An introduction to the « cellular ansatz »

From RSK to the PASEP

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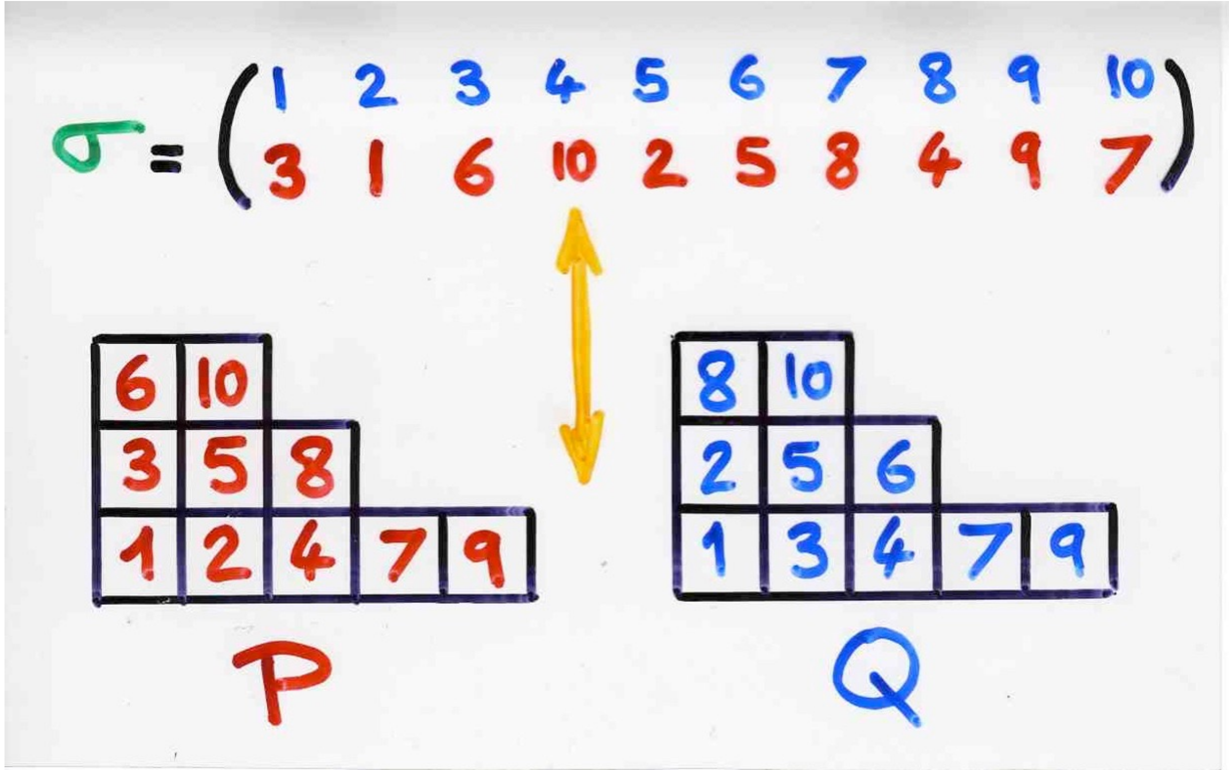
From lecture Tianjin-2

The Robinson-Schensted correspondence

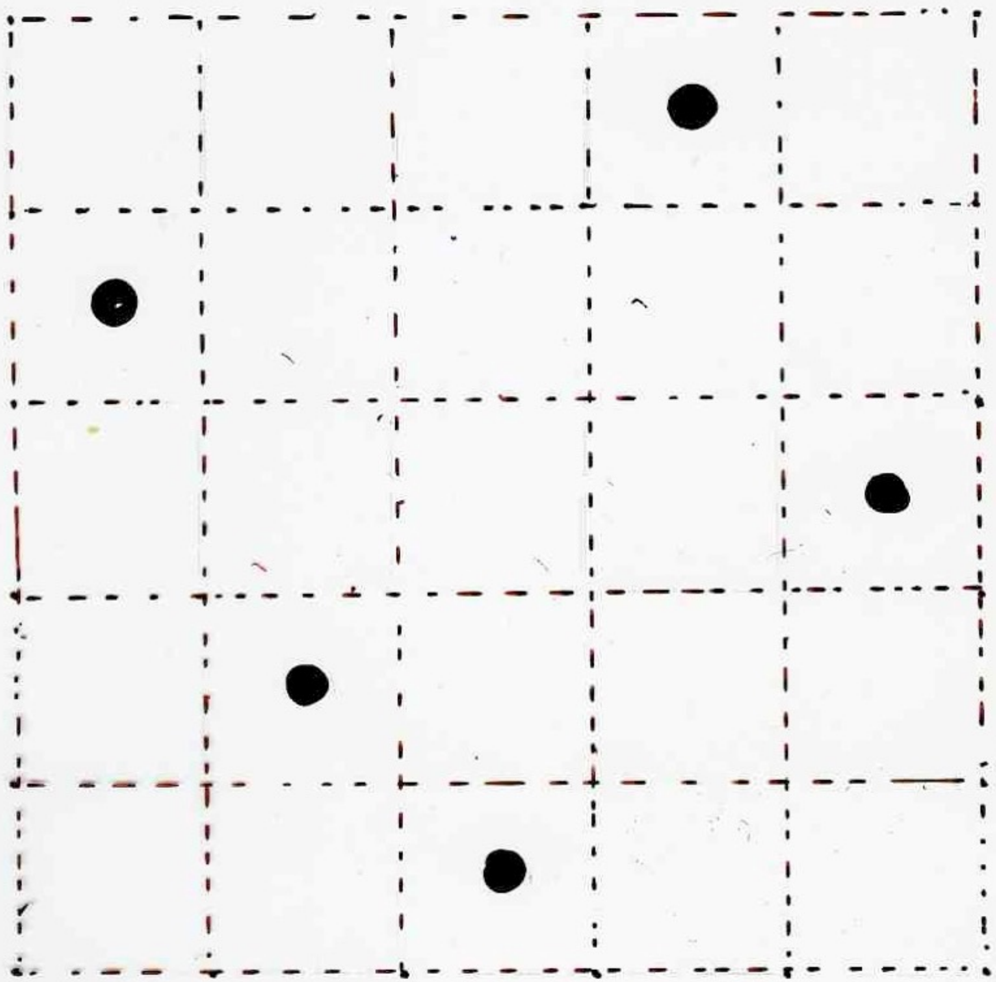
G. de B. Robinson, 1938

- Schensted insertions algorithm — C. Schensted, 1961
- Geometric version — X.V. 1976
- Growth diagrams — S. Fomin, 1986, 1994
- Edge local rules



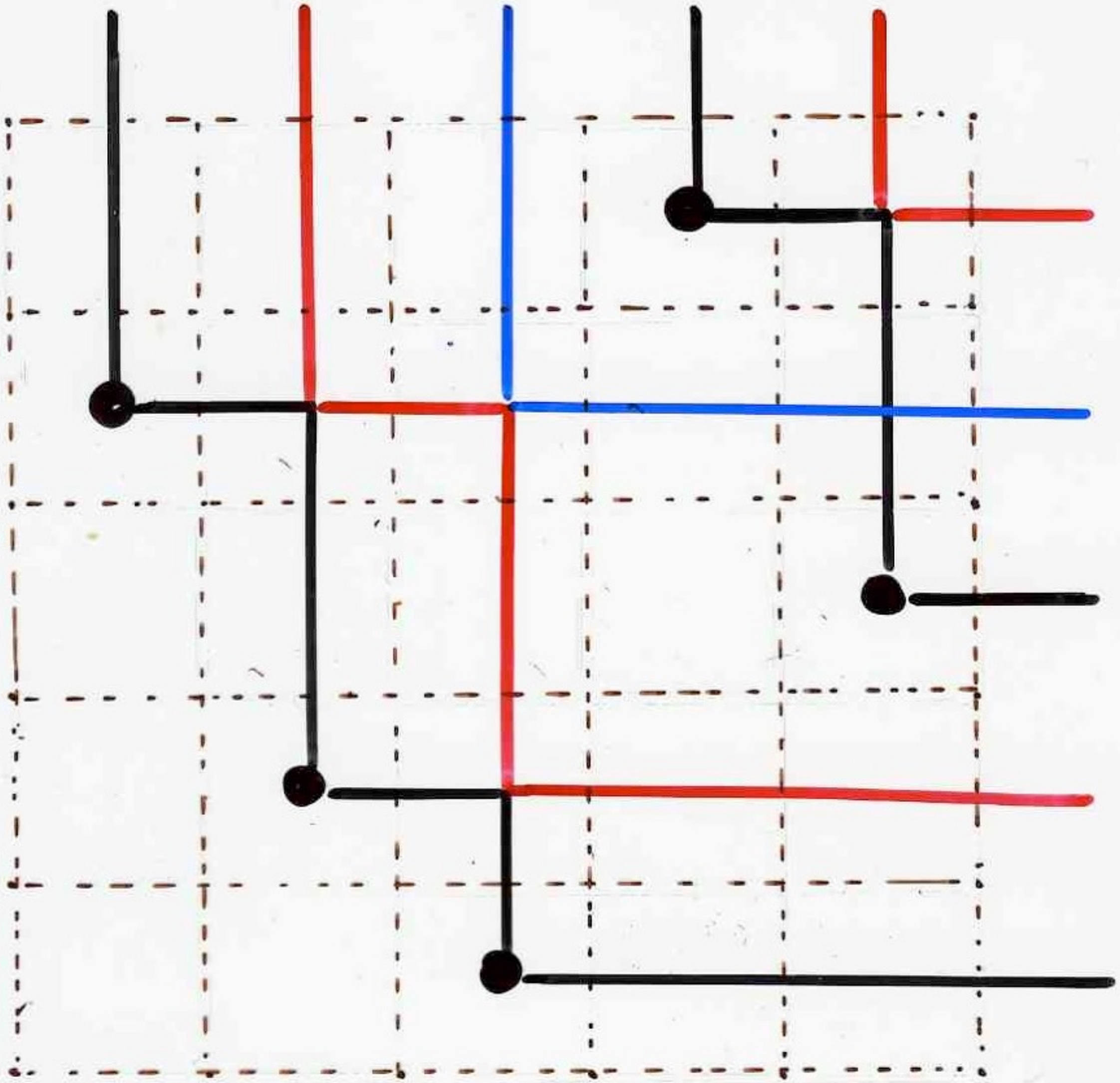
The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape

Permutation on a grid



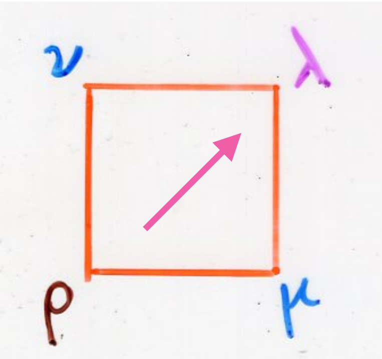
$\sigma = 4, 2, 1, 5, 3$

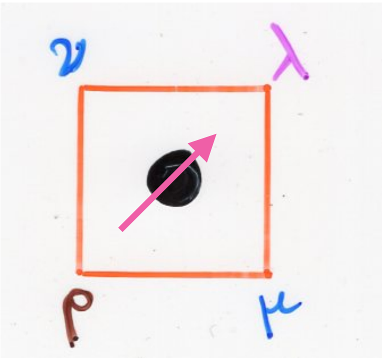
geometric version of RS with "light" and "shadow lines"

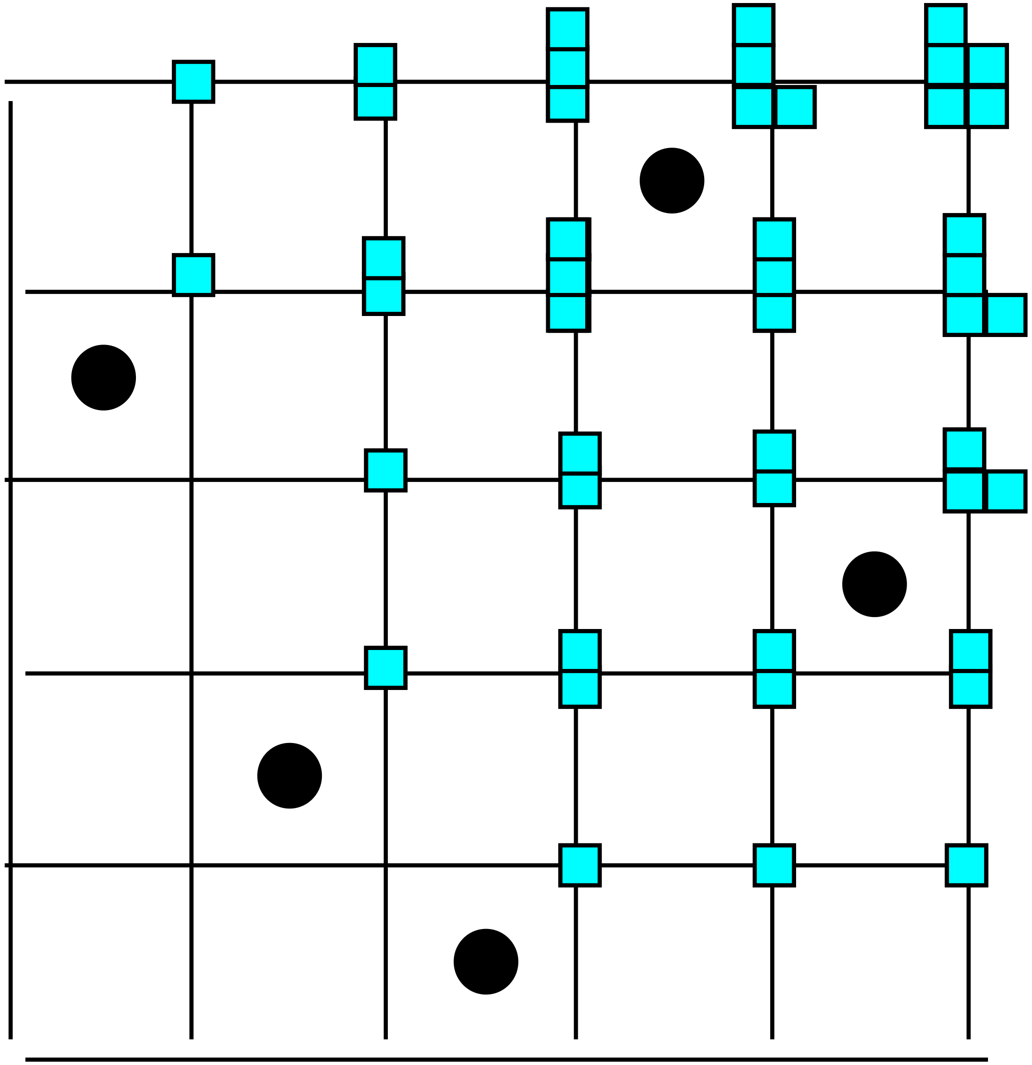


$\sigma = 4, 2, 1, 5, 3$

"growth diagrams"

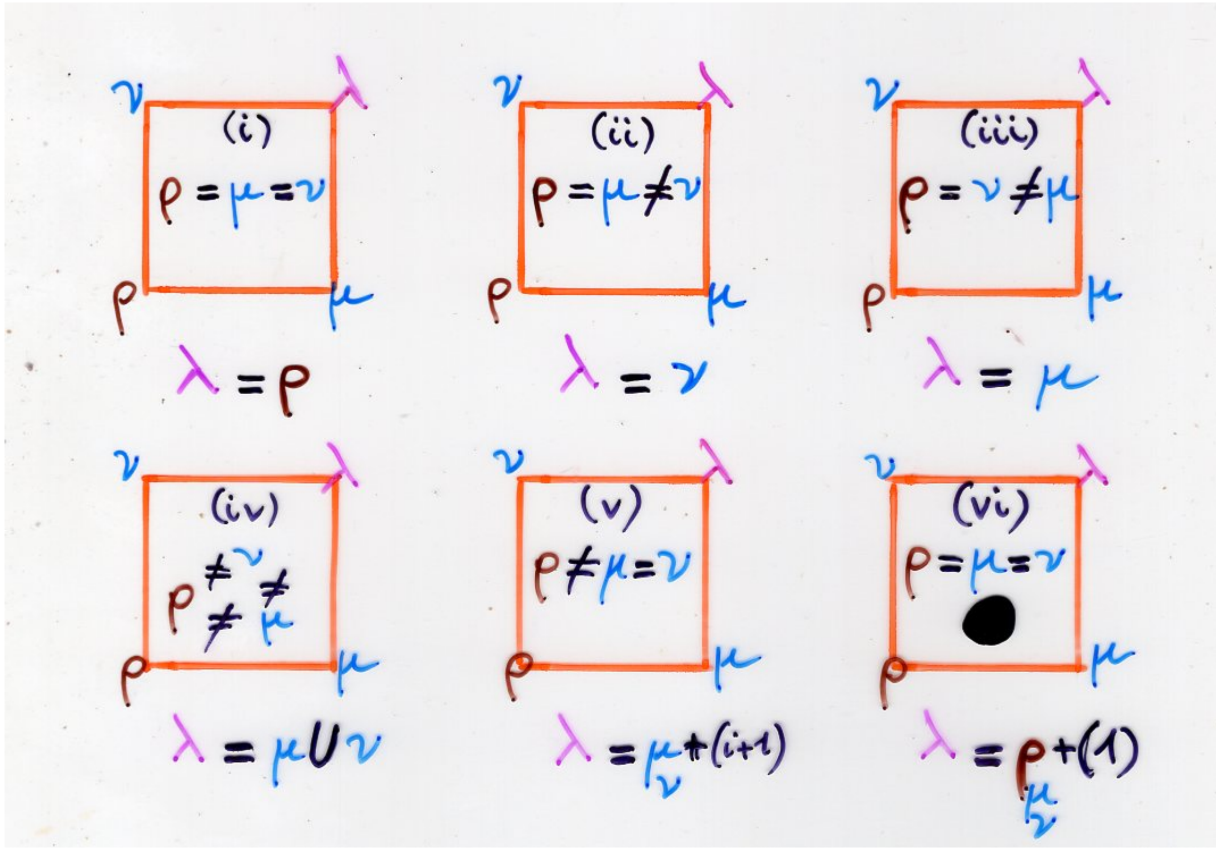




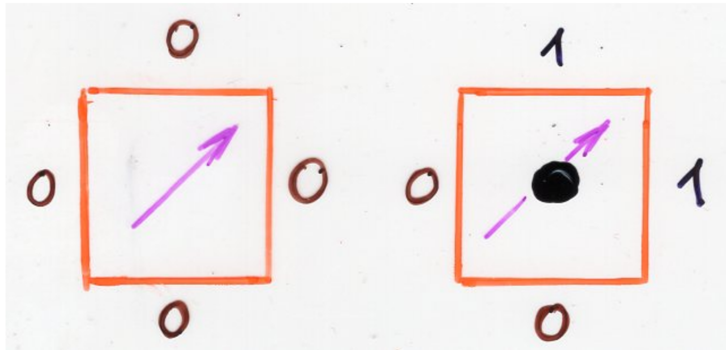


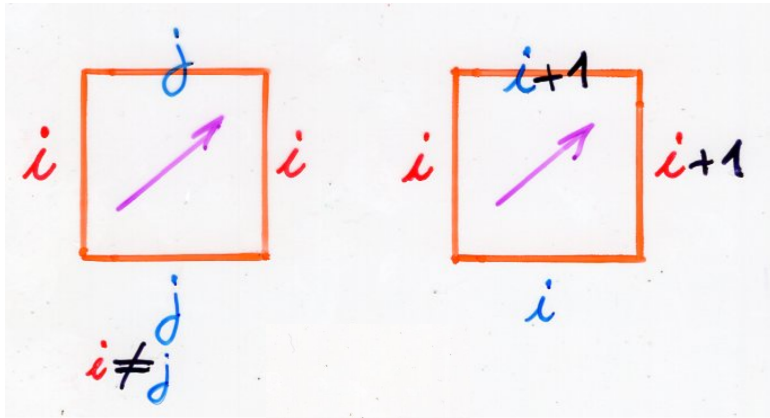
$\sigma = 4, 2, 1, 5, 3$

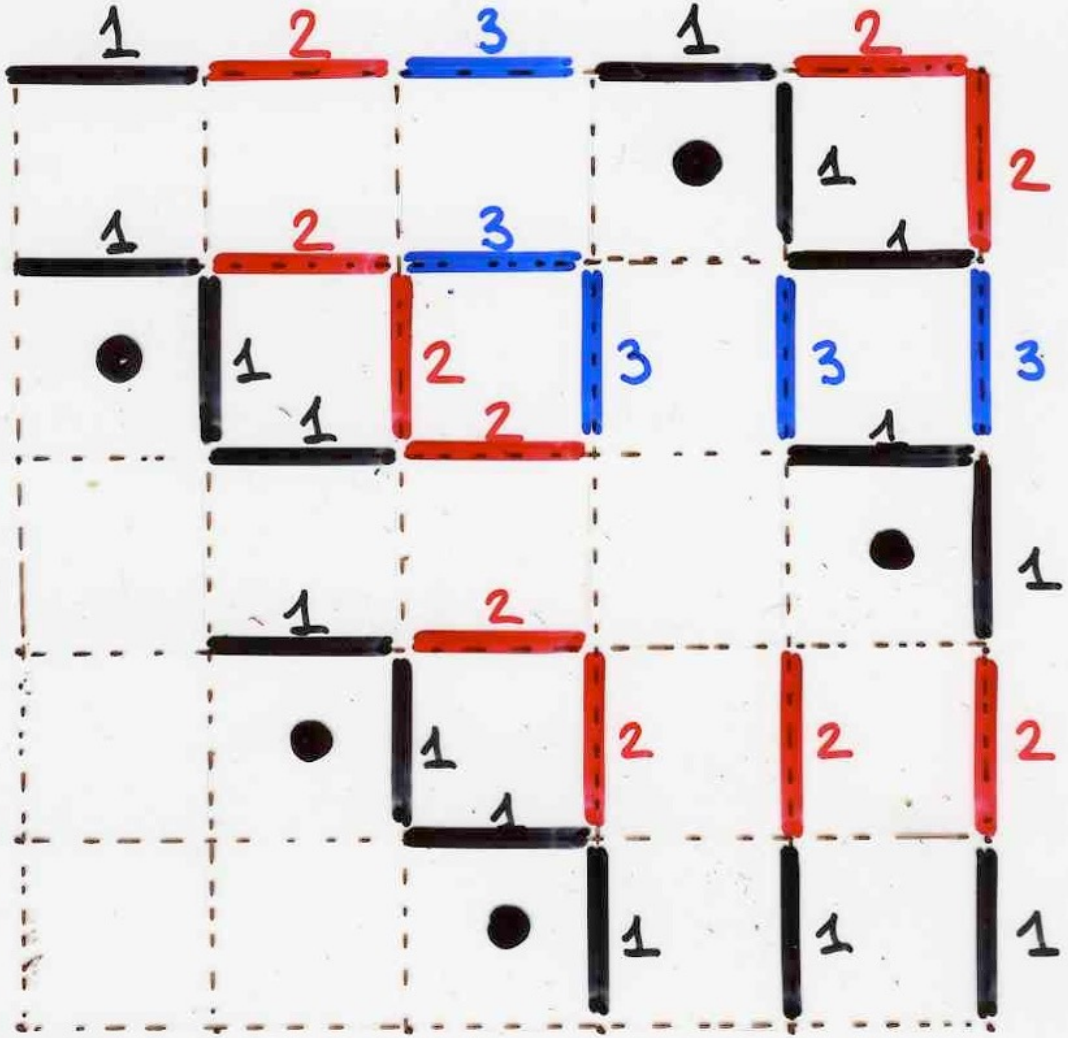
"local rules" on the vertices



"local rules" on the edges

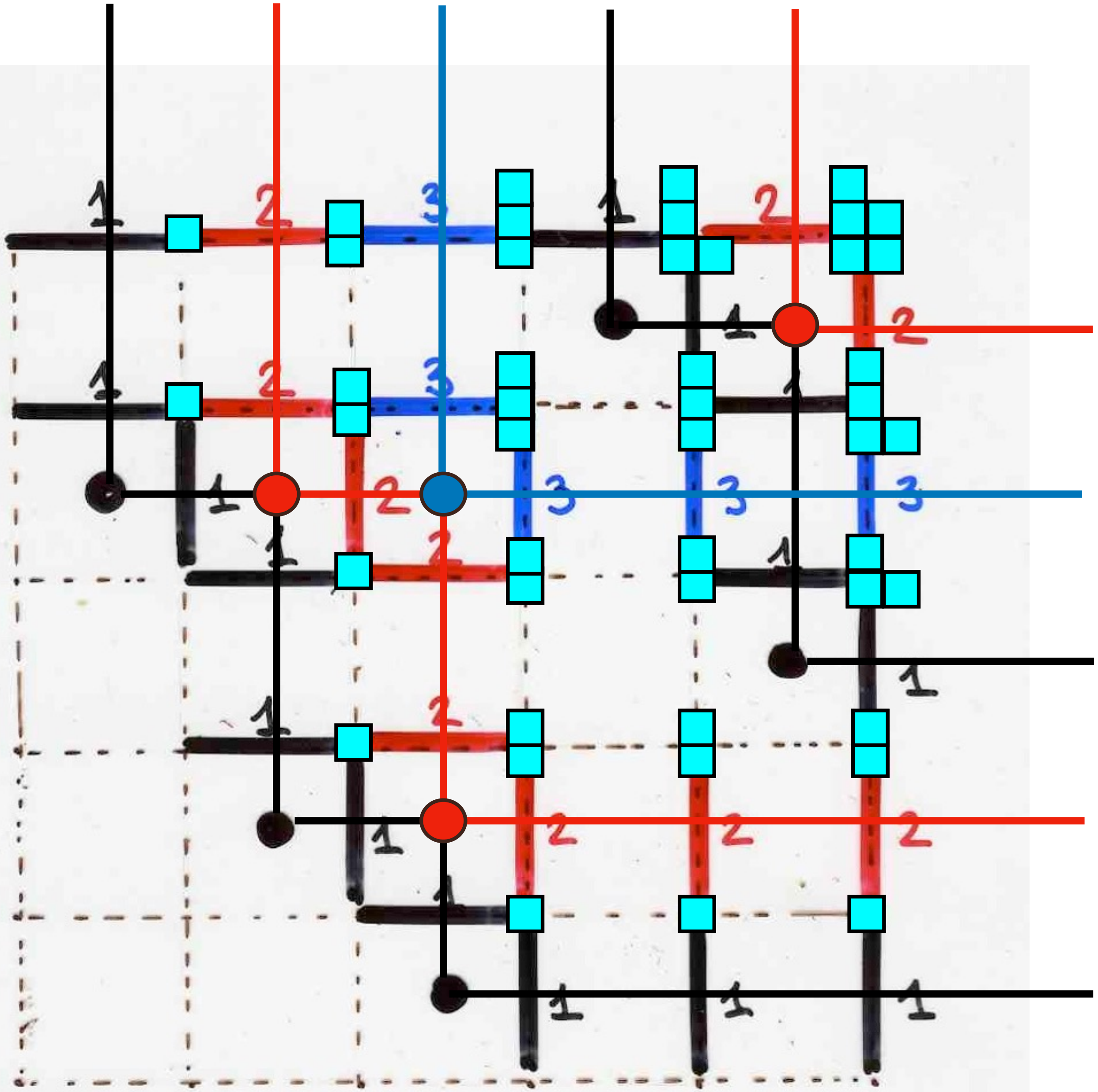






"local rules" on the edges

$\sigma = 4, 2, 1, 5, 3$



$\sigma = 4, 2, 1, 5, 3$

Fomin growth diagrams

Combinatorial representation of

the algebra

$UD = DU + Id$

$$UD = DU + Id$$

commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering in physics

$UD = DU + Id$ commutations

$UD \rightarrow DU$ $\quad UD \rightarrow Id$ rewriting rules

$UUDD$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2UD \\
&= (DUDU + DU) + 2(DU + Id) \\
&= (DDUU + 2DU) + 2(DU + Id) \\
&= DDUU + 4DU + 2\,Id
\end{aligned}
$$

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

Combinatorial representation of

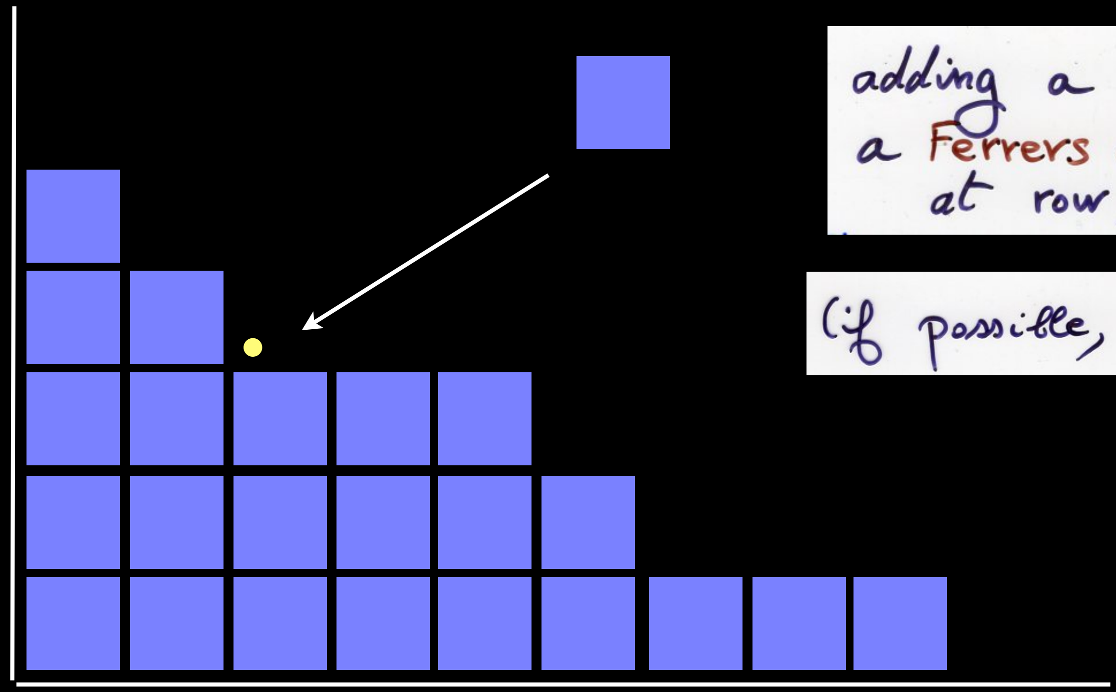
the algebra

$UD = DU + Id$

notations

operator U_i

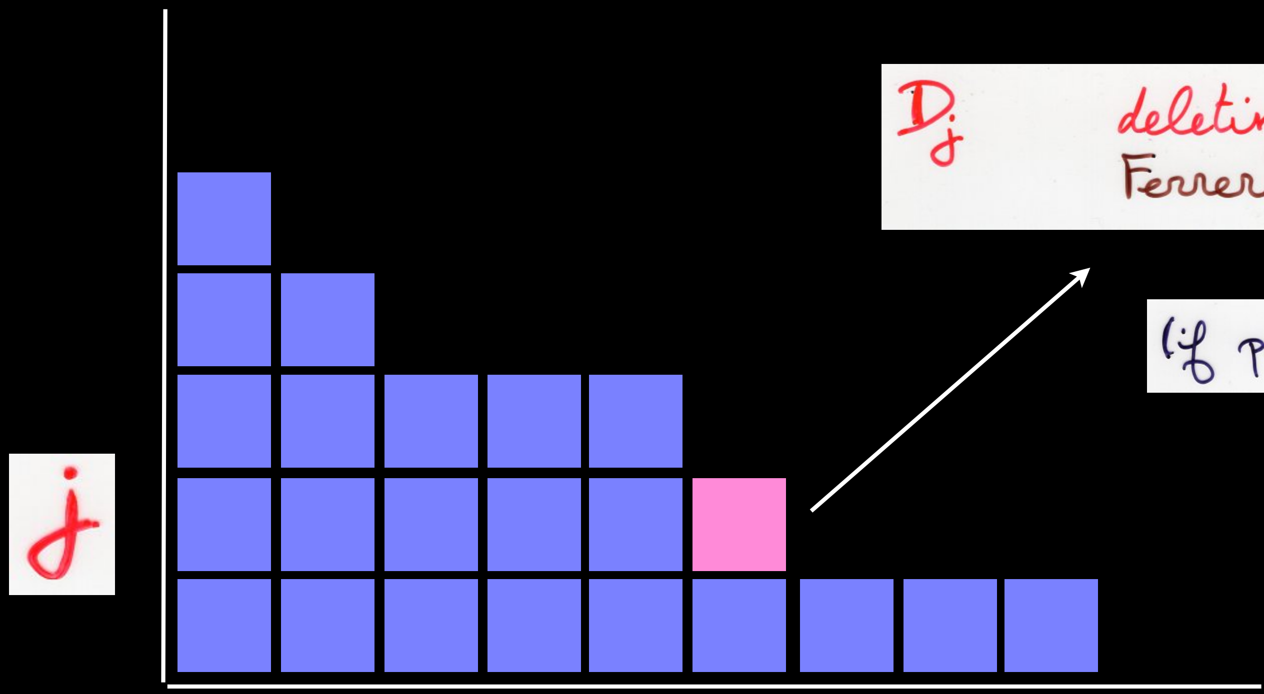
i



adding a cell in a Ferrers diagram ρ at row i

(if possible, else $U_i(\rho)=0$)

notations

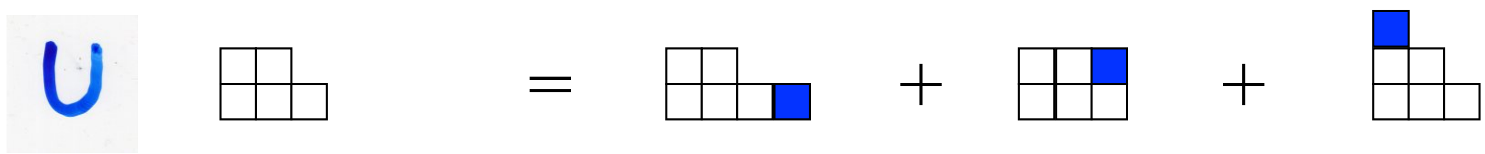


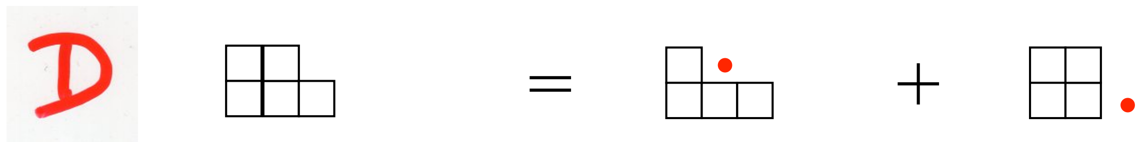
D_j deleting a cell in a Ferrers diagram ρ at row j

(if possible, else $D_j(\rho) = 0$)

$$U = \sum_{i \geq 1} U_i \qquad\qquad D = \sum_{i \geq 1} D_i$$

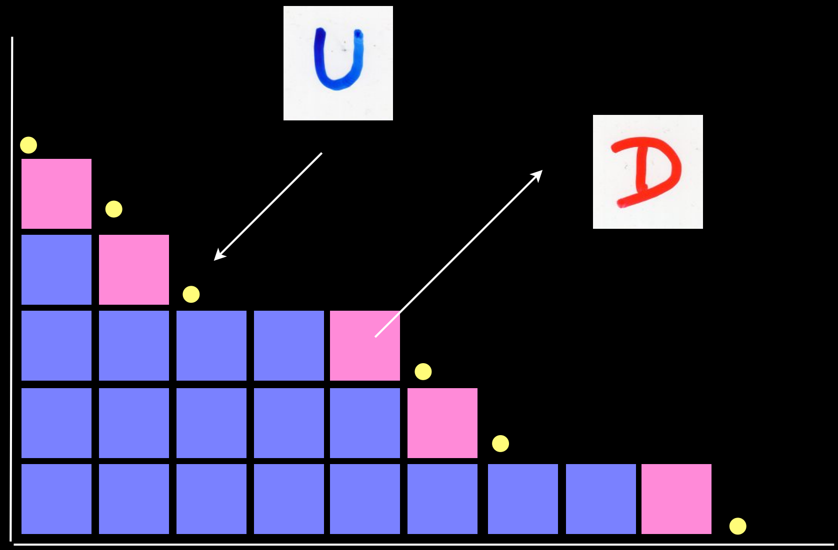
U and D are operators acting on the vector space generated by Ferrers diagrams





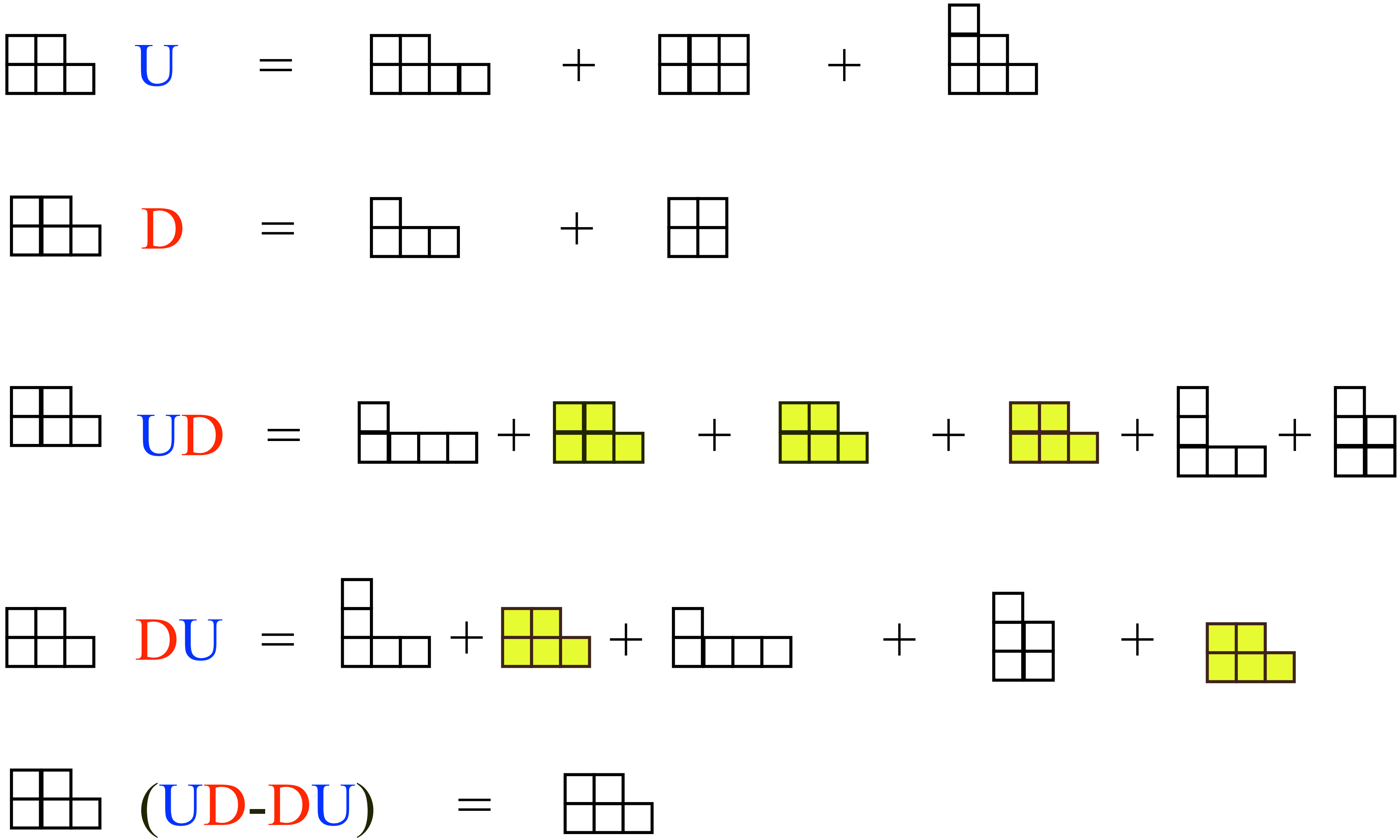
$$UD = DU + Id$$

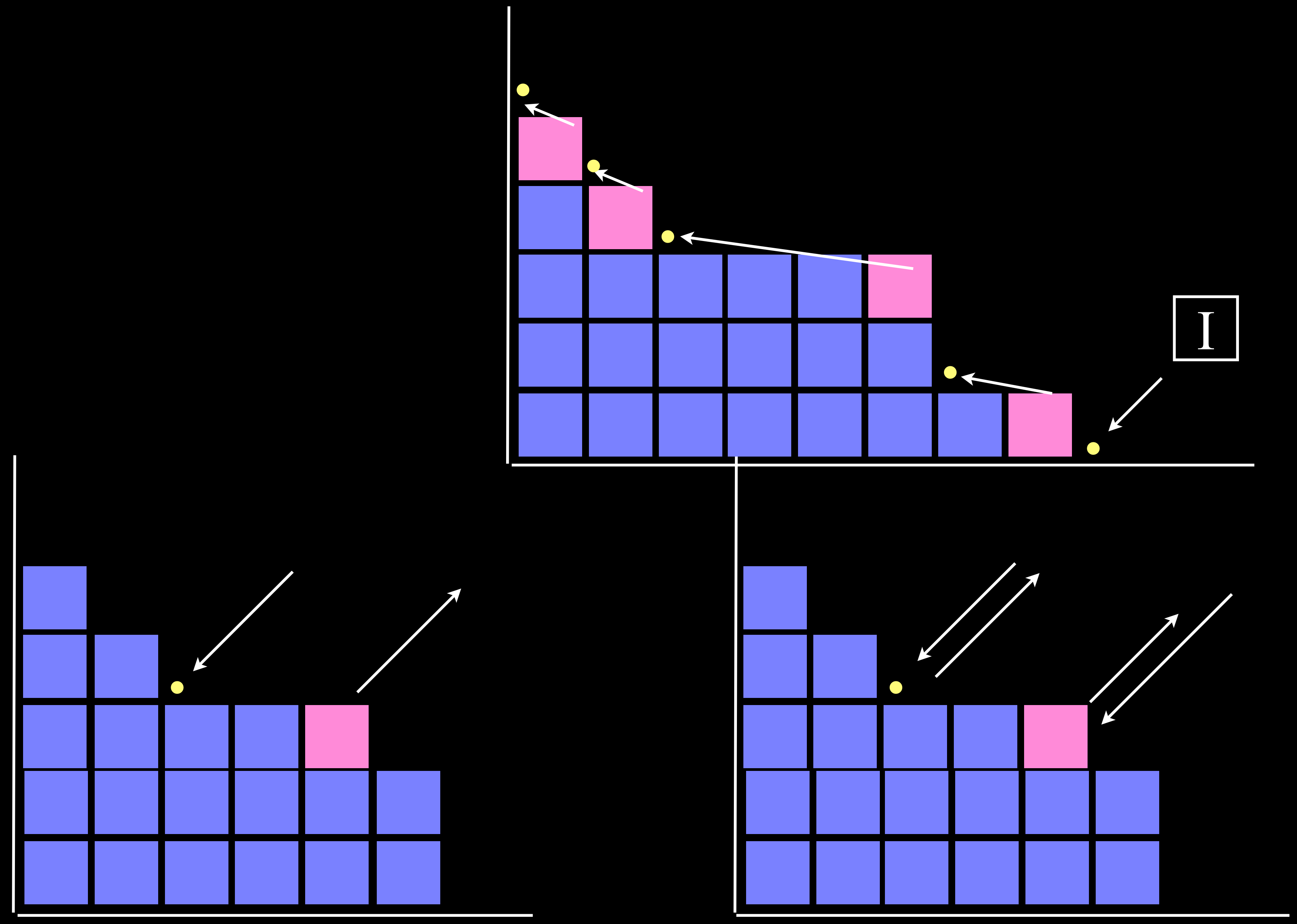
operators U and D



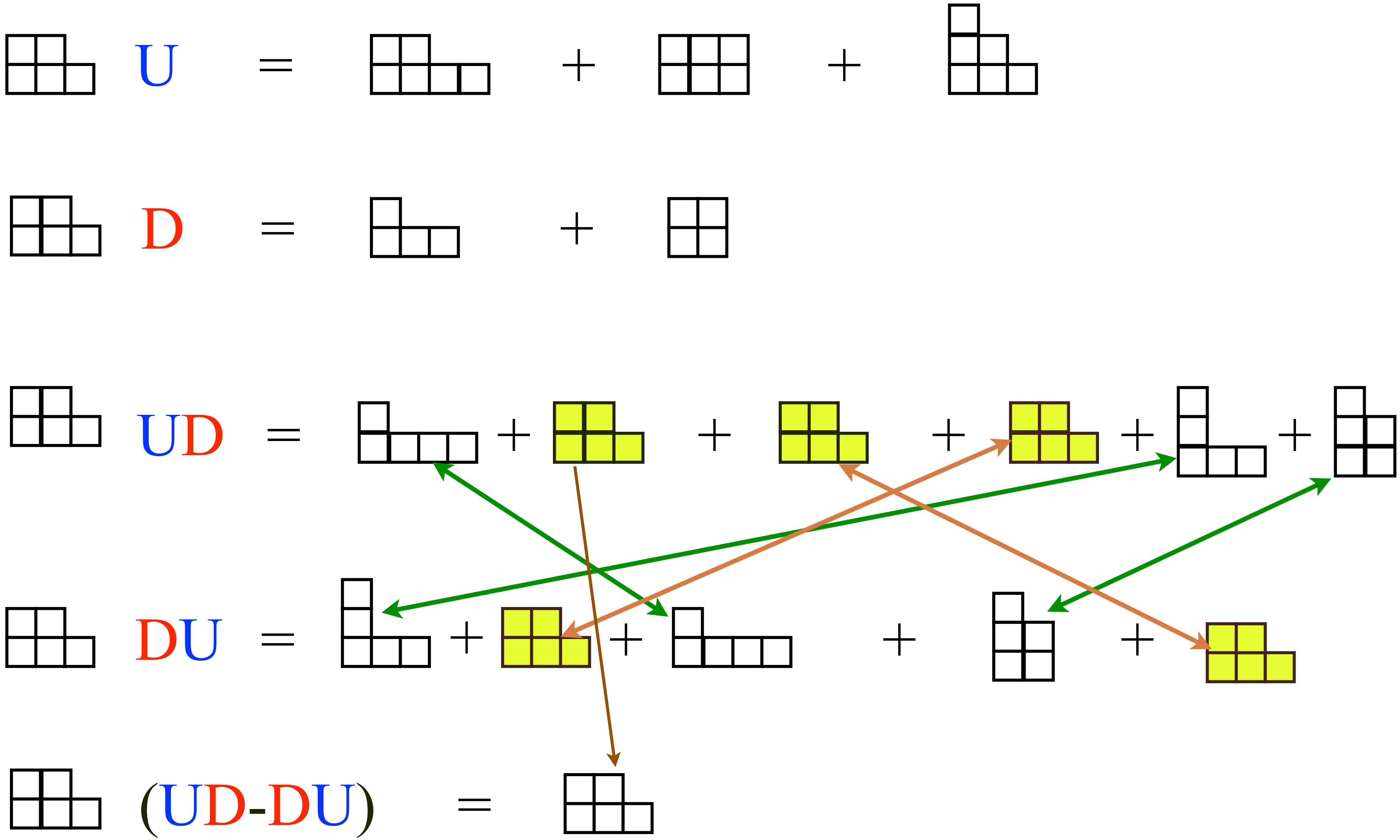
Young lattice

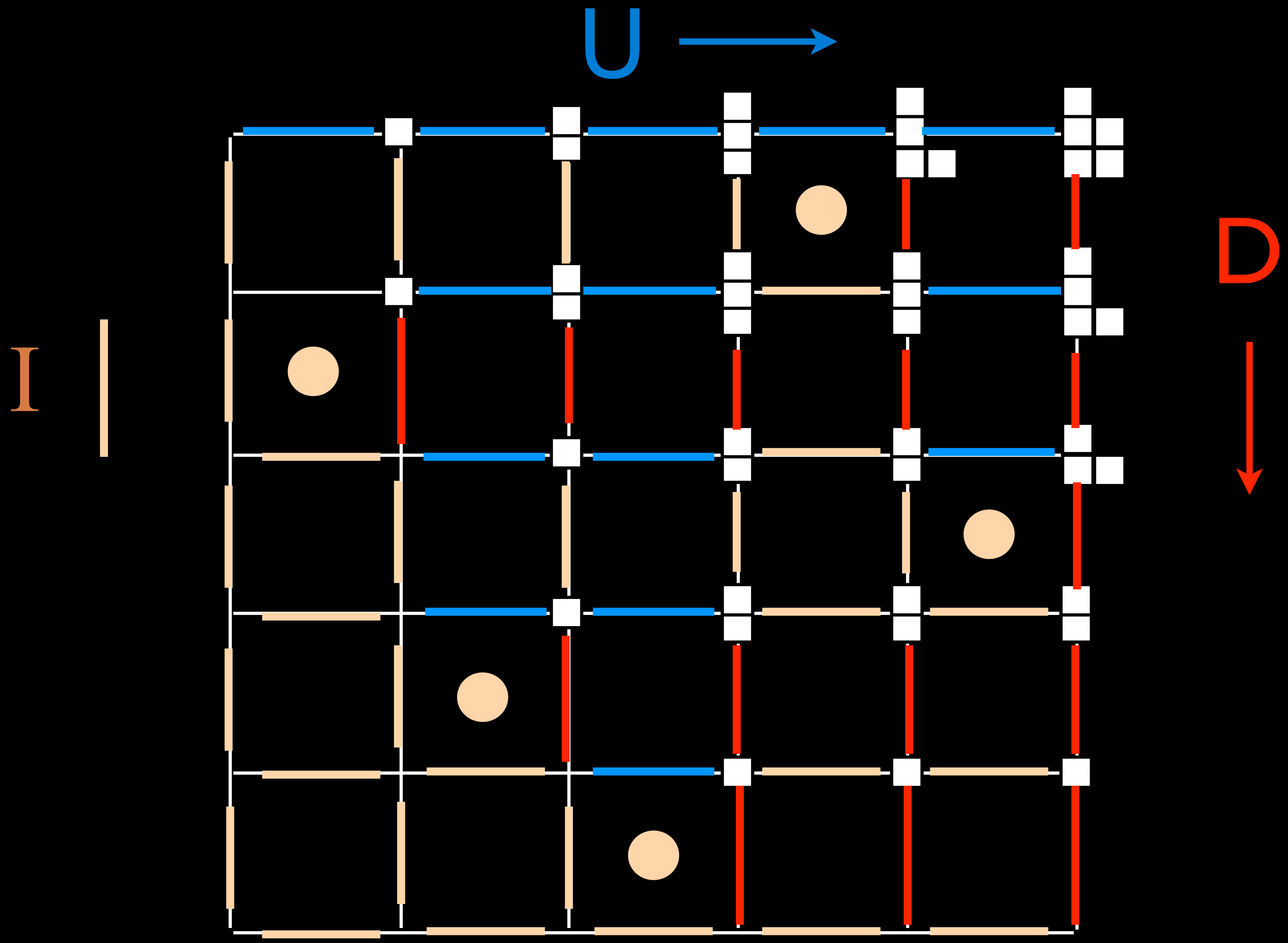
U adding / D deleting a cell in a Ferrers diagram


U
D
UD
DU
(UD-DU)



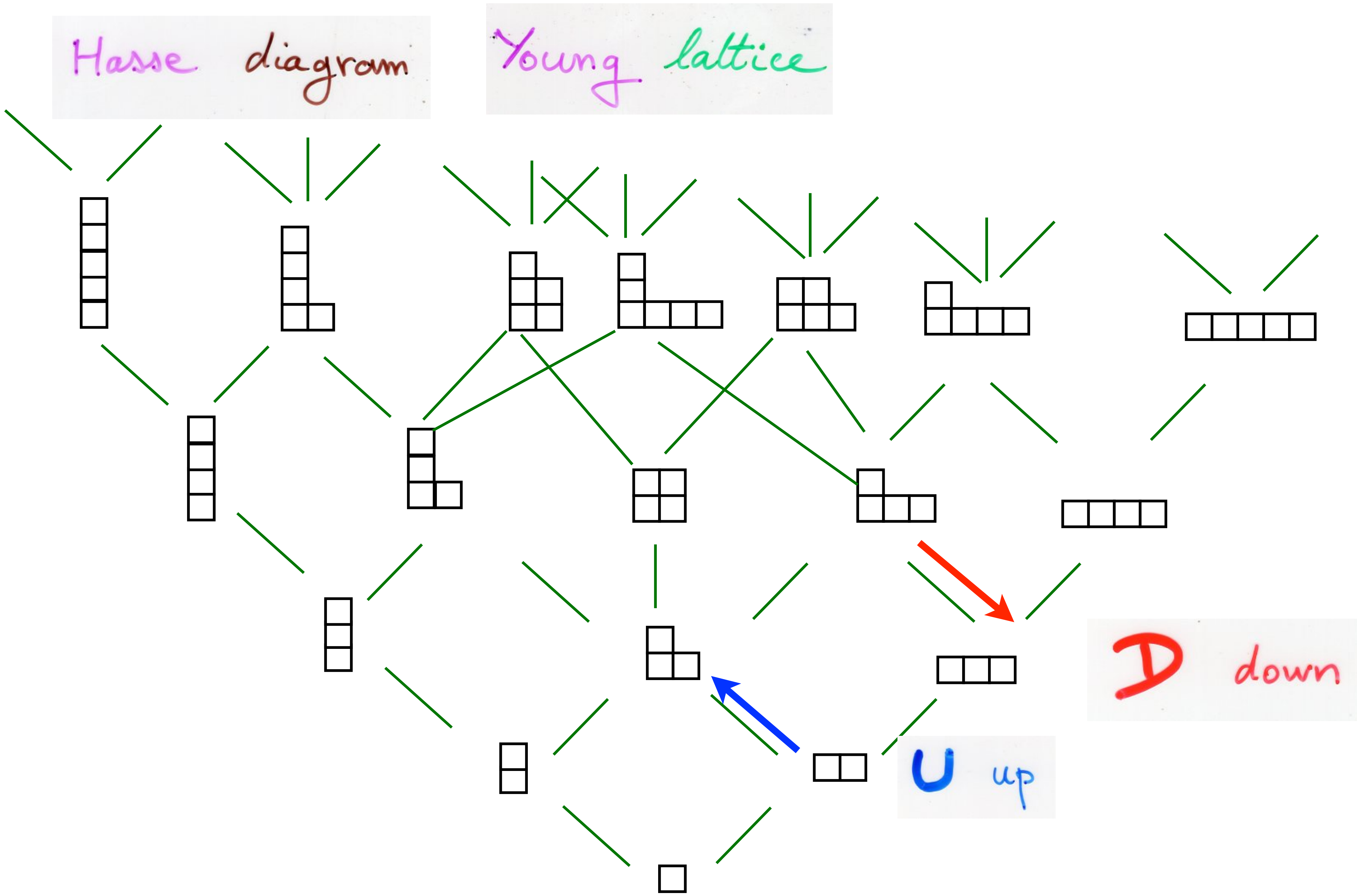
I


U
D
UD
DU
(UD-DU)



Fomin's "local rules"
"growth diagrams"

I



Hasse diagram
Young lattice
D down
U up

differential poset

Fomin (1992, 1995)

Stanley (1988, 1990)

Roby (1991)

$$UD = DU + I$$

U up

D down

"The cellular ansatz"

X.V., ABjC III

quadratic algebra Q

$UD = DU + Id$

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

bijections

RSK ⟷

representation of Q by combinatorial operators

pairs of Young tableaux

(i) first step

(ii) second step

quadratic algebra Q

Q-tableaux

$UD = DU + Id$

permutations

Planarization of the rewriting rules

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutations

$UD = DU + I_d$

commutations

$UD \to DU$ $UD \to I_d$

rewriting rules

homogenization of the system of commutations relations

$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases}$$

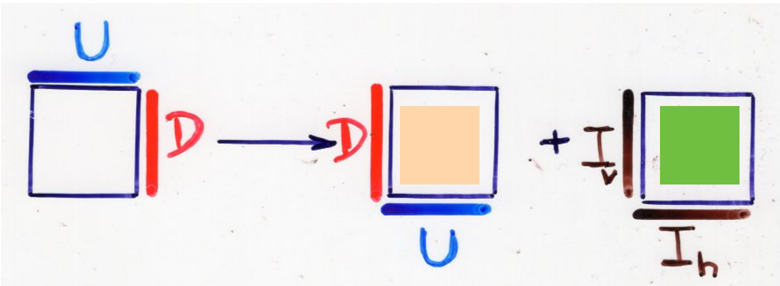
$UD \to I_v I_h$

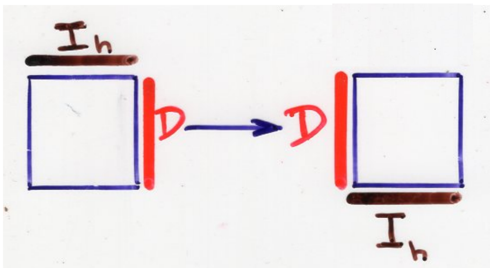
rewriting rules

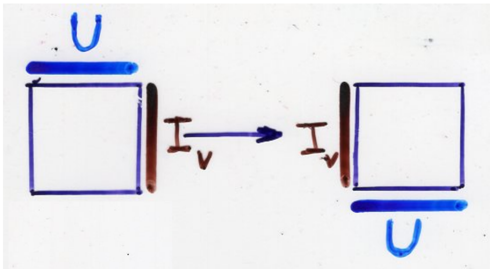
$$\begin{cases} UD \to DU \\ UI_v \to I_vU \\ I_hD \to DI_h \\ I_hI_v \to I_vI_h \end{cases} \qquad UD \to I_vI_h$$

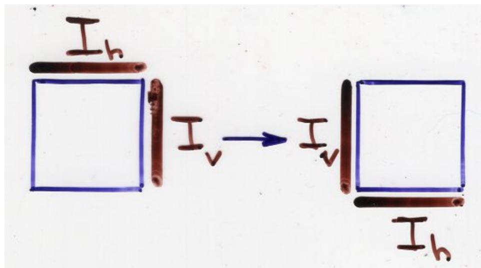
rewriting rules

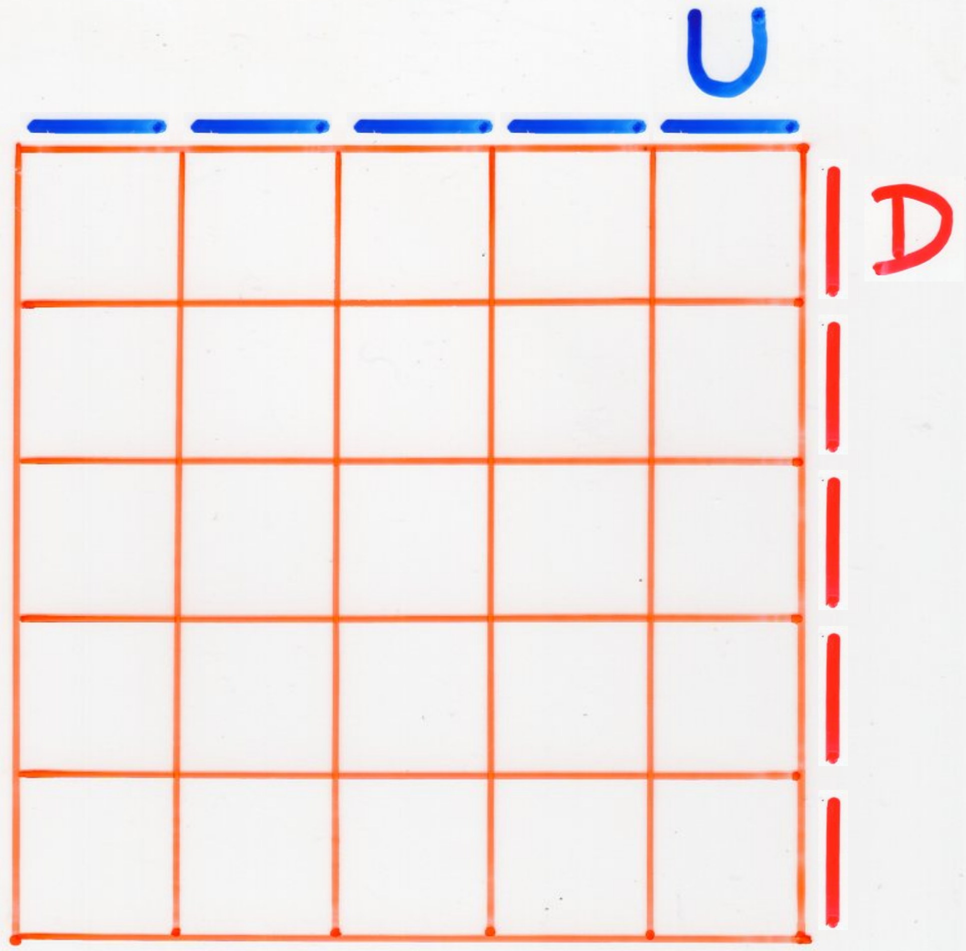
planarization of the rewriting rules

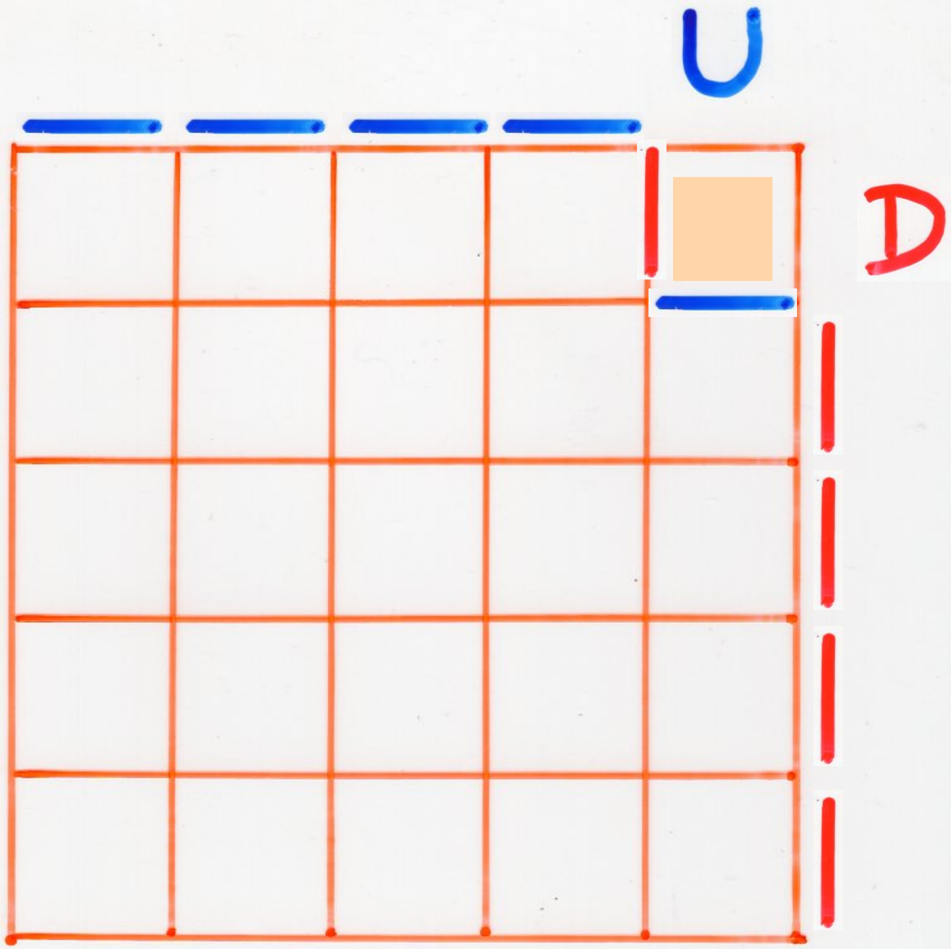


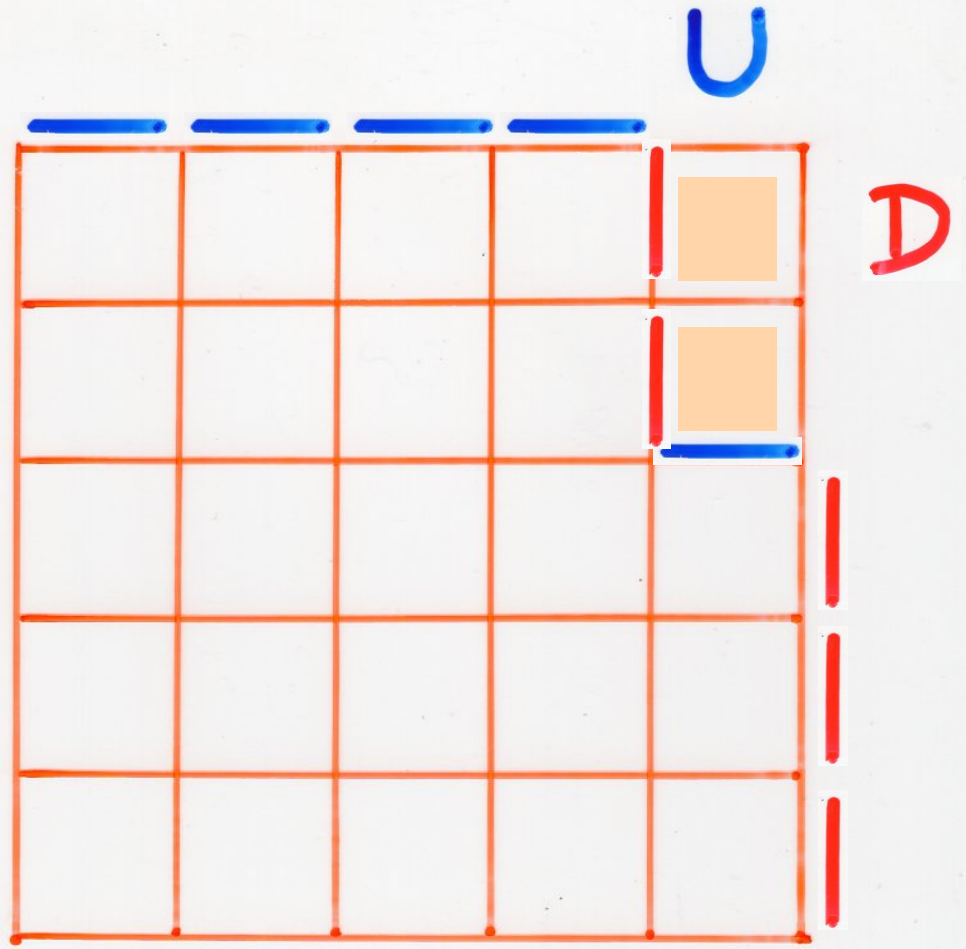


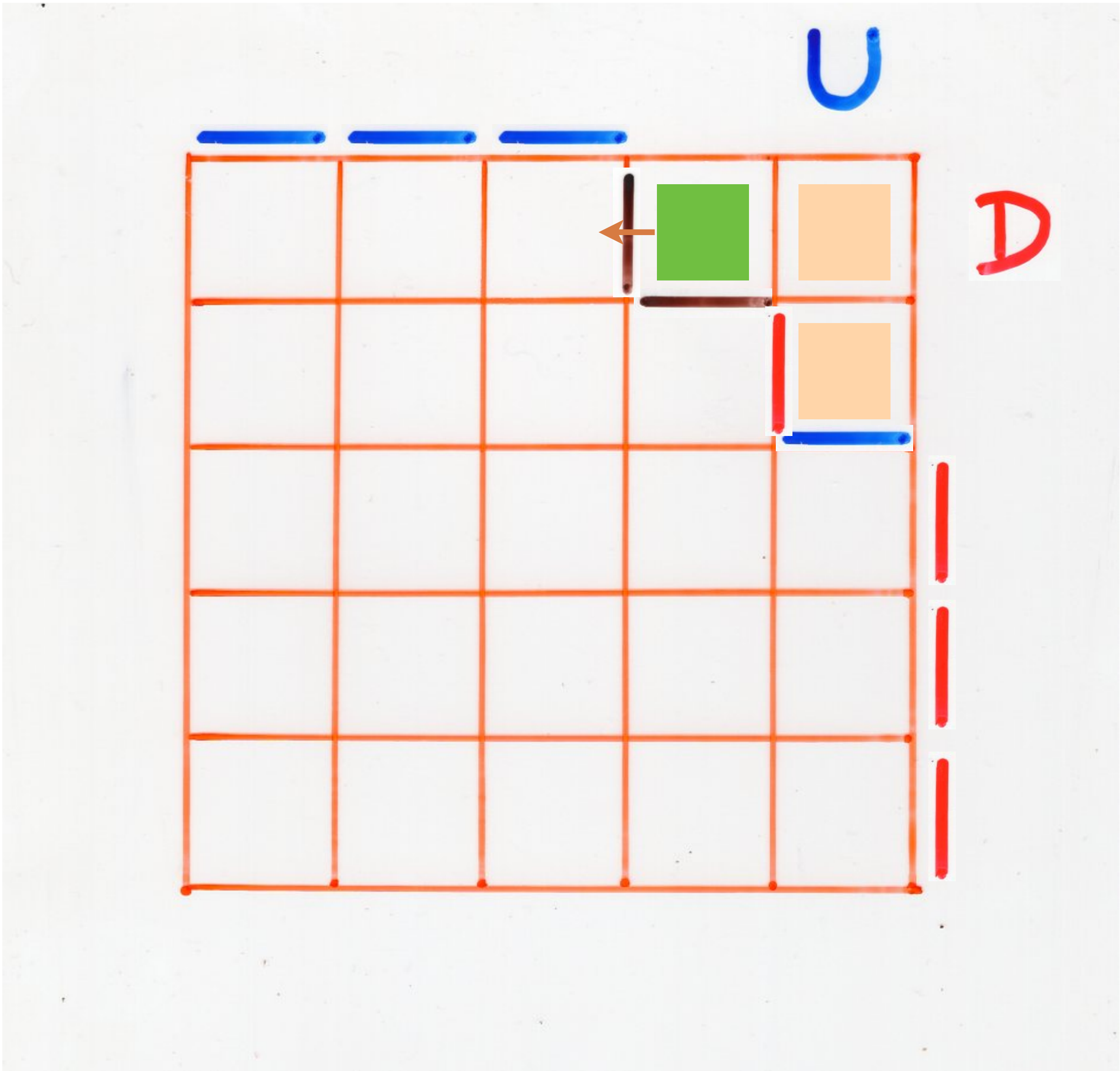





U
D

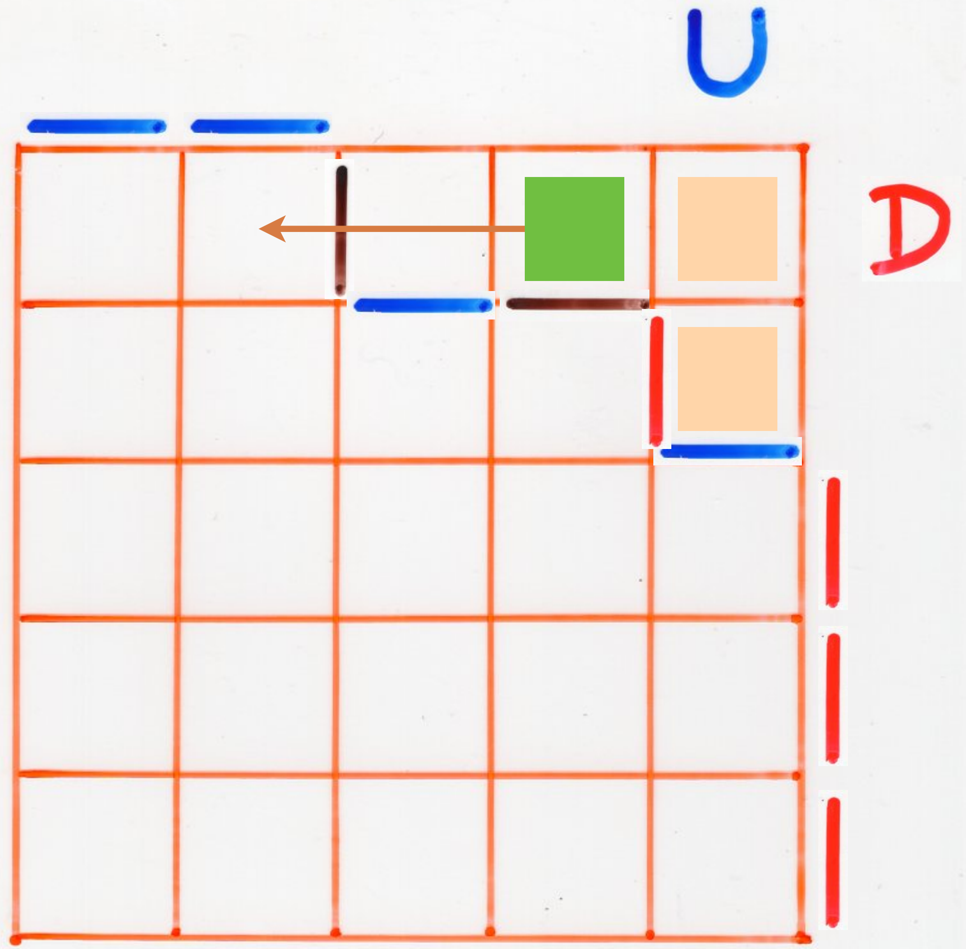

U
D

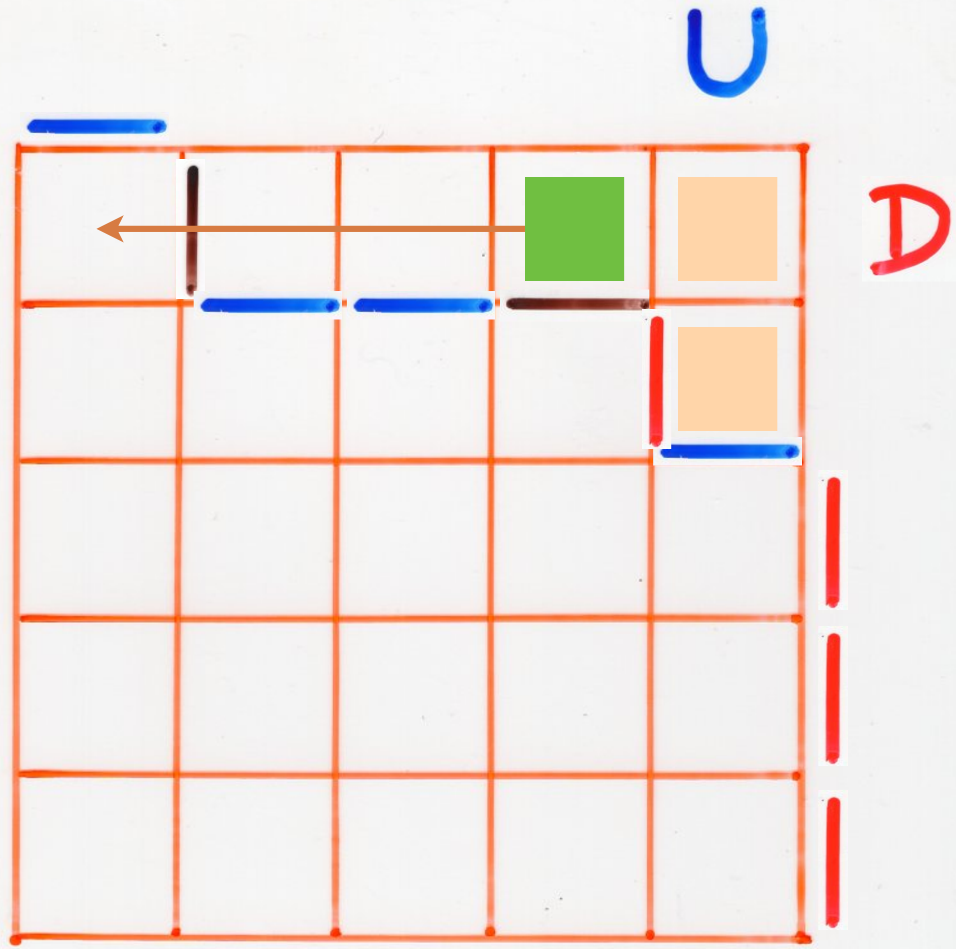

U
D

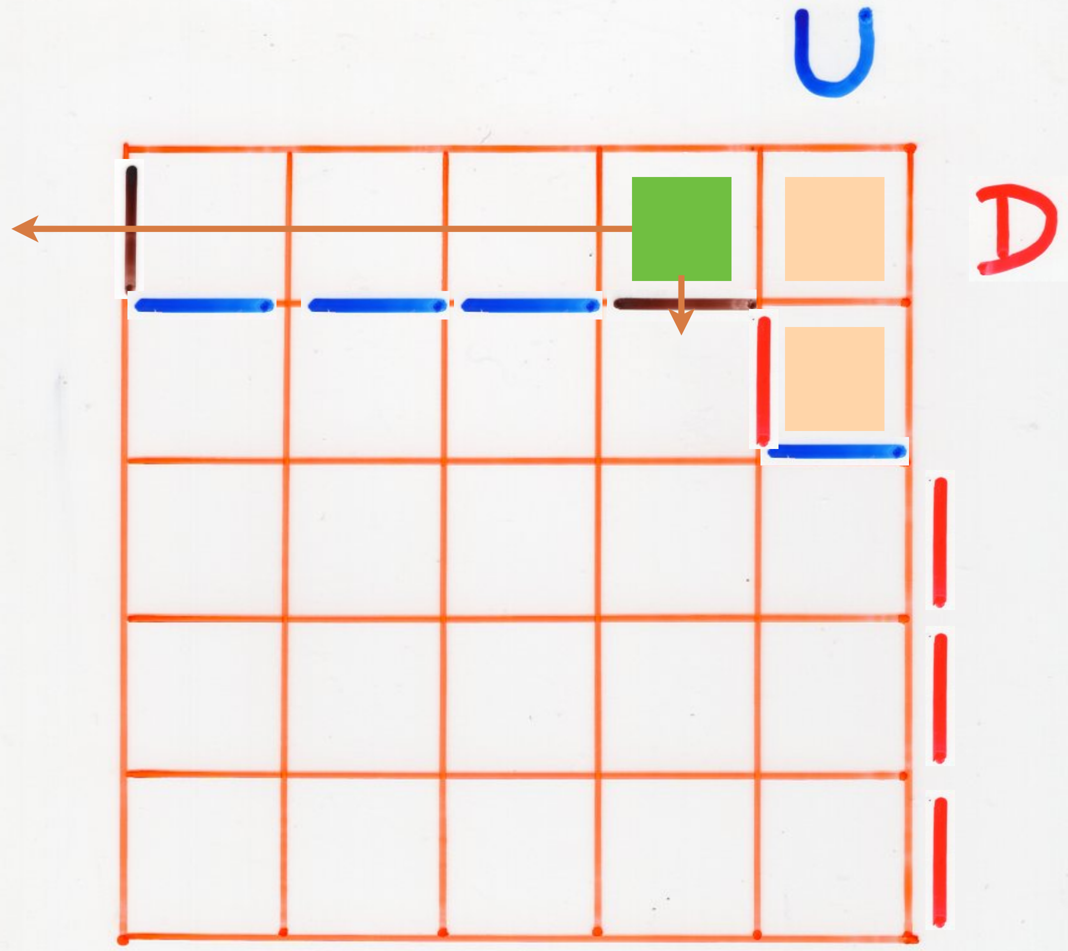


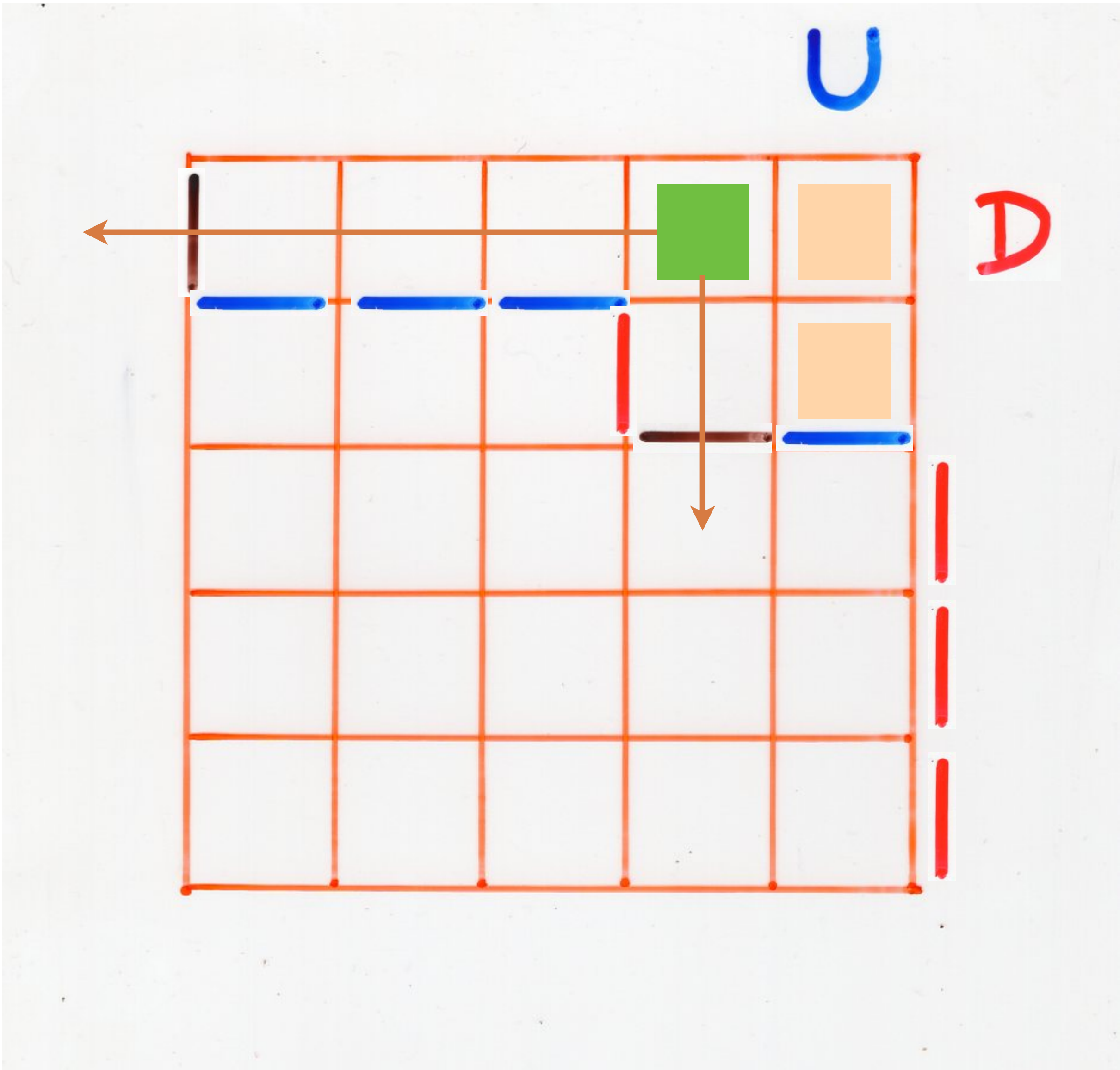
U
D

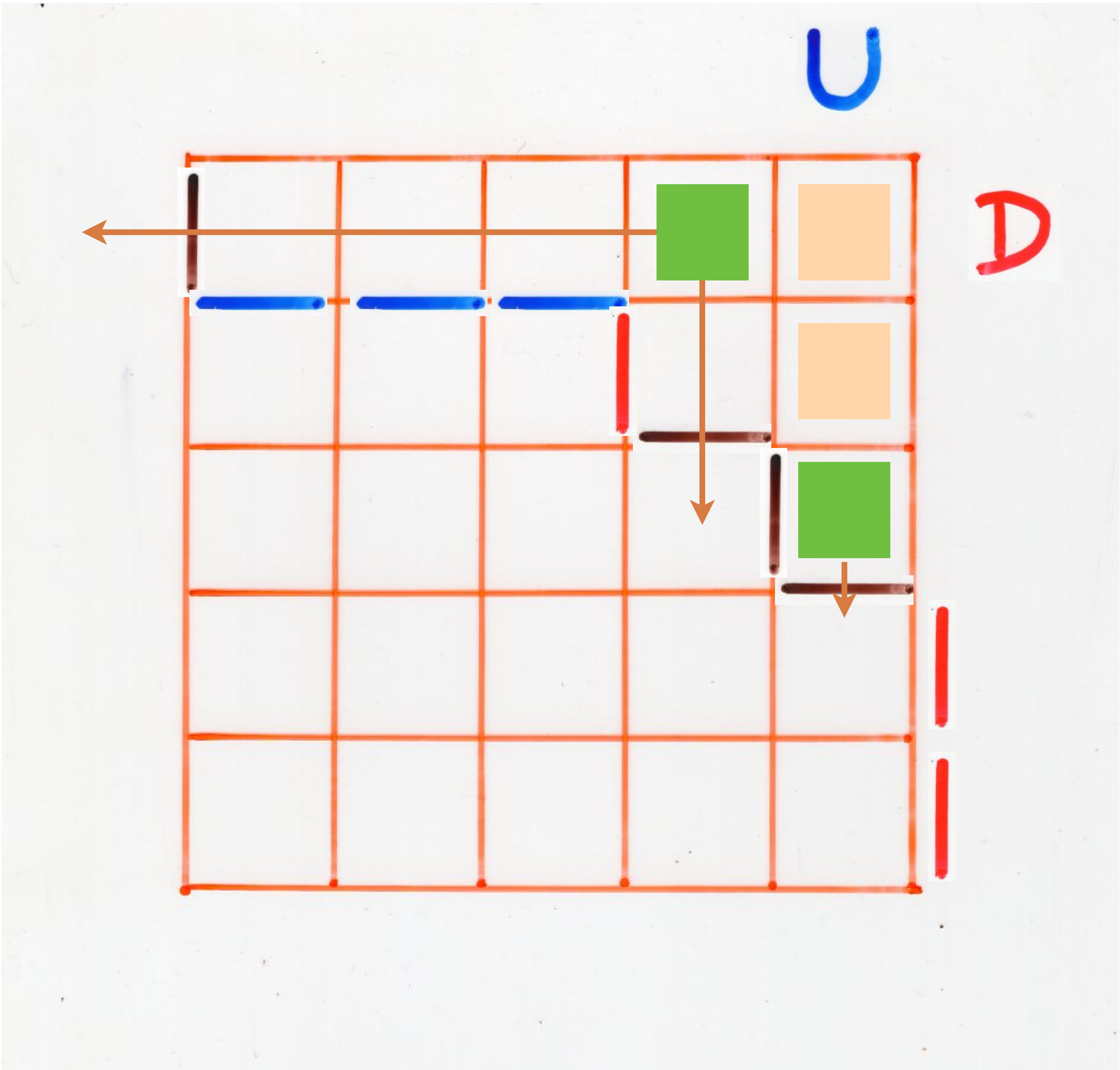
I_h
I_v

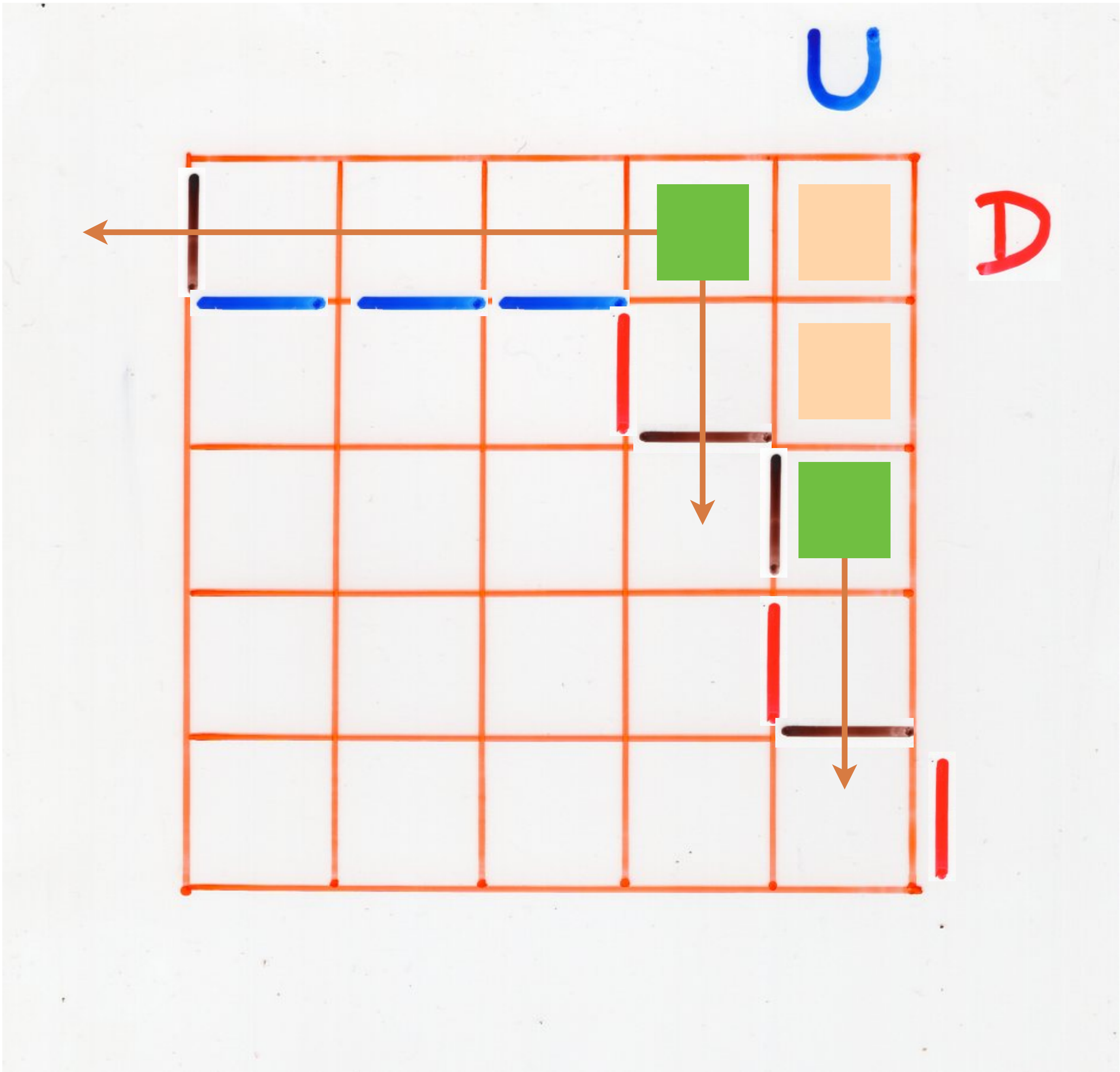

U
D


U
D


U
D

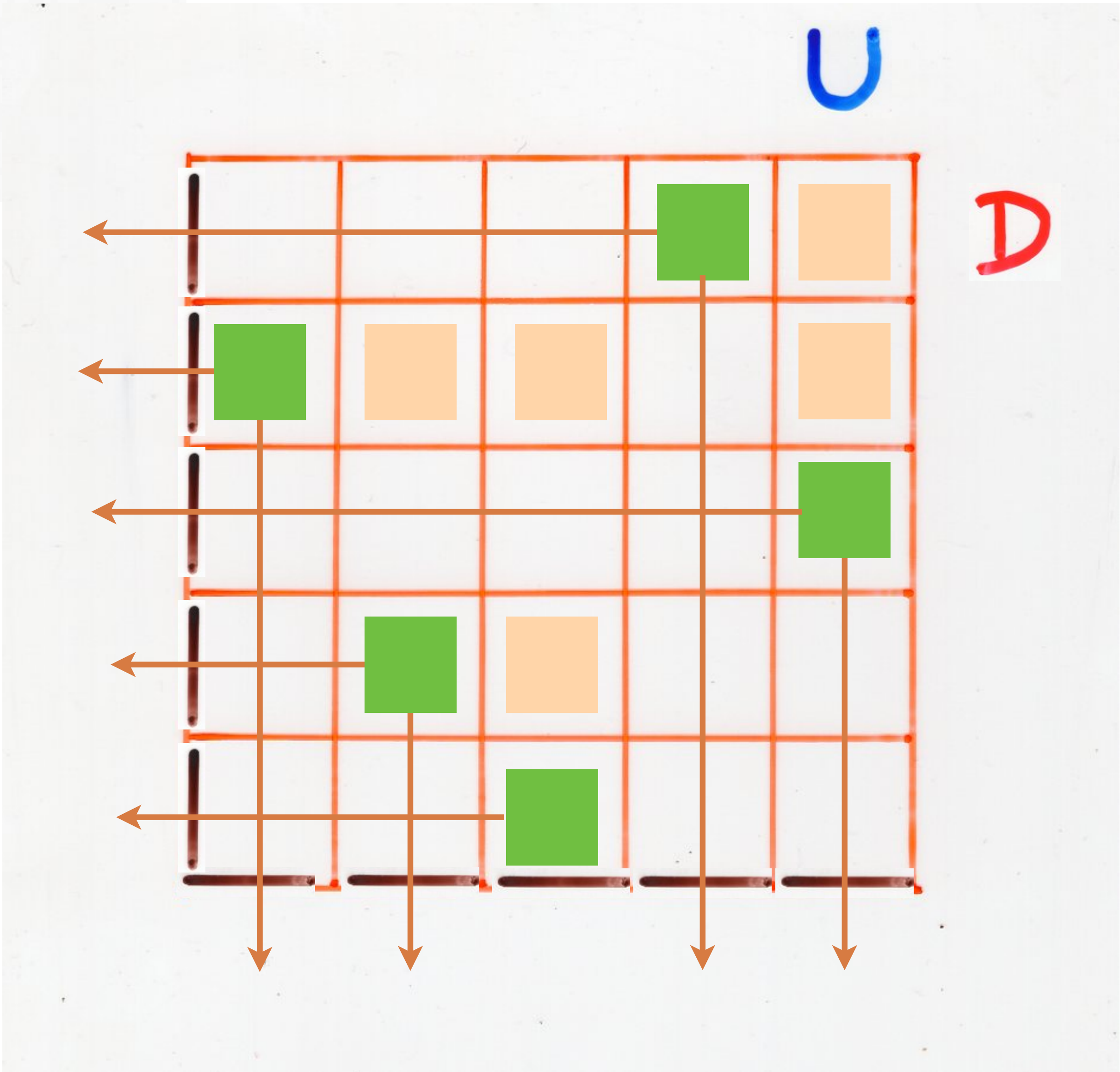

U
D

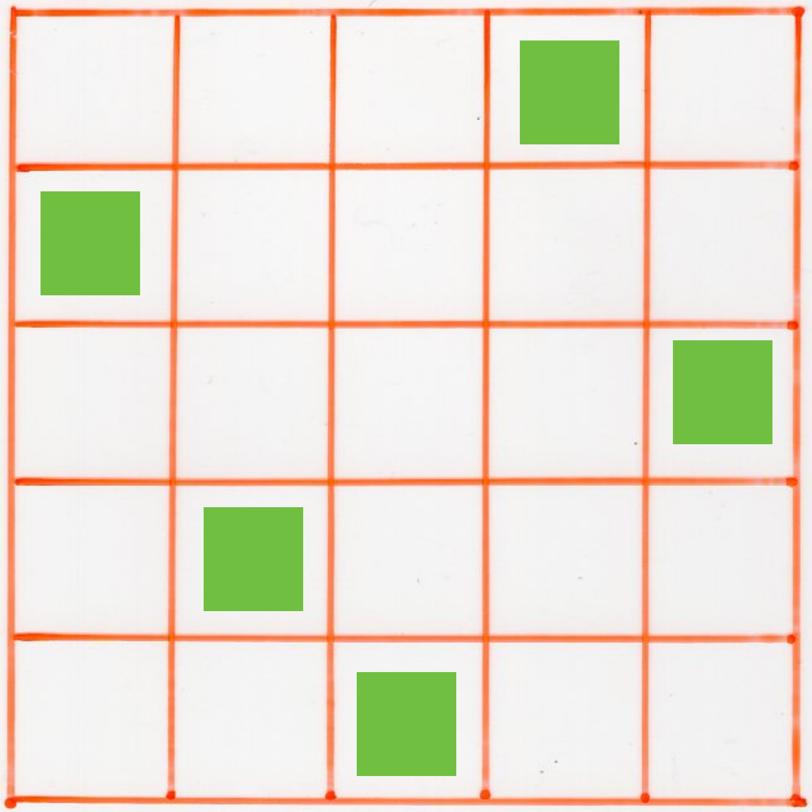

U
D


U
D

$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

complete Q-tableau





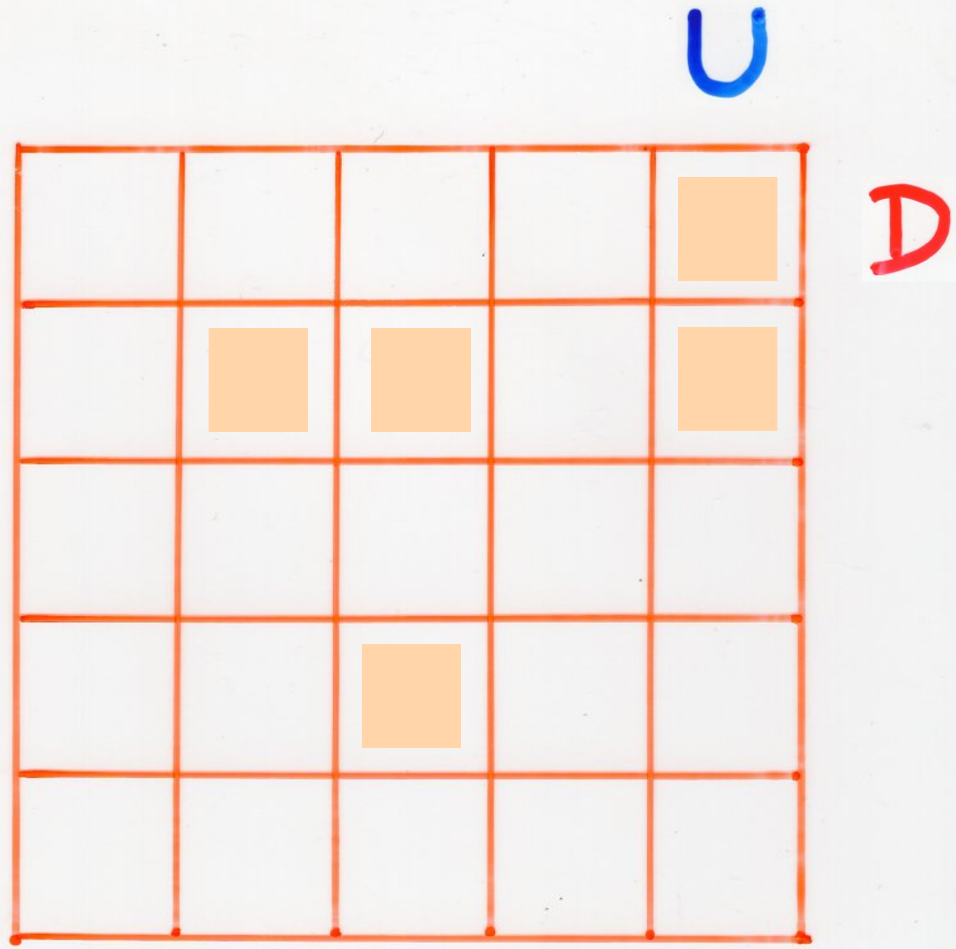
quadratic algebra **Q**

$UD = DU + Id$

permutation as a **Q**-tableau

$$UD = q\,DU + I_d$$

$$UD \rightarrow DU$$



Rothe diagram of a Permutation (1800)

"The cellular ansatz"

quadratic algebra Q

UD = DU + Id

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

bijections

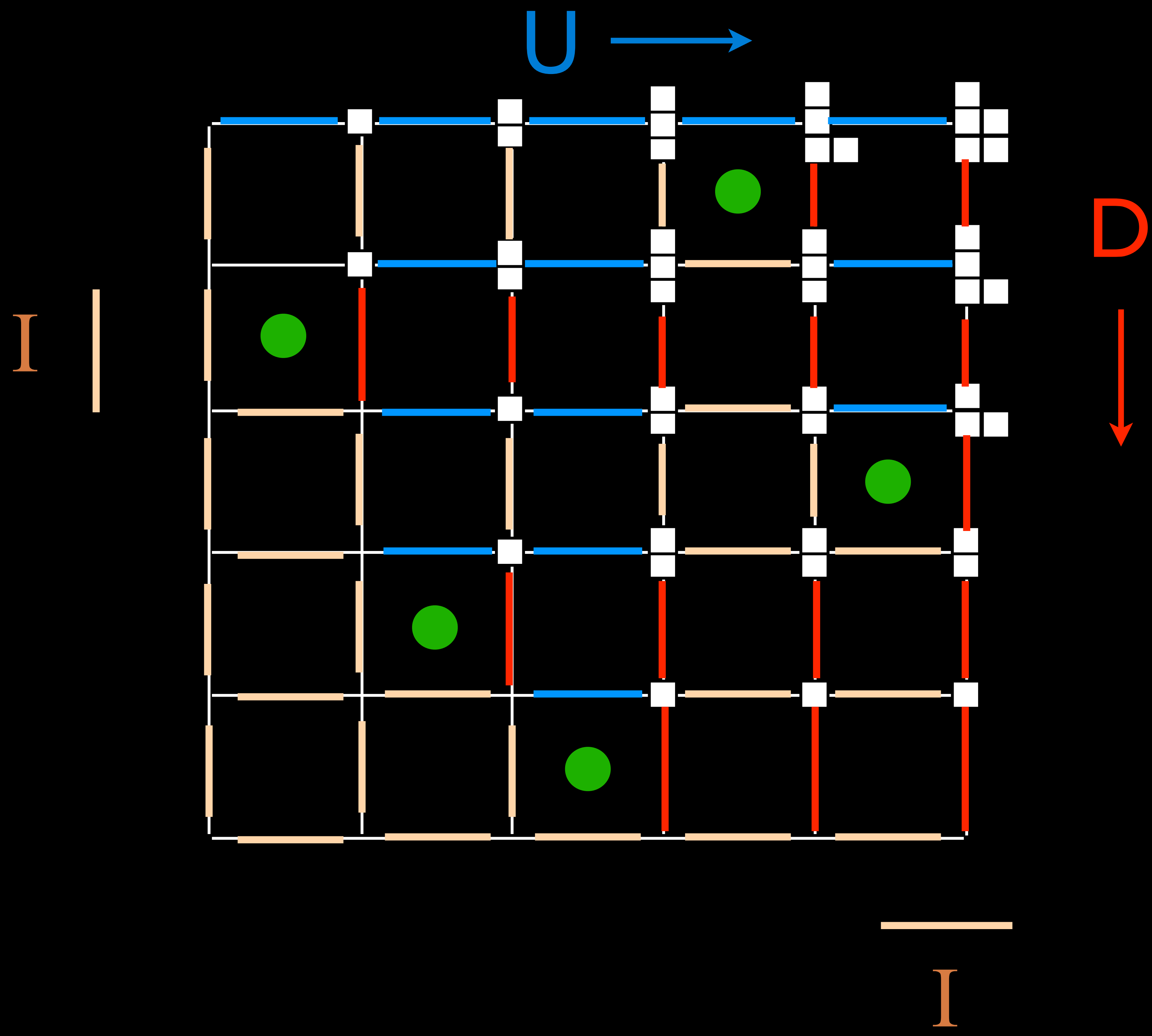
RSK ⟷

representation of Q by combinatorial operators

pairs of Young tableaux

(i) first step

(ii) second step



U
D
I
I

"The cellular ansatz"

quadratic algebra Q

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Q-tableaux

combinatorial objects on a 2D lattice

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RSK ⟷

pairs of Young tableaux

(i) first step

(ii) second step

$DE = qED + E + D$

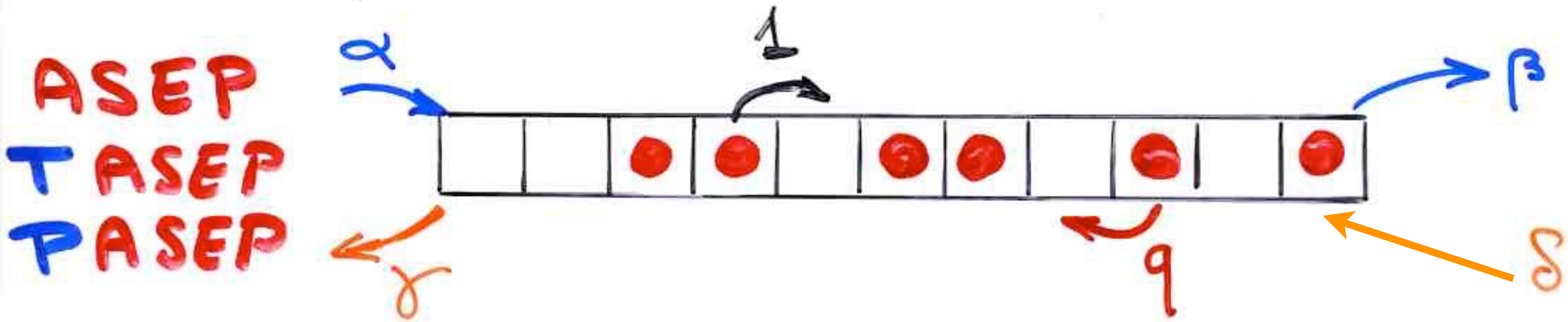
alternative tableaux

EXF ⟷

permutations

The PASEP

toy model in the physics of
dynamical systems far from equilibrium

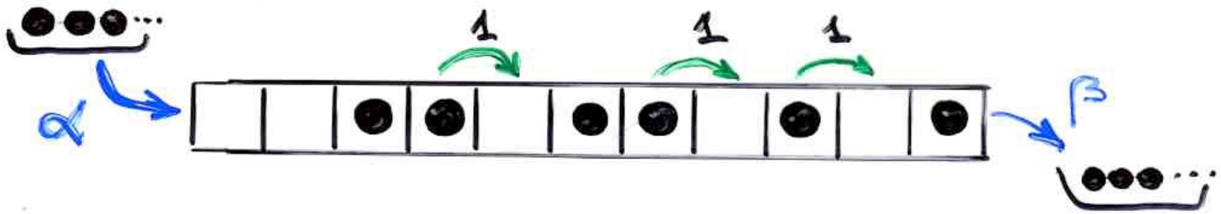


computation of the
"stationary probabilities"

$q = 0$ TASEP (α, β)

TASEP

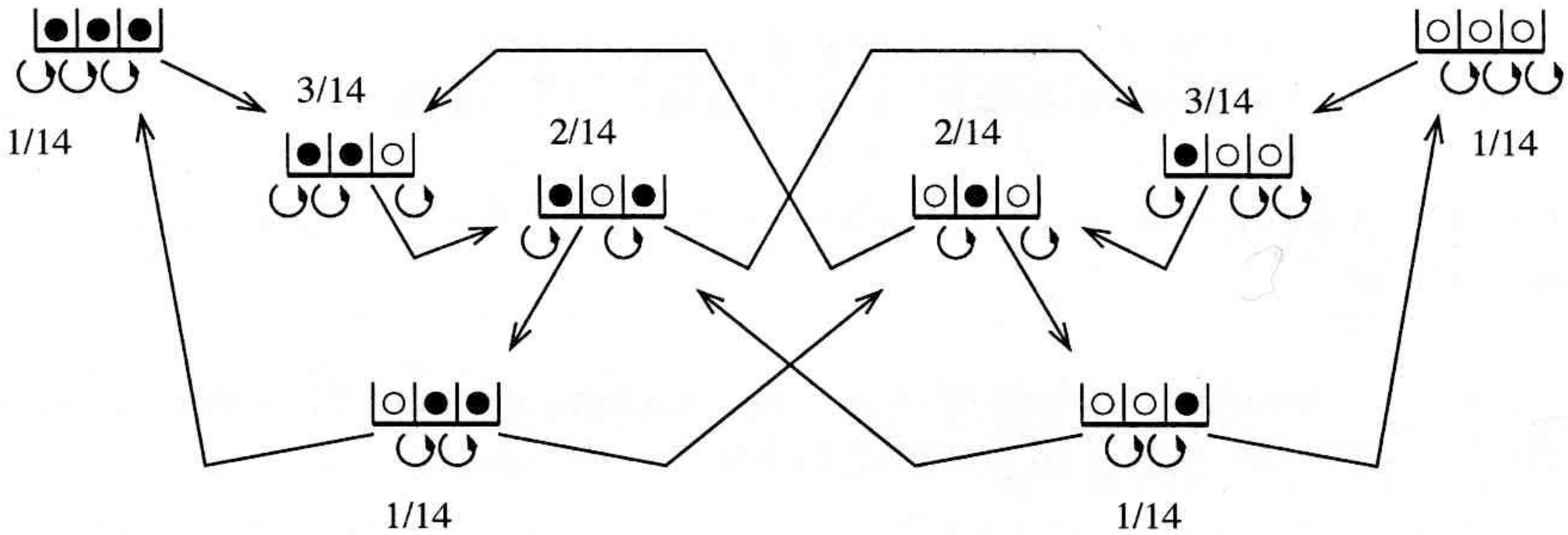
"totally asymmetric exclusion process"

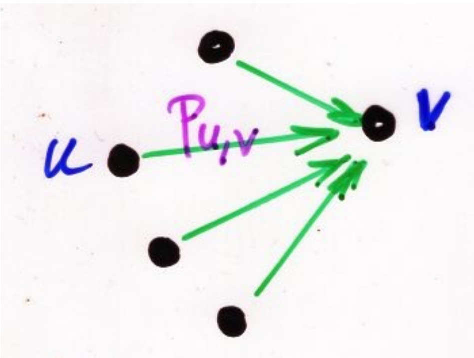


$q = 0$ TASEP (α, β)

$\gamma = \delta = 0$ $\alpha = \beta = 1$

"stationary probabilities"





$$p_v^\infty = \sum_{u \in S} p_u^\infty \, p_{u,v}$$

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

colum vector V

row vector W

$$\begin{cases} DE = qED + E + D \\ \langle W | (\alpha E - \gamma D) = \langle W | \\ (\beta D - \delta E) | V \rangle = | V \rangle \end{cases}$$

The PASEP algebra

$$DE = qED + E + D$$

alternative tableaux

The PASEP algebra

$$DE = qED + E + D$$

$$DDE(DE)EDE$$

$$q\,DDE(ED)EDE \qquad DDE(E)EDE \qquad DDE(D)EDE$$

The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

tableau

unique

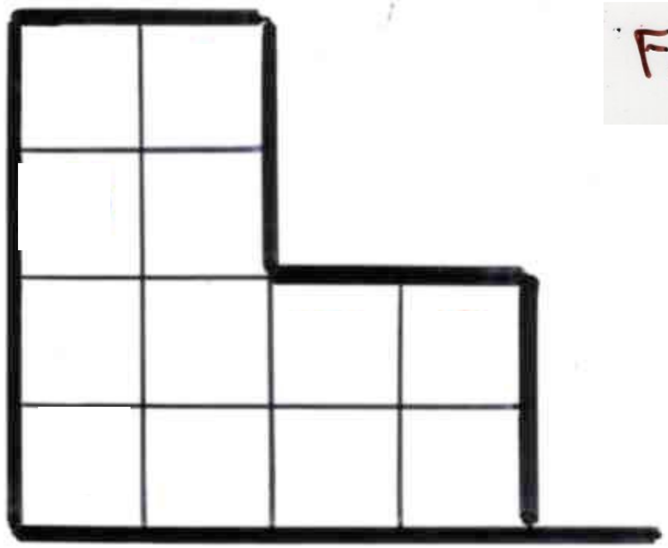
analog of the normal ordering

alternative tableaux

(X.V. 2008)

alternative tableau

Definition



Ferrers diagram **F**

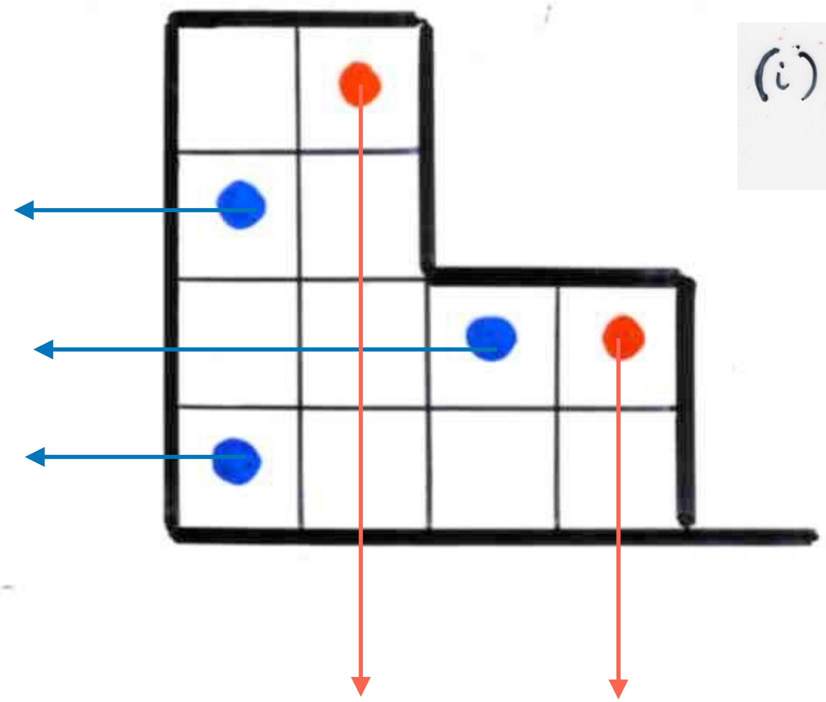
with possibly empty rows or columns

size of **F**

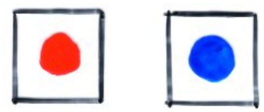
$\mathbf{n}$ = (number of rows) + (number of columns)

alternative tableau

Definition



(i) some cells are coloured red or blue



(ii) • no coloured cell at the left of a blue cell
• no coloured cell below a red cell

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

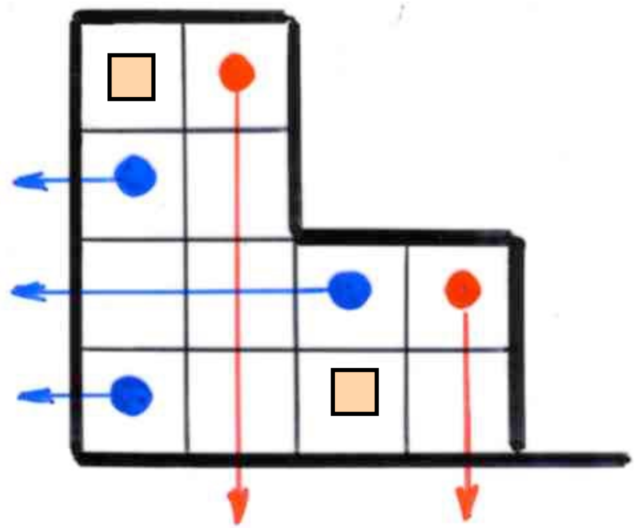
alternative tableau

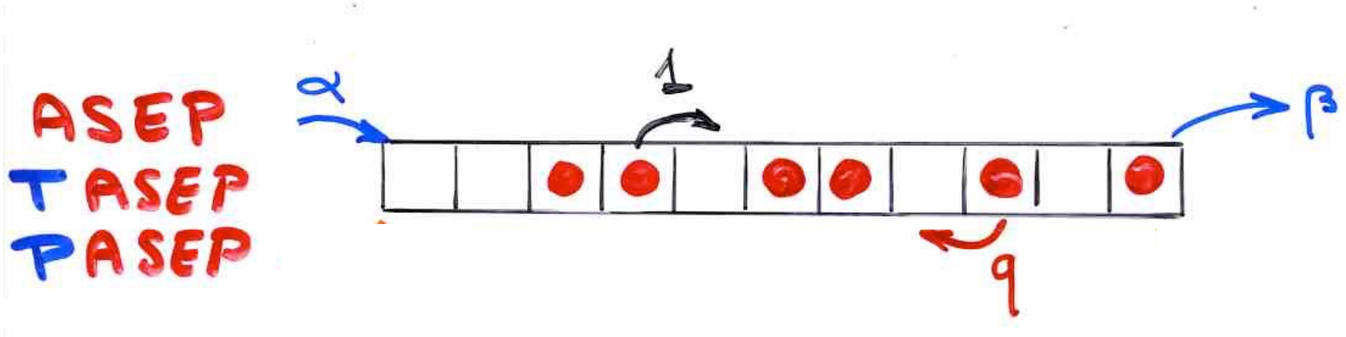
unique

$k(T)$ = nb of cells ☐

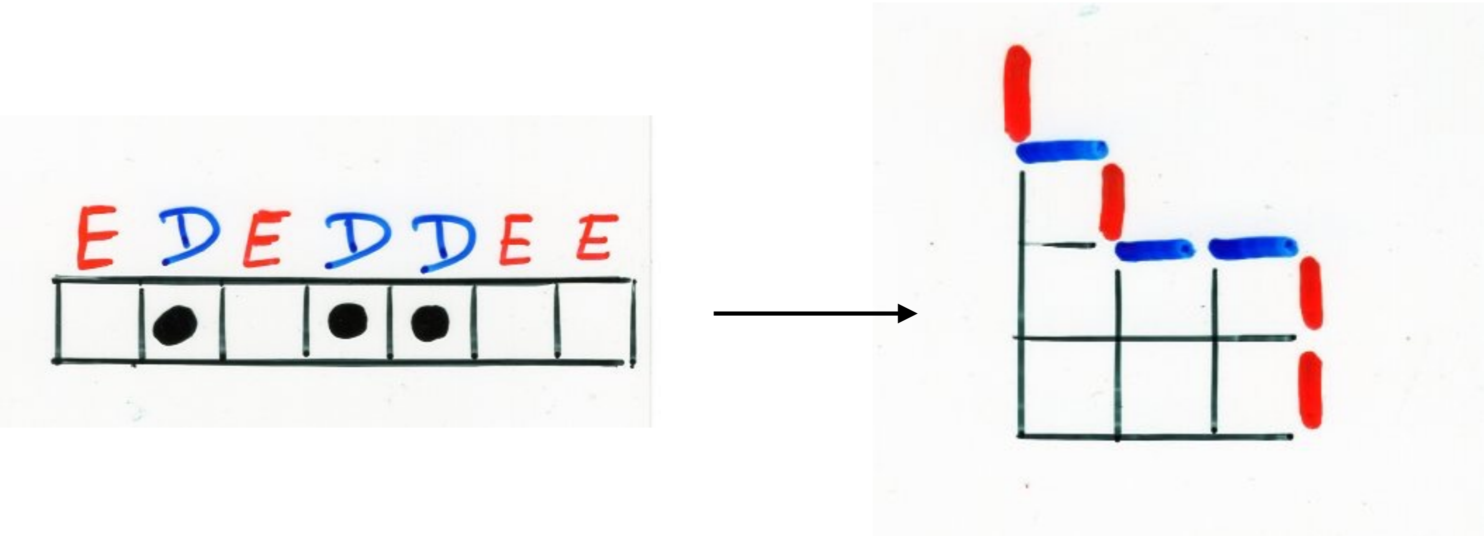
$i(T)$ = nb of rows without ●

$j(T)$ = nb of columns without ●





computation of the "stationary probabilities"



Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

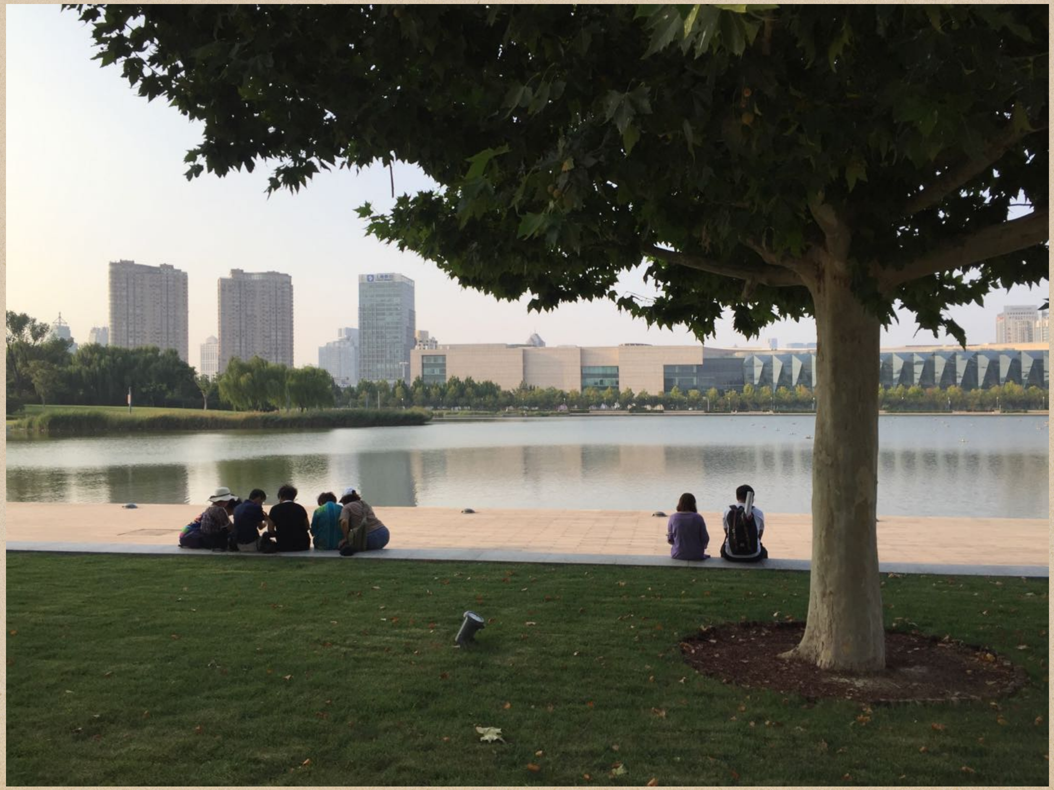
$k(T)$ = nb of cells ☐

$i(T)$ = nb of rows without ● (blue)

$j(T)$ = nb of columns without ● (red)

Alternative tableau as a Q-tableau

Idea of proof



The PASEP algebra

$$DE = qED + E + D$$

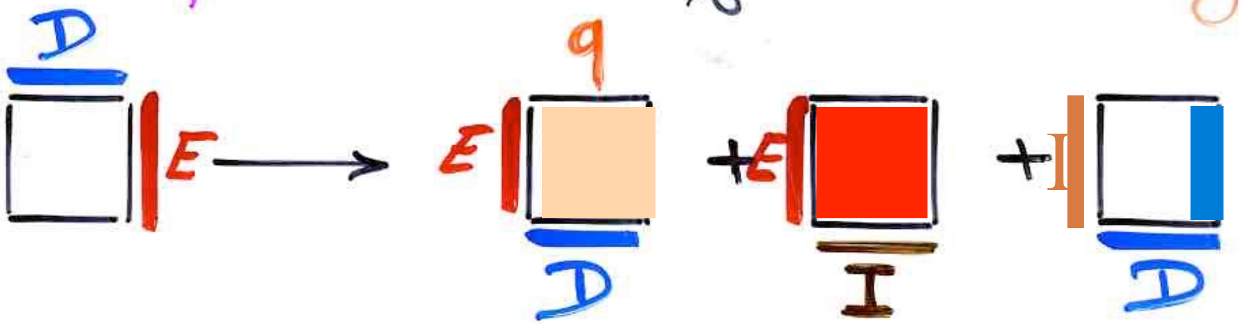
$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

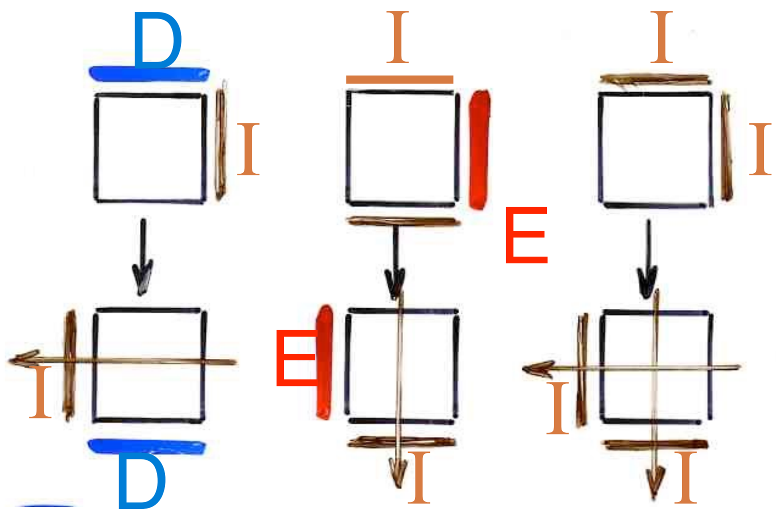
$$I_hE = EI_h$$

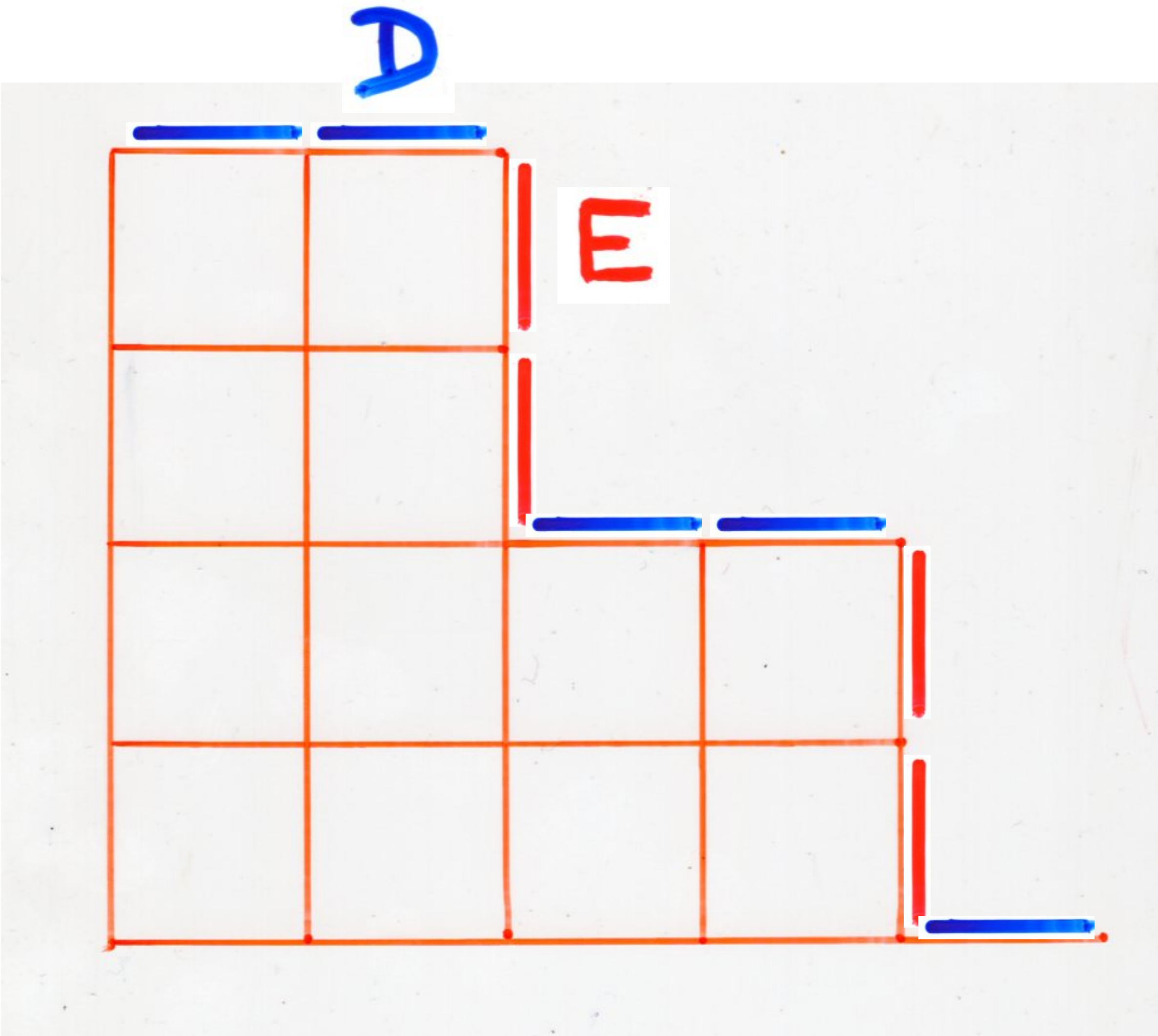
$$I_hI_v = I_vI_h$$

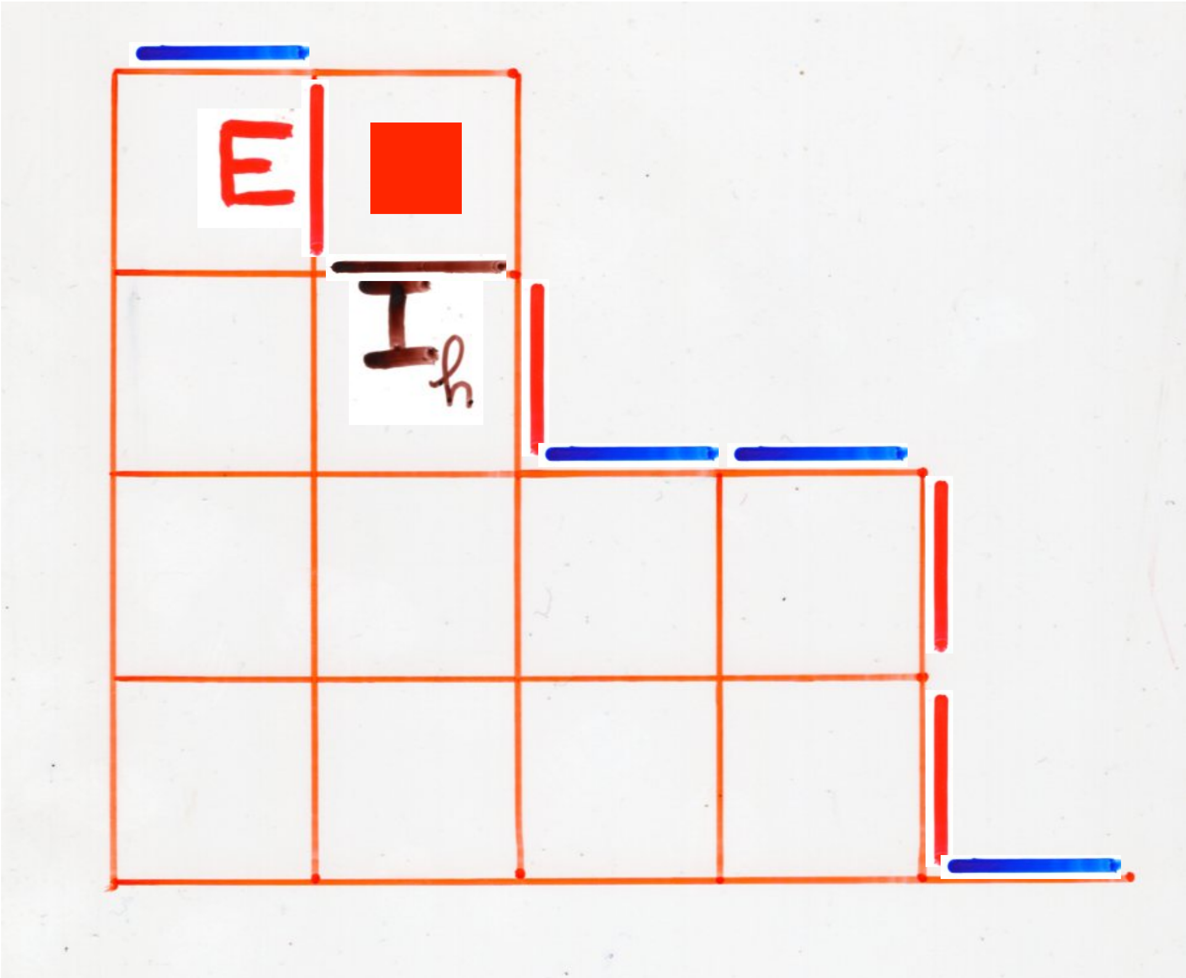
"planarization" of the rewriting rules

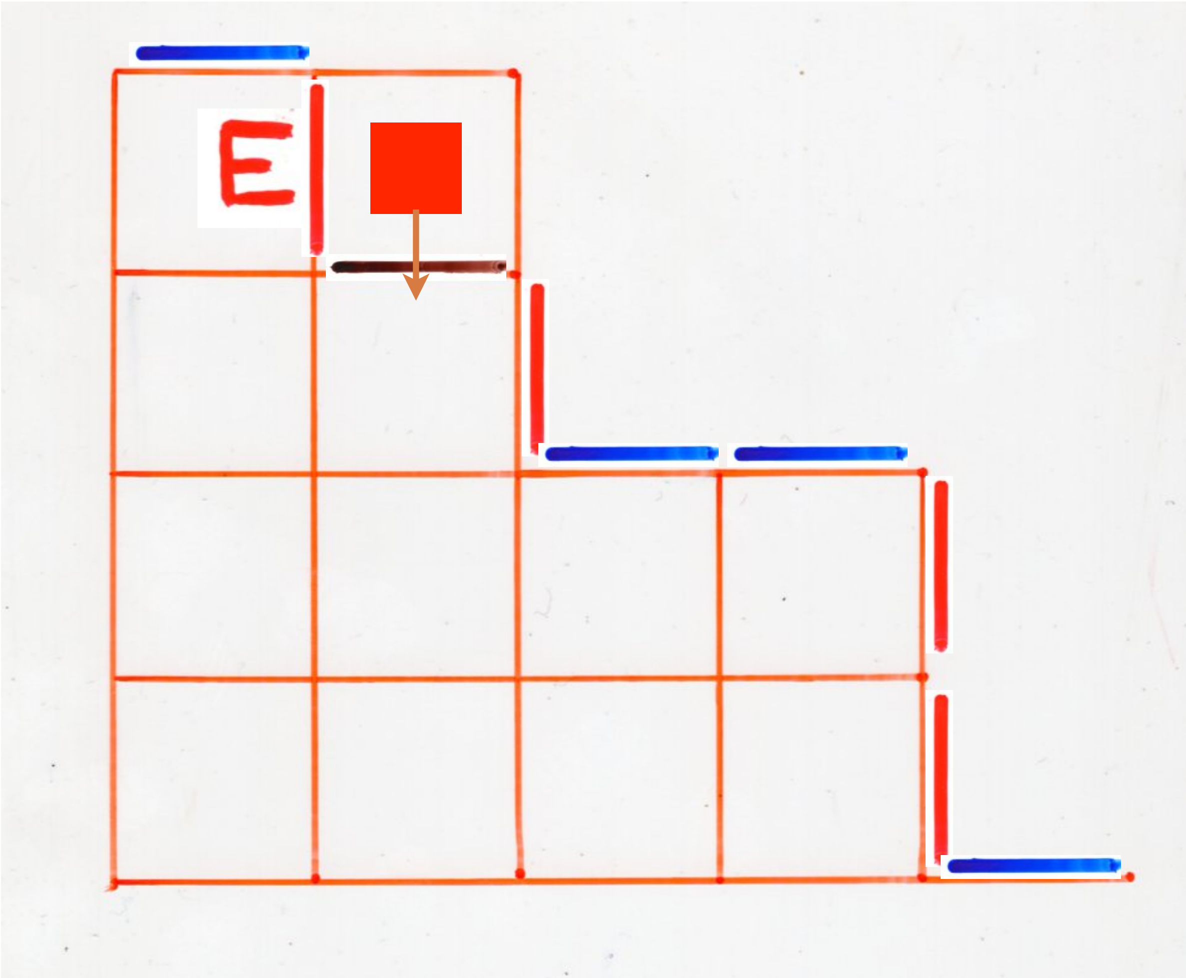


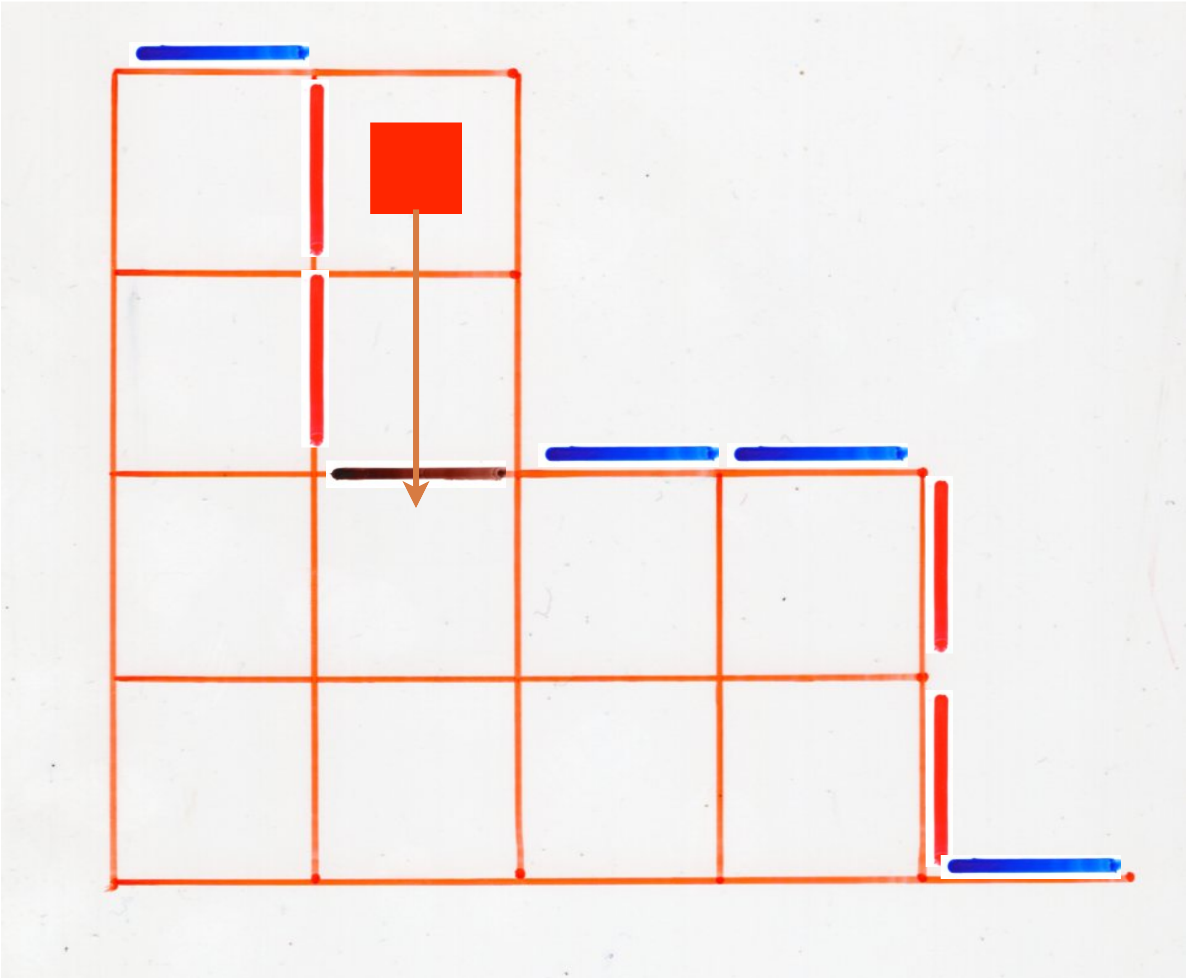
I identity

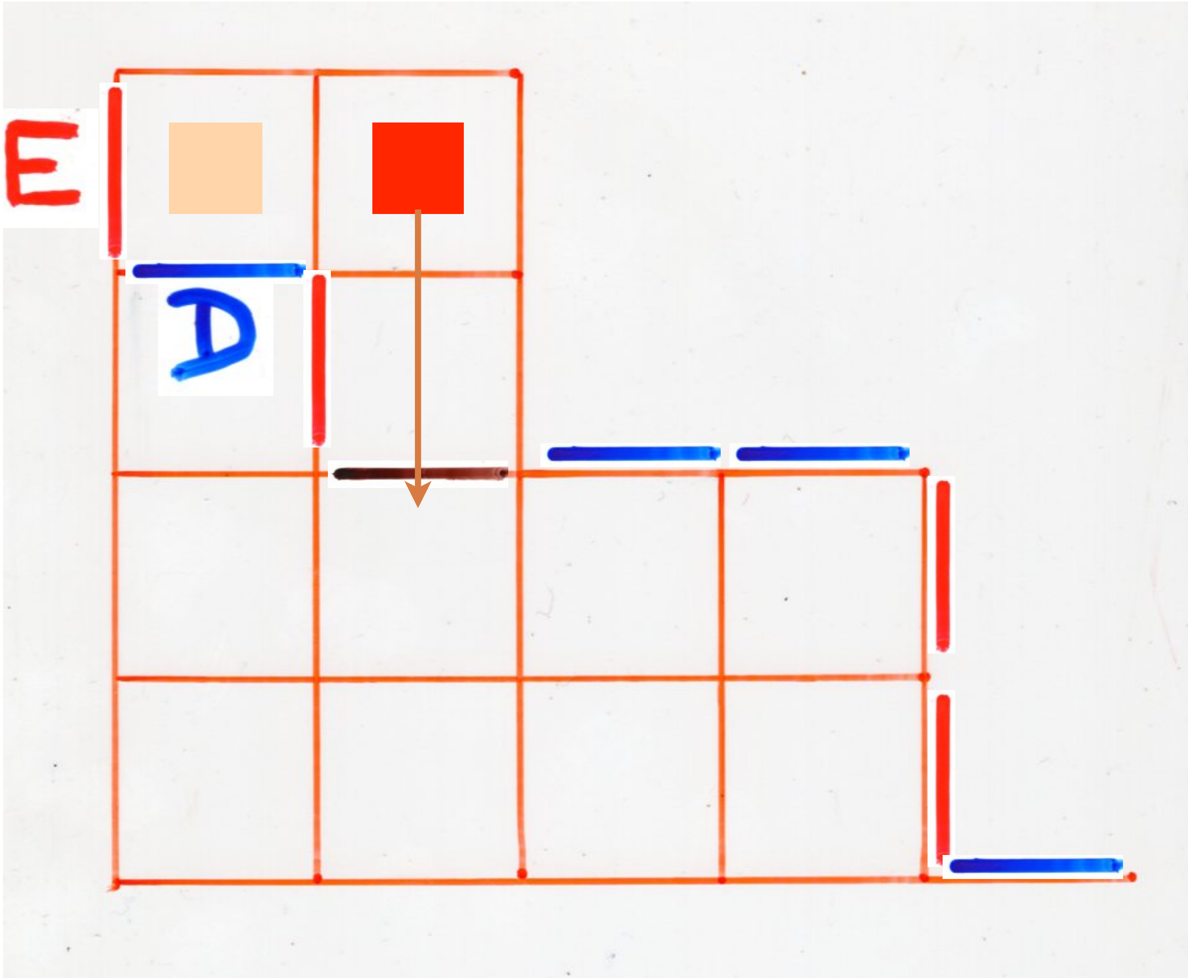


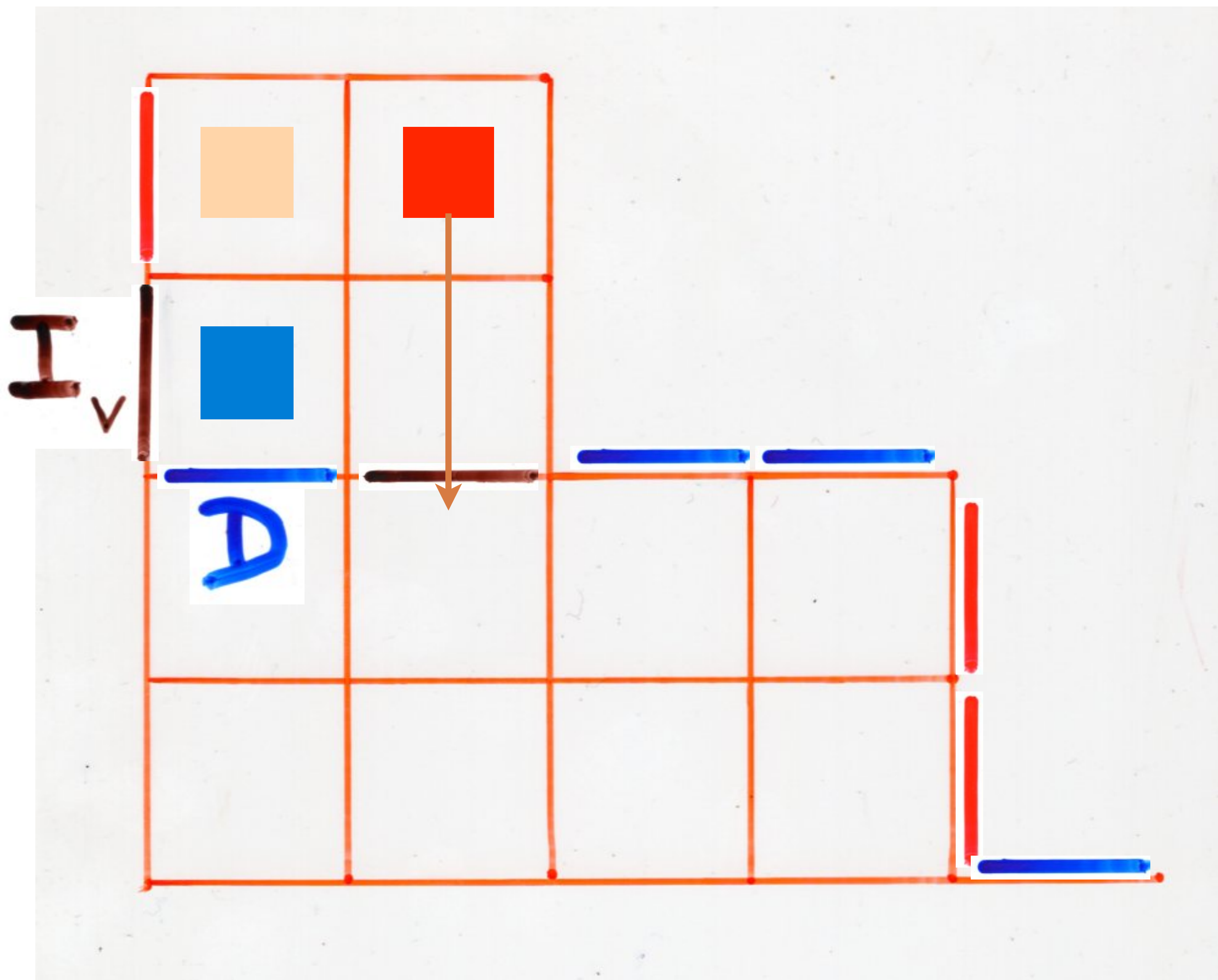

D
E


E
I
h


E




E
D


I
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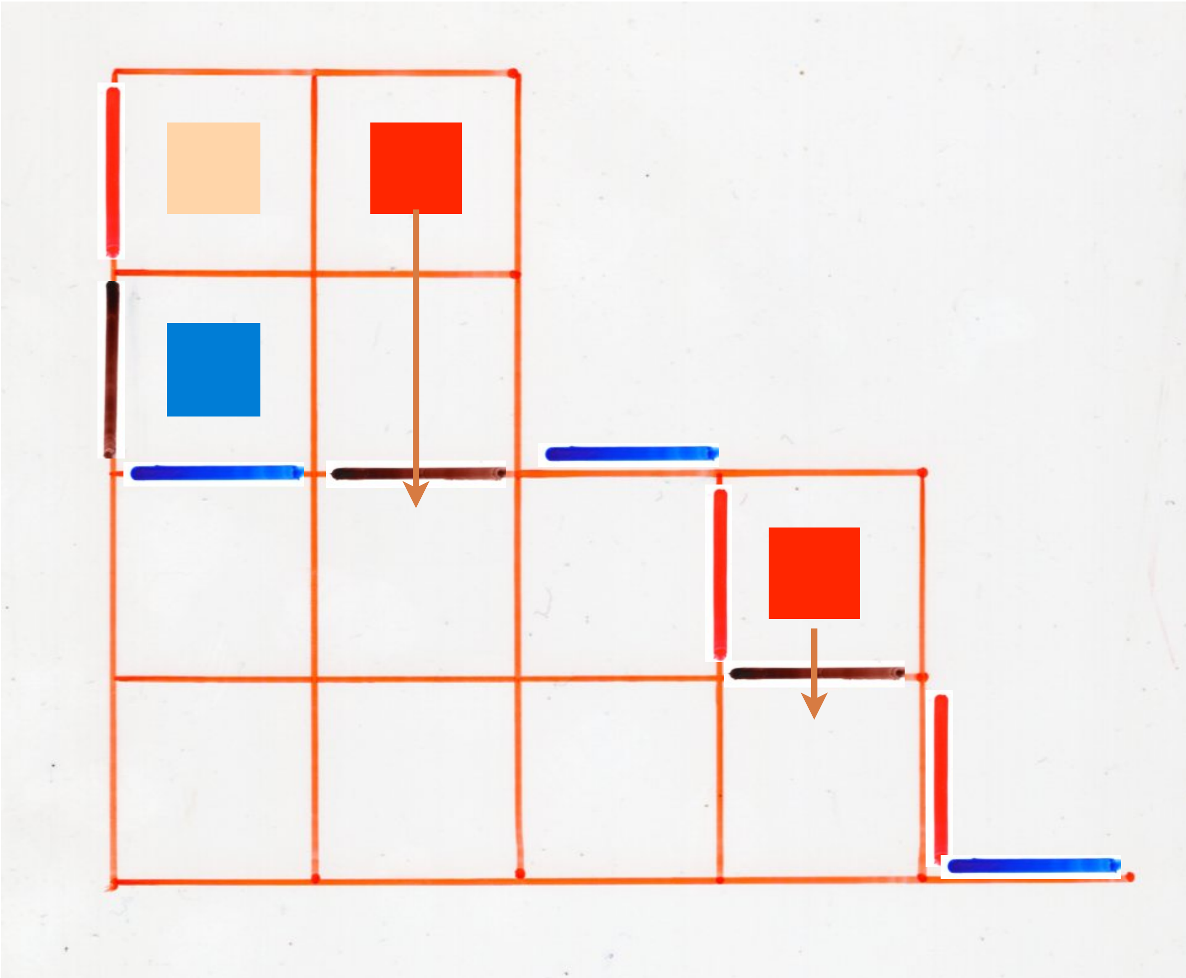
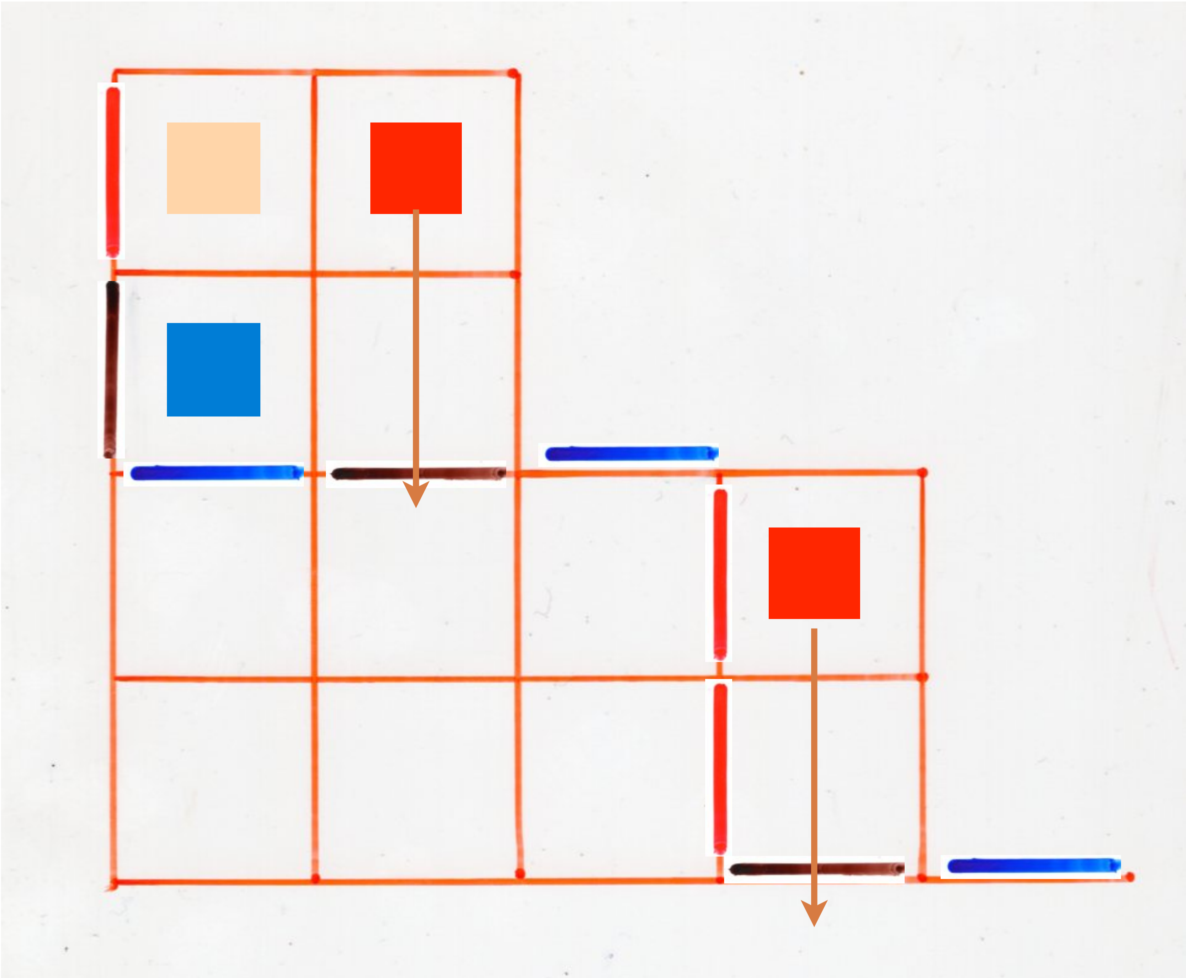
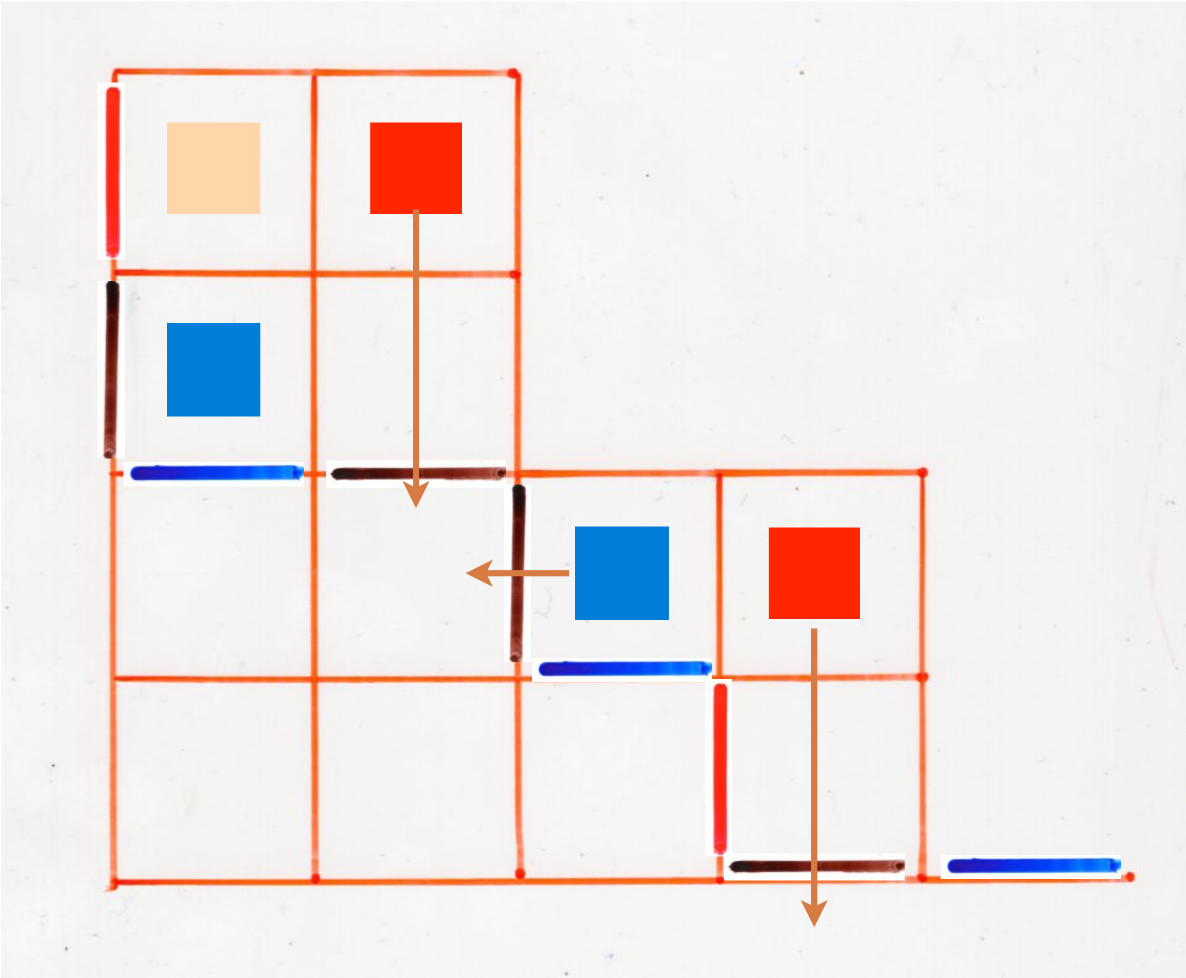
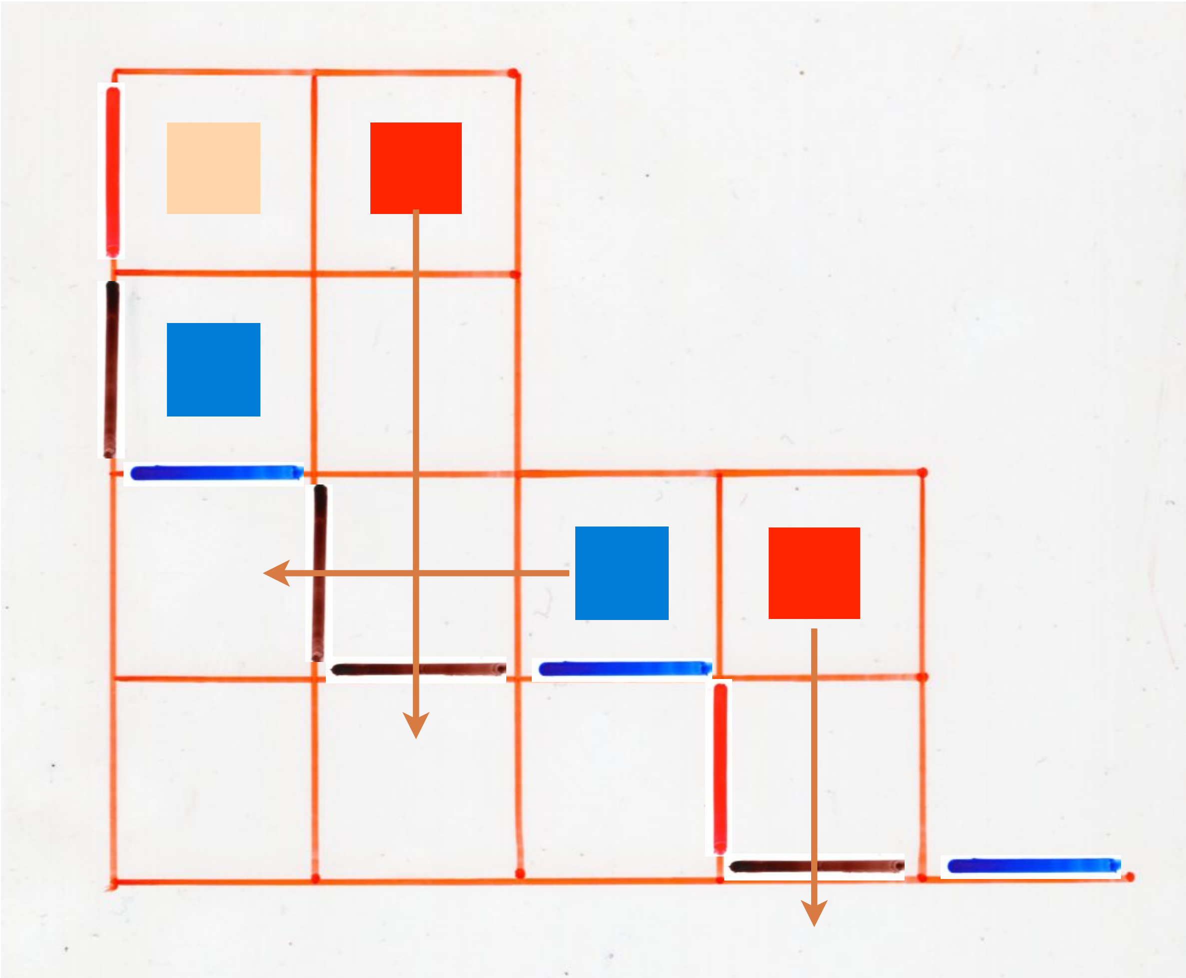
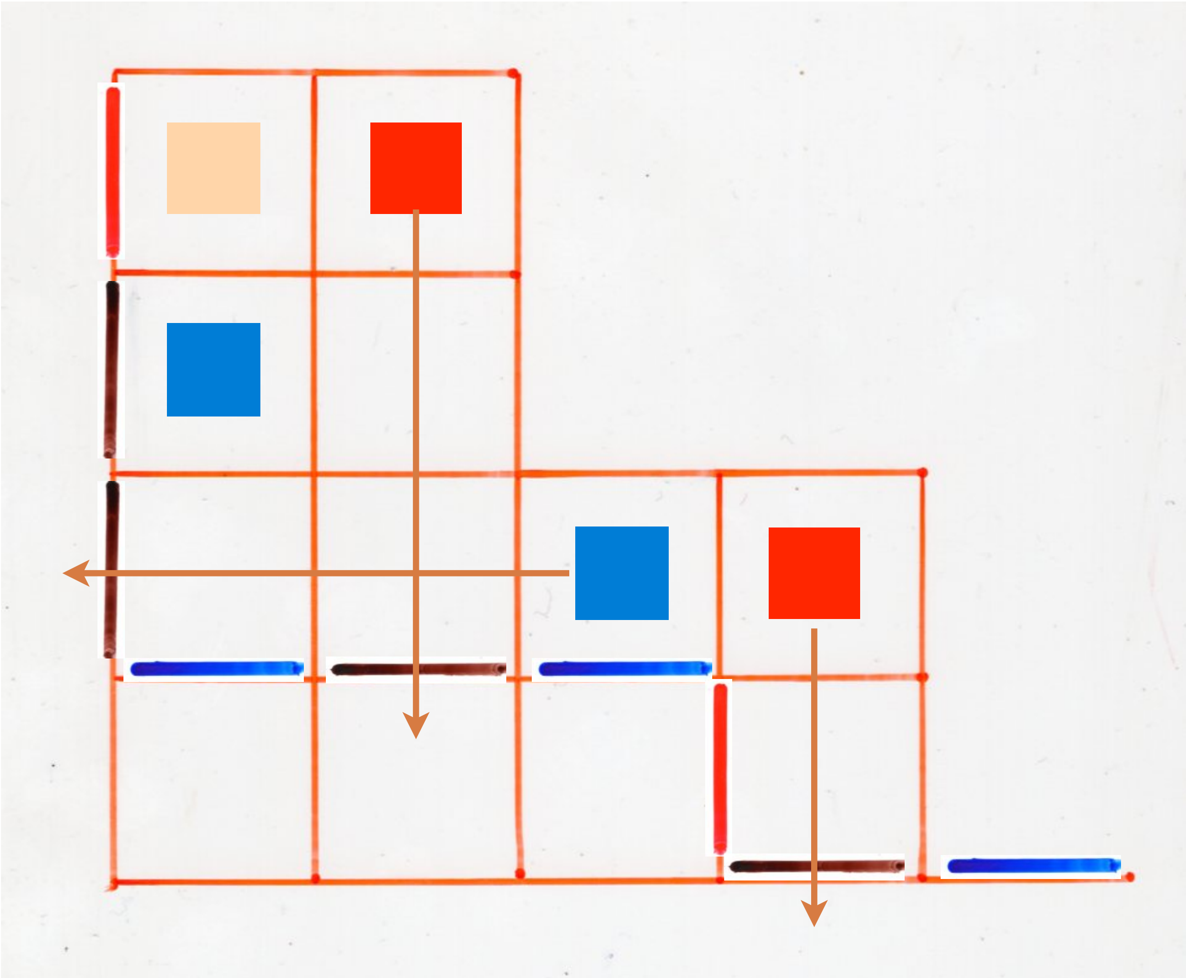
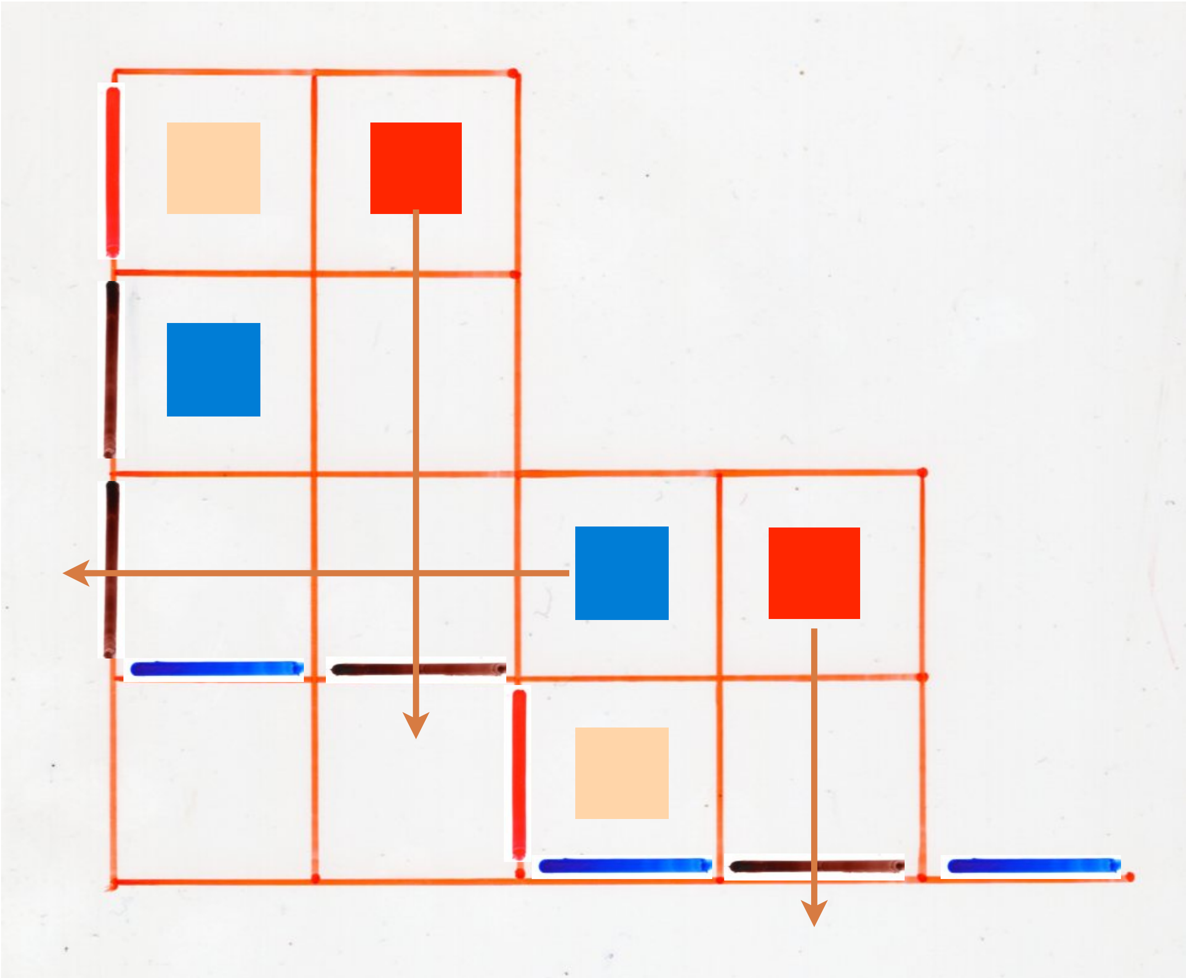


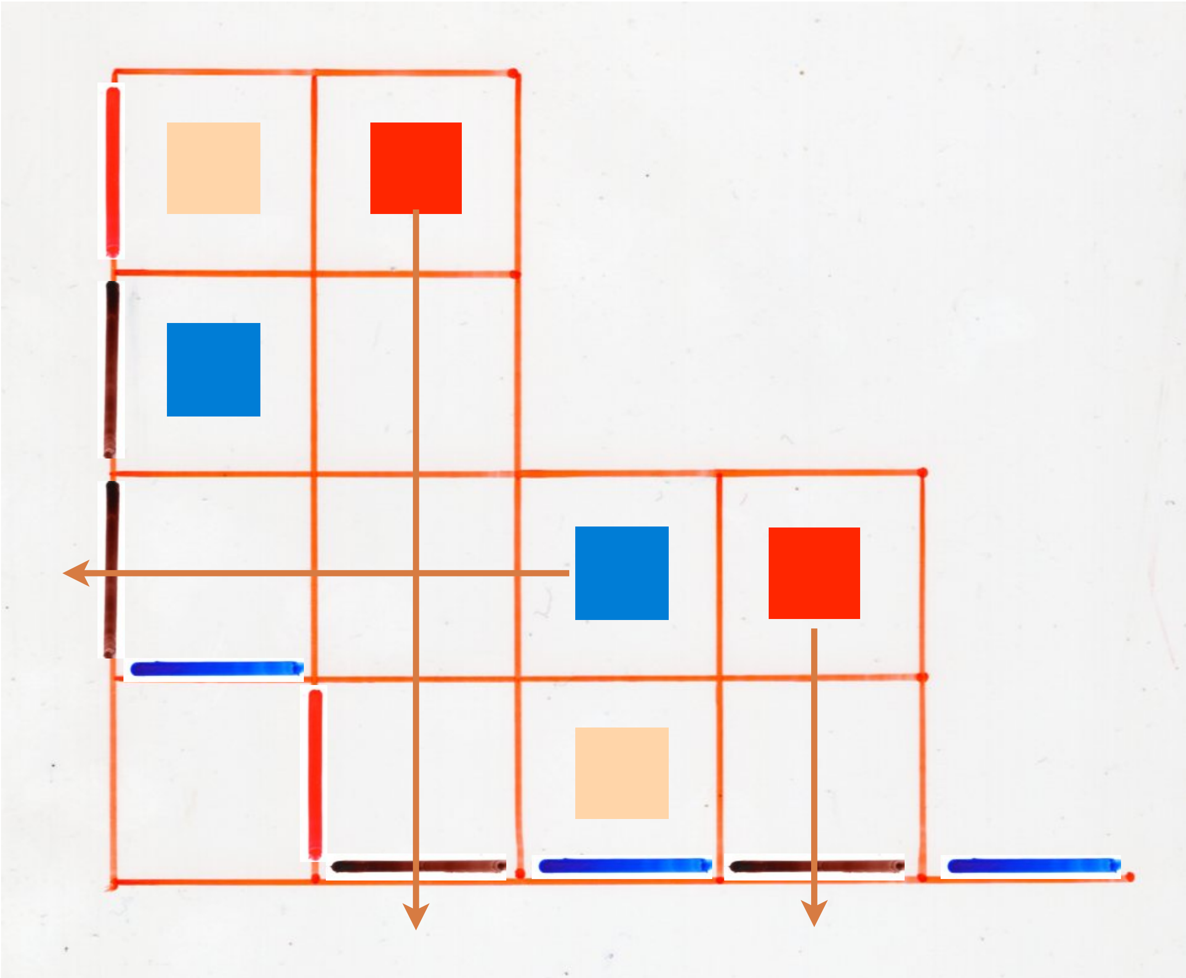


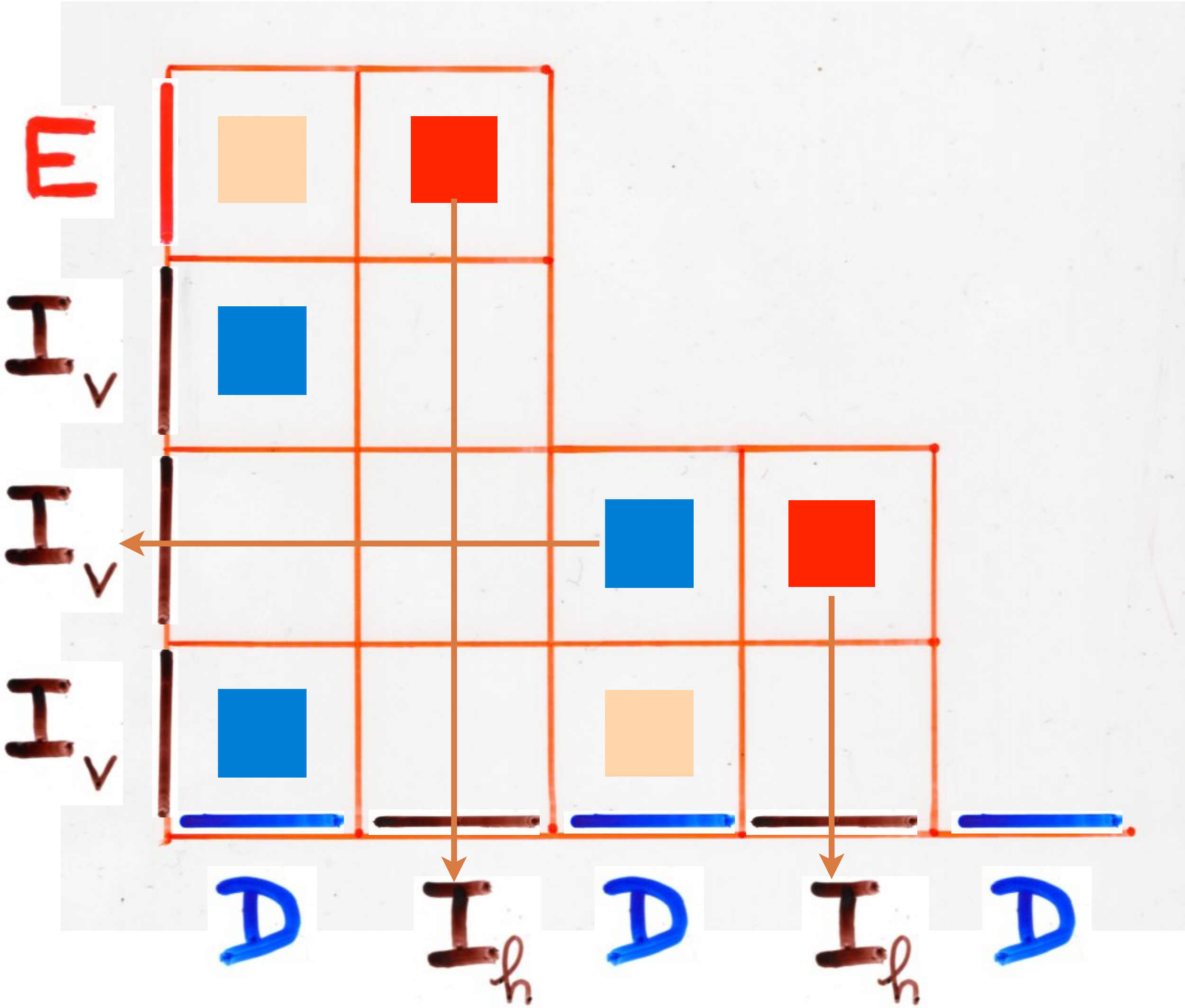











E
I
v
I
v
I
v
D
I
h
D
I
h
D

The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

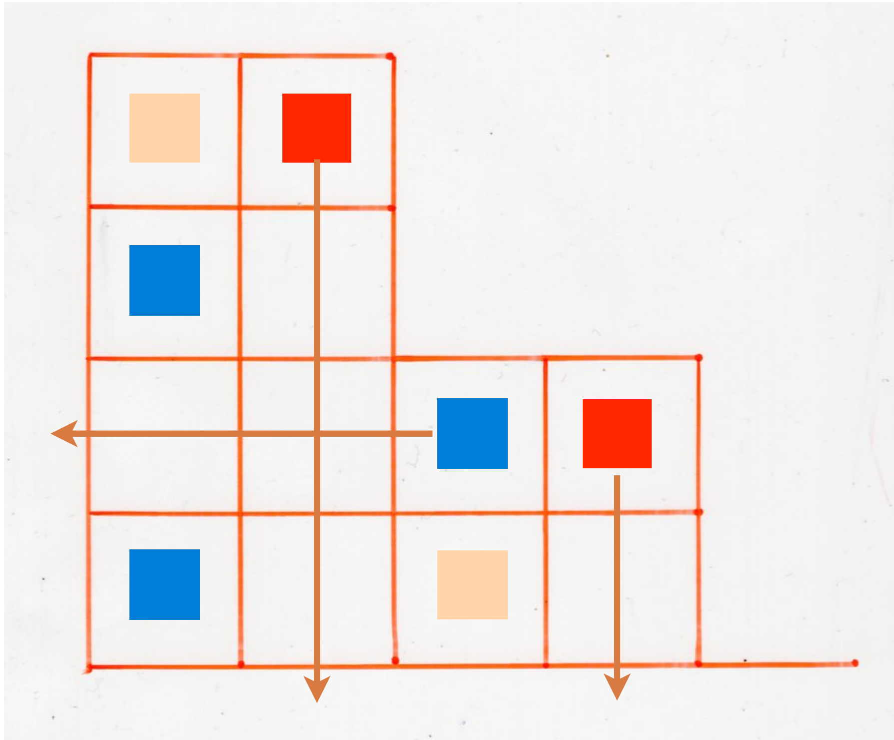
tableau

unique

complete Q-tableau

alternative tableau

$$DE = qED + EI_h + I_v D$$
$$DI_v = I_v D$$
$$I_h E = E I_h$$
$$I_h I_v = I_v I_h$$



The PASEP algebra

$$DE = qED + E + D$$

$$DE = \square\, q\, ED + E\, I_h + I_v\, D$$

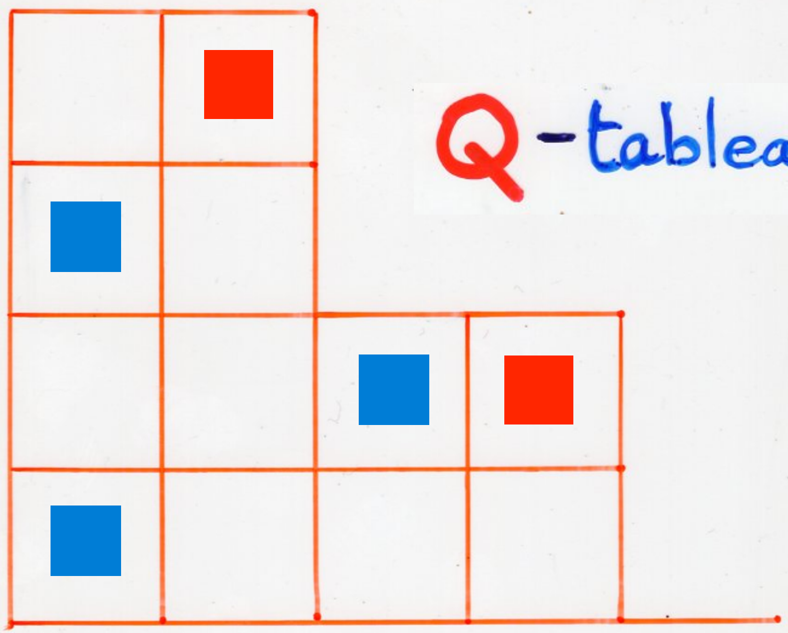
$$D\, I_v = \square\, I_v\, D$$

$$I_h\, E = \square\, E\, I_h$$

$$I_h\, I_v = \square\, I_v\, I_h$$

alternative tableau

Q-tableaux



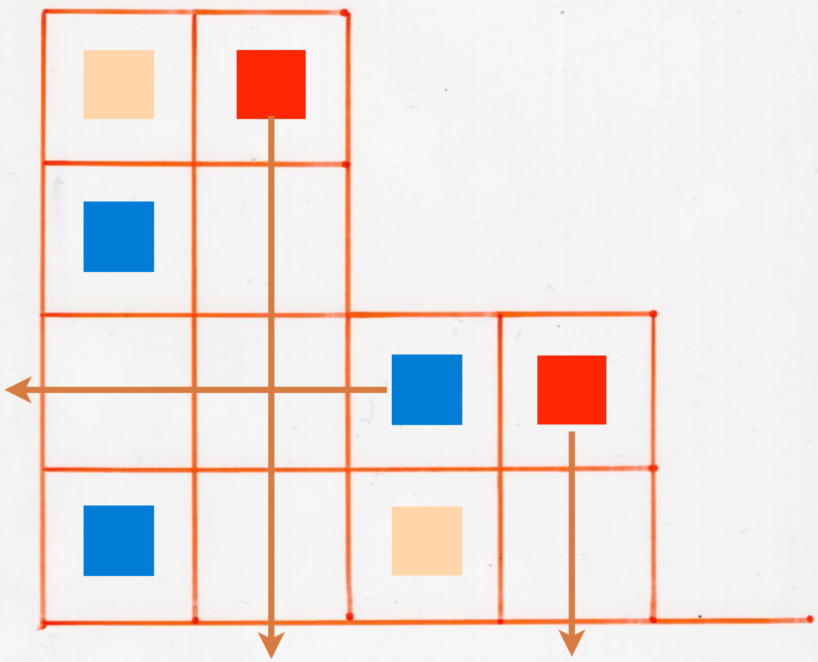
$$DE = qED + E\, I_h + I_v\, D$$

$$D\, I_v = I_v\, D$$

$$I_h\, E = E\, I_h$$

$$I_h\, I_v = I_v\, I_h$$

complete Q-tableau



"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

$UD = DU + Id$

permutations

RSK ⟷

pairs of Young tableaux

towers placements

Physics

(i) first step

(ii) second step

$DE = qED + E + D$

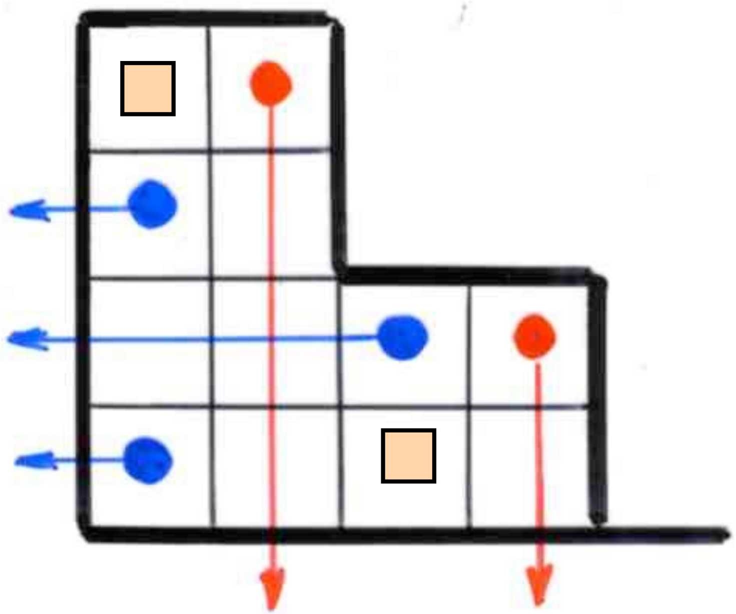
alternative tableaux

commutations

rewriting rules

planarization

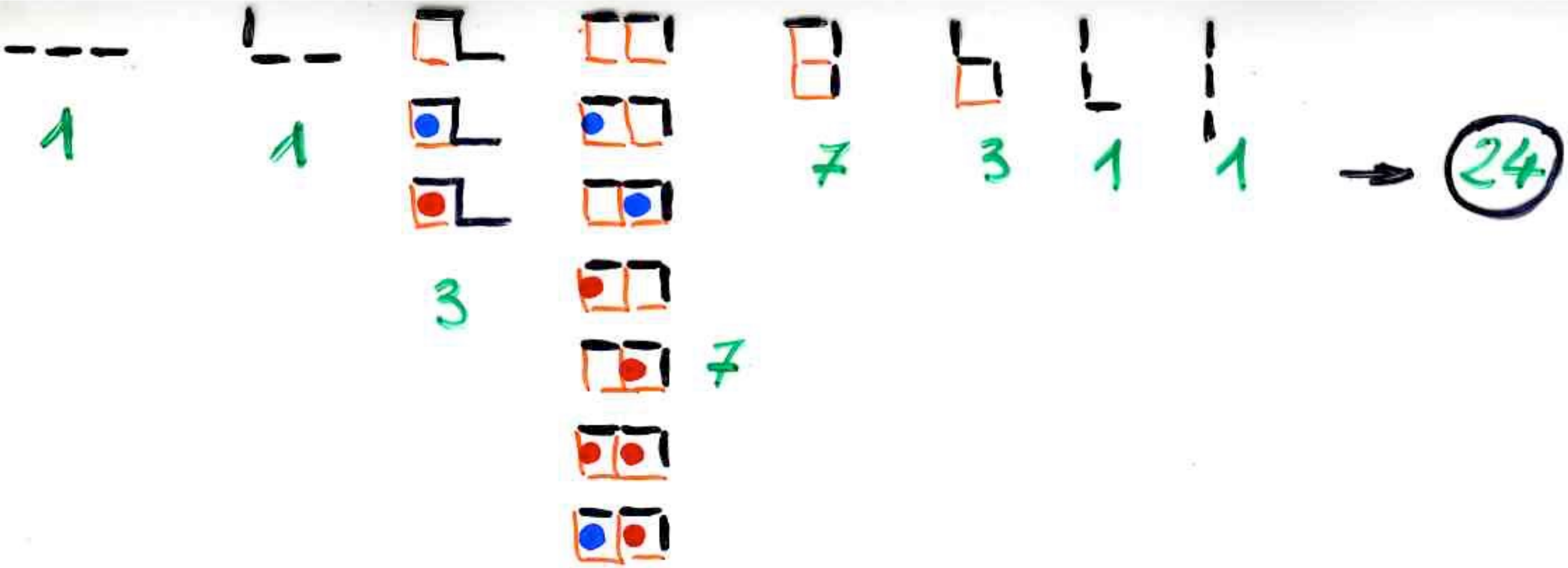
Enumeration of alternative tableaux



Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

ex: n=2

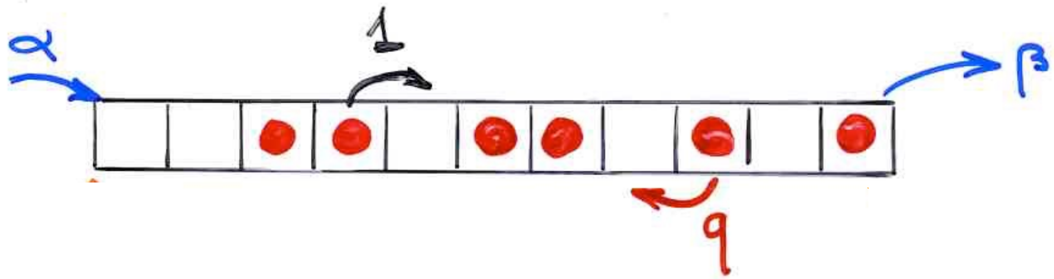


$q = 0$

The PASEP algebra

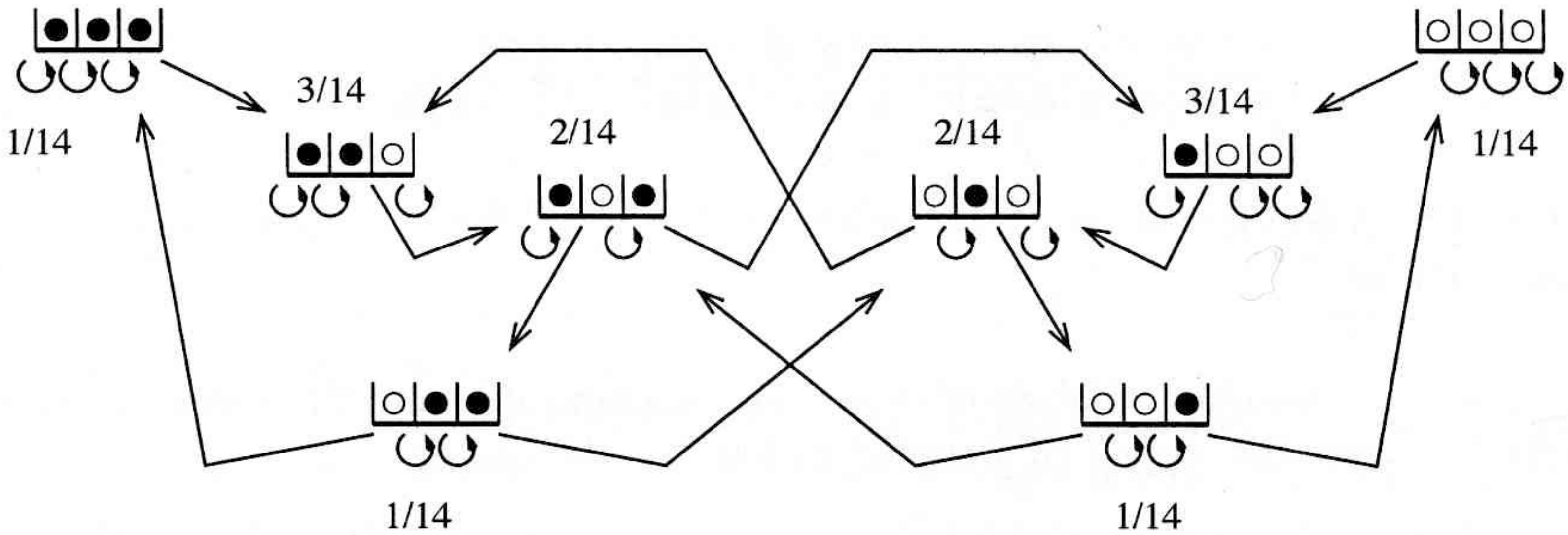
$DE = E + D$

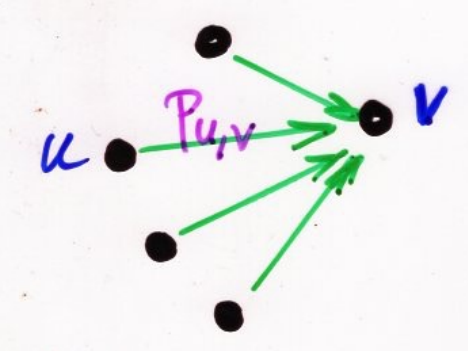
TASEP



$q = 0$ TASEP (α, β)

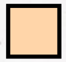
$\gamma = \delta = 0$ $\alpha = \beta = 1$



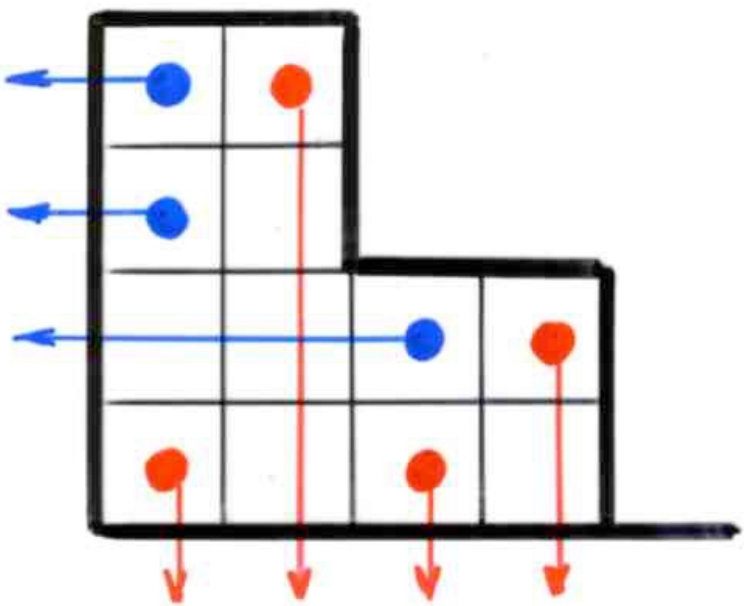


$$p_v^{\infty} = \sum_{u \in S} p_u^{\infty} \, p_{u,v}$$

Definition Catalan alternative tableau

alternative tableau **T** without cells 

i.e. every empty cell is below a red cell
or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

$q = 0$

TASEP

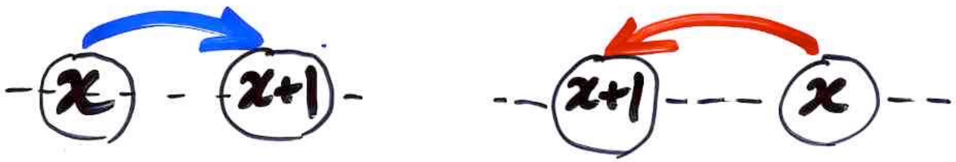
(α, β)

The "exchange-fusion" algorithm

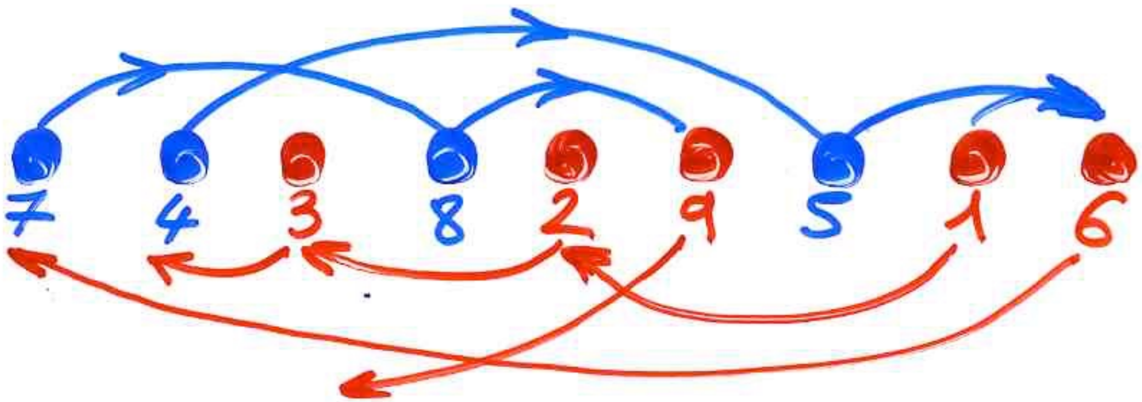
x advance in a permutation σ

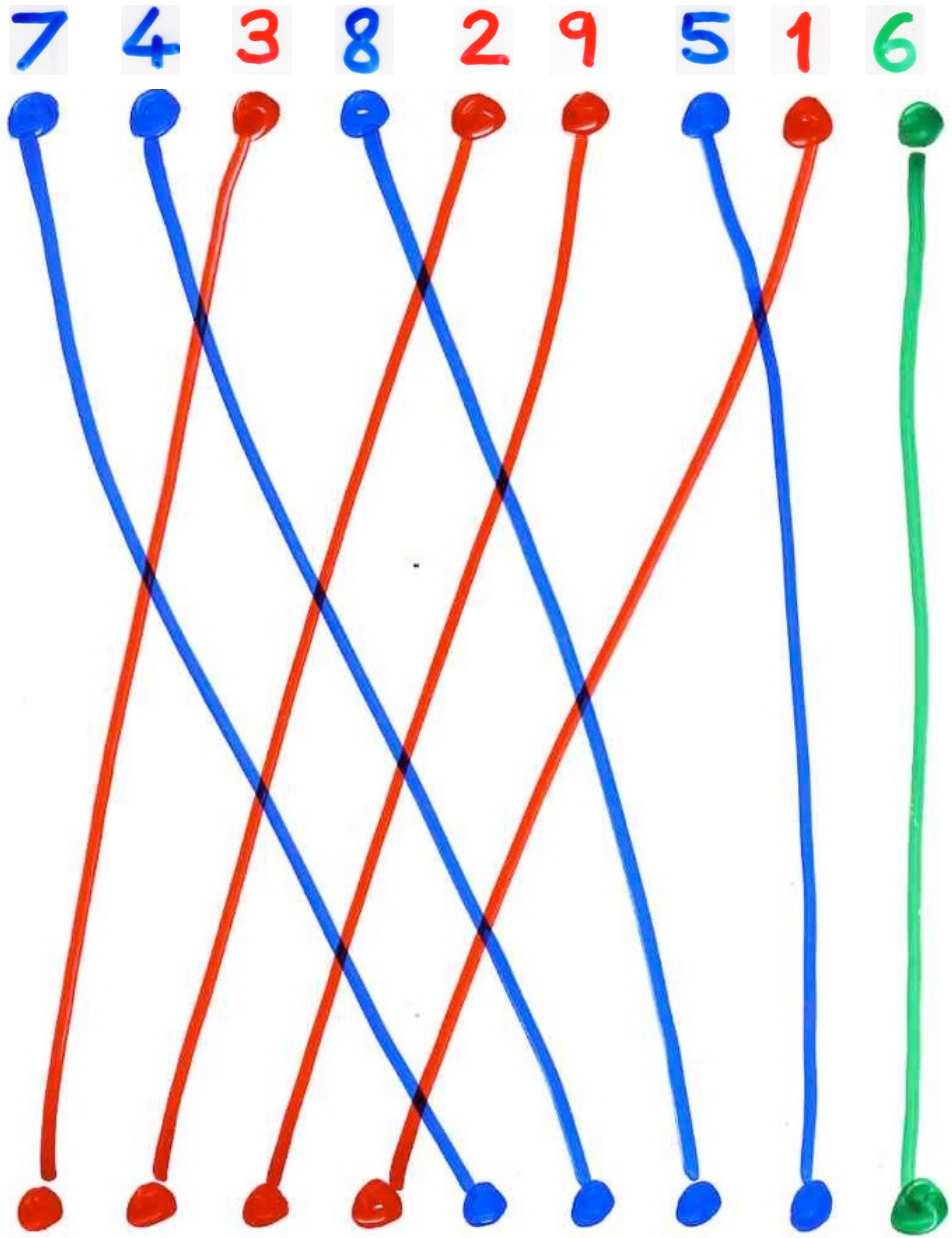
iff $x = \sigma(i)$, $x+1 = \sigma(j)$

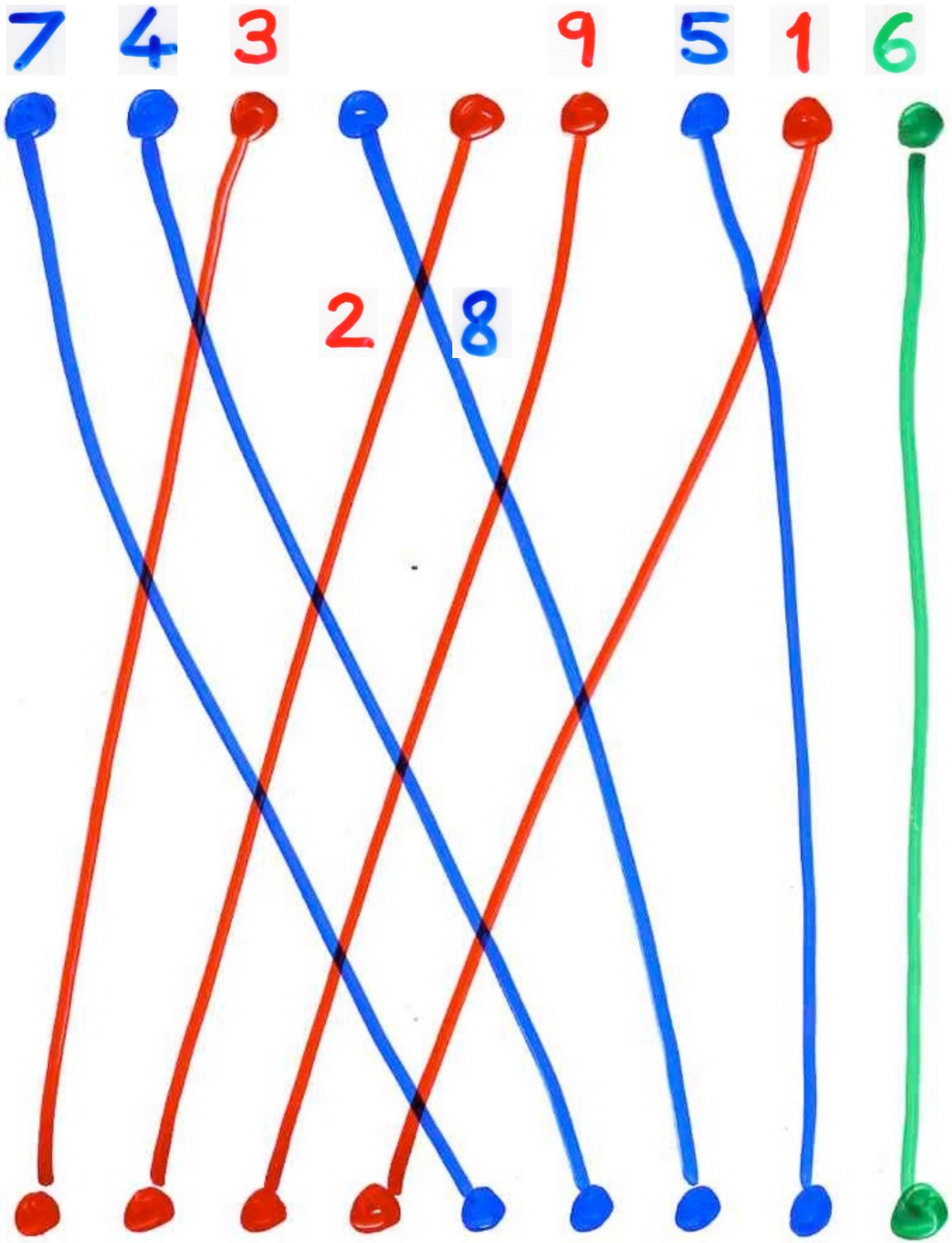
with $i < j$

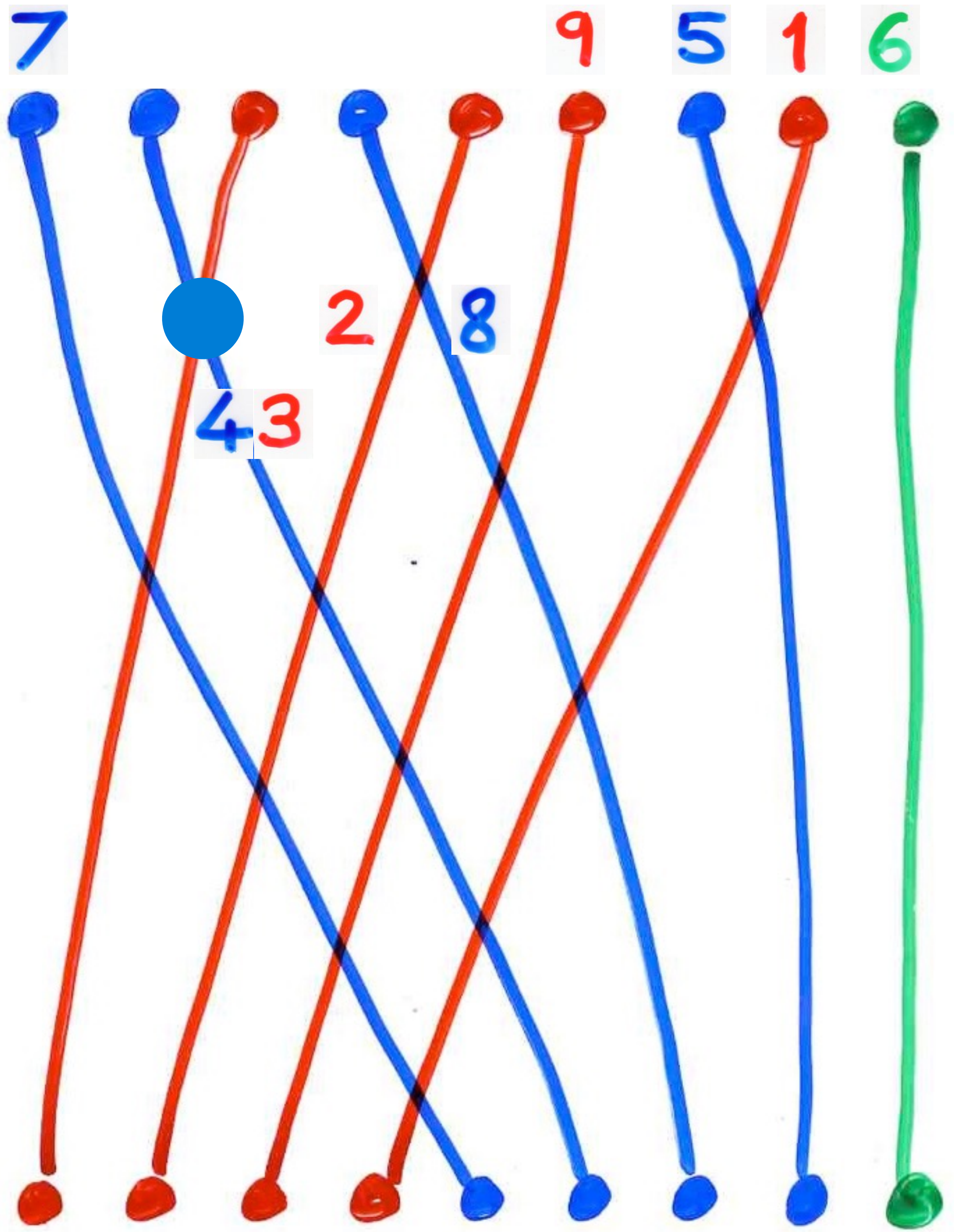


$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$




7 4 3 8 2 9 5 1 6


7
4
3
9
5
1
6
2
8



7
9
5
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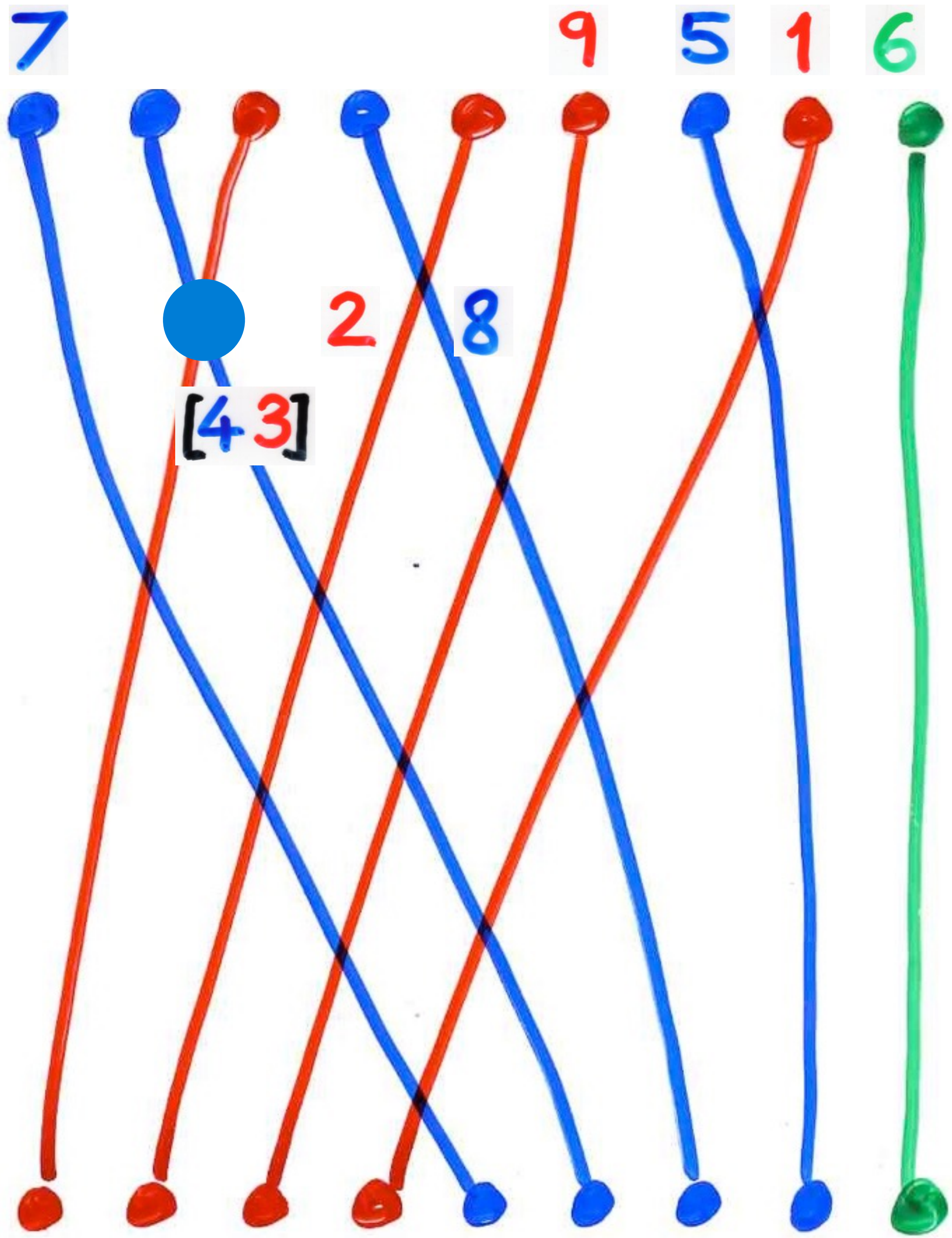
4
3



7
9
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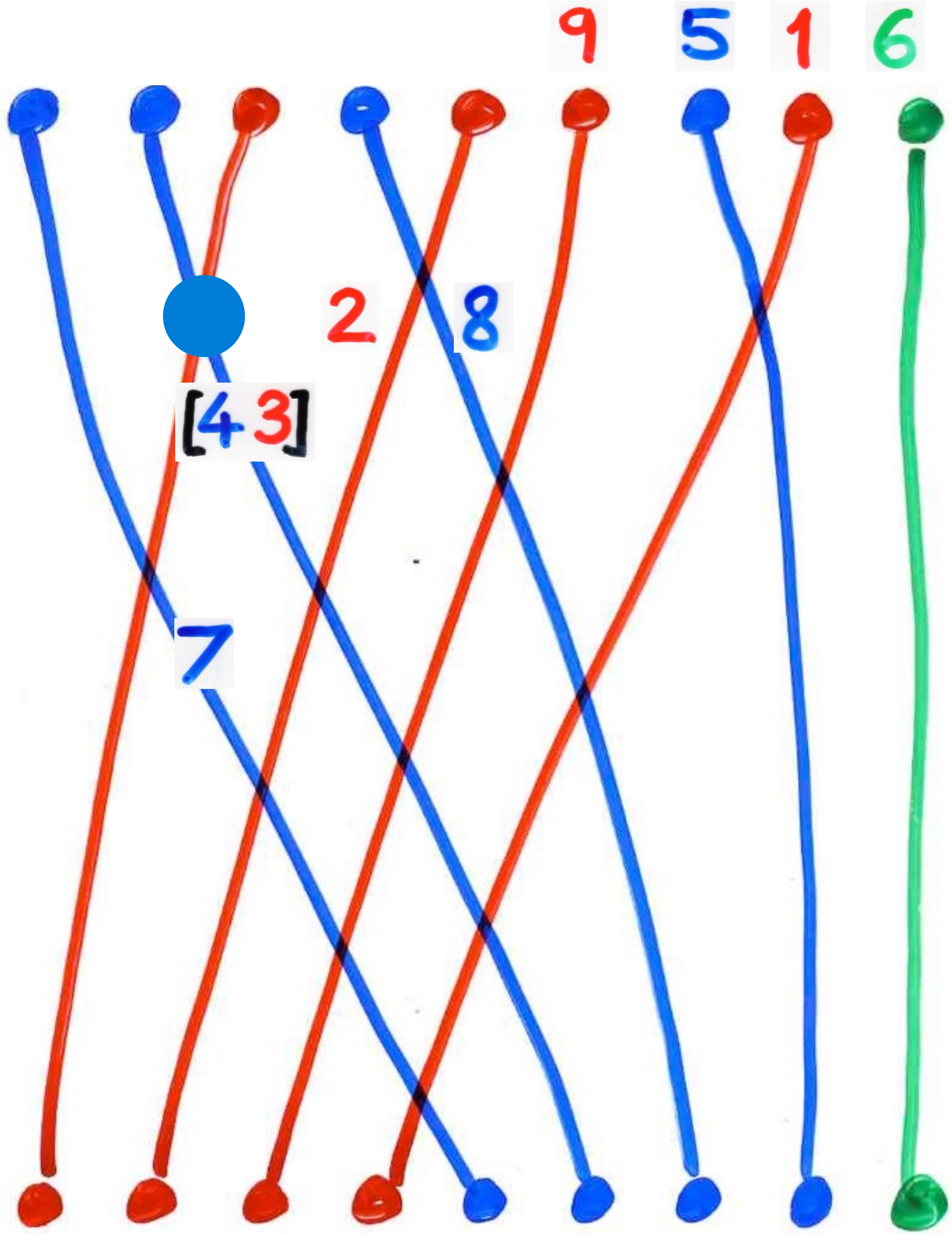
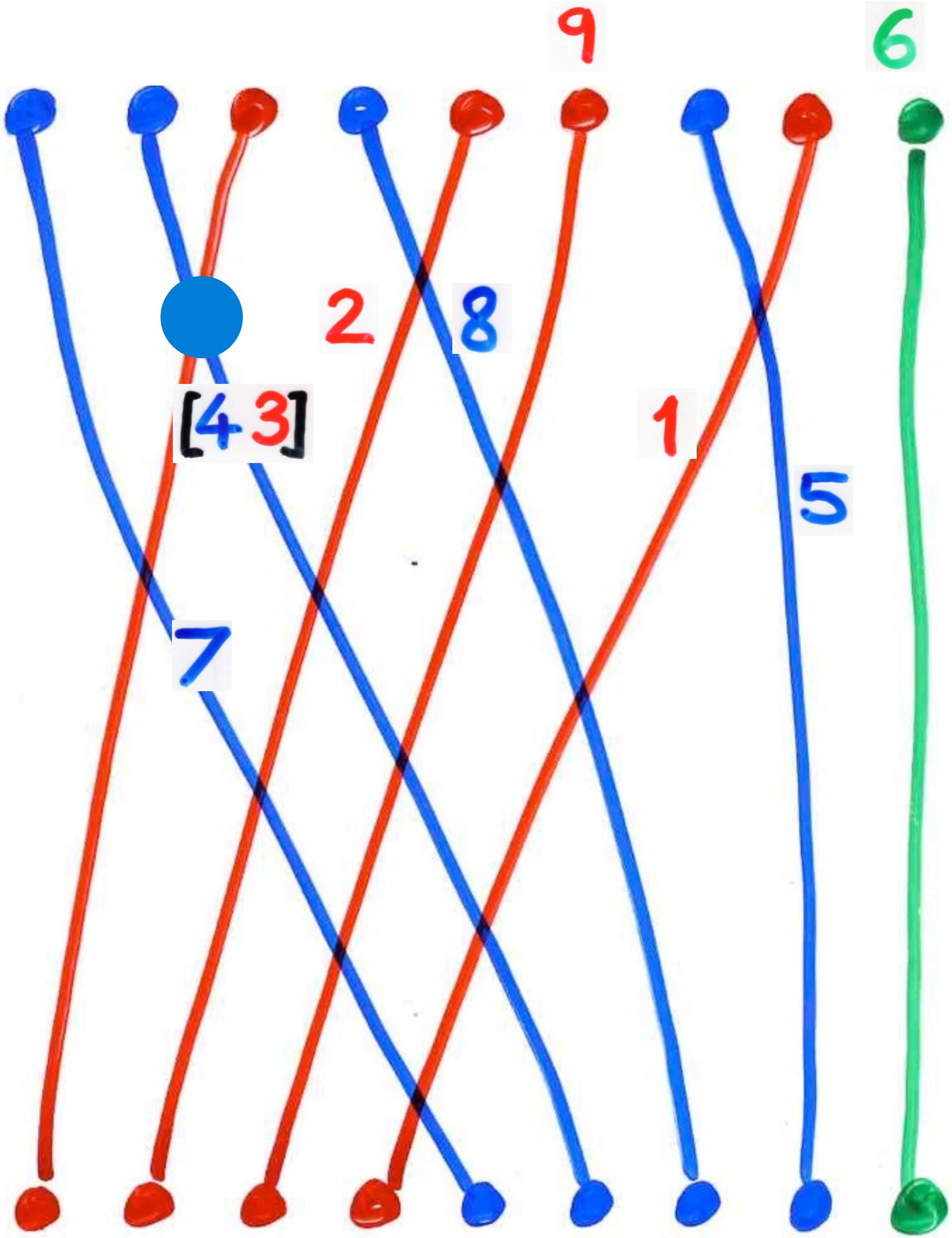
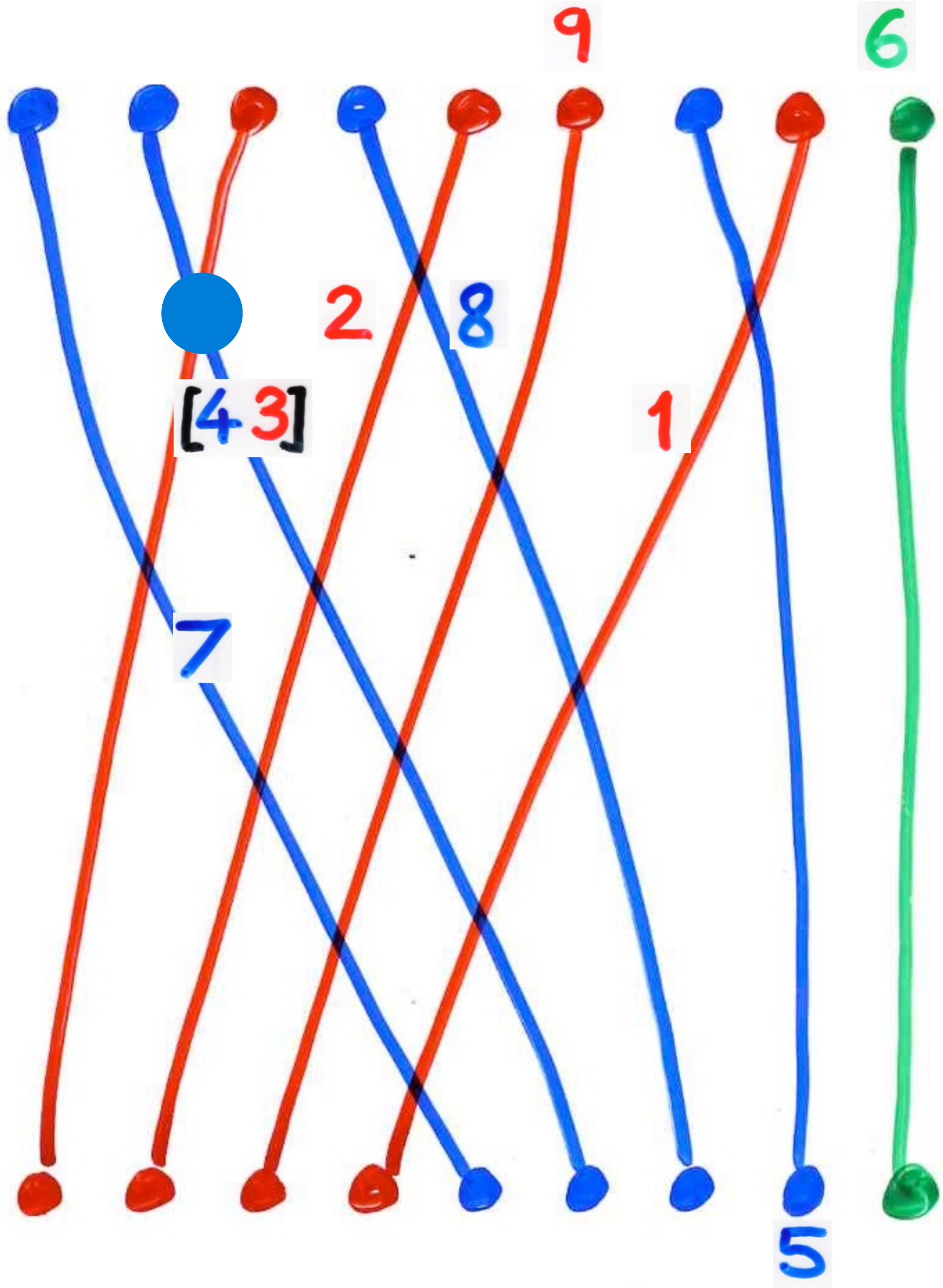
2
8
[4 3]



9
5
1
6
2
8
[43]
7



9
6
2
8
[43]
1
5
7


9
6
2
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[43]
1
7
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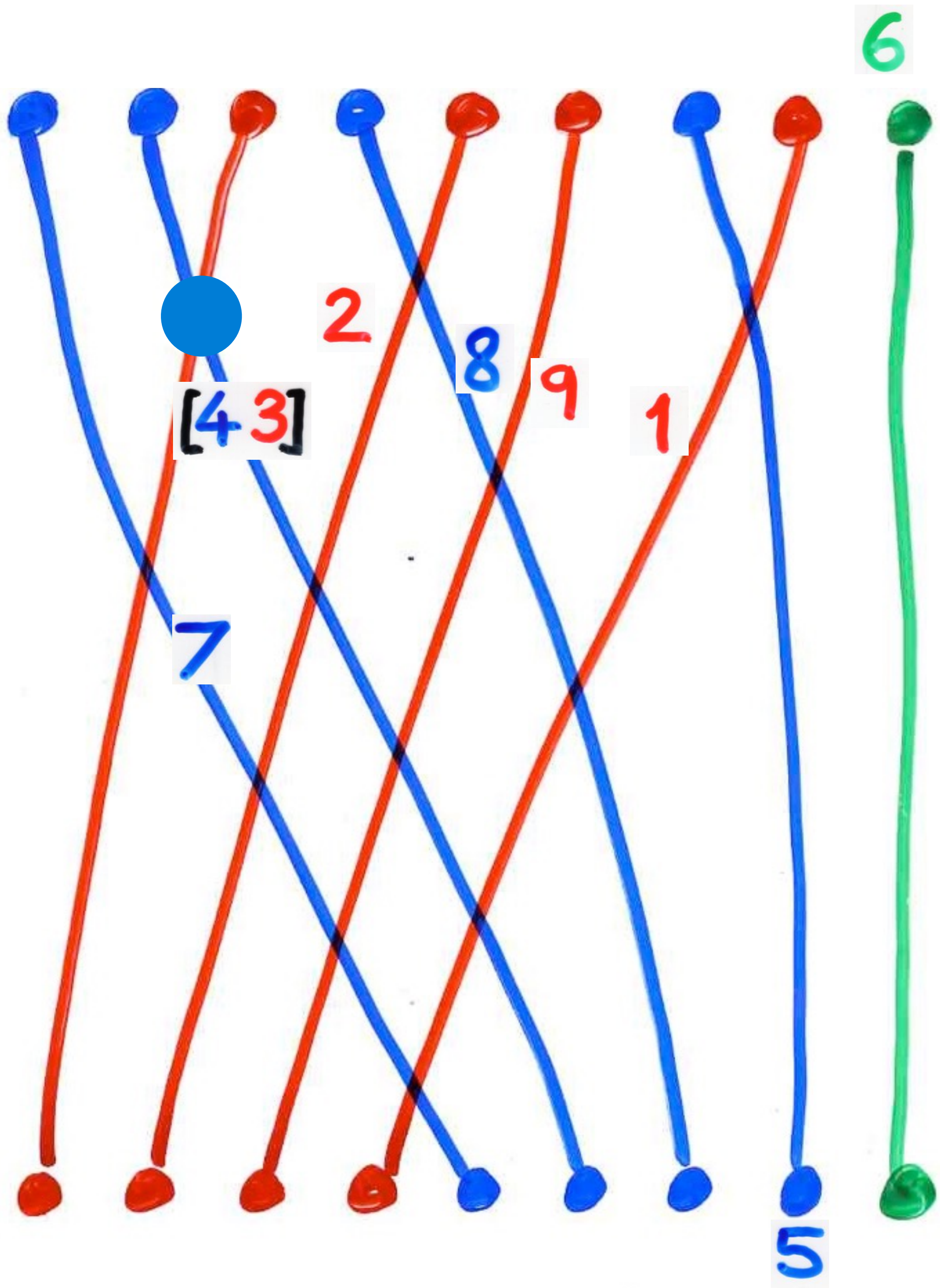

6
2
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[43]
7
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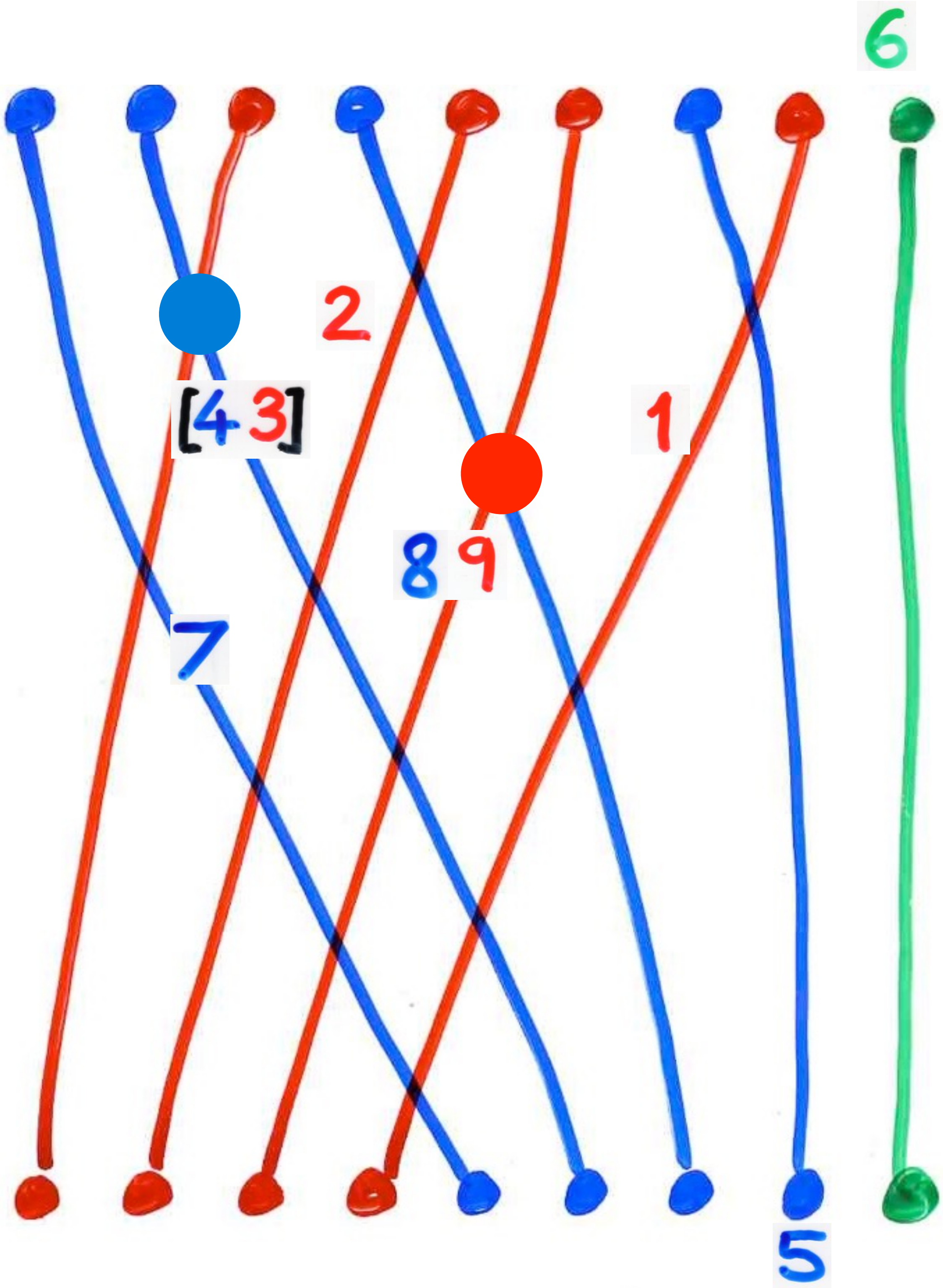

6
2
[4 3]
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8 9
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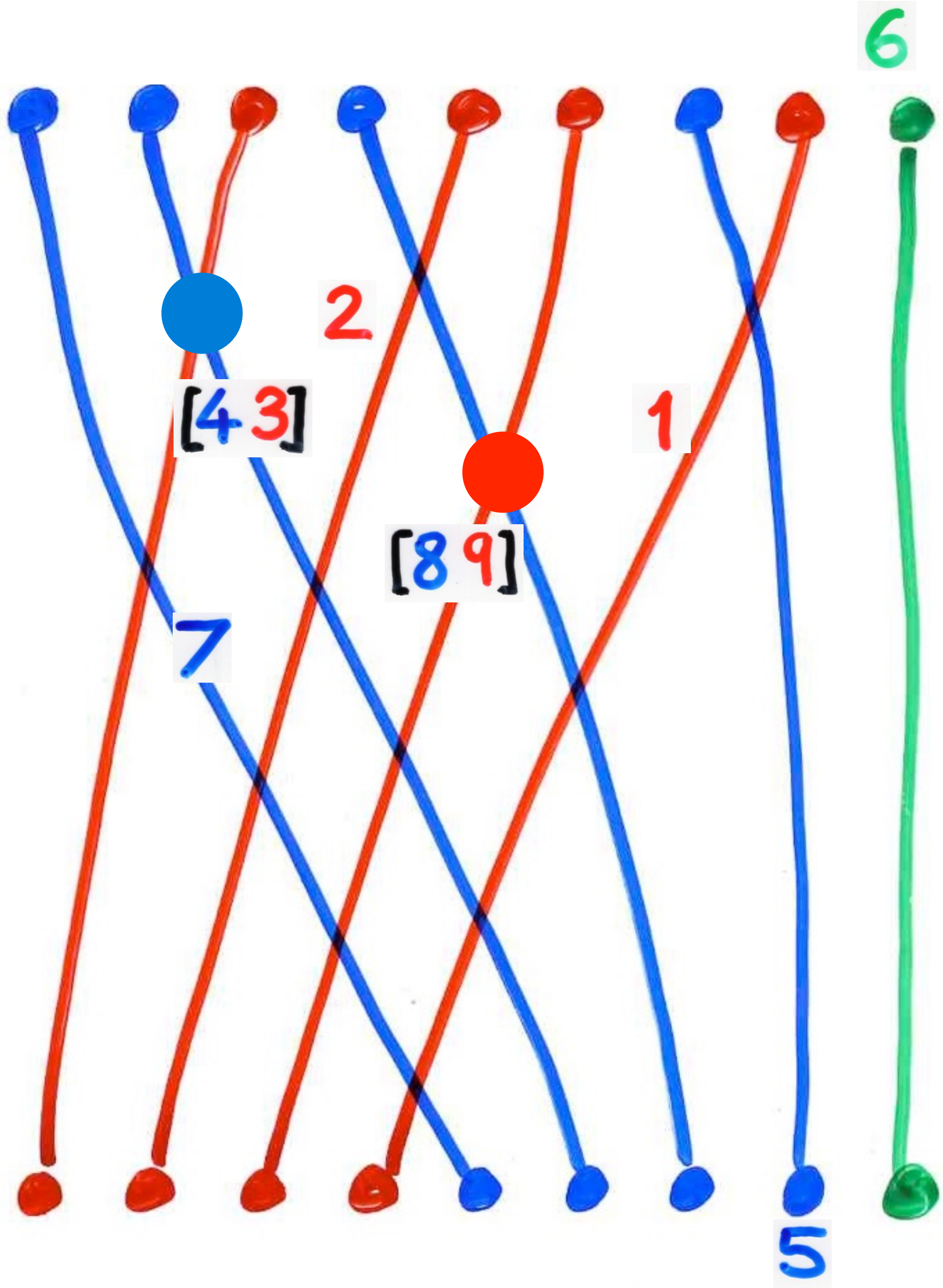
6
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[4 3]
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[8 9]
7
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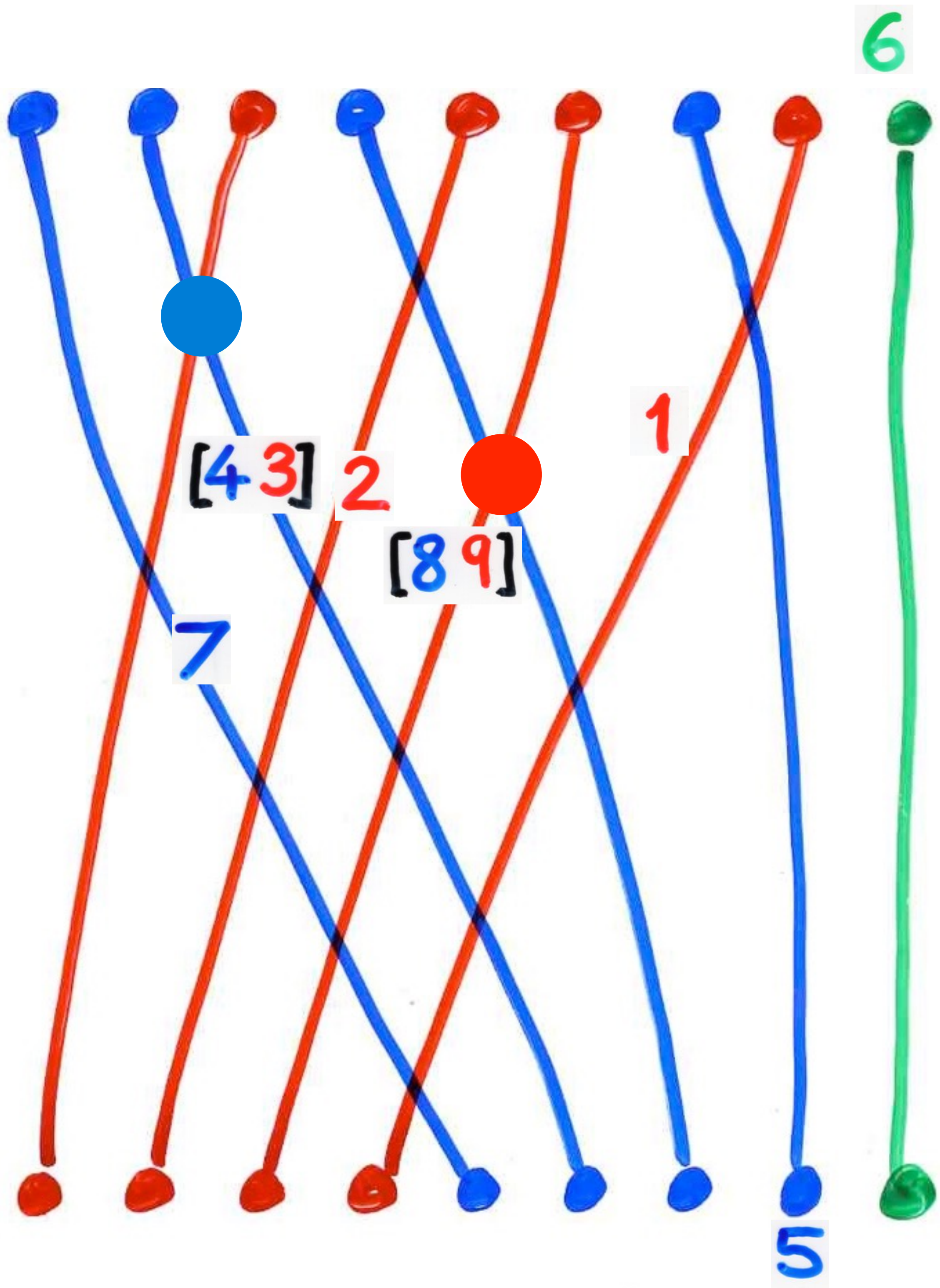

6
[4 3]
2
1
[8 9]
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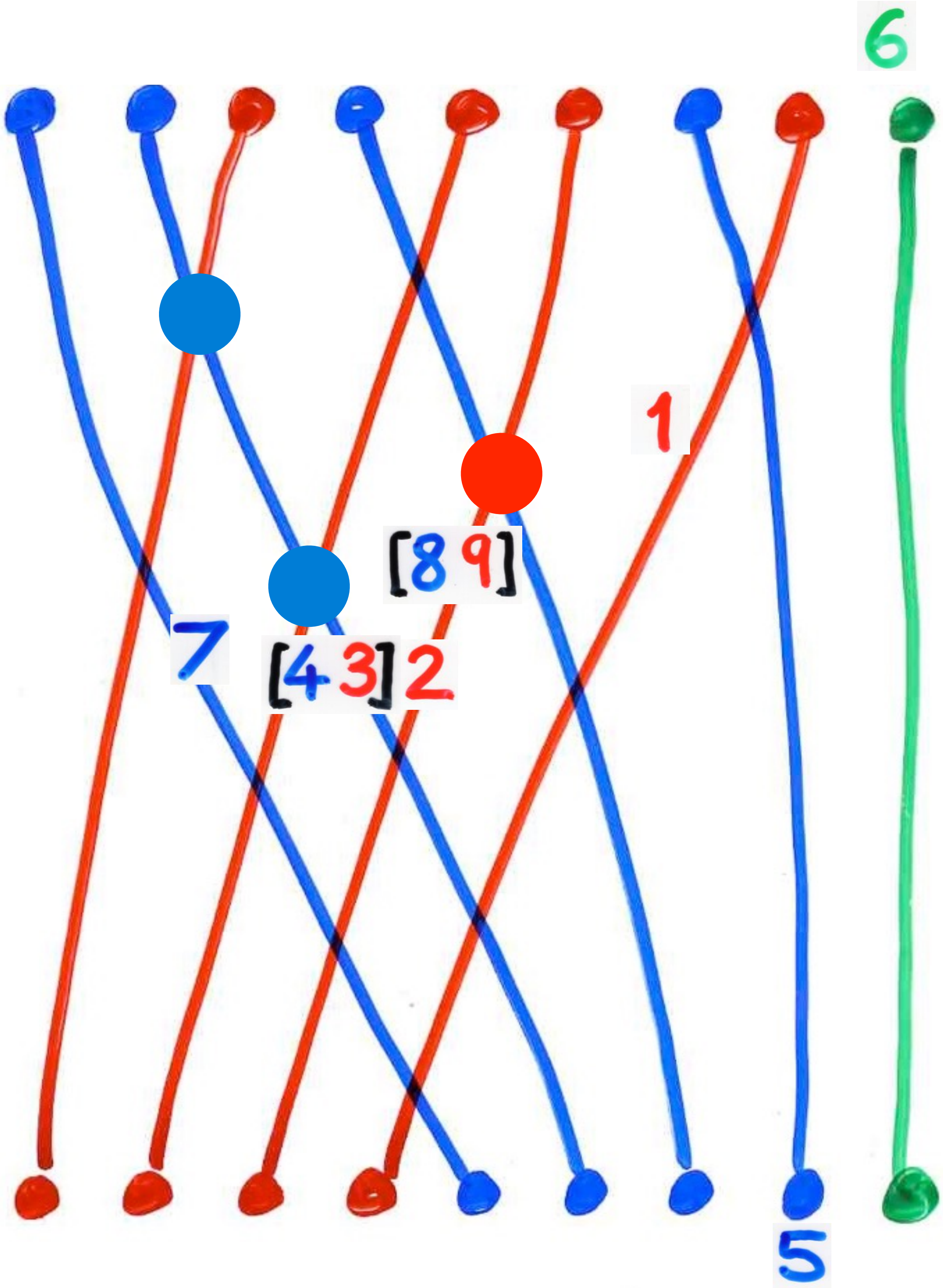
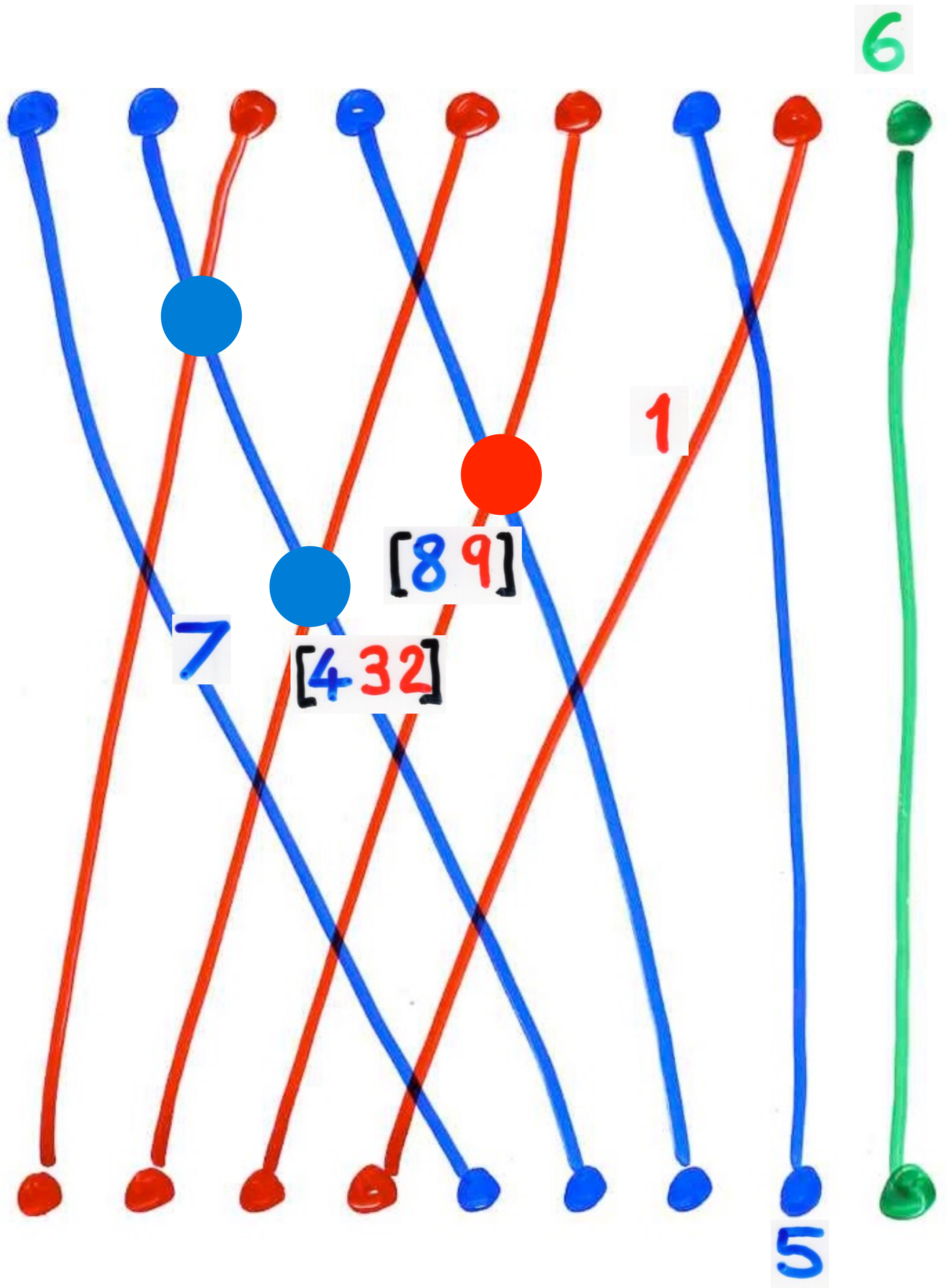

6
1
[8 9]
7
[4 3] 2
5


6
1
[8 9]
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[4 3 2]
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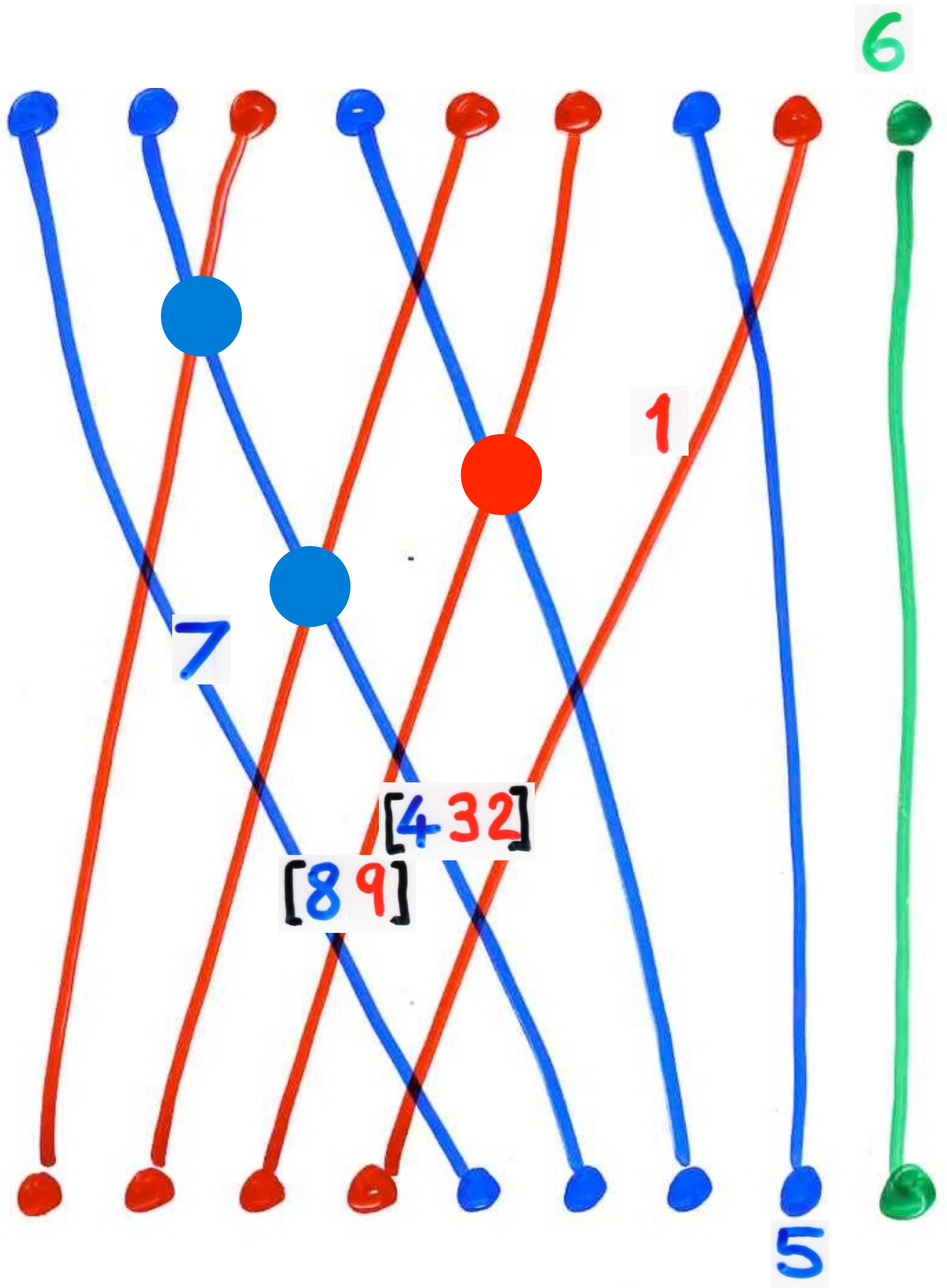
6
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7
[4 3 2]
[8 9]
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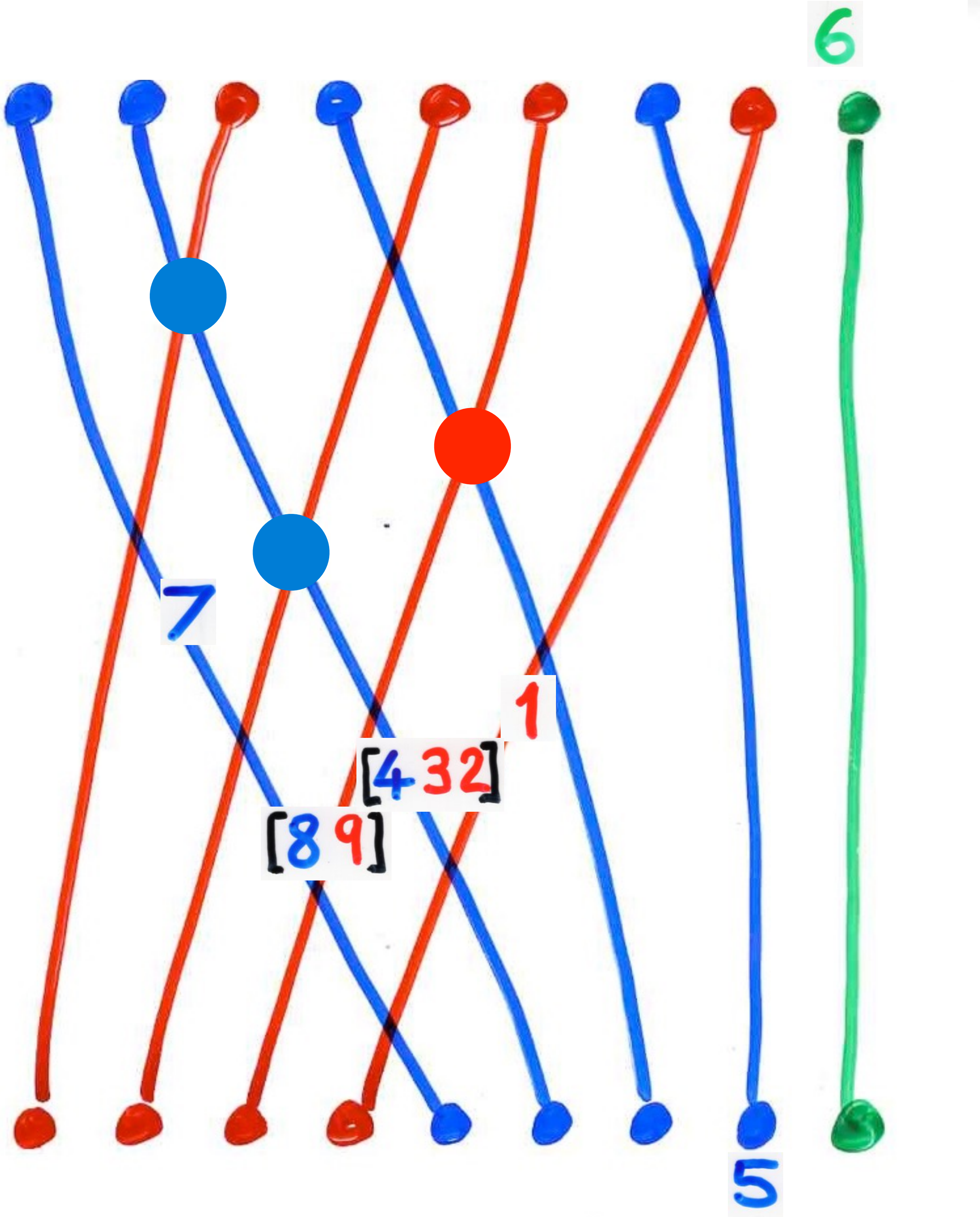
6
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[4 3 2]
[8 9]
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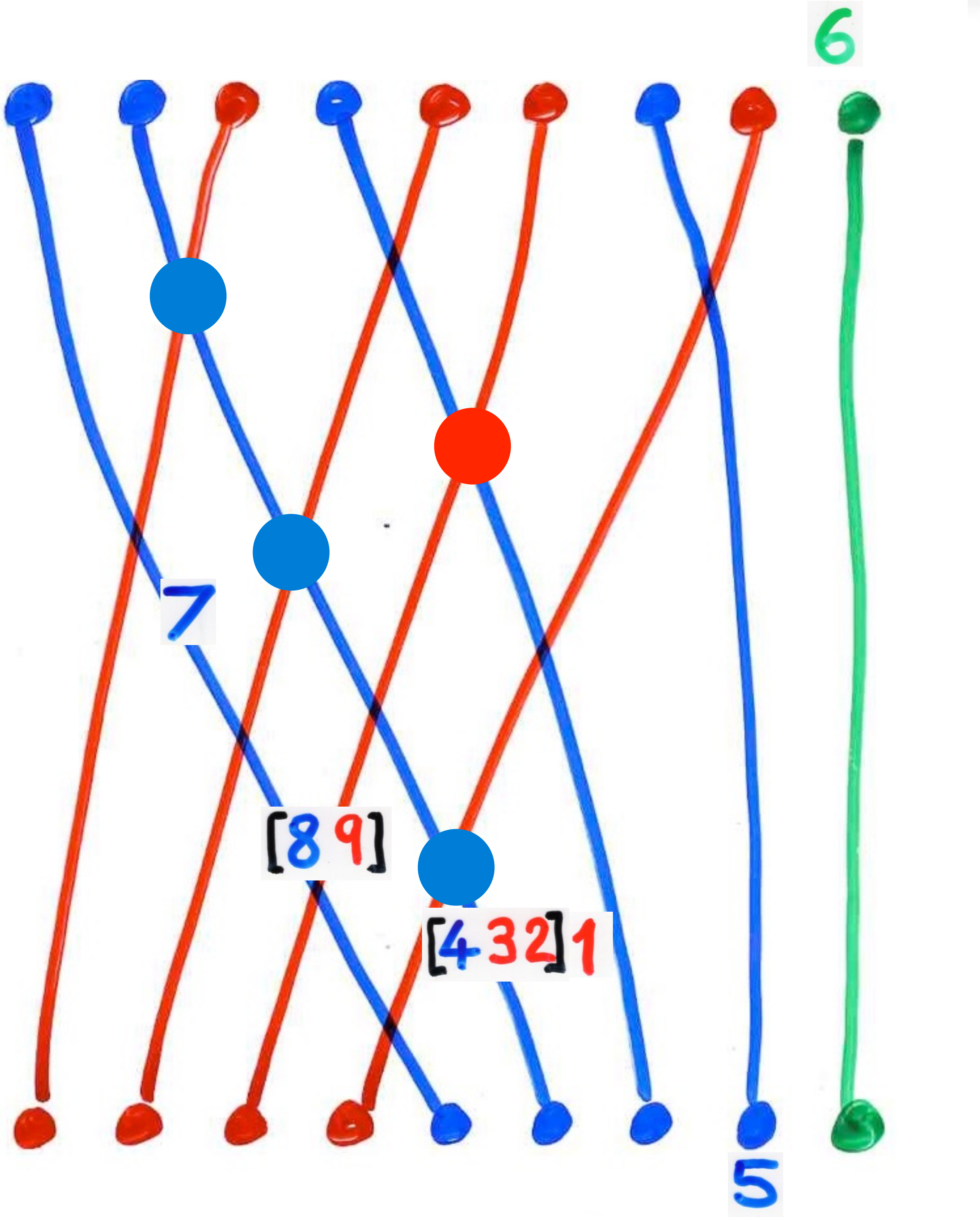
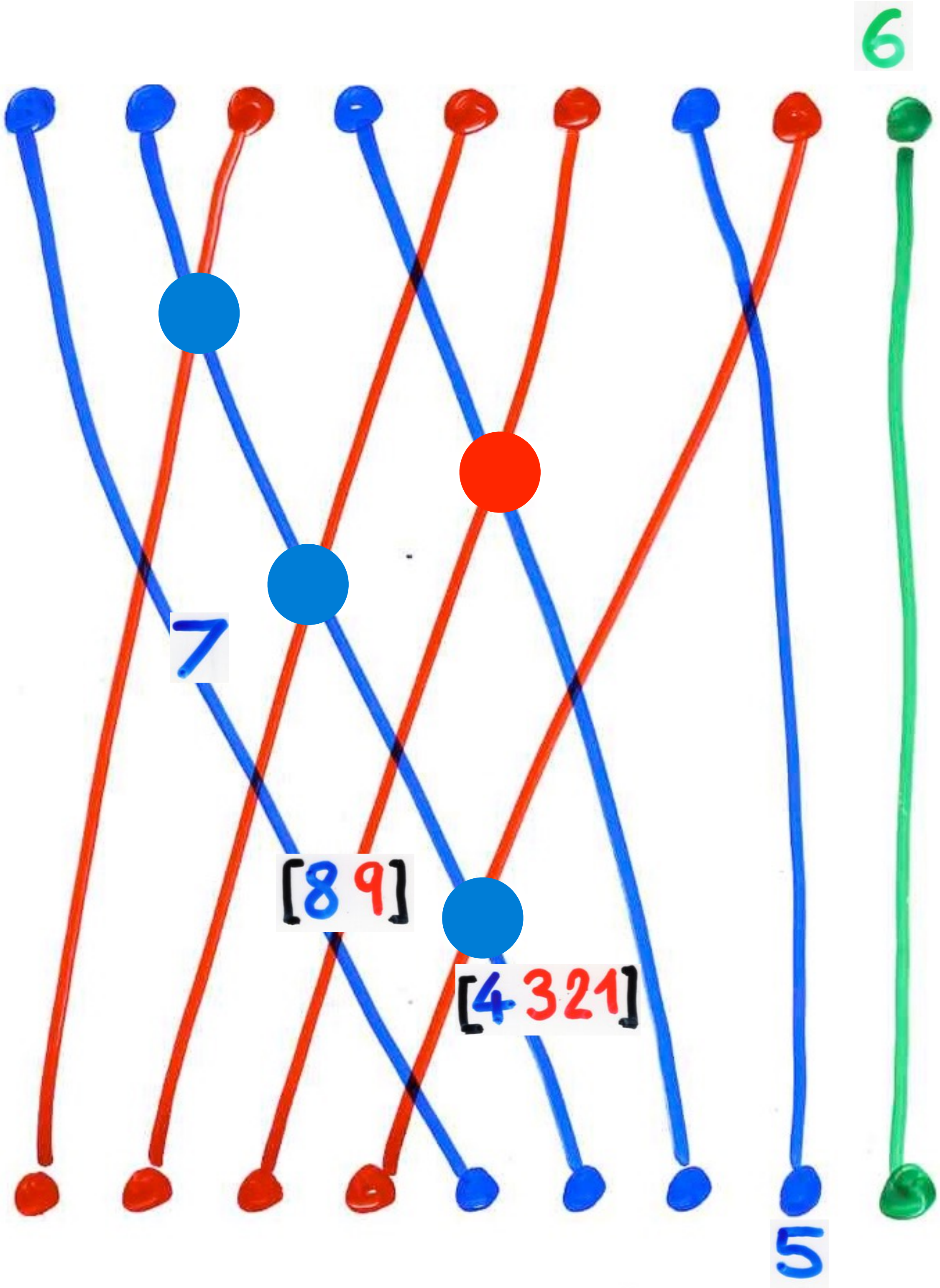


6
7
[8 9]
[4 3 2] 1
5


6
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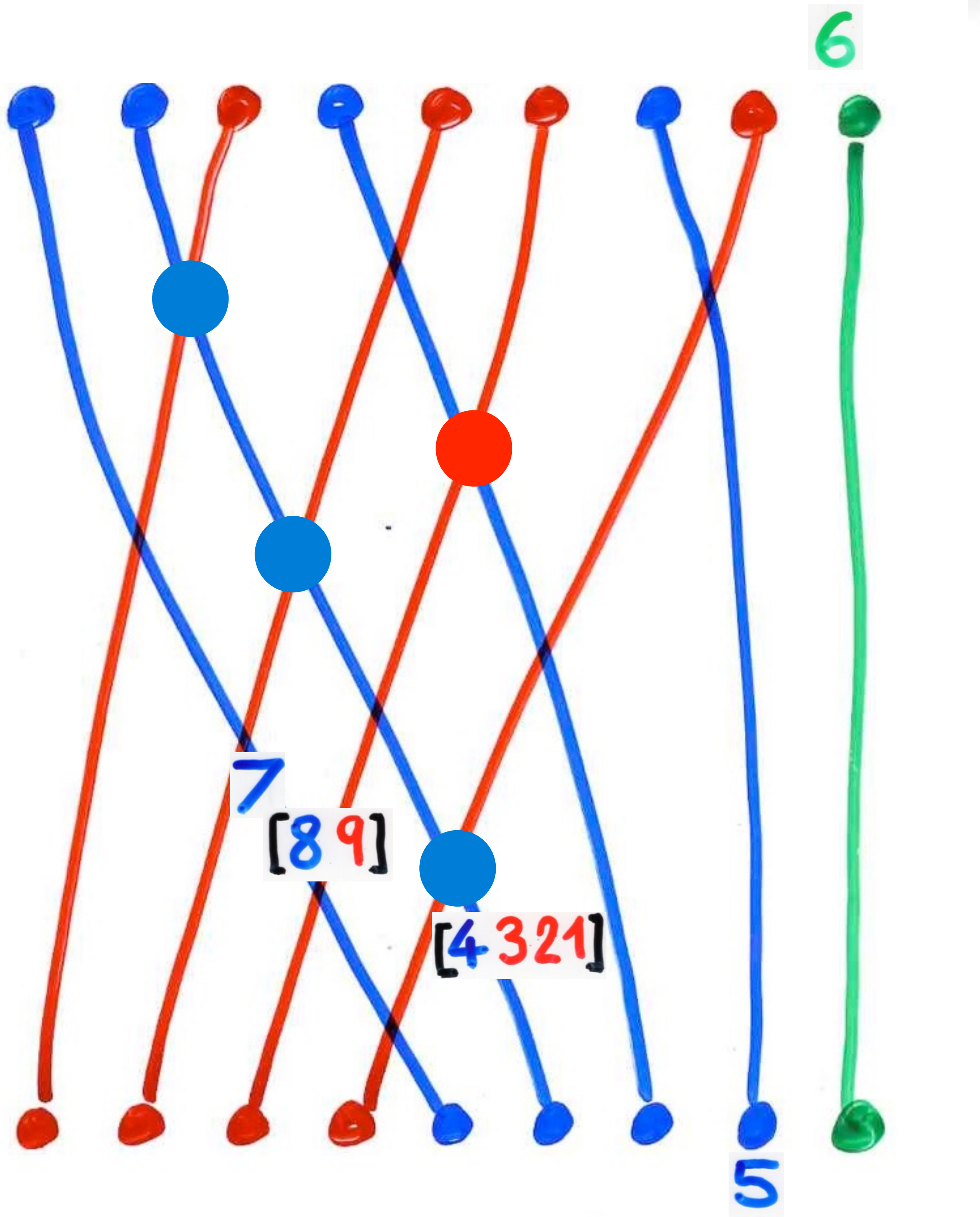
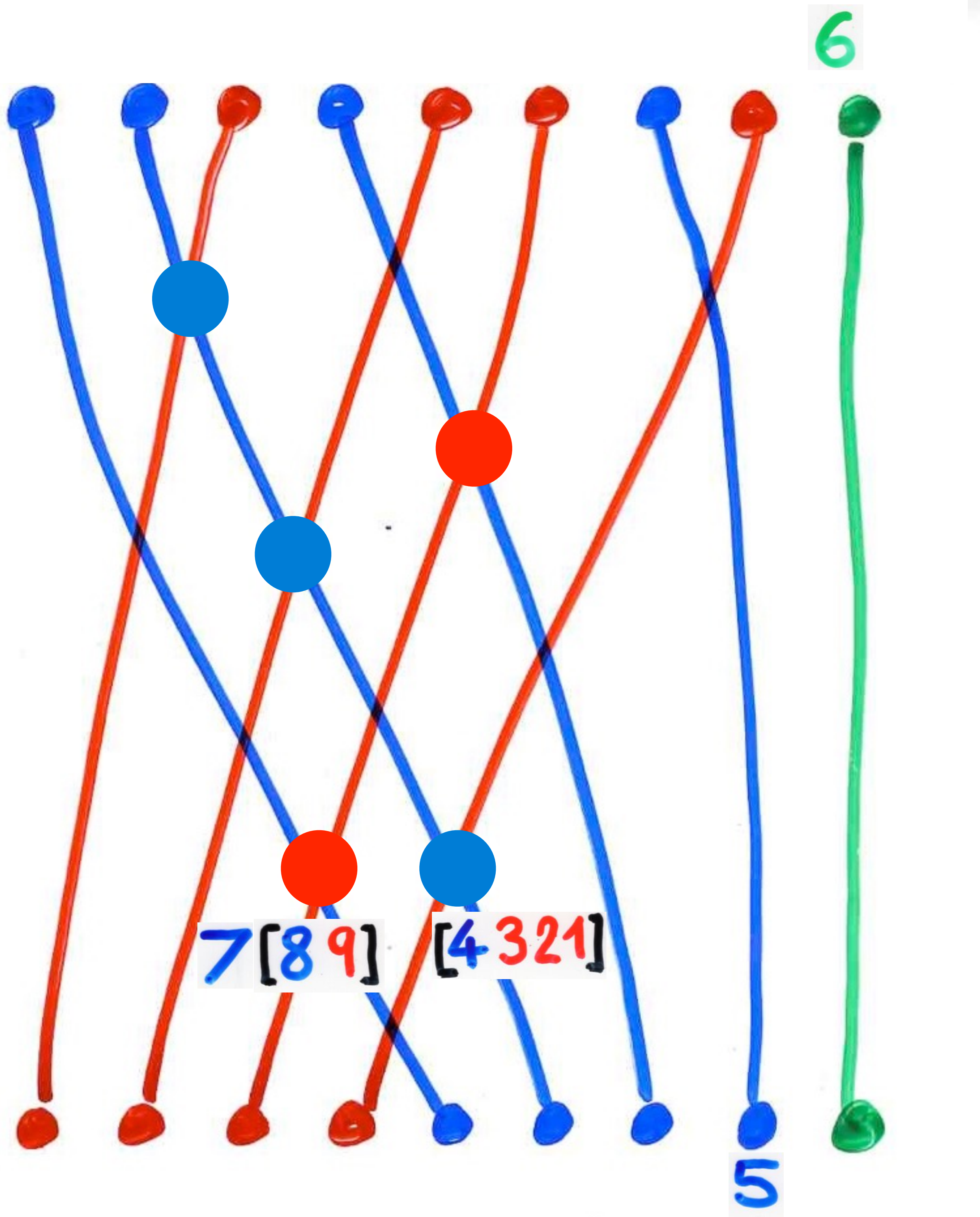
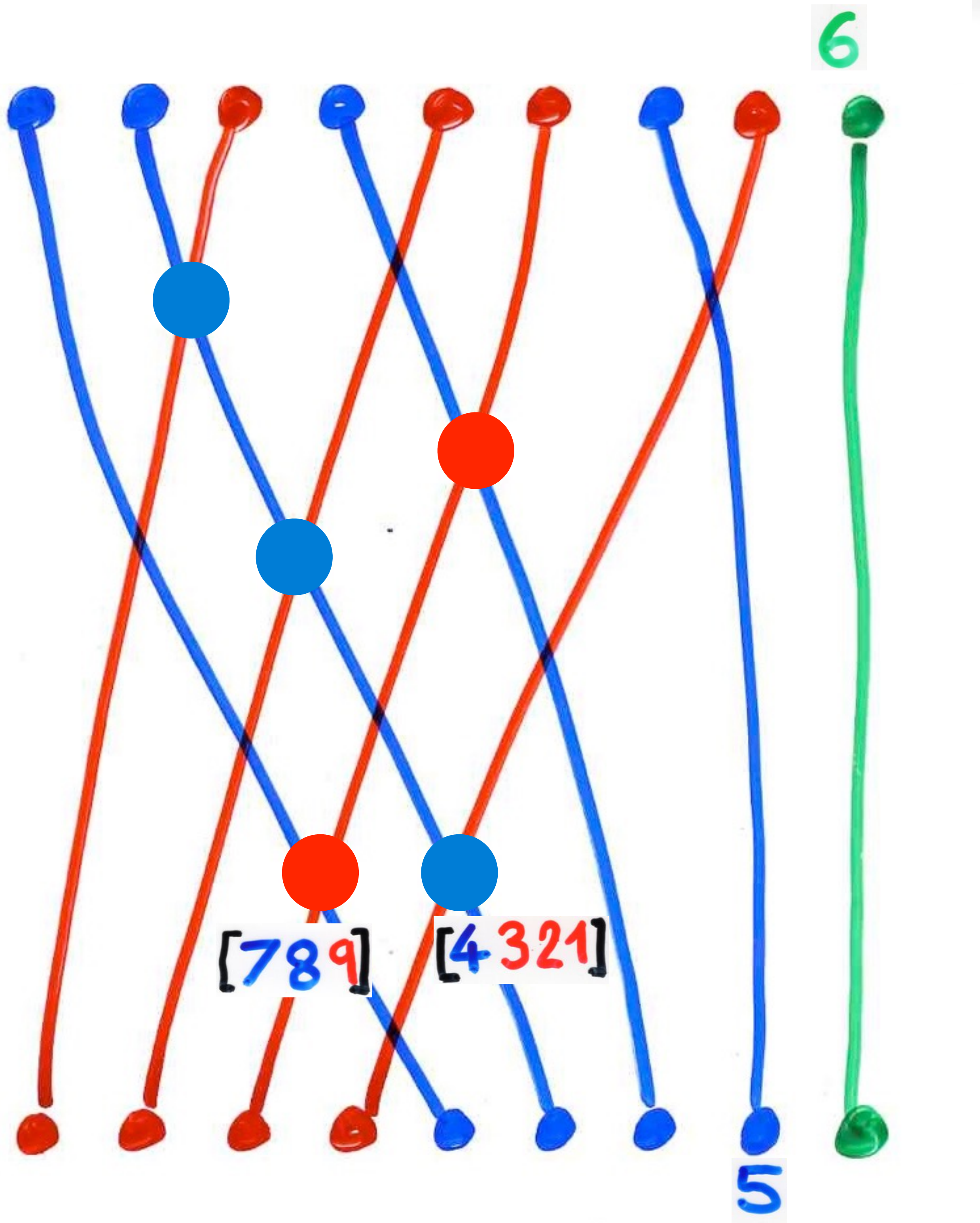
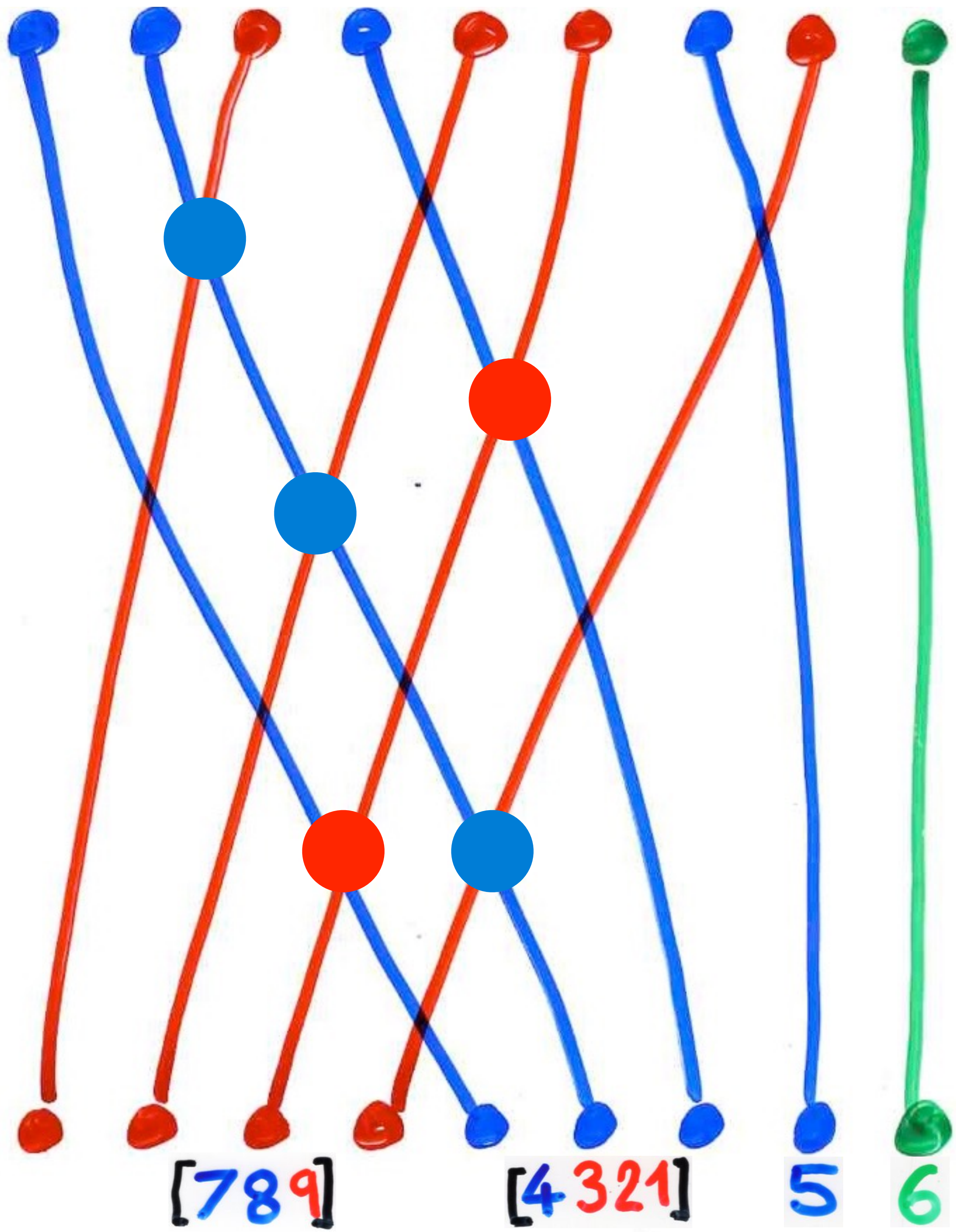
[8 9]
[4 3 2 1]
5



6
7
[8 9]
[4 3 2 1]
5

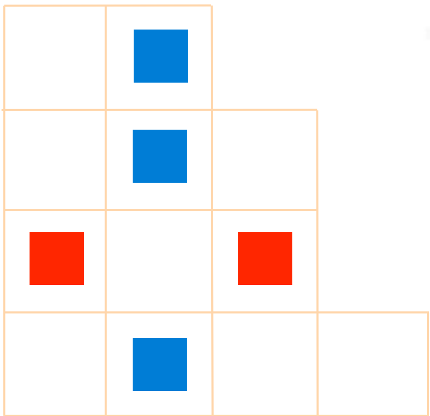

6
7 [8 9]
[4 3 2 1]
5

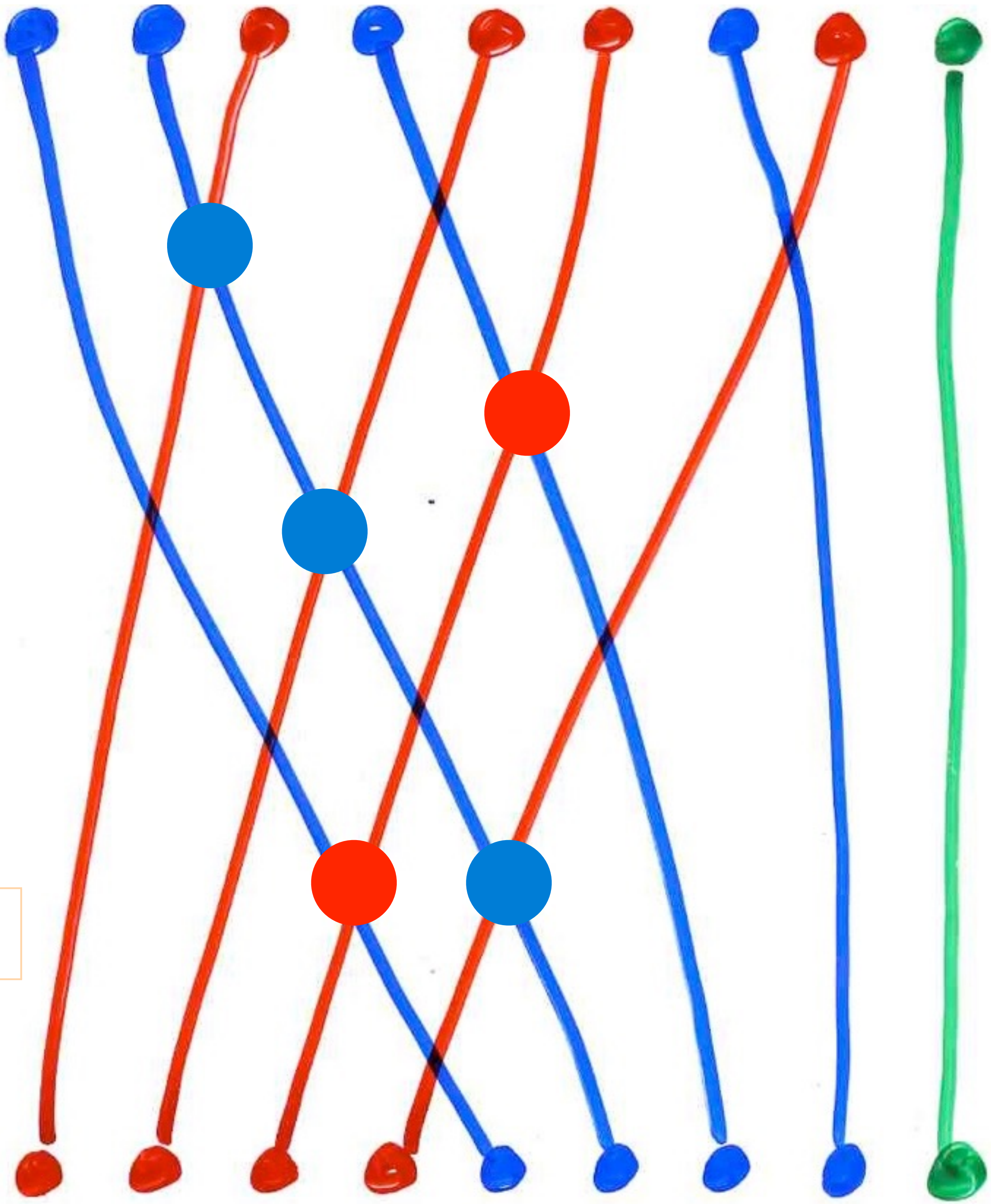

6
[789]
[4321]
5



[789]
[4321]
5
6

"exchange-
fusion"
algorithm





"exchange-fusion" algorithm (X.V. 2008)

EXF

alternative tableaux ⟷ permutations

equivalent to a bijection
Corteel, Nadeau (2007)

(with permutation tableaux)

Steingrimsson, Williams
(2005, 2007)

Postnikov

"The cellular ansatz"

quadratic algebra Q

$UD = DU + Id$

Physics

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

representation of Q by combinatorial operators

bijections

RSK ⟷

pairs of Young tableaux

(i) first step

(ii) second step

$DE = qED + E + D$

alternative tableaux

EXF ⟷

permutations

commutations

rewriting rules

planarization

alternative tableaux $\overset{\text{EXF}}{\longleftrightarrow}$ permutations

This bijection can be obtained

- from a combinatorial representation of the PASEP algebra (X.V., 2008)

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurence of ○○ or ●●)

4 operators A, S, J, K

4 operators $A, S, J, K,$ $\quad u \in B$

$$\langle u | A = \sum_{\substack{\text{letter } \circ \\ \text{of } u}} v \quad , \quad v \text{ obtained by: } \circ \rightarrow \circ\bullet\circ$$

$$\langle u | S = \sum_{\circ \text{ of } u} v \quad \quad v \text{ obtained by: } \circ \rightarrow \bullet \quad (\text{and } \bullet\bullet \rightarrow \bullet \quad \bullet\bullet\bullet \rightarrow \bullet)$$

$$\langle u | J = \sum_{\circ \text{ of } u} v \quad \quad v, \quad \circ \rightarrow \bullet\circ \quad (\text{and } \bullet\bullet \rightarrow \bullet)$$

$$\langle u | K = \sum_{\substack{\circ \\ \text{of } u}} v \quad \quad v, \quad \circ \rightarrow \circ\bullet \quad (\text{and } \bullet\bullet \rightarrow \bullet)$$

$$\circ\bullet\circ\bullet \,|\, S = \bullet\circ\bullet + \circ\bullet$$

$$D = A + J$$
$$E = S + K$$

claim:

$$DE = ED + E + D$$

AS = SA + J + K

AK = KA + A

JS = SJ + S

JK = KJ

D = A + J

E = S + K

$$D = A + J$$

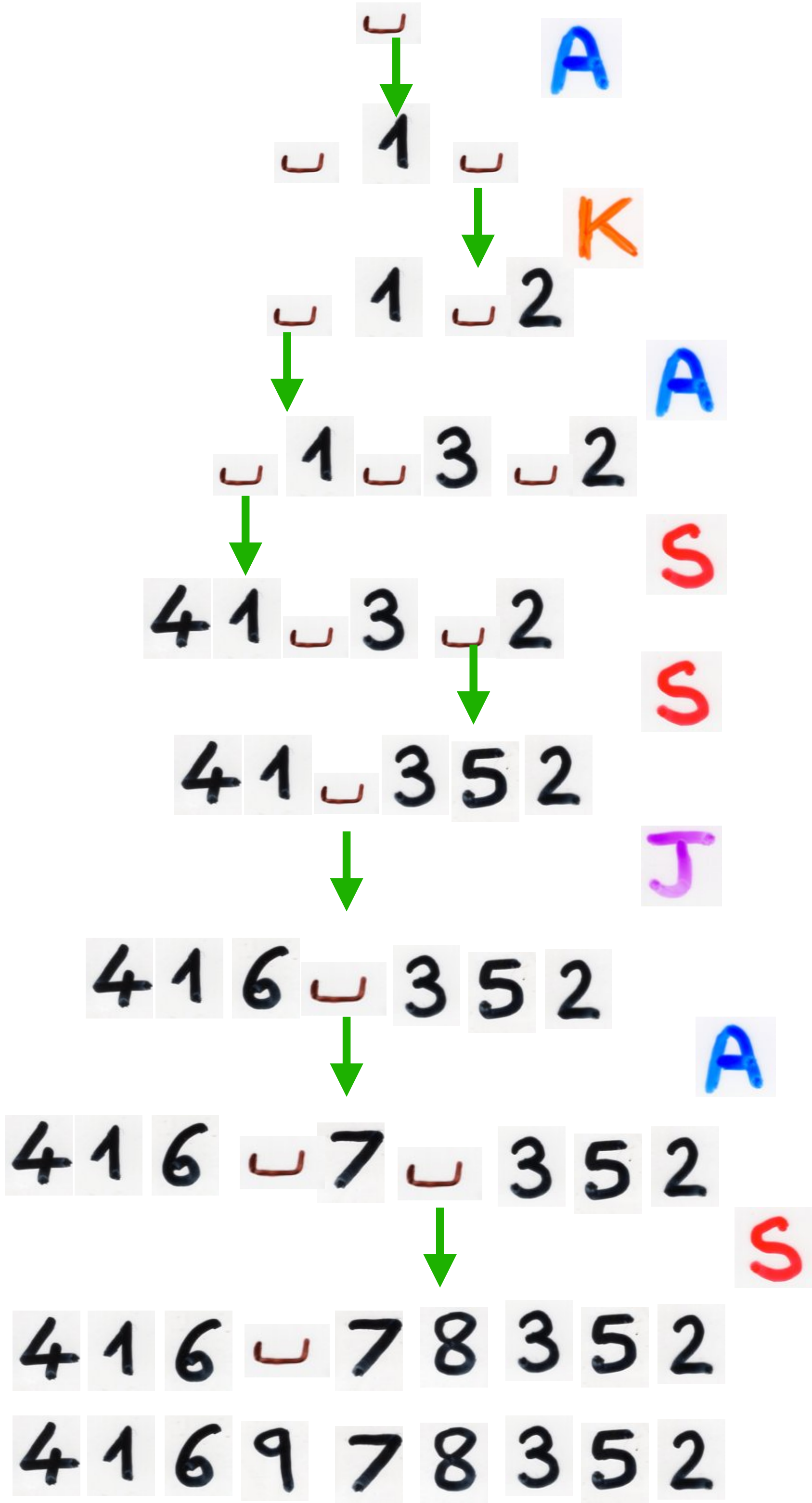
$$E = S + K$$

$$
\begin{aligned}
DE &= (A+J)(S+K) \\
&= AS + AK + JS + JK \\
&= \underbrace{(SA + KA + SJ + KJ)}_{\underbrace{(S+K)(A+J)}_{ED}} + \underbrace{J + K + A + S}_{E + D}
\end{aligned}
$$

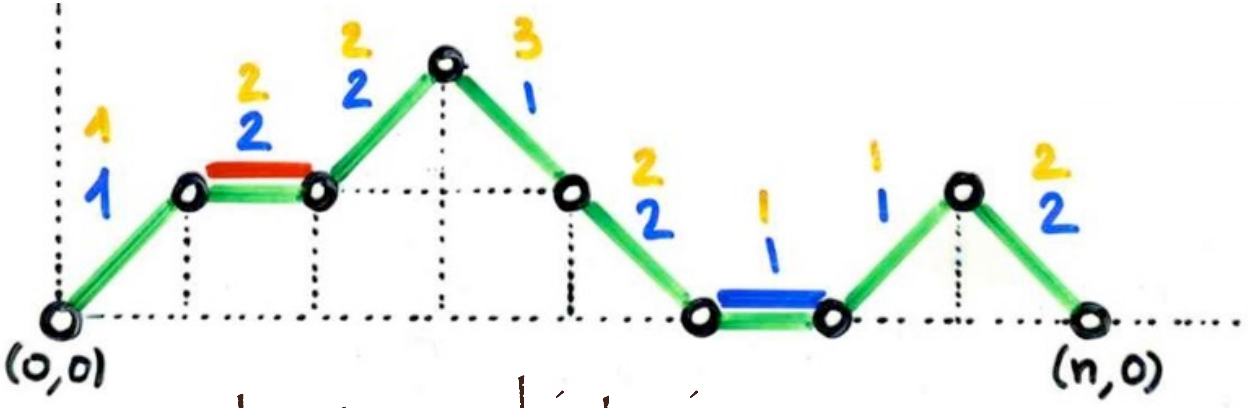
$$DE = ED + E + D$$

claim:

$$
\begin{aligned}
&\langle 0 | (D+E)^n | \circ \bullet \rangle \\
+ &\langle 0 | (D+E)^n | \bullet \circ \rangle \\
+ &\langle 0 | (D+E)^n | \bullet \circ \bullet \rangle = (n+1)!
\end{aligned}
$$



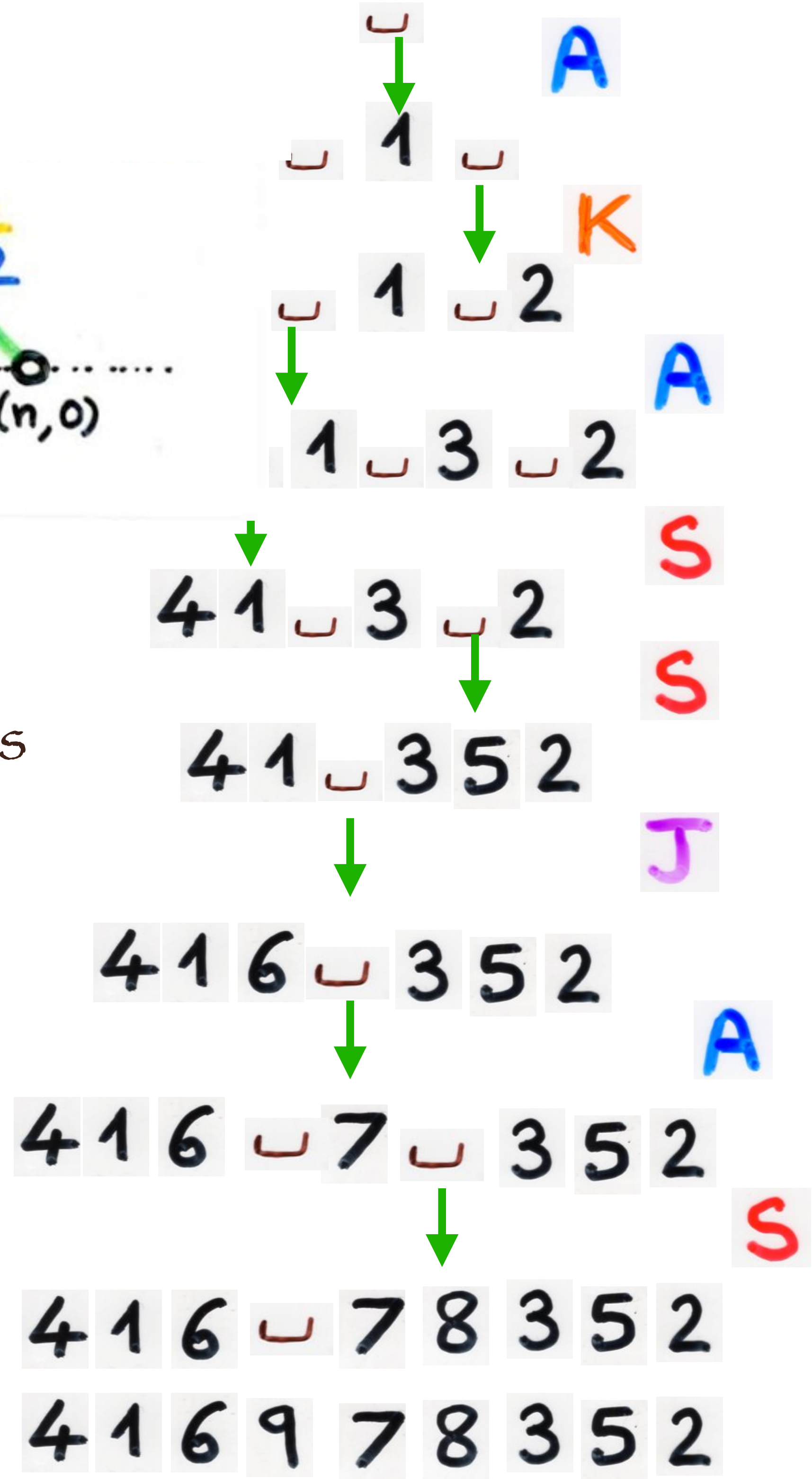
A
1
K
1 2
A
1 3 2
S
4 1 3 2
S
4 1 3 5 2
J
4 1 6 3 5 2
A
4 1 6 7 3 5 2
S
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2



Laguerre histories

FV bijection

Permutations



Moments of Laguerre polynomials
combinatorial theory of
orthogonal polynomials

Tianjin lecture 4

the bijection

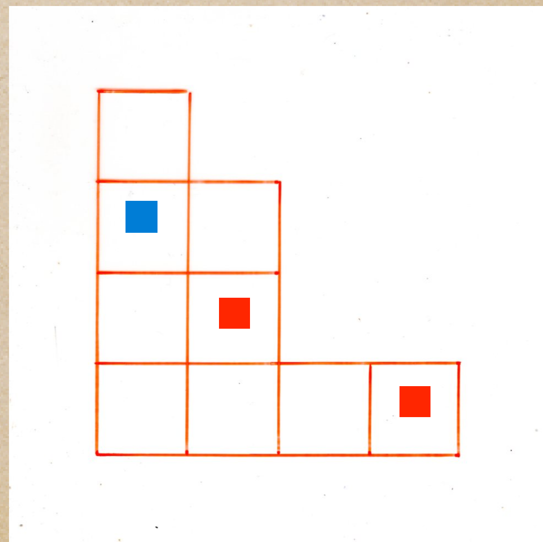
permutations — alternative tableaux

(Laguerre histories)

with local rules

(commutation diagrams)

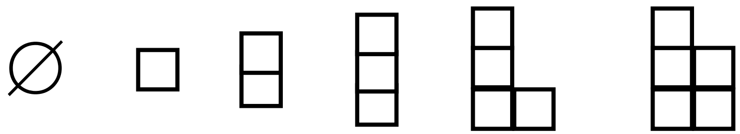
4 1 6 9 7 8 3 5 2 →



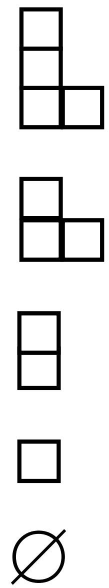
$$\langle 0|(D+E)^n|\circ\bullet\rangle + \langle 0|(D+E)^n|\bullet\circ\rangle + \langle 0|(D+E)^n|\bullet\circ\bullet\rangle = (n+1)!$$

Analogy with growth diagrams for
the Robinson-Schensted correspondence

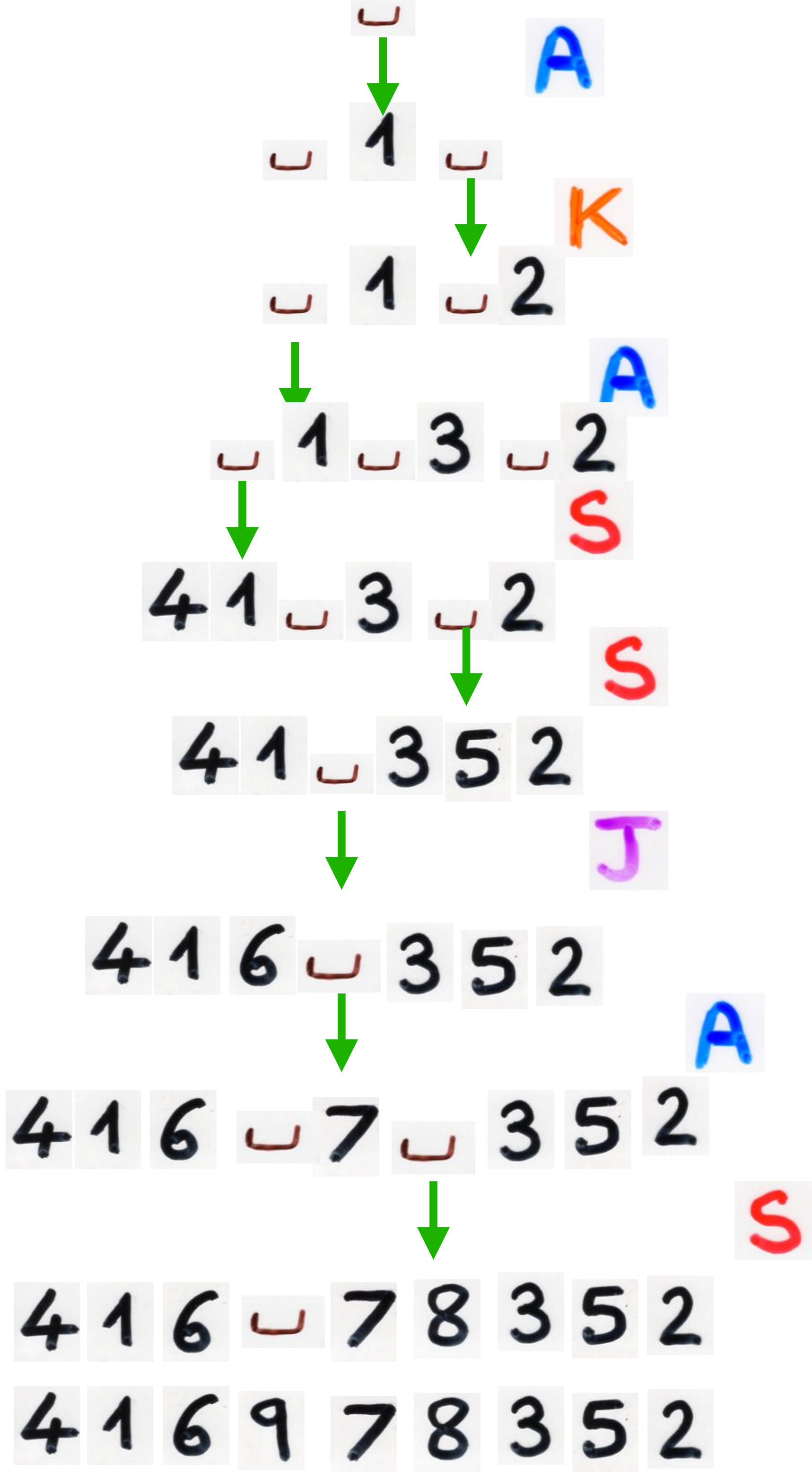
$$UD = DU + I$$



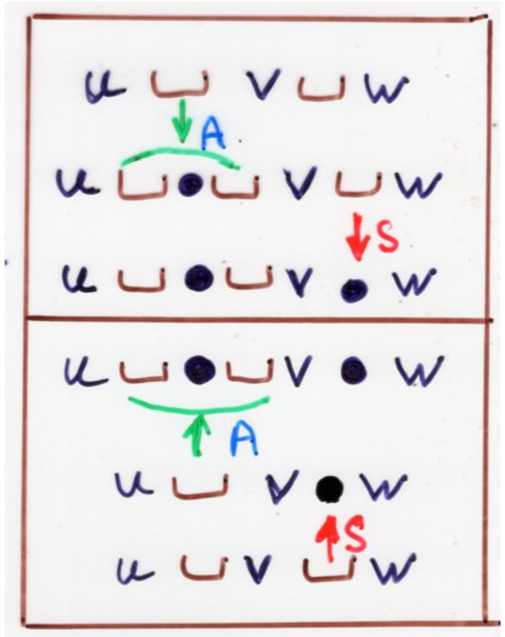
$$\langle \emptyset | U^n D^n | \emptyset \rangle = \sum_{\substack{\lambda \\ \text{partition} \\ \text{of } n}} \left(f_\lambda \right)^2$$

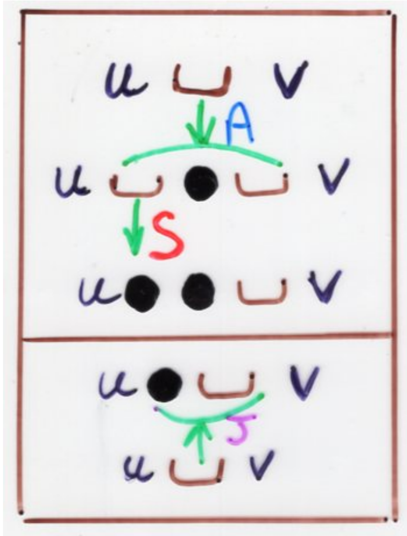


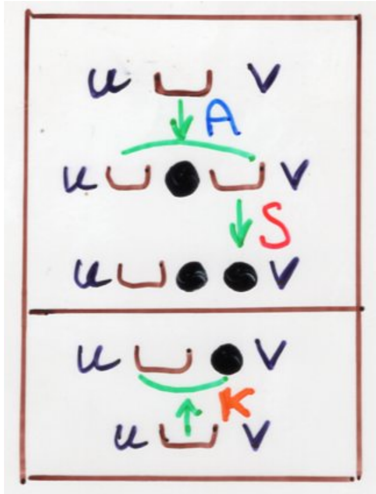
$$= n!$$



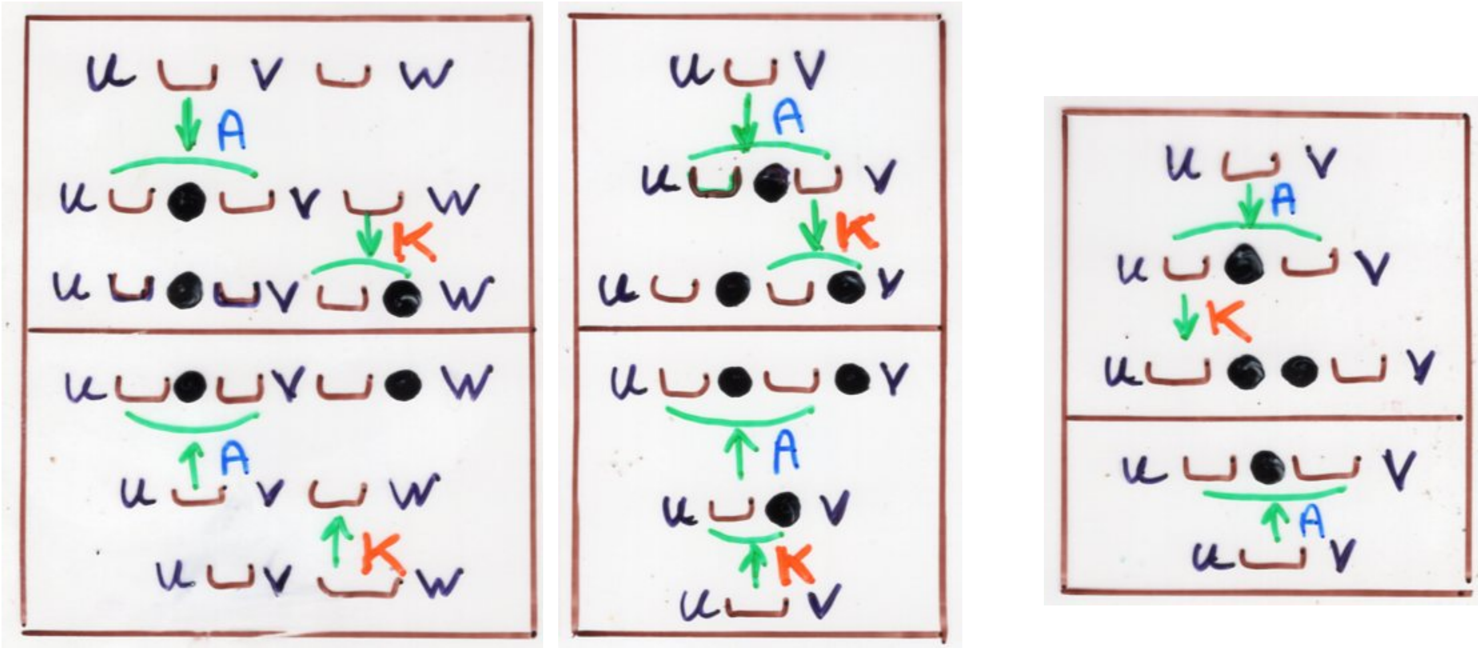
␣
A
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2



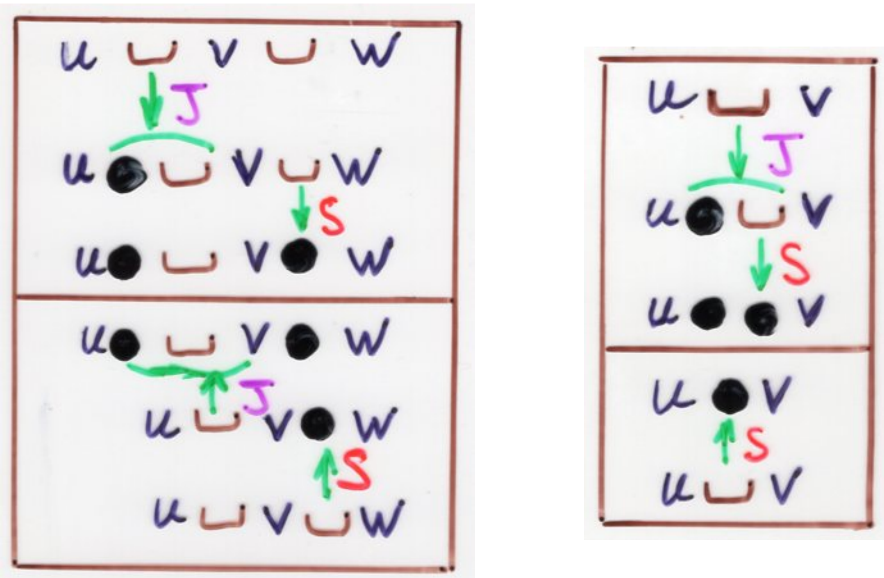




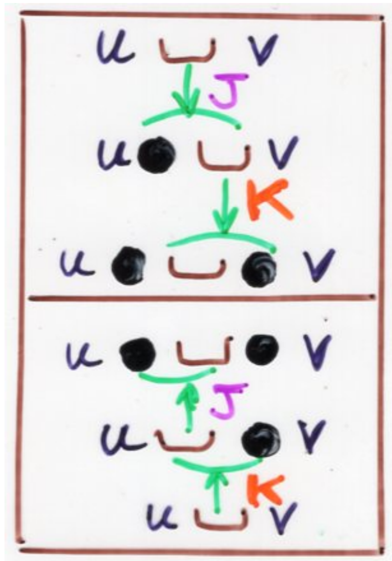
$$AS = SA + J + K$$



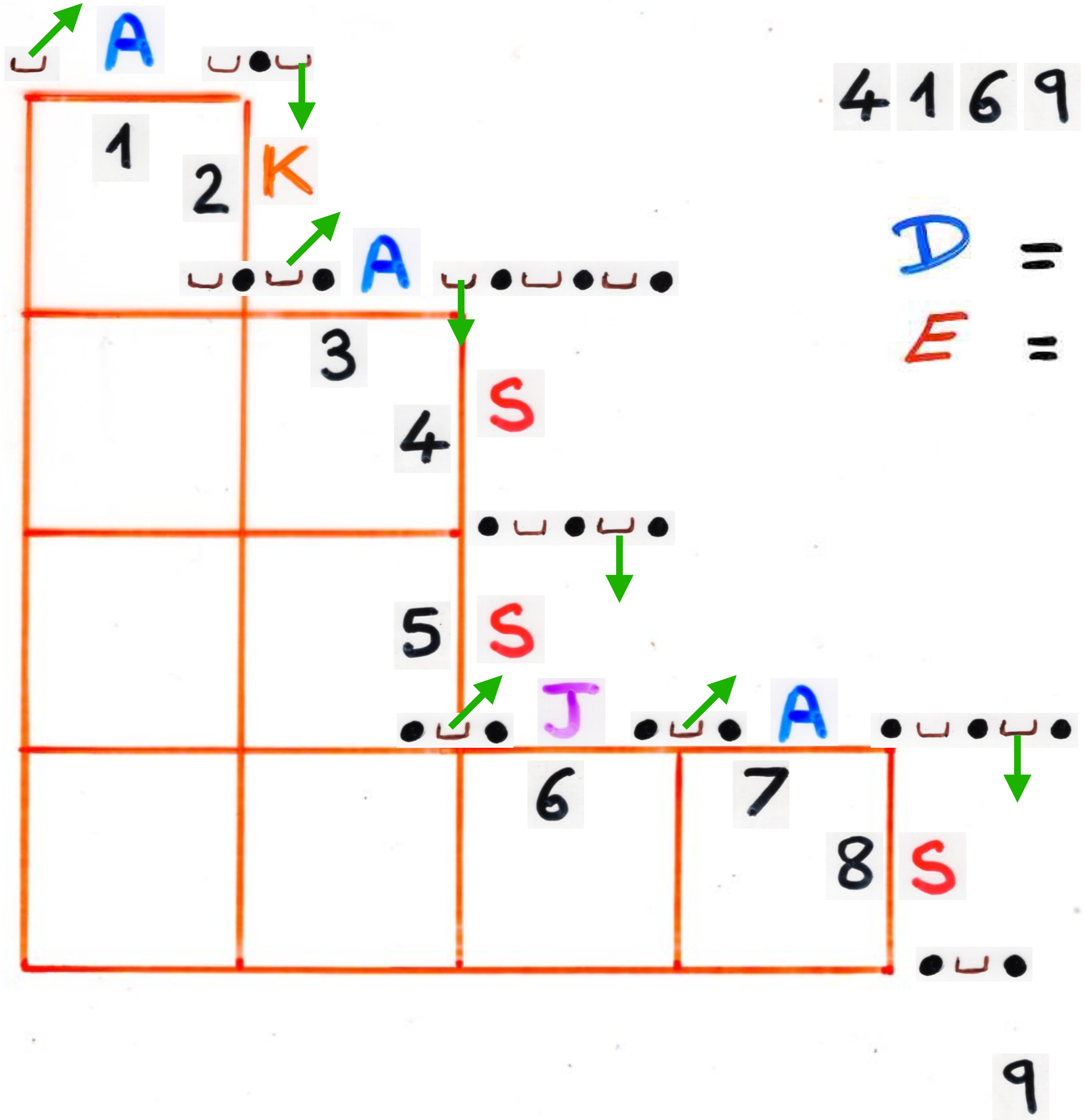
AK = KA + A



JS = SJ + S



JK = KJ



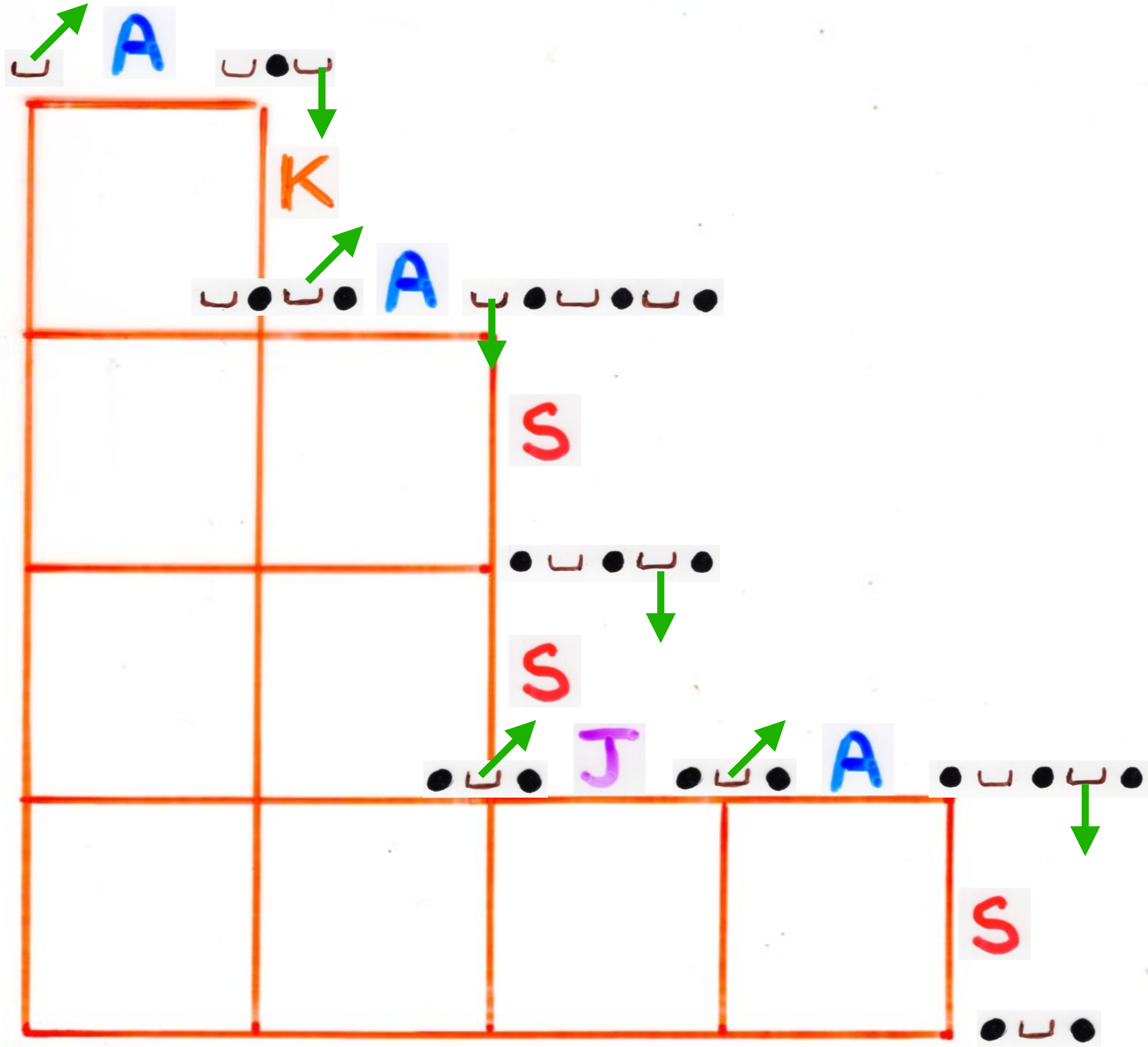
4 1 6 9 7 8 3 5 2

$D = A + J$

$E = S + K$

D

|E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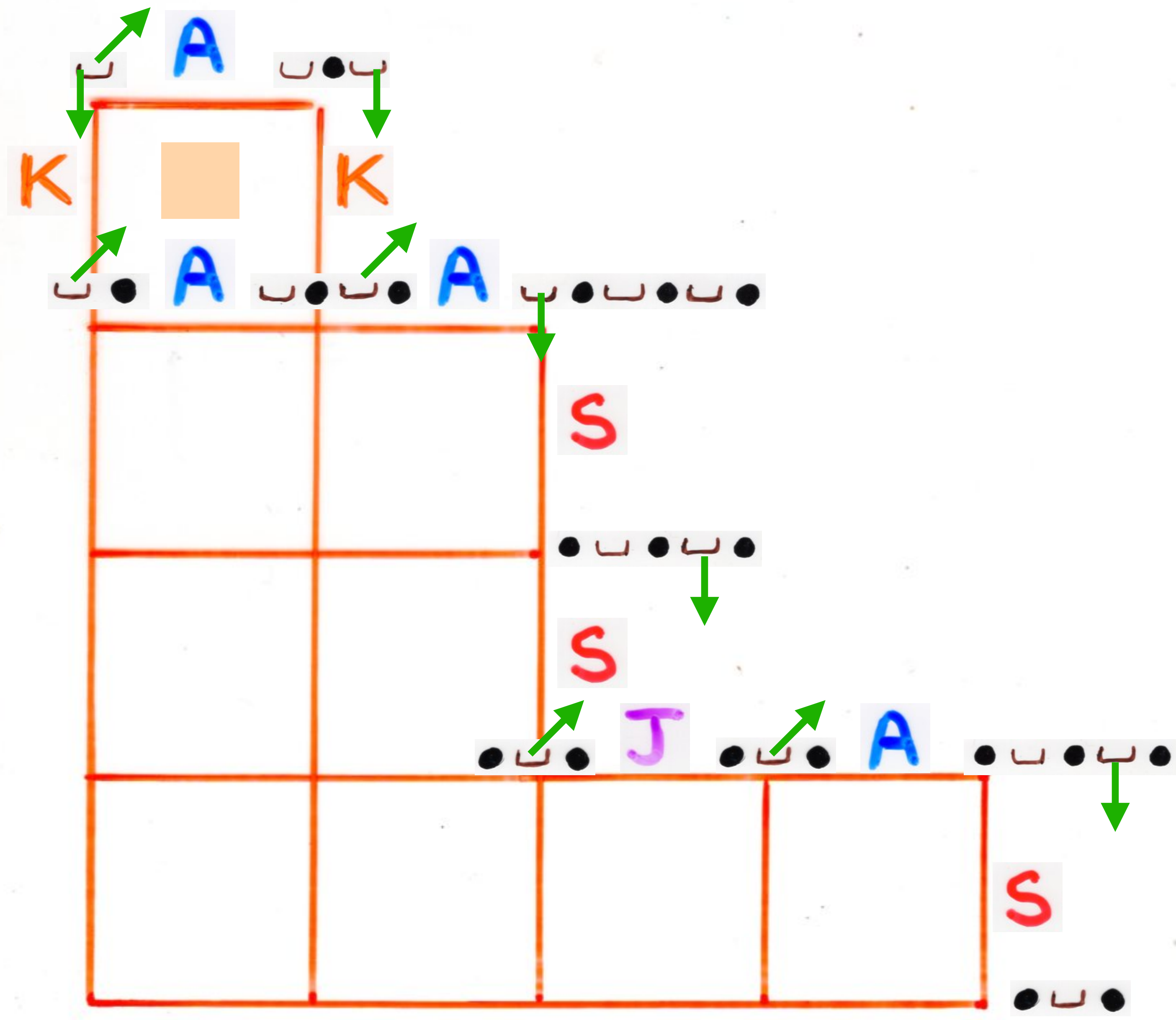
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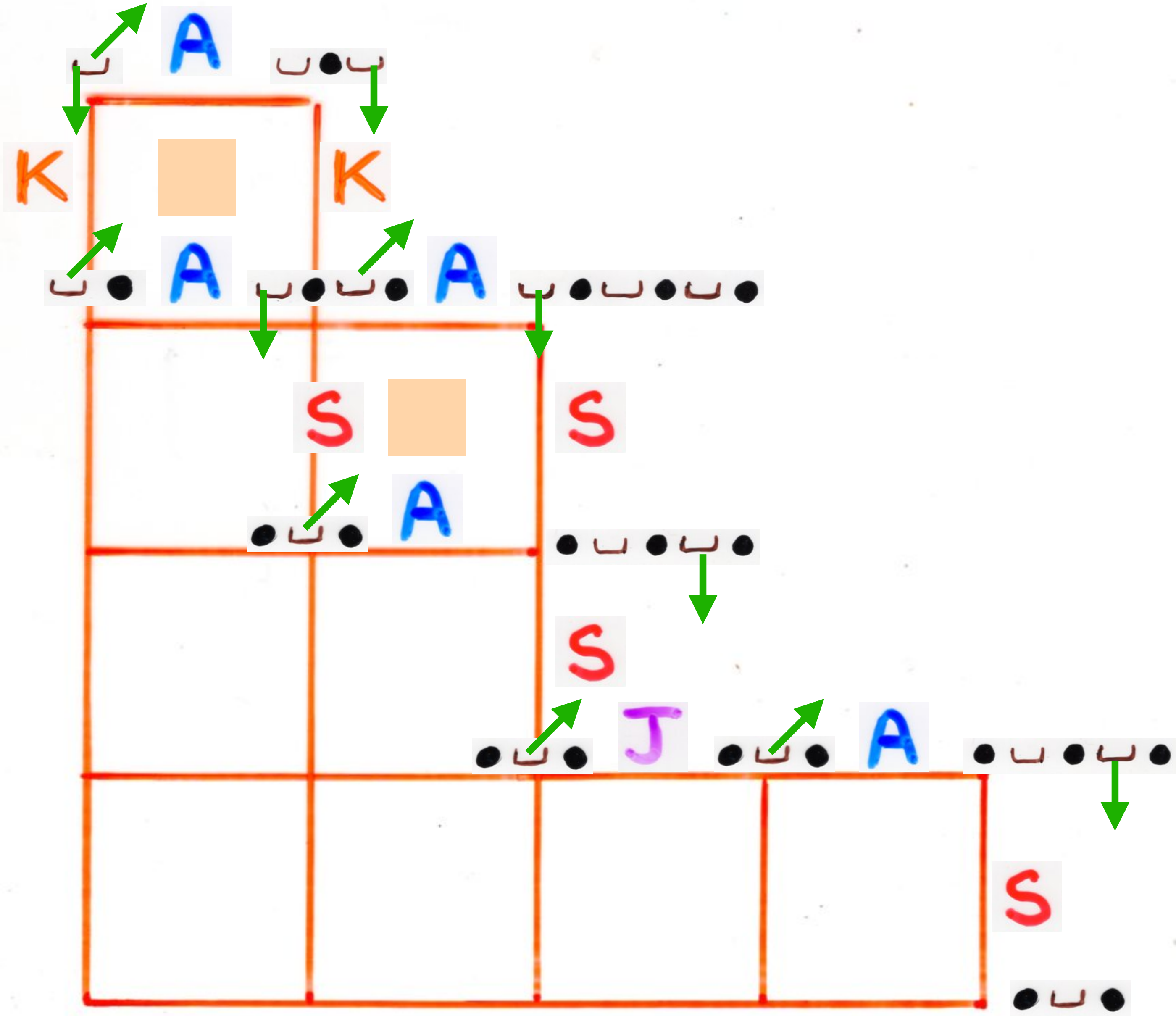
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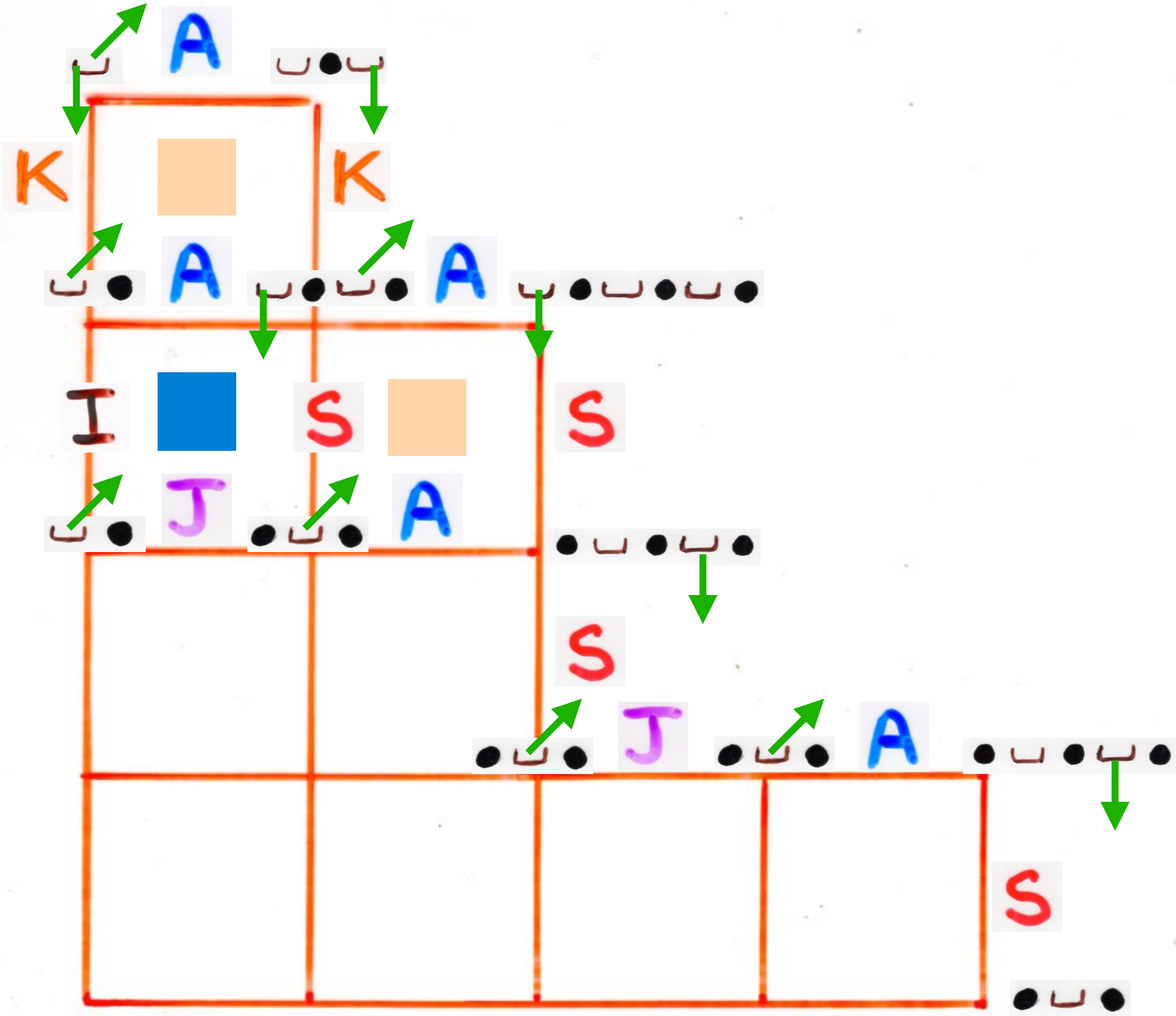

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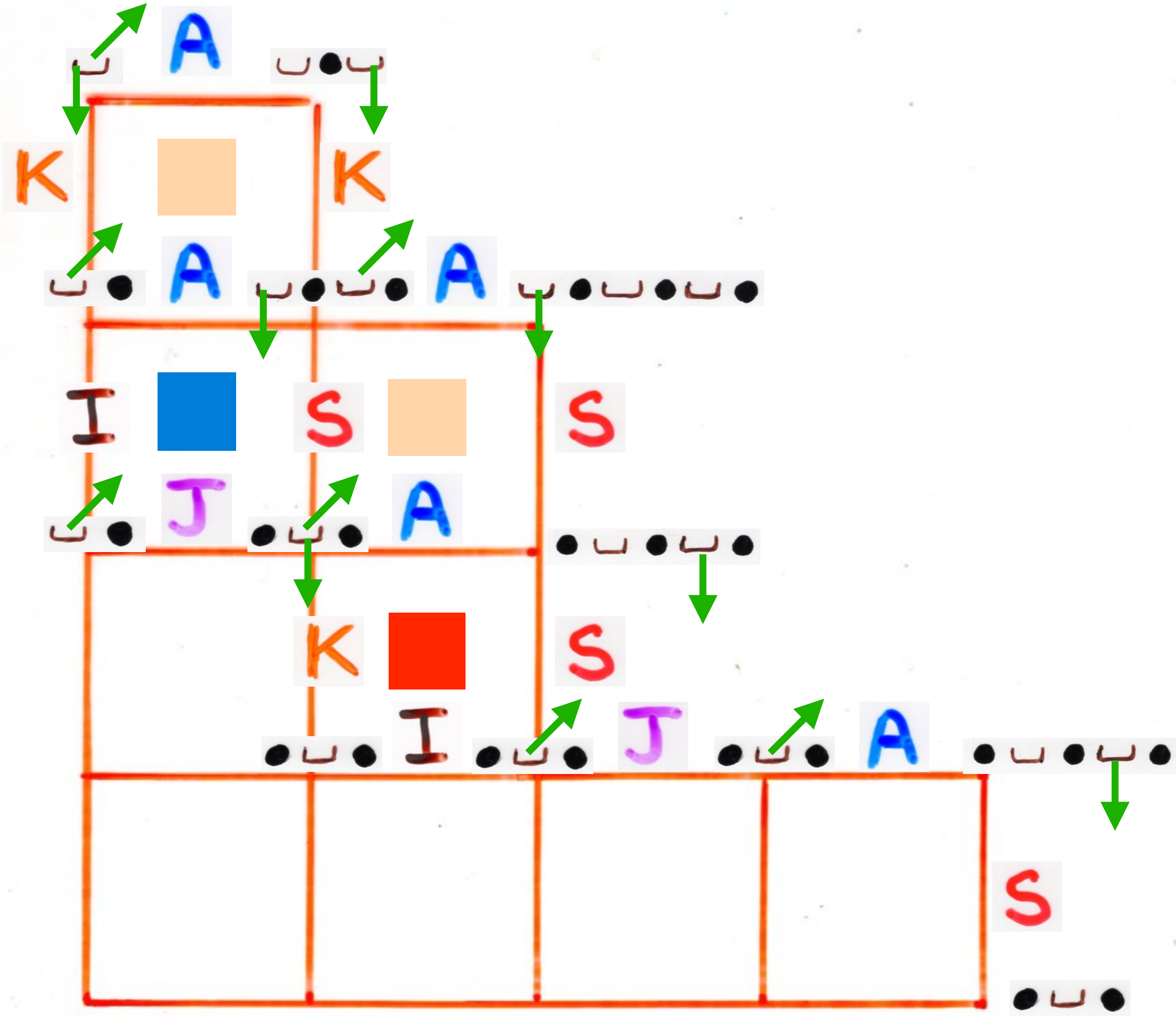
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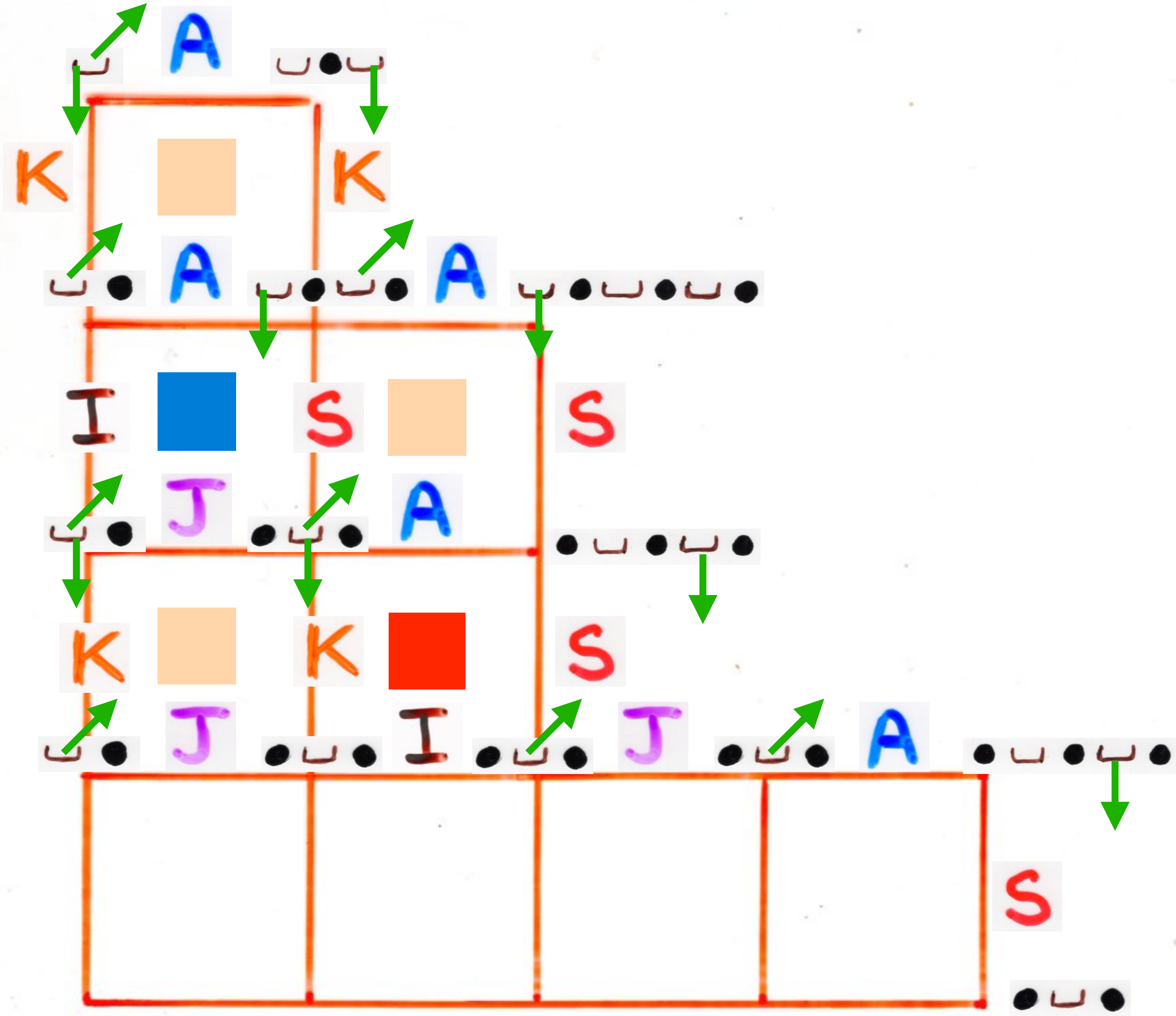

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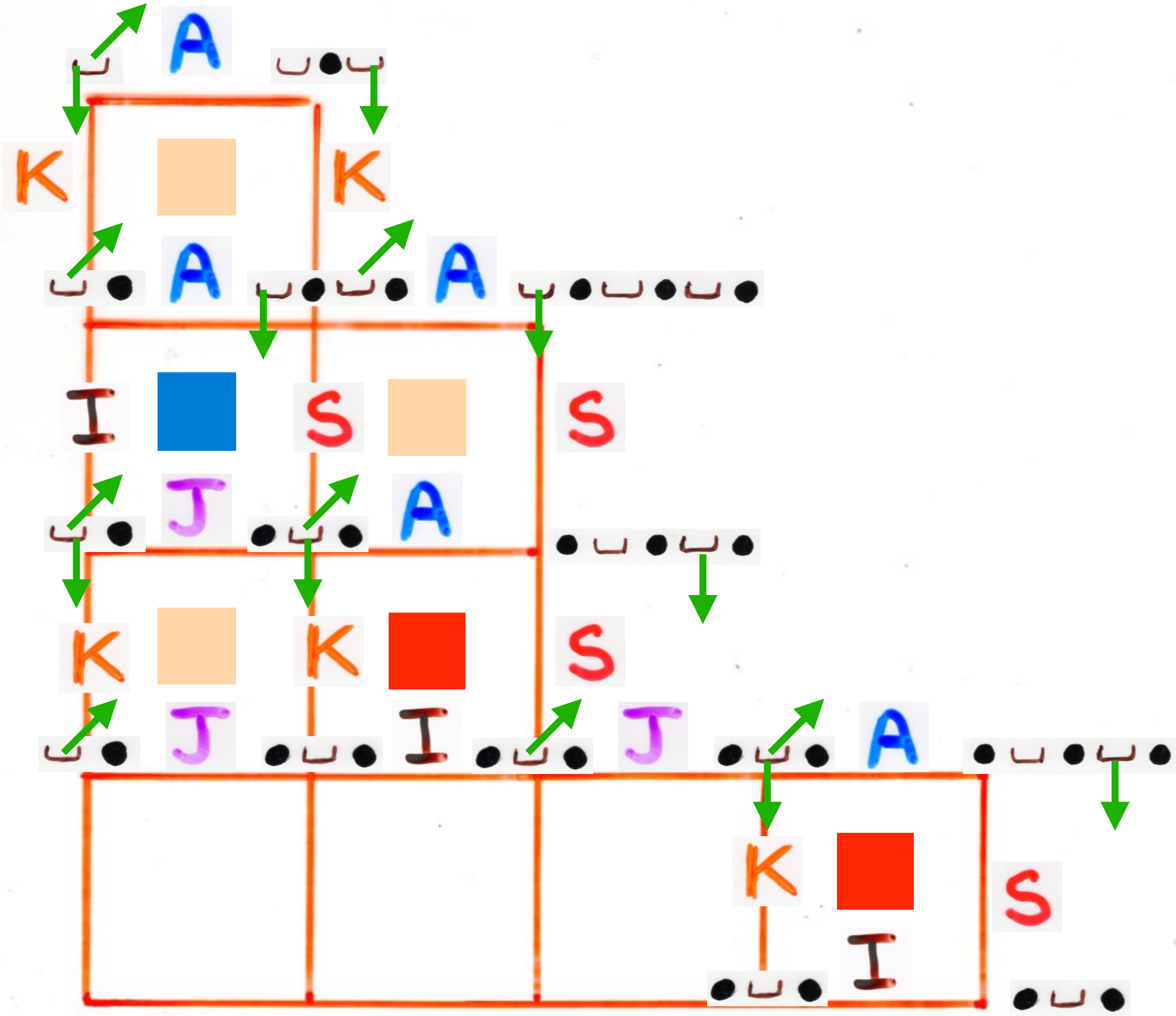

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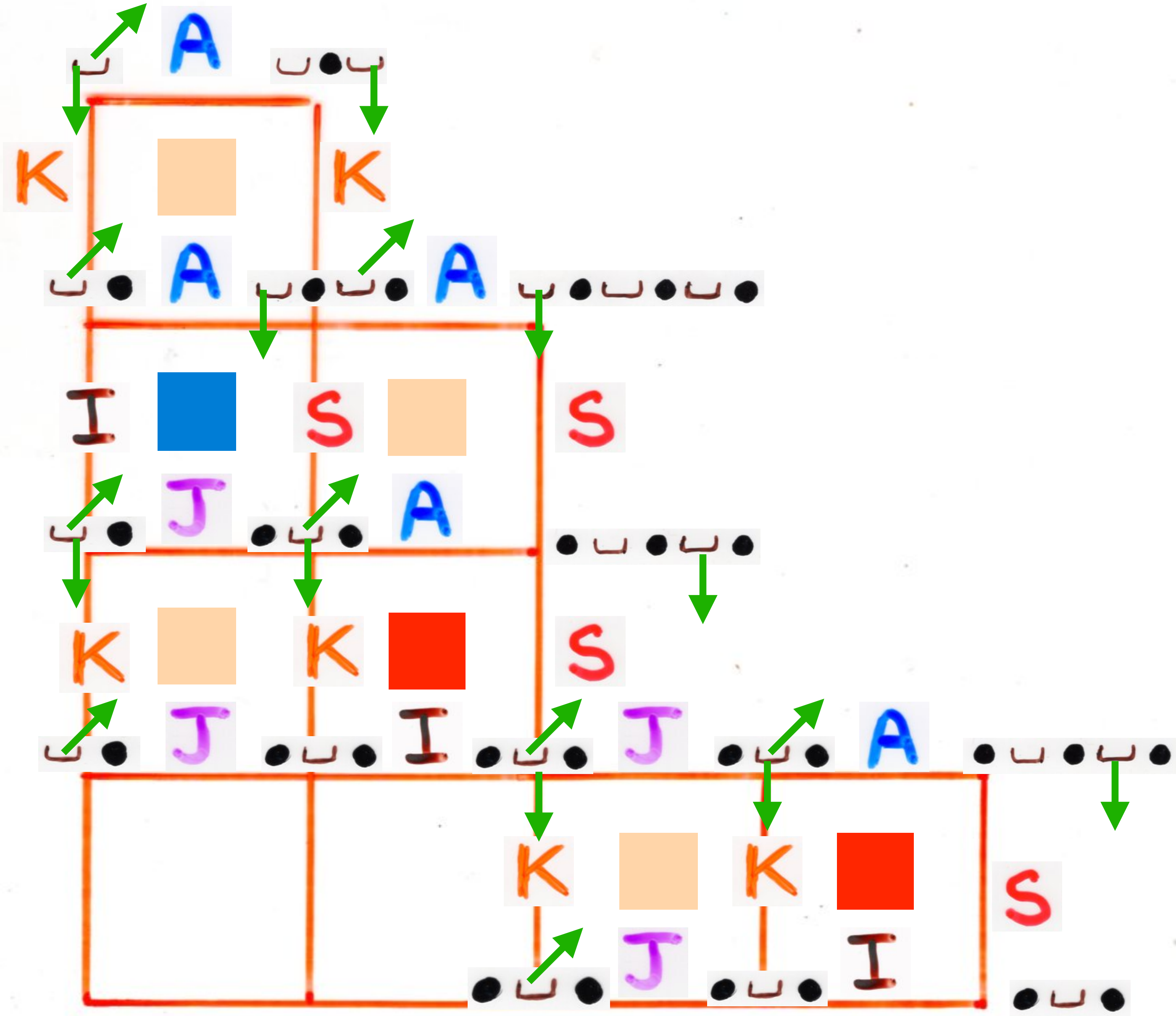
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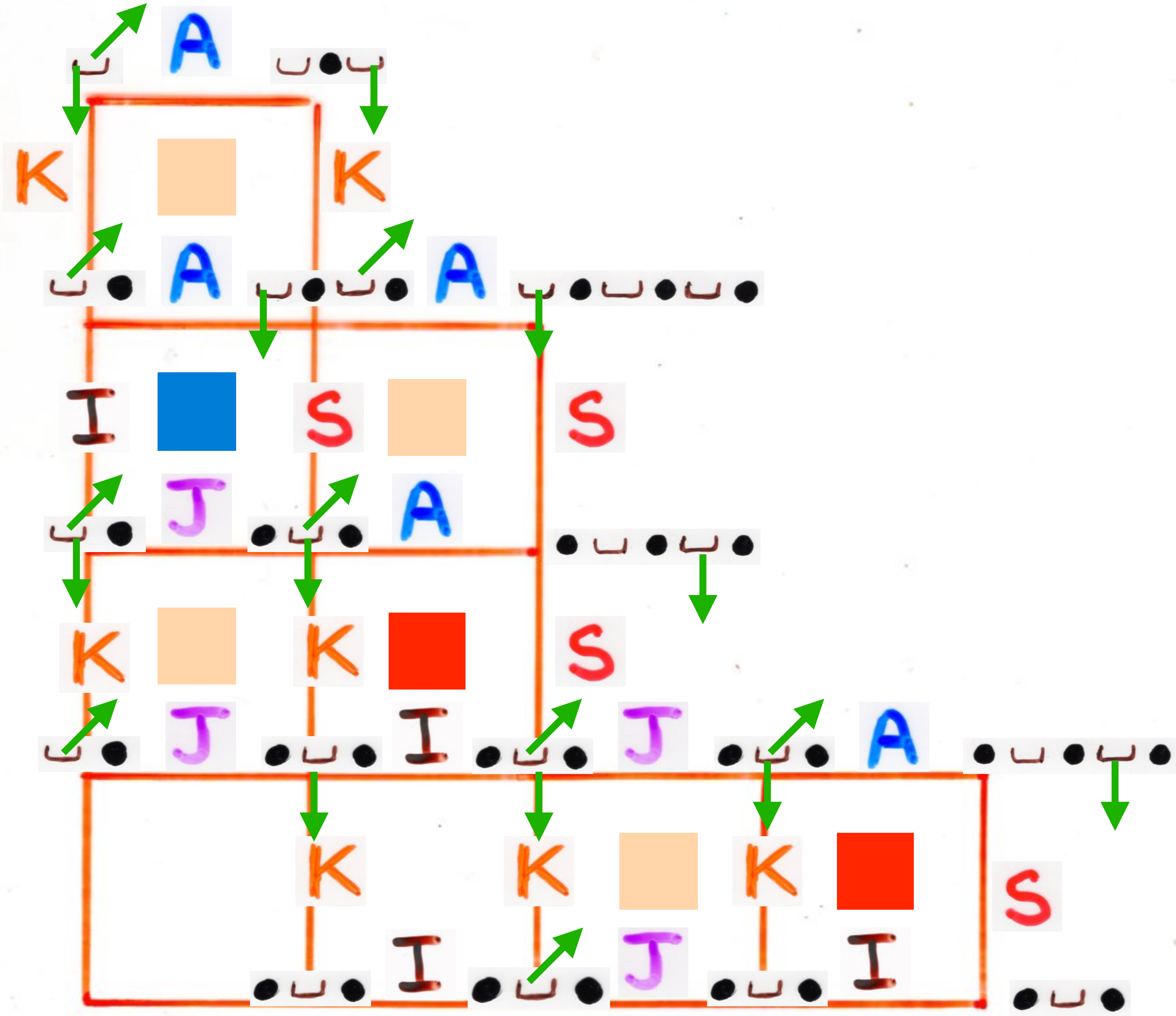
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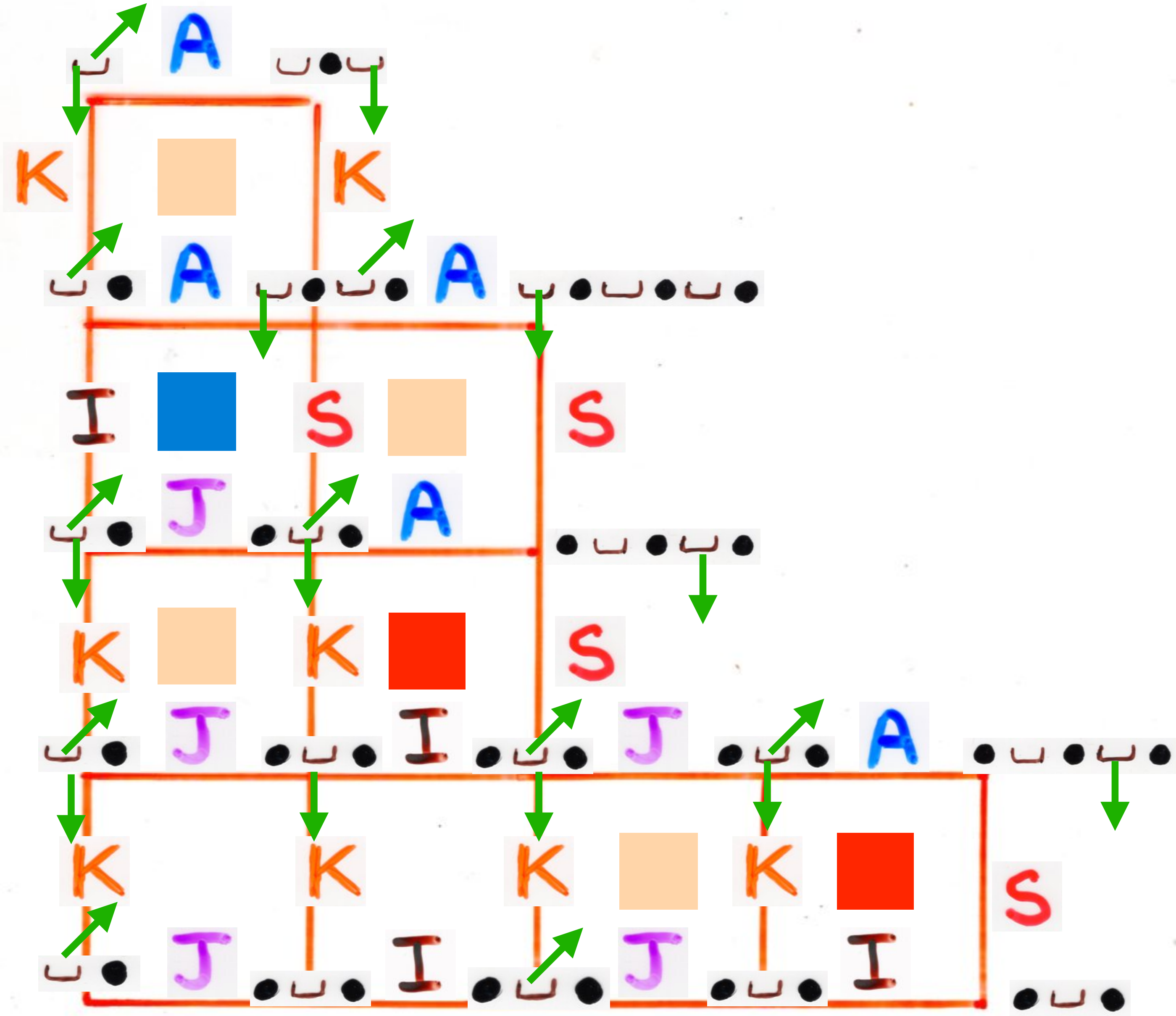
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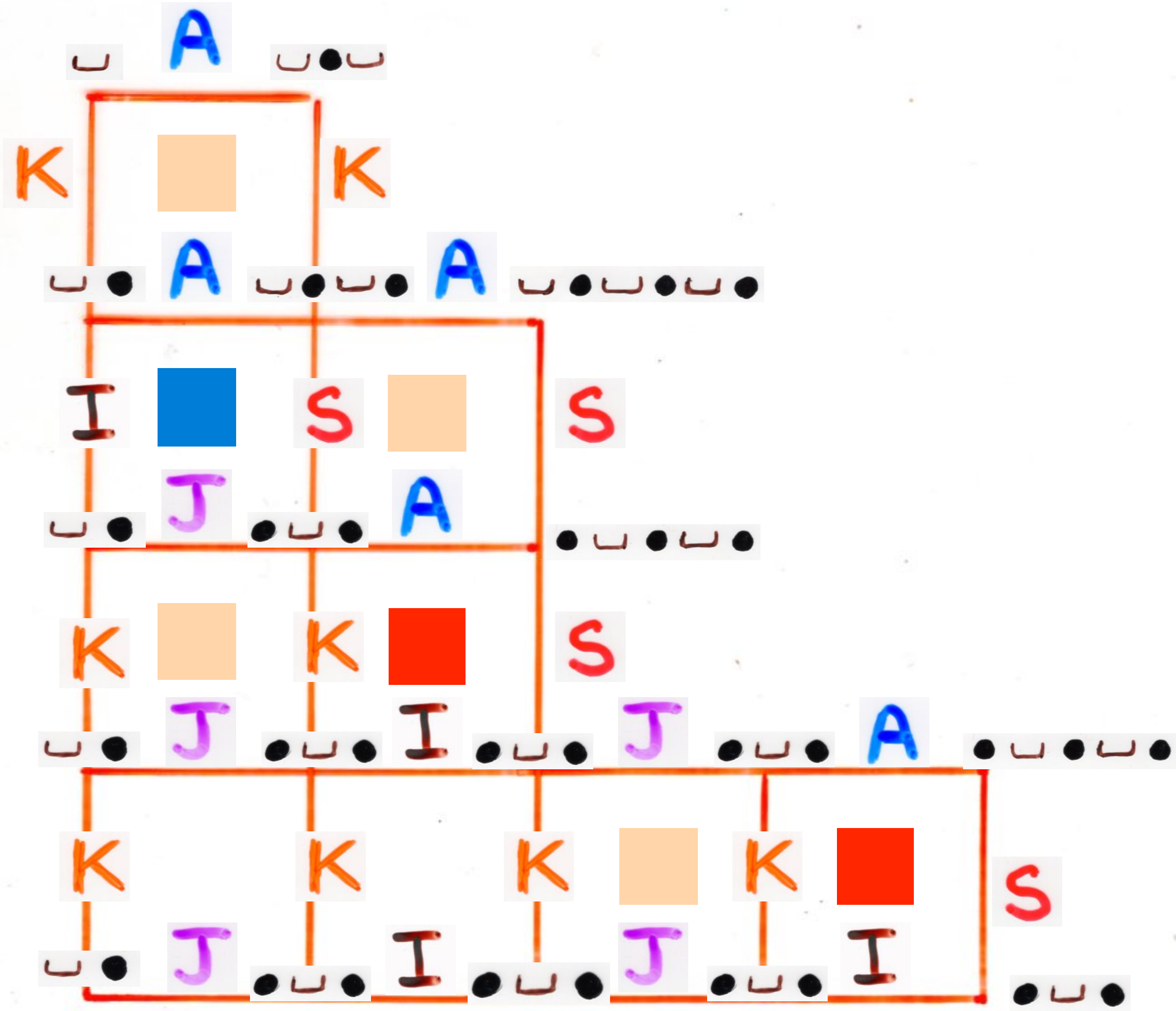

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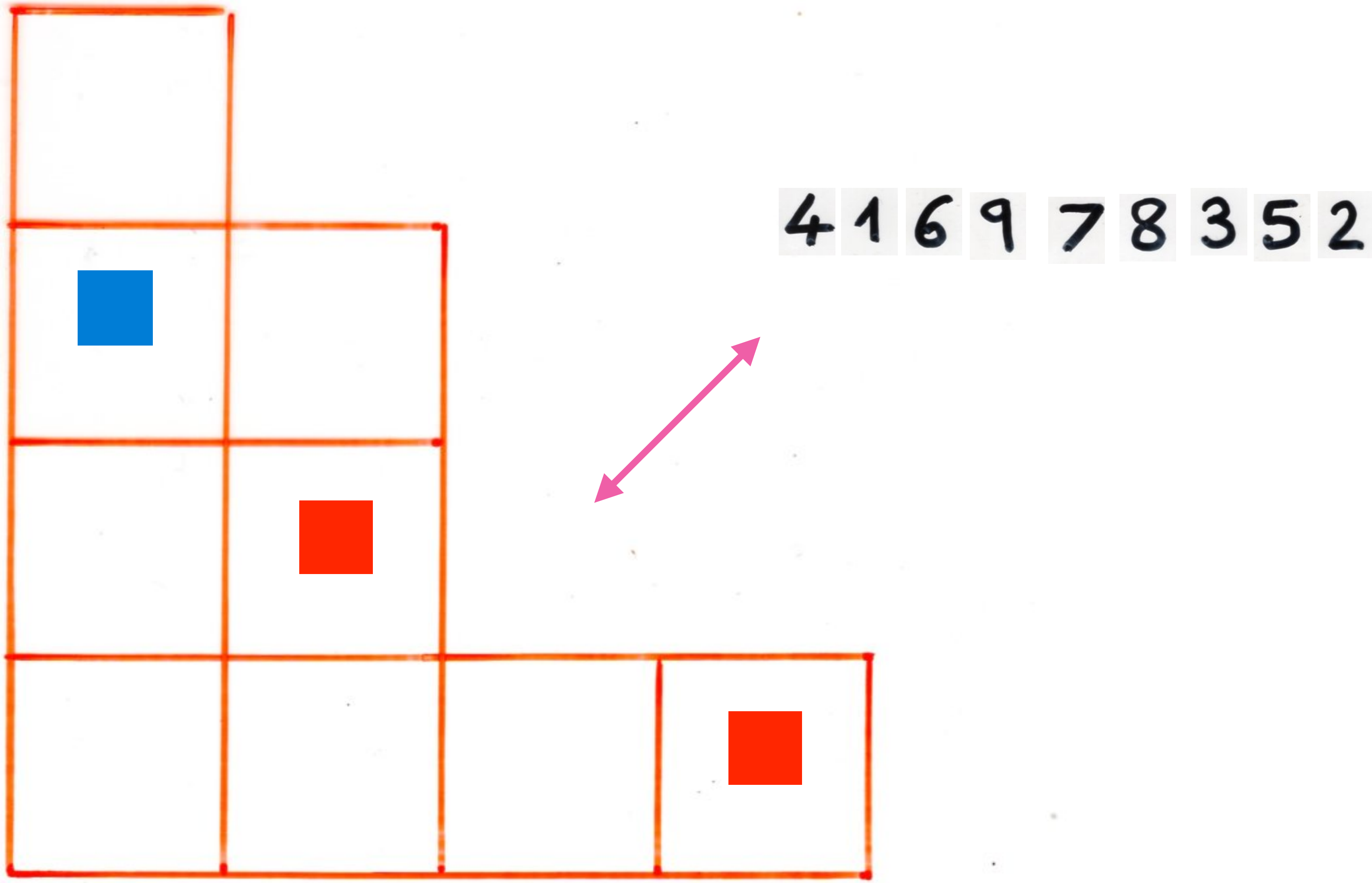

4 1 6 9 7 8 3 5 2

two bijections
one theorem

Prop.

T alternative tableau

→ σ : "exchange-fusion" inverse algorithm

→ τ : "local" algorithm from $DE = ED + E + D$

$$\sigma = \tau^{-1}$$

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

bijections

combinatorial objects on a 2D lattice

$UD = DU + Id$

permutations

towers placements

Physics

RSK ⟷

pairs of Young tableaux

(i) first step

(ii) second step

$DE = qED + E + D$

alternative tableaux

EXF ⟷

permutations

commutations

rewriting rules

planarization

ASM alternating sign matrices

tilings

non-crossing paths

8-vertex model

⟷ ?

"The cellular ansatz"

(iii) third step

quadratic algebra Q

"duplication"

Q-tableaux ⟷ edge local rules

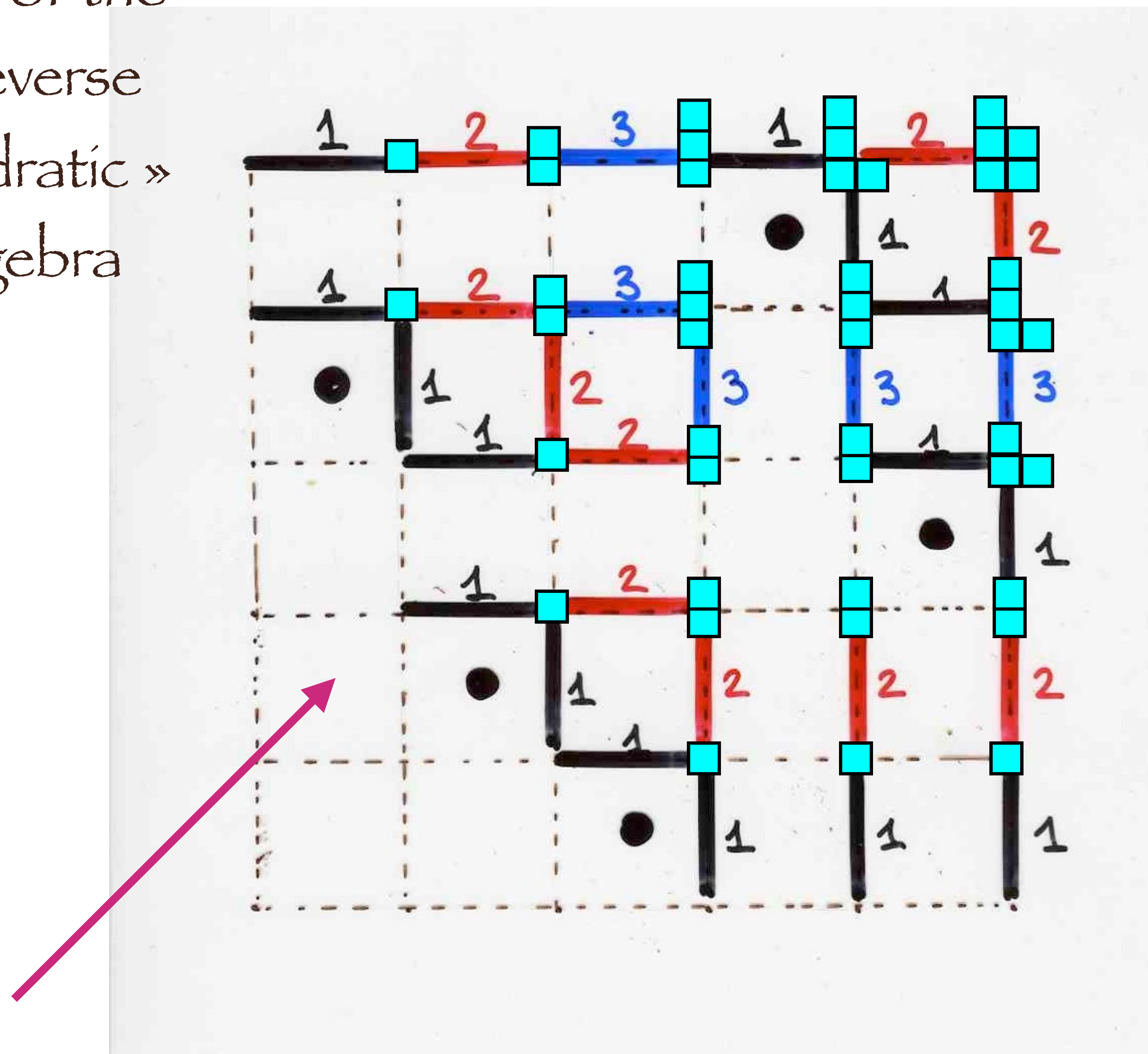
$UD = DU + Id$

permutations $\overset{RSK}{\longleftrightarrow}$ pairs of Young tableaux

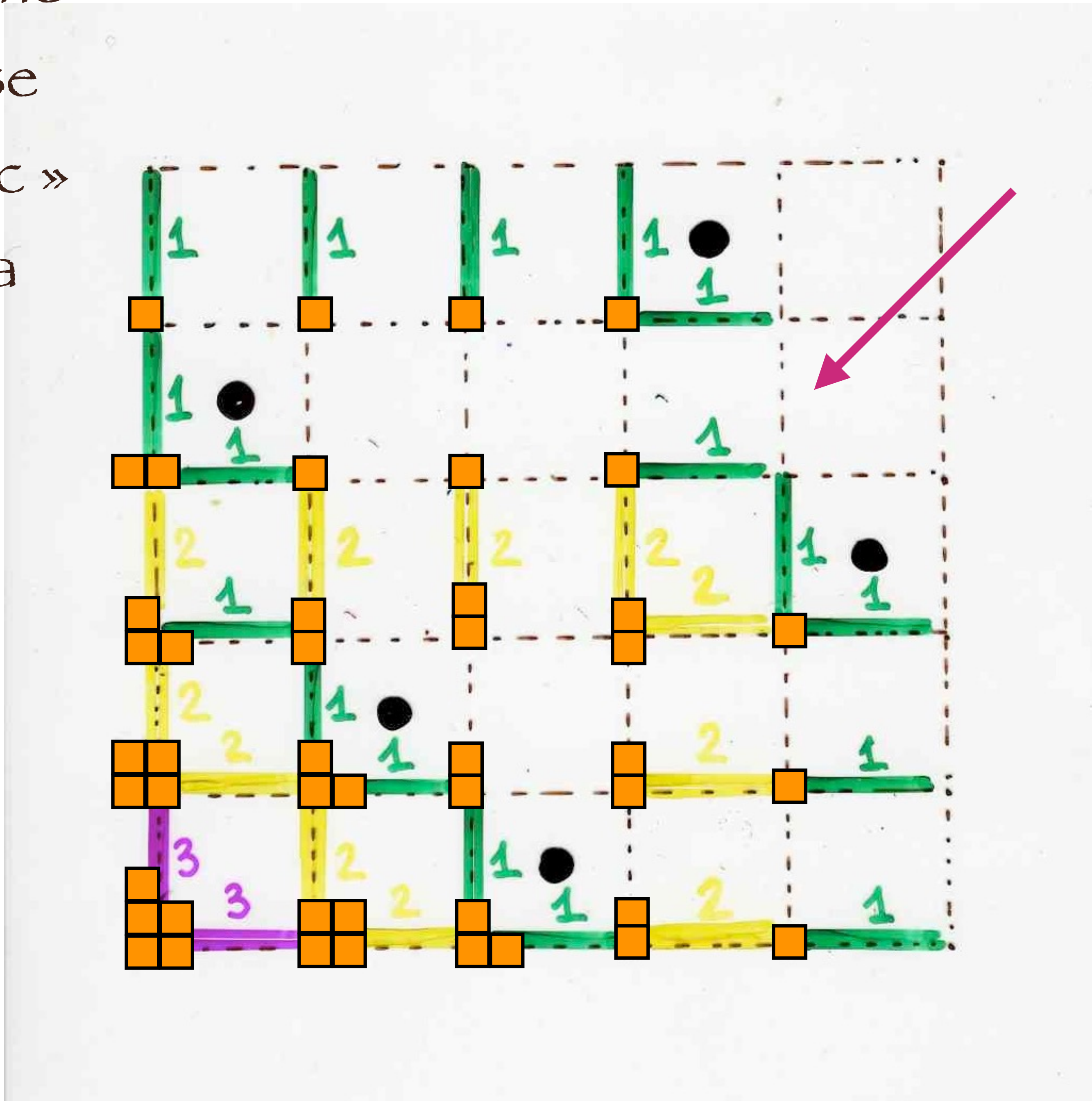
duplication of equations
in the « reverse quadratic » algebra

Here the « reverse quadratic » algebra
is isomorphic to the quadratic algebra itself

Idea of the « reverse quadratic » algebra



Idea of the « reverse quadratic » algebra



combinatorial representation
of the quadratic algebra

$DE = qED + E + D$ → EXF "Laguerre histories"

alternative tableaux

Local rules on vertices

Equivalence ↕

Local rules on edges

$$AS = SA + J + K$$
$$AK = KA + A$$
$$JS = SJ + S$$
$$JK = KJ$$

$$D = A + J$$
$$E = S + K$$

with duplication of equations
in the « reverse quadratic » algebra

"The cellular ansatz"

(iii) third step

"duplication"

quadratic algebra **Q**

edge local rules

Q-tableaux ⟷

$DE = qED + E + D$ duplication in the quadratic algebra

alternative tableaux ⟷ The « Adela bijection »

alternative tableaux ⟷ The « Tamil bijection »

duplication in the « reverse » quadratic algebra

Tianjin lectures 2019

www.viennot.org/tianjinlectures.html

more material in « ABjC »

www.viennot.org

The Art of Bijective Combinatorics, Part III

The cellular ansatz:
bijective combinatorics and quadratic algebra

Thank you !