An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc
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Chapter 1
Ordinary generating functions

IMSc

12 January 2016

From the previous lecture

7 January 2016

Operations on combinatorial objects

Def. class of valued combinatorial objects

$\mathcal{A} = (A, v)$ $\quad A$ finite or enumerable set

$$v : \underset{\text{valuation}}{A \longrightarrow \mathbb{K}[X]}$$

(*) for w monomial of $\mathbb{K}[X]$,
let $A_w = \left\{ \begin{array}{l} \alpha \in A, \text{ coeff. of } w \\ \text{in } v(\alpha) \text{ is } \neq 0 \end{array} \right\}$

then for every monomial w,
A_w is finite

$v(\alpha)$ weight or valuation of α

$\{v(\alpha),\ \alpha \in A\}$ is summable

<u>Def.</u> $$f_\alpha = \sum_{\alpha \in A} v(\alpha)$$

generating power series

of objects $\alpha \in A$ weighted by v

$$f_\alpha \in \mathbb{K}[[x]]$$

<u>ex</u>:

$X = \{t\} \cup Y \qquad v(\alpha) = w(\alpha)\, t^n$

$|\alpha| = n$, size of α

is the number of $\alpha \in A$ such that $v(\alpha) = w(\alpha)\, t^n$

$\alpha = (A, v_A)$ $\quad \beta = (B, v_B)$

- **sum** $\qquad \alpha + \beta = \gamma$

$A \cap B = \emptyset$ $\qquad = (C, v_C)$

- $C = A \cup B$ (disjoint union)
- $v_C/_A = v_A \qquad v_C/_B = v_B$

Lemma $\quad f_\gamma = f_\alpha + f_\beta$

- **product** $\mathcal{A} \cdot \mathcal{B} = \mathcal{C}$

$$= (C, v_C)$$

  - $C = A \times B$
  - $(\alpha, \beta) \in C$ $\quad v_C(\alpha, \beta) = v_A(\alpha)\, v_B(\beta)$

ex: "size" $|(\alpha, \beta)| = |\alpha| + |\beta|$

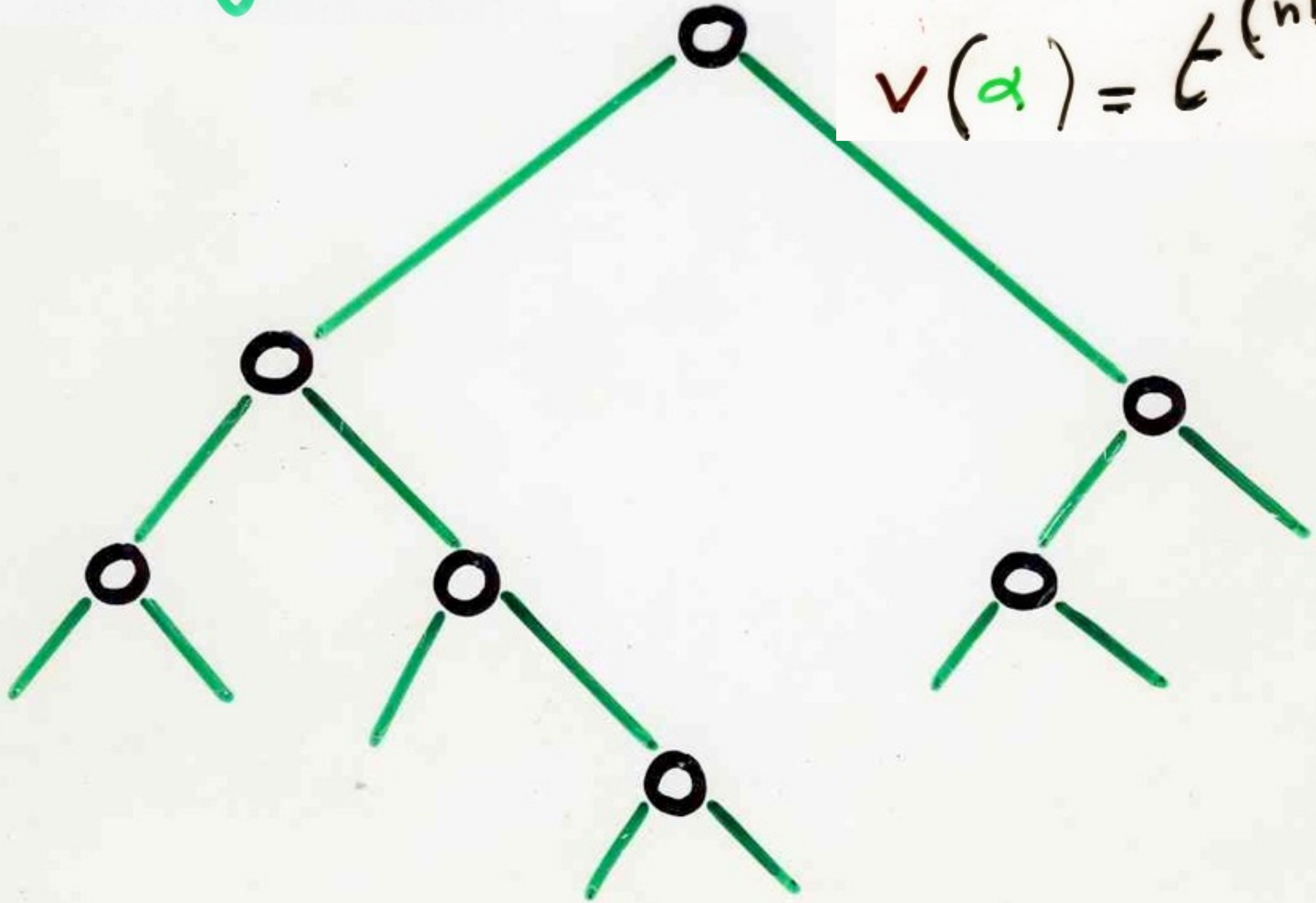
ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

binary tree

$\mathcal{B} = (BT, v)$

$v(\alpha) = t^{(\text{nb of internal vertices})}$



binary tree

$$f_\beta = \sum_{\substack{\alpha \\ \text{binary} \\ \text{trees}}} v(\alpha)$$

$$= \sum_{n \geqslant 0} C_n t^n$$

Catalan numbers

C_n = number of binary trees having n internal vertices

(or $n+1$ leaves = external vertices)

$$\underset{\text{family of binary trees}}{\mathcal{B}} = \{\bullet\} + (\mathcal{B} \times \underset{\text{root}}{\bullet} \times \mathcal{B})$$

$$y = 1 + t\,y^2$$

$$y = f_{\mathcal{B}}$$

algebraic equation

• sequence

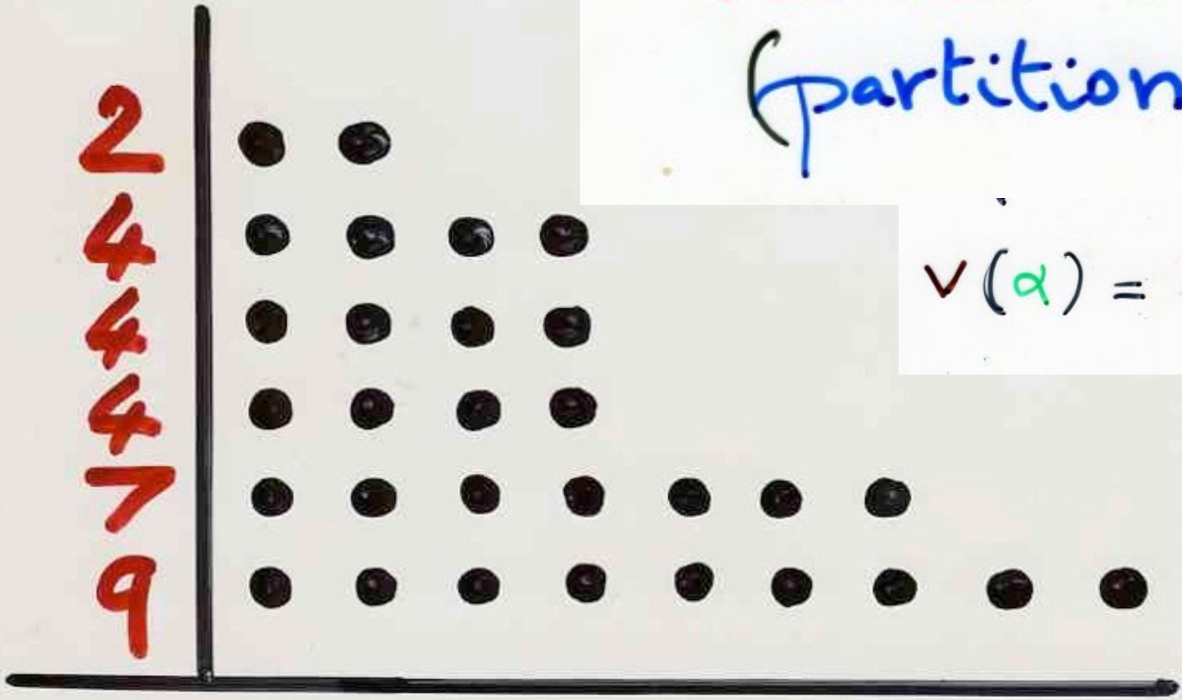
$\mathcal{A} = (A, v_A)$ $\qquad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \dots + \mathcal{A}^n + \dots$$
$$= \mathcal{A}^*$$

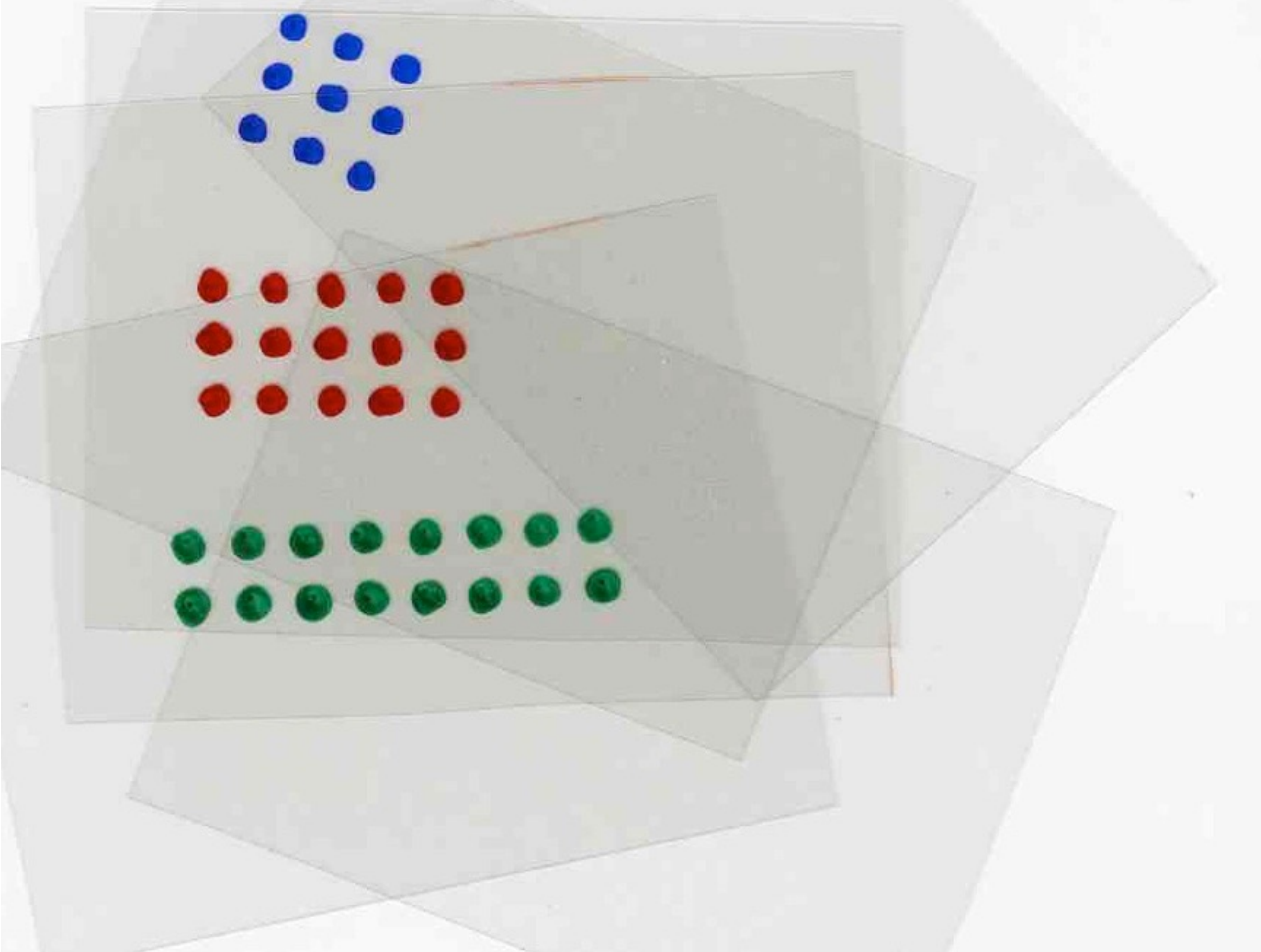
Lemma $\displaystyle f_{\mathcal{A}^*} = \frac{1}{1 - f_{\mathcal{A}}}$

Ferrers diagram (partitions of integers)

$$v(\alpha) = q^{(\text{nb of cells of } \alpha)}$$



$$30 = 2+4+4+4+7+9$$



$$\underbrace{\bullet\ \bullet\ \bullet\ \bullet\ \bullet}_{i} \qquad\qquad 9^{i}$$

$$\prod_{i \geq 1} \frac{1}{(1-q^{i})}$$

generating function for the number of partitions of an integer n

operations on combinatorial objects

derivative

$\mathcal{A} = (A, v_A)$ class of weighted combinatorial objects satisfying $(*)$ with valuation v of the type

$$v_A(\alpha) = w_A(\alpha)\, t^{|\alpha|}$$

$$A_n = \{\alpha \in A,\ v(\alpha) = w(\alpha)\, t^n\}$$

$|\alpha| = n$ size of $\alpha \in A$

<u>Definition</u> $\mathcal{C} = \mathcal{A}^\bullet$ class of pointed objects

$\mathcal{C} = (C, v_C)$ with

- $C = \biguplus A_n \times [1, n]$ (disjoint union)
- $v_C(\gamma) = v_A(\alpha)$ for $\gamma = (\alpha, i)$ with $1 \leqslant i \leqslant |\alpha| = n$

Lemma

$$\mathcal{L}\,a^{\bullet} = t\,\frac{d}{dt}\,\mathcal{L}\,a$$

Proof

$$f_\alpha = \sum_{\alpha \in A} w_A(\alpha)\, t^{|\alpha|}$$

$$= \sum_{n \geq 1} \sum_{\substack{\alpha \in A \\ |\alpha| = n}} w_A(\alpha)\, t^{n}$$

$$f_{\alpha^\bullet} = \sum_{\substack{\gamma = (\alpha, i) \\ 1 \leq i \leq n = |\alpha|}} w_A(\alpha)\, t^{|\alpha|}$$

$$= \sum_{n \geq 1} \sum_{\substack{\alpha \in A \\ |\alpha| = n}} n\, w_A(\alpha)\, t^{n}$$

$$f_{\alpha^\bullet} = t \frac{d}{dt} f_\alpha$$

operations on combinatorial objects

derivative

example:
heaps of dimers

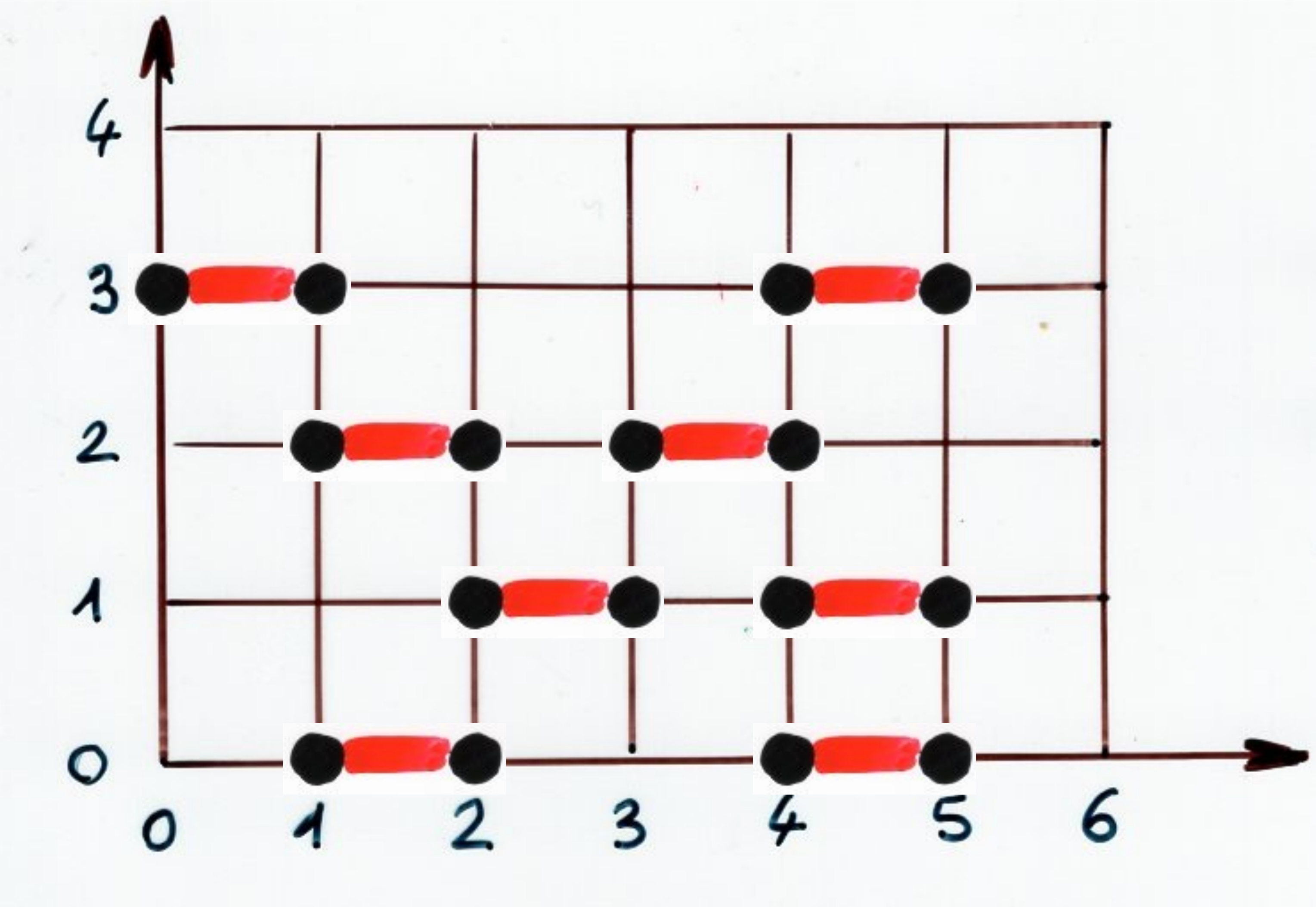
Def Heap of dimers on $\mathbb{N}$

finite set E of horizontal edges (or dimers) of the lattice $\mathbb{N}\times\mathbb{N}$ such that

(i) they are 2 by 2 disjoints

(ii) if $((i,k),(i+1,k)) \in E$ with $k \geq 1$, then $\exists$ a dimer $((j,k-1),(j+1,k-1))$ with $j = i-1$ or i or $i+1$

k is the level of the dimer



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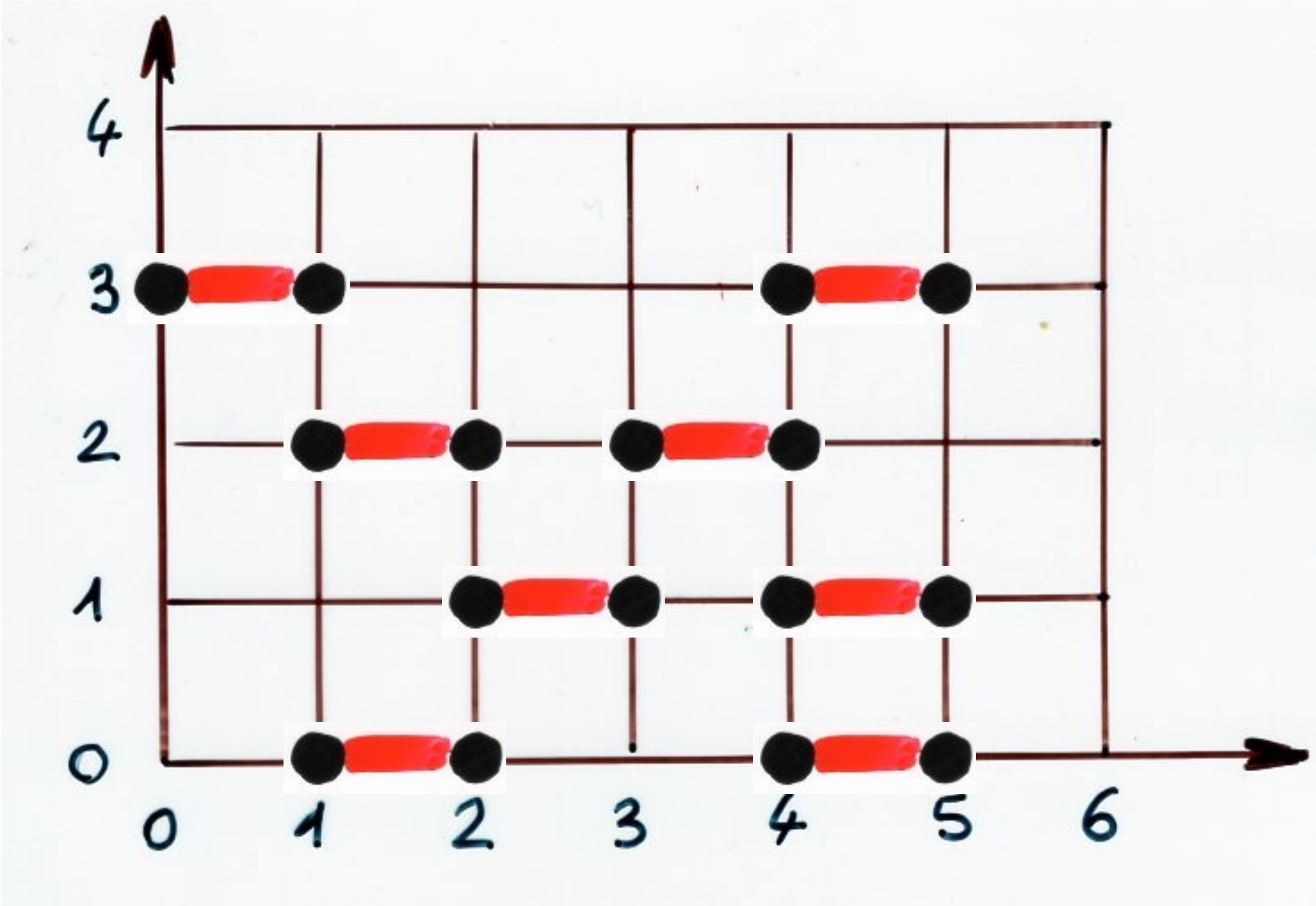

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the weight of the dimer $d = ((i, k), (i+1, k))$
is of the form: $v(d) = w(i)t$ $\quad w(i) \in \mathbb{K}[x]$

k is the level of the dimer

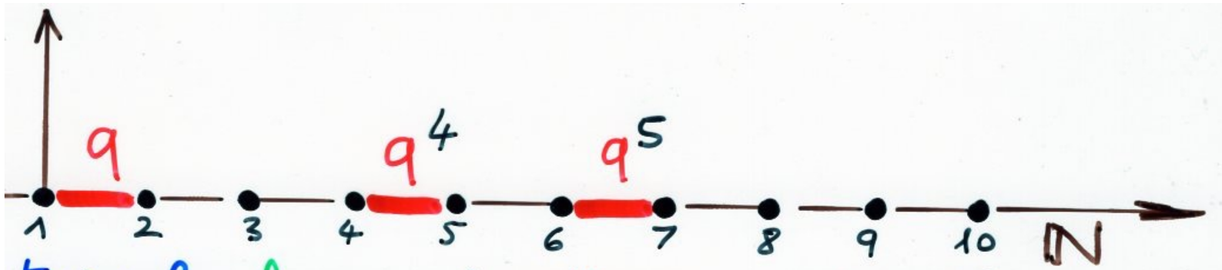
the weight of the heap E is

$$v(E) = \prod_{d \in E} v(d)$$

Suppose $y = \sum_{\substack{E \\ \text{heaps}}} v(E)$ is well defined

for example:

- $w(i) = 0$ for $i \geq k$
  (heaps on the segment $[0, k]$)
- $w(i) = q^{i}$



trivial heap of dimers on $\mathbb{N} - \{0\}$

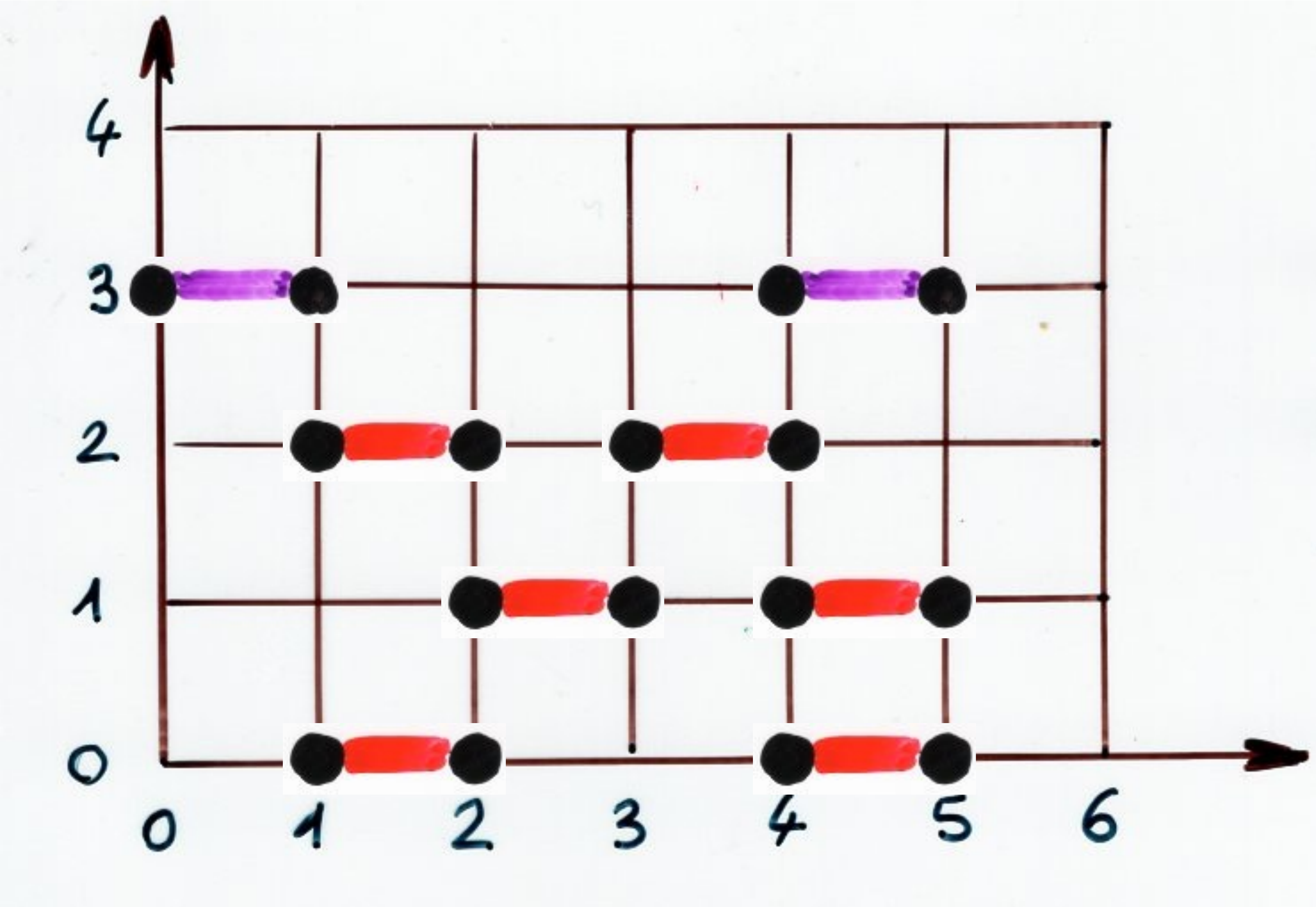
$=$ **D**- partition

logarithmic Lemma

$$t \frac{d}{dt} \log \Big(\sum_{\substack{E \\ \text{heap}}} v(E) \Big) = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

Def Pyramid is a heap having only one maximal piece

Def A maximal piece (dimer) of a heap E is a dimer $((i, k), (i+1, k))$ such that there is no dimer of E $((j, \ell)(j+1, \ell))$ with
$j = i-1, i$ or $i+1$ and $\ell > k$.



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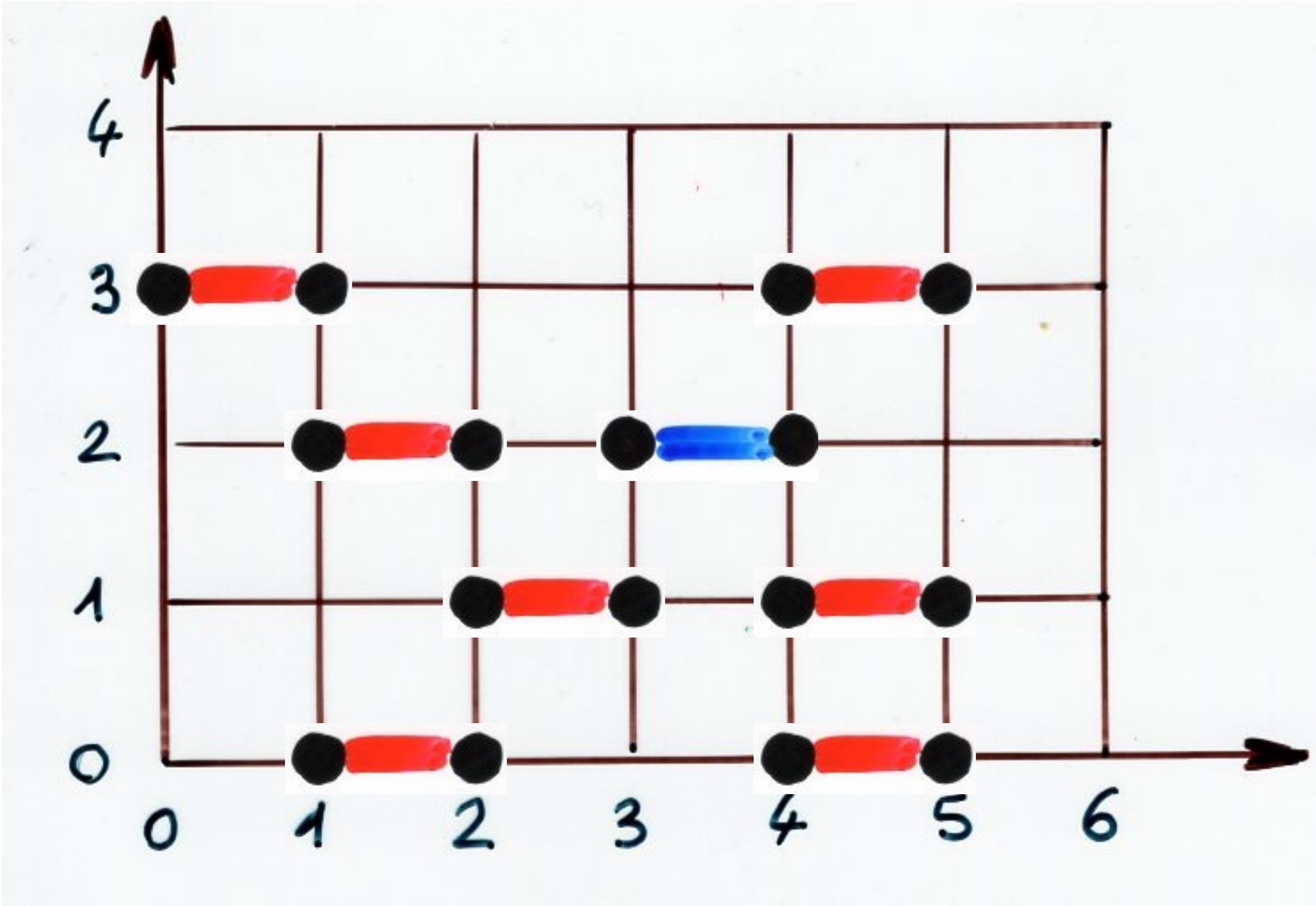
proof of the logarithmic lemma for heaps of dimers



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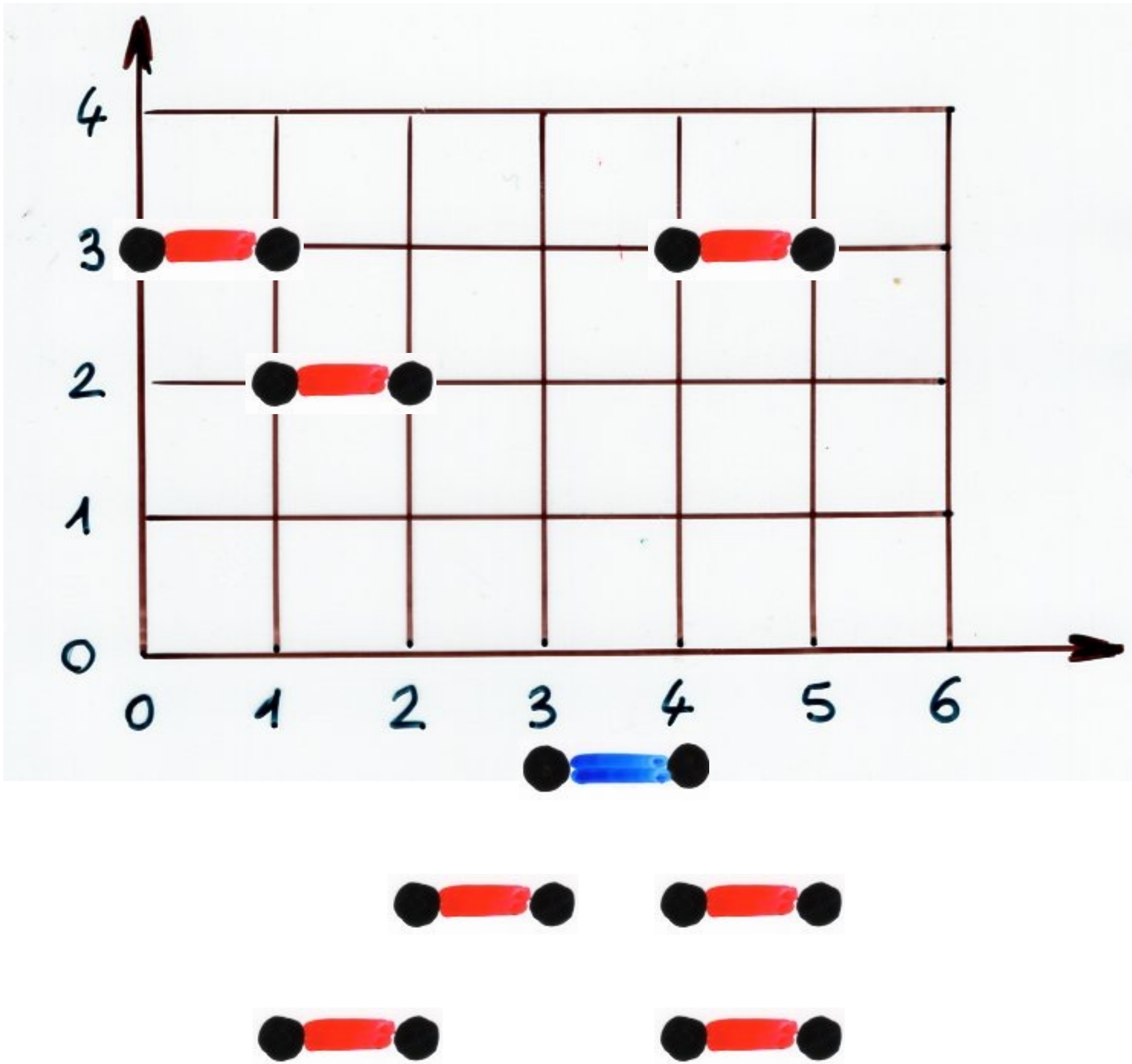
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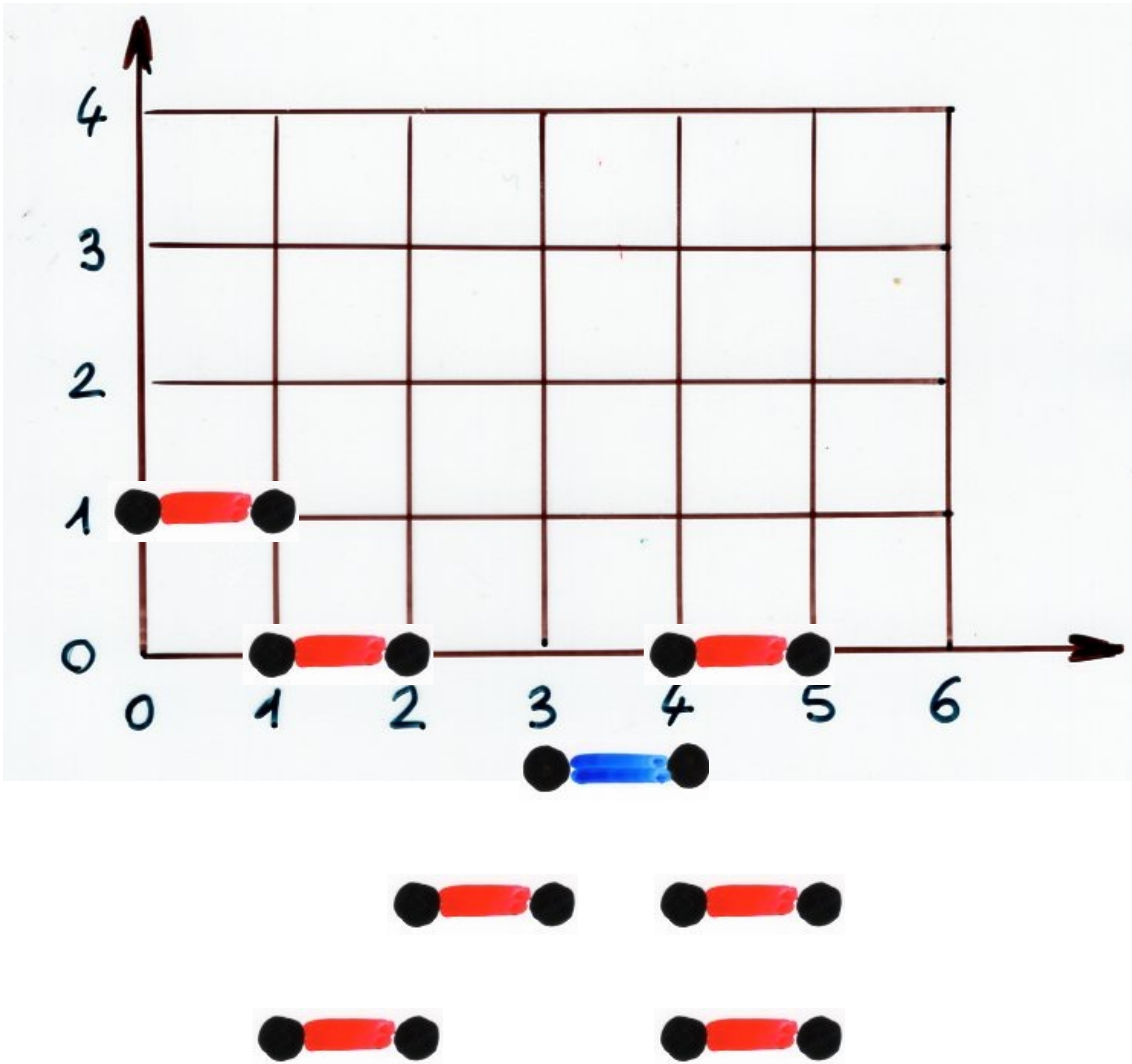
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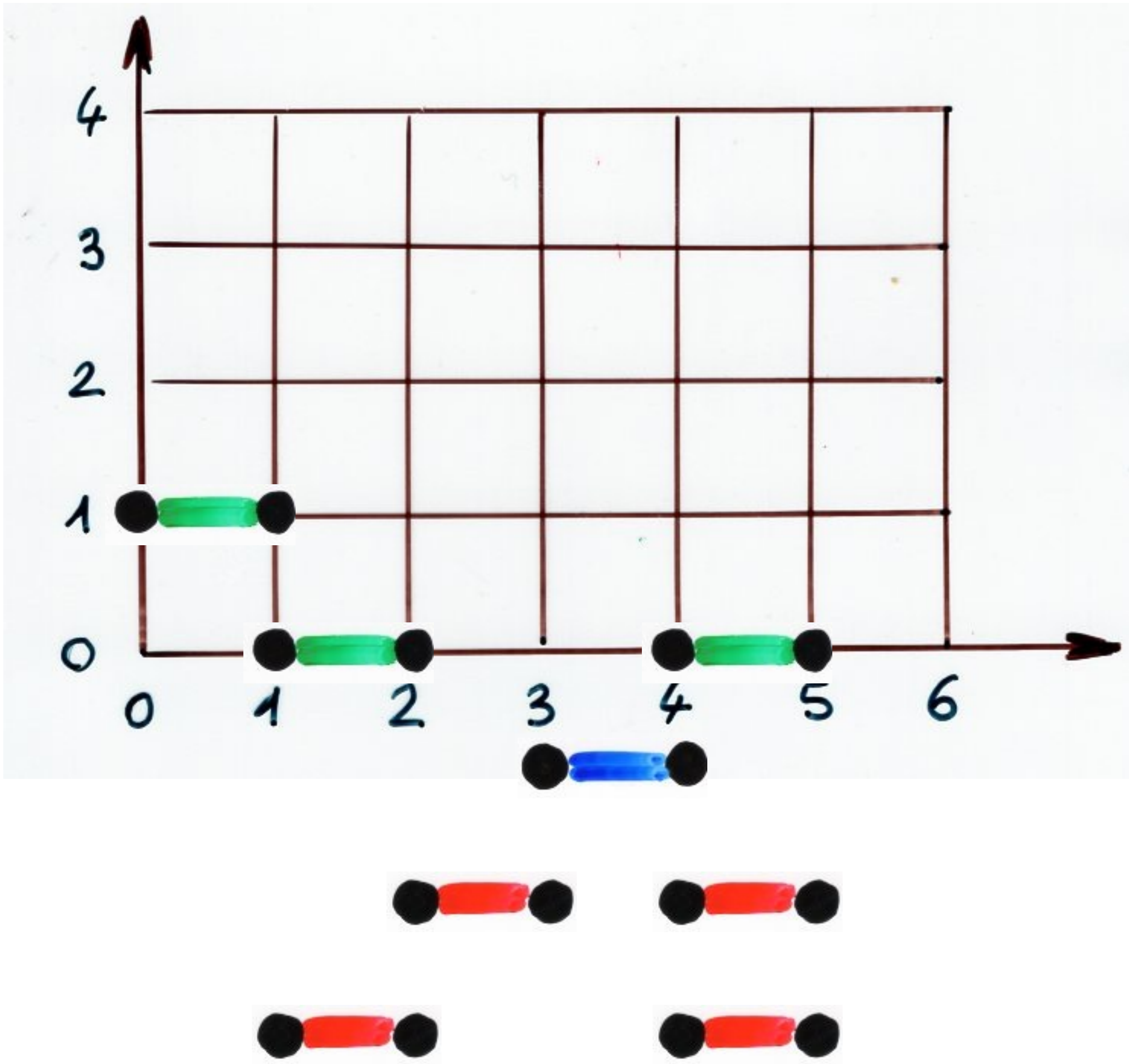
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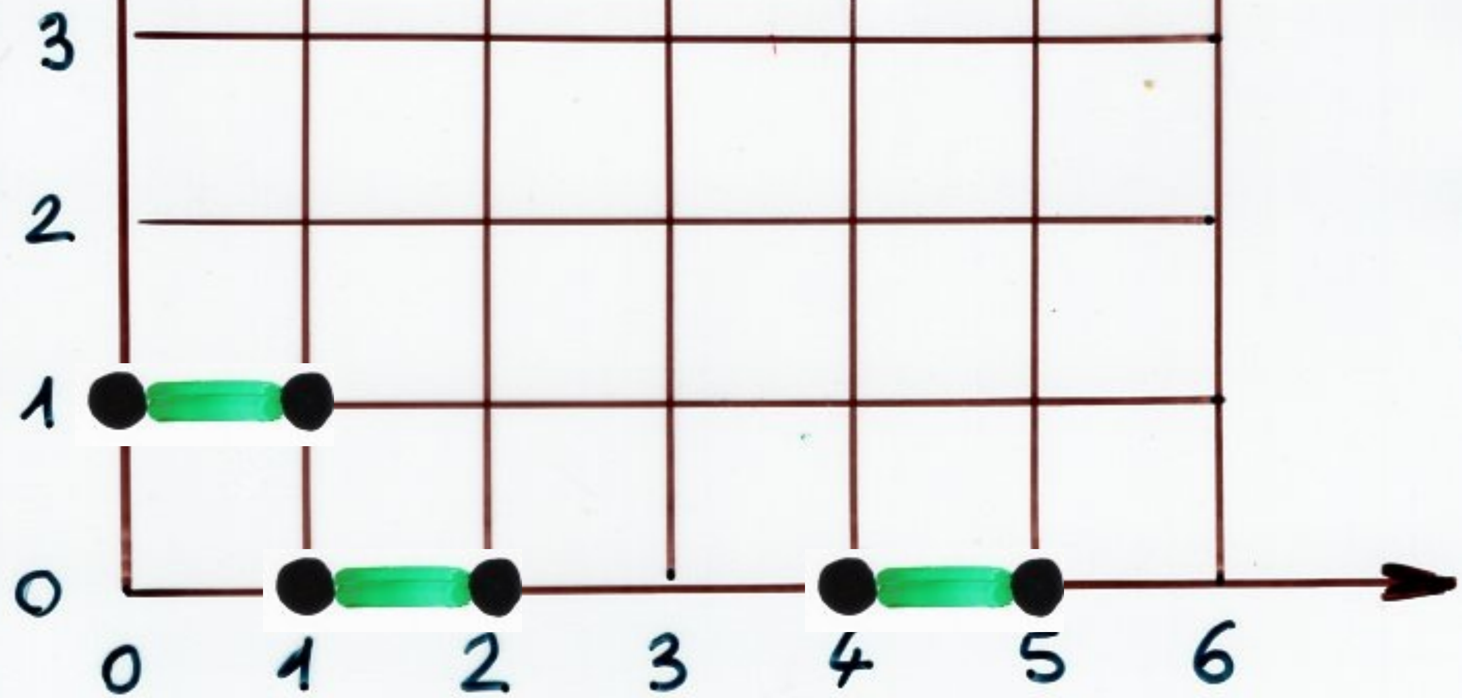
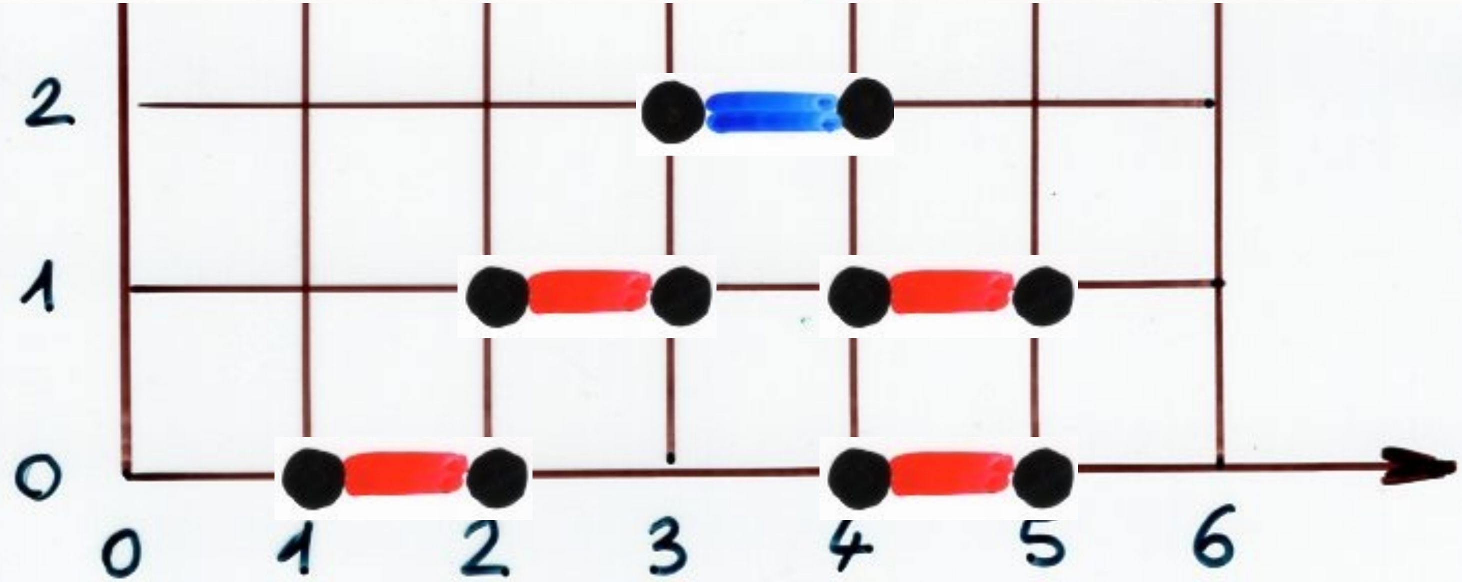


Pointed heap = Pyramid × heap



Pointed heap = Pyramid $\times$ heap

$$t\,y' = z\,y$$

$$z = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

$$\frac{t\,y'}{y} = z$$

logarithmic Lemma

$$t\frac{d}{dt}\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\Big) = \sum_{\substack{P\\ \text{pyramid}}} v(P)$$

$\square$

heaps of pieces :

much more general theory

- Cartier-Foata commutation monoid
- trace monoid in computer science
- visualization with heaps
  (→ course next year)

heap (of pieces) = "empilement" (de pièces)

D. Knuth (2015)
Vol 4, Fascicle 6
"Satisfiability"
The Art of Computer Programming

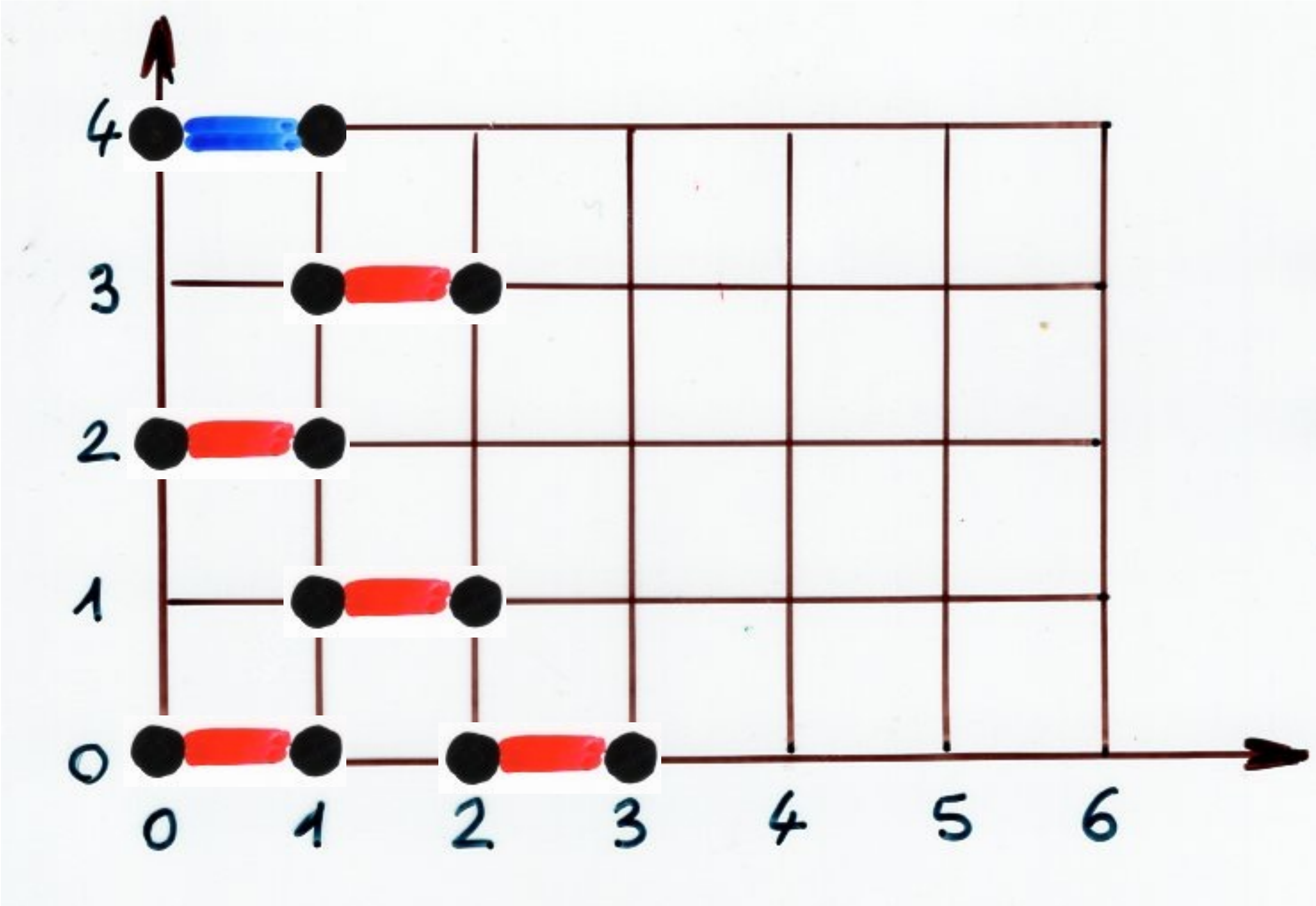
exercises

pyramids and algebraic generating functions

<u>exercise</u> semi-pyramid of dimers on $\mathbb{N}$

the unique maximal piece has projection $[0,1]$

prove that the number of such semi-pyramid with n dimers is the Catalan number C_n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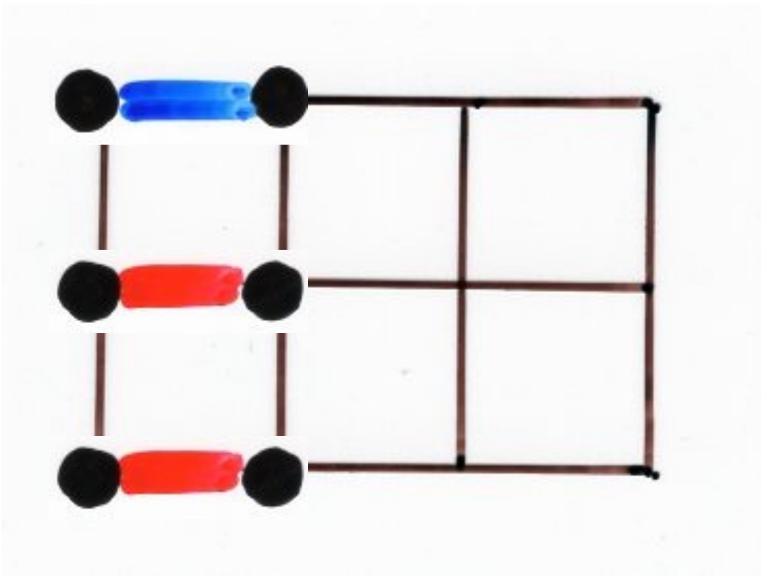
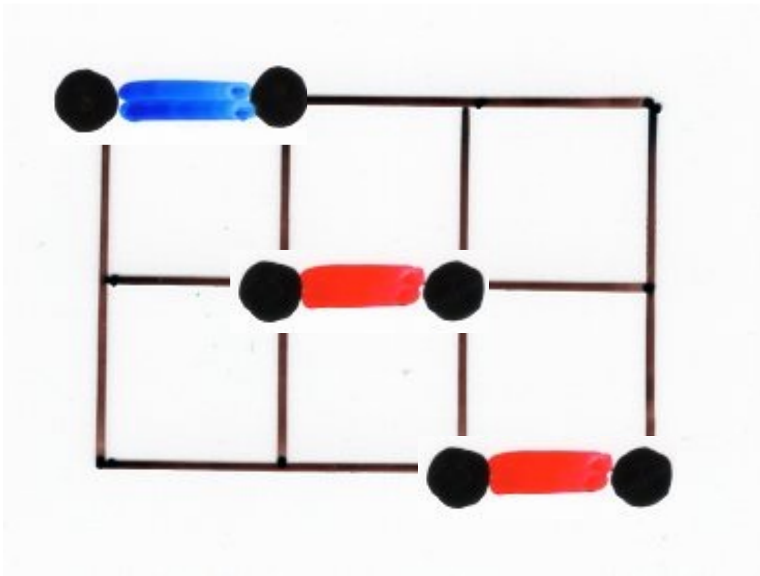
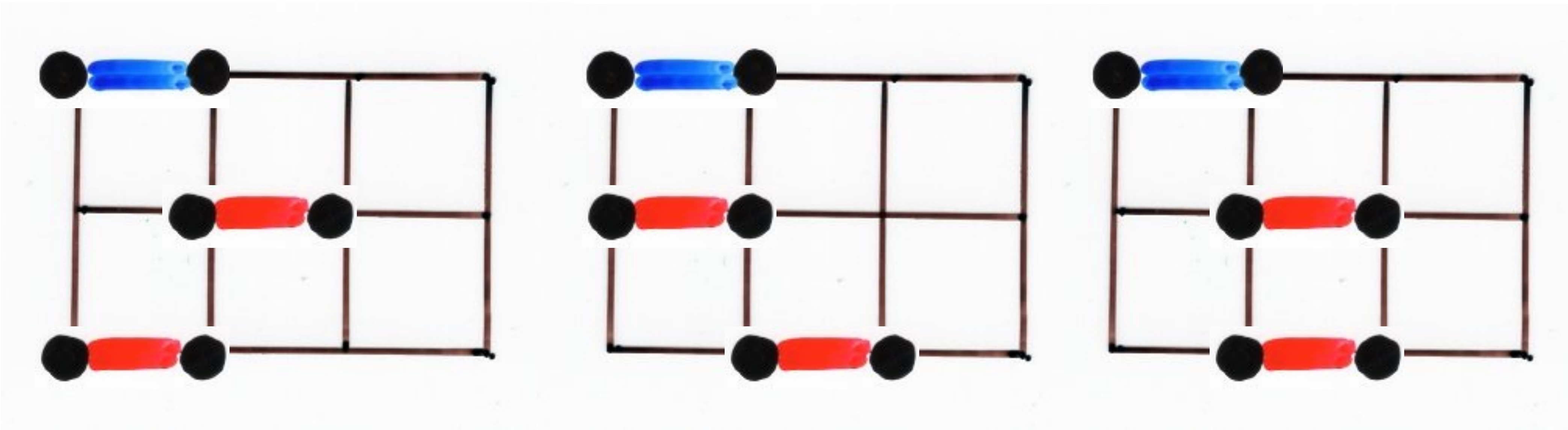






$C_3 = 5$

exercise
(more difficult)

Pyramid of dimers on $\mathbb{N}$ up to a translation enumerated by $\frac{1}{2}\binom{2n}{n}$

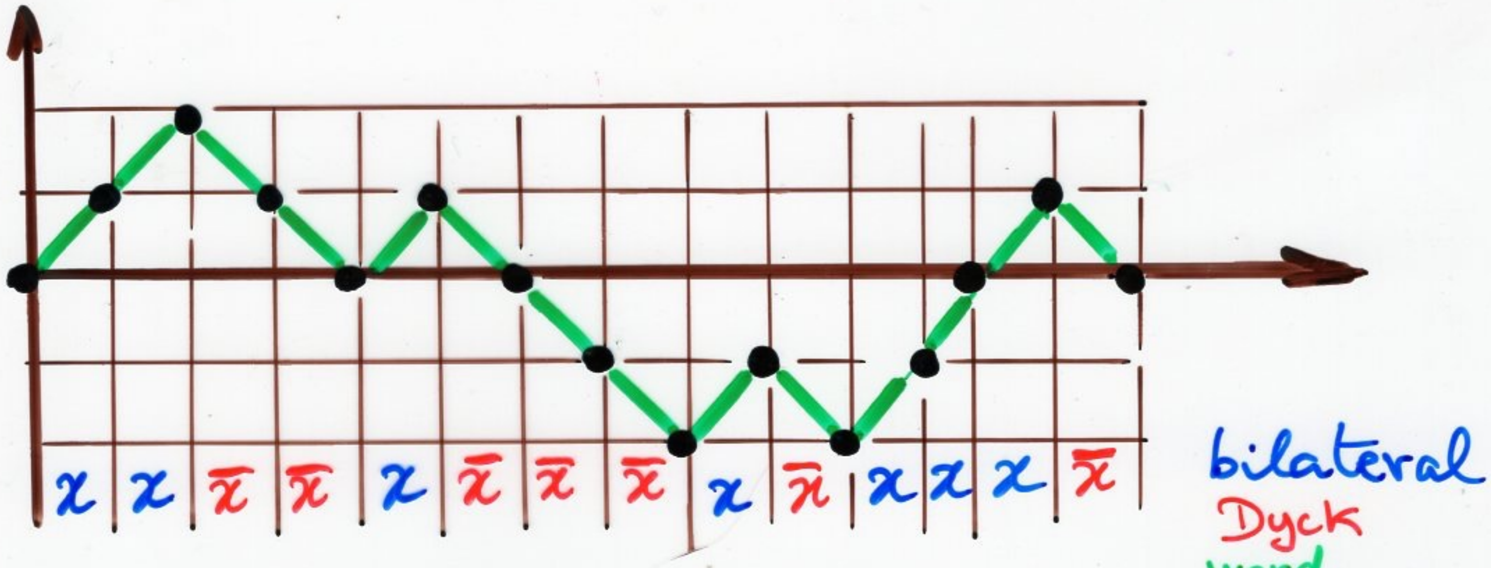
bilateral Dyck paths

(next exercise)

same algebraic system of equations than bilateral Dyck paths

or bijection ?

bilateral Dyck path



number of bilateral Dyck paths ω of length $2n$ $= \binom{2n}{n}$

exercise

bilateral Dyck paths

- find an algebraic system of equations satisfied by the generating function

- deduce that $y = \dfrac{1}{\sqrt{1-4t}}$

- go back to execise on pyramids of dimers $\frac{1}{2}\binom{2n}{n}$

operations on combinatorial objects

substitution

example:
Strahler number of binary trees

$\mathcal{A} = (A, v_A) \qquad \mathcal{B} = (B, v_B)$

$$v_A(\alpha) = w_A(\alpha)\, t^{|\alpha|}$$

composite class $\quad \mathcal{C} = \mathcal{A}(\mathcal{B})$

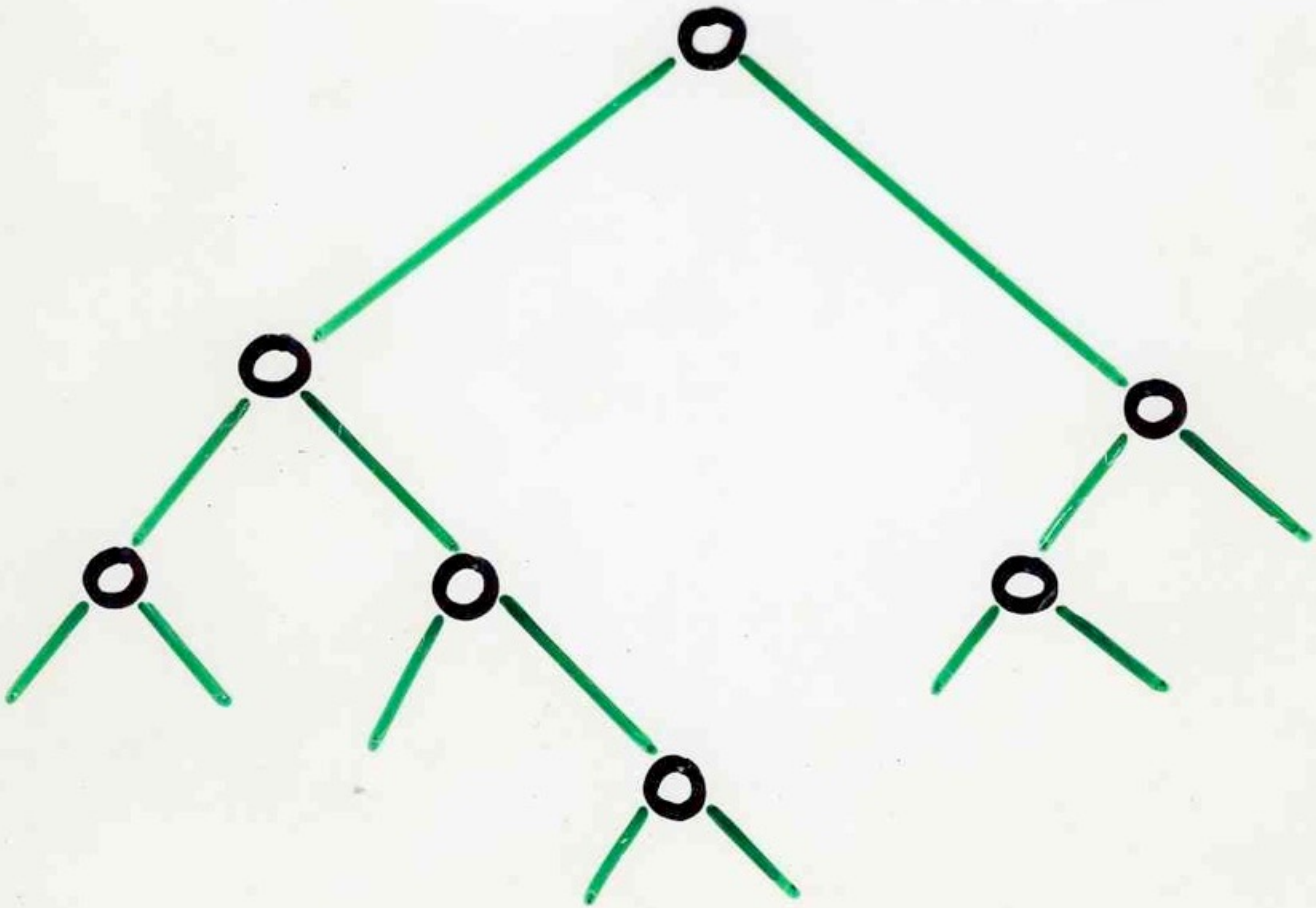
$$C = \sum_{n \geq 0} A_n \times B^n \qquad\qquad A_n = A_{t^n}$$

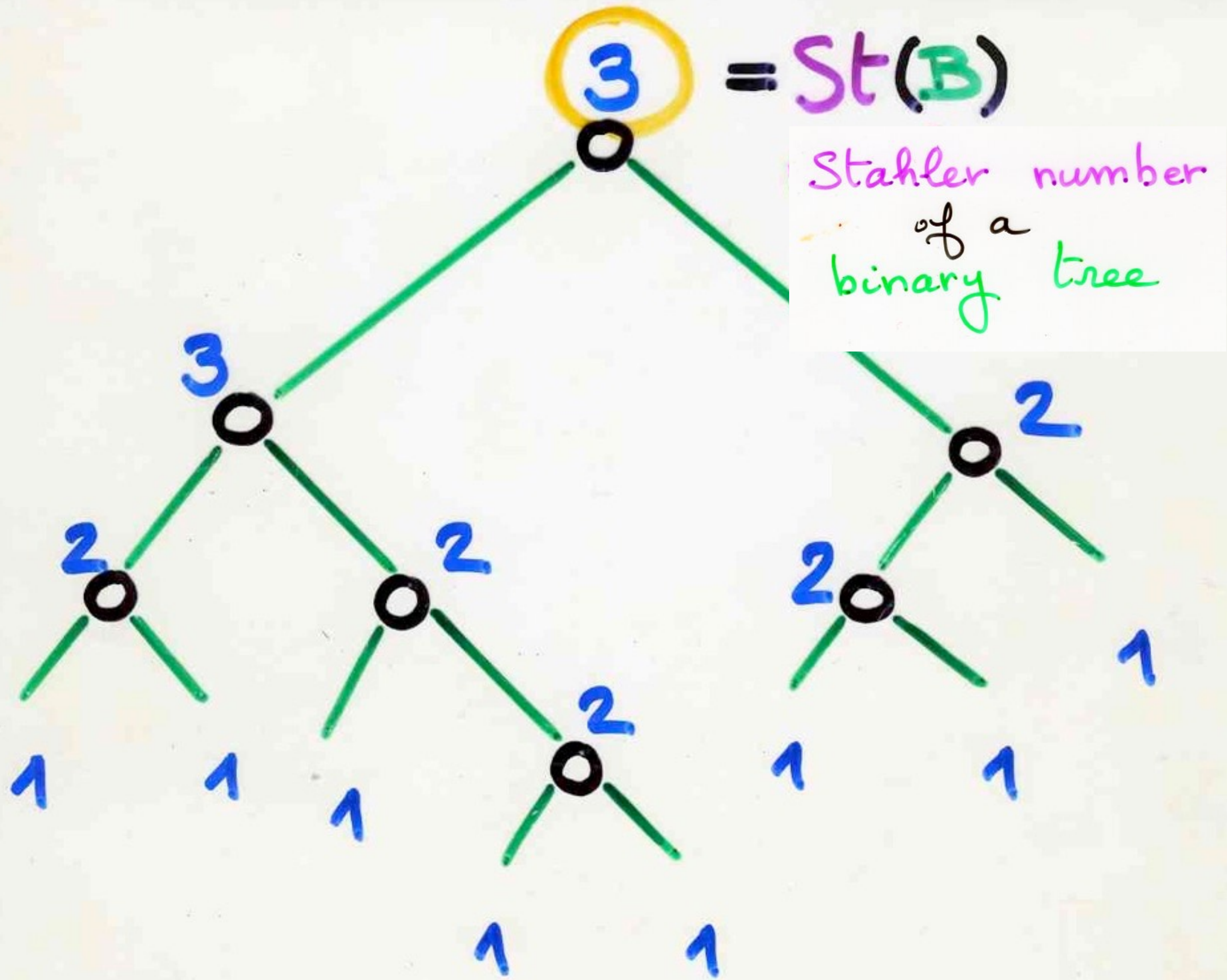
$$\gamma = (\alpha; \beta_1, \ldots, \beta_n) \in A \times B^n$$

$$v_C(\gamma) = w_A(\alpha)\, v_B(\beta_1) \cdots v_B(\beta_n)$$

- composition (substitution)

Lemma $\quad f_{\mathcal{C}} = f_{\mathcal{A}}(f_{\mathcal{B}})$




3 = St(B)
Stahler number
of a
binary tree
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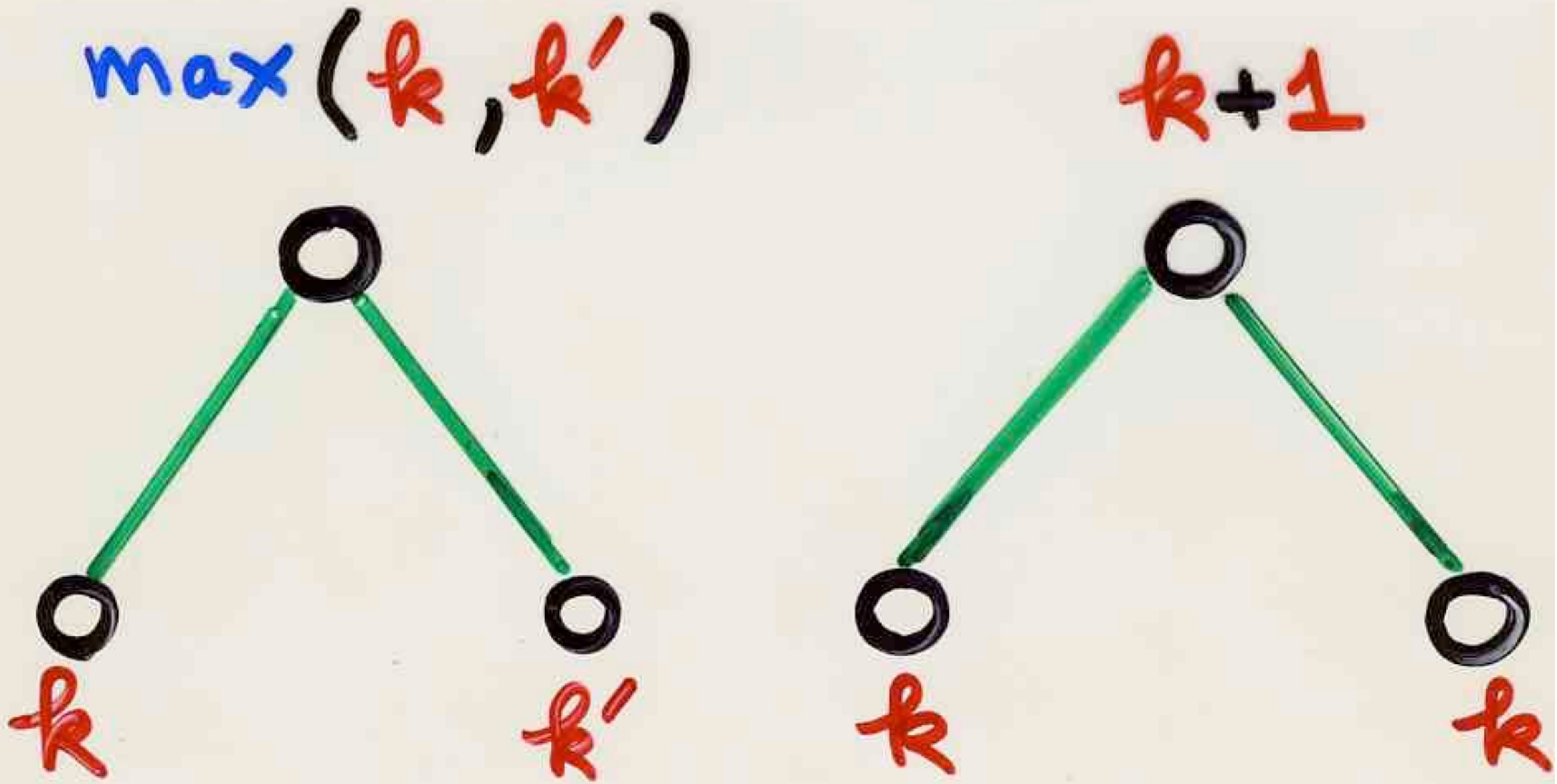


max(k, k')
k+1
k
k'
k
k

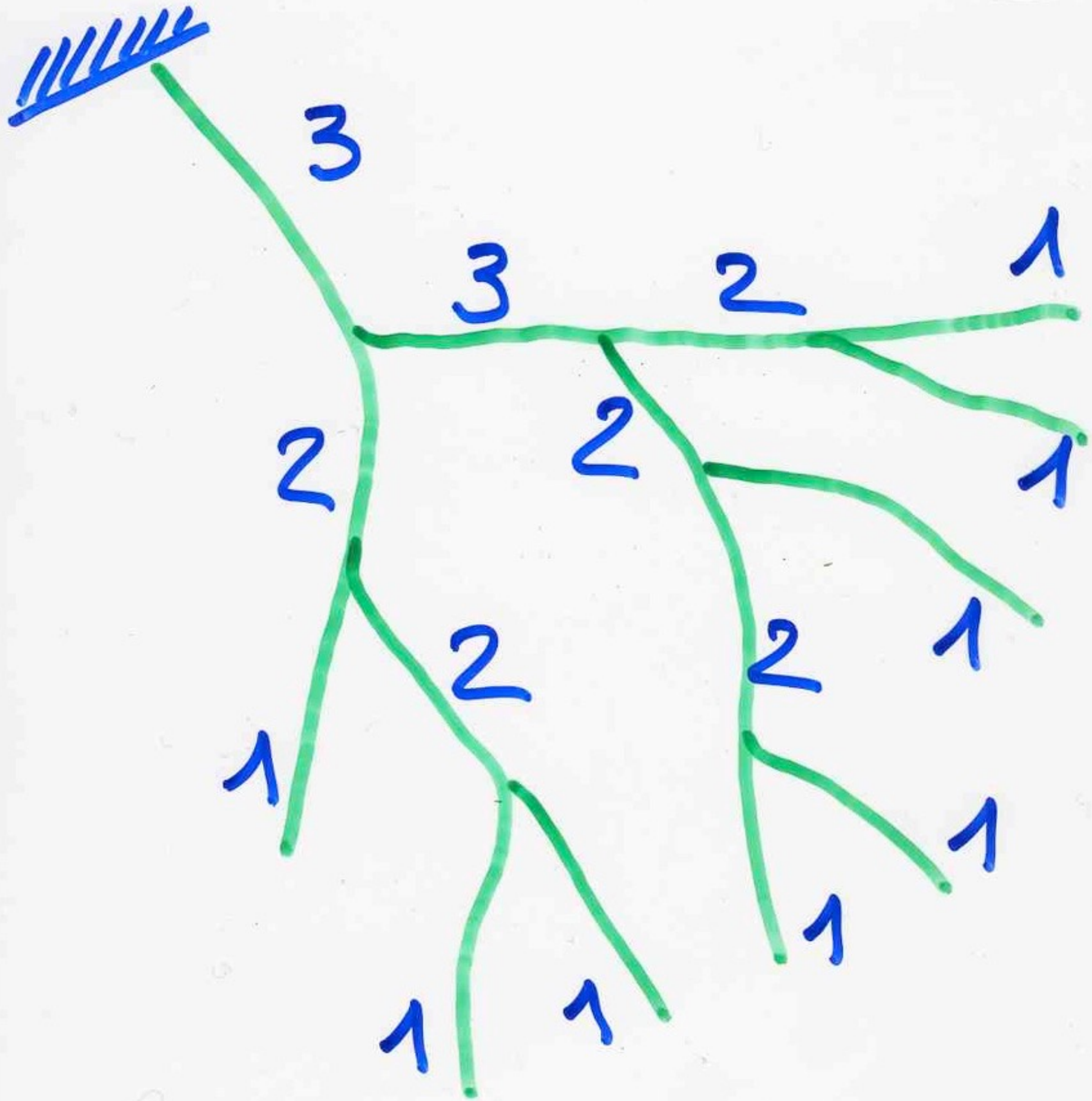
Horton (1945)

Strahler (1952)

Hydrogeology

morphology of rivers network

Order of a river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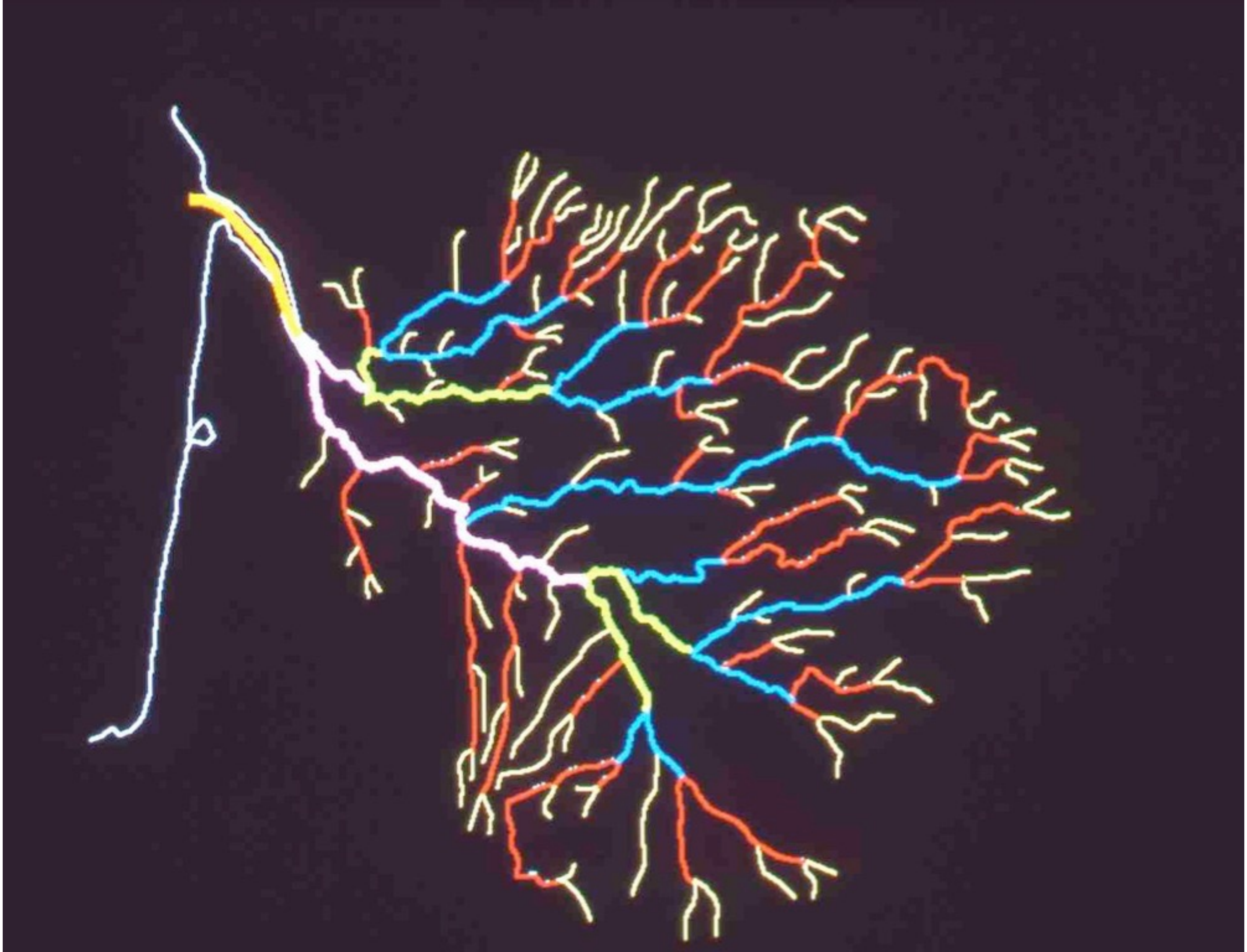


$$S_{n,k} = \left\{ \begin{array}{l} \text{number of binary trees } B \\ \text{with } n \text{ internal vertices} \\ \text{and } \quad St(B) = k \end{array} \right.$$

$$S_k(t) = \sum_{n \geq 0} S_{n,k}\, t^n$$

$$S_{k+1}(t) = t\, S_k^2(t) + 2t\, S_{k+1}(t) \left[\sum_{1 \leq i \leq k} S_i(t) \right]$$

experimental combinatorics

SAGE

MAPLE

.....

O. E. I. S.

The Online Encyclopedia of Integer Sequences

$$S_1 = 1$$

$$S_2 = \frac{t}{1-2t}$$

$$S_3 = \frac{t^3}{1-6t+10t^2-4t^3}$$

$$S_4 = \frac{t^7}{1-14t+78t^2-220t^3+330t^4-252t^5+84t^6-8t^7}$$

1, 3, 21, 987, ...

O. E. I. S.
The Online Encyclopedia
of Integer Sequences

1, 2, 3, 5, 8, 13, 21, 34, 55,

89, 144, 233, 377, 610, 987,

1, 2, 3, 5, 8, 13, 21, 34, 55,
F_1 F_3 F_7

89, 144, 233, 377, 610, 987,
F_{15}

Fibonacci !

F_{2^k-1}

$$S_1 = 1$$

$$S_2 = \frac{t}{1-2t}$$

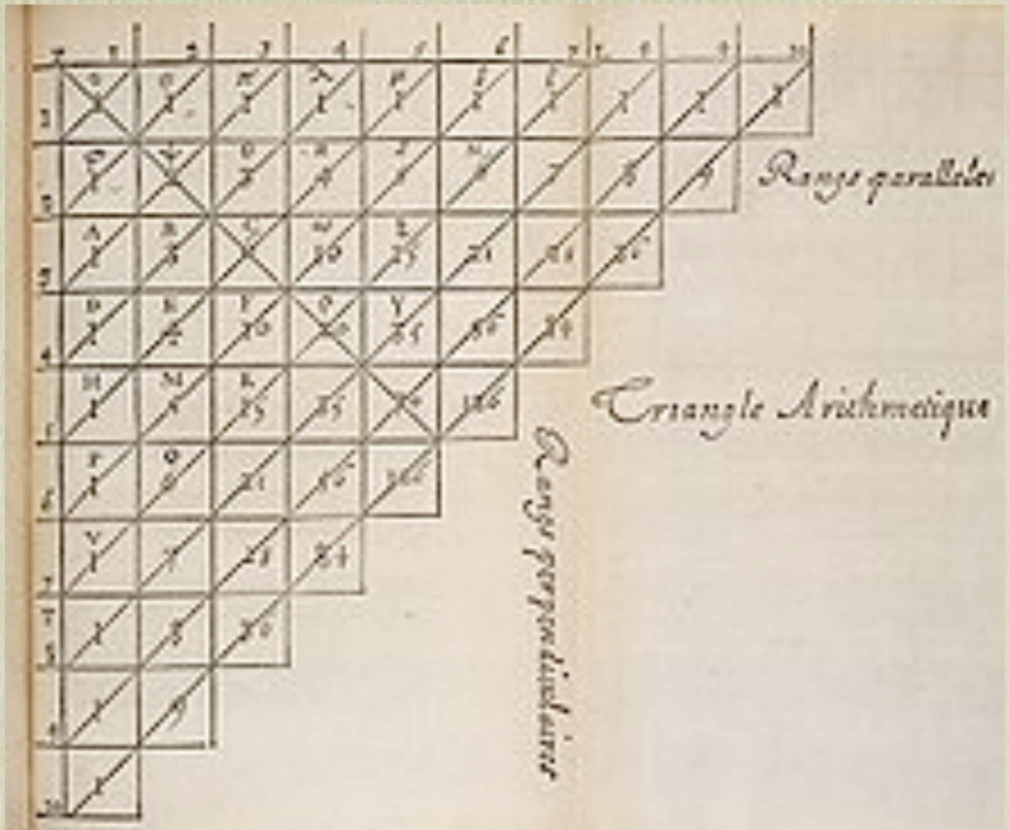
$$S_3 = \frac{t^3}{1-6t+10t^2-4t^3}$$

$$S_4 = \frac{t^7}{1-14t+78t^2-220t^3+330t^4-252t^5+84t^6-8t^7}$$

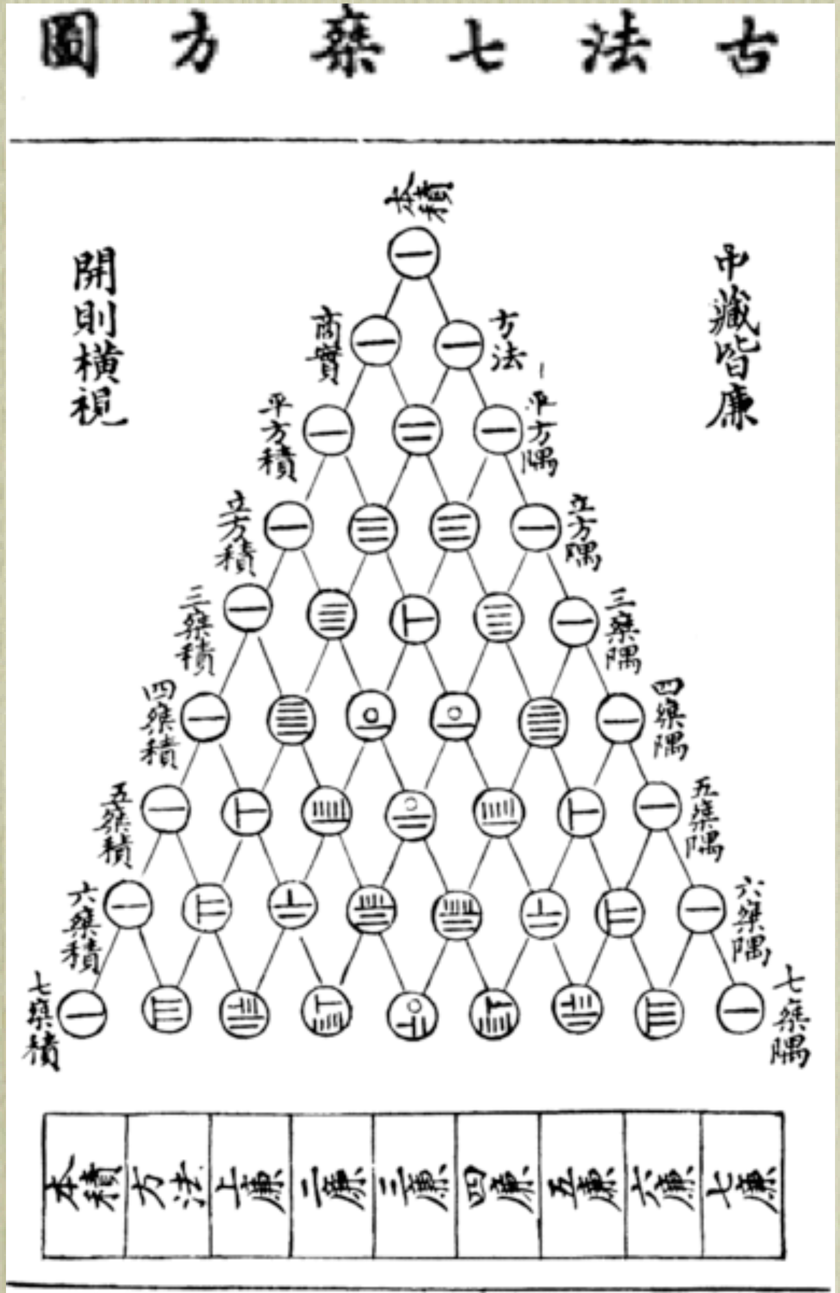
Pascal triangle

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1
1	10	45	120	210	252	210	120	45	10
1	11	55	165	330	462	462	330	165	55
1	12	66	220	495	792	924	792	495	220
1	13	78	286	715	1287	1716	1716	1287	715
1	14	91	364	1001	2002	3003	3432	3003	2002
1	15	105	455	1365	3003	5005	6435	6435	5005



Pascal triangle
binomial coefficients



Yang Hui triangle
(11th, 12th century)

in Persia
Omar Khayyam
(1048-1131)

in India
Chandas Shastra by Pingala
2nd century BC

relation with Fibonacci numbers
(10th century or earlier ?)

addition +

	1								
1	1	1							
1	1	2	1						
2	1	3	3	1					
3	1	4	6	4	1				
5	1	5	10	10	5	1			
8	1	6	15	20	15	6	1		
13	1	7	21	35	35	21	7	1	
21	1	8	28	56	70	56	28	8	1

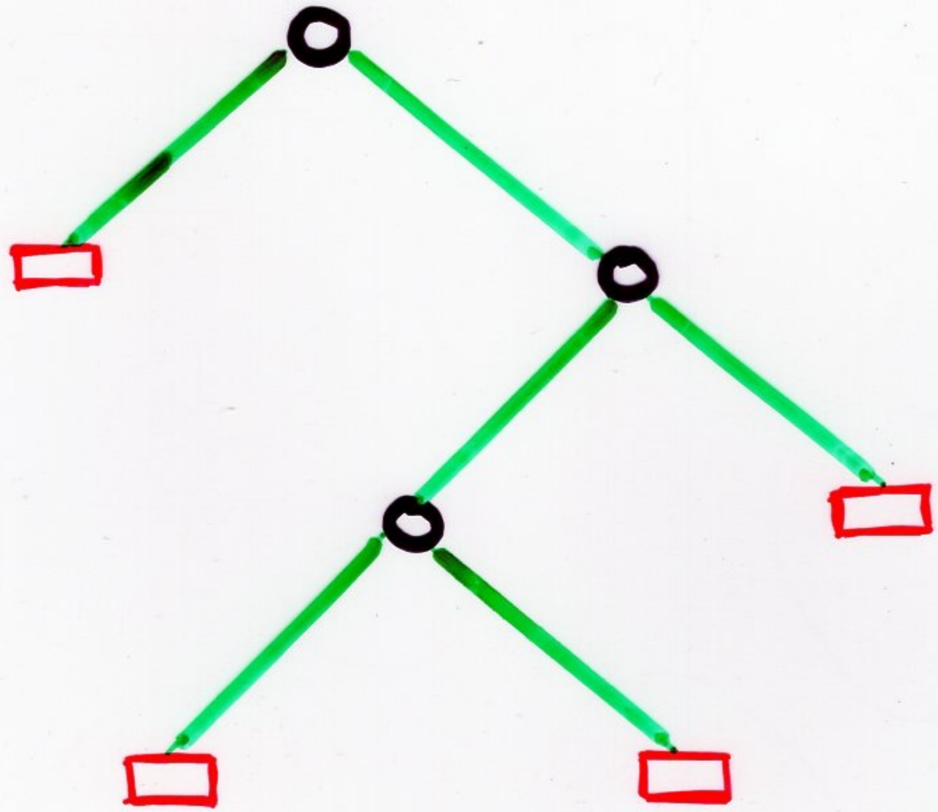
$$S(t,x) = \sum_{k\geq 0} S_k(t)\, x^k$$

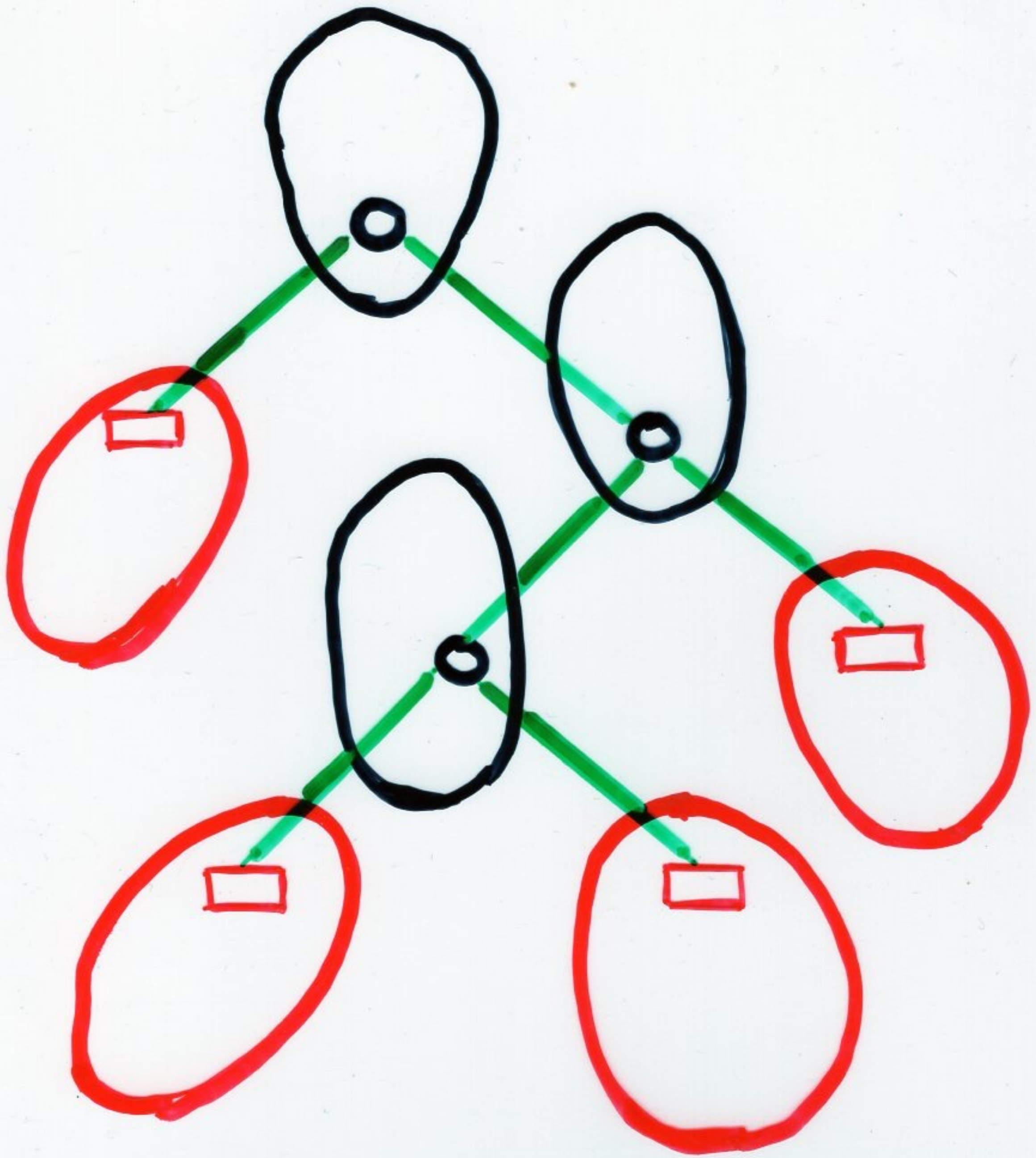
$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

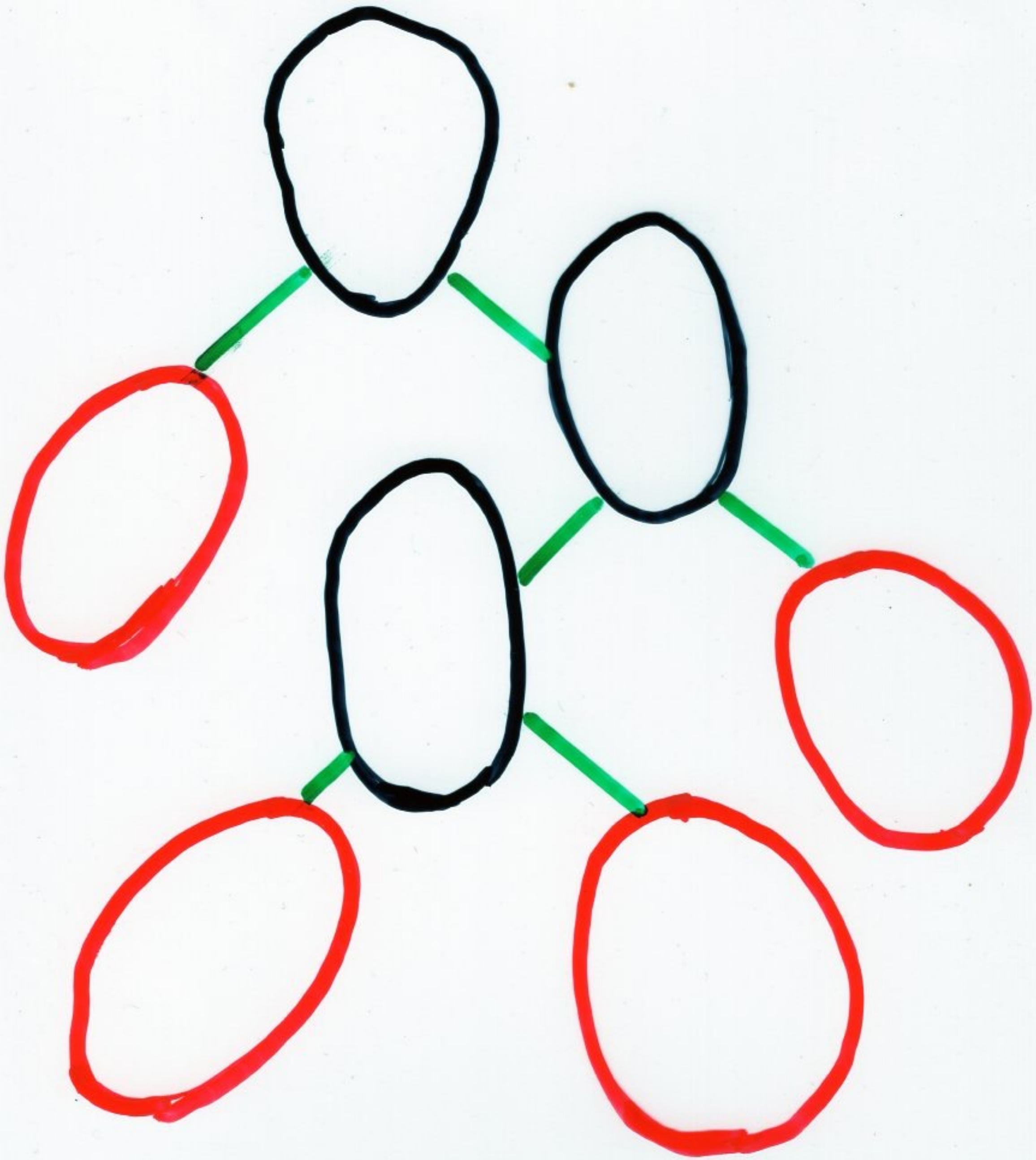
$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

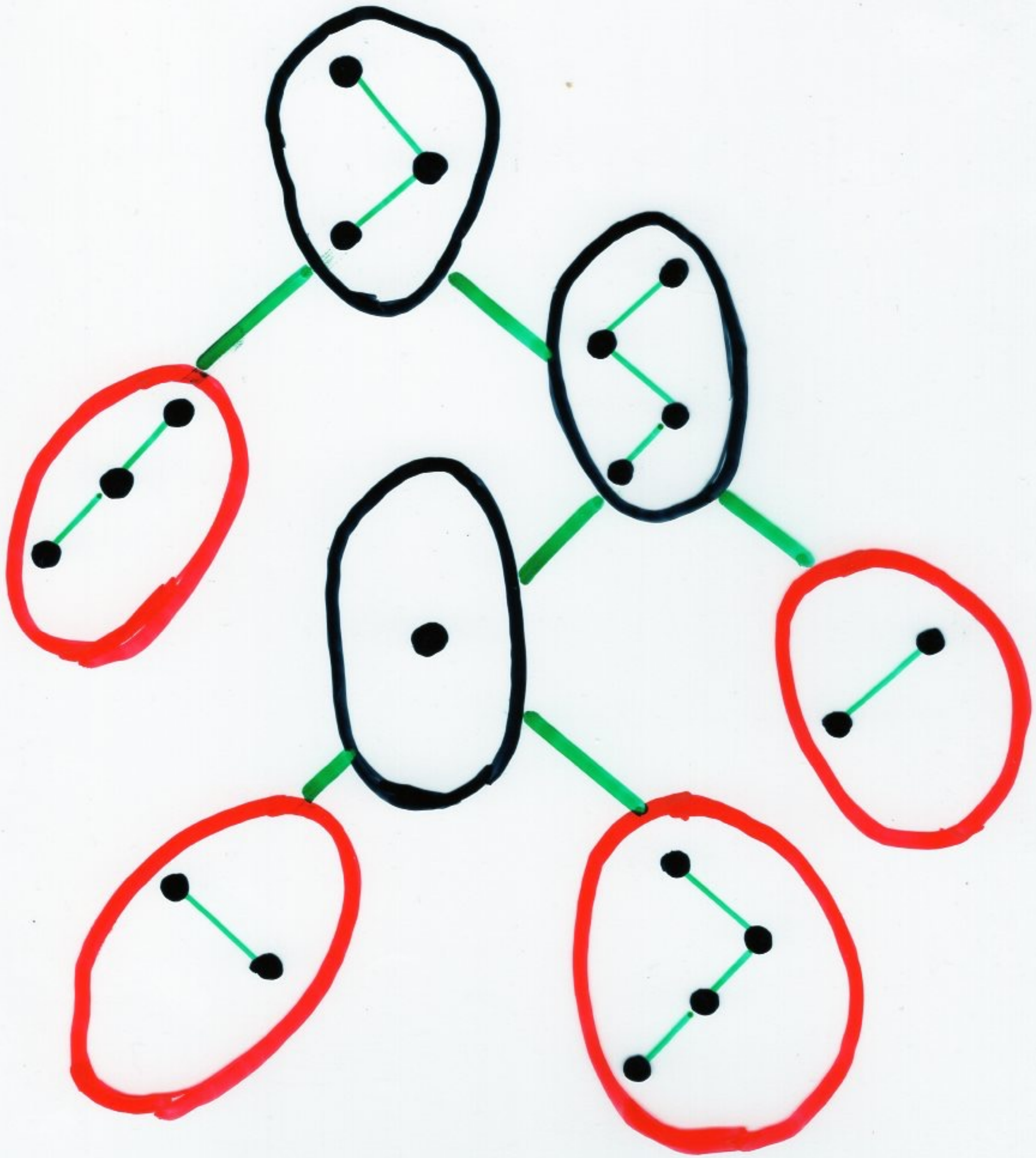
Françon (1984)

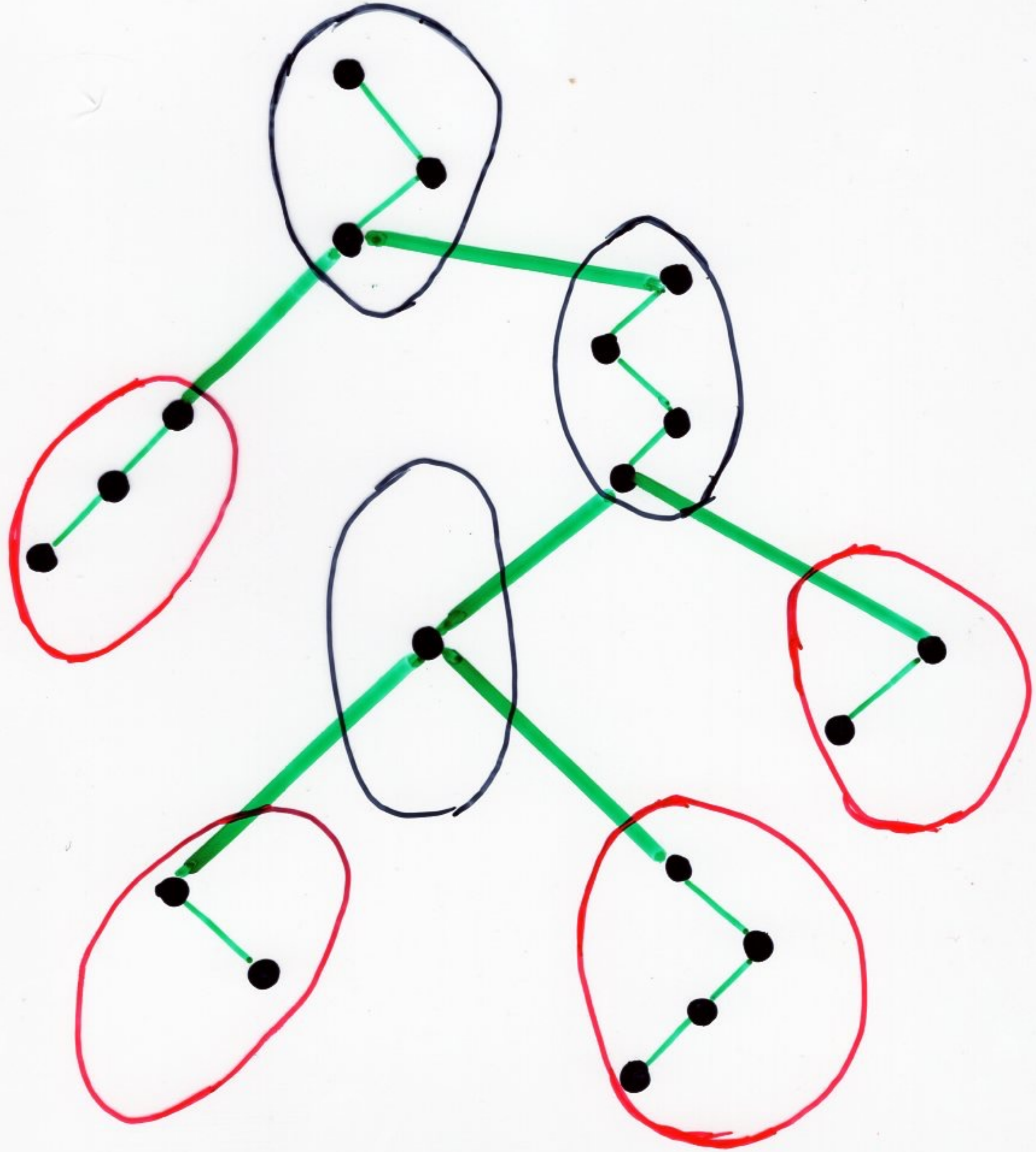
Knuth (2005)

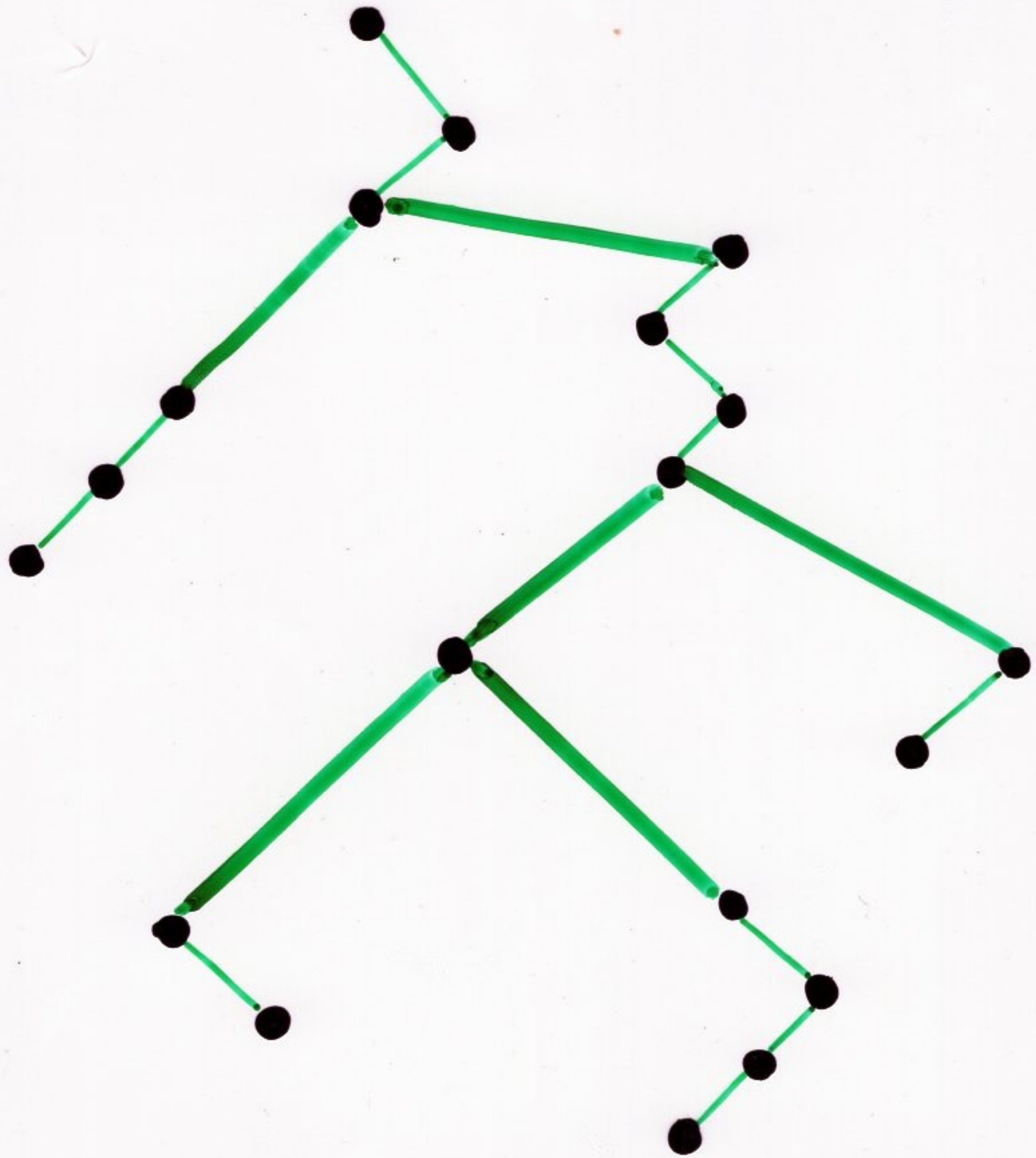


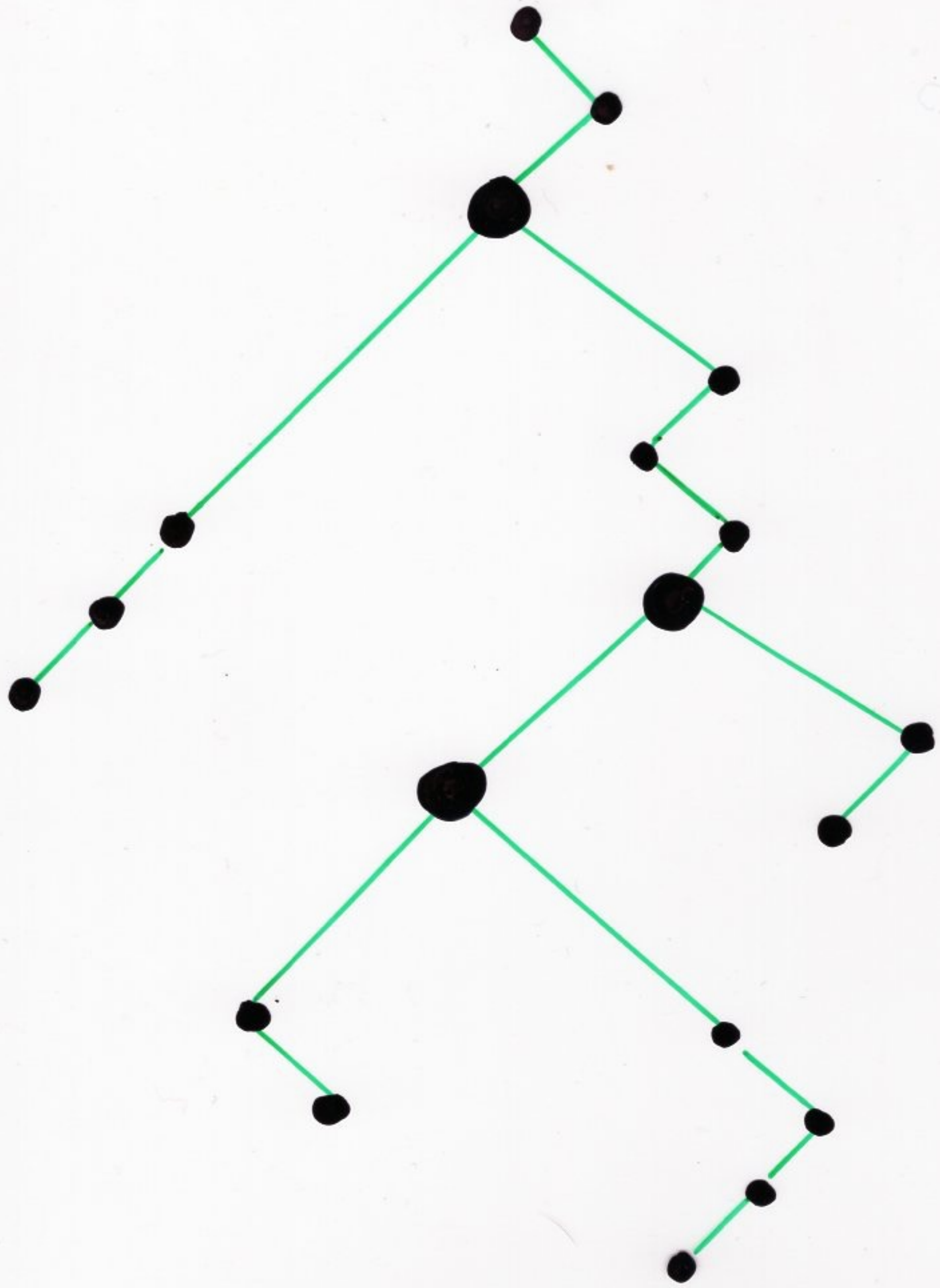


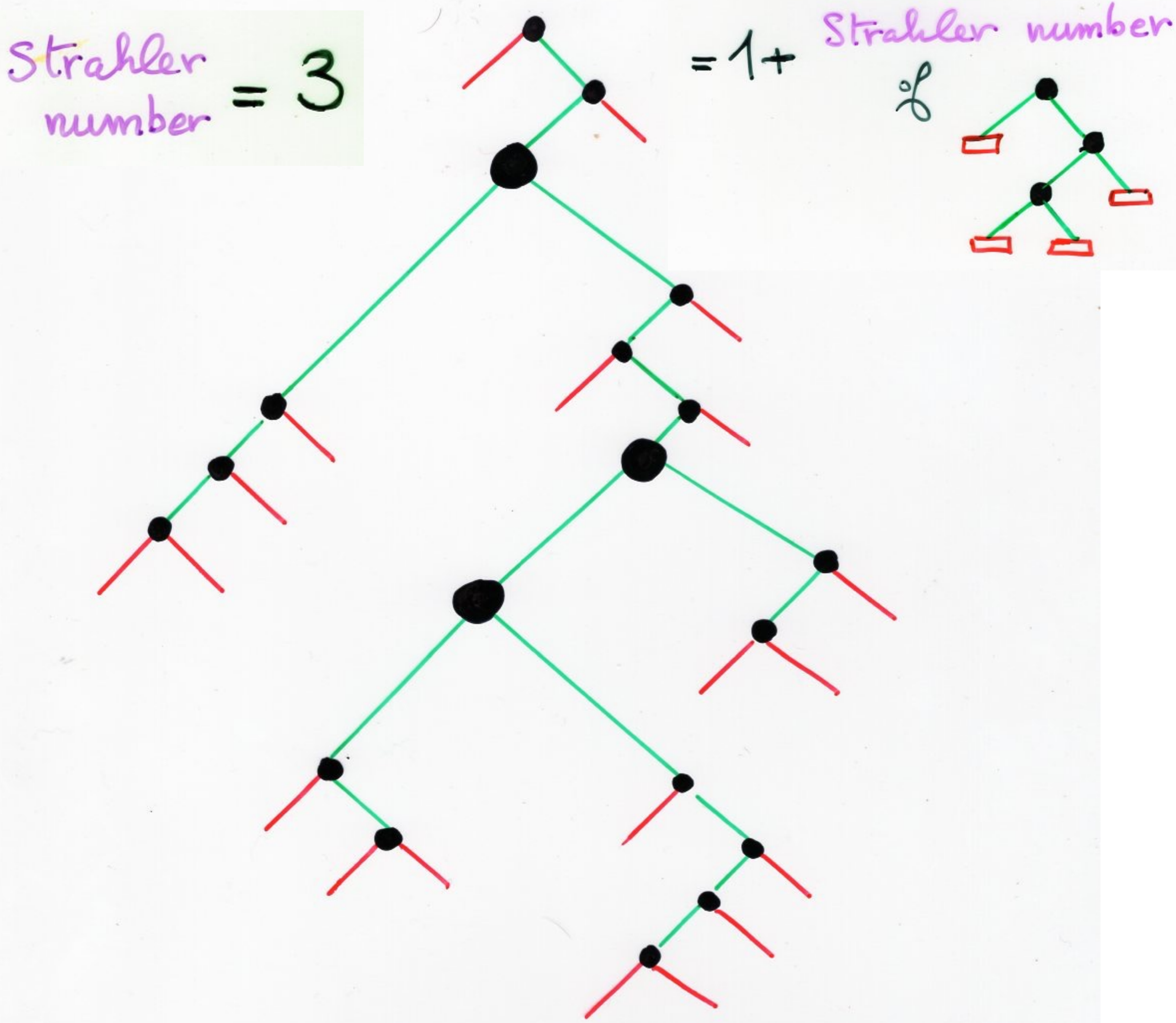




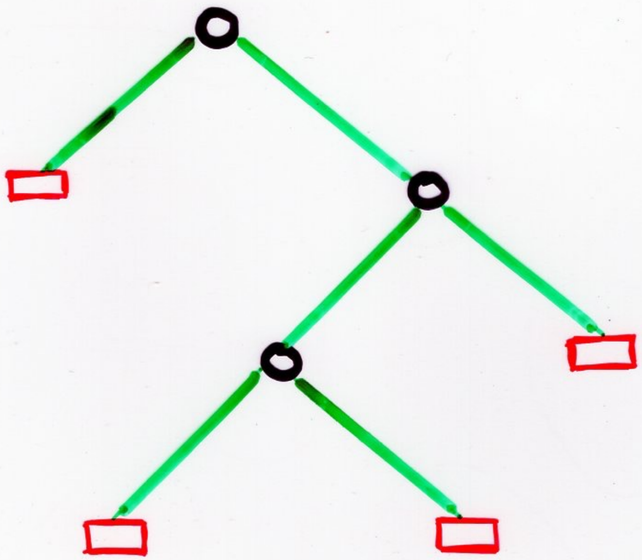







Strahler number = 3
= 1 + Strahler number of

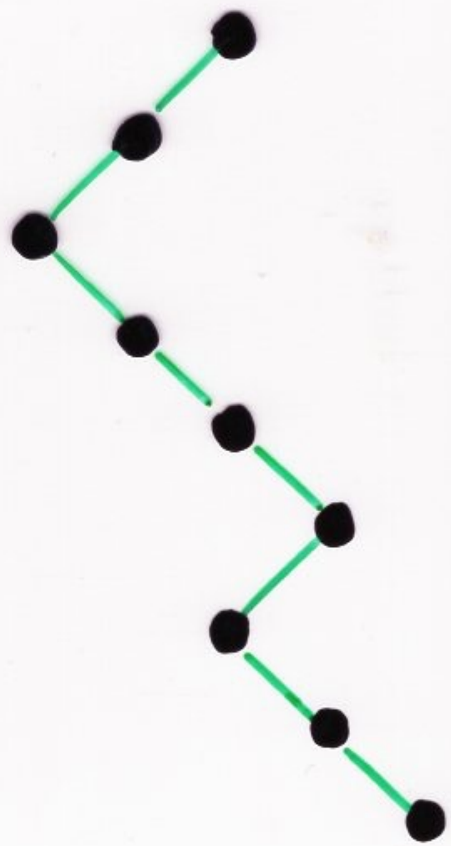
$$S(u,x) \rightarrow u\,S(u^2,x)$$



$$S(u,x) \rightarrow u\,S(u^2,x)$$

$$u \leftarrow \frac{t}{1-2t}$$

substitution



$$S(t,x) = \sum_{k \geq 0} S_k(t)\, x^k$$

$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

Françon (1984)

Knuth (2005)

generating functions

rational

algebraic

D-finite

rational power series

$\cap\!|$

algebraic power series

$\cap\!|$

P-recursive
(D-finite) power series

$$\sum_{n\geqslant 0} a_n t^n = \frac{N(t)}{D(t)}$$

$$P(y,t) = 0$$

$$P_k(n)a_{n+k} + P_{k-1}(n)a_{n+k-1} + \cdots + P_0(n)a_n = 0$$

rational power series $\Leftrightarrow$ recurrence relation with $P_0, \ldots, P_k$ constants

- Rat

$$\sum_{n\geq 0} F_n t^n = \frac{1}{1-t-t^2}$$

Fibonacci numbers

$$F_{n+1} = F_n + F_{n-1}$$

- Alg

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

$$y = 1 + t y^2$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

- D-finite

$$a_n = n!$$

$$a_n = n\, a_{n-1}$$

rational generating functions

Rational generating function

$$\sum_{n \geq 0} a_n t^n = \frac{N(t)}{D(t)}$$

$N(t)$
$D(t)$ polynomials in t

Path (or walk)

$$\omega = (s_0, s_1, \ldots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

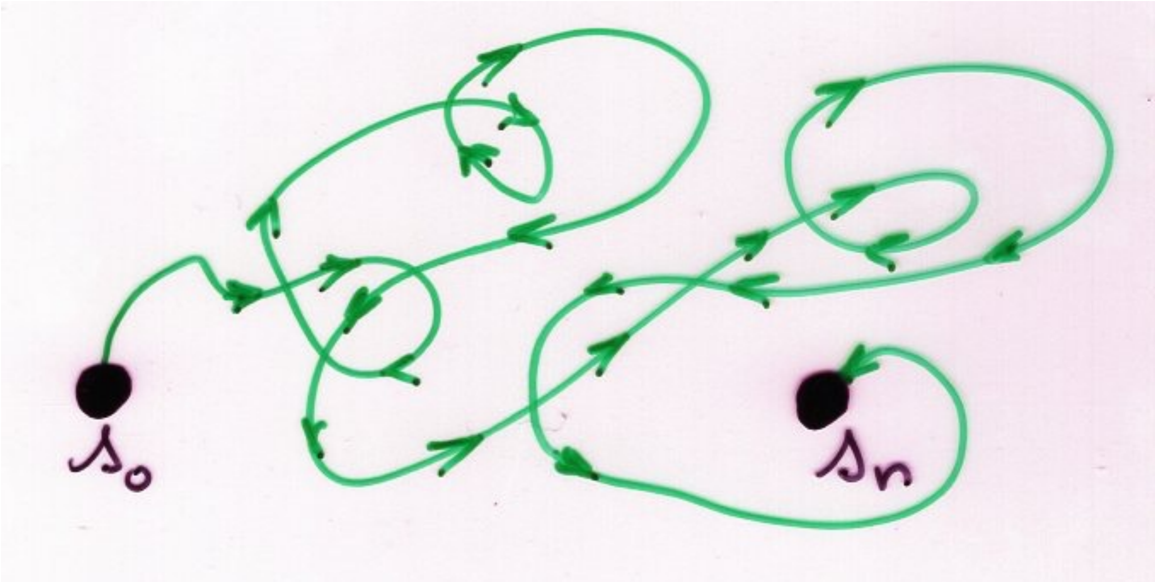
length n

(s_i, s_{i+1}) elementary step

valuation (weight)

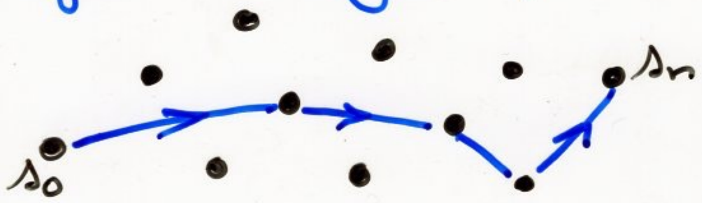
$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

$$v : S \times S \to \mathbb{K}[x]$$



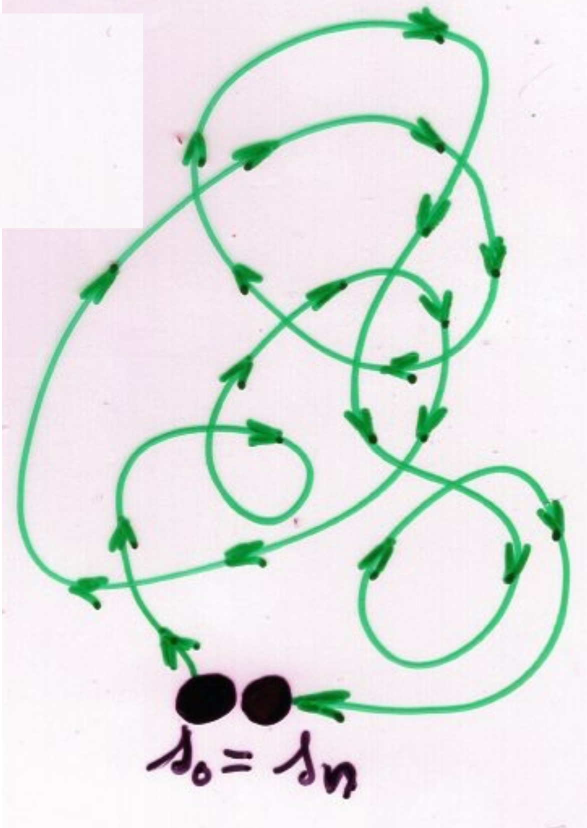
s_0
s_n

self-avoiding path (or walk)

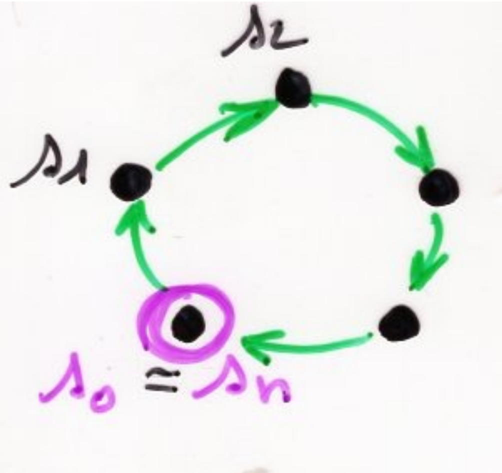


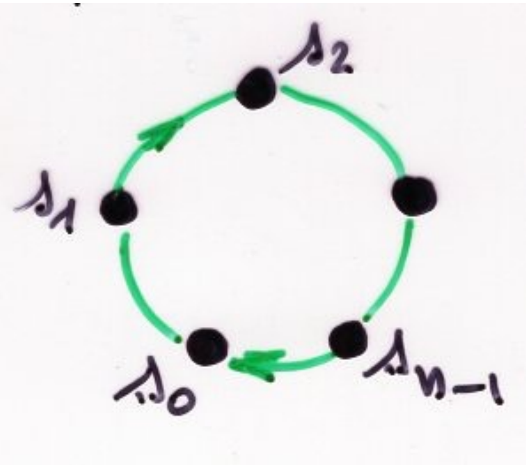
all vertices $s_0, s_1, .., s_n$ are disjoint

path $\omega = (s_0, s_1, .., s_n)$ with $s_0 = s_n$
is a circuit or (lacet)



elementary circuit $\omega = (s_0, \dots, s_n)$
with $s_0 = s_n$, all vertices are disjoint
except $s_0 = s_n$.

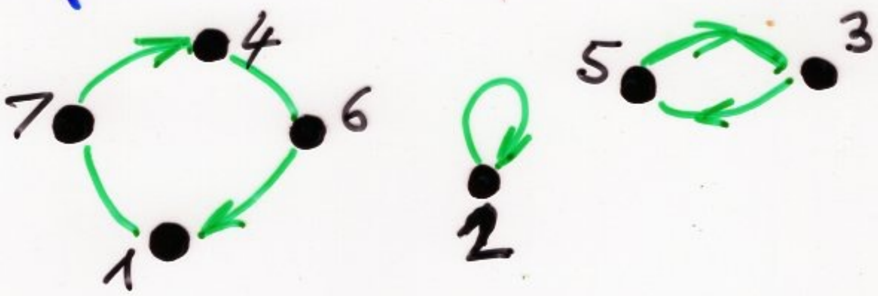




Cycle = elementary circuit up to a
circular permutation of the vertices

Cycles

of a permutation σ



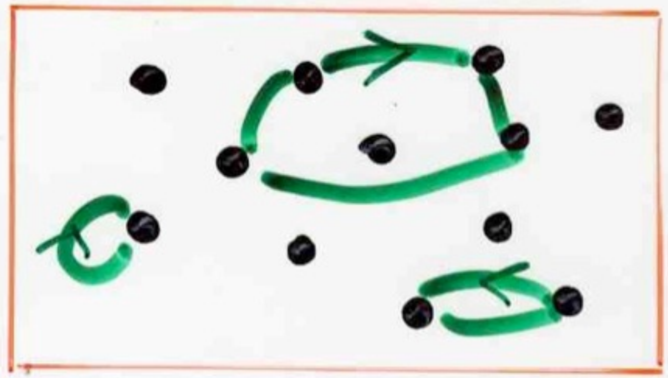
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}$$

Proposition

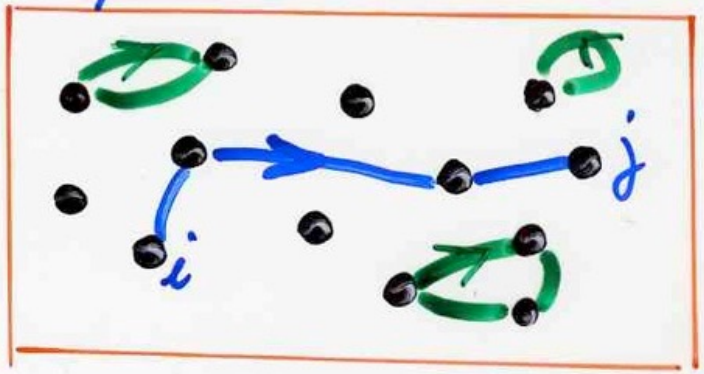
$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} V(\omega) = \frac{N_{i,j}}{D}$$

$$D = \sum_{\{\gamma_1, \ldots, \gamma_r\}} (-1)^r \, v(\gamma_1) \ldots v(\gamma_r)$$

$\{\gamma_1, \ldots, \gamma_r\}$ 2 by 2 disjoint cycles



$$N_{ij} = \sum_{\{\eta\,;\,\gamma_1, \ldots, \gamma_r\}} (-1)^r \, v(\eta) \, v(\gamma_1) \ldots v(\gamma_r)$$



η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \ldots, \gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η

Transition matrix methodology in Physics linear algebra

proof

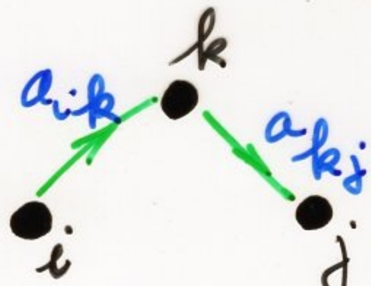
Lemma $S = \{1, 2, \ldots, n\}$

$A = (a_{i,j})$ $n \times n$ matrix

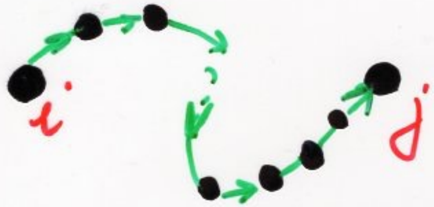
$$(I - A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

$(A^2)_{ij}$



$(A^m)_{ij}$



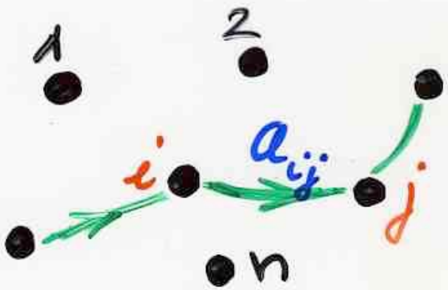
$|\omega| = m$

$$(A^m)_{ij} = \sum_{|\omega| = m} v(\omega)$$

$$\left(I_n - A\right)^{-1}_{ij} = \frac{\mathrm{cof}_{ji}\left(I_n - A\right)}{\det\left(I_n - A\right)}$$

$$I_n + A + A^2 + \dots + A^n + \dots$$

$$A = \left(a_{ij}\right)$$

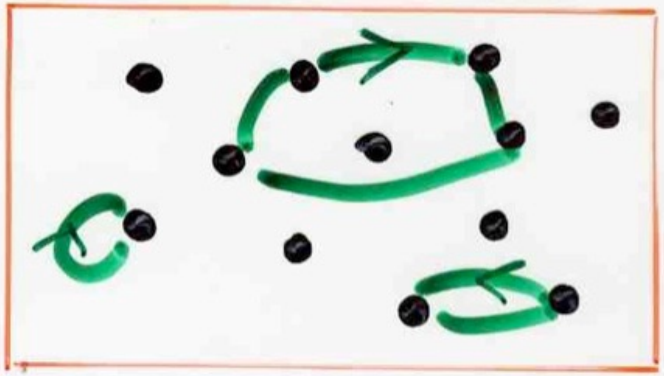


$$\det(A) = \sum_{\sigma} (-1)^{\mathrm{inv}(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

permutations of $\mathfrak{S}_n$

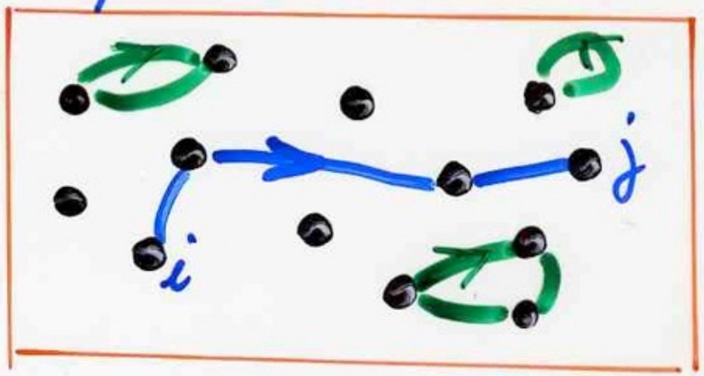
$$\det(I_n - A) = \sum_{\{\gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$

2 by 2 disjoint cycles



$$\mathrm{cof}_{ji}(I_n - A)$$

$$N_{ij} = \sum_{\{\eta\,;\,\gamma_1, \ ,\gamma_r\}} (-1)^r\, v(\eta)\; v(\gamma_1) \cdots v(\gamma_r)$$

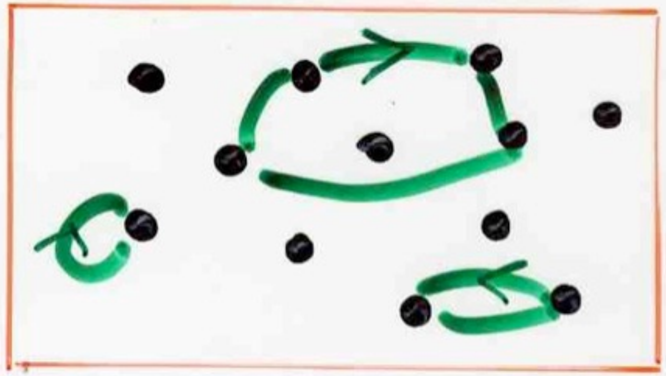


η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} V(\omega) = \frac{N_{i,j}}{D}$$

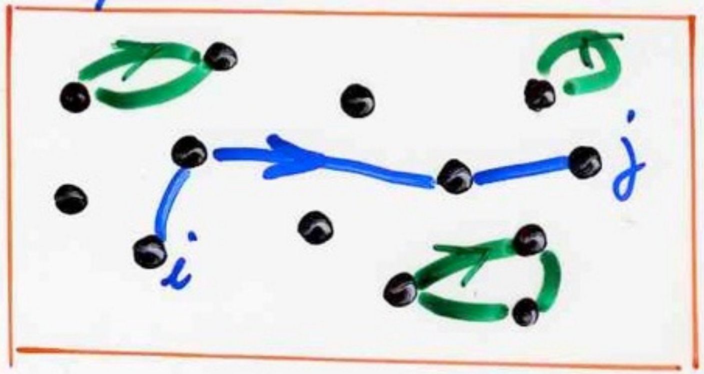
$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint} \\ \text{cycles}}} (-1)^r \, v(\gamma_1) \dots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta ; \gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\eta) \, v(\gamma_1) \dots v(\gamma_r)$$

η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η



$\square$