Tianjin-5

Tilings, determinants and non-crossing paths

Tianjin University
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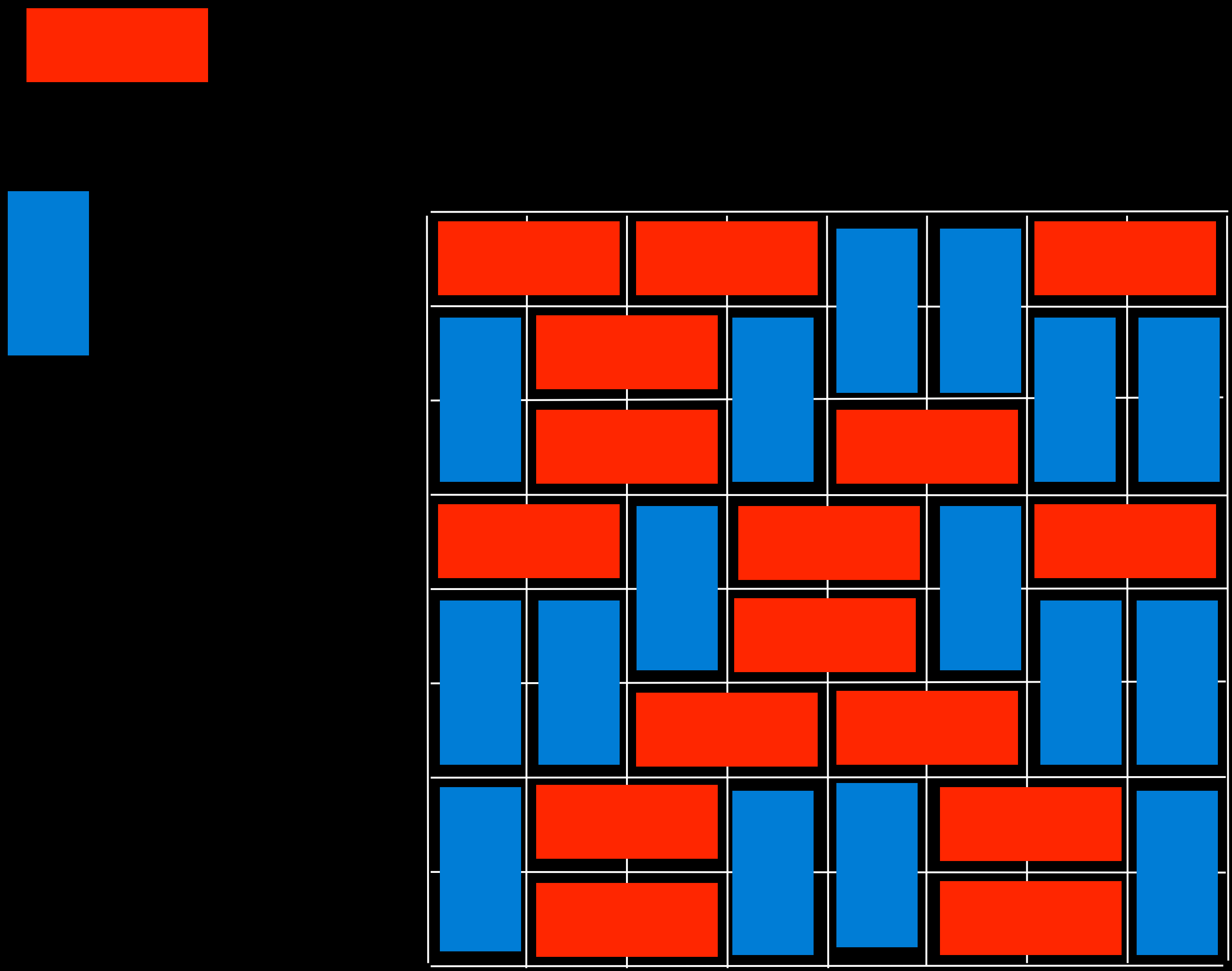
Tilings



number of tilings on a 8 x 8 chessboard

= 12 988 816

number of tilings with dimers of a $m \times n$ rectangle

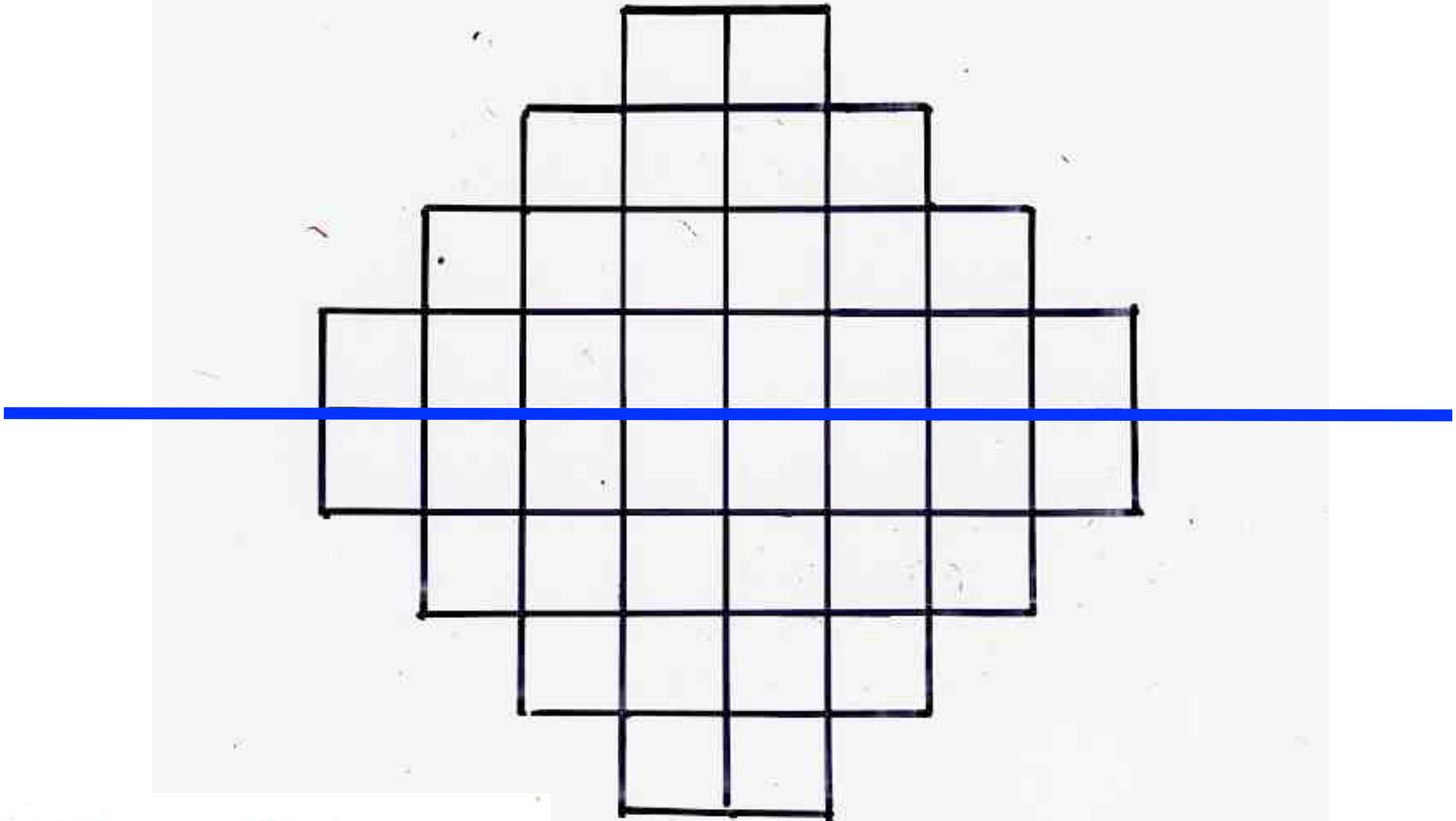
$$4^{mn} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

it is an integer !!

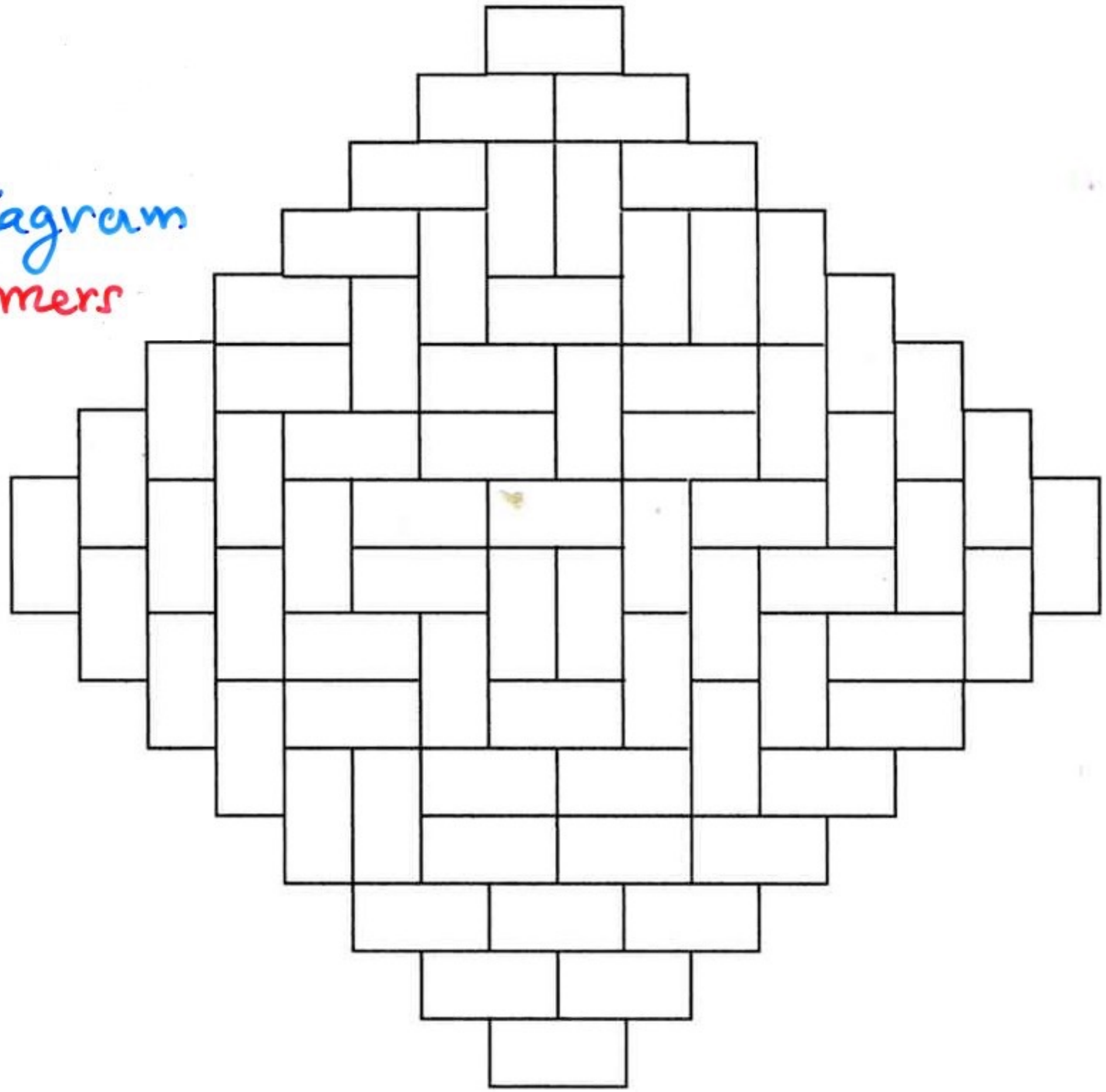
for the chessboard $m=n=8$: 12 988 816

Aztec tilings

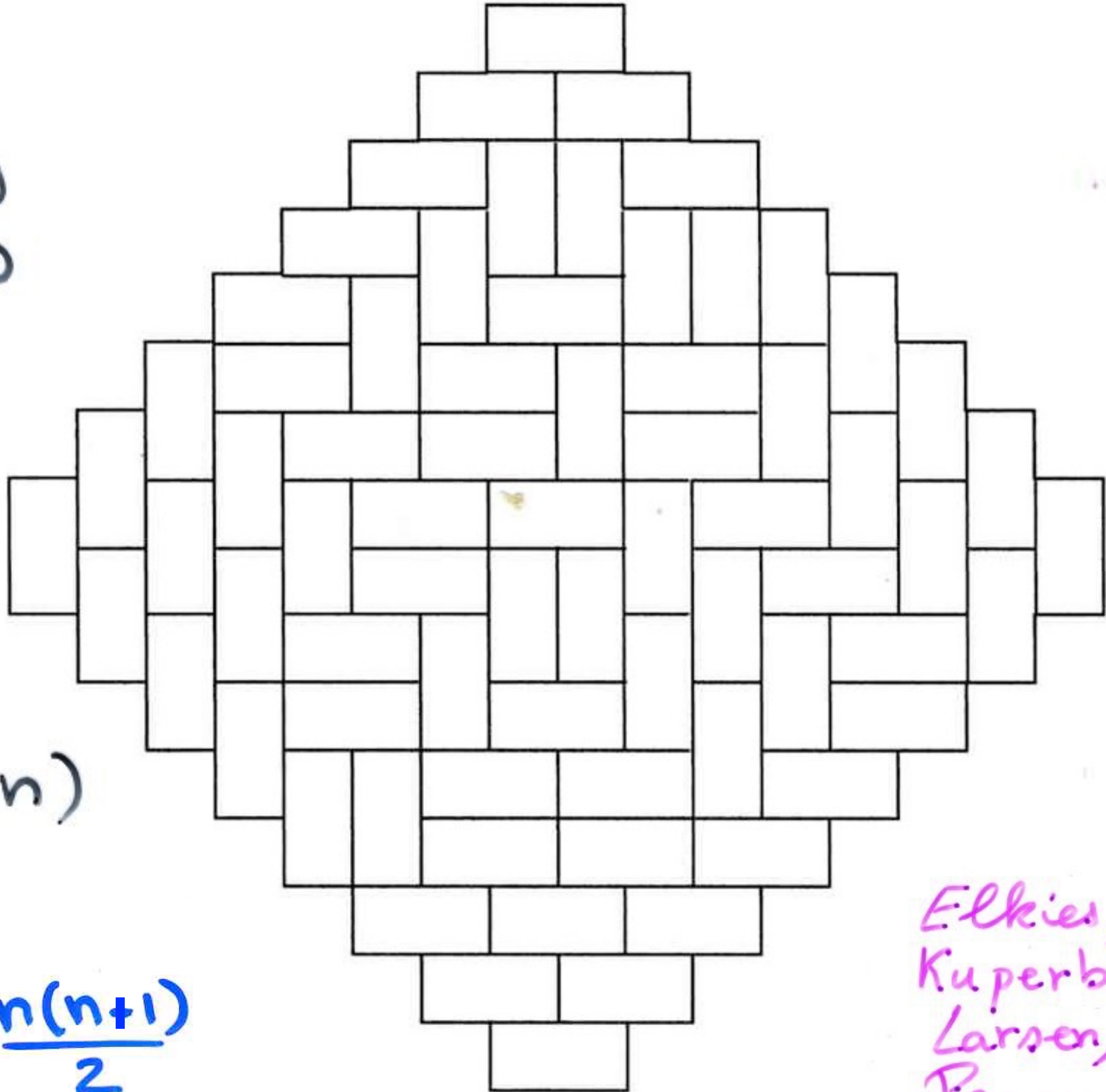


Aztec diagram

tilings
of the
Aztec diagram
with dimers



number of *tilings*



$$2^{(1+2+\dots+n)}$$

$$2^{\frac{n(n+1)}{2}}$$

Elkies, Kuperberg, Larsen, Propp (1992)

The LGV Lemma

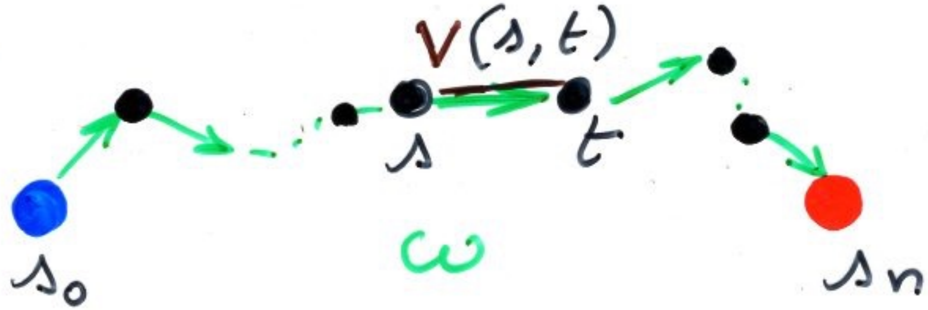

non- intersecting
configuration
of paths
determinant

Path $\quad \omega = (s_0, s_1, \dots, s_n) \quad s_i \in S$

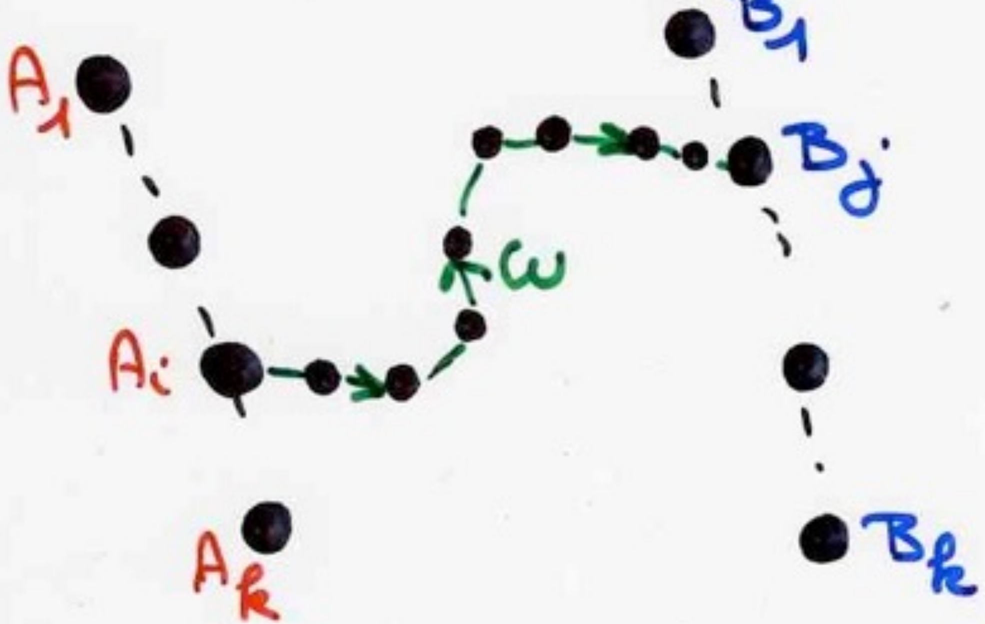
<u>notation</u> $\quad s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation $\quad v : S \times S \longrightarrow \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \cdots v(s_{n-1}, s_n)$$



weighted path



$A_1, \dots, A_k$

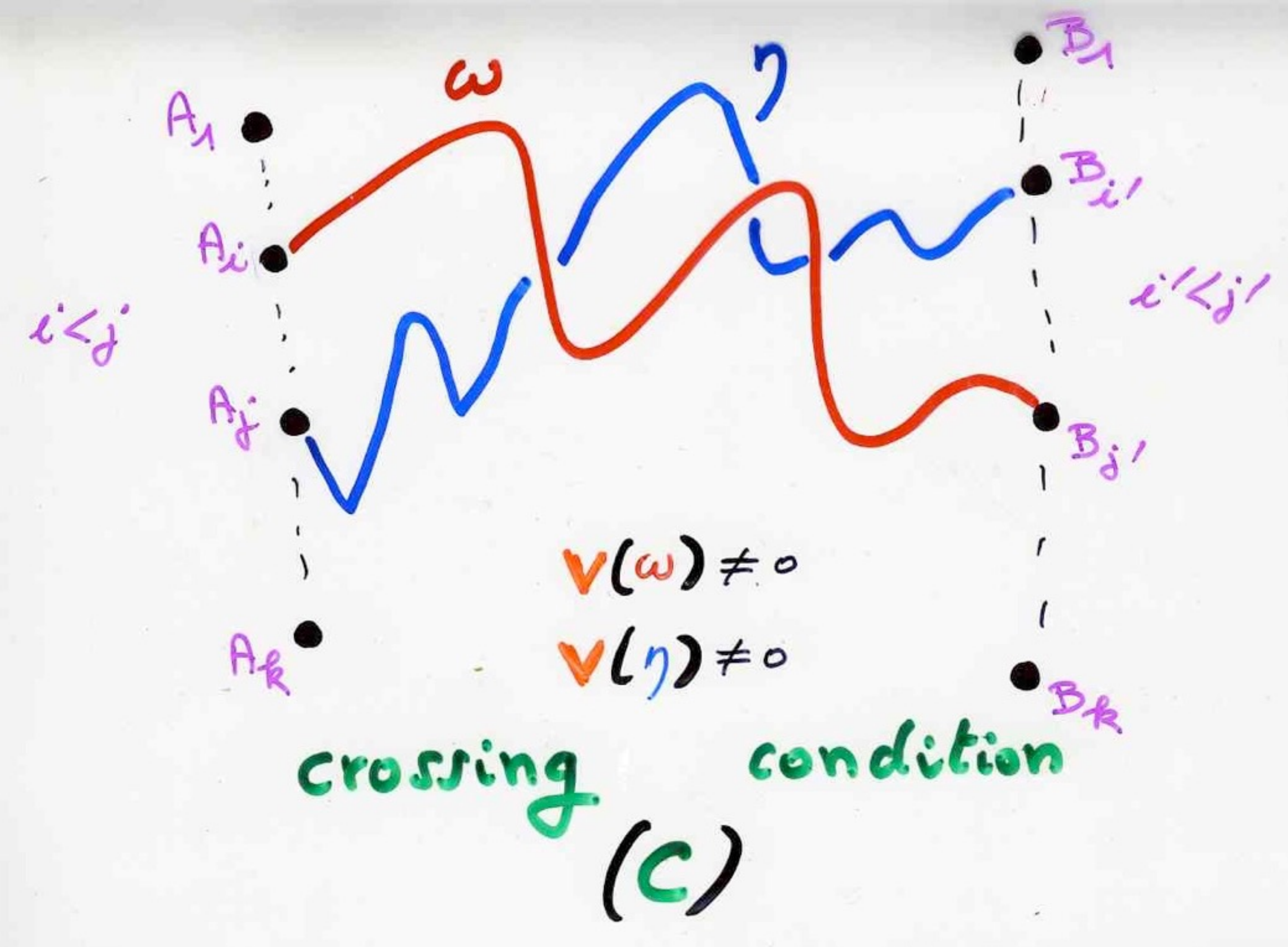
$B_1, \dots, B_k$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\mathbf{det}\,(a_{ij}) = \sum_{(\sigma;\,\omega_1,\dots,\omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1)\dots(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$



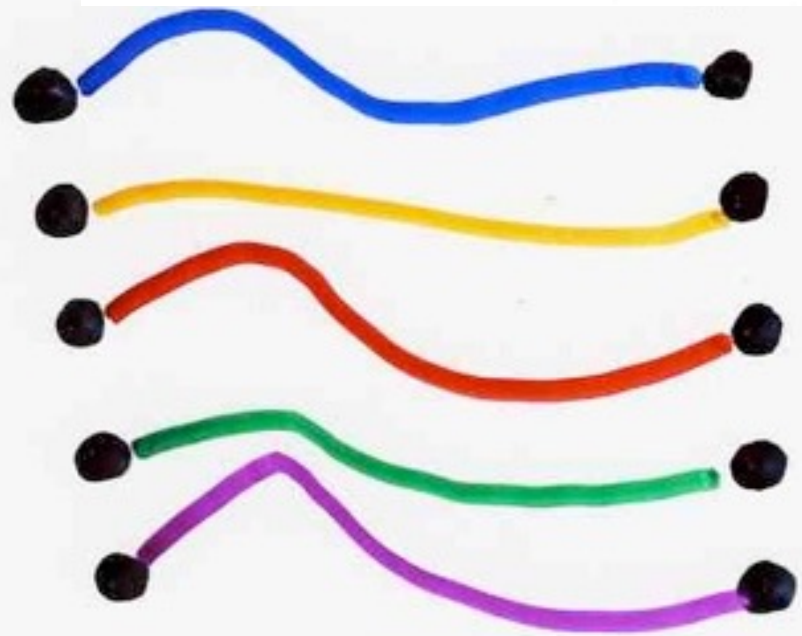
ω
η
A_1
A_i
A_j
A_k
$i < j$
B_1
$B_{i'}$
$B_{j'}$
B_k
$i' < j'$
$v(\omega) \neq 0$
$v(\eta) \neq 0$
crossing condition
(C)

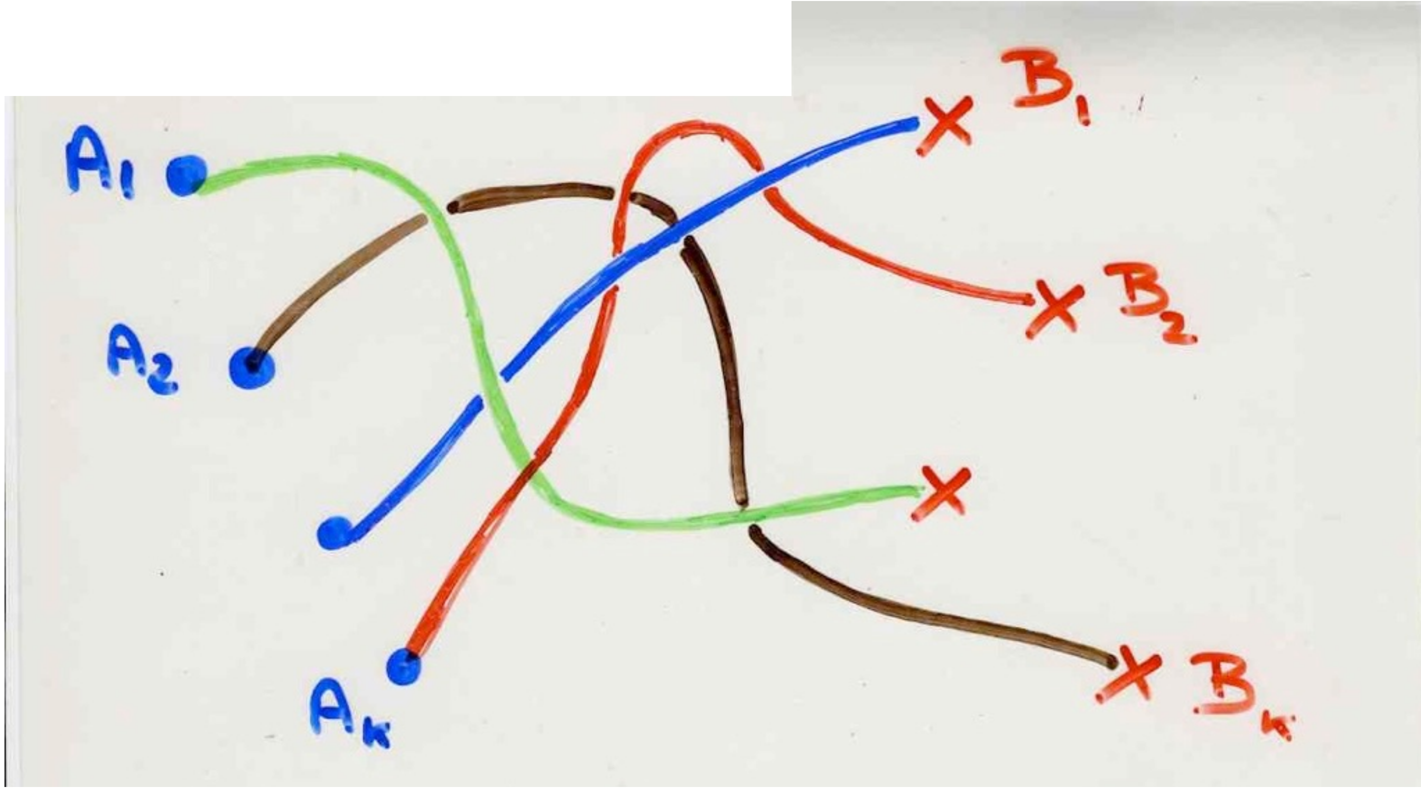
Proposition (LGV Lemma) (C) crossing condition

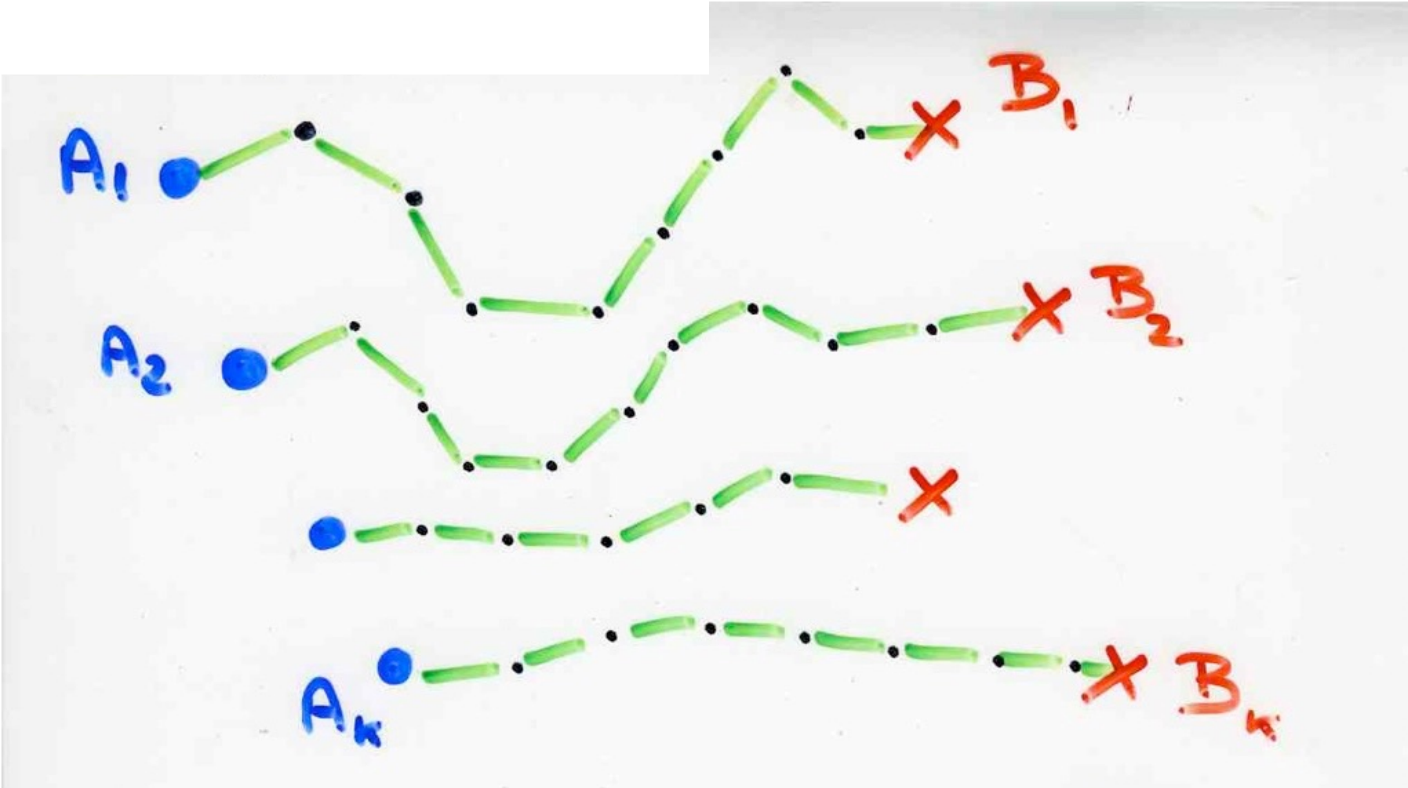
$$\det(a_{ij}) = \sum_{(\omega_1, \ldots, \omega_k)} v(\omega_1) \ldots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

non-intersecting

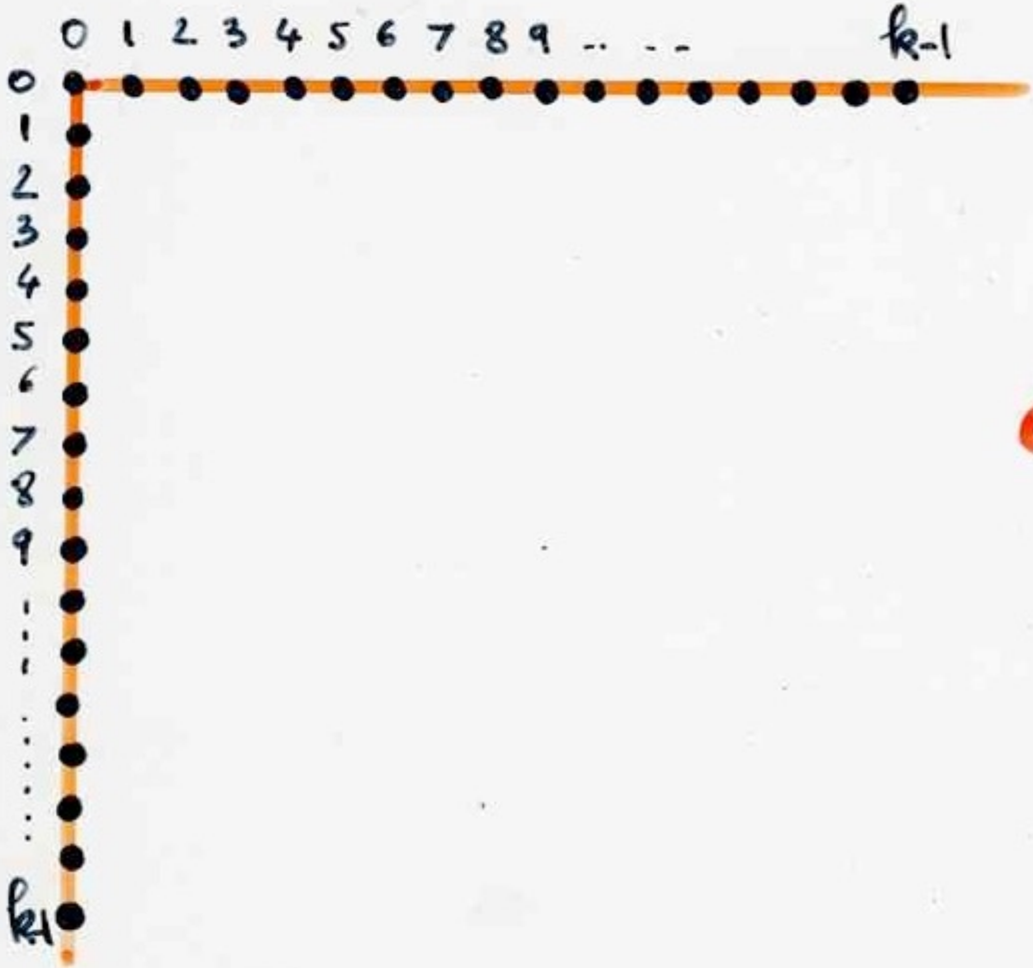



A1
A2
Ak
B1
B2
Bk

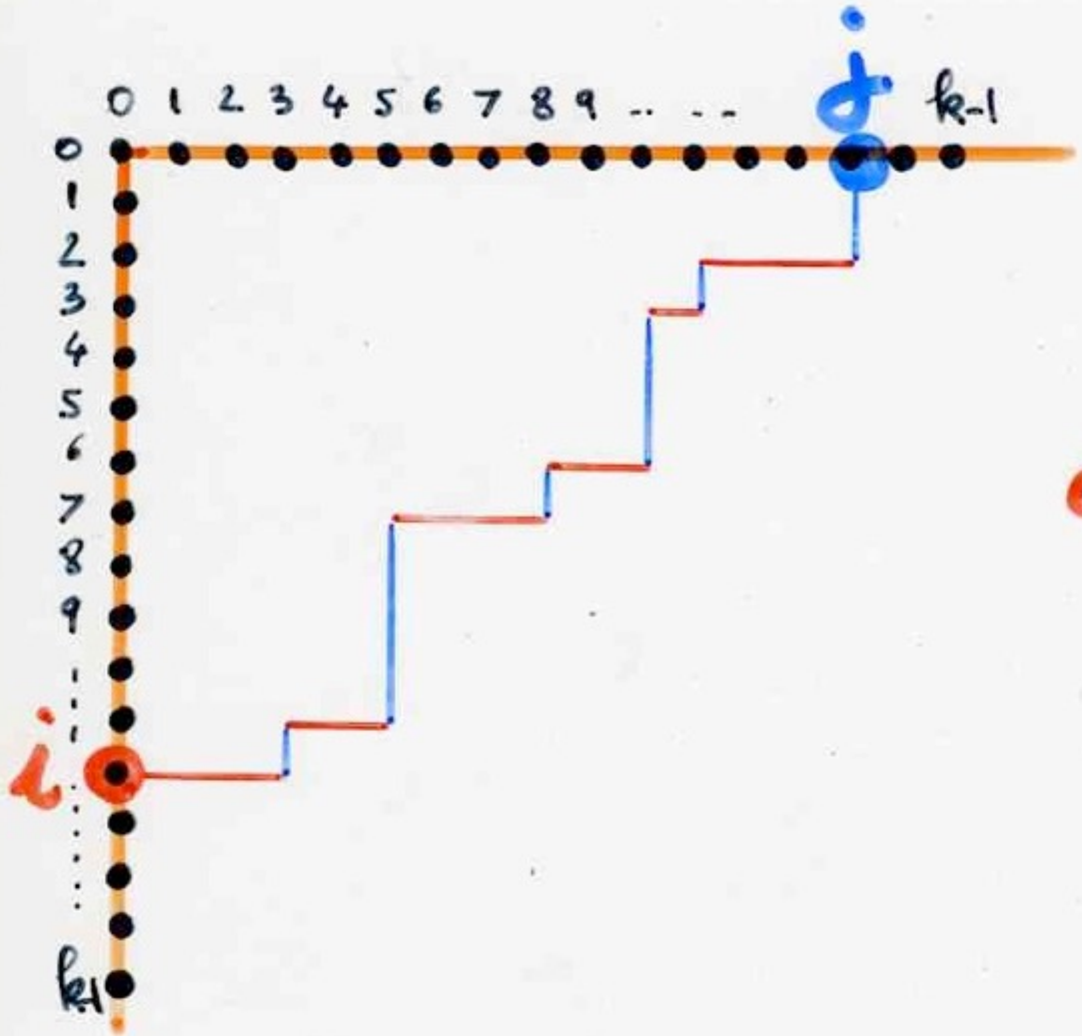


A1
B1
A2
B2
Ak
Bk

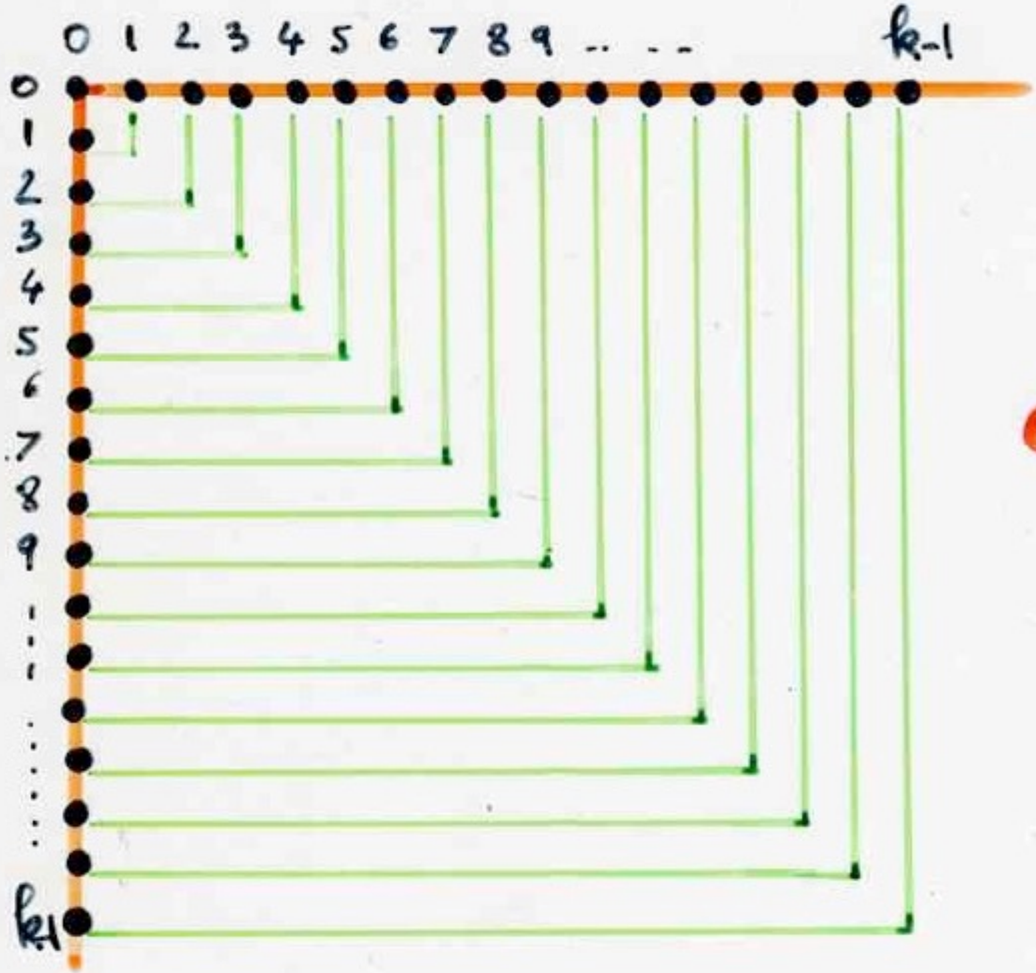
a simple example



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



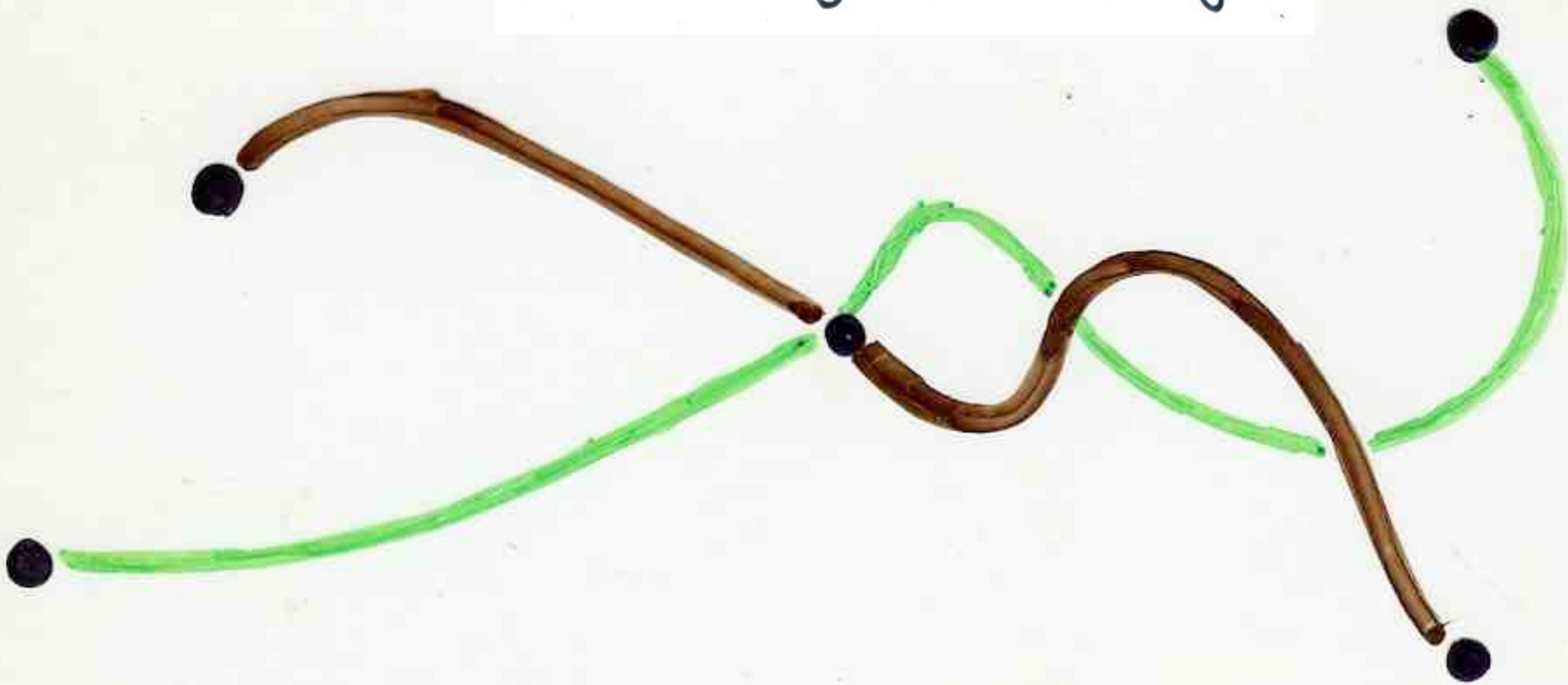
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



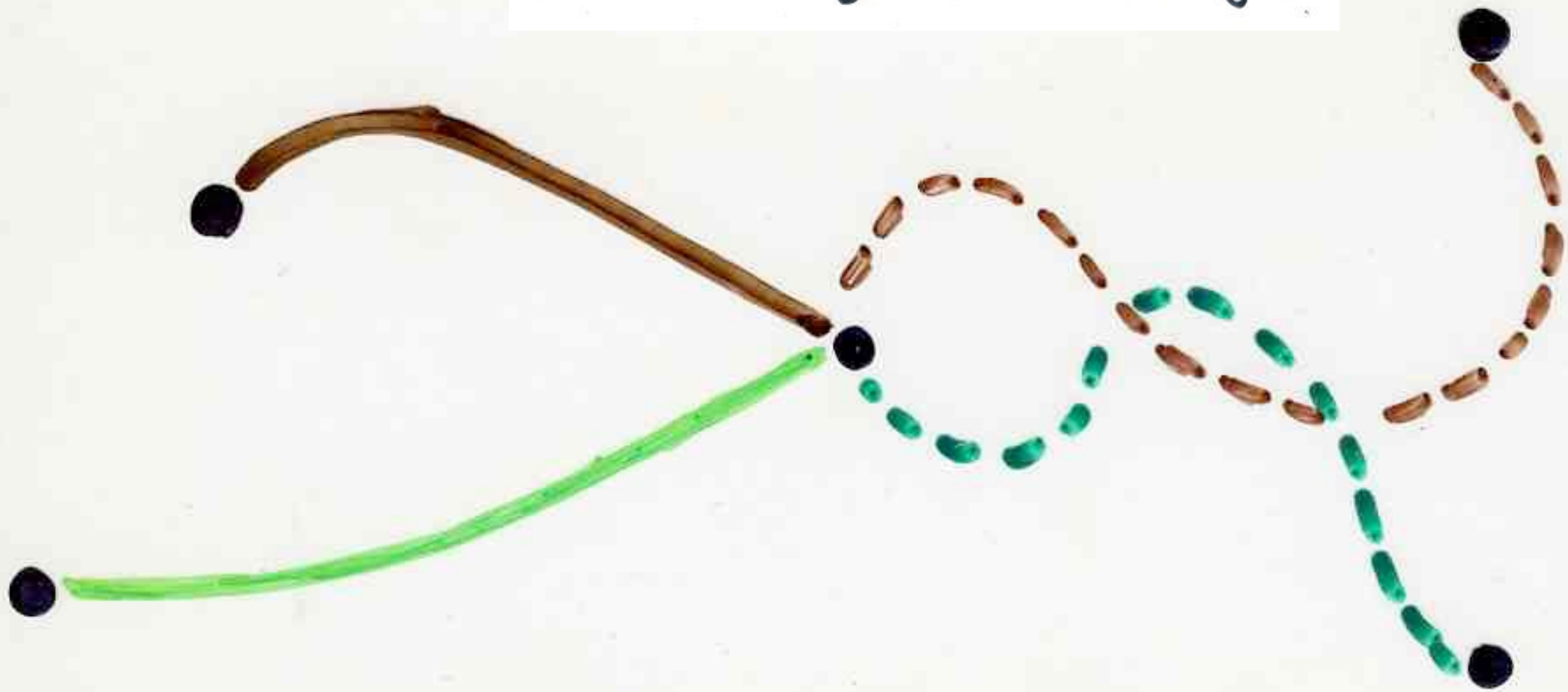
$$\det\begin{bmatrix}1&1&1&1&1&1&\cdots\\1&2&3&4&5&&\\1&3&6&10&&&\\1&4&10&&&&\\1&5&&&\binom{i+j}{i}&&\\1&&&&&&\\\vdots&&&&&&\end{bmatrix}_{k\times k} = 1$$

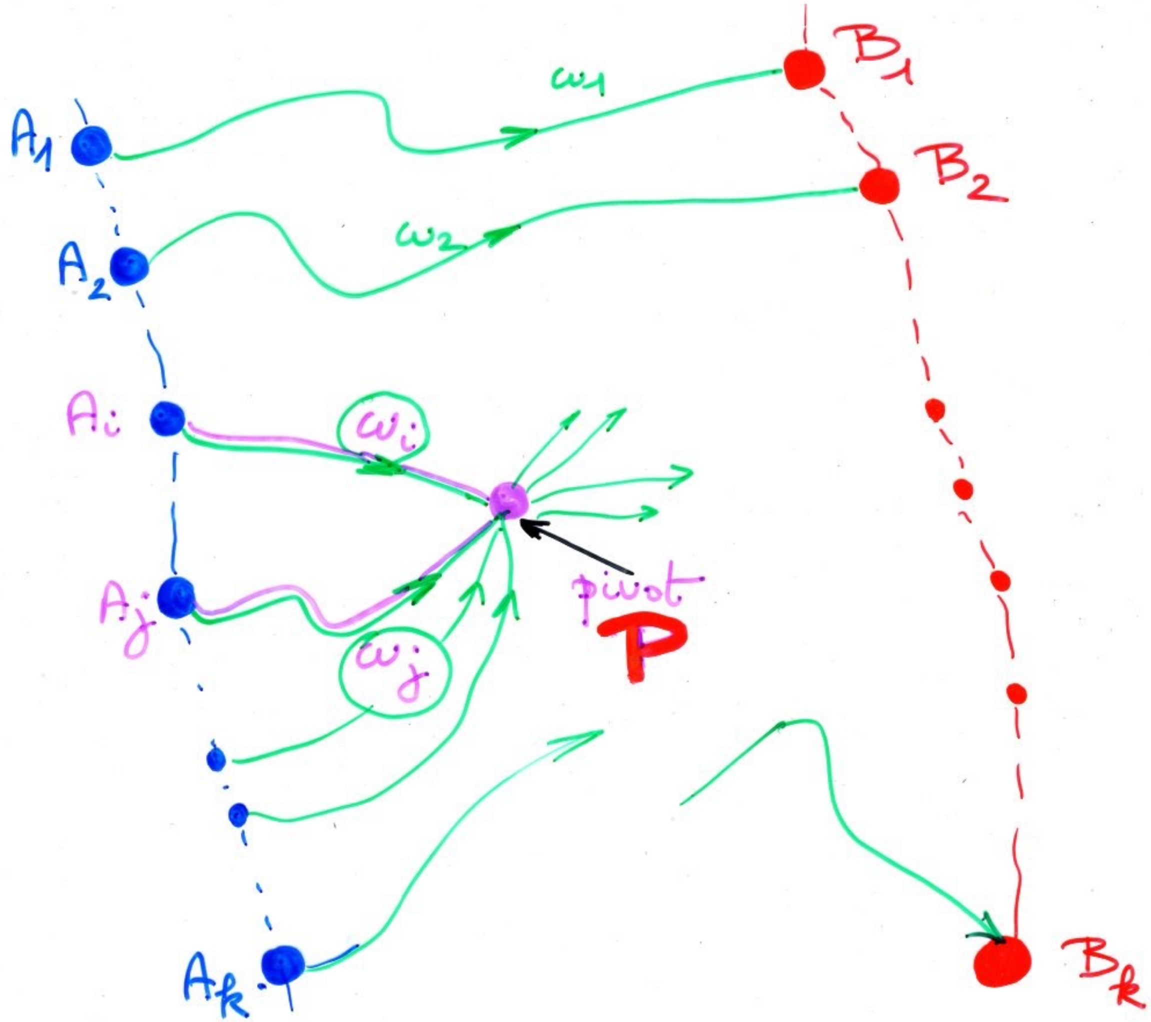
proof of LGV Lemma

idea of the proof:



idea of the proof:





A_1
A_2
A_i
A_j
A_k
B_1
B_2
B_k
ω_1
ω_2
ω_i
ω_j
pivot
P

Proof: Involution ϕ

$$E = \left\{ (\sigma ; (\omega_1, \dots, \omega_k)) \; ; \; \begin{array}{l} \sigma \in S_n \\ \omega_i : A_i \rightsquigarrow B_{\sigma(i)} \end{array} \right\}$$

$NC \subseteq E$ non-crossing configurations

$$\phi : (E - NC) \to (E - NC)$$

$$\phi(\sigma ; (\omega_1, \dots, \omega_k)) = (\sigma' ; (\omega'_1, \dots, \omega'_k))$$

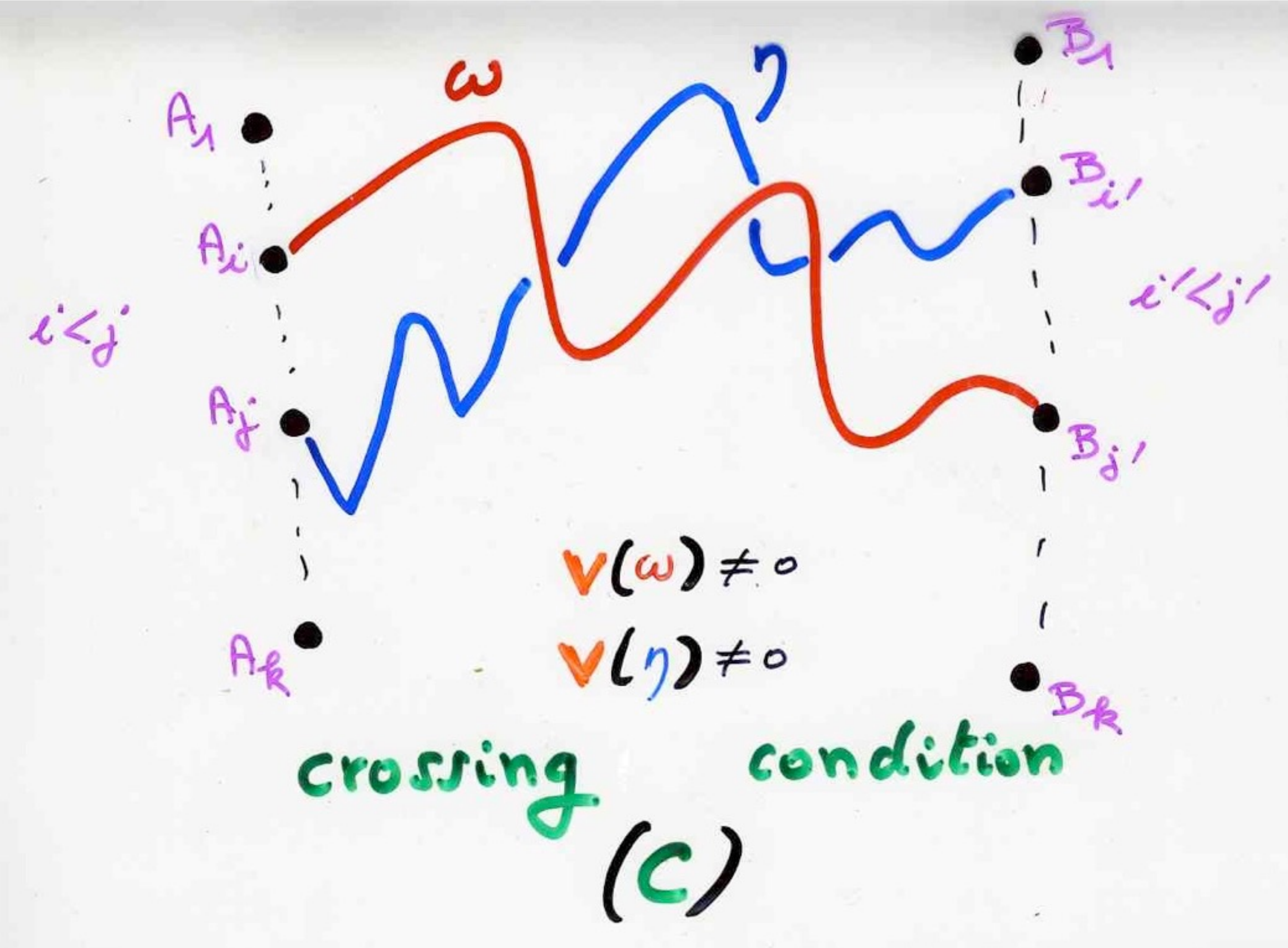
$$\begin{cases} (-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')} \\ v(\omega_1) \dots v(\omega_k) = v(\omega'_1) \dots v(\omega'_k) \end{cases}$$

LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{inv(\sigma)}\, v(\omega_1) \dots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting


ω
η
A_1
A_i
A_j
A_k
B_1
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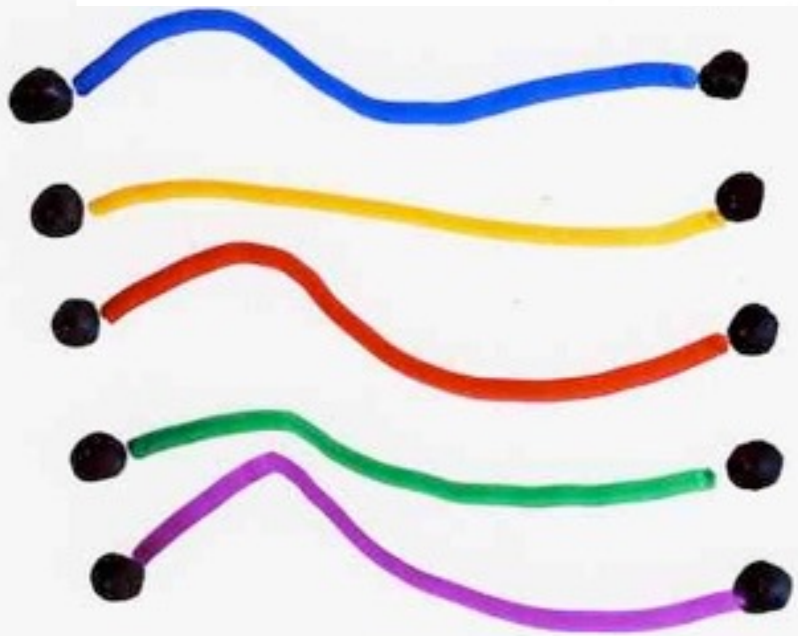
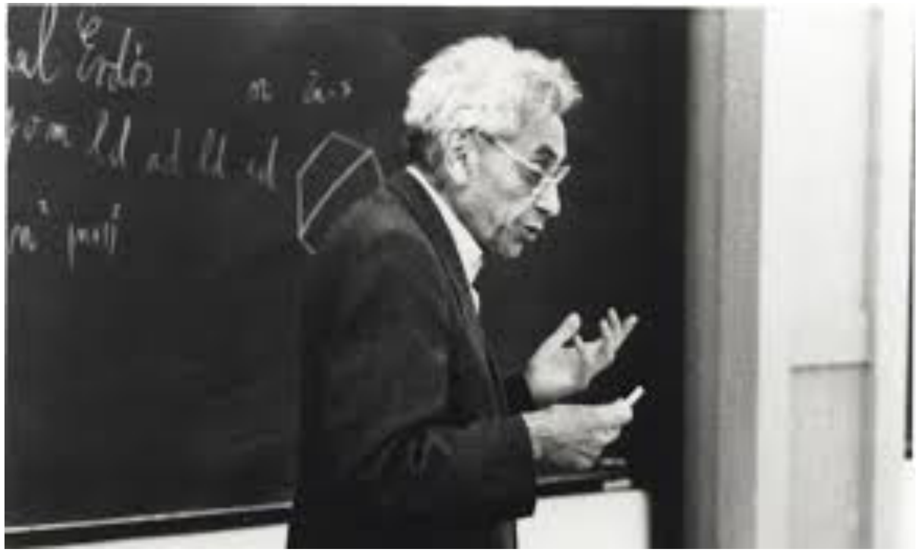
non-intersecting



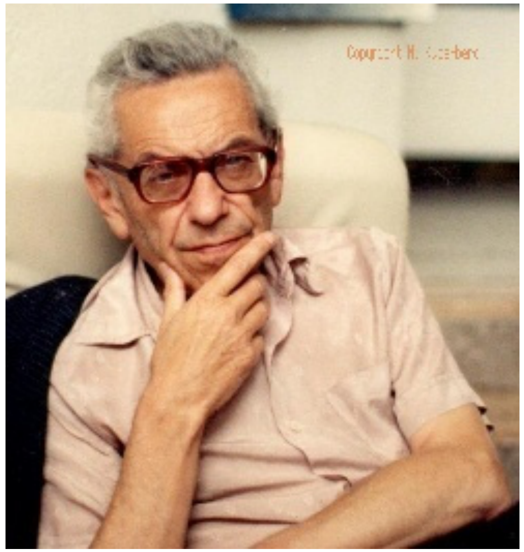
Lattice paths and determinants Chapter 29

Why « LGV **Lemma** » ?



Paul Erdös liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems,

Erdös also said that you need not believe in God but, as a mathematician, you should believe in The Book.



The essence of mathematics is proving theorems — and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside–Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma — including its proof — should be beautiful!

The « Flajolet Lemma »

$$\sum_{n\geqslant 0}\mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

combinatorial interpretation of a Jacobi continued fraction with weighted Motzkin paths

Why « **LGV** Lemma » ?

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

[4]Lindström used the term “pairwise node disjoint paths”. The term “non-intersecting,” which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

[5]By a curious coincidence, Lindström’s result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling’s bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the “Lindstrom–Gessel–Viennot theorem.” It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called “Slater determinant” in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an “ancestor” of the Lindstro¨m–Gessel–Viennot determinant.

[6]There exist however also several interesting applications of the general form of the Lindstro¨m–Gessel–Viennot theorem in the literature, see [10, 16, 51].

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

combinatorics

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

I. Gessel and X.G.V., *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

I. Gessel and X.G.V., *Determinants, paths and plane partitions, preprint (1989)*

statistical physics: (wetting, melting)

Fisher, Vicious walkers, Botzmann lecture (1984)

combinatorial chemistry:

John, Sachs (1985)

Gronau, Just, Schade, Scheffler, Wojciechowski (1988)

probabilities, birth and death process,

Karlin , McGregor (1959)

quantum mechanics: Slater determinant

Slater(1929) (1968), De Gennes (1968)

Relation with Lecture 3

A introduction to the combinatorial theory of orthogonal polynomials and continued fractions

Hankel determinants

and

orthogonal polynomials

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)\,P_k(x) - \lambda_k\,P_{k-1}(x)$$

for every $k \geq 1$

(formal) Favard's Theorem

3-terms linear recurrence relation

$\Rightarrow$ orthogonality

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

moments

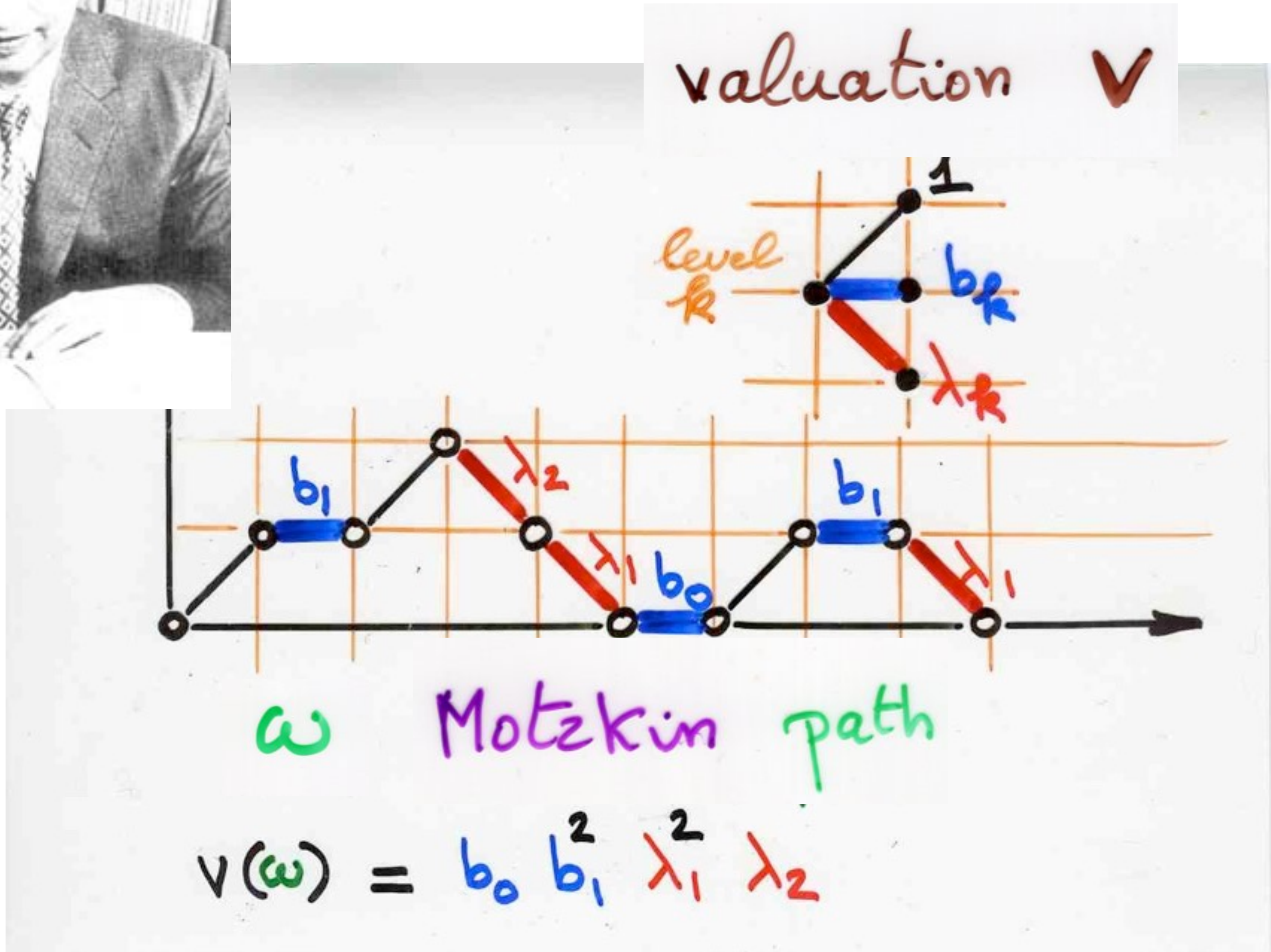
$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

length

$$f(x^n) = \mu_n$$


valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$

Hankel determinant

any minor of the matrix

$$\mathbf{H}\left(\{\mu_n\}_{n\geq 0}\right)$$

$$\begin{array}{c|cccccc}
 & & & & & j & \\
\hline
 & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \\
 & \mu_1 & \mu_2 & \mu_3 & & & \\
 & \mu_2 & \mu_3 & & & & \\
 & \mu_3 & & & & & \\
i & & & & & \mu_{i+j} & \\
\end{array}$$

LGV Lemma

determinant $\longleftrightarrow$ configuration of non-intersecting paths

Hankel determinant

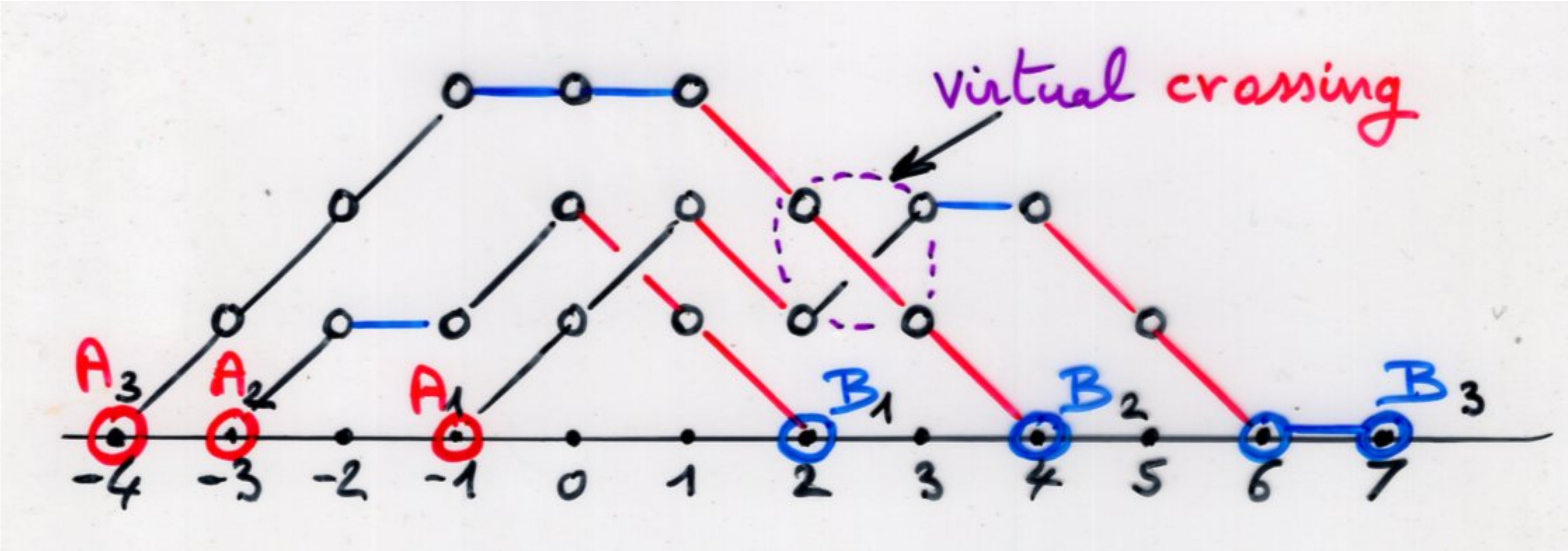
any minor of

$$\begin{array}{c|ccccc} & & & & & j \\ \hline & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ & \mu_1 & \mu_2 & \mu_3 & \cdots & \\ & \mu_2 & \mu_3 & \cdots & & \\ & \mu_3 & & & & \\ i & & & & & \mu_{i+j} \end{array}$$

$$H\begin{pmatrix} \alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k \end{pmatrix}$$

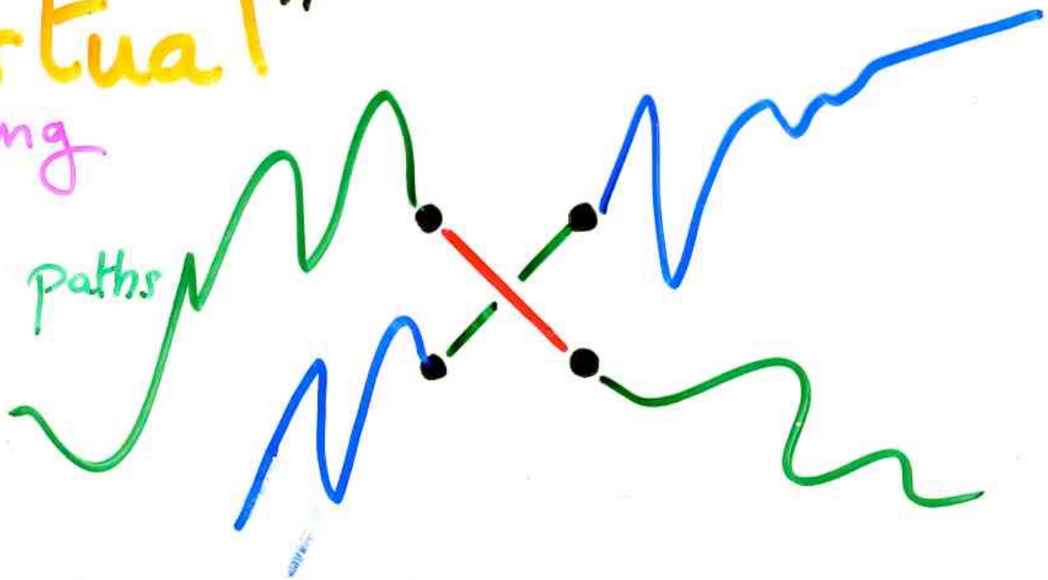
$$0 \le \alpha_1 < \cdots < \alpha_k$$

$$0 \le \beta_1 < \cdots < \beta_k$$



$$H\begin{pmatrix} 1, & 3, & 4 \\ 2, & 4, & 7 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1, & 2, & 3 \\ 3, & 1, & 2 \end{pmatrix}$$

"virtual" crossing of Motzkin paths



LGV Lemma. general form

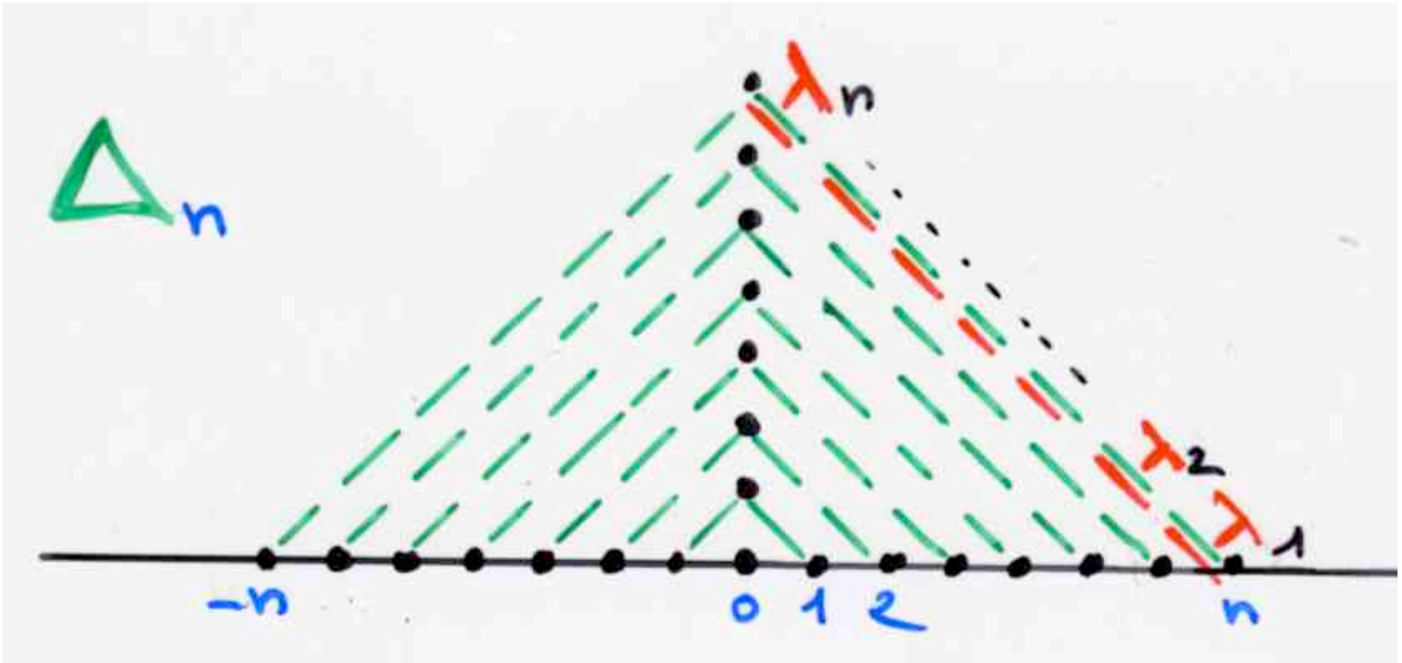
$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \ldots, \omega_k)} (-1)^{inv(\sigma)}\, v(\omega_1) \ldots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting

$$\Delta_n = \det\begin{bmatrix}\mu_0 & \mu_1 & \cdots & \mu_n\\ \mu_1 & \mu_2 & \cdots & \mu_{n+1}\\ \vdots & \vdots & & \vdots\\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n}\end{bmatrix}$$

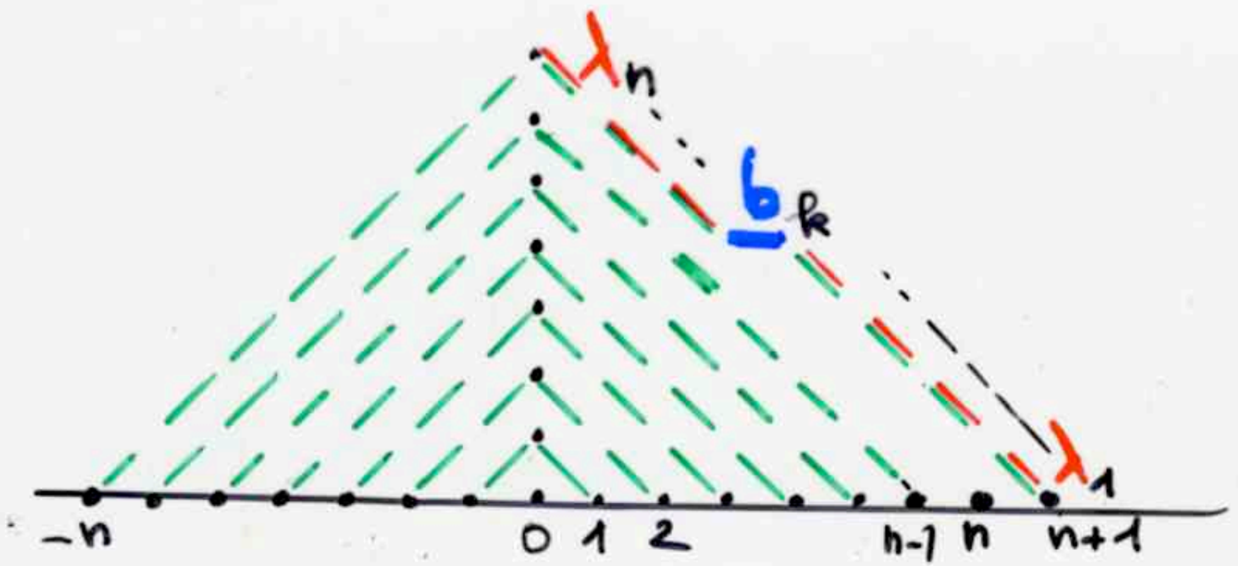
$$\lambda_n = \frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}}$$



$$\Delta_n = (\lambda_1)^n(\lambda_2)^{n-1}\cdots(\lambda_{n-1})^2\,\lambda_n$$

$$\chi_n = \det \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}$$

$$b_n = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}$$



$$\chi_n = (b_0 + \cdots + b_n)\,\Delta_n$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cdots\cdots\cfrac{}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$$J(t;b,\lambda)$$

Jacobi

continued fraction

$$b=\{b_k\}_{k\geq 0} \qquad \lambda=\{\lambda_k\}_{k\geq 1}$$

continued fractions

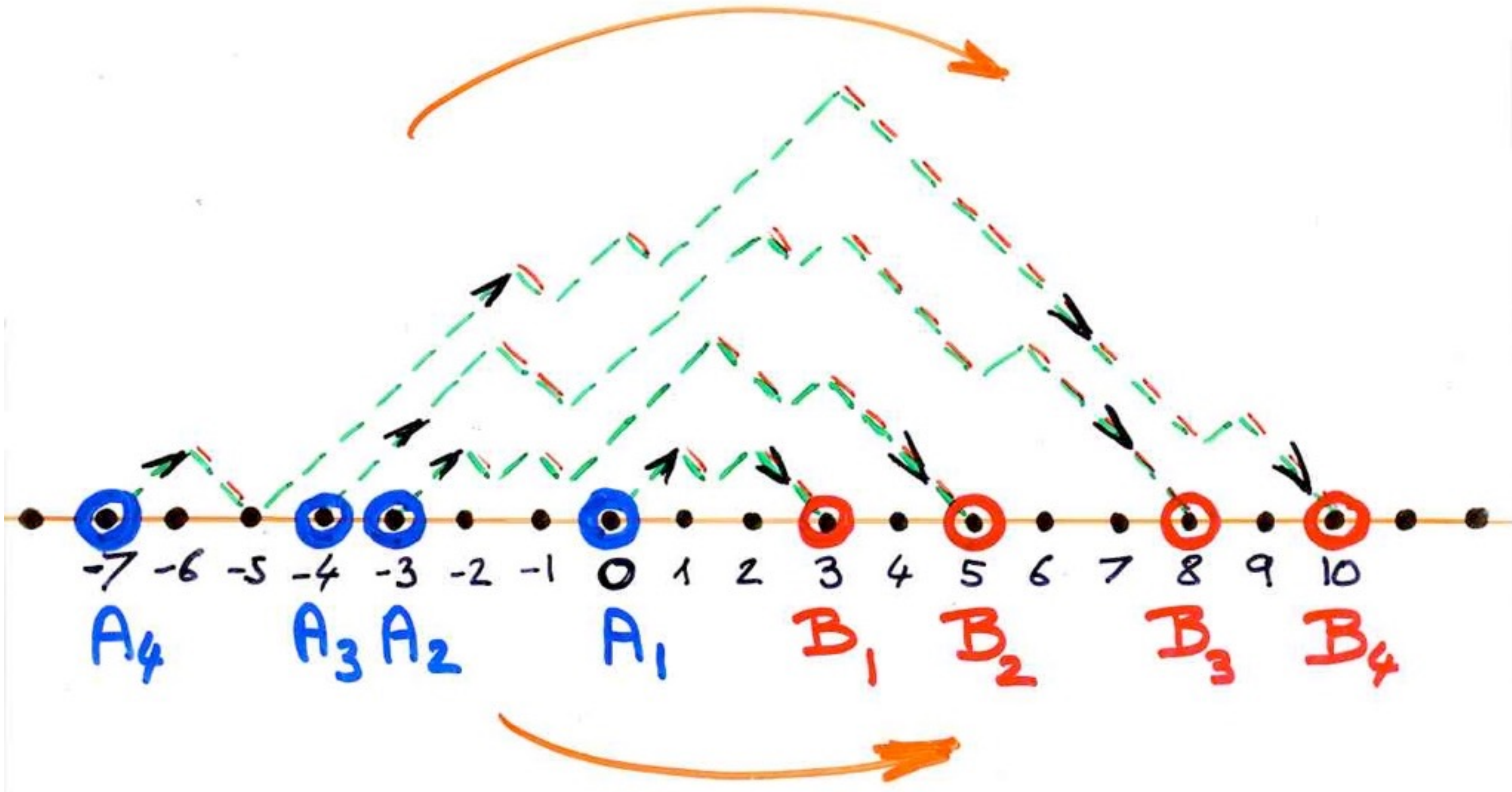
Stieltjes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

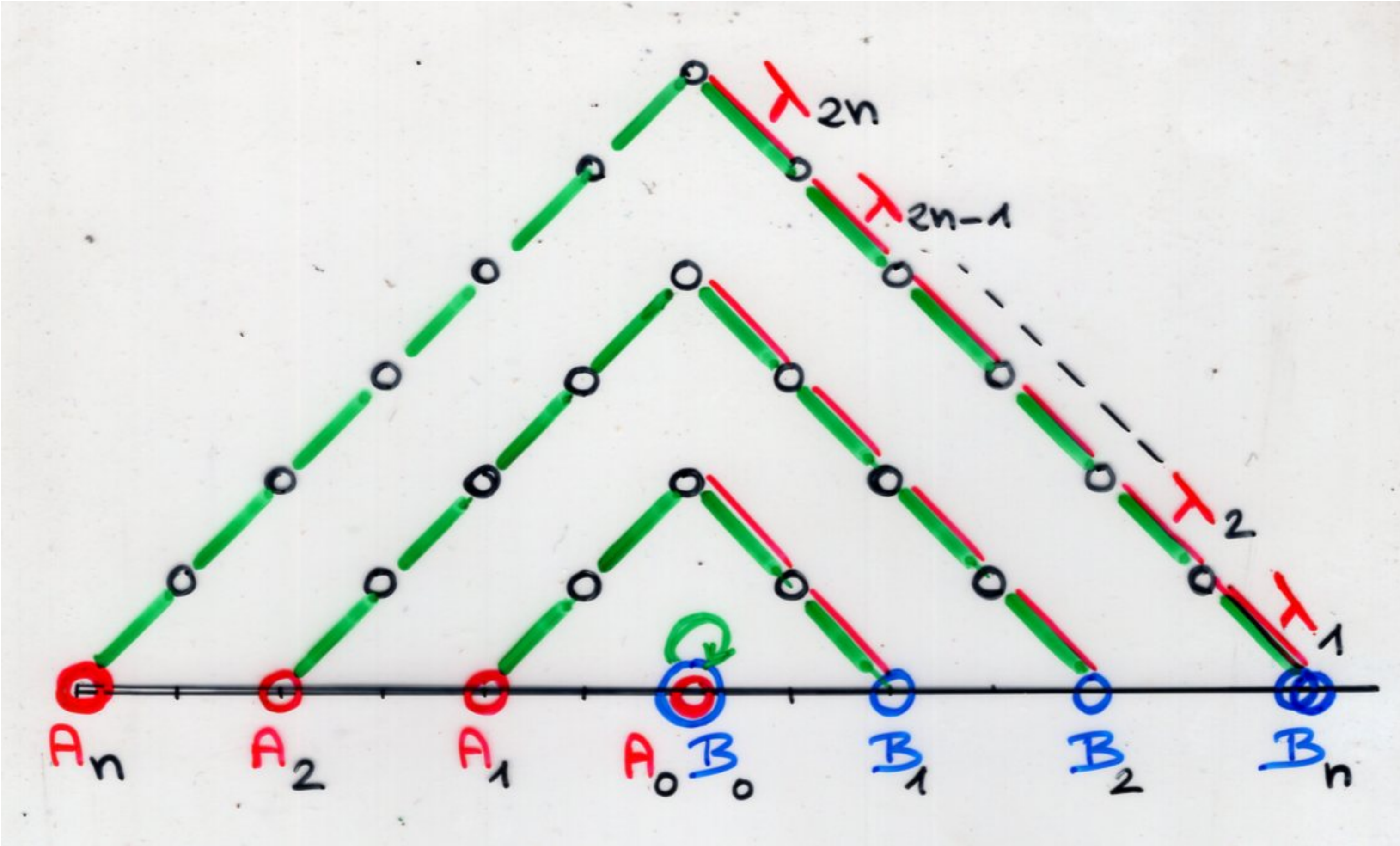
$$S(t;\lambda)$$

$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$

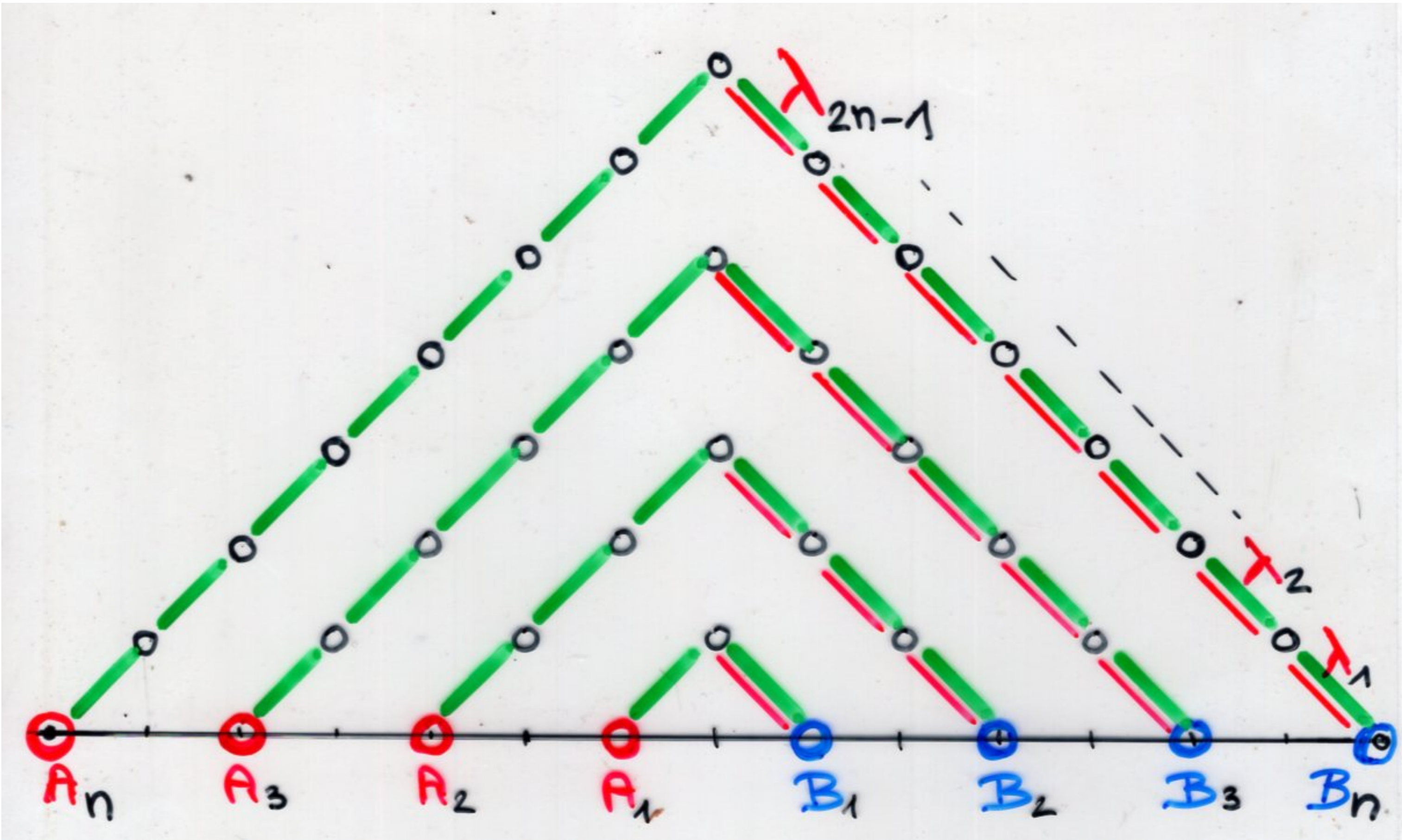
Dyck paths



$$\Delta_n^{(0)}(\nu) = H_\nu\begin{pmatrix} 0, 1, \ldots, n \\ 0, 1, \ldots, n \end{pmatrix}$$



$$\Delta_n^{(1)}(\nu) = H_\nu\begin{pmatrix}1, \dots, n \\ 1, \dots, n\end{pmatrix}$$

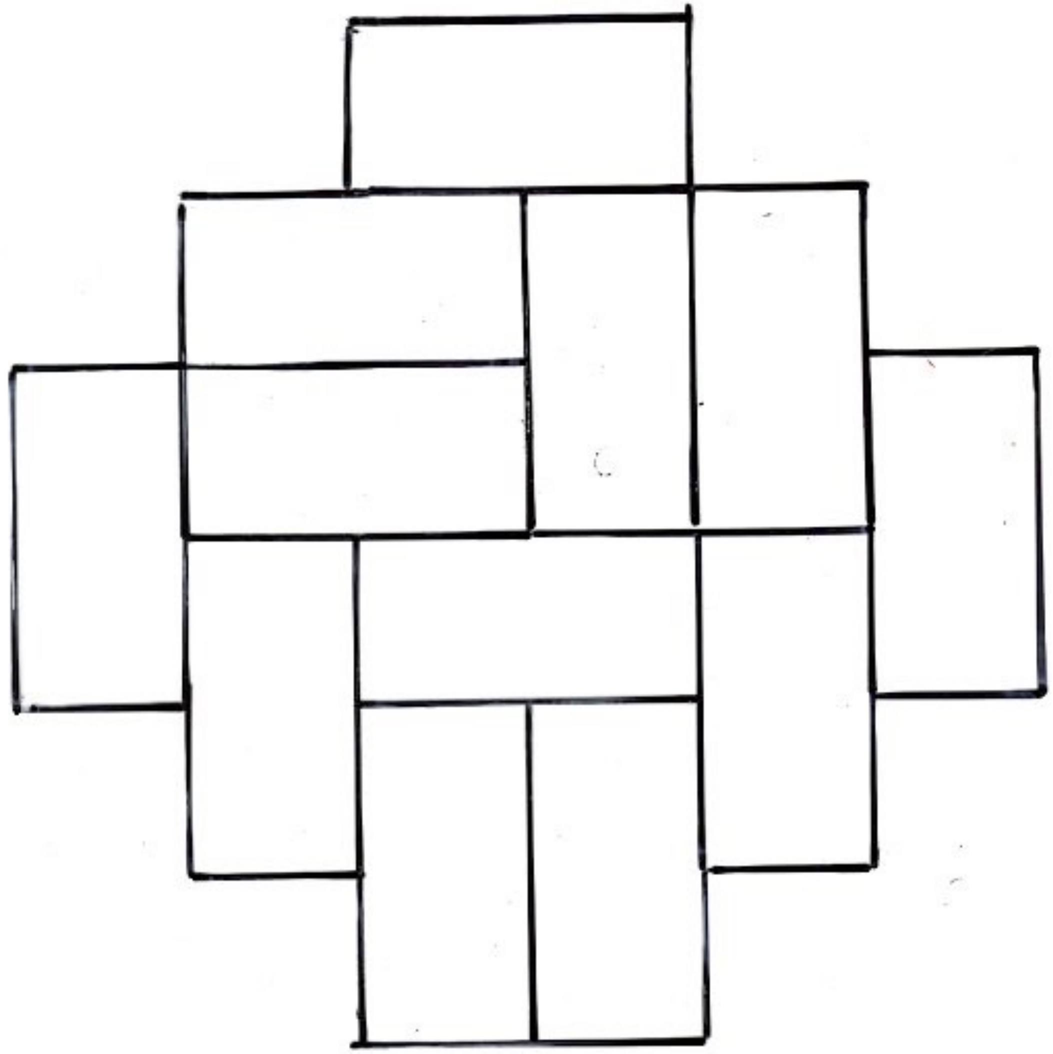


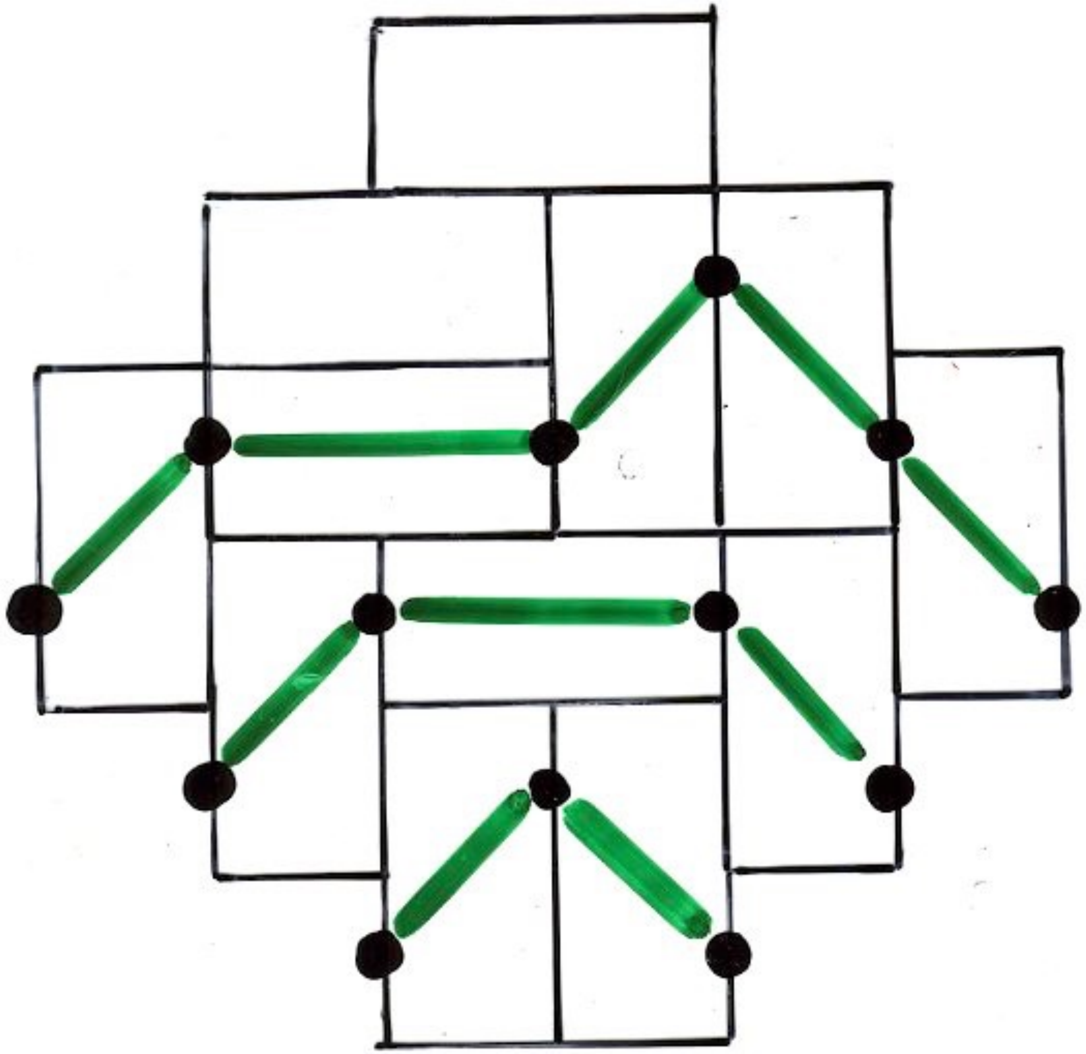
bijection

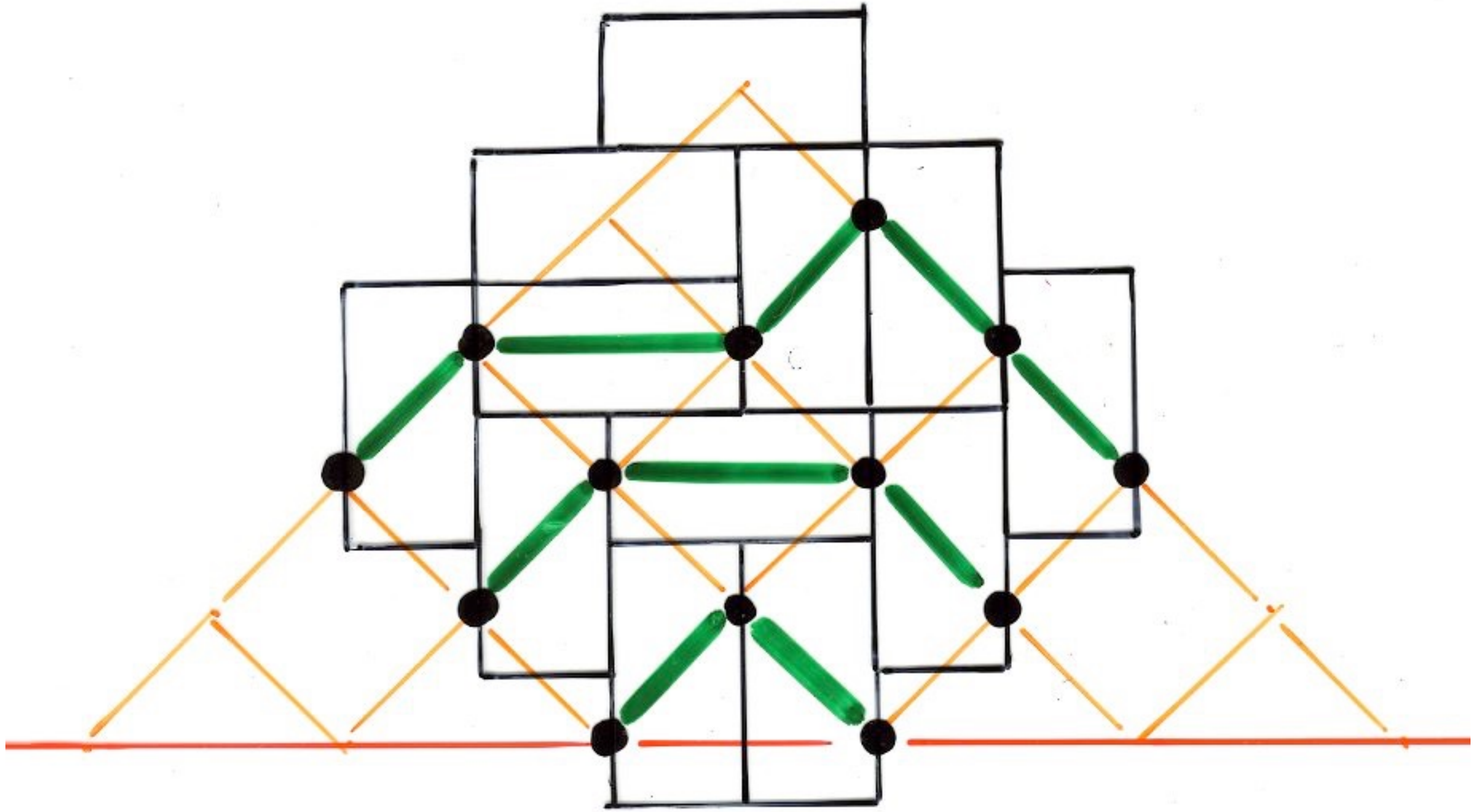
Aztec tilings

$$\updownarrow$$

non-intersecting paths
related to a Hankel determinant

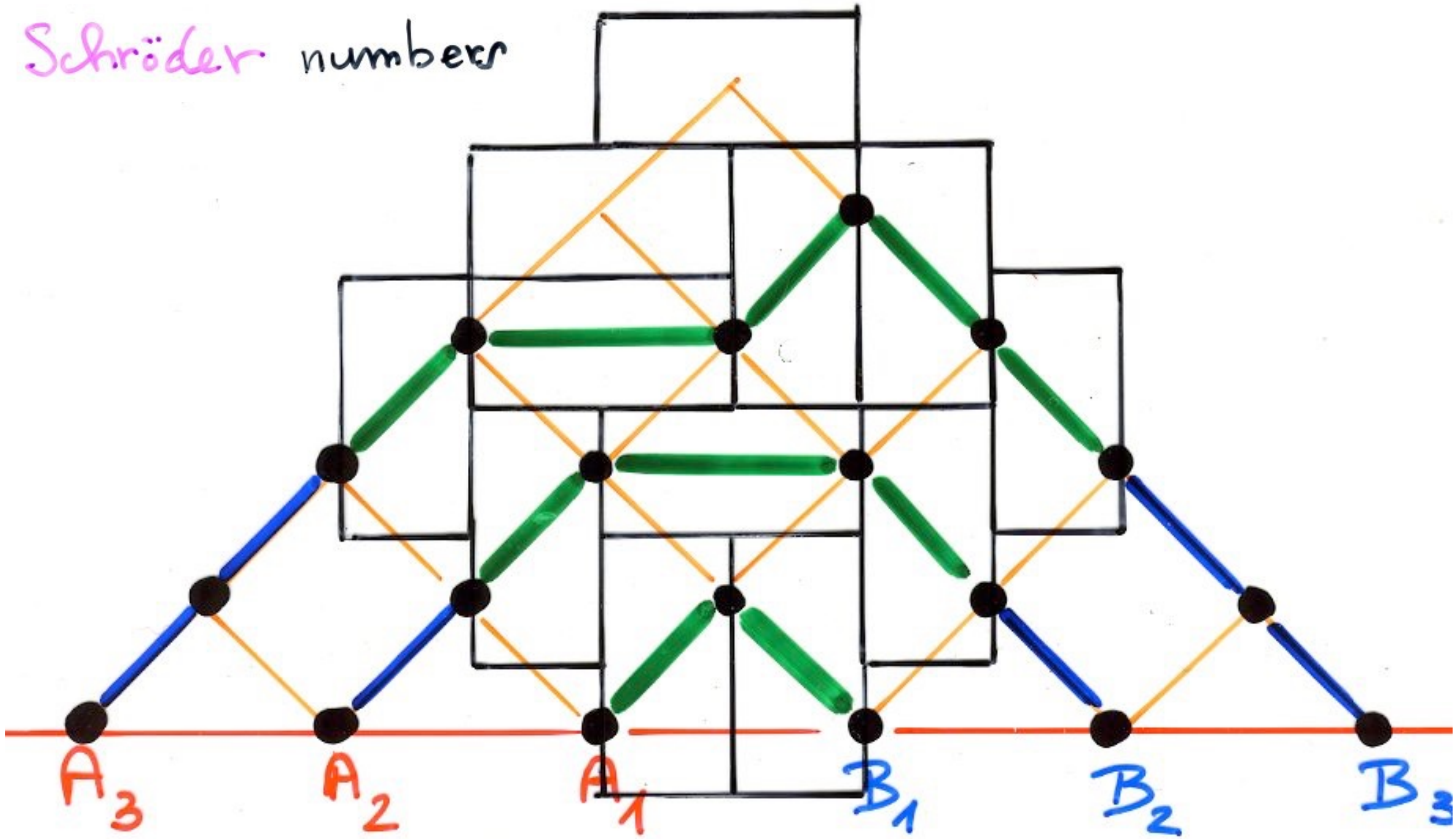






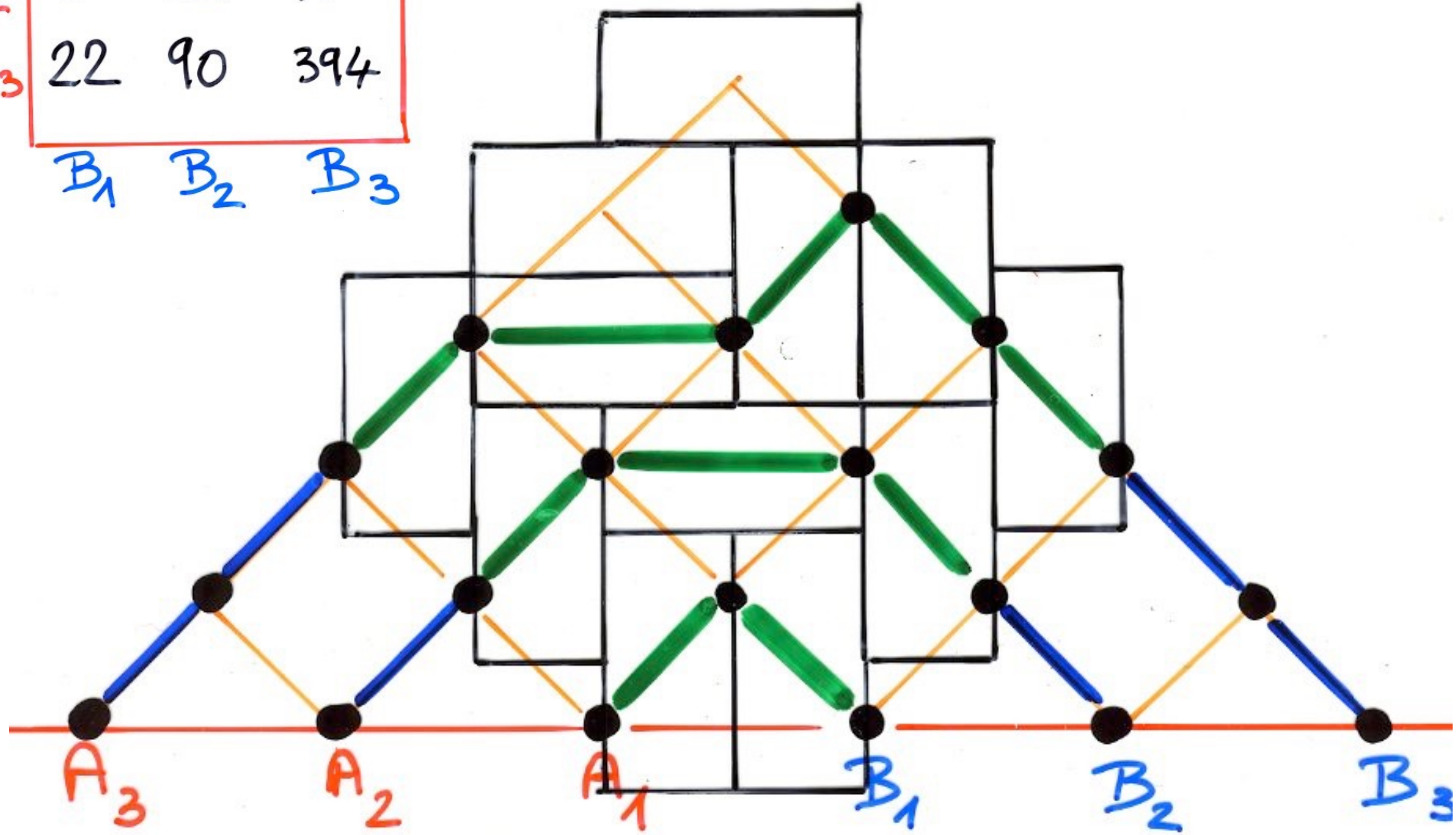
Schröder paths

Schröder numbers



$$\begin{array}{c|ccc|}
A_1 & 2 & 6 & 22 \\
A_2 & 6 & 22 & 90 \\
A_3 & 22 & 90 & 394 \\
\hline
 & B_1 & B_2 & B_3
\end{array}$$

Hankel determinant



$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

$$= 44 - 36$$

$$
\begin{aligned}
\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} &= (2 \times 22) - (6 \times 6) \\
&= 44 - 36 \\
&= 8 = 2^3
\end{aligned}
$$

(!)

$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$+\,17336 \qquad +\,11880 \qquad +\,11880 \;\to\; 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$-\,16200 \qquad -\,14184 \qquad -\,10648 \;\to\; -41032$$

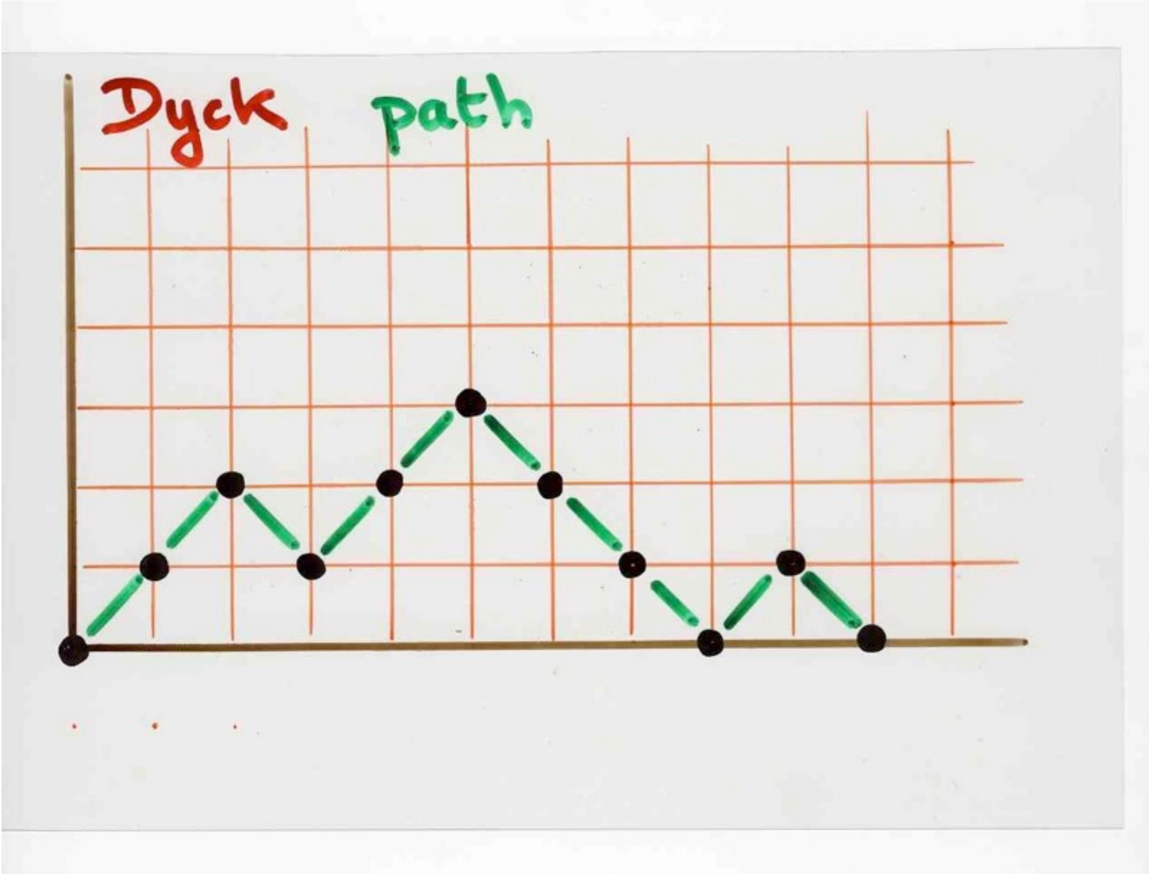
$$\overline{64}$$

$$= 2^6 \qquad (!!)$$

- Dyck paths

- 2-colored Motzkin paths

- Staircase polygons

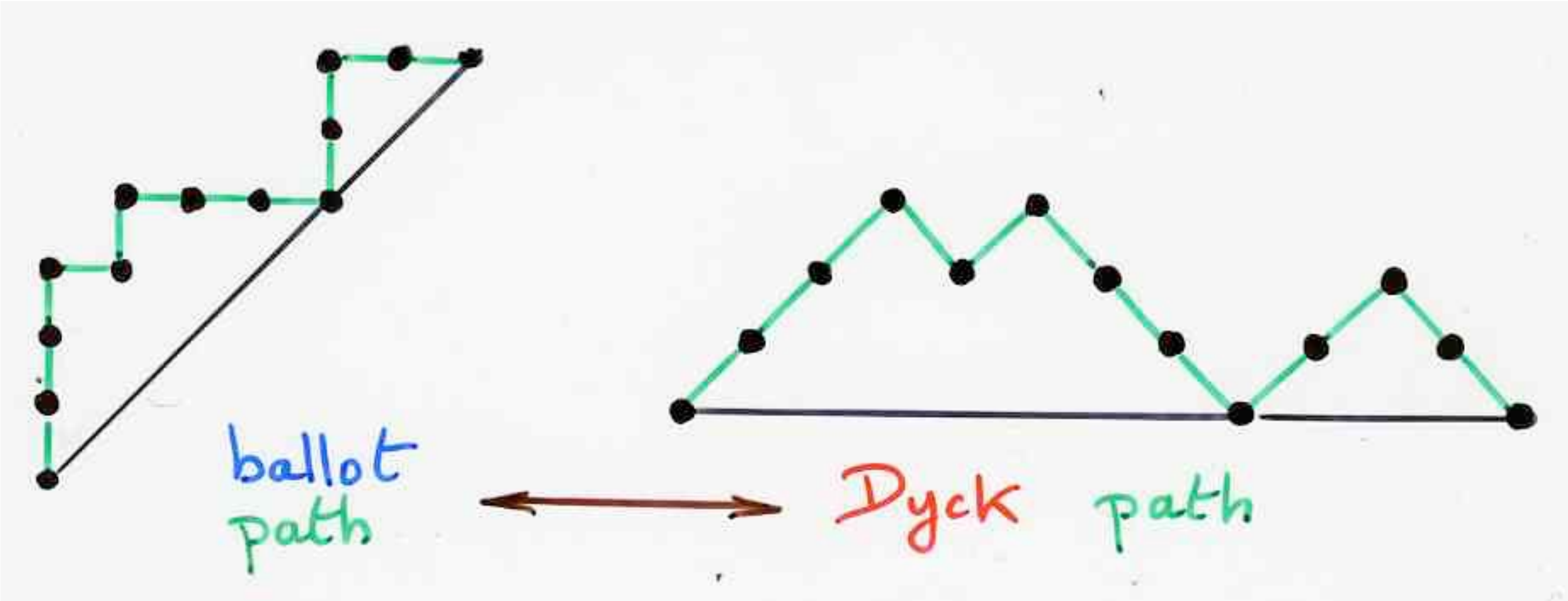


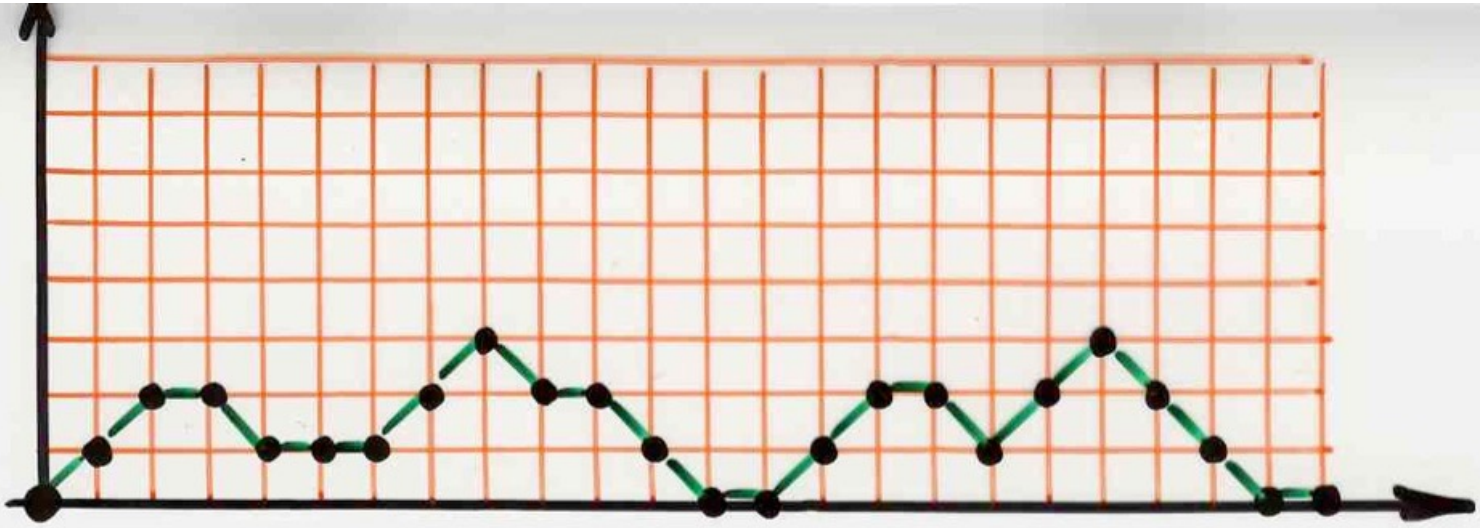
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan
numbers

$$y = 1 + t y^2$$

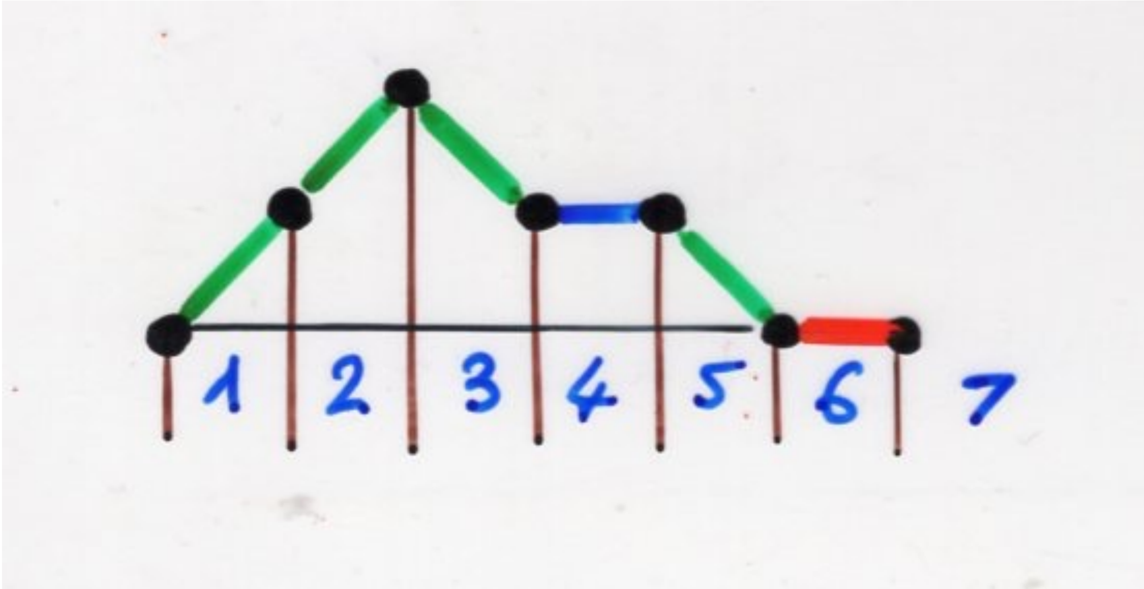
vocabulary: ballot path
Dyck path





$$m = 1 + t\,m + t^2 m^2$$

2-colored
Motzkin
path



$a_n = C_{n+1}$

number of
such paths
of length n

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan
numbers

bijection

Dyck paths

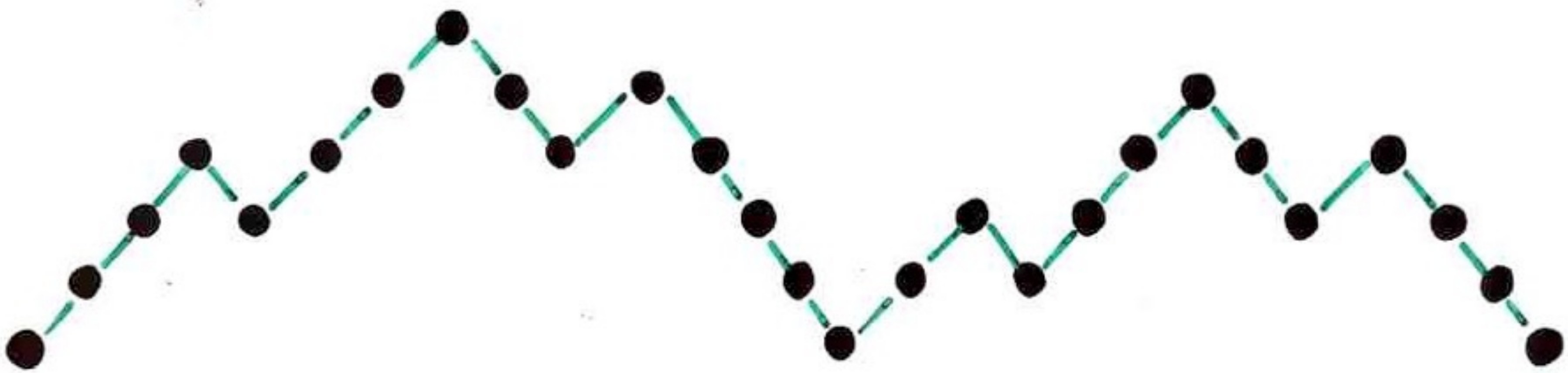
$$\updownarrow$$

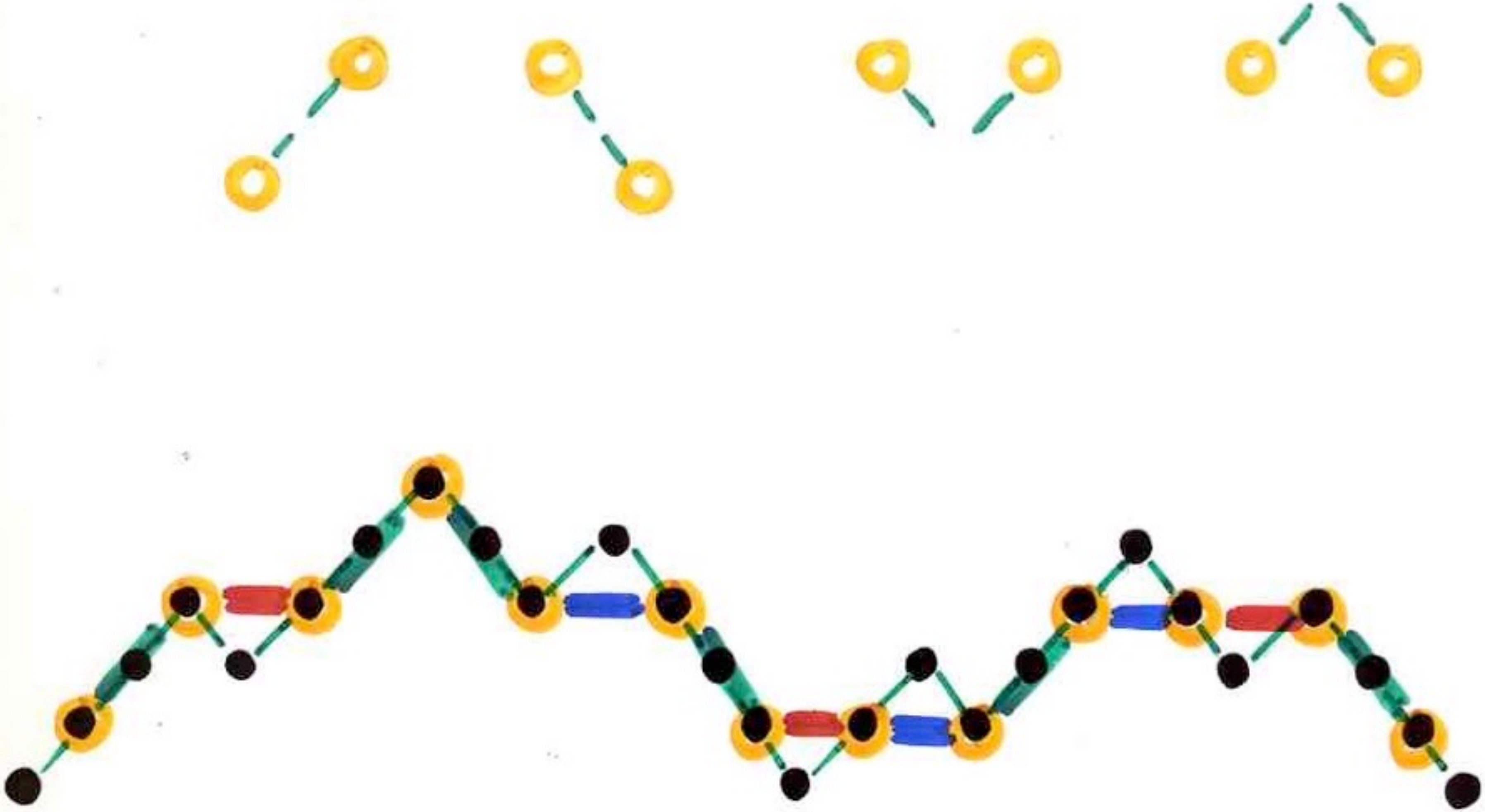
2-colored Motzkin paths

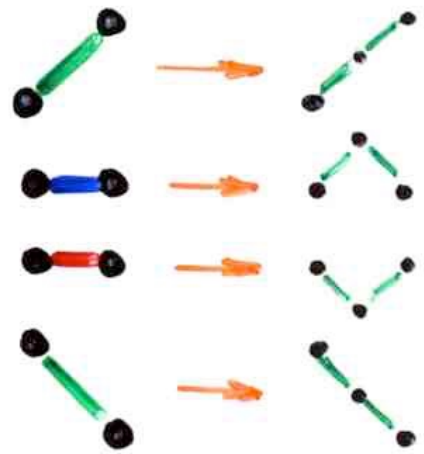
bijection

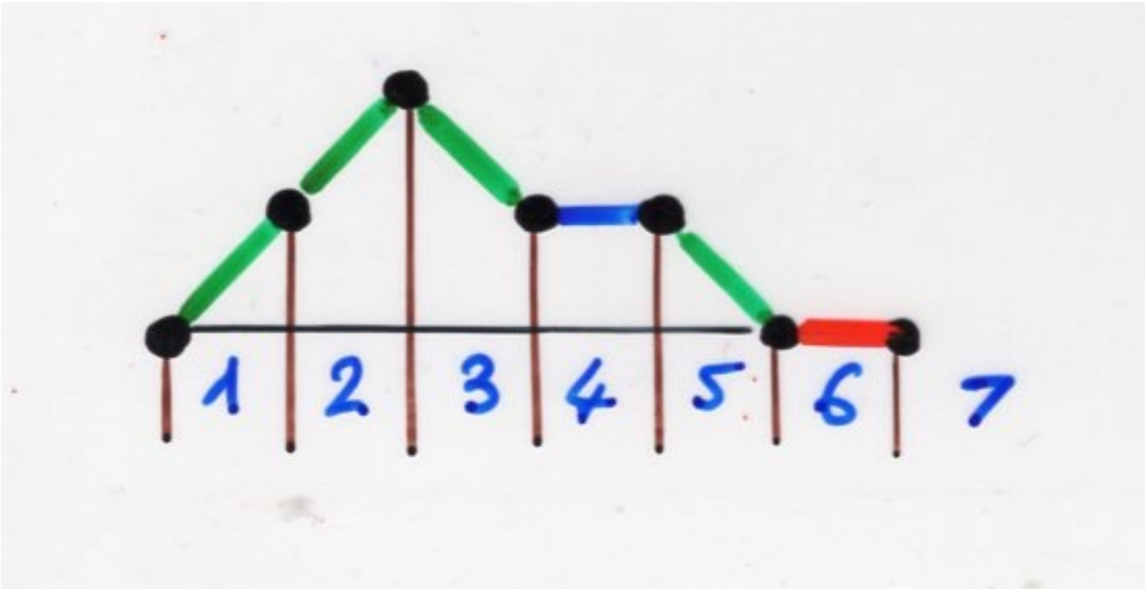
2-colored Motzkin paths

Dyck paths







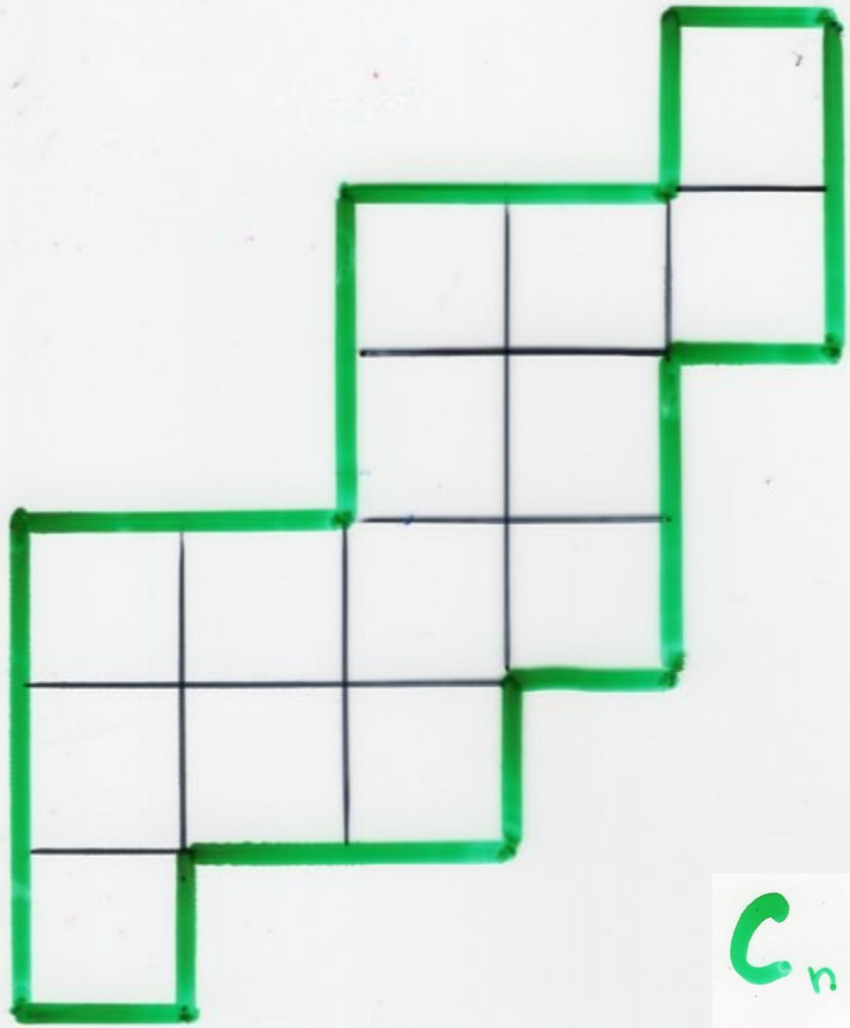

1
2
3
4
5
6
7

exercise

Touchard identity

$$C_{n+1} = \sum_{0 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i \, 2^{2n-2i}$$

staircase polygons



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

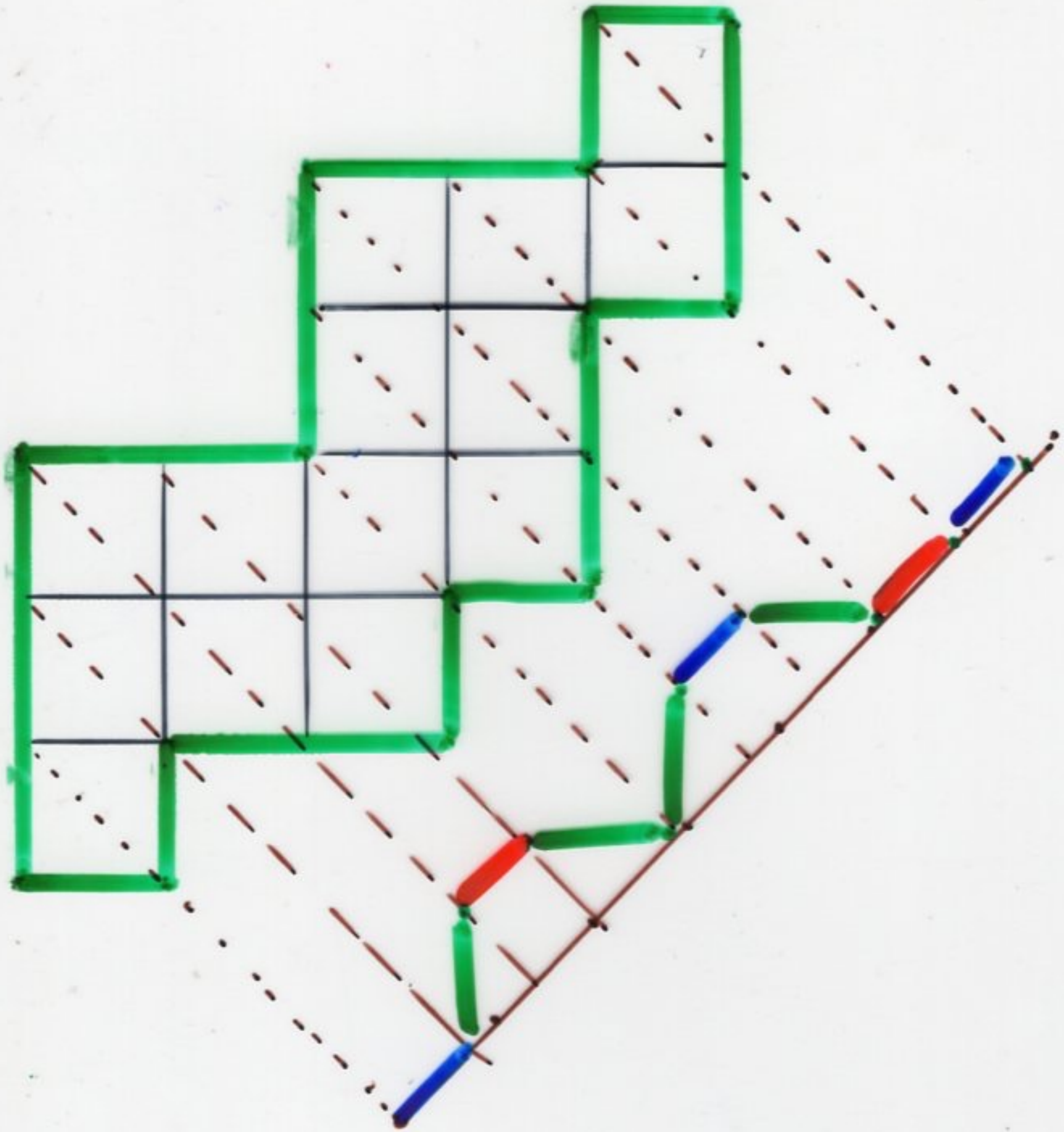
Catalan numbers

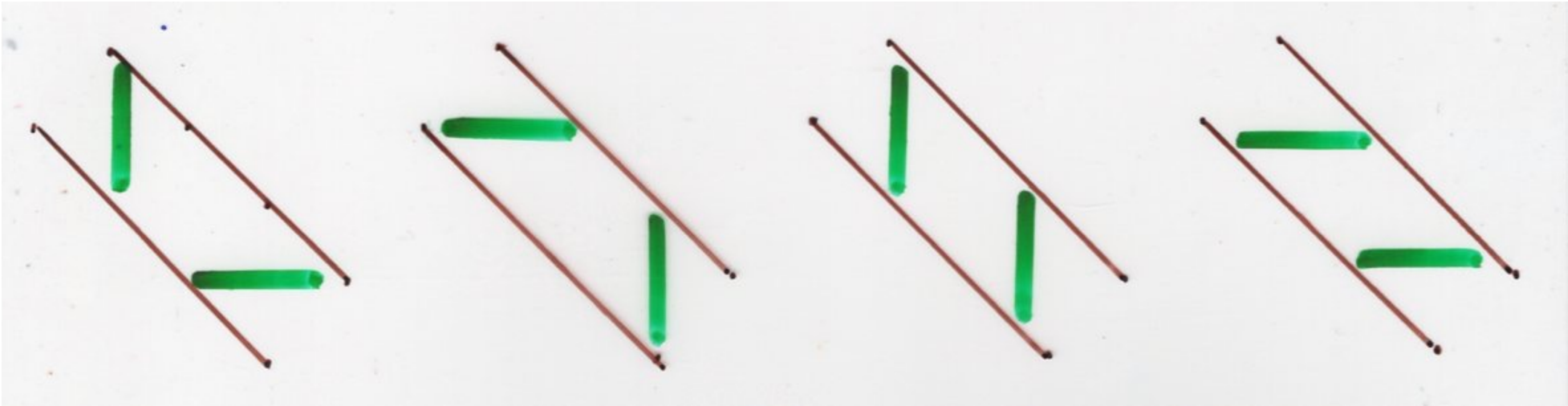
bijection

staircase polygons

$\updownarrow$

2-colored Motzkin paths



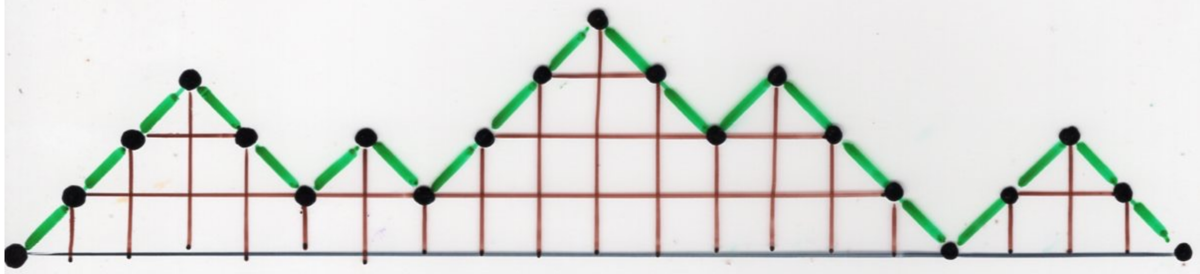


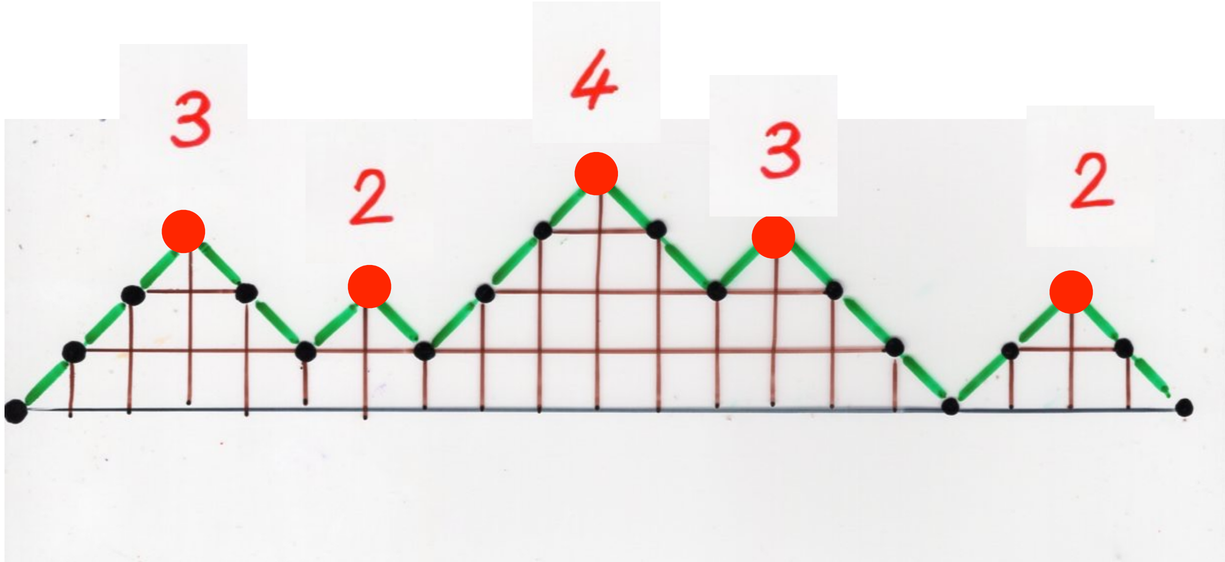
bijection

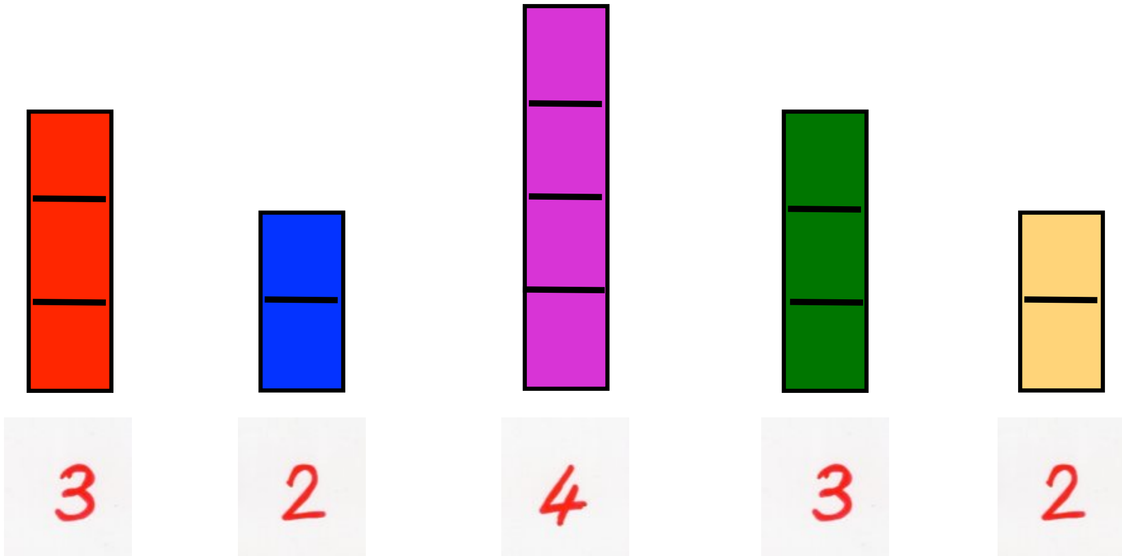
staircase polygons

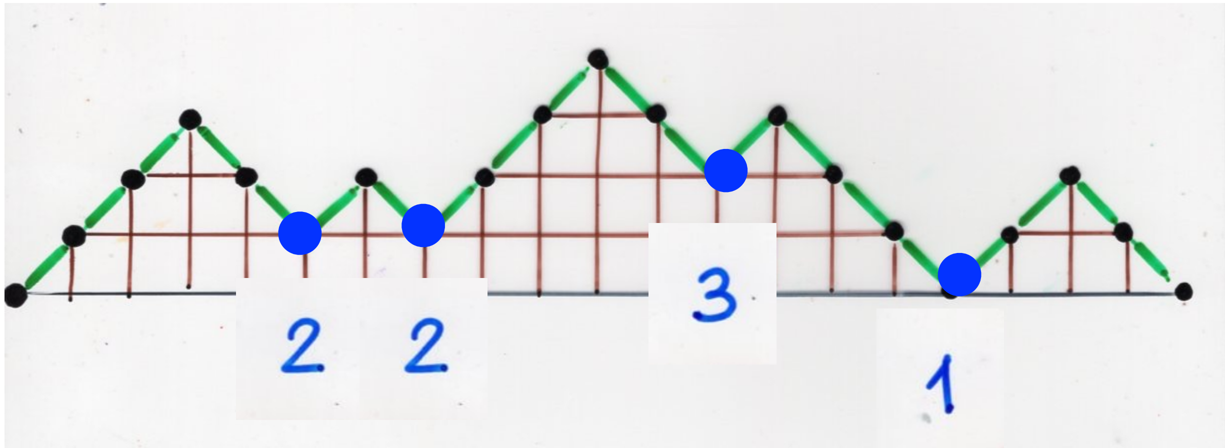
$\updownarrow$

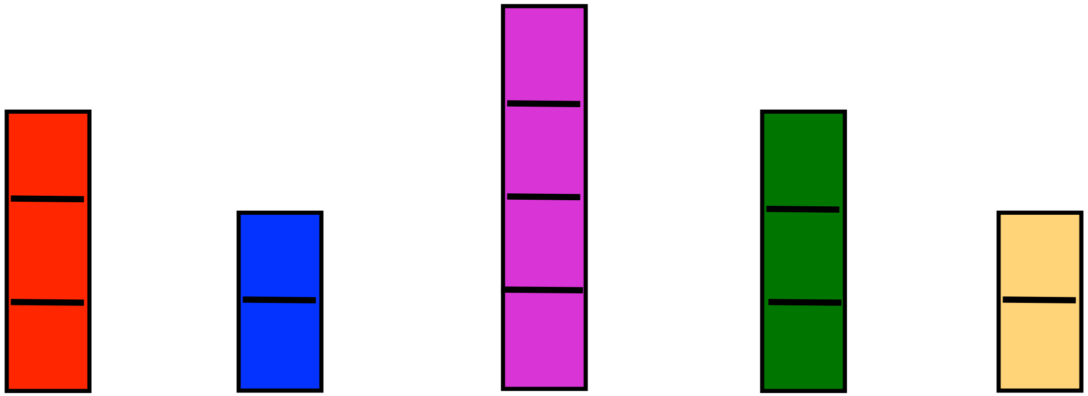
Dyck paths

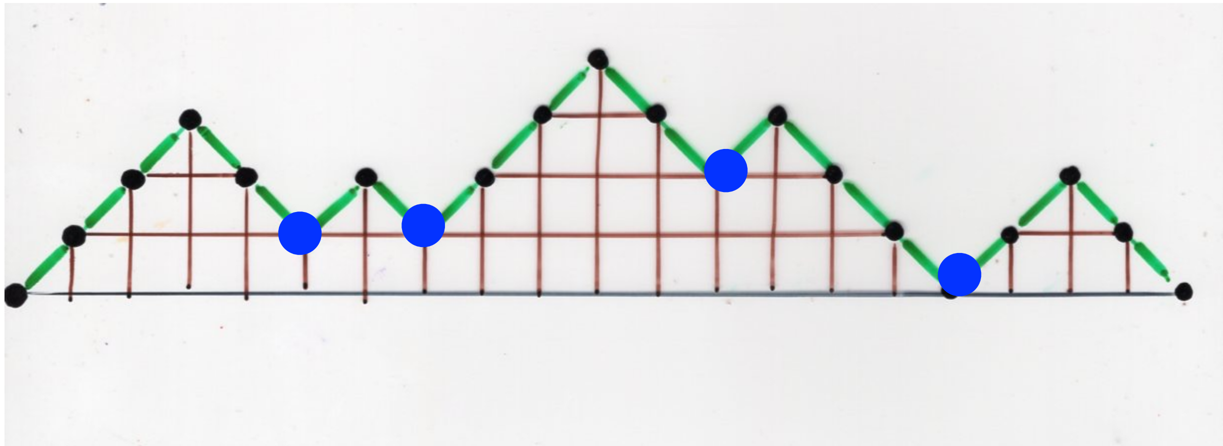


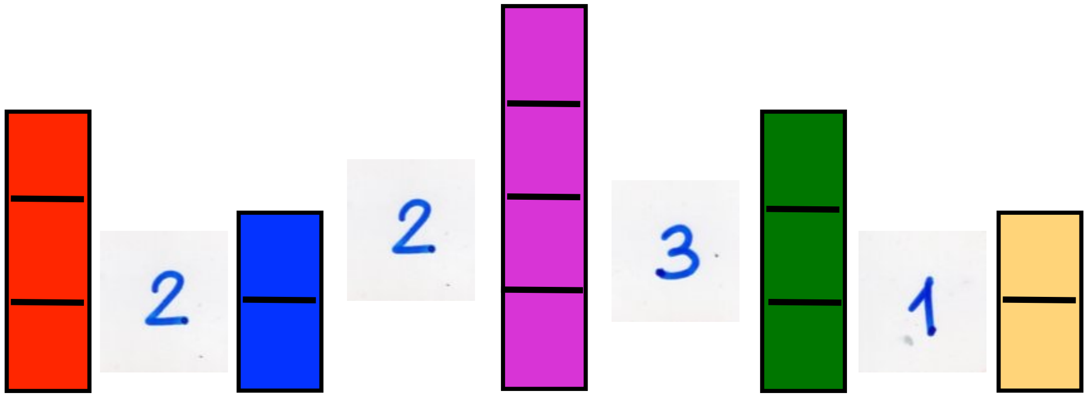

3
2
4
3
2

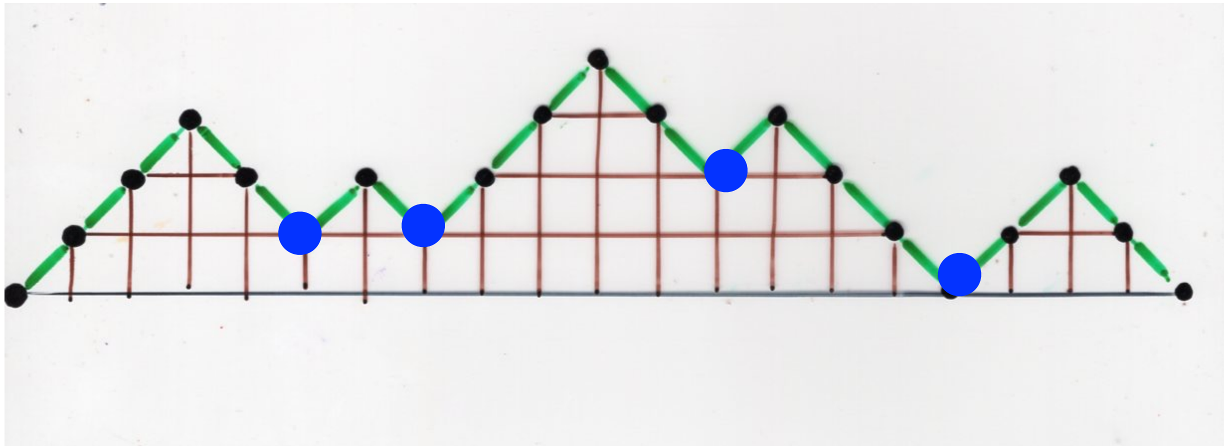

3
2
4
3
2

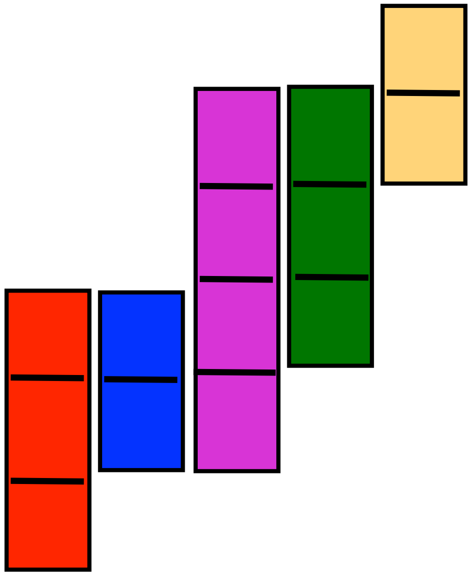

2
2
3
1

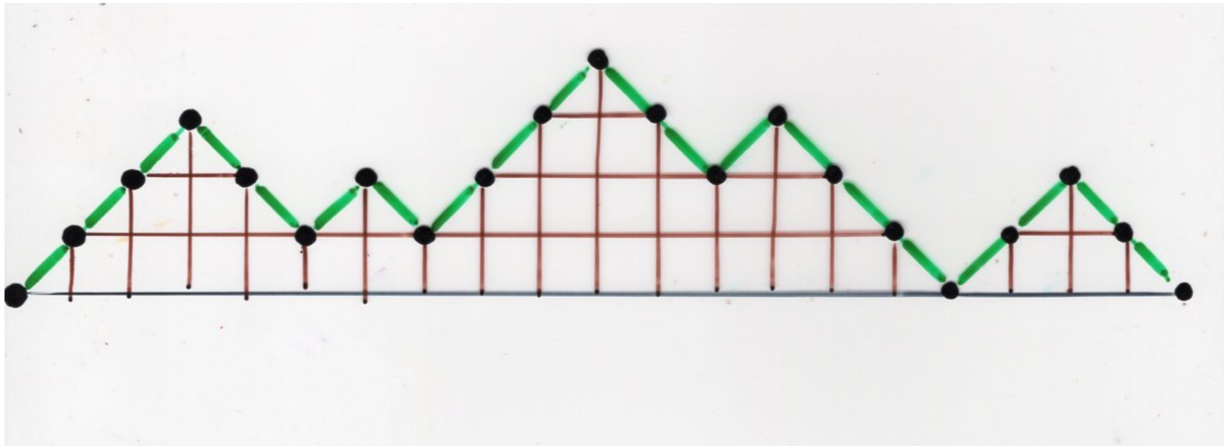


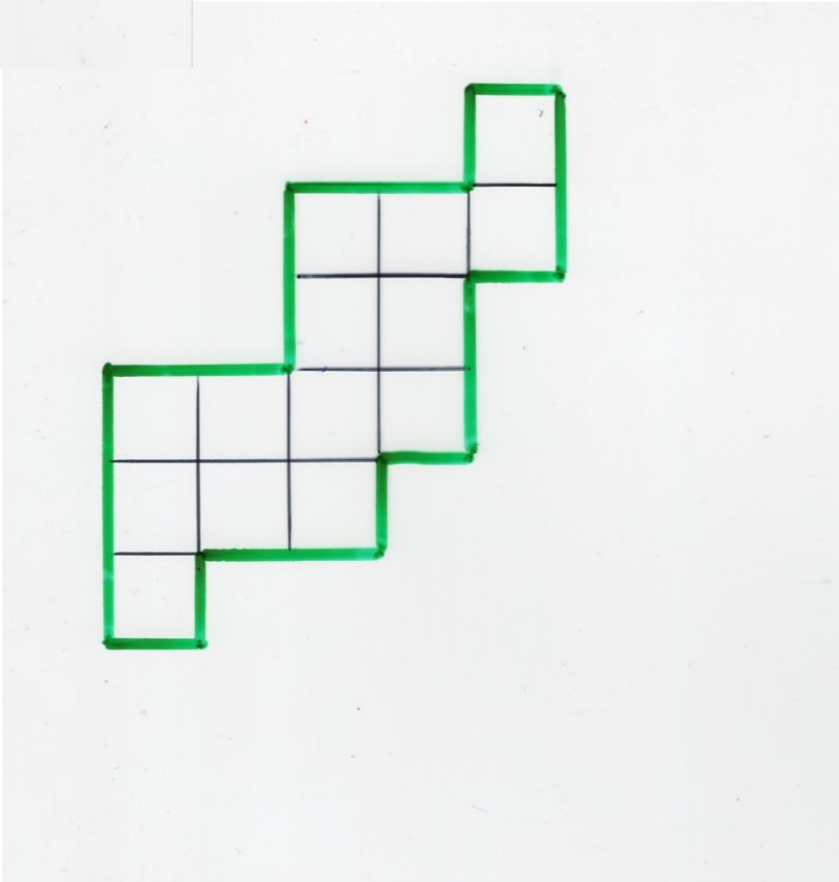



2
2
3
1



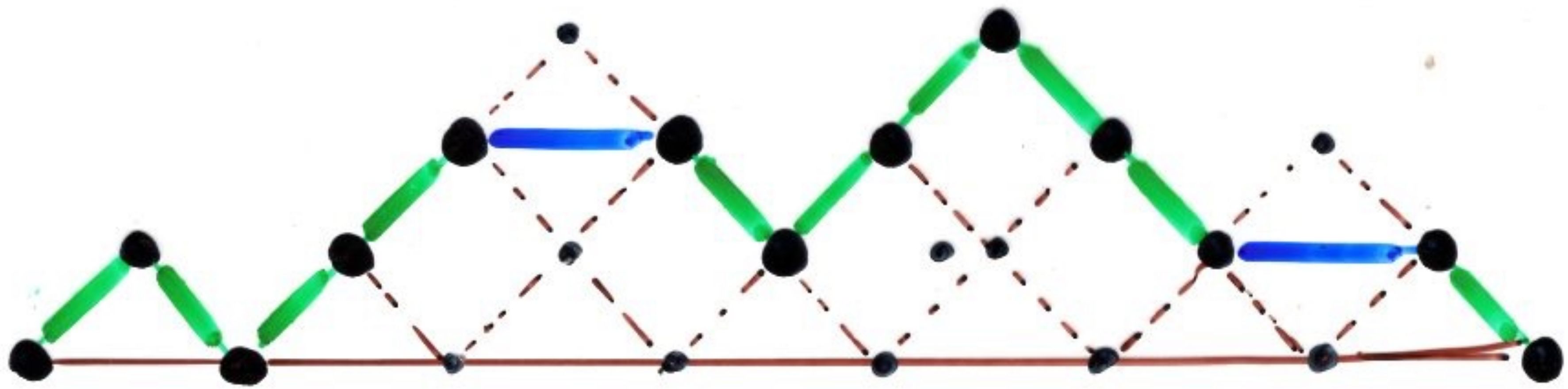


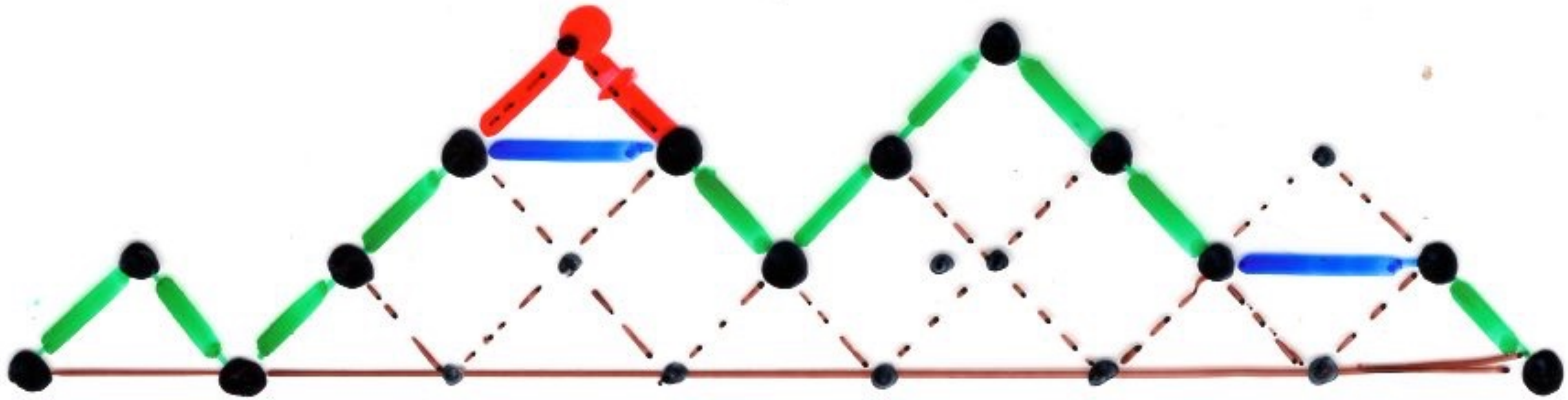


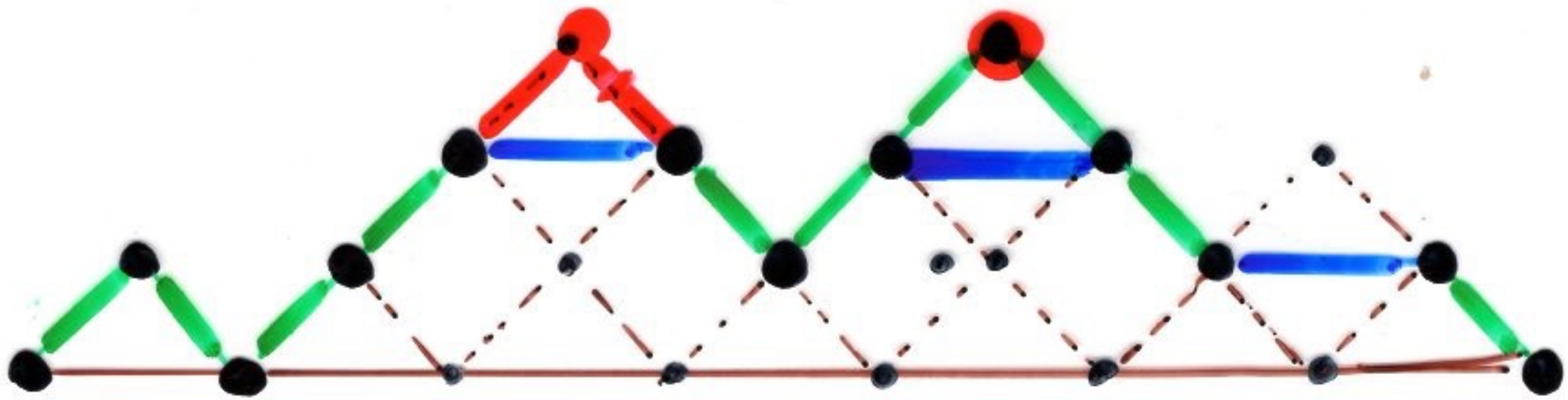


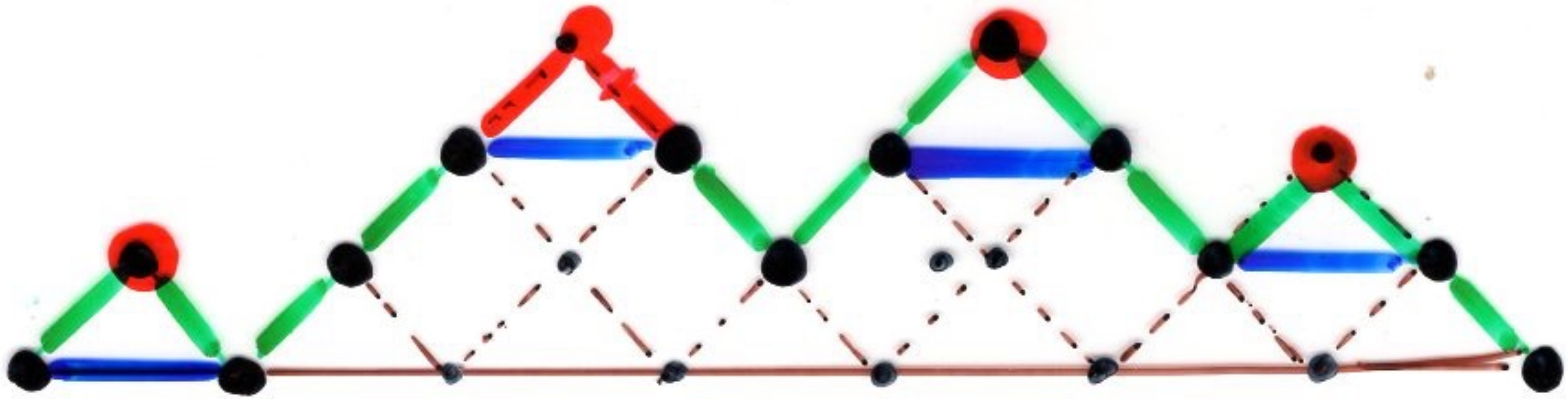
« bijective computation » of the Hankel determinant of Schröder numbers

giving the number of tilings of the Aztec diagram



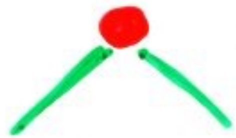






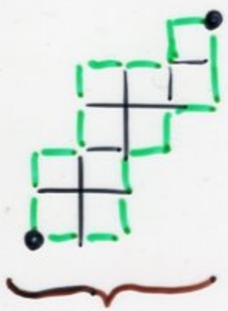
$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} 2^{\text{peak}(\omega)}$$

$\text{peak}(\omega) =$ number of peaks of the path ω



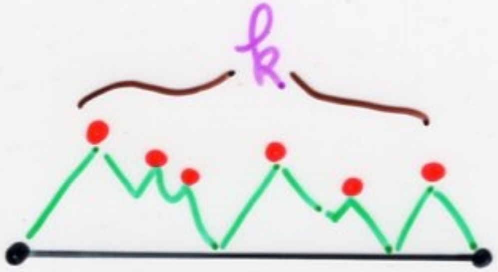
(β)- distribution

→ Ch 2c the Catalan garden



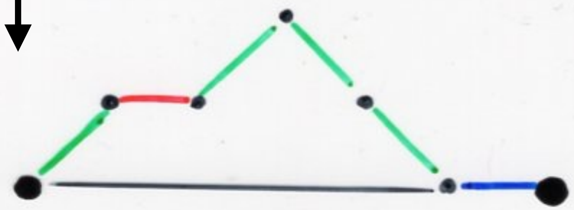
k

- staircase polygons
  number of columns



- Dyck path
  number of peaks

(β)- distribution $\quad \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$

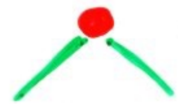


- number of / or — in a 2-colored Motzkin path

- Dyck path
  number of SE steps from odd to even level

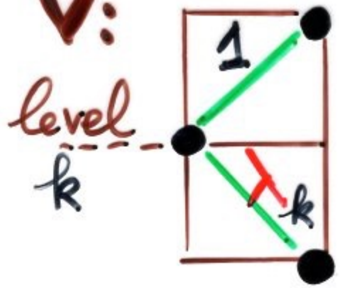
$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} 2^{\text{peak}(\omega)}$$

peak(ω) = number of peaks of the path ω



$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} v(\omega)$$

v:

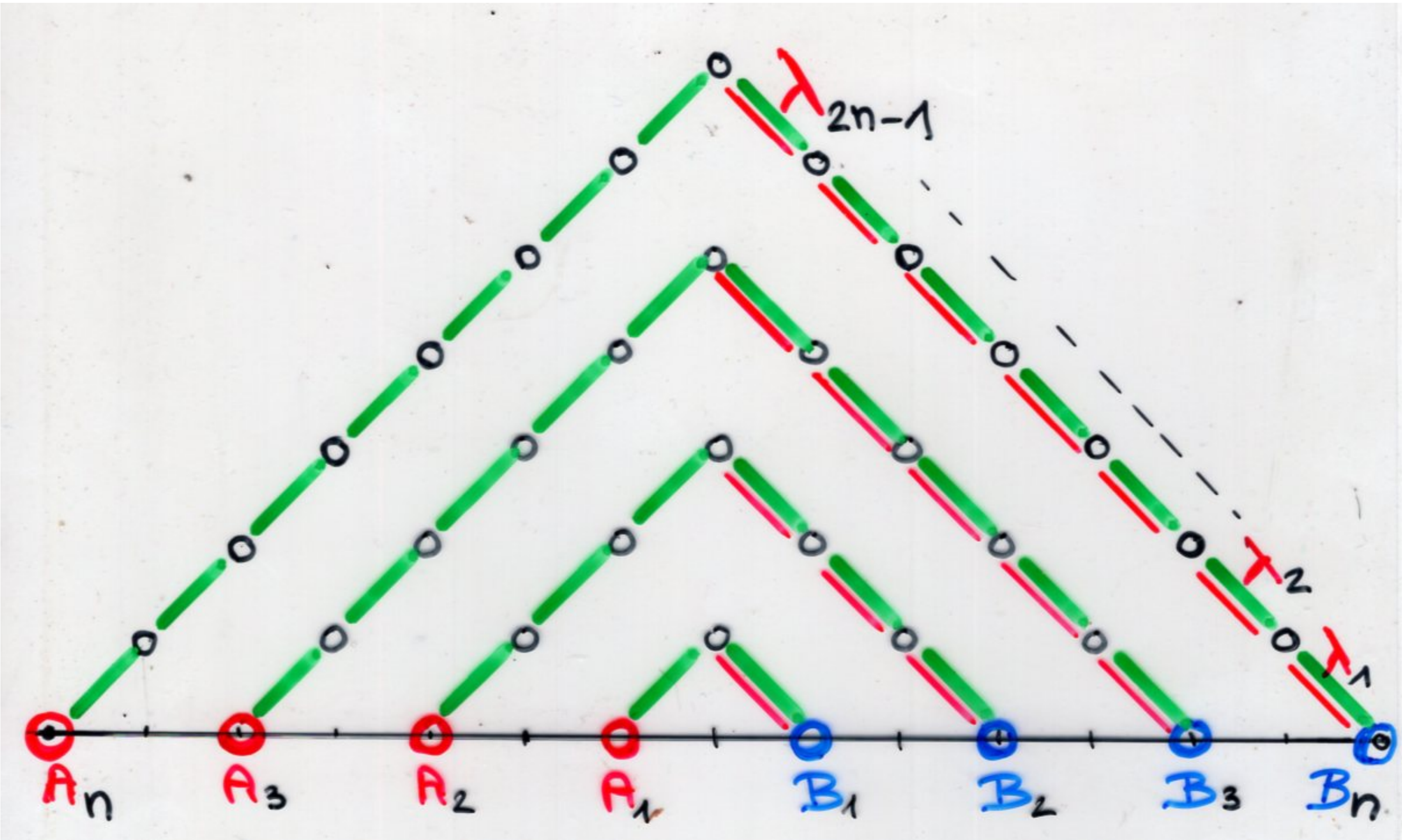


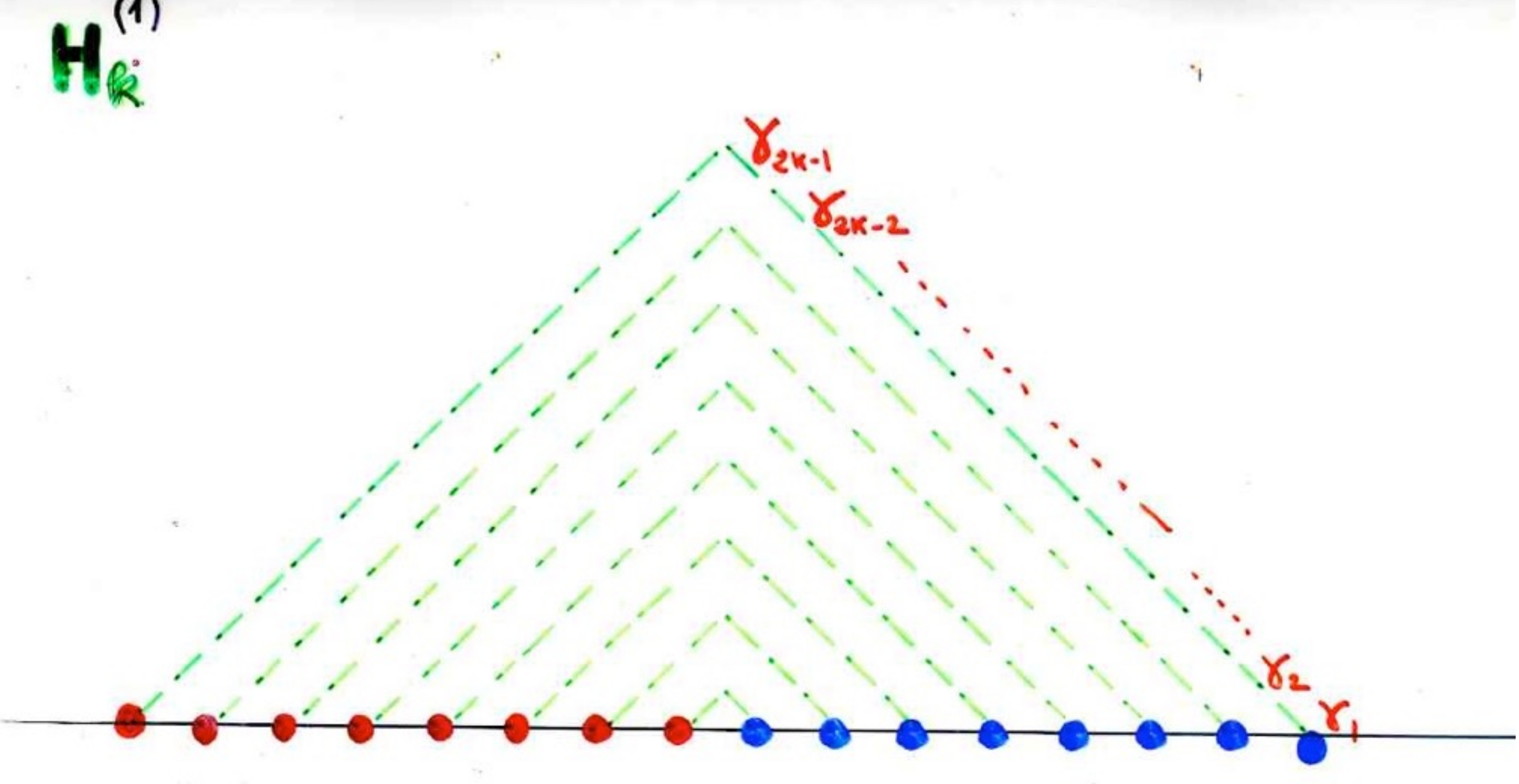
$$\lambda_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$$

(β)- distribution

$\rightarrow$ Ch 2c the Catalan garden

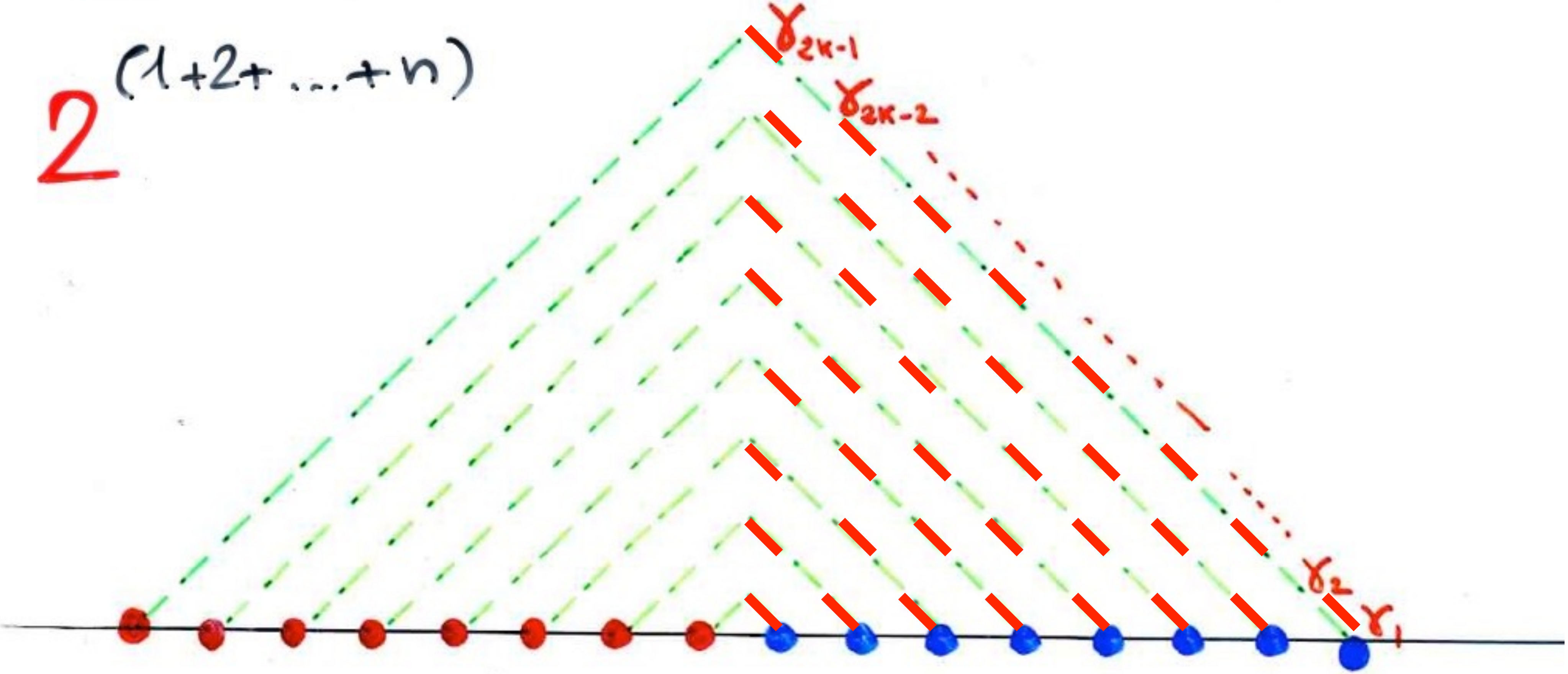
$$\Delta_n^{(1)}(\nu) = H_\nu\left(\begin{matrix}1, \dots, n\\ 1, \dots, n\end{matrix}\right)$$



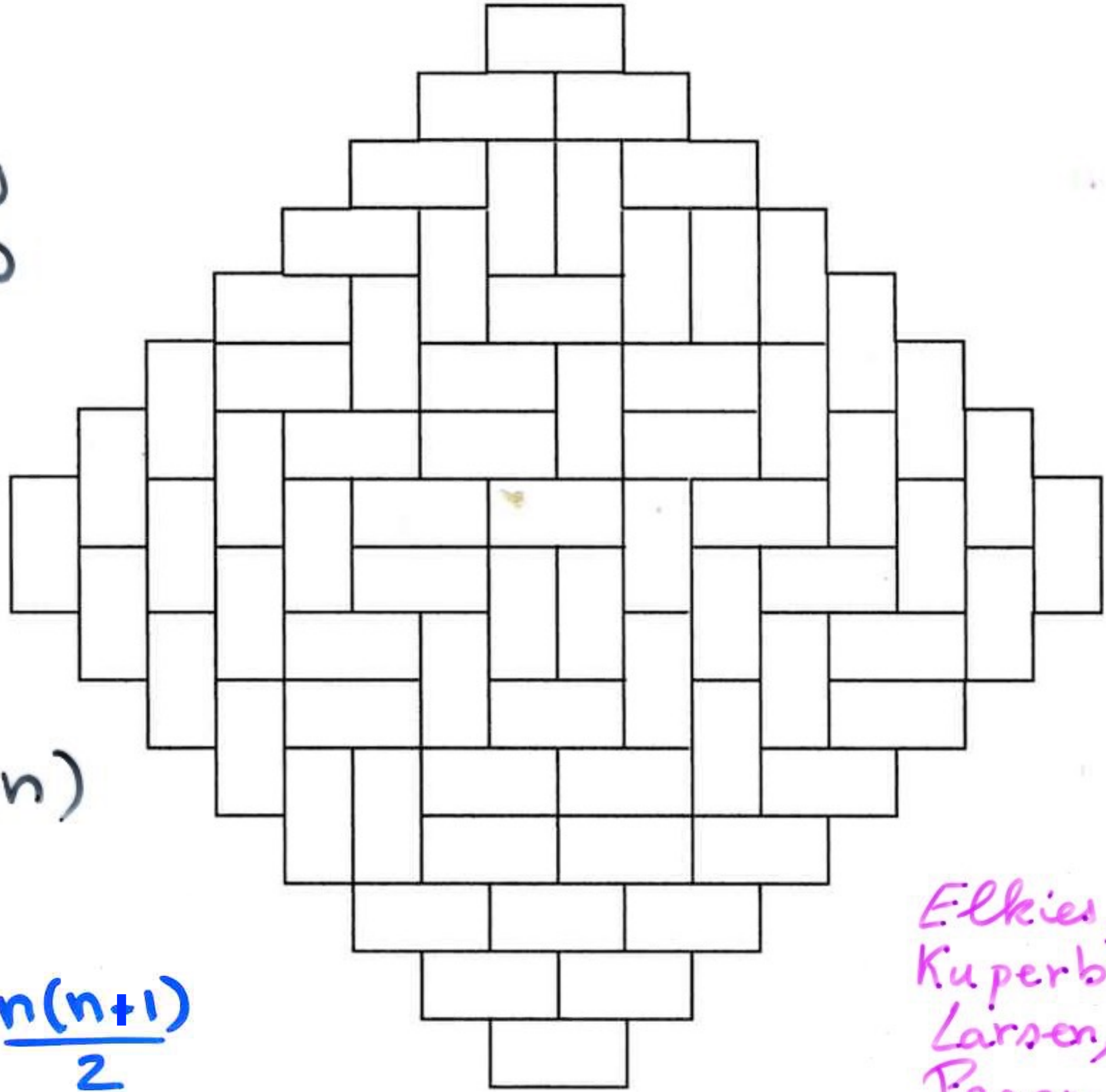

$H_k^{(1)}$
γ_{2k-1}
γ_{2k-2}
γ_2
γ_1

$H_k^{(1)}$

$2^{(1+2+\ldots+n)}$



number of tilings



$$2^{(1+2+\dots+n)}$$

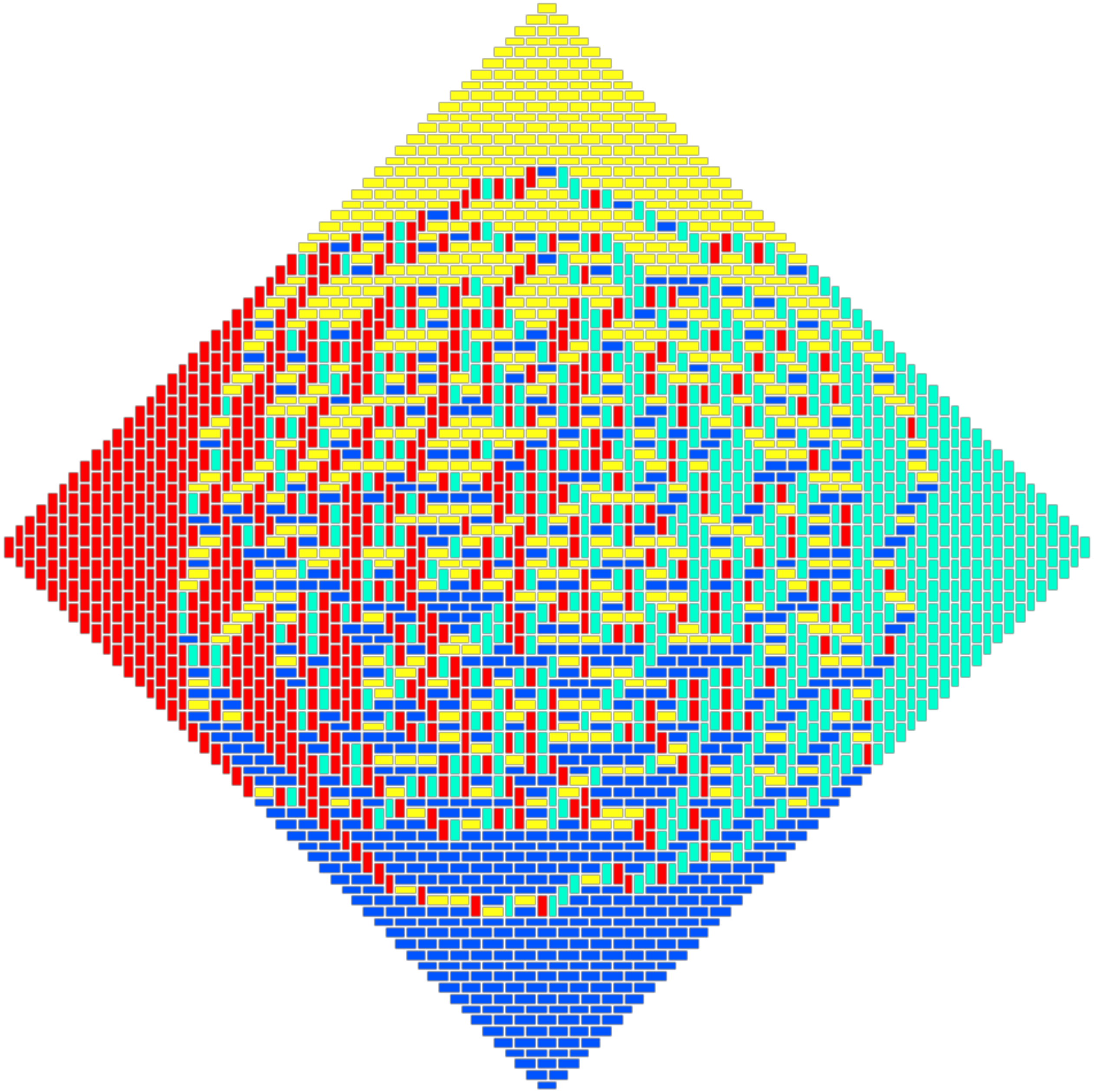
$$2^{\frac{n(n+1)}{2}}$$

Elkies,
Kuperberg,
Larsen,
Propp
(1992)

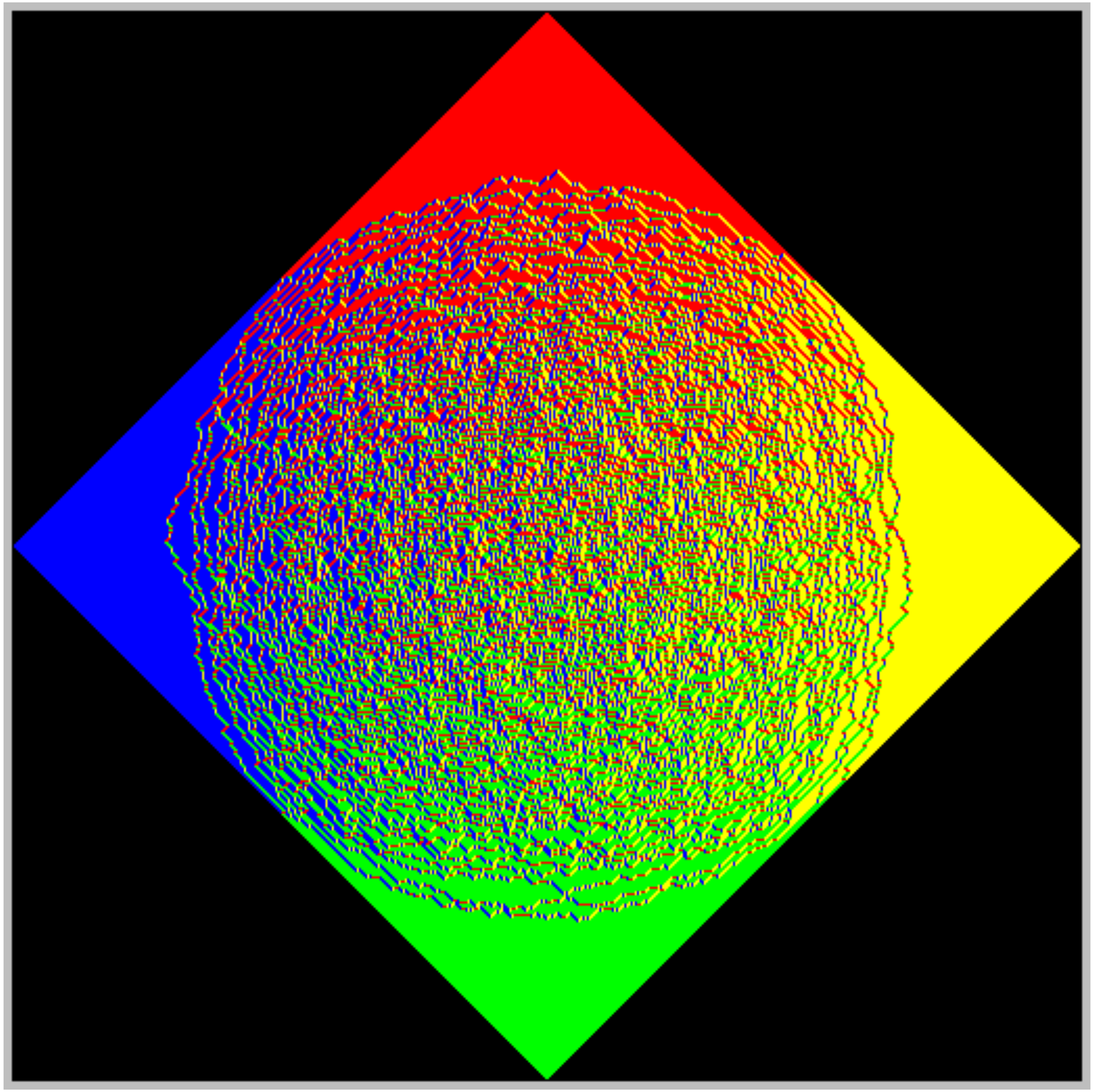
Random Aztec tilings

The arctic circle theorem

random Aztec tiling



The "arctic circle" theorem



Relation with alternating sign matrices (ASM)

Def - **ASM** alternating sign matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

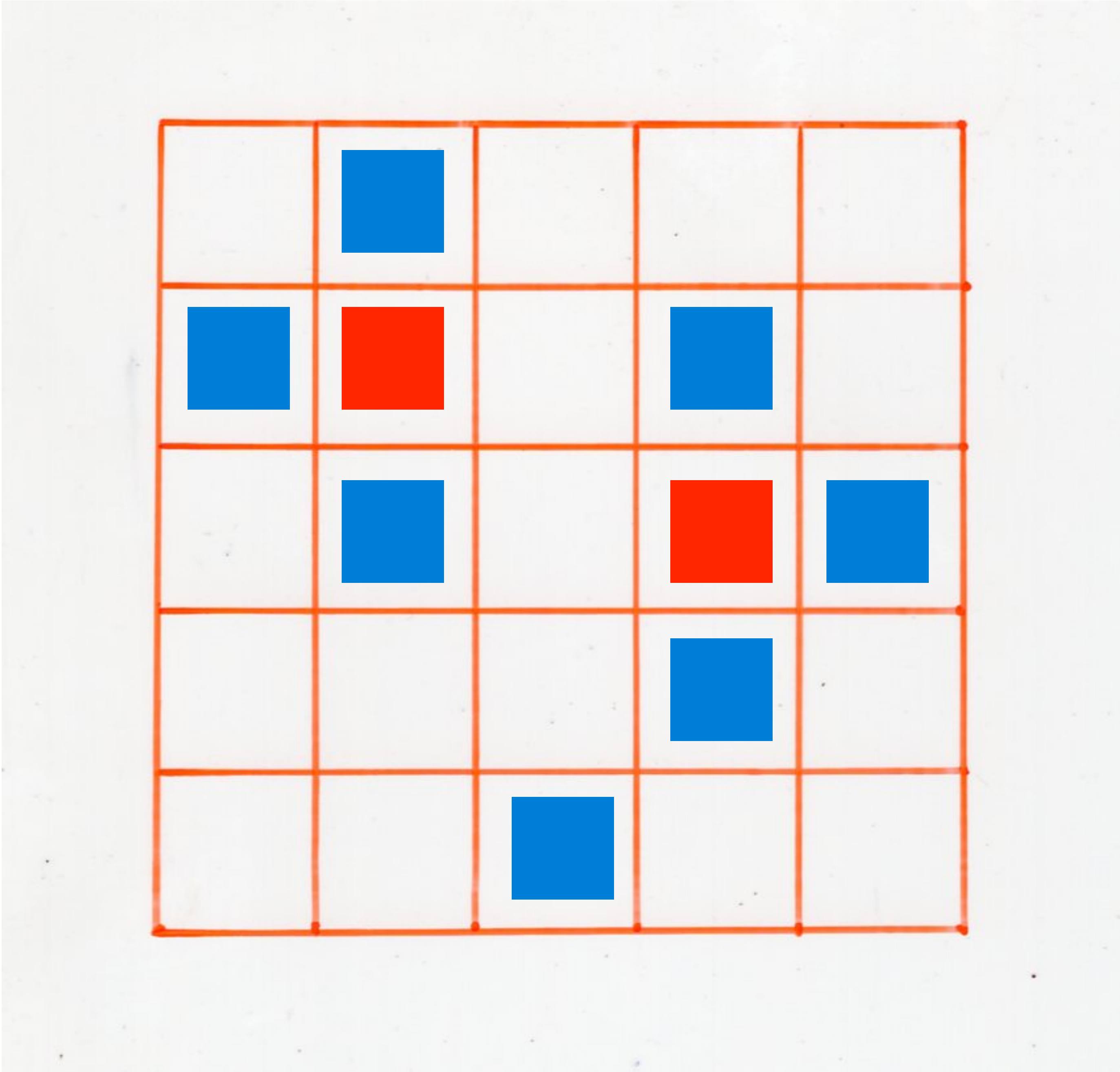
(i) entries: $0, 1, -1$

(ii) sum of entries in each $\left\{\begin{matrix}\text{row} \\ \text{column}\end{matrix}\right.$ $= 1$

(iii) non-zero entries alternate in each $\left\{\begin{matrix}\text{row} \\ \text{column}\end{matrix}\right.$

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \underset{\text{permutations}}{6}$$

$$1, \; 2, \; 7, \; 42, \; 429, \; \ldots$$

1, 2, 7, 42, 429, ...

$$\frac{1!\ \ 4!\ \ [\ldots]\ \ (3n-2)!}{n!\ \ (n+1)\ [\ldots]\ \cdot\ (n+n-1)!}$$

?

alternating sign matrix

(ex –) conjecture

$A_n(x)$ enumeration of ASM according to the number of (-1)

$$A_n(2) = 2^{n(n-1)/2}$$

tilings of the Aztec diagram with dimers

Relation with lecture 3

An introduction to the « cellular ansatz »

From RSK to the PASEP

alternating sign matrices (ASM)
and a quadratic algebra

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

$UD = DU + Id$

permutations

RSK $\longleftrightarrow$

pairs of Young tableaux

Physics

towers placements

(i) first step

(ii) second step

$DE = qED + E + D$

alternative tableaux

EXF $\longleftrightarrow$

permutations

commutations

ASM alternating sign matrices

tilings

rewriting rules

$\longleftrightarrow$?

non-crossing paths

planarization

8-vertex model

claim:

ASM
alternating sign
matrices

are

Q-tableaux

A, A', B, B',

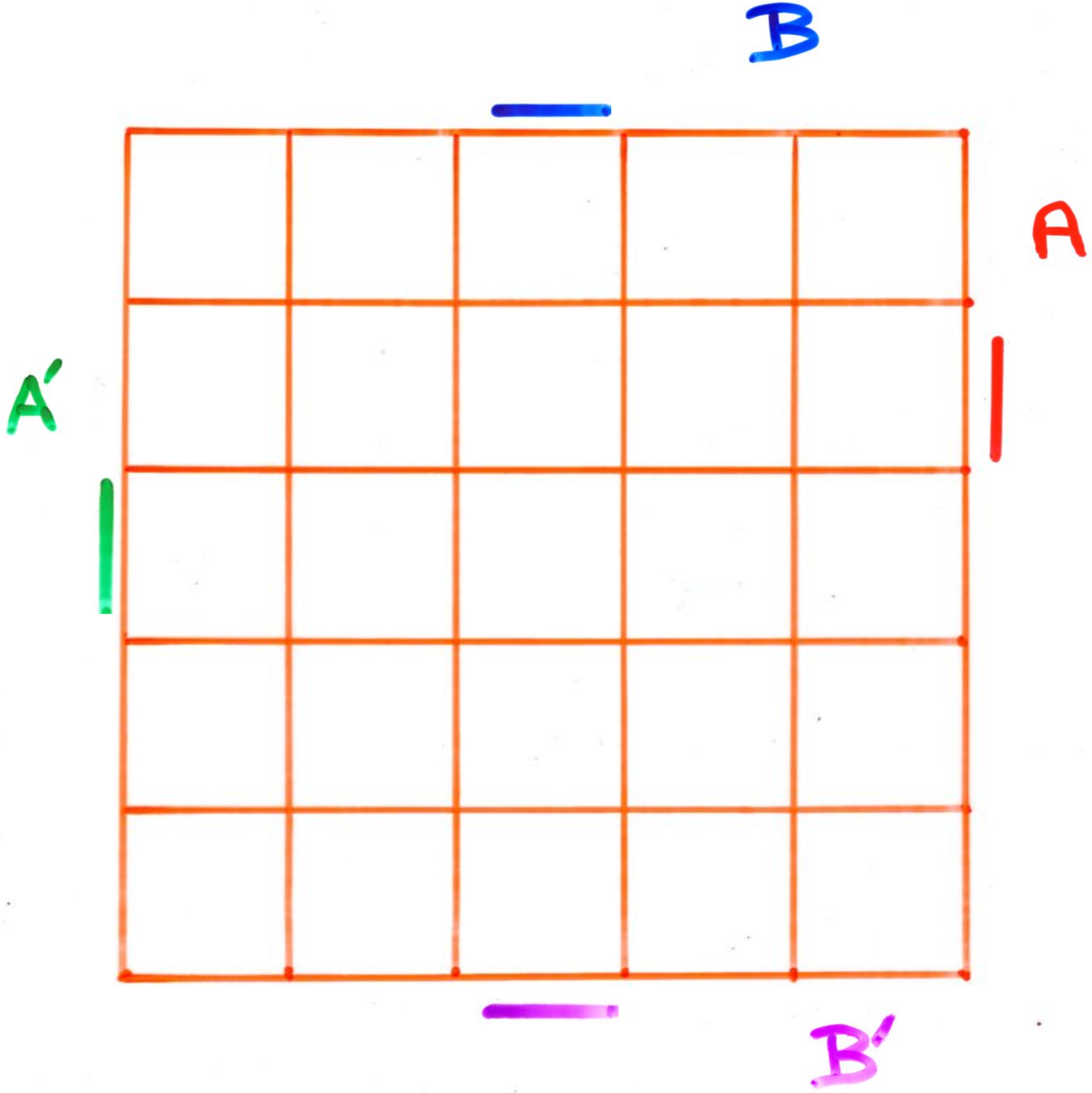
for

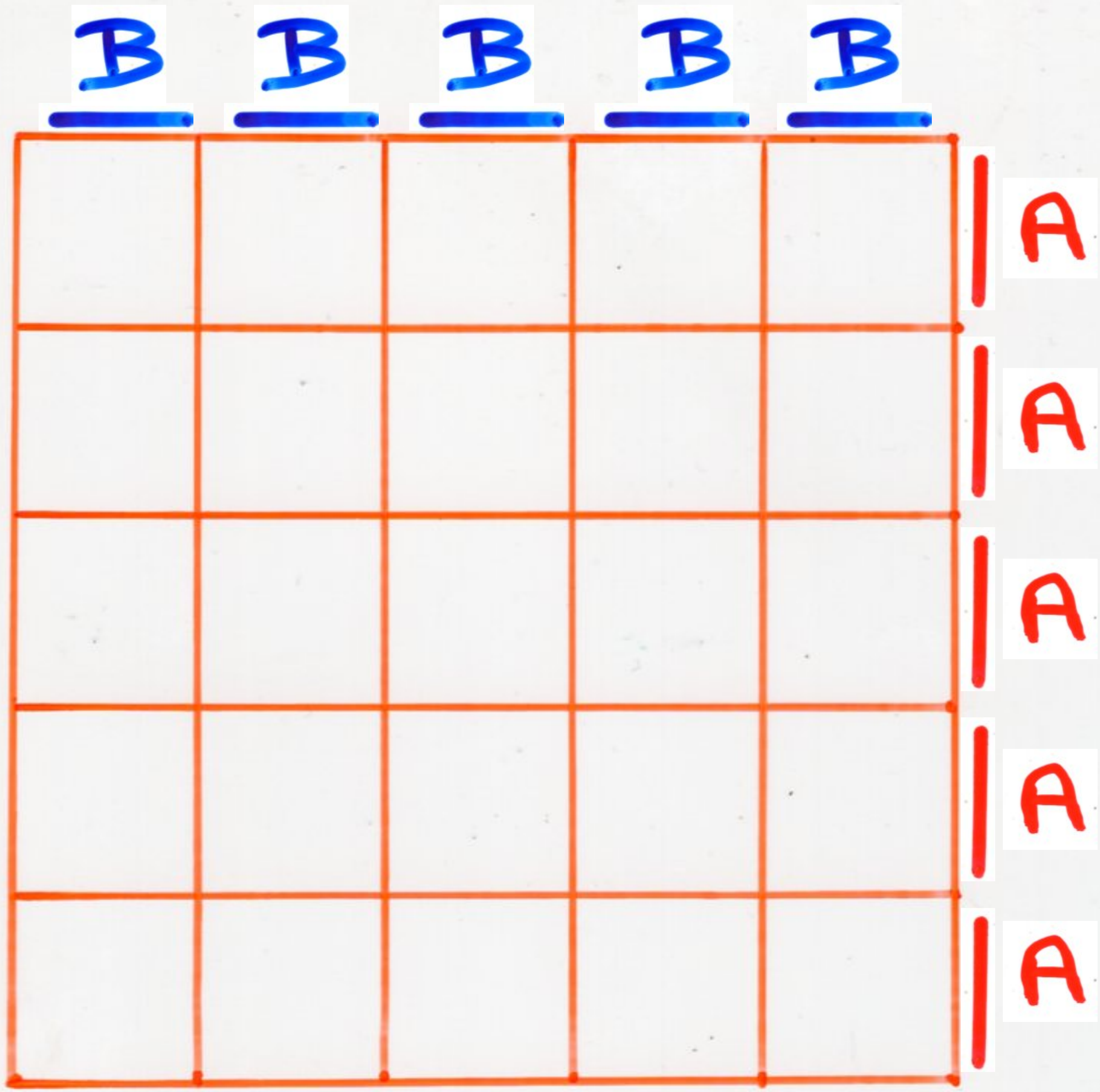
quadratic
algebra Q

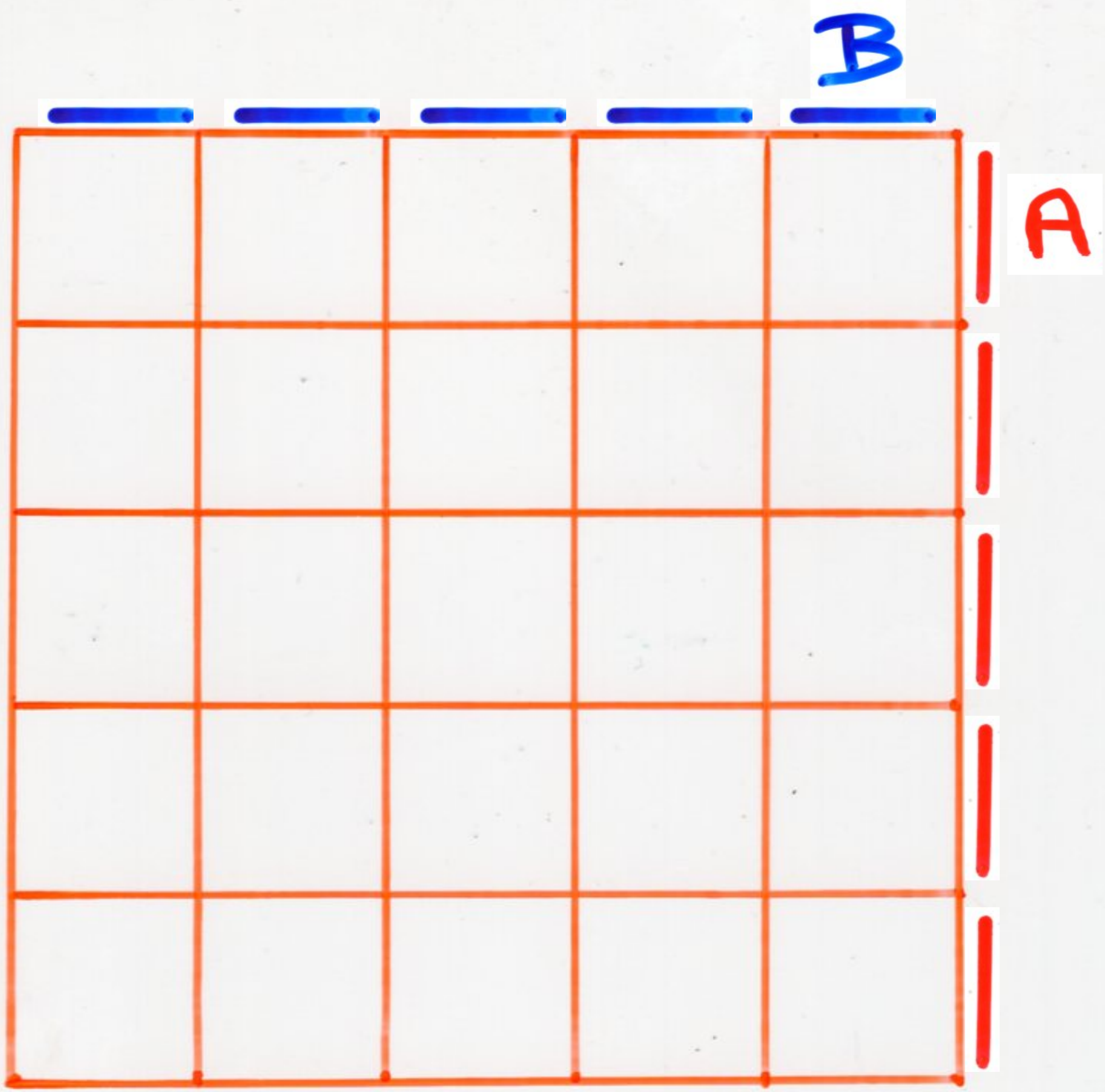
commutations

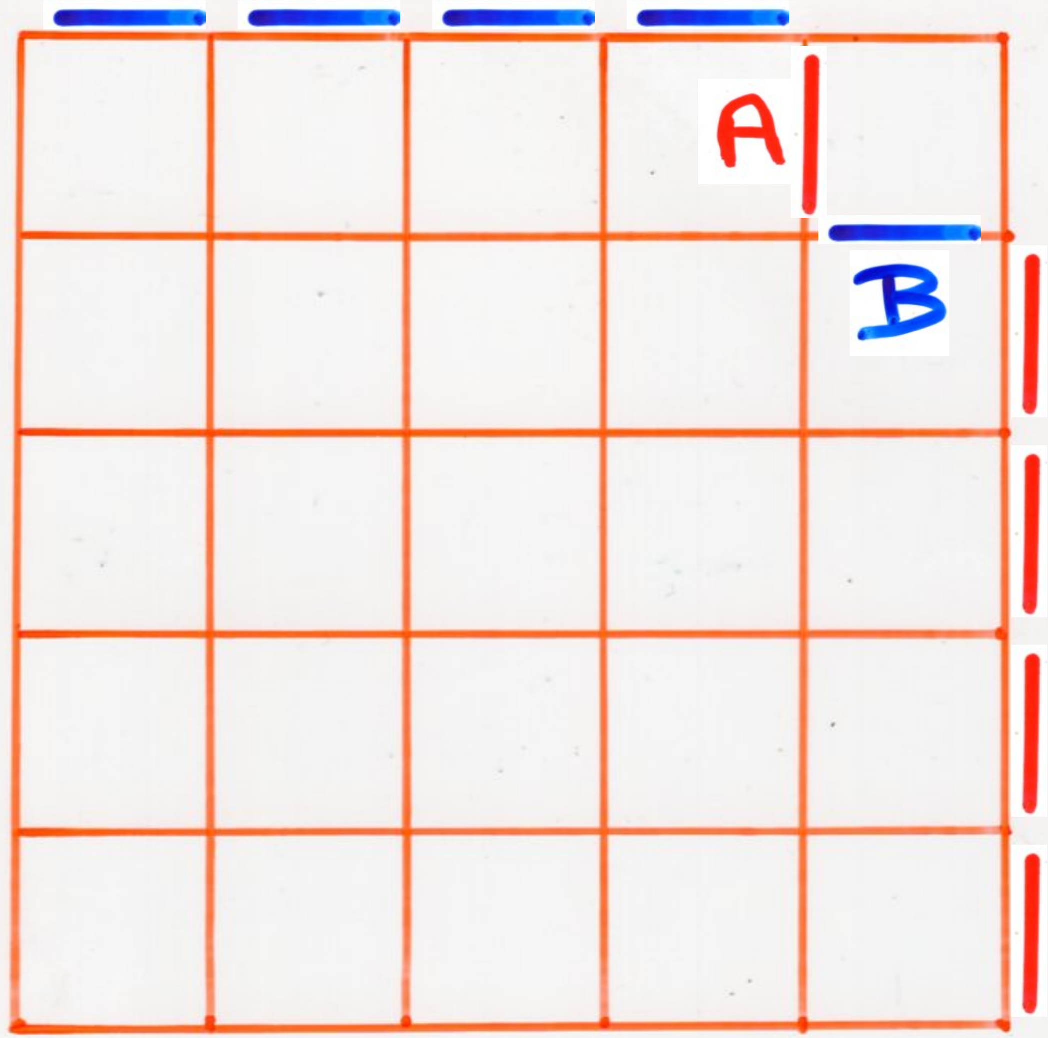
$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

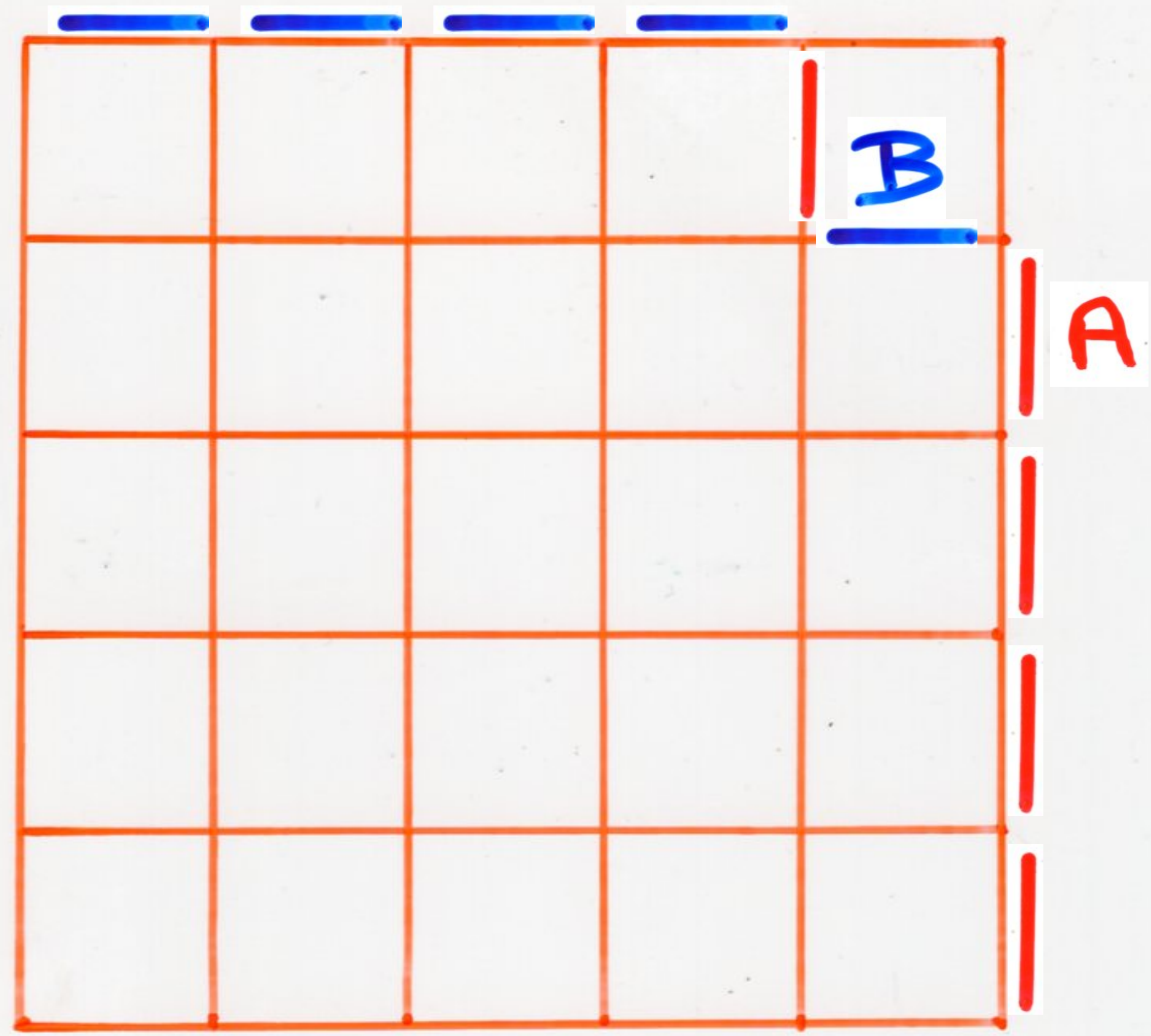
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

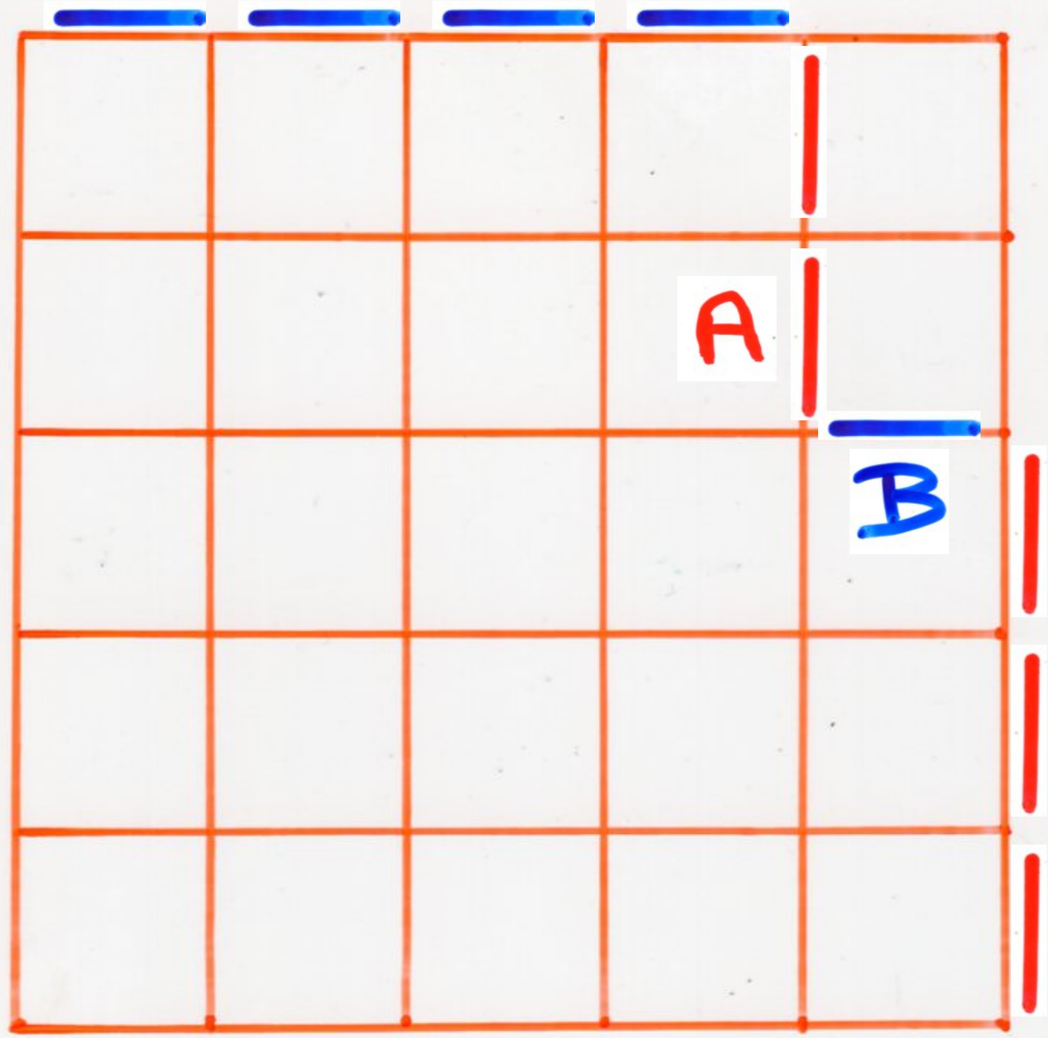

B
A
A′
B′

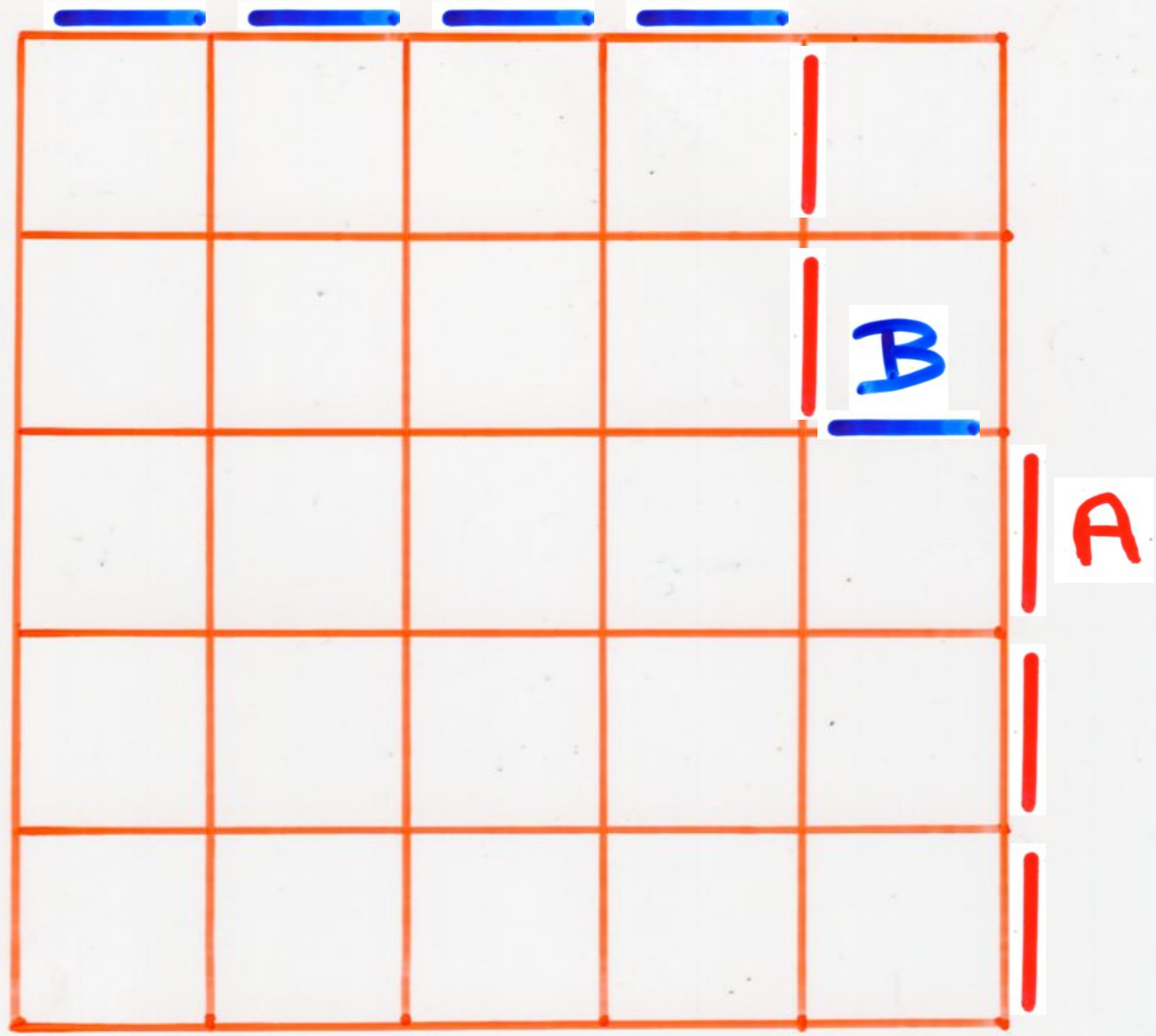

B B B B B
A
A
A
A
A

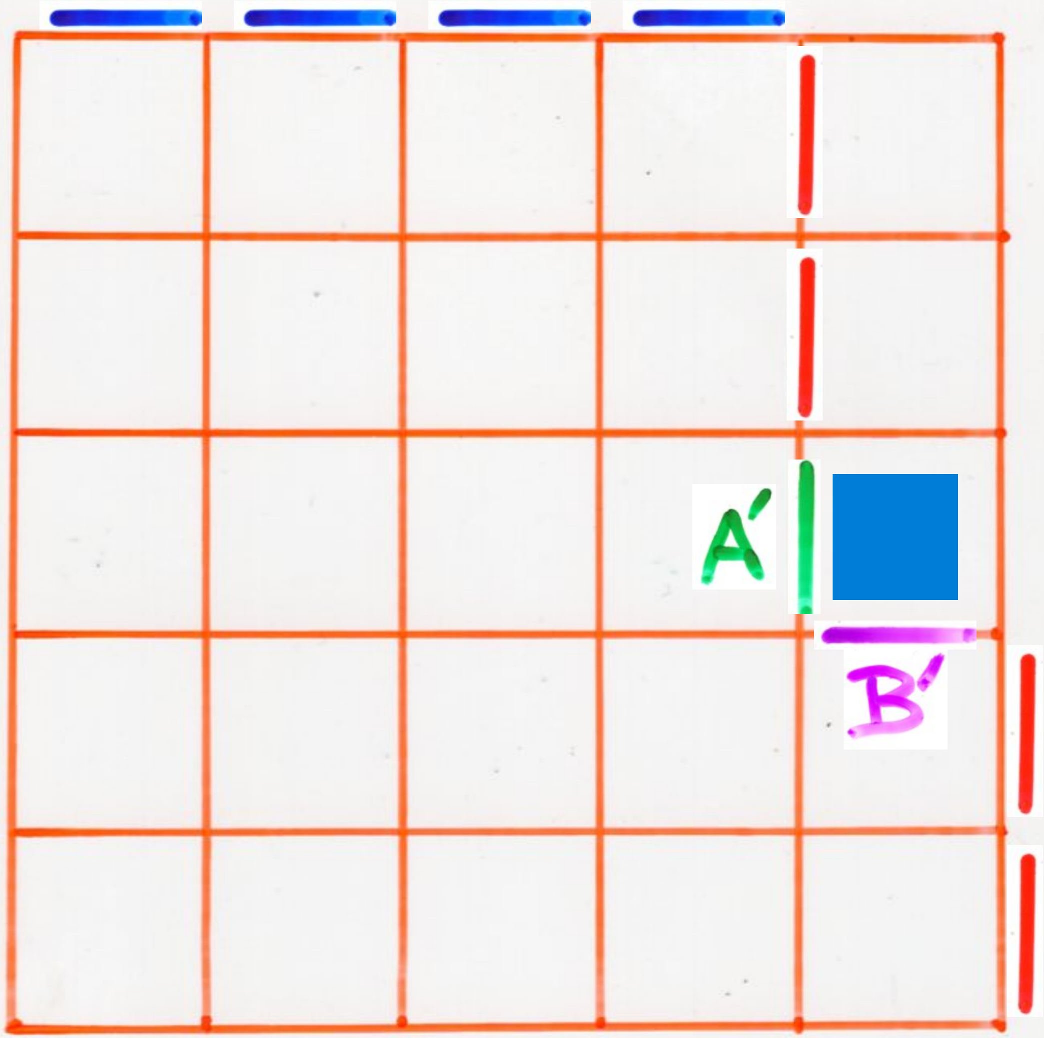

B
A

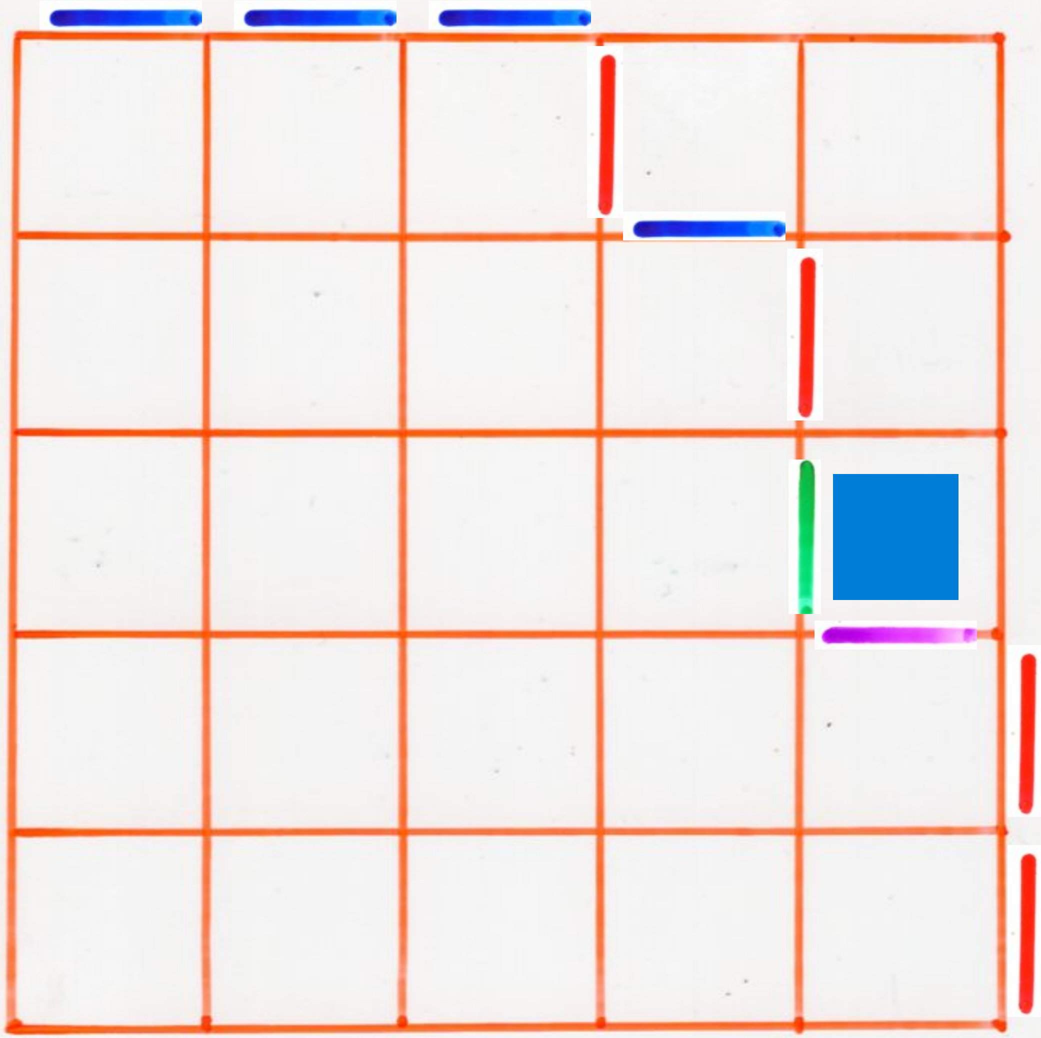

A
B

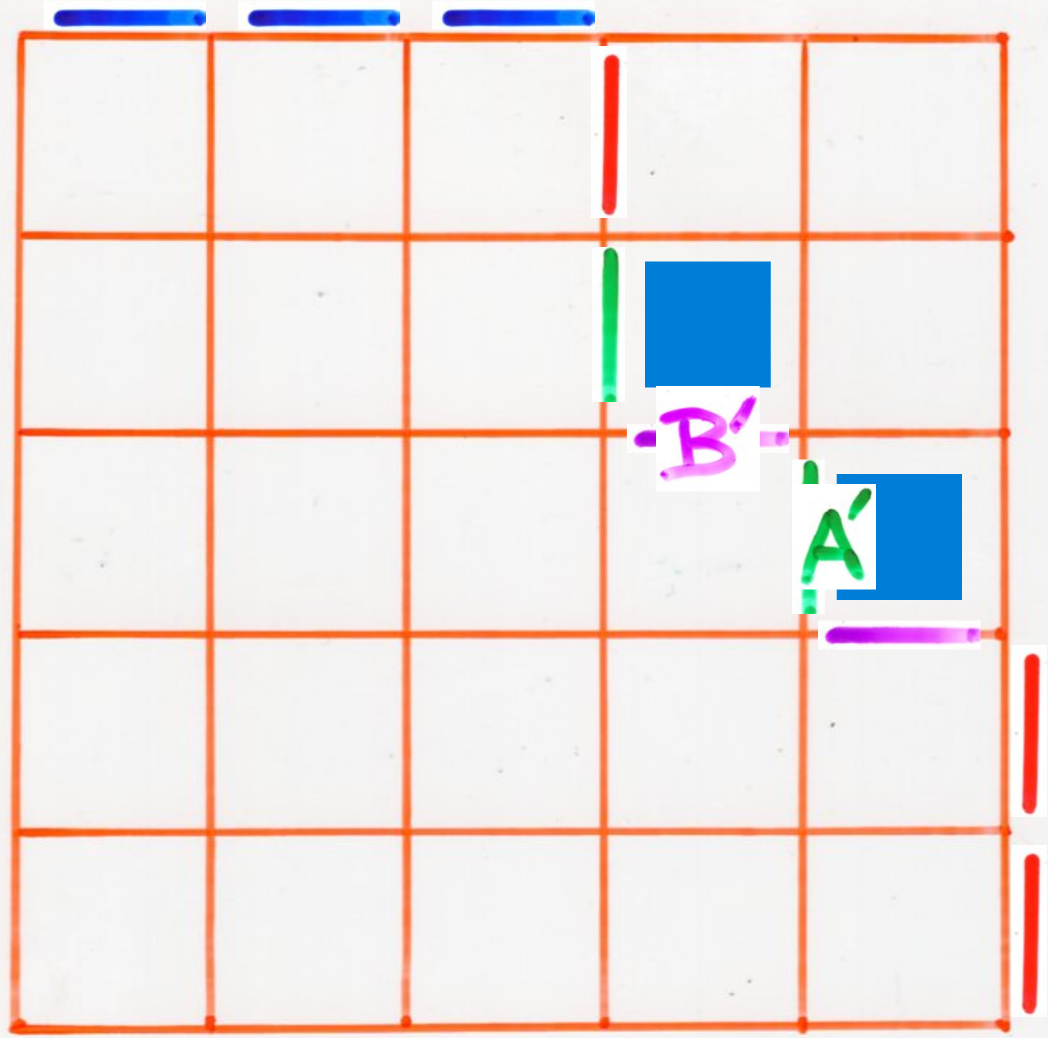

B
A

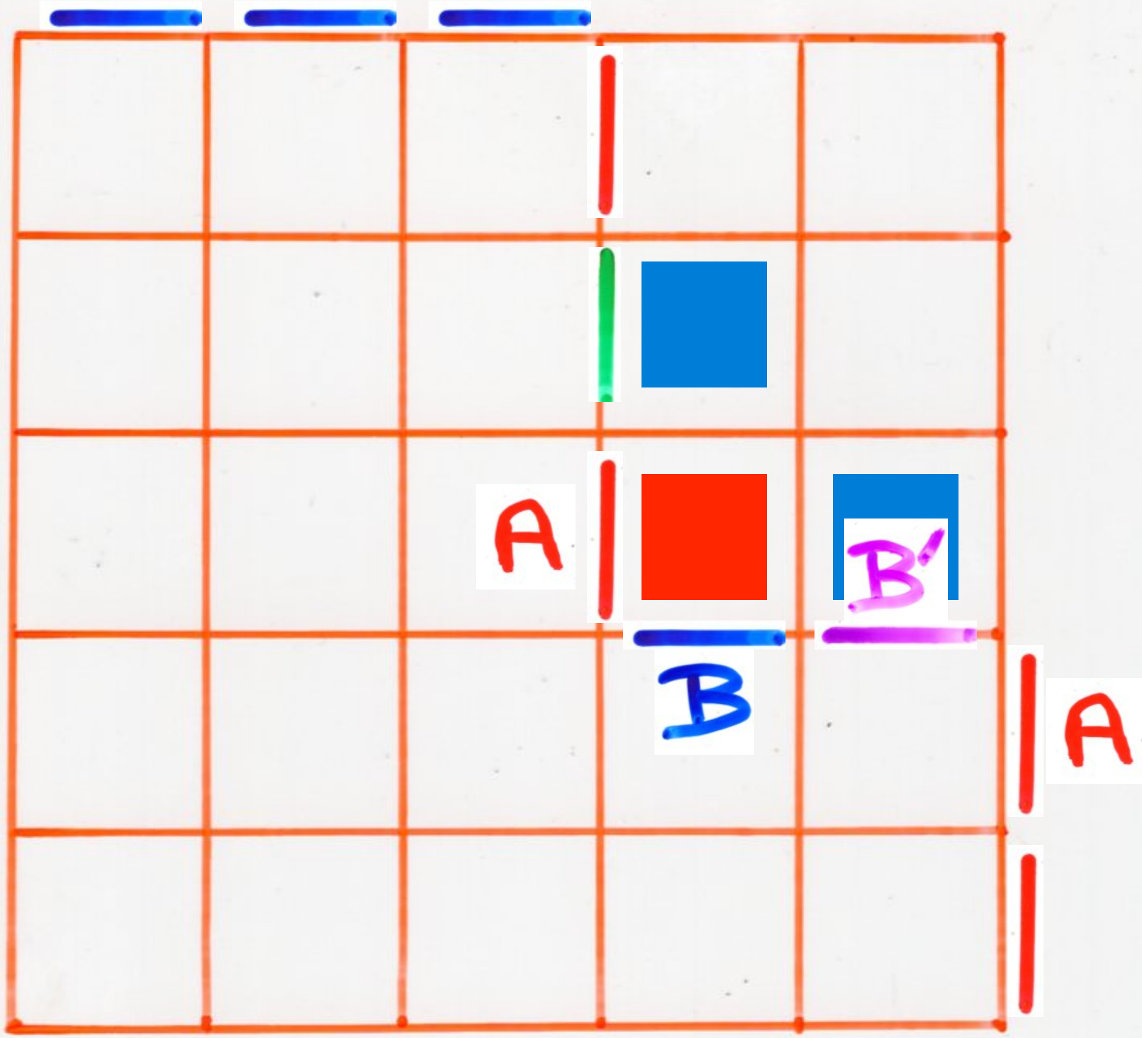

A
B

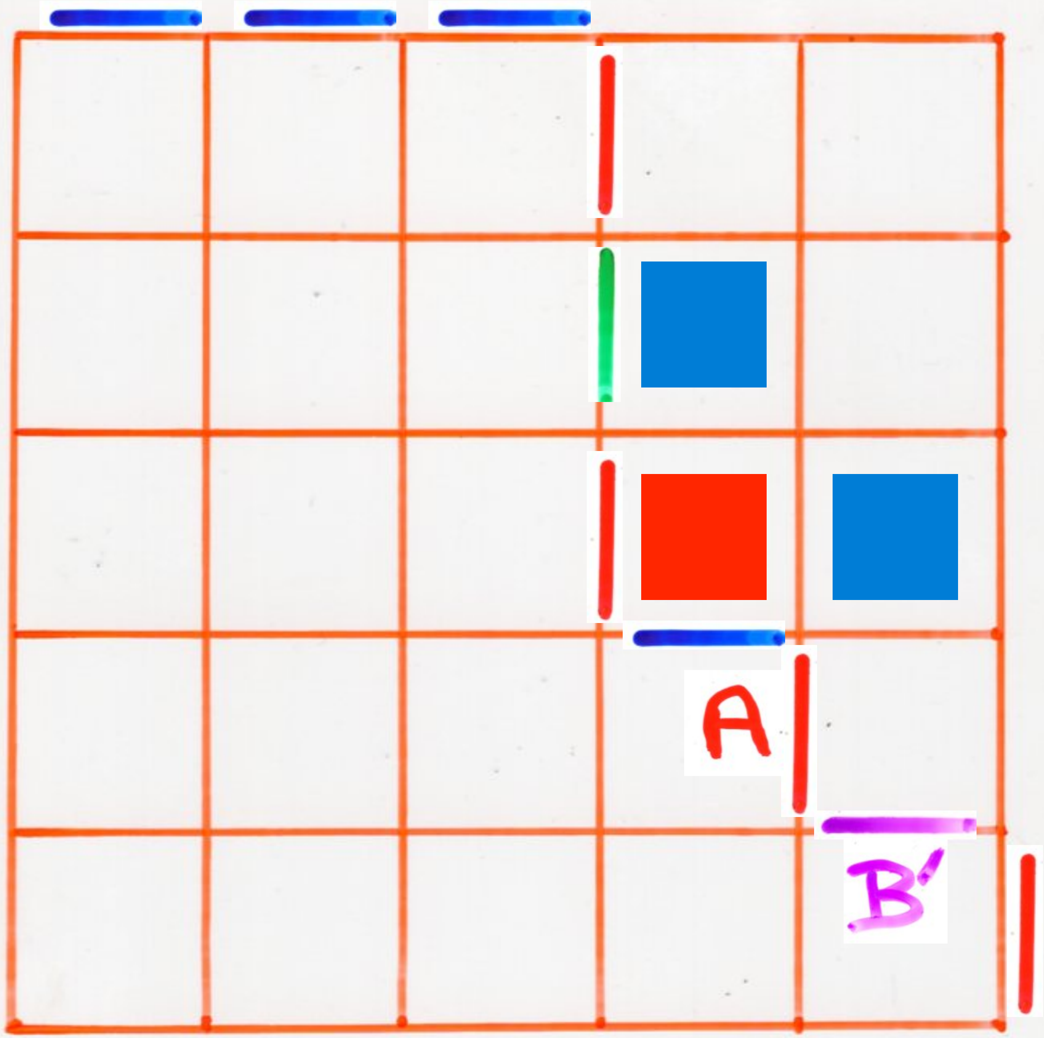

B
A

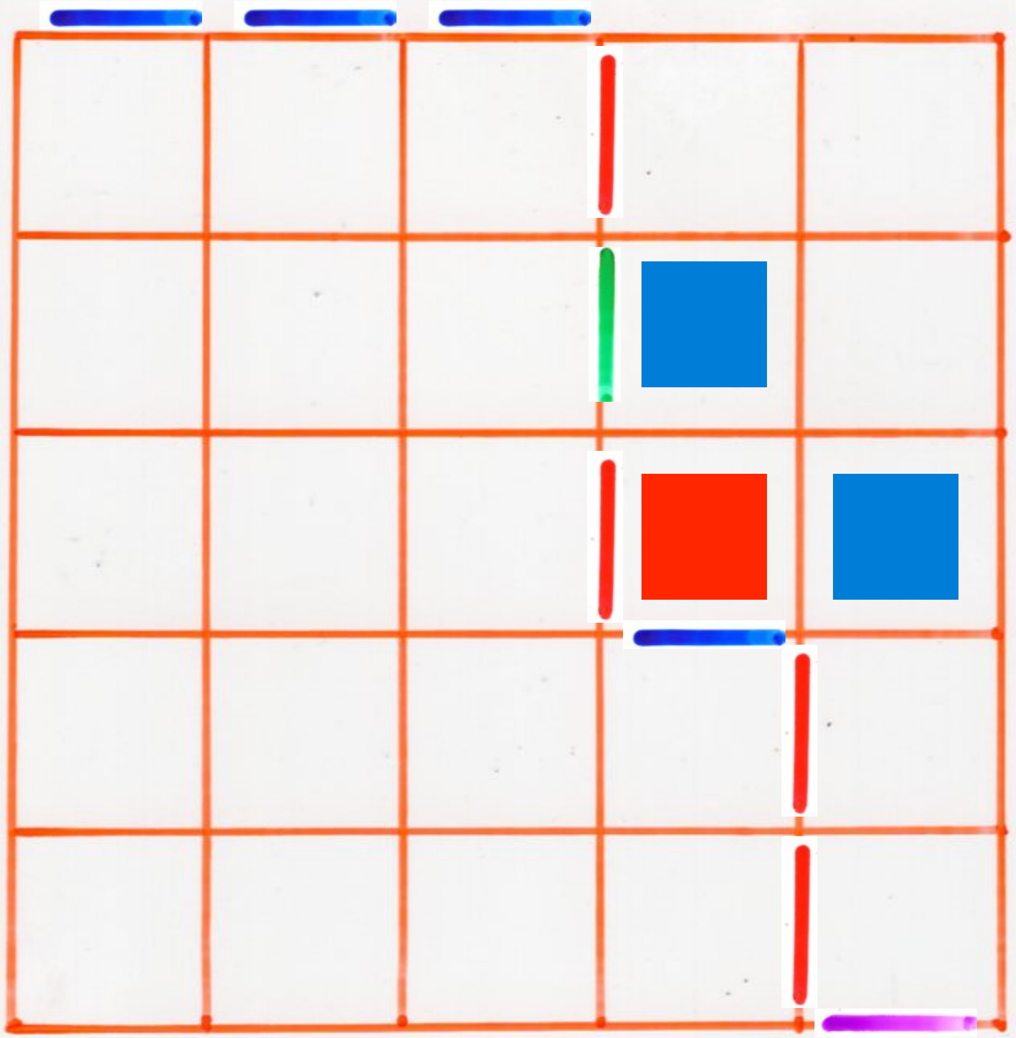

A′
B′

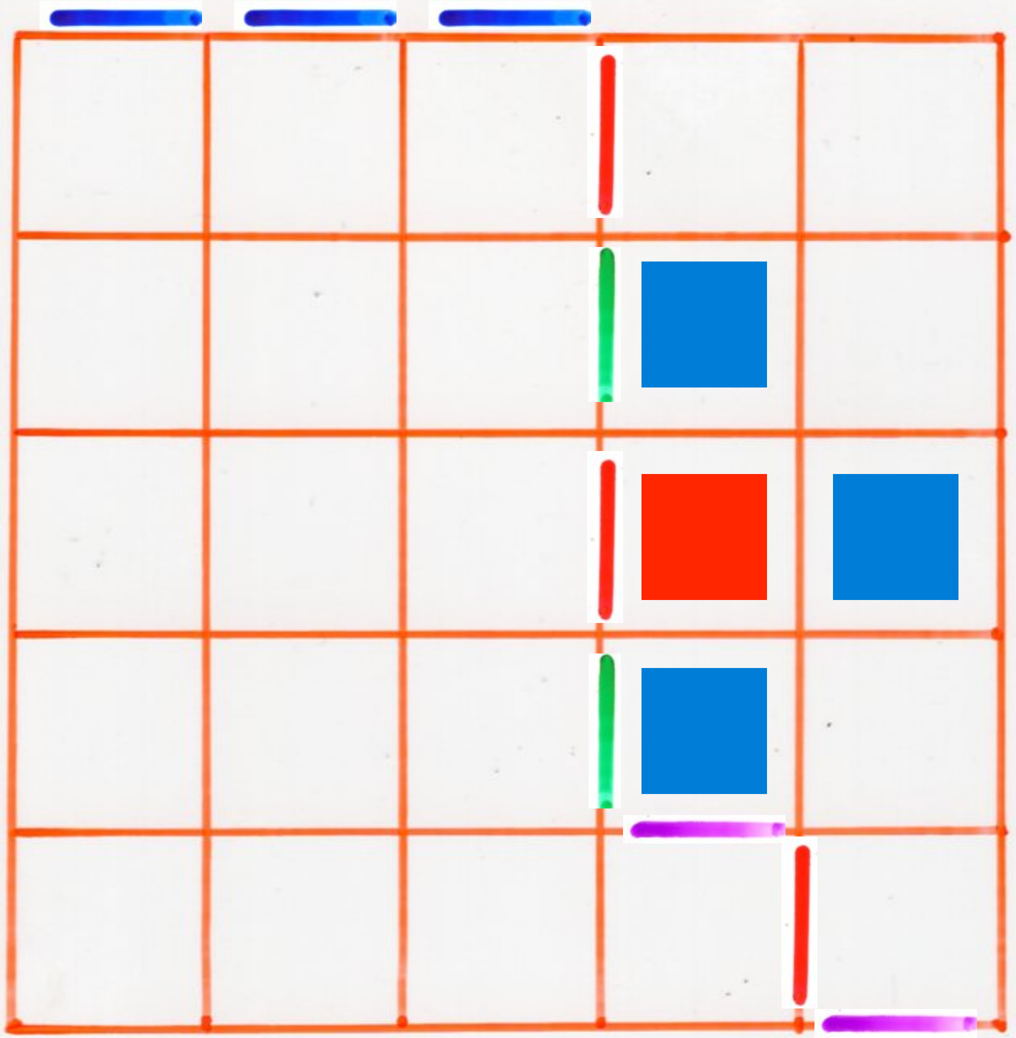


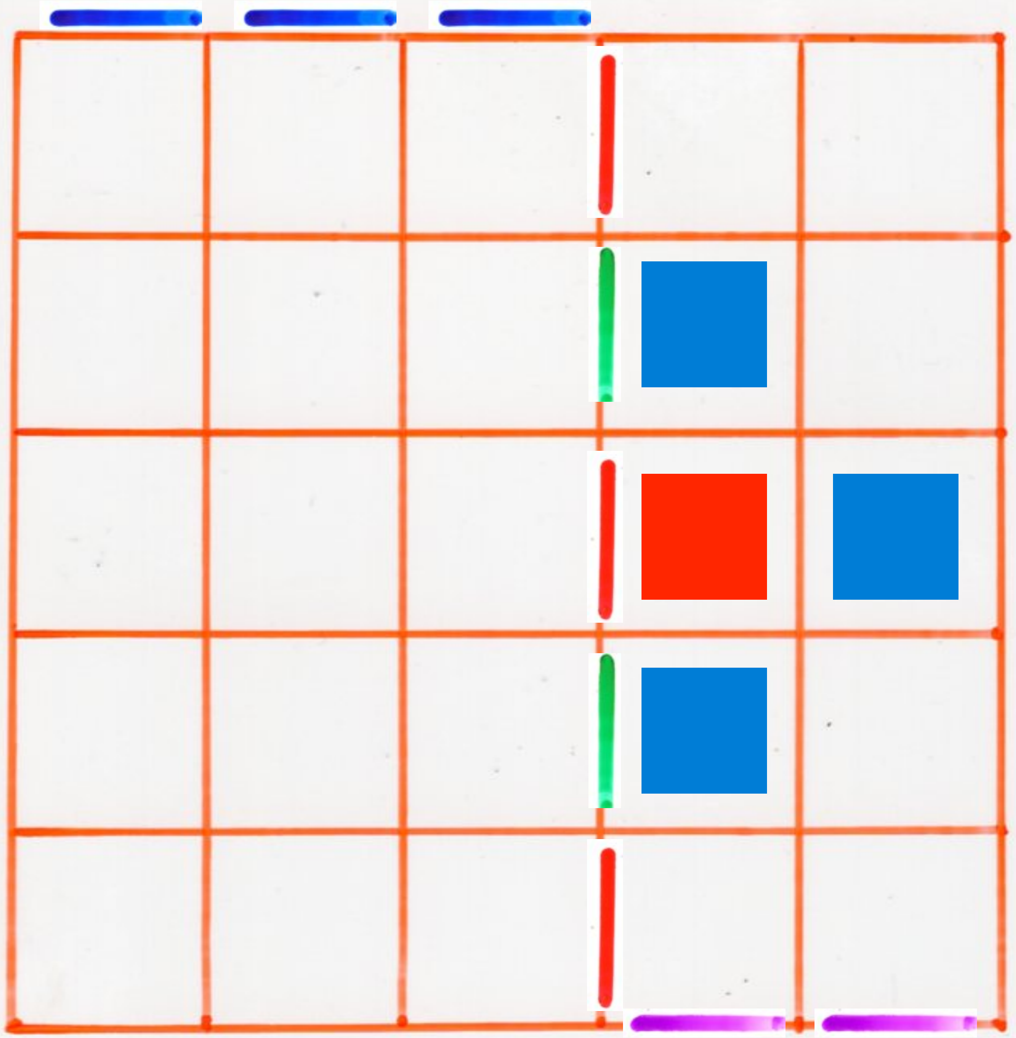

B′
A′

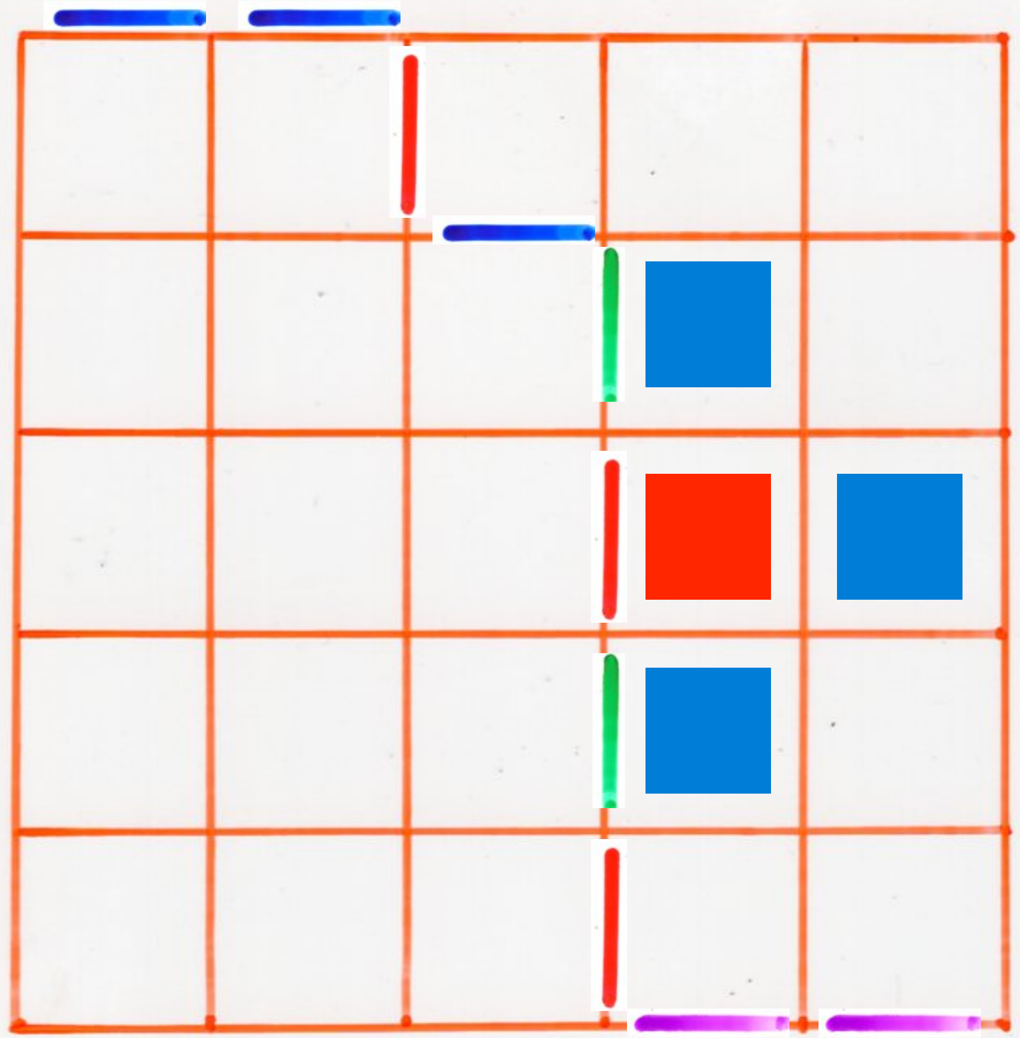

A
B'
B
A

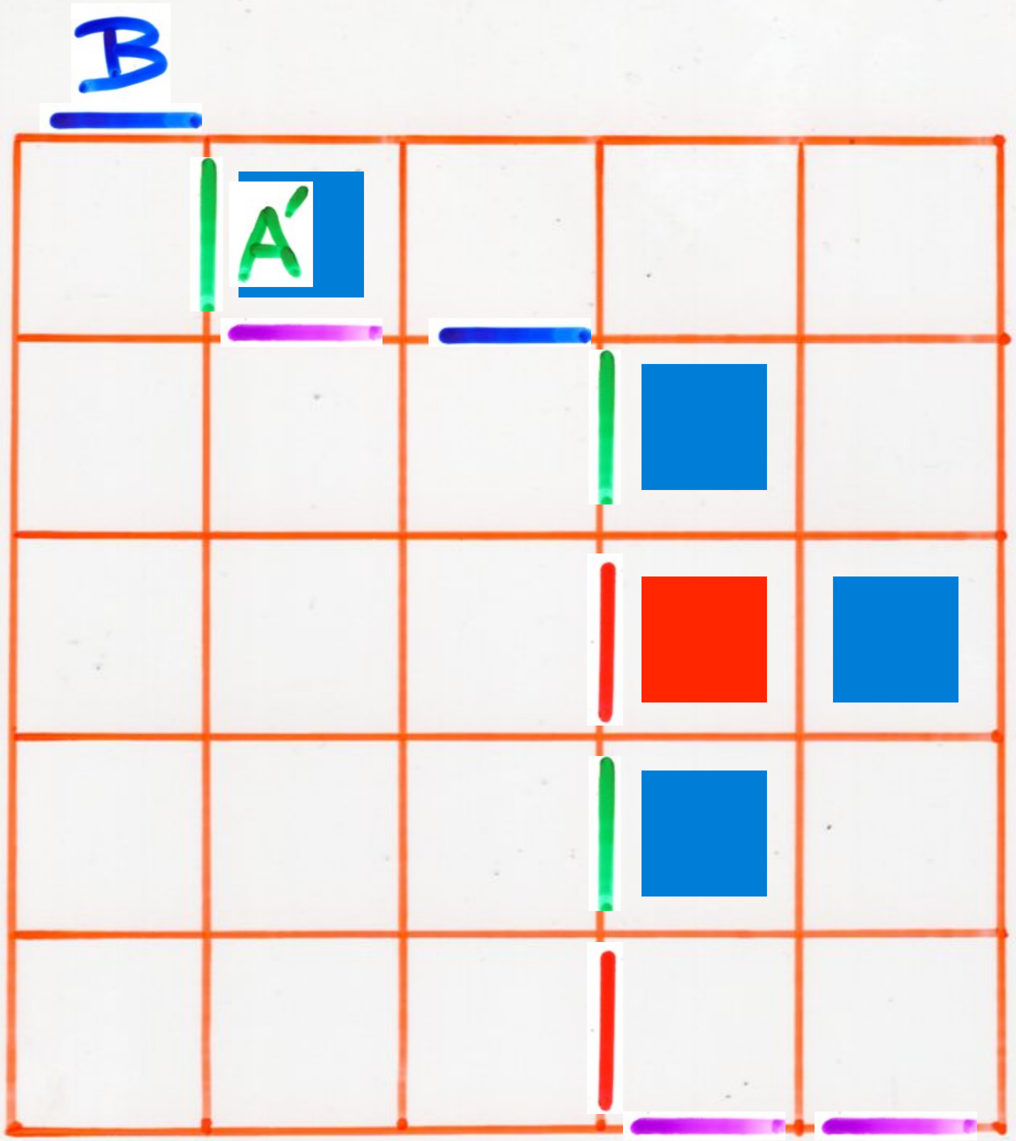

A
B'

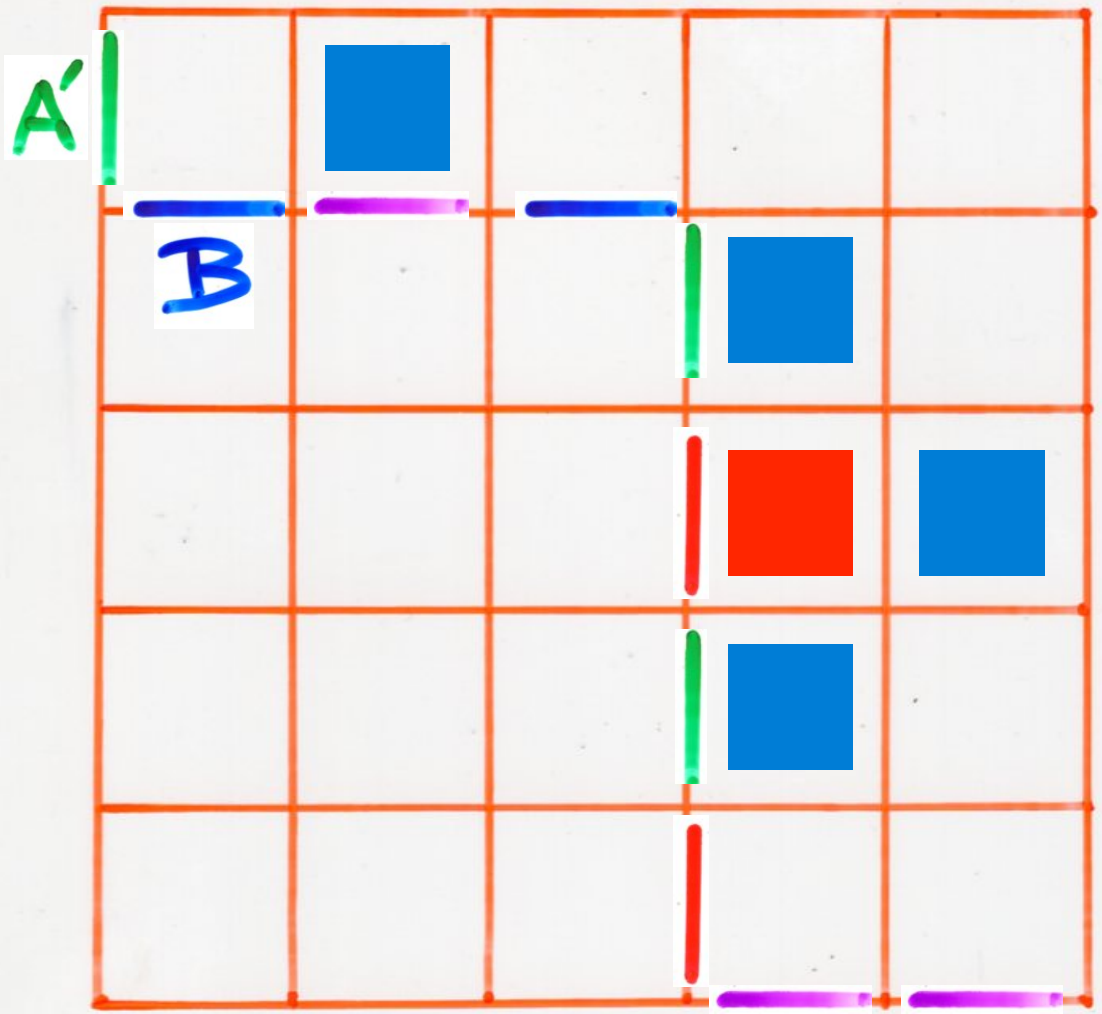


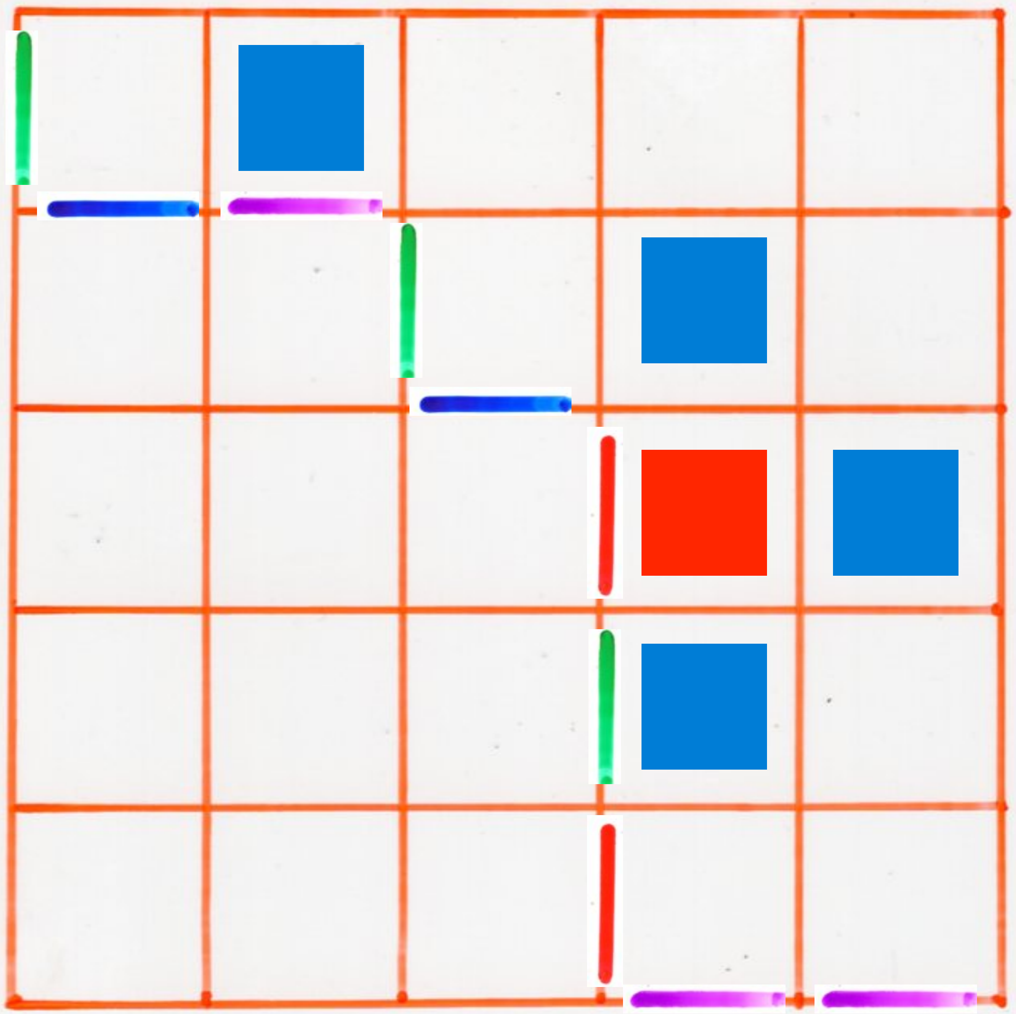


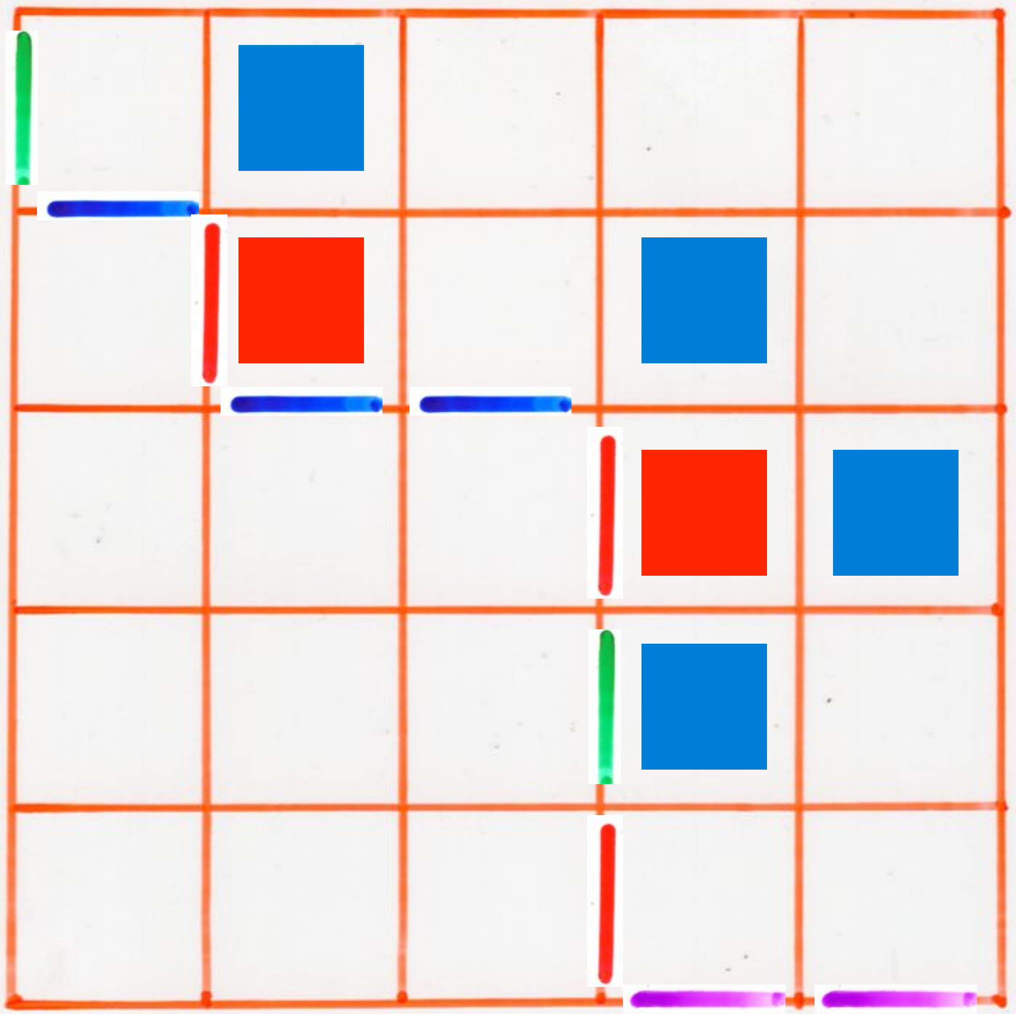


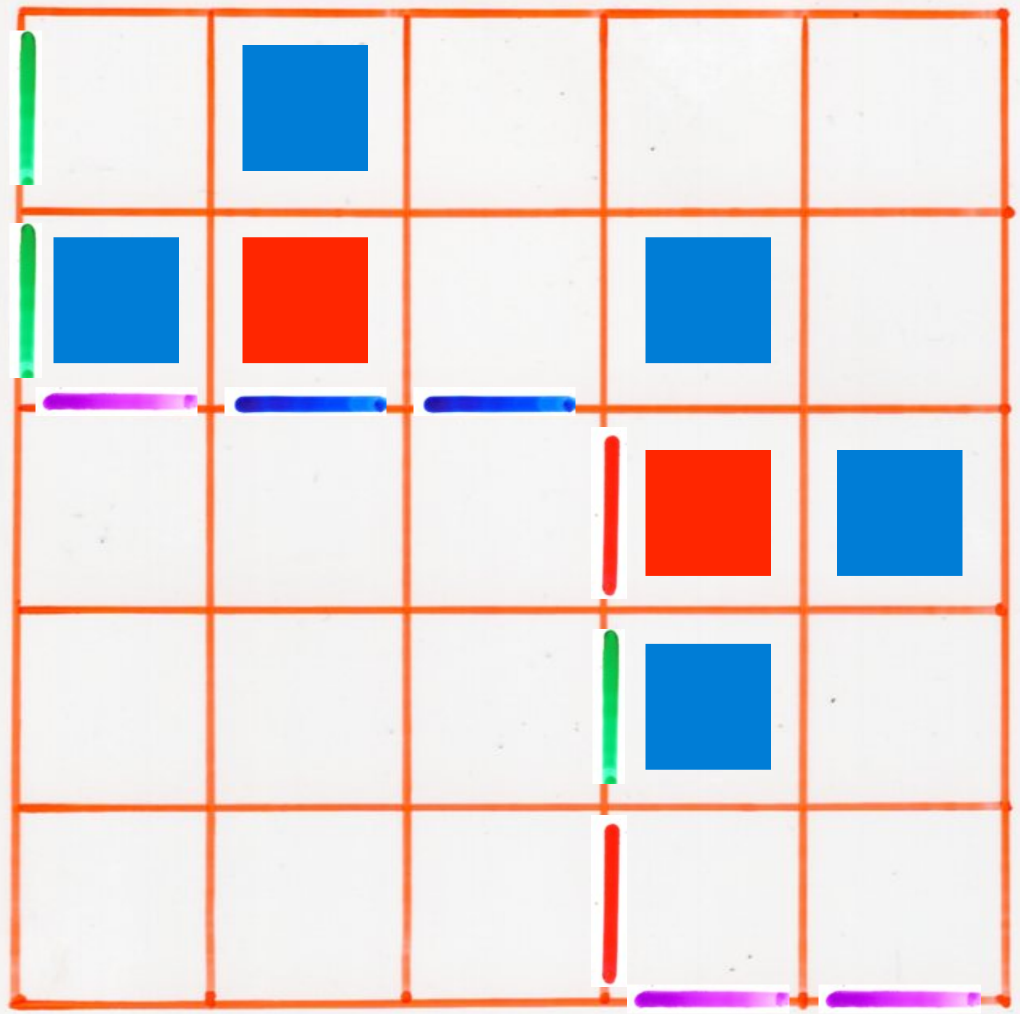


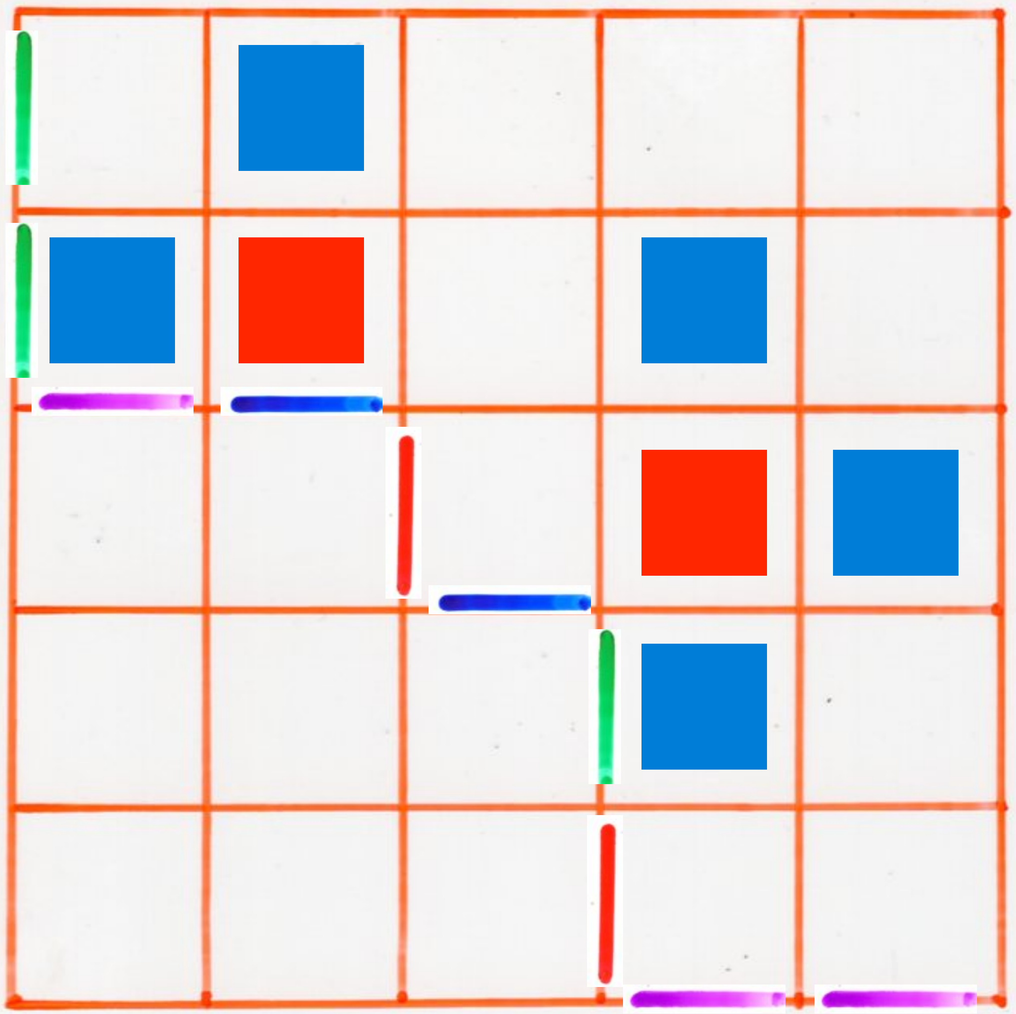

B
A´

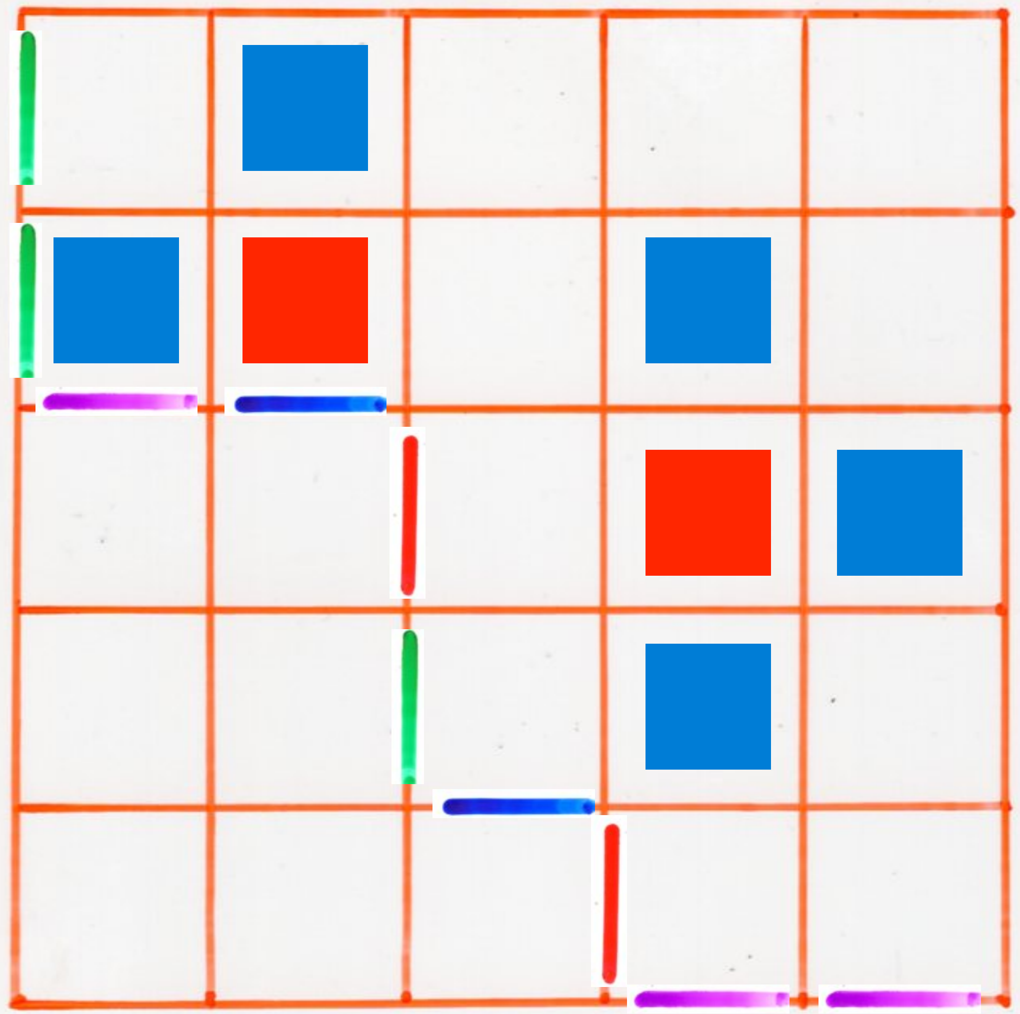

A′
B

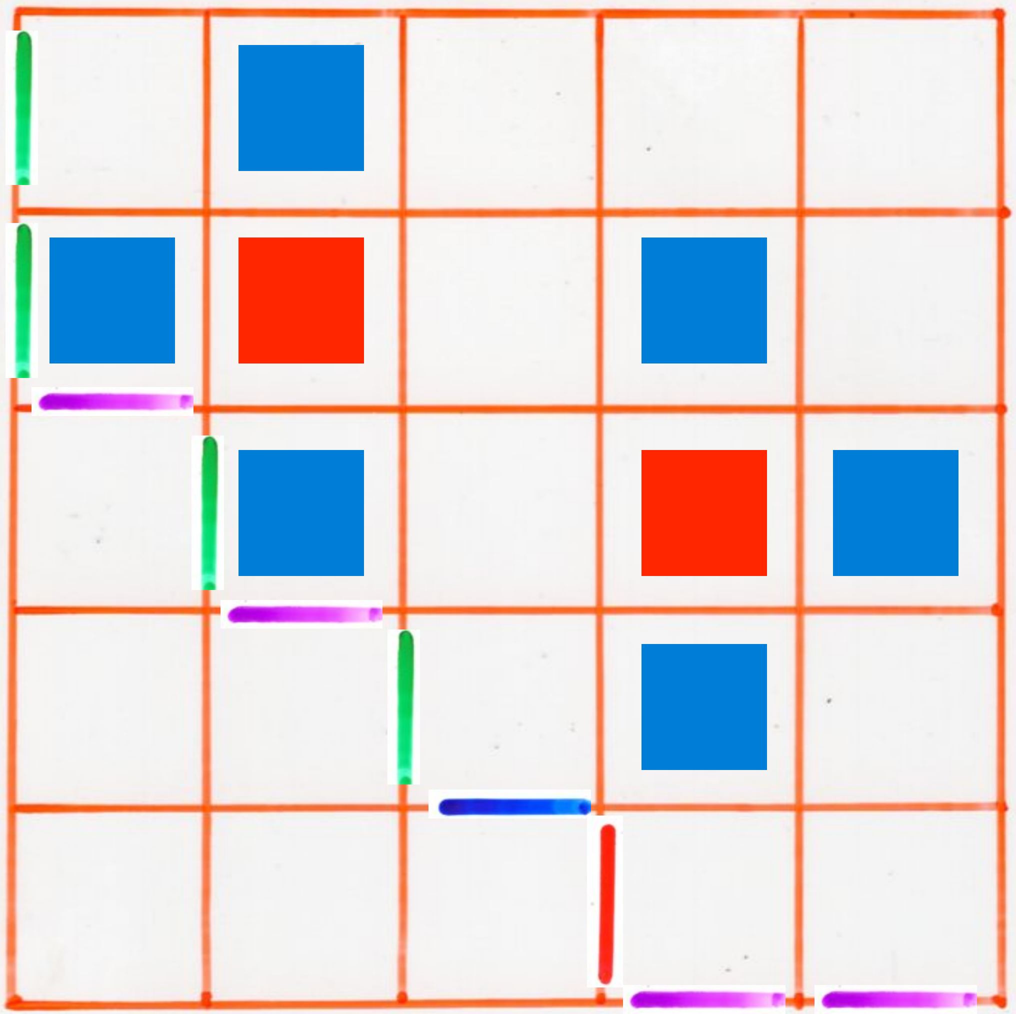


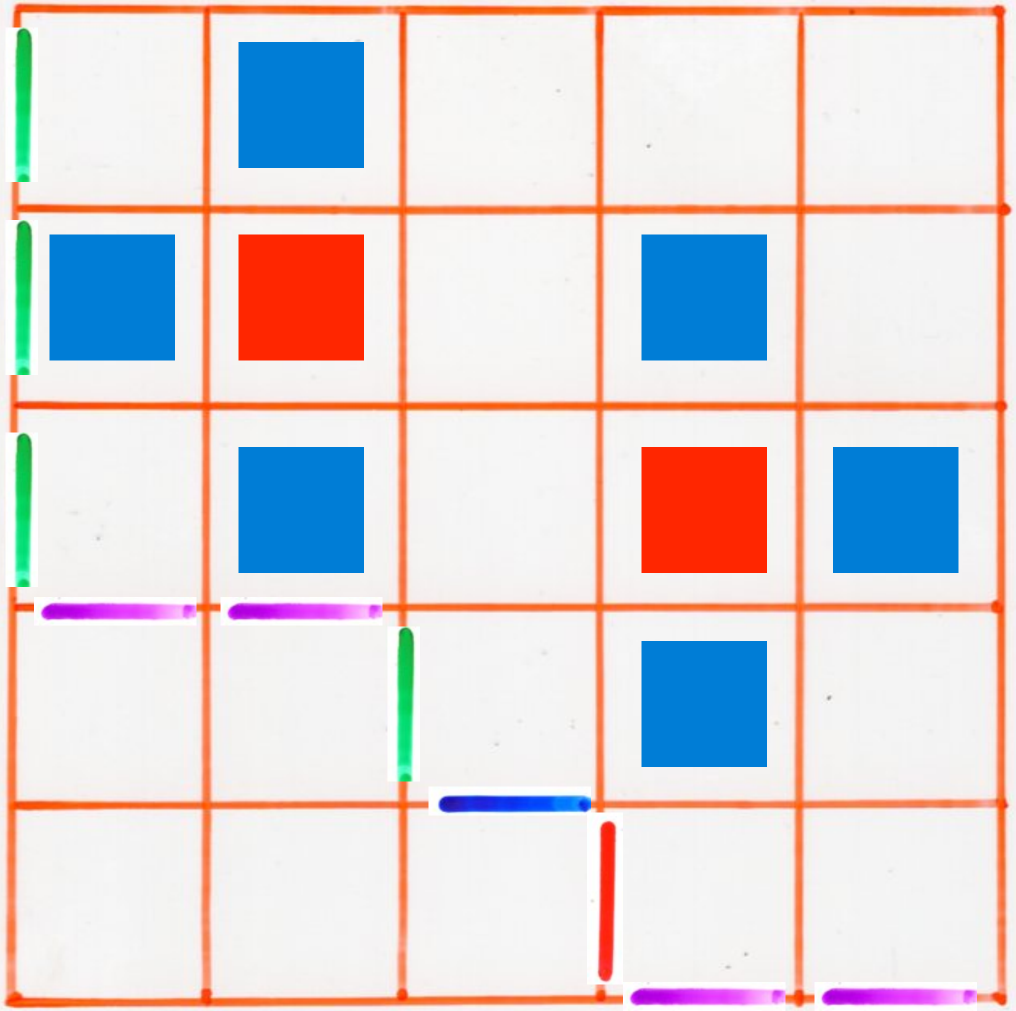


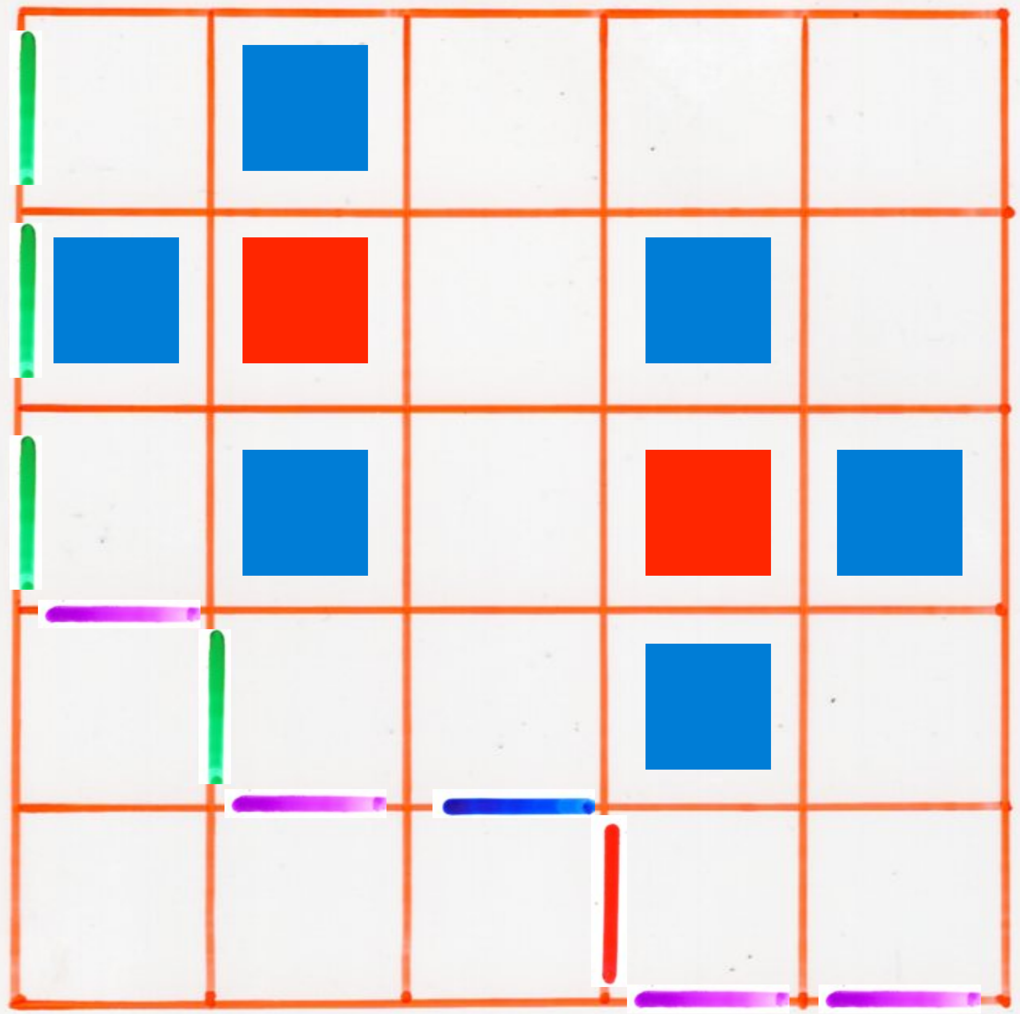


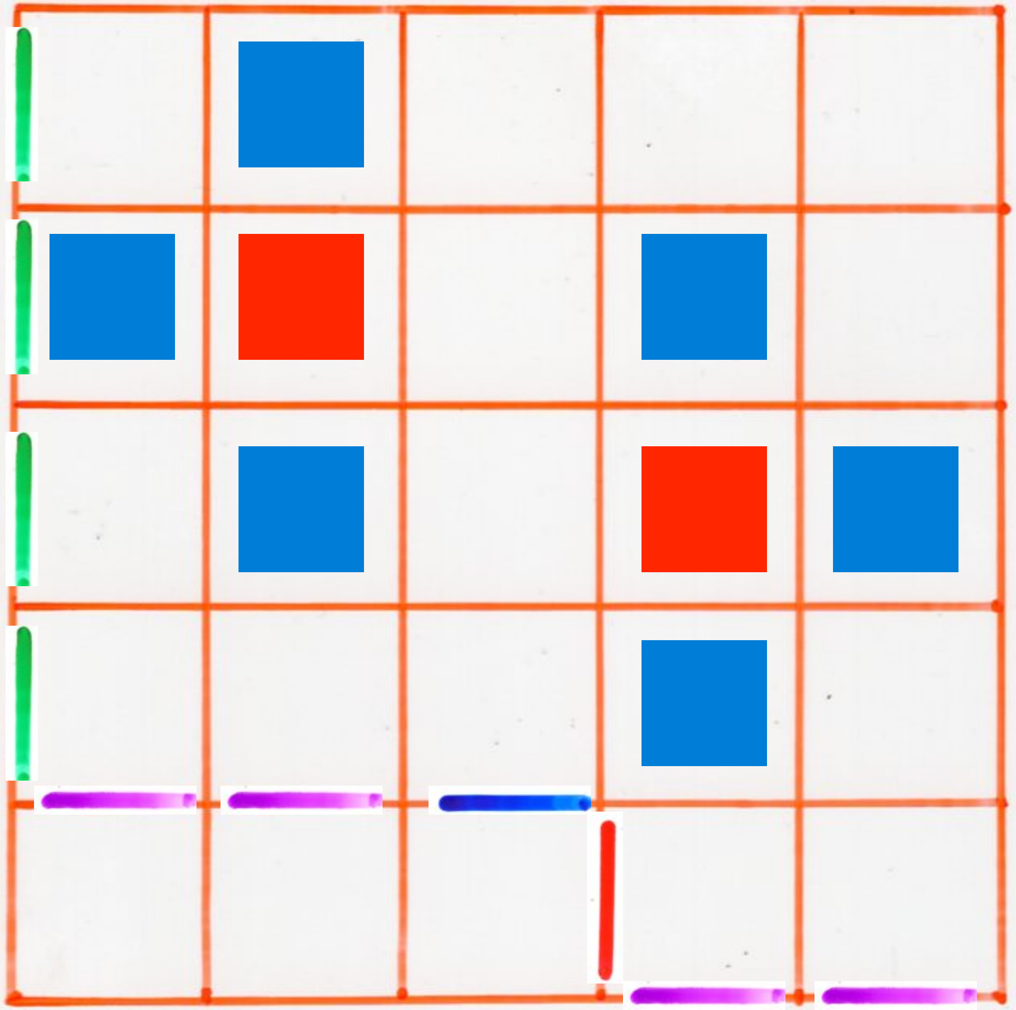


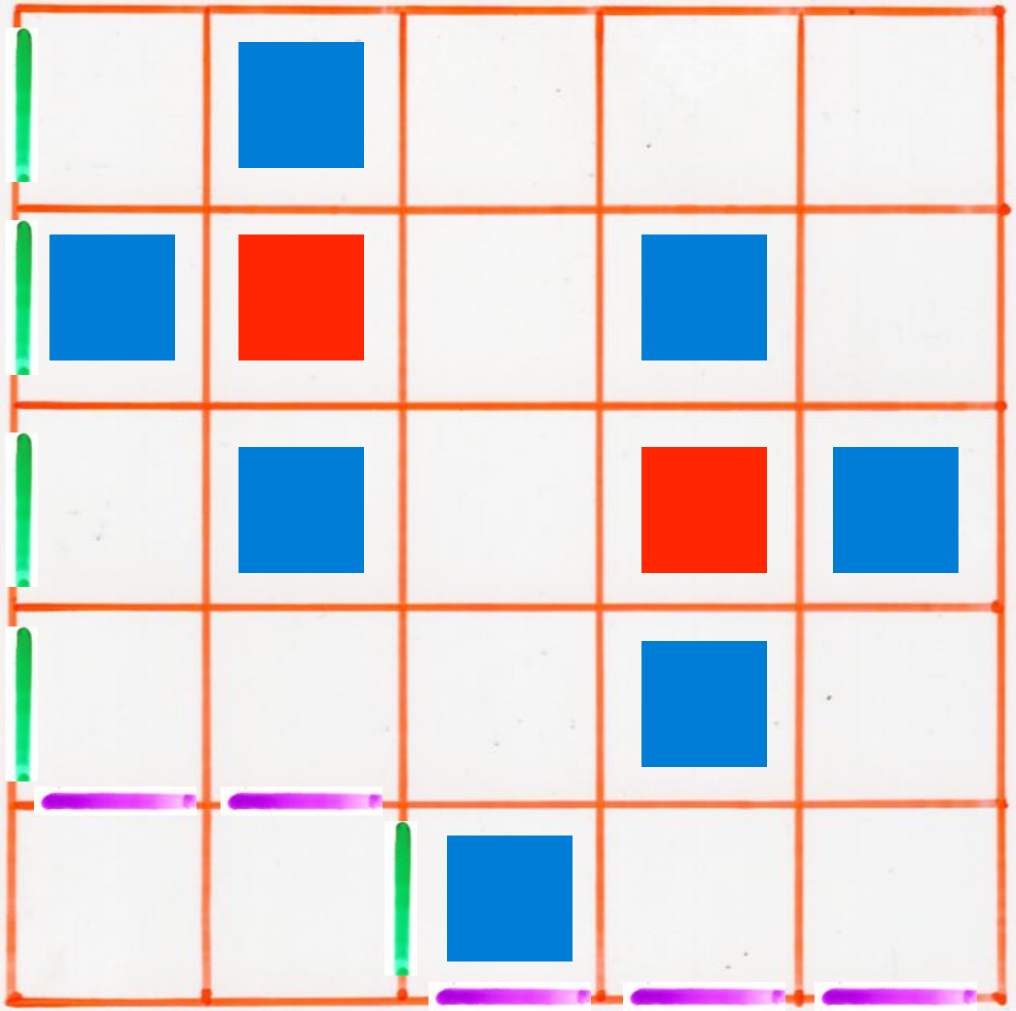


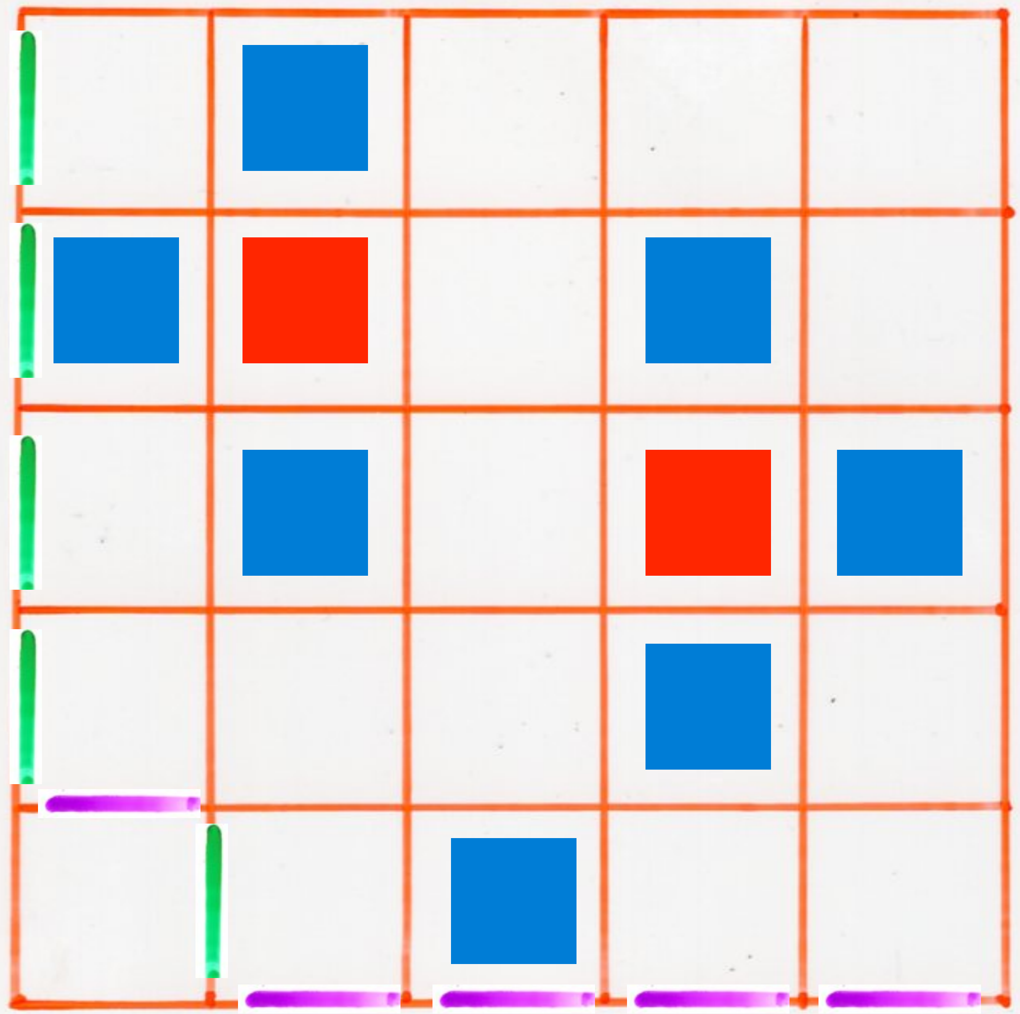


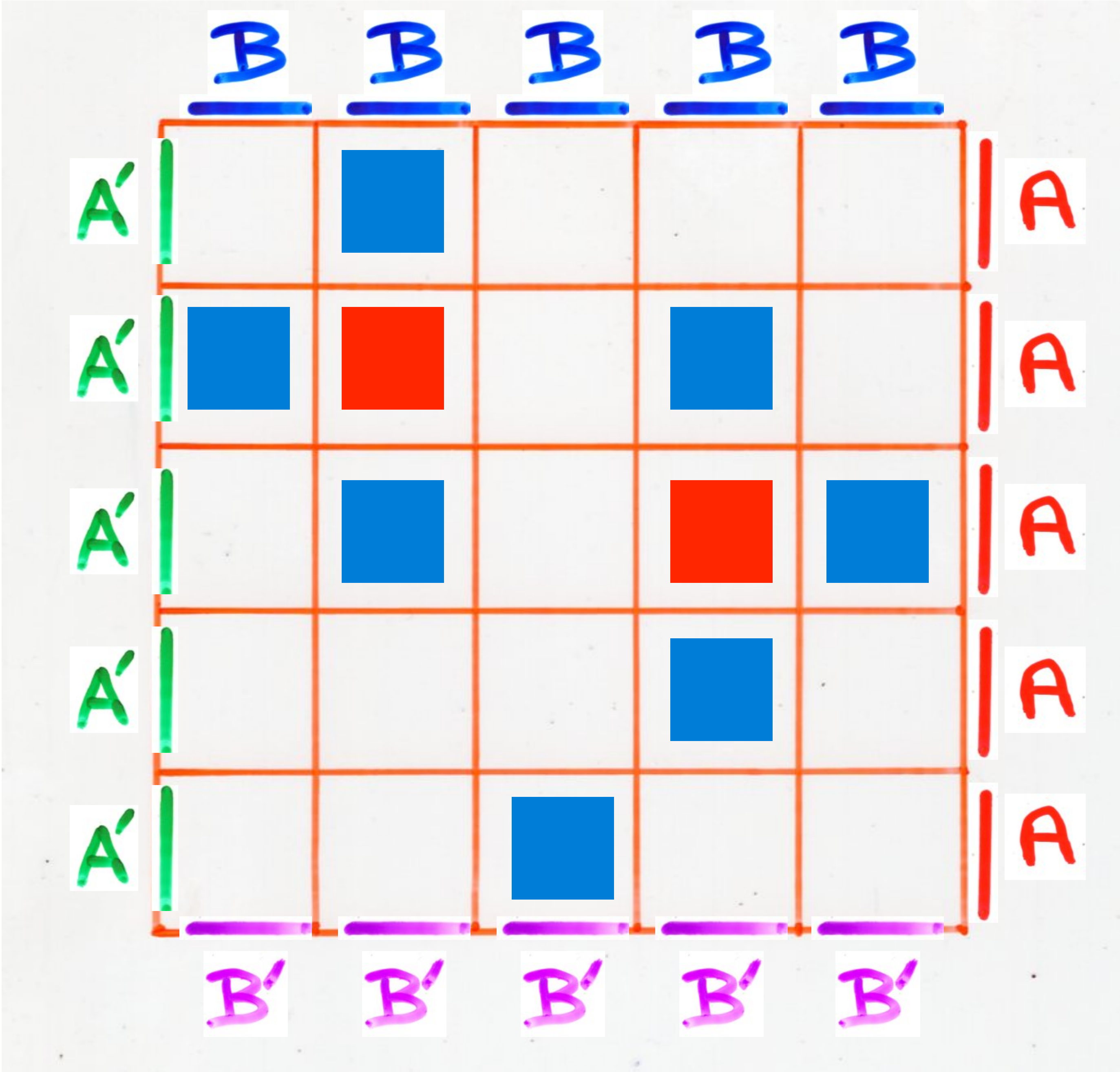











B B B B B
A′ A′ A′ A′ A′
A A A A A
B′ B′ B′ B′ B′

Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum c(u,v;w) \underbrace{u(A,A')}_{\substack{\text{word} \\ \text{in } A,\, A'}} \underbrace{v(B,B')}_{\substack{\text{word} \\ \text{in } B,\, B'}}$$

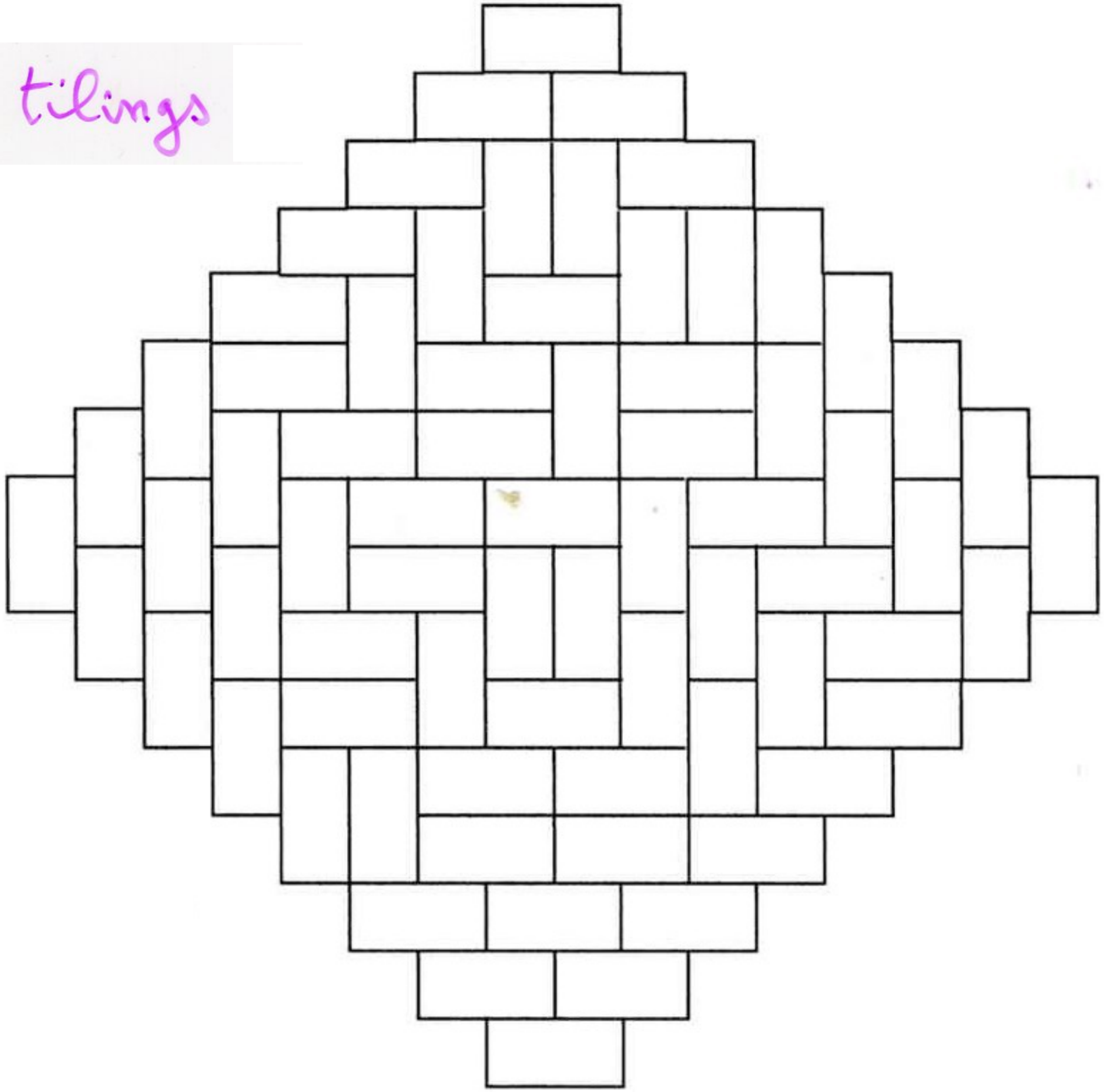
Prop. For $w = B^n A^n$

$u = A'^n$, $v = B'^n$

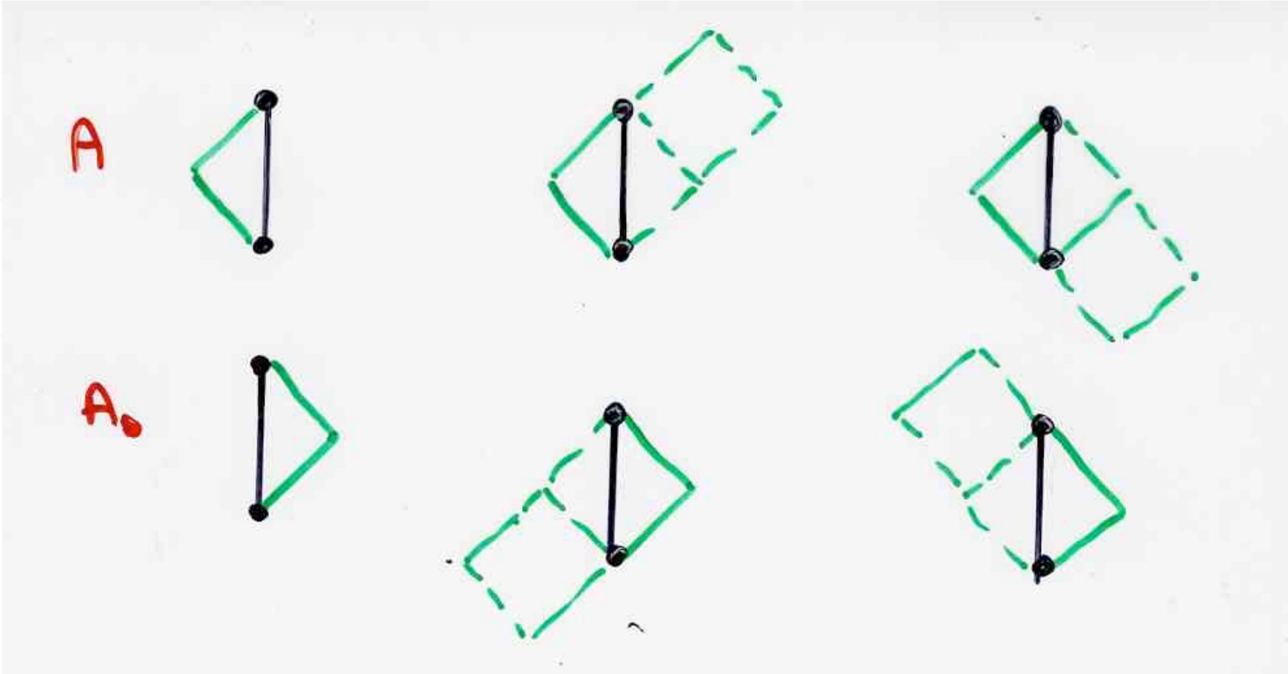
$c(u,v;w) =$ the number of $n \times n$ ASM (alternating sign matrices)

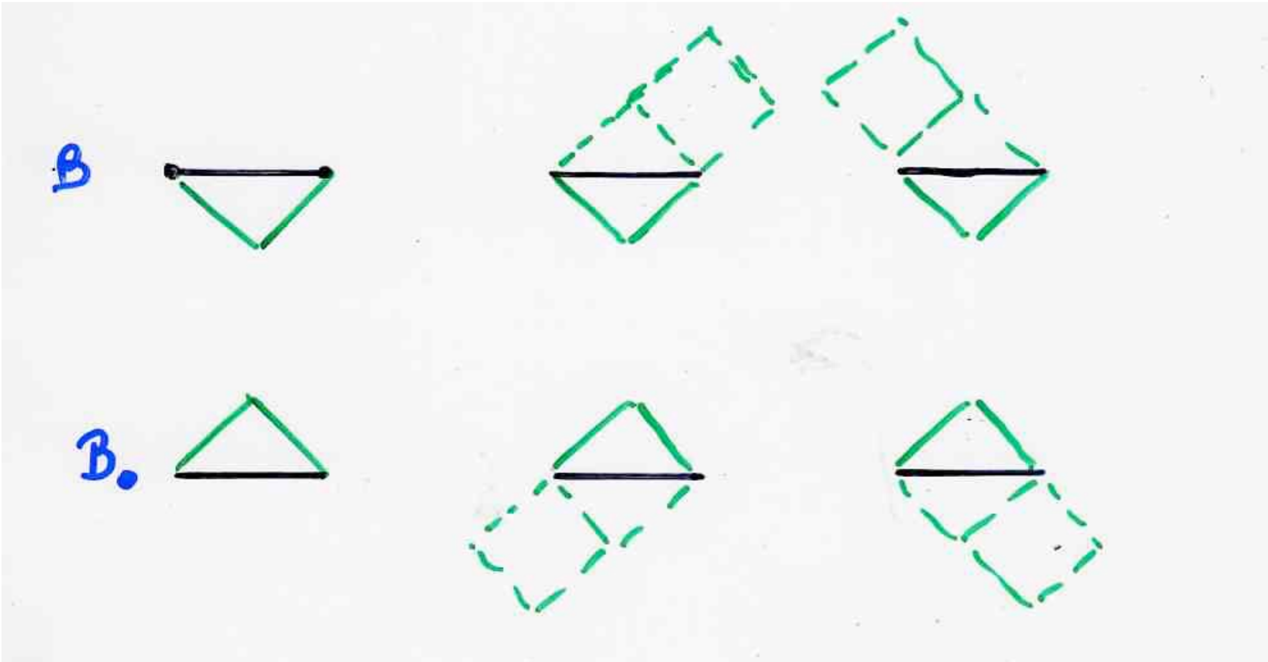
Aztec tilings
as a Q-tableau

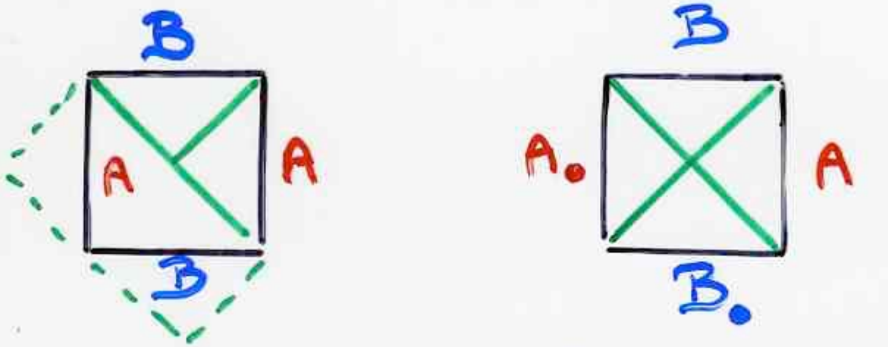
Aztec tilings



Aztec tilings



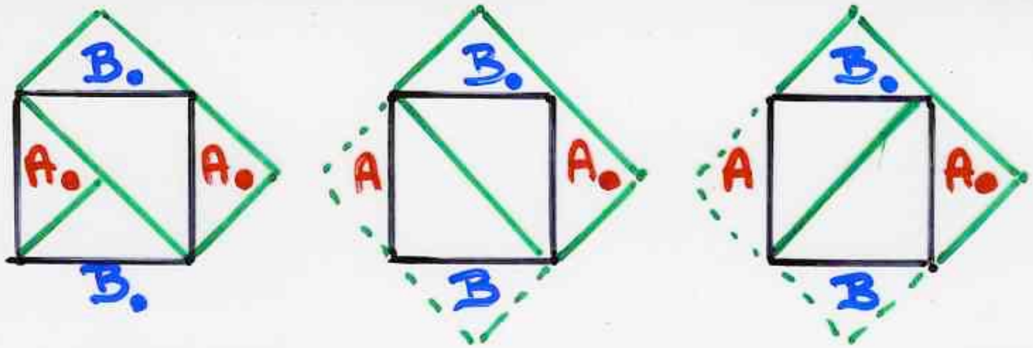




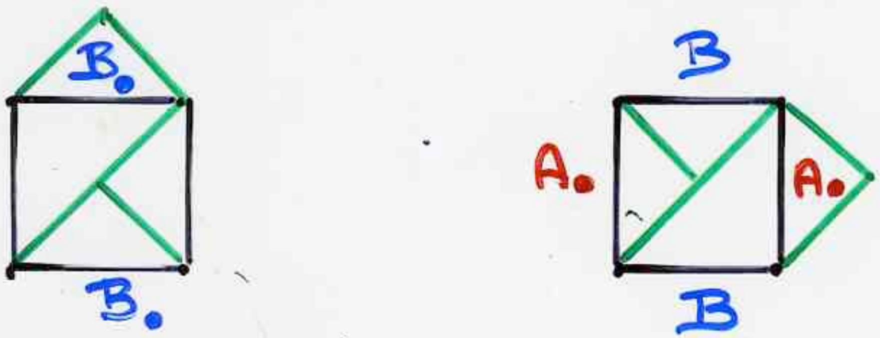
$$BA = AB + A_\bullet B_\bullet$$

rewriting rules for tilings

(Aztec lattice)



$$B_\bullet A_\bullet = A_\bullet B_\bullet + 2\,AB$$



$$B_\bullet A = A B_\bullet \qquad B A_\bullet = A_\bullet B$$

Aztec tilings

$$\begin{cases} BA = AB + A_\bullet B_\bullet \\ B_\bullet A_\bullet = A_\bullet B_\bullet + 2\, AB \\ B_\bullet A = A B_\bullet \\ B A_\bullet = A_\bullet B \end{cases}$$

ASM

alternating sign matrices

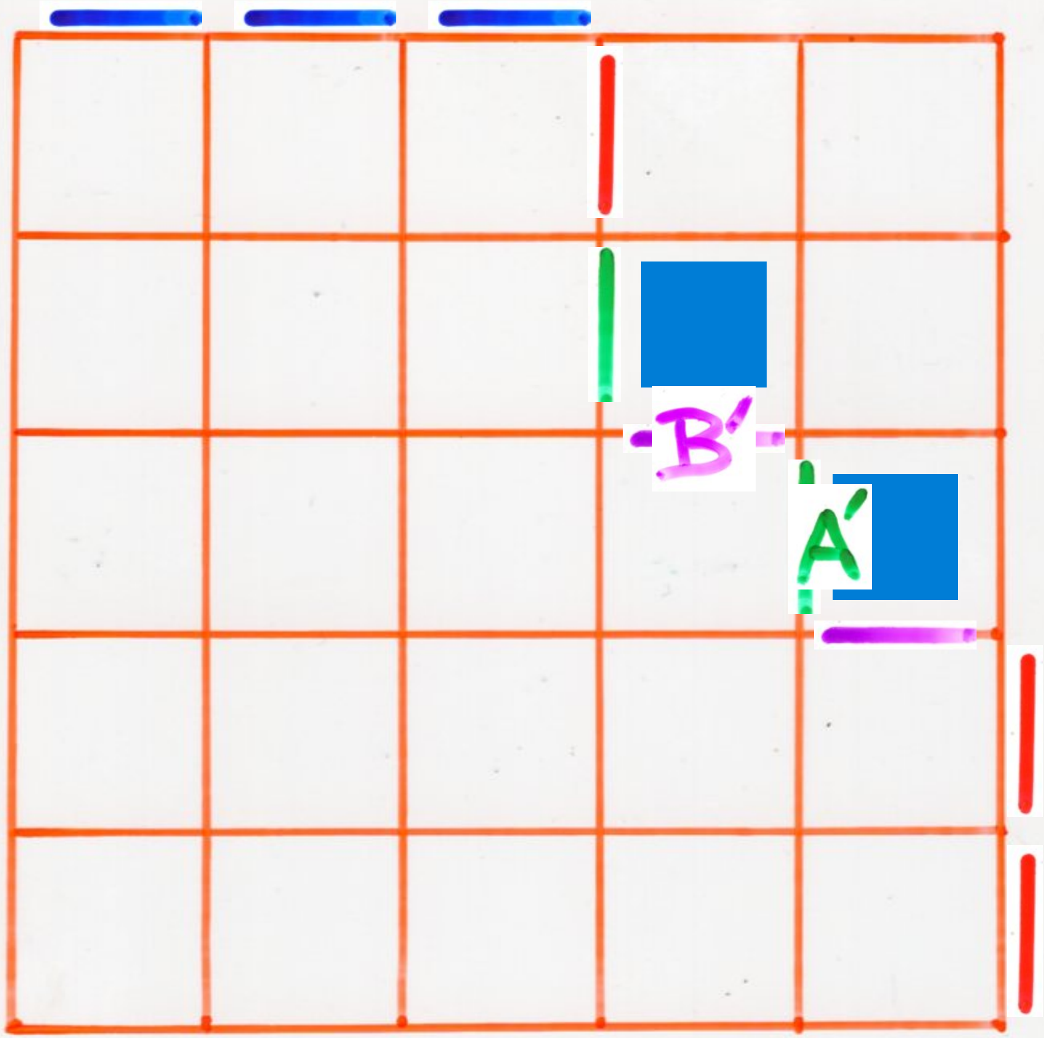
commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

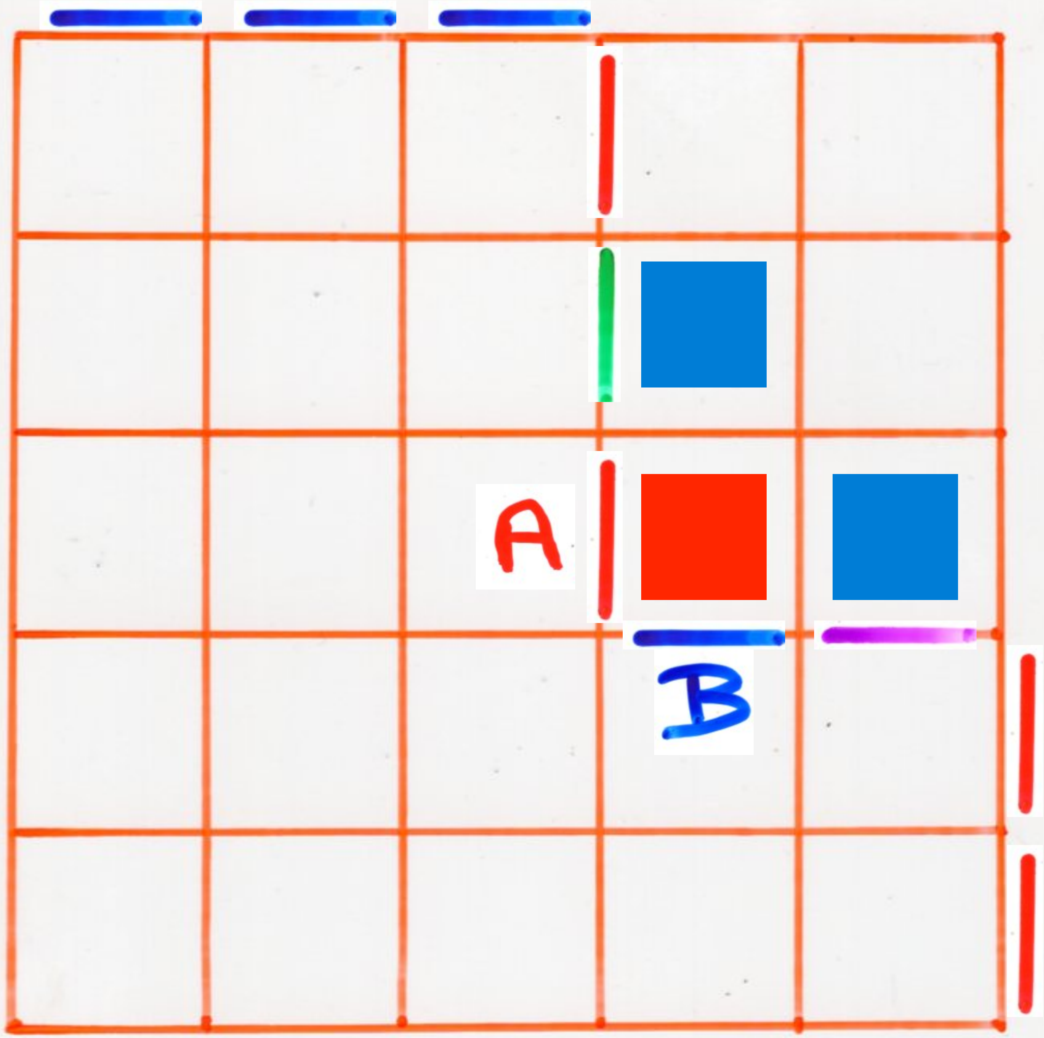
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

$A_n(x)$

enumeration of ASM according to the number of (-1)



B′
A′


A
B

Aztec tilings

$$\begin{cases} B\,A = A\,B + A_{\bullet}\,B_{\bullet} \\ B_{\bullet}\,A_{\bullet} = A_{\bullet}\,B_{\bullet} + 2\,A\,B \\ B_{\bullet}\,A = A\,B_{\bullet} \\ B\,A_{\bullet} = A_{\bullet}\,B \end{cases}$$

commutations

ASM
alternating sign matrices

$$\begin{cases} B\,A = A\,B + A'B' \\ B'A' = A'B' + A\,B \end{cases}$$

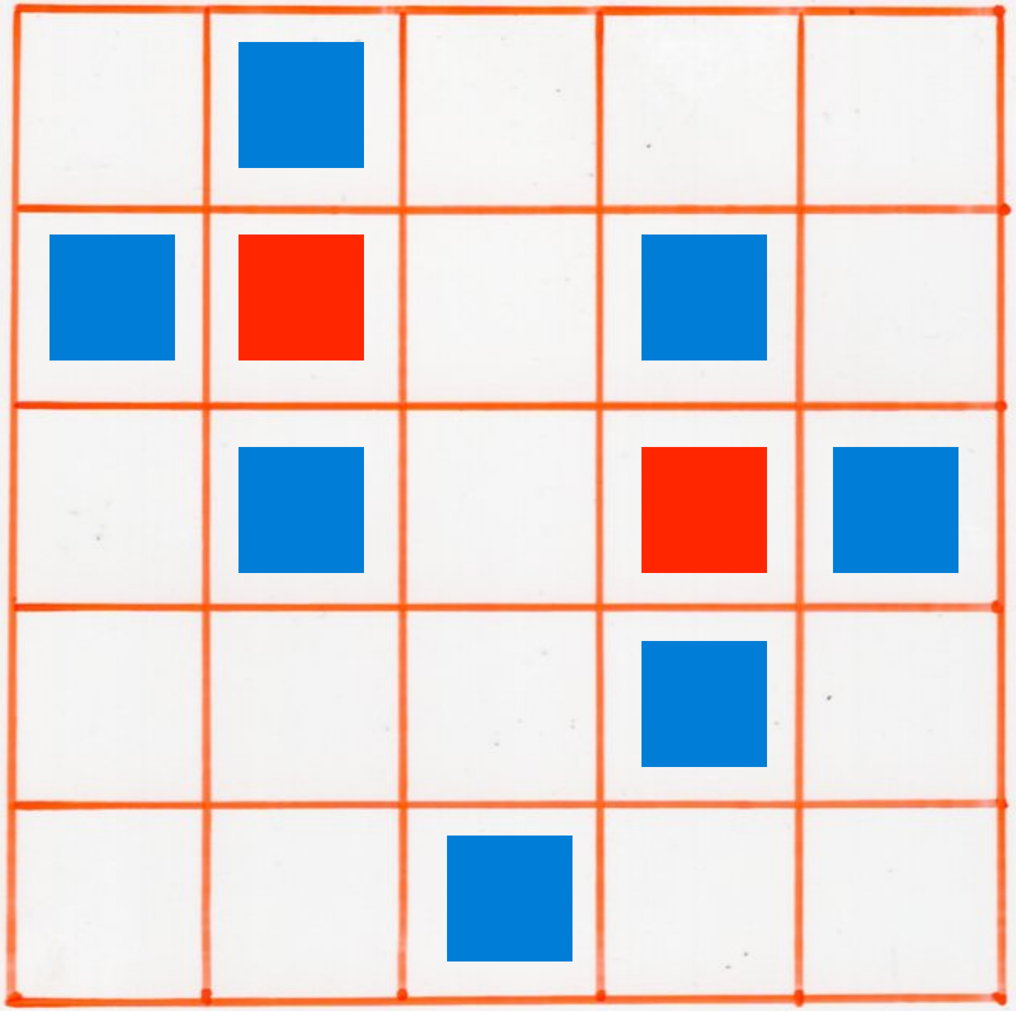
$$\begin{cases} B'A = A\,B' \\ B\,A' = A'B \end{cases}$$

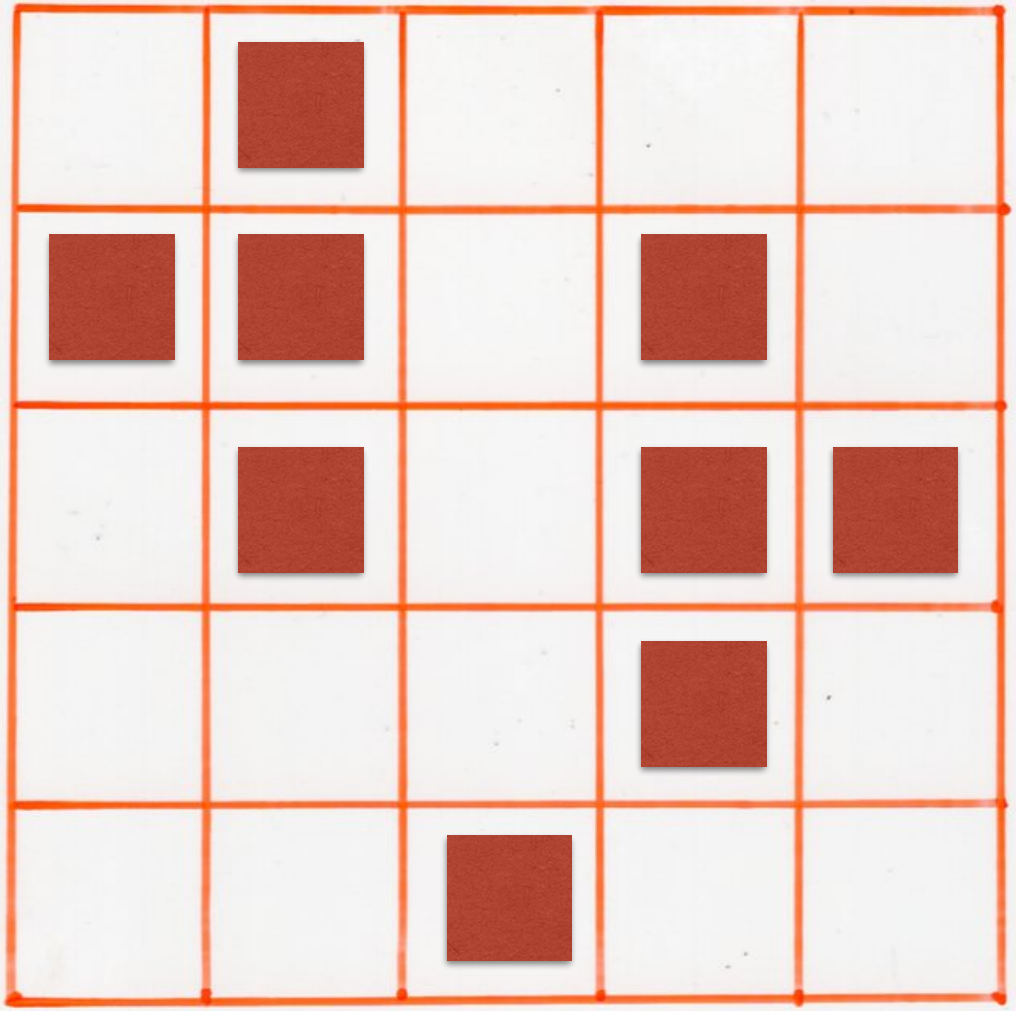
$A_n(x)$

enumeration of ASM according to the number of (-1)

$$A_n(2) = 2^{n(n-1)/2}$$

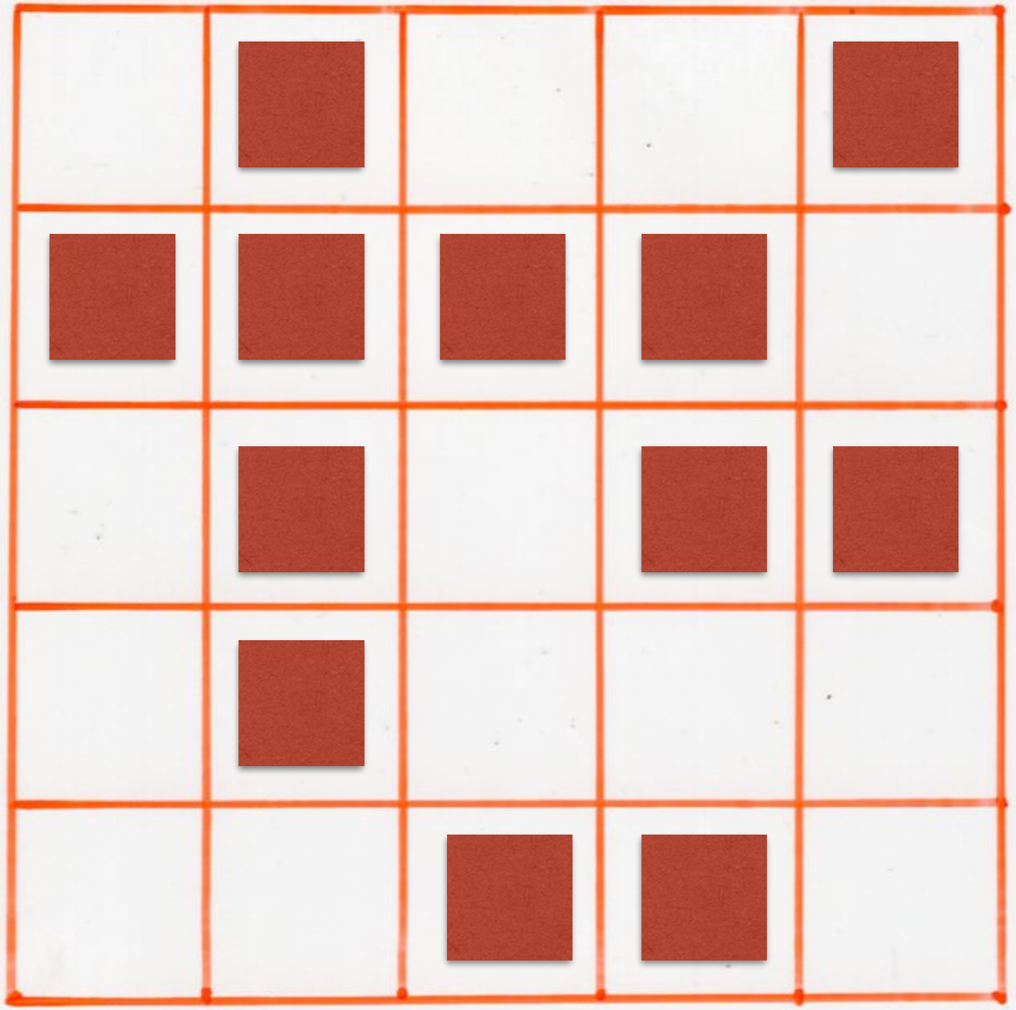
tilings
of the
Aztec diagram
with dimers





Q-tableaux

$2^{(n^2)}$



4 generators $B_\bullet \quad A_\bullet \quad B \quad A$

8 parameters $q_{xy}, \ t_{xy} \quad \begin{cases} x = \bullet, \circ \\ y = \bullet, \circ \end{cases}$

Q-tableaux

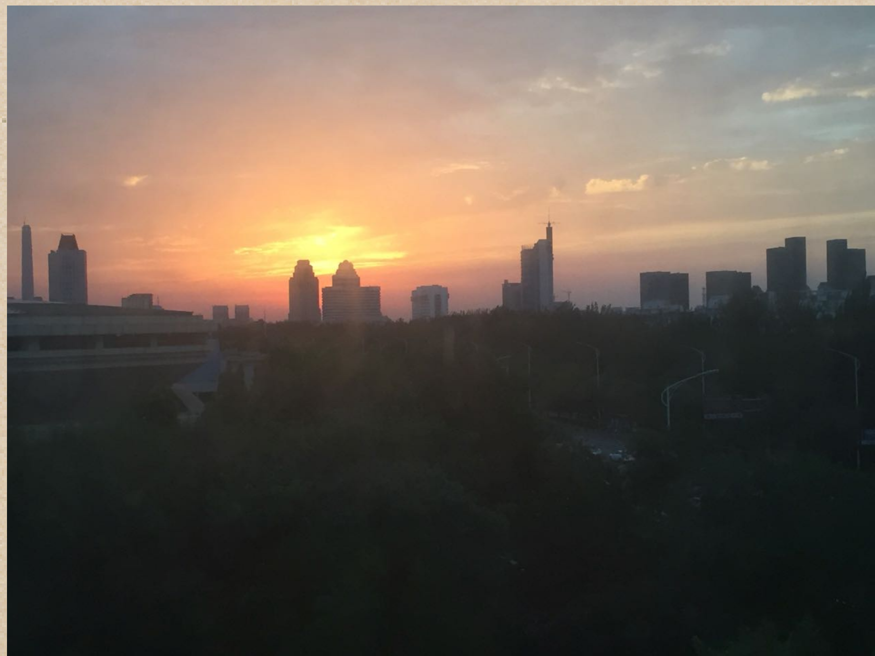
for

quadratic algebra **Q**

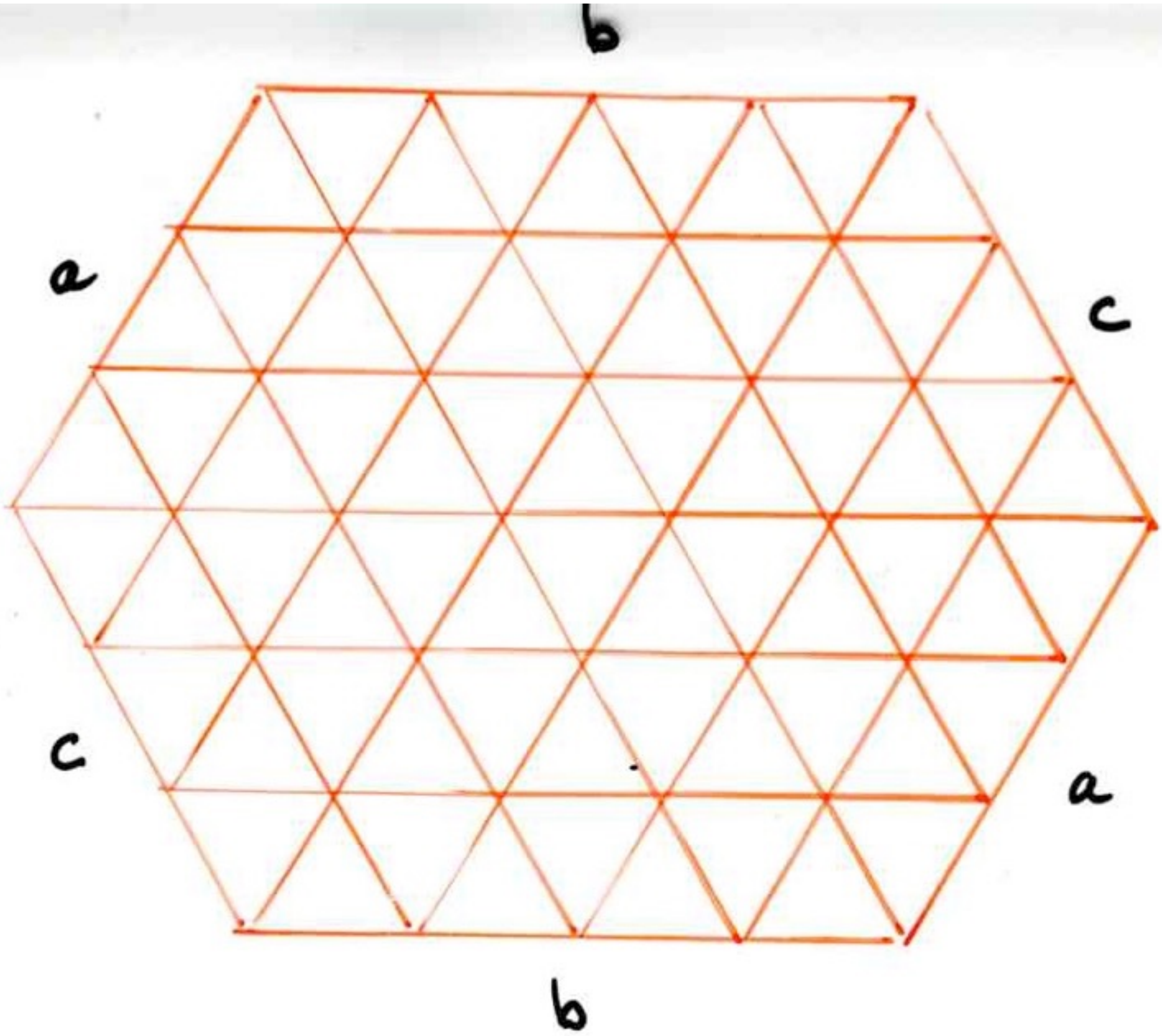
$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = q_{\bullet\bullet}\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + t_{\bullet\circ}\, A_\bullet\, B \\ B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + t_{\circ\bullet}\, A\, B_\bullet \end{cases}$$

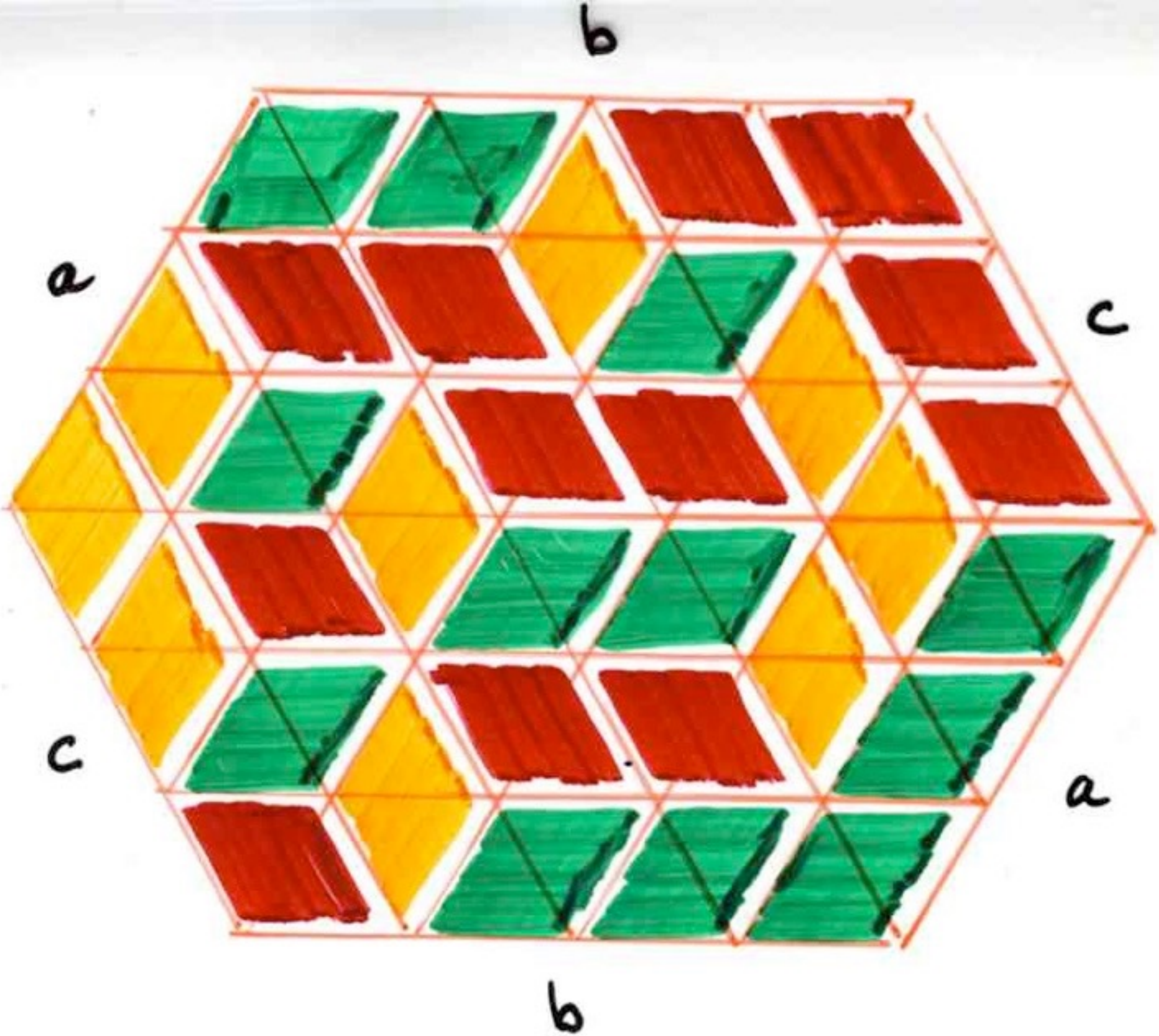
XYZ- quadratic algebra

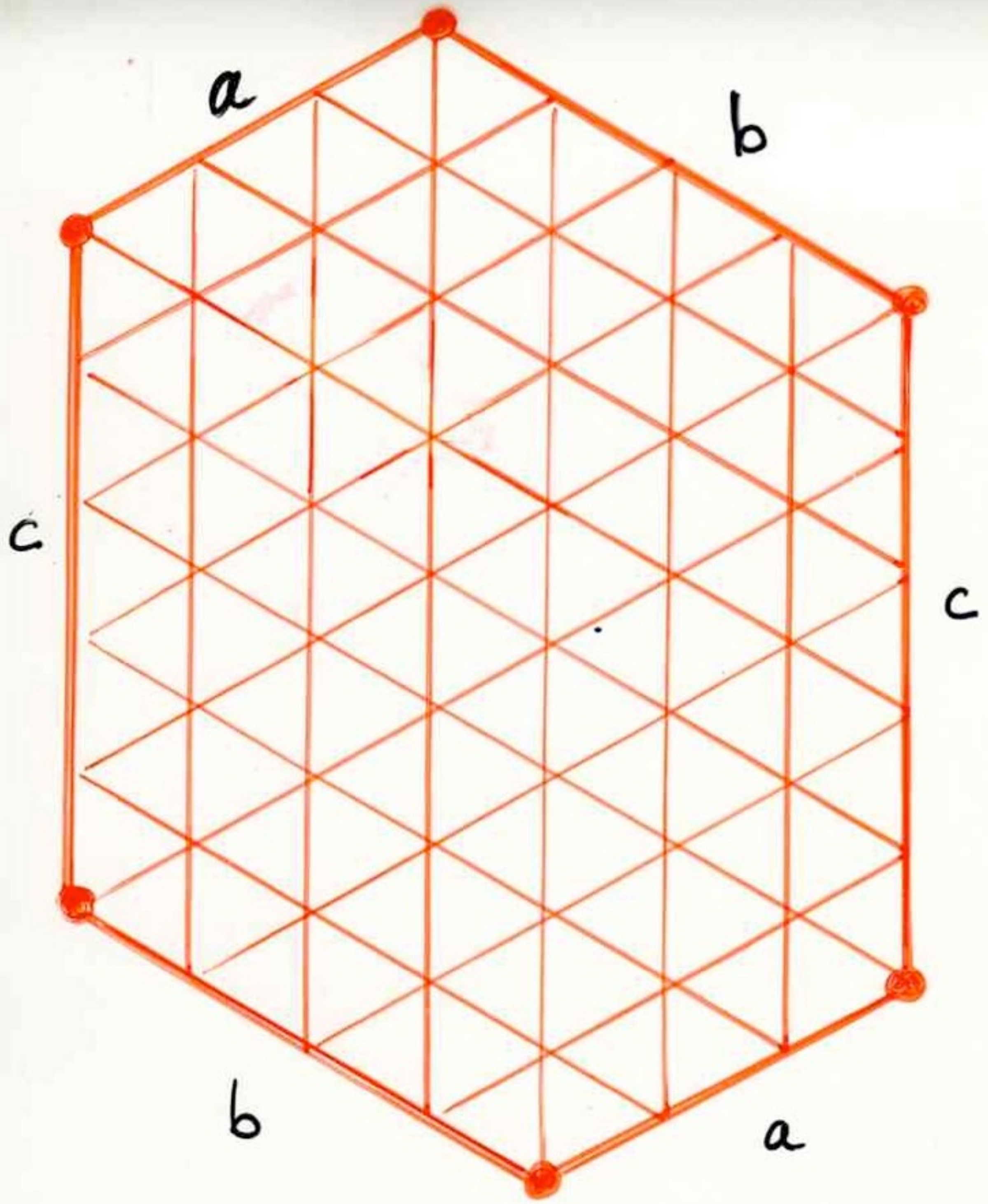
Another example of tilings enumerated with a determinant interpreted by non-crossing configurations ofpaths



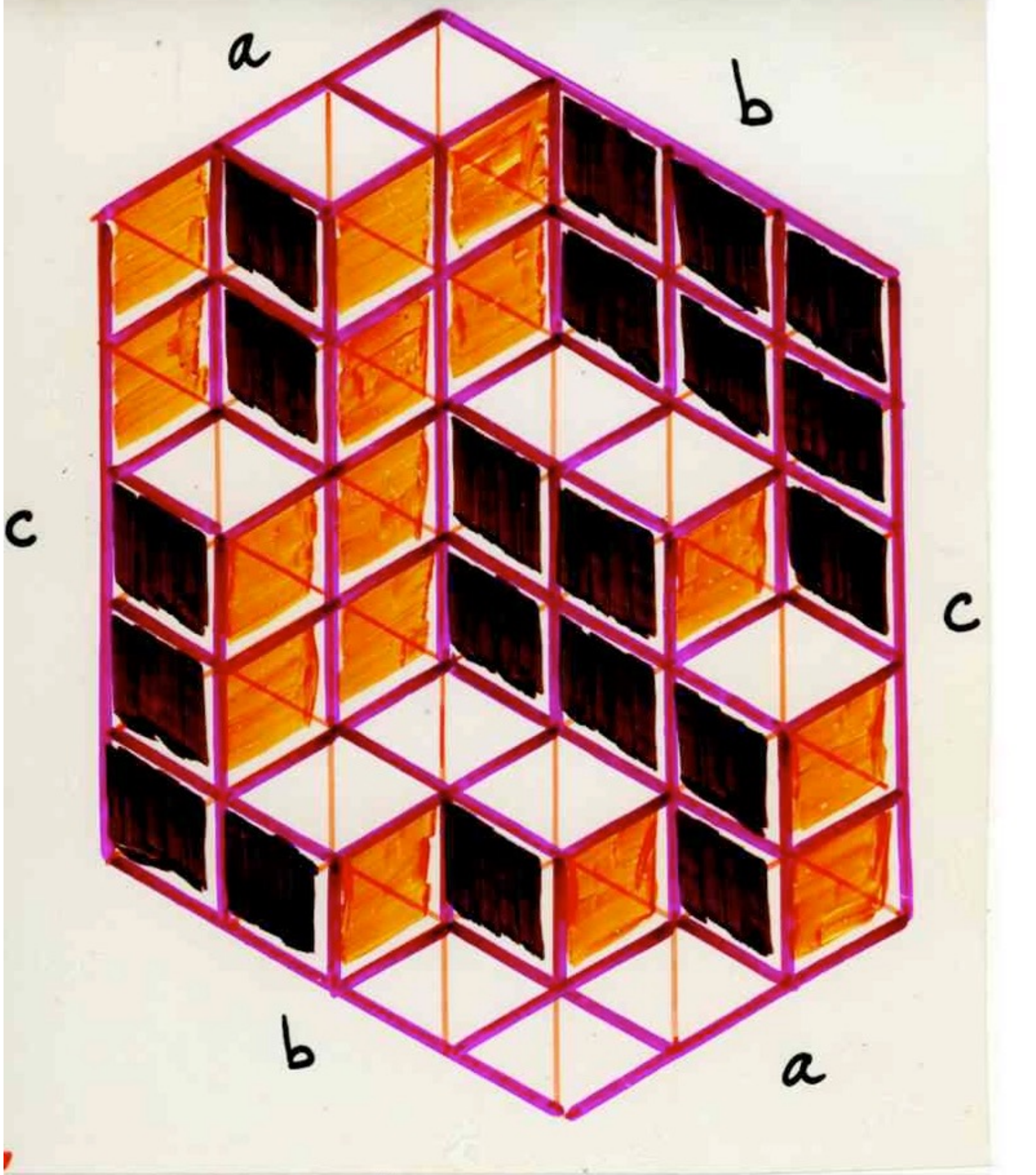
Tilings on triangular lattice and plane partitions

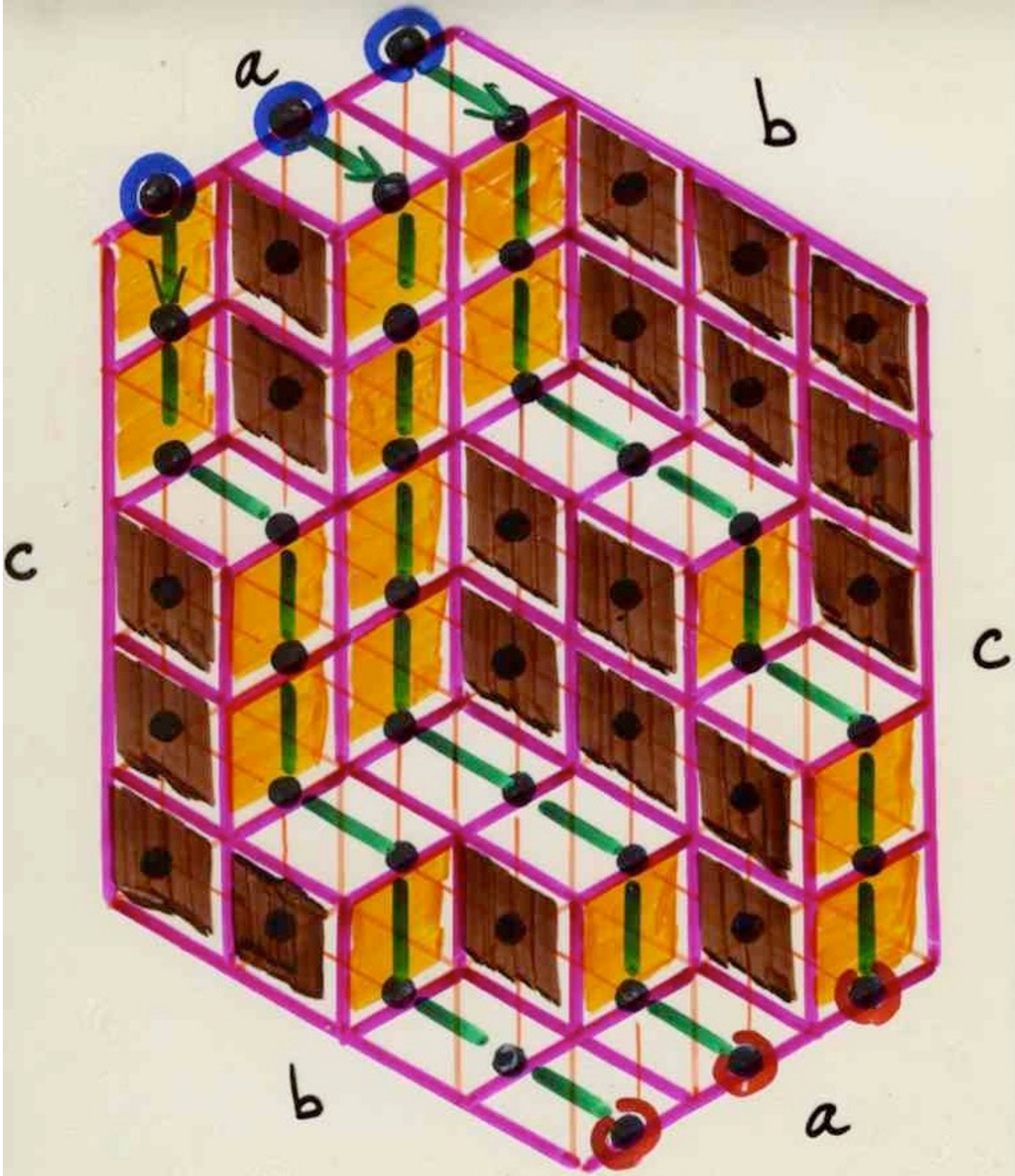

b
a
c
c
a
b

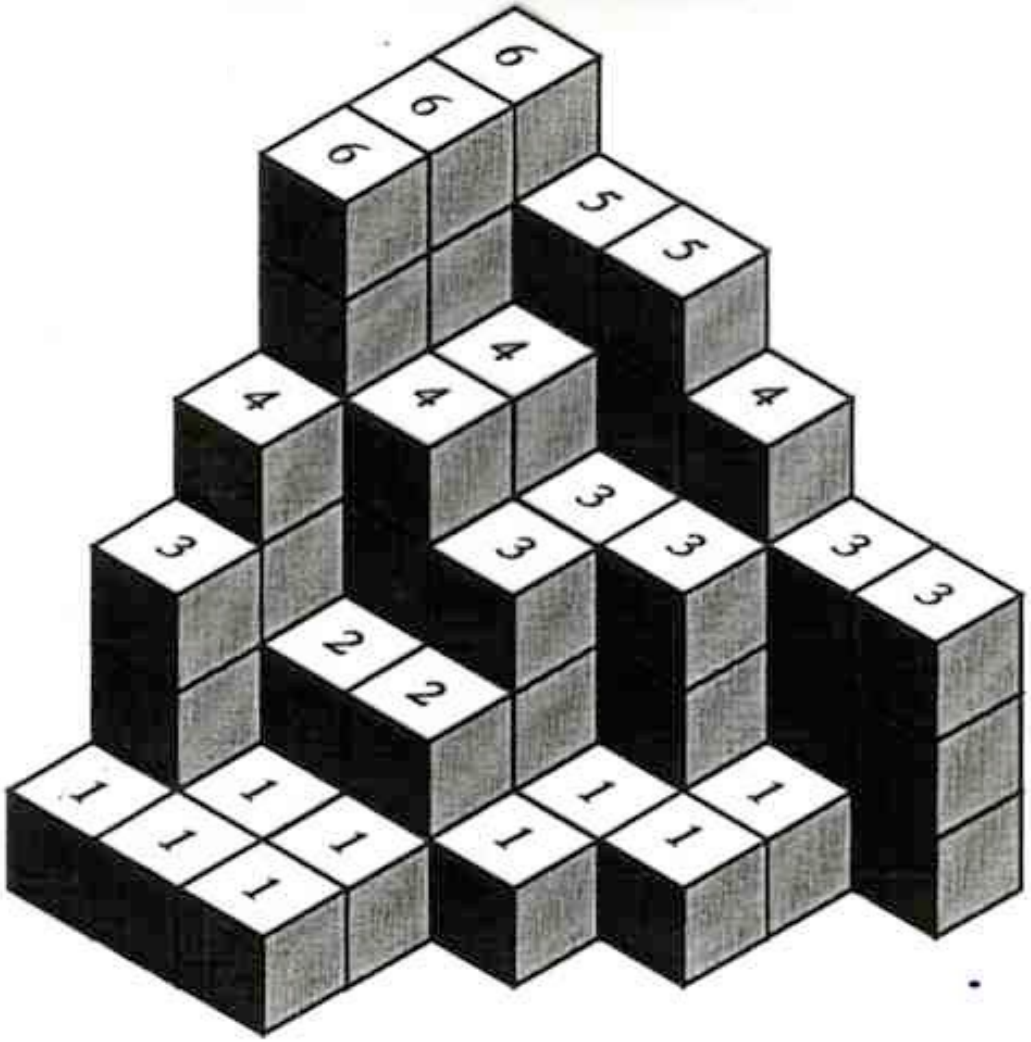

b
a
c
c
a
b



a
b
c
c
b
a


a
b
c
c
b
a


a
b
c
c
b
a

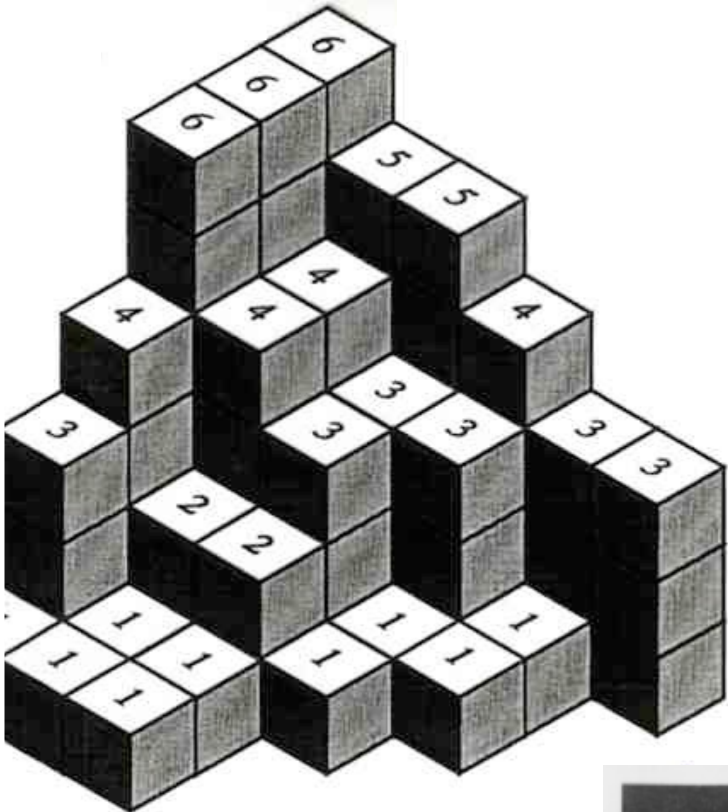


6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

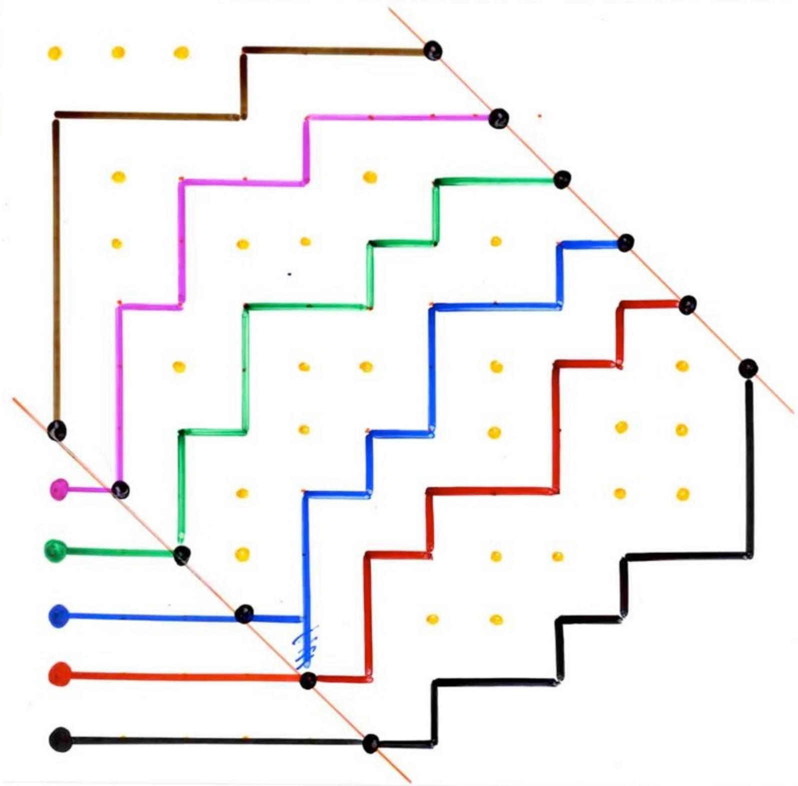
plane partitions

3D Ferrers diagrams

in a box $\mathcal{B}(a,b,c)$

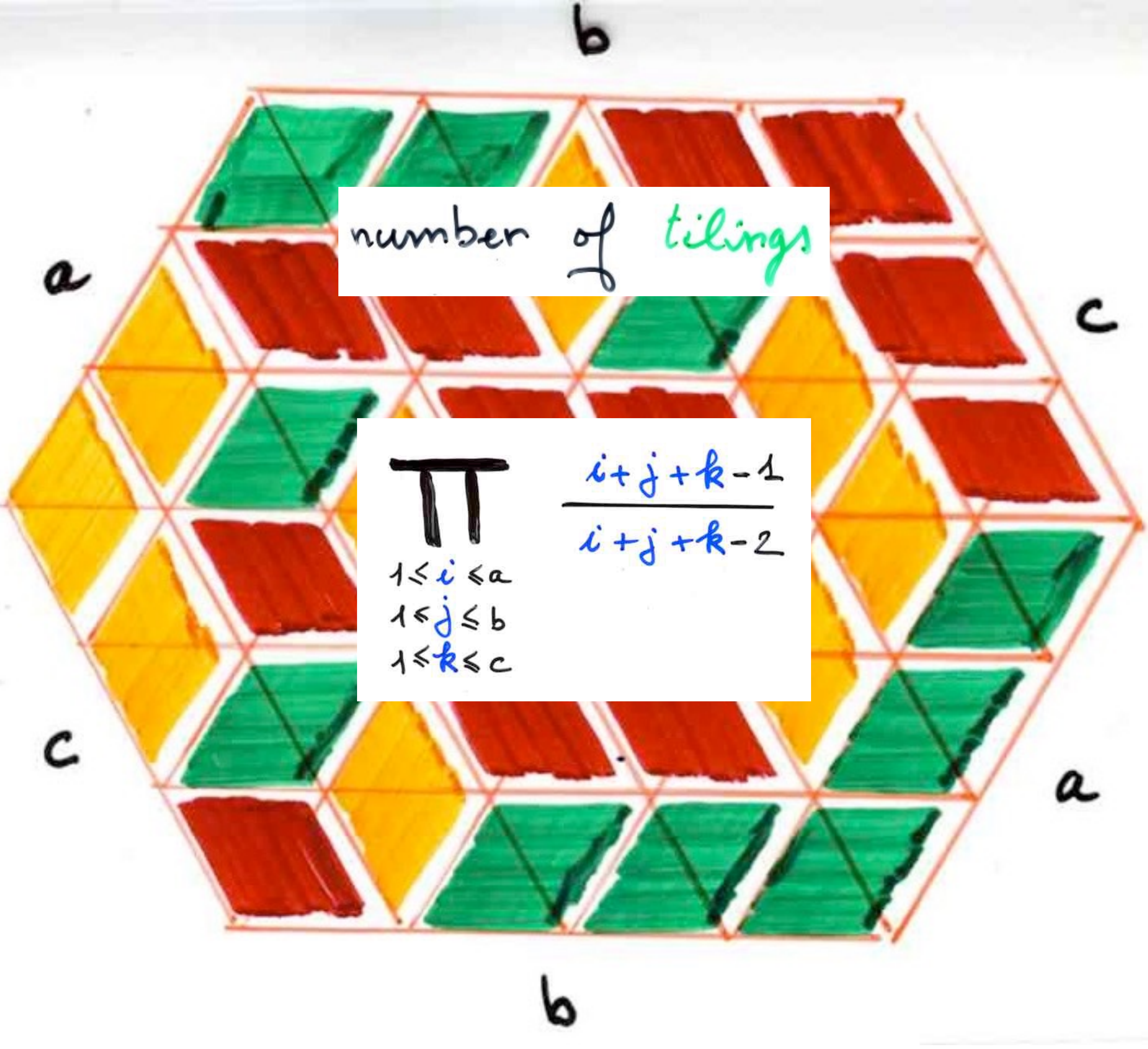


6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



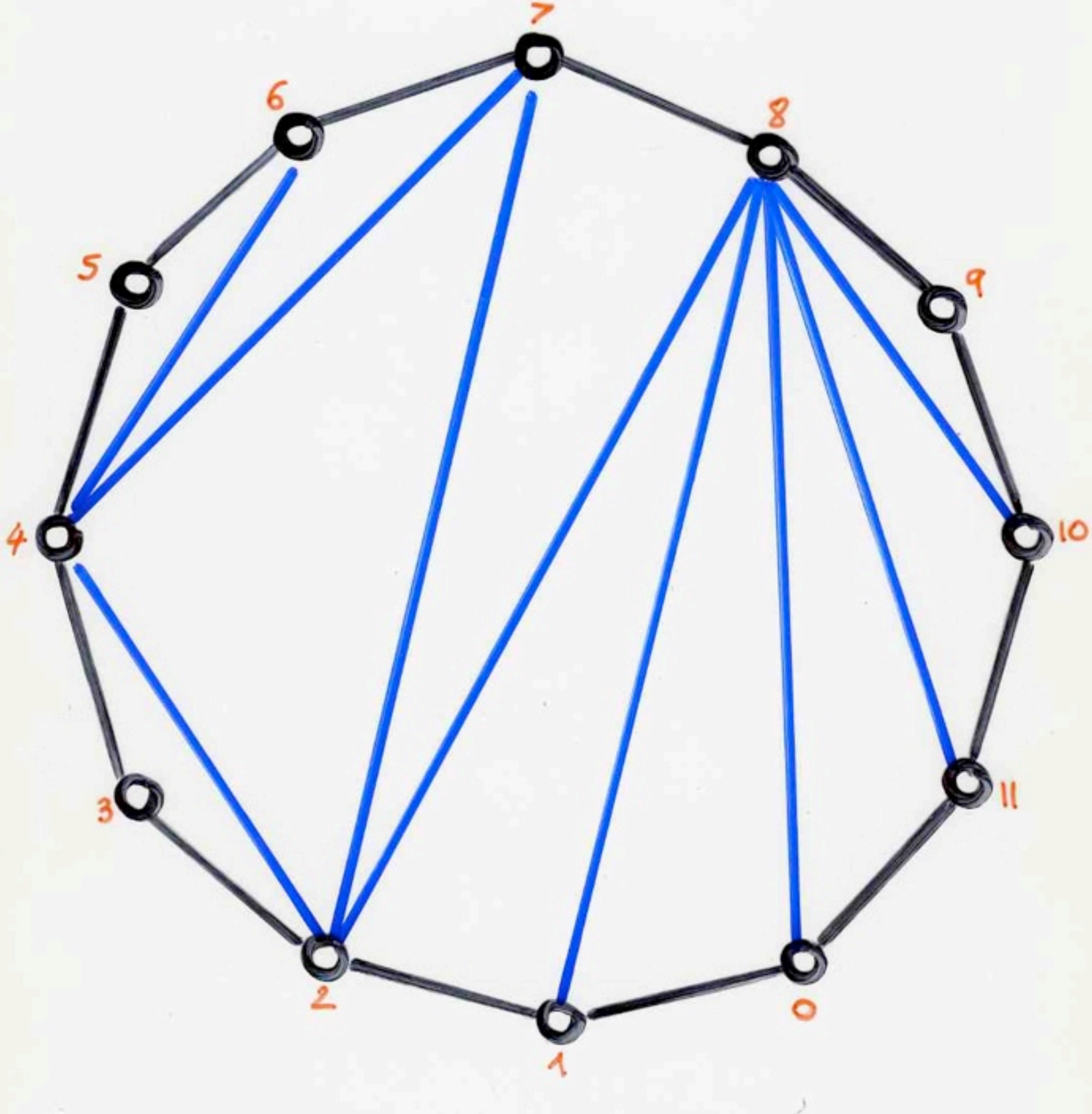
$$\prod_{\substack{1\leq i\leq a\\ 1\leq j\leq b\\ 1\leq k\leq c}} \frac{i+j+k-1}{i+j+k-2}$$

number of tilings



$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

The Catalan garden


7
6
8
5
9
4
10
3
11
2
0
1


EULER

getheilet und solches kann auf 5 verschiedene Arten geschehen, nehmlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{da}{db}$; V. $\frac{ec}{eb}$

Ferner kann ein Sechseck durch 3 Diagonales in 4 Triangula zerschnitten und dieses kann auf 14 verschiedene Arten geschehen.

Nun ist die Frage generaliter, da ein Polygonum von n Seiten durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen könne.

Setze ich nun die Anzahl dieser verschiedenen Arten $= x$, so habe ich per Inductionem gefunden

wann $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14.\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2.\ 6.\ 10.\ 14.\ 18.\ 22.\ \ldots\ (4n-10)}{2.\ 3.\ 4.\ 5.\ 6.\ 7.\ \ldots\ (n-1)}$$

oder es ist

$1 = \frac{2}{2}$; $2 = 1.\frac{6}{3}$; $5 = 2.\frac{10}{4}$; $14 = 5.\frac{14}{5}$; $42 = 14.\frac{18}{6}$; $132 = 42\,\frac{22}{7}$;

dass also aus einer jeden Zahl die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen $1,\ 2,\ 5,\ 14,\ 42,\ 132$, etc. habe ich auch diese Eigenschaft angemerket, dass

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

also wann $a = \frac{1}{4}$, so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4$.

Alle die meinige lassen sich zu Ew. Hochwohlgeb. beständigen Gewogenheit gehorsamst empfehlen, und ich habe die Ehre mit der schuldigsten Hochachtung lebenslang zu verharren

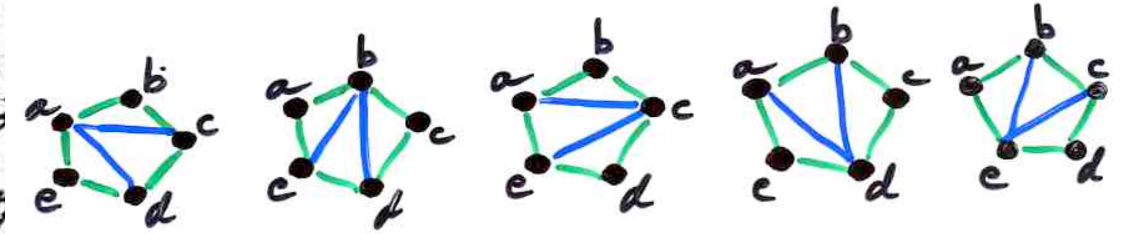
Ew. Hochwohlgebohrnen

Berlin d. 4ten Sept 1751.

gehorsamster Diener

L. Euler

Durch die Diagonalen I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V. $\frac{ec}{eb}$



durch $n-3$ Diagonalen in $n-2$ Triangula ...

... $= x$

$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

wann $n = \ldots$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$5 = 2 \cdot \frac{10}{4}$; $14 = 5 \cdot \frac{14}{5}$; $42 = 14 \cdot \frac{18}{6}$; ...

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a^2}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$1+2a+5a^2+14a^3+42a^4+\text{etc} = 4.$$

also wenn $a=\frac{1}{4}$ so ist $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc}$

$$a = \frac{1}{4}$$

Berlin d. 4. Sept 1751.

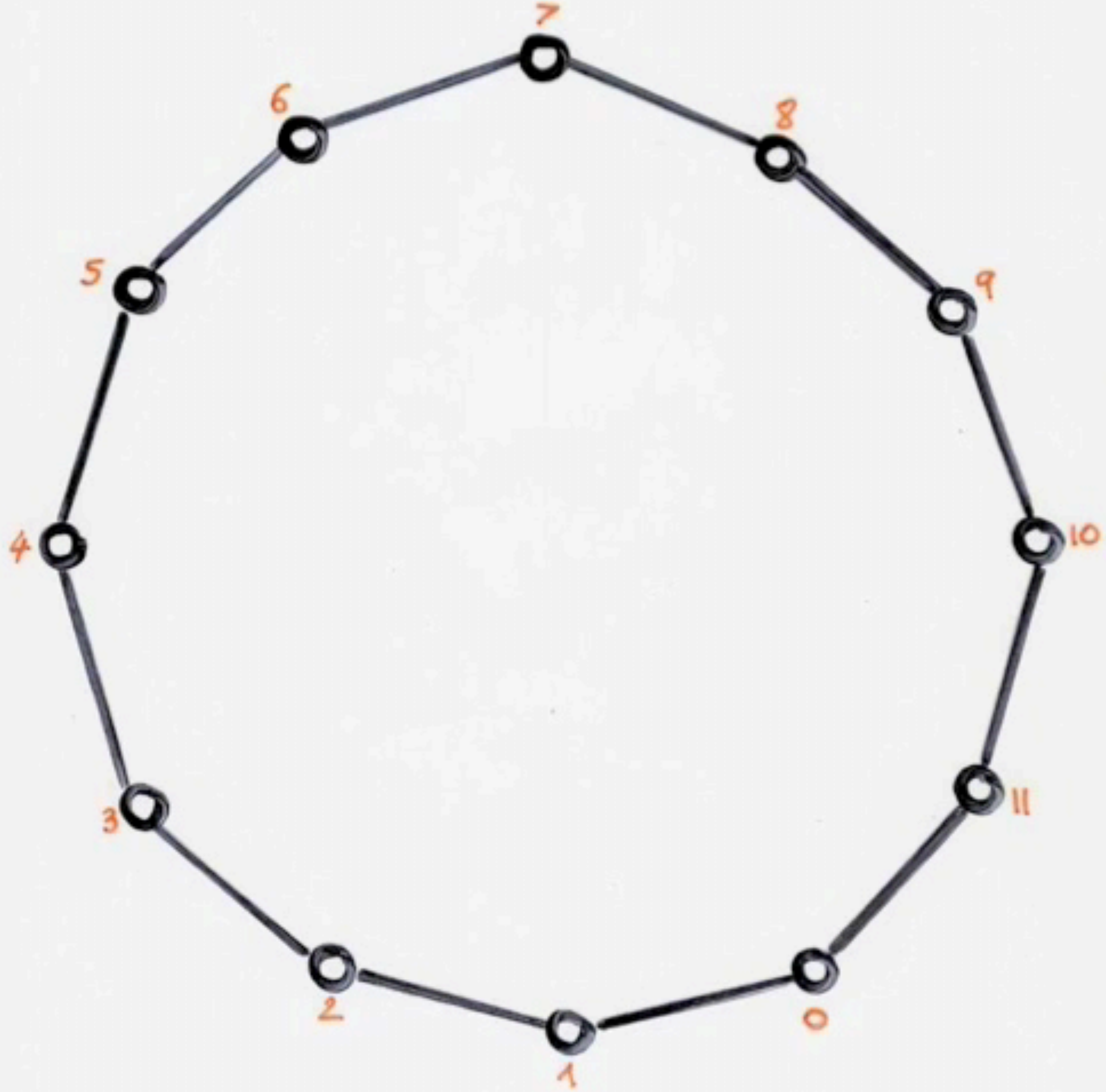
4 Sept 1751
Berlin

L. Euler

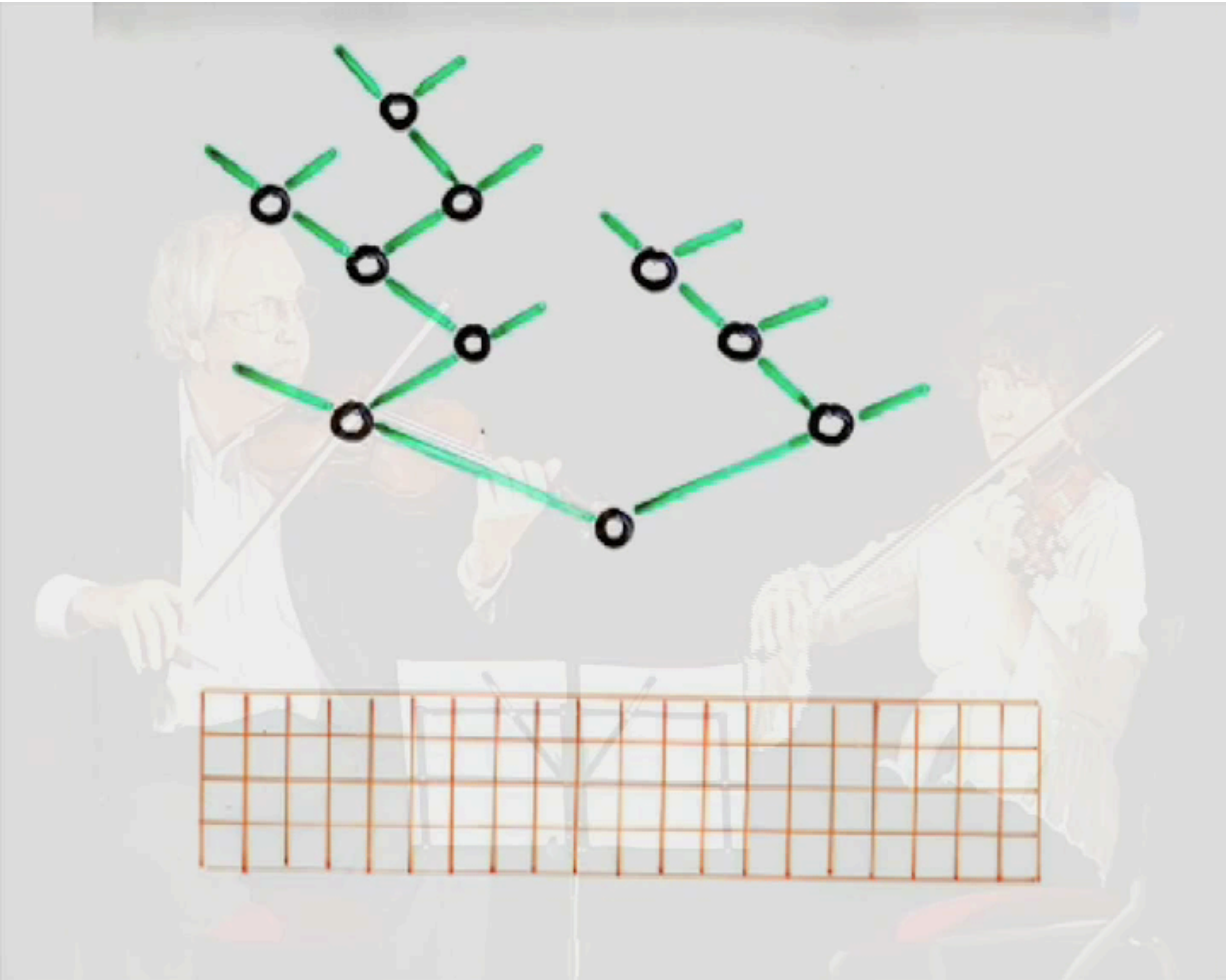
From triangulations

to

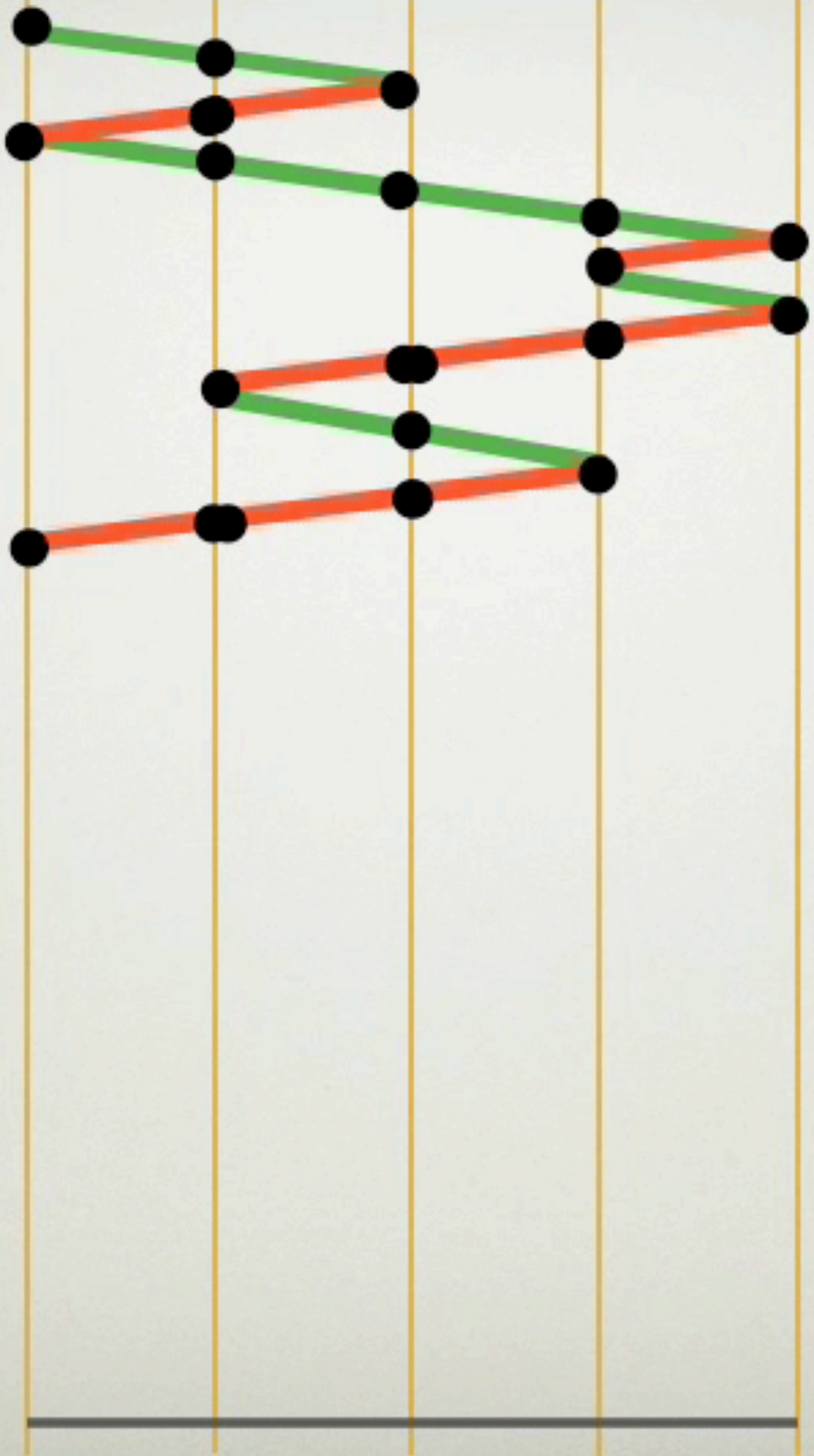
binary trees


7
6
8
5
9
4
10
3
11
2
0
1

From binary trees
to Dyck paths



From Dyck paths
to semipyramids of dimers



Thank you !