Some remarks on Baxter permutations and Baxter matrices

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In [Kn21] Don Knuth defined an extension of *Baxter permutations* to matrices, under the name *Baxter matrices.*

Extending Baxter permutations to Baxter matrices have a certain analogy with the extension RS to RSK, that is the Robinson-Schensted correspondence (between permutations and pairs of standard Young tableaux) and RSK introduced by Don Knuth more than 50 years ago [Kn70], between matrices and pairs of Young tableaux. This note is to give an idea in order to continue this analogy.

Remark In this note, we will often refer to the so-called "video-book" *The Art of Bijective Combinatorics* (or ABjC for short), a collection of videos, slides and a website (www.viennot.org/abjc.html) which enable one to navigate inside the videos in the same way as turning the pages of a book. The discussion "book or video-book", inconvenients and advantages, is detailed in the paper [Ma,Vi21]. As it is completely unusual to refer to a video-book instead of a book , we will follow the suggestions given in the preface of each Part of the video-book.

1 Laguerre histories

First we need to go back to the first bijective proof for the formula giving the number of Baxter permutations. This go back to 1981 [Vi81]. This reference is just an abstract of a talk given at the 3rd session of the "Séminaire Lotharingien de Combinatoire" (SLC). A complete description can be found in Chapter 4b of [ABjC, Part I, Ch4].

The key ingredient is a bijection between permutations and the so-called « *Laguerre histories* » which is described in [ABjC, Part I, Ch4], chapter 4b, pp3-41. Laguerre histories are certain pairs (γ_c, f) where γ_c are the so-called *2-colored Motzkin paths* (enumerated by the Catalan numbers) and f is the « *choice function* ».

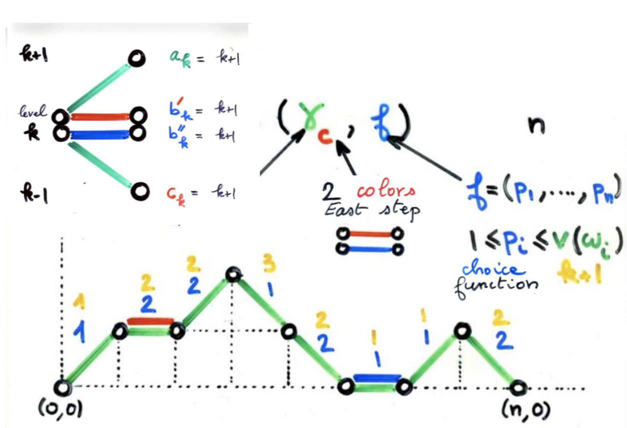

Figure 1: A Laguerre history (γ_c, f), ABjC, Part I, p9

A bijection between Baxter permutations and triple of non-intersecting paths is described pp107-122 of chapter 4b [ABjC, Part I, Ch4]. Then the Baxter property for a permutation is characterised by the choice function f which can take only two possible values, depending on the context (*peak, trough, double rise, double descent*) of the permutation. Then γ_c is well known to be in bijection with a pair of non-intersecting lattice paths (or *staircase polygons*, or *parallelogram polyominoes*) and the choice function for Baxter permutations

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becomes a third path between the two previous paths, giving a bijection between Baxter permutations and triple of non-crossing paths.

Then the bijective proof of the formula for the number of permutations follows in [ABjC, part I, Ch5], chapter 5a, pp43-86, with consideration of binomial determinants, Young tableaux, contents/hook-length formula and the so-called LGV Lemma (Lindström-Gessel-V.).

2 Laguerre histories and data structure dictionary

The idea, similar to the passage RS to RSK would be to extend Laguerre histories to matrices. One possibility is to look at Laguerre histories in a different way by introducing the notion of *« Laguerre heaps of segments »*, defined in [ABjC, Part IV, Ch3], Chapter 3b, p104.

In fact Part IV of ABjC is devoted to the combinatorial theory of orthogonal polynomials and continued fractions where Laguerre histories play a central role. Laguerre histories are again defined in [ABjC, IV, Ch2], see Chapter 2a, pp56-76, the notion of *« restricted Laguerre histories »* pp19-23 (enumerated by $n!$ instead of $(n+1)!$ for general Laguerre histories). Some people prefer to describe the bijection permutations — Laguerre histoires with *increasing binary trees*, see pp52-72 of the same Chapter 2a.

In [ABjC, IV, Ch2], pp78-91 of Chapter 2c, we explain the origin of the notion of *« histories »* in relation with data structure in computer science , which is fundamental in the theory of Flajolet, Françon and Vuillemin about computing the average cost of a data structure for a sequence of primitive operation (series of papers in the 80's). Restricted Laguerre histories and Laguerre polynomials correspond to the data structure « dictionary ».

3 Laguerre heaps of segments

Now here is the point where such history for the data structure dictionary can be described with the new notion of « Laguerre heaps of segments ». See pp95-115 of Chapter 2c, [ABjC, IV, Ch2] and the description of the bijection between permutations pp116-133. One can see the picture of a « Laguerre heap of segments » on page 115 . Underlying this new notion there is the notion of heaps of segments.

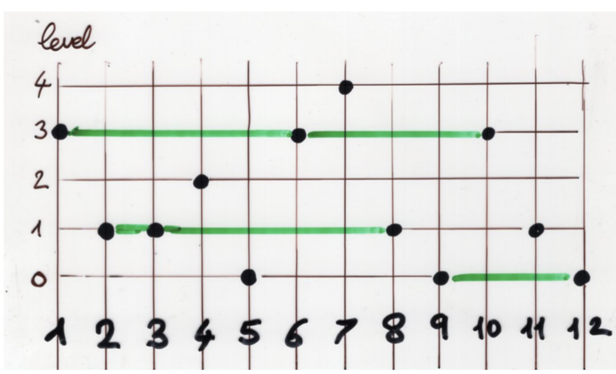

Figure 2: A Laguerre heap of segments, ABjC, Part IV, Ch2c, p115

The general theory of heaps of pieces is described in part II of ABjC. I had a long discussion with Don Knuth at Mittag-Leffler Institute about the terminology « heaps of pieces » in combinatorics and the data structure « heaps » in computer science, following by the notion of « Kepler towers » and the use of the French name *« empilements »* for Volume 4B of TAOCP.

In term of « Laguerre heaps of segments » the (restricted) Laguerre history can be seen by cutting with vertical lines the segments of the heaps. Supposed at time t you intersect k segments. At time $(t+1)$ there are 4 cases: one new segment is added (with $k+1$ possibilities)(corresponding to adding a new element in the dictionary), a segment is closed (with k possibilities)(corresponding to deleting a element in the dictionary), a point is added in an existing segment (with k possibilities)(corresponding to asking a question with « positive » answer), a single point is added between some segments (with $k+1$ possibilities)(corresponding to a « negative answer » in the data structure dictionary). Starting and ending with no segments, you get exactly the description of a restricted Laguerre histories. (general Laguerre histories would correspond to a choice function being $(k+1)$ everywhere.

4 Baxter heaps of segments

Now come the characterisation of Laguerre histories coming from Baxter permutations. When you cut by a vertical line a Laguerre heaps of segments, the variation of the number of segments you cut corresponds exactly to γ_c, the 2-colored Motzkin paths described above. The number of possibilities (choice function) must be reduced to two (or one if there are only one possibility). There are different ways to defined such « *Baxter heaps of segments* », such as taking for each cut the maximum or minimum position in the cut.

In fact, Bishal Deb, my former student in India from CMI, Chennai, who followed the courses ABjC, II and III (now in thesis with Alan Sokal in London), remarked that the notion of « Laguerre heaps of segments » is easily equivalent to the notion of *non-crossing arc diagrams* Introduced by Nathan Reading in [Rea14] , where some special cases are enumerated by Catalan numbers and . . . . Baxter numbers, corresponding to the choice function reduced to one (Catalan) or to two (Baxter) choices.

Let's come back to our purpose, extending Baxter permutation to Baxter matrices. Start from a permutation written on an n by n grid (one point in each row and column). You can easily put it in bijection with a Laguerre heaps of segments, as explained on pages 116-133 Chapter 2c, [ABjC, IV, Ch2]. In this construction one is looking the grid from bottom to top, line by line. The bijection with Laguerre histories would correspond to look the grid left to right, column by column as described above.

5 Geometric RSK

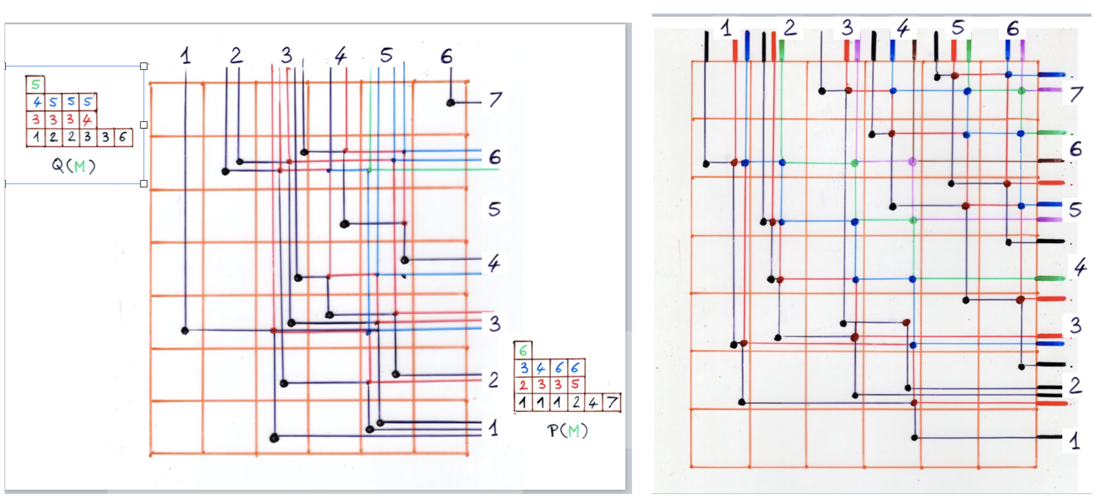

Figure 3: ABjC, Part III, Ch1e, Geometric RSK, p53 (left) and dual geometric RSK, p82 (right)

Going from permutation to matrices, would correspond to put several points in the same cell. But we can recognize the similar construction for RSK. For RS, there is a well-known « *geometric* » construction with « *light and shadow* », equivalent to the description in [Kn70]. A geometric way to define RS for matrices, is to transform matrices into permutations (more precisely a grid with at most one point in row and column). This is done by « moving the points » in a certain way. There are 4 possibles ways to move the points. I prefer this description to the so-call « matrix balls » of Fulton. Descriptions are given on pp43-55 and pp77-89 (dual RSK) of Chapter 1e of [ABjC, Part III, Chapter 1]. Thus the geometric description of RSK for matrices is reduced to the geometric description for permutations.

6 Extended heaps and Laguerre histories

One can do the same for the extension of Laguerre histories fo matrices. The matrix become a permutation, by replacing each number m in position (i, j) in the matrix, by m points in a cell of a grid. Then you move the points exactly in the same way as for the geometric construction of RSK. There are 4 different ways, such that in each row (resp. column) the points form an increasing (resp. decreasing) sequence. Then apply the same bijection as above giving a Laguerre heaps of segments reading the row from bottom to top. Reading the column from left to right one get the extension to matrices of the Laguerre histories for permutations.

Here we give an example on Figure 4 corresponding to a 5 by 9 matrix with entries only 0 or 1. It seems that the best is to choose the movement of the points such that the sequence of points in each row are decreasing (from left to right, in green on Figure 4) and are increasing in each column (from bottom to top, in red on Figure 4). This choice is the "opposite" as the one for the geometric construction of dual RSK.

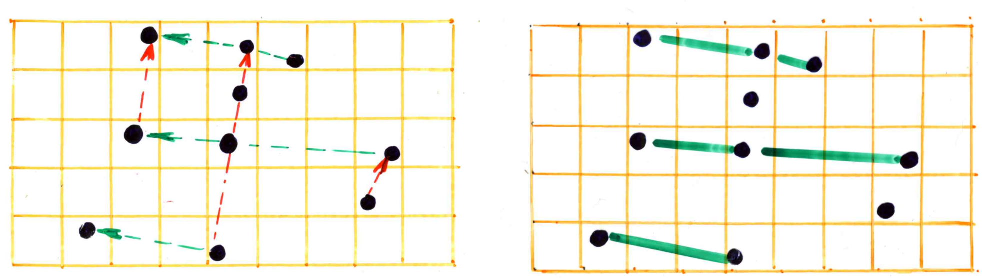

Figure 4: matrix to set of points (left), extended Laguerre heap of segment (right)

The remaining work would be to define such extended Laguerre histories (with only two choices for the choice function) and see if they can correspond to Don Knuth's definition of Baxter matrices. In fact D. Knuth mentioned that there is no need of general matrices with integers entries, but only matrices with 0 and 1.

Final remark

The notion of *Laguerre heaps of segments* plays a central role in the Epilogue of ABjC *"The essence of bijections: from growth diagrams to Laguerre heaps of segments for the PASEP"*. This Epilogue is a repetition (in 1h50') at IMSc, Chennai, of the (high speed) talk given at Strobl in September 2028 for the 60th birthday of Christian Krattenthaler. It is supposed to be self-contained.

References

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