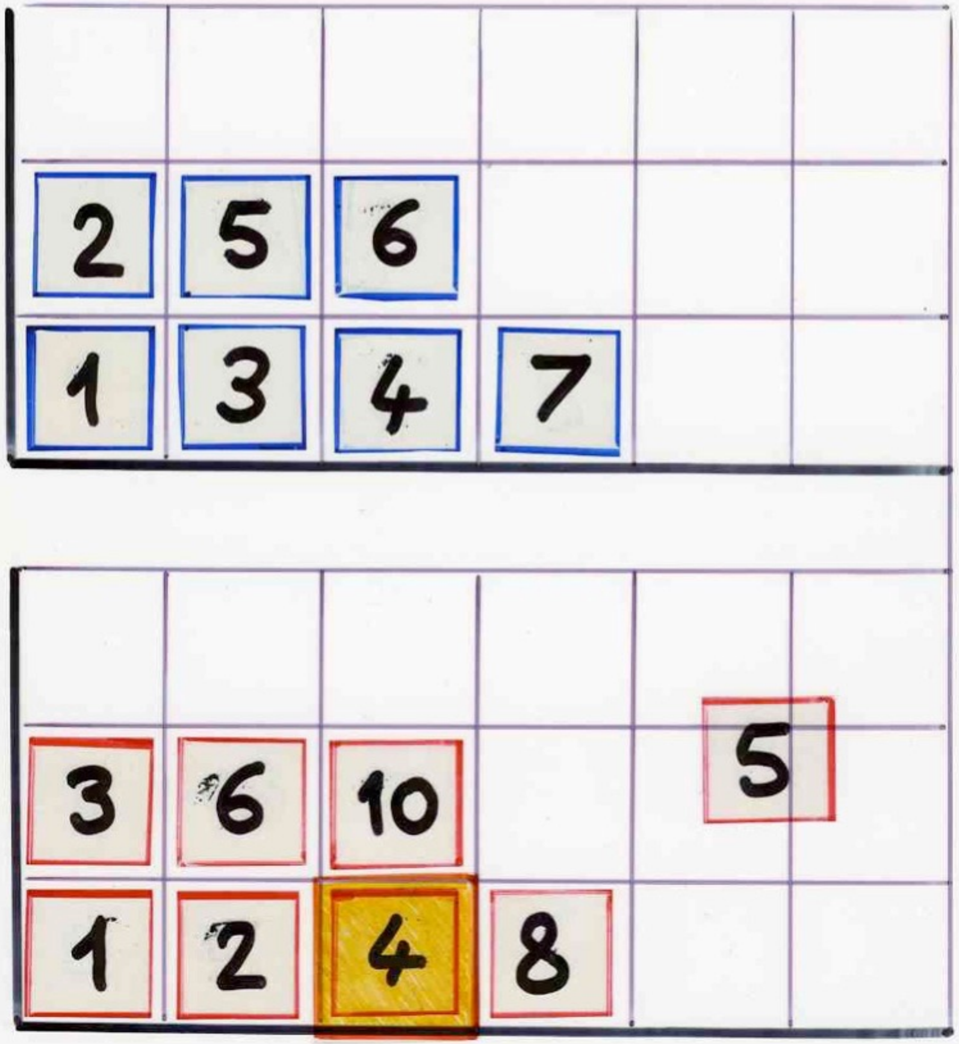
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10				
1	2	5	8			4

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

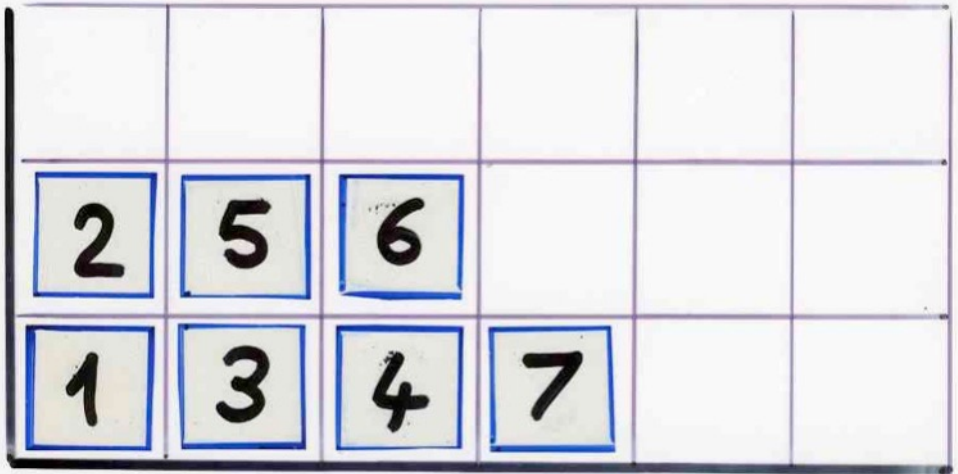


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

3	6	10		5	
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				
1	2	4	8	9		7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						10
3	5	8				
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

$$? \longleftrightarrow (Q, P)$$

The group of permutations

$$\sigma \longleftrightarrow (P, Q)$$

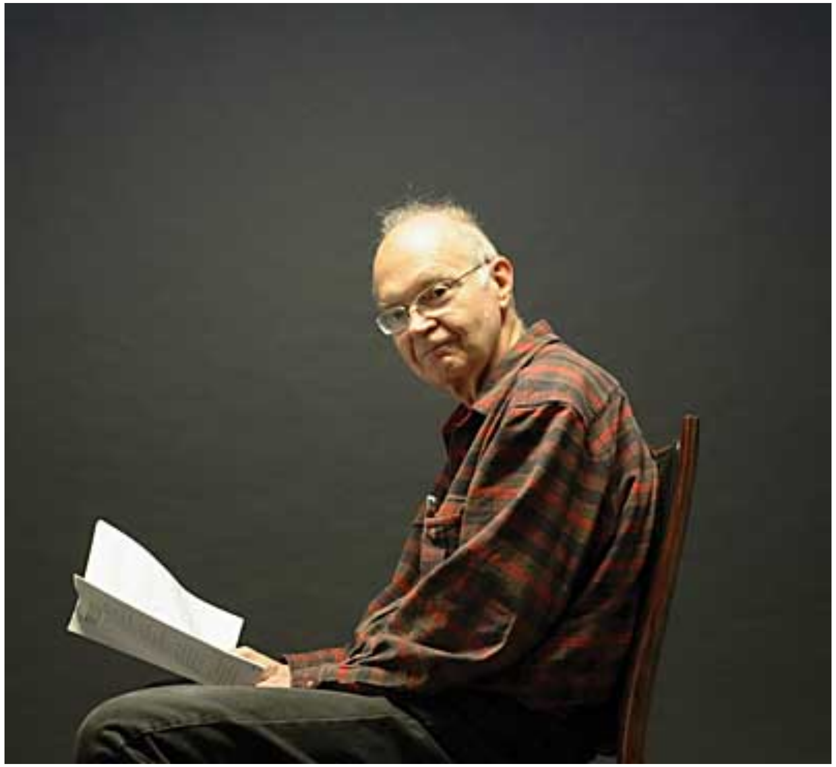
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

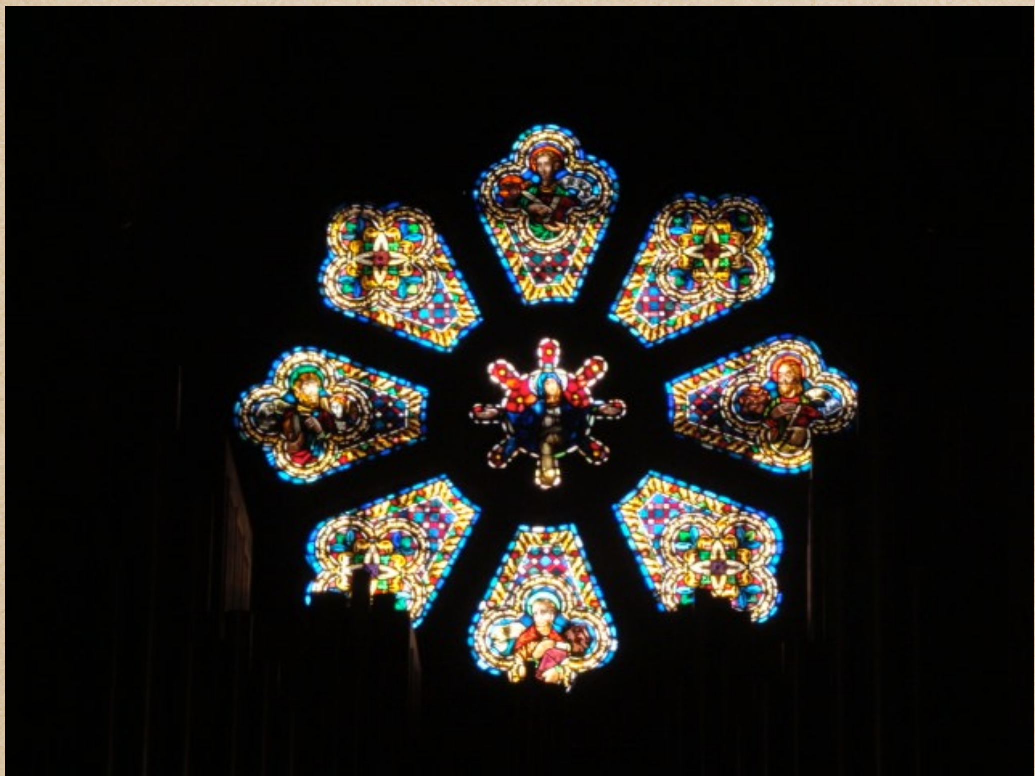
Vol 3, *"The art of computer programming"*



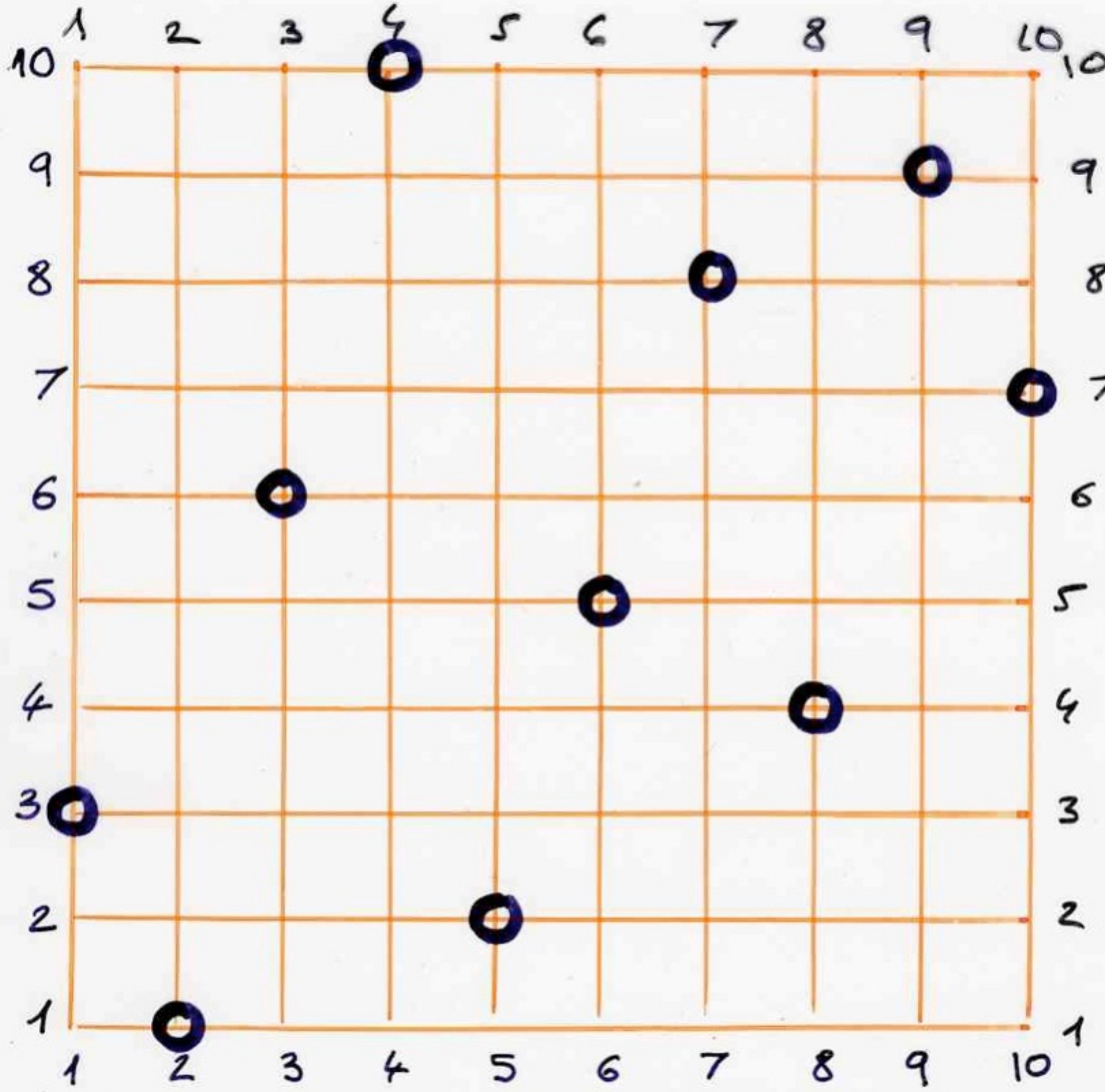
Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

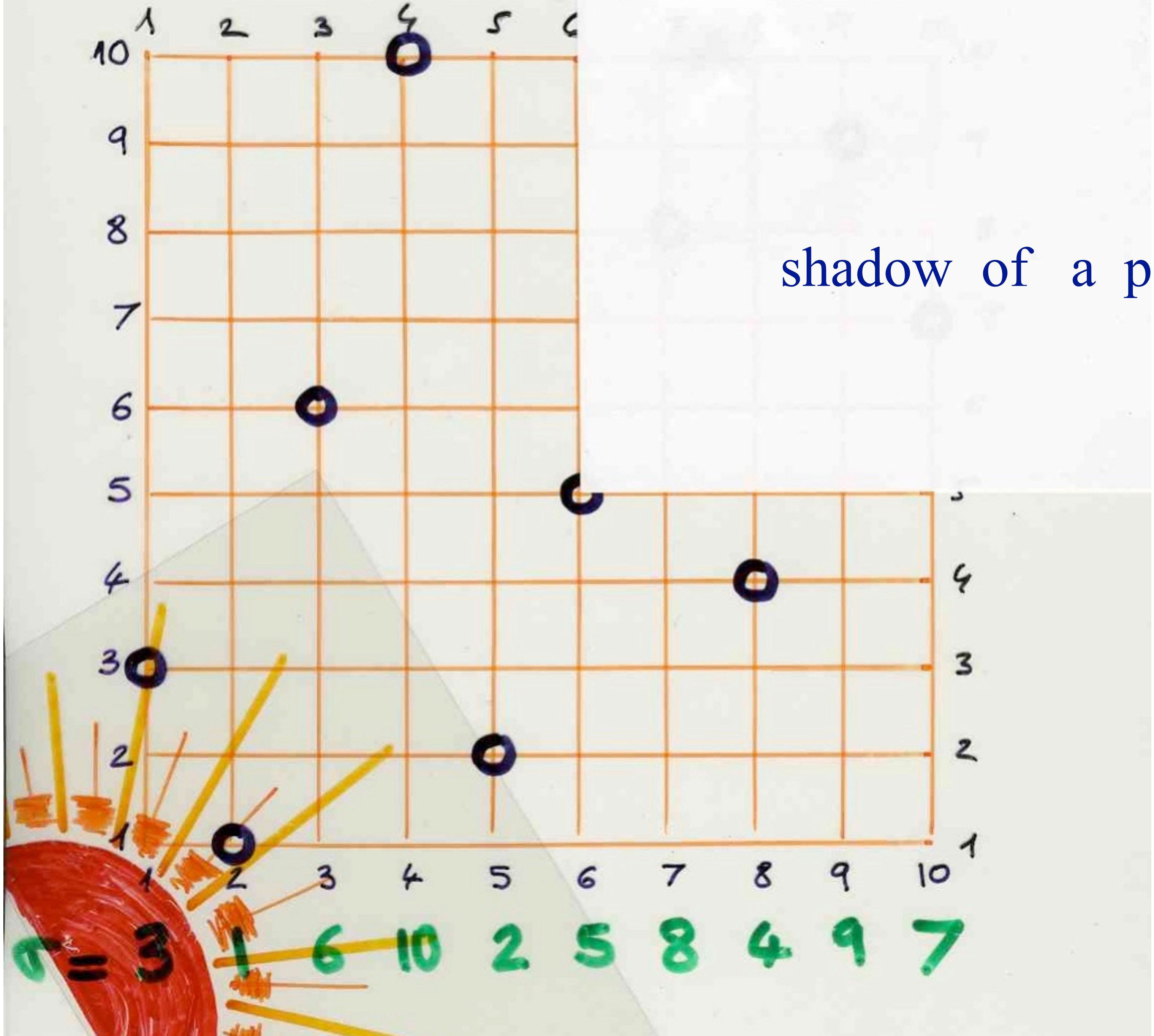
A geometric version of RSK with "light" and "shadow lines"



X.V., 1976

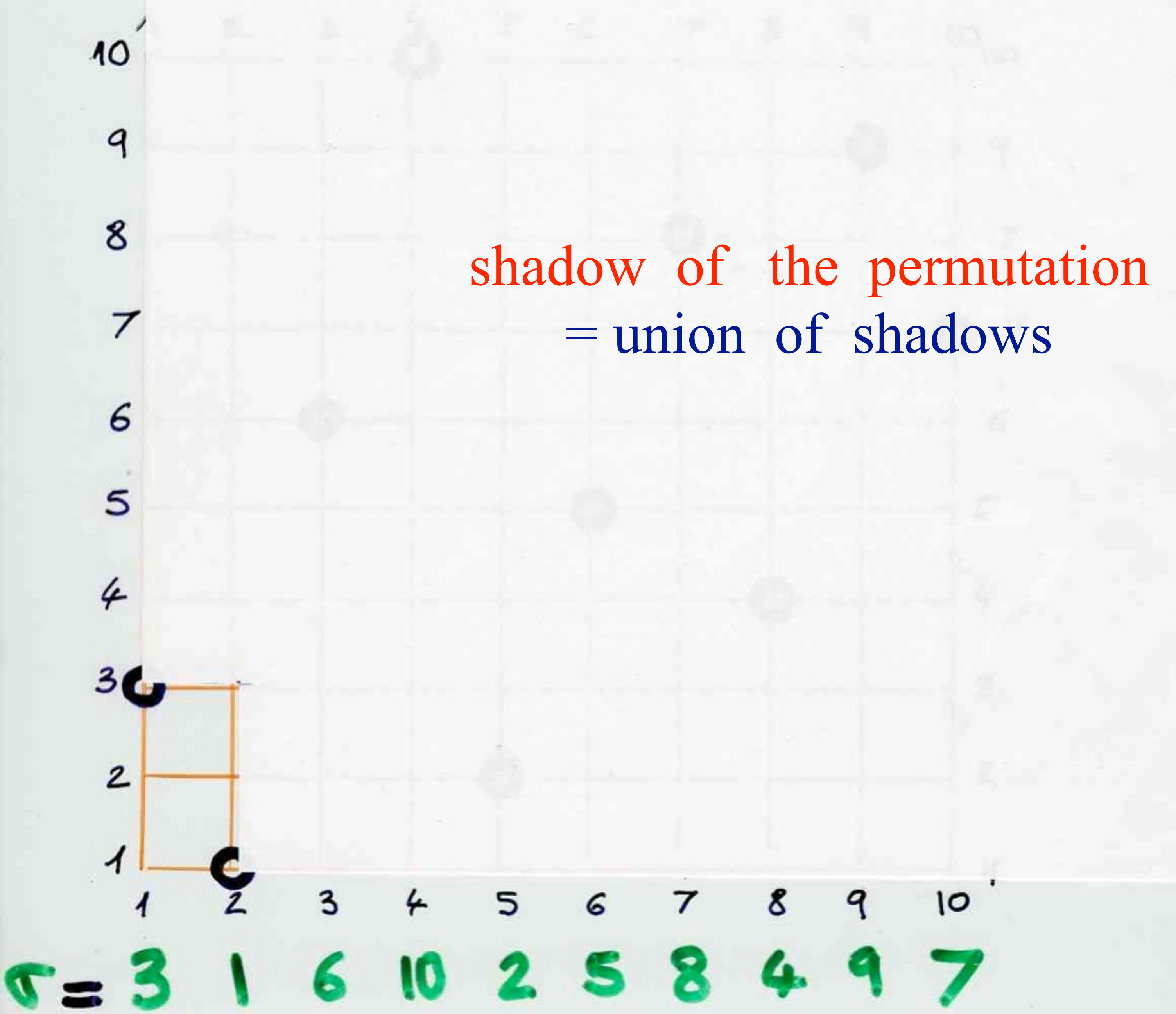


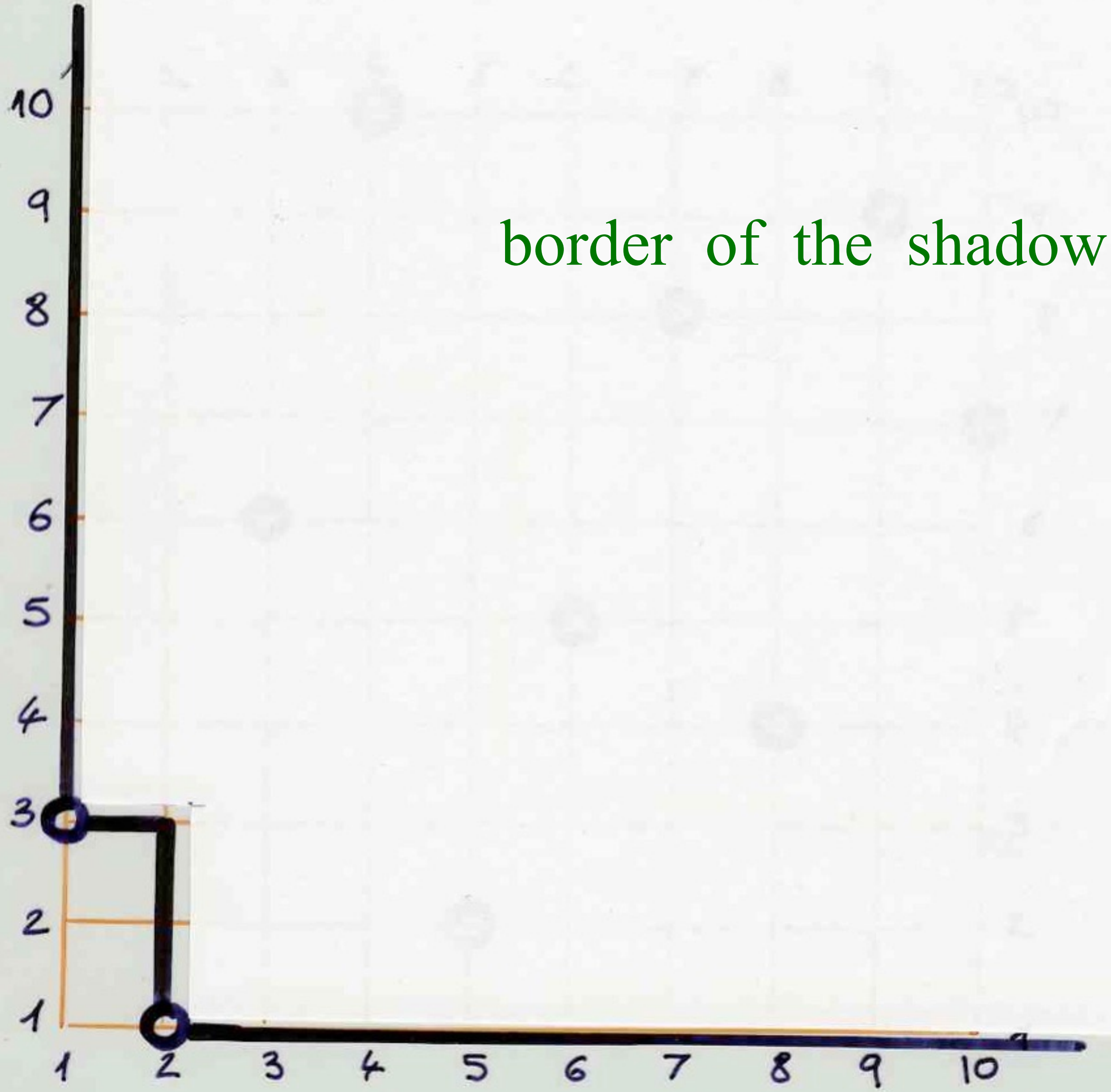
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



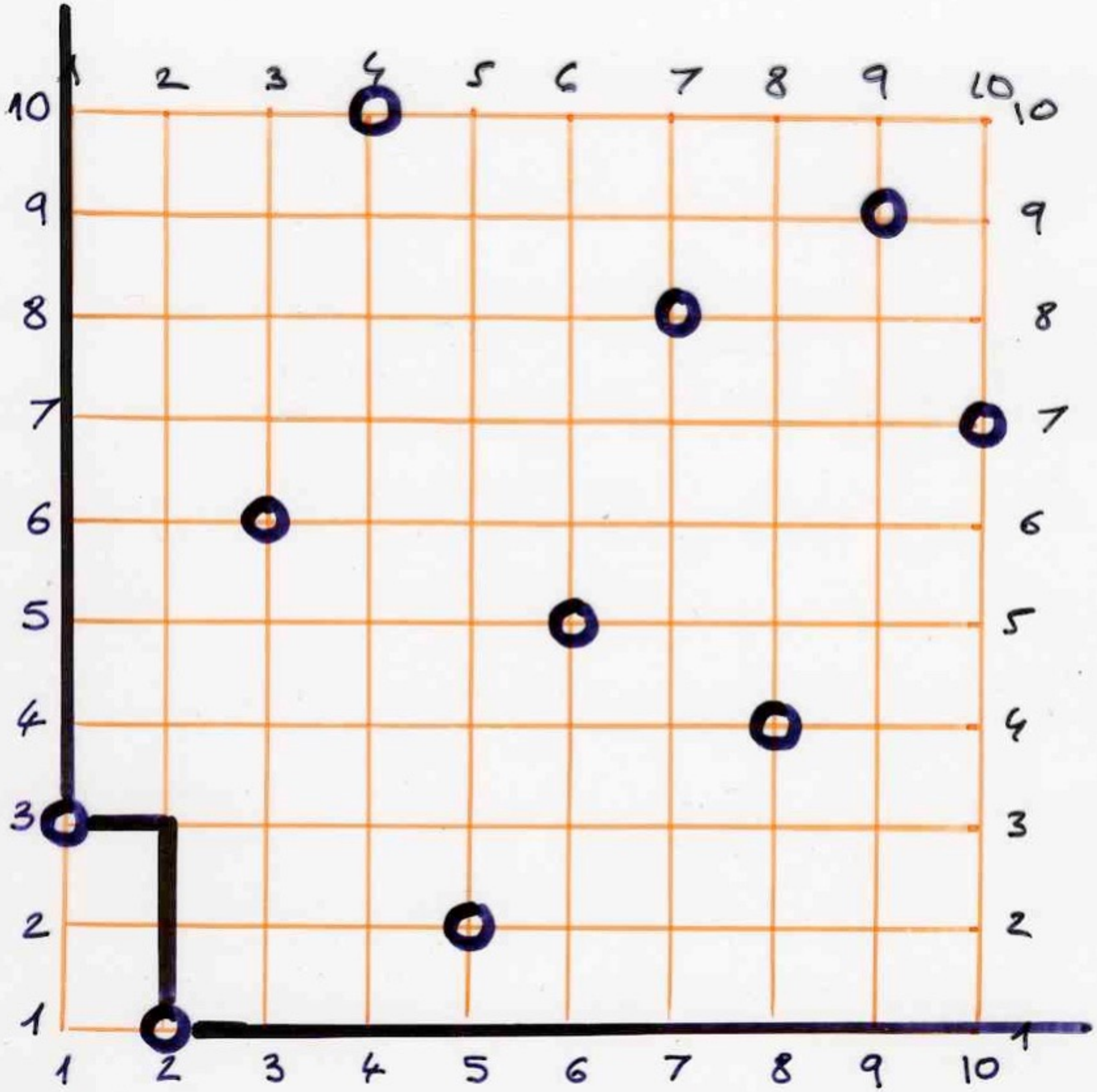
shadow of a point

shadow of the permutation
= union of shadows

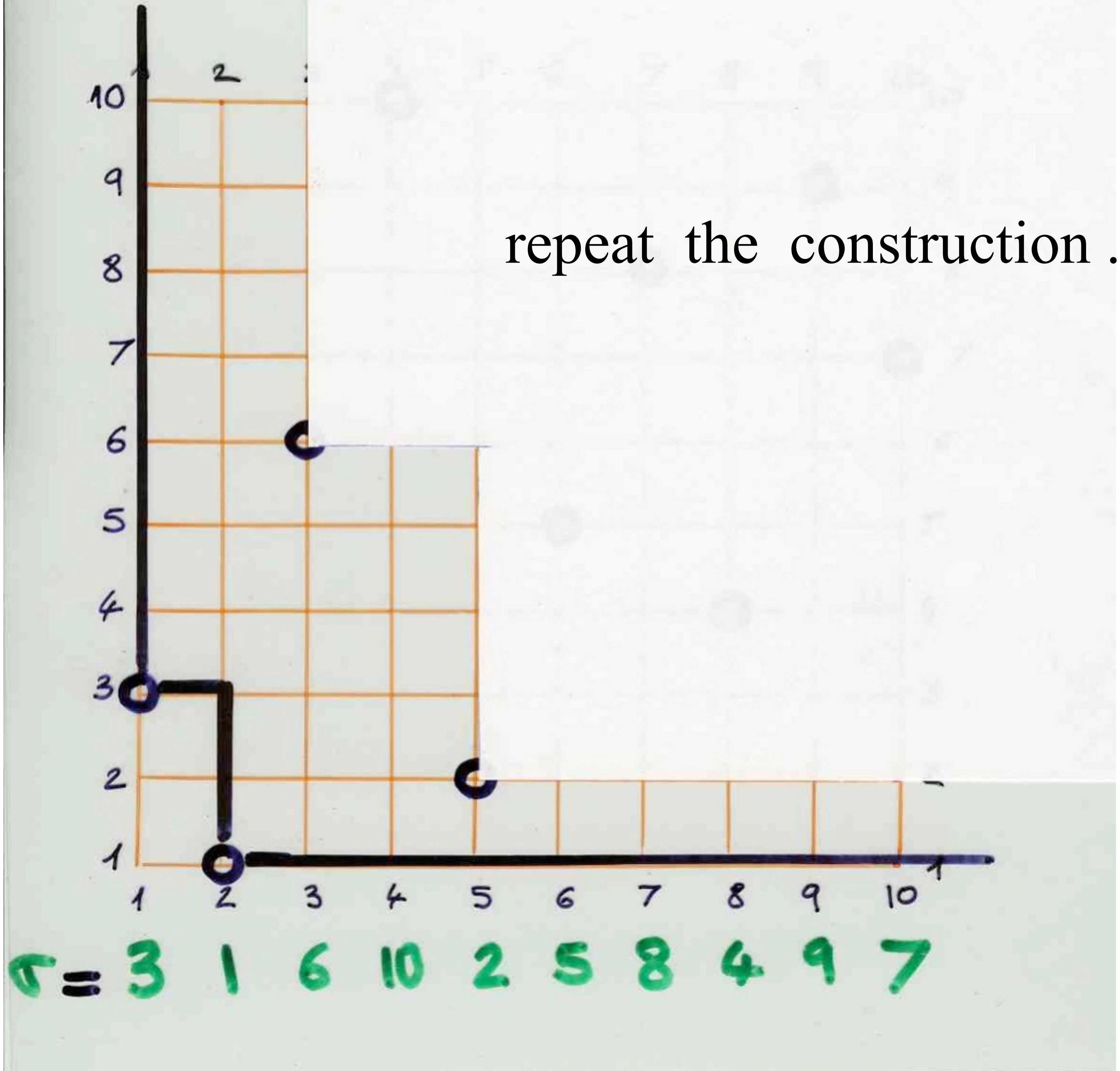




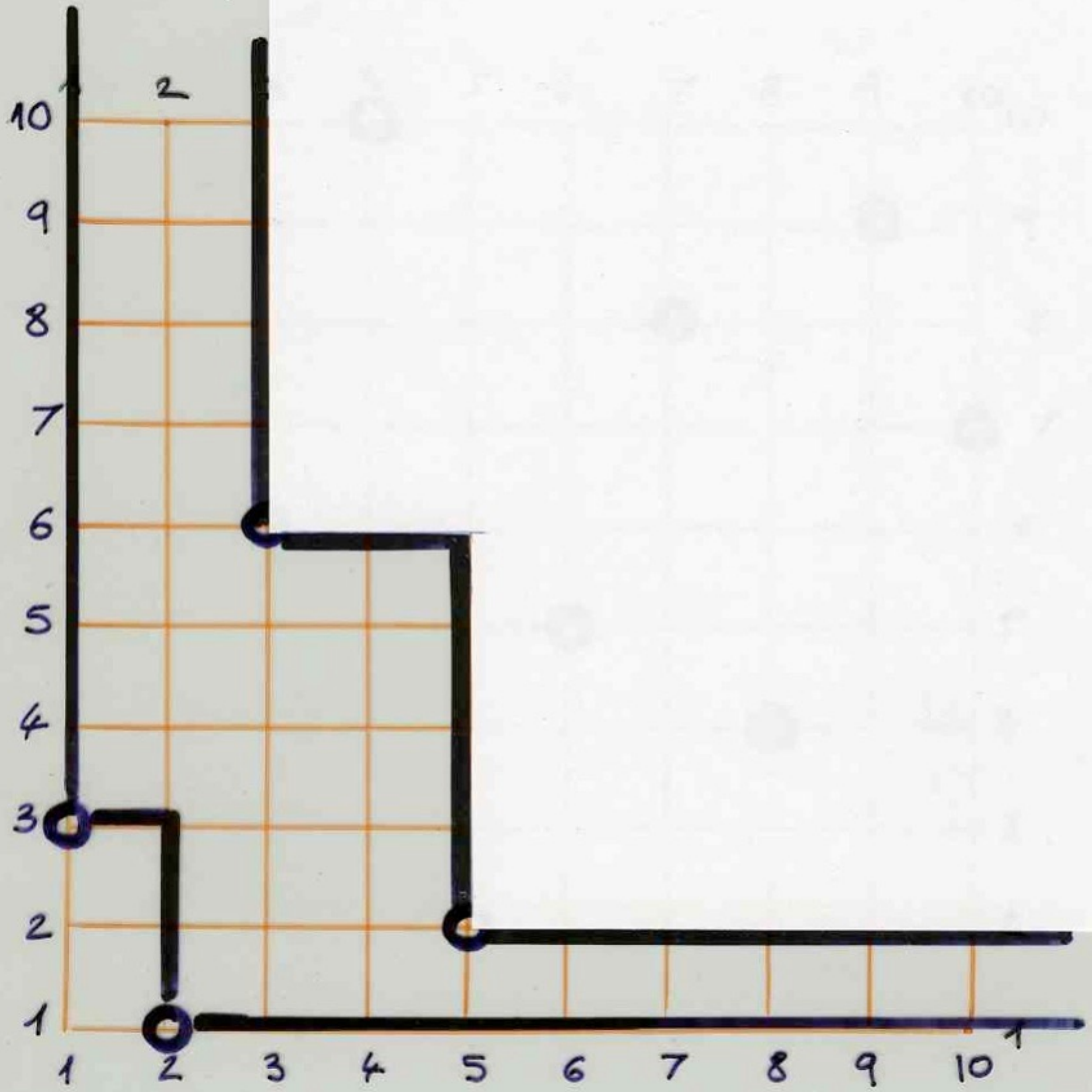
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



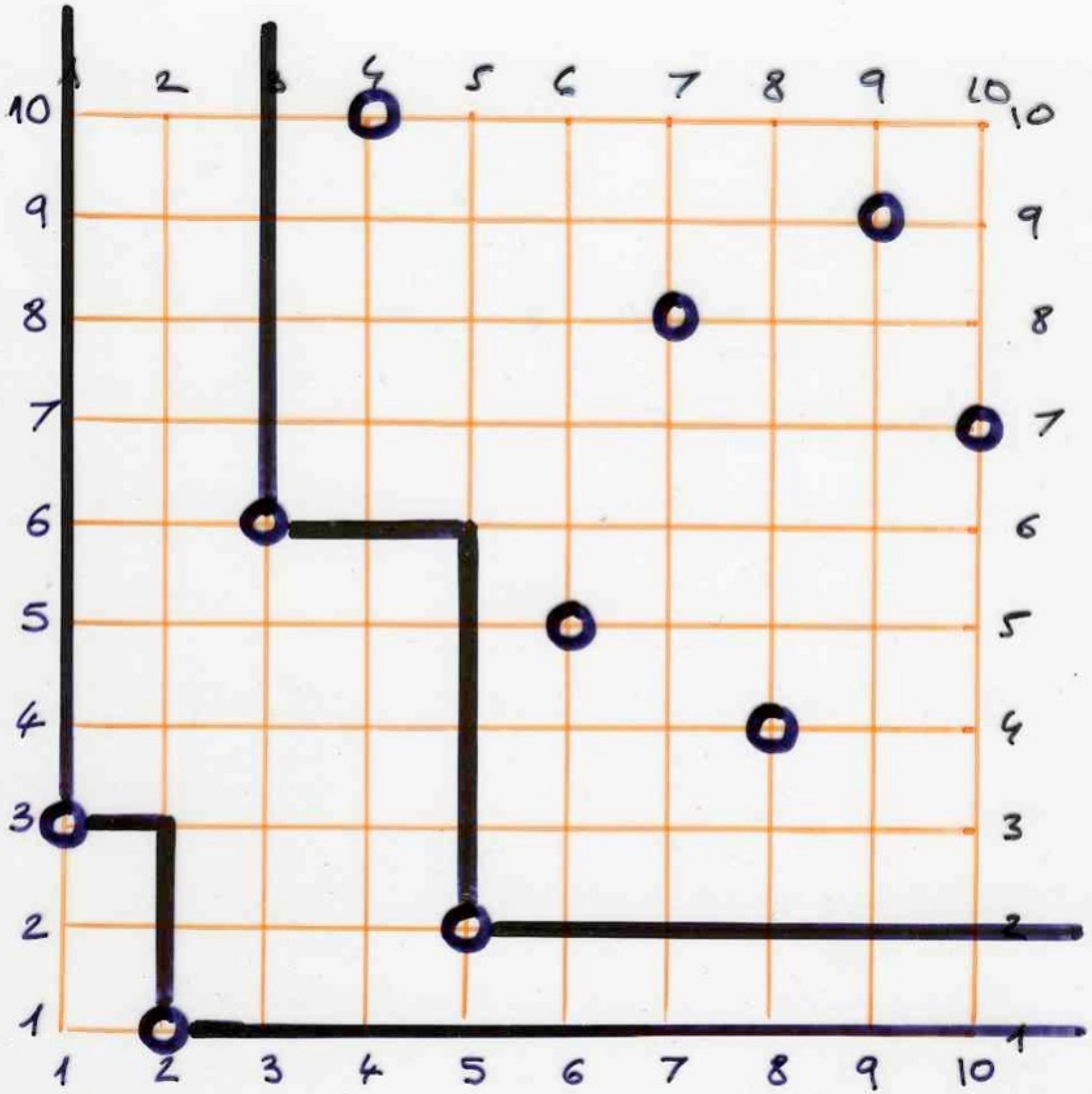
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



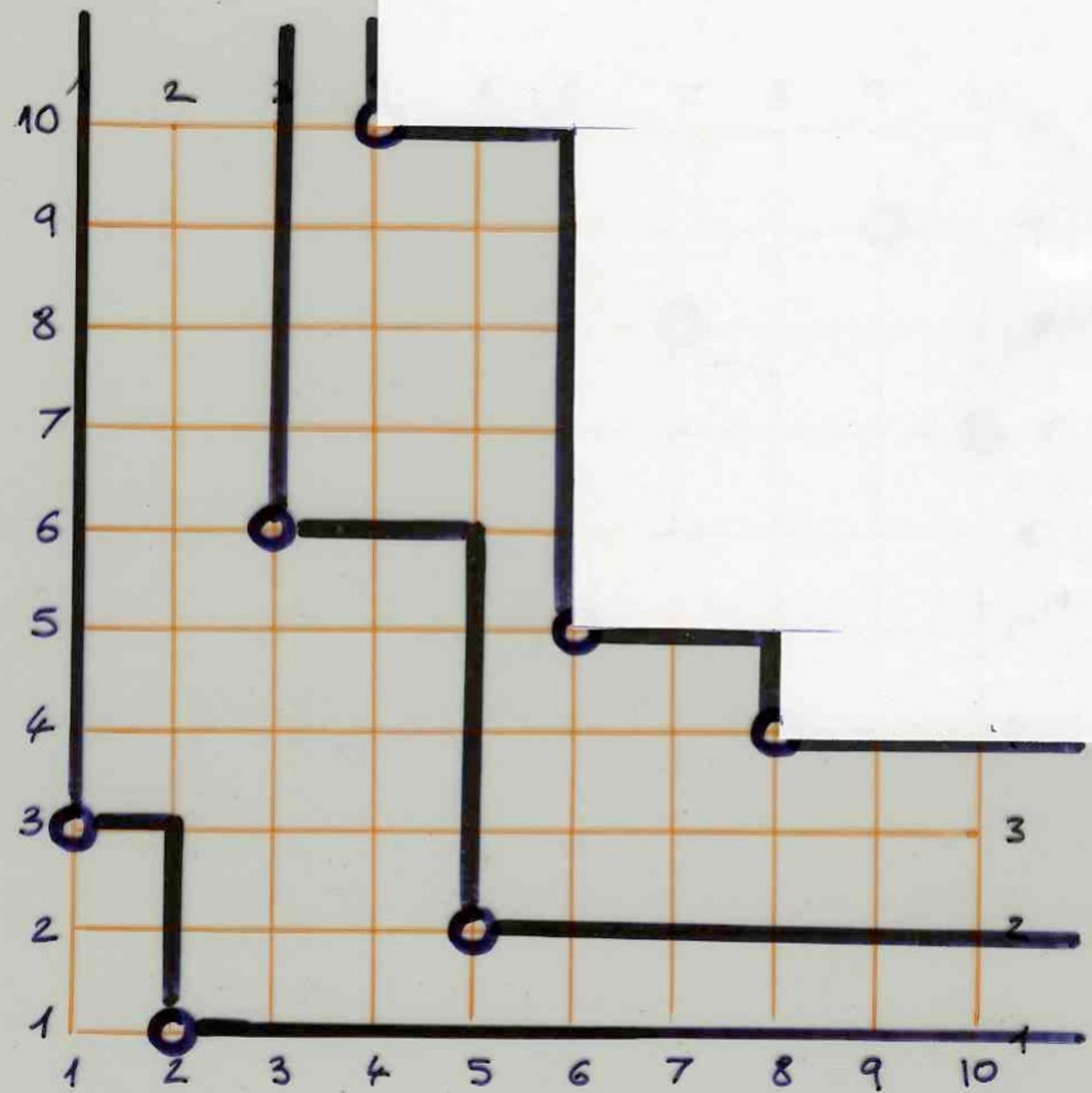
repeat the construction ...



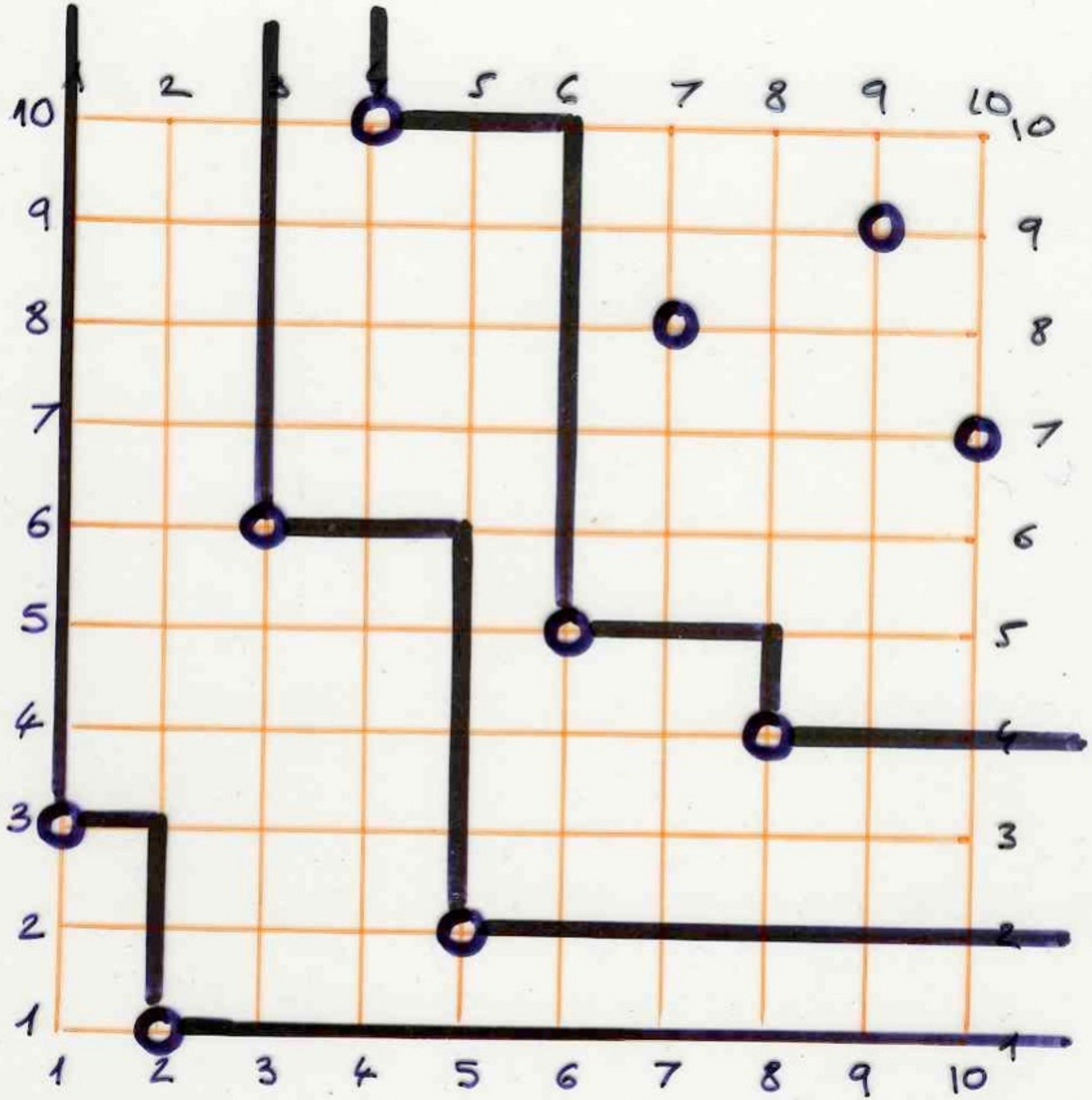
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



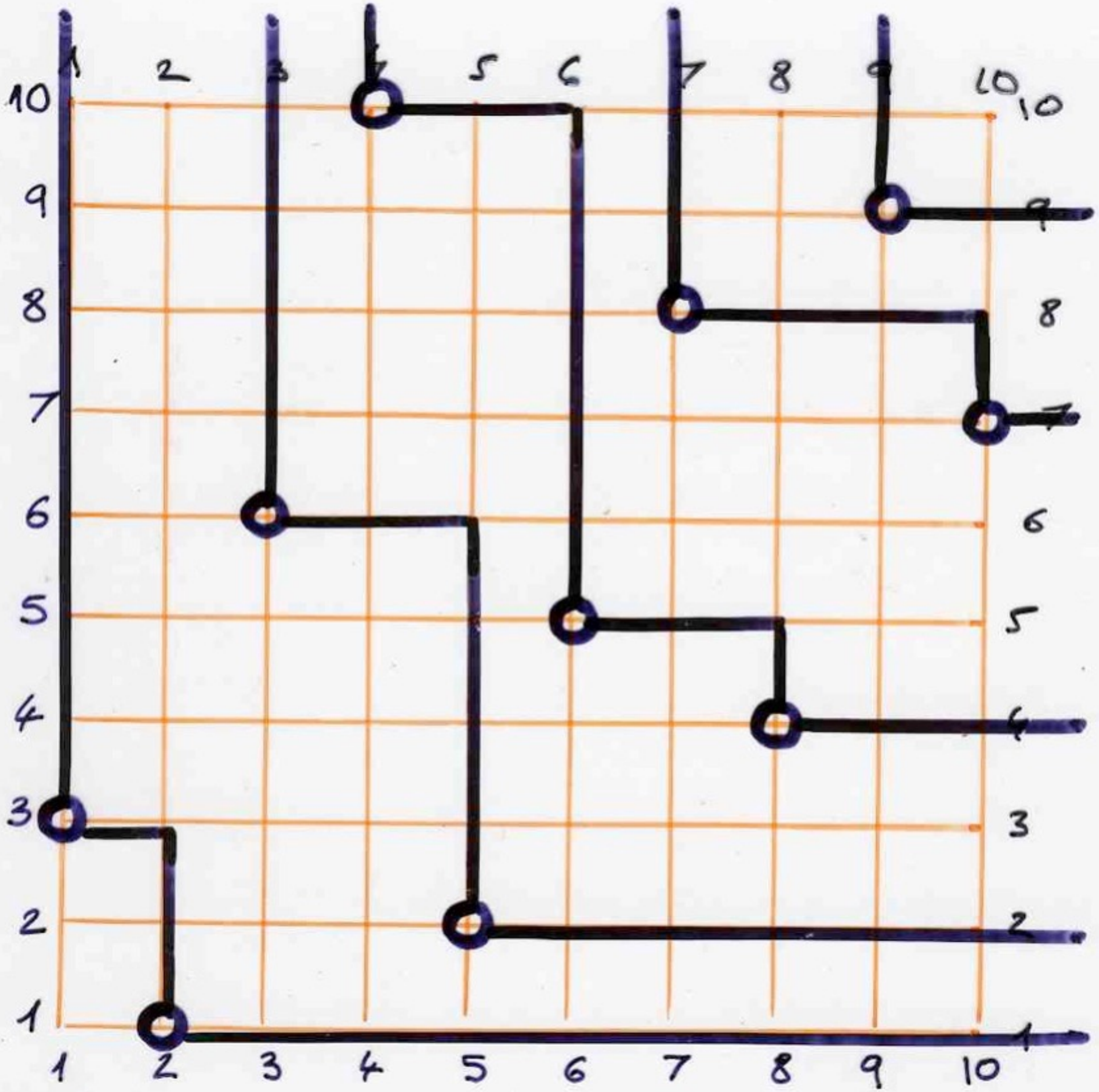
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



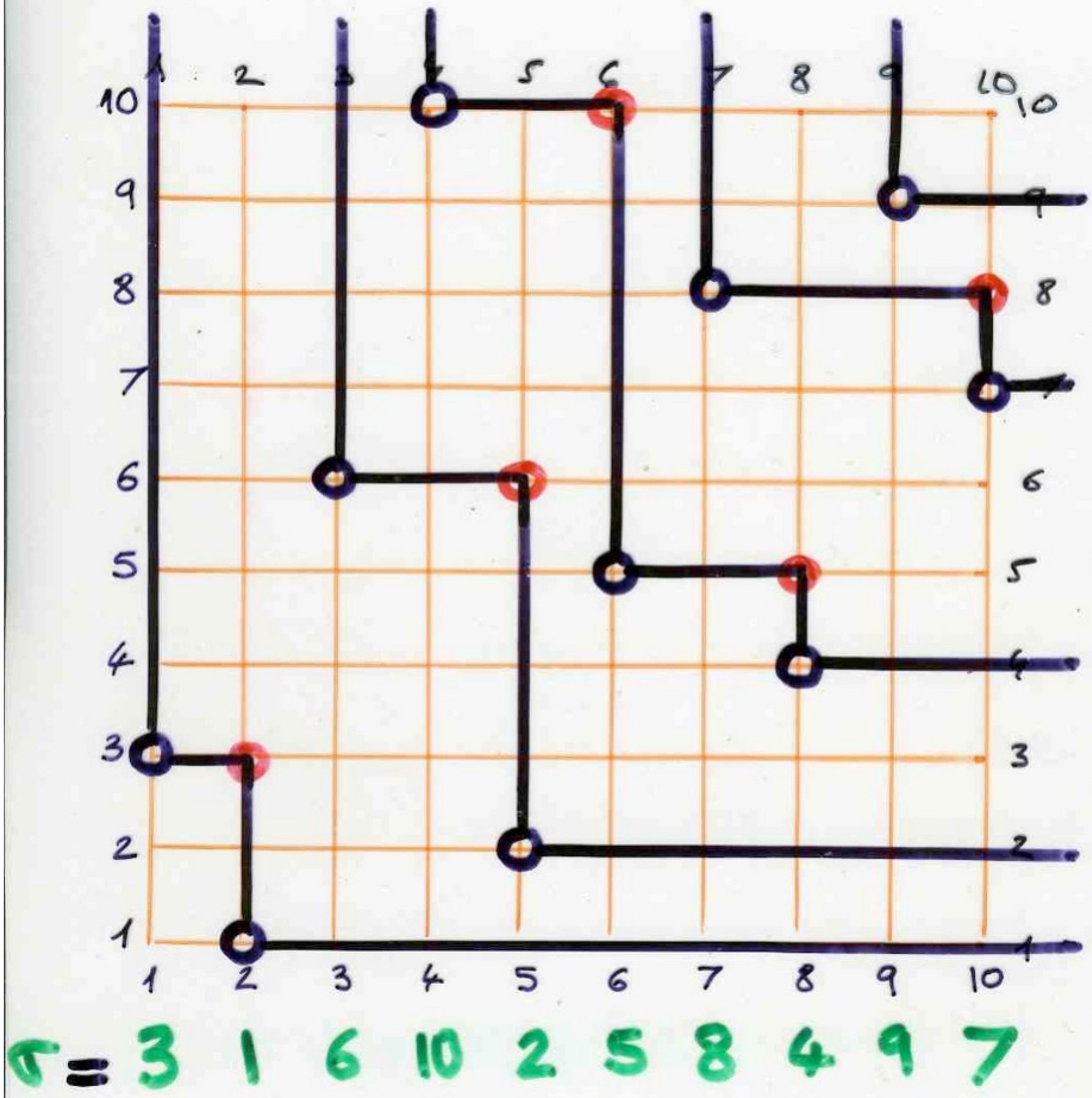
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



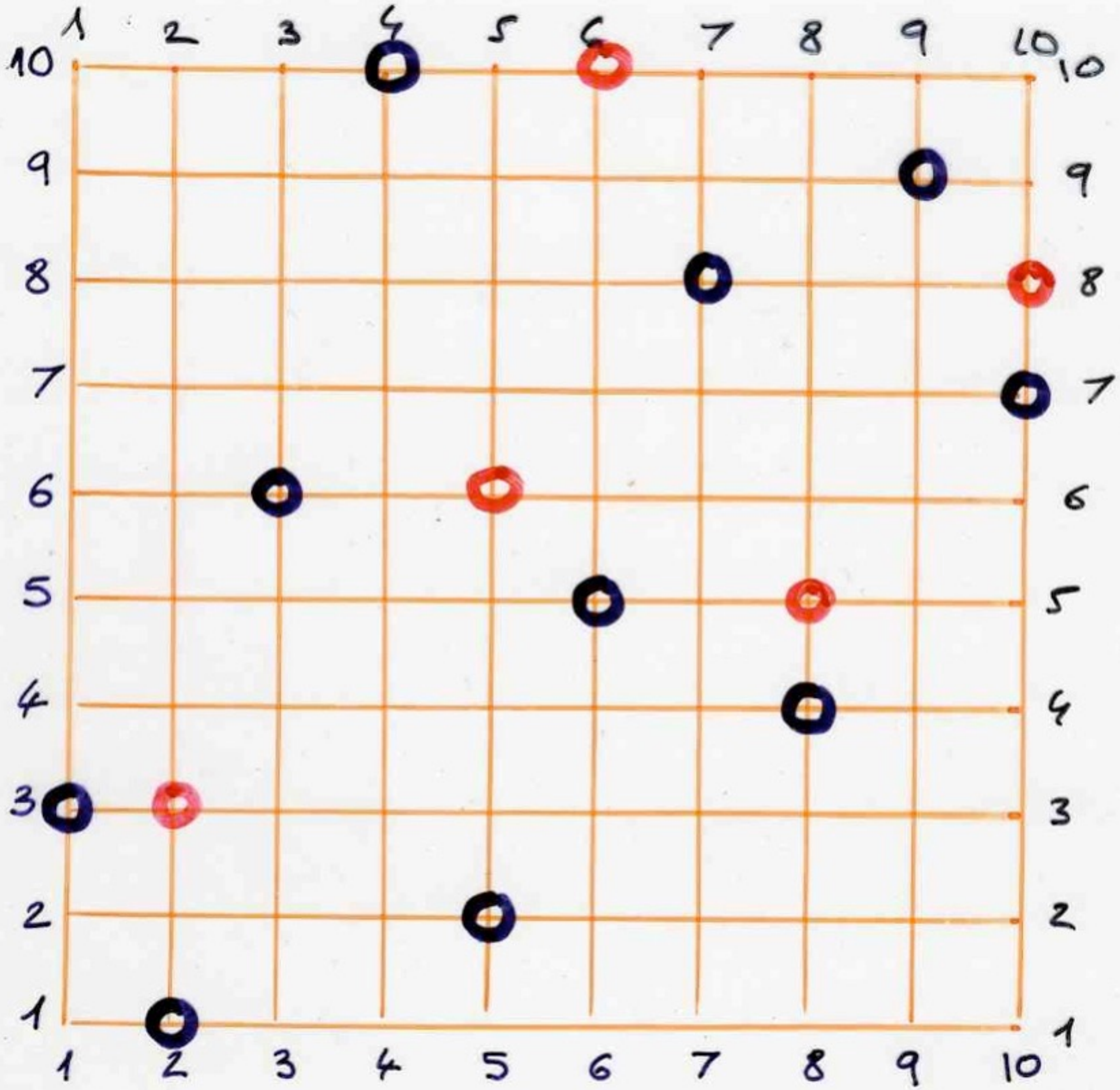
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma =$ 3 1 6 10 2 5 8 4 9 7

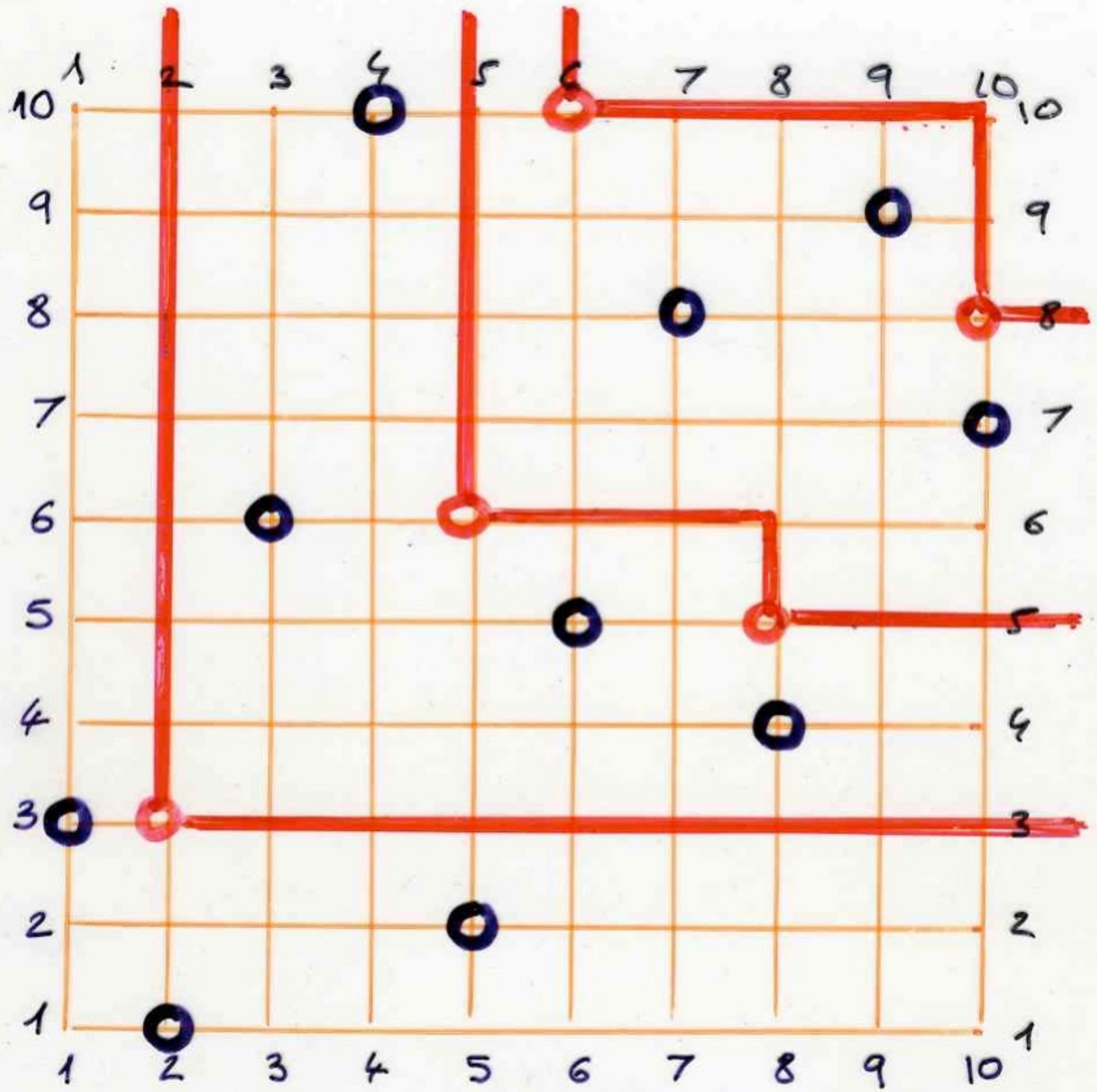


red
points

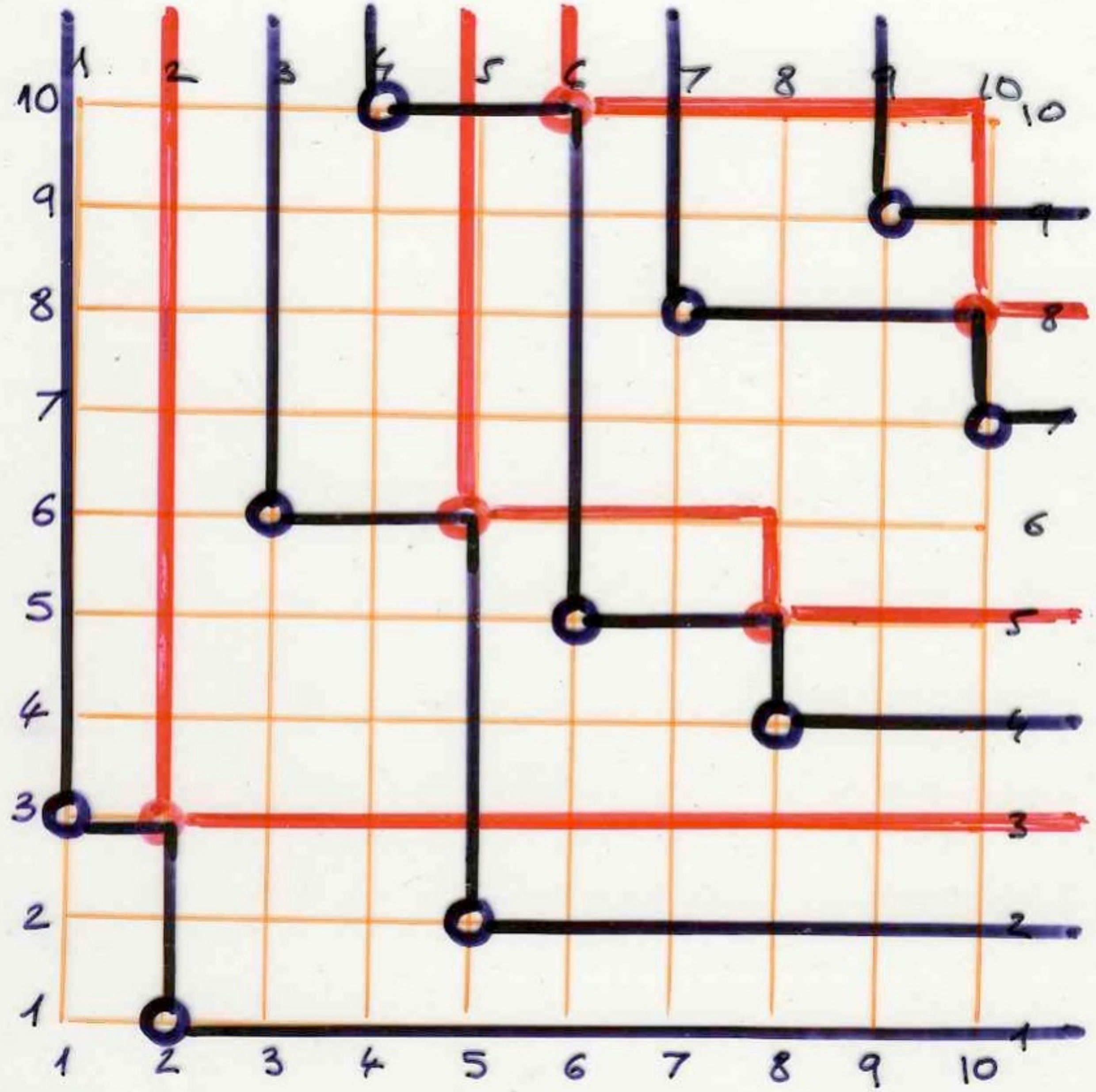


$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

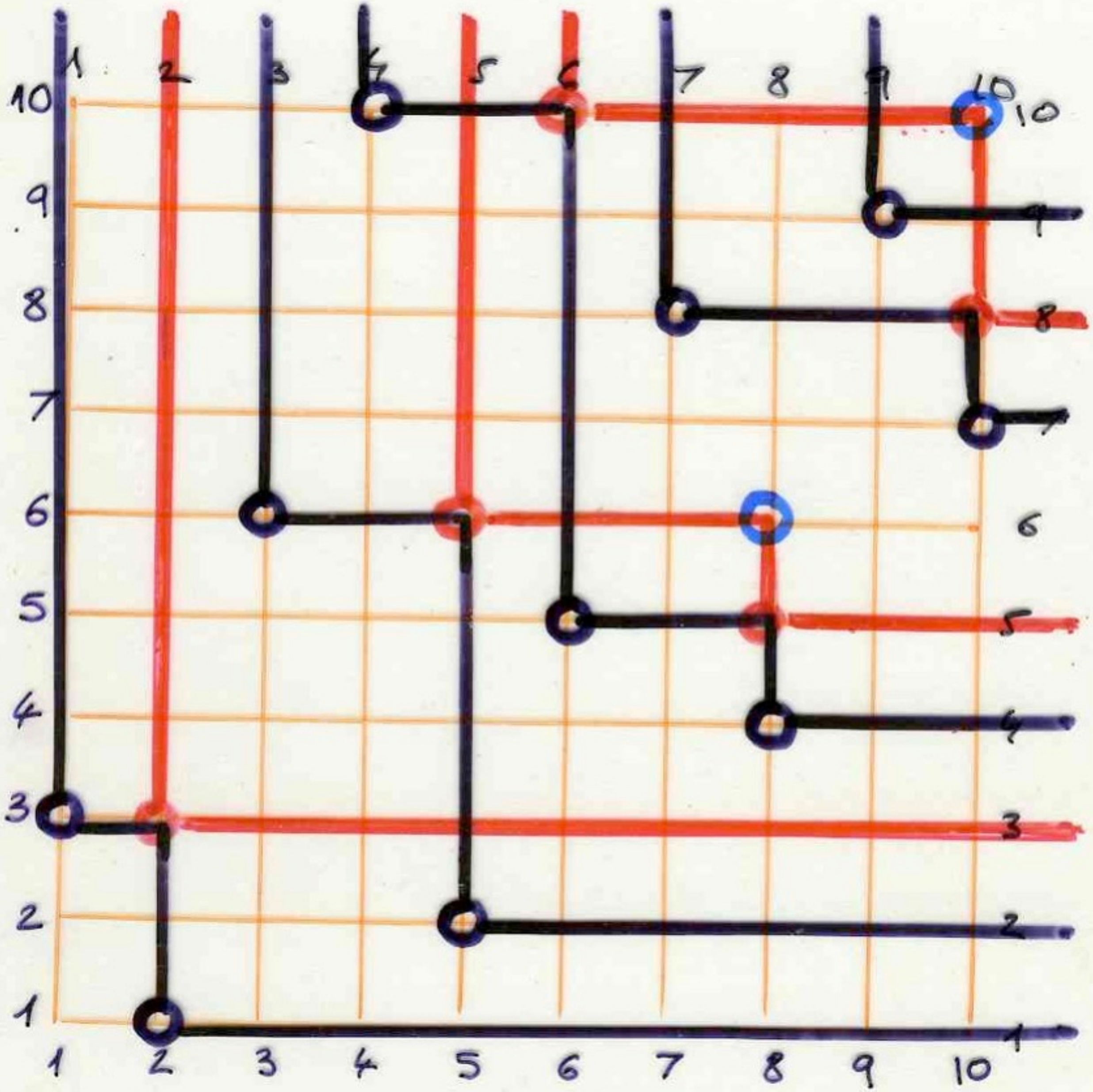
Repeat with the red points
the construction of sucessives shadows



$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



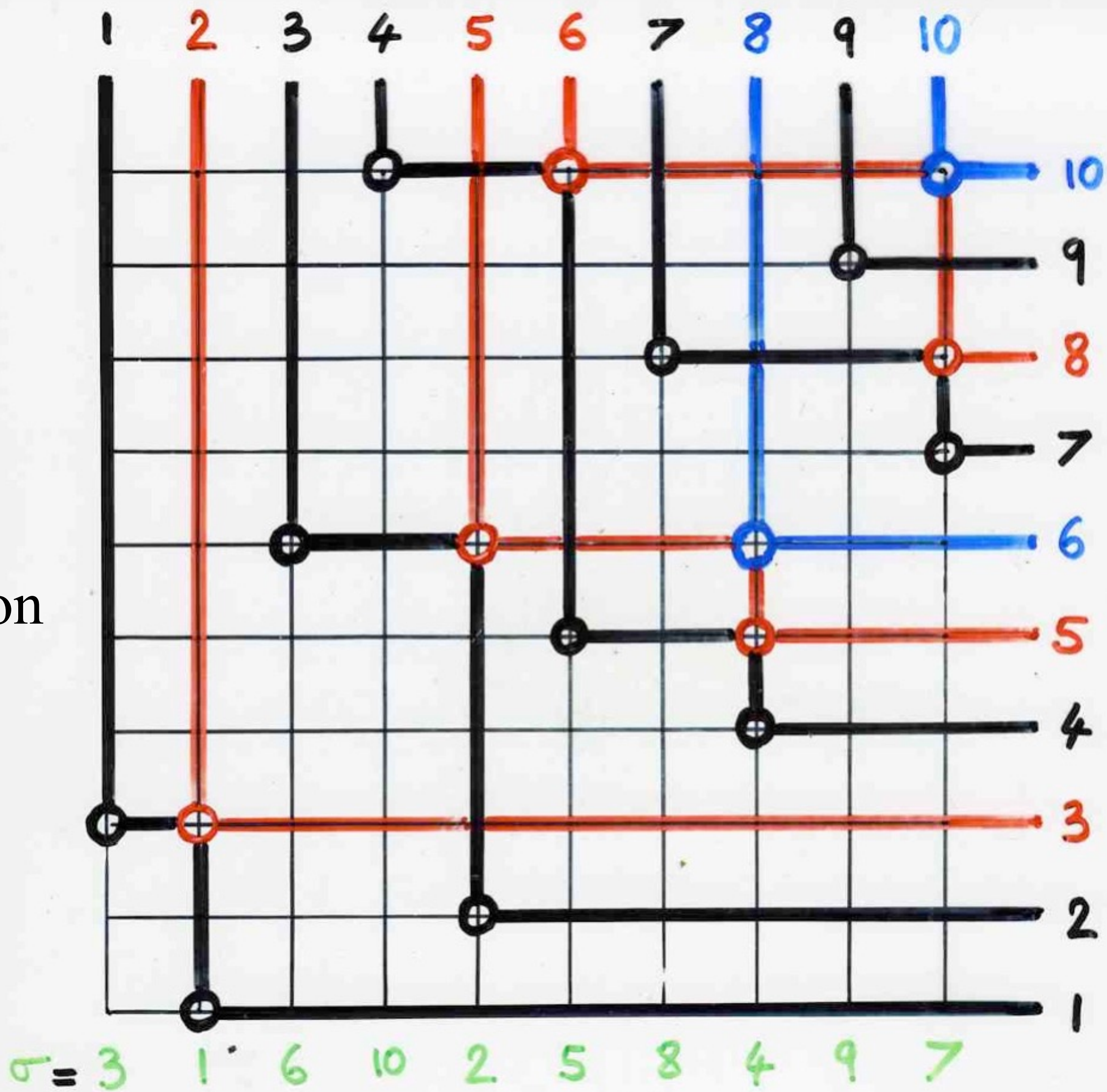
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

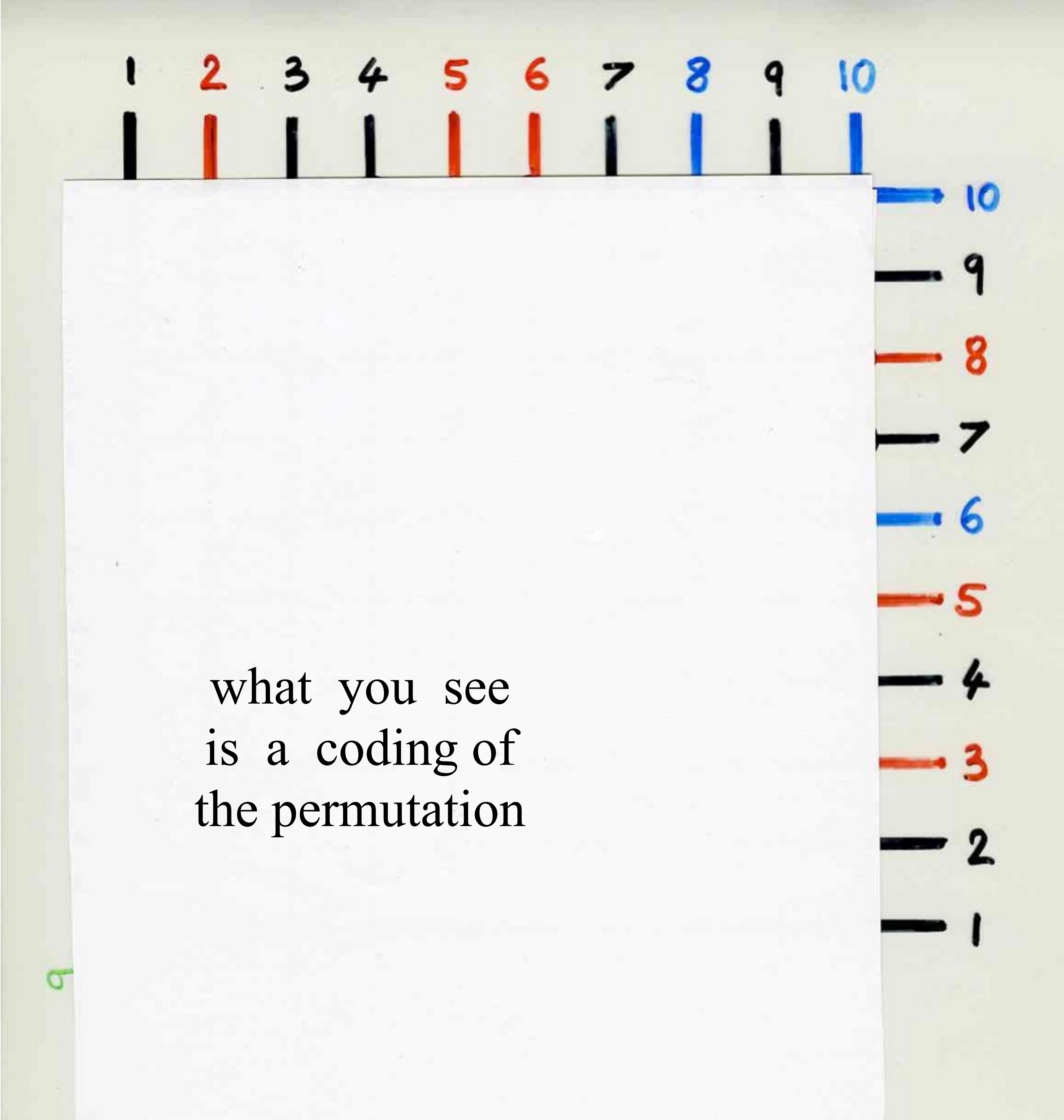


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

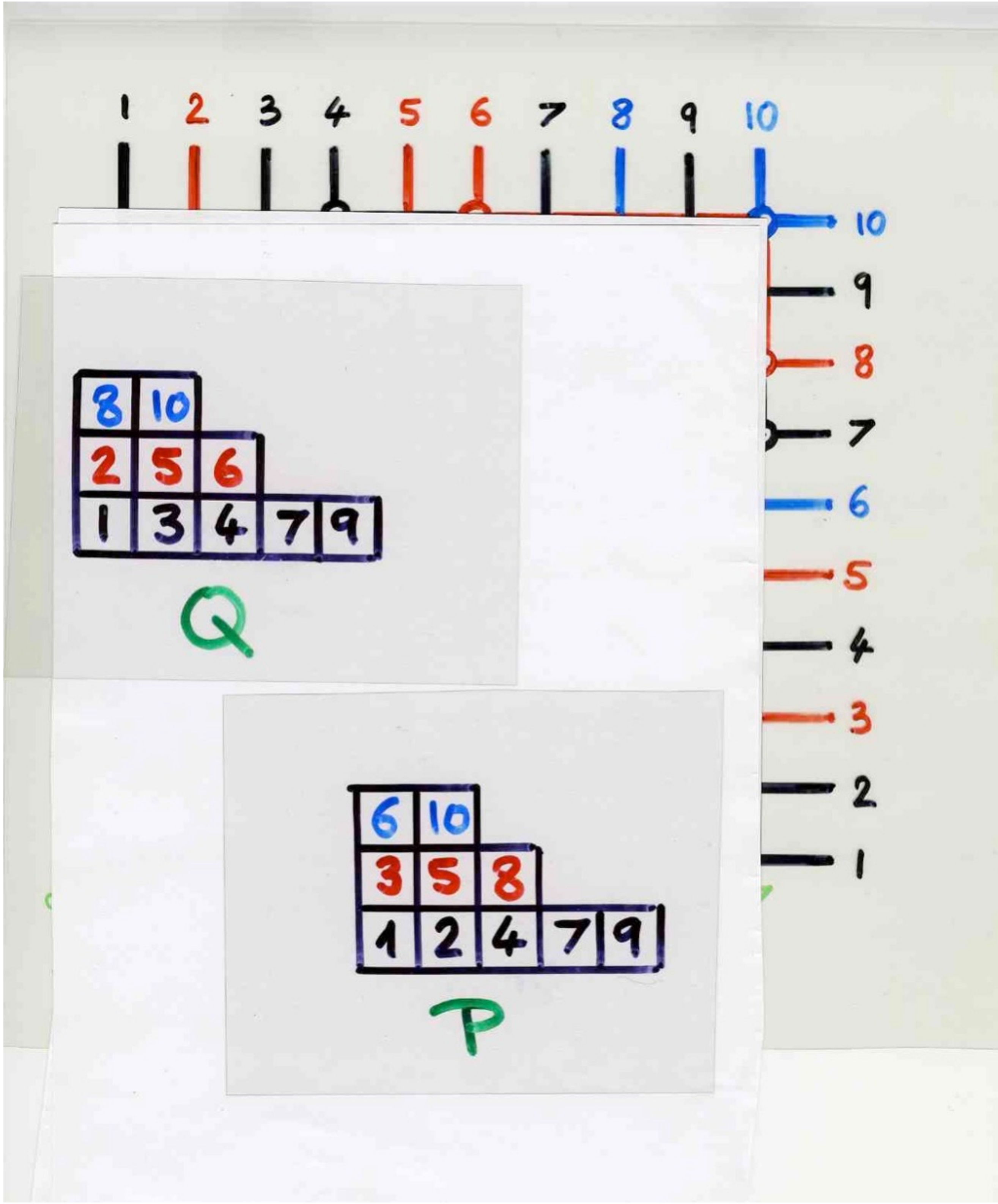
blue points

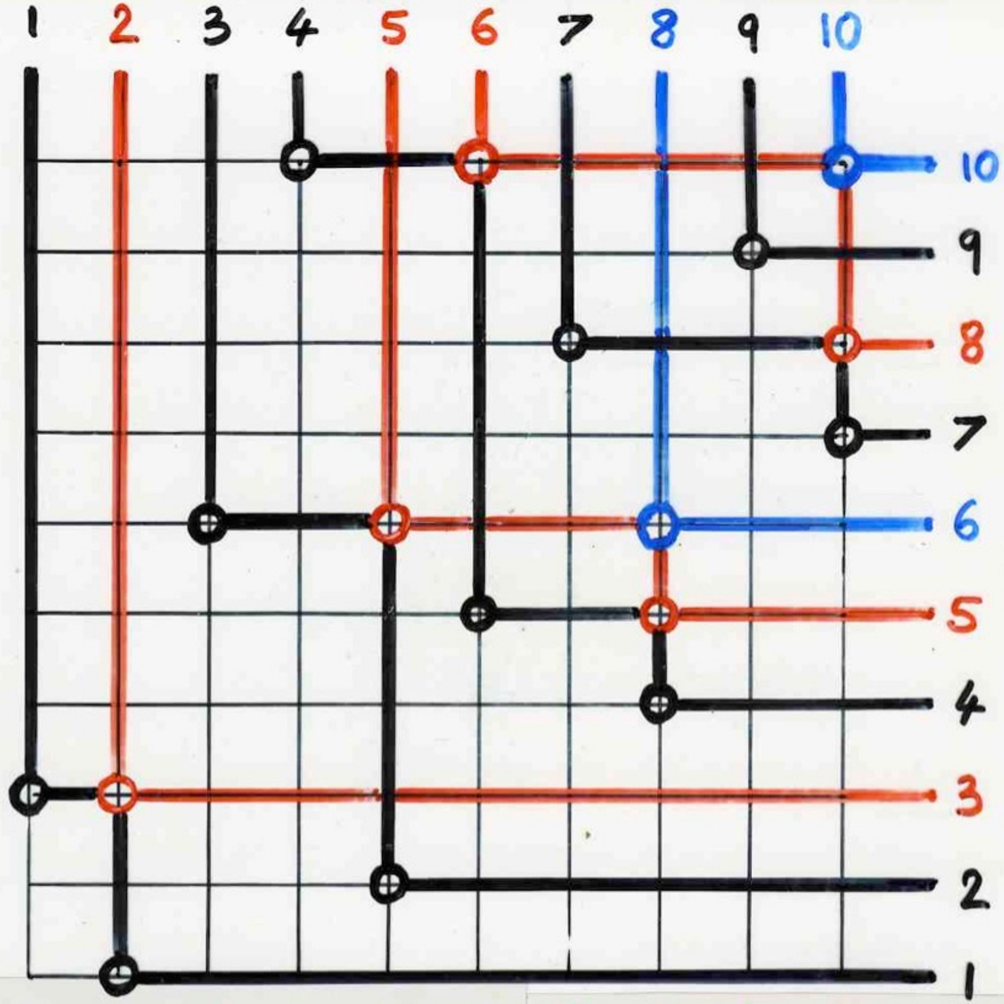
no
green
points:
end
of the
construction





what you see
is a coding of
the permutation


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

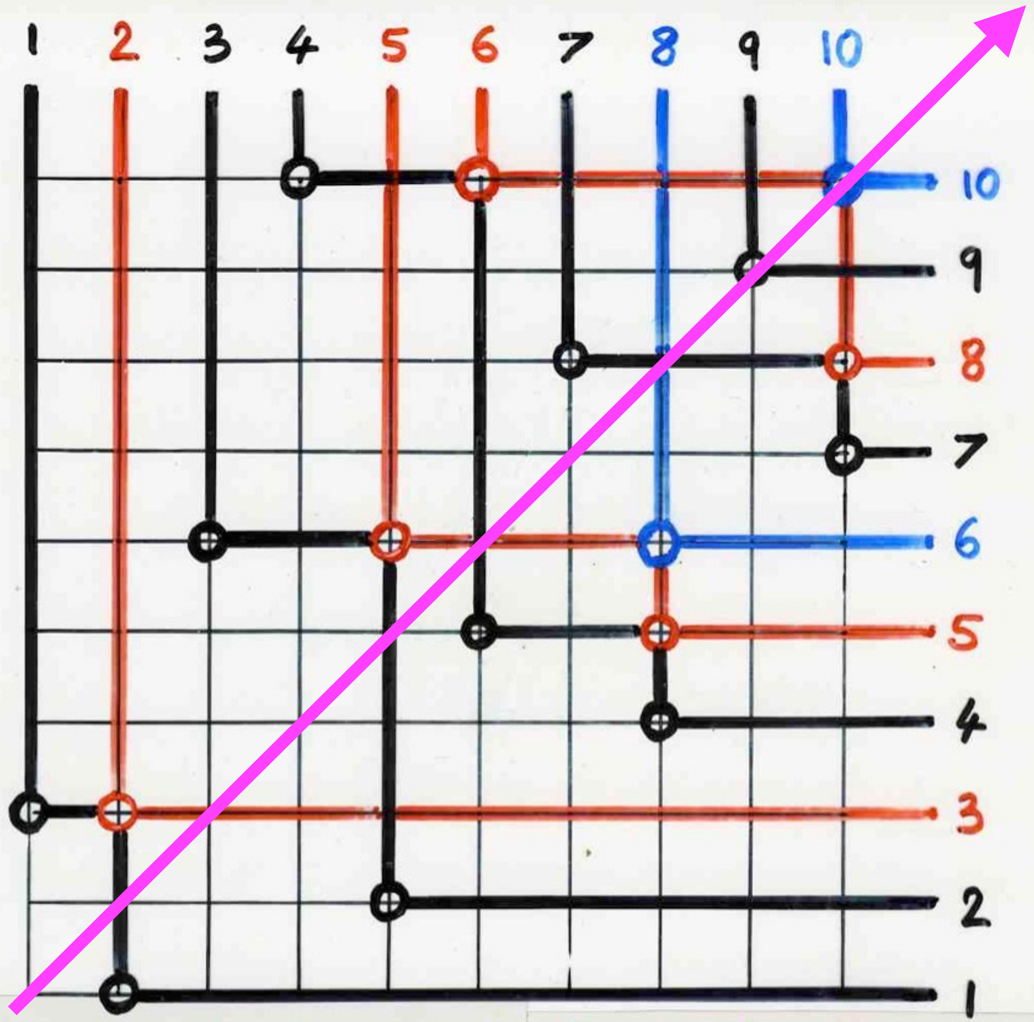
P

8	10			
2	5	6		
1	3	4	7	9

Q

$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

some exercises

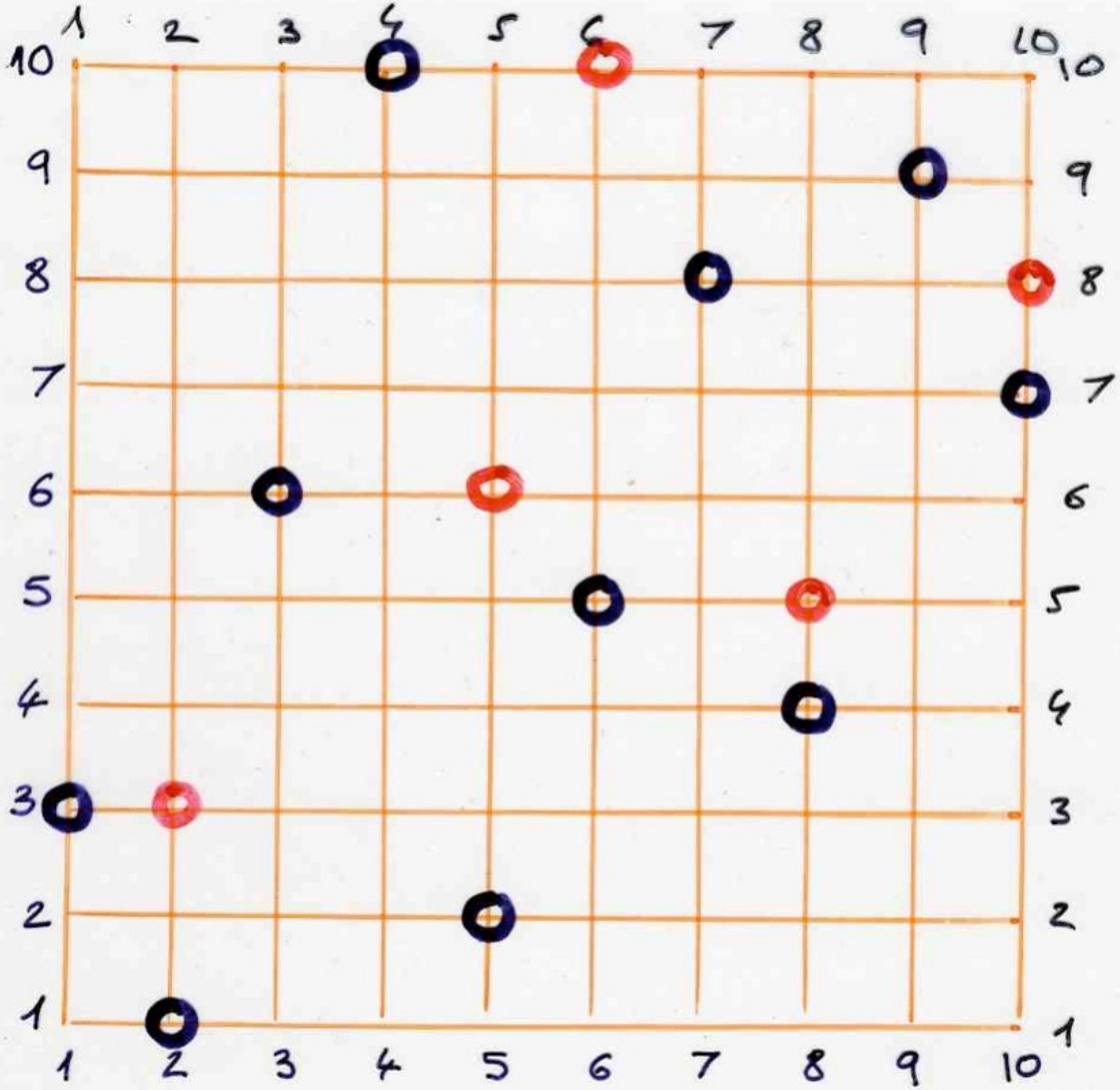
exercise

permutation $\sigma \in \mathfrak{S}_n$

black points $(i, \sigma(i)) \longrightarrow Sq(\sigma)$ squeleton

set of red points

Give a procedure to construct σ knowing $Sq(\sigma) \subseteq [1,n] \times [1,n]$



$$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$$

exercise

permutation $\sigma \in \mathfrak{S}_n$

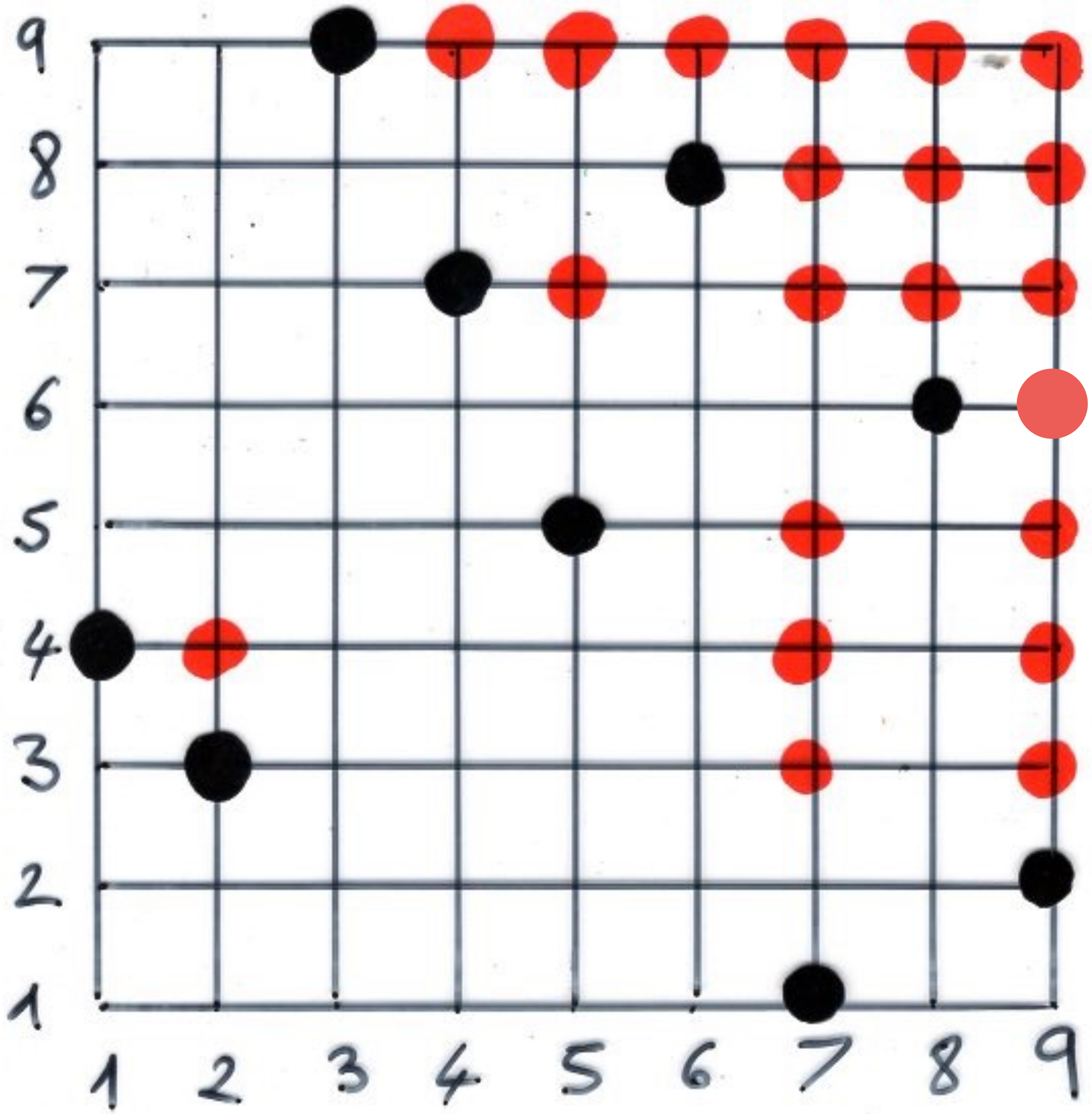
black points $(i, \sigma(i)) \longrightarrow Sq(\sigma)$ squeleton

set of red points

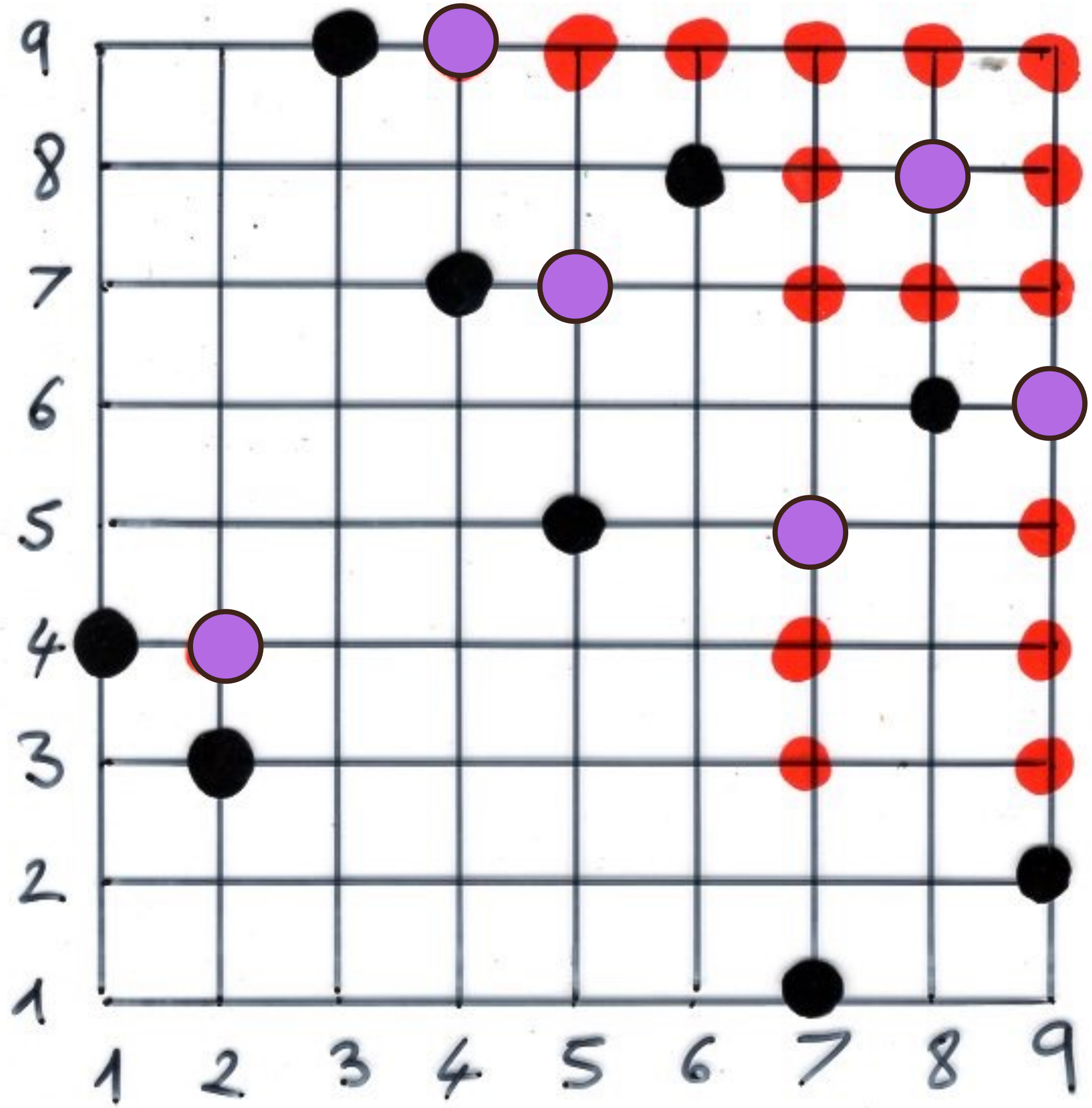
Characterisation of the red points

$$Sq(\sigma) \subseteq [1,n] \times [1,n]$$

It is the set of "wining positions" in a Nim game on the Rothe diagram of σ


9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9

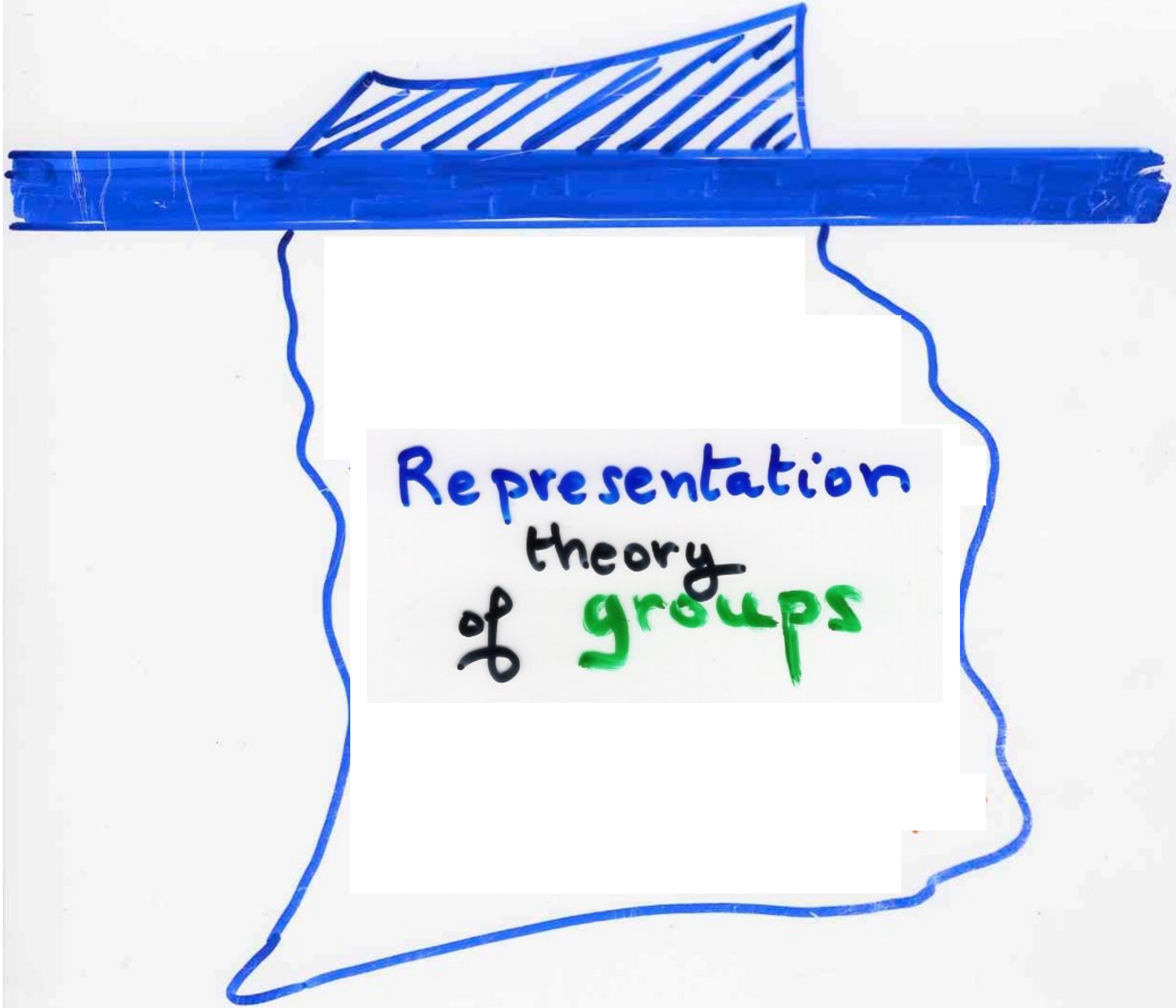
exercise



more about groups theory

The Robinson-Schensted correspondence

The Robinson-Schensted correspondence



Representation theory of groups

see a group G as a (sub)-group of matrices

$$G \to \text{Matrices}$$
$n \times n$, coeff. in $\mathbb{R}$

see G as a group of transformations

Important in Physics

standard model of particles

4 fundamental forces $\left.\begin{cases}\text{electro-magnetism}\\ \text{strong}\\ \text{weak}\end{cases}\right\} + \text{gravity}$

for every group

into
irreducible
representation $\xrightarrow[\text{decomposition}]{}$ representations

analogy [every number n $= p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

prime numbers
decomposition

Case of the group $\mathfrak{S}_n$ permutations

irreducible
representations $\longleftrightarrow$ partition λ
of n

dimension
of the irreducible
representation $= f_\lambda$
(= order of the
matrices)

number of Young tableaux
with shape λ

in (finite) group theory :

$$|G| = \sum_{R} (\deg R)^2$$

order of the group

R irreducible representation

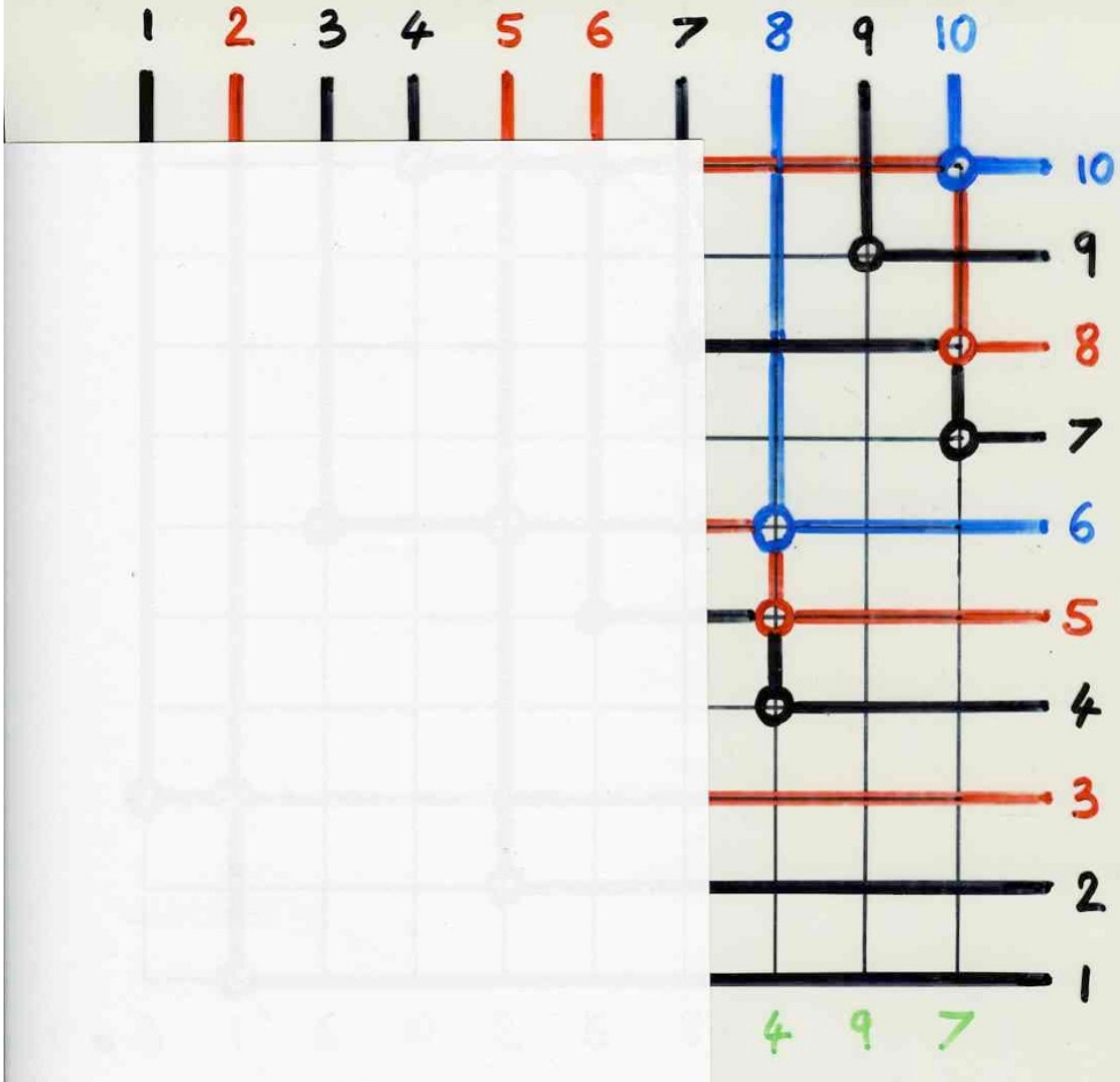
for the symmetric group $\mathfrak{S}_n$:

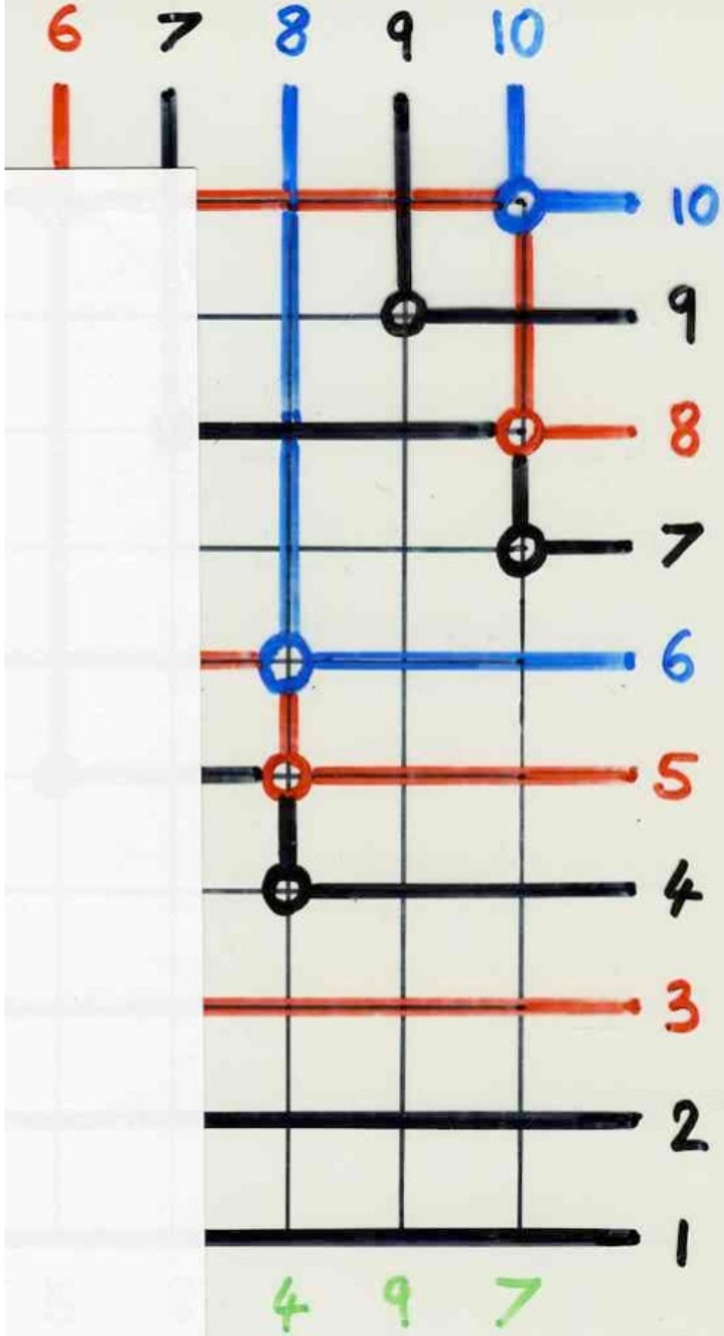
$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ partitions of n

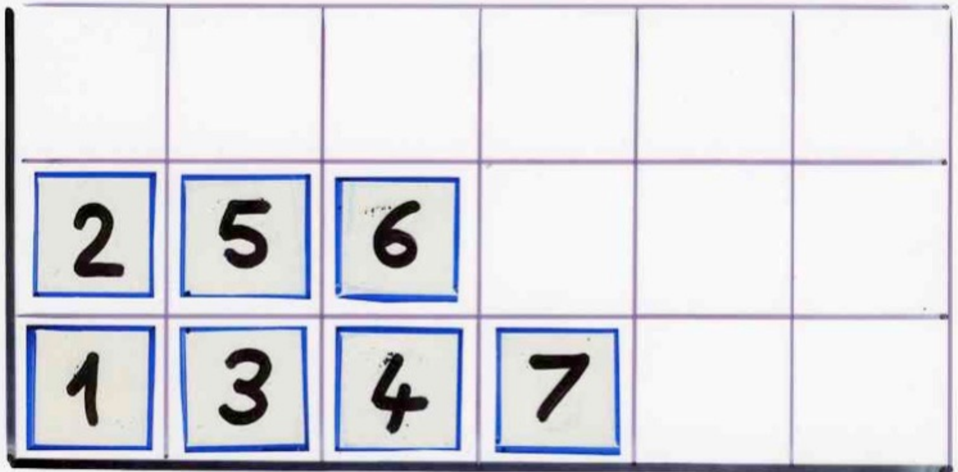
proof of the equivalence

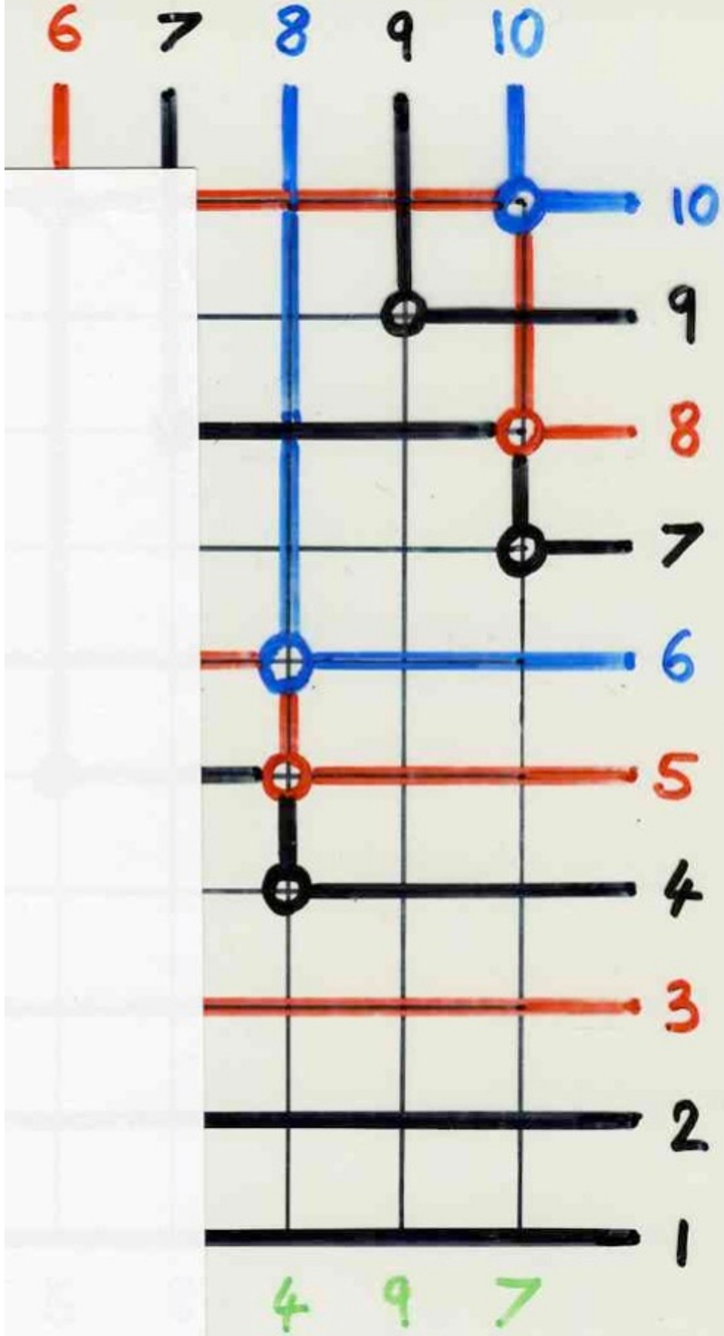
insertions --- geometric constrution


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
4 9 7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





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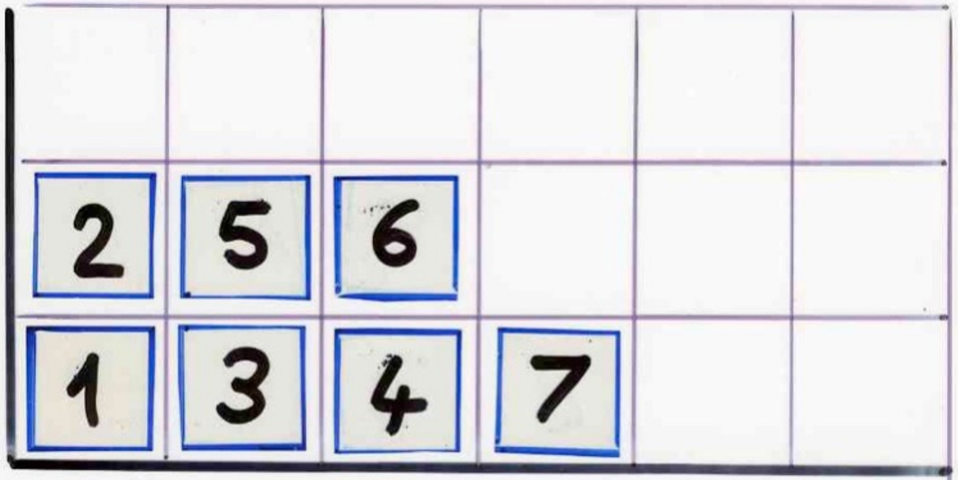

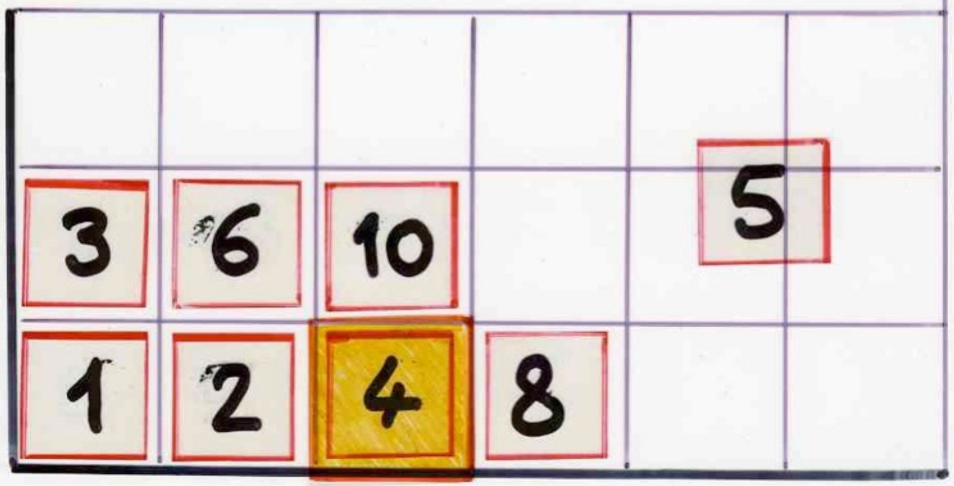

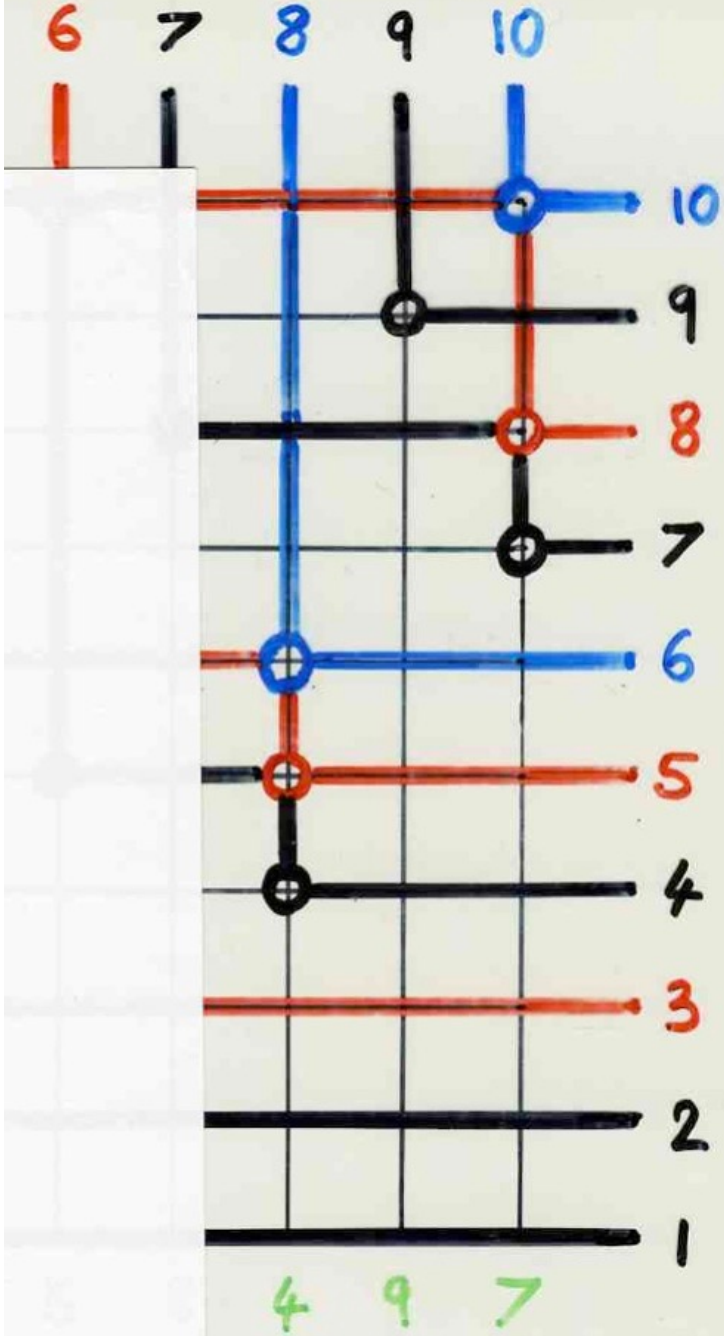

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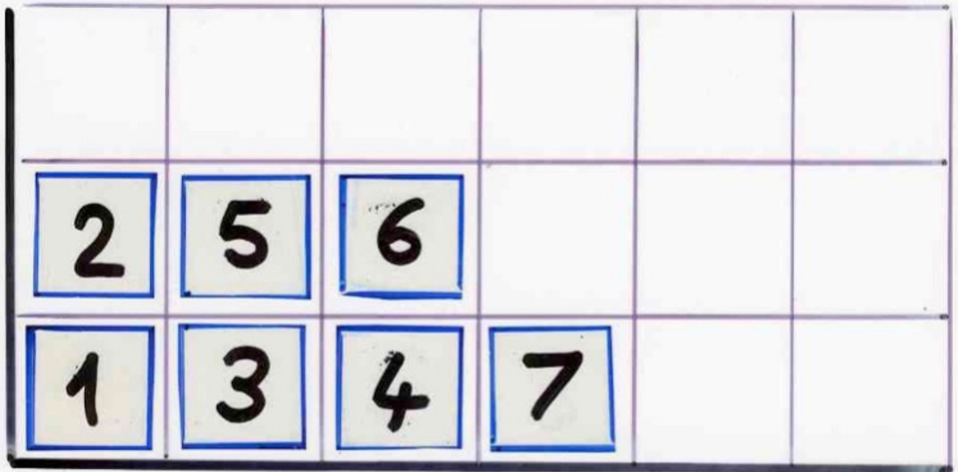

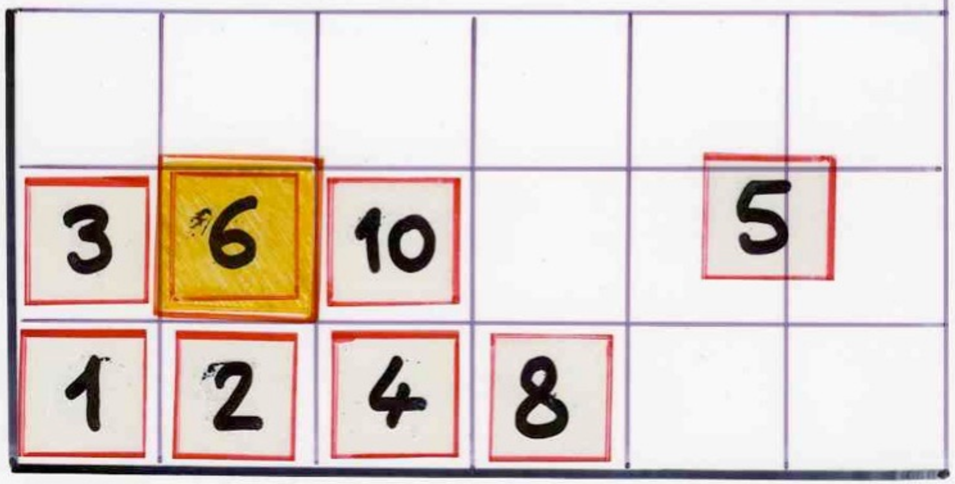

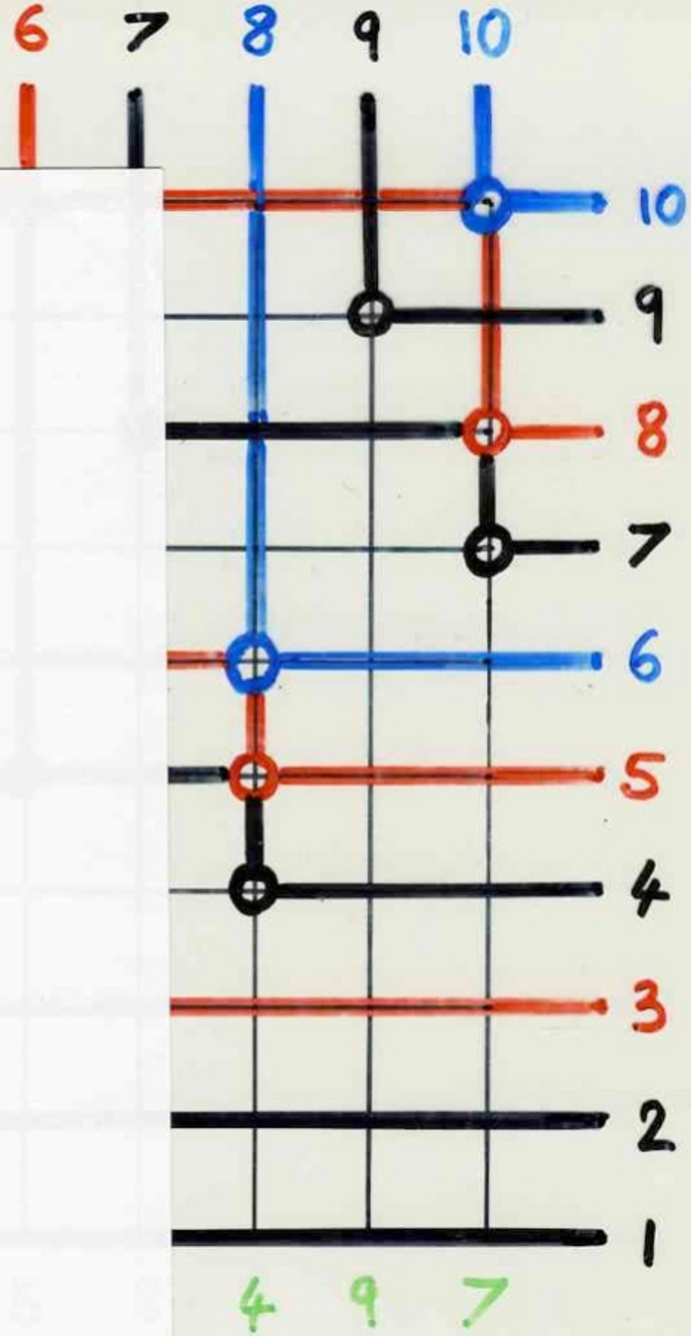

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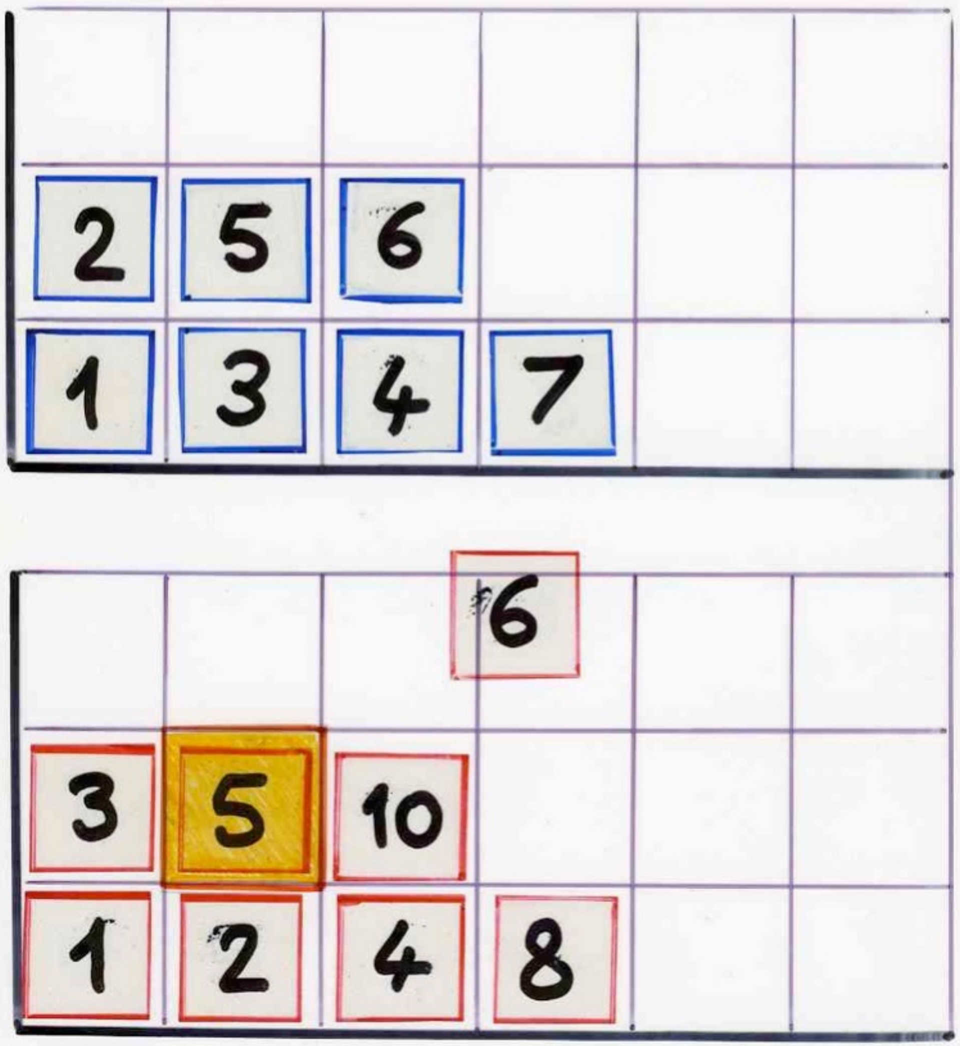
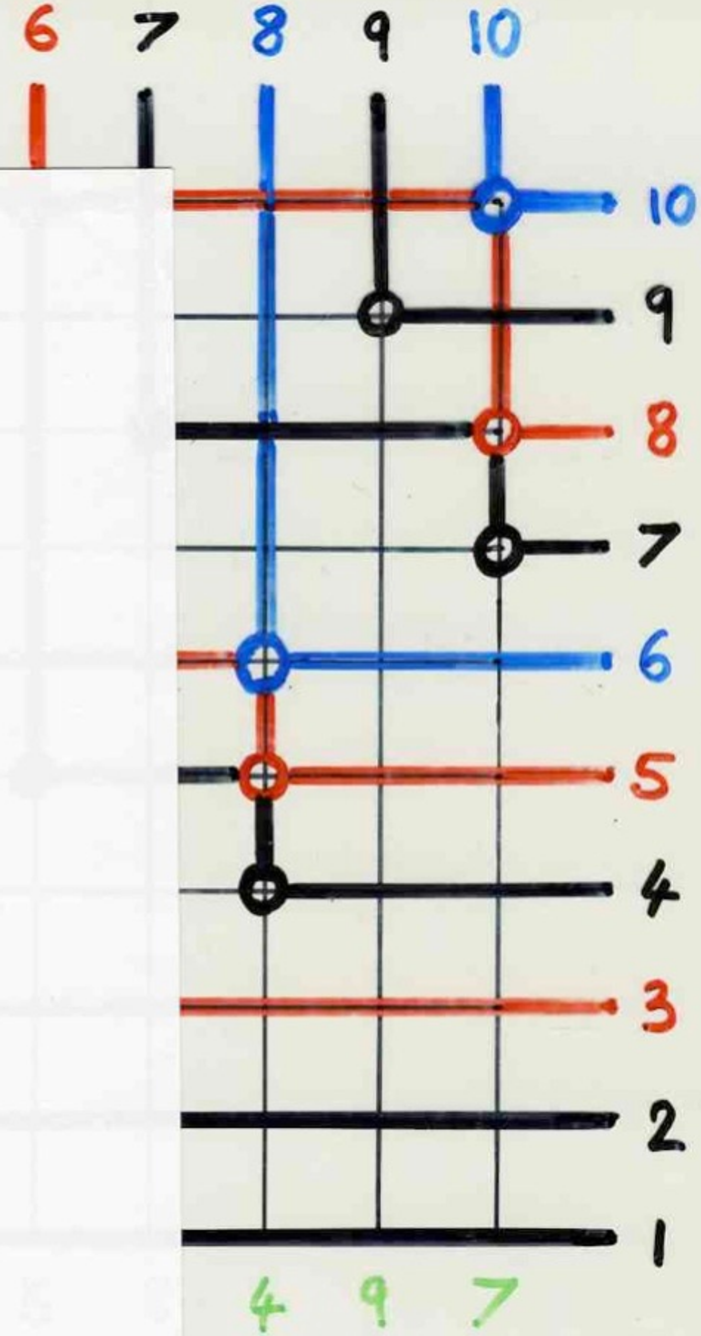




1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

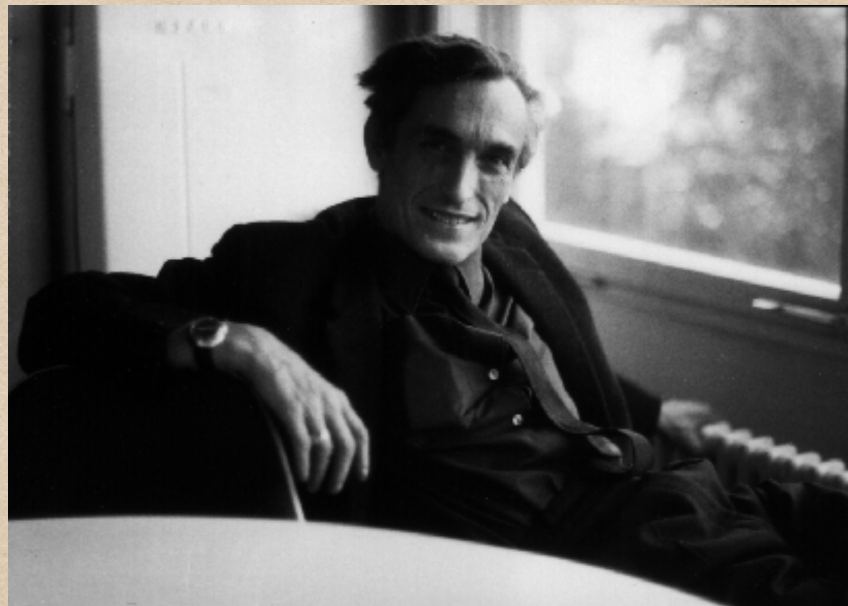
8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

Jeu de taquin

(without proof)

see slides Ch4c complements



M.P. Schützenberger

duality

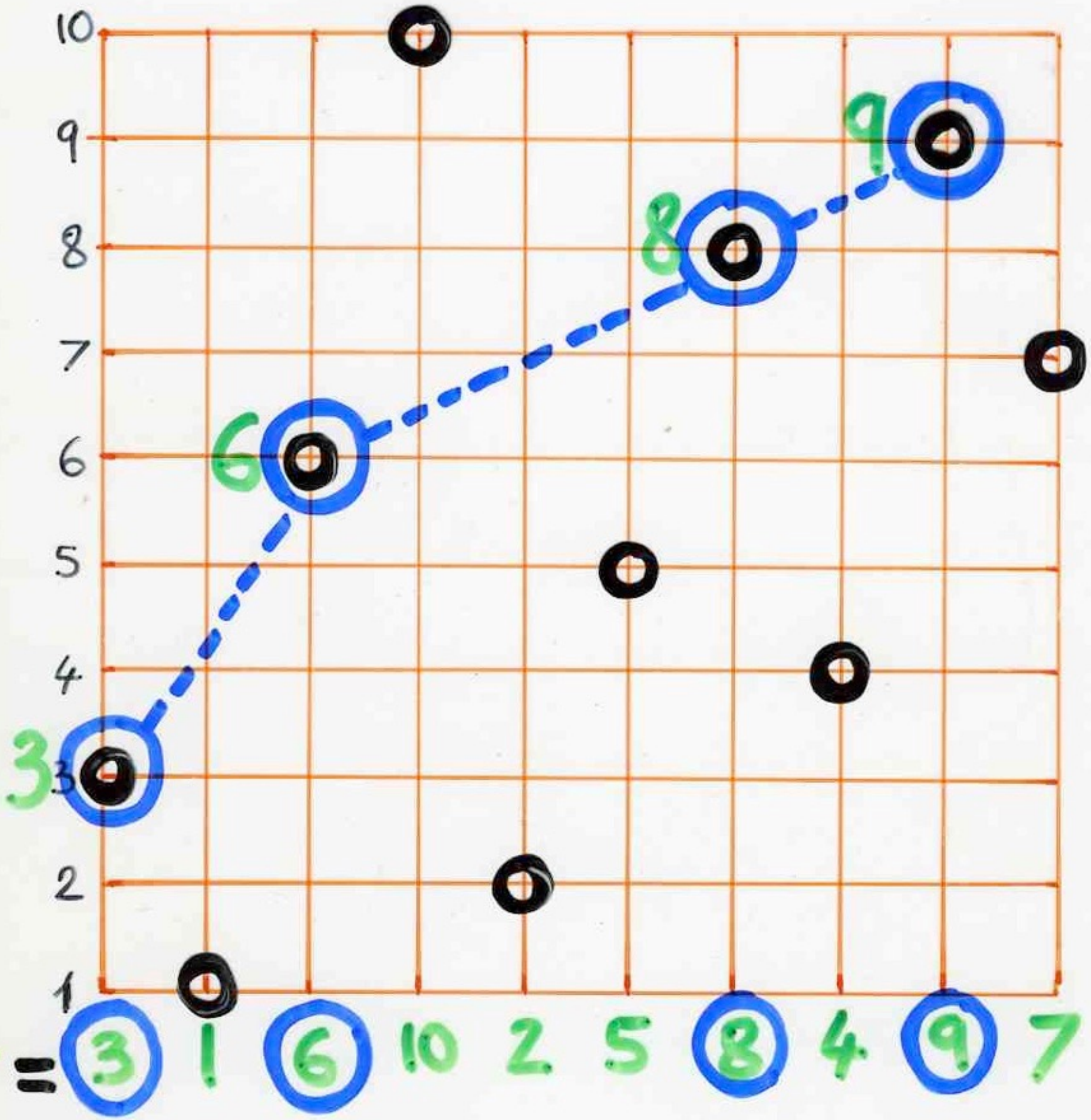
M.P. Schützenberger, 1963, 1972

(without proof)

see slides Ch4c complements

application:

increasing and decreasing subsequences



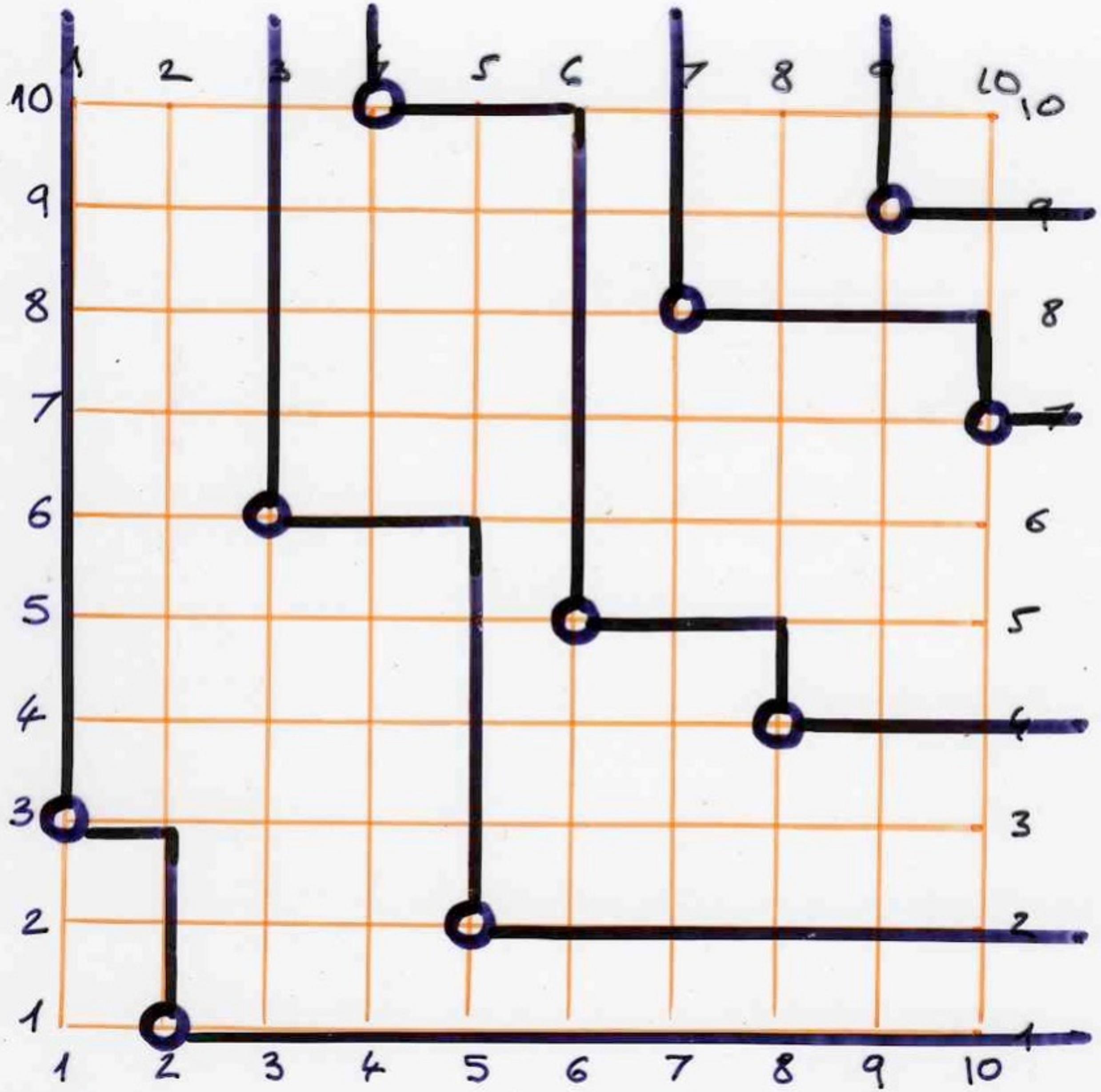
Increasing subsequence

Prop. permutation $\sigma \longleftrightarrow (P, Q)$

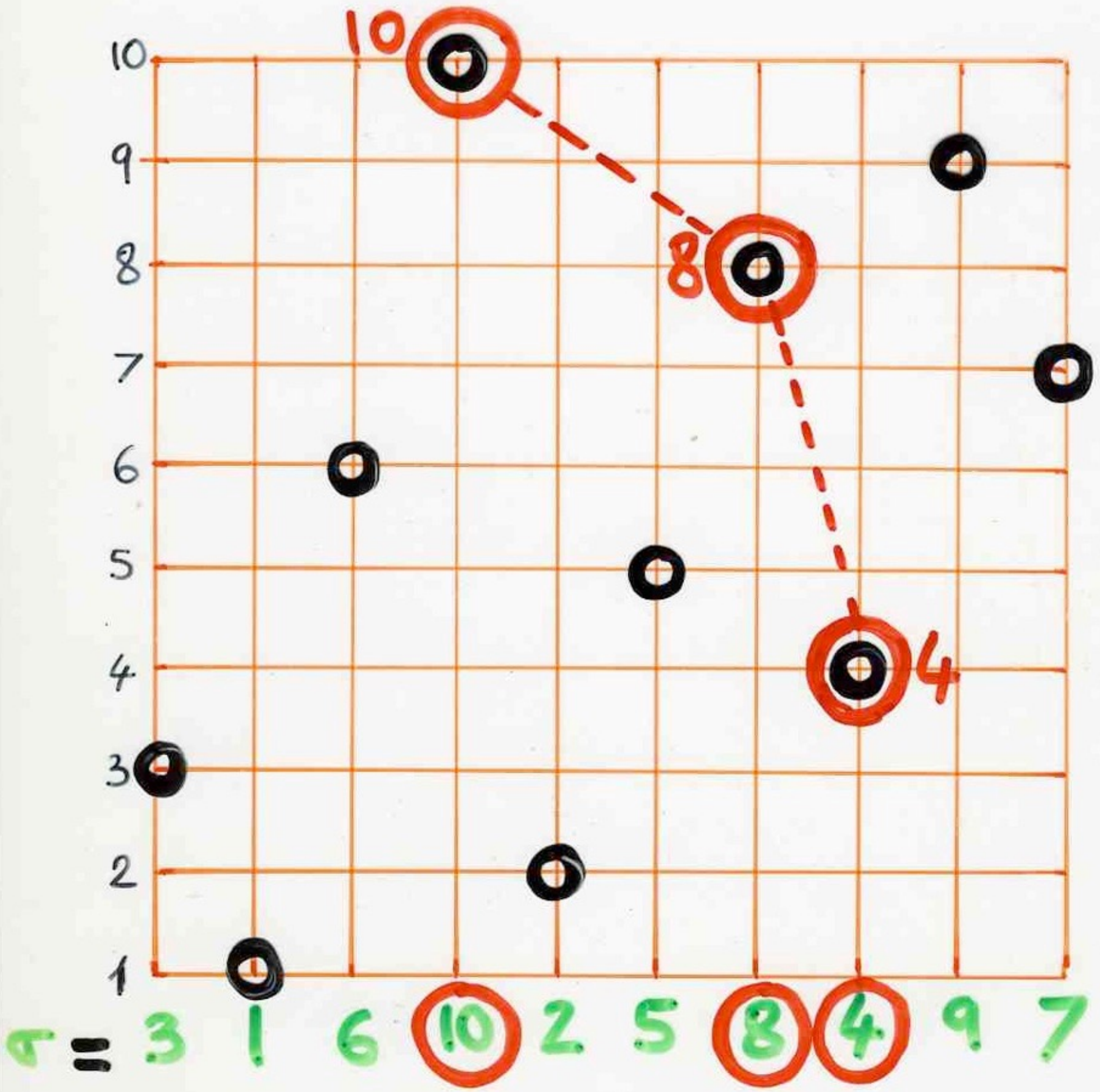
- The number of elements in the first row of the Young tableaux P and Q is the maximum length of increasing subsequences of σ.

exercise

Prove the proposition
(using the first set of lines
in the "light-shadow" algorithm



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



Decreasing subsequence

Prop. $\sigma \longleftrightarrow (P, Q)$

permutation

- The number of elements in the first column of the Young tableaux P and Q is the maximum length of decreasing subsequences of σ.

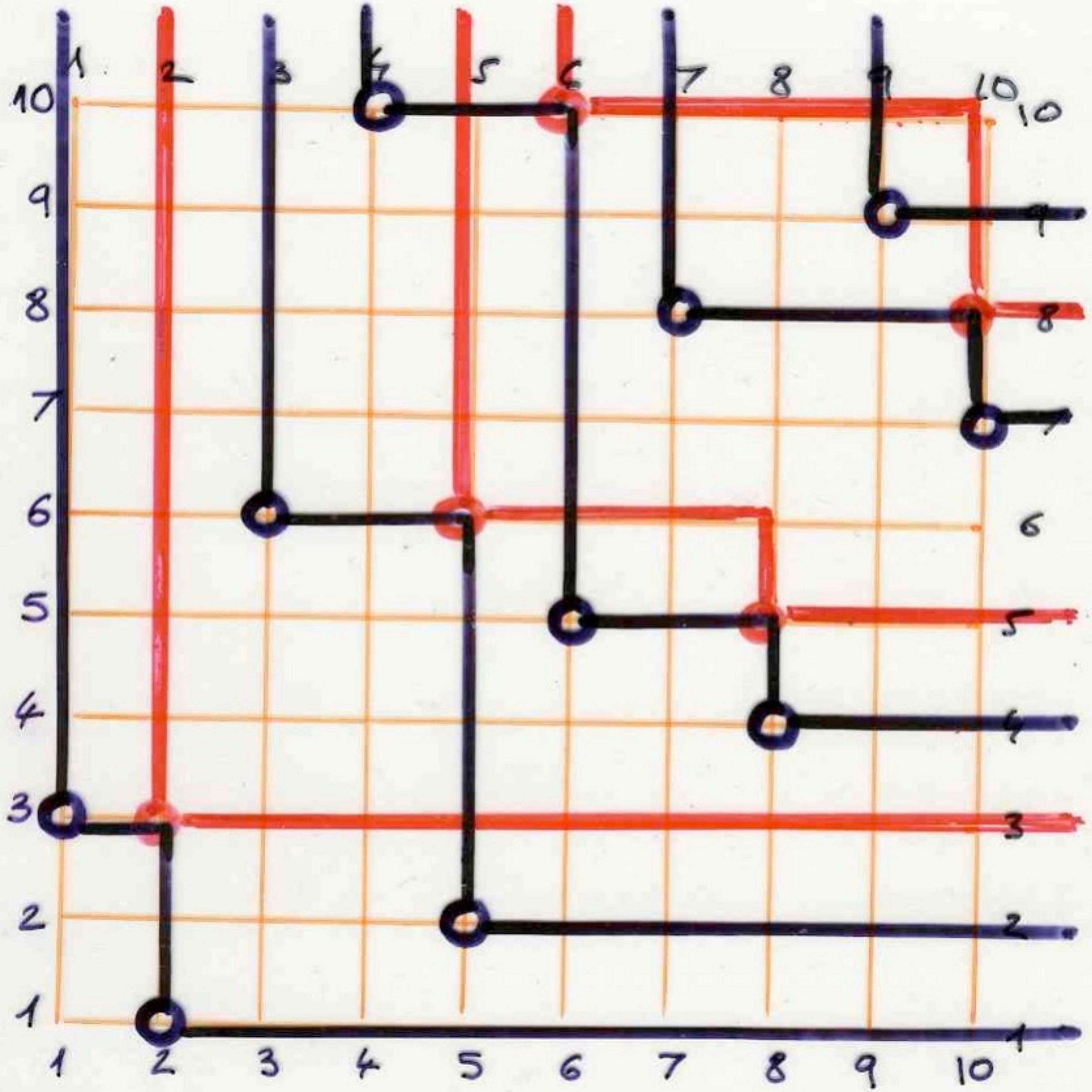
exercise

permutation $\sigma \in \mathfrak{S}_n$

black points $(i, \sigma(i)) \longrightarrow Sq(\sigma)$ squeleton

set of red points

Prove the proposition with the following hint:
if $(x_1, .., x_k)$ is a decreasing sequence of points of $Sq(\sigma)$, construct with "light-shadow" a decreasing sequence of points of $\{(i, \sigma(i)\}$ with $k+1$ elements



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

Erdös, Szekeres (193-)

$\sigma \in \mathfrak{S}_n$ $\qquad n \geqslant N^2$

$\exists$ increasing / decreasing subsequence $|\tau| \geqslant N$

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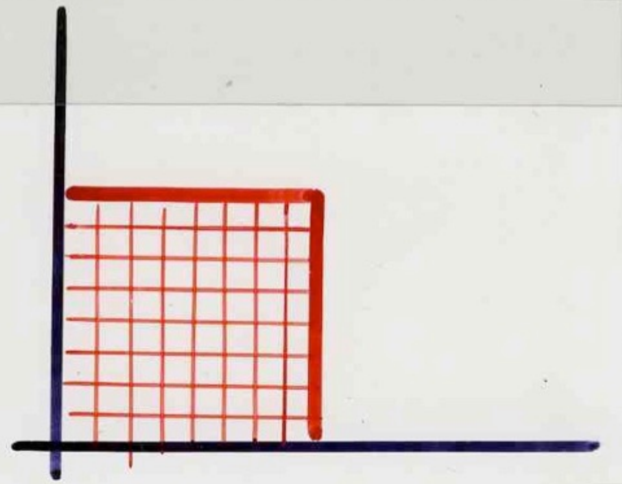

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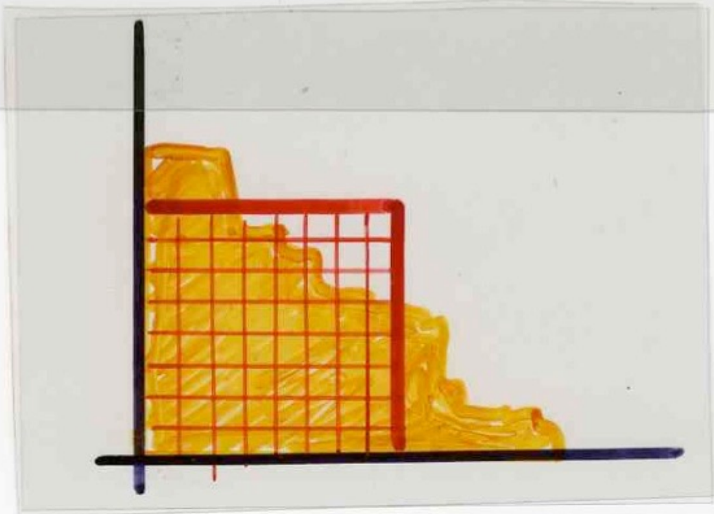
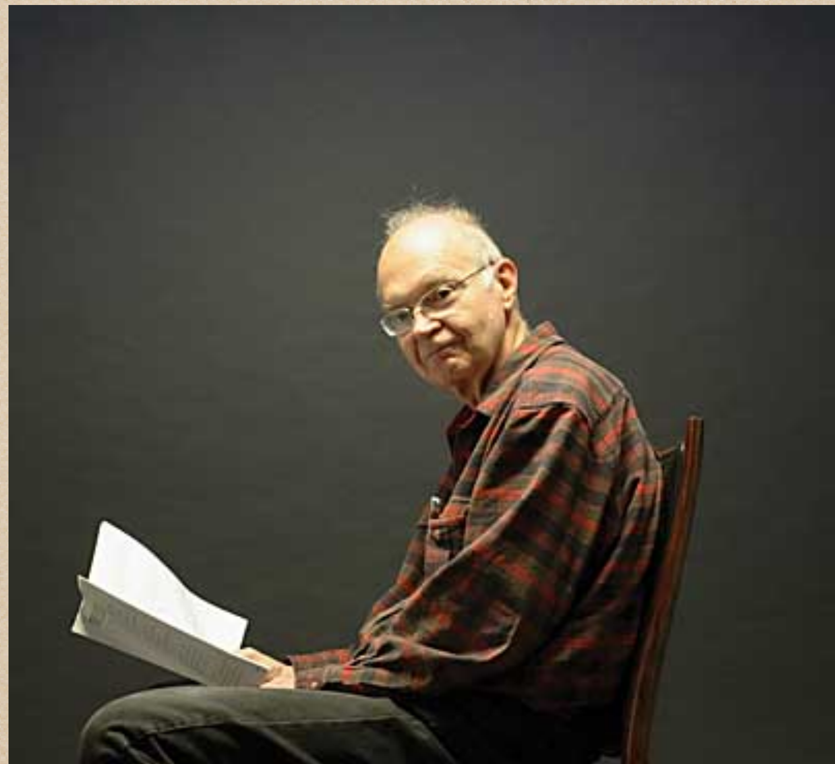


Knuth's transpositions

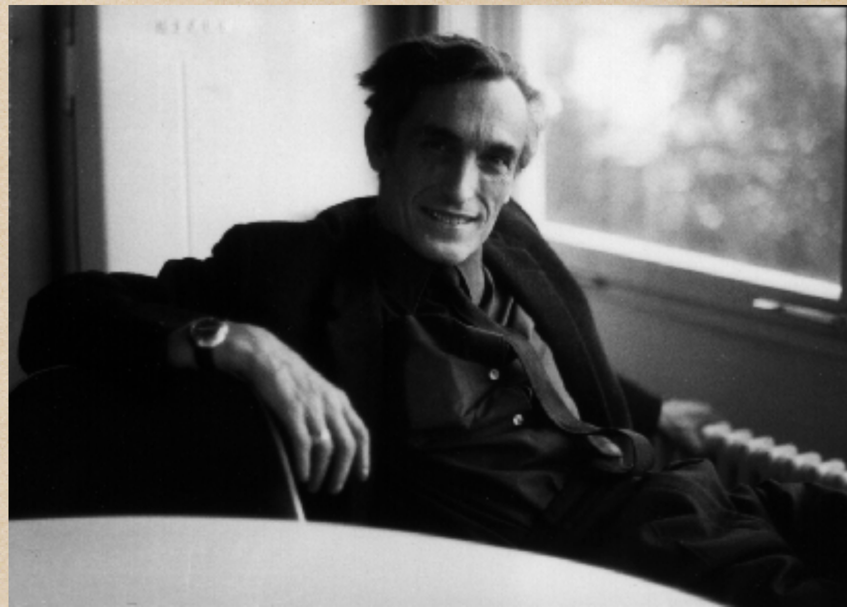
D. Knuth, 1970



see slides Ch4c complements

plactic monoid

A. Lascoux



M.P. Schützenberger

see slides Ch4c
complements

Schur functions

see slides Ch4c
complements