An introduction to enumerative and bijective combinatorics with binary trees

12 exercises

TESSELLATE
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In this series of slides, « ABjC » refers to the video-book
The Art of Bijective Combinatorics, www.viennot.org/abjc.html

exercise 1

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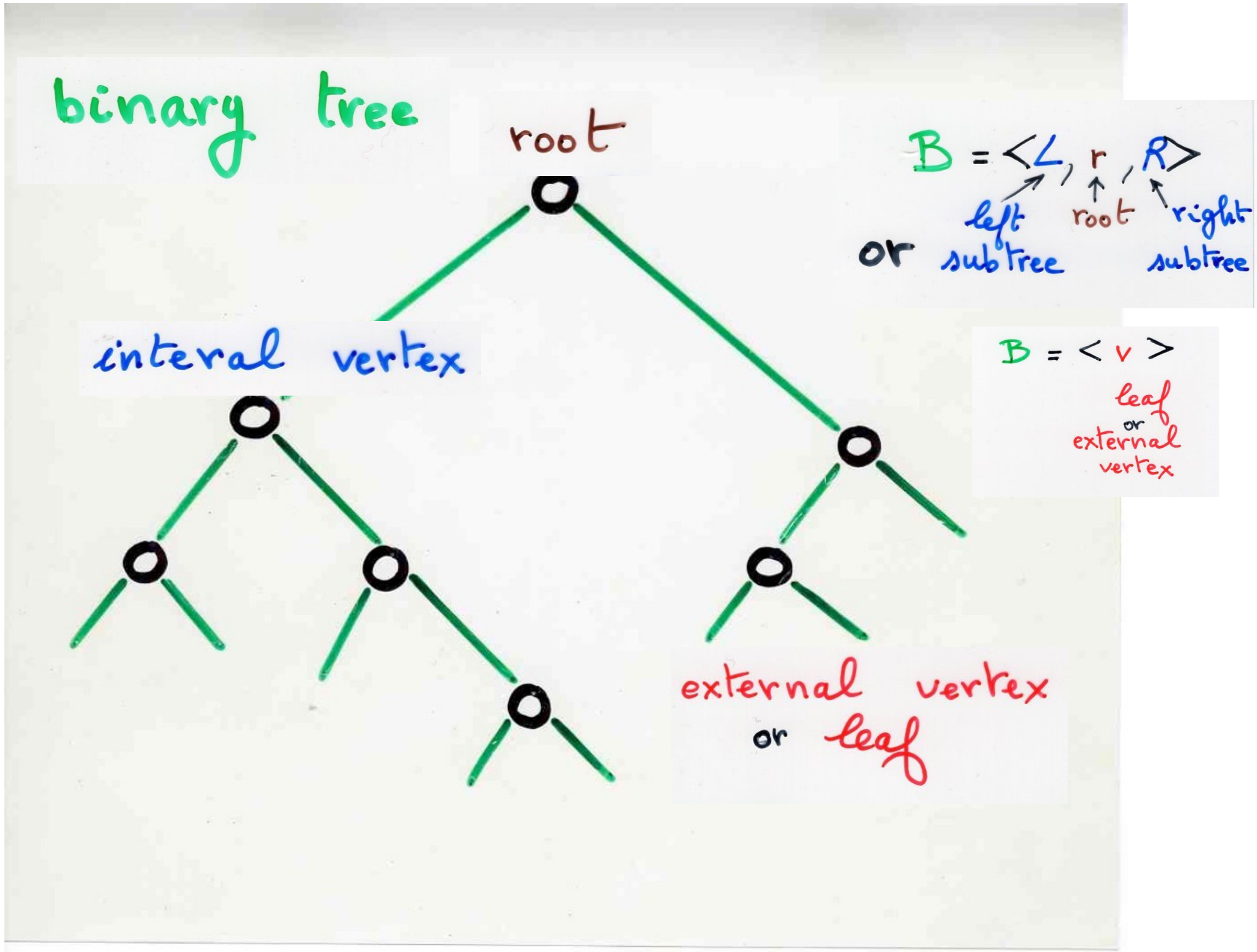
exercise 1

B binary tree having n internal vertices $\Rightarrow$ $(n+1)$ external vertices



Exercise 1

Proof by recurrence on the number of vertices of the binary tree B.

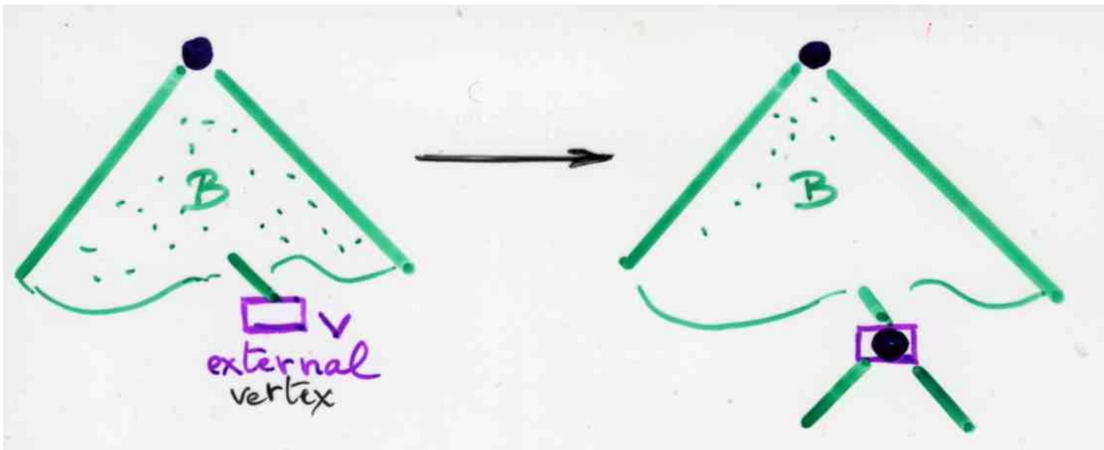
Let d(B) be the difference between the number of external and internal vertices of B.
For B = (L, r, R) where r is the root, L the left subtree and R the right subtree, then

$$d(B) = d(L) + d(R) - 1.$$

From the recurrence hypothesis, d(B) = 1 +1-1 = 1.
The property is true for the binary tree B reduced to a single external vertex B = (v), d(B)=1-0=1.

Q.E.D.

Another proof is the following:
For a binary B =(L, r, R), choose an external leaf v and replace v by the binary tree reduced to a single internal vertex (with 2 external vertices) as shown on the figure below. In this process the parameter d(B) is invariant.



exercise 2

exercise 2

describe the reverse bijection

$\sigma \longrightarrow f$

$$\mathfrak{S}_n \longrightarrow \mathfrak{F}_n$$

permutations

subexceedant function

Correction to the slide in the talk:
one should write *subexceedant* function

1

1

1 2

1 2

3 1 2

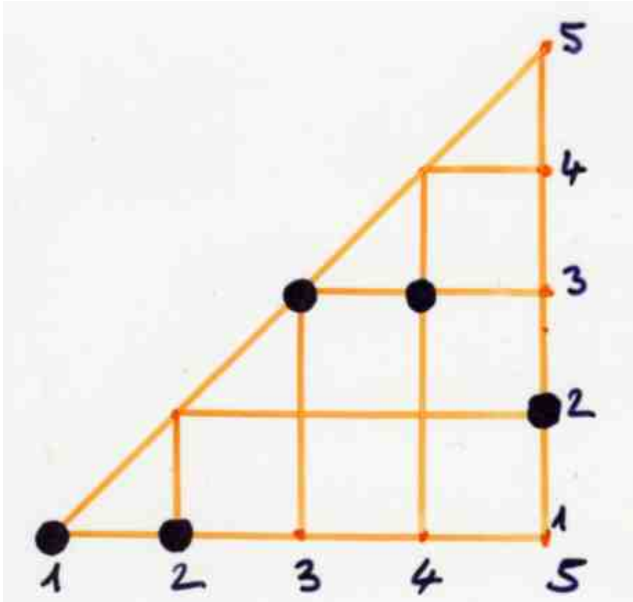
3 1 2

3 4 1 2

3 4 1 2

$\sigma = 3\ 4\ 1\ 5\ 2$

?



$$f: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 2 \end{array}$$

Let σ be a permutation, i be an integer, $1 \leq i \leq n$, and denote $x = \sigma(i)$.
Define $f(x)$ to be the number of j, $1 \leq i < j \leq n$ with $\sigma(j) < \sigma(i)$.
With the notation of the talk, the reverse bijection g is defined by $g(x)=1+f(x)$.

The pair (i,j) with $\sigma(j) < \sigma(i)$ is called an *inversion pair* of the permutation σ and f is called the *inversion table* of the permutation. Here, with the notation of the talk, you need to shift by one the values $f(x)$ of f.

See « ABjC », Part I, Chapter 4a, pp 23-32. http://www.viennot.org/abjc1-ch4.html

$$\sigma = 3\ 4\ 1\ 5\ 2 \longrightarrow$$

	1	2	3	4	5
f	0	0	2	2	1
g	1	1	3	3	2

slide 29 of « ABjC », Part I, Chapter 4a

$\sigma \in \mathfrak{S}_n \longrightarrow f \in \mathcal{F}_n$

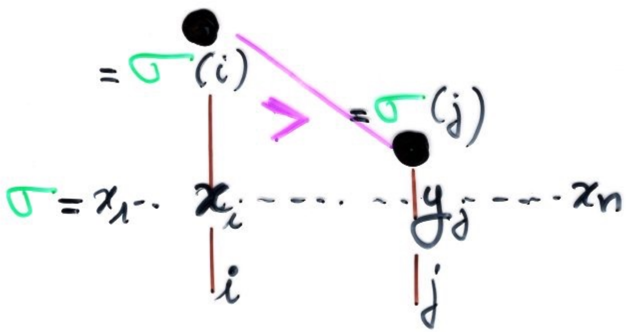
Inversion table

$$\begin{array}{l} \sigma = 7\ 2\ 3\ 6\ 8\ 5\ 1\ 4 \\ \hline \quad\;\; 6\ 1\ 1\ 3\ 3\ 2\ 0\ 0 \end{array}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

$1 \leqslant x \leqslant n \qquad x = \sigma(i)$

$f(x) =$ number of j, $i < j \leqslant n$ with $\sigma(j) < \sigma(i)$



In this correction of exercise 2
f is in fact g

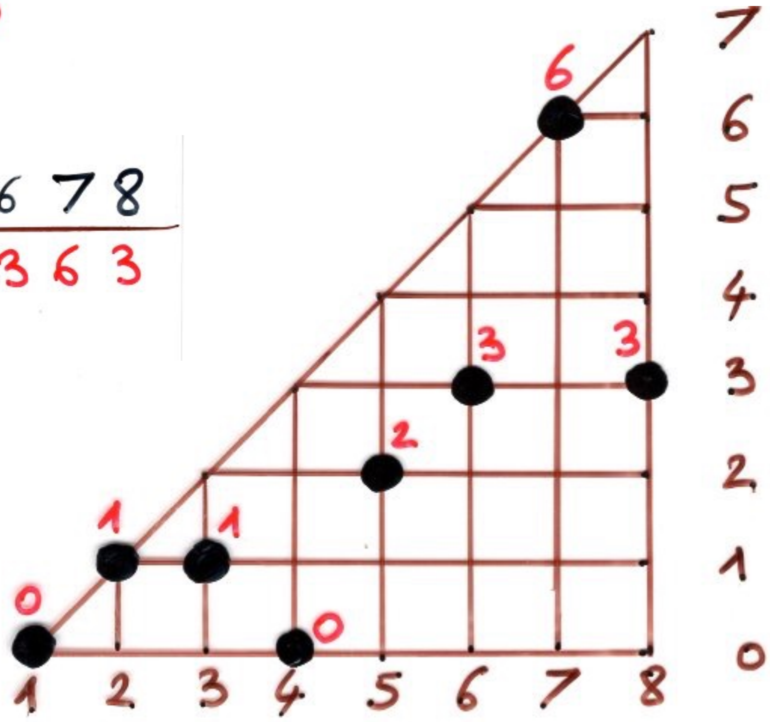
slide 30 of « ABjC », Part I, Chapter 4a

In this correction of exercise 2
f is in fact g

Inversion table

$$\sigma = \frac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3



exercise 3

exercise 3

The number of increasing binary trees is $n!$ with n vertices

Define an *elementary* binary tree, as a binary tree having a single internal vertex (and two external vertices). Any binary tree is obtained by adding successively an *elementary* binary tree on an external vertex, as described in exercise 1. If B is an increasing binary tree with n vertices, then B has (n+1) external vertices and there are (n+1) ways to add to B an elementary binary tree. The single internal vertex (= the root) of this *elementary* binary tree will be labelled (n+1). We get an increasing binary tree with (n+1) internal vertices. Conversely, from the labels 1, 2, …, (n+1) of the vertices of this increasing binary tree, one can recover the different choices made during the insertion process.

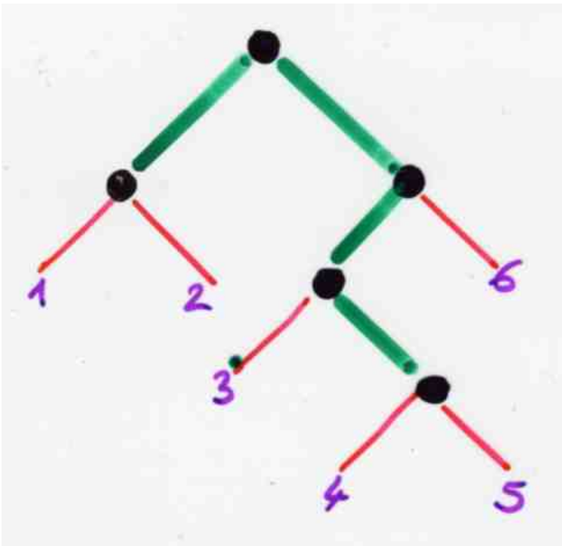
We can conclude that the number of increasing binary trees with n vertices is n!

It is better to really define a bijection between increasing binary trees and subexceedant functions.

A possible bijection with subexceedant functions is defined by numbering the external vertices 1, 2,…, n. For example one possible order is obtained by numbering the external vertices « *from left to right* ». It is defined recursively by the following:

For a binary tree B = (L, r, R), first visit the external vertices of L , then visit the external vertices of R.

Example:

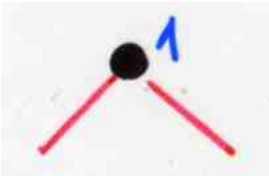


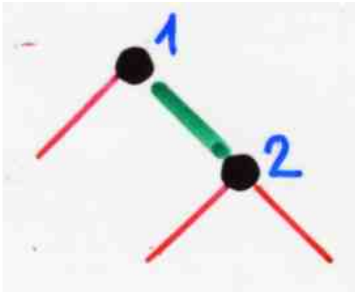
ordering
the external vertices
from "left to right"

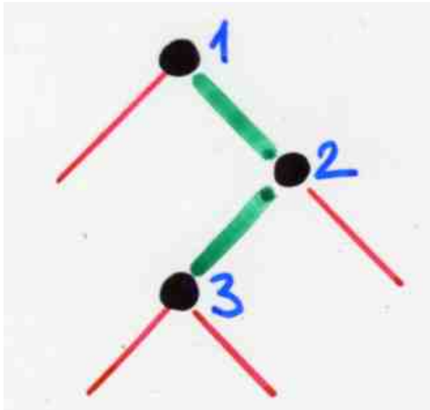
also called symmetric order
or inorder

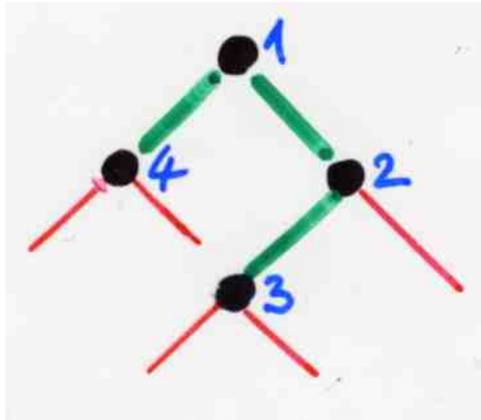
Then the bijection between increasing binary trees and subexceedant functions is obtained by labelling the vertices 1, 2, …, n in the insertion process according to the ordering of the external vertices « *from left to right* ». (also called *inorder* or *symmetric* order).

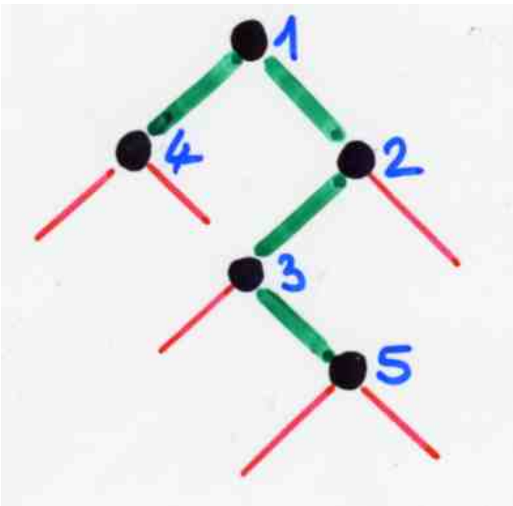
Example (with f given in the talk)







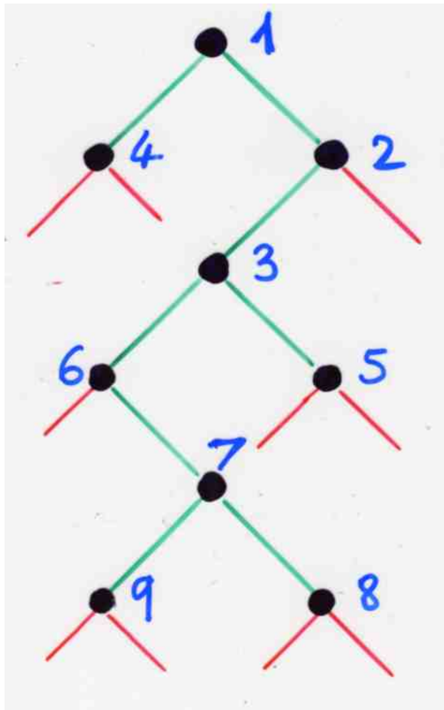




	1	2	3	4	5
f:	1	2	2	1	4

In fact a direct bijection between increasing binary trees and permutations can be defined in the. following way. Again define an order (called *inorder* or *symmetric* order) on the internal vertices of a binary tree, in the same way as above:
for a binary tree $B = (L, r, R)$, first visit the vertices of L, then visit the root r, then visit the internal vertices of R. Reading the internal vertices of B. according to this order, we get a permutation σ (written as a word). The map $B \longrightarrow \sigma$ is a bijection (called the *projection* of the increasing binary tree B).

example:



For a description of the reverse bijection, and more details, see:
« ABjC », Part I, Chapter 4a, pp 74-92.
http://www.viennot.org/abjc1-ch4.html

exercise 4

exercise 4

find a bijective proof for the identity

$$\sum_{j-1 \leqslant \ell \leqslant i+j-1} \binom{\ell}{j-1} = \binom{i+j}{j}$$

An exemple of this identity with i = 5, j = 3

$$\binom{8}{5} = 56$$

56 = 1 + 3 + 6 +10 +15 +21

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

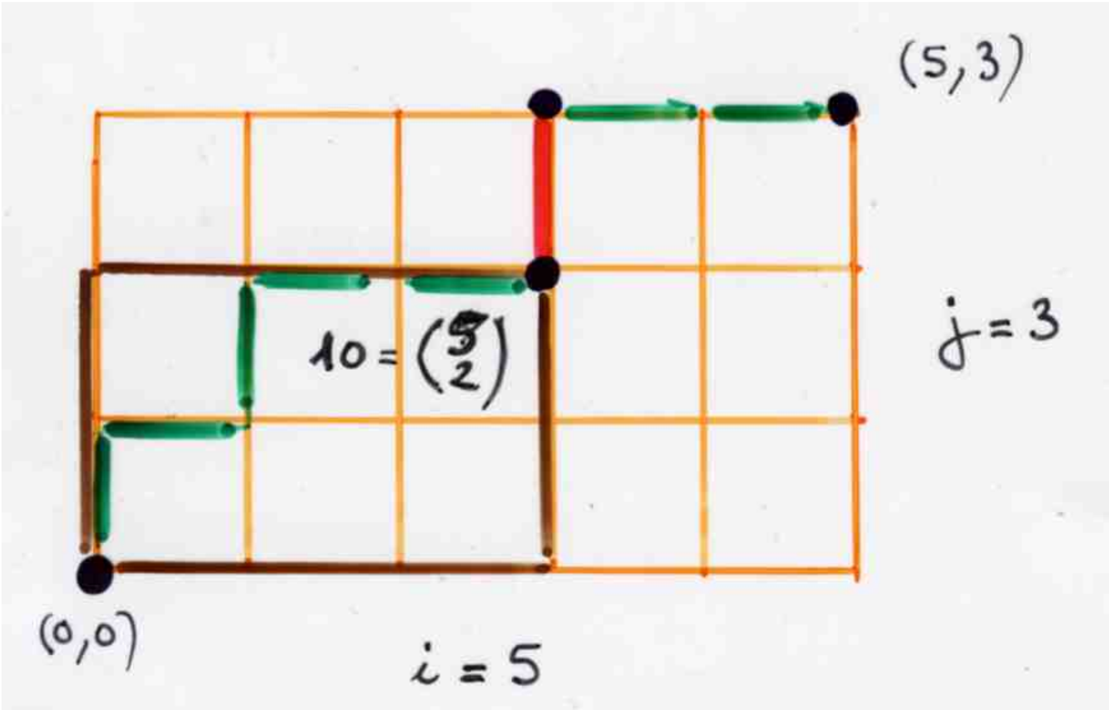
$$\sum_{j-1 \leqslant \ell \leqslant i+j-1} \binom{\ell}{j-1} = \binom{i+j}{j}$$

Proof:

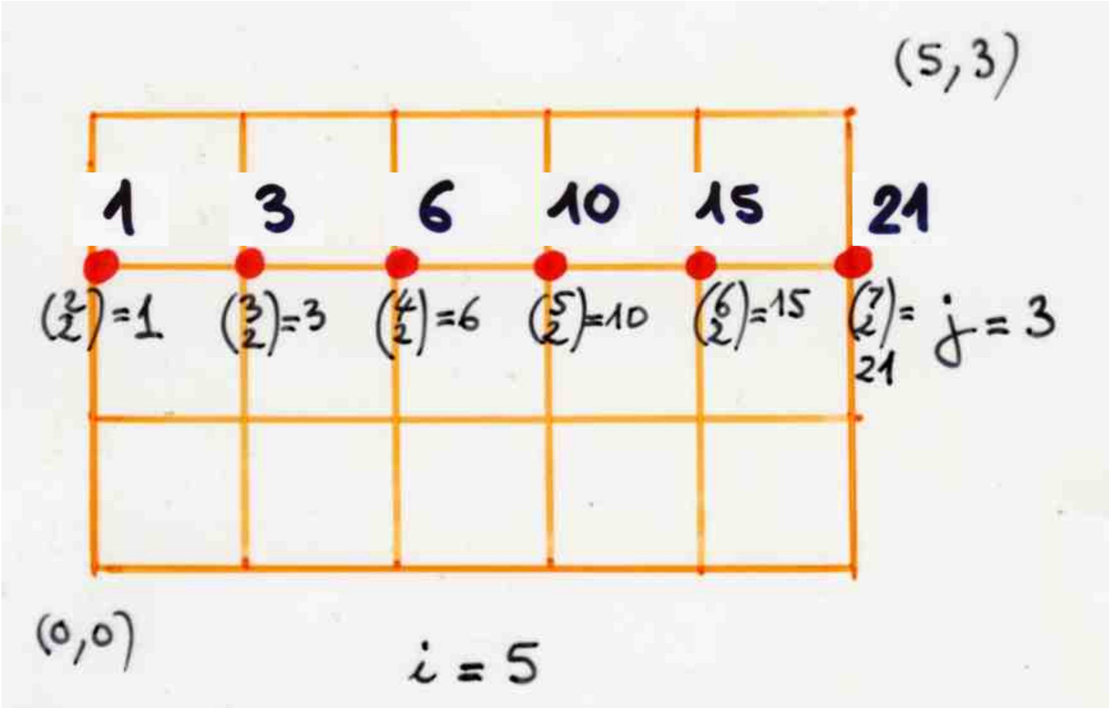
As explained in the talk,we use the interpretation of binomial coefficients with paths on a grid, going from (0,0) to (i,j).

For any such path ω, let l be the abcissa of the last North step of the path. This last North step goes from the point $(l, j-1)$ to the point (l, j). (see Figure below). The number of such paths is the binomial coefficient appearing in the left hand-side of the identity.

Q.E.D.



$$\binom{8}{5} = 56$$

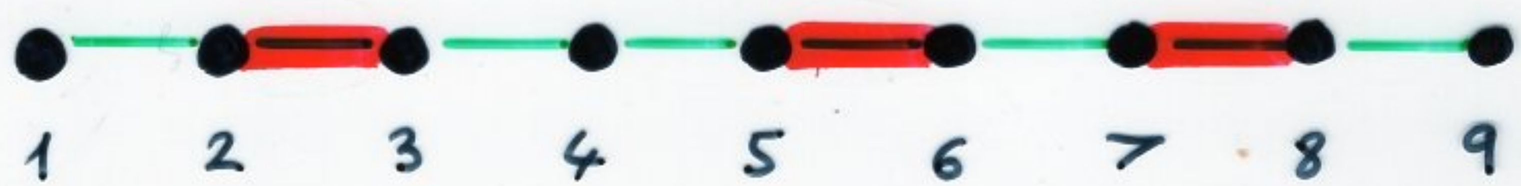


(5,3)
1
3
6
10
15
21
$\binom{2}{2}=1$
$\binom{3}{2}=3$
$\binom{4}{2}=6$
$\binom{5}{2}=10$
$\binom{6}{2}=15$
$\binom{7}{2}=21$
$j=3$
(0,0)
$i=5$

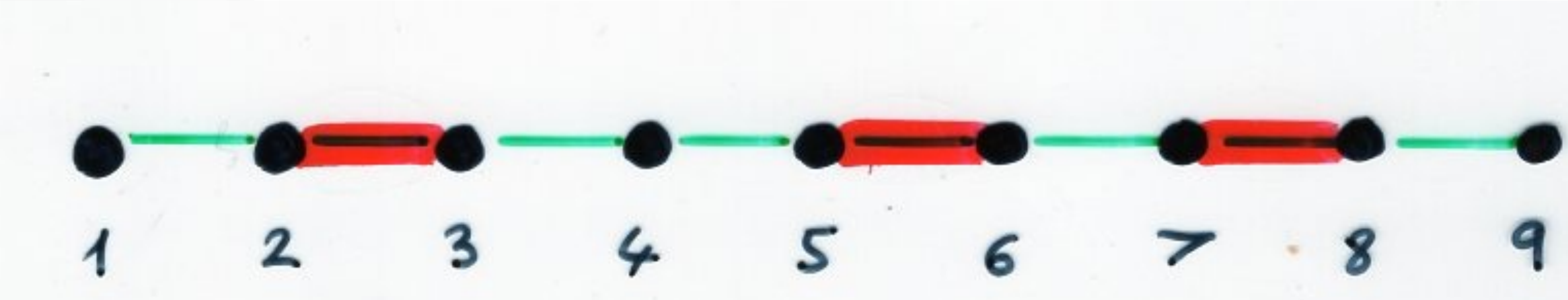
exercise 5

exercise 5

a) Prove that the number of matchings of the segment $[1, n]$ is the Fibonacci number F_n



exercise 5



$a_{n,k}$ = number of matchings of $\{1, 2, \ldots, n\}$ with k dimers

b) Give a bijective proof for

$$a_{n,k} = \binom{n-k}{k}$$

thus

$$\sum_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1
1
2
3
5
8
13
21

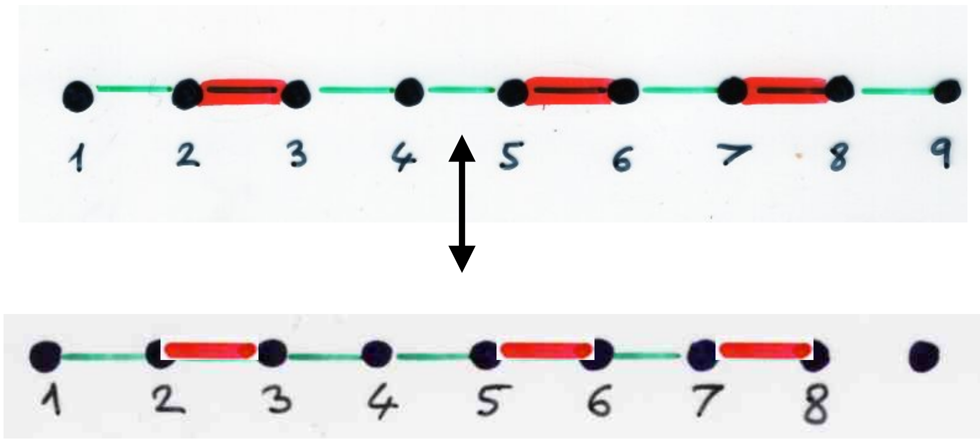
$$\sum_{0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

Proof of a):
The set of matchings of the segment [1, n] can be divided into two disjoint classes: the matchings where (n) is an isolated point (i.e. (n) is not contained in an edge of the matching) and the matchings where [(n-1), n] is an edge of the matching. The first class is in bijection with matchings of the segment [(n-1), n]. The second class is in bijection with matchings of the segment [(n-2), n].

Thus the number of matchings of the segment [1, n] satisfies the same linear recurrence relation as the Fibonacci numbers. The initial conditions are the same. The number of such matchings is thus the Fibonacci number. (see Figures below). .

$$F_n = F_{n-1} + F_{n-2} \qquad (n \geq 2)$$

$$F_1 = 1 \, , \, F_2 = 2$$

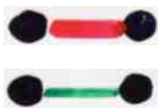


F_n

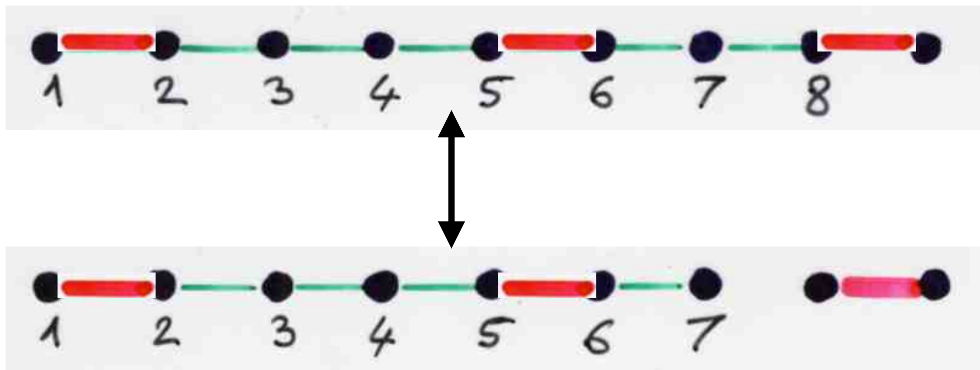
F_{n-1}

$F_1 = 1$

$F_2 = 2$



F_n



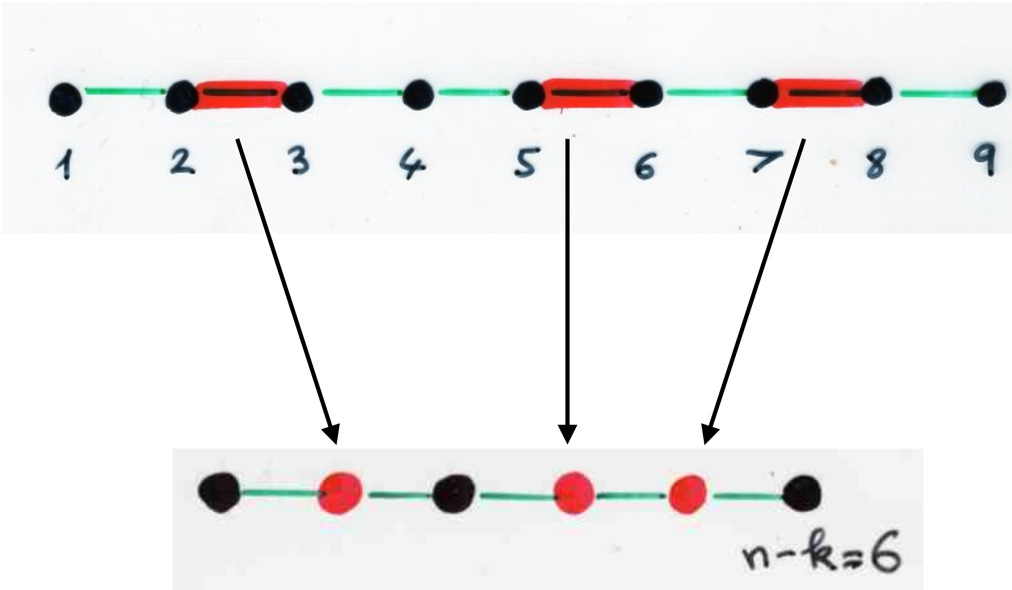
F_{n-2}

Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

$F_1 = 1$, $F_2 = 2$

Proof of b):
There is a bijection between matchings of [1, n] with k edges and subsets with k elements of a set with (n-k) elements. It suffice to « shrink » each edge of the matching into a point (see Figure below).



F_n

$n = 9$
$k = 3$

thus

$$\sum_{0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$$

exercise 6

exercise 6

Prove (by calculus) that

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{(n+1)}\binom{2n}{n}$$

Elementary calculus gives:

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)(2n-1)\cdots(2n-n+1)}{n(n-1)\cdots 2\times 1} - \frac{(2n)(2n-1)\cdots(2n-n+2)}{(n-1)\cdots 2\times 1}$$

$$= \frac{(2n)(2n-1)\cdots(2n-n+2)}{(n-1)\cdots 2\times 1}\left(\frac{2n-n+1}{n} - 1\right)$$

$$\frac{n+1-n}{n} = \frac{1}{n}$$

$$= \frac{(2n)(2n-1)\cdots(\cdot\, n+2)}{n(n-1)\cdots(2)\times 1}$$

$$= \frac{1}{(n+1)}\;\frac{(2n)(2n-1)\cdots(n+1)}{n(n-1)\cdots 2\times 1}$$

$$= \frac{1}{(n+1)}\binom{2n}{n}$$

exercise 7

exercise 7

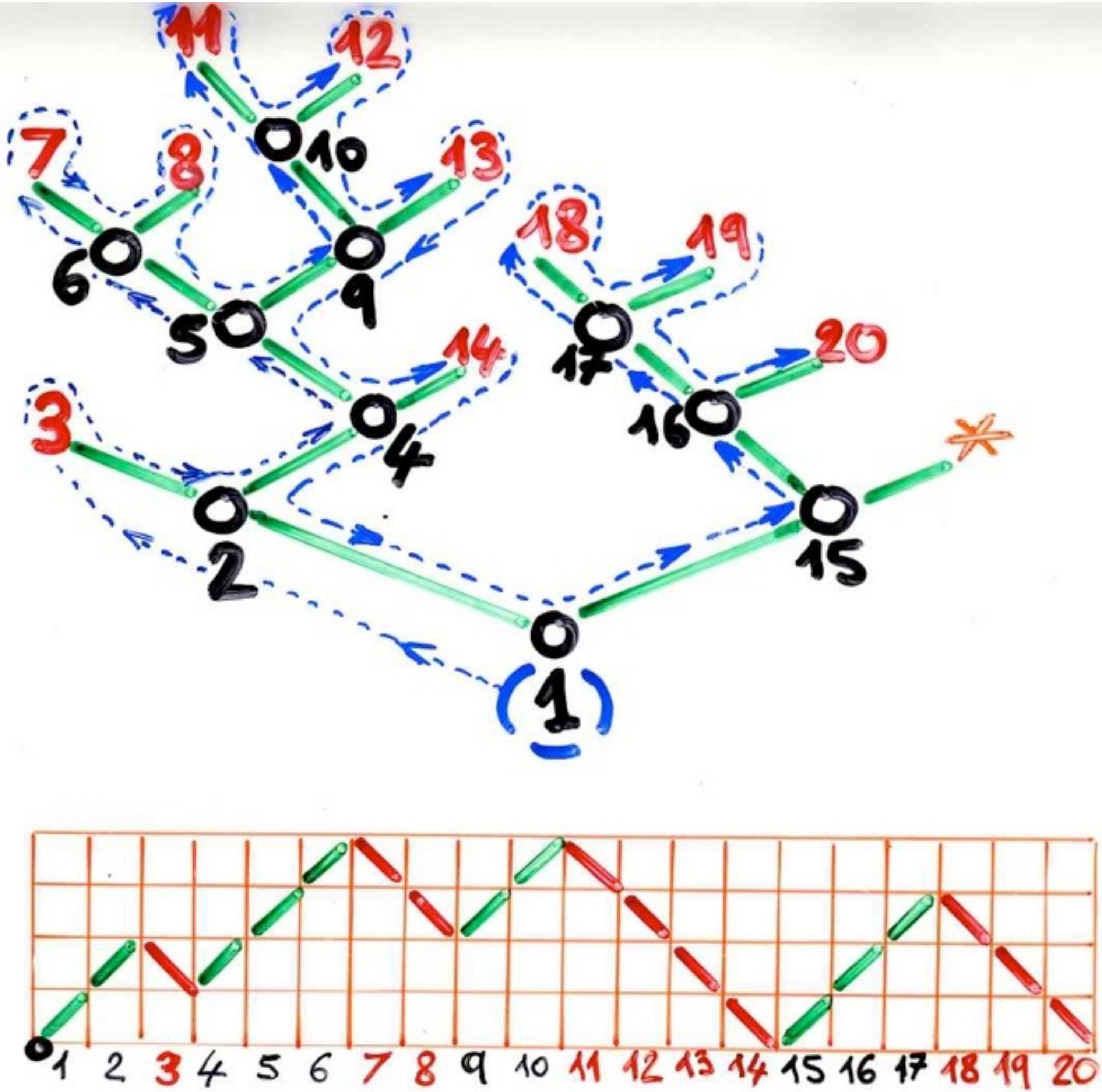
Describe the reciprocal bijection

Dyck path $\longrightarrow$ binary tree

length $2n$

n internal vertices
$(n+1)$ external vertices

C_n



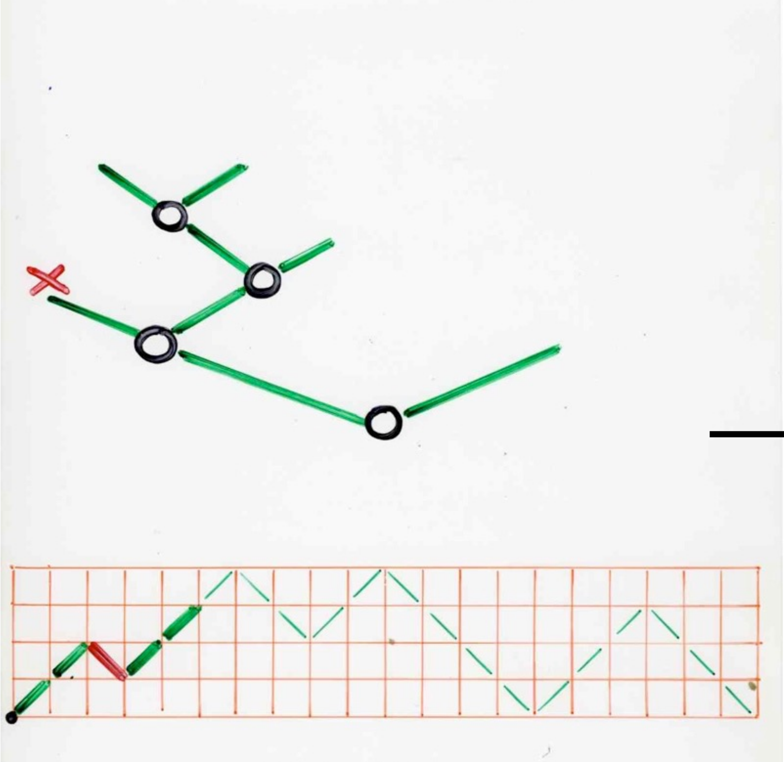
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

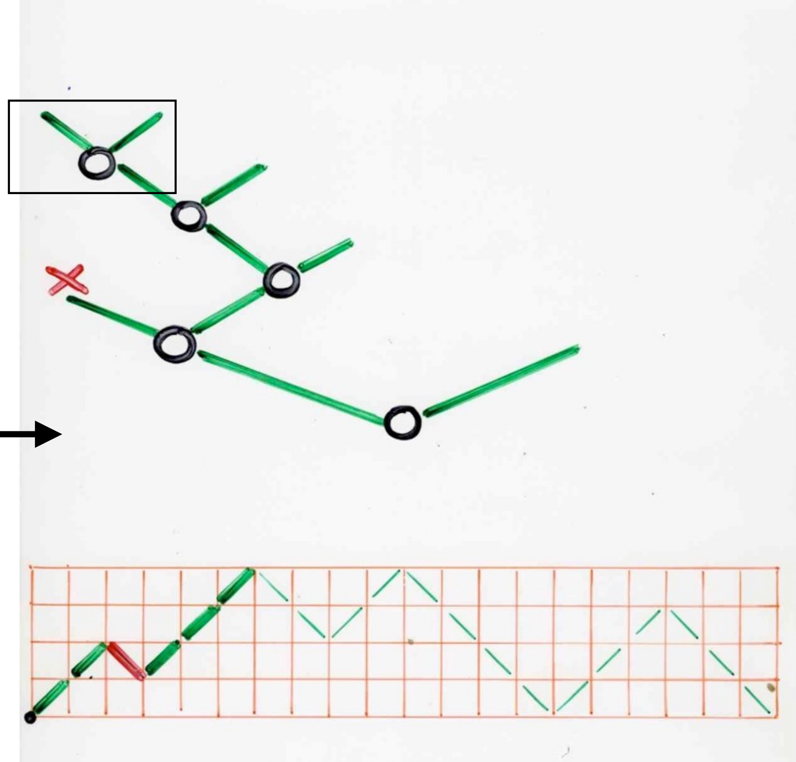
The reciprocal bijection is constructed by the following algorithm.

Follow the Dyck path from left to right. Suppose one associate a binary tree B after following the first i steps. Some external vertices are called « *closed* »,some are called « *open* ». The open external vertices are ordered « *from left to right* » (see exercise 3).

If step $(i+1)$ is North-East, then add an « *elementary* » binary tree on the first *open* external vertex. Then the two new external vertices are labelled « *open* ».

If step $(i+1)$ is South-East, then *close* the first *open* external vertex of B.

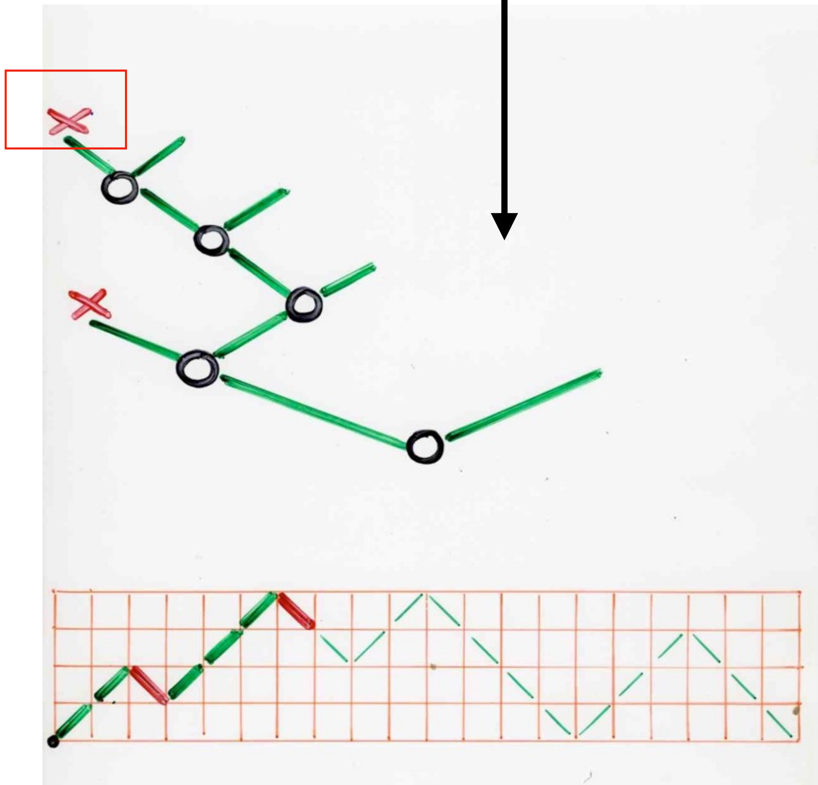




An example is given on the Figure.
Closed external vertices are labelled with a red cross.

Step 6 of the Dyck path is North-East.
Step 7 of the Dyck path is South-East.

An example of the complete reverse bijection is given on the set of slides STEM21_exo7 available on the web page www.viennot.org/abjc-stems21.html



exercise 8

In this series of slides, « ABjC » refers to the video-book
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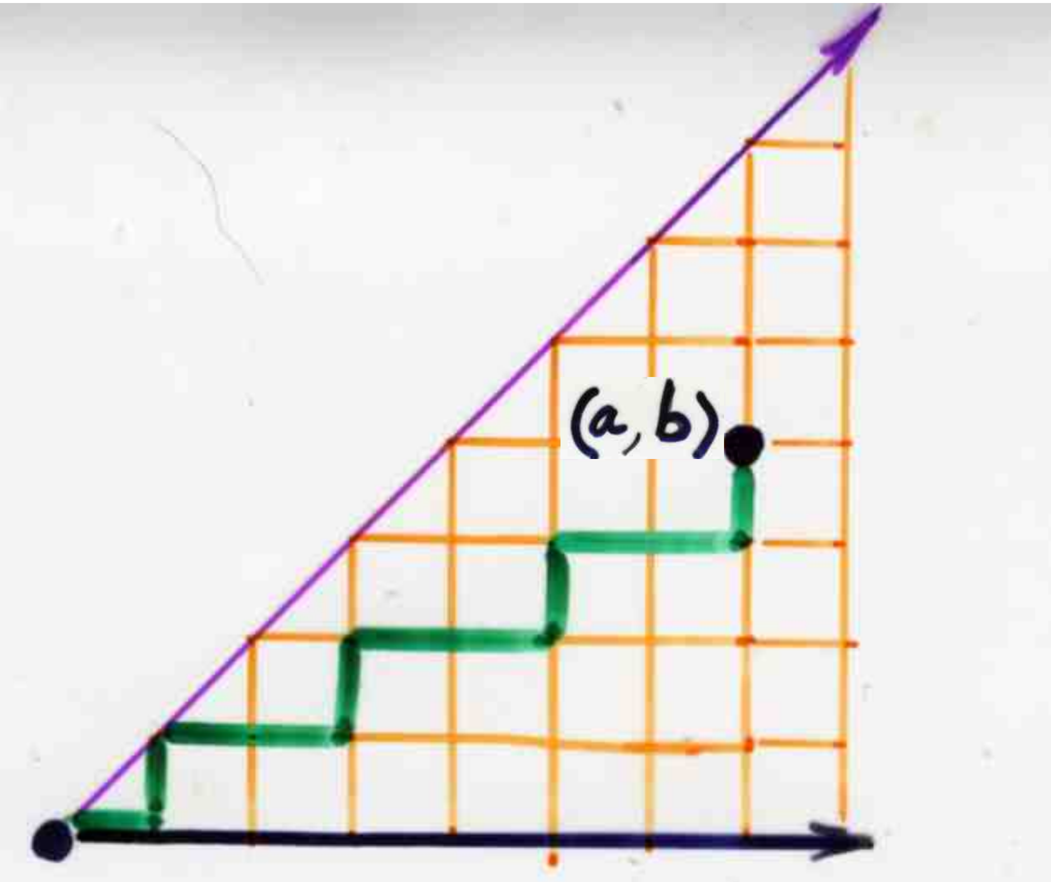
"ballot numbers"

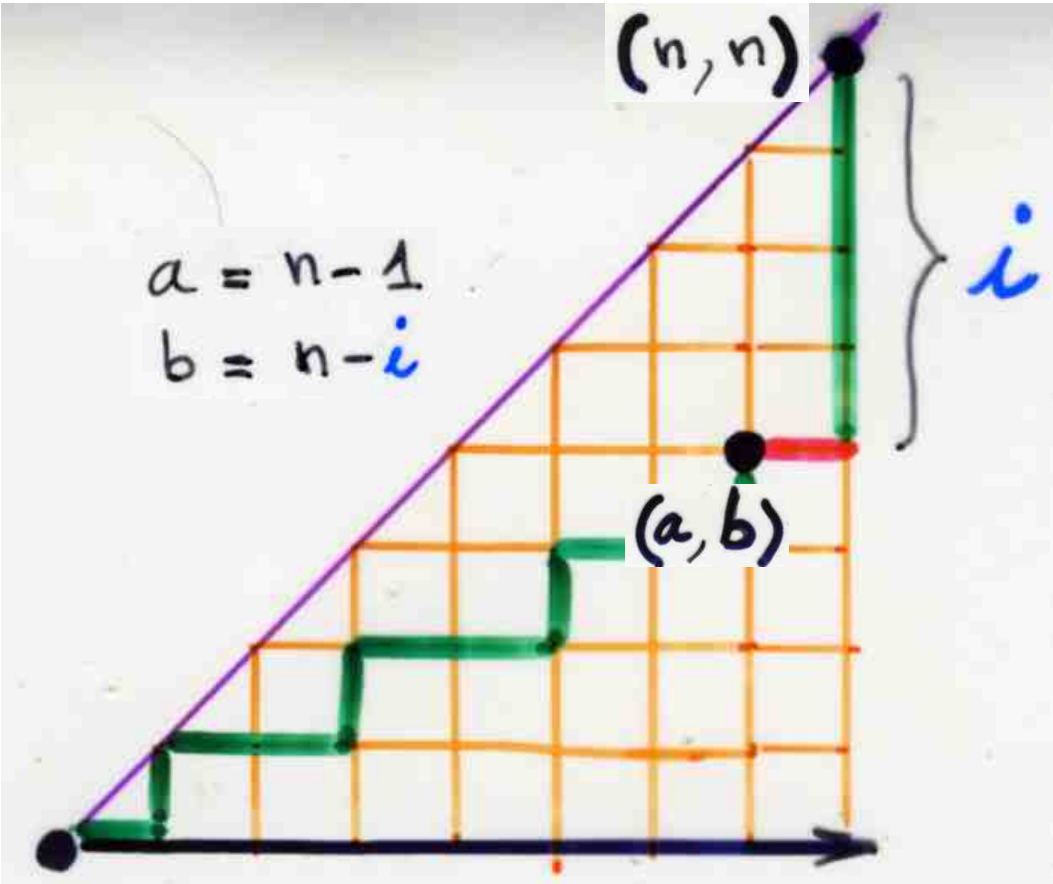
exercise 8



Exercise 8 is giving a formula for the famous « *ballot problem* », solved at the end of the 19th Century (J.Bertrand, D.André, H. Delannoy, …), which can be formulated in term of paths:

find a formula for the number of paths with elementary steps North and East, going from (0,0) to (a,b) and located below the diagonal.

exercise 8



Giving a formula for the number of paths with elementary steps North and East, going from (0,0) to (a,b) and located below the diagonal,

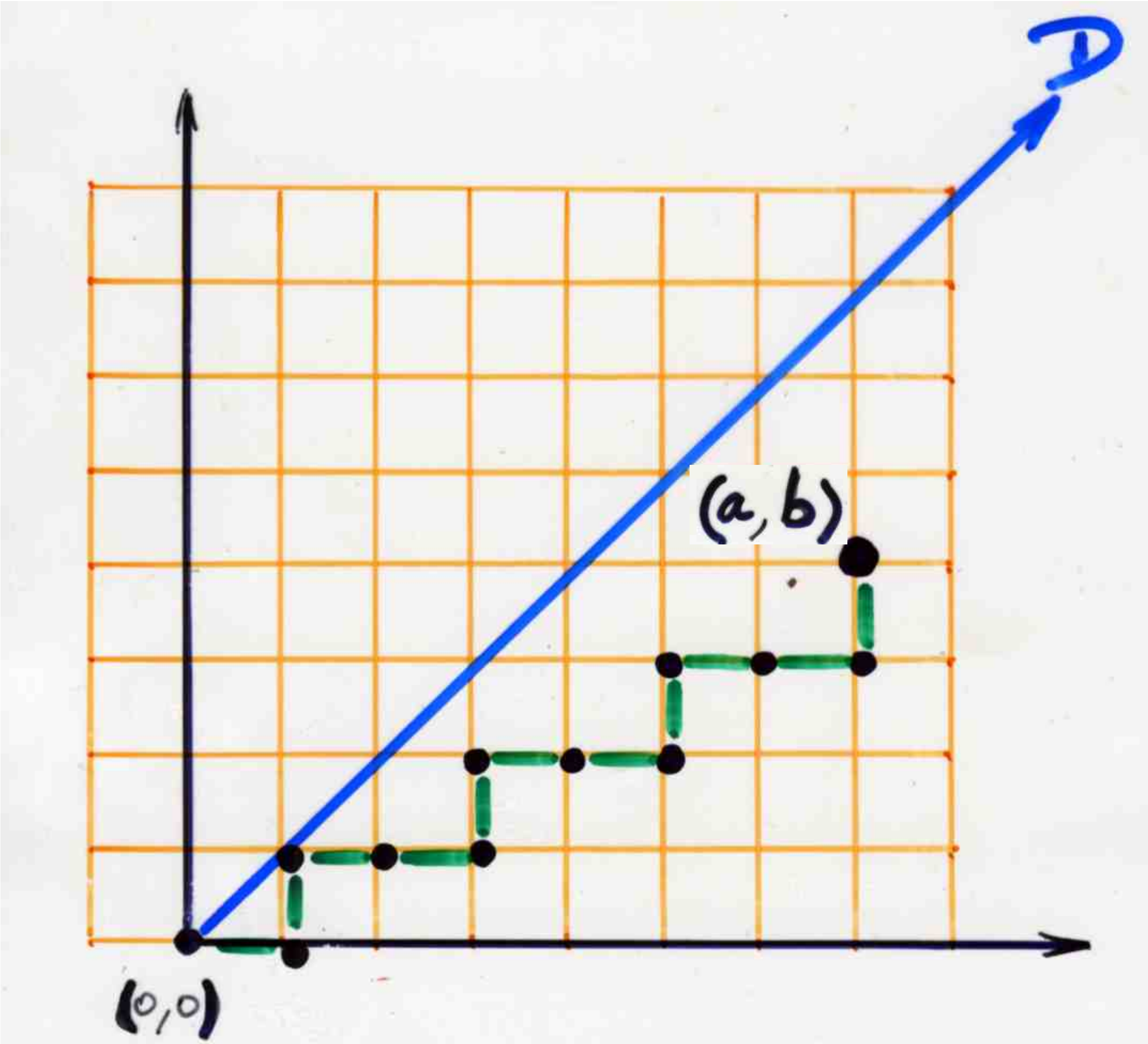
is obviously equivalent to give a formula for the number of Dyck paths of length 2n such that the length of the longest sequence of North-East steps is equal to i.

This number is:

$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$

$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

See the extension with i and j and the proof in ABjC, PartI, Ch 2c, pp 40-47

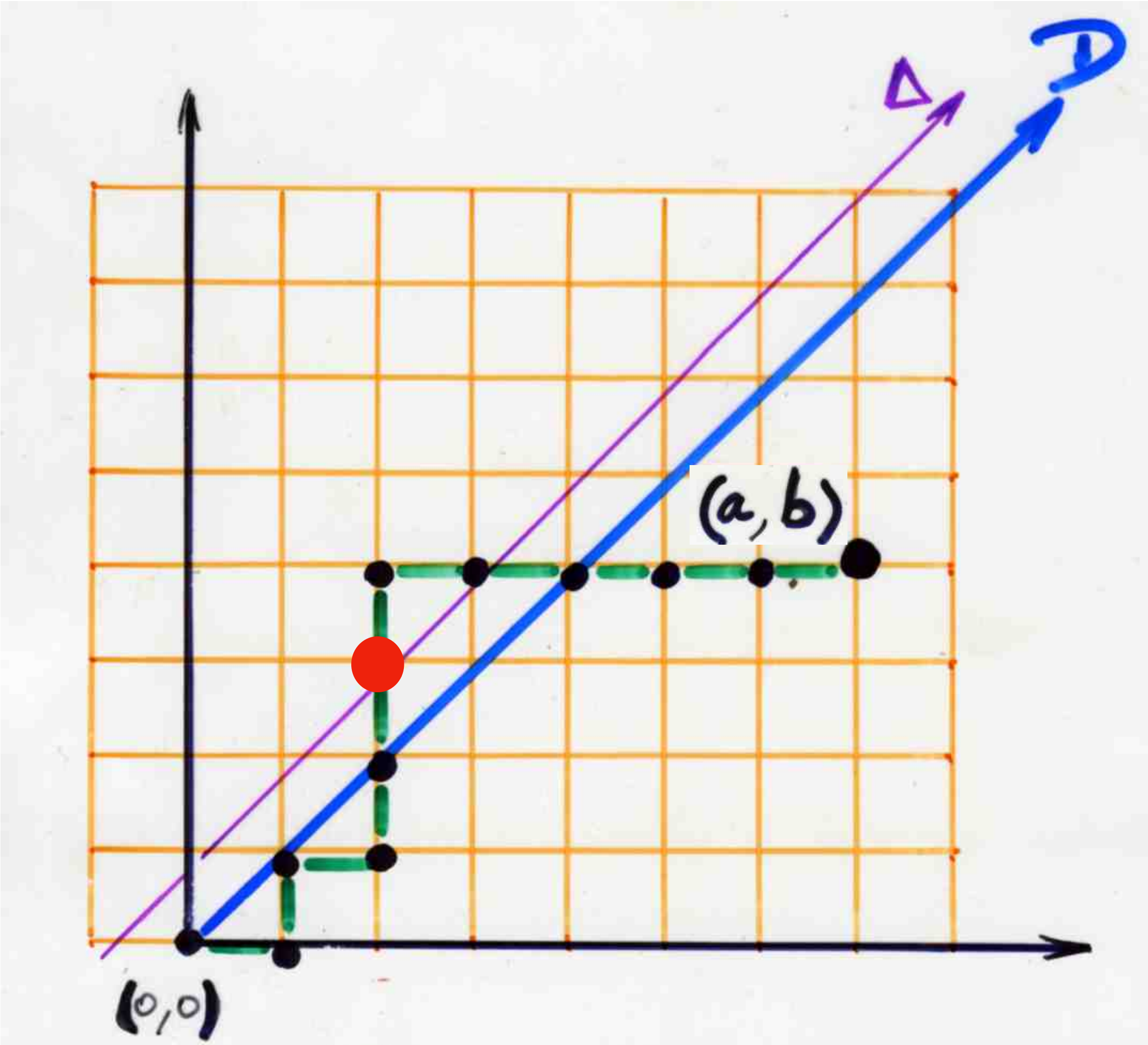


Solving the problem of the enumeration of such paths going from (0,0) to (a,b) and located below the diagonal D is just an extension of the proof given in the talk for the case of Catalan numbers (a,b) = (n,n). We use the same « reflection » principle.

The total number of paths going from (0,0) to (a,b) is

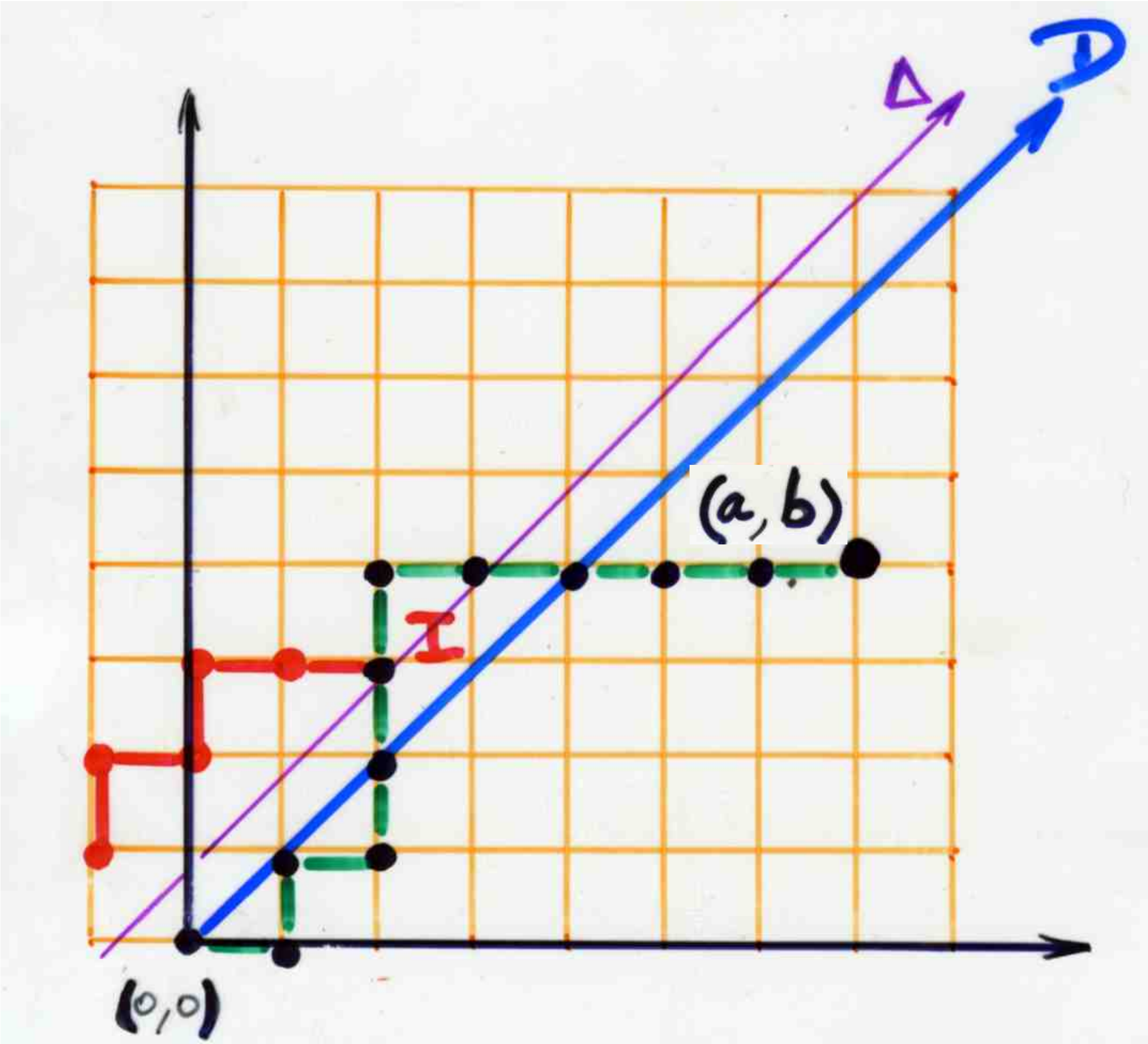
$$\binom{a+b}{a}$$

From this total we subtract the number of «bad » paths, i.e. paths which cross the diagonal D. Such paths are in bijection with paths going from (-1,1) to (a,b). The bijection is given in the following Figures.



A « bad » crossing the diagonal D. Such path ω will intersect the line Δ, a line parallel to main diagonal D, passing through the point (0,1).

We denote by I the first intersection of the path ω with Δ.



We denote by I the first intersection of the path ω with Δ. The path ω is divided into two portions: ω' from $(0,0)$ to I and η from I to (a,b).

The initial portion ω' of the path ω from $(0,0)$ to I is reflected according to the line Δ. (in red on the Figure). We get a path ω'' going from $(-1,1)$ to I. Gluing ω'' and η, we get a path $R(\omega)$ going from $(-1,1)$ to (a,b).

It is easy to see that the map R is a bijection between « bad » paths and paths going form $(-1,1)$ to (a,b).

Thus, the number of bad paths is the binomial coefficient $\binom{a+b}{b-1}$

and the number of good paths is $\binom{a+b}{b} - \binom{a+b}{b-1}$

We have to compute $\binom{a+b}{b} - \binom{a+b}{b-1}$

$$\binom{a+b}{b} = \frac{(a+b)(a+b-1)\cdots(a+b-(b-1))}{(b)!}$$

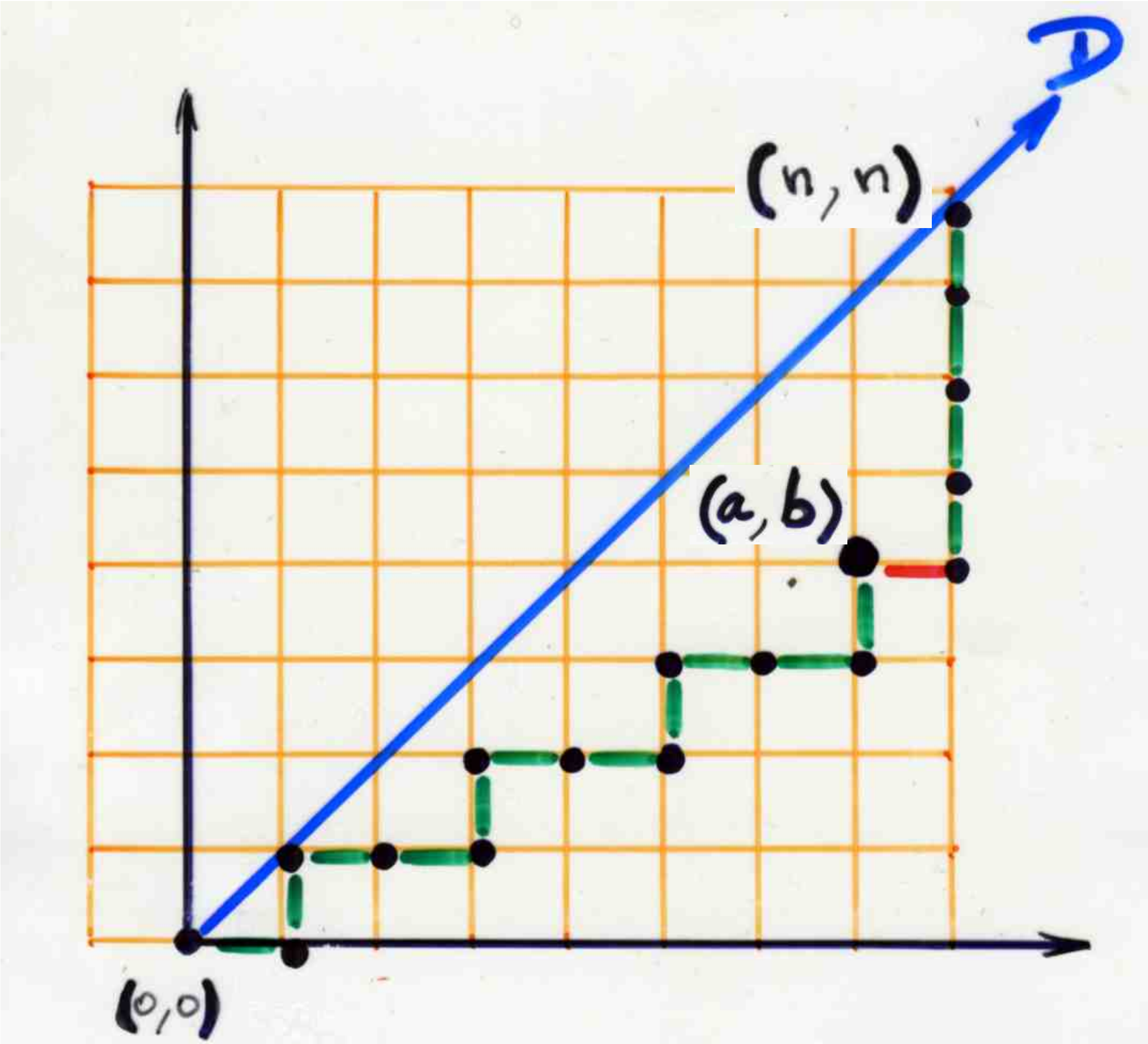
$$\binom{a+b}{b-1} = \frac{(a+b)(a+b-1)\cdots(a+b-(b-2))}{(b-1)!}$$

$$\binom{a+b}{b} - \binom{a+b}{b-1} = \binom{a+b}{b}\left[1 - \frac{b}{(a+1)}\right]$$

$(a+b-(b-1))$

$$= \binom{a+b}{b}\frac{a+1-b}{a+1}$$

This formula solves the « ballot » problem.



The formula for the number of paths with elementary steps North and East, going from (0,0) to (a,b) and located below the diagonal,

is obviously equivalent to give a formula for the number of Dyck paths of length 2n such that the length of the longest sequence of North-East steps is equal to i.

The second part of the exercise is to prove the following equality

$$\binom{a+b}{b}\frac{a+1-b}{a+1} = \frac{i}{2n-i}\binom{2n-i}{n} \quad \text{where} \quad \begin{aligned} a &= n-1 \\ b &= n-i \end{aligned}$$

with $\begin{array}{l} a = n-1 \\ b = n-i \end{array}$

$$\binom{a+b}{b}\frac{a+1-b}{a+1} = \binom{2n-i-1}{n-1}\frac{i}{n}$$

For any integers m and n, it is easy to prove (by calculus or with a trivial bijection) that:

$$\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$$

for $m = 2n-i$ we have

$$\binom{2n-i}{n} = \frac{2n-i}{n}\binom{2n-i-1}{n-1}$$

Going back to the value of $\binom{a+b}{b}\frac{a+1-b}{a+1}$ we get:

$$\binom{2n-i-1}{n-1}\frac{i}{n} = \frac{n}{2n-i}\binom{2n-i}{n}\times\frac{i}{n}$$

$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

Q.E.D.

exercise 9

exercise 9

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

$$\Longleftrightarrow$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

with $C_1 = 1$

Suppose $$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

$$C_n = \frac{(2n)(2n-1)\cdots(2n-n+1)}{(n+1)\,n\,\cdots\,2\times 1}$$

$$= \frac{(2n)(2n-1)\cdots(n+2)}{n(n-1)\cdots 2\times 1}$$

$$C_{n+1} = \frac{(2n+2)(2n+1)\cdots(n+3)}{(n+1)(n)\cdots 2\times 1}$$

$$\frac{C_{n+1}}{C_n} = \frac{(2n+2)(2n+1)\cdots(n+3)}{(2n)(2n-1)\cdots(n+2)} \times \frac{n(n-1)\cdots 2\cdot 1}{(n+1)\,n\,\cdots 2\cdot 1}$$

$$= \frac{(2n+2)(2n+1)}{(n+2)}\,\frac{1}{(n+1)}$$

$$= \frac{2(2n+1)}{(n+2)}$$

$$2(2n+1)\,C_n = (n+2)\,C_{n+1}$$

Conversely, if $2(2n+1)\,C_n = (n+2)\,C_{n+1}$ with $C_1 = 1$

$$C_n = \frac{2\,(2n-1)}{(n+1)}\,C_{n-1}$$

$$= \frac{2\,(2n-1)}{(n+1)} \times \frac{2\,(2n-3)}{n}\,C_{n-2}$$

$$= \underbrace{\frac{2\,(2n-1)}{(n+1)} \times \frac{2\,(2n-3)}{n} \times \cdots\cdots \times \frac{2\times 5}{4} \times \frac{2\times 3}{3}}_{(n-1)\ \text{products.}}\; C_1$$

$$= \underbrace{\frac{2n\,(2n-1)}{n\,(n+1)} \times \frac{2\,(n-1)(2n-3)}{(n-1)\ n} \times \cdots \times \frac{2\times 3\times 5}{3\times 4} \times \frac{2\times 2\times 3}{2\times 3} \times \frac{2\times 1}{2\times 1}}_{n\ \text{products}} \times C_1$$

$C_1 = 1$

$$= \frac{2n\,(2n-1)(2n-2)(2n-3) \times \cdots \times 6\times 5\times 4\times 3\times 2\times 1}{\big(n\,(n-1)\cdots 3\times 2\times 1\big)\ \big((n+1)\,n\times\cdots\times 4\times 3\times 2\big)}$$

$$= \frac{2n\,!}{n\,!\ (n+1)\,!}$$

exercise 10

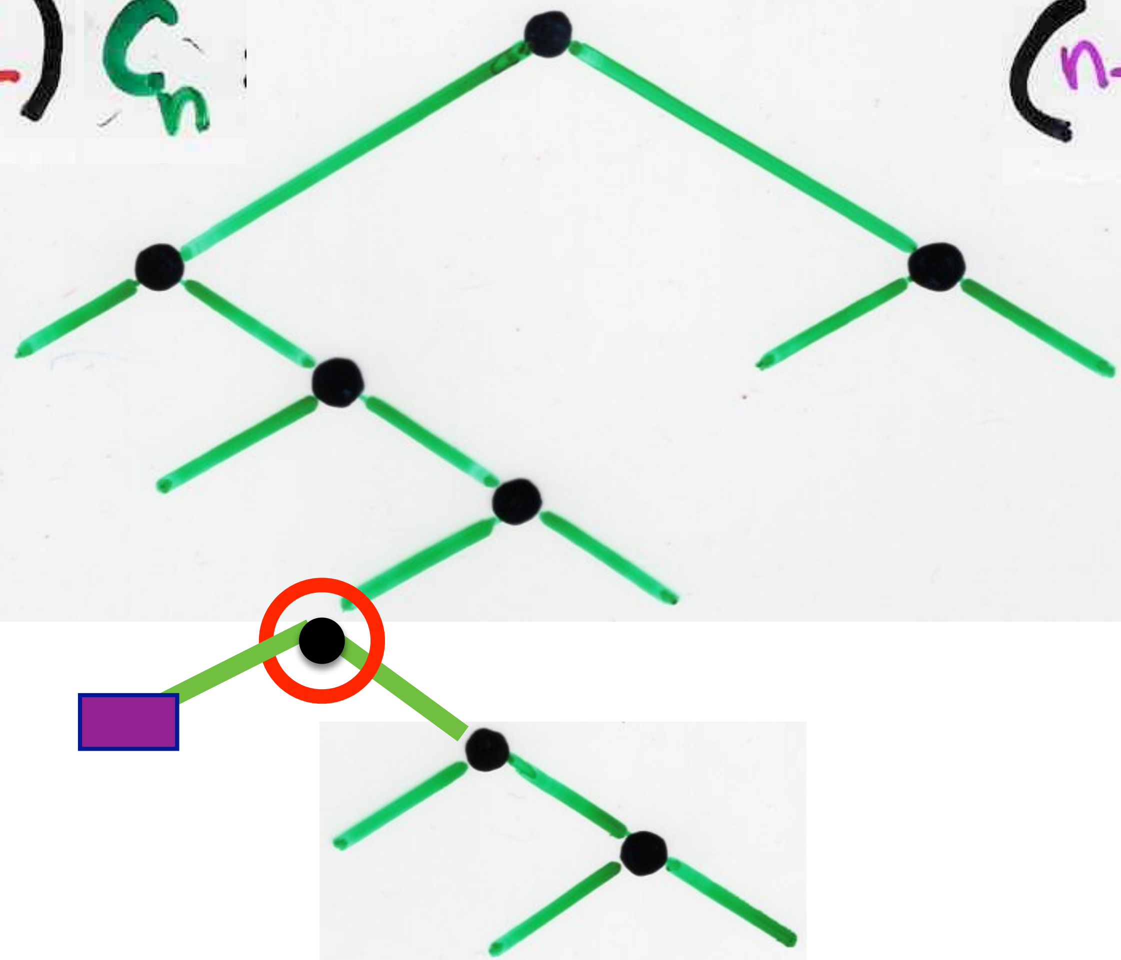
exercise 10

Describe the reciprocal bijection

(external vertex , $\mathcal{B}$ binary tree (n+1) vertices) $\longrightarrow$

($\{\ell, r\}$, vertex (internal or external) , $\mathcal{B}$ binary tree n vertices)

$2\,(2n+1)\,C_n$ $\qquad$ $(n+2)\,C_{n+1}$



An example of the reciprocal bijection is given on the set of slides STEM21_exo10 available on the web page www.viennot.org/abjc-stems21.html

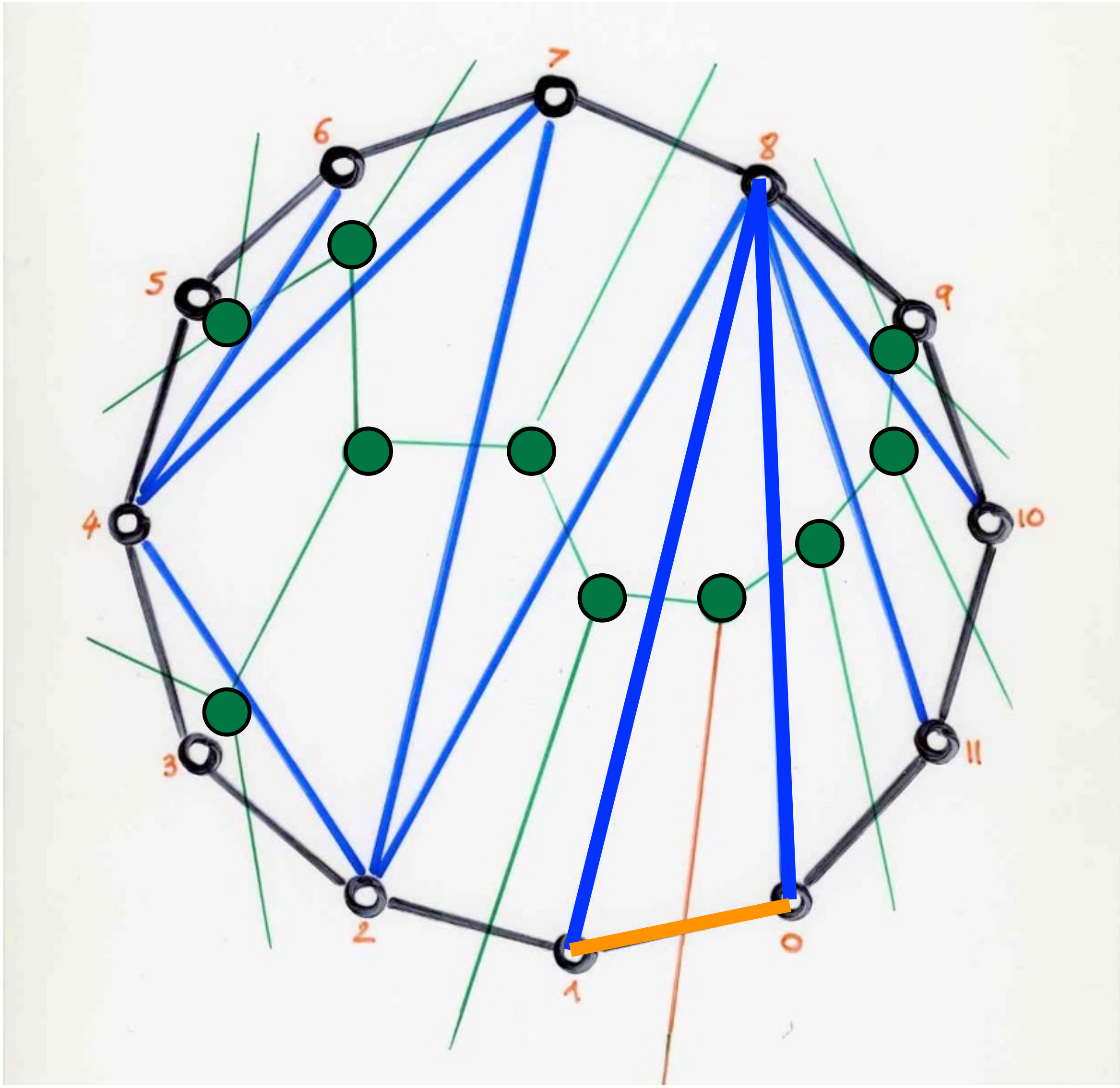
exercise 11

exercise 11

describe the reverse bijection

binary trees ⟶ triangulations

An example of the algorithmic construction of the reciprocal bijection is given on the set of slides STEM21_exo11 available on the web page www.viennot.org/abjc-stems21.html



Let B be a binary tree with n (internal) vertices. First we label these vertices by the integers 1 ,2, …, n such that the labels are increasing when one goes from the root to any external vertex of B (i.e. we get an *increasing* binary tree, see exercise 3).

To the root of B (labelled « 1 ») we associate a triangle, labelled by 1, with one edge labelled « *inactive* » and called the « *root edge* » (coloured in orange on the figures). In the algorithmic construction, to each vertex of B we associate a triangle. One of the edge will get a label « *inactive* » (in black on the figures), the two other being labelled « *active* » (in blue on the figures). The triangles are embedded in a plane and we can define the *left edge* (resp. *right edge*) as being the first (resp. second) *active* edge when turning clockwise around the triangle, starting from the (unique) *inactive* edge. (see Figure below).

During the construction, after reading the vertices labelled 1, 2,…, i of the binary tree B, we get a triangulation of a polygon with $i+2$ edges, all of them are active, except the root edge, the other edges of the triangles (i.e. the diagonals of the triangulation) are inactive. Each triangle is labelled by an integer j, $1 \leq j \leq i$.

Step $(i+1)$. In the increasing binary tree B, the vertex labelled $(i+1)$ is the left (resp. right) son of the unique vertex labelled j. On the triangle labelled j, we add a new triangle on the left (resp right) active edge. This edge become inactive. This triangle is added « outside » of the polygon and is labelled $(i+1)$.

At the end after n steps, we get a triangulation of a (convex) polygon having $(n+2)$ edges, one of them being distinguished as the root. The polygon is « labelled », we mean that it is not defined up to a rotation. This triangulation is independent of the increasing labelling ot the binary tree B.

exercise 12

In this series of slides, « ABjC » refers to the video-book
The Art of Bijective Combinatorics, www.viennot.org/abjc.html

exercise 12

partition function

$$Z_n = \sum \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

$$\left[\bar{\alpha}^{i} + \bar{\alpha}^{(i-1)}\beta + \ldots + \bar{\alpha}^{(i-j)}\bar{\beta}^{j} + \ldots + \bar{\beta}^{i}\right]$$

This last exercise is to give a proof for the expression of Z_n the partition function of the TASEP (with parameter α and β), a toy model in the physics of systems far from equilibrium.

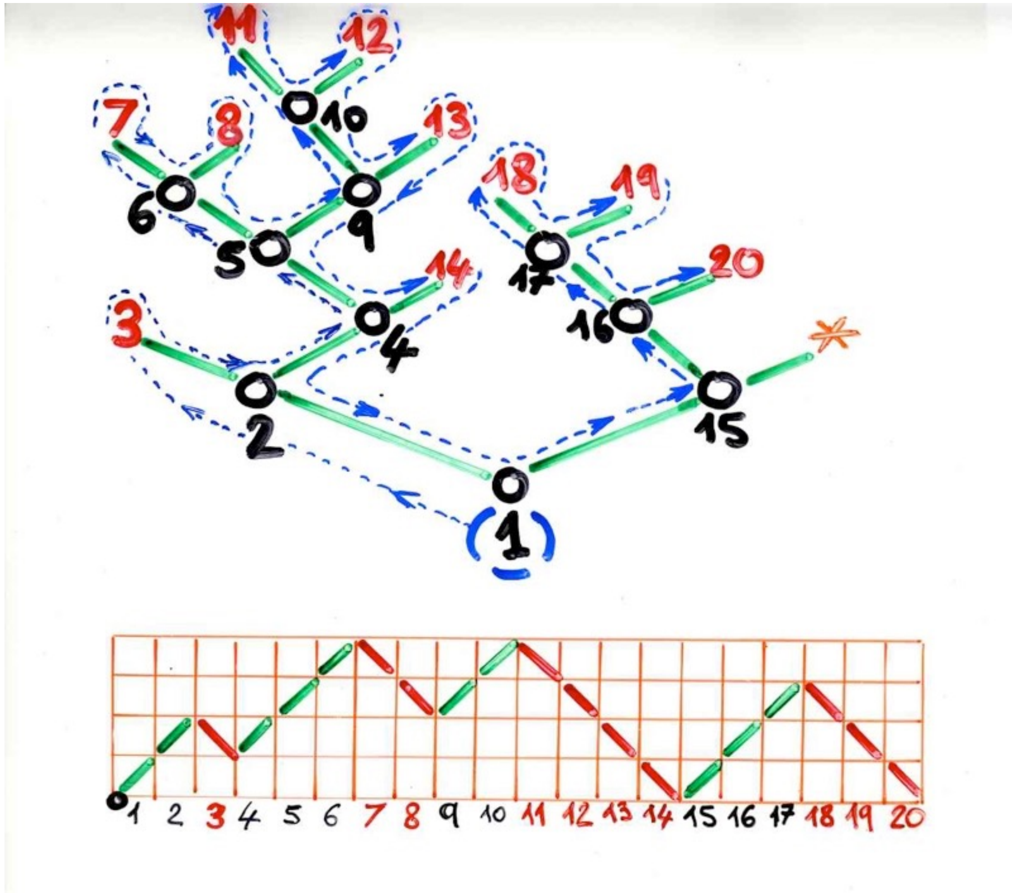
In the talk I gave an expression of this partition function in term of binary trees. For a binary tree T, lb(T) (resp. rb(T)) denotes the length (number of edges) of the principal left (resp. right) branch, that is the number of edges going always left (resp.right) from the root of T.

$$Z_n = \sum_{\substack{T \\ \text{binary trees} \\ n \text{ vertices}}} \bar{\alpha}^{lb(T)} \bar{\beta}^{rb(T)}$$

$\bar{\alpha} = \alpha^{-1}$

$\bar{\beta} = \beta^{-1}$

$\bar{\alpha} = \alpha^{-1}$ $\bar{\beta} = \beta^{-1}$



In the bijection between binary trees and Dyck paths given in the talk (see the violin video), the parameter lb(T), length of the principal left branch of T becomes the length of the maximal sequence of North-East steps of the Dyck paths.

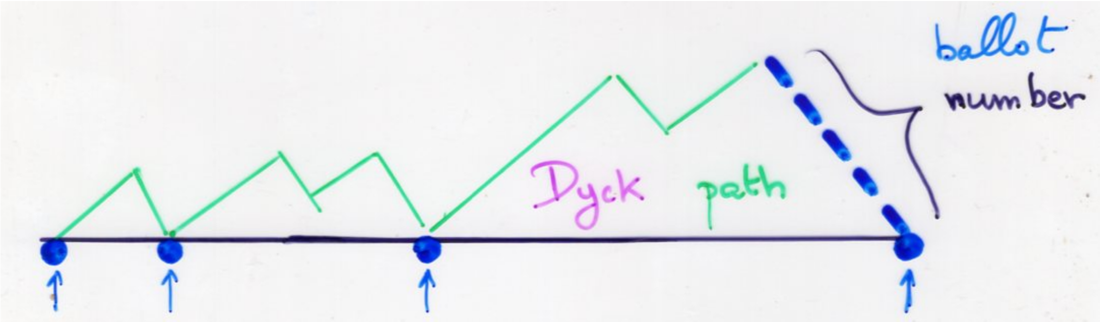
On the figure $n = 10$, $lb(T) = 2$

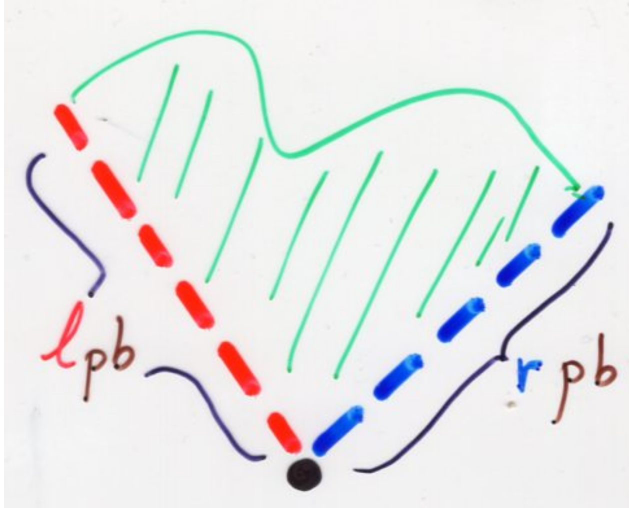
From exercise 8 we deduce that the number of binary trees having n vertices with $lb(T) = i$ is

$$\frac{i}{2n-i}\binom{2n-i}{n}$$

Thus this last exercise is to prove that the summation giving Z_n involving the distribution of two parameters (length of the left and right branch) can be reduced to the distribution of a single parameter.

$$\frac{i}{(2n-i)}\binom{2n-i}{n}$$



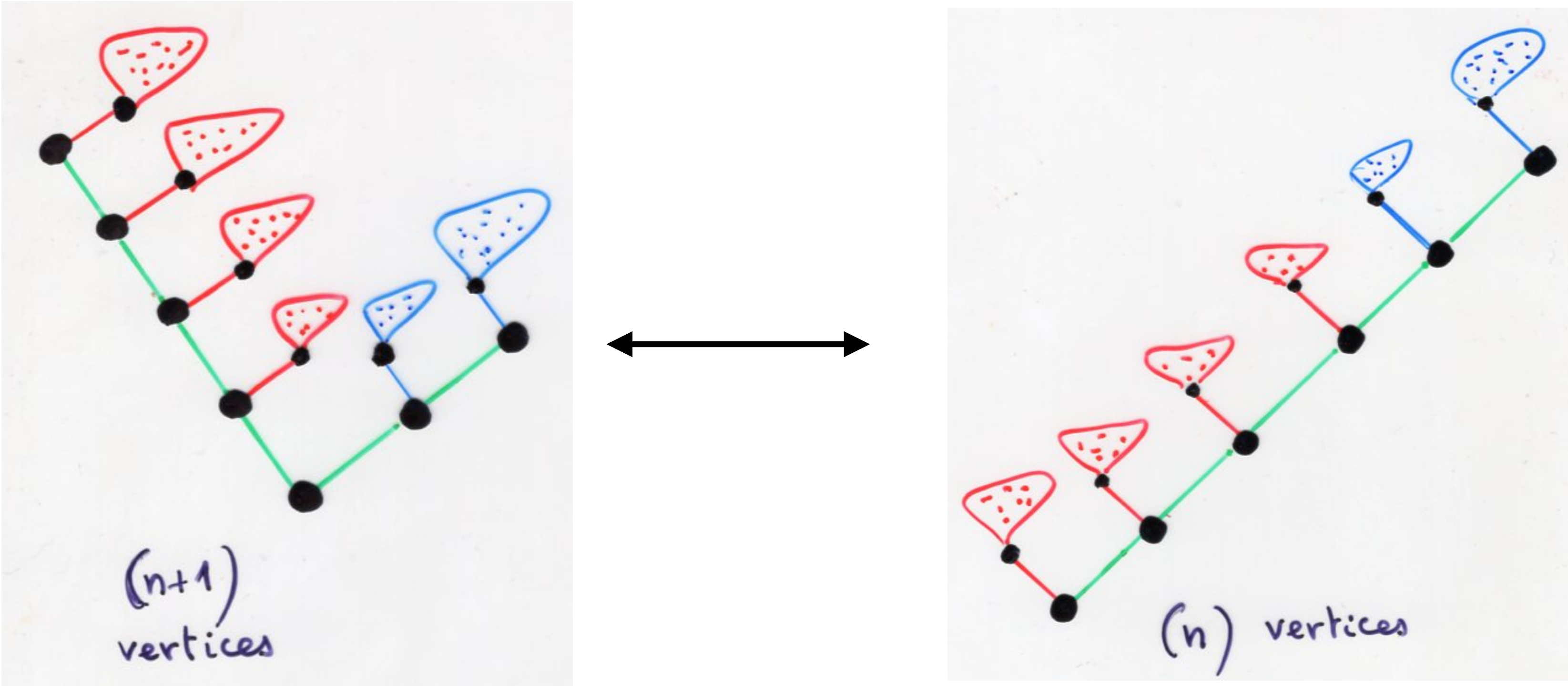


$$Z_n = \sum_{1\leq i\leq n} \frac{i}{(2n-i)}\binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

Here we use the right principal branch of the binary tree (in blue) corresponding to the longest sequence of South-step of the Dyck path, as in ABjC, PartIII, Ch 4a, p112-114

$$\left[\bar{\alpha}^{i} + \bar{\alpha}^{(i-1)}\beta + \ldots + \bar{\alpha}^{(i-j)}\bar{\beta}^{j} + \ldots + \bar{\beta}^{i}\right]$$



There exists a bijection between binary trees with (n+1) vertices and binary trees where the internal vertices of the right branch are labelled red or blue, starting from the root with a sequence of red vertices (may be empty), followed by a sequence of blue vertices (may be empty).

See ABjC, PartIII, Ch 4a, p112-114

Thus

$$\sum_{B} \alpha^{-lpb(B)} \beta^{-rpb(B)}$$

$$\left[\bar{\alpha}^{(i)} + \bar{\alpha}^{(i-1)}\bar{\beta} + \dots + \bar{\alpha}\,\bar{\beta}^{(i-1)} + \bar{\beta}^{(i)}\right]$$

$$\frac{i}{(2n-i)}\binom{2n-i}{n}$$

$$Z_n = \sum \frac{i}{2n-i}\binom{2n-i}{n}\frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

See ABjC, PartIII, Ch 4a, p112-114