An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc
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Chapter 1
Ordinary generating functions

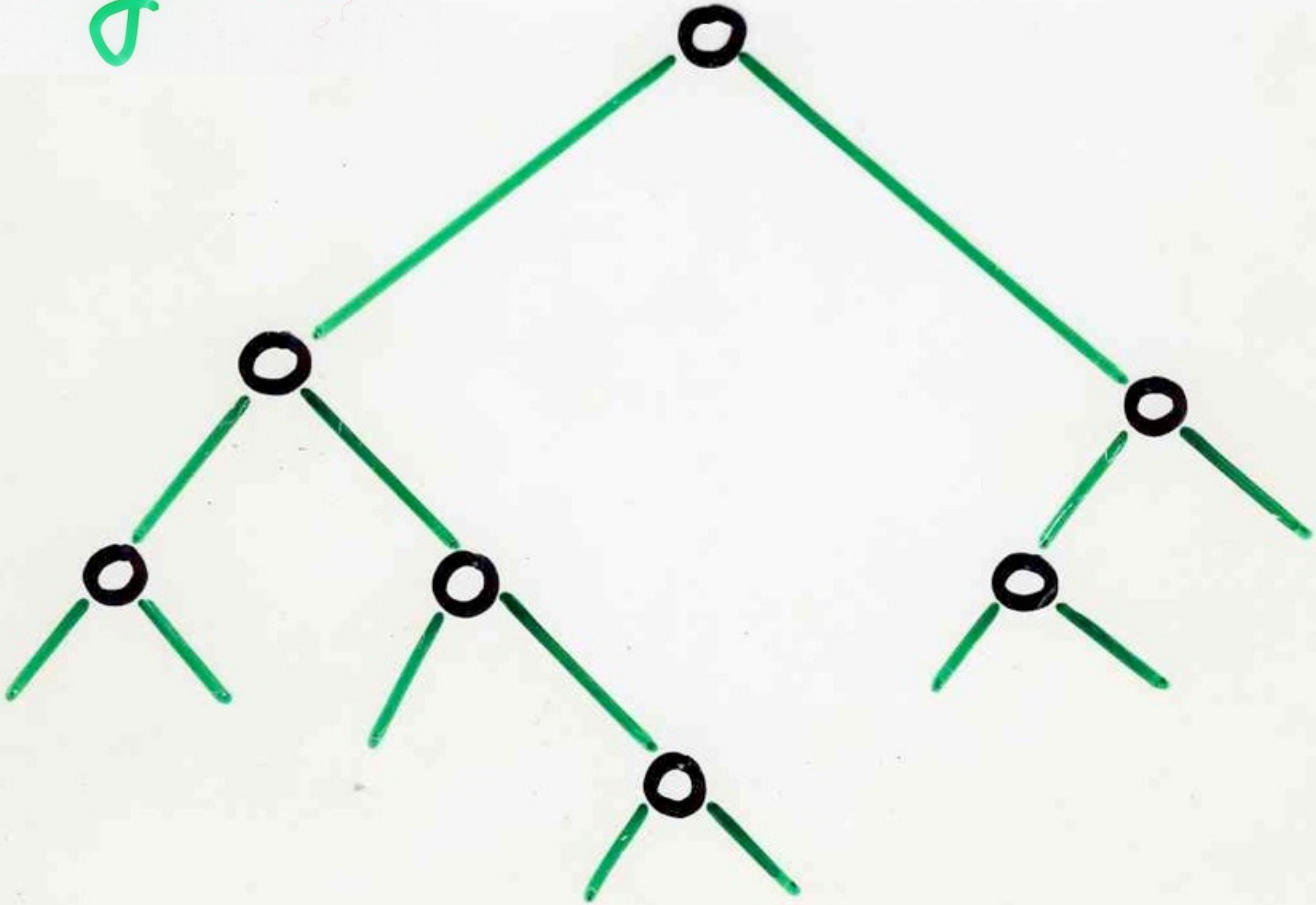
IMSc

7 January 2016

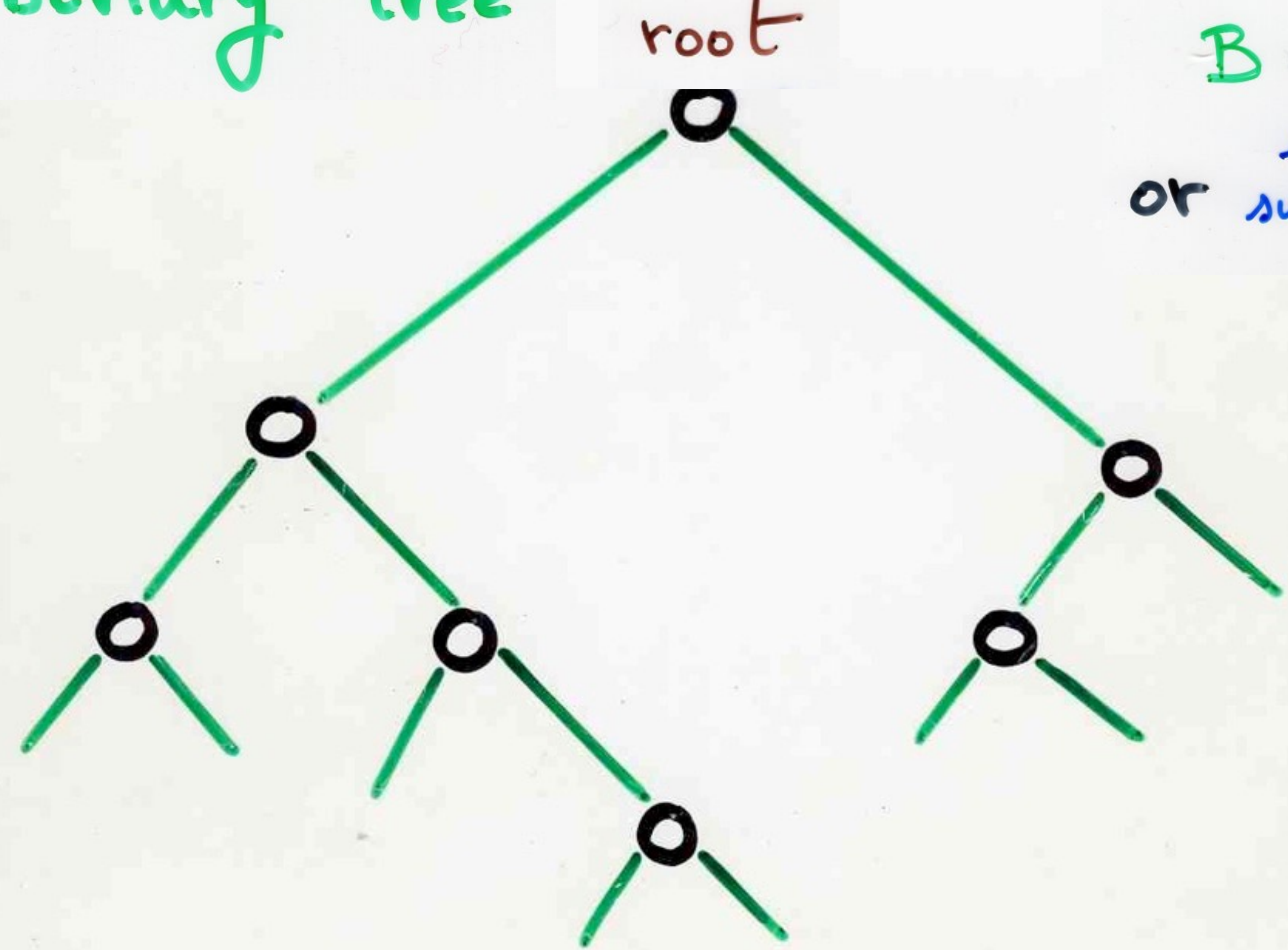
a simple example:

binary tree

binary tree



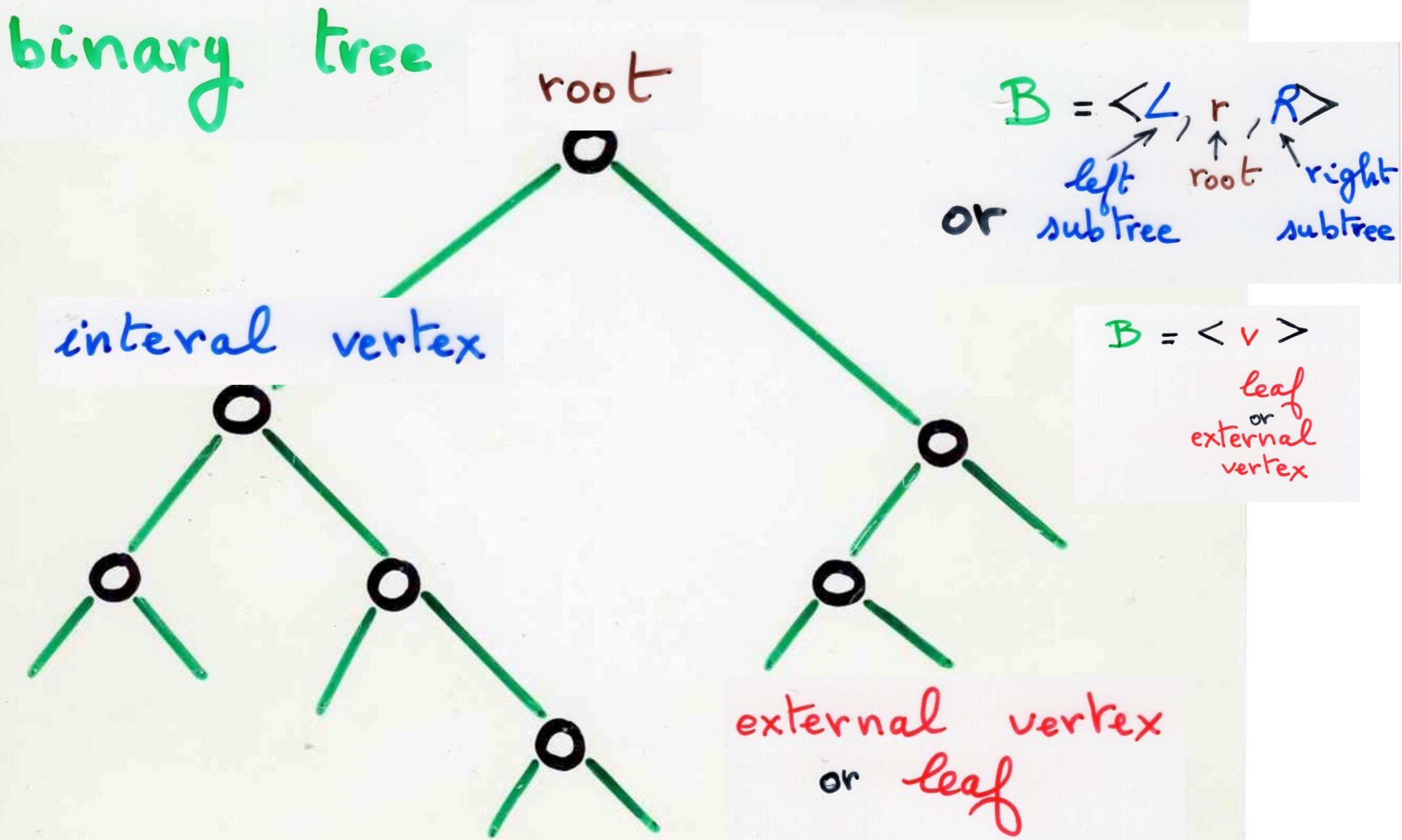
binary tree



$$B = \langle \underset{\text{left subtree}}{L},\ \underset{\text{root}}{r},\ \underset{\text{right subtree}}{R} \rangle$$

or

$$B = \langle \underset{\text{leaf or external vertex}}{v} \rangle$$


binary tree
root
B = <L, r, R>
left subtree
root
right subtree
or
B = < v >
leaf
or
external vertex
interal vertex
external vertex
or leaf

C_n = number of binary trees having n internal vertices

(or $n+1$ leaves = external vertices)

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5$

$1, 1, 2, 5, 14, 42, \ldots$

$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

$$132 \qquad 1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\; n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

classical
enumerative
combinatorics

3 (1838)

Note sur une Équation aux différences finies ;

Par E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\ \dots\ \frac{4n-6}{n}P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\ 4.\ 5\dots\ n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3)$$
$$= \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1}C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned}\frac{1}{n+1}C_{2n,n} &= \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1}\\ &+ \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \dots + \frac{1}{n}C_{2n-2,n-1}.\end{aligned}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale ; savoir :

$$1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924\dots;$$

qu'on les divise respectivement par

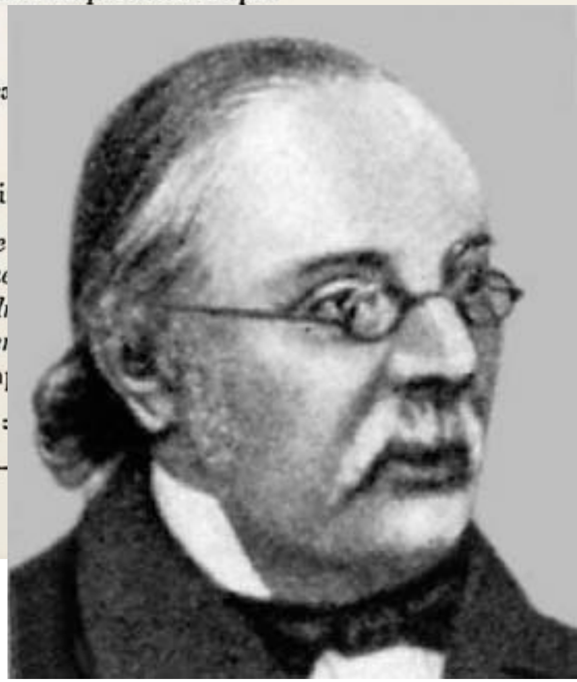
on obtiendr

(A)

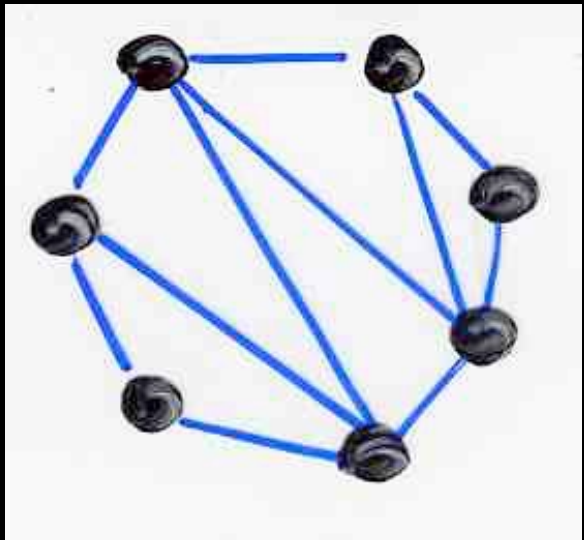
lesquels joui

Un terme *nme des*
produits qu *ème, et*
dans un ord *n multi-*
pliant les te
Par exem

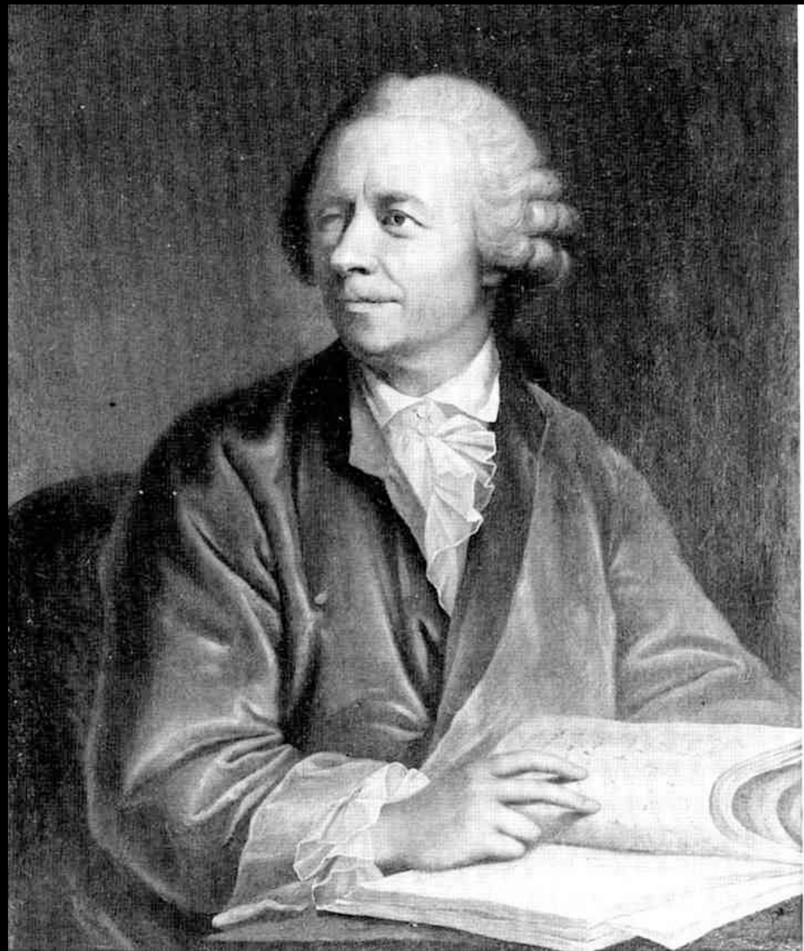
132



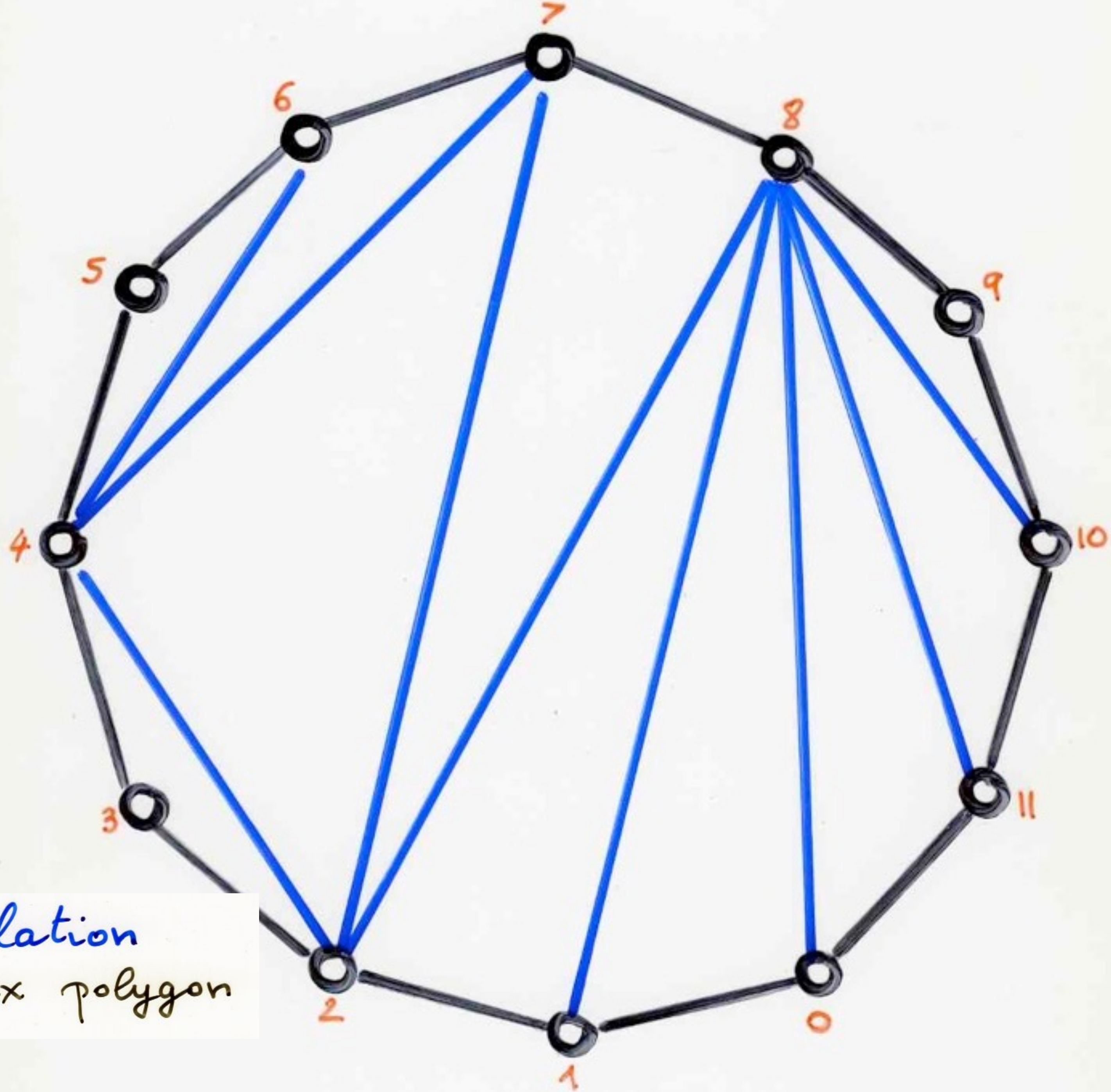
Eugène Catalan (1814-1894)



triangulation
of a convex polygon



Leonhard Euler (1707- 1783)



triangulation
of a convex polygon

[illegible] getheilet, und solches kan auf 5 verschiedene Arten geschehen nemlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V. $\frac{ec}{eb}$

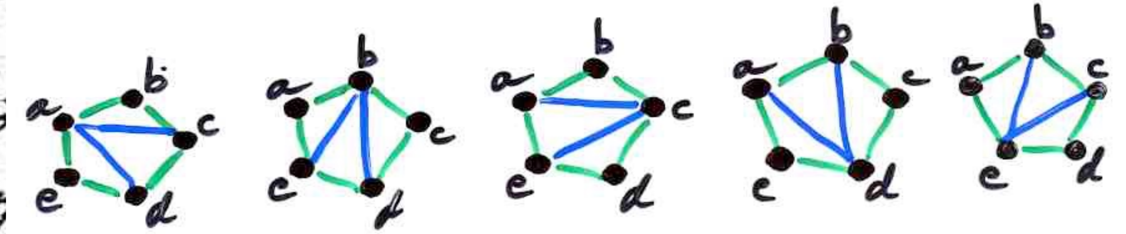
Ferner kan ein [illegible]

zertheilet, und [illegible]

Nun ist die Frage [illegible]

durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten werden, auf

wie vielerley verschiedene Arten solches geschehen kann.



$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

wann $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter

sey

$$x = \frac{2.\,6.\,10.\,14.\,18.\,22.\ \ldots\ (4n-10)}{2.\,3.\,4.\,5.\,6.\,7.\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = 2 \cdot \frac{6}{...}$; $5 = 2 \cdot \frac{42}{4}$; $14 = 5 \cdot \frac{14}{5}$; $42 = 14 \cdot \frac{...}{6}$; [illegible]

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-a^2}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$a=\tfrac{1}{4}: \quad 1+\tfrac{2}{4}+\tfrac{5}{4^2}+\tfrac{14}{4^3}+\tfrac{42}{4^4}+\text{etc} = 4.$$

$$a = \frac{1}{4}$$

4 Sept 1751
Berlin

d. 4. Sept 1751.

L. Euler

intuitive introduction to

ordinary generating functions

formal power series

1 1 2 5 14 42

Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynomial

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

$$+ \cdots$$

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \cdots$$
$$+ C_n t^n + \cdots$$

$$f(t) = \sum_{n \geq 0} a_n t^n$$

generating function

Formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

a little exercise

$$\frac{1}{1-(t+t^2)} = \ ?$$

$$\frac{1}{1-(t+t^2)} = \ ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \dots$$

$$\sum_{i \geq 0} (t+t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \ldots$$

$$+ (t^5 - - - -$$

$$\ldots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \ldots$$
$$+ (t^5 - - - -$$

$$1 \quad 2 \quad 3 \quad 5 \quad 8$$

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$$t + t + t + \ldots t + \ldots$$

$$1 + 1 + 1 + \ldots$$

$$t + t + t + \ldots\; t + \ldots$$

$$1 + 1 + 1 + \ldots$$

formal power series algebra

formalisation

Formal power series algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \ldots]$

$$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$\mathbb{K}[t]$ polynomials algebra

$$(a_0, a_1, a_2, \cdots, a_n, \cdots)$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable t and coefficients in $\mathbb{K}$)

algebra of formal power series

- sum $\quad f+g=h, \quad a_n+b_n=c_n$
- product $\quad fg=h, \quad c_n=\sum_{\substack{p+q=n \\ p,q\geq 0}} a_p b_q$
- product (by a scalar) $\quad \lambda f=h, \quad c_n=\lambda a_n$

$$f=\sum_{n\geq 0} a_n t^n, \quad g=\sum_{n\geq 0} b_n t^n, \quad h=\sum_{n\geq 0} c_n t^n$$

generating power series
of the coefficients (numbers a_n)

$$\sum_{n \geq 0} a_n t^n = f(t)$$

(ordinary generating function)

exponential generating function

$$\sum_{n \geq 0} a_n \frac{t^n}{n!}$$

summable family

$$\sum_{i \in I} f_i(t)$$

Def. for every n, the set of $i \in I$ such that the coefficient of t^n in the power series $f_i(t)$ is $\neq 0$, is a finite set.

example

$$\sum_{i \geqslant 0} (t + t^2)^i =$$

$1 + (t + t^2)$

$(t^2 + 2t^3 + t^4)$

$(t^3 + 3t^4 + 3t^5 + t^6)$

$(t^4 + 4t^5 + 6t^6 + \dots$

$+ (t^5 - - - -$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

example

$$f(t) = \sum_{n \geq 0} a_n t^n$$

justification of the notation

$$(a_0, a_1, a_2, \dots, a_n, \dots)$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

summable family

infinite product

$$\sum_{i \in I} f_i^{(t)}$$

$$\prod_{i \in I} \left(1 + g_i^{(t)}\right)$$

example

$$\prod_{i \geq 1} \frac{1}{(1-q^i)}$$

other operations

- substitution

$$f(t) = \sum_{n\geq 0} a_n t^n \quad , \quad g(t) = \sum_{n\geq 0} b_n t^n$$

$$b_0 = 0$$

$$f \circ g(t) \; ; \; f(g(t)) = \sum_{n\geq 0} a_n (g(t))^n$$

- Inverse

$$\frac{1}{1-f} = 1 + f + f^2 + \dots + f^n + \dots$$

(si $\operatorname{ord}(f) \geq 1$)

- derivative

$$f' \qquad \frac{df}{dt} = \sum_{n \geq 1} n\, a_n t^{n-1}$$

exponential $\quad \exp(t) = \sum_{n\geq 0} \frac{t^n}{n!}$

logarithm $\quad \log(1-t)^{-1} = \sum_{n\geq 1} \frac{t^n}{n}$

binomial power series

$$(1+t)^{\alpha} = \sum_{n\geq 0} \binom{\alpha}{n} t^n$$

$$= \sum_{n\geq 0} \alpha(\alpha-1)\ldots(\alpha-n+1)\frac{t^n}{n!}$$

$\mathrm{ord}(f)\geq 1 \qquad \exp(f) \qquad \log(1+f) \qquad (1+f)^{\alpha}$

formal power series
in several variables

$$f(t_1, t_2, \dots, t_p) = \sum_{n_1, \dots, n_p} a_{n_1, \dots, n_p} t_1^{n_1} t_2^{n_2} \cdots t_p^{n_p}$$

$\mathbb{K}[t_1, \dots, t_p]$

$\mathbb{K}[[t_1, \dots, t_p]]$

algebra

opérations

$\partial / \partial t_i$

free monoid X^*

$X = \{a, b, c, ..\}$ alphabet

$x \in X$ letter word $u = x_1 x_2 \dots x_n$

X^* free monoid generated by X

- concatenation $u = x_1 x_2 .. x_n$ $v = y_1 .. y_m$

  $uv = x_1 \dots x_n y_1 \dots y_m$
- neutral element e (empty word)

length of a word $u = x_1 \dots x_n$

$|u| = n = \sum_{x \in X} |u|_x$ number of occurrence of x in u

$w = uv$ $\quad u$ left factor

$\quad v$ right factor

factor

$w = u_1 u_2 \cdots u_k$ $\quad u_i$ word

factorization of w

subword

$w = \underbrace{x_1 \cdots x_{i_1} \cdots x_{i_2} \cdots x_{i_k} \cdots x_n}$

$u = x_{i_1} x_{i_2} \quad x_{i_k}$

language

$L \subseteq X^*$

$\mathbb{K}\langle X\rangle$ algebra of non-commutative polynomials in variables X

$\mathbb{K}\langle\langle X\rangle\rangle$ algebra of non-commutative power series in variables X and coefficients in $\mathbb{K}$

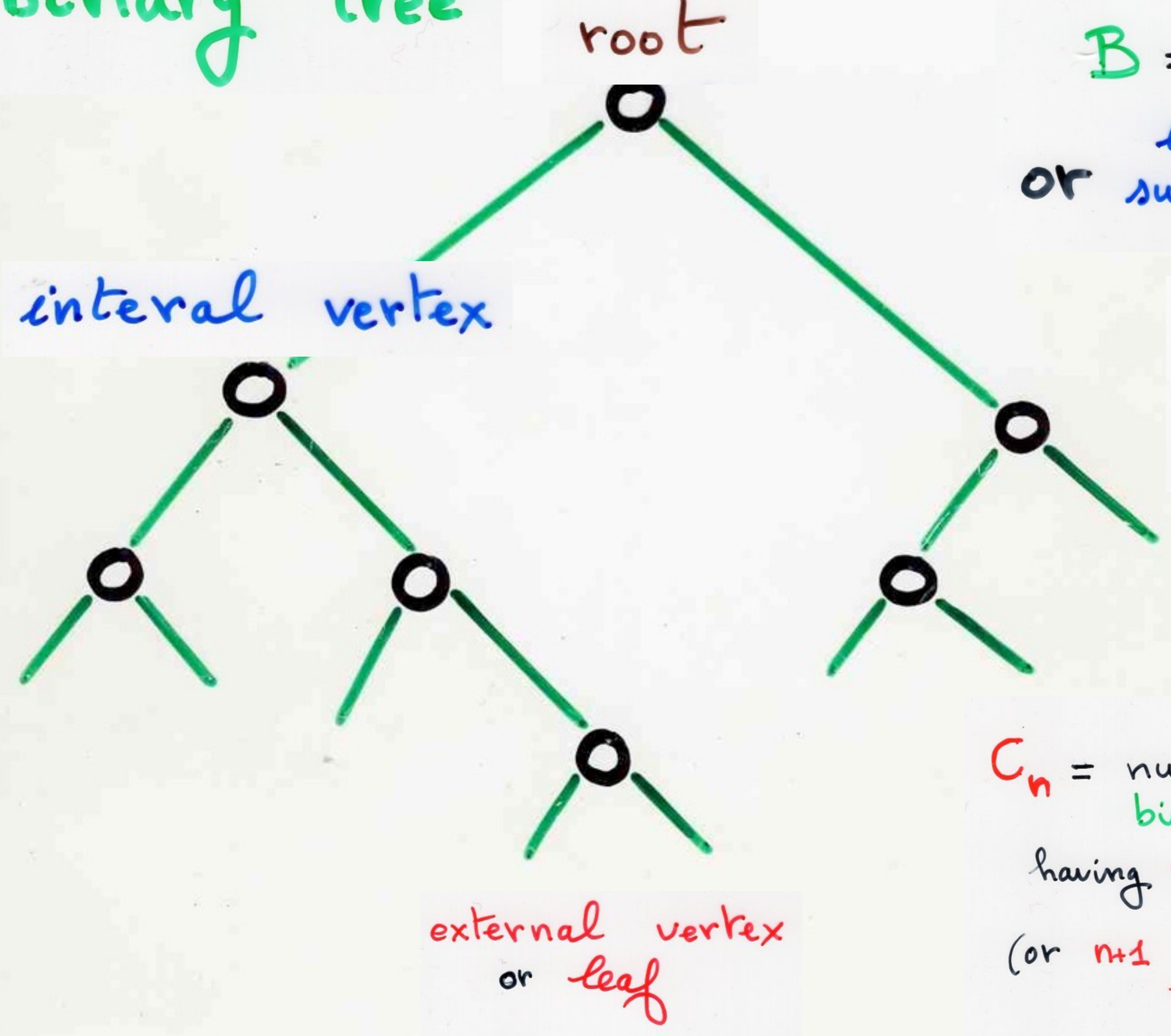
$$\sum_{w\in X^*} c_w\, w$$

$$c_w \in \mathbb{K}$$

operations on combinatorial objects

intuitive introduction with binary trees

binary tree



$B = \langle L, r, R \rangle$

left subtree, root, right subtree

or

$B = \langle v \rangle$

leaf or external vertex

C_n = number of binary trees having n internal vertices (or $n+1$ leaves = external vertices) recurrence

$$C_{n+1} = \sum_{i+j=n} C_i \, C_j$$

$C_0 = 1$

$\Longleftrightarrow$

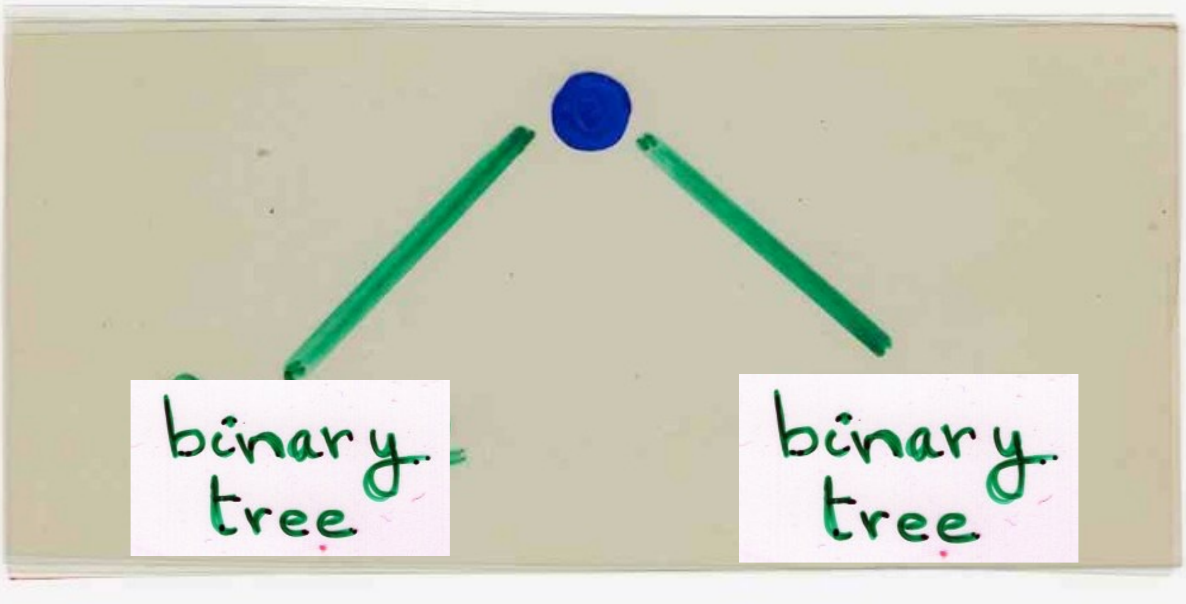
$$y = 1 + t\,y^2$$

algebraic equation

classical
enumerative
combinatorics

modern
enumerative
combinatorics

binary tree $=$ $\bullet$ $+$



$$\underset{\text{binary tree}}{B} = \{\bullet\} + \left(B \times \underset{\text{root}}{\bullet} \times B\right)$$

$$y = 1 + t\,y^2$$

algebraic equation

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$(1+u)^{m} =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^{2} + \frac{m(m-1)(m-2)}{3!}u^{3} +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

operations on combinatorial objects

formalisation

Operations on combinatorial objects

Def class of valued combinatorial objects

$\mathcal{A} = (A, v)$ $\quad A$ finite or enumerable set

$v : \underset{\text{valuation}}{A \longrightarrow \mathbb{K}[X]}$

$(*)$ for w monomial of $\mathbb{K}[X]$,
let $A_w = \left\{ \alpha \in A, \text{ coeff. of } w \text{ in } v(\alpha) \text{ is } \neq 0 \right\}$

then for every monomial w,
A_w is finite

$v(\alpha)$ weight or valuation of α

$\{v(\alpha), \alpha \in A\}$ is summable

Def. $f_A = \sum_{\alpha \in A} v(\alpha)$

generating power series

of objects $\alpha \in A$ weighted by v

ex: objects of size n

$X = \{t\}$ $\qquad v(\alpha) = t^n$

n is the size of α,

$a_n = |A_{t^n}|$ (finite set)

$= $ number of objects $\alpha \in A$ of size n

$$f_A = \sum a_n t^n$$

ex: more generally

$X = \{t\} \cup Y \qquad v(\alpha) = w(\alpha)\, t^n$

in general $a_0 = 1$, only one "empty" object

ε with weight $v(\varepsilon) = 1$

$\alpha = (A, v_A)$ $\quad \beta = (B, v_B)$

- **sum** $\qquad \alpha + \beta = \gamma$

$$= (C, v_C)$$

$A \cap B = \emptyset$

- $C = A \cup B$ (disjoint union)
- $v_C / A = v_A \qquad v_C / B = v_B$

Lemma $\quad f_\gamma = f_\alpha + f_\beta$

- **product** $\mathcal{A} \cdot \mathcal{B} = \mathcal{C}$

$$= (C, v_C)$$

- $C = A \times B$
- $(\alpha, \beta) \in C$ $\qquad v_C(\alpha, \beta) = v_A(\alpha)\, v_B(\beta)$

ex: "size" $|(\alpha, \beta)| = |\alpha| + |\beta|$

ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

• sequence

$\mathcal{A} = (A, v_A)$ $\qquad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \dots + \mathcal{A}^n + \dots$$

$$= \mathcal{A}^*$$

Lemma $\displaystyle f_{\mathcal{A}^*} = \frac{1}{1 - f_{\mathcal{A}}}$

symbolic method

Philippe Flajolet (1948–2011)

(with Robert Sedgewick)

Analytic Combinatorics

(Cambridge Univ. Press, 2008)

operations on combinatorial objects

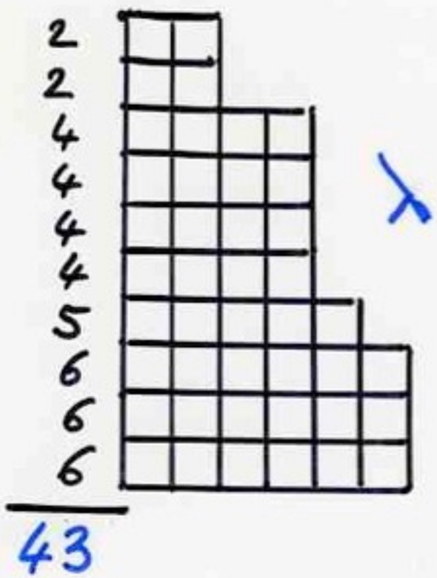
example: integers partitions

q-series

partition of an integer $\mathbf{n}$

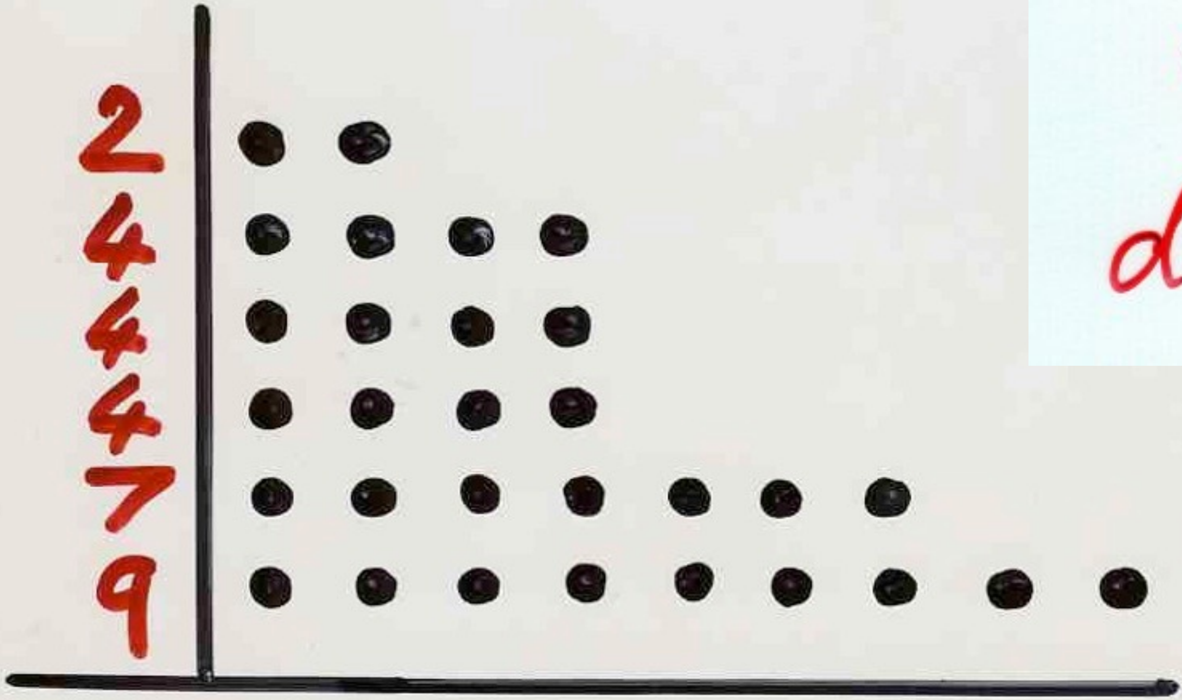
$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram

Ferrers diagrams



$$30 = 2+4+4+4+7+9$$

1	$1+1$	$1+1+1$	$1+1+1+1$	$1+1+1+1+1$
	2	$2+1$	$2+1+1$	$2+1+1+1$
		3	$3+1$	$2+2+1$
			$2+2$	$3+1+1$
			4	$3+2$
				$4+1$
				5

$$1, \quad 2, \quad 3, \quad 5, \quad 7$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

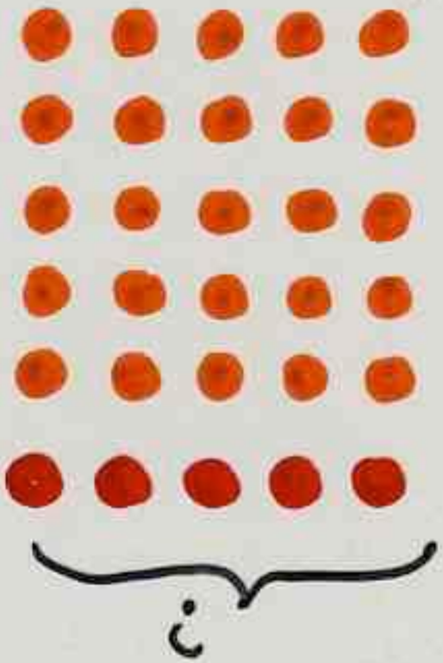
$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

generating function

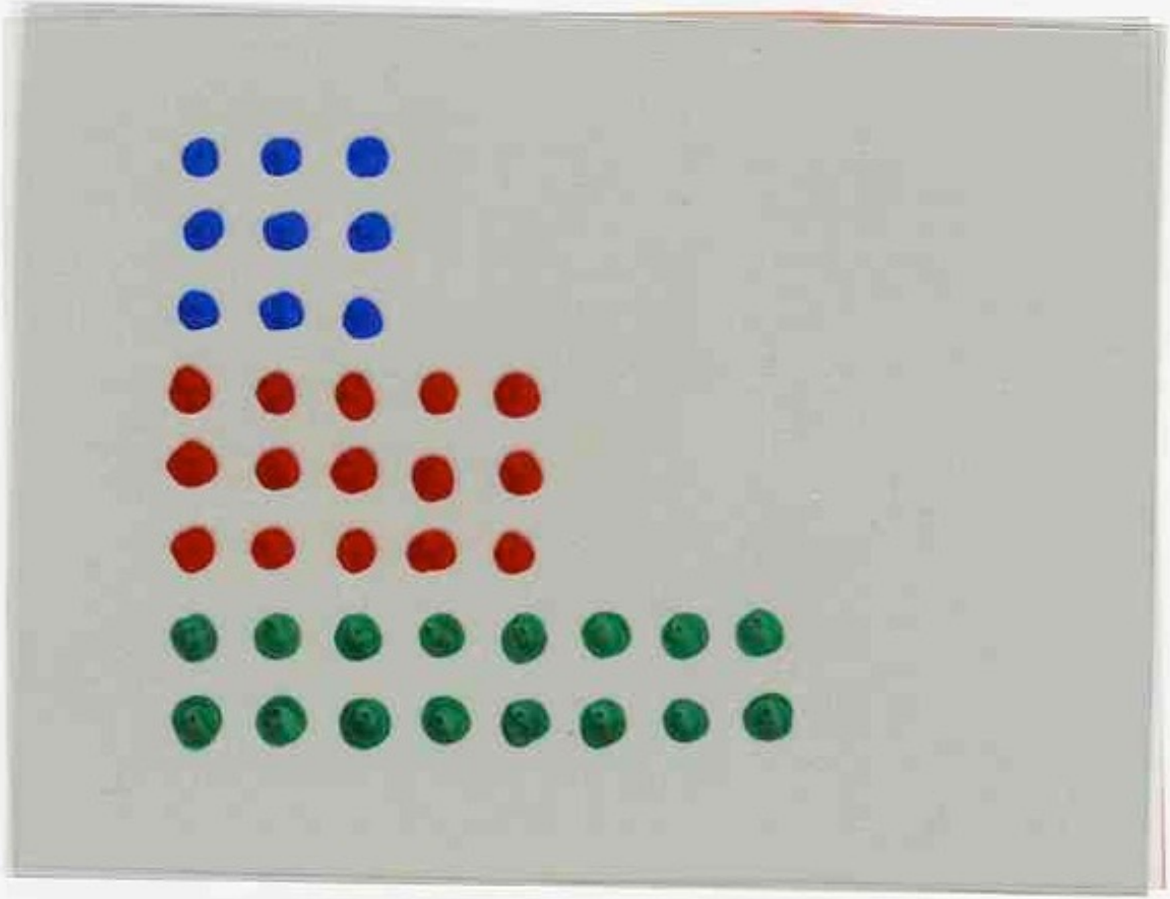
for (integer) partitions

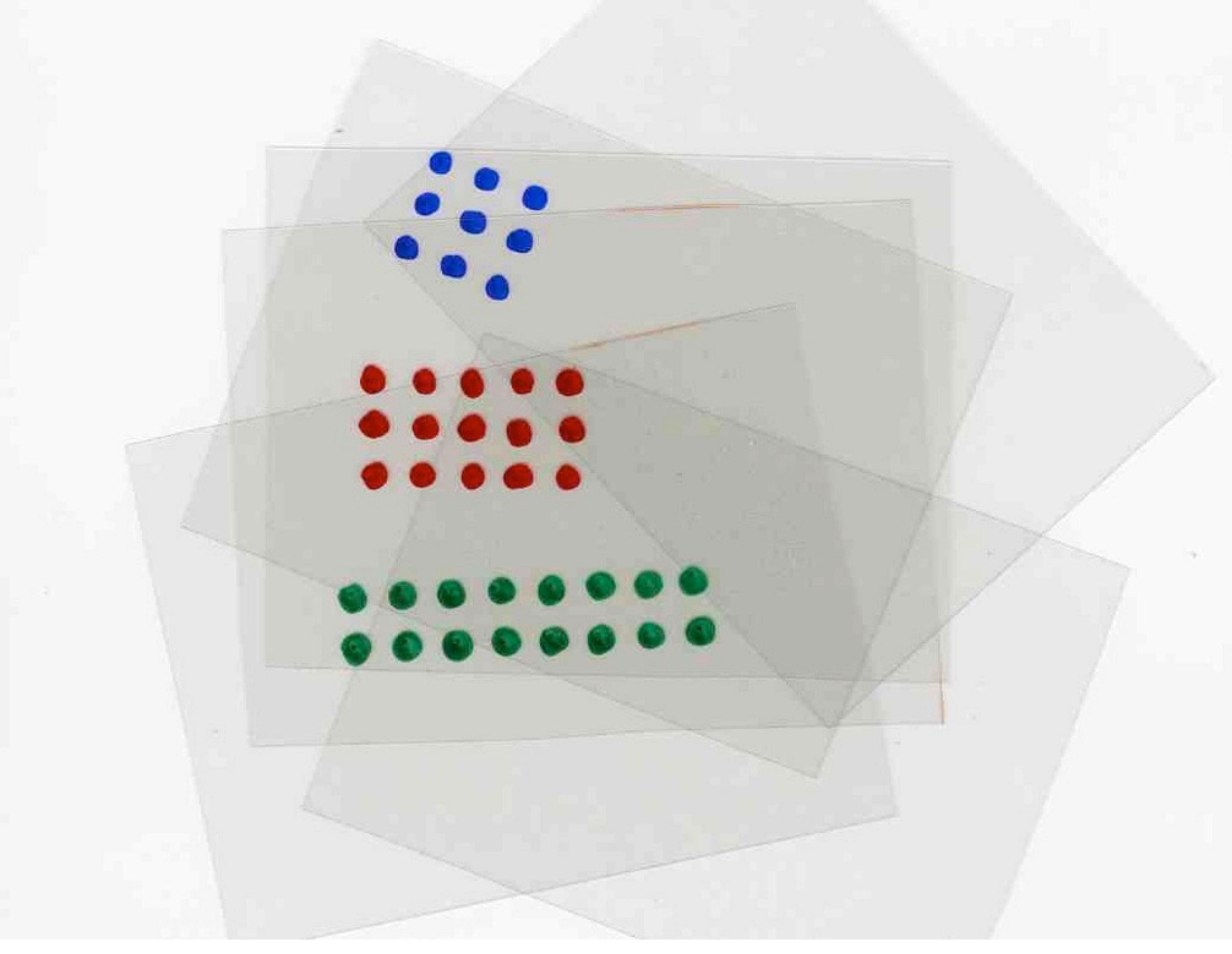
$$\sum_{n \geqslant 0} a_n q^n$$

$\underbrace{\bullet\bullet\bullet\bullet\bullet}_{i}$ $\qquad$ q^{i}



$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\cdots(1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$\prod_{i\geq 1} \frac{1}{(1-q^i)}$$

generating function
for the number of
partitions of an integer n

exercise

ex 1

$$\sum_{n\geq 0} p(n, I)\, q^n = \prod_{i\in I} \frac{1}{1-q^i}$$

partitions

parts $\lambda_j \in I$

ex 2

D- partition $\lambda_i - \lambda_{i+1} \geq 2$

$\lambda = (\lambda_1, \ldots, \lambda_k)$ $(1 \leq i < k)$

generating function for D- partitions

$$\sum_{m\geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

hint: find a bijection between:

partitions of n with at most m parts $\longleftrightarrow$ D-partitions of $n + m^2$ having exactly m parts

Rogers-Ramanujan identities

Ramanujan's home
Sarangapani Street
Kumbakonam

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

Rogers - Ramanujan identities

$$R_I \qquad \sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i \equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D-partitions

$$\begin{cases} 9 \\ 8+1 \\ 7+2 \\ 6+3 \\ 5+3+1 \end{cases}$$

partitions

parts $\equiv 1, 4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \quad \begin{cases} 1+\dots+1 \end{cases}$$

R_{II} $$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D-partitions

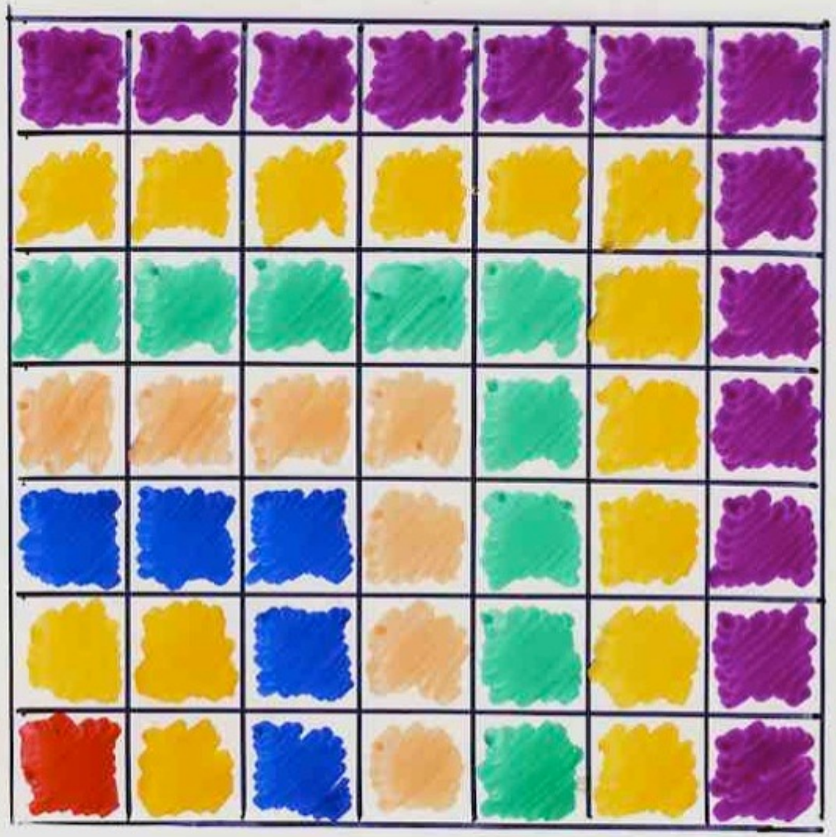
parts $\neq 1$

Partitions

parts $\equiv 2, 3$ mod 5

$$\begin{cases} 7+2 \\ 6+3 \\ 9 \end{cases} \qquad \begin{cases} 2+2+2+3 \\ 3+3+3 \\ 7+2 \end{cases}$$

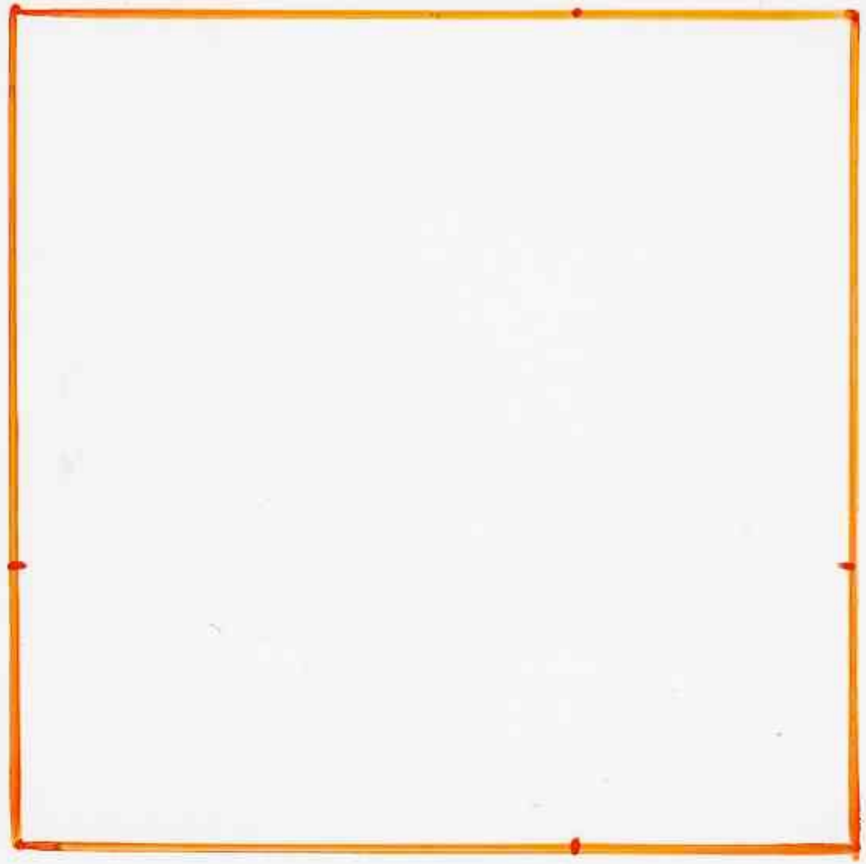
bijective proof of an identity

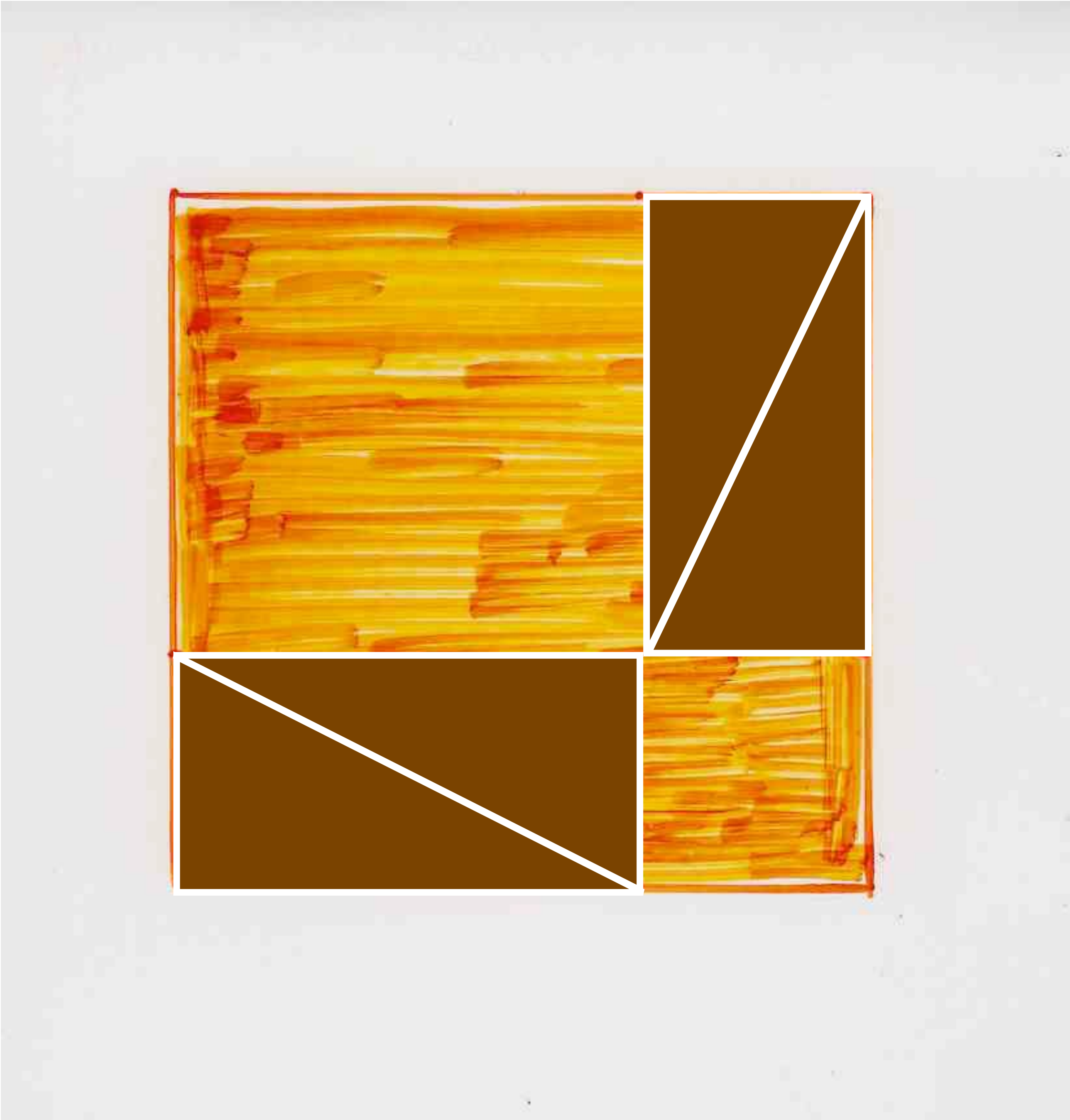


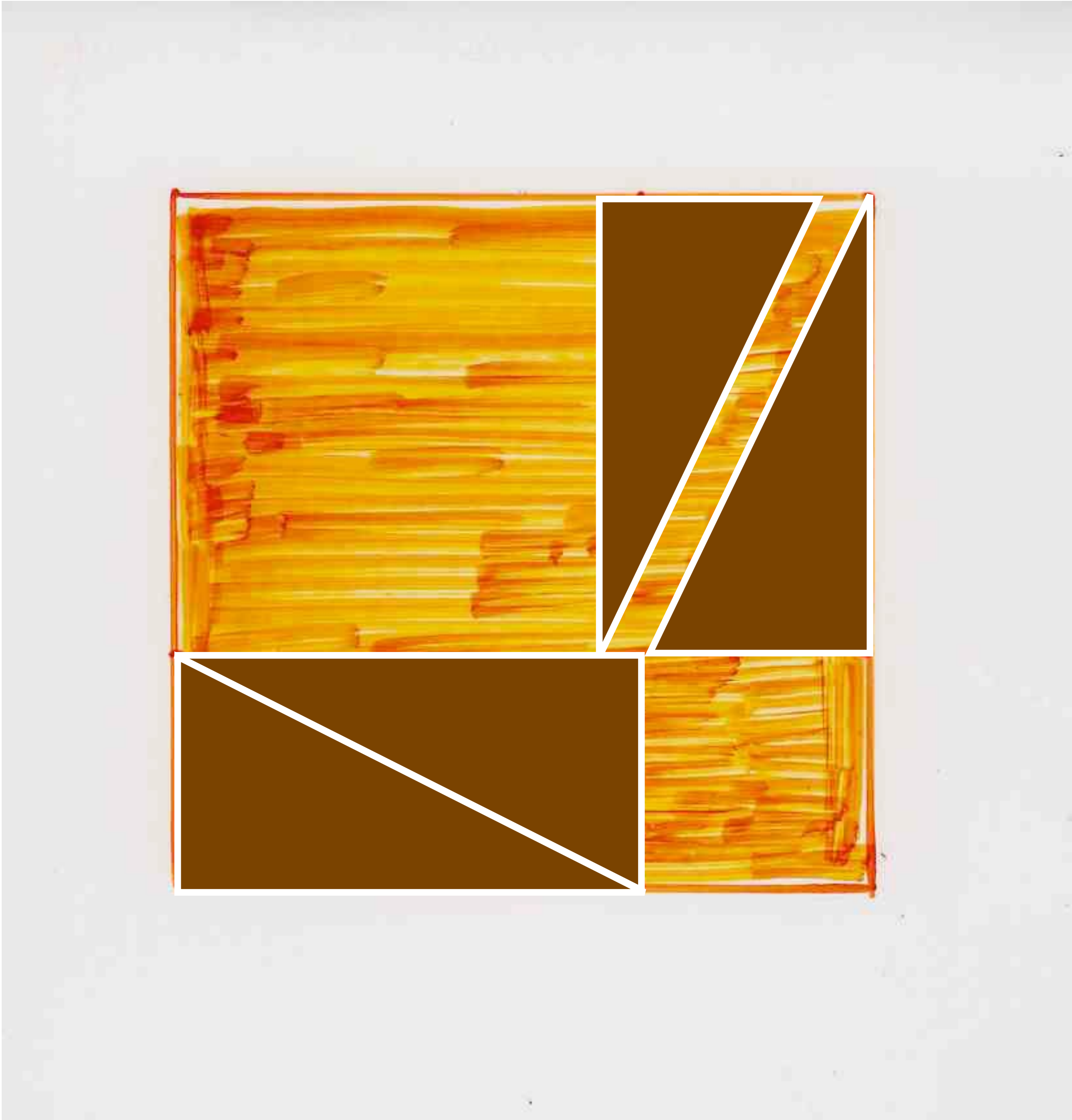
$$n^2 = 1 + 3 + \dots + (2n-1)$$

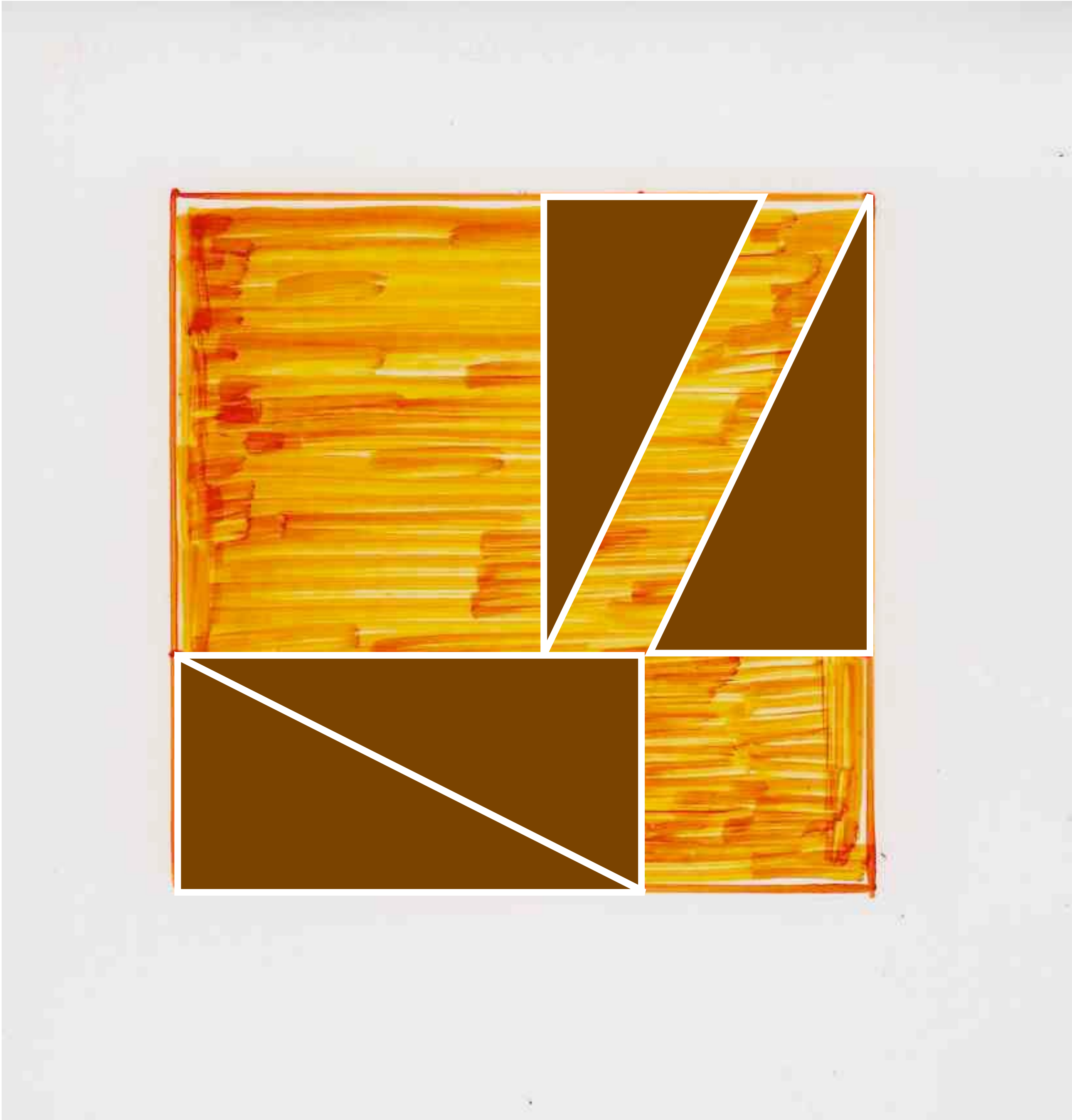
« visual proof »

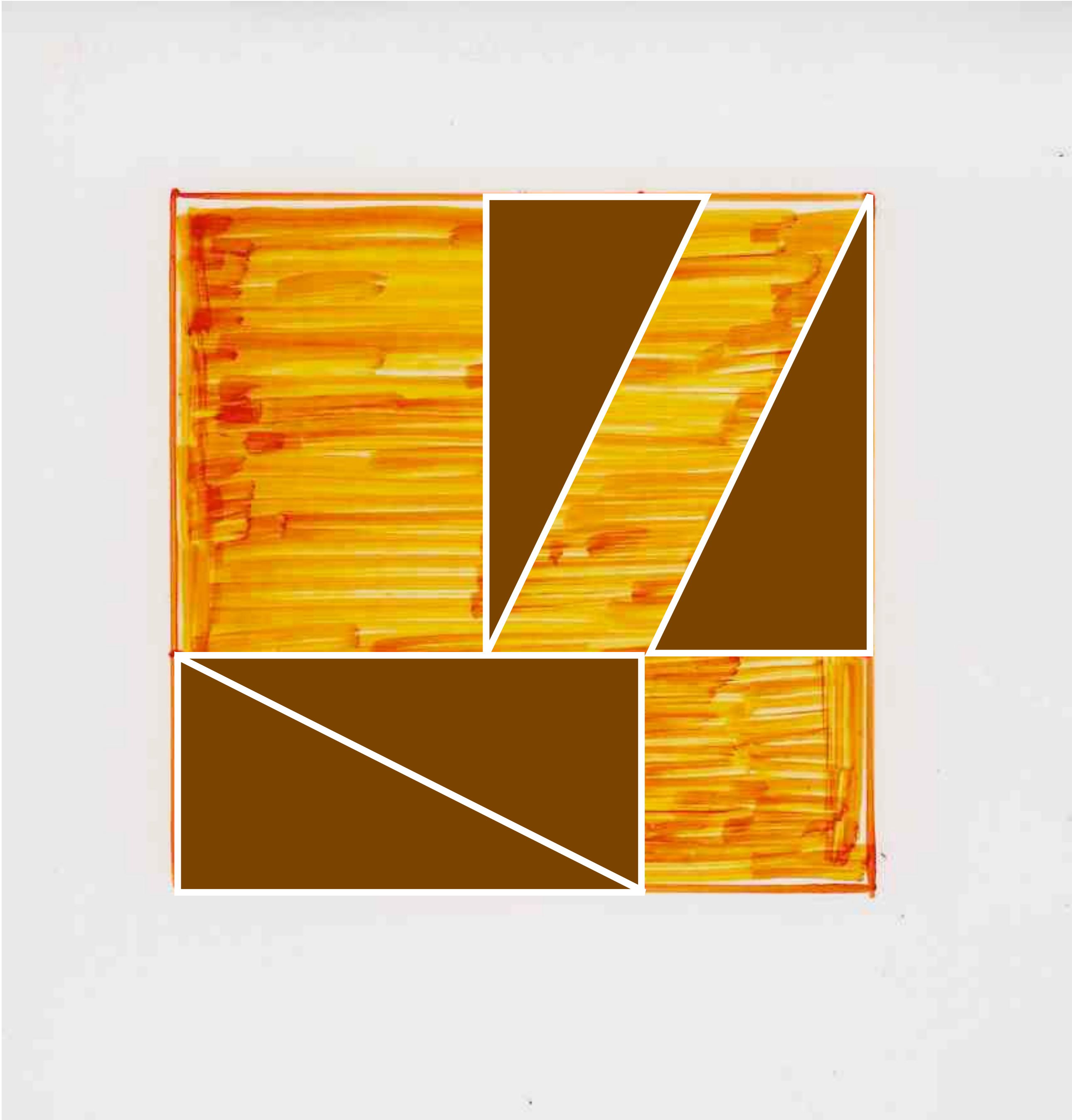
Pythagoras ....

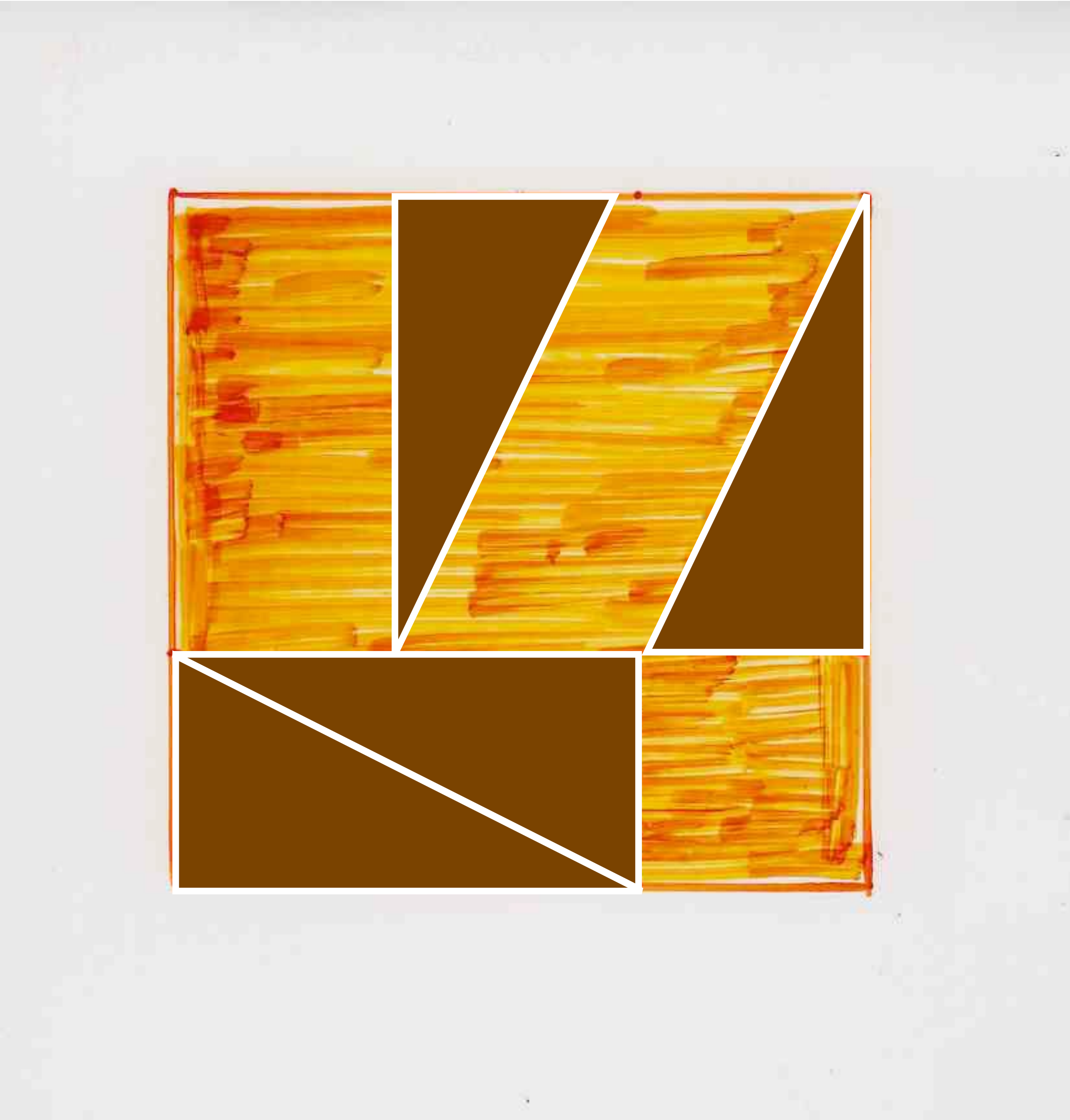


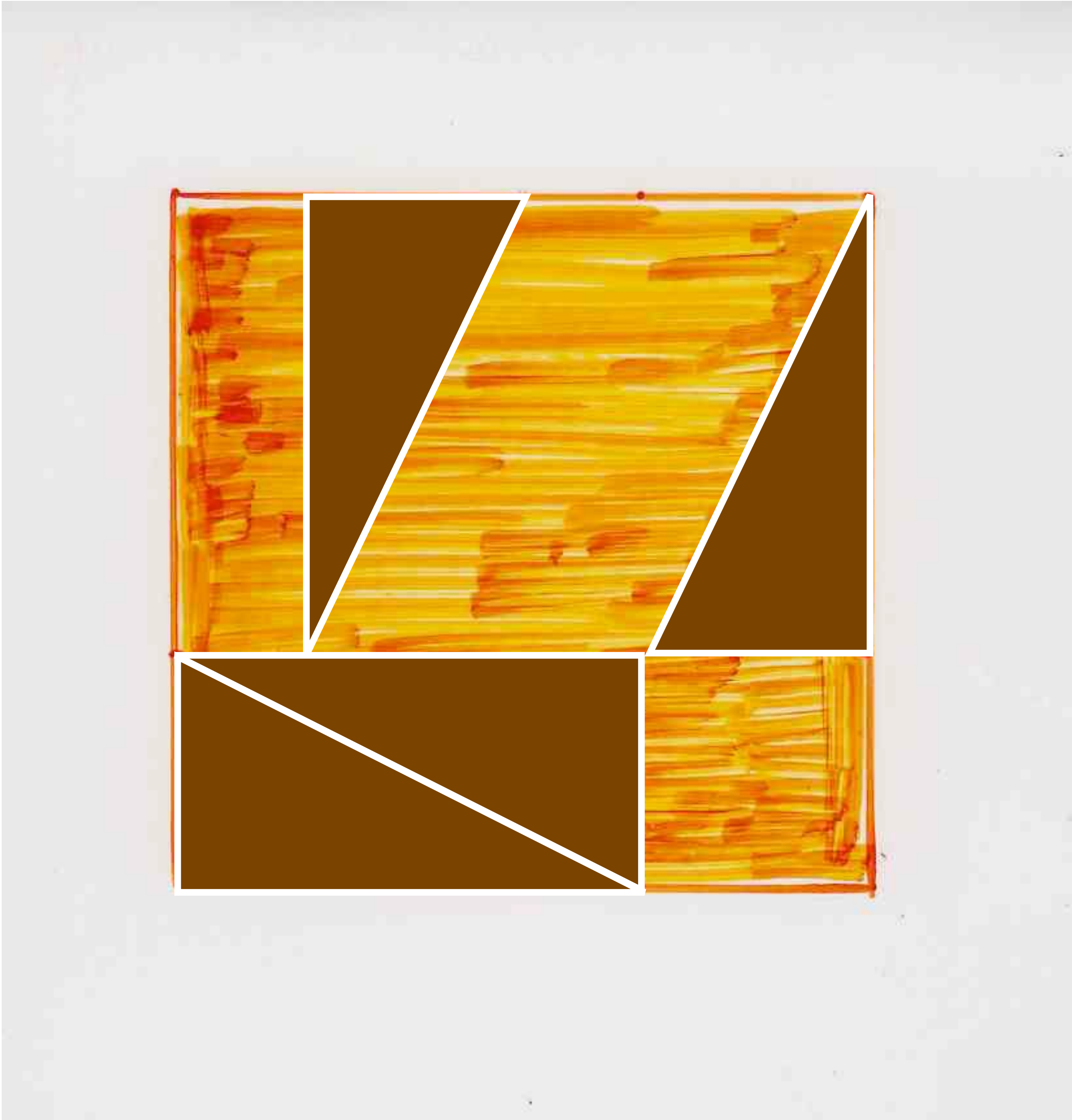


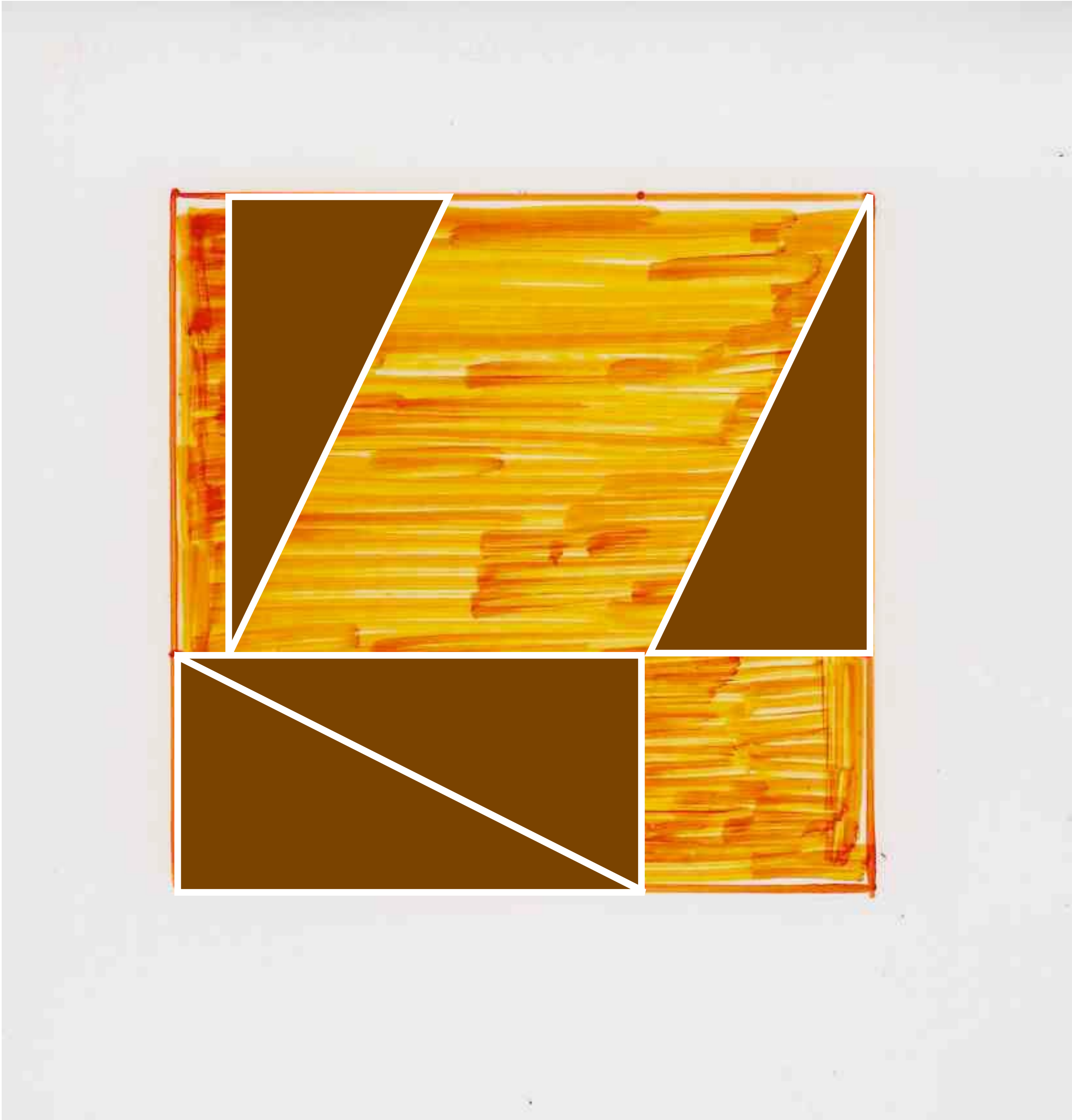


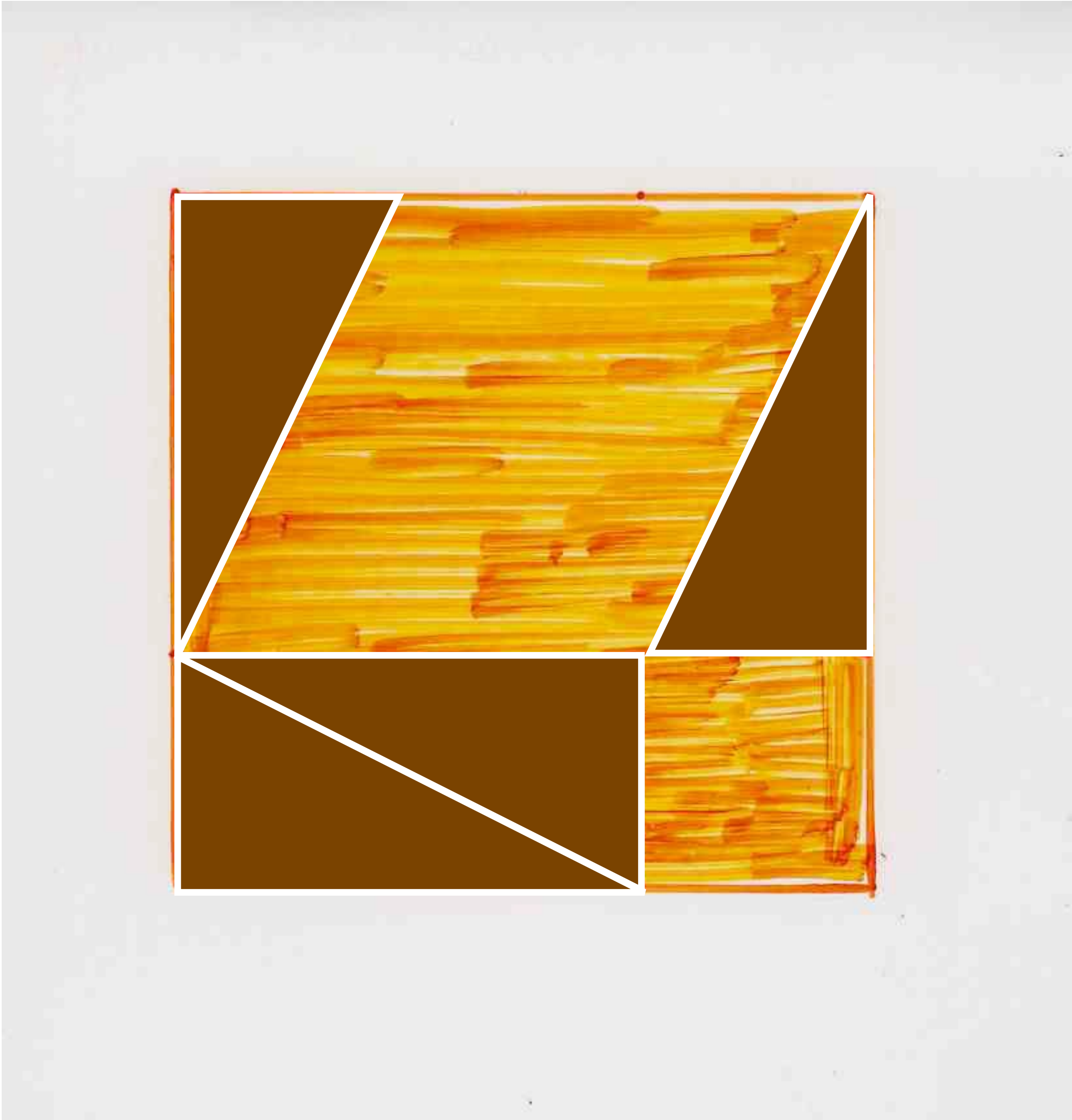


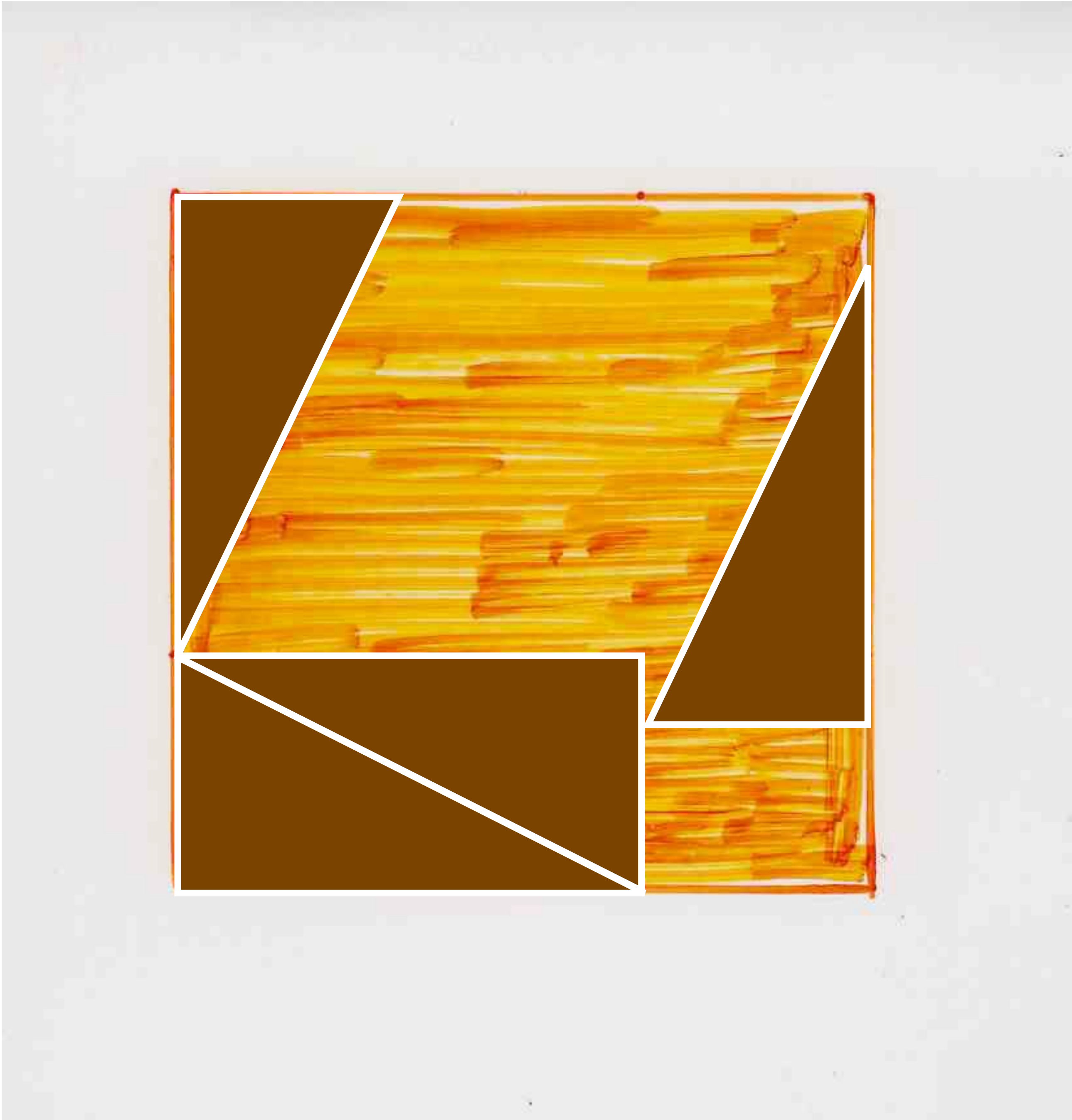


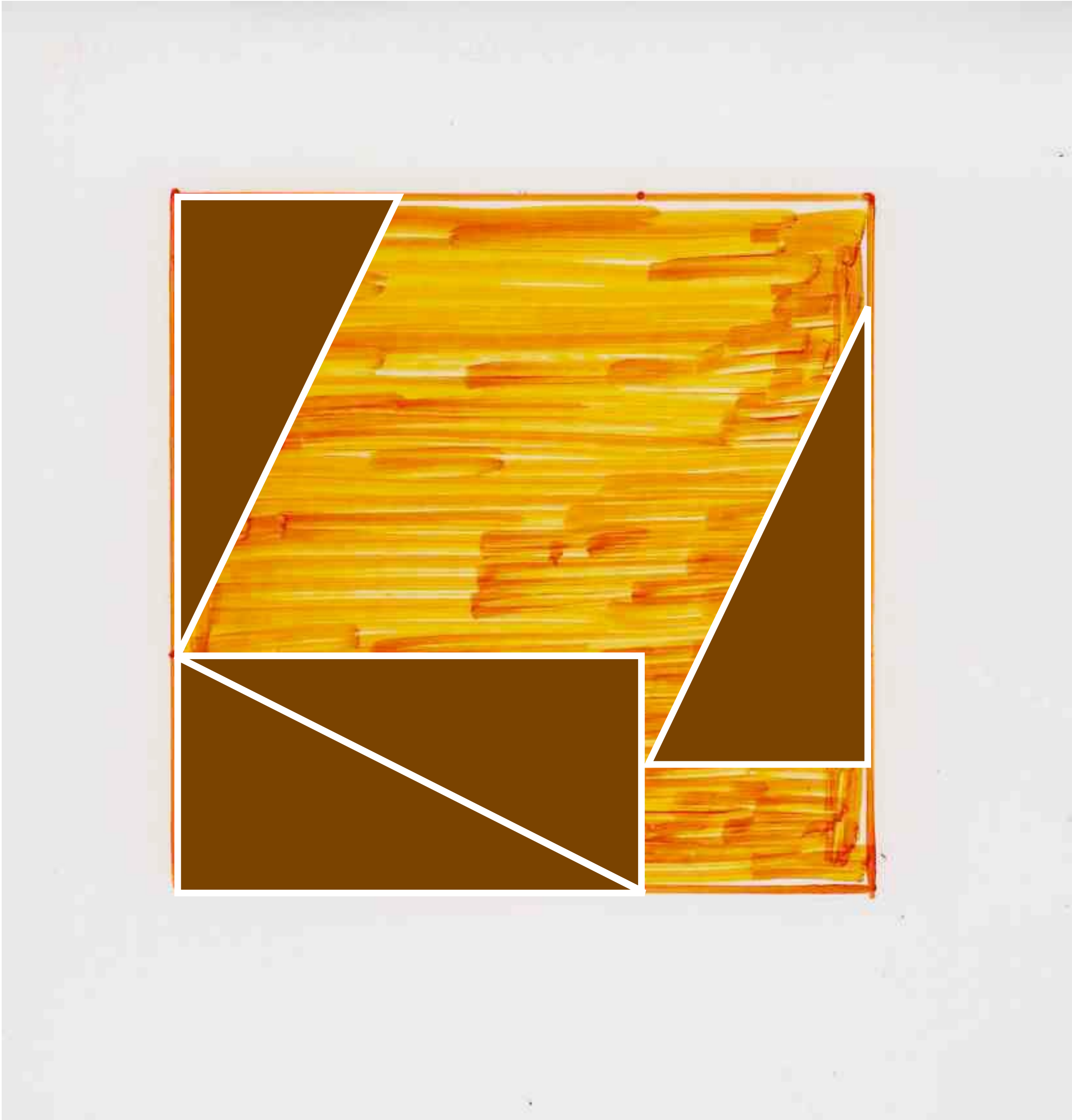


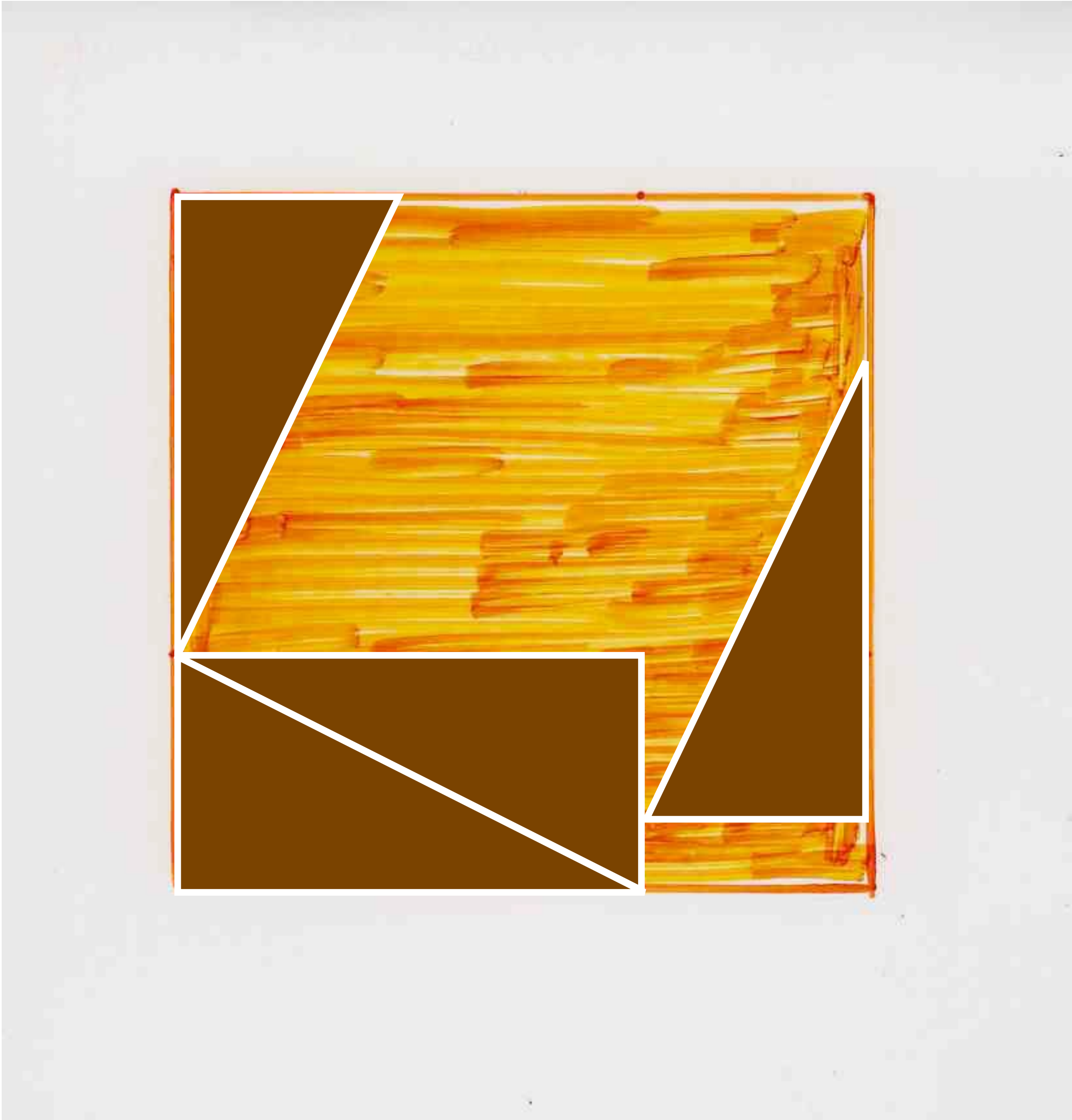


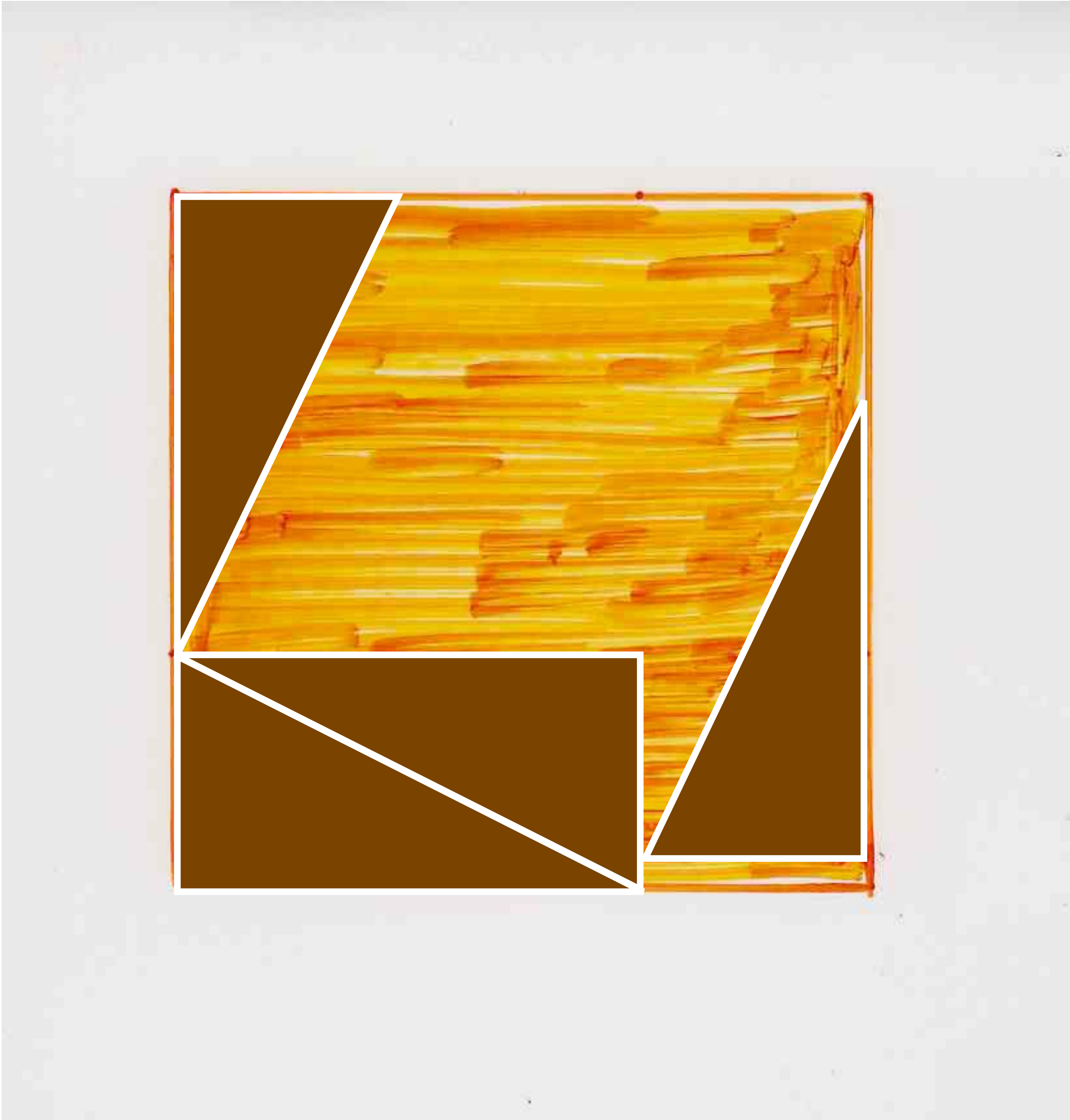


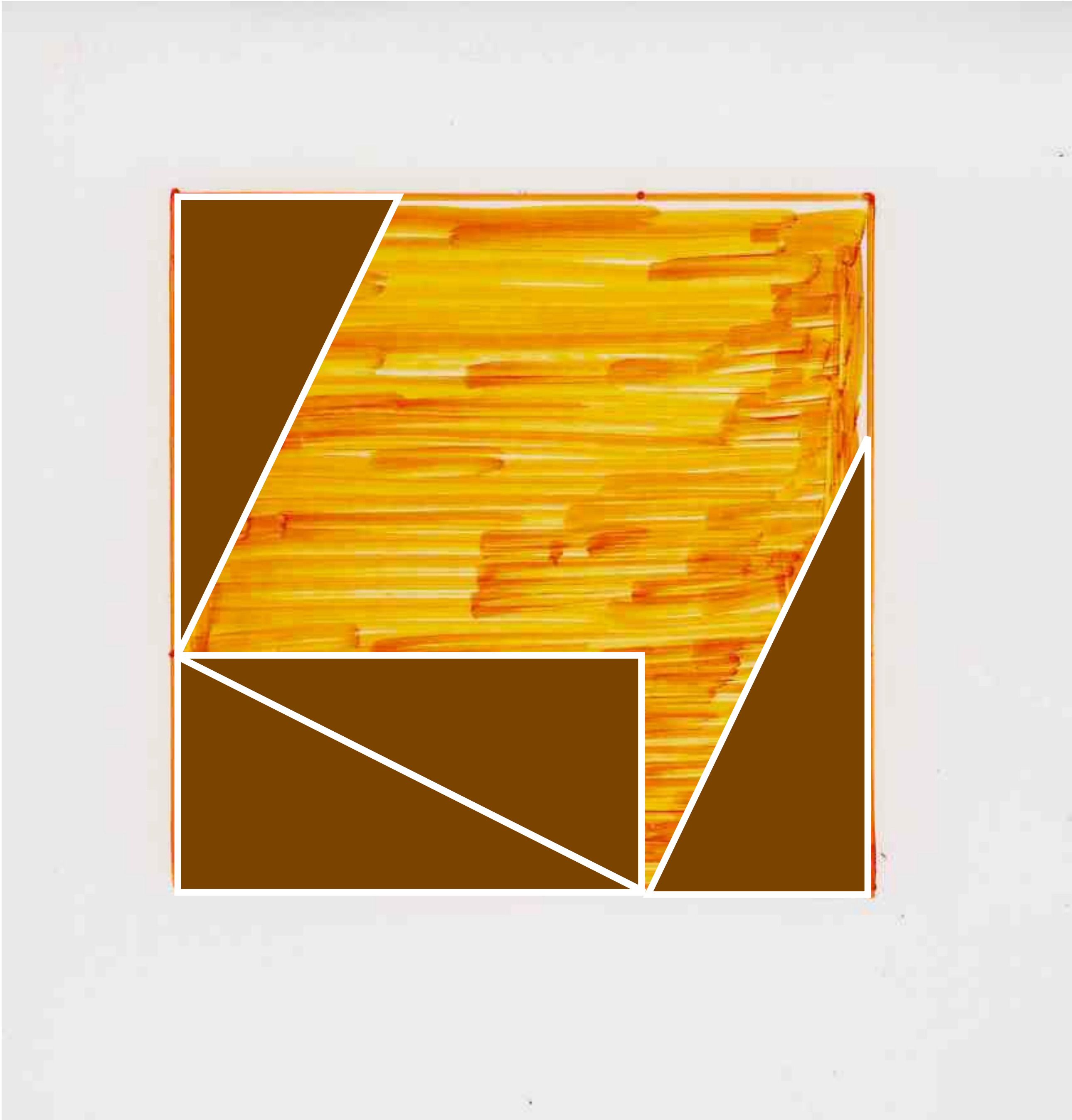


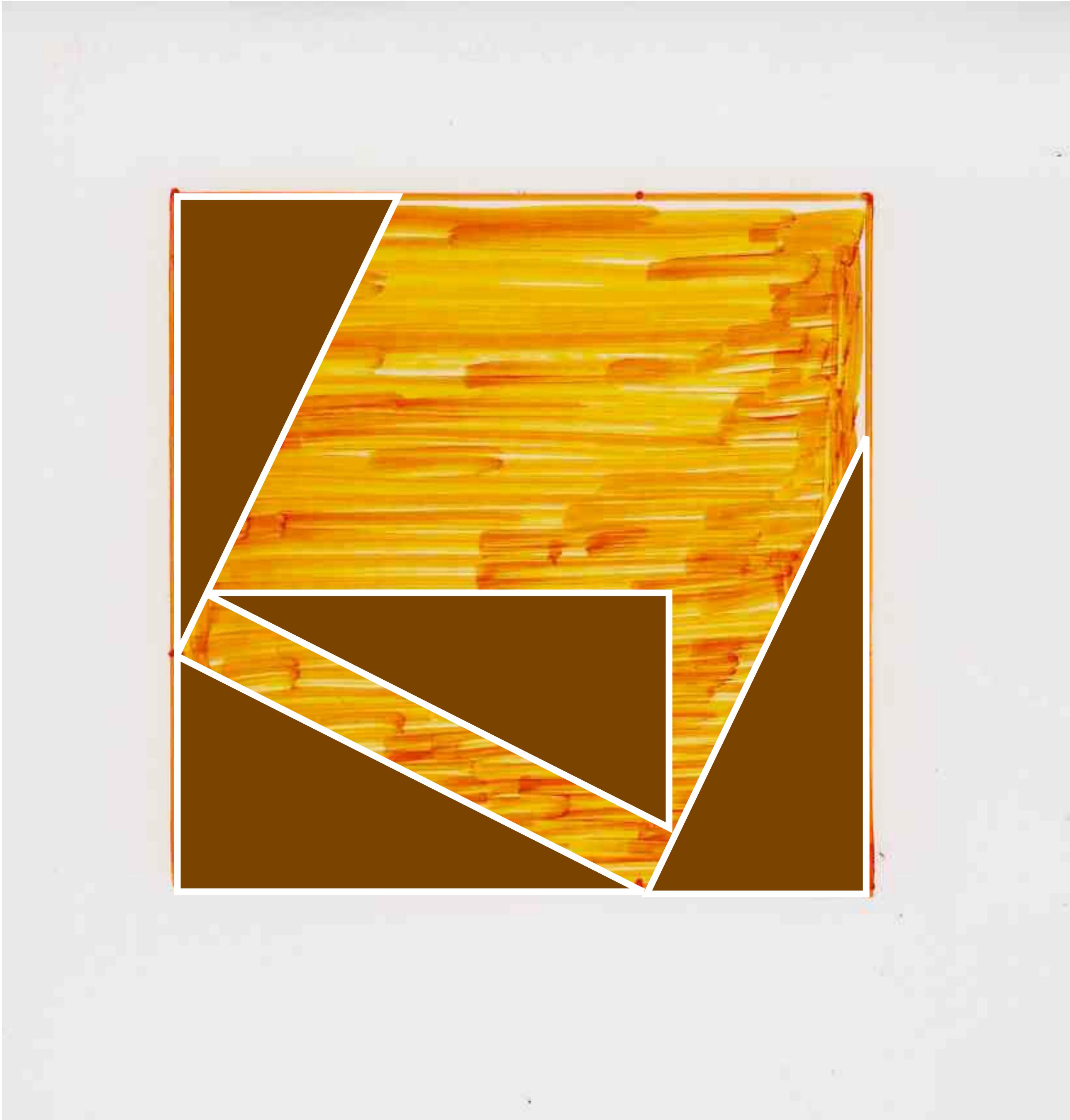


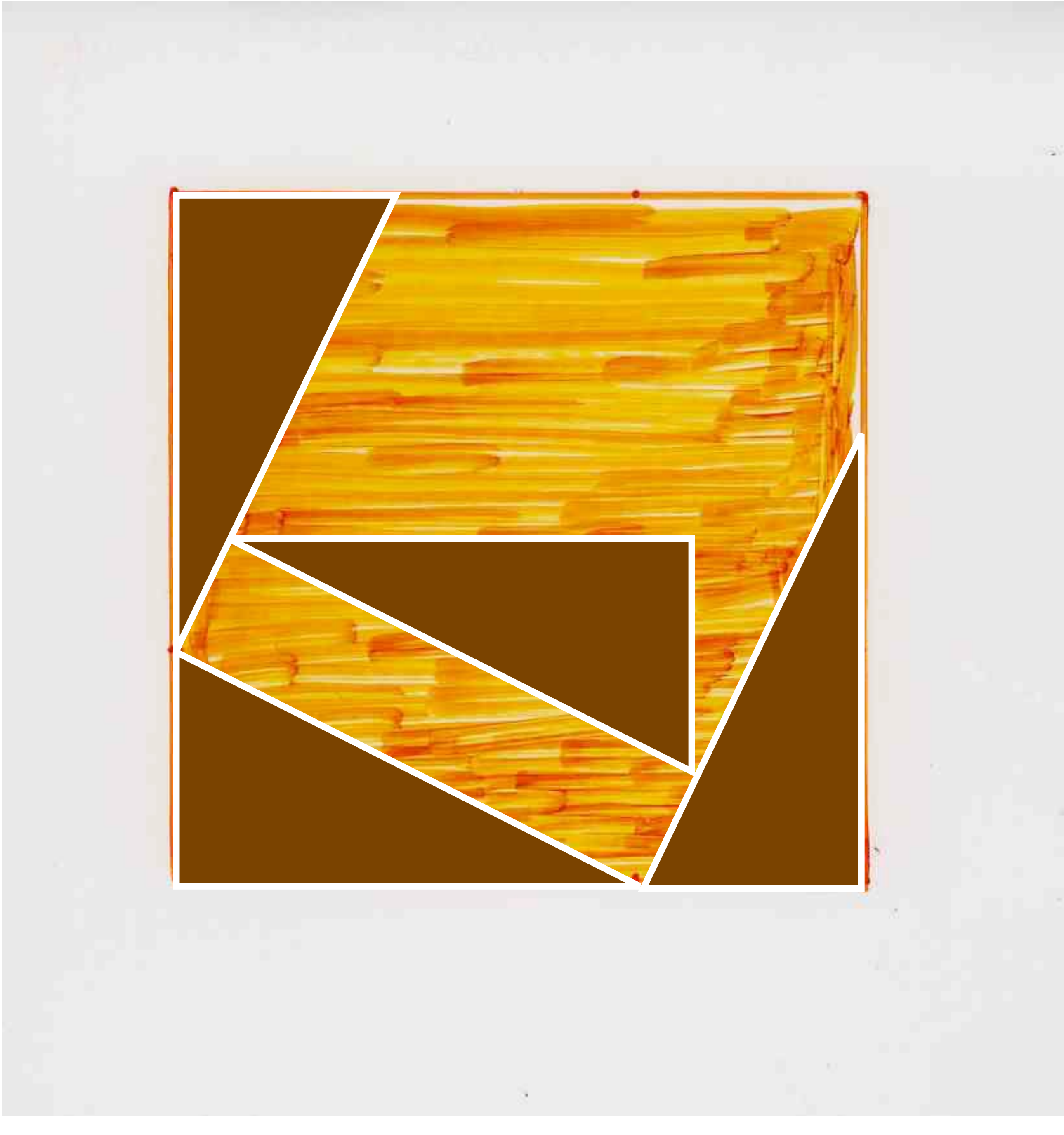


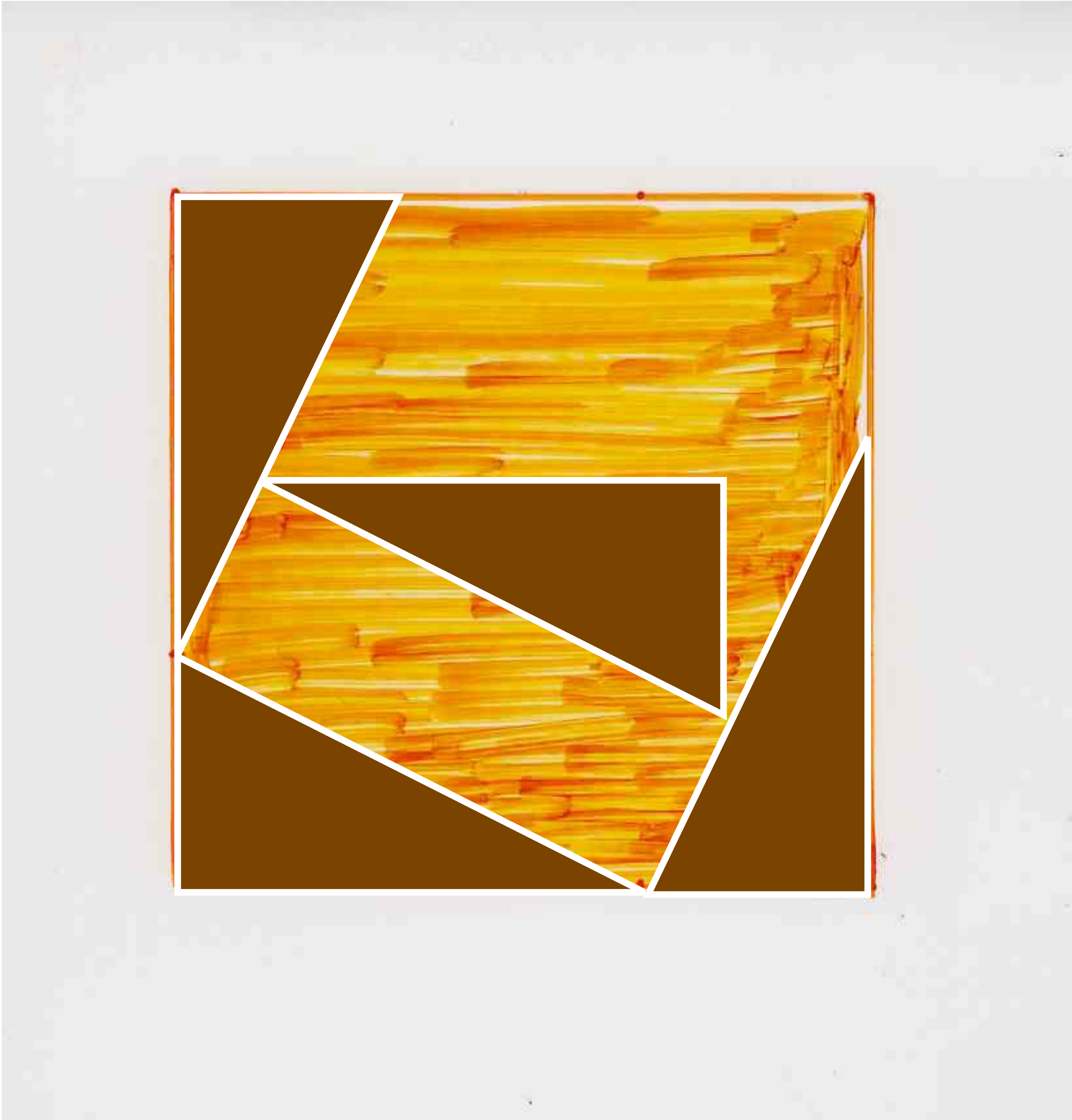


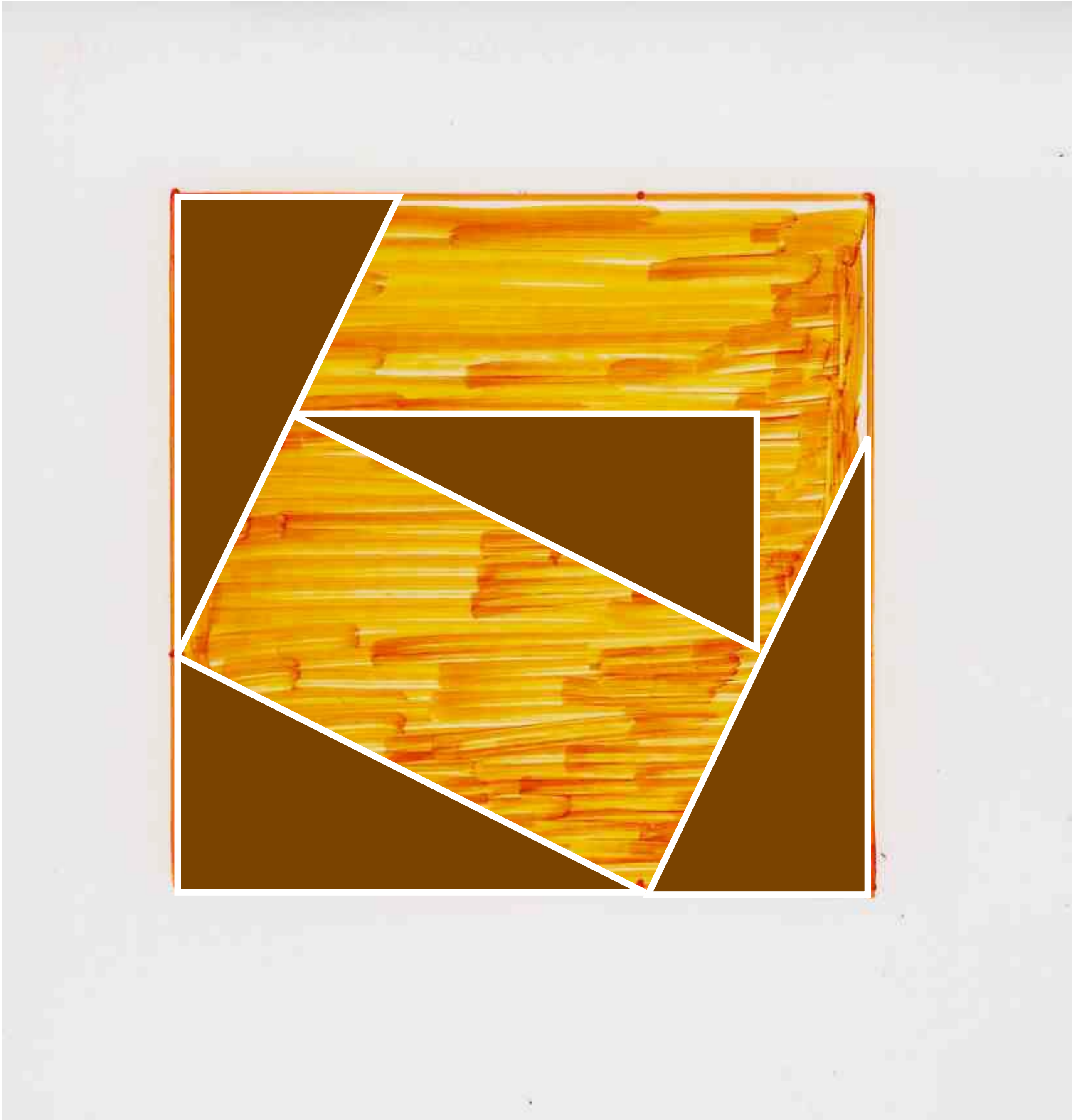


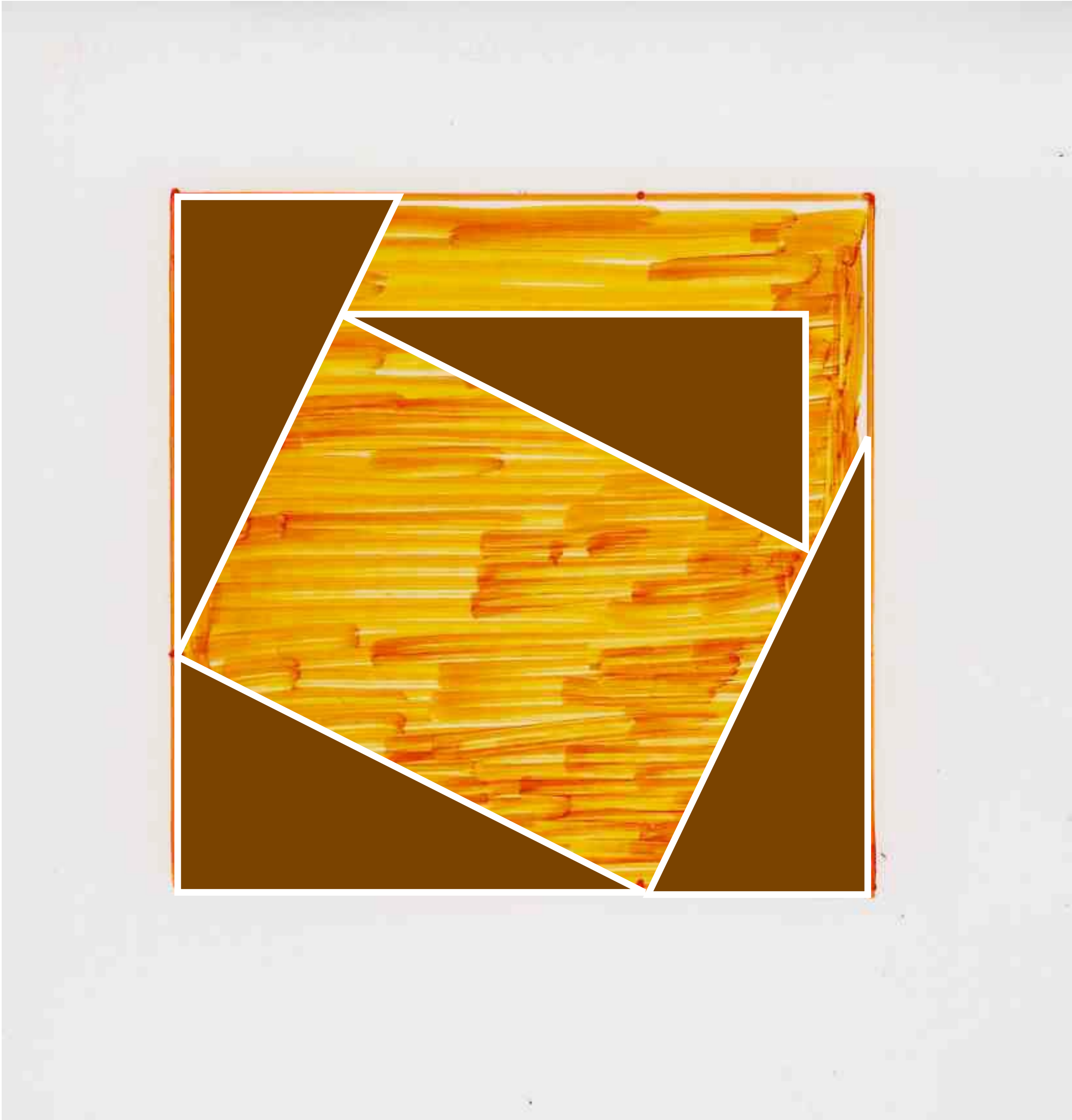


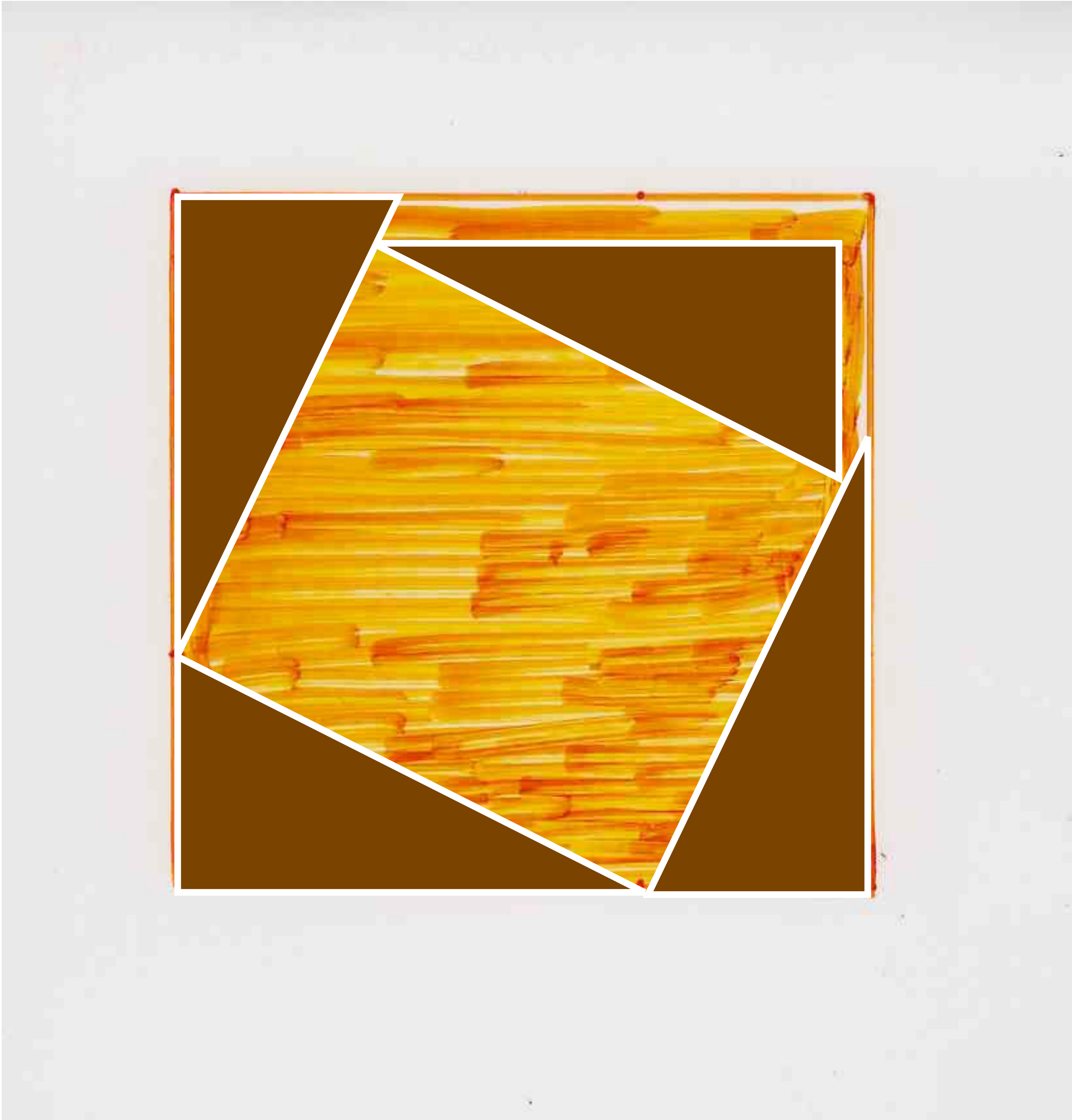


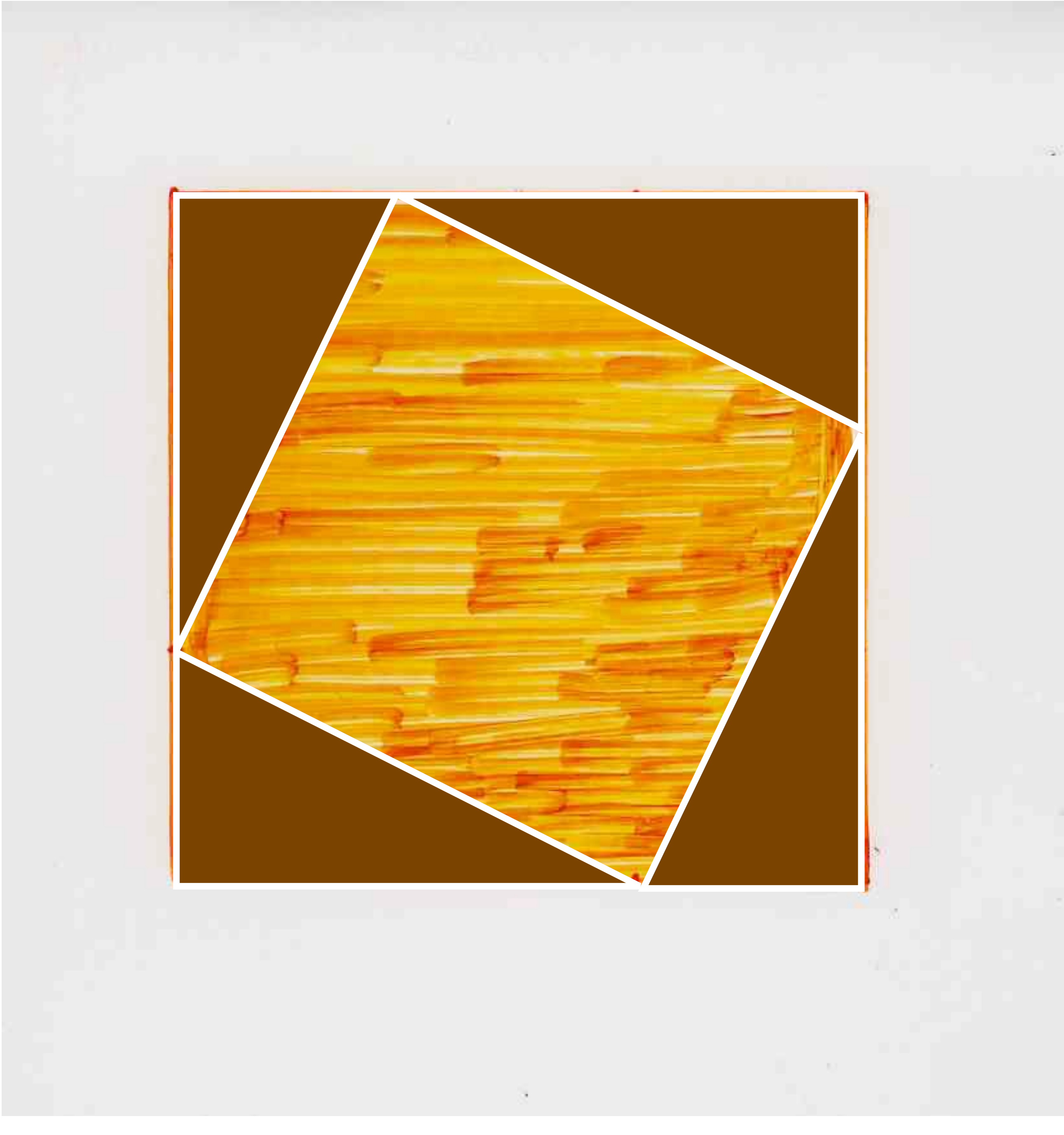






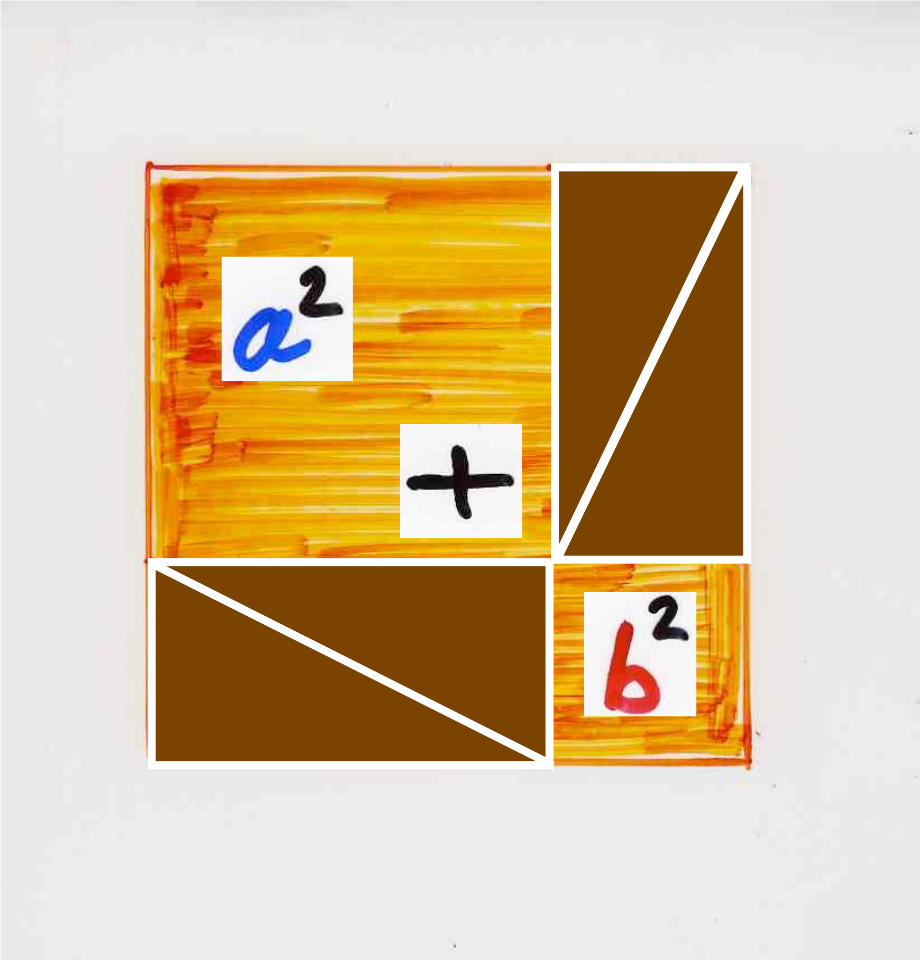


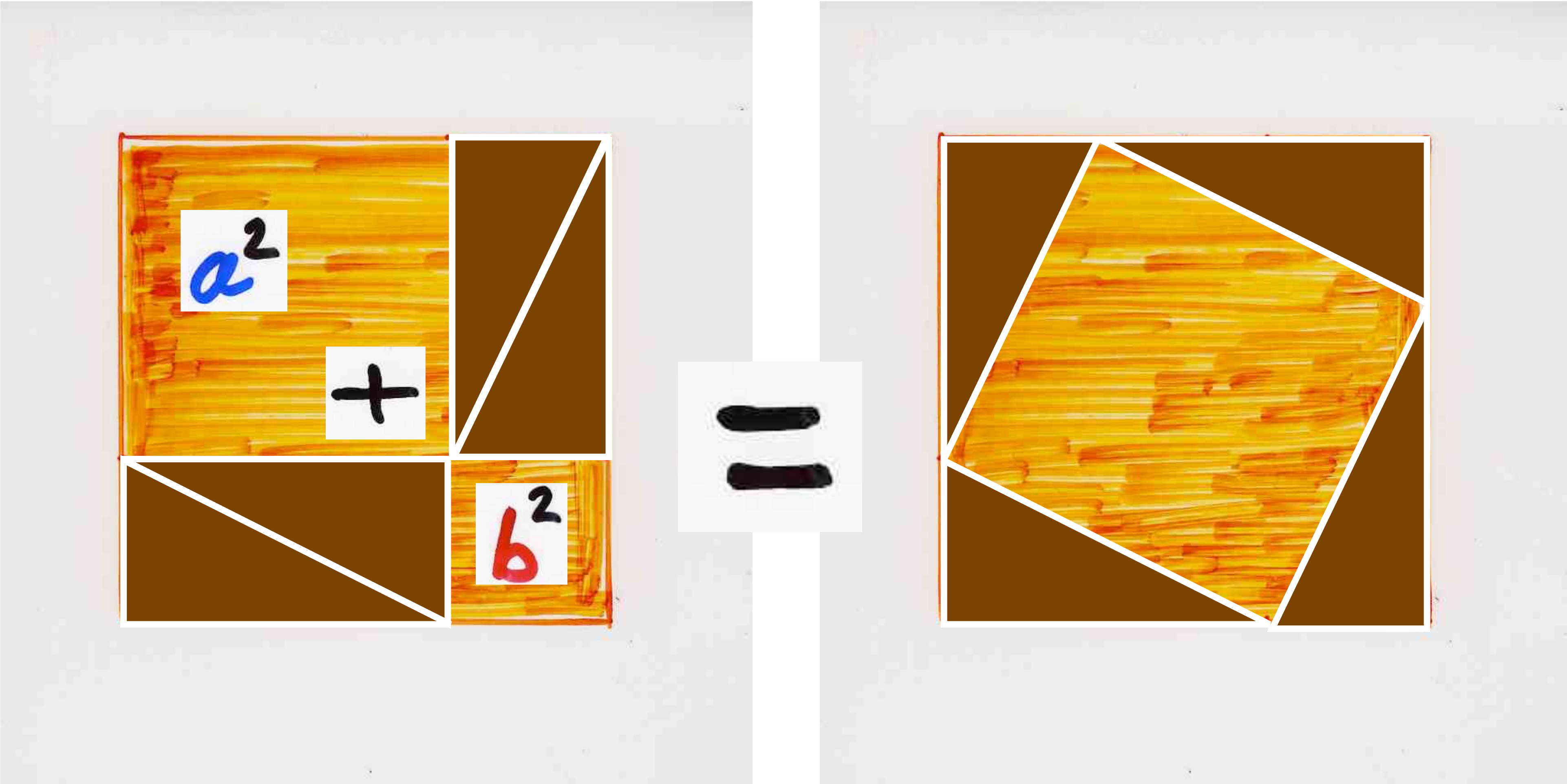


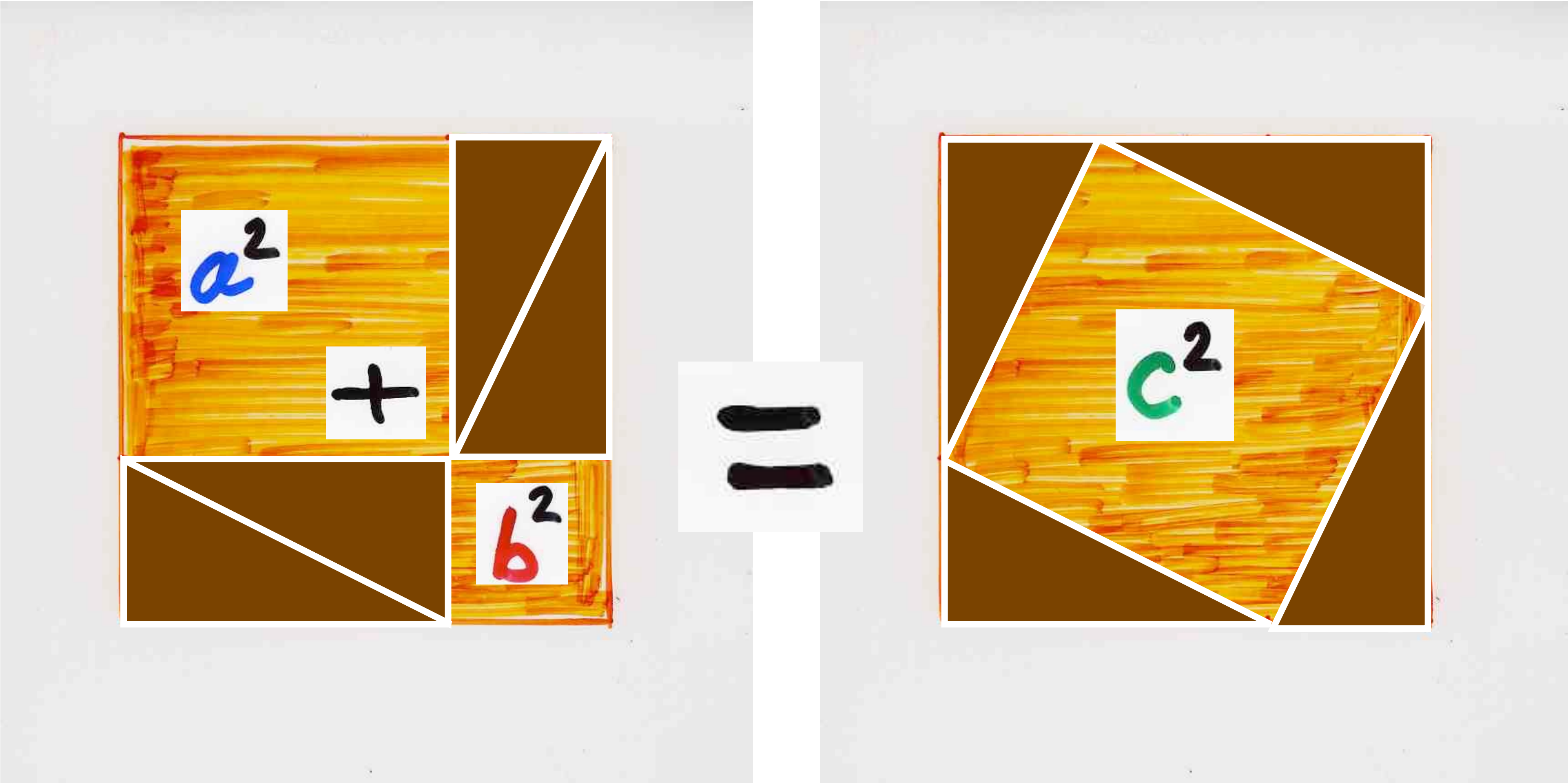




a2


a^2
+
b^2



a^2
+
b^2
=


a^2
+
b^2
=
c^2

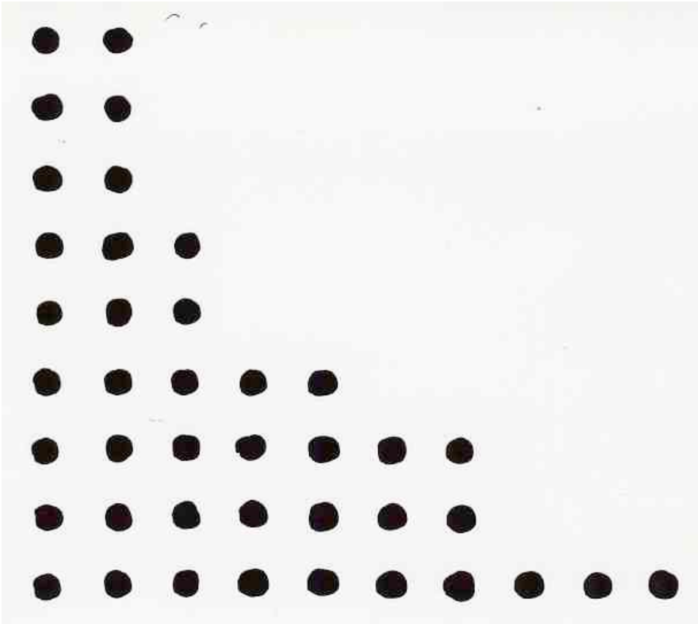
bijective proof of an identity

The "bijective paradigm"

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

right handside

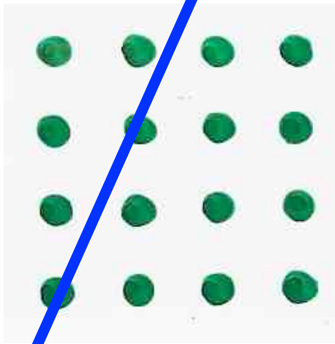


$$= \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

Ferrers diagram (= partition of an integer) left handside

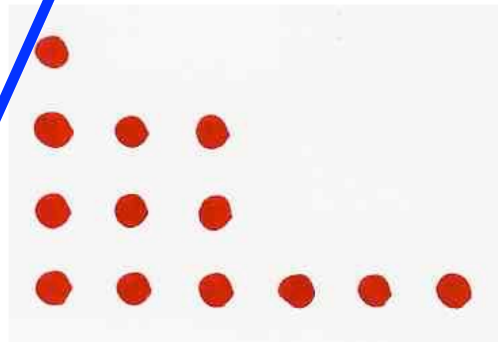
$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$q^{m^2}$$



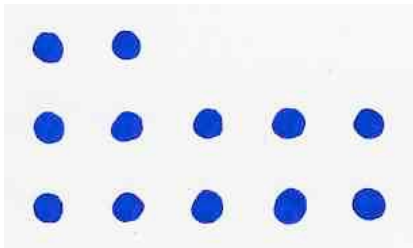
$m \times m$ square

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$



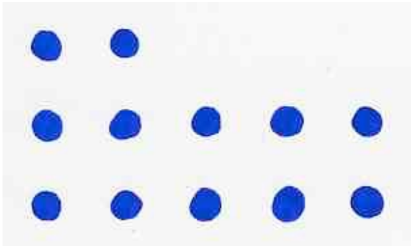
$\Big\}$ at most m rows

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

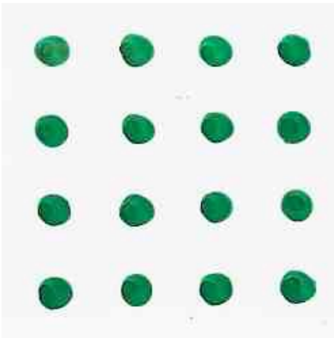


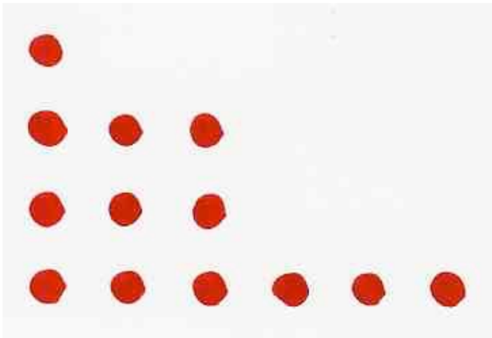
$\Big\}$ at most m rows

$m \times m$ square



at most m rows



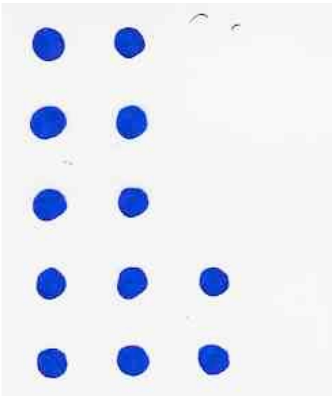


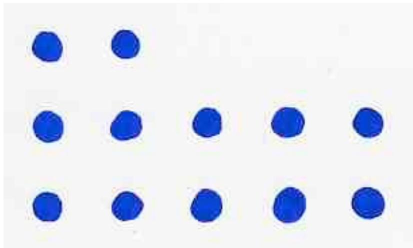
at most m rows

at most m columns

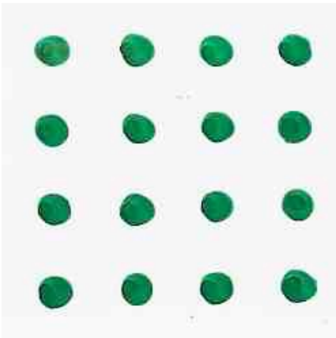
symmetry

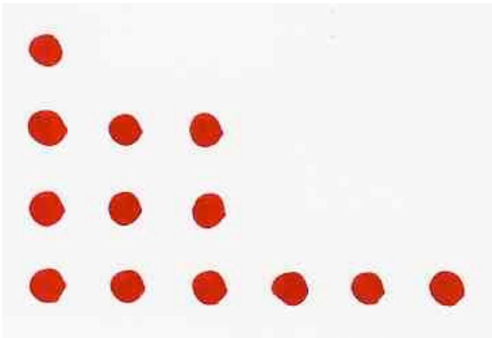
diagonal



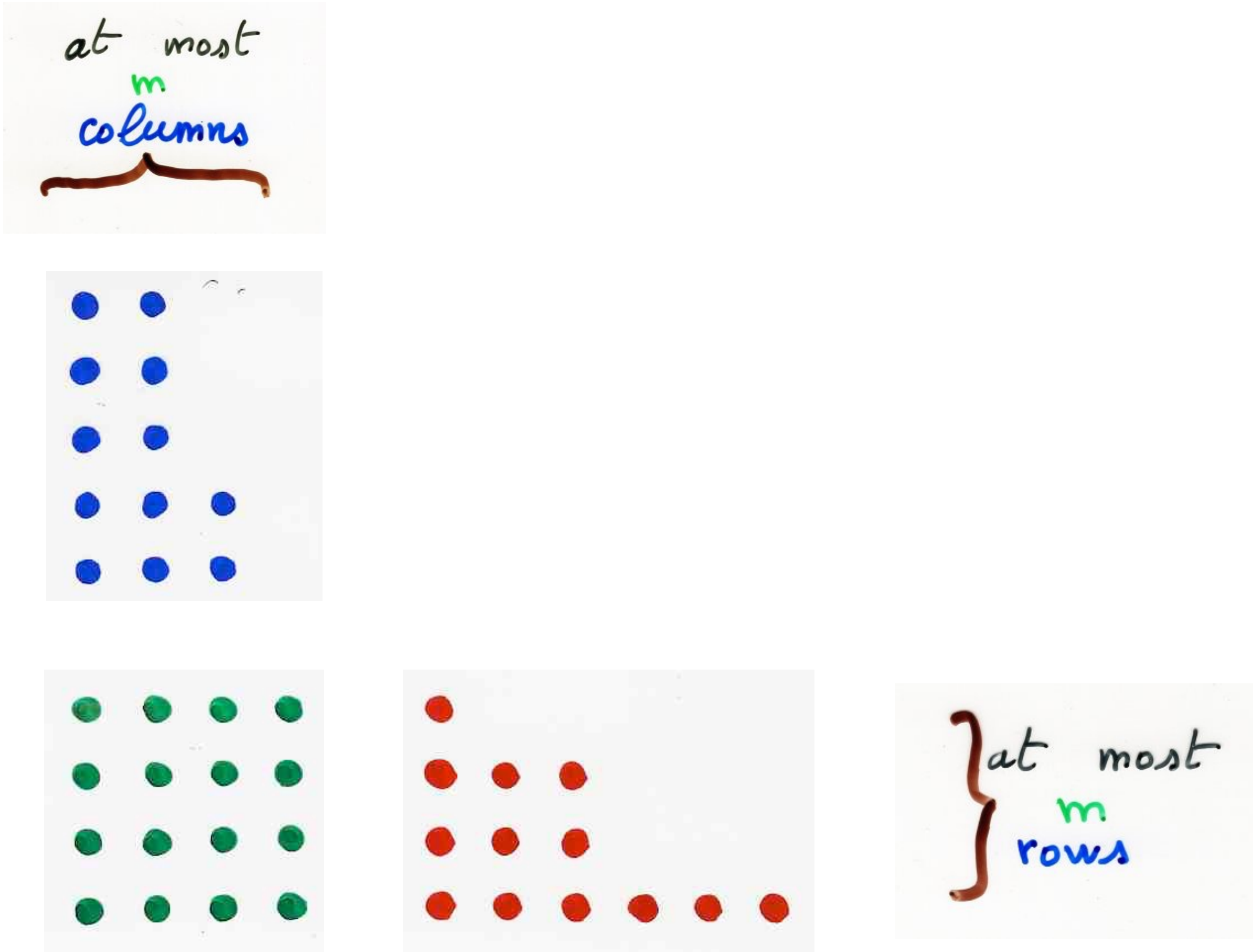


at most m rows



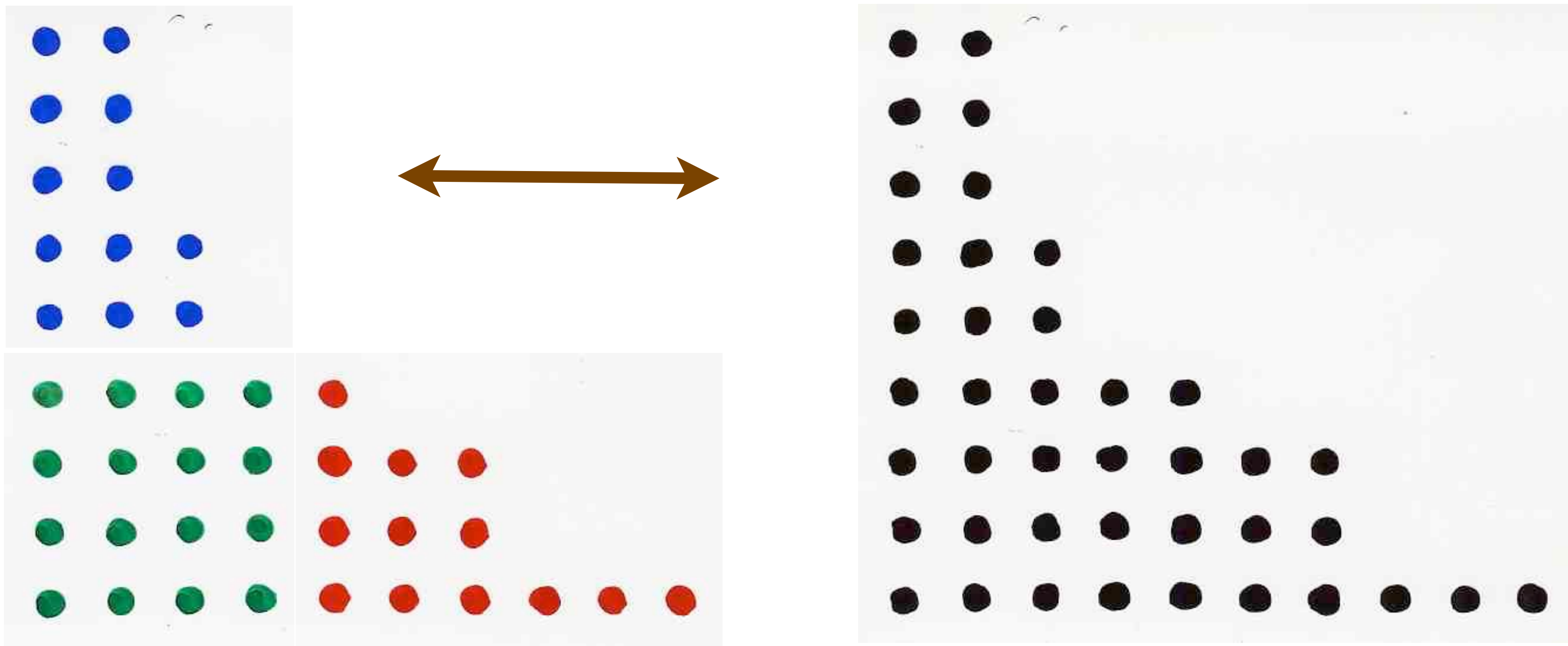


at most m rows

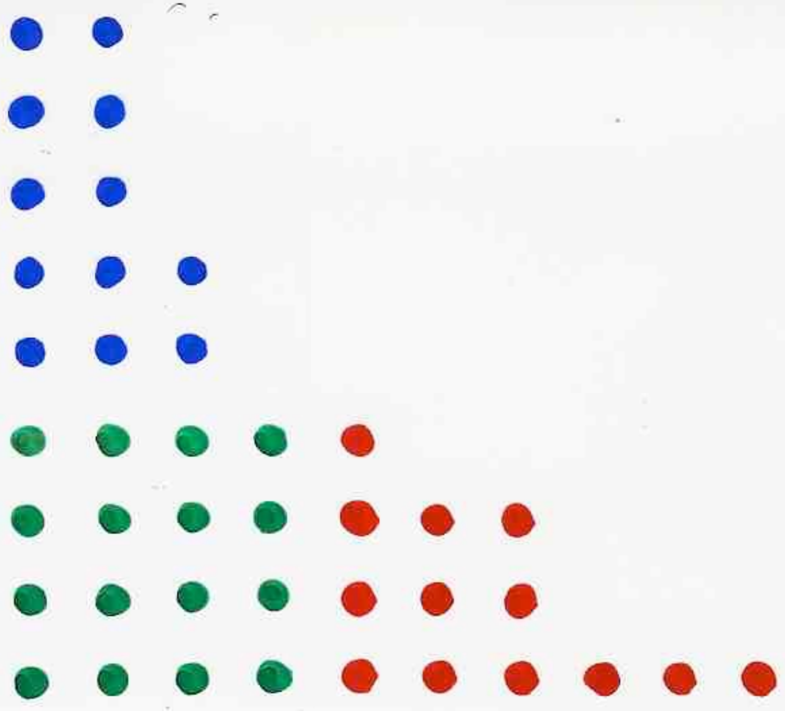

at most
m
columns
at most
m
rows

left handside

right handside



The identity means:



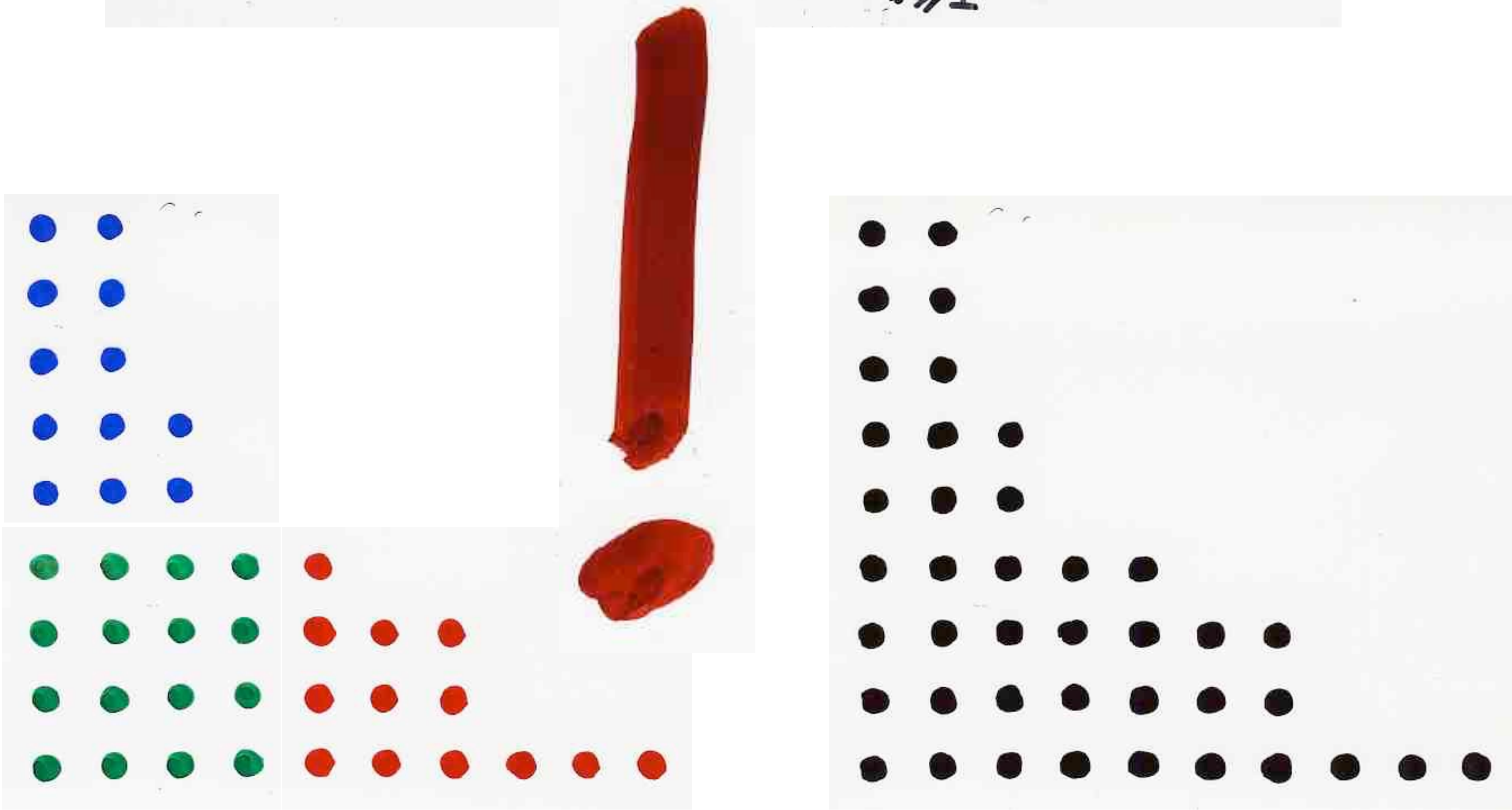
extract the biggest square ⊆ Ferrers diagram

What remains

- diagram having at most m rows
- diagram having at most m columns

m size of the square

$$\sum_{m\geqslant 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geqslant 1} \frac{1}{(1-q^i)}$$



"drawing" calculus...

computing with "drawings"

(figures)

二

=

$\longleftrightarrow$

=

better
understanding

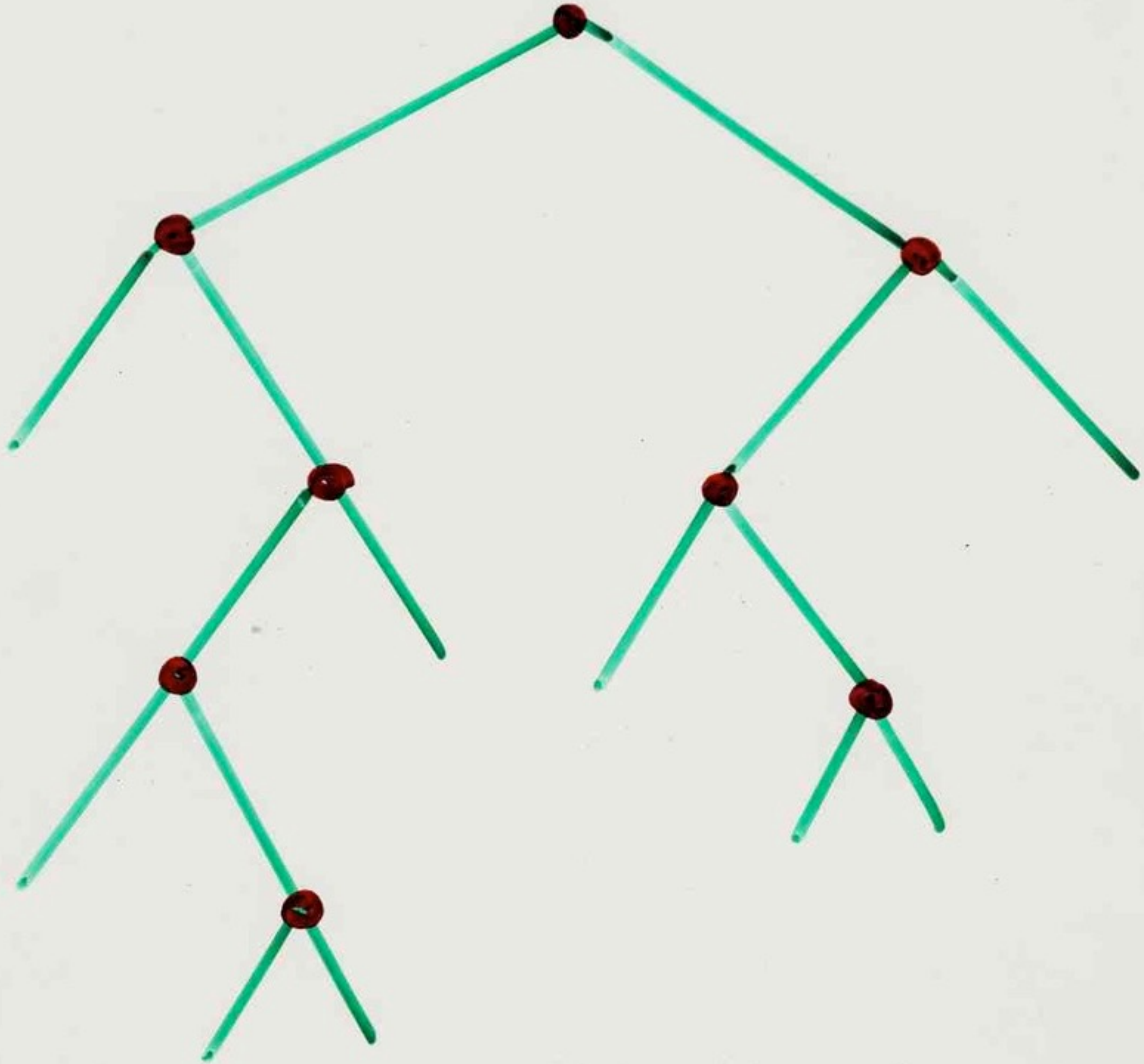
$\longleftrightarrow$

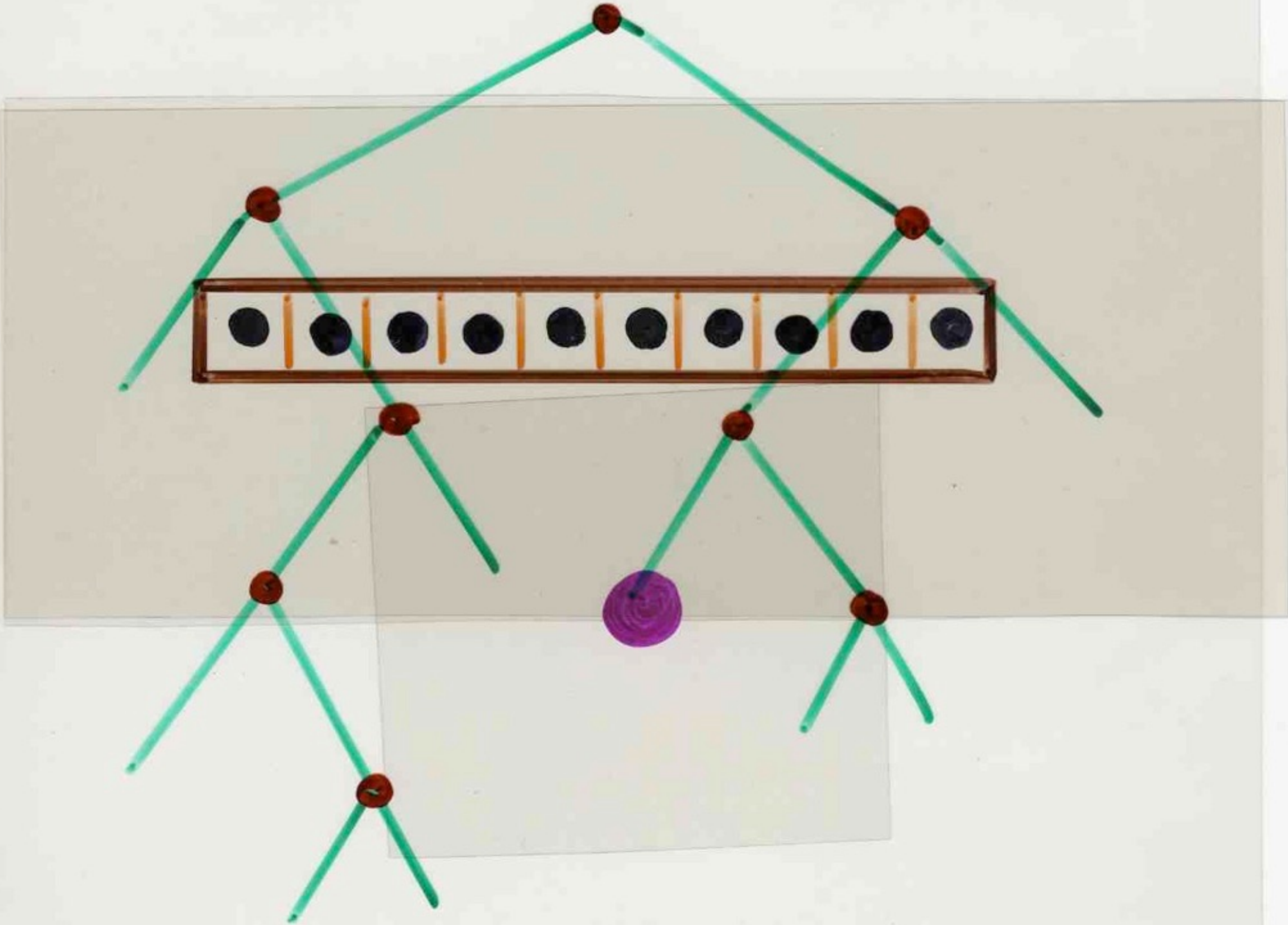
bijective combinatorics

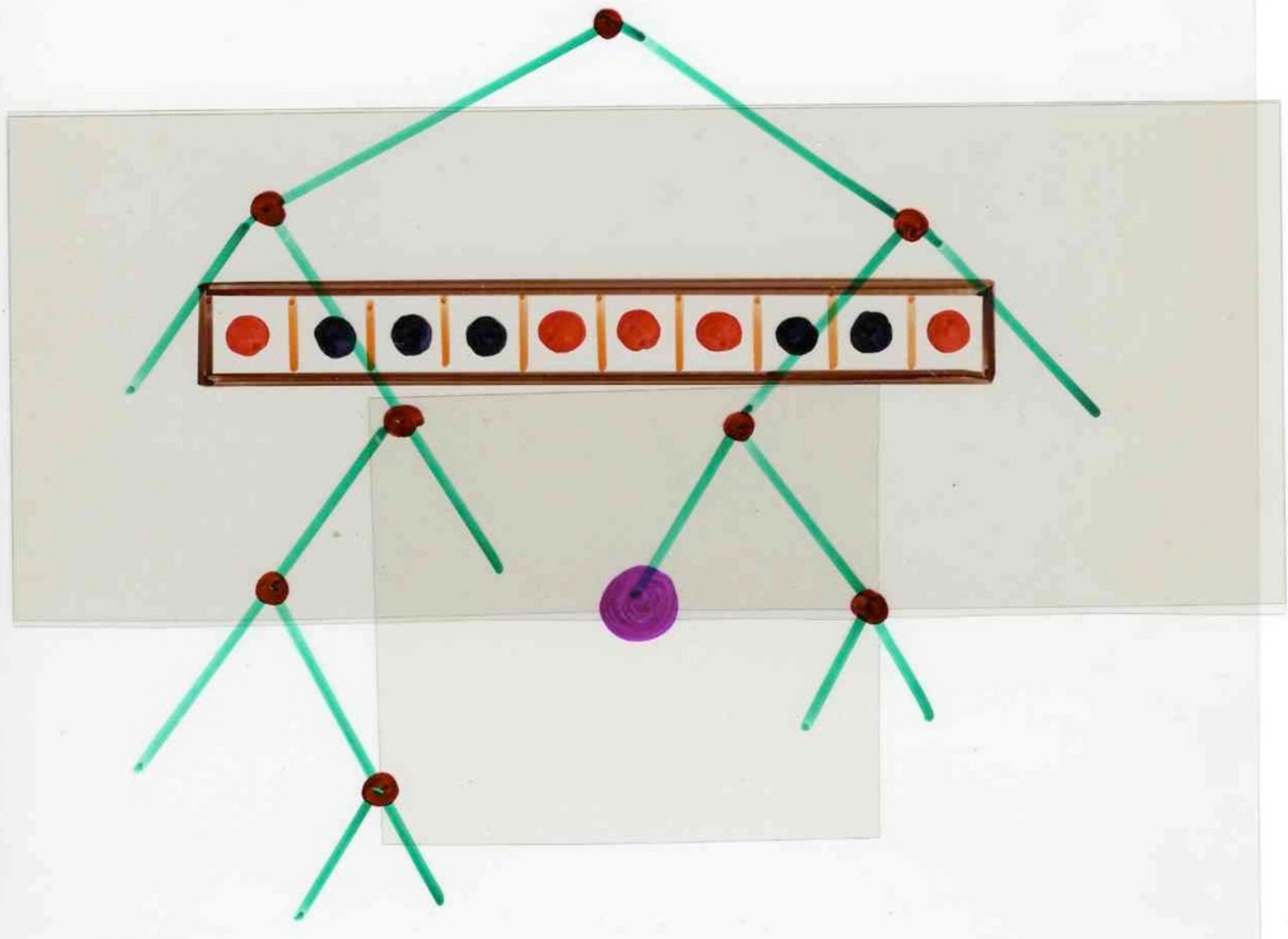
example: Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$(n+1)\,C_n = \binom{2n}{n}$$







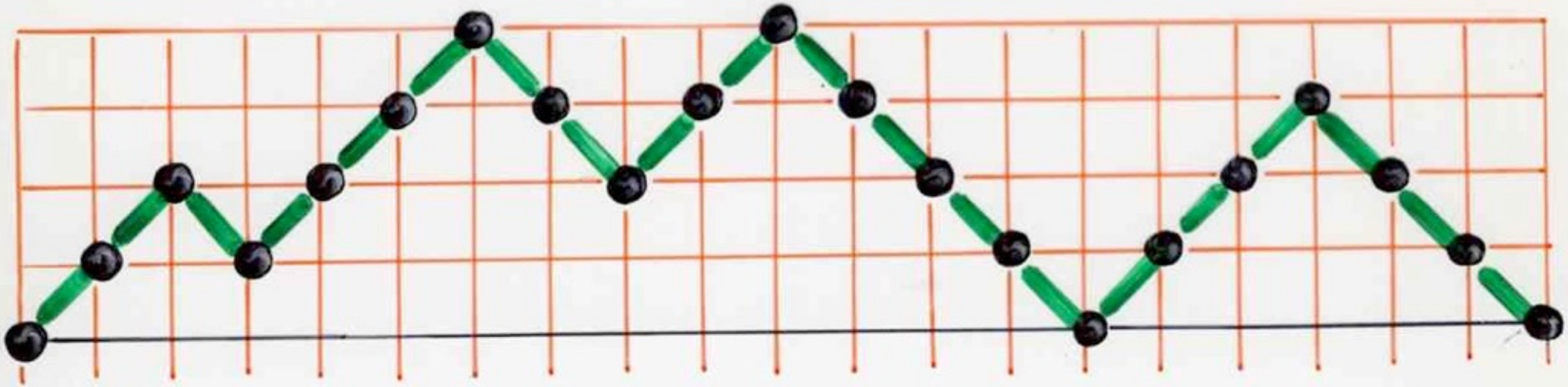
$$C_n = \frac{1}{(2n+1)}\binom{2n+1}{n}$$

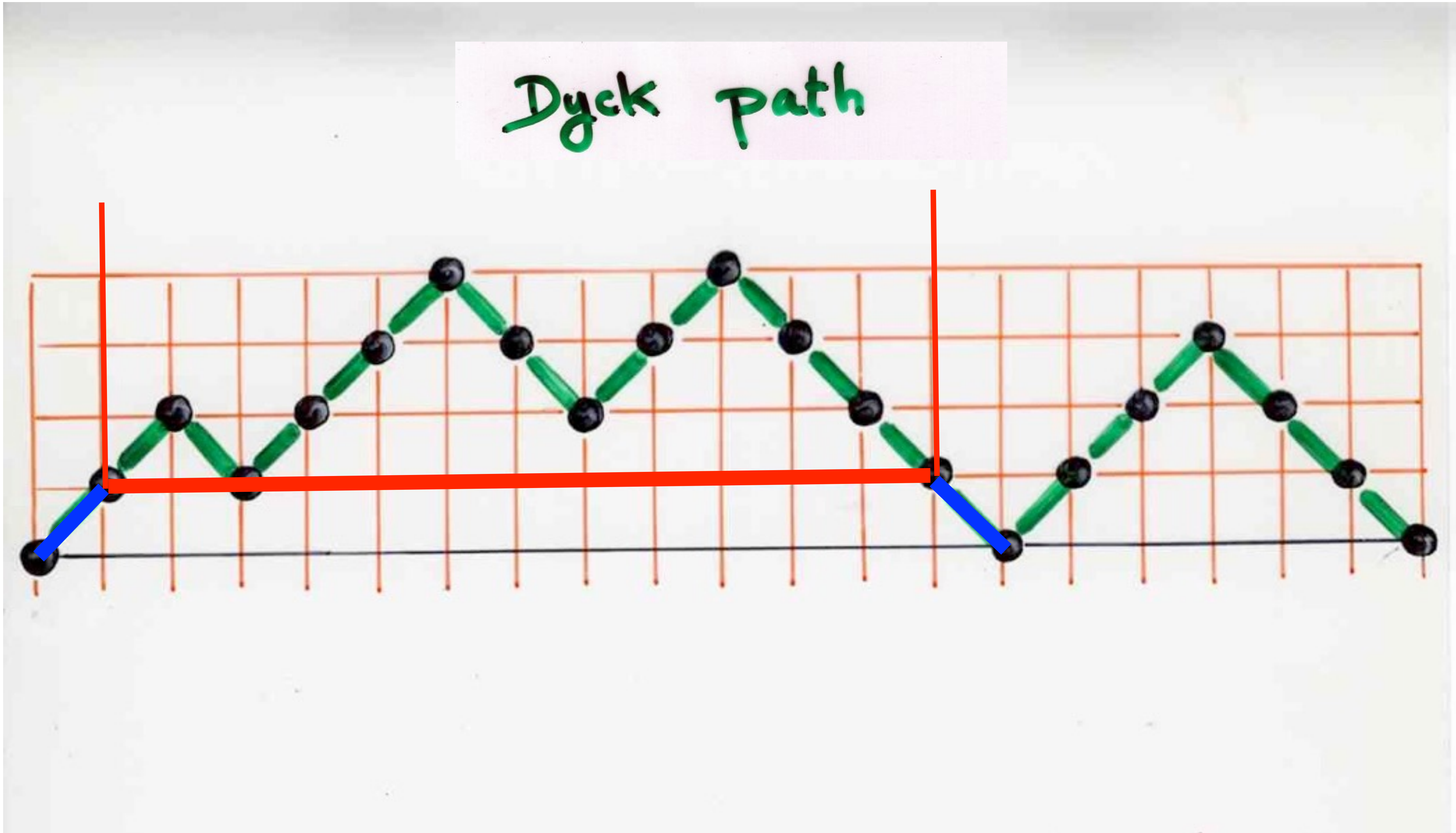
$$(2n+1)\,C_n = \binom{2n+1}{n}$$

$$2\,(2n+1)\,C_n = (n+2)\,C_{n+1}$$

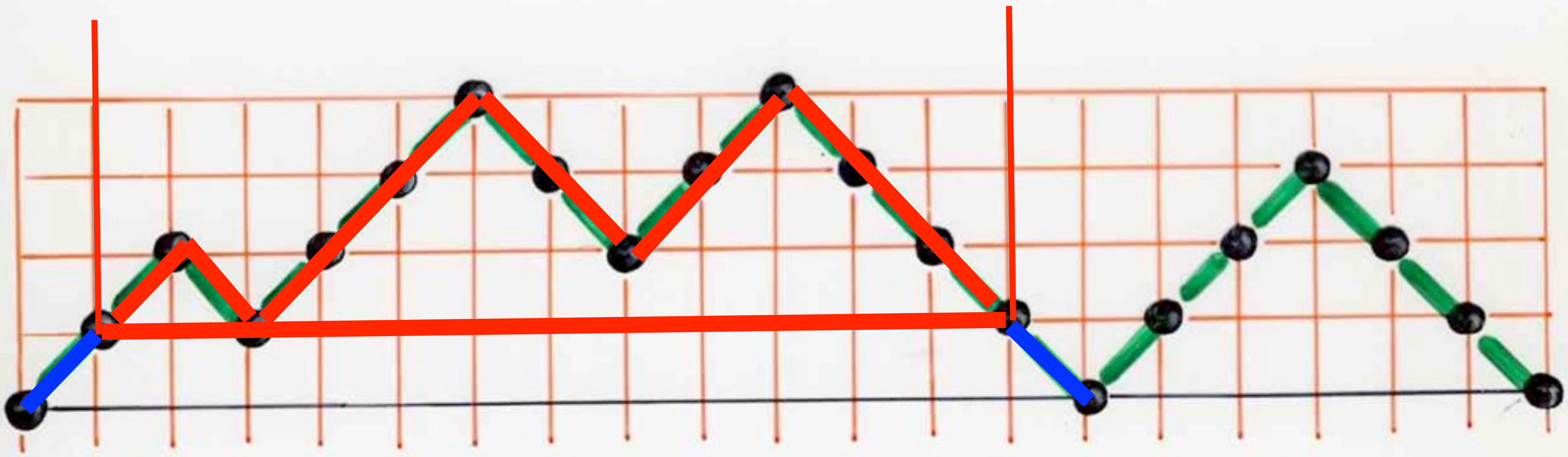
Dyck paths

Dyck path



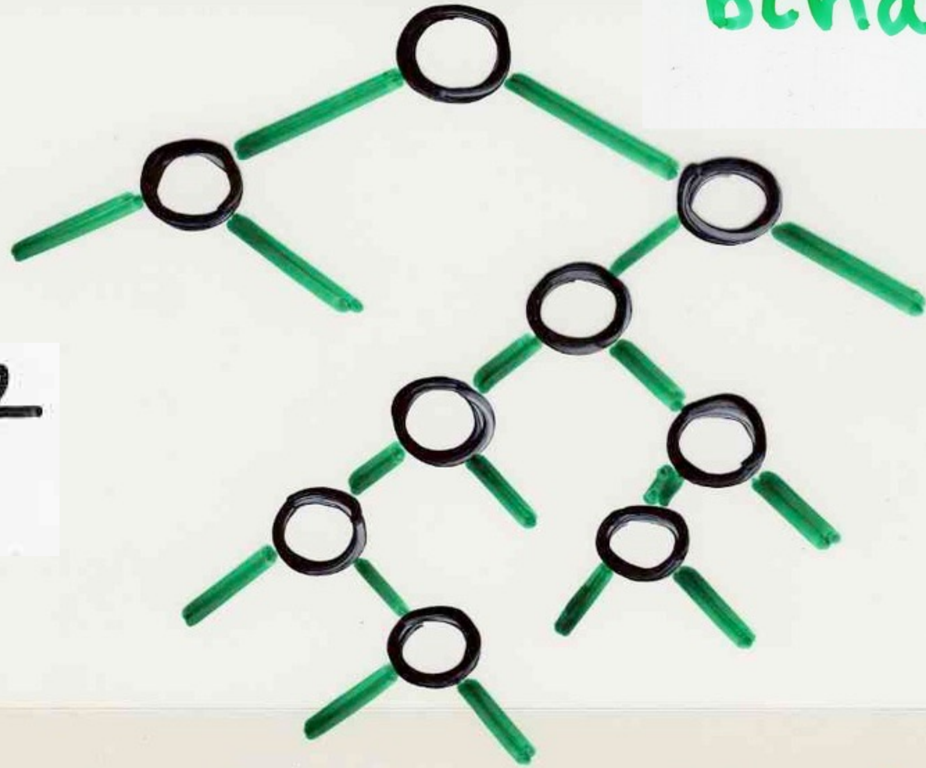

Dyck path

Dyck path



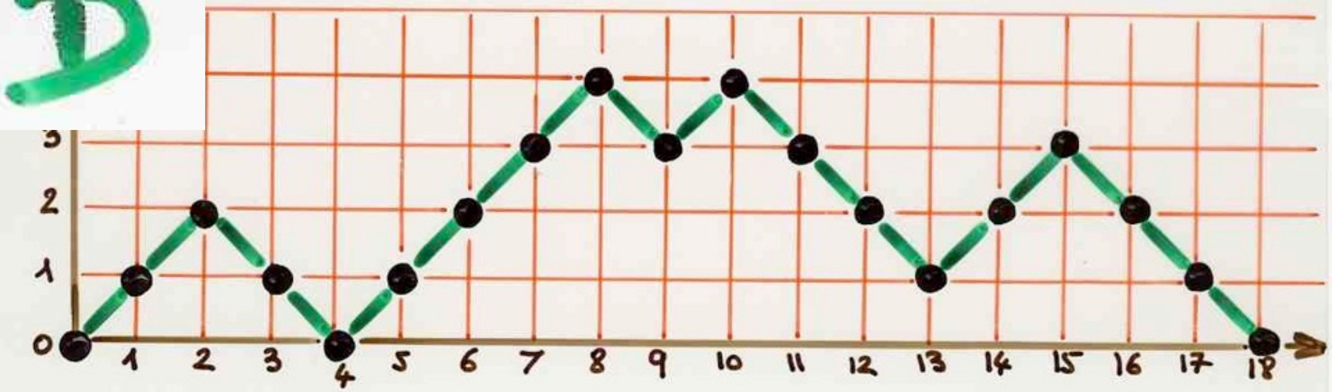
$$D = 1 + t D^2$$

binary tree



$$A = 1 + tA^2$$

Dyck path



$$D = 1 + tD^2$$

The number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{(n+1)} \binom{2n}{n}$

$$\left.\begin{array}{l} \bullet\ \text{binary trees} \\ \bullet\ \text{triangulations} \\ \quad \text{(of a convex polygon)} \\ \bullet\ \text{Dyk paths} \end{array}\right\} \begin{array}{l} 3\ \text{"incarnations"} \\ \text{of Catalan} \\ \text{numbers} \end{array}$$