The lattice Tamari(v) is a maule

IMSc, Chennai

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slides (version2)

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https://www.imsc.res.in/~viennot

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IMSc, Chennai
26 February 2018

Maule: tilings, Young and Tamari lattices under the same roof

IMSc, Chennai
19 February 2018

Maule

X **cloud** is a finite subset of the square lattice $\mathbb{Z} \times \mathbb{Z}$

Definition

Γ- move

X cloud. Let $\alpha, \beta, \gamma \in X$ in Γ-position, that is

$$\begin{array}{ccc} \alpha & \cdots & \beta \\ \vdots & & \\ \gamma & & \end{array}$$

Suppose that all the vertices of the rectangle

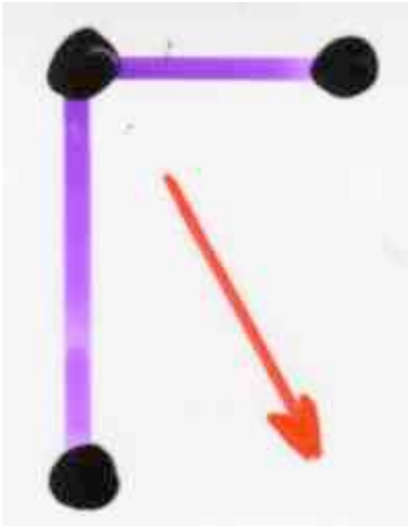
$$\begin{array}{ccc} \alpha & \cdots & \beta \\ \vdots & & \vdots \\ \gamma & \cdots & \delta \end{array}$$

, except α, β, γ, are empty (denoted $\times$)

Definition Γ- move

X, Y clouds $\quad Y = \Gamma(X)$

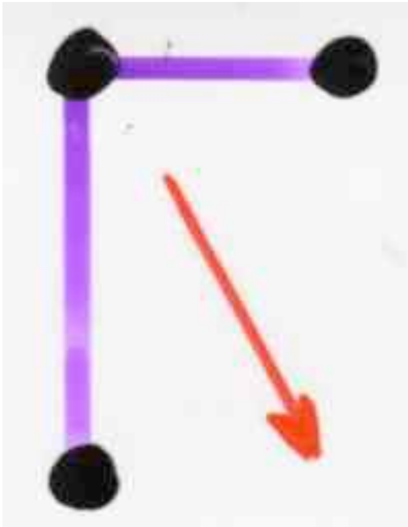
Γ- move



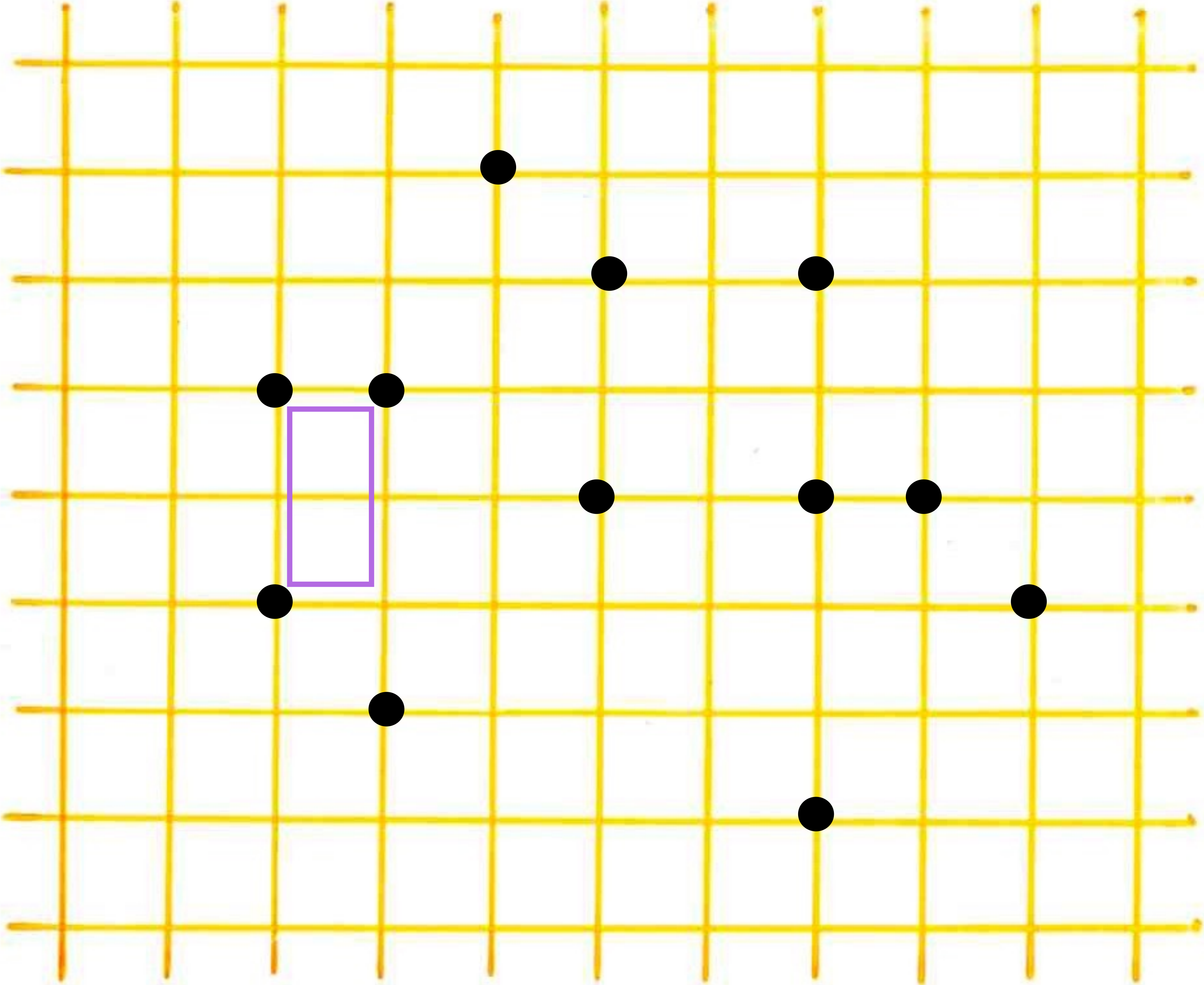
Definition Γ- move

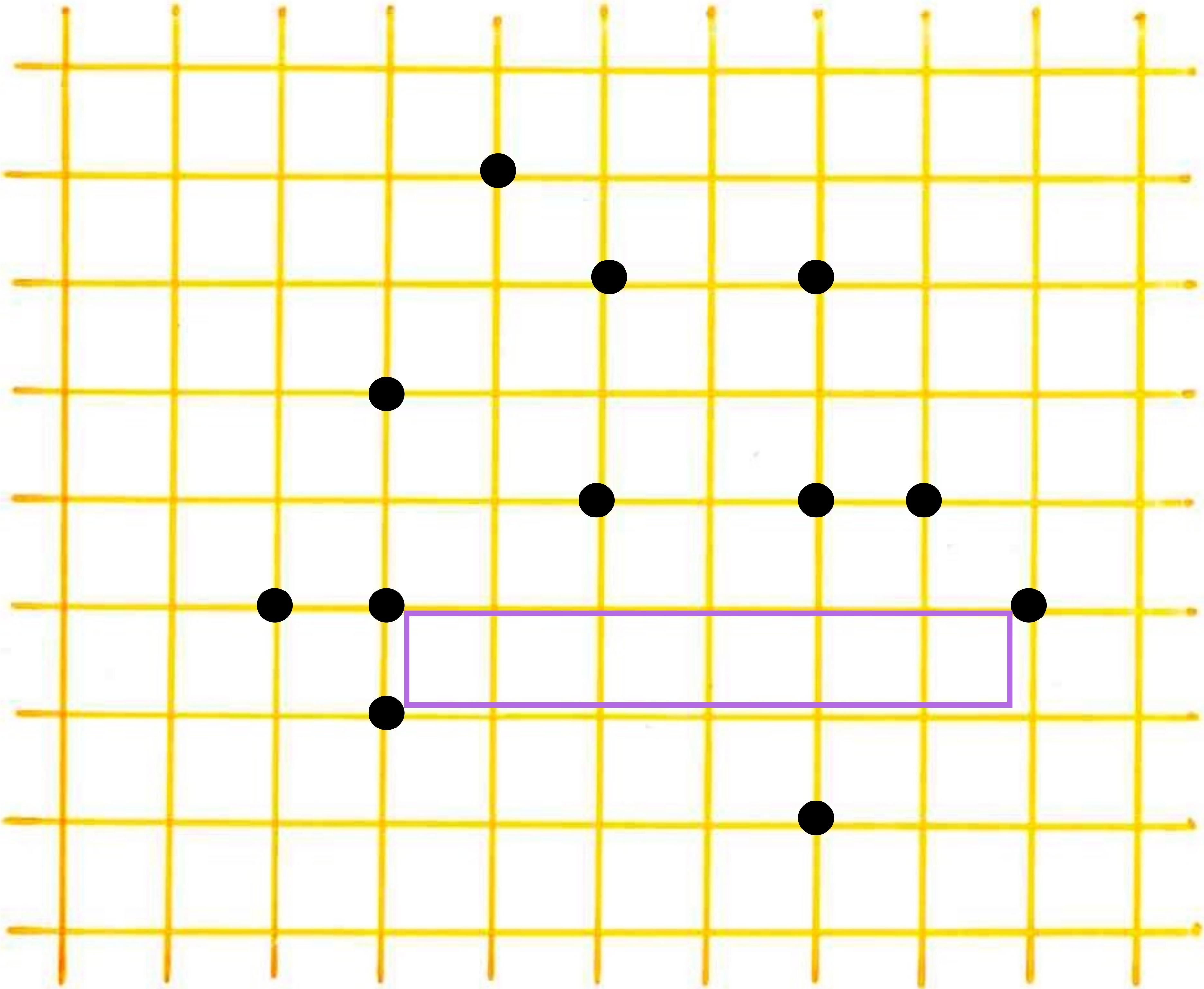
X, Y clouds $Y = \Gamma(X)$

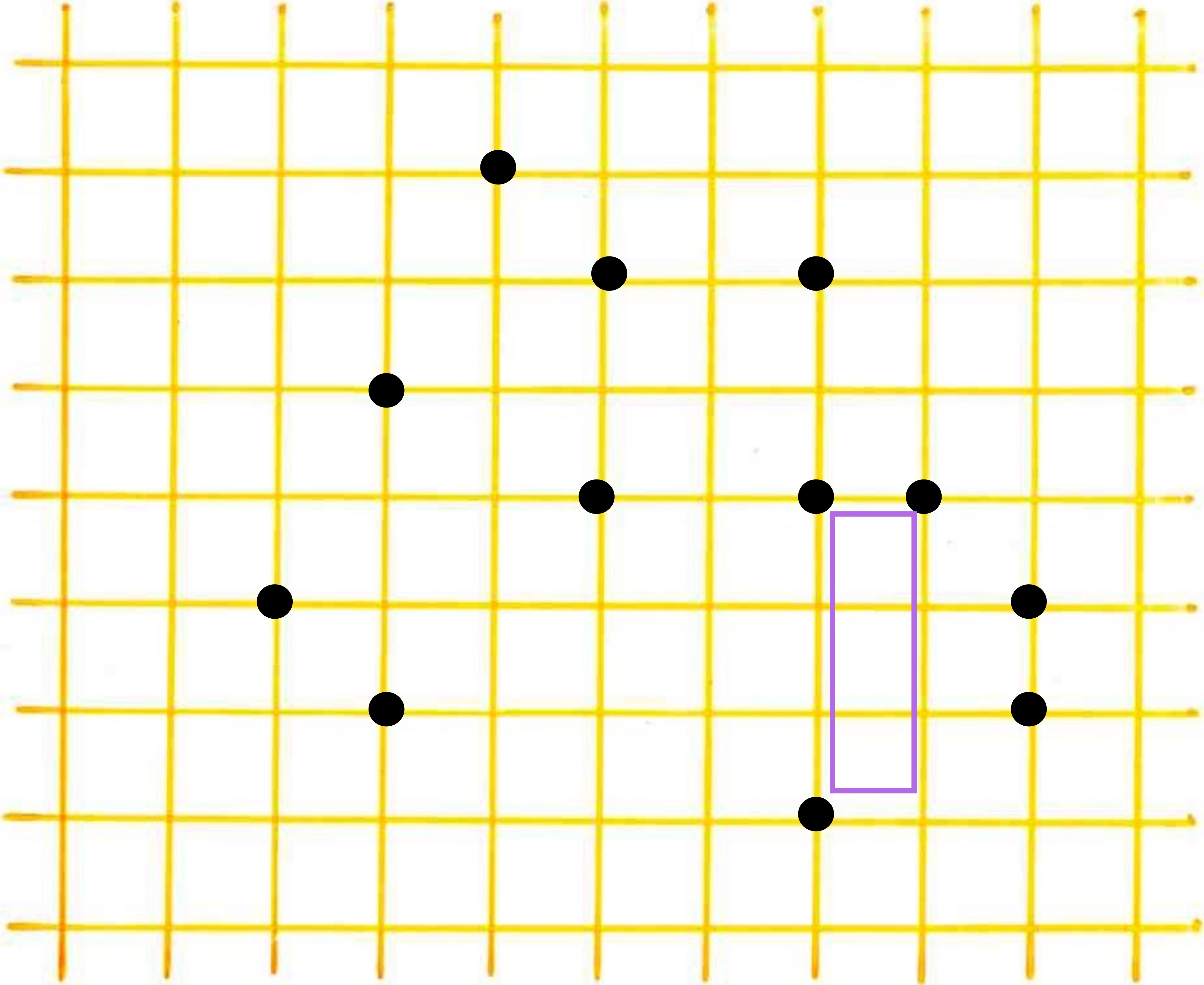
Γ - move

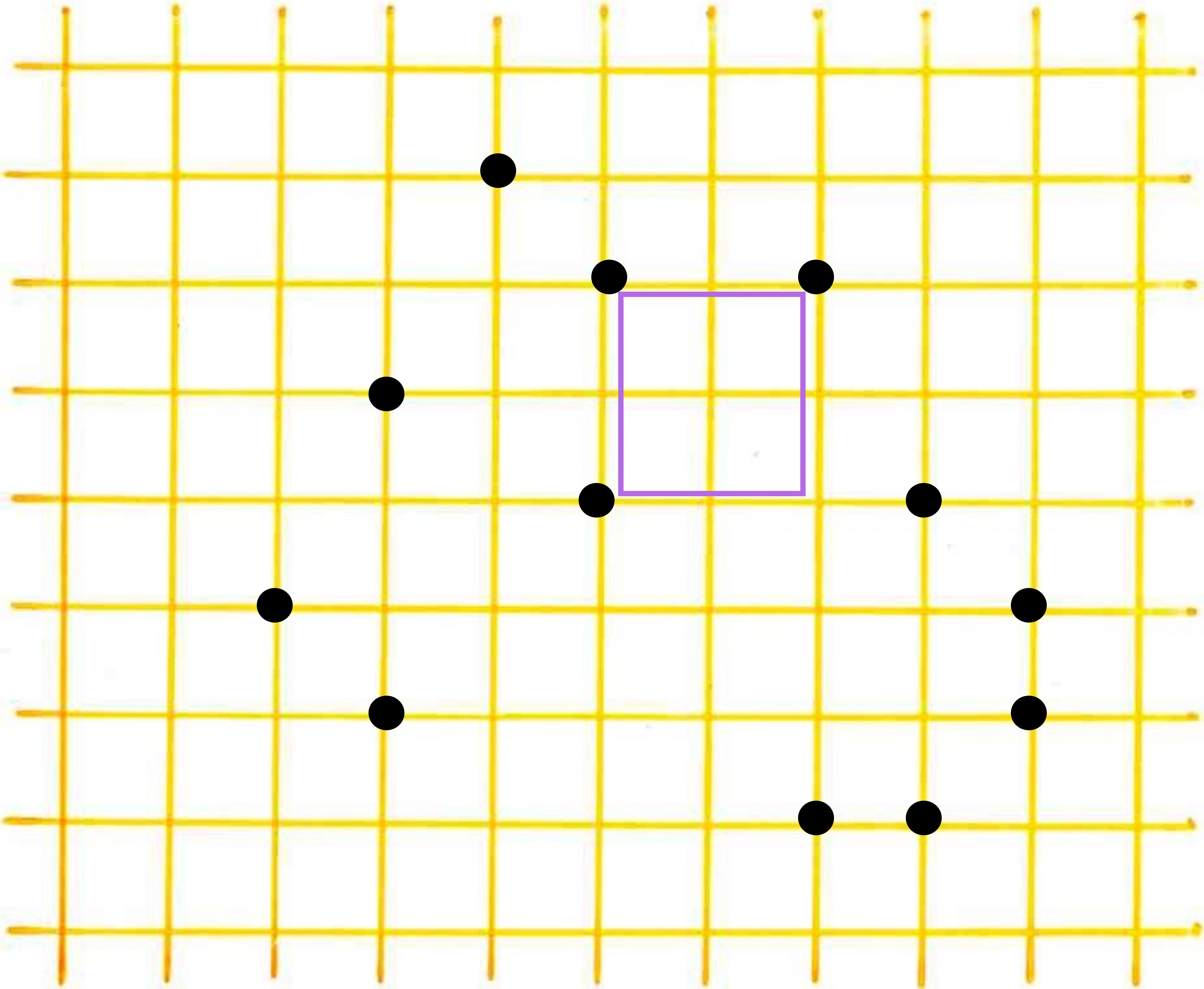


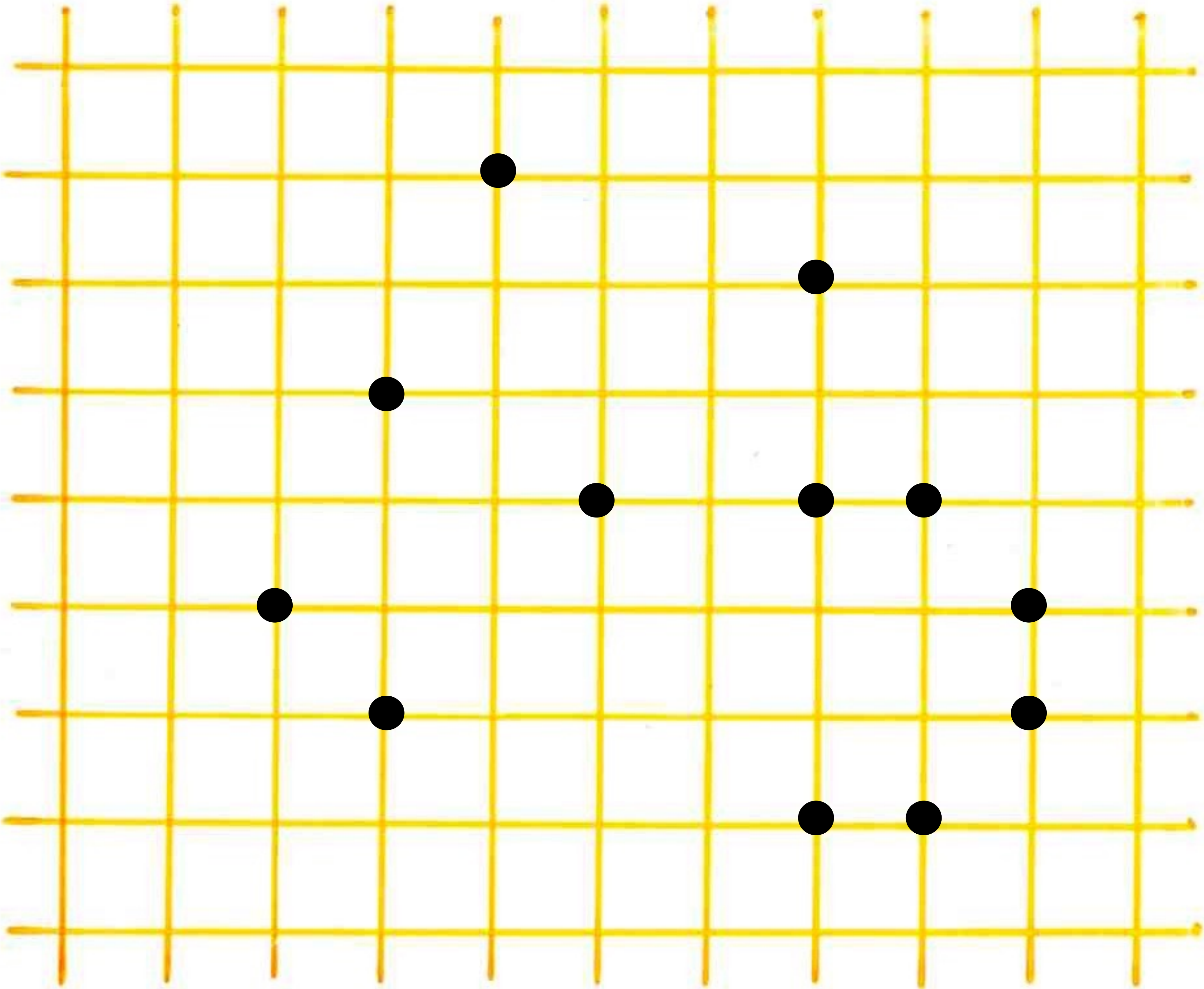
notation $X \xrightarrow{\Gamma} Y$











Main definition The poset Maule(X) is the set of all clouds obtained from X by a succession of Γ-moves, (i.e. $X \xrightarrow{\Gamma^*} Y$) equipped with the order relation $Y \xrightarrow{\Gamma^*} Z$ for $Y, Z \in$ Maule(X).

Remark Maule

- name of an area in Chile where this research was started, thanks to an invitation of Luc Lapointe (Talca Univ.)
- also the name of the river crossing this area

Mapuche name : pronounce Ma-ou-lé

signification: rainy

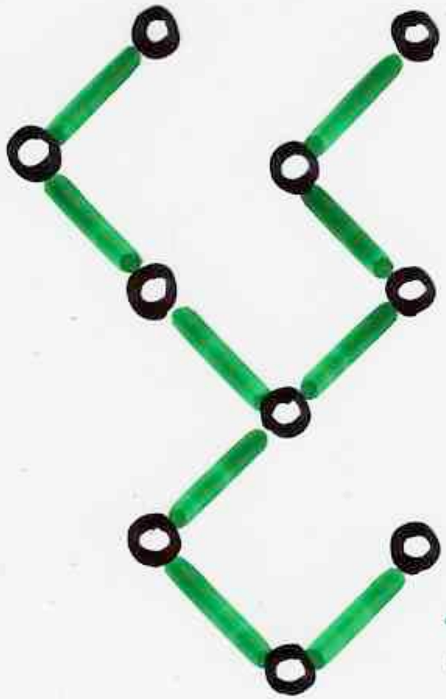
Maule
Area
(Chile)

Maule valley

Luc
Lapointe
(Talca Univ.)

Tamari lattice

definition



B

a binary tree B

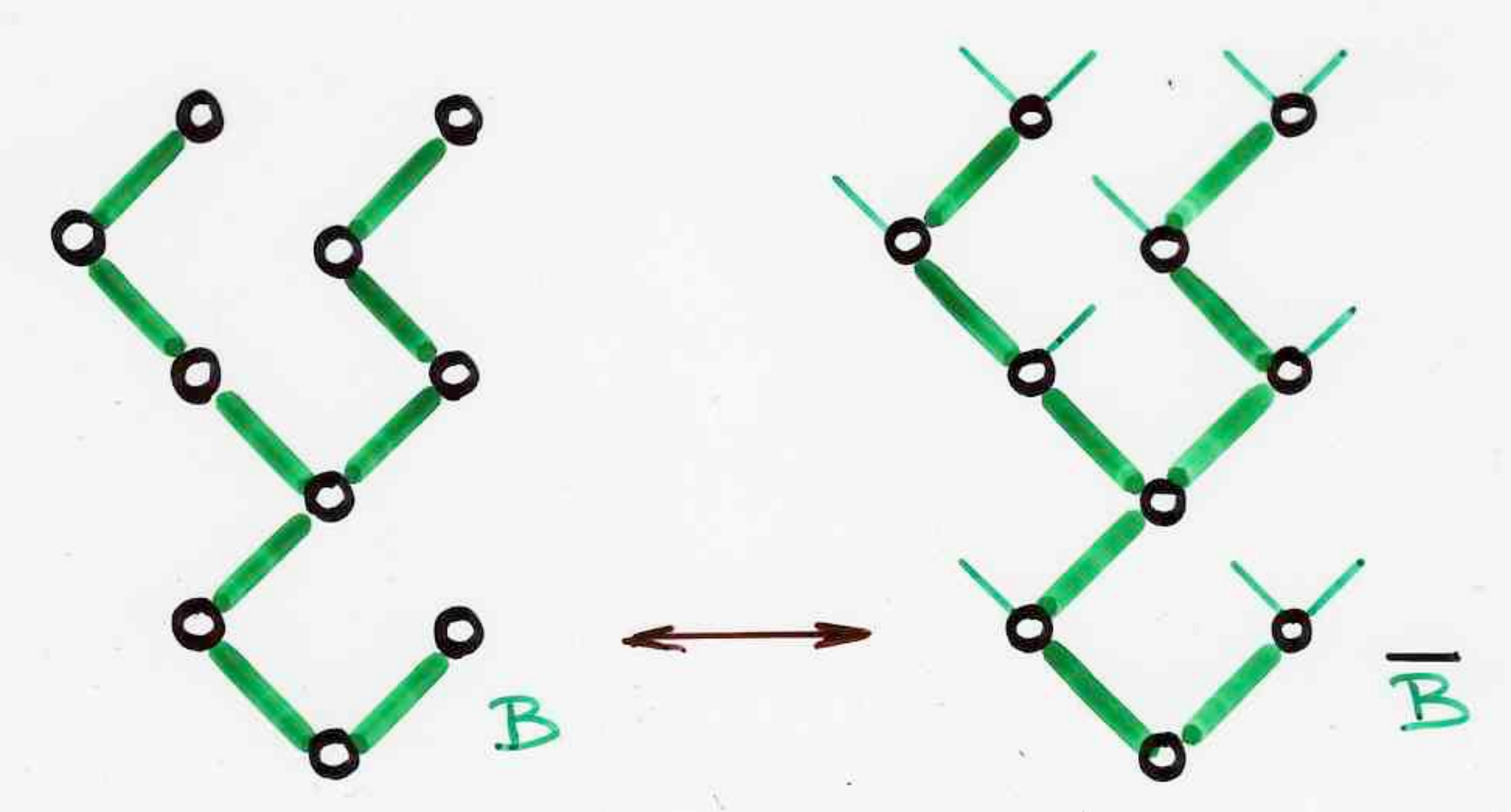
$$\begin{cases} B = (L, r, R) \\ \text{or} \\ B = \emptyset \end{cases}$$

L, R binary trees

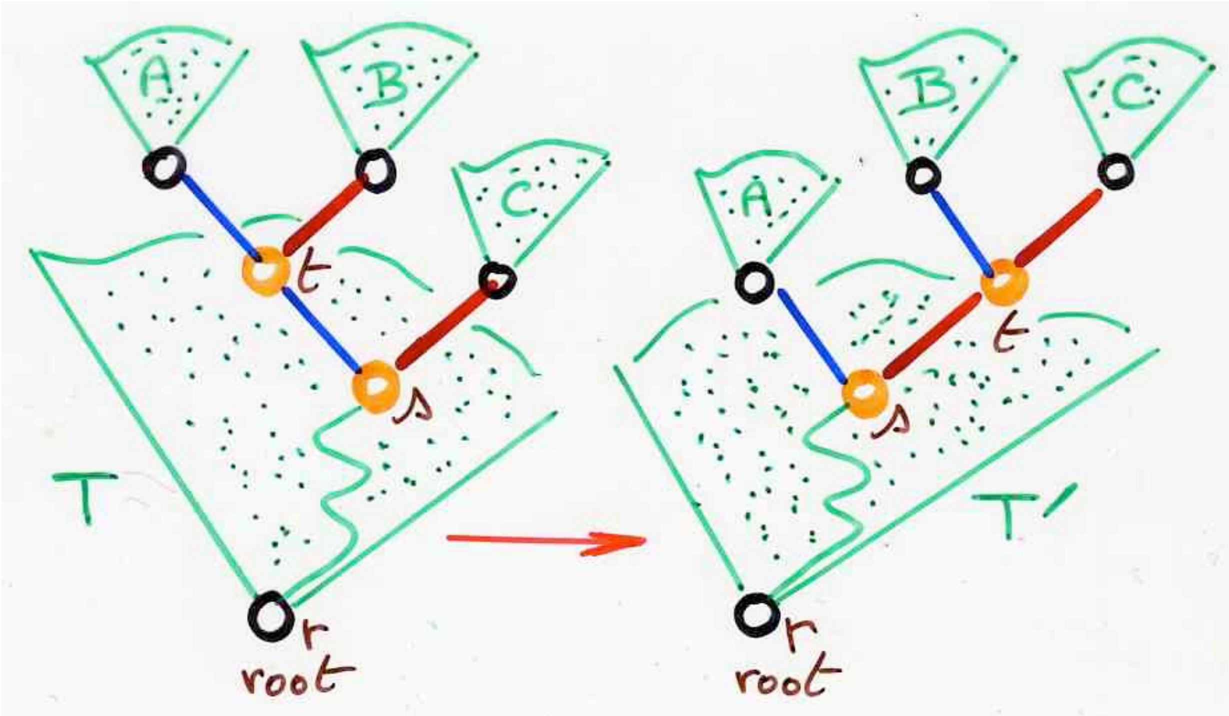
r root

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Catalan numbers

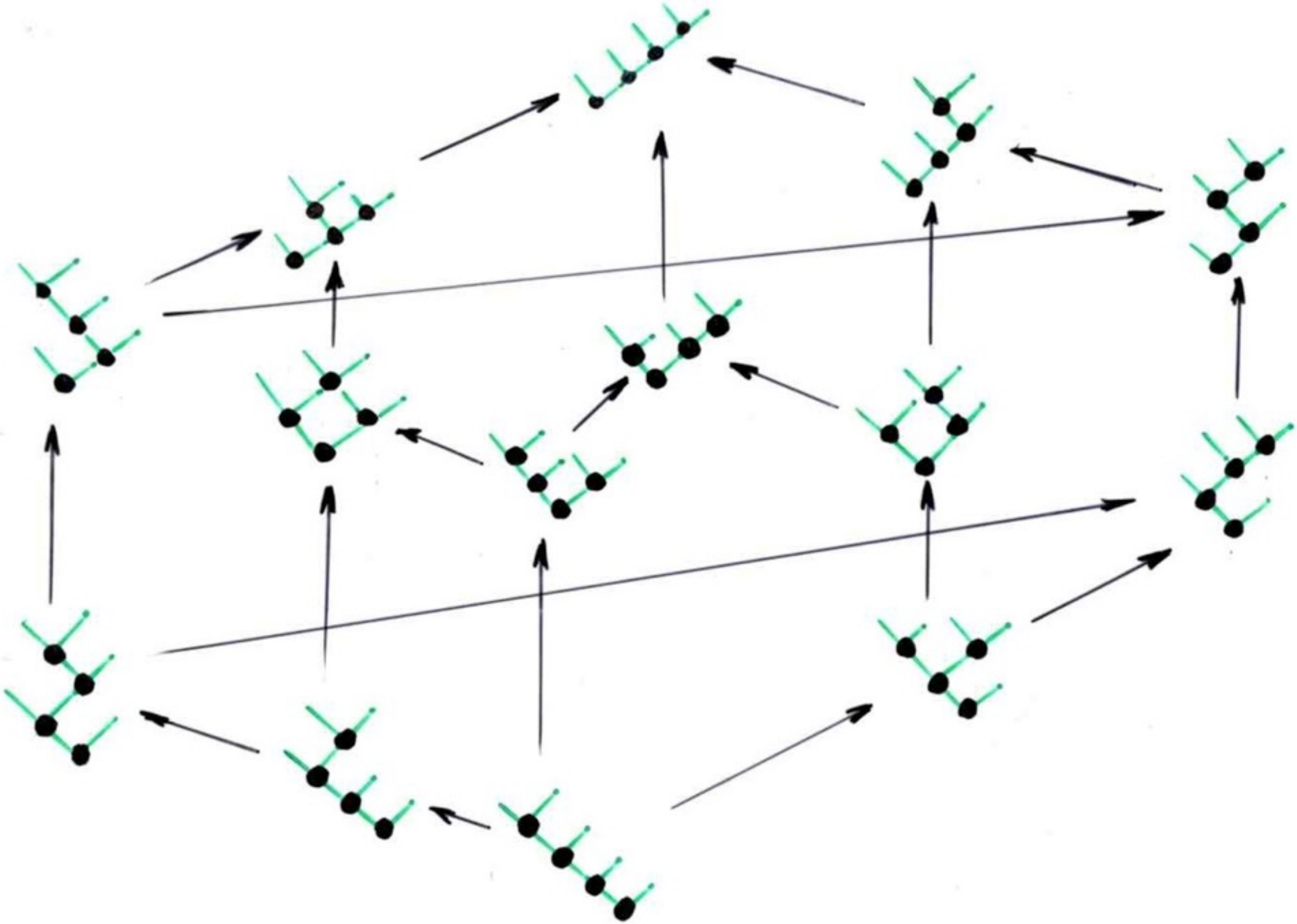


a binary tree B
and its associated complete binary tree $\overline{B}$
(full)



Tamari lattice

Rotation in a binary tree: the covering relation in the Tamari lattice

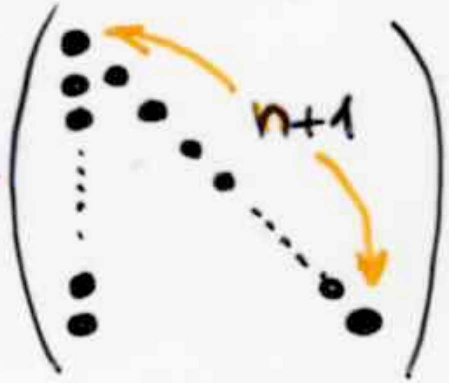


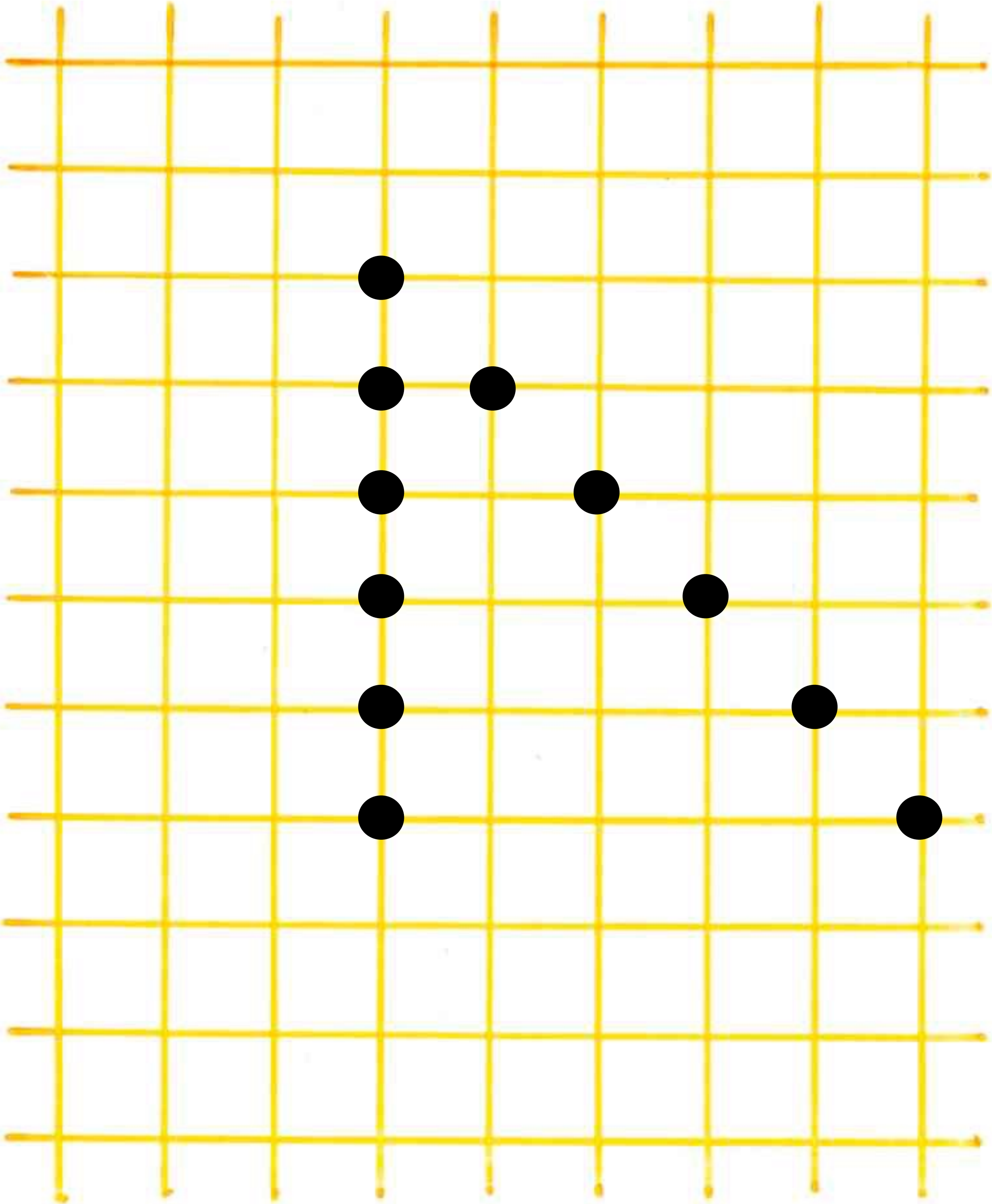
Tamari lattice

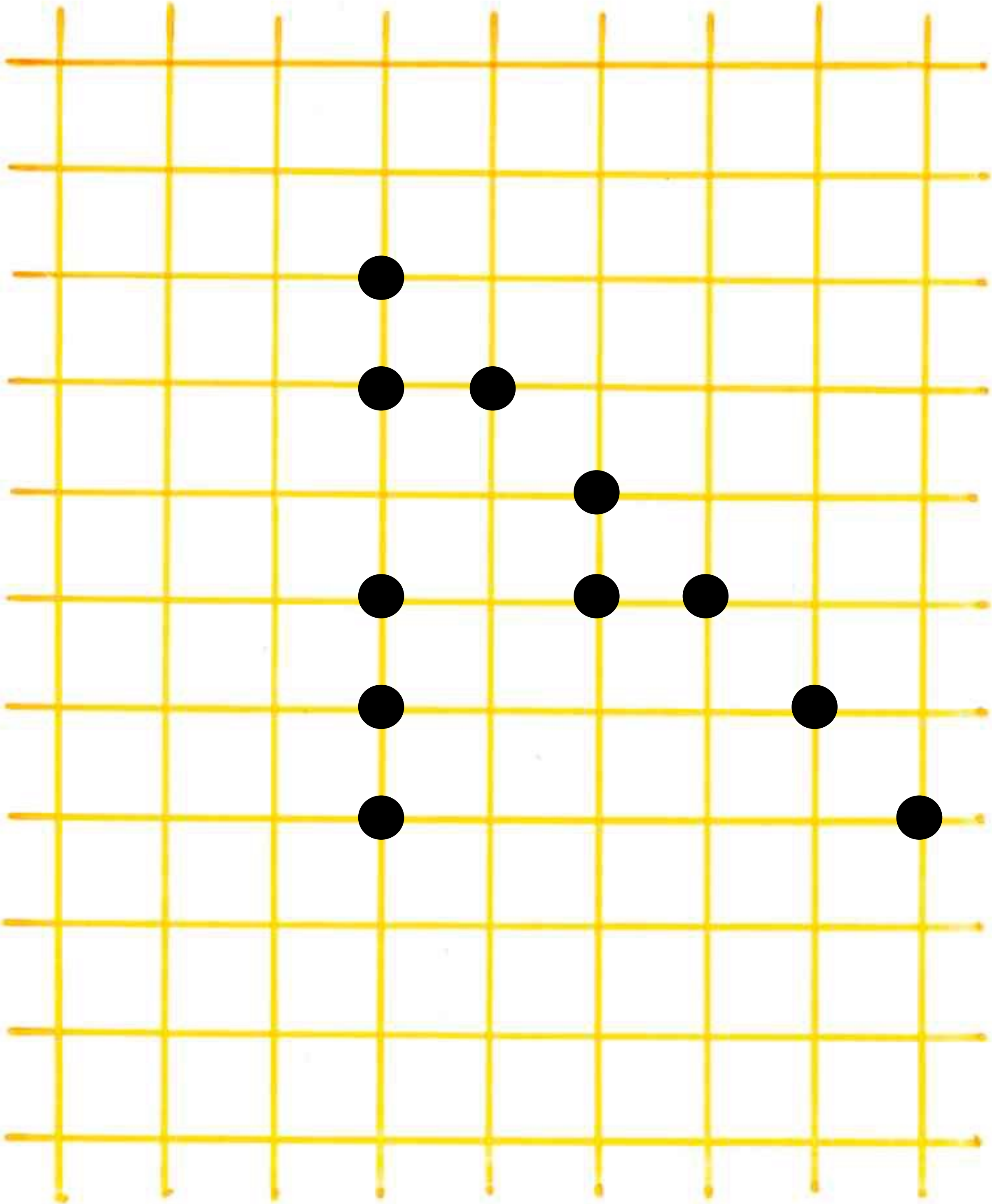
as a maule

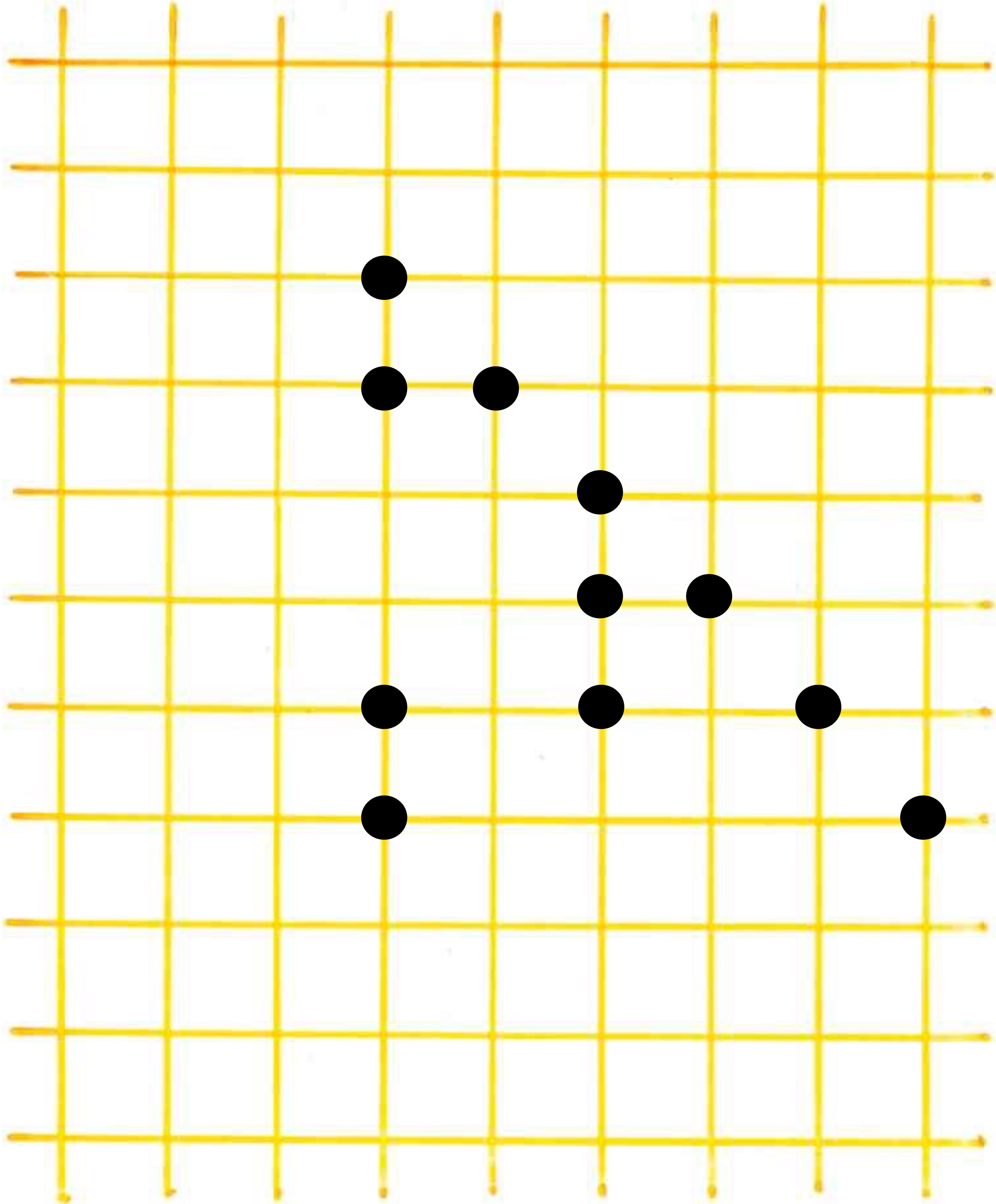
Proposition

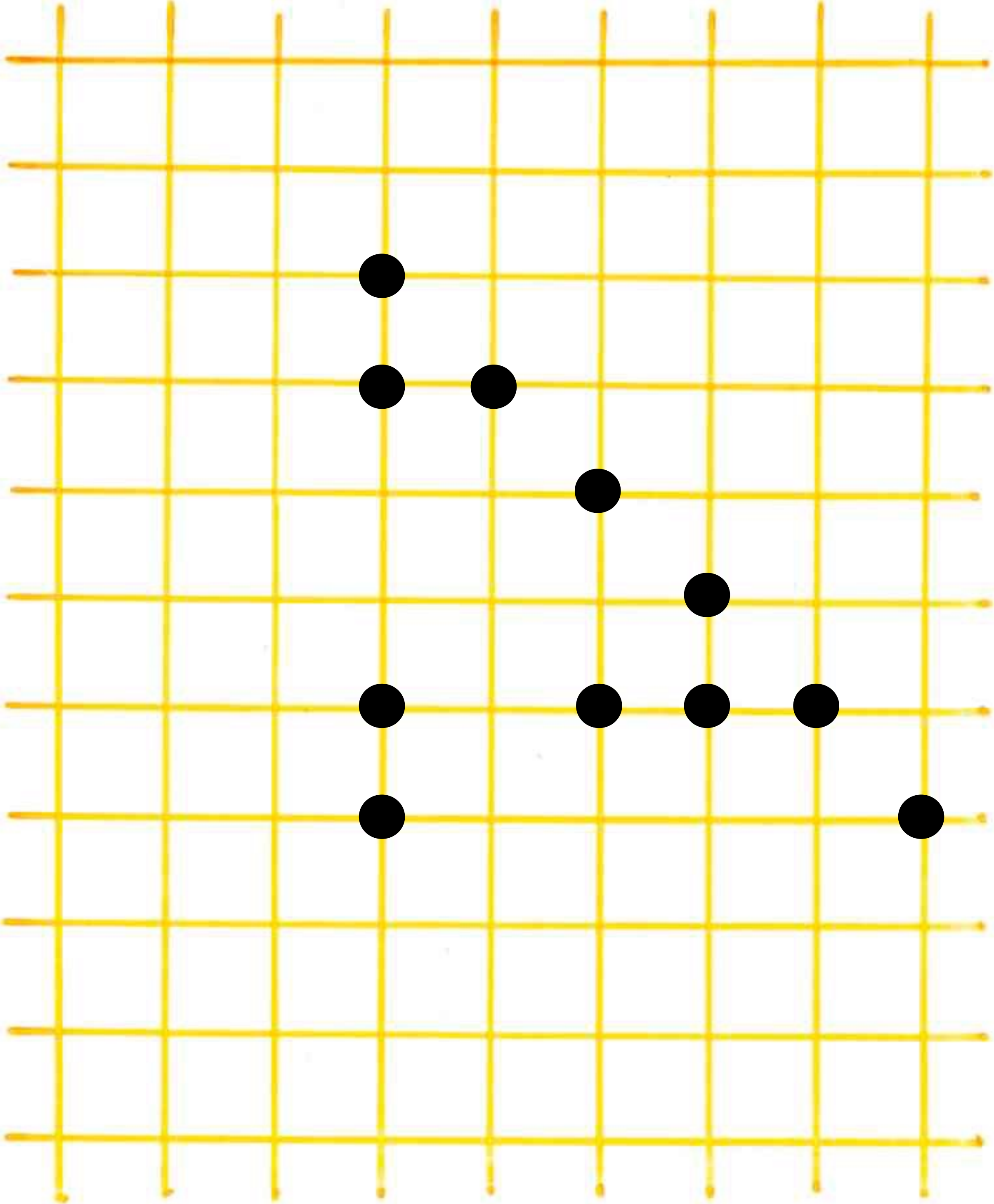
Tamari(n) = Maule

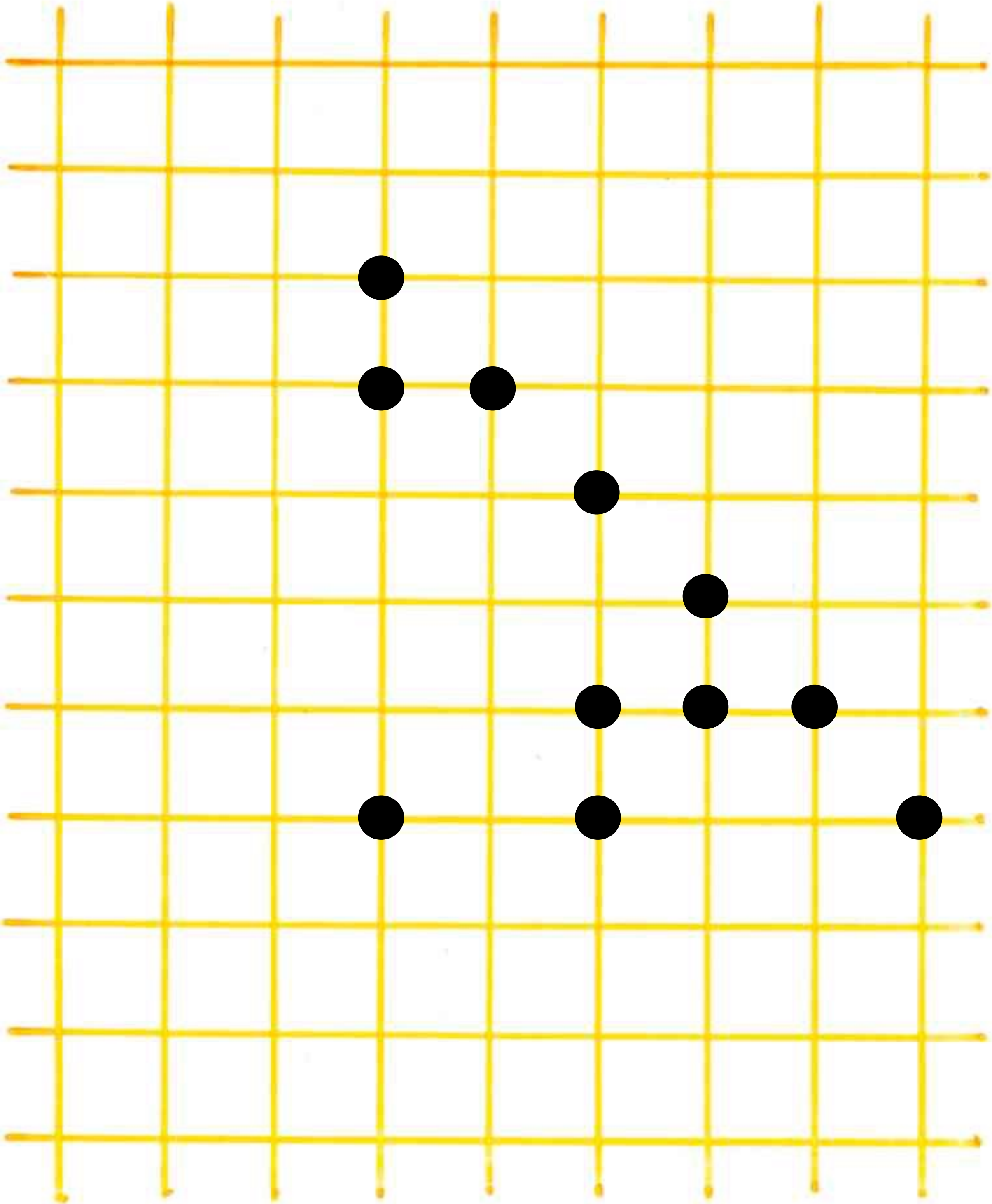


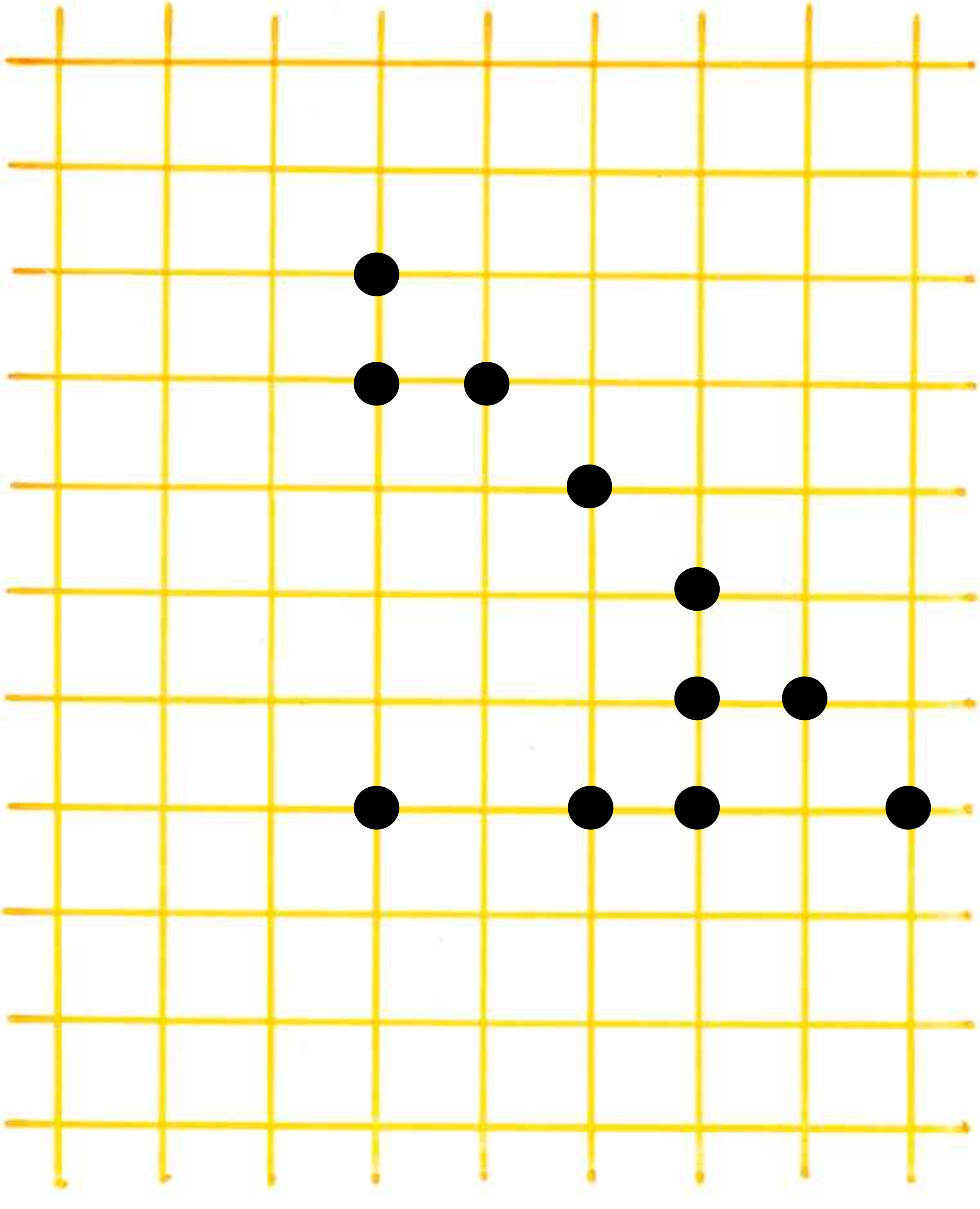


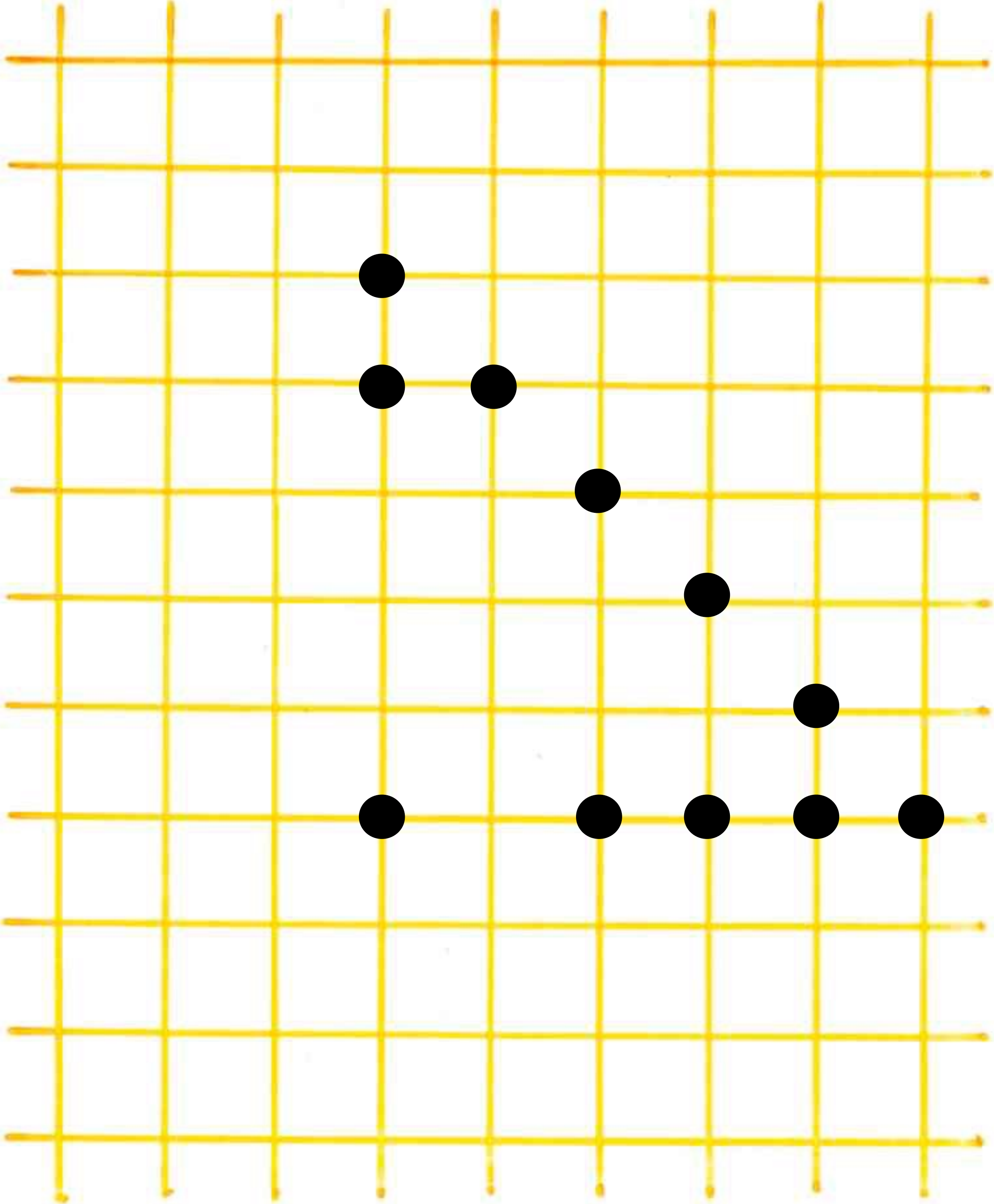


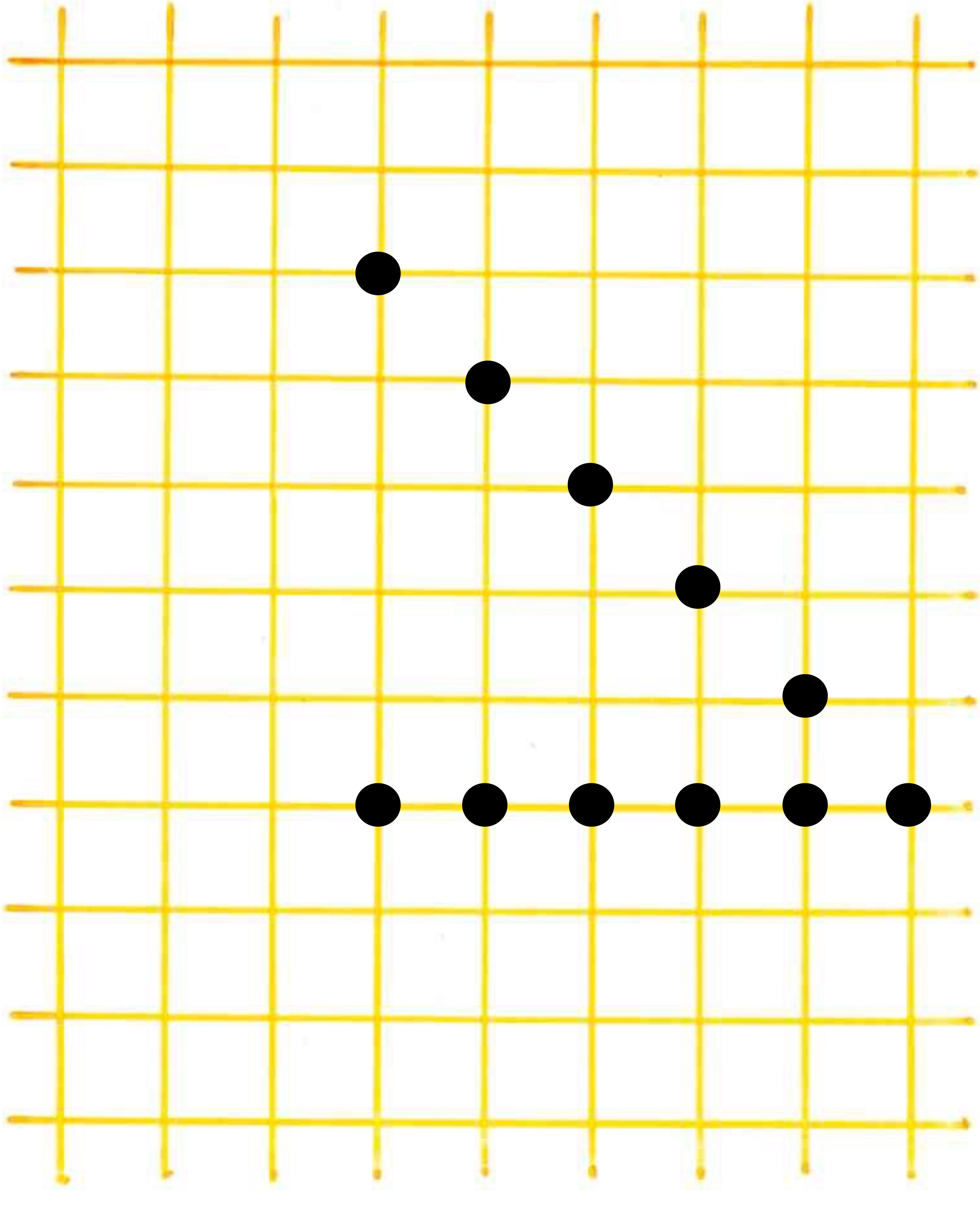


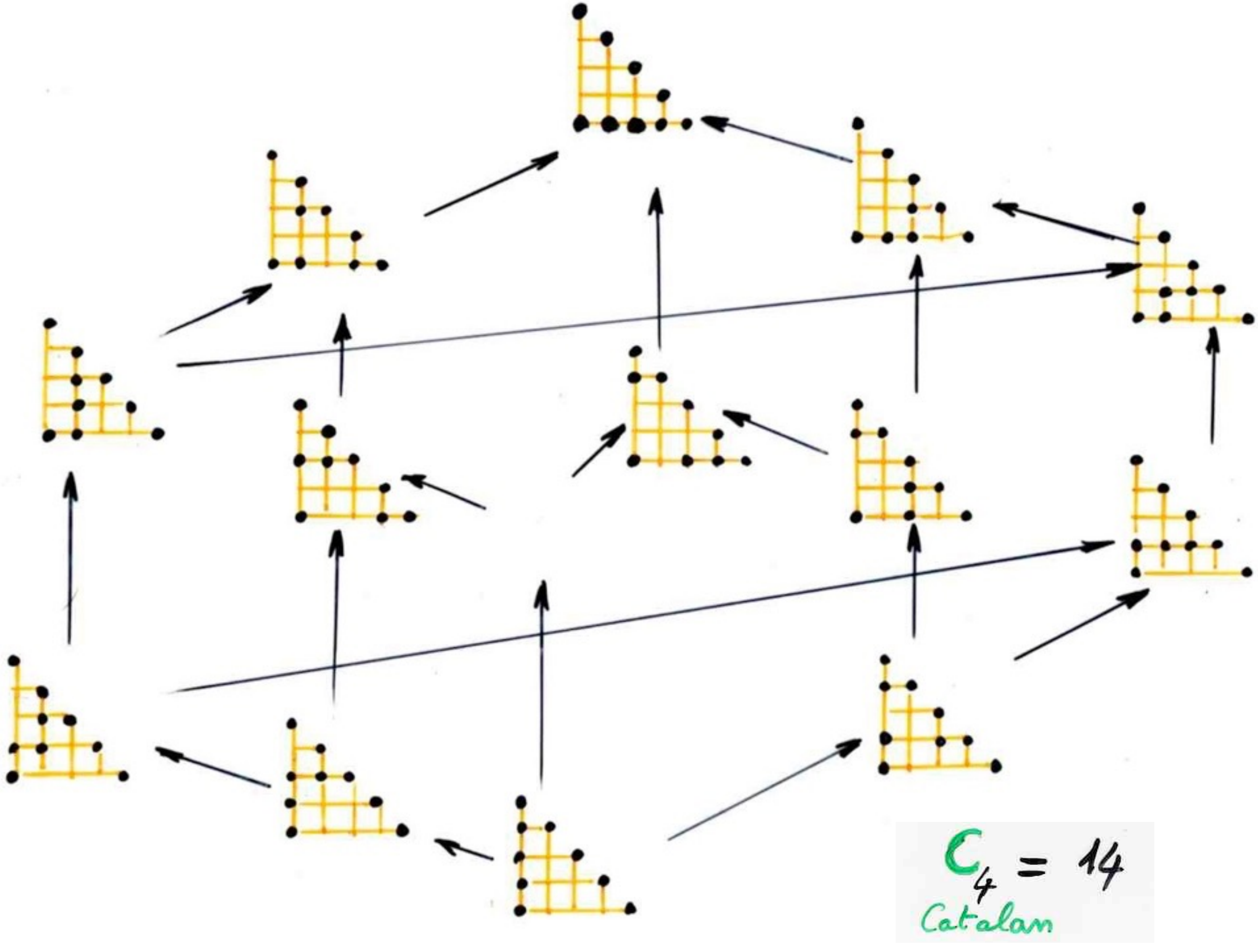


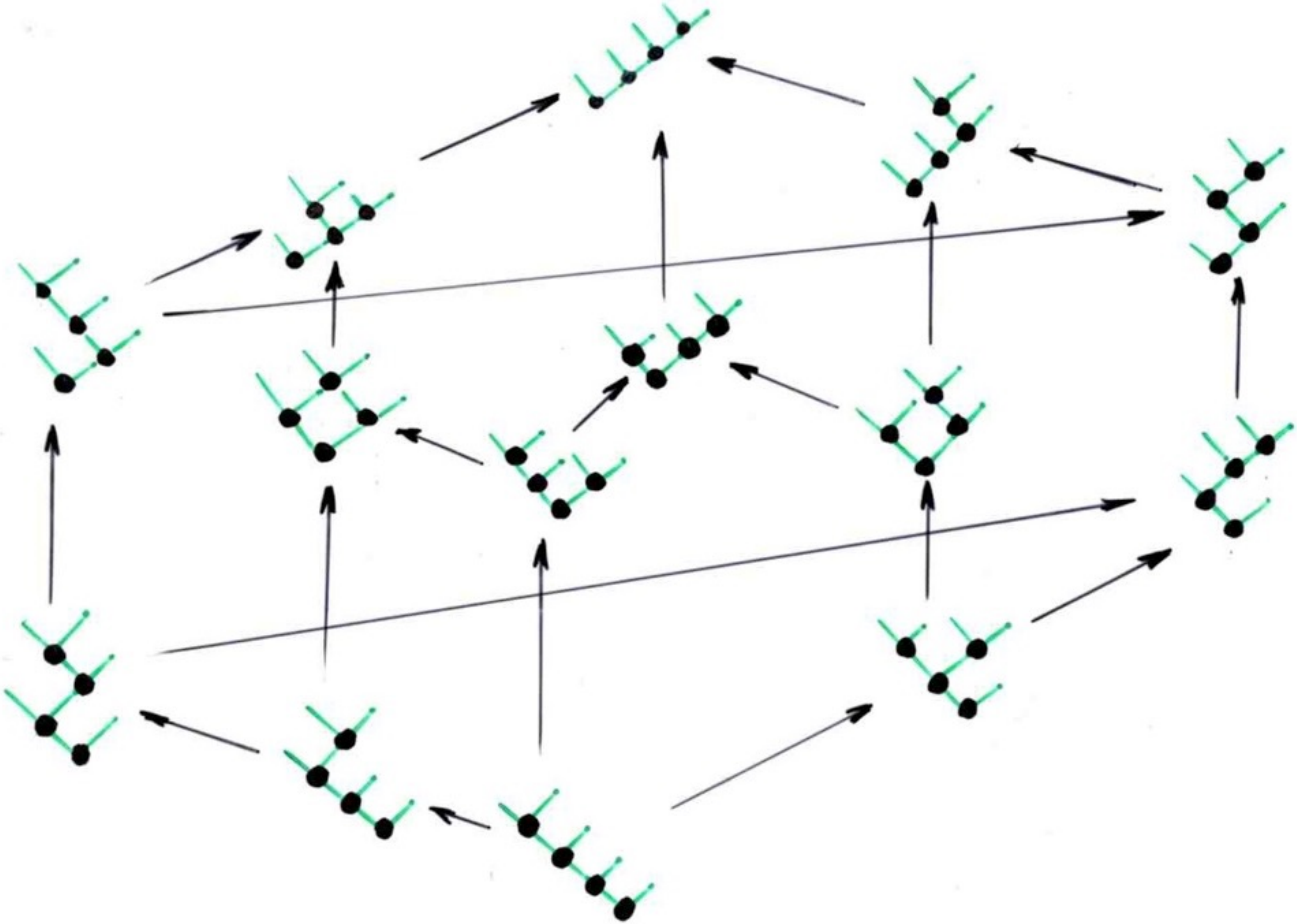






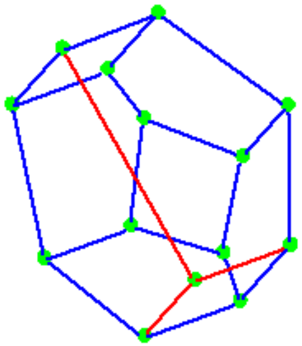


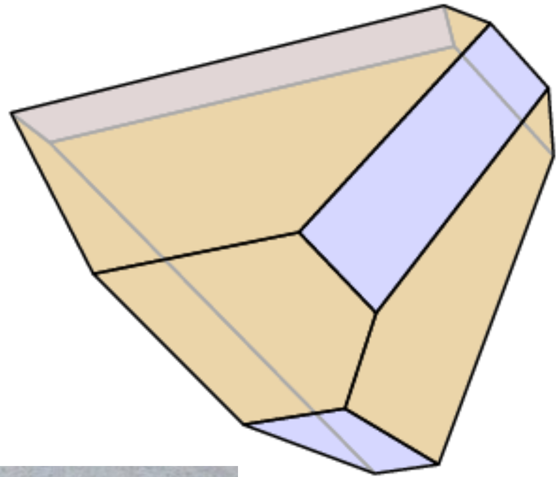

$C_4 = 14$
Catalan



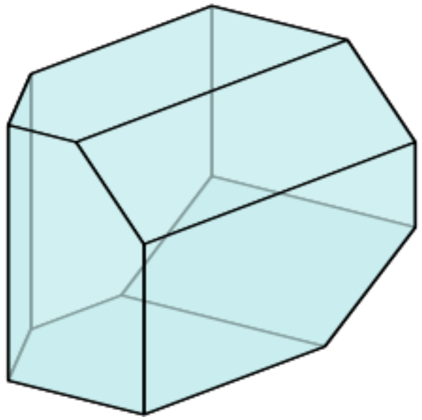
geometric realization of The Tamari lattice

Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?

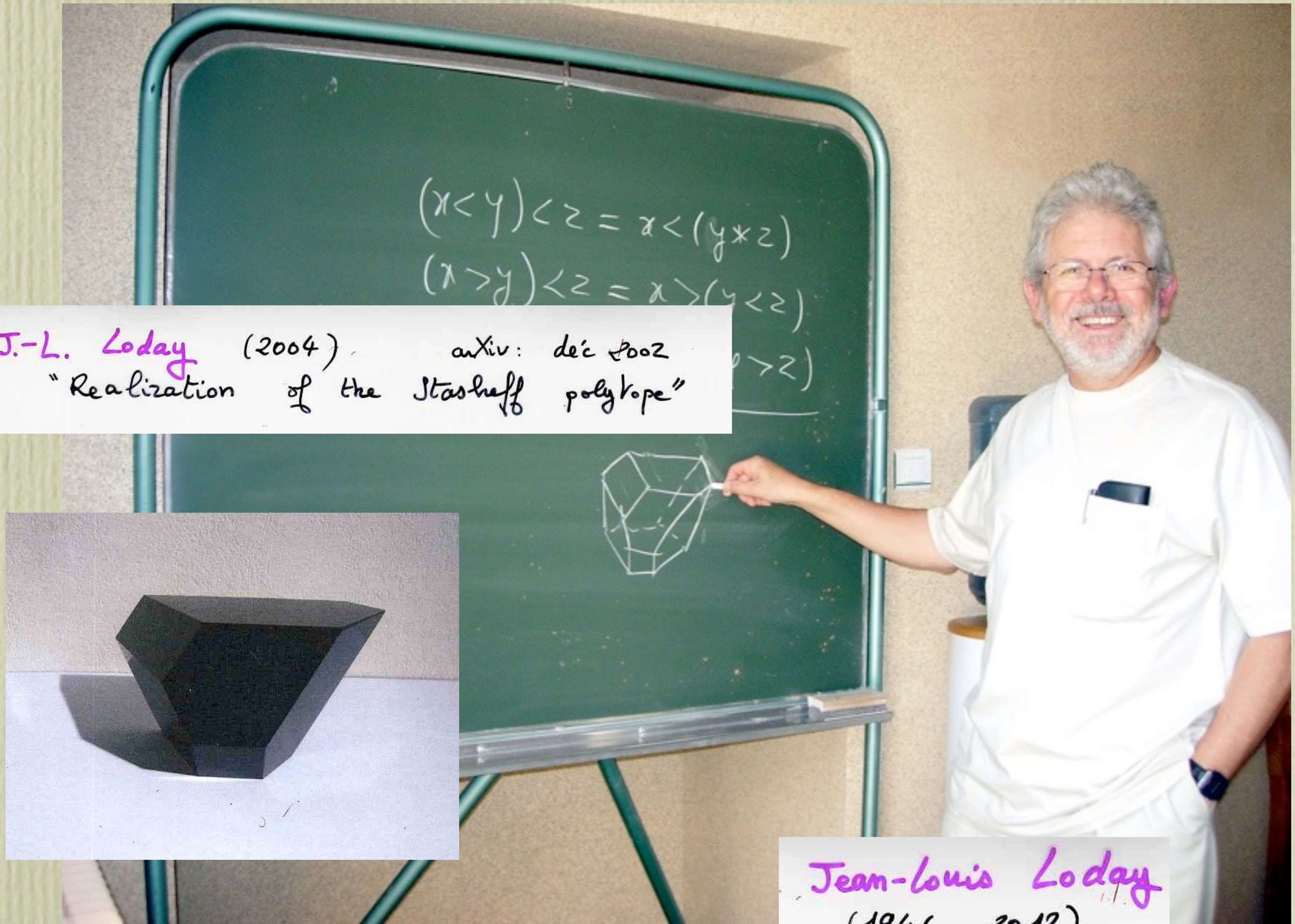




associahedron



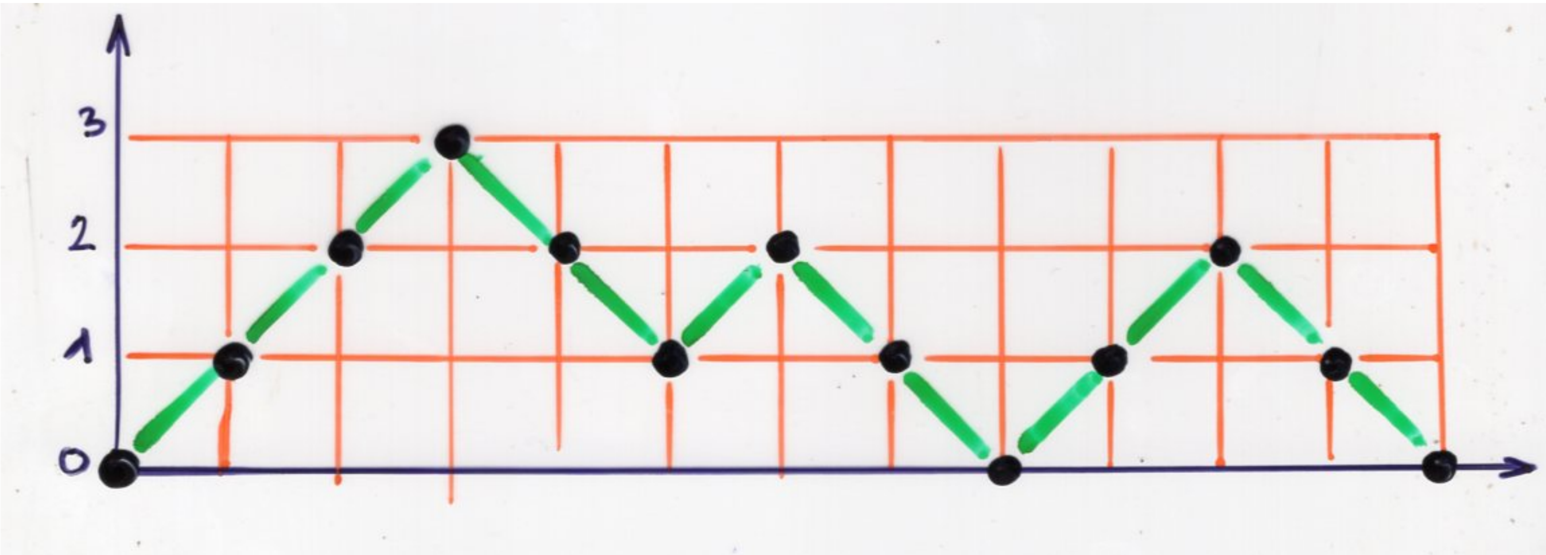
J.-L. Loday (2004) arXiv: déc 2002
"Realization of the Stasheff polytope"



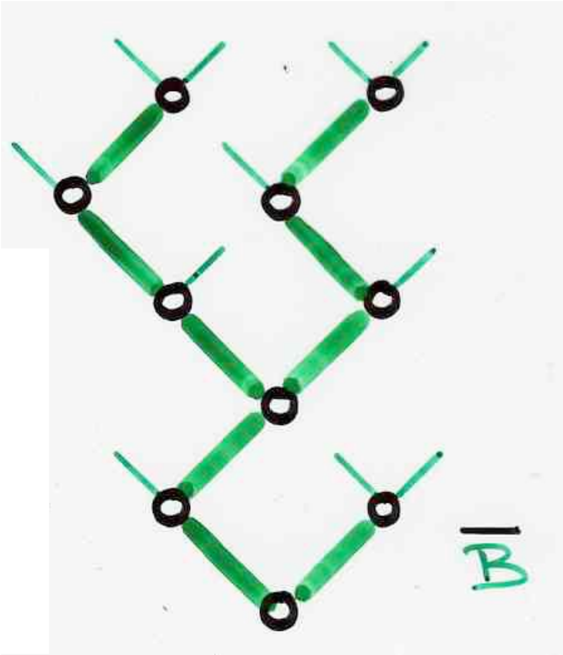
Jean-Louis Loday
(1946 – 2012)

the Tamari lattice in term of Dyck paths

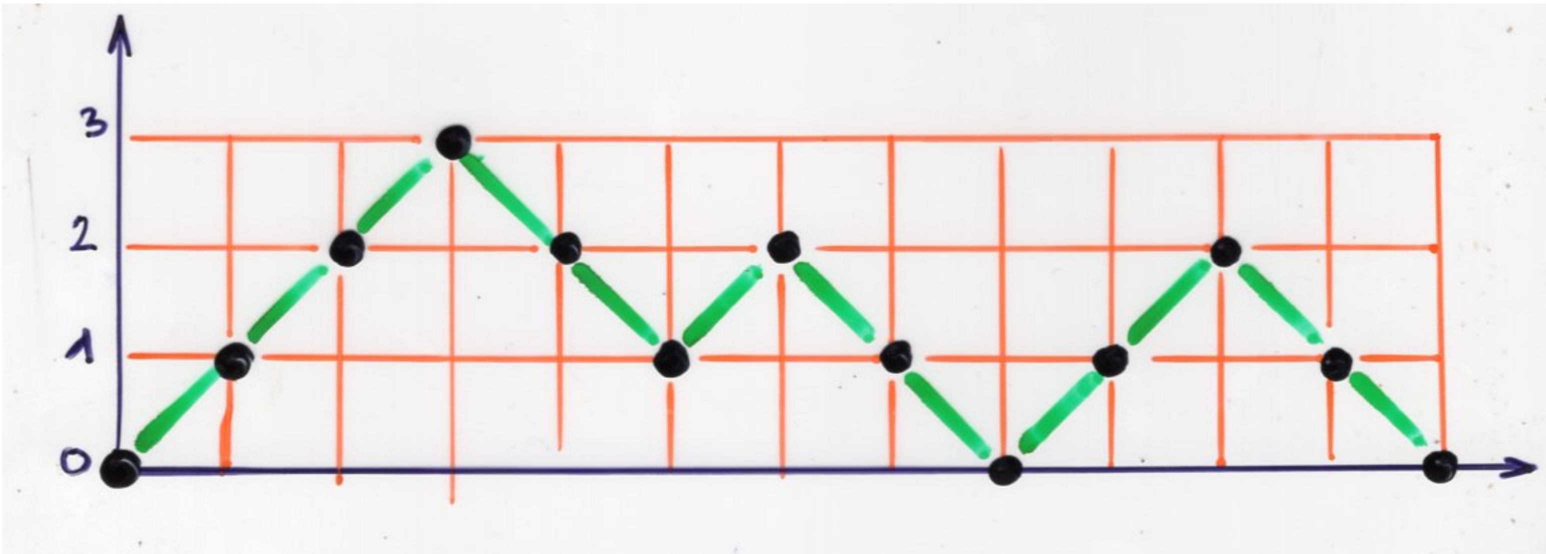
Dyck path

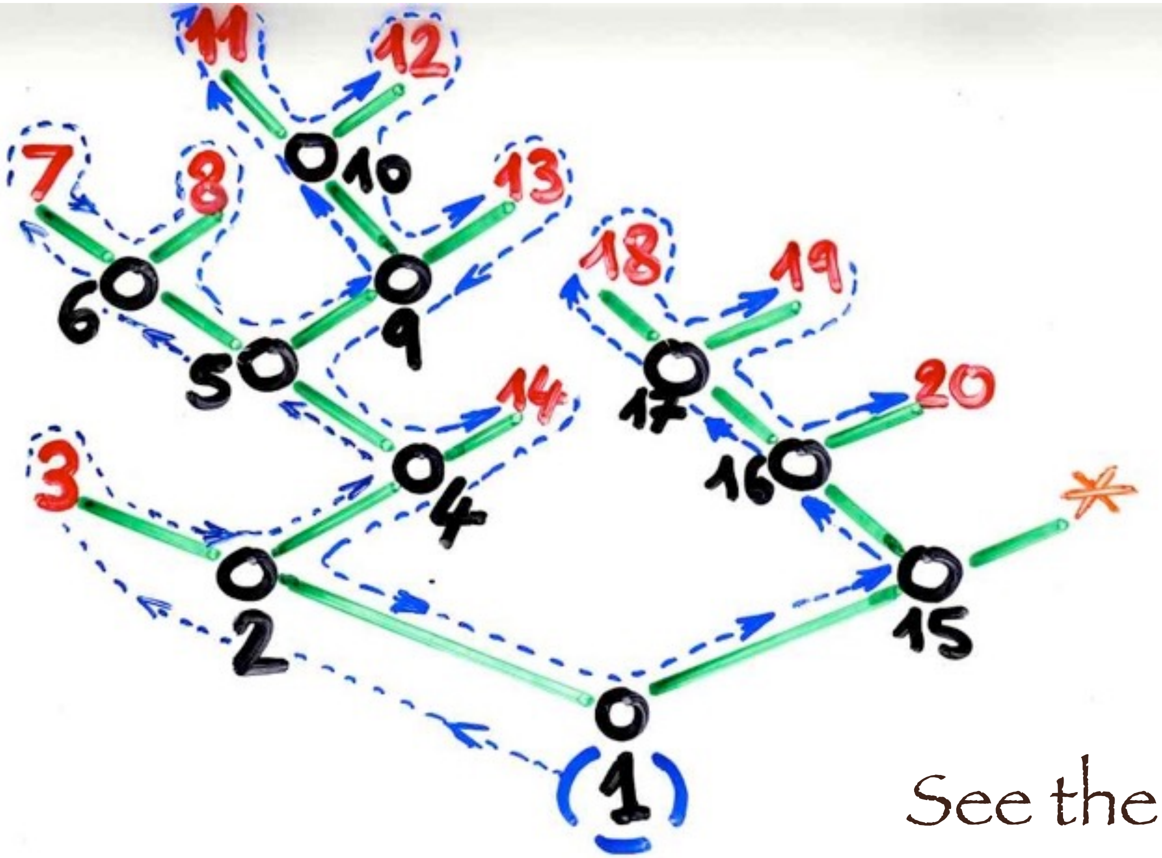



complete binary trees
Binary trees
Dyck paths

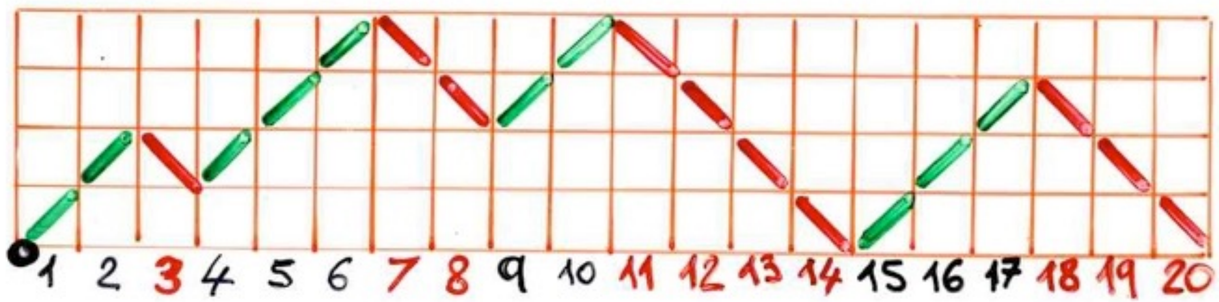


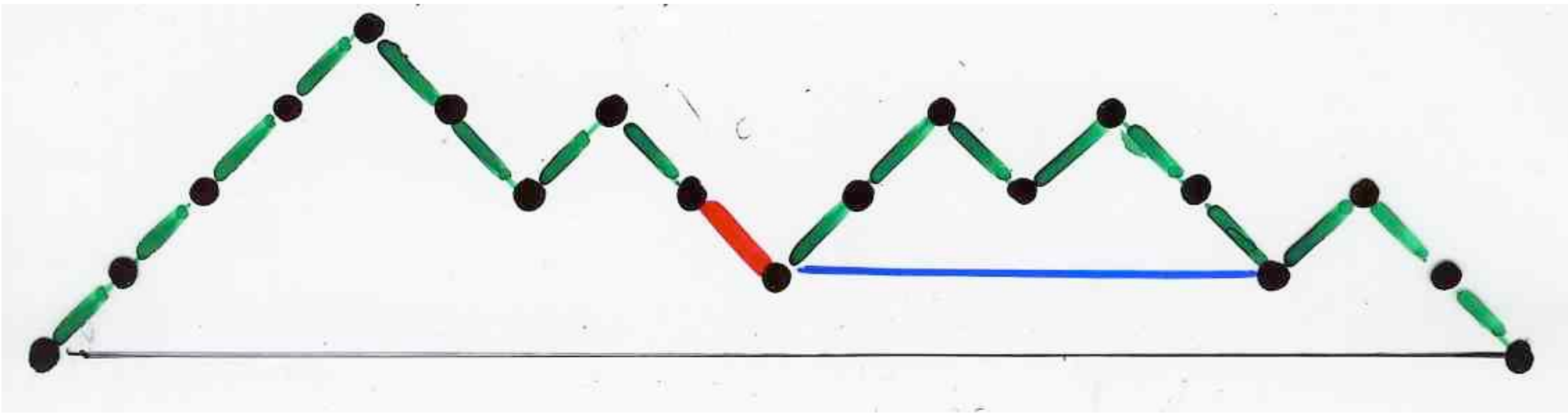
From binary trees to Dyck paths

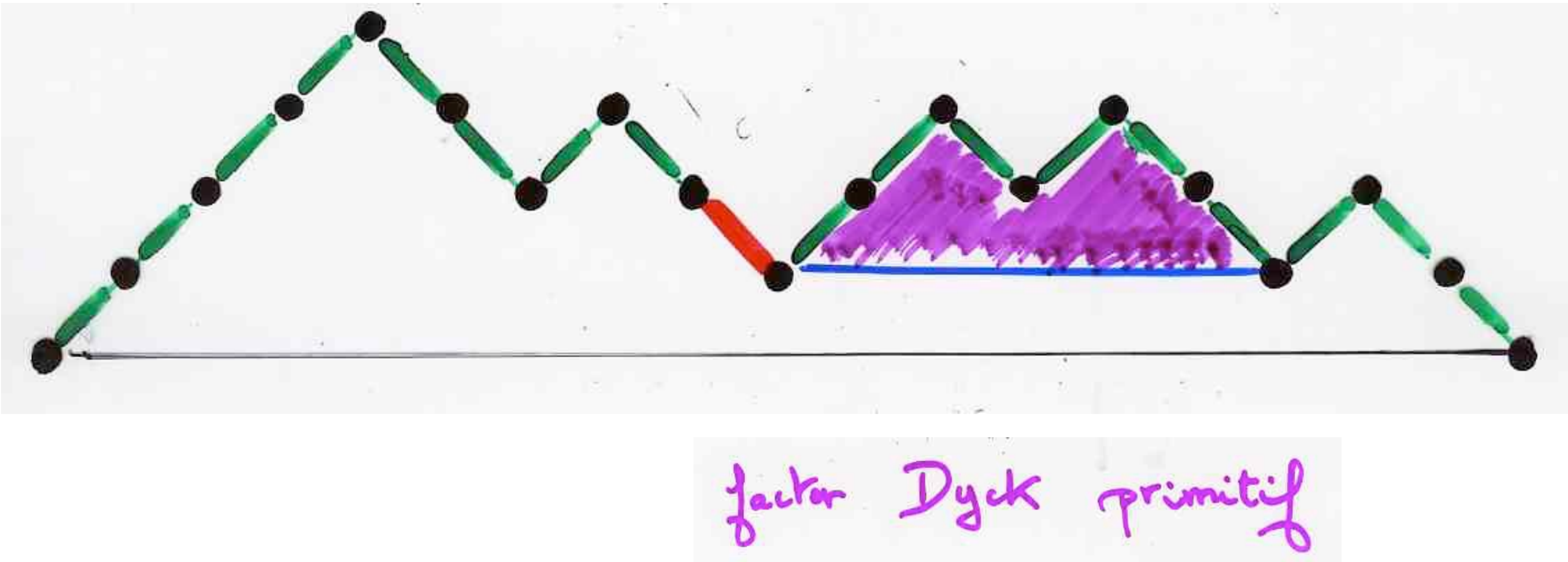




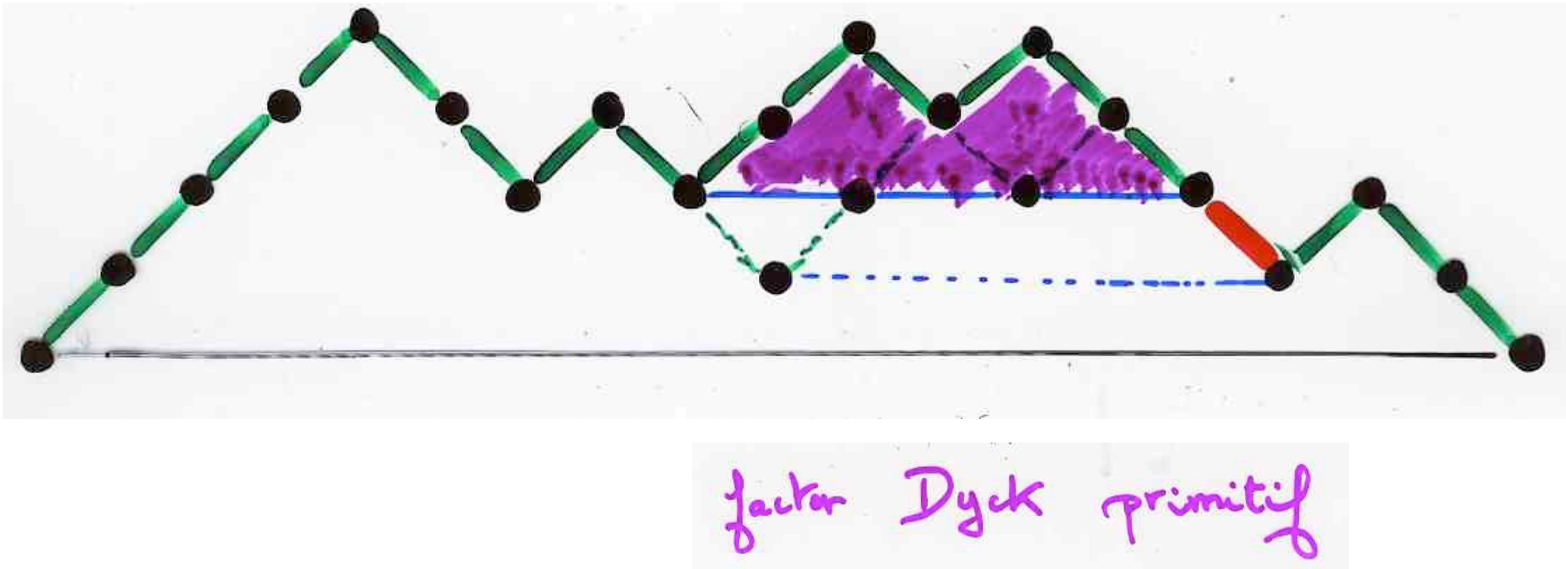
See the video with violinists



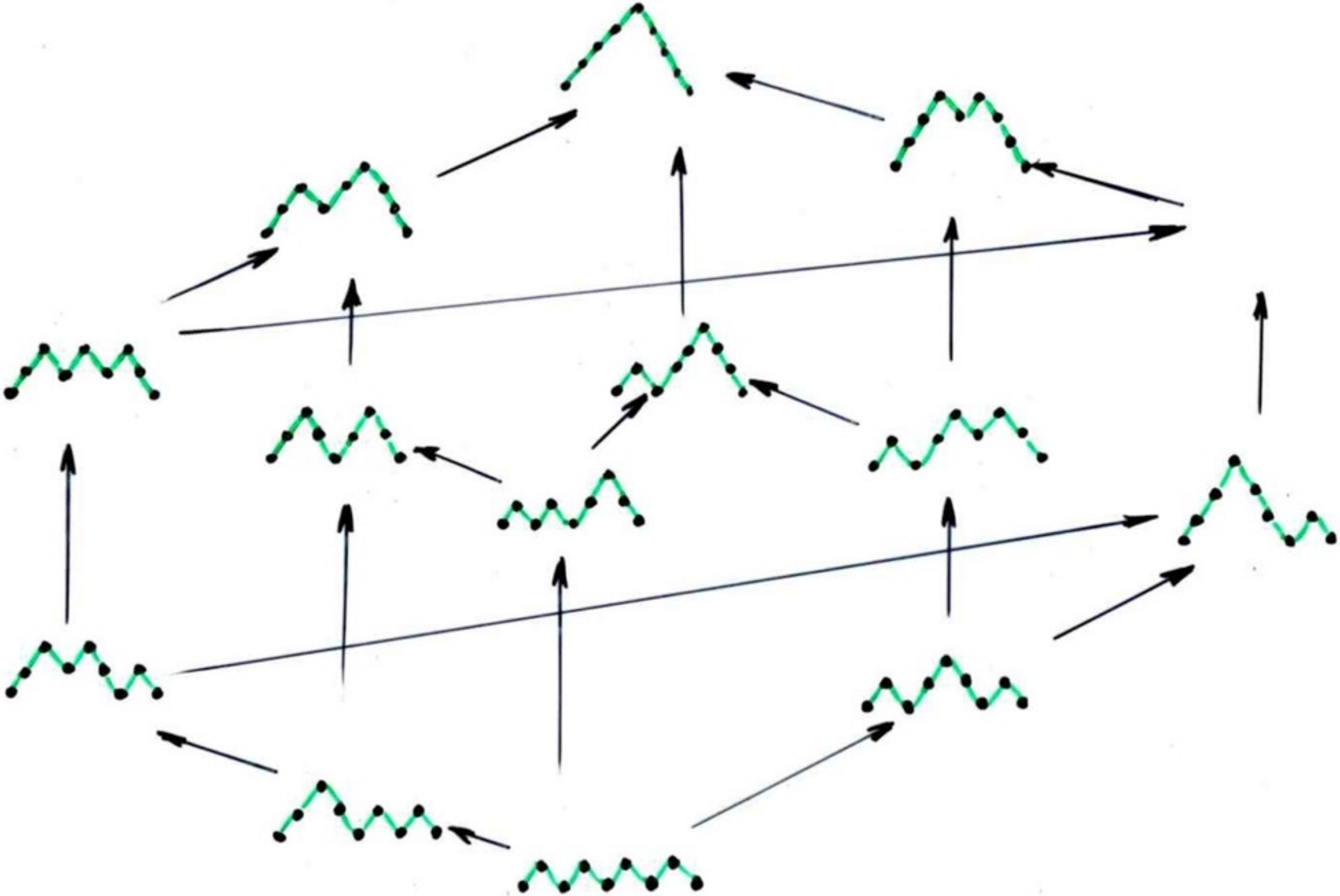


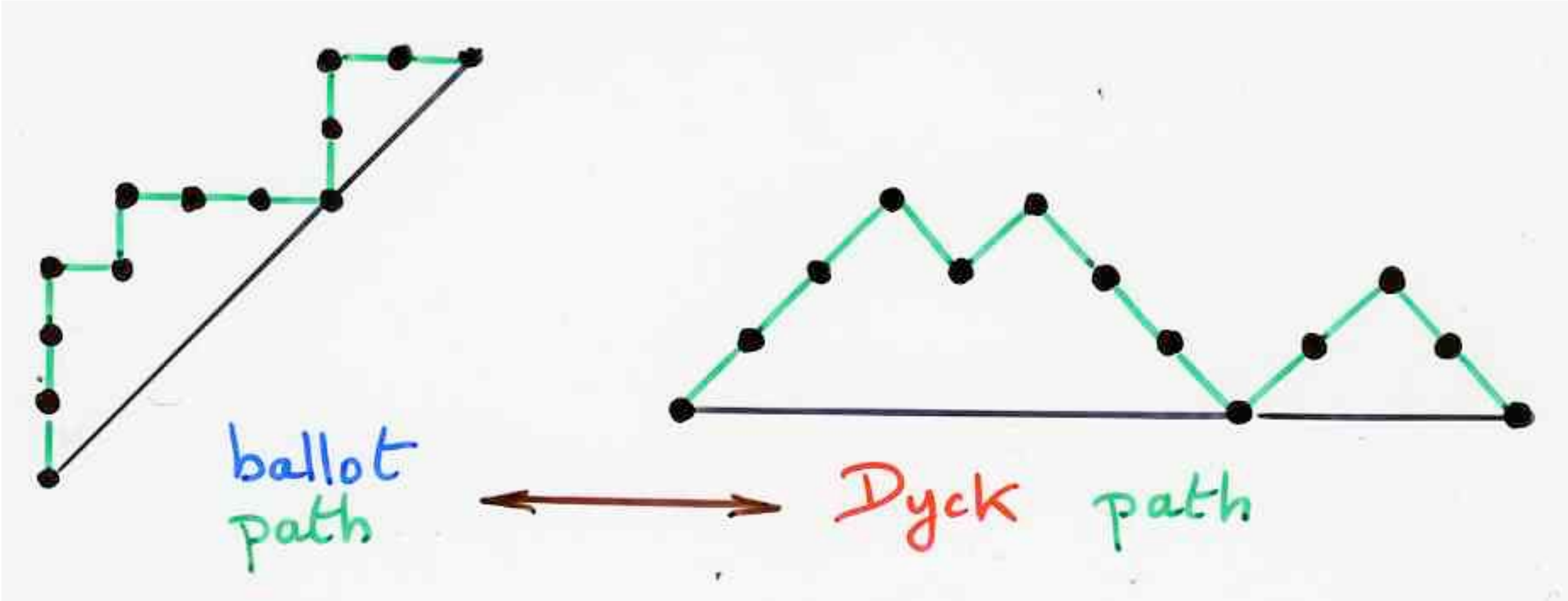


The analog of the rotation in a binary tree in term of the associated Dyck path (via the classical bijection binary trees —- Dyck paths).

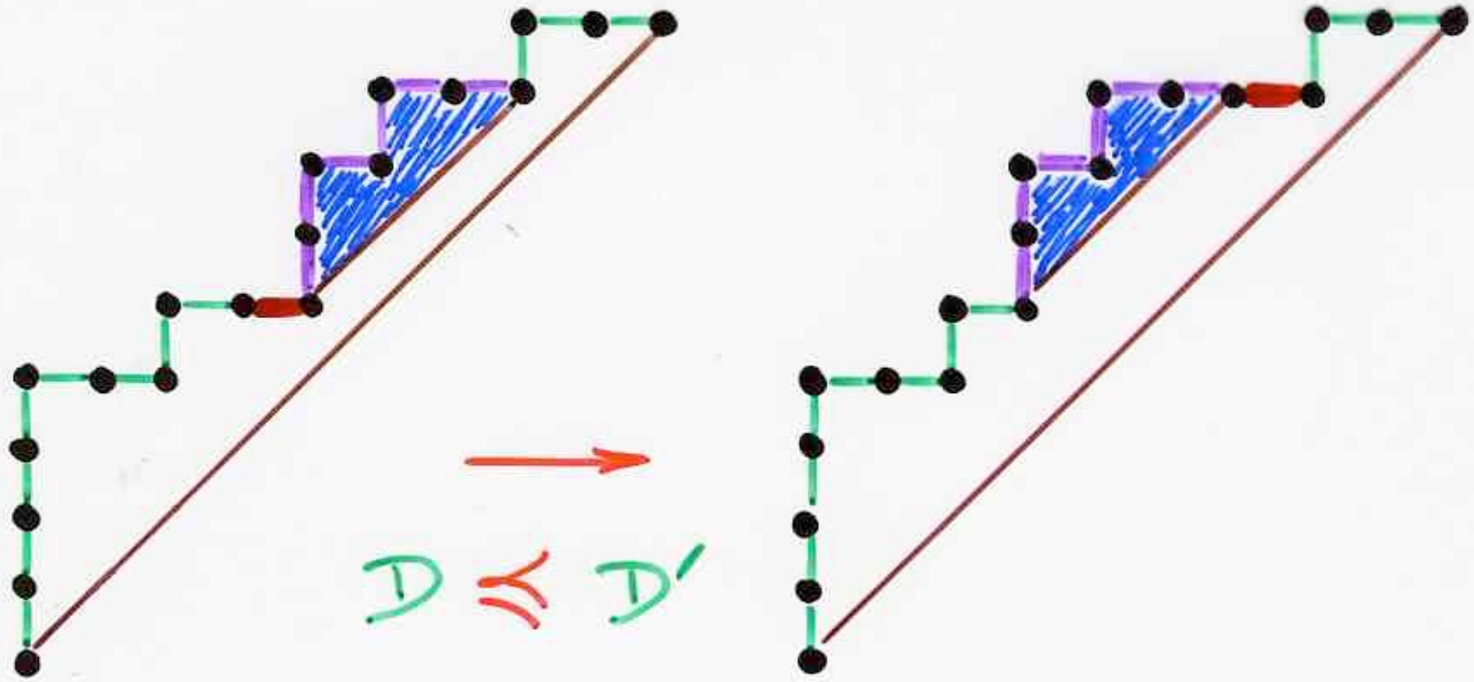


The analog of the rotation in a binary tree
in term of the associated Dyck path.





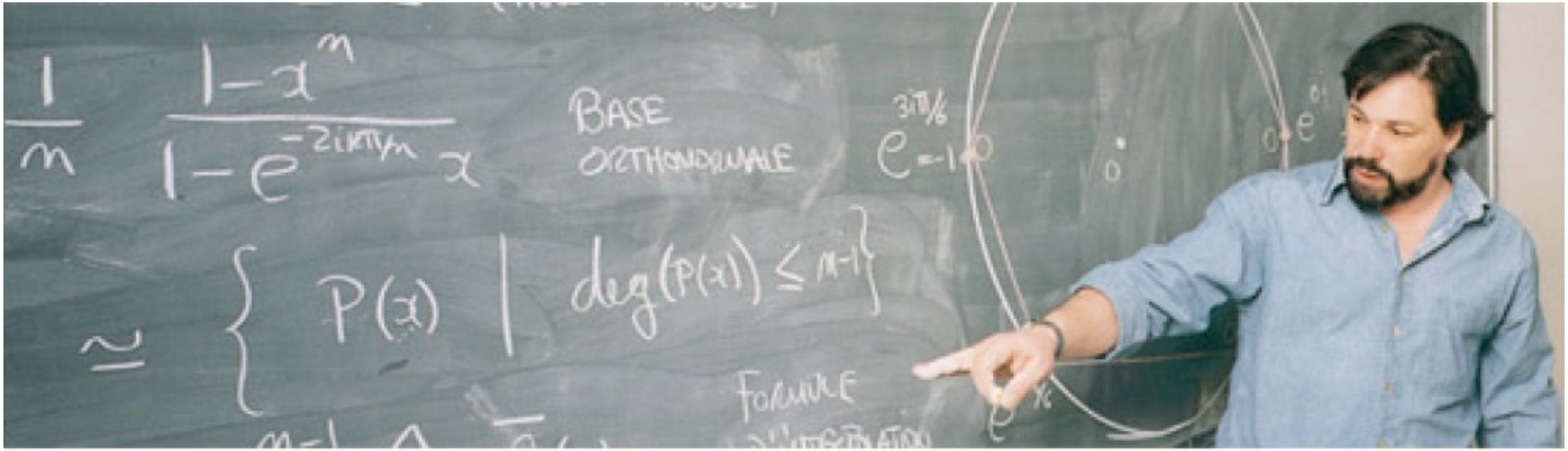
vocabulary: ballot path
Dyck path



the Tamari covering relation
for ballot (Dyck) path

relation with diagonal coinvariant spaces

The m-Tamari lattice



François Bergeron

Adriano Garsia

diagonal
coinvariant
spaces

$X = (x_{i,j})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ matrix of variables

$\sigma \in \mathfrak{S}_n$ symmetric group

$\sigma(X) = (x_{i,\sigma(j)})_{\substack{1 \le i \le k \\ 1 \le j \le n}}$ action on $\mathbb{C}[X]$

diagonal coinvariant spaces

$DR_{k,n} = \mathbb{C}[X]/J$ $DR^{\varepsilon}_{k,n}$

Armstrong, Garsia, Haglund, Haiman, Hicks, Lee, Li, Loehr, Morse, Remmel, Rhoades, Stout, Xin, Warrington, Zabrocki, ...

\+ ......

$k=3$ $\quad DR^{\varepsilon}_{3,n}$

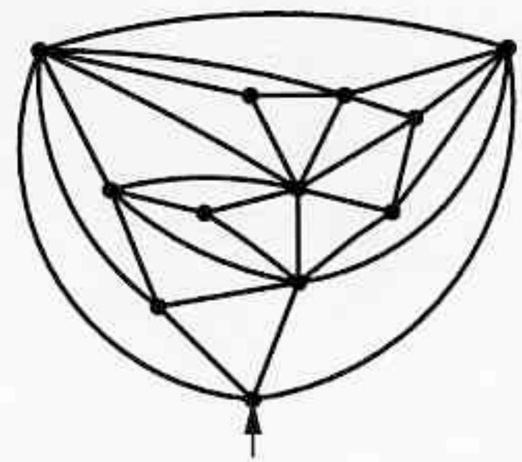
dimension $\dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$

Haiman (conjecture) (1990)

number of intervals of Tamari_n

Chapoton (2006)

triangulation



Bijective proof FPSAC 2007

Bernardi, N. Bonichon

higher diagonal coinvariant spaces

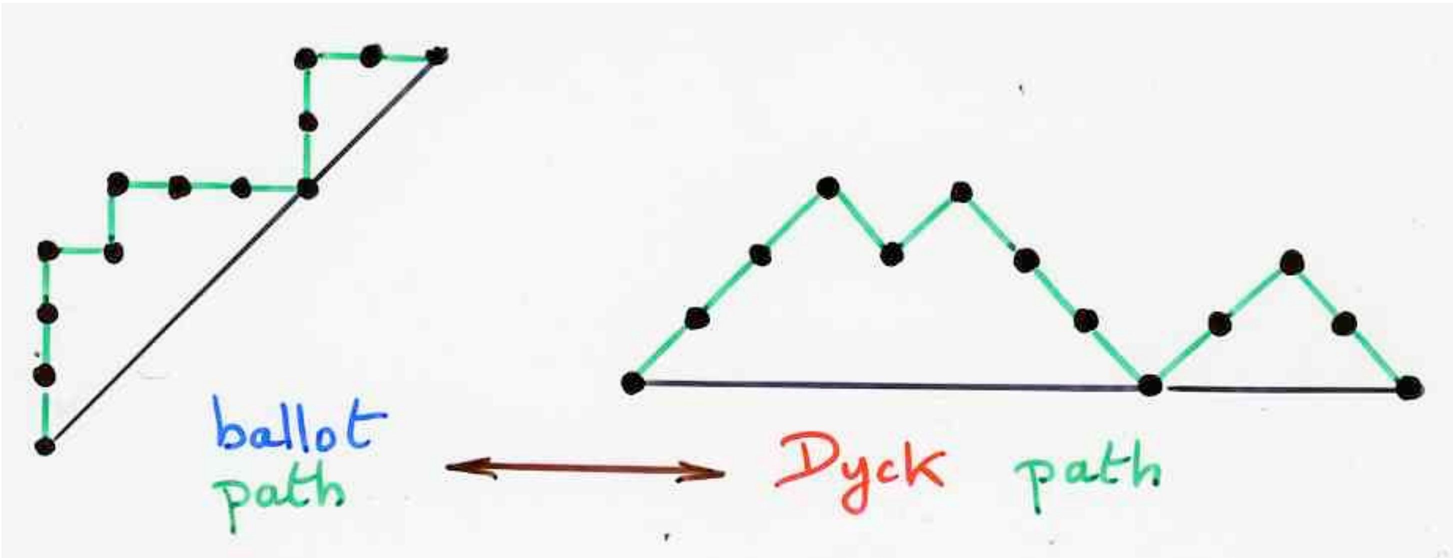
$DR^{m\,\varepsilon}_{k,n}$ $\quad$ $DR^{m}_{k,n}$

$k=2$ Garsia Haiman

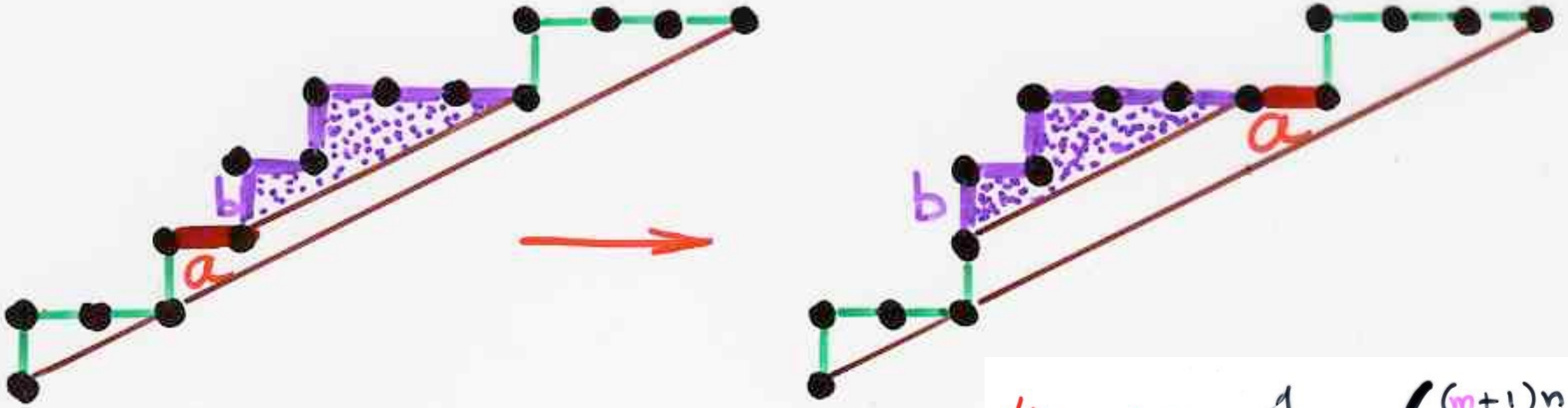
$DR^{m\,\varepsilon}_{2,n}$

dimension $\dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{mn}$

m - ballot paths



F. Bergeron (2008) introduced the m-Tamari lattice



dimension $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$

m-ballot paths

the covering relation in the m-Tamari lattice
(m = 2)

F. Bergeron (2008) introduced the m-Tamari lattice

conjecture
$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1} \qquad (m+1)^n(mn+1)^{n-2}$$

nb of intervals $\qquad$ nb of labelled intervals

M. Bousquet-Mélou, E. Fusy, L.-F. Préville-Ratelle (2011)

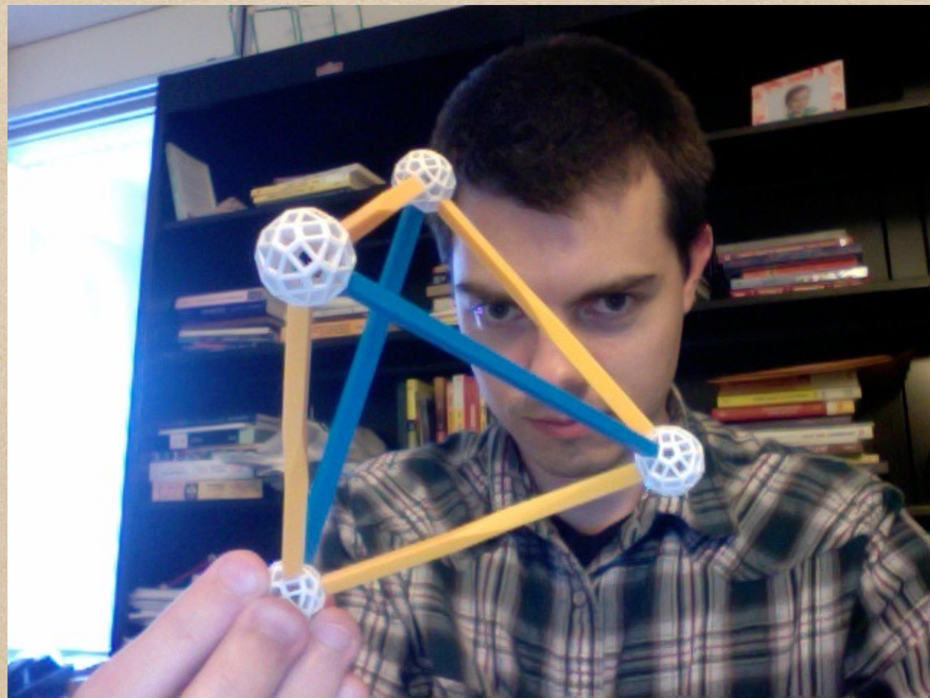
nb of intervals of m-Tamari lattices

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$ F. Bergeron

M. Bousquet-Mélou, G. Chapuy, L.-F. Préville-Ratelle (2011)

nb of labelled intervals $(m+1)^n(mn+1)^{n-2}$

Rational Catalan Combinatorics



Rational Catalan Combinatorics

D. Armstrong

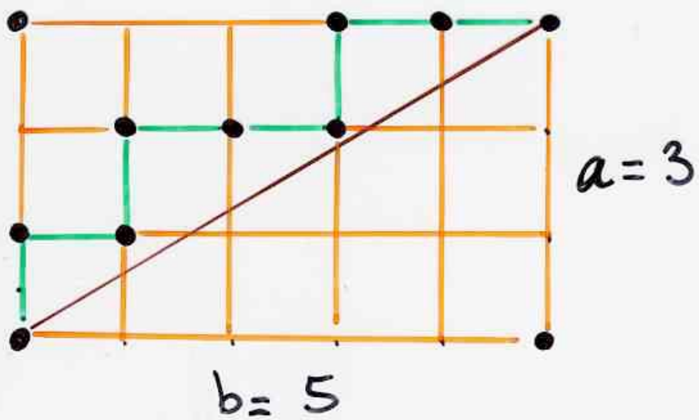
$$Cat(a,b) = \frac{1}{a+b}\binom{a+b}{a,\,b}$$

number of (a,b)- ballot paths $= Cat(a,b)$

Grossman (1950)

Bizley (1954)

rational ballot (Dyck) paths

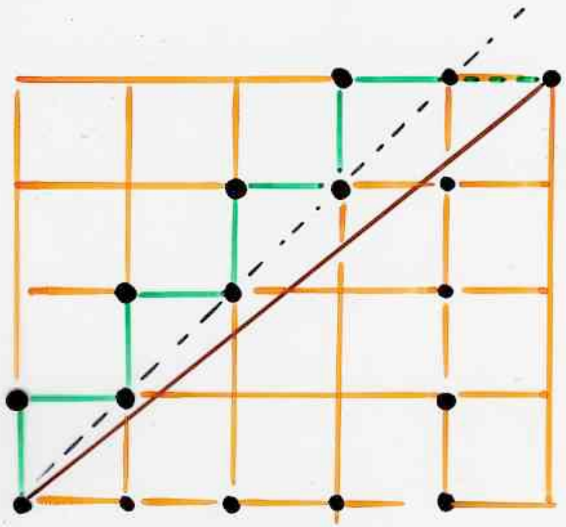


$(a,b) = (3,5)$

$(a, b) = (n, n+1) \longrightarrow C_n$ Catalan nb

$(a, b) = (n, mn+1) \longrightarrow \dfrac{1}{(m+1)n+1}\dbinom{(m+1)n+1}{n}$

Fuss-Catalan nb

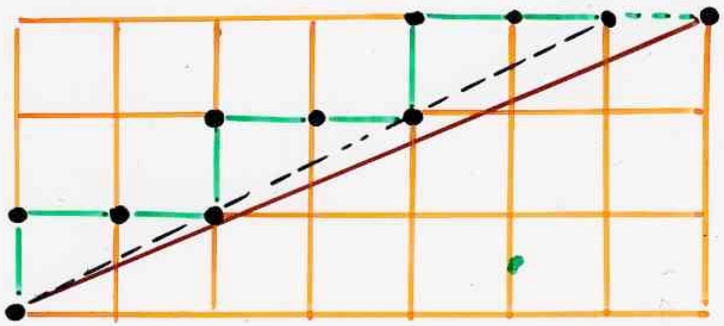


$n = 4$

$n+1 = 5$

$(a, b) = (4, 5)$

Catalan



$n = 3$

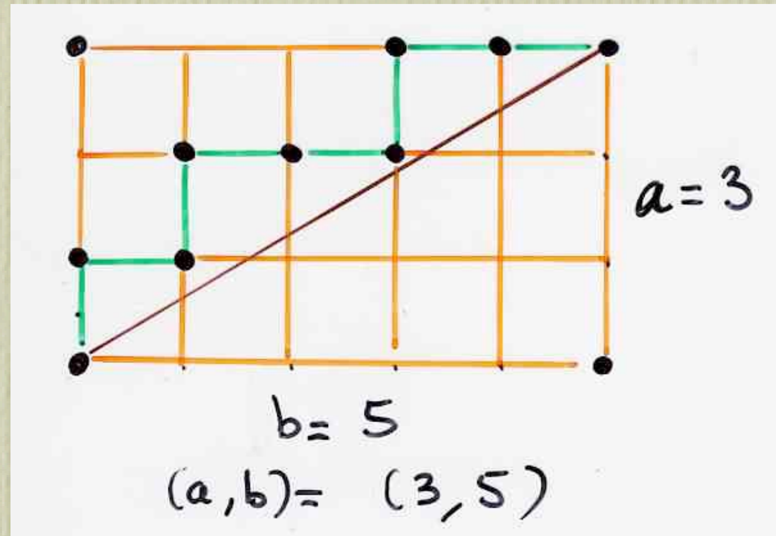
$m = 2 \qquad mn+1 = 7$

$(a, b) = (3, 7)$

Fuss-Catalan

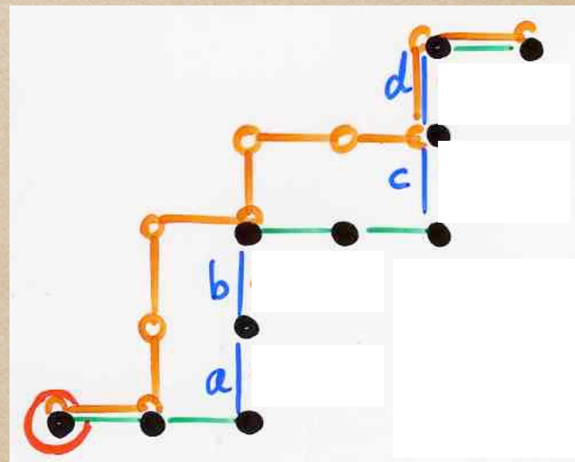
question:

define an (a,b)- Tamari lattice ?



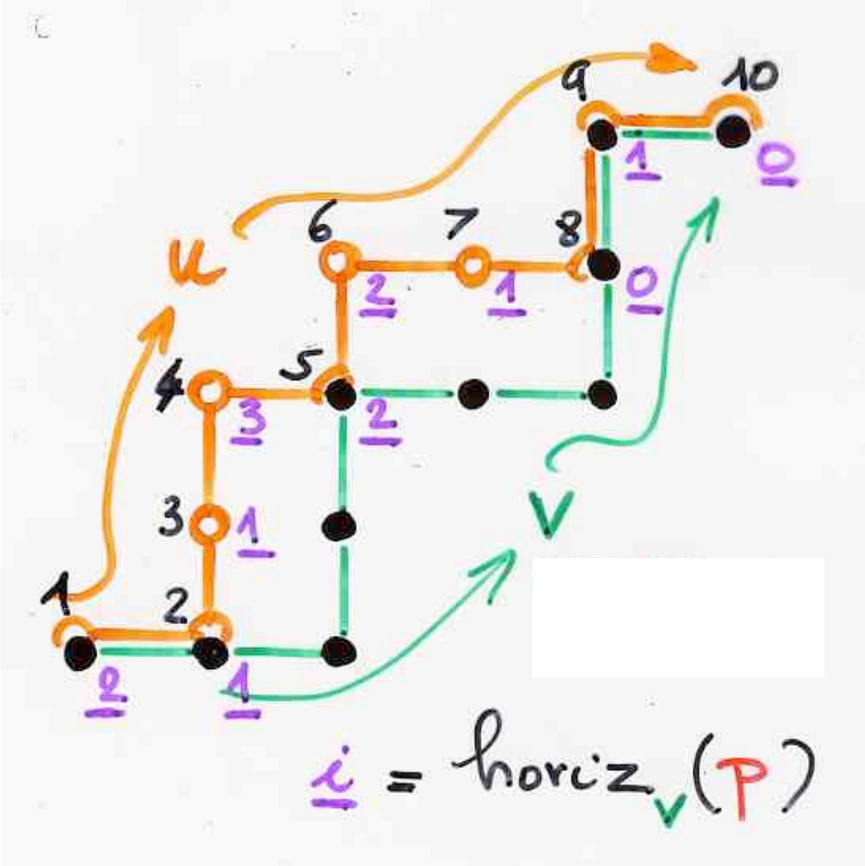
Préville-Ratelle, X.V. (2015)(2017)

Tamari lattice Tamari(V)



Transactions AMS, 369 (2017) 5219-5239

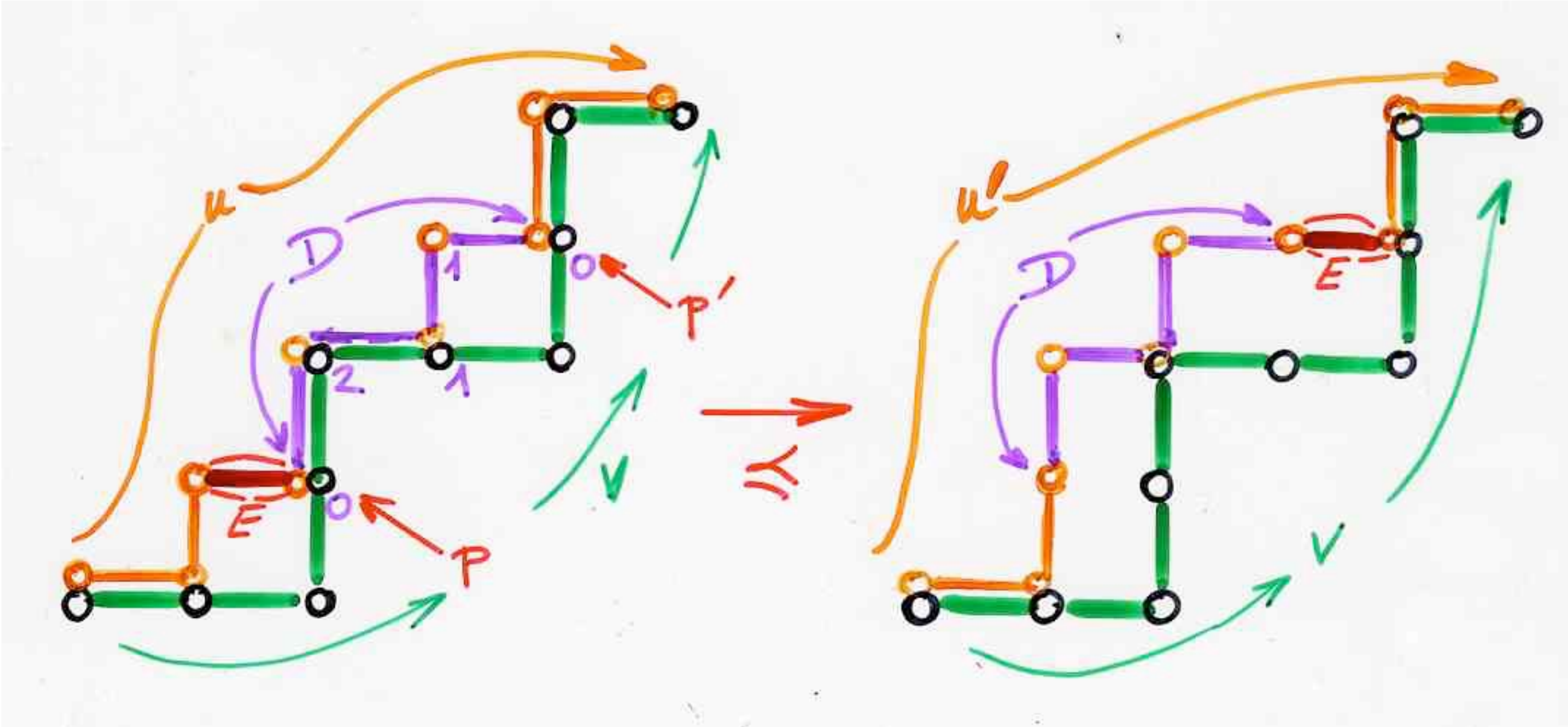
FPSAC'2015, Daejon, Korea



For each vertex of the path u,
we associate a number (in purple),
as the distance from this vertex
to the rightmost vertex of the path v.

a pair (u, v) of paths
with the "horizontal distance"
$\mathrm{horiz}_v(p)$

the covering relation in the poset T_V

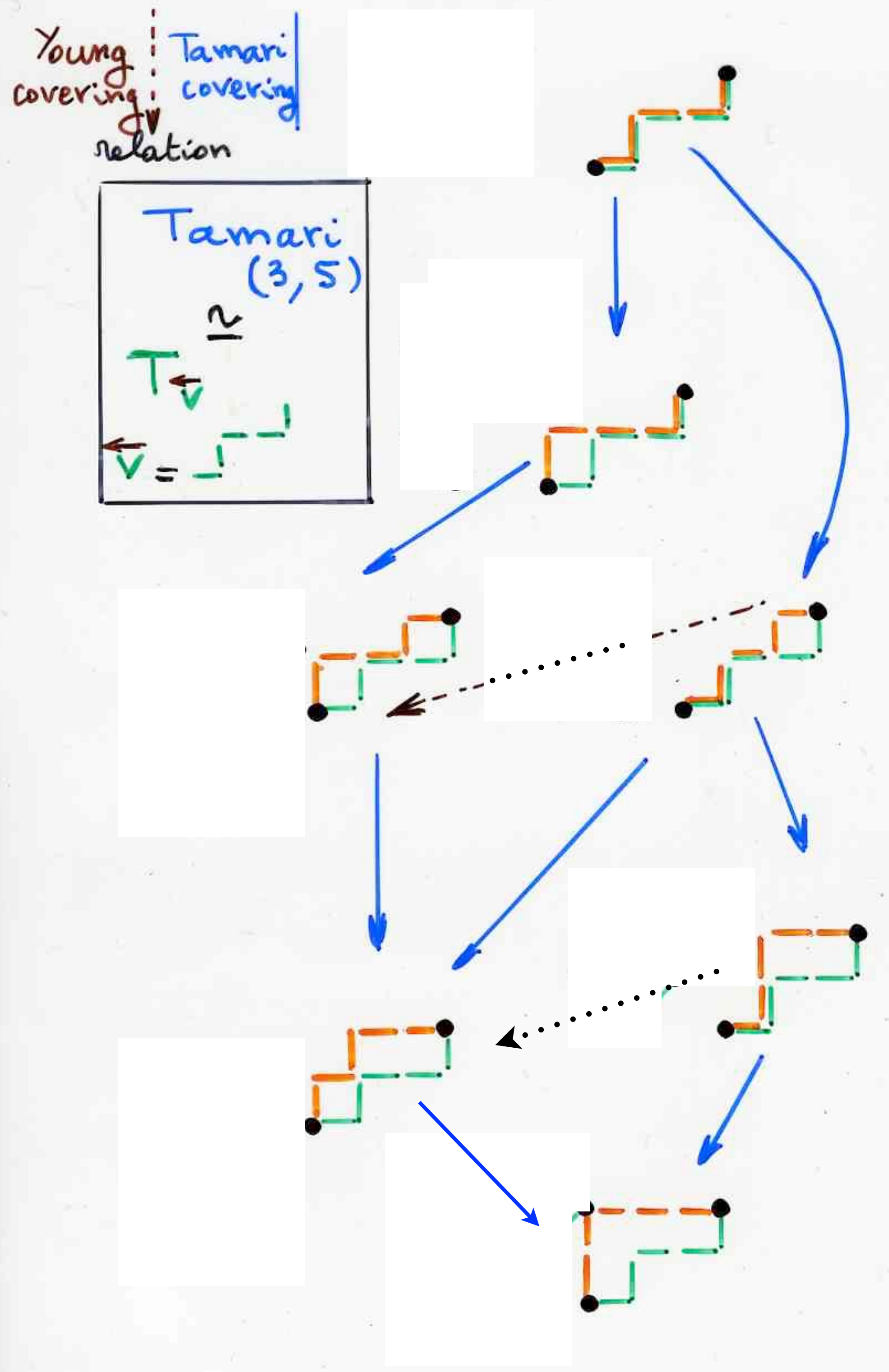


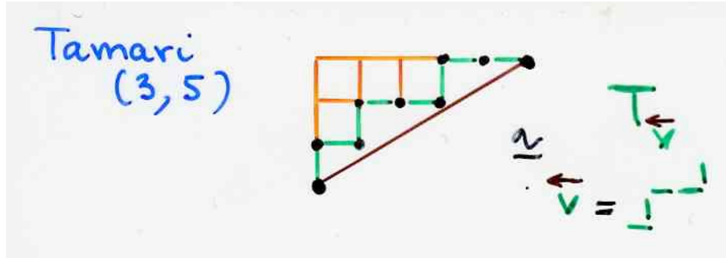
Take an East step of the path u (here in red), take the associated purple integer k associated to the vertex p at the end of the East step (here k=0). Then take the longest portion of the path u such that all the associated purple numbers are strictly bigger than k, until one get a vertex p' with purple number = k. We get the portion D of the path u (in purple on the figure). Then exchange the selected East step with the portion D.

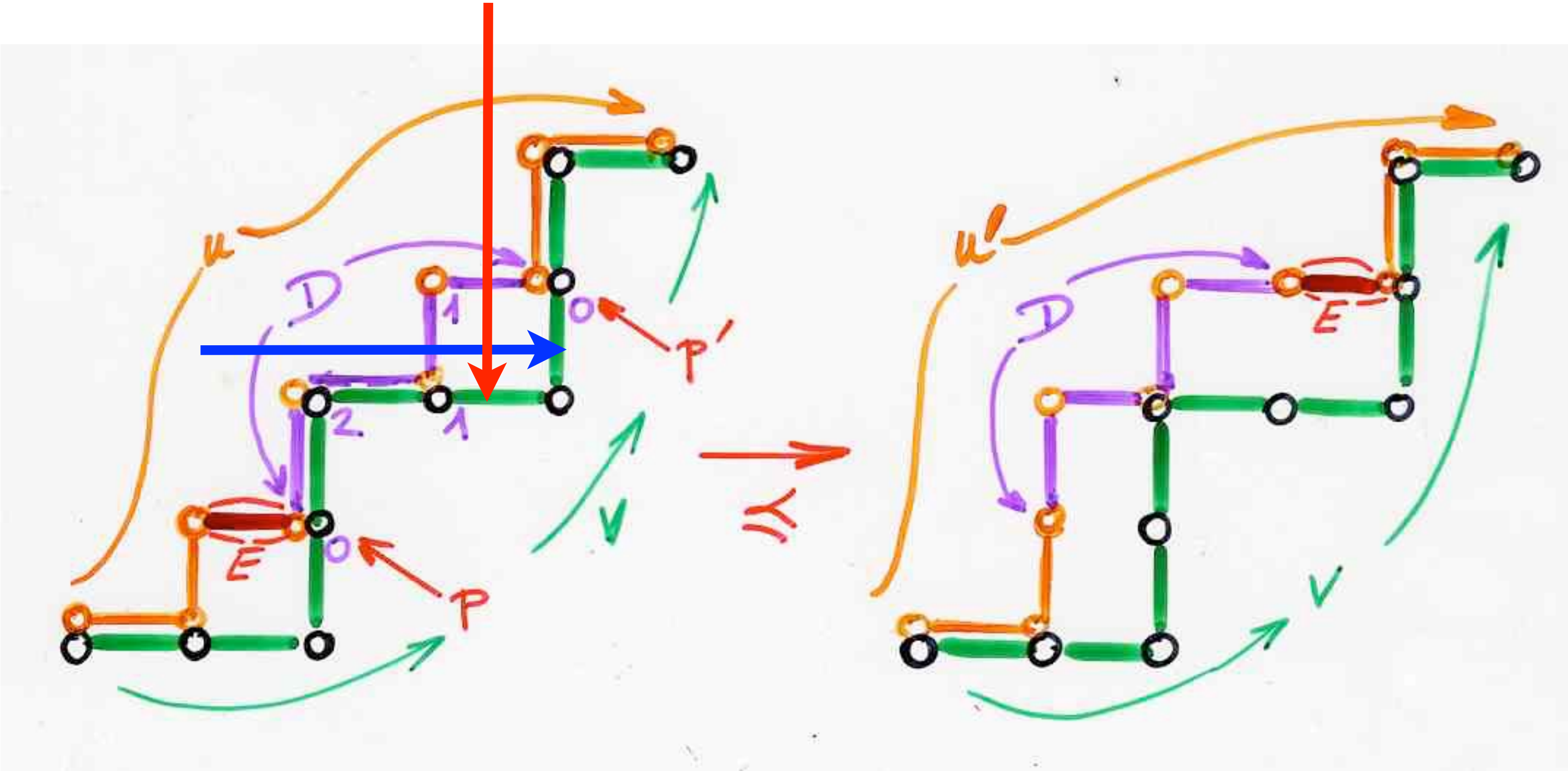
Thm 1. For any path v

T_v is a lattice

Tamari(v)


Young covering relation
Tamari covering
Tamari (3, 5) ≃ T_←v
←v =

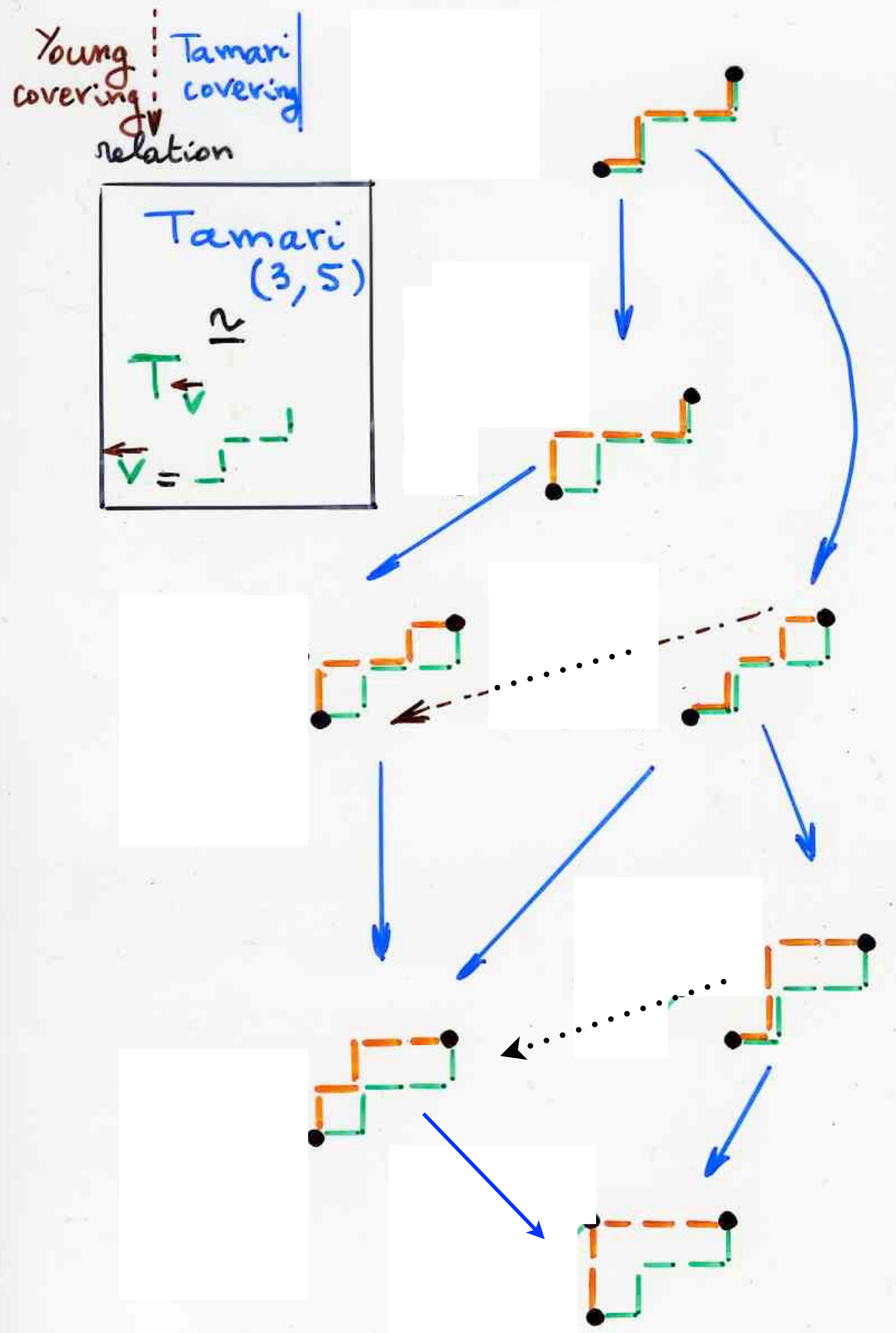

Tamari (3, 5)
≃ T_←v
←v =



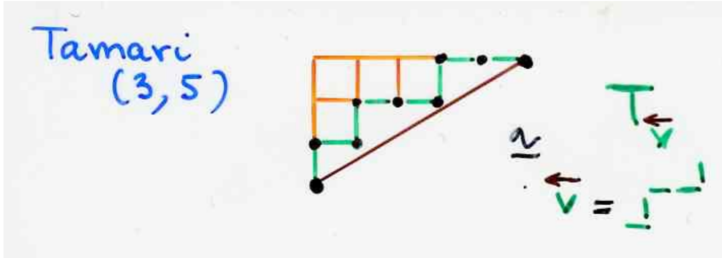
«row covering relation»

«column covering relation»

mirror image, exchange N and E



Young covering relation
Tamari covering
Tamari (3,5)
≅
T
v =



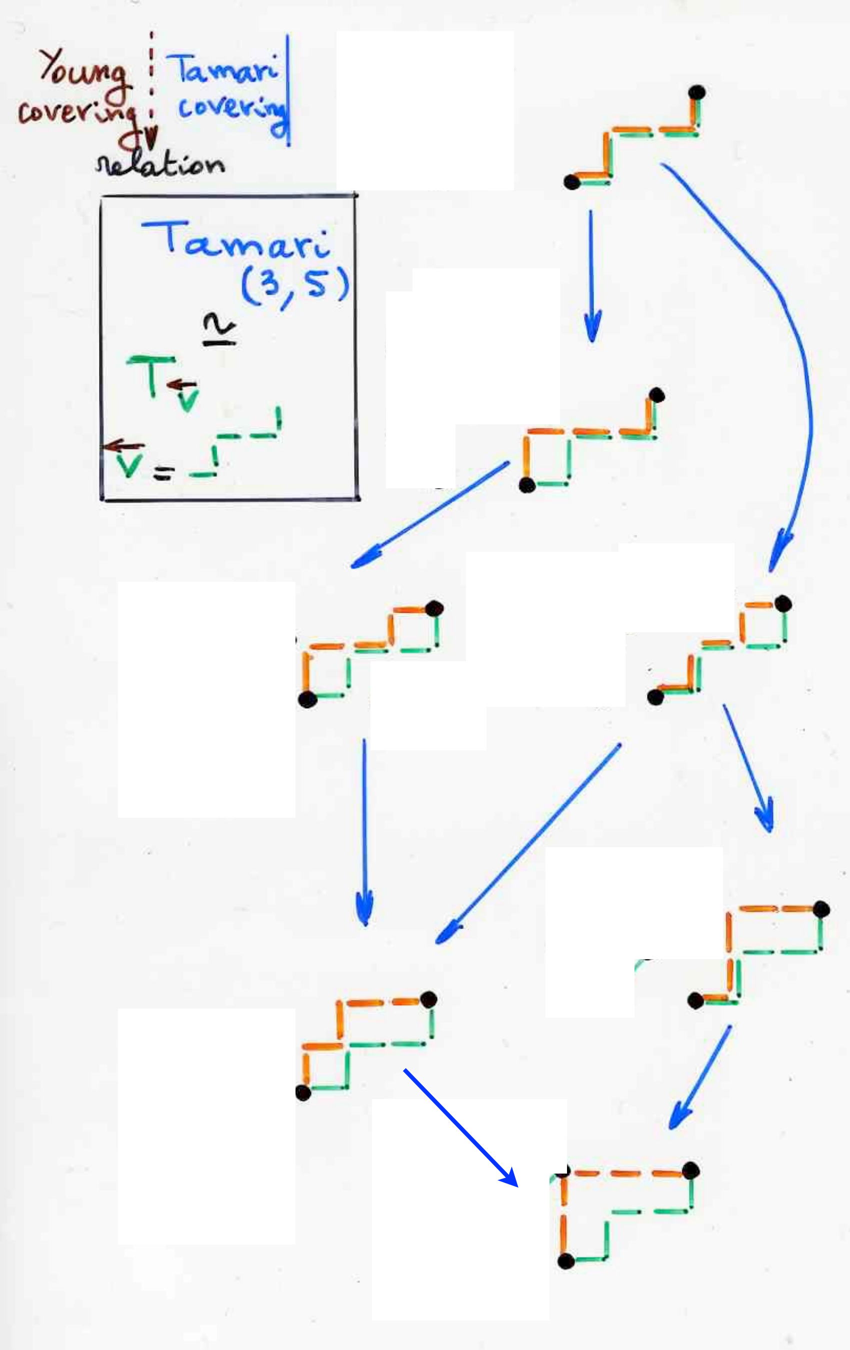
Tamari (3,5)
≅
T
v =

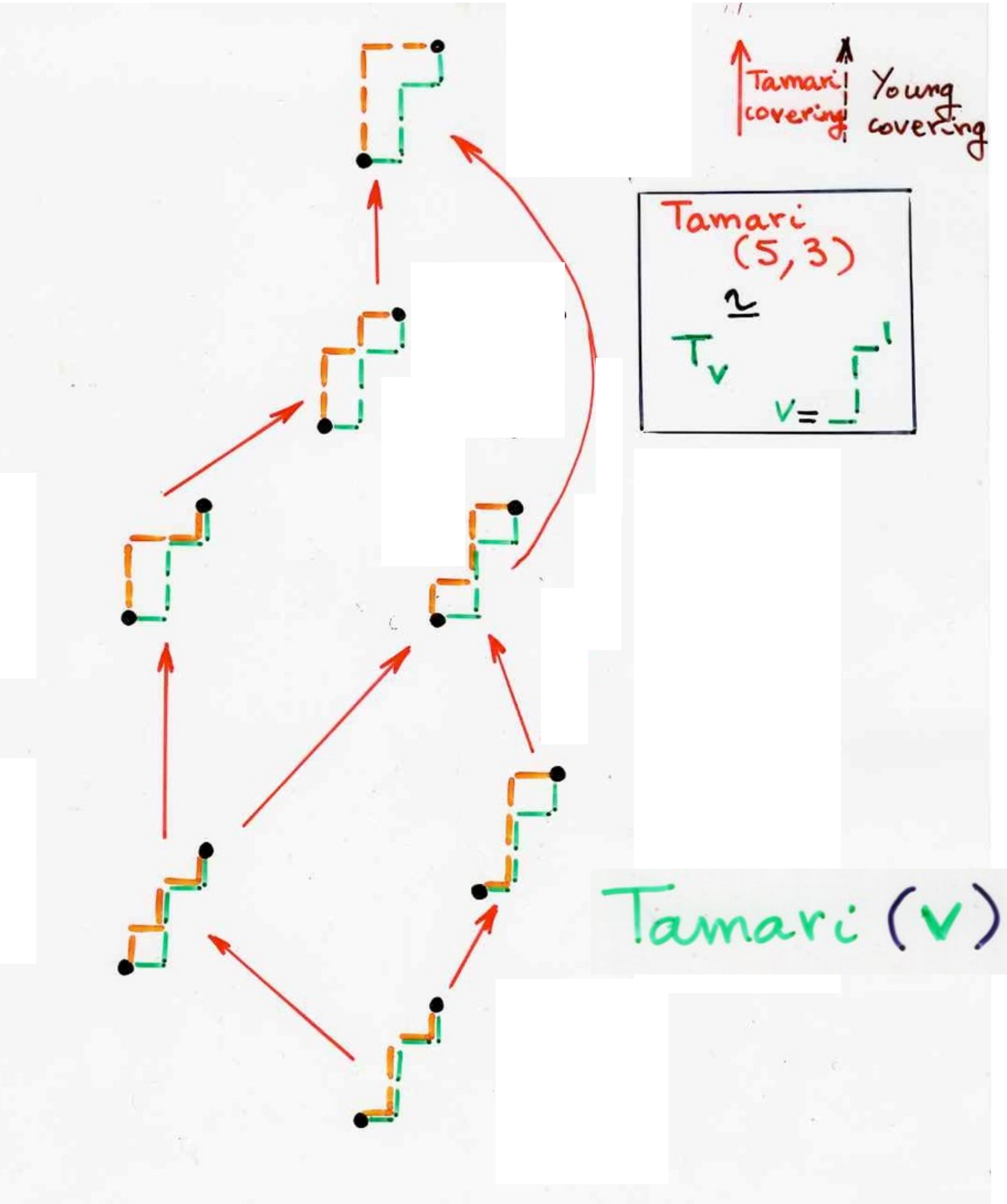
Thm 1. For any path v, T_v is a lattice

Tamari (v)

Thm 2. The lattice T_v is isomorphic to the dual of $T_{\overleftarrow{v}}$

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$





Thm 1. For any path v

T_v is a lattice

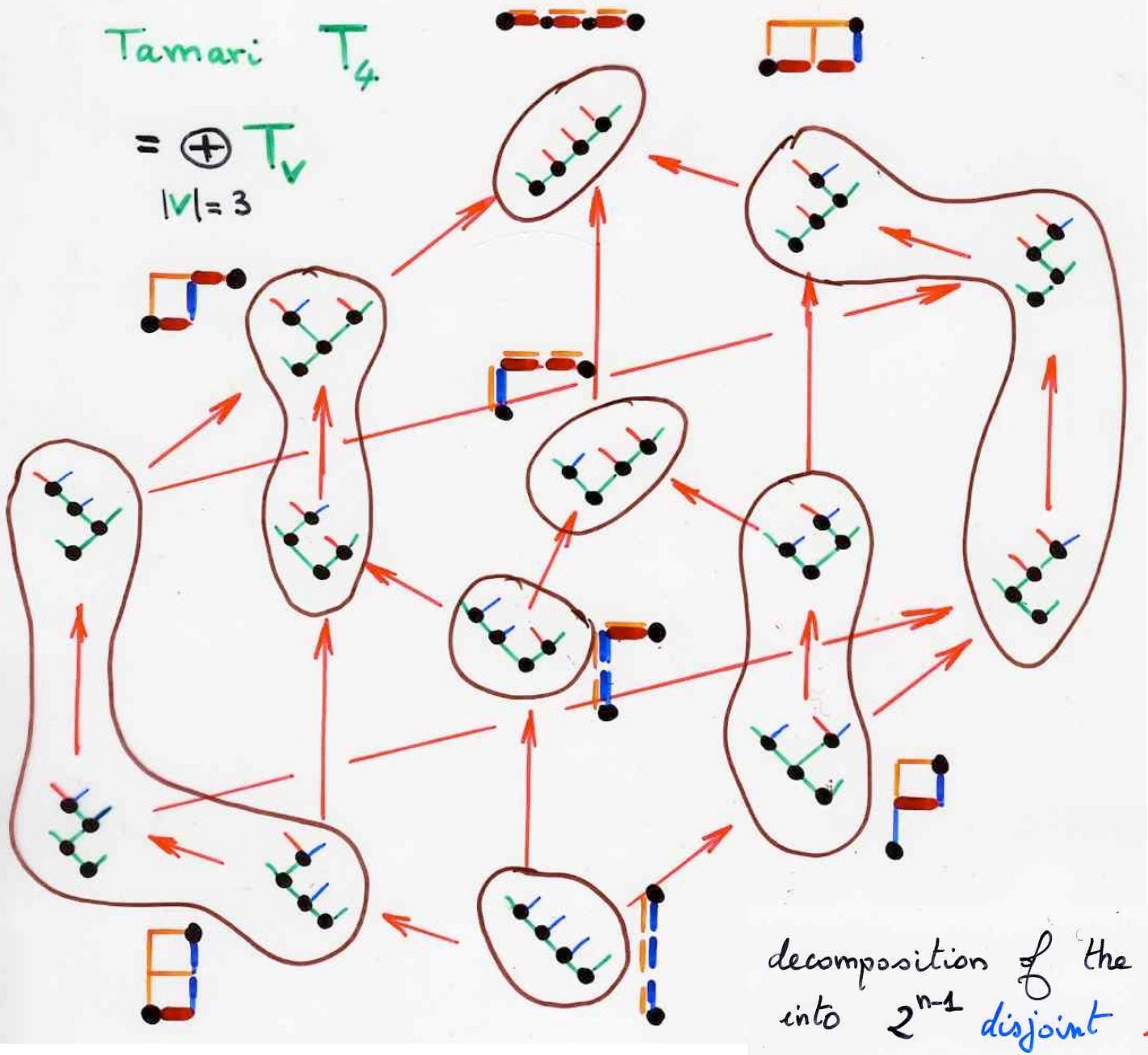
Thm 2. The lattice T_v is isomorphic to the dual of $T_{\overleftarrow{v}}$

Tamari (v)

Thm 3. The usual Tamari lattice T_n can be partitioned into intervals indexed by the 2^{n-1} paths v of length $(n-1)$ with $\{E, N\}$ steps,

$$T_n \cong \biguplus_{|v|=n-1} I_v ,$$

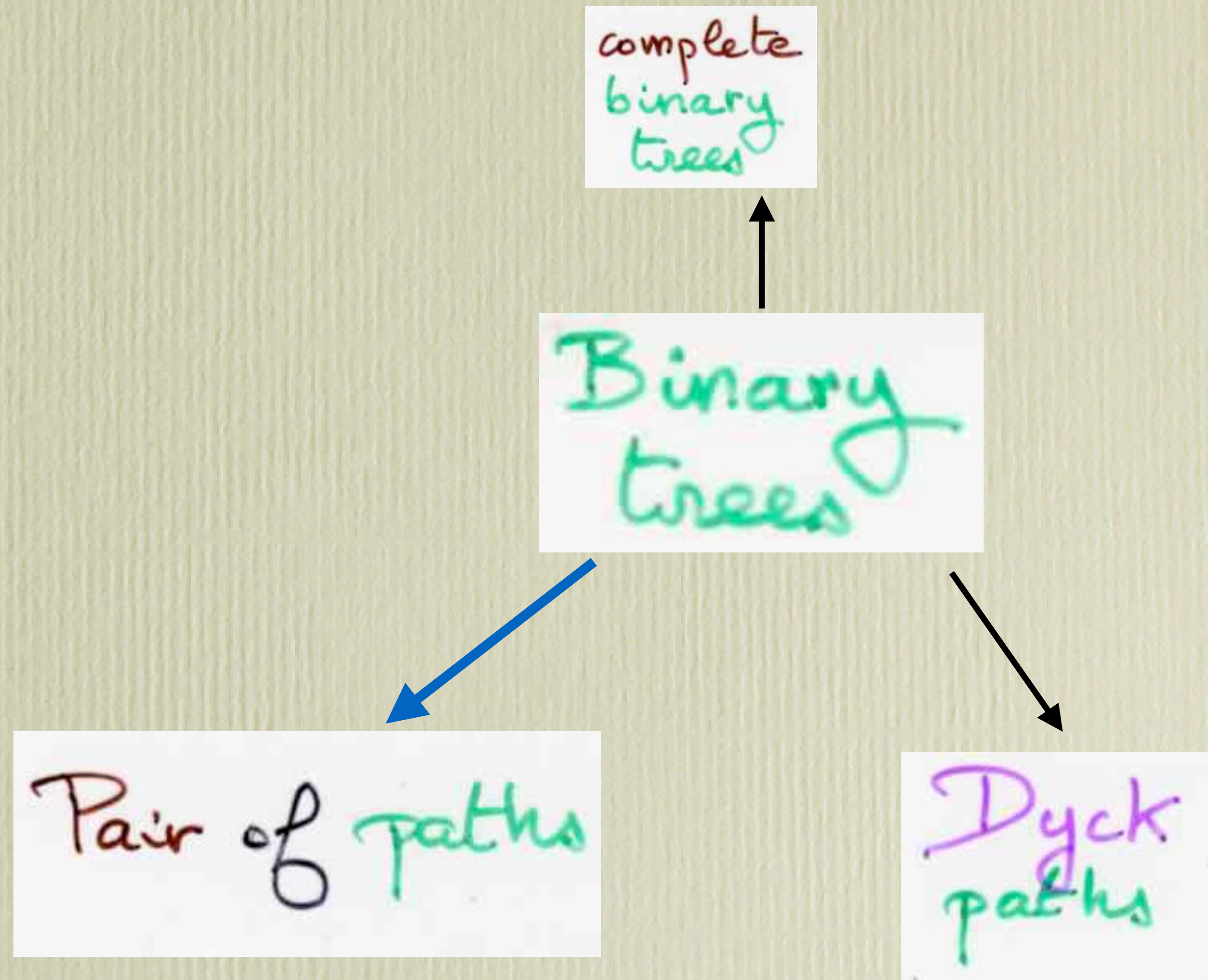
where each $I_v \cong T_v$.

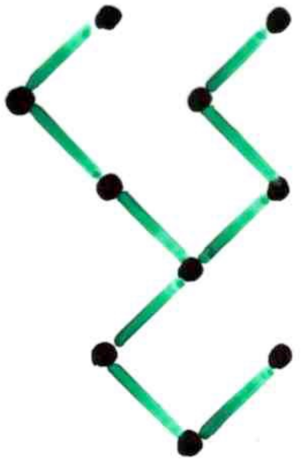


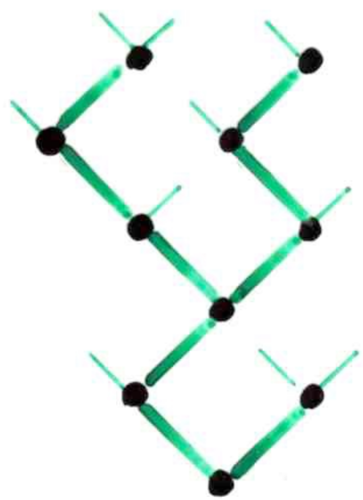
decomposition of the lattice T_n
into 2^{n-1} disjoint intervals

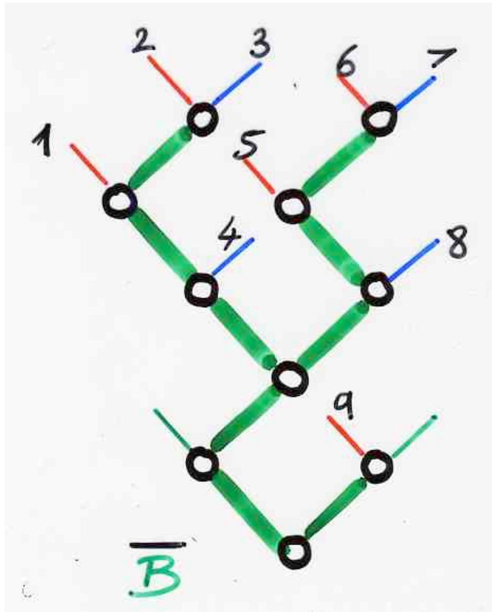
proof with a bijection

binary tree B $\longrightarrow$ pair of paths (u,v)

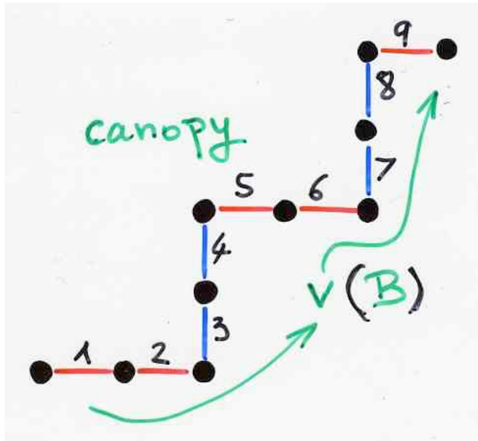

complete binary trees
Binary trees
Pair of paths
Dyck paths



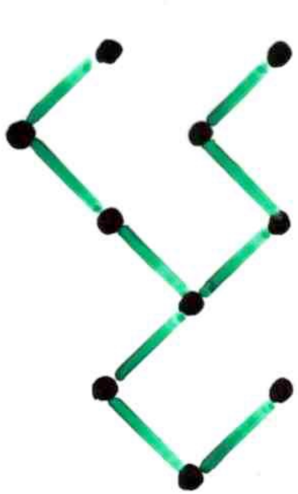


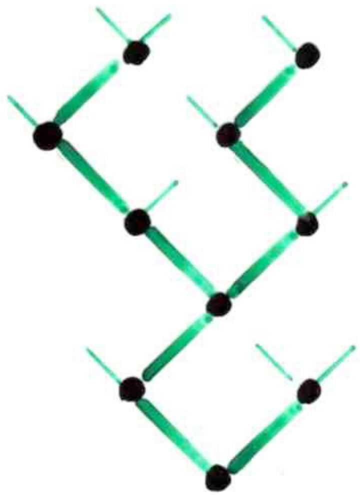


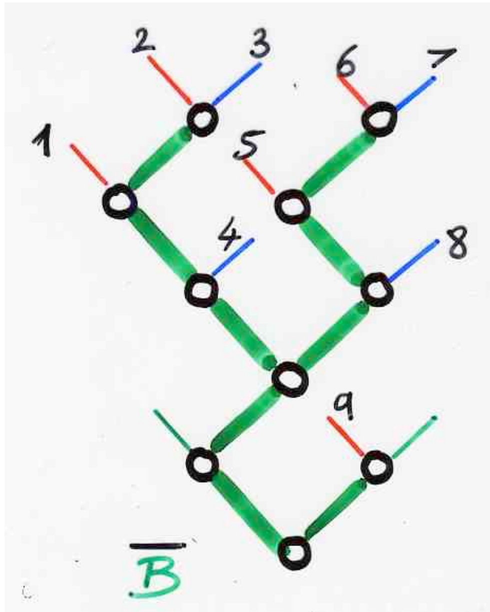
Loday, Ronco (1998) (2012)



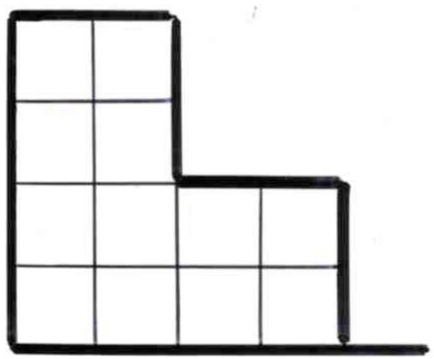
the path v is the canopy of the binary tree B

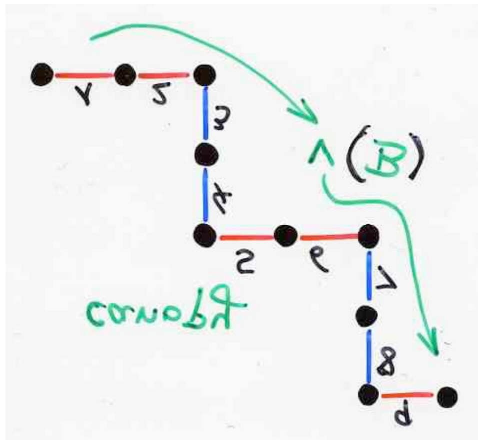


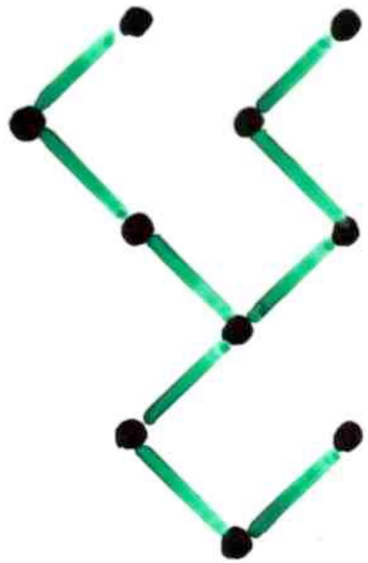


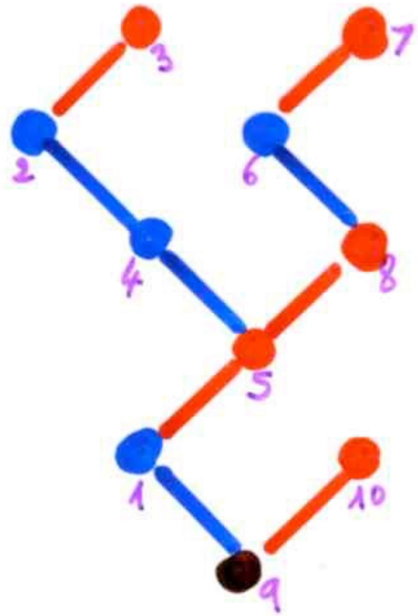


which gives a Ferrers diagram
(in french notation)



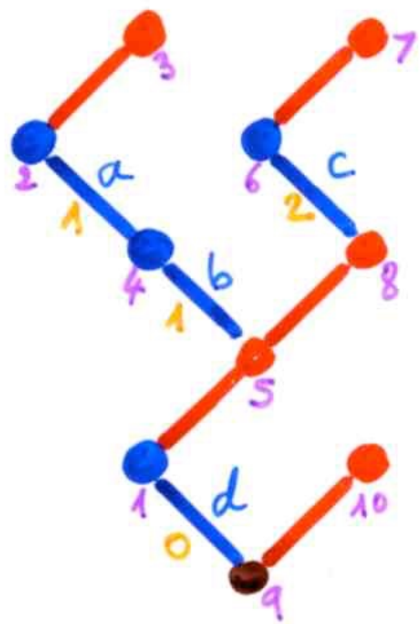






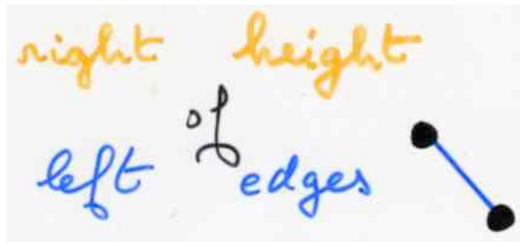
inorder
(=symmetric order)

The left edges (in blue) of the binary tree are ordered according to the in-order (= symmetric order) of the first vertex of the edge.
Here the order is a, b, c, d.

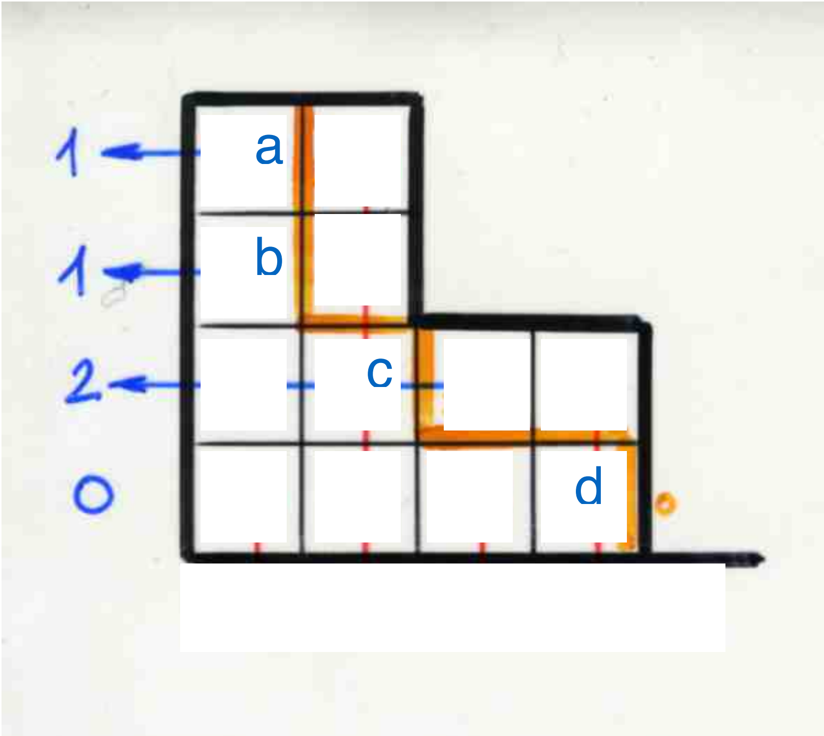


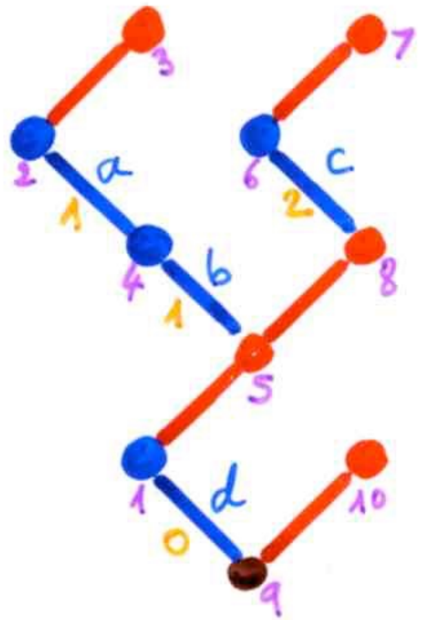
Then the right height of a left edge is the number of right edges (in red) needed to reach the vertices of that left edge.

we get the vector:

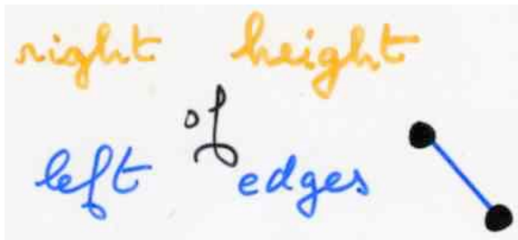


A path u (here in yellow) is uniquely defined by the following process: the South steps are ordered from top to down and associated to the order of the blue edges a,b, c, d. The distance from each North step of u to the North-East border (the path v) is given by the corresponding blue number (the right height of the left edge)





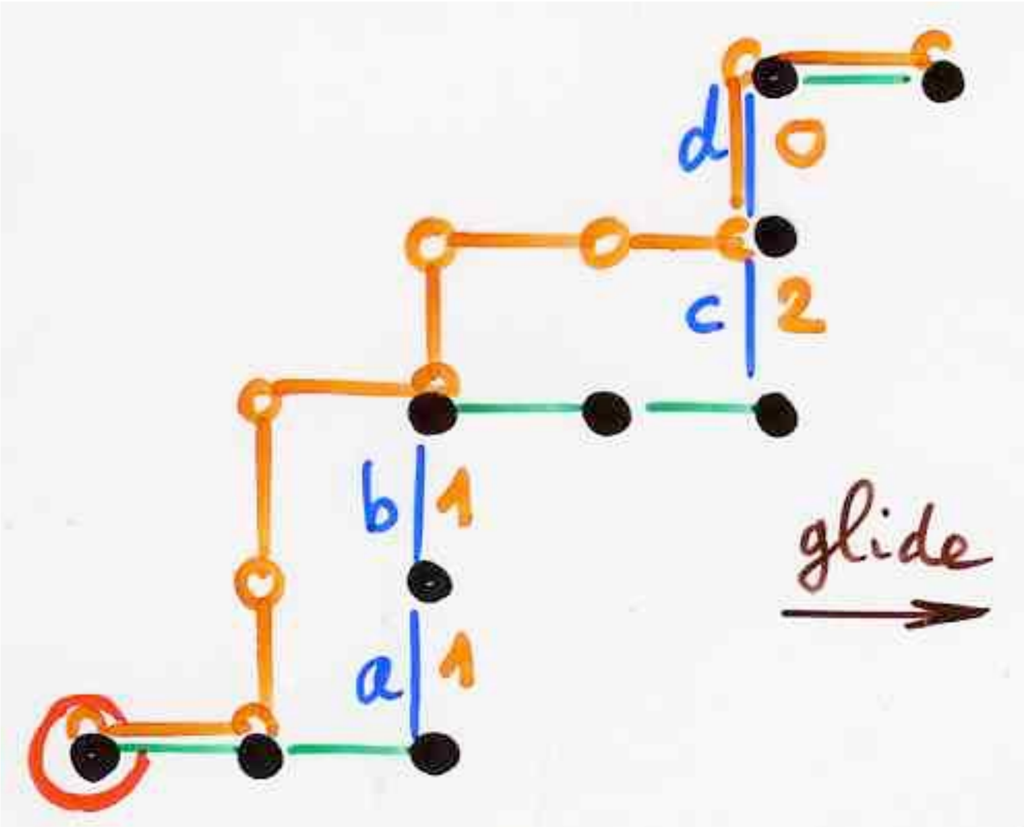
a	b	c	d
1	1	2	0



reverse bijection

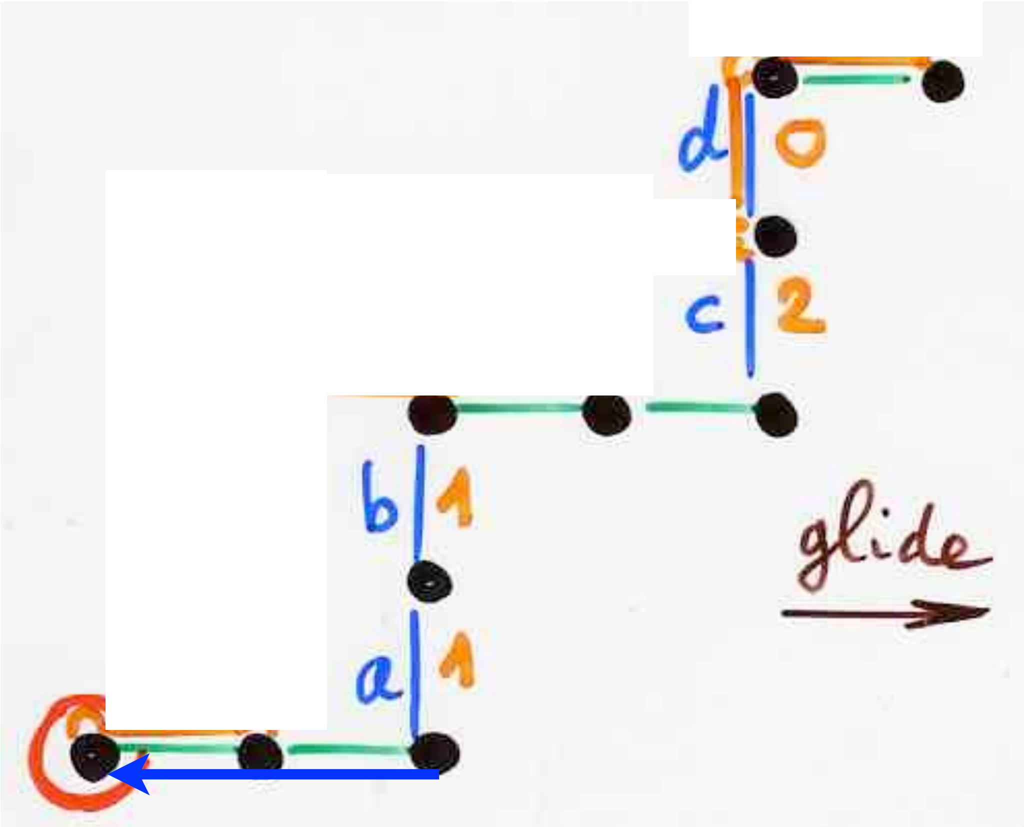
pair of paths (u,v) $\longrightarrow$ binary tree B

the «push-gliding» algorithm



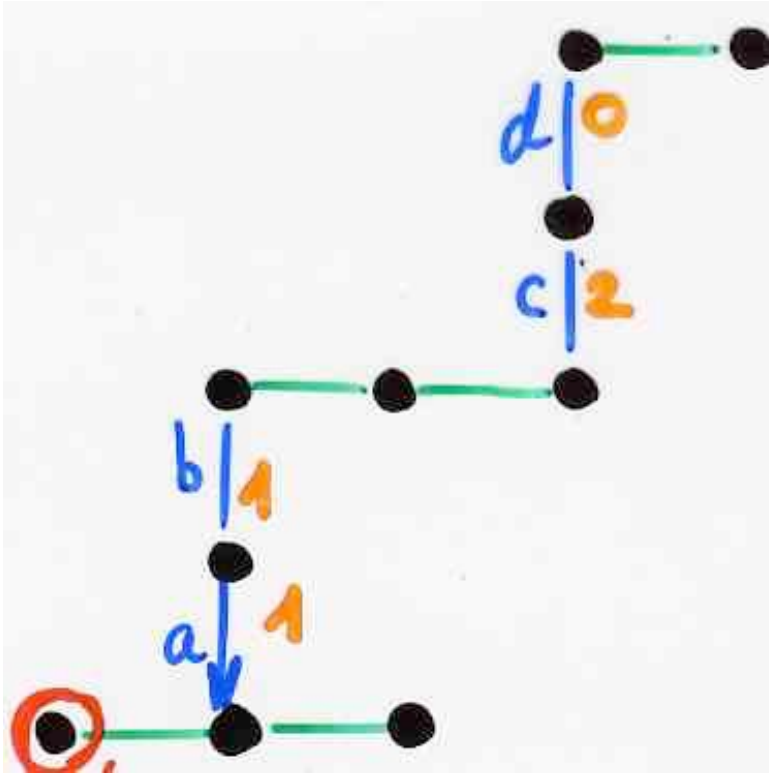
reverse bijection

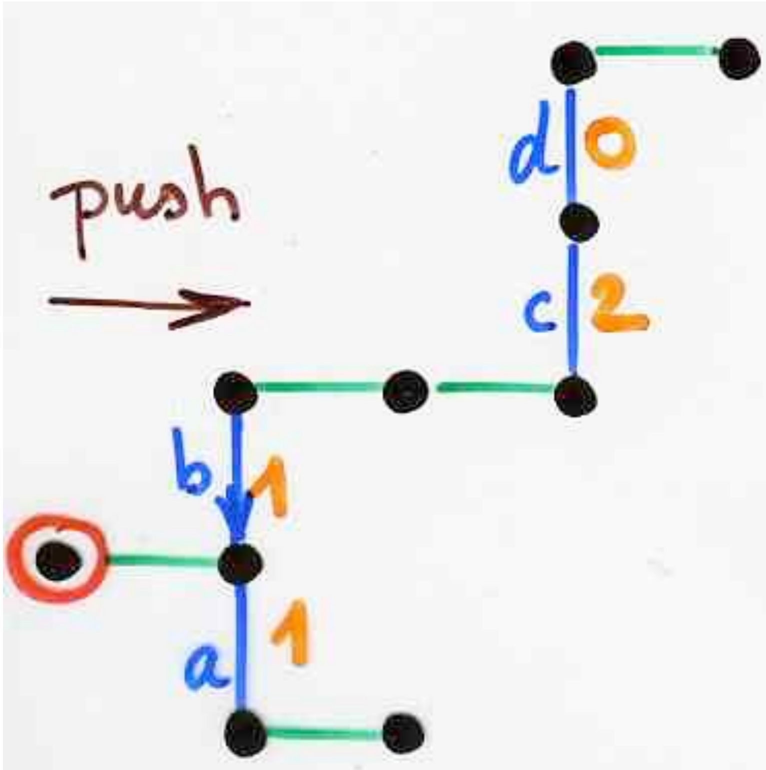
the "push-gliding" algorithm

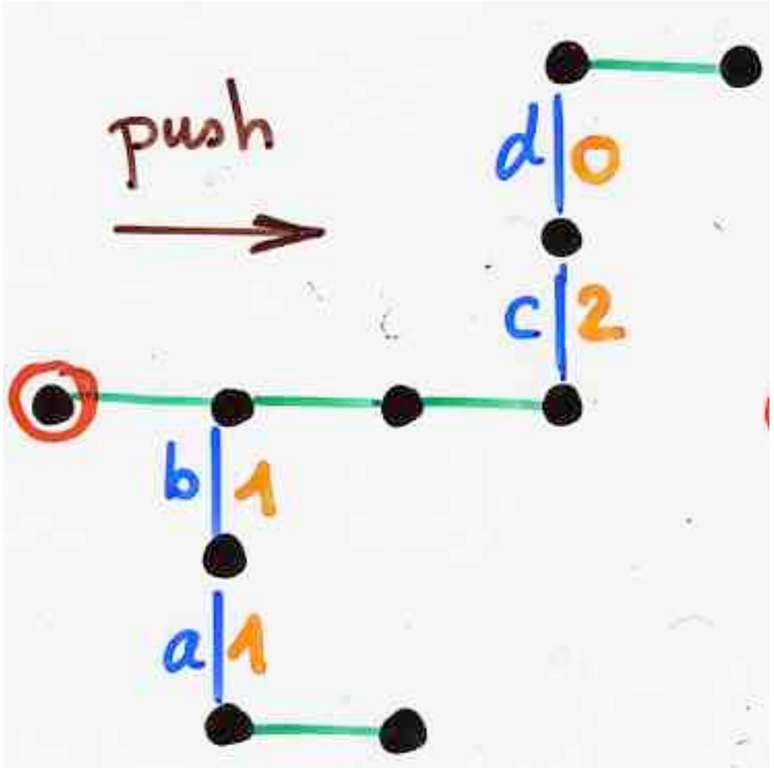


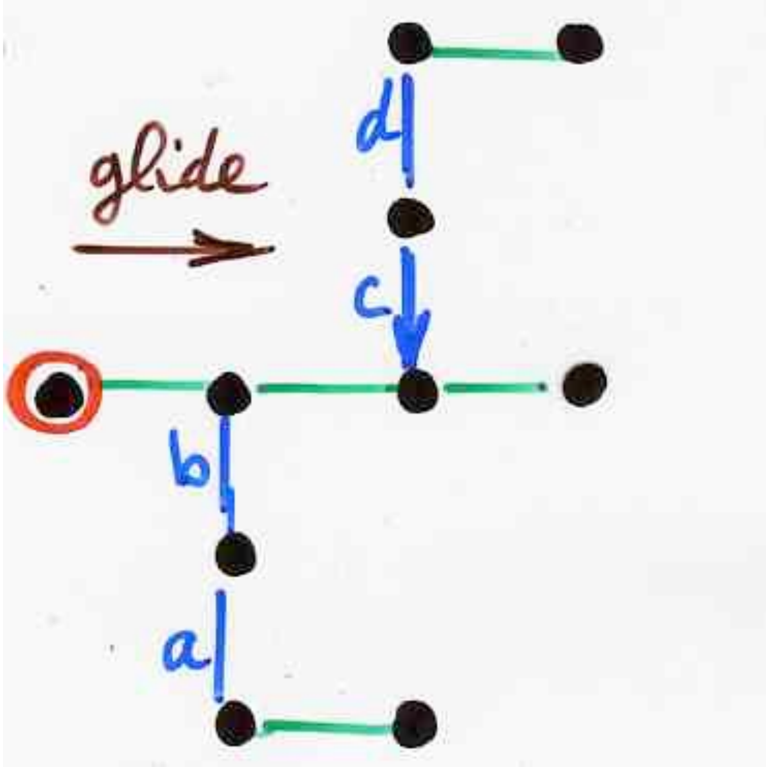
reverse bijection

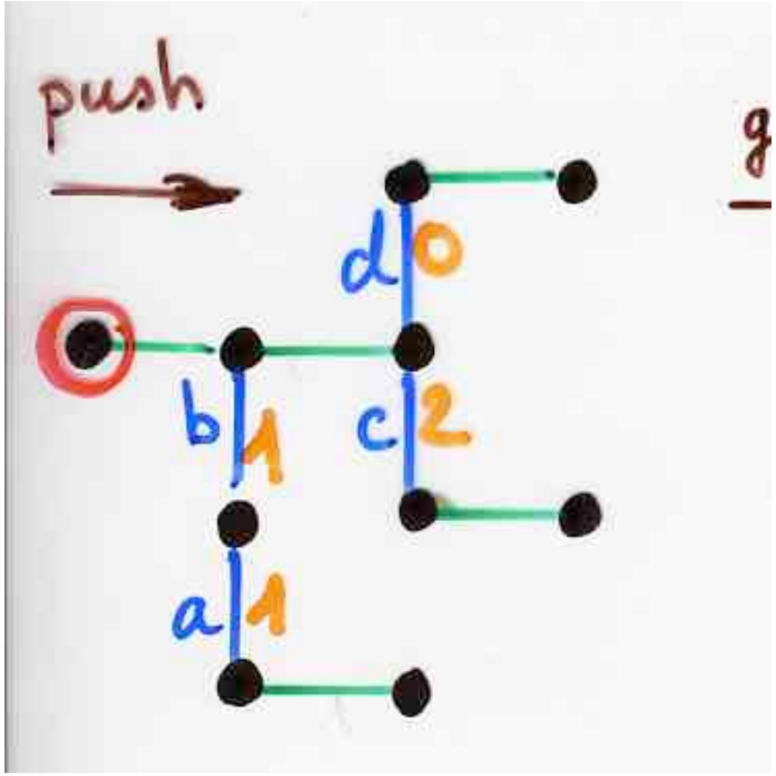
the "push-gliding" algorithm

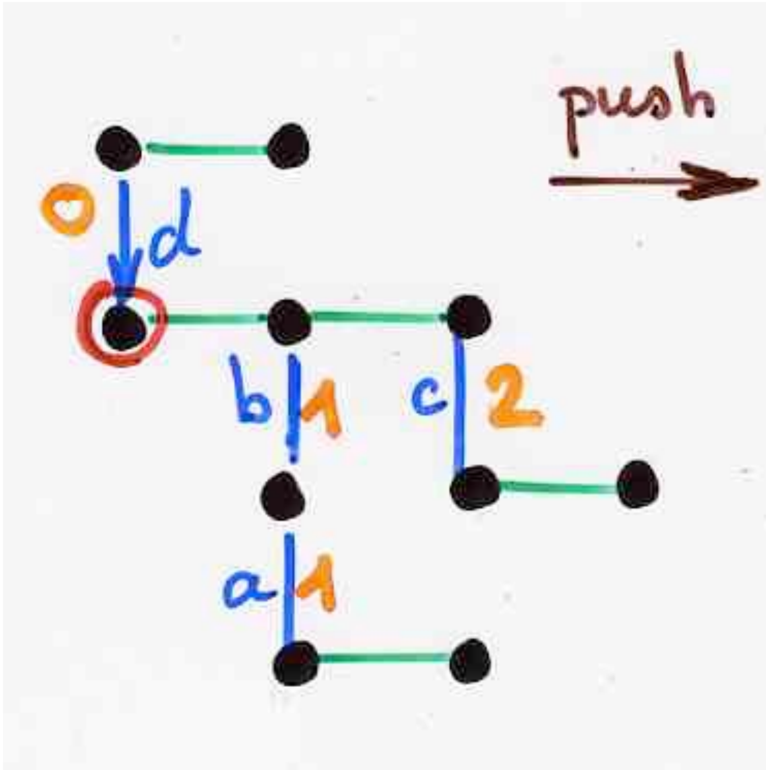

d
0
c
2
b
1
a
1


push
d
0
c
2
b
1
a
1

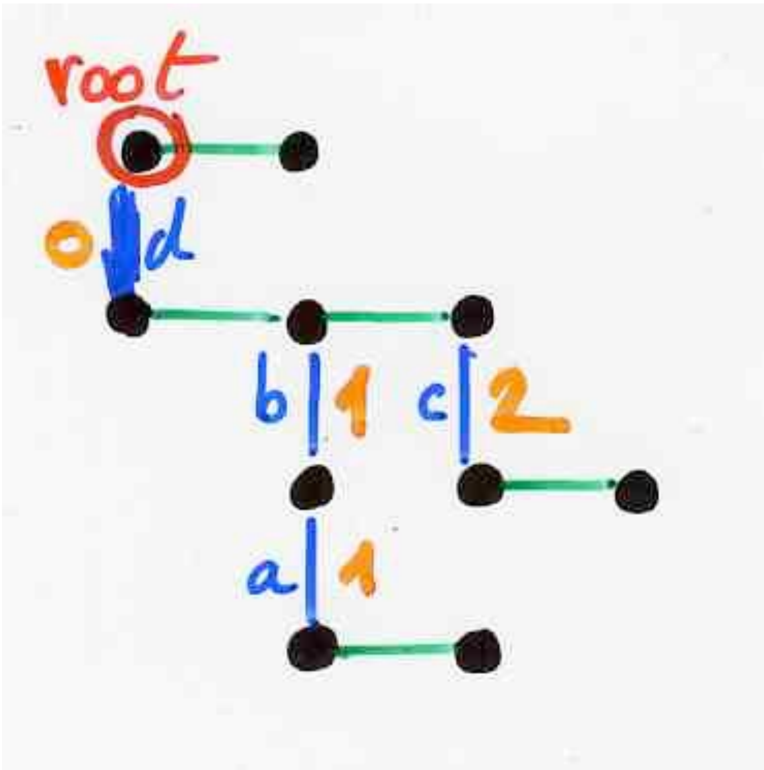

push
d
0
c
2
b
1
a
1


glide
d
c
b
a

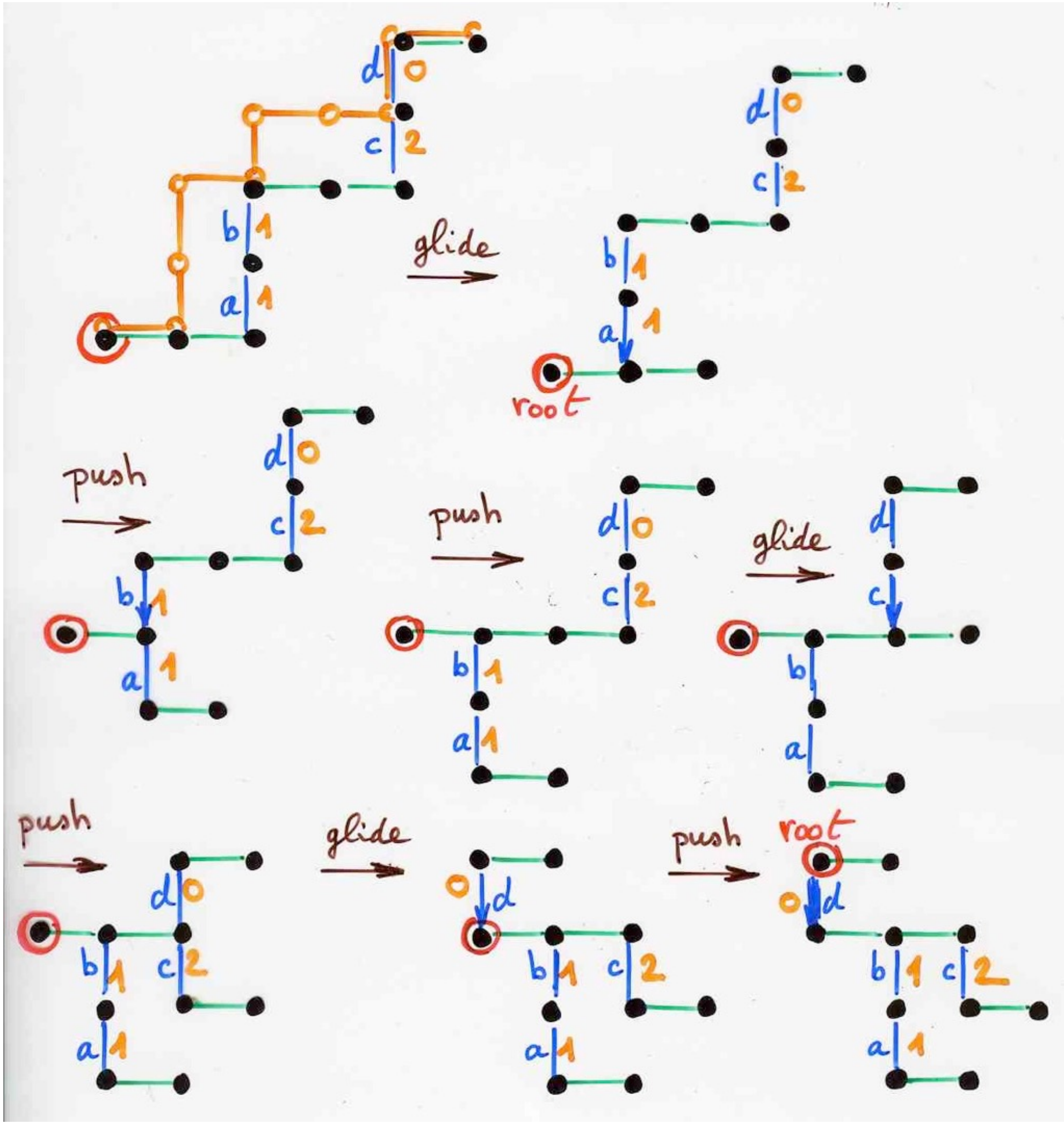

push
d
0
b
1
c
2
a
1
g



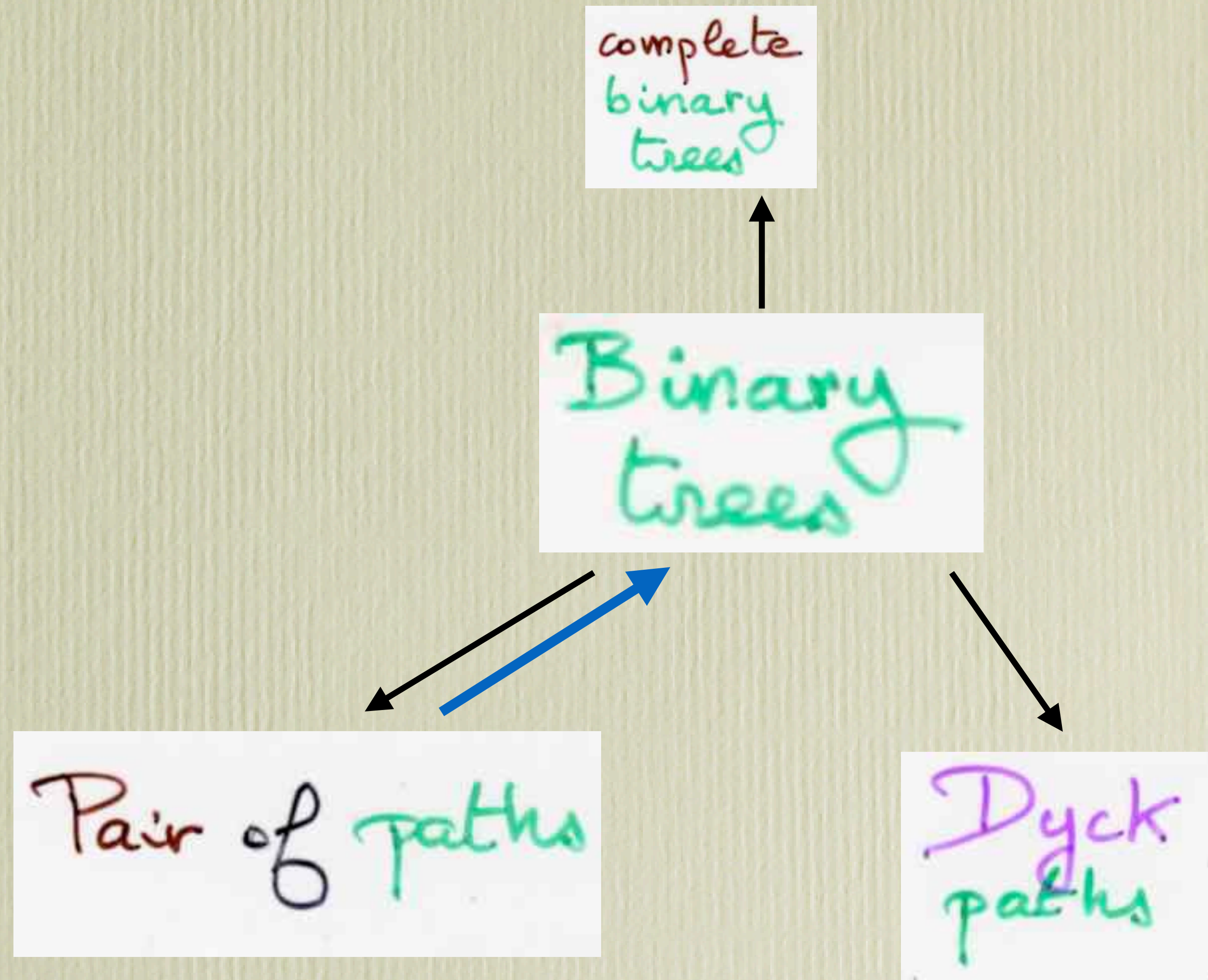
push
0
d
b
1
c
2
a
1



root
0
d
b
1
c
2
a
1

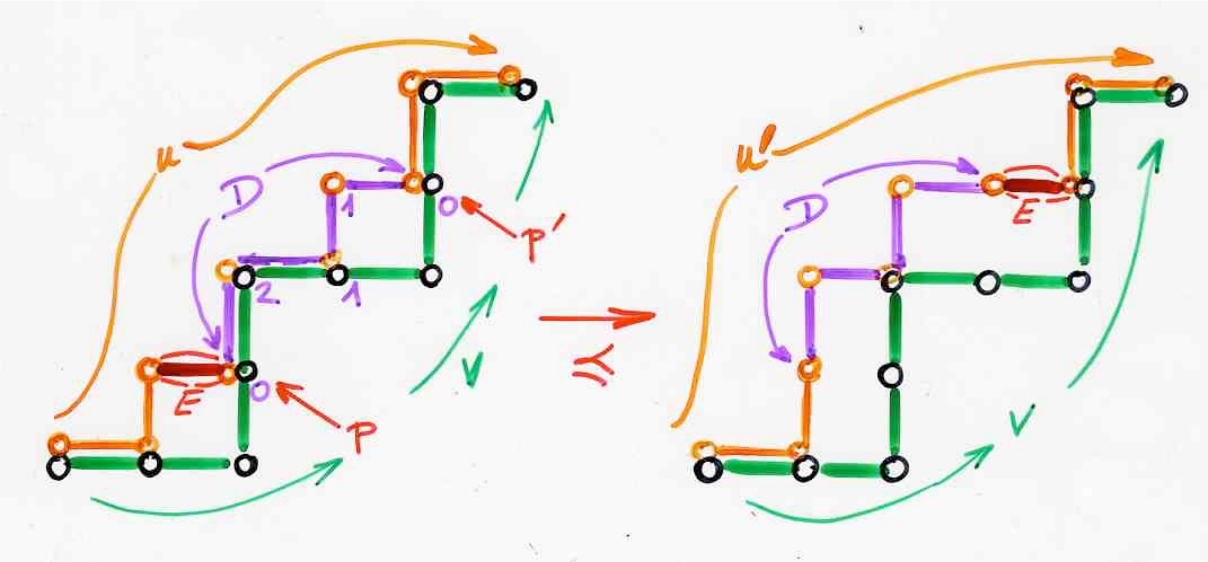


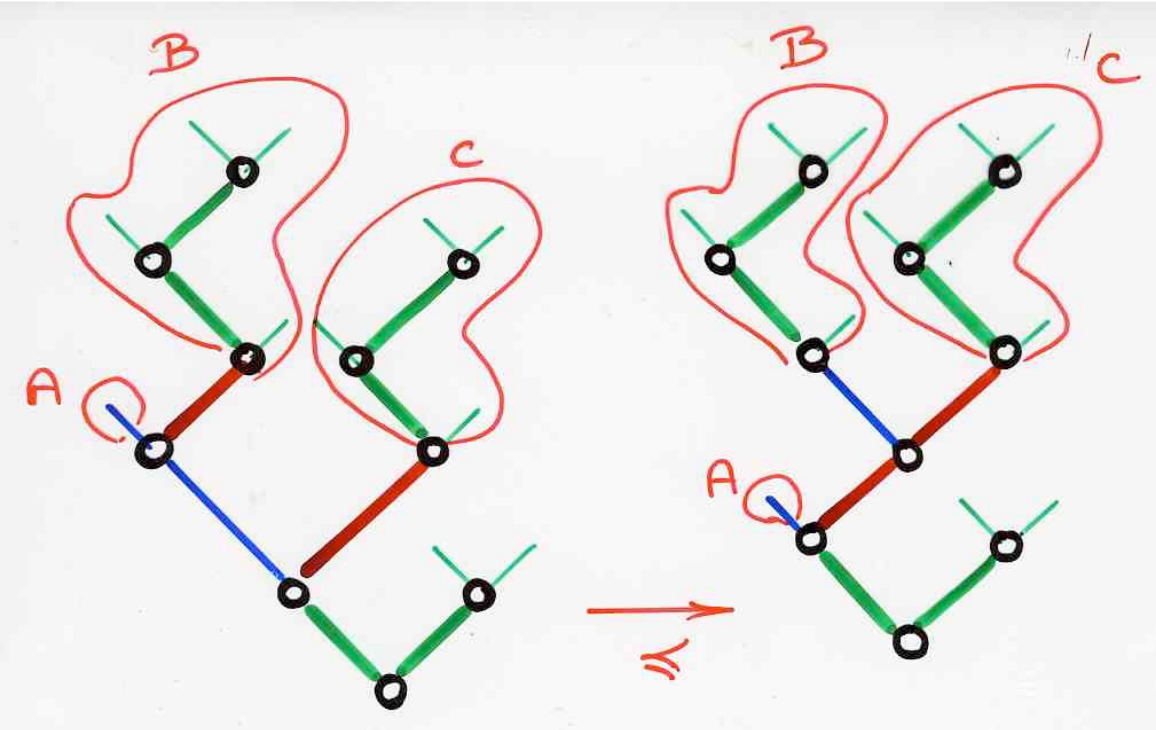
d 0
c 2
b 1
a 1
glide
root
push
push
glide
push
glide
push
root


complete binary trees
Binary trees
Pair of paths
Dyck paths

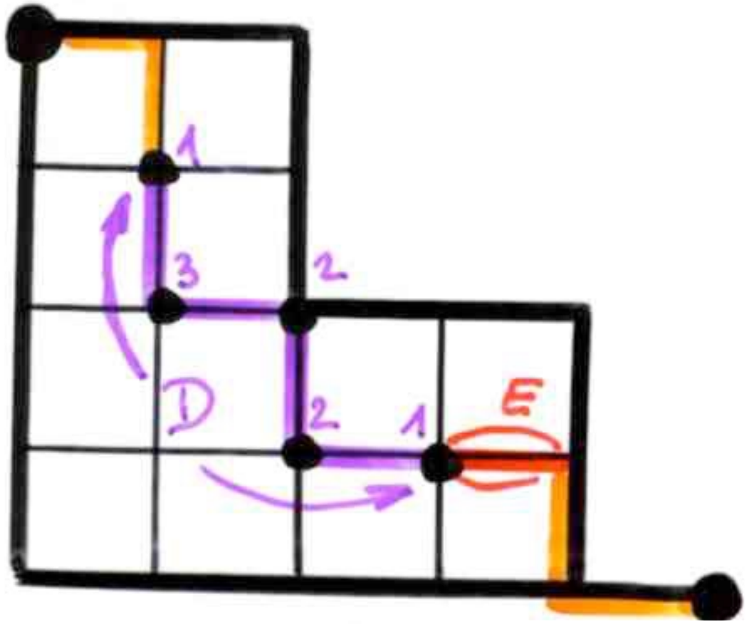
idea of the proof of

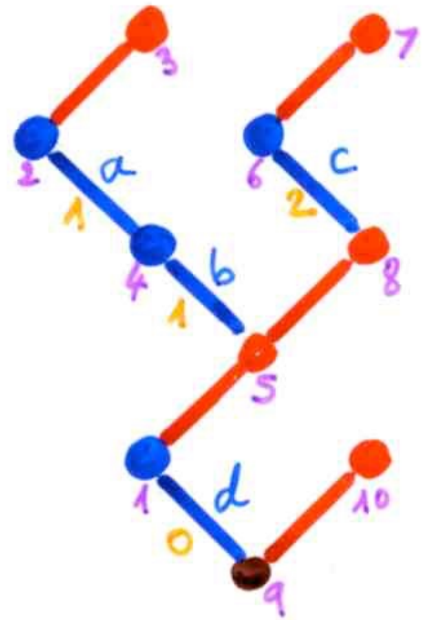
Theorems 1,2 3



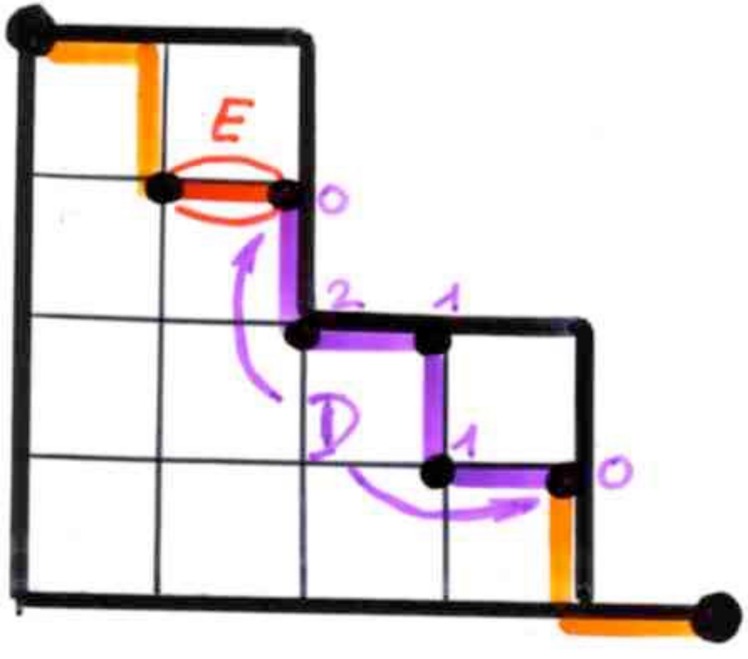


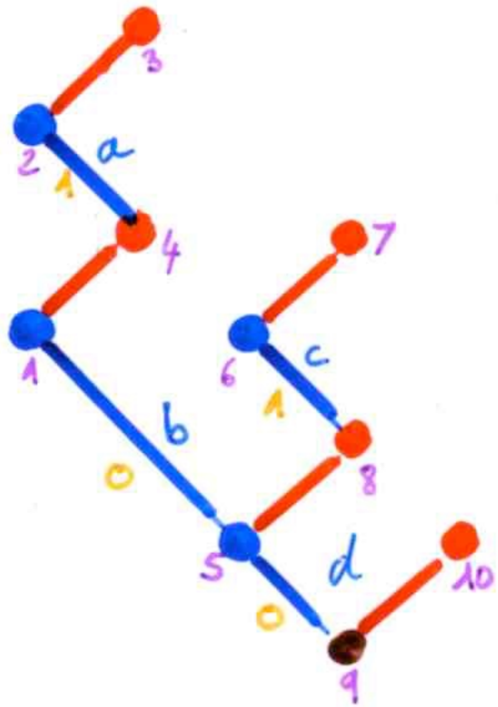
the covering relation in T_V
and the corresponding rotation
in (ordinary) T

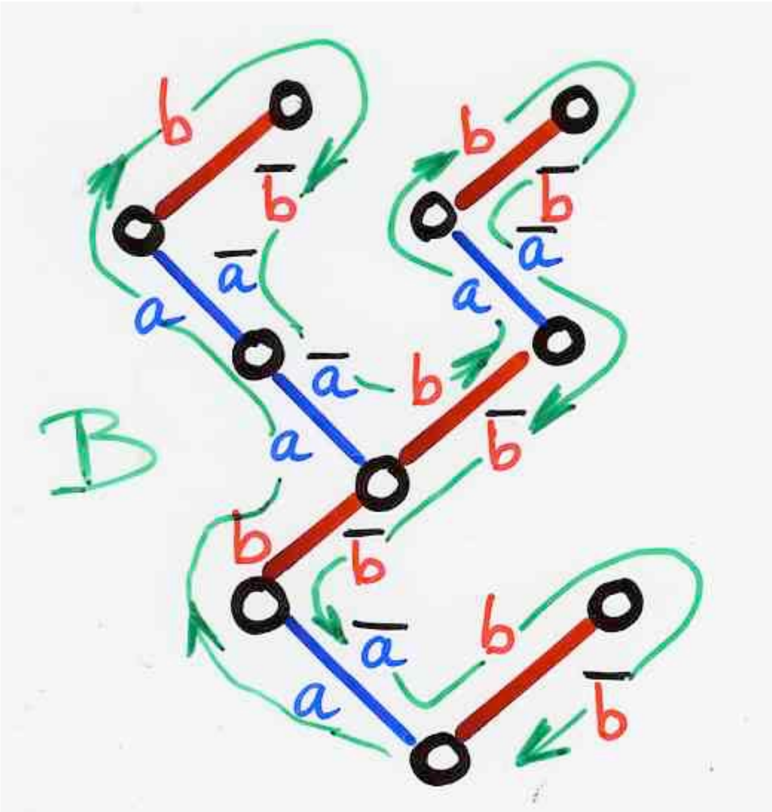

1
3
2
D
2
1
E


3
7
a
c
2
6
1
2
b
4
8
1
5
d
1
10
0
9

an example


E
0
2
1
D
1
0


3
a
2
1
4
7
c
1
6
1
b
8
0
5
d
10
0
9



walk around a binary tree B

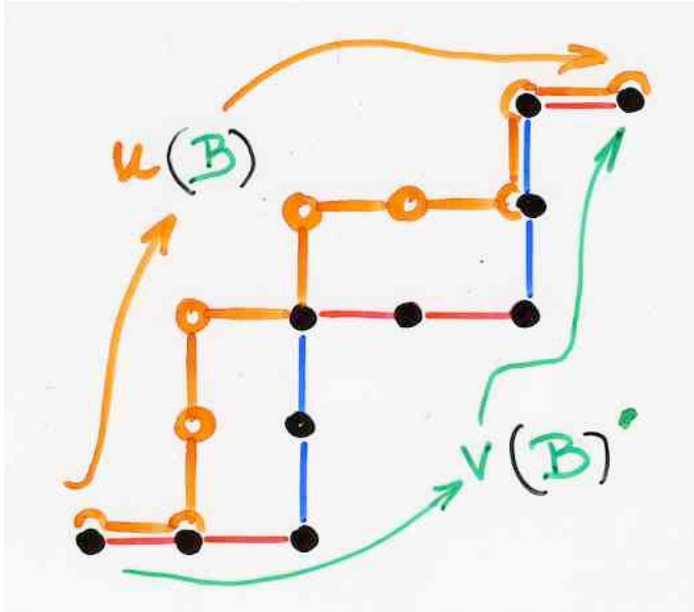
the words $\left\{ \begin{array}{l} w(B) \\ u(B) \\ v(B) \end{array} \right.$

$$w(B) = a\, b\, a\, a\, b\, \bar{b}\, \bar{a}\, \bar{a}\, b\, a\, b\, \bar{b}\, \bar{a}\, \bar{b}\, \bar{b}\, \bar{a}\, b\, \bar{b}$$

$$u(B) = \bar{b}\, \bar{a}\, \bar{a} \quad \bar{b}\, \bar{a}\, \bar{b}\, \bar{b}\, \bar{a} \quad \bar{b}$$

$$v(B) = b \quad b \quad \bar{a}\, \bar{a}\, b \quad b \quad \bar{a} \quad \bar{a}\, b$$

$$\bar{a} \rightarrow N \qquad \left.\begin{array}{c} b \\ \bar{b} \end{array}\right\} \rightarrow E$$



the pair (u, v) of paths associated to a binary tree B

v (B) is the Canopy

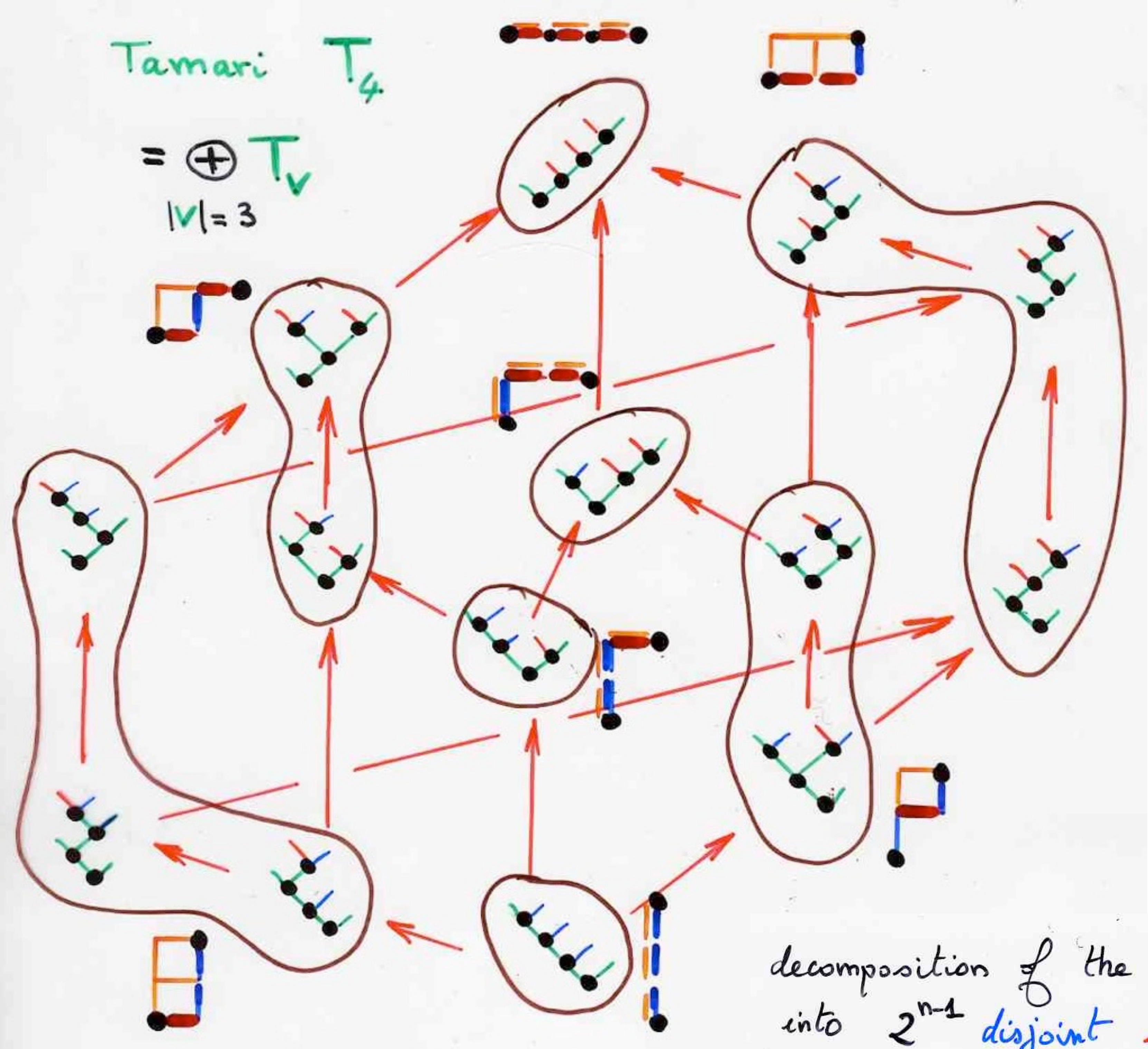
Thm 1. For any path v
T_v is a lattice

Thm 2. The lattice T_v
is isomorphic to the dual of $T_{\overleftarrow{v}}$

Thm 3. The usual Tamari lattice T_n
can be partitioned into intervals
indexed by the 2^{n-1} paths v of
length $(n-1)$ with $\{E, N\}$ steps,

$$T_n \cong \biguplus_{|v|=n-1} I_v ,$$

where each $I_v \cong T_v$.



decomposition of the lattice T_n into 2^{n-1} disjoint intervals combinatorial structures

hyperculbe associahedron permutahedron

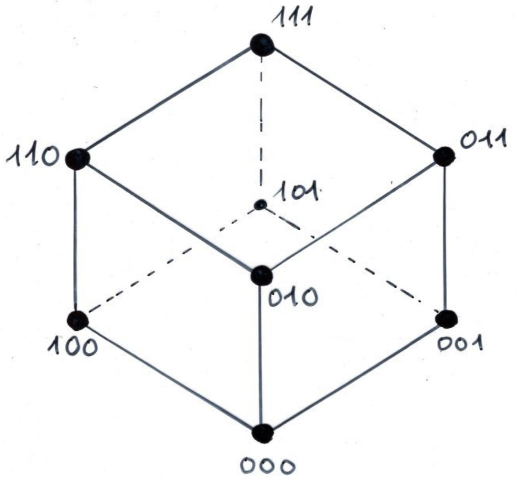
Boolean lattice inclusion $\longleftrightarrow$ Tamari order $\longleftrightarrow$ weak Bruhat order

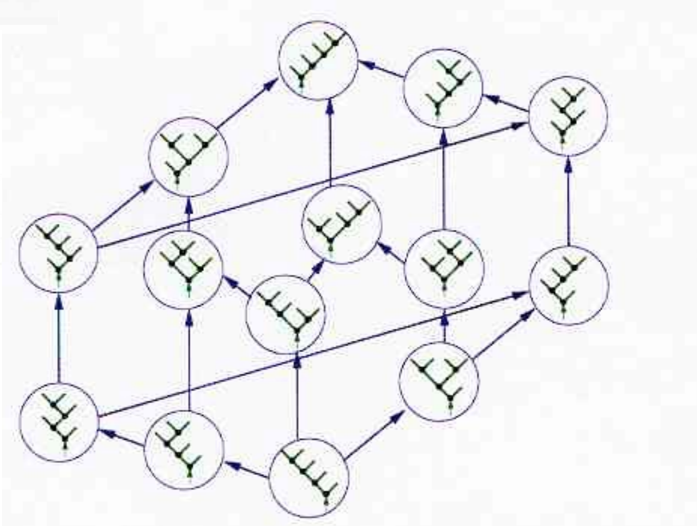
dim 2^n C_n $n!$

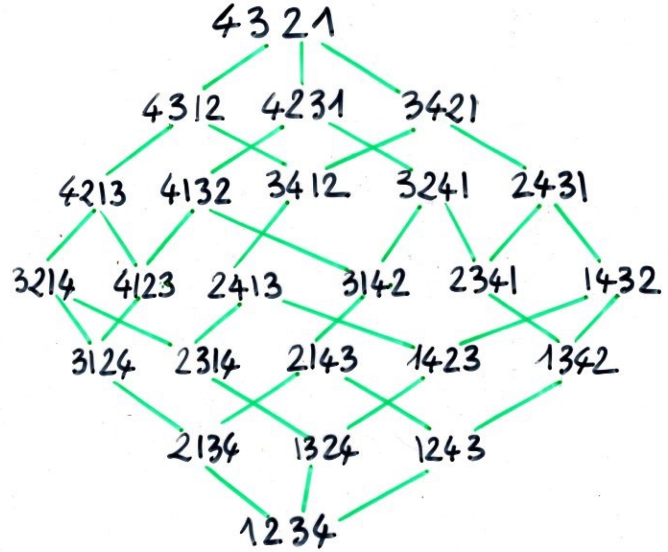
Catalan

algebraic structures

Hopf algebra





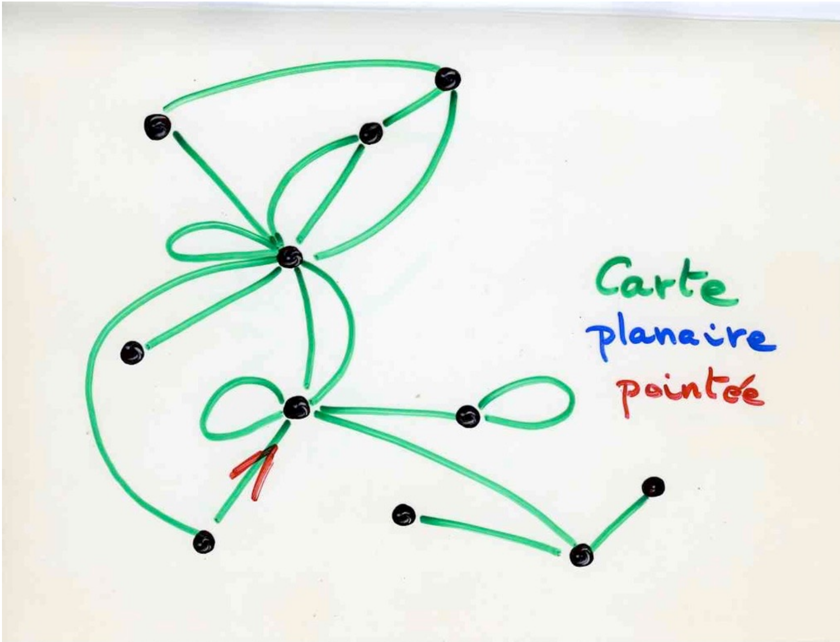


Fang, Préville-Ratelle (2015)

The total number of intervals in all T_V, $|V|=n$ is the number of rooted non-separable planar maps with $(n+1)$ edges

Tamari (V)

$$\frac{2\,(3n+3)!}{(n+2)!\,(2n+3)!}$$

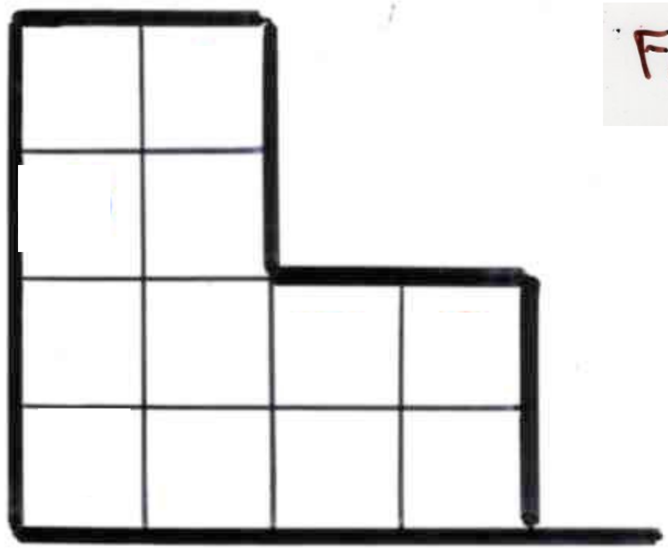


proof with a bijection

Tamari(v) lattice as a maule

alternative tableau

Definition



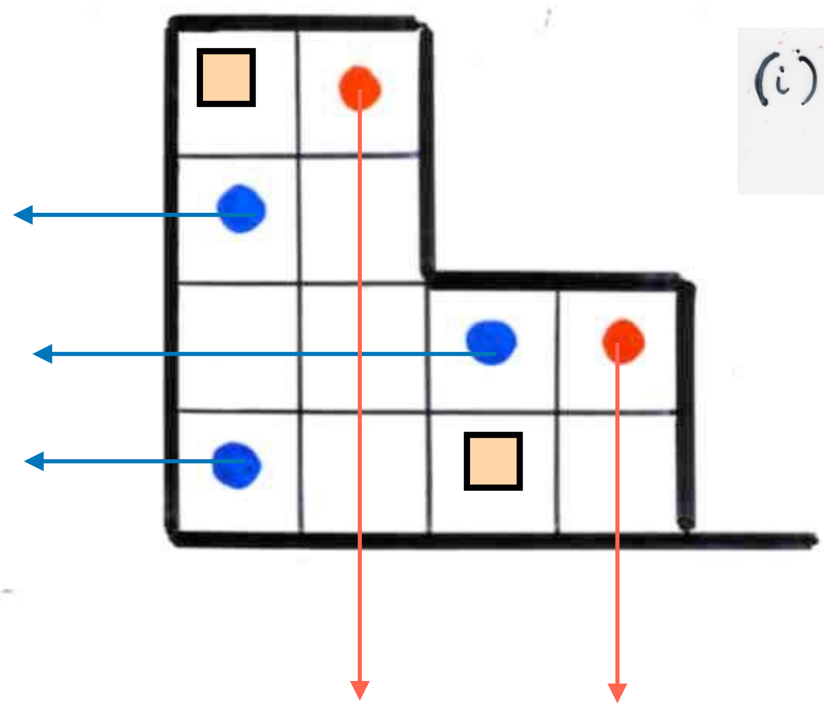
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

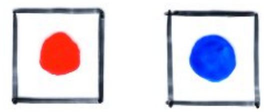
$\mathbf{n} = (\text{number of rows}) + (\text{number of columns})$

alternative tableau



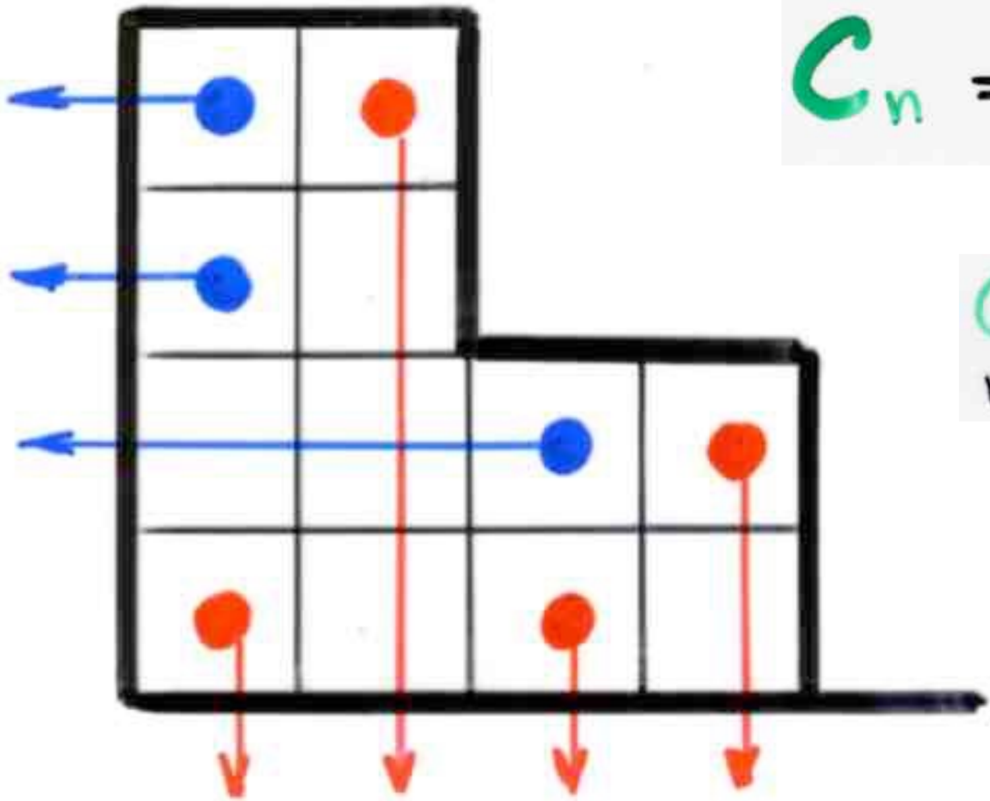
Definition

(i) some cells are coloured red or blue



(ii)
- no coloured cell at the left of a blue cell
- no coloured cell below a red cell

Def Catalan alternative tableau T
alt. tab. without cells $\square$
i.e: every empty cell is below a red cell or on the left of a blue cell



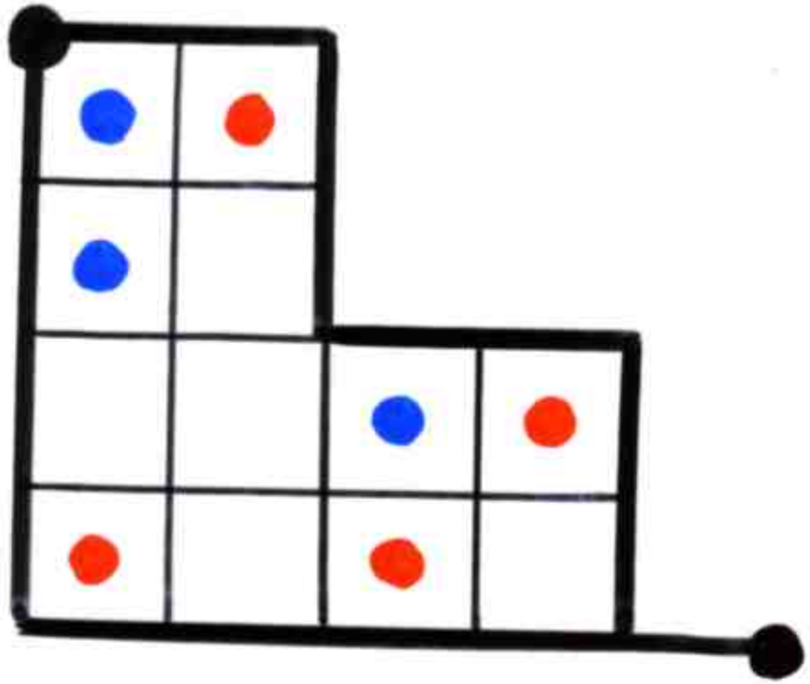
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

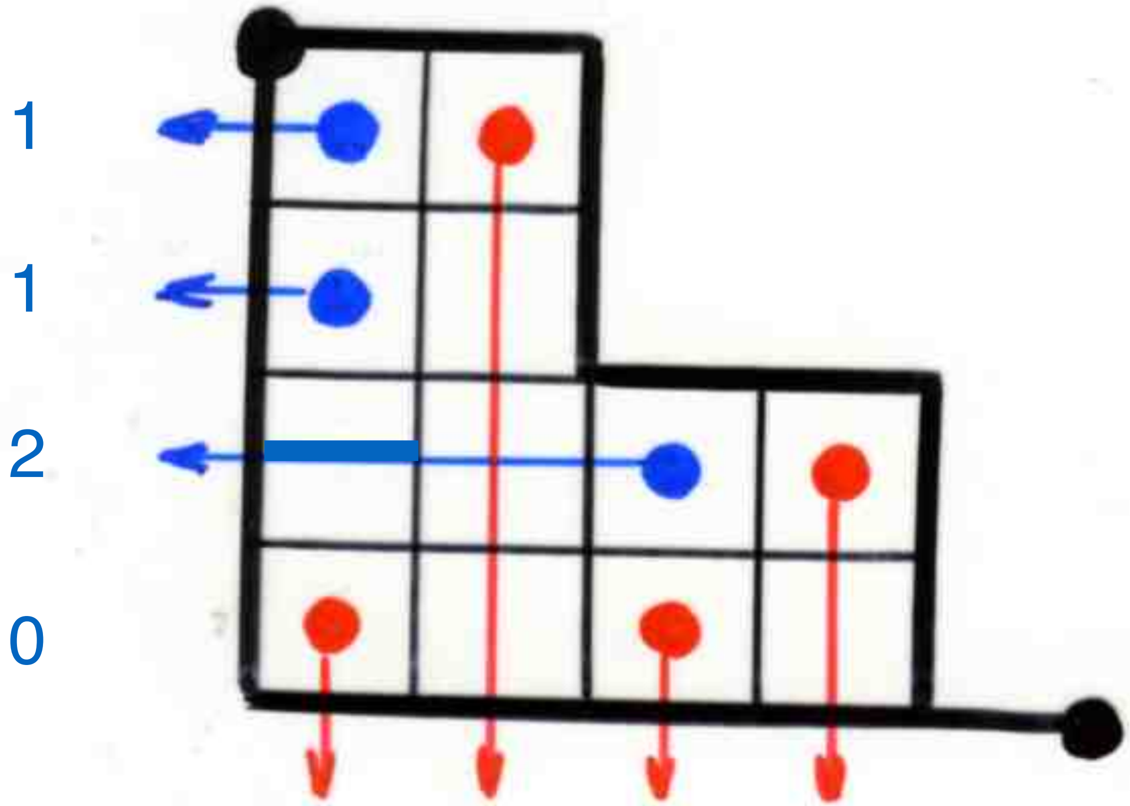
bijection

Catalan alternative tableaux

pair of paths

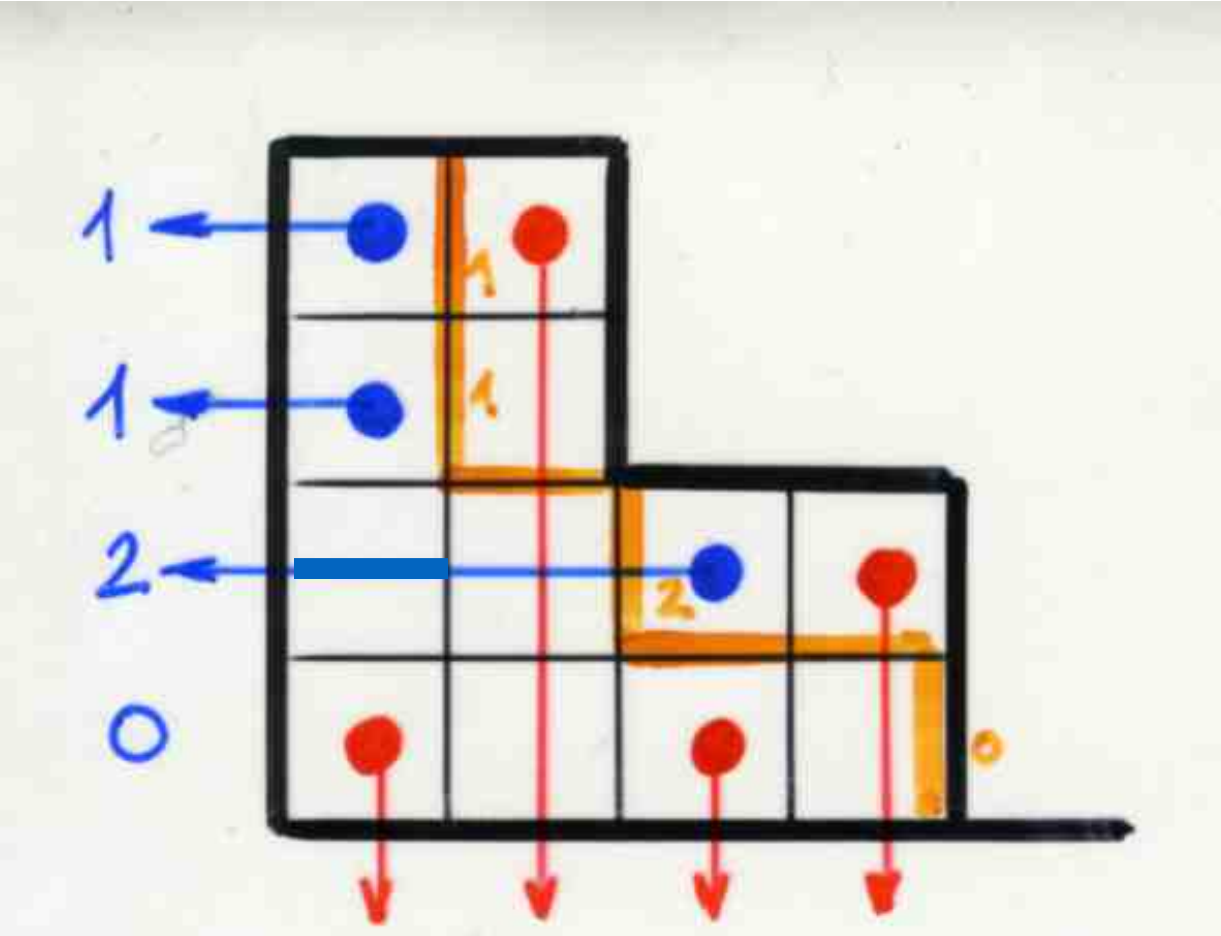


For each row of a Catalan alternative tableau we associate a blue number by the following rule:

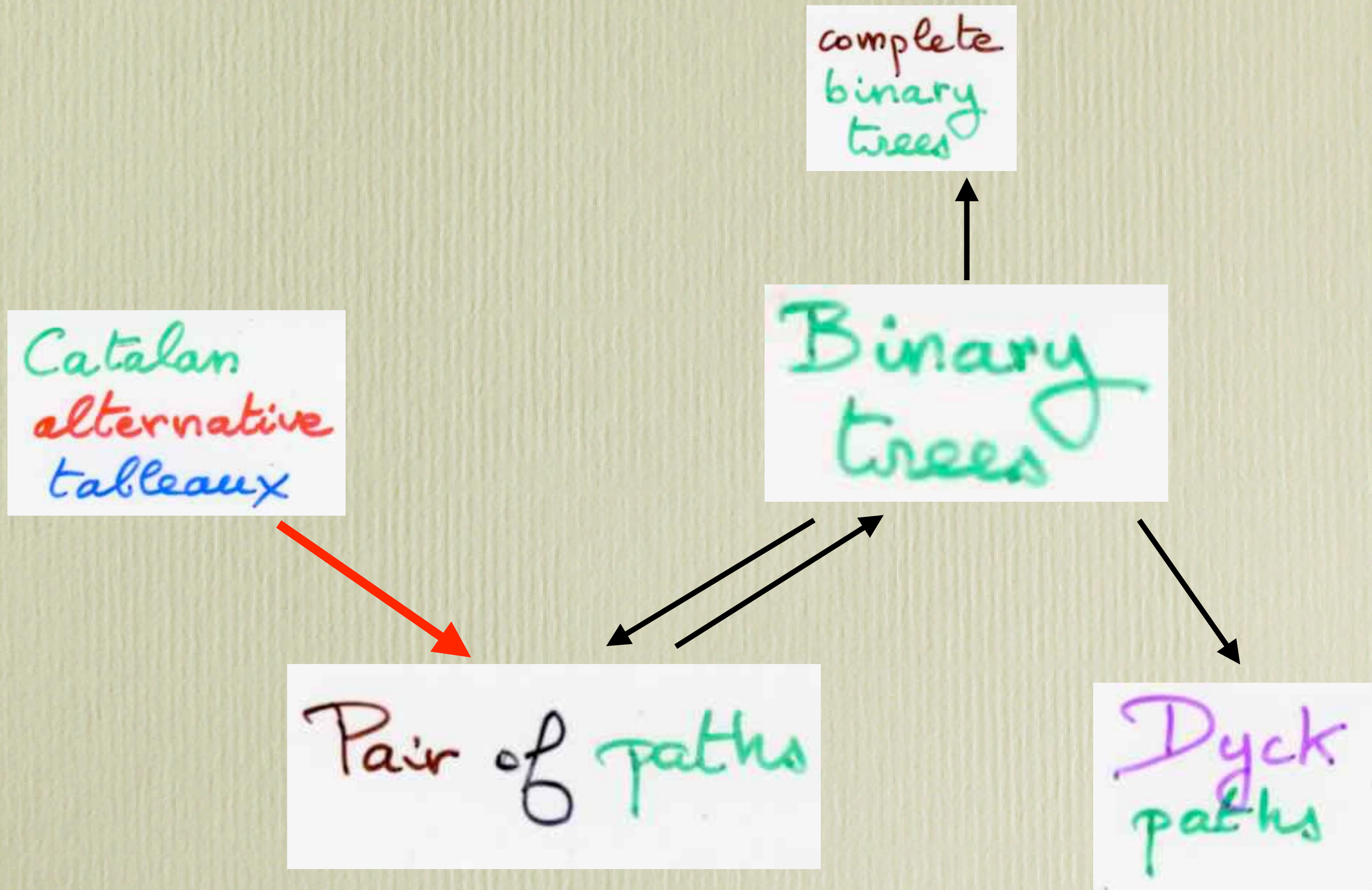


- 0 if there are no blue point in the row
- 1 + the number of cells in the row which are of the type 
  (i.e. there is a blue point at its right, but no red point above)

We get a vector P of blue numbers (here P= 1, 1, 2, 0), which we call the **Adela row vector** (see slides 116-119).



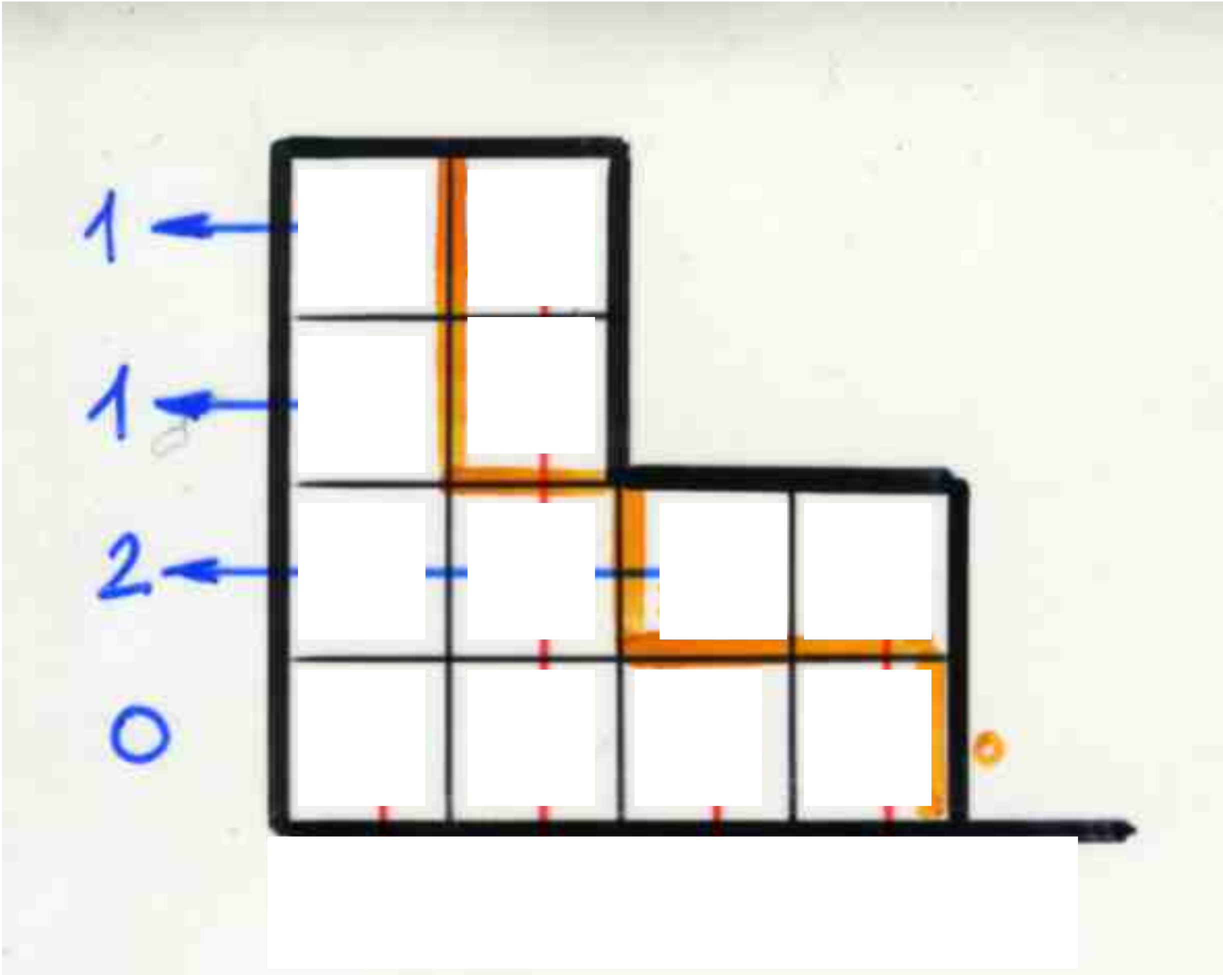
From this vector P, we define a path u (in yellow) such that the distance of each South step of u to the North-East border is given by the corresponding blue number (analog rule in slide 29)


complete binary trees
Catalan alternative tableaux
Binary trees
Pair of paths
Dyck paths

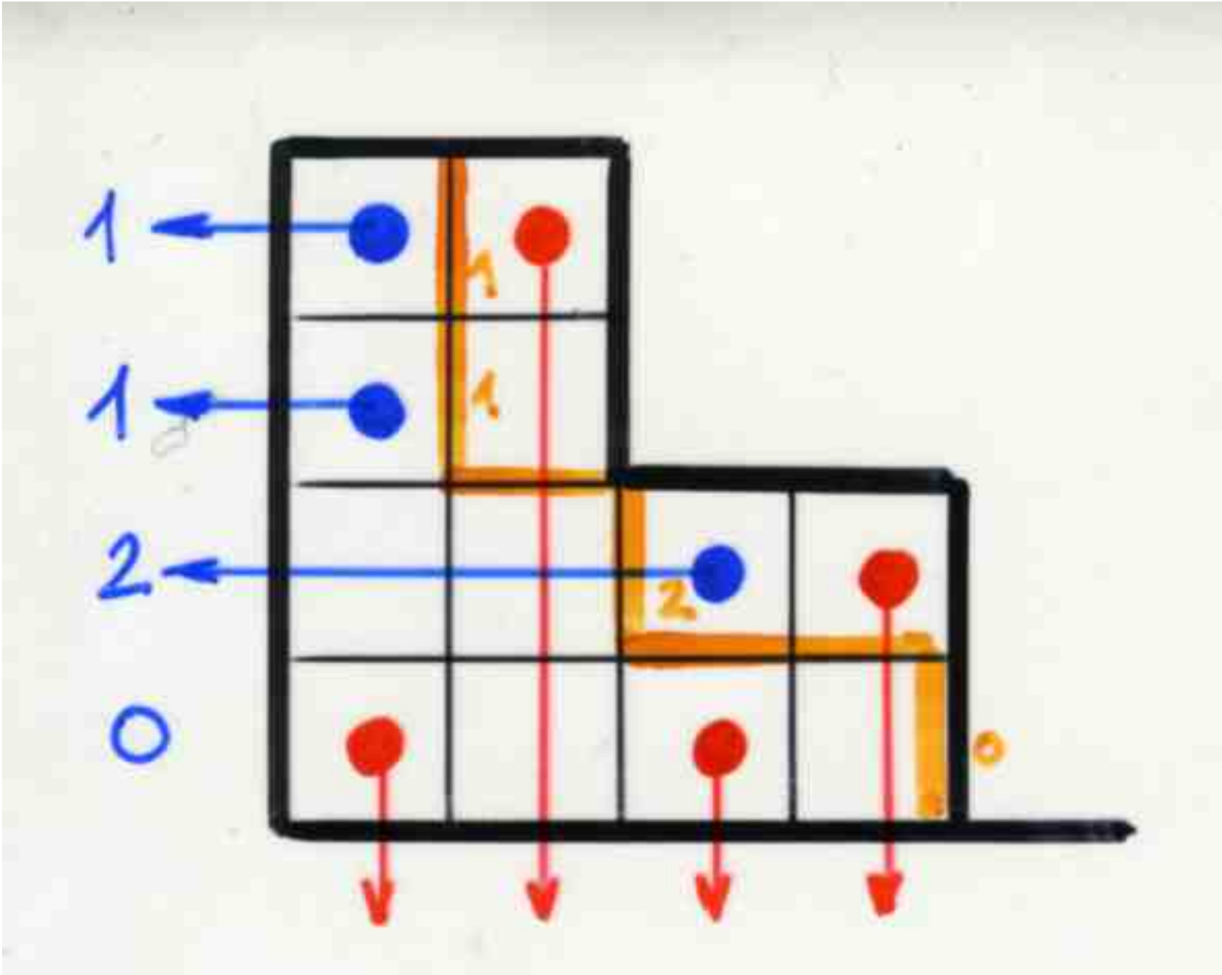
reverse bijection

pair of paths

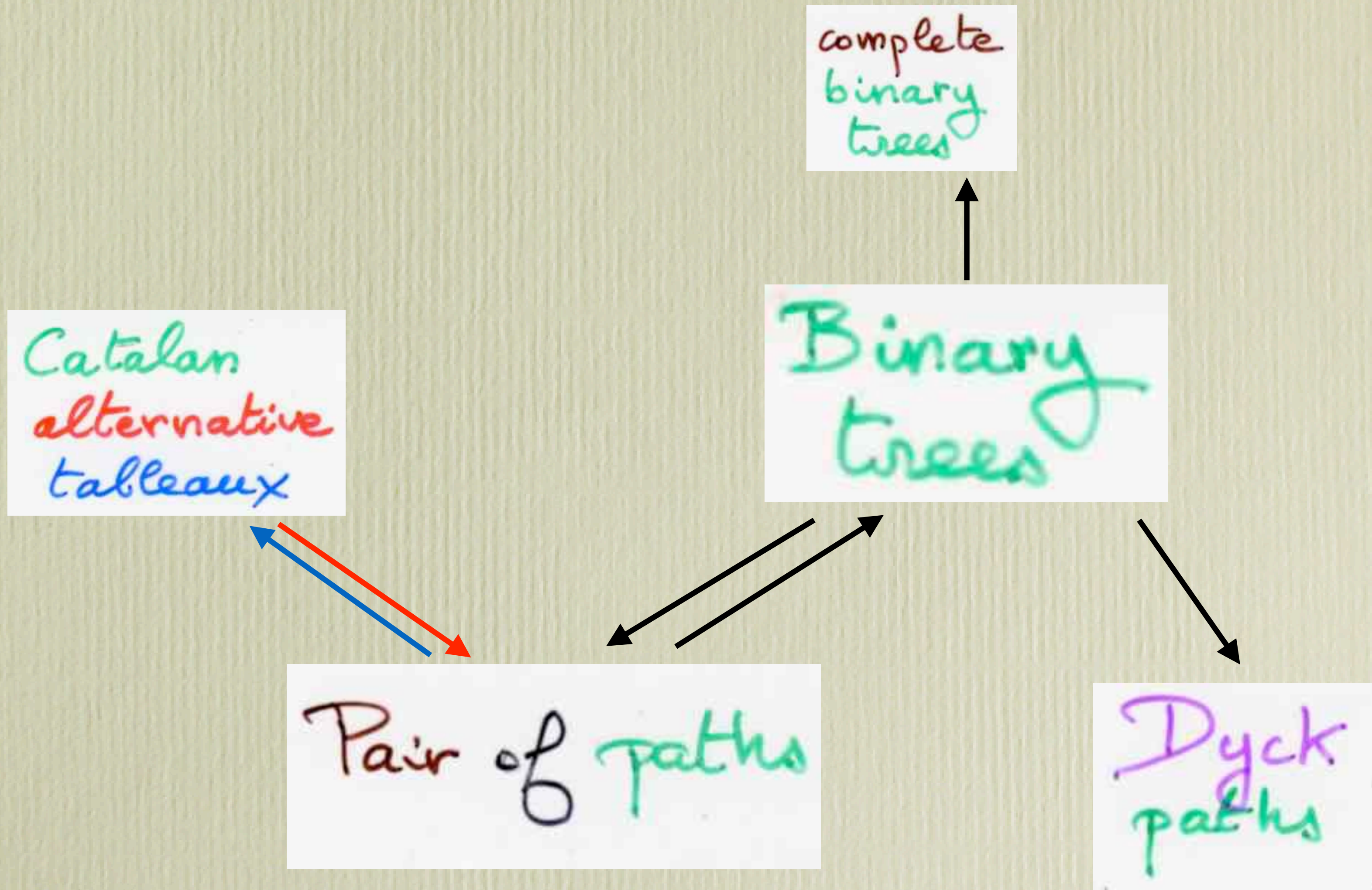
Catalan alternative tableaux



From the path u we get the blue numbers as the distance in each row of the South step of u to the border of Ferrers diagram (path v). We get a vector V (here V = 1, 1, 2, 0)

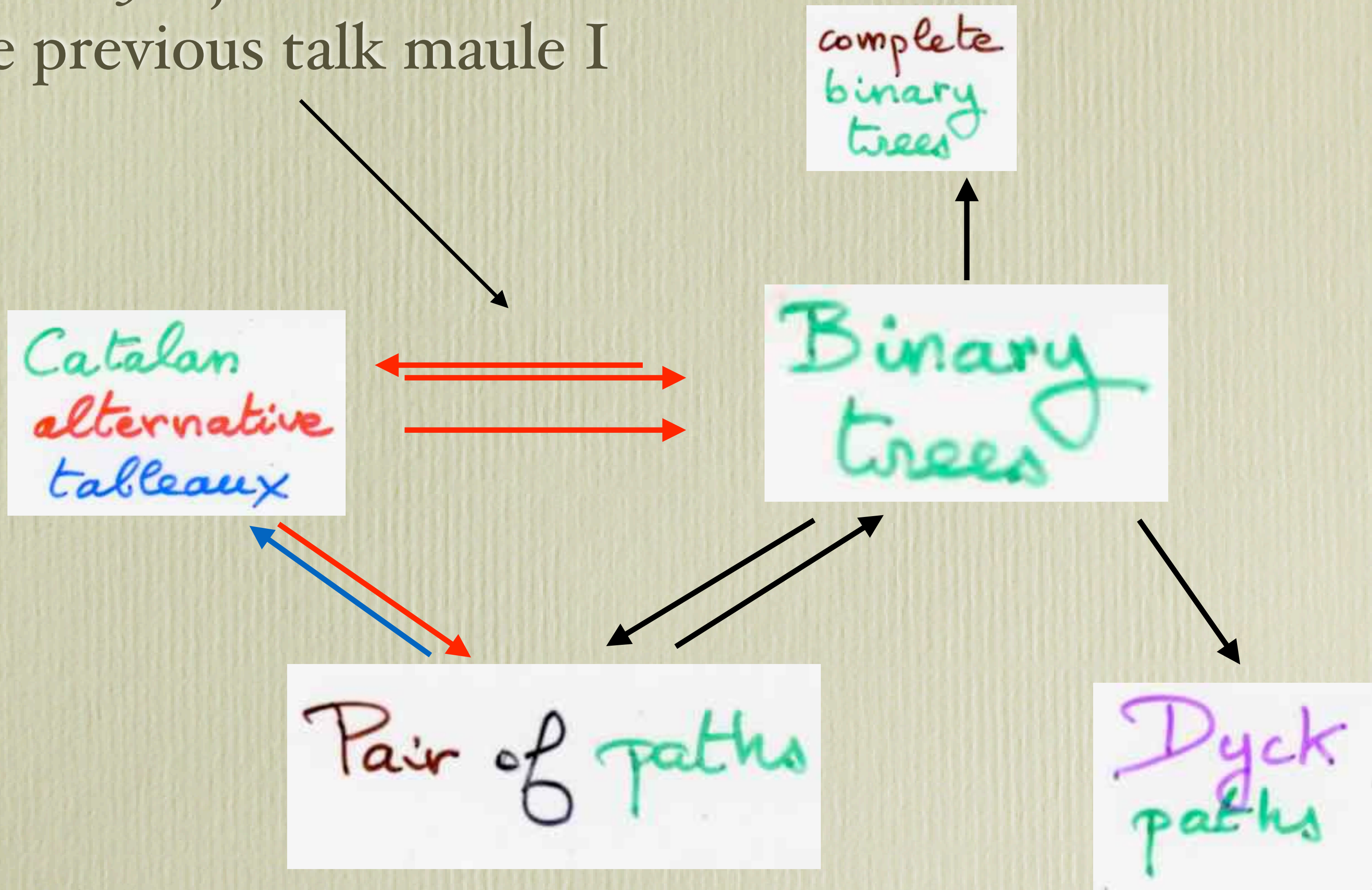


Then there is a unique Catalan alternative tableau whose Adela row vector P (see definition slide 39) is equal to V. This tableau can be obtained by filling the rows from top to down with first a (possible) blue point and then the red points in a unique way from V.

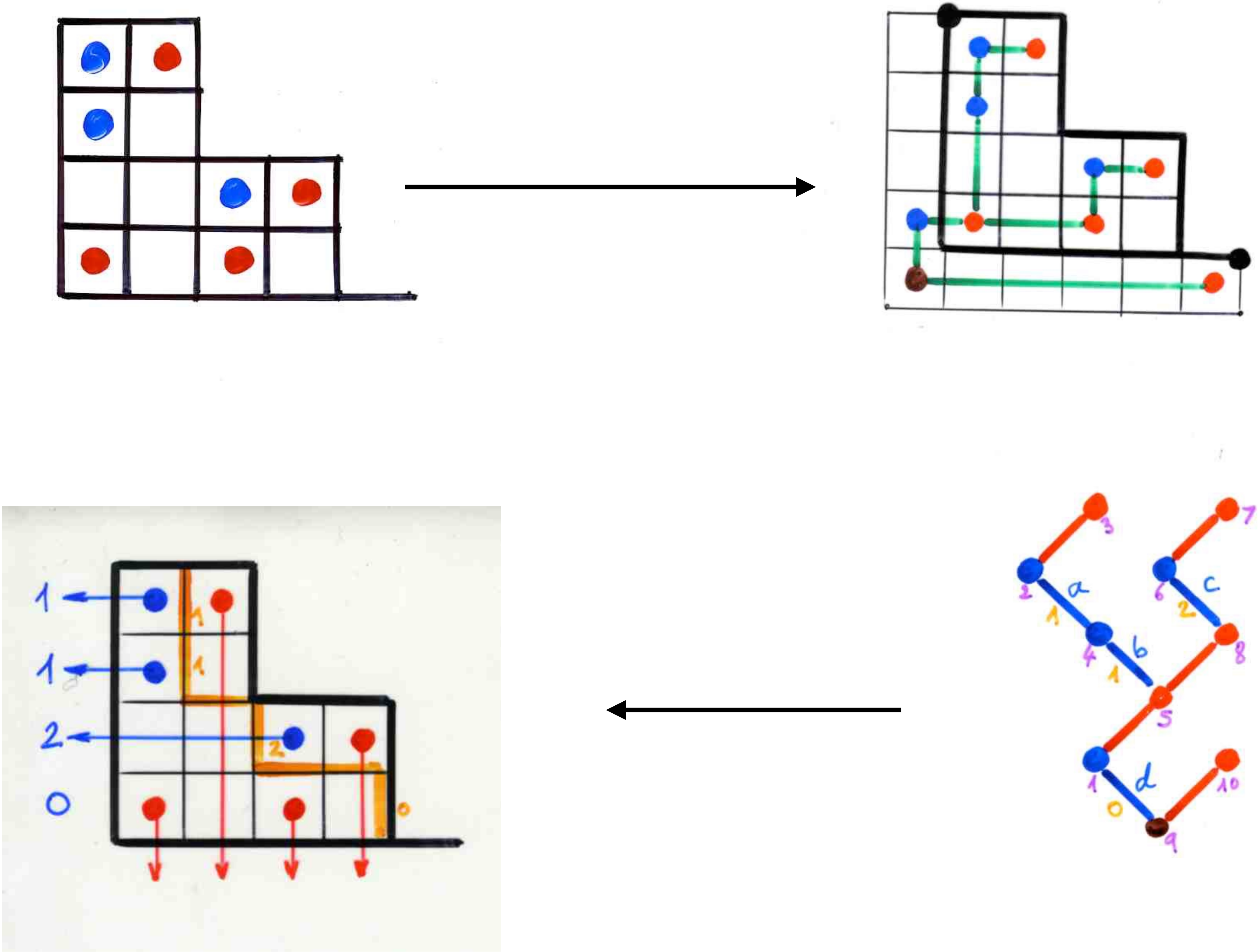


commutative diagram

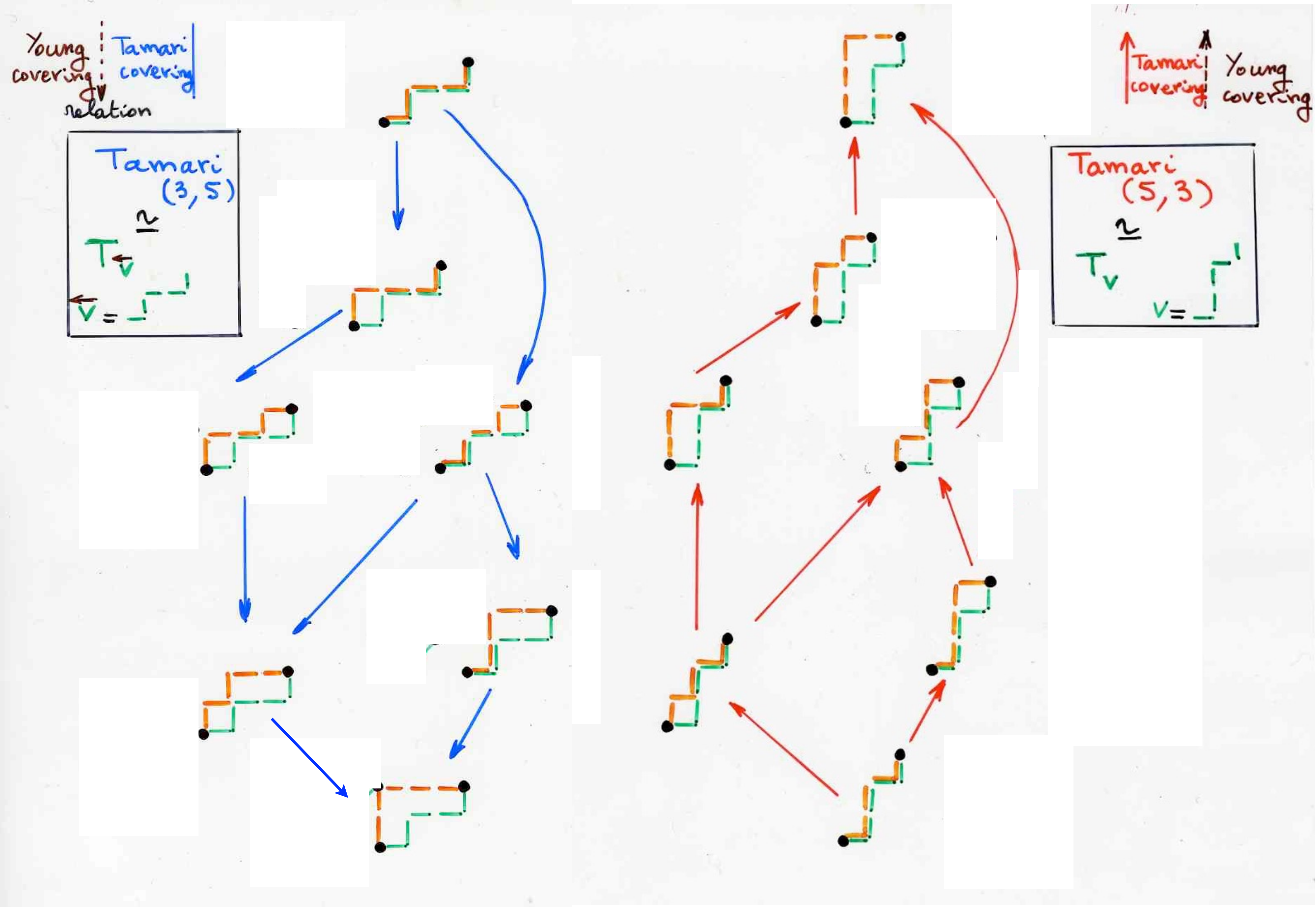
3 bijections
see previous talk maule I

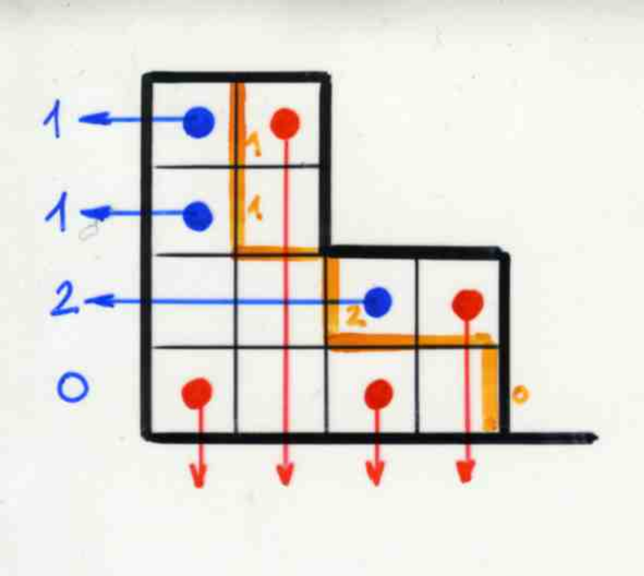


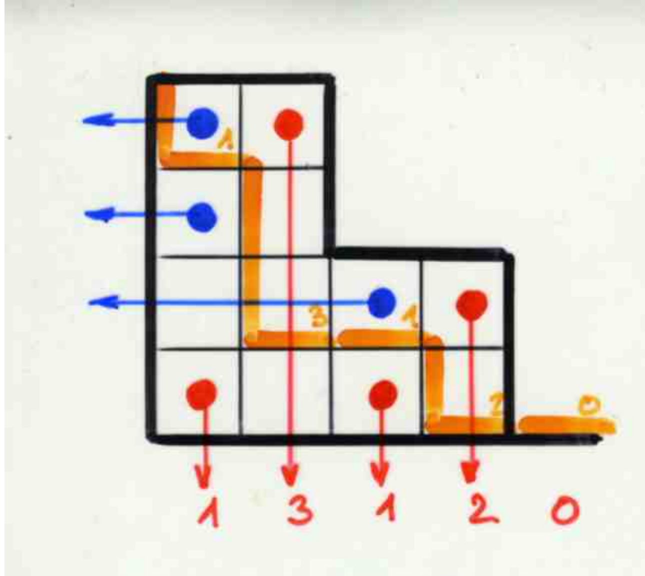
commutative diagram

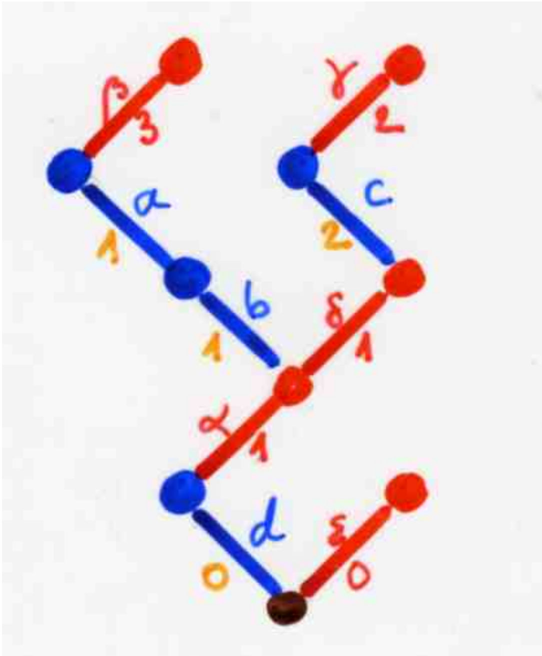

1
1
2
0
a
b
c
d

Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$









a	b	c	d
1	1	2	0

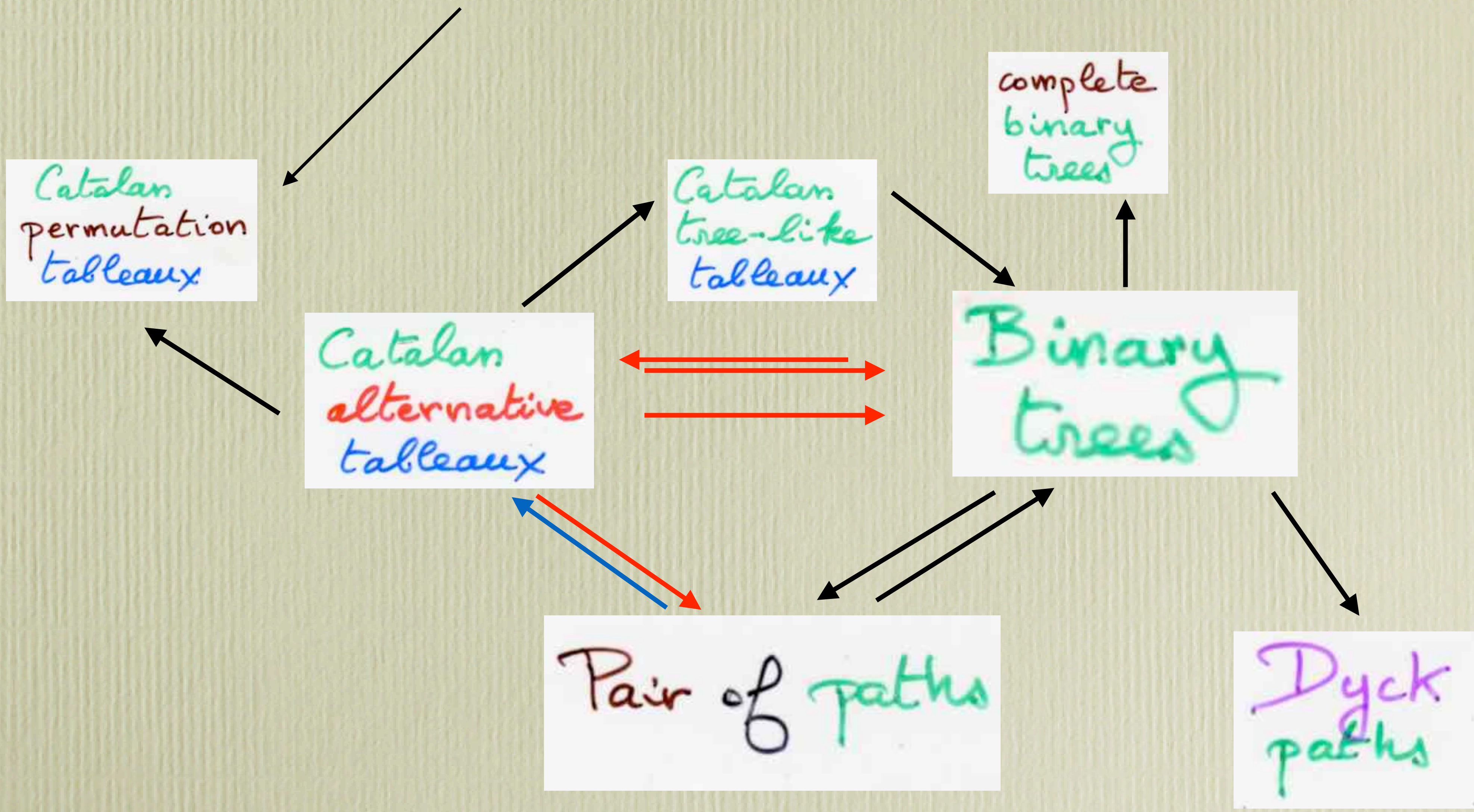
α	β	γ	δ	ε
1	3	1	2	0

right height of left edges

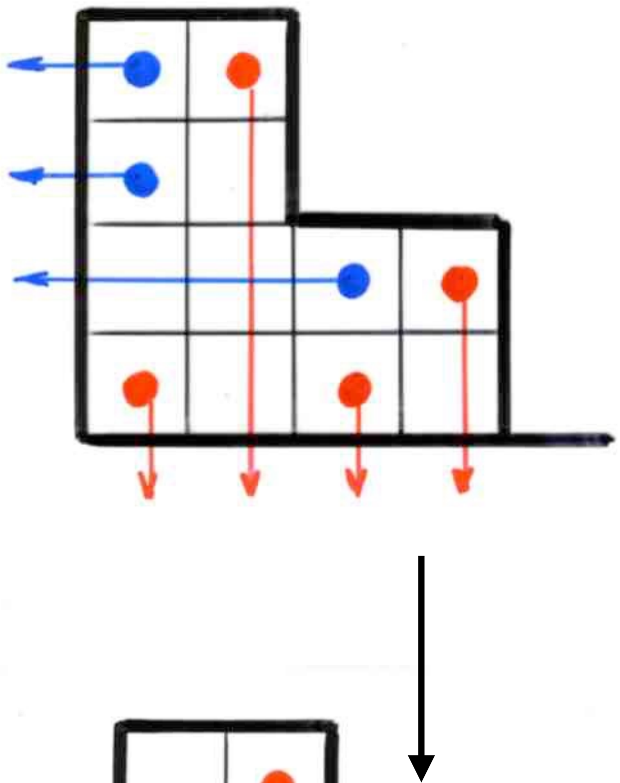
inorder (= symmetric order)

left height of right edges

see previous talk maule I

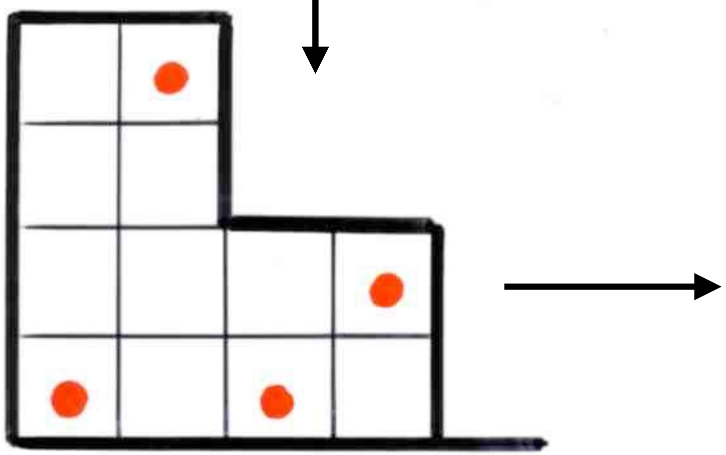


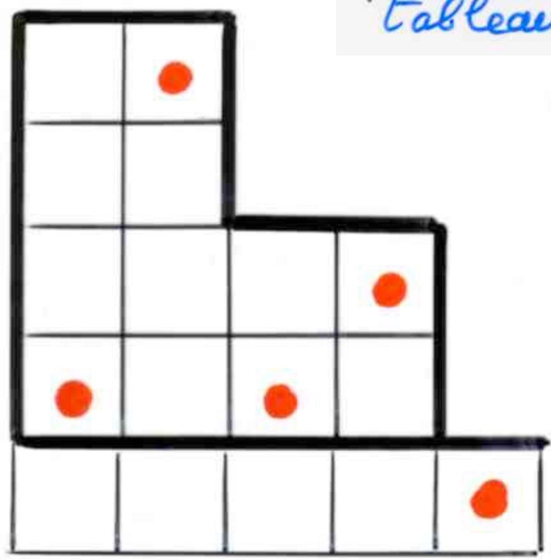
commutative diagram

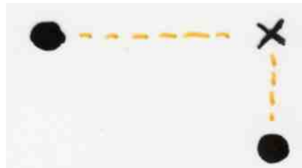


see previous talk maule I

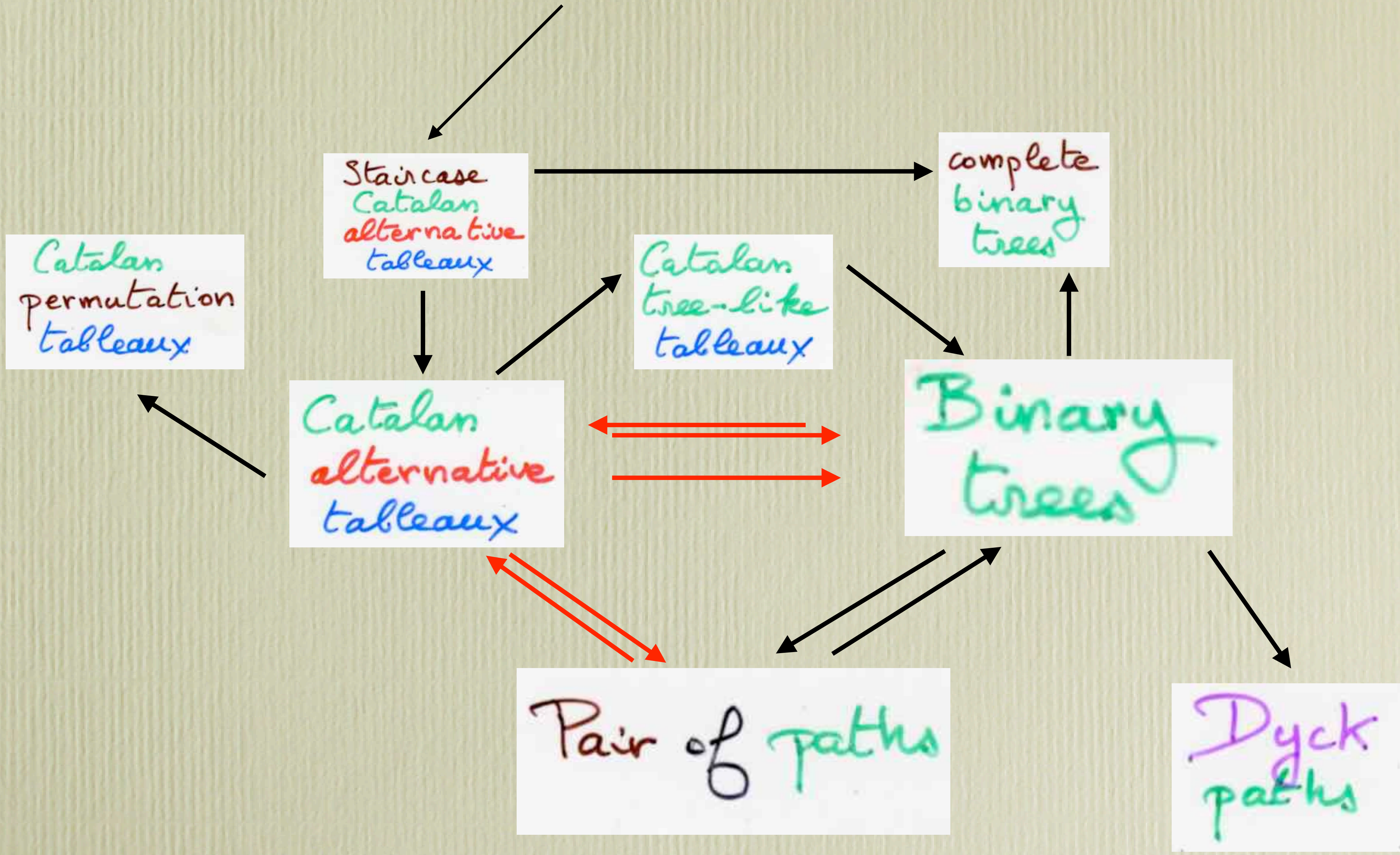
Catalan
permutation
tableaux





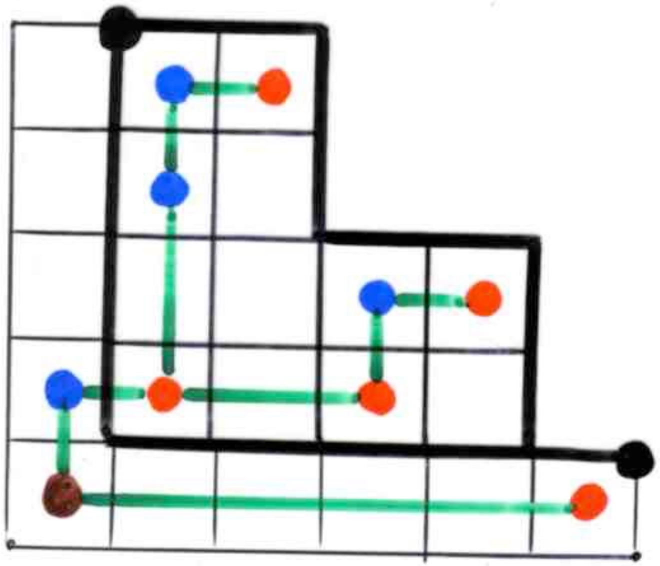


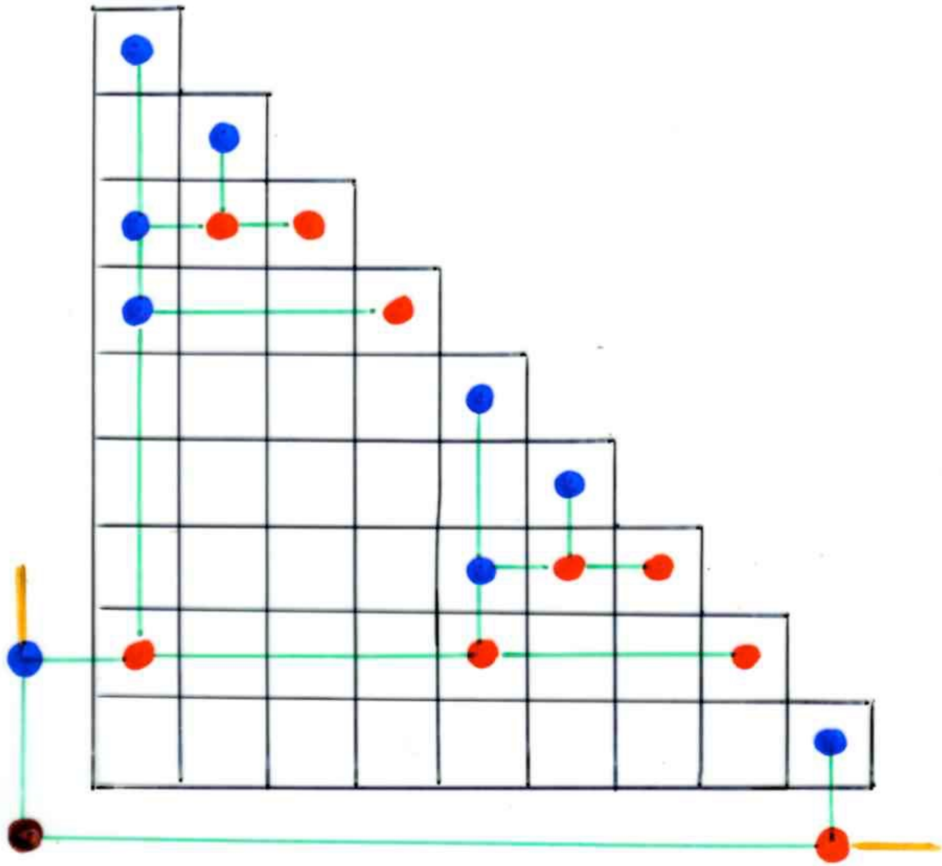
see previous talk maule I



commutative diagram

see previous talk maule I



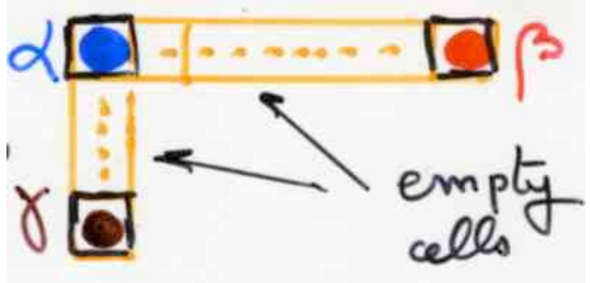


The lattice Tamari(v) is a maule

equivalence Gamma move
and
covering relation in Tamari(v)

Main Lemma

In a Catalan alternative tableau let α, β, γ be 3 colored cells in a Γ position (α is necessarily blue and γ red)



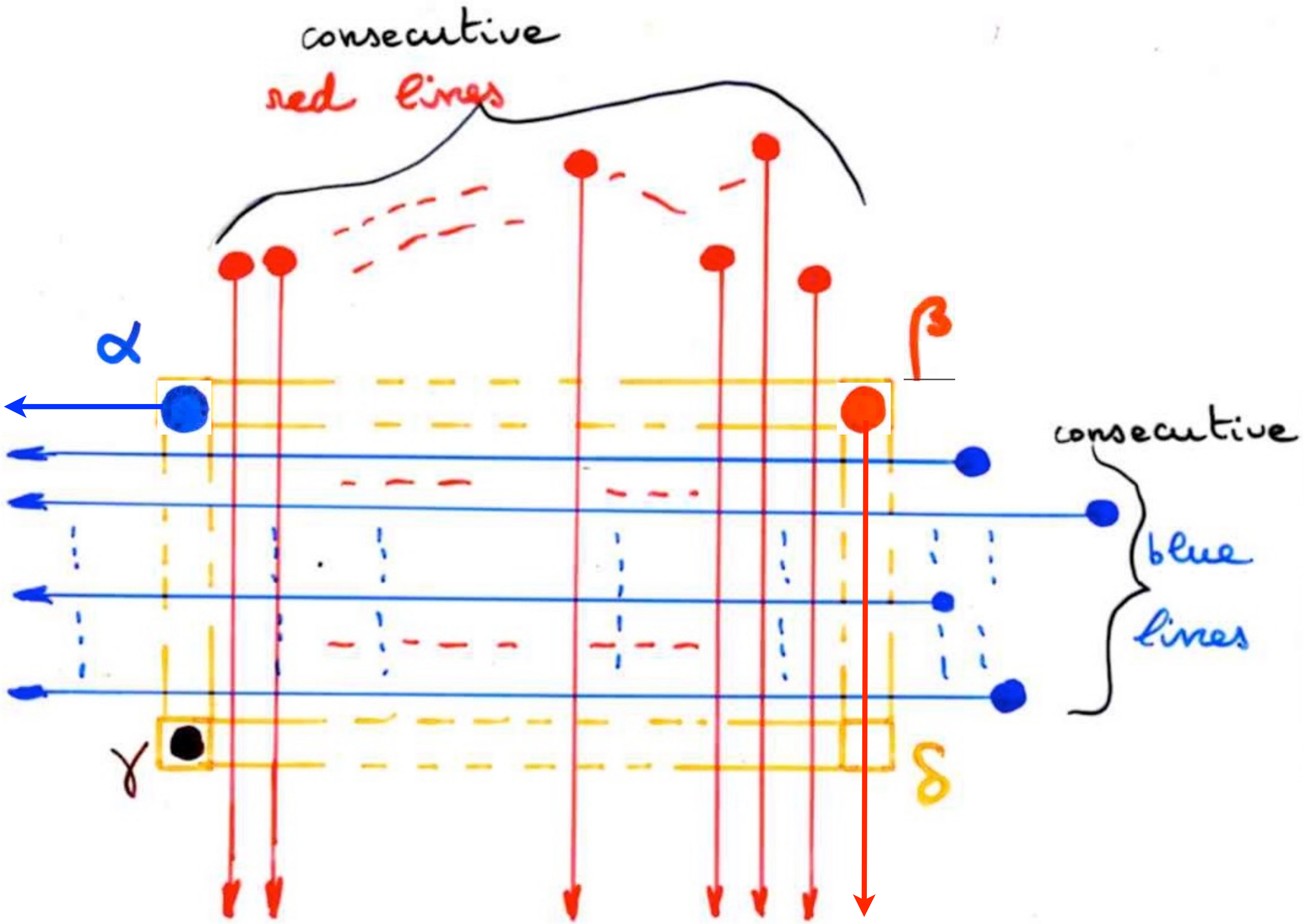
such that there is no colored cell between α and β and between α and γ.

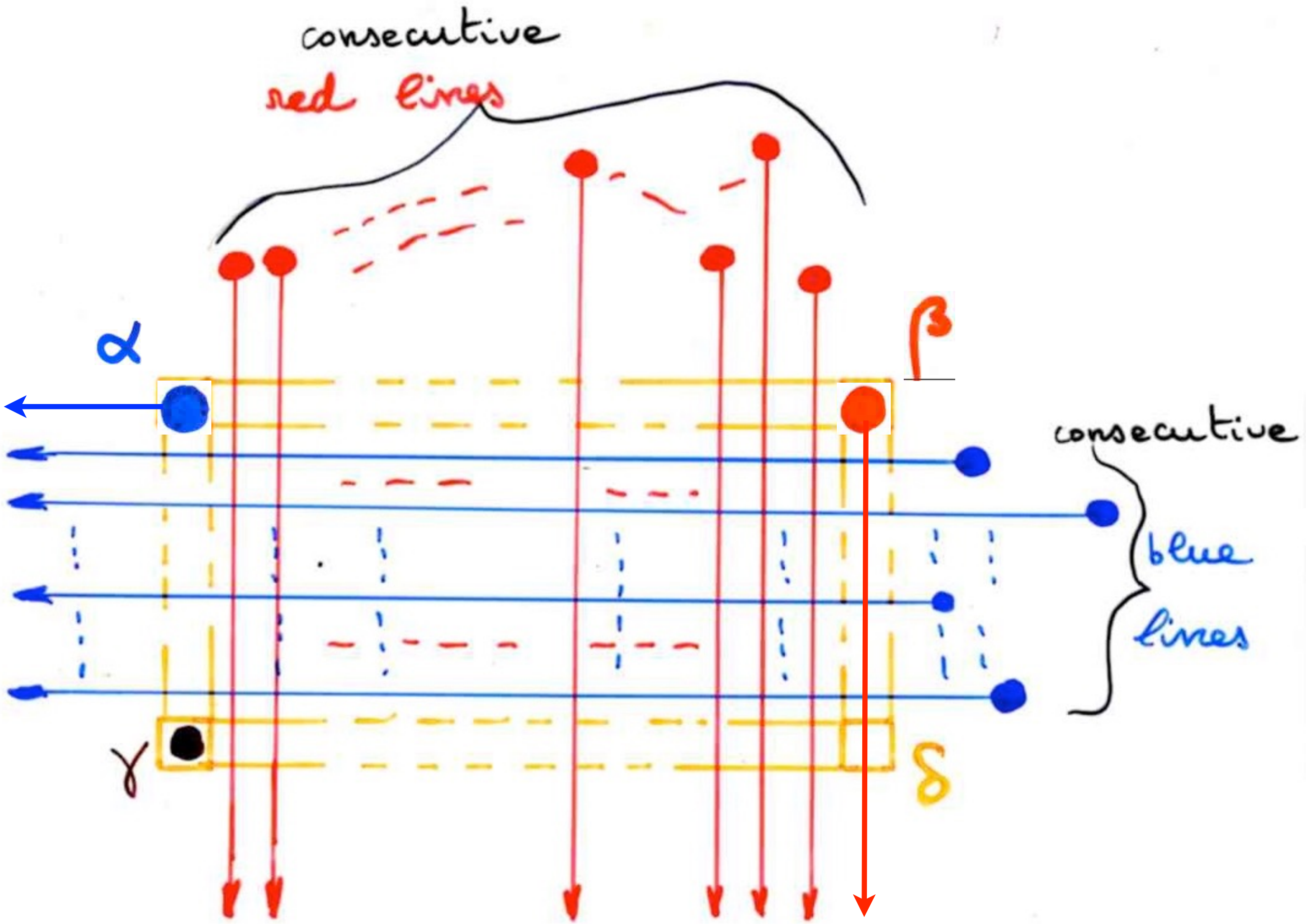
Then the cells of the whole rectangle

$$\begin{tikzcd} \alpha \arrow[r, dash] \arrow[d, dash] & \beta \arrow[d, dash] \\ \gamma \arrow[r, dash] & \delta \end{tikzcd}$$

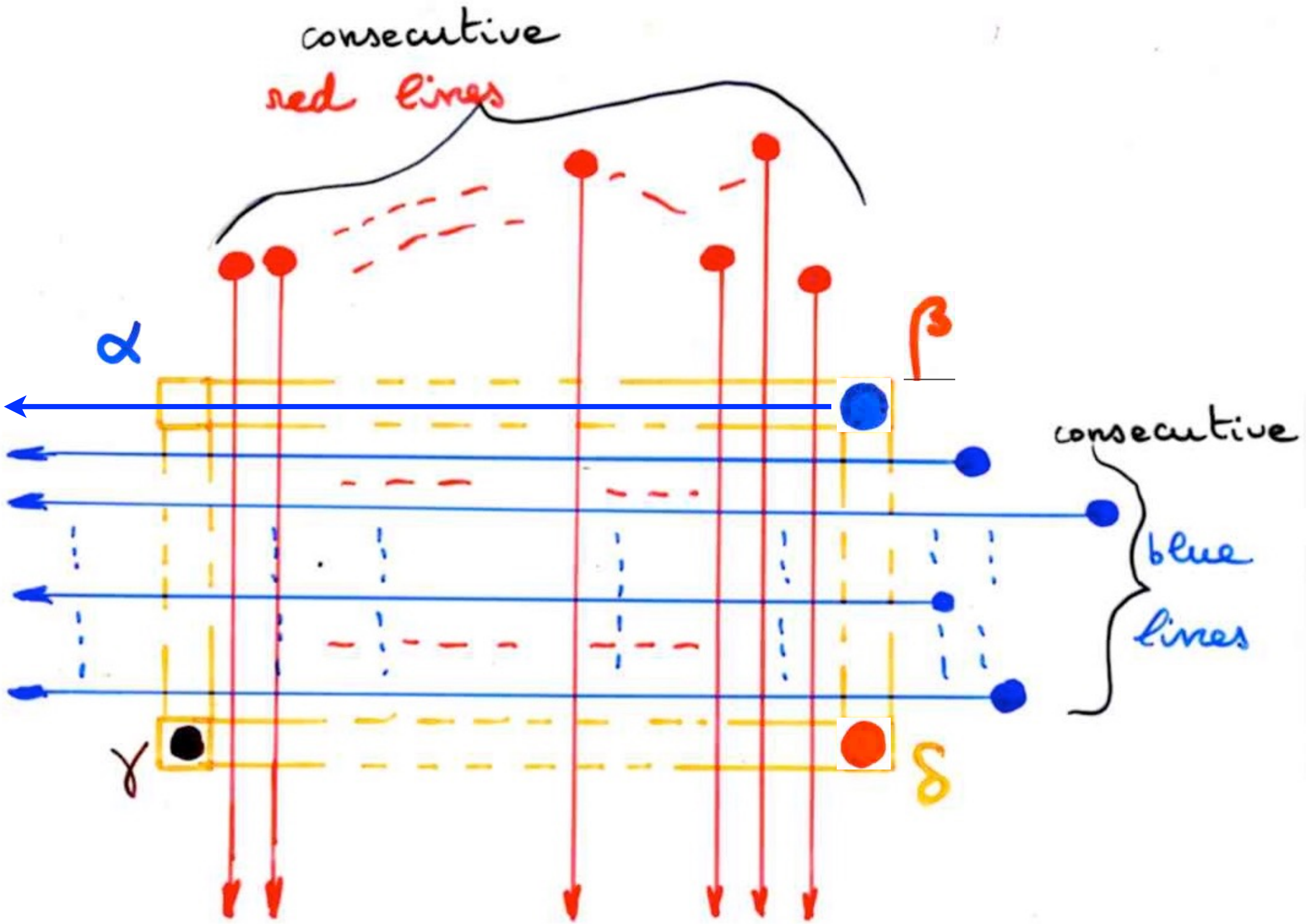
are empty (except α, β, γ)

Moreover we have the following configuration of blue cells and lines, with red cells and lines :

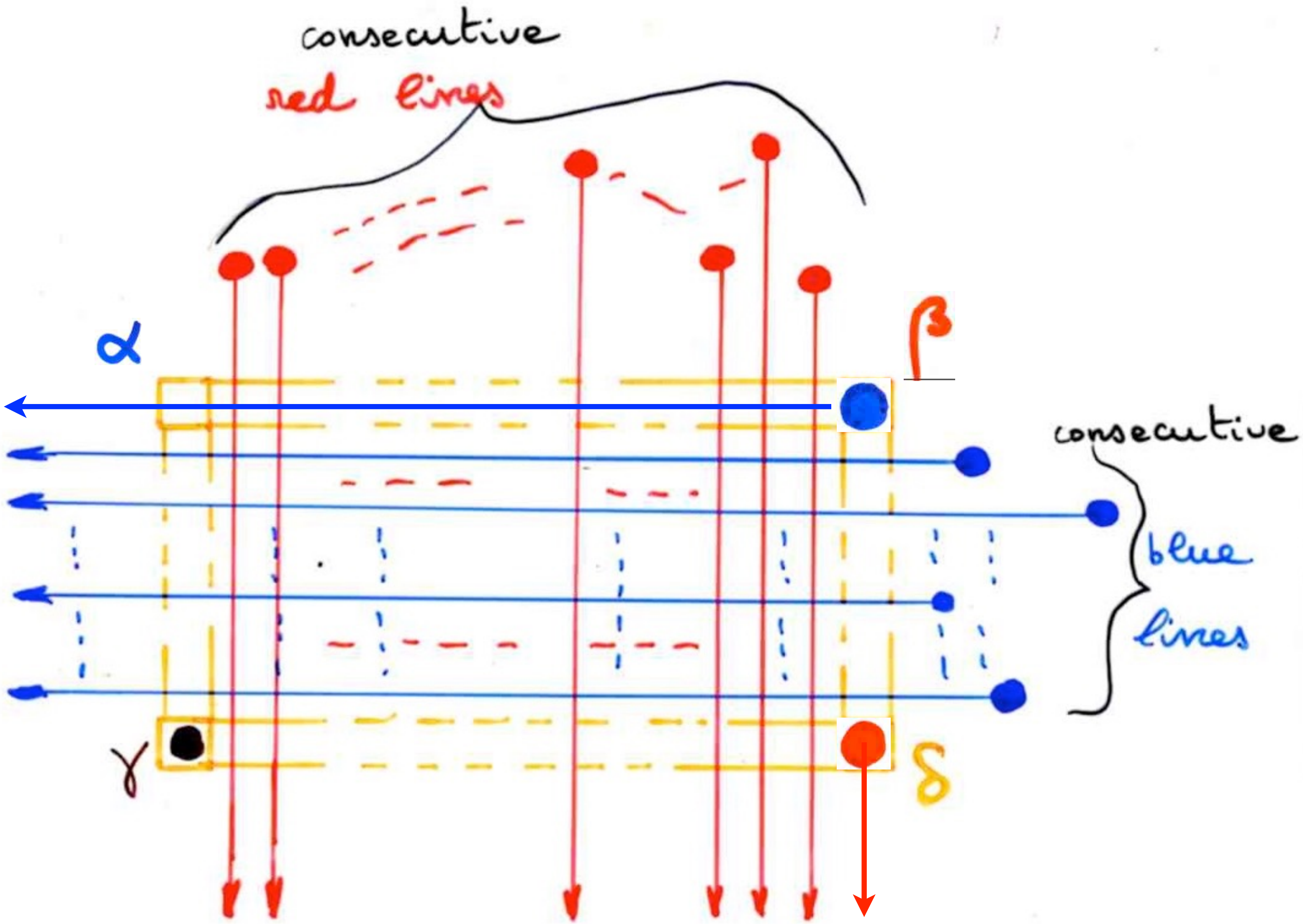

consecutive
red lines
α
β
consecutive
blue
lines
γ
δ

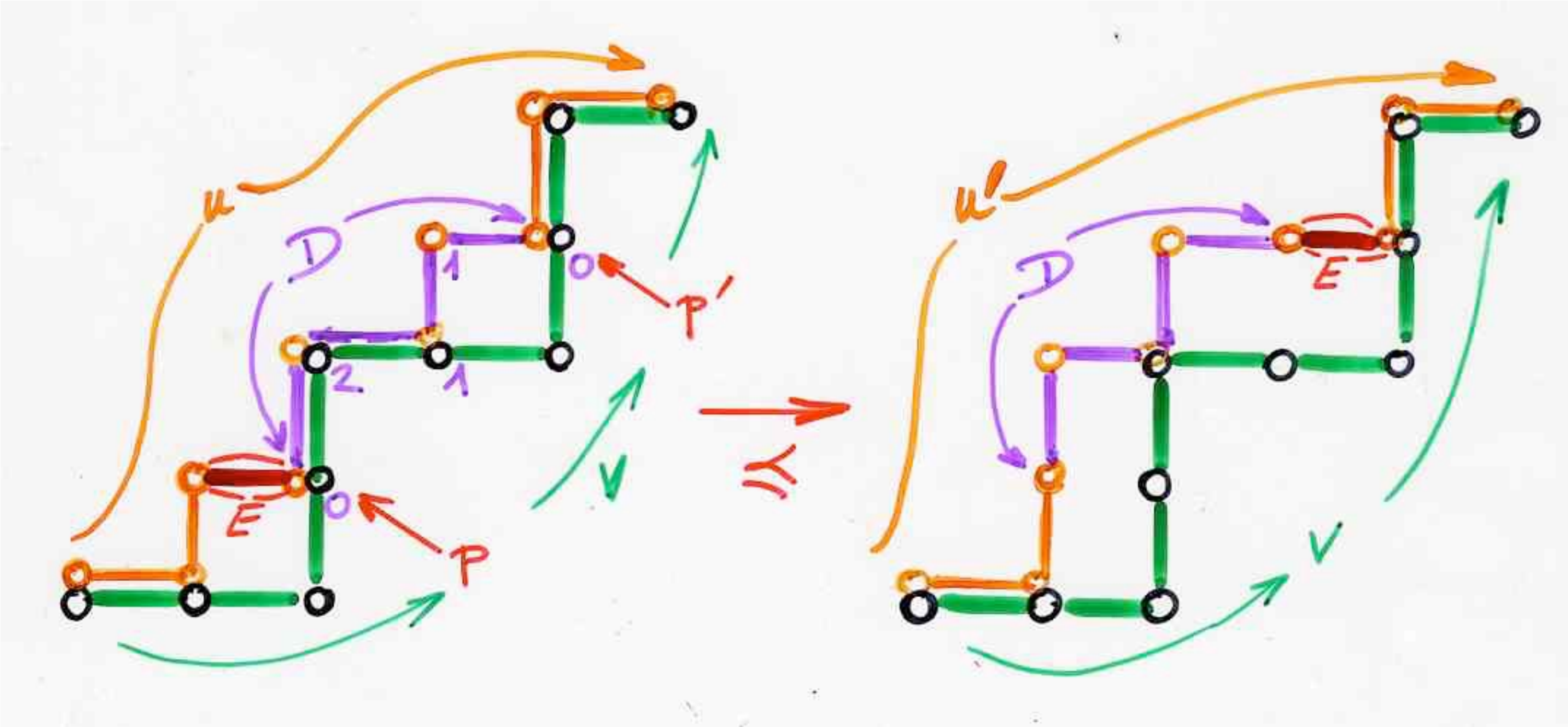


from the main Lemma, slides 121-122, part I
A possible Γ-move in a Catalan alternating tableau T

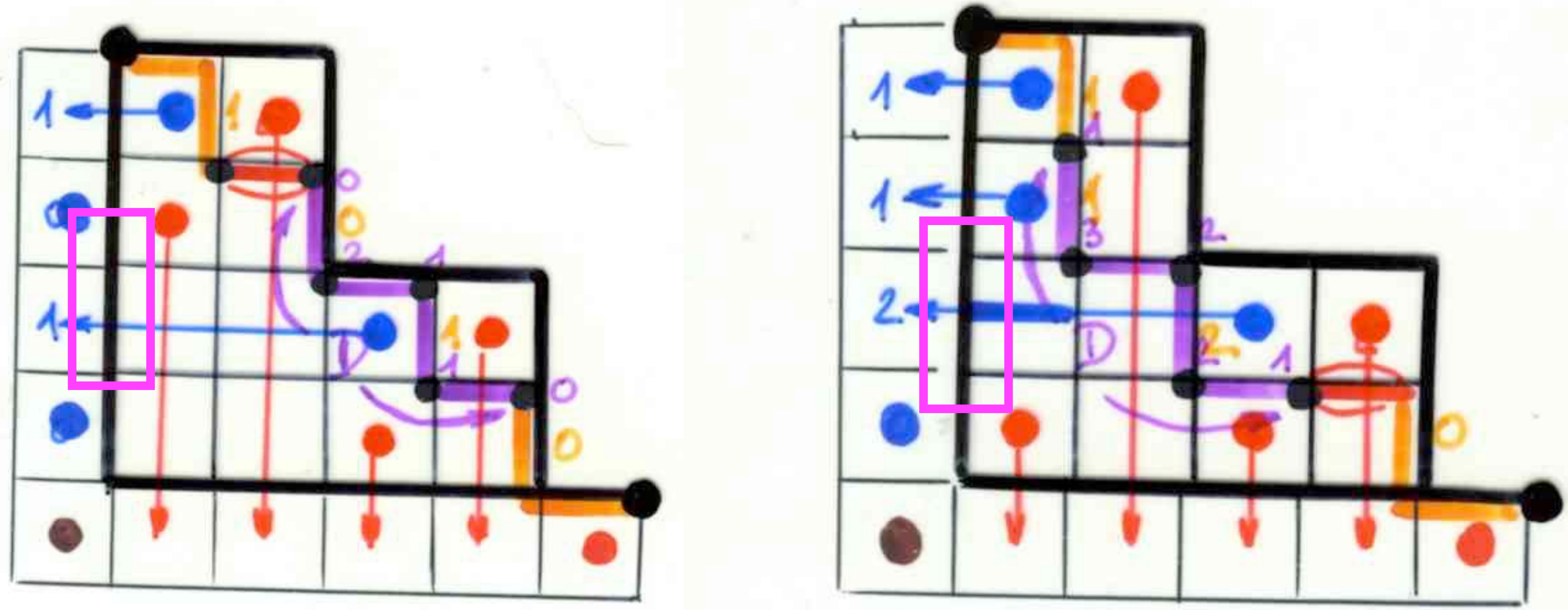


For a Γ-move in a Catalan alternating tableau T, the elements of the Adela row vector P (definition slide 39) will increase by one for all the rows of the rectangle defined by α, β, γ , δ (except the row γ δ). In all other rows, the coordinates will remain invariant.

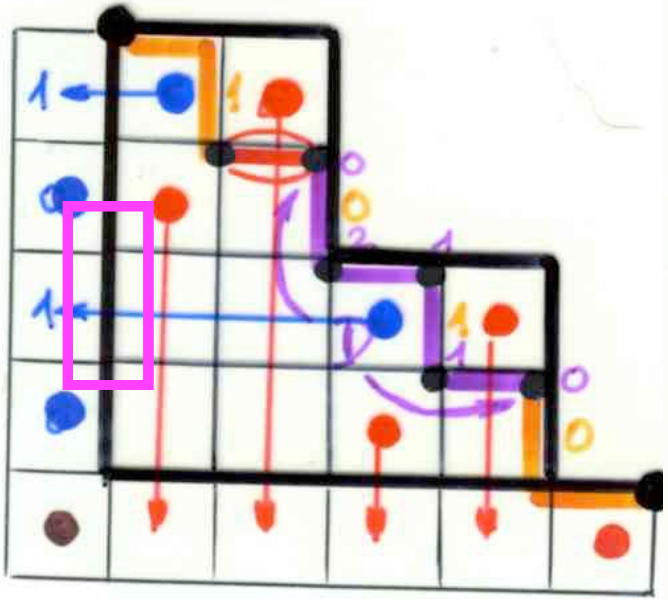

consecutive
red lines
consecutive
blue lines
α
β
γ
δ

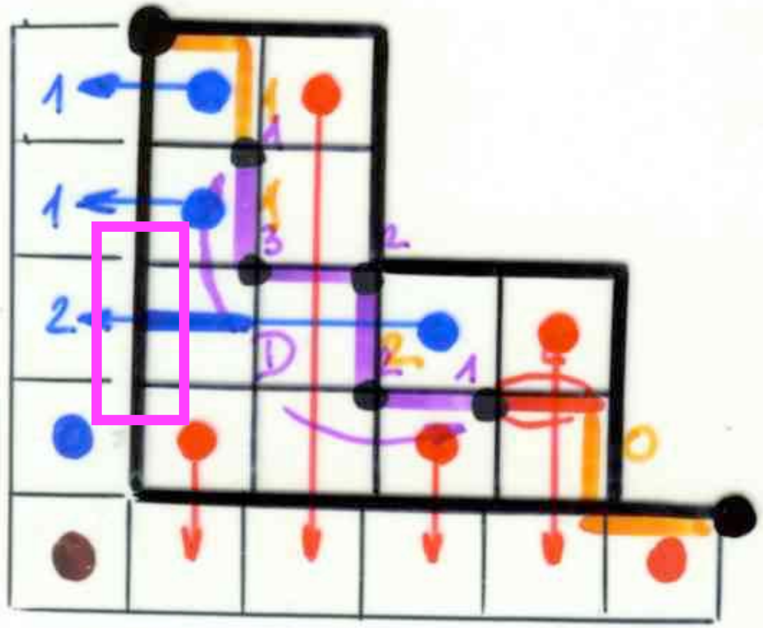


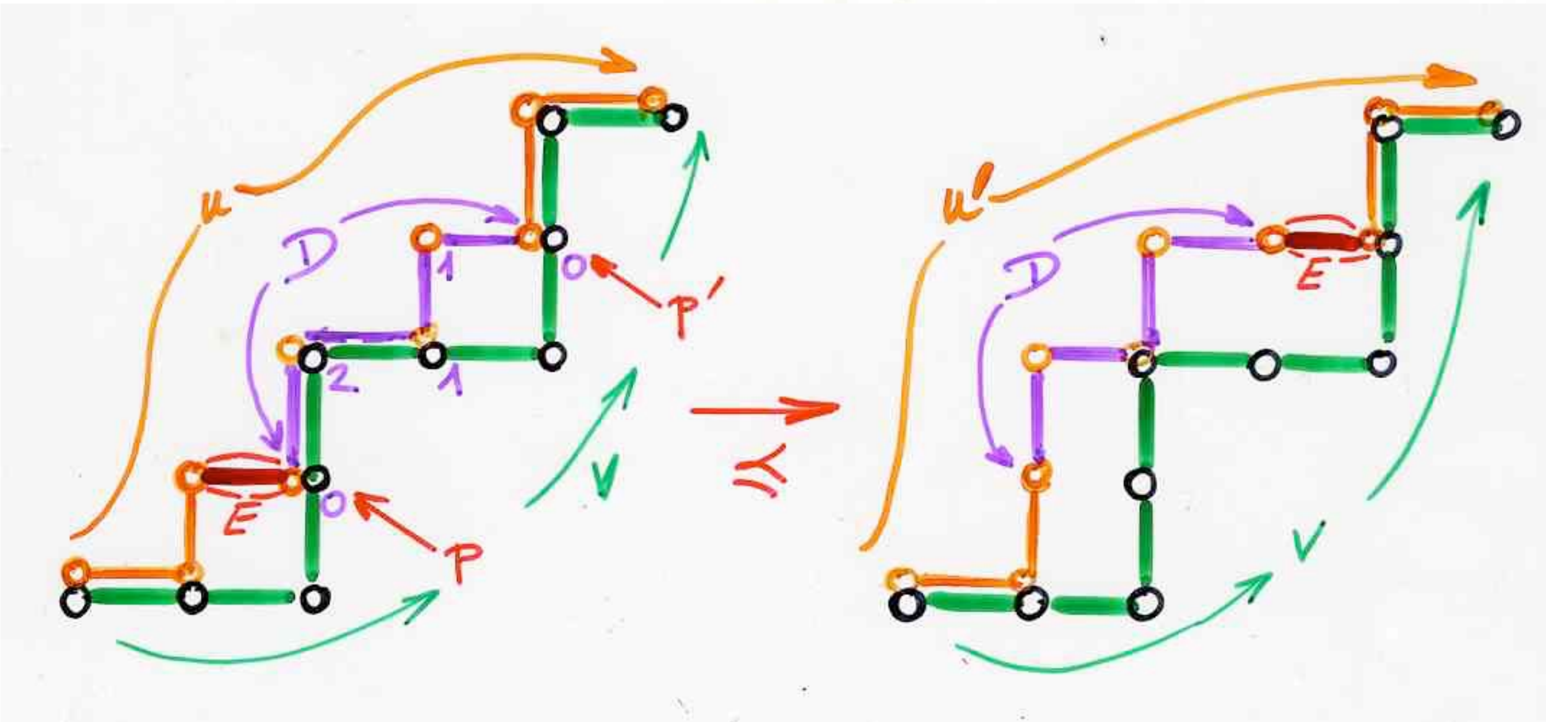
the covering relation
in the poset T_v



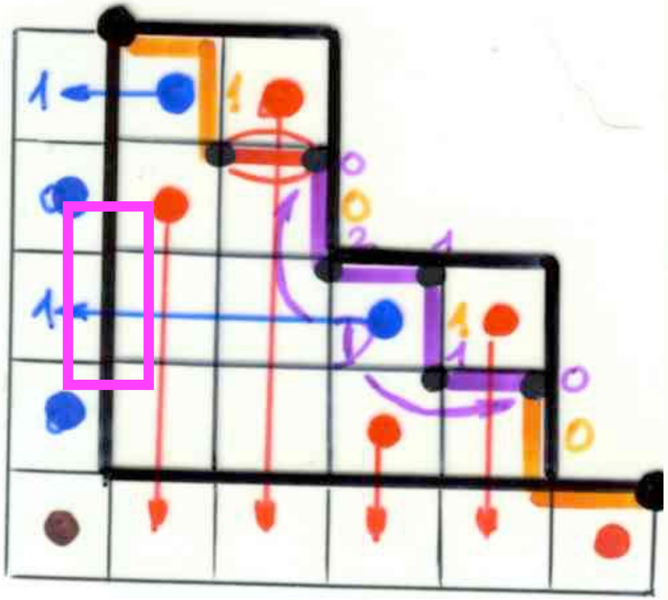
Such possible Γ-move in a Catalan alternating tableau T, related to the rectangle defined by α, β, γ , δ, corresponds exactly to a possible flip in the pair of paths (u,v). The rows of the rectangle α, β, γ , δ (except the row γ , δ) correspond to the North steps of the portion D of the path u (in purple on the figure)



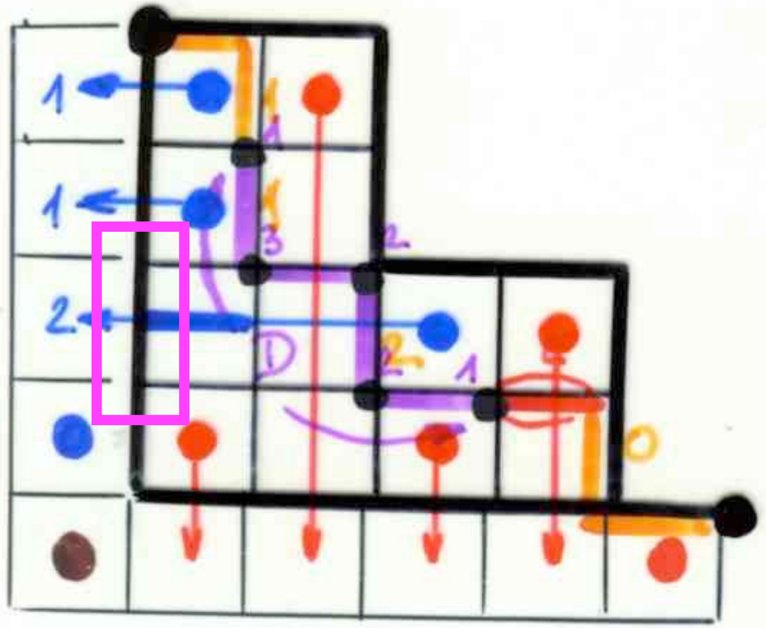




the covering relation in the poset T_v (also denoted by Tamari(v))



equivalence between a flip defining the covering relation of Tamari(v) and a Γ-move

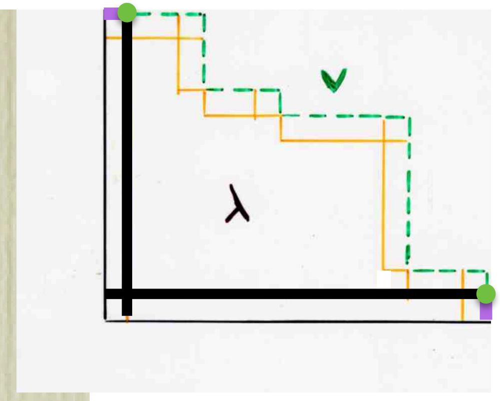


the covering relation in the poset T_v

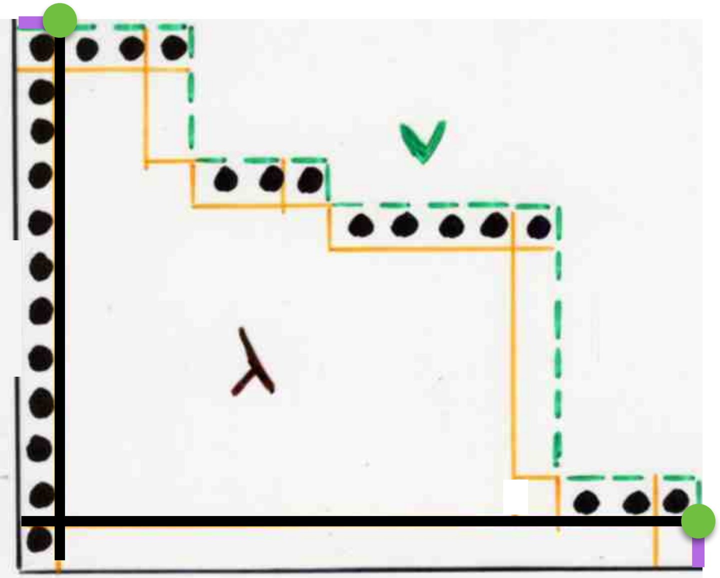
(also denoted by Tamari(v))

Main theorem

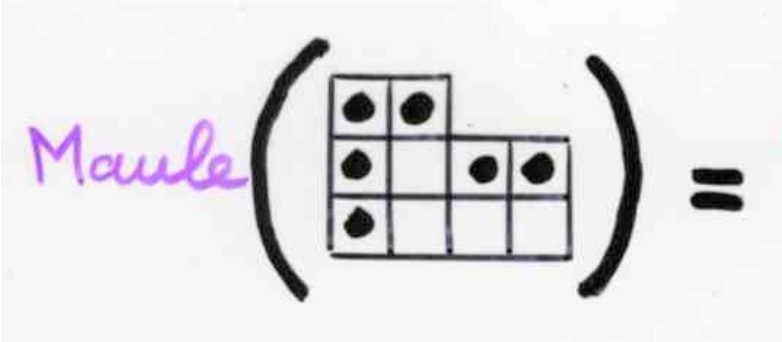
Ferrers diagram λ with profile ν

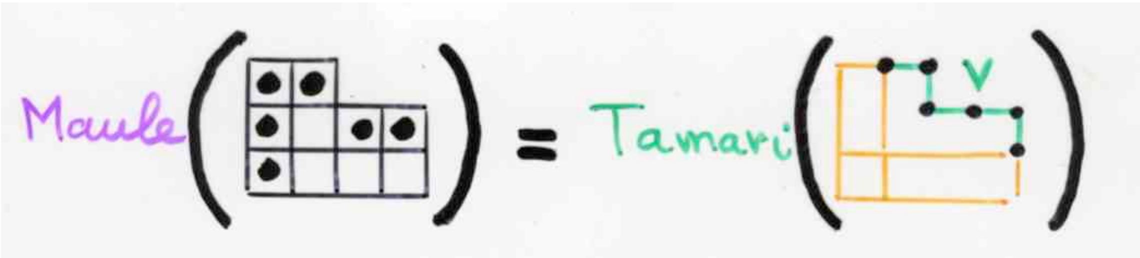


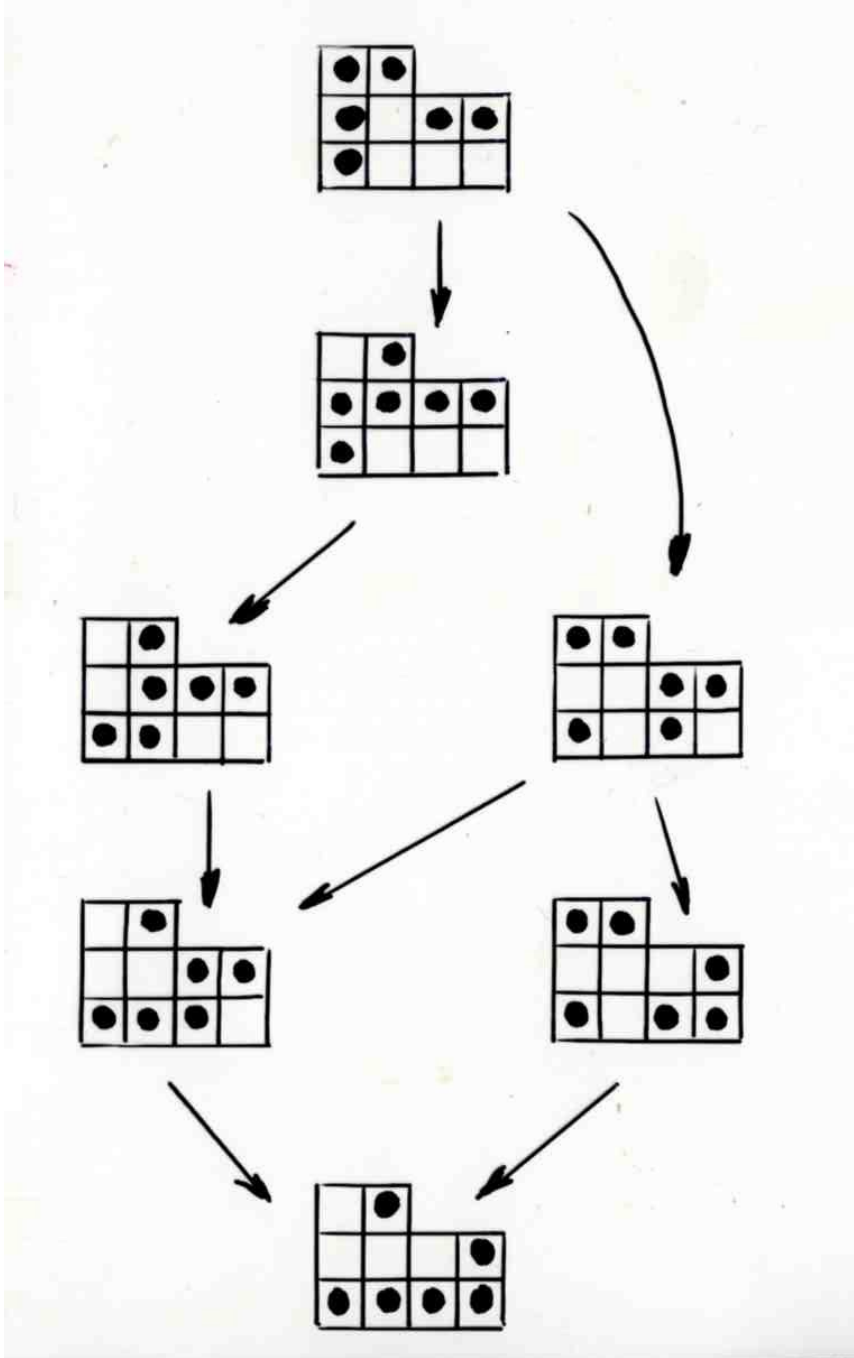
Let $X(\lambda) = X(\nu)$ be the cloud

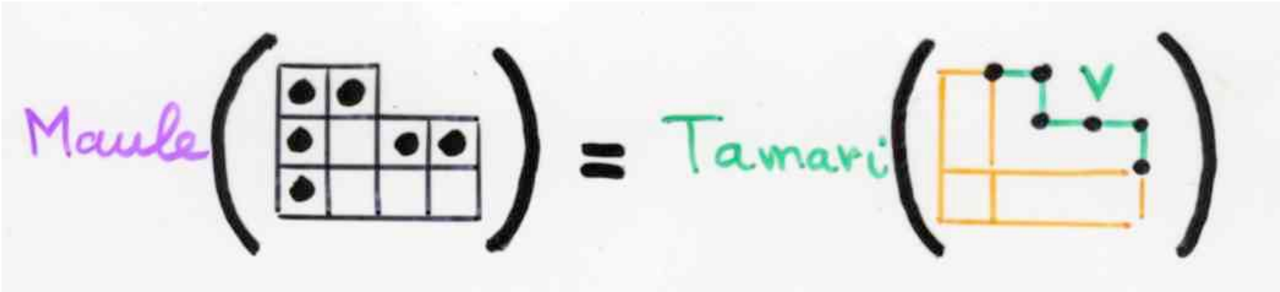


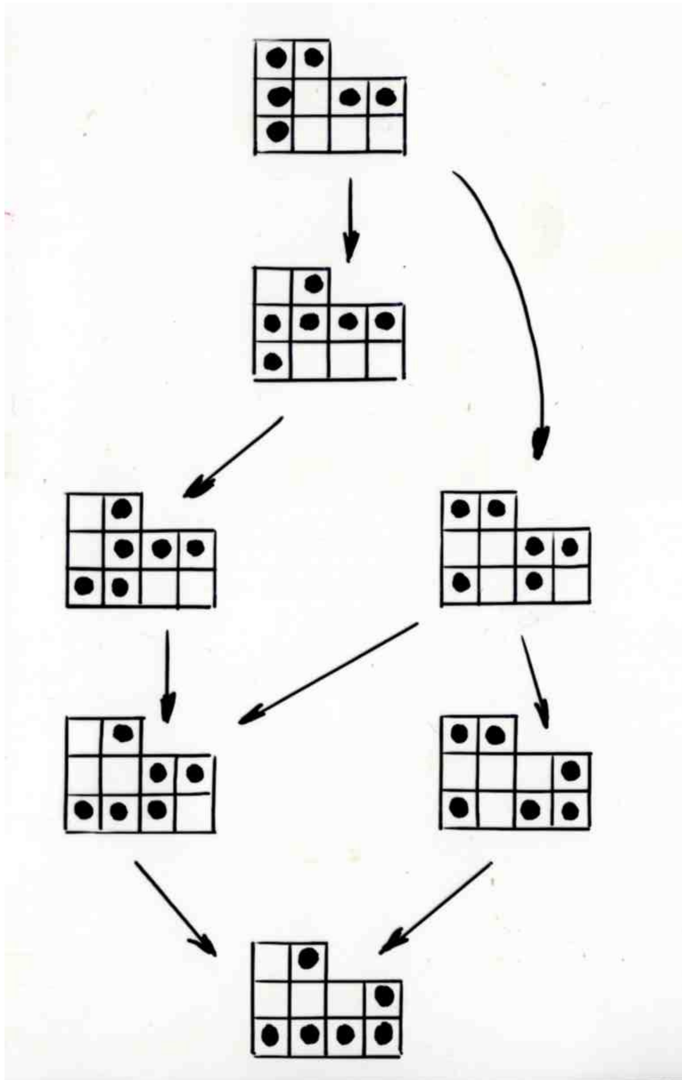
$$\text{Tamari}(\nu) = \text{Maule}(X(\nu))$$

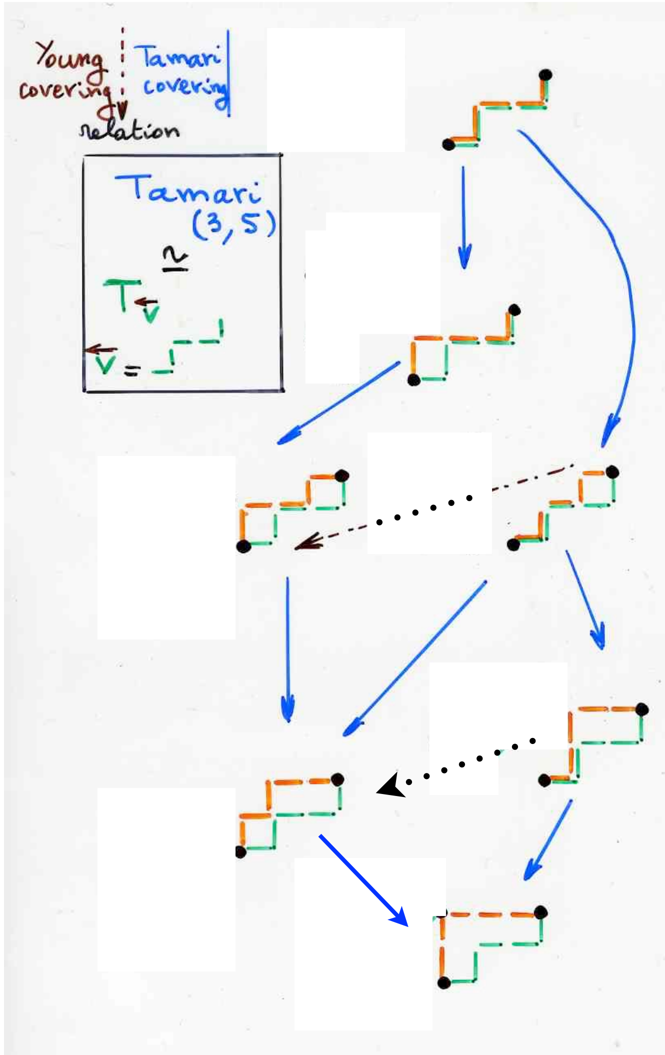

Maule

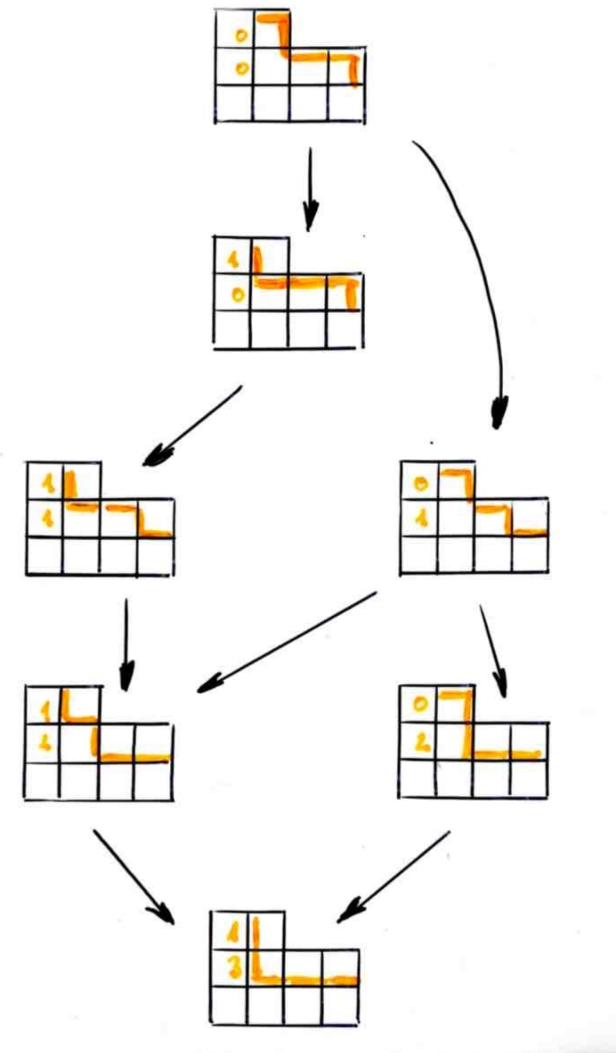

Maule
Tamari
V

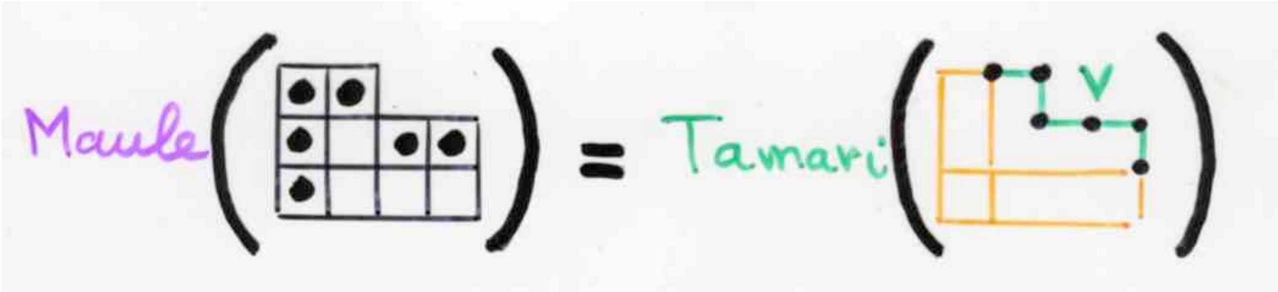


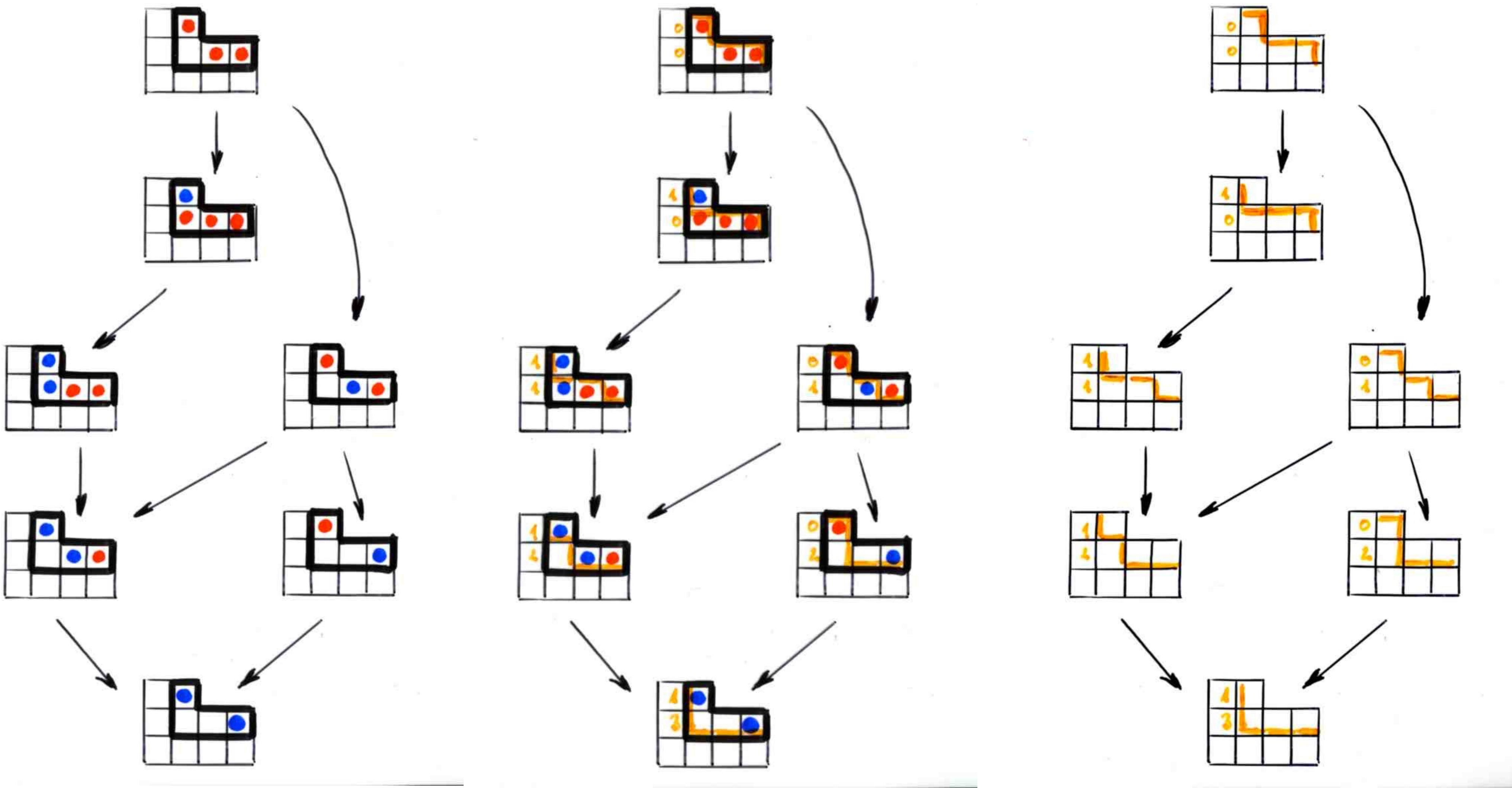

Maule
=
Tamari
v



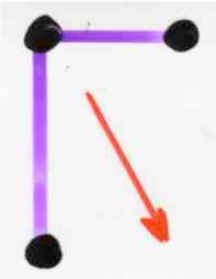

Young covering relation
Tamari covering
Tamari (3,5)
≅
$T_{\overleftarrow{v}}$
$\overleftarrow{v}$ =



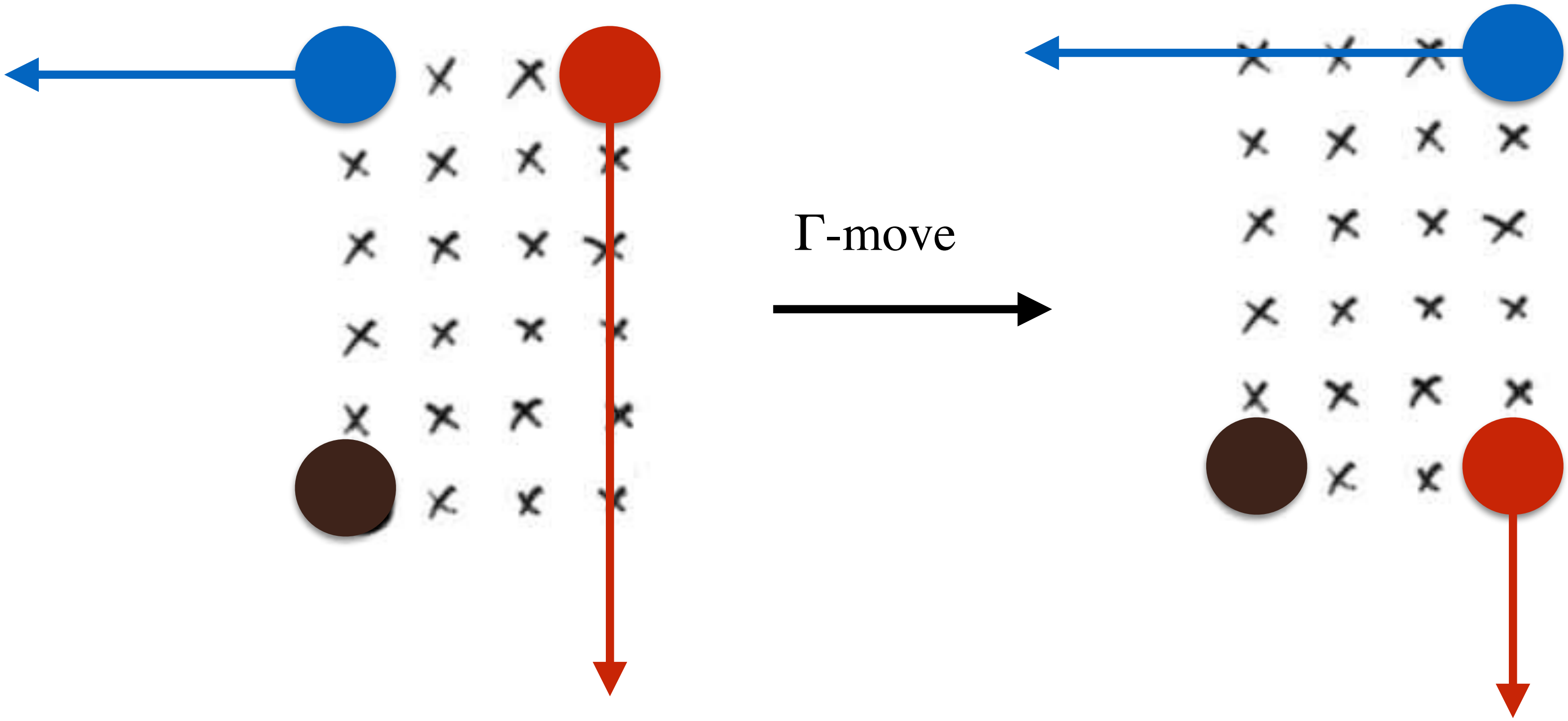

Maule
=
Tamari
v



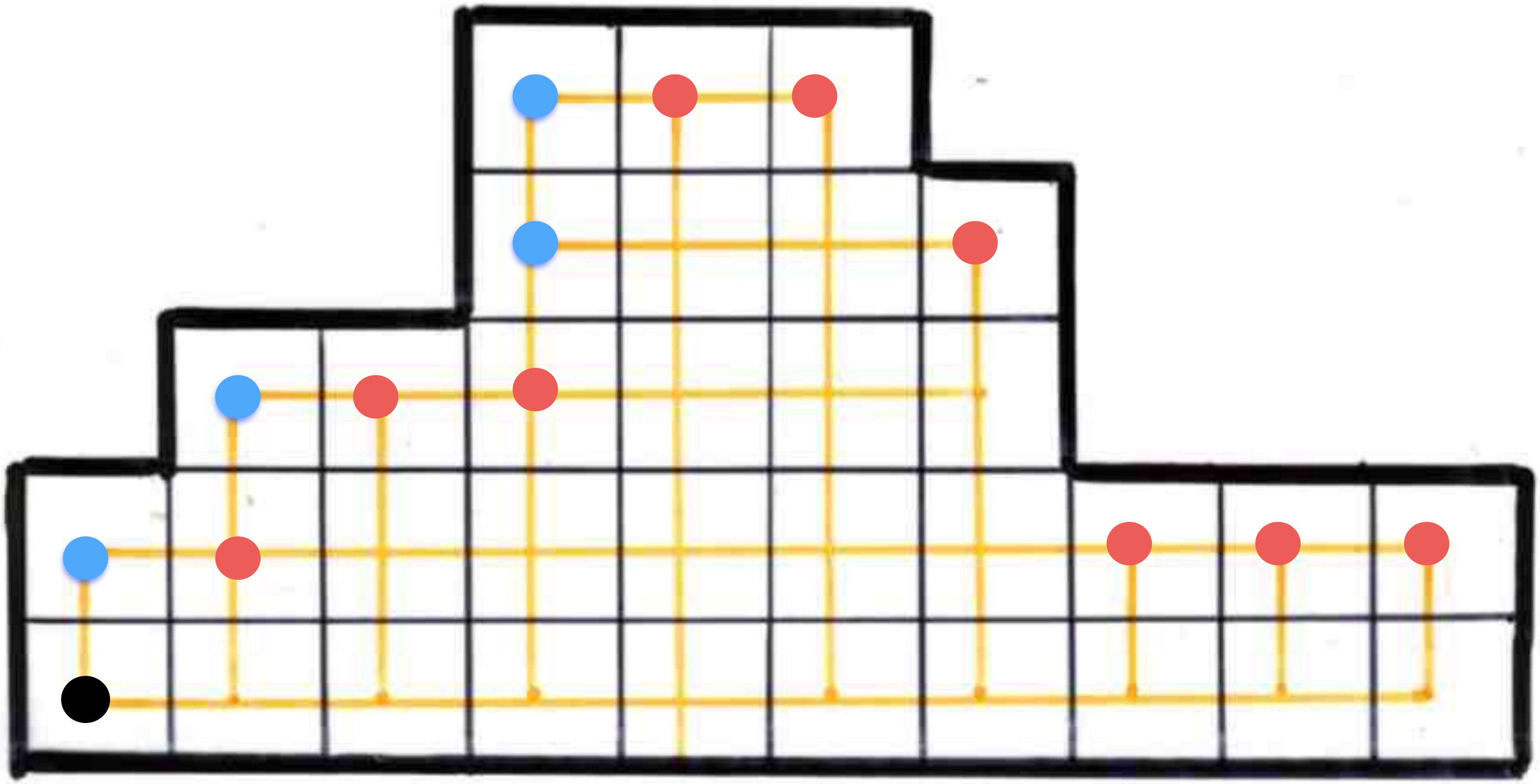
A mixture of Young Y(u) lattice
and
Tamari(v) lattice

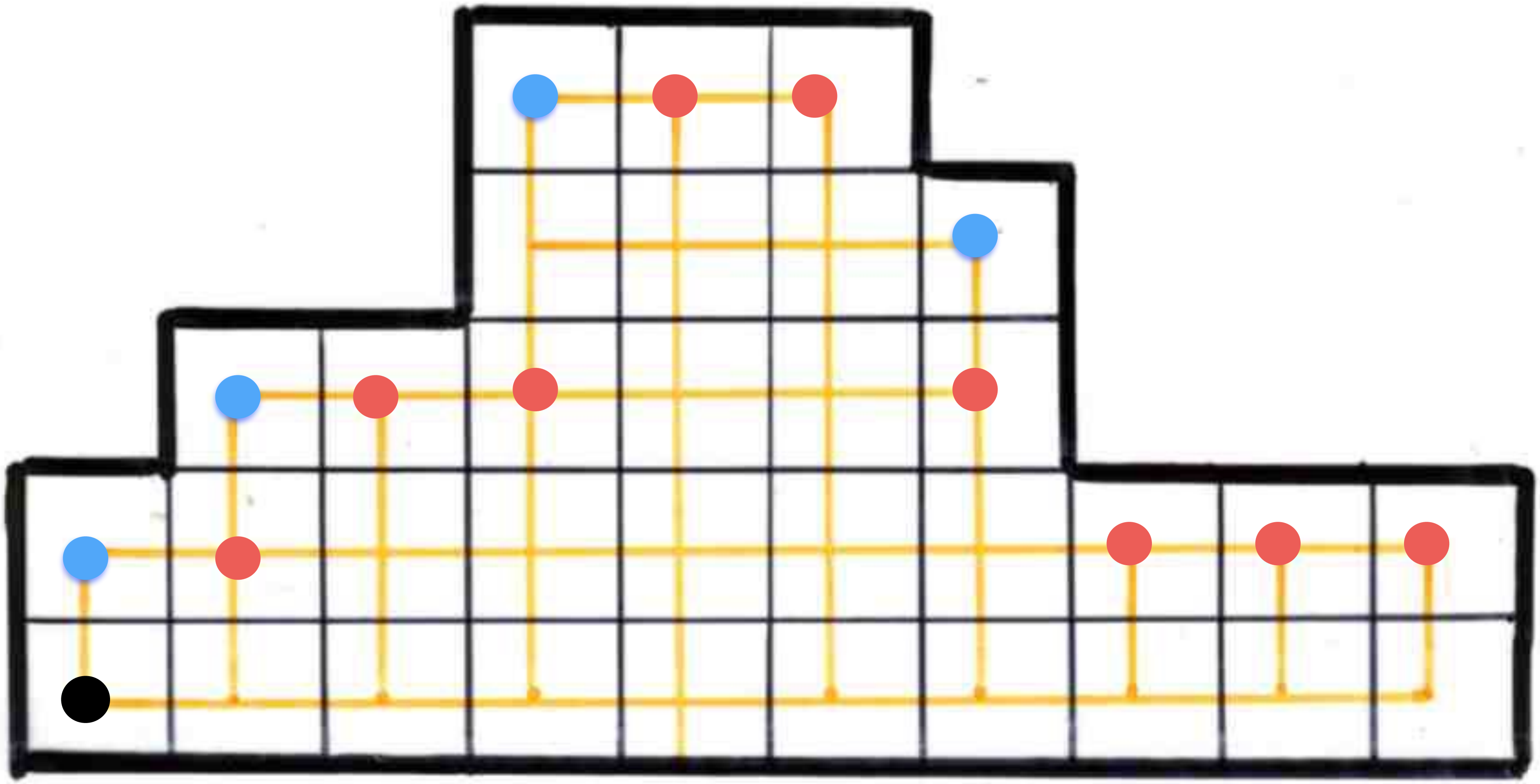


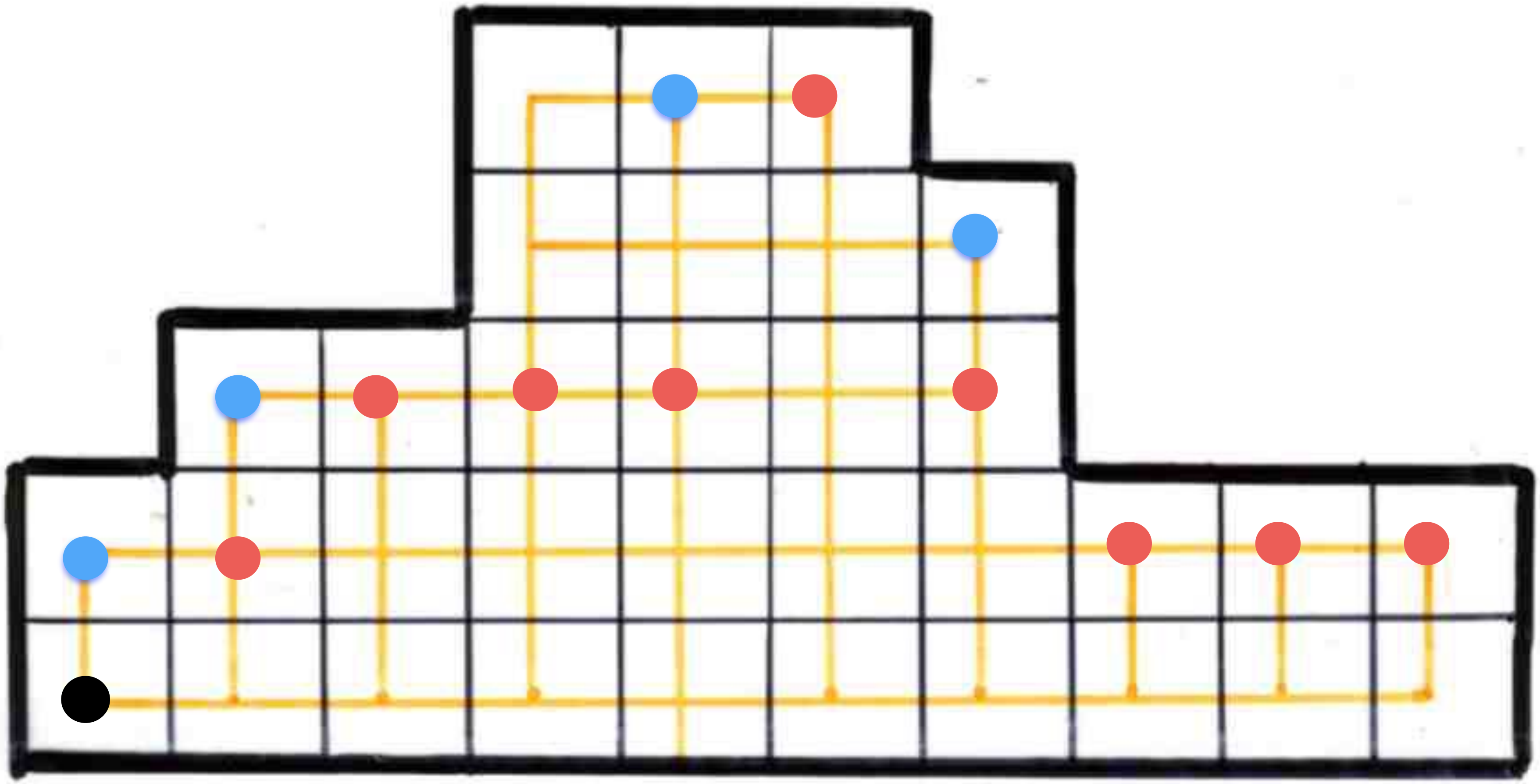
When the elements of the cloud X can be coloured in two colors blue and red satisfying the conditions defining the alternative tableaux, instead of seeing a Γ-move as the jump of a single particle, we can see it as the movement of two particles, a blue going to the right and a red going down (as on slides 156-157, part I and 118-119, part II)

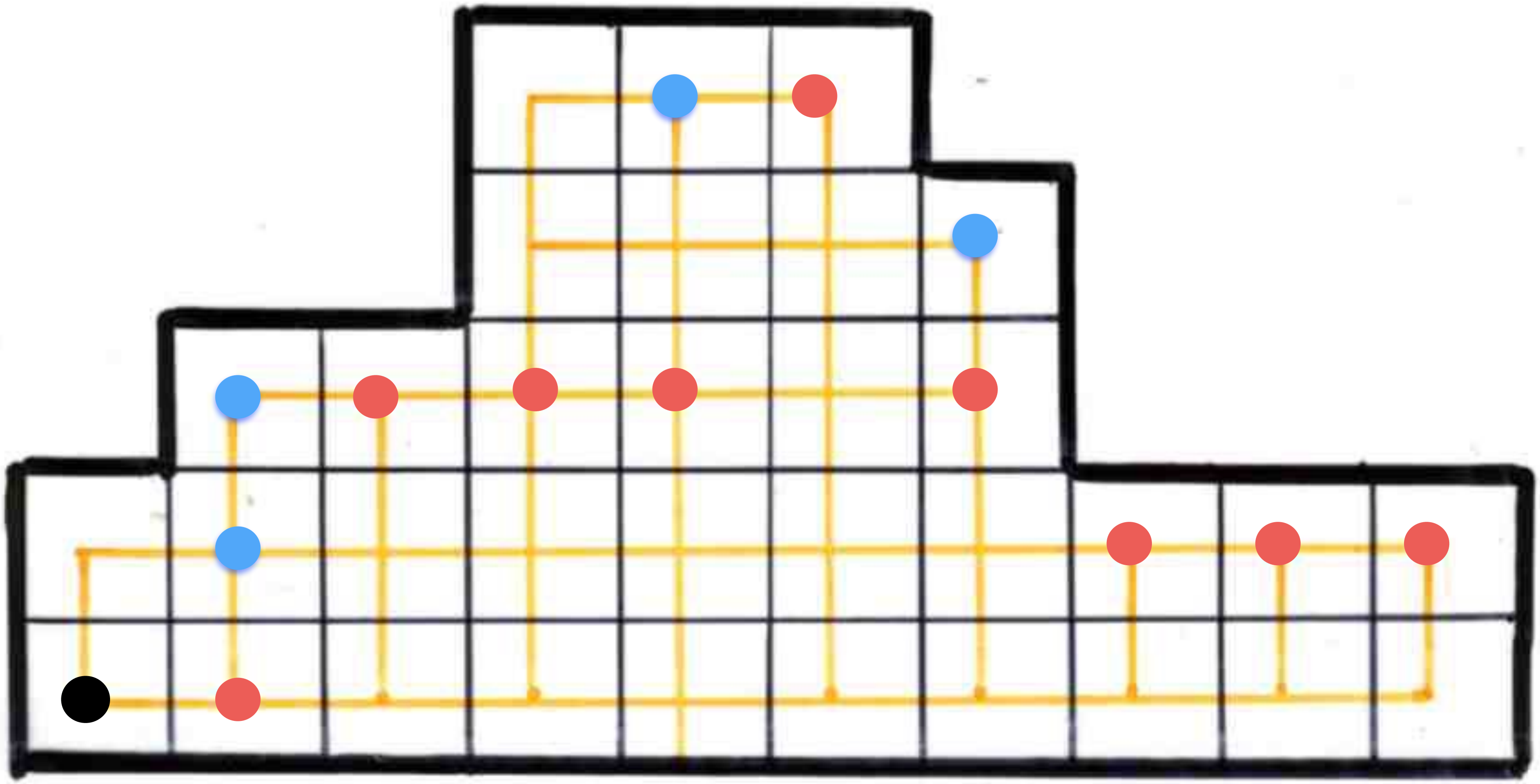


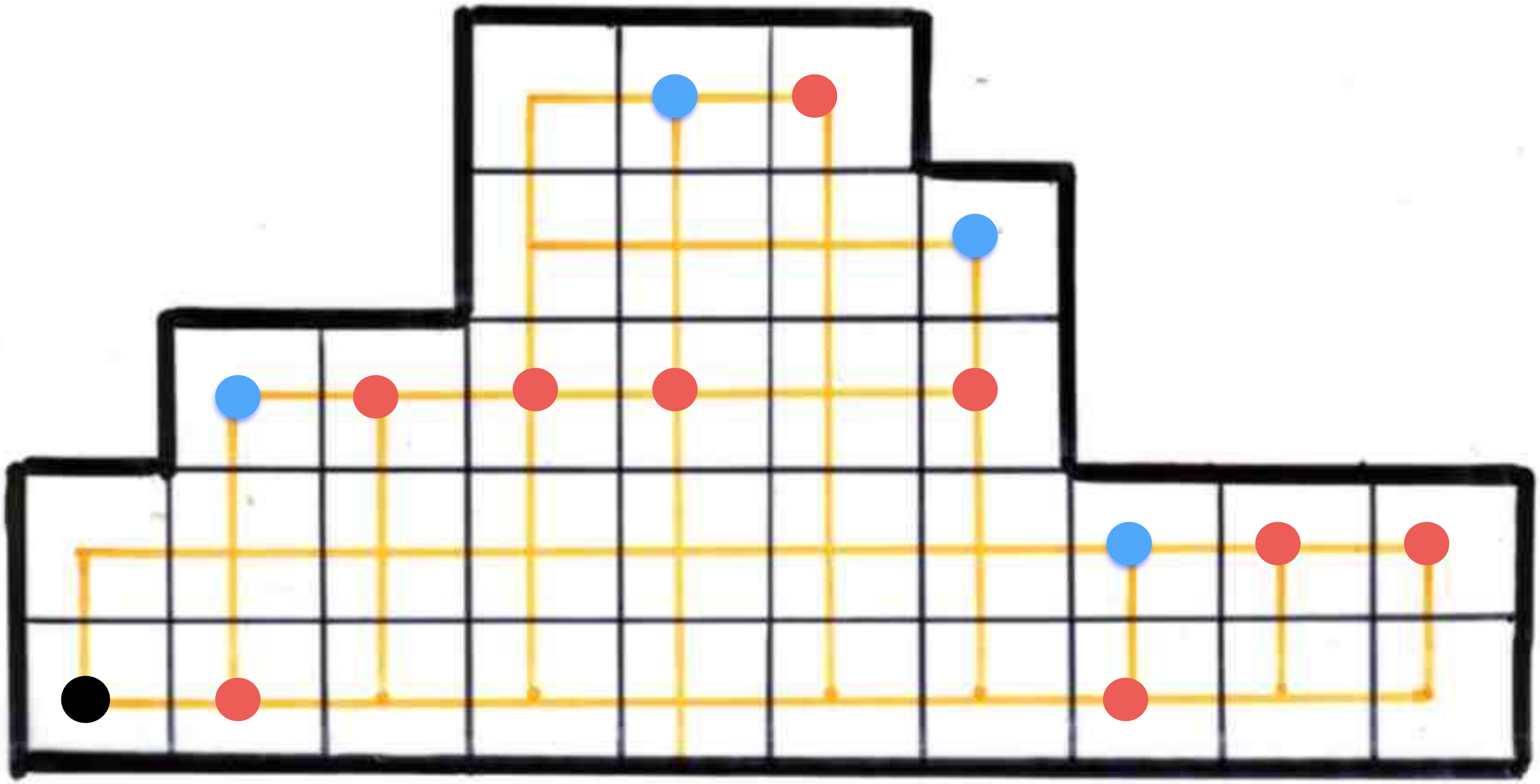
This is what we do in the following sequence of Γ-moves.

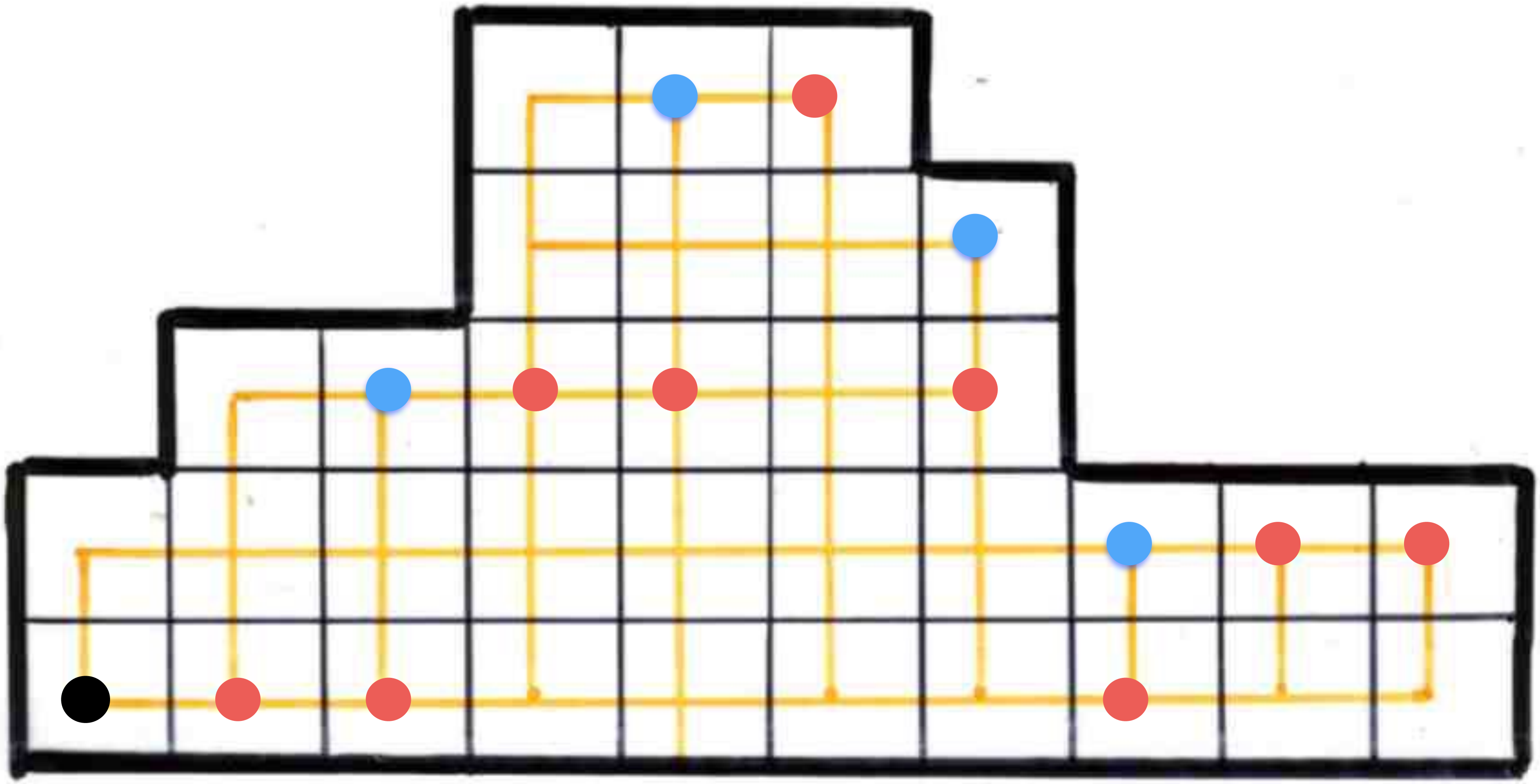


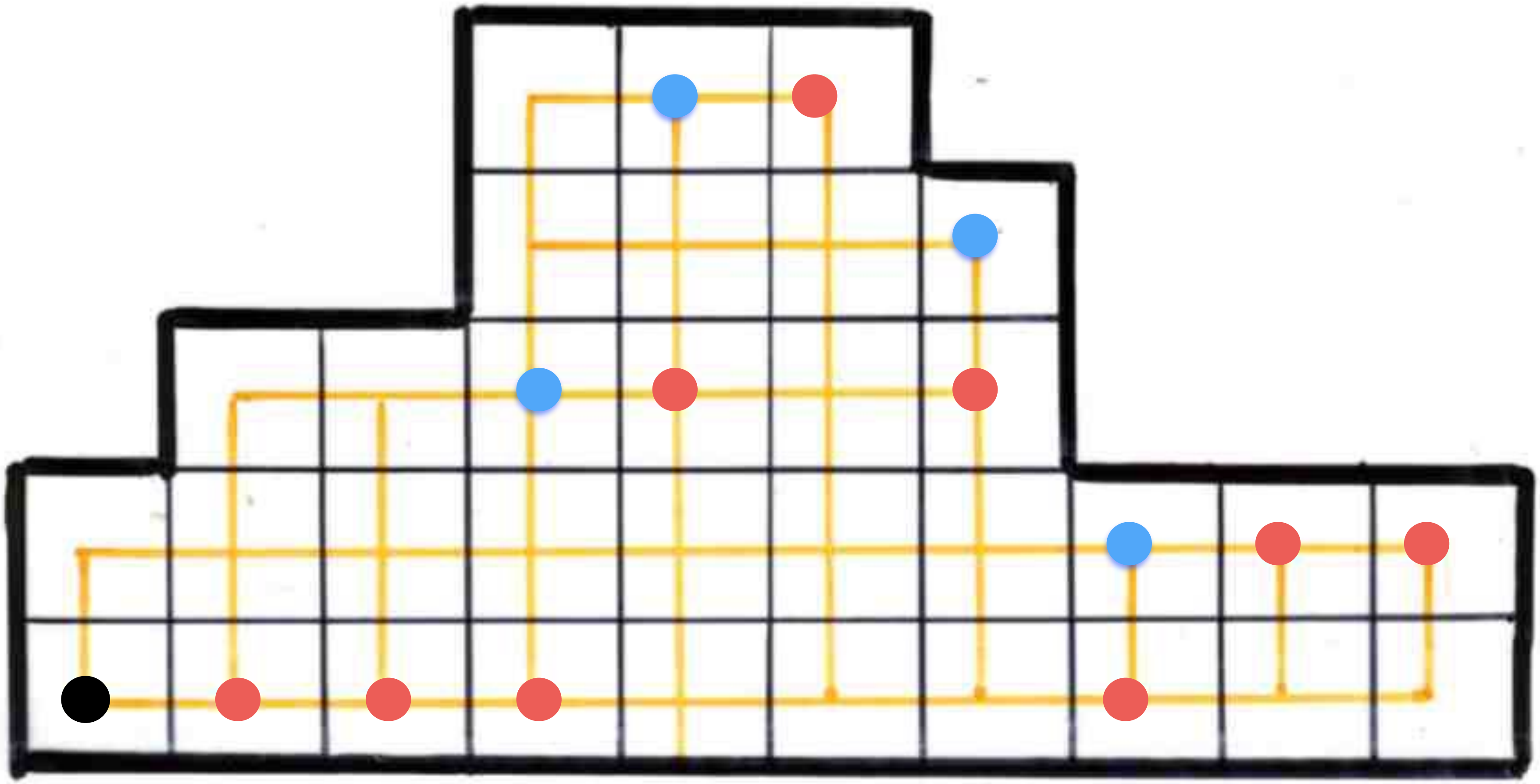


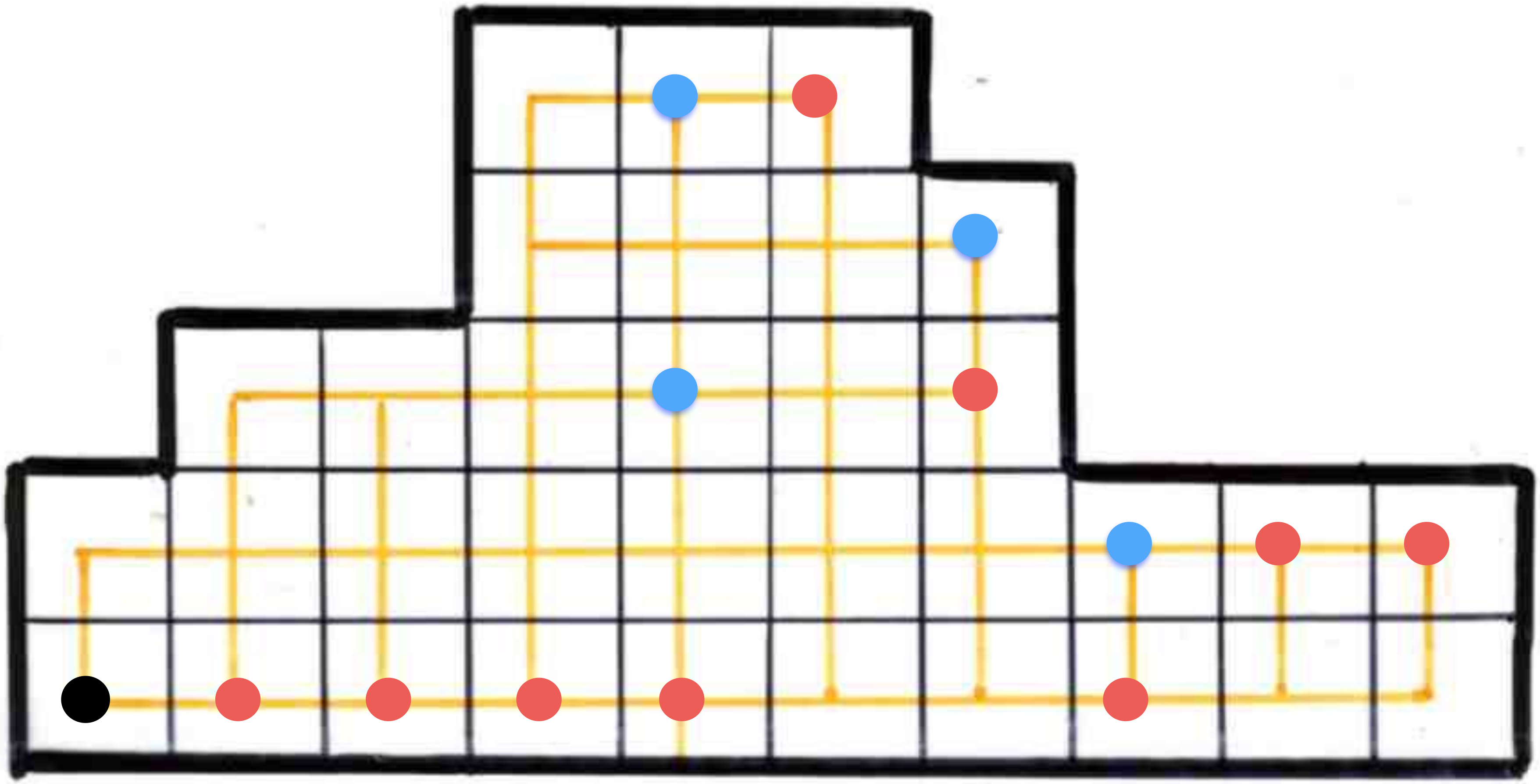


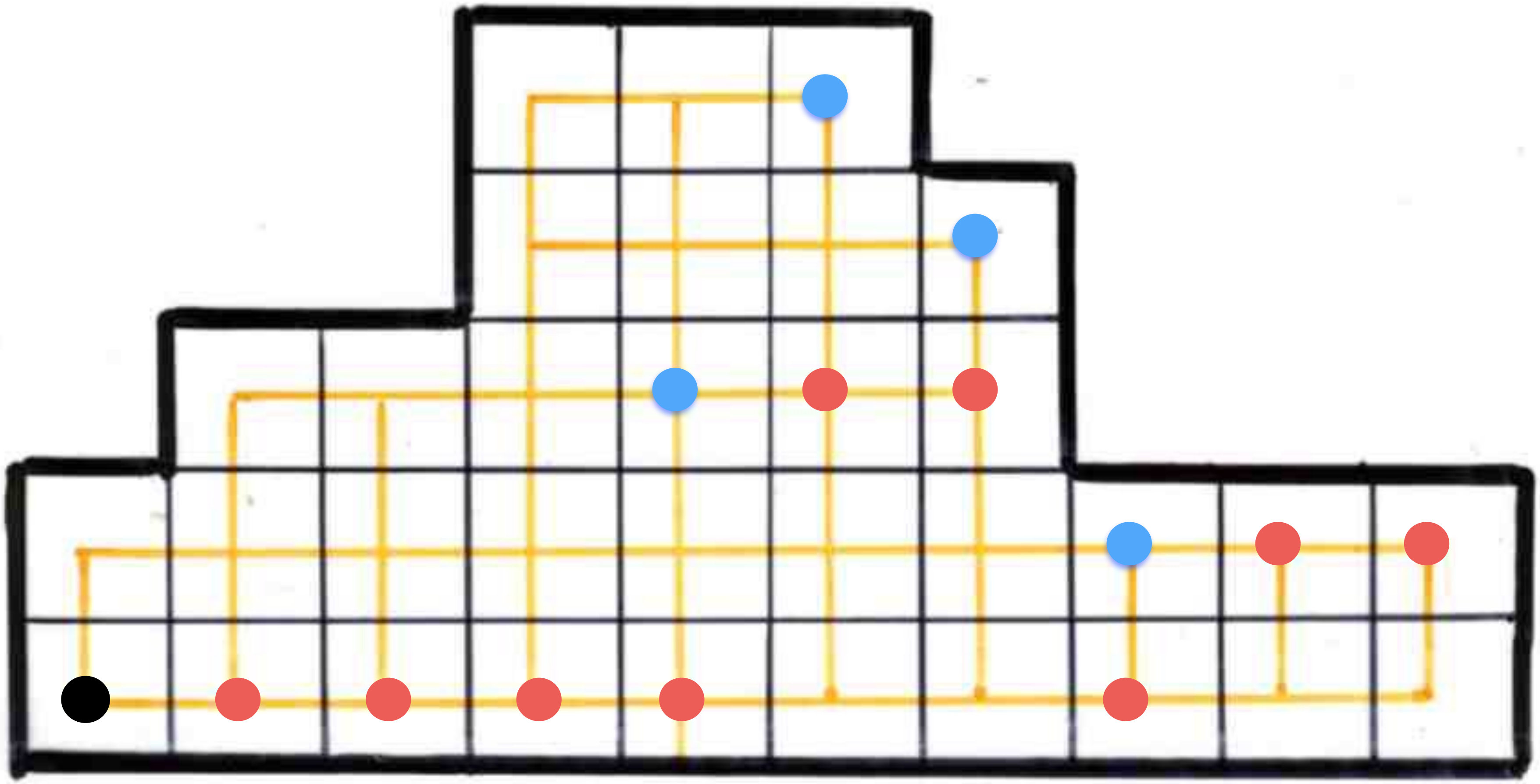


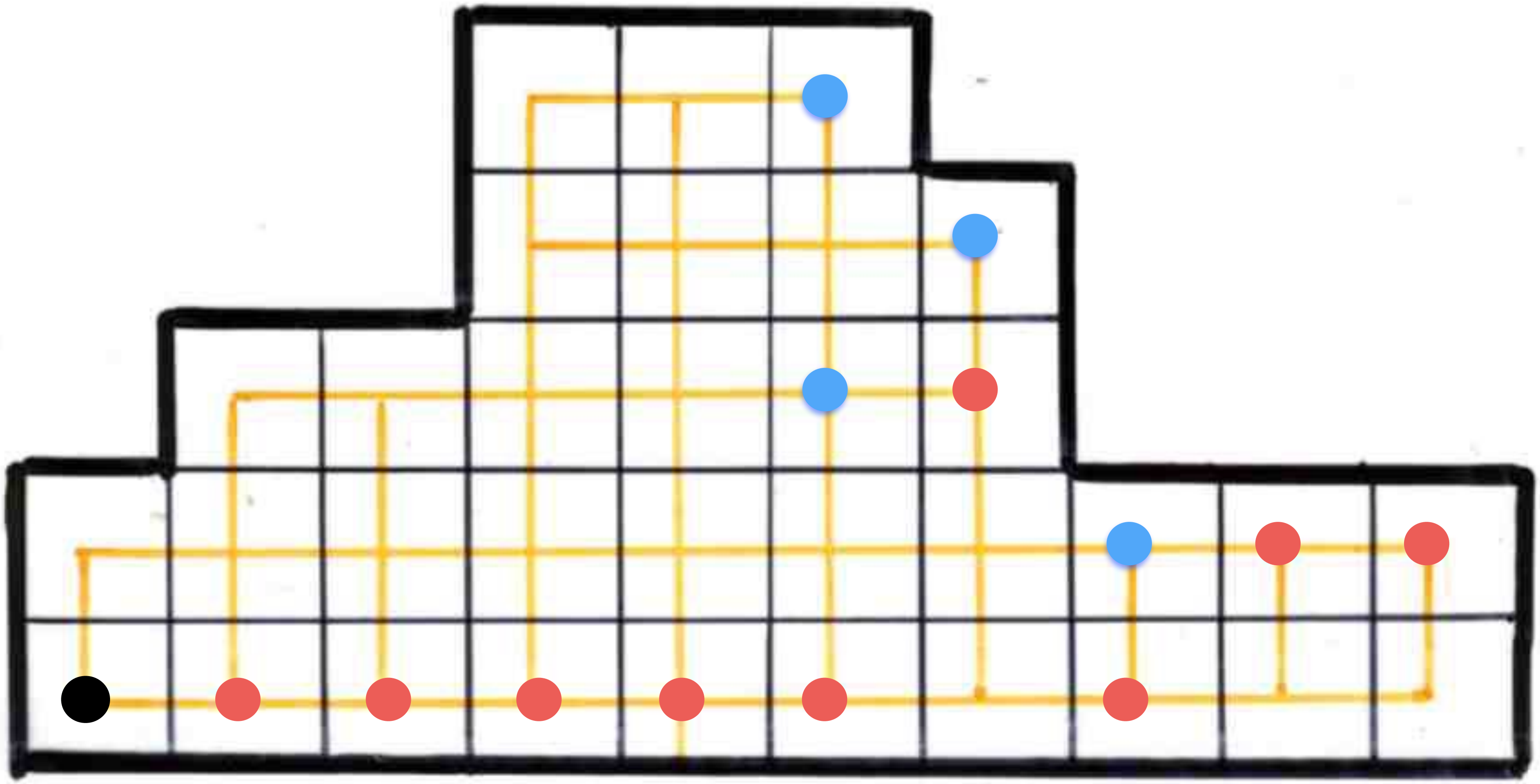


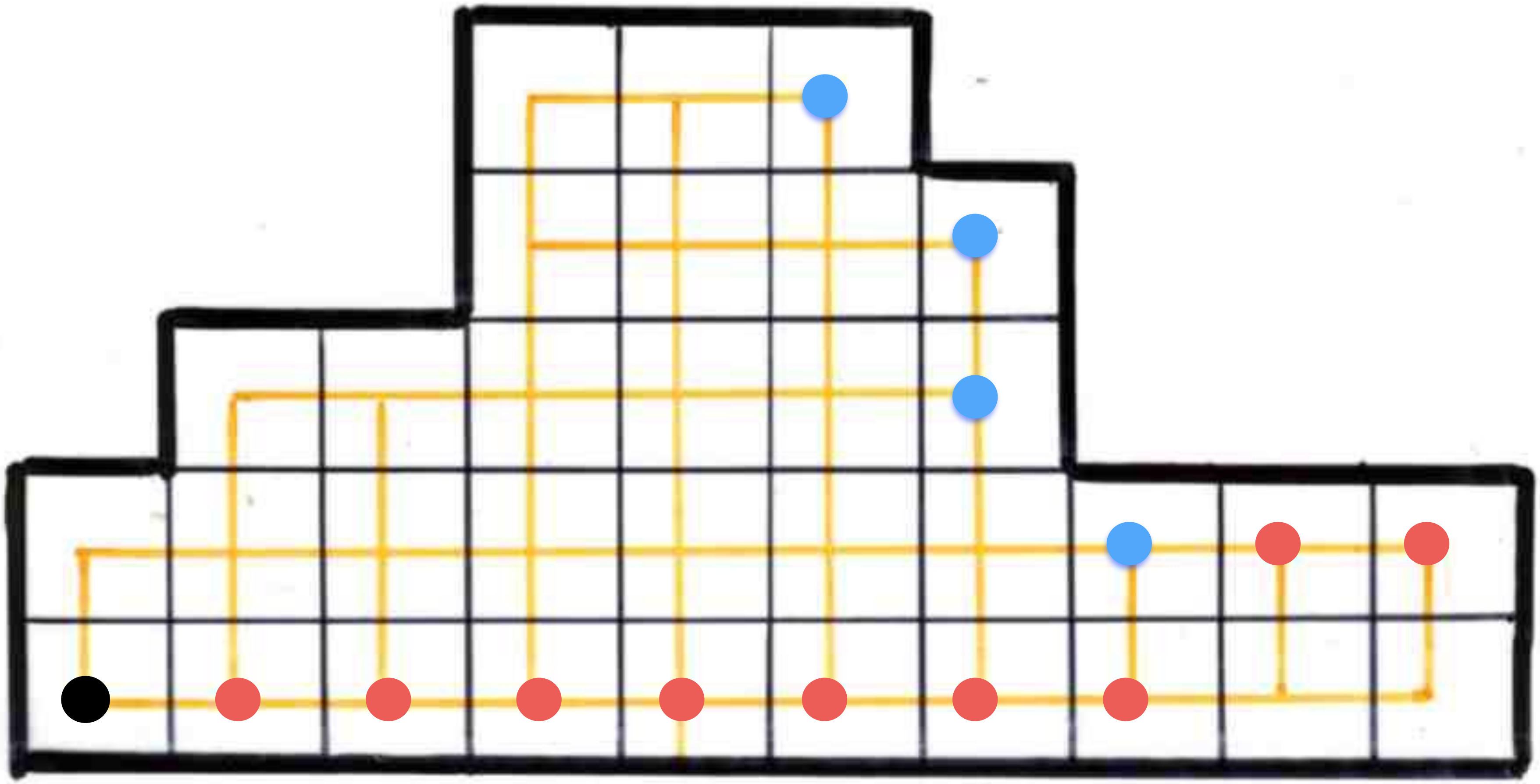


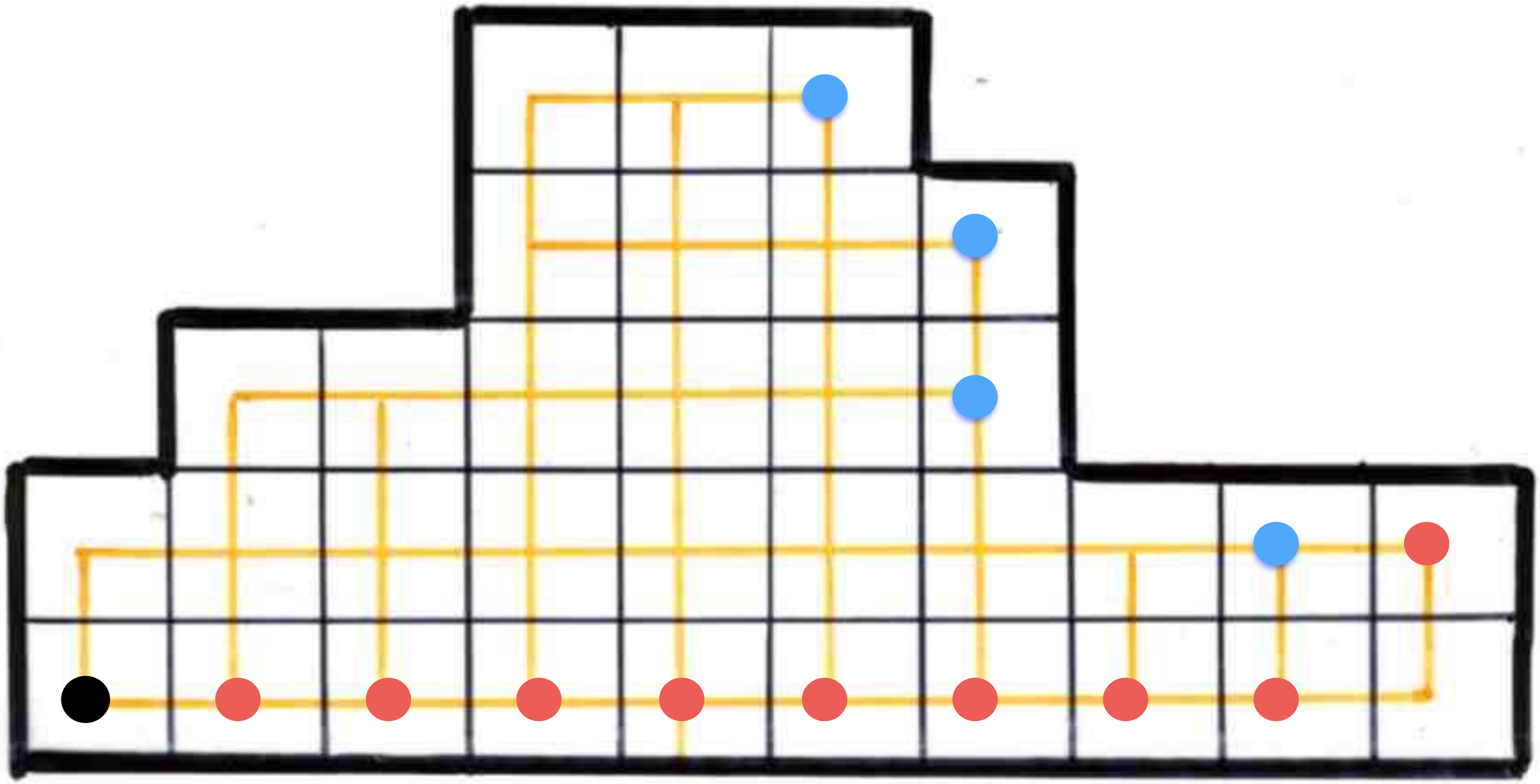


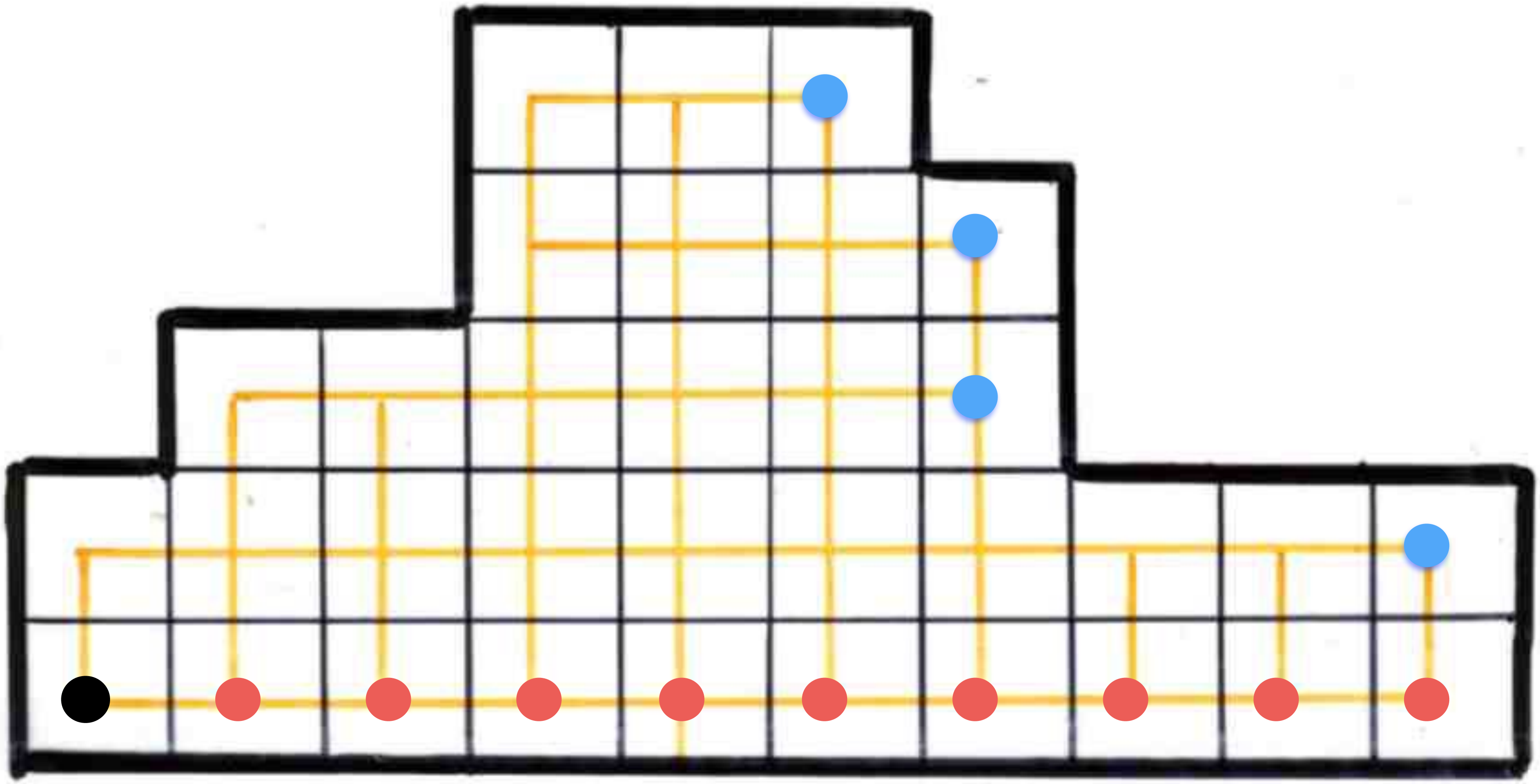












number of maximal chains in Tamari(n) ?

Nelson (2016) Ph.D.

number of chains with maximum length

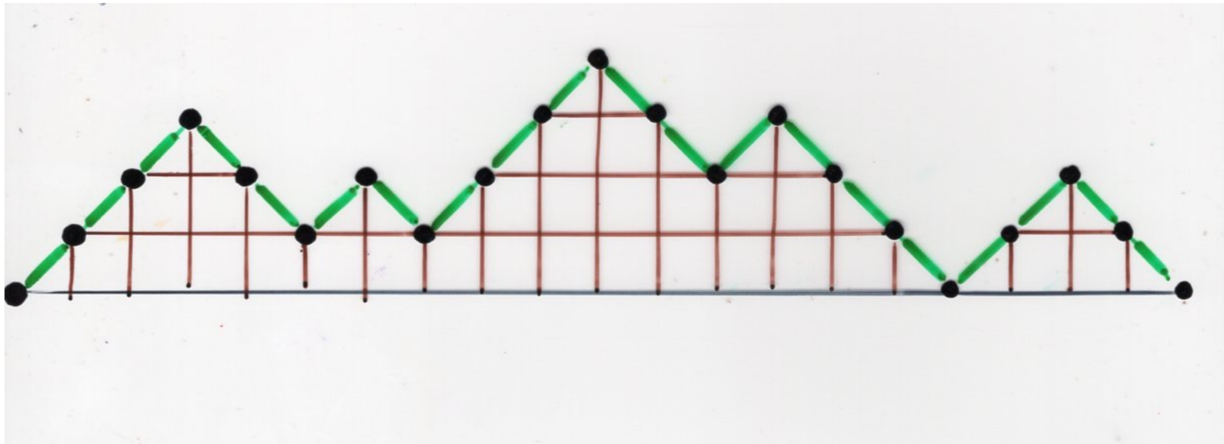
Fishel, Nelson (2014)
bijection with standard shifted tableaux of staircase shape

a festival of bijections

Pair of paths

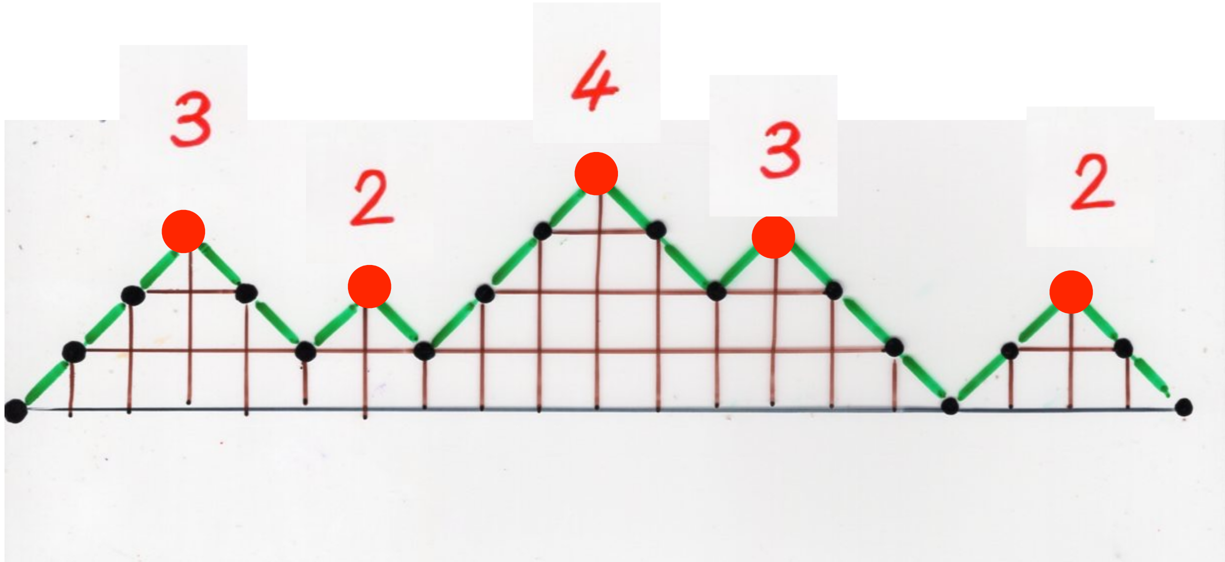
Dyck paths

Staircase polygons

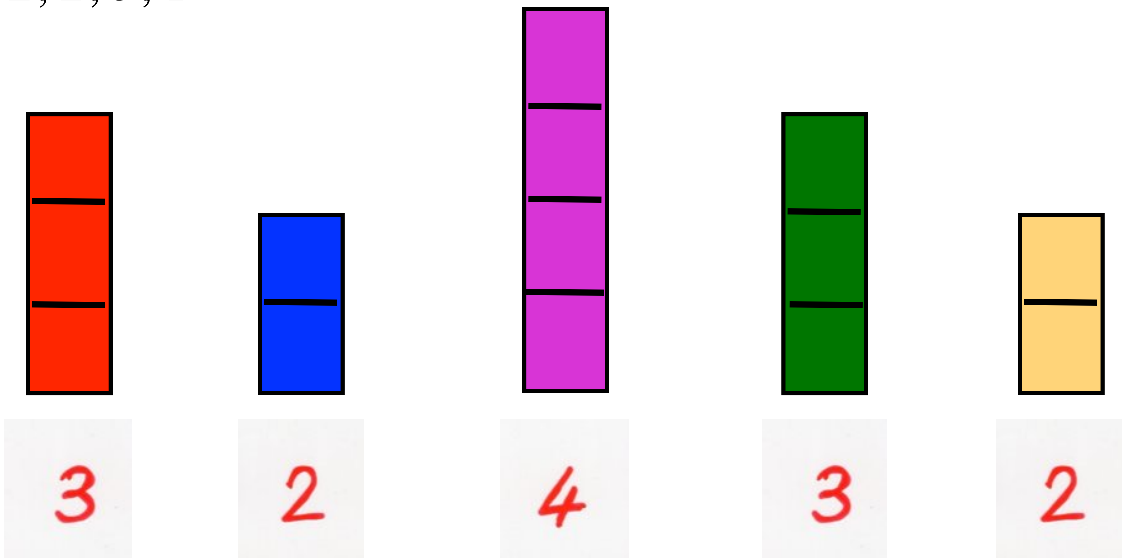


We described a bijection between Dyck paths and pairs (u,v) of paths, defined first by M. Delest and X.V., for the enumeration of convex polygons, with a formulation given by J.M. Fedou.

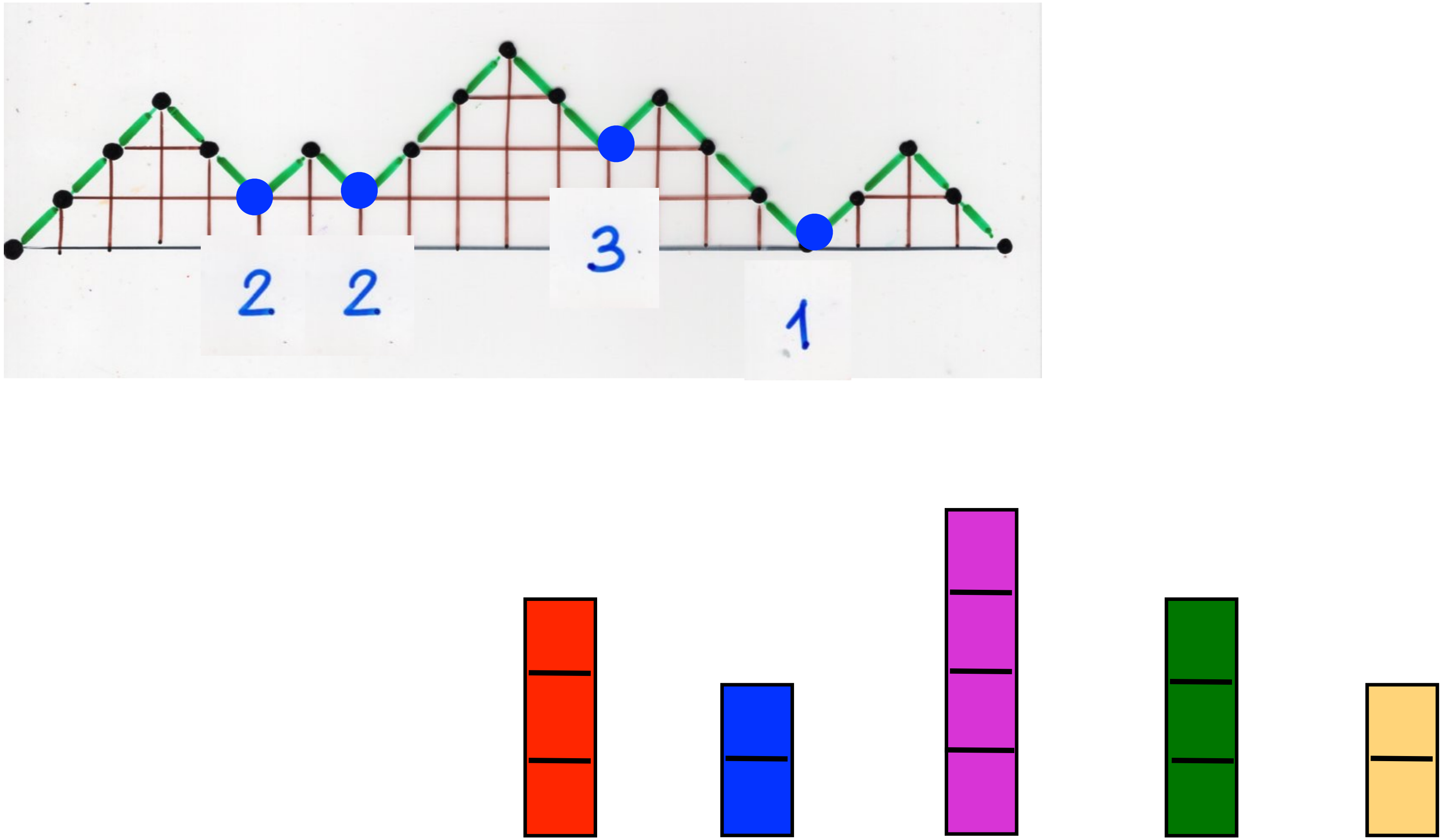
M. Delest and X.V., Algebraic languages and polyominoes enumeration, Theoretical Computer Science, 34 (1984) 169-206



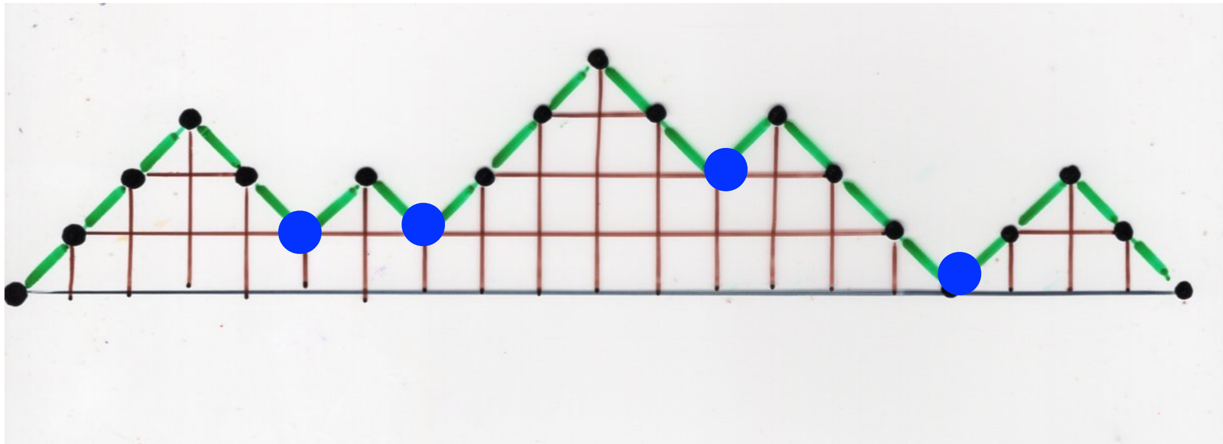
Height of the peaks: 3, 2, 4, 3, 2
1 + height of the valley: 2, 2, 3, 1

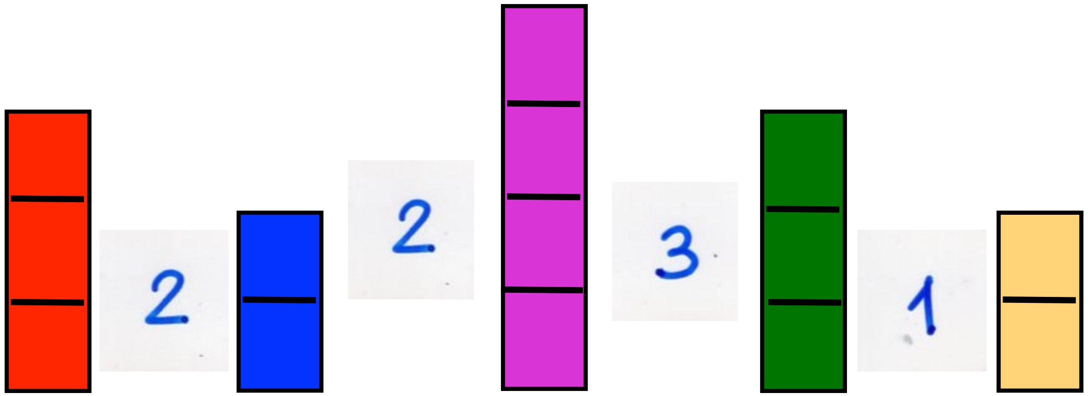


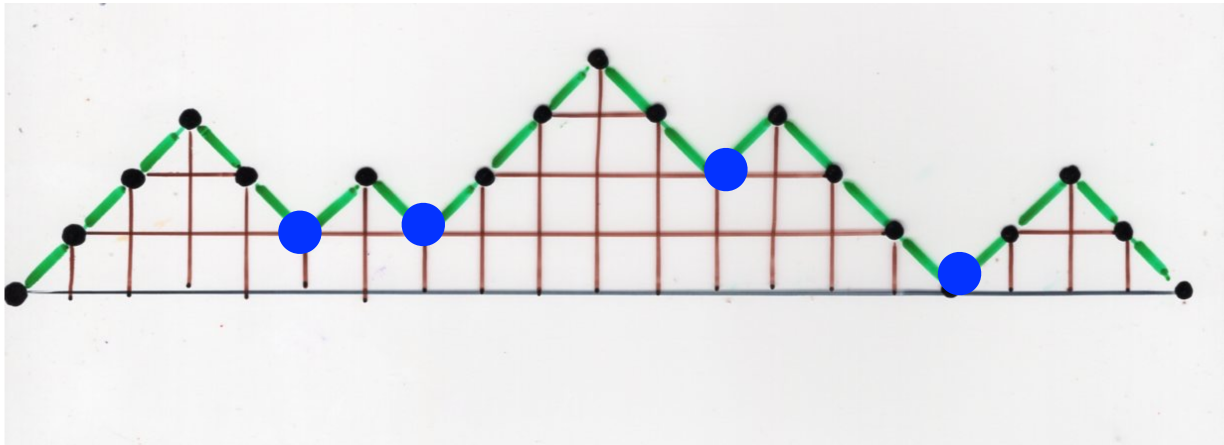
A sequence of columns from the red numbers

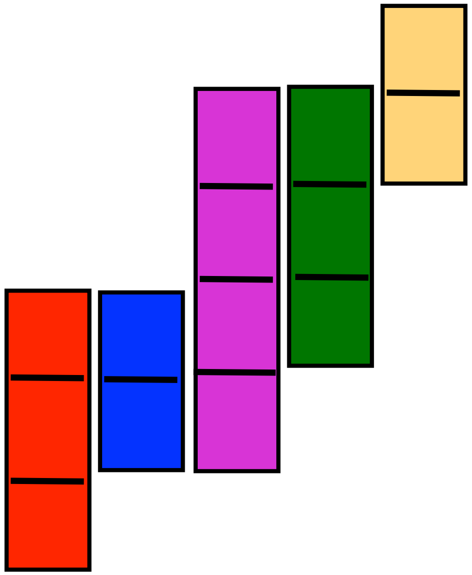


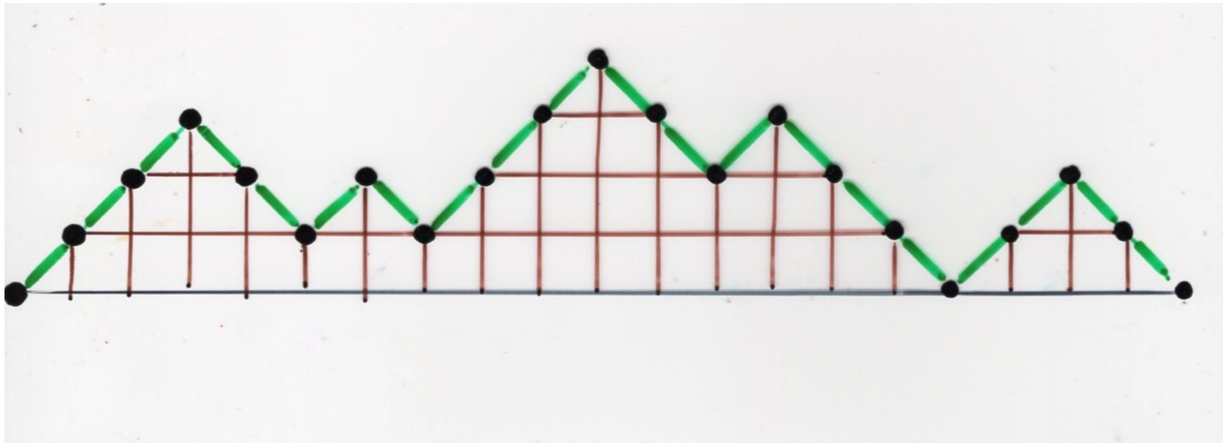
gluing the columns according to the blue numbers

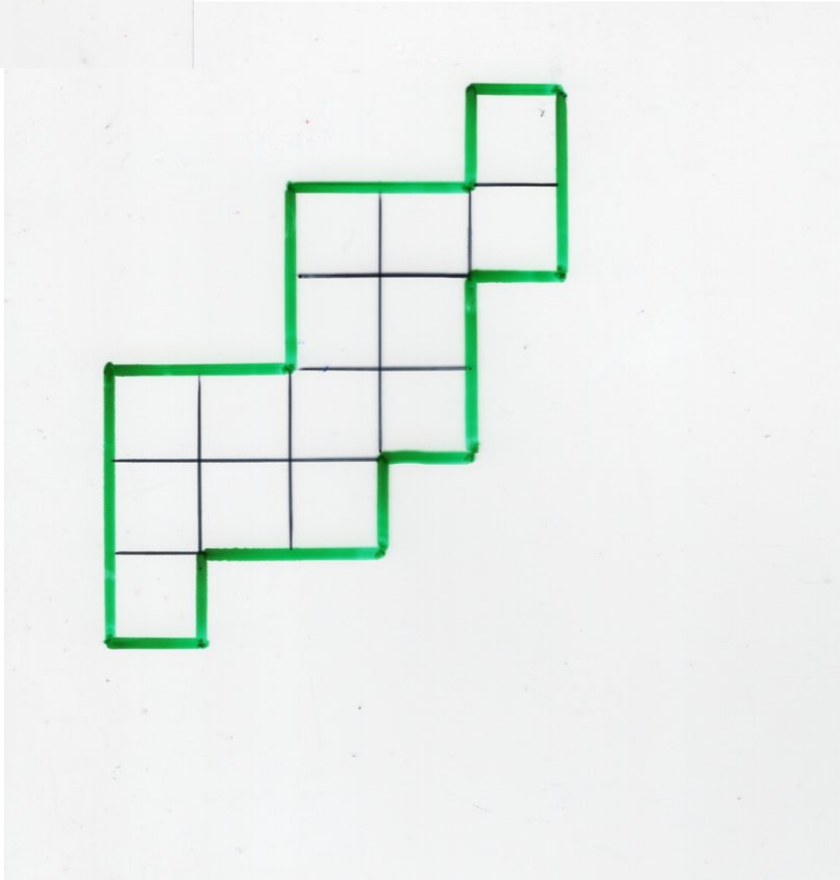


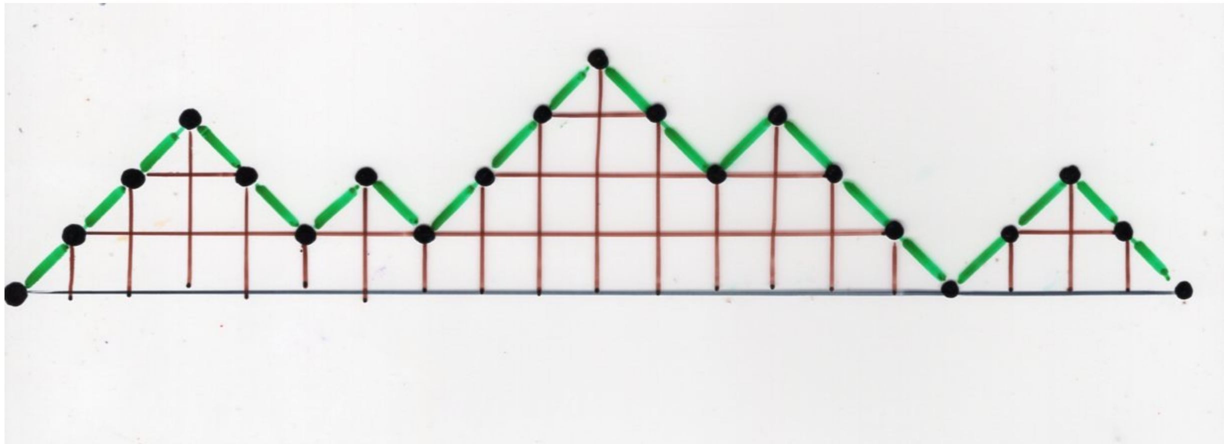

2
2
3
1

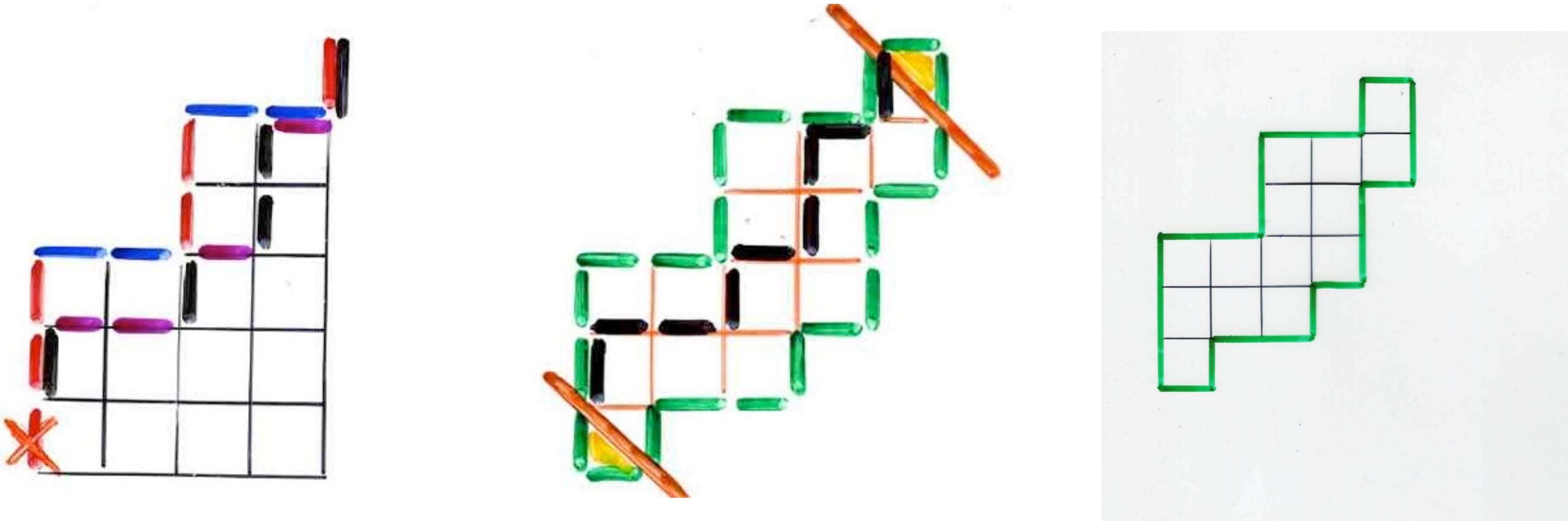




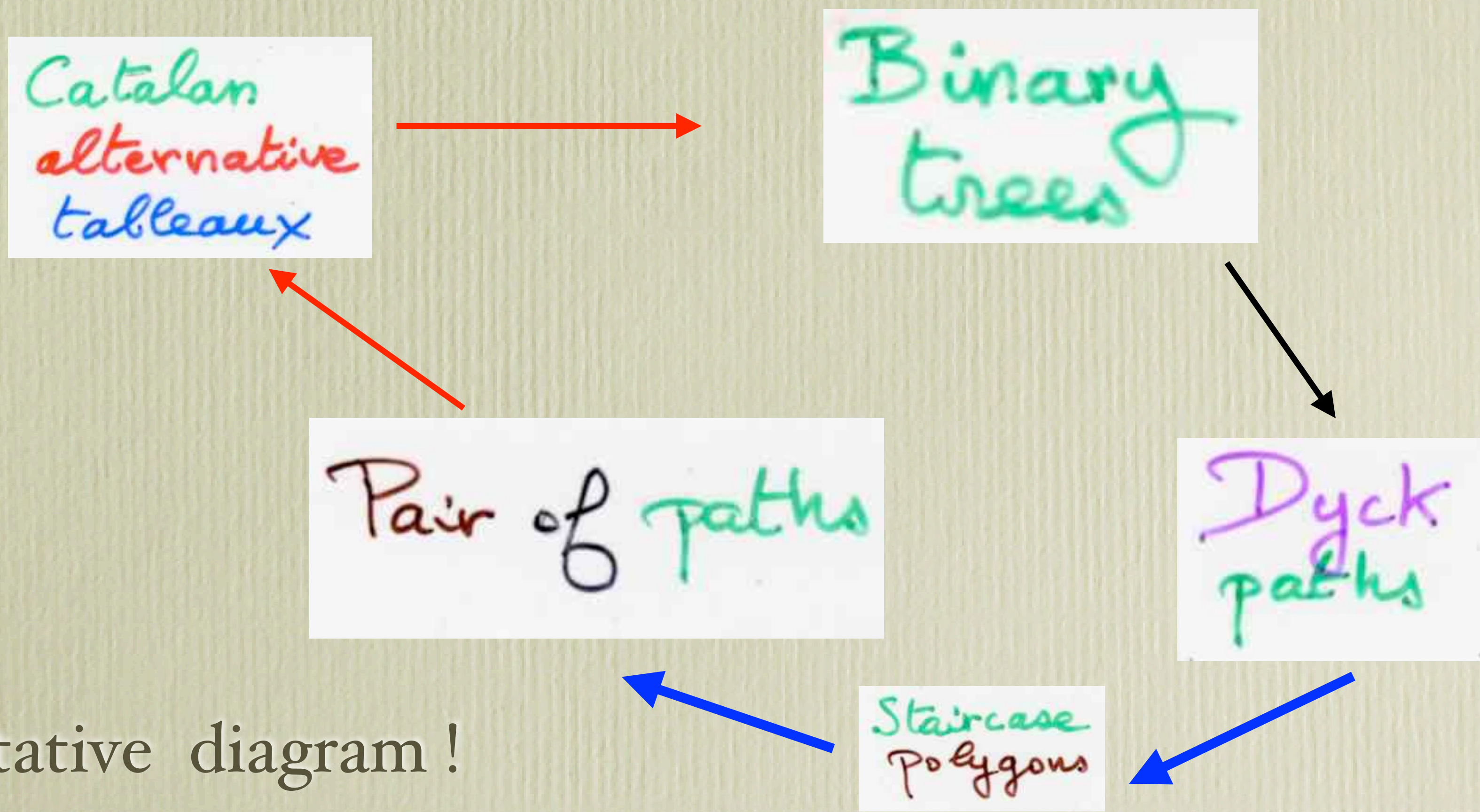




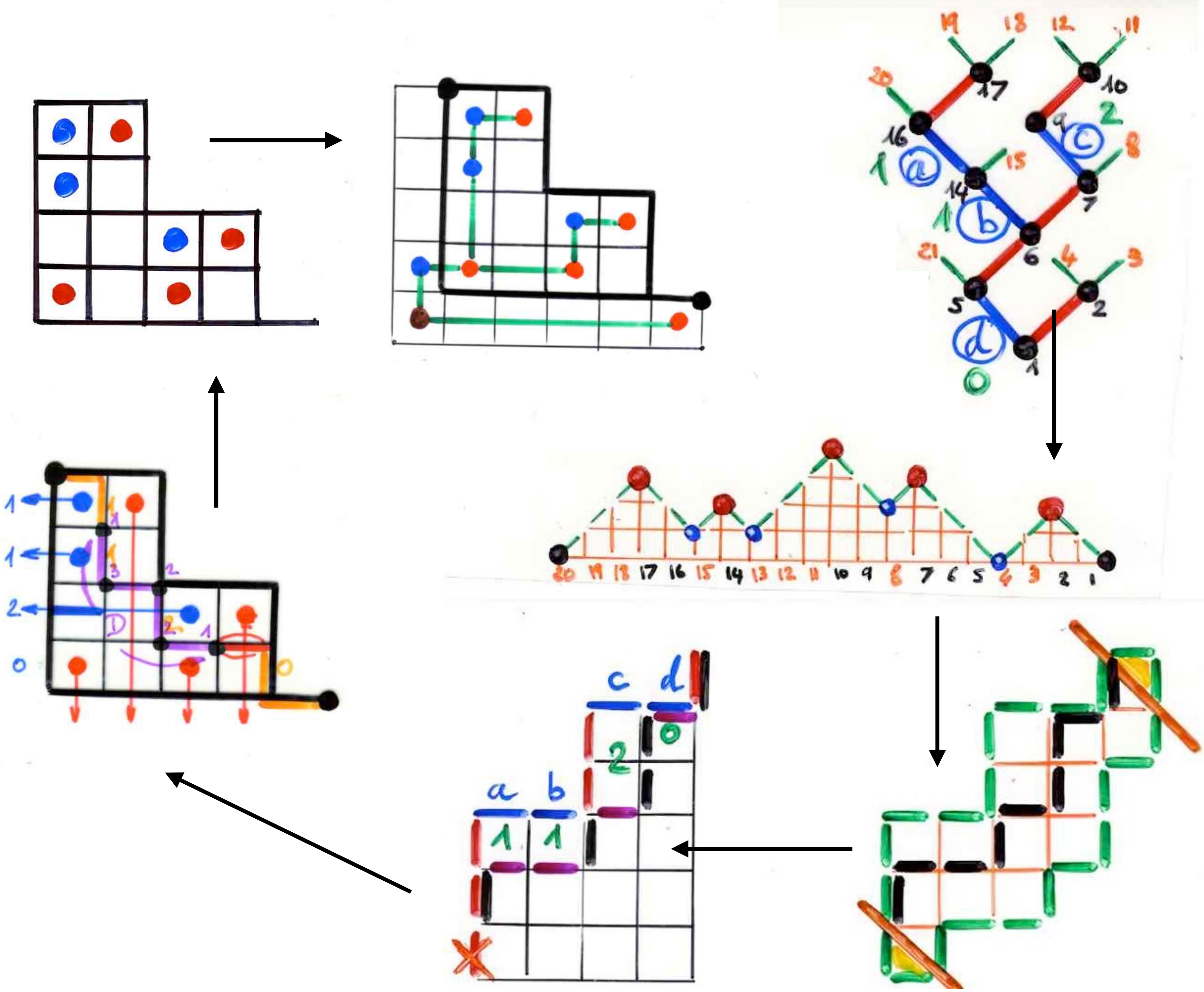


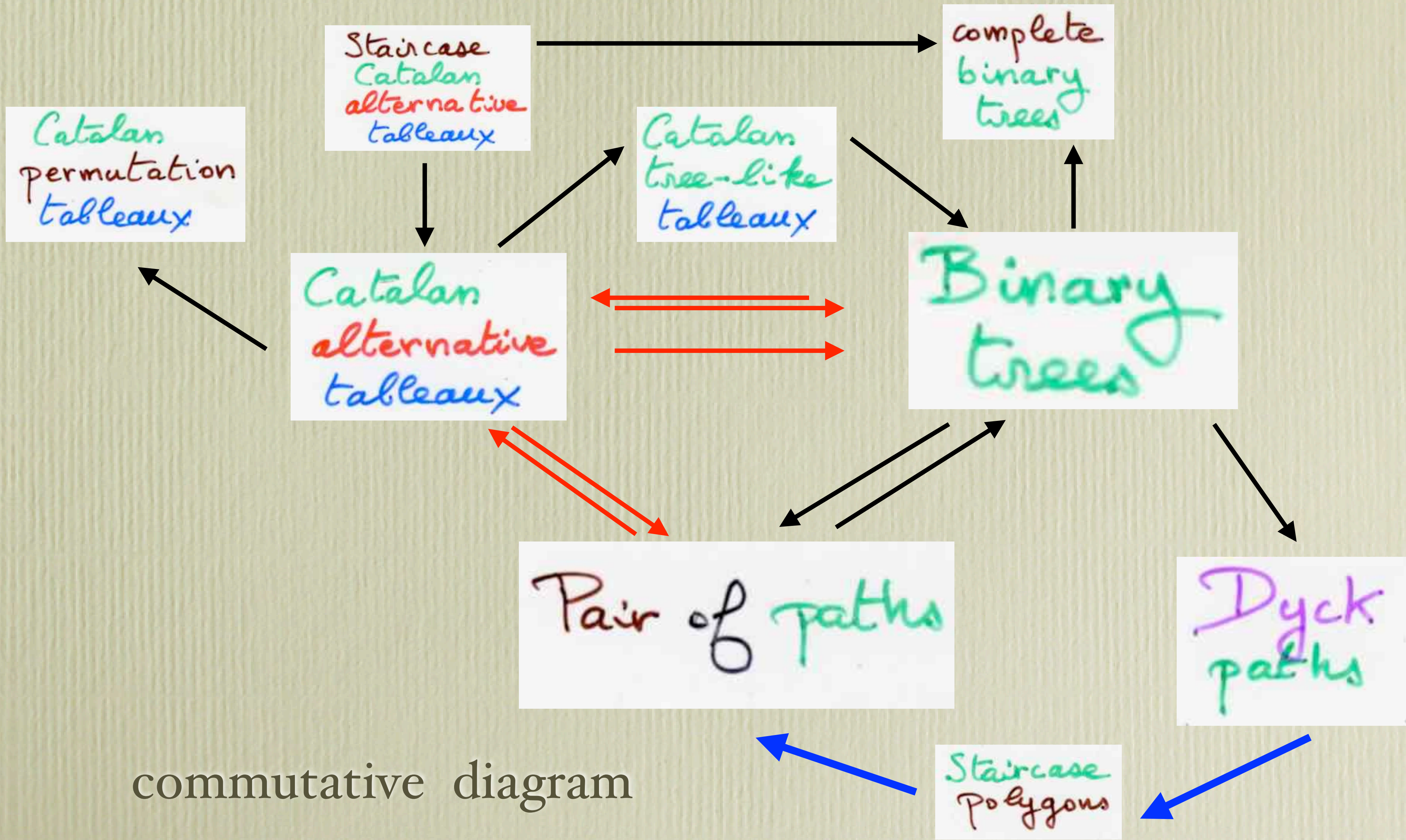


sliding the SE border up one step



commutative diagram !


20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
a b c d

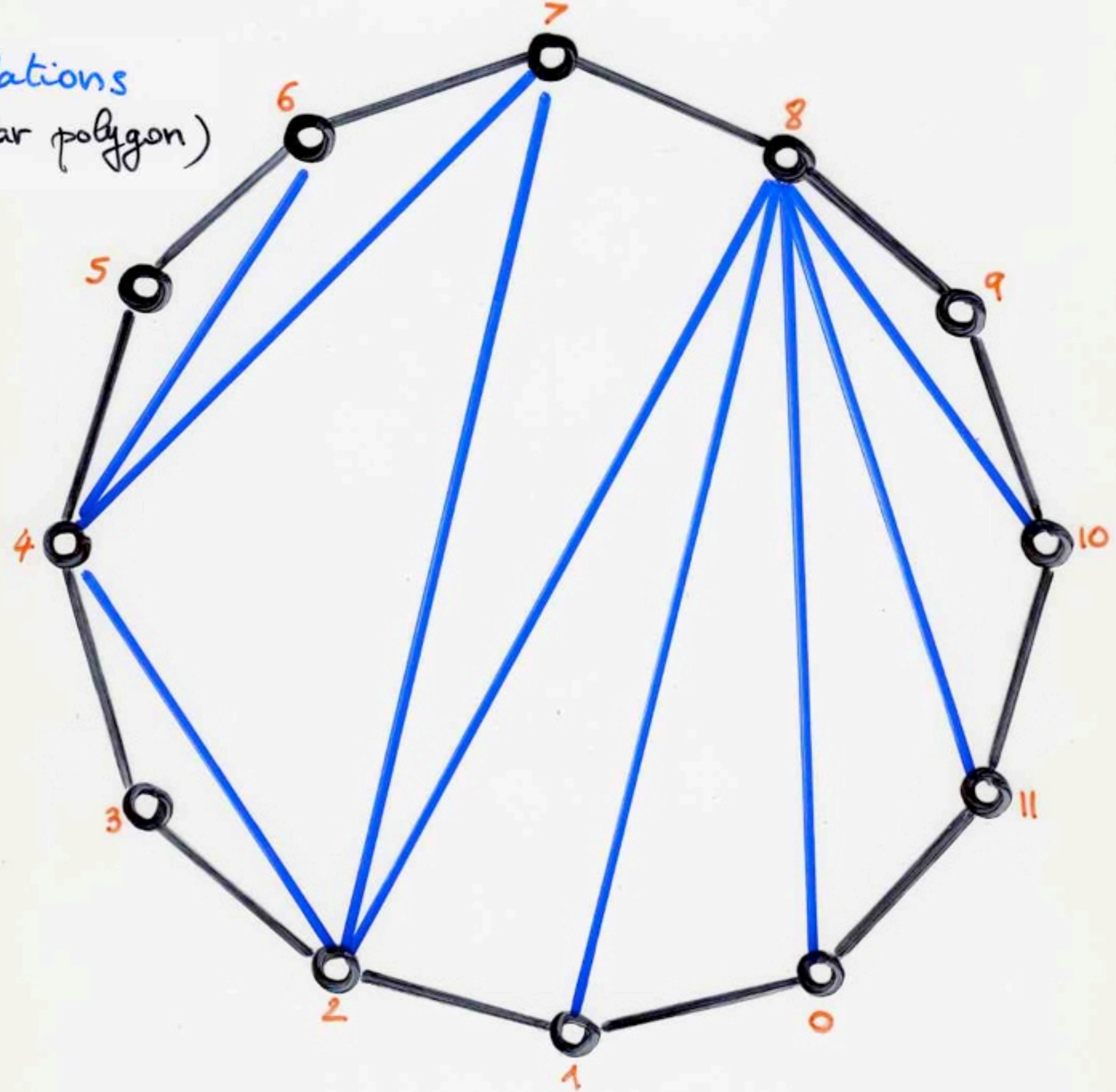


commutative diagram

the work of

Ceballos, Padrol, Sarmiento
(2016) (2017)

Triangulations
(of a regular polygon)




EULER

getheilt und solches kan auf 5 erley verschiedene Arten geschehen nemlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V. $\frac{ec}{eb}$

Ferner kan ein Sechseck durch 3 Diagonales in 4 Triangula zerschnitten und dieses kan auf 14 verschiedene Arten geschehen.

Nun ist die Frage generaliter, da ein Polygonum von n Seiten durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen könne.

Setze ich nun die Anzahl dieser verschiedenen Arten $= x$

so habe ich per Inductionem gefunden

wann $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluß gemacht, daß generaliter sey

$$x = \frac{2.\ 6.\ 10.\ 14.\ 18.\ 22. \ldots (4n-10)}{2.\ 3.\ 4.\ 5.\ 6.\ 7. \ldots (n-1)}$$ oder es ist

$1 = \frac{2}{2}$; $2 = 1.\frac{6}{3}$; $5 = 2.\frac{10}{4}$; $14 = 5.\frac{14}{5}$; $42 = 14.\frac{18}{6}$; $132 = 42\frac{22}{7}$;

daß also aus einer jeden Zahl die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, daß diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen $1, 2, 5, 14, 42, 132$, etc. habe ich auch diese Eigenschaft angemerket, daß

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

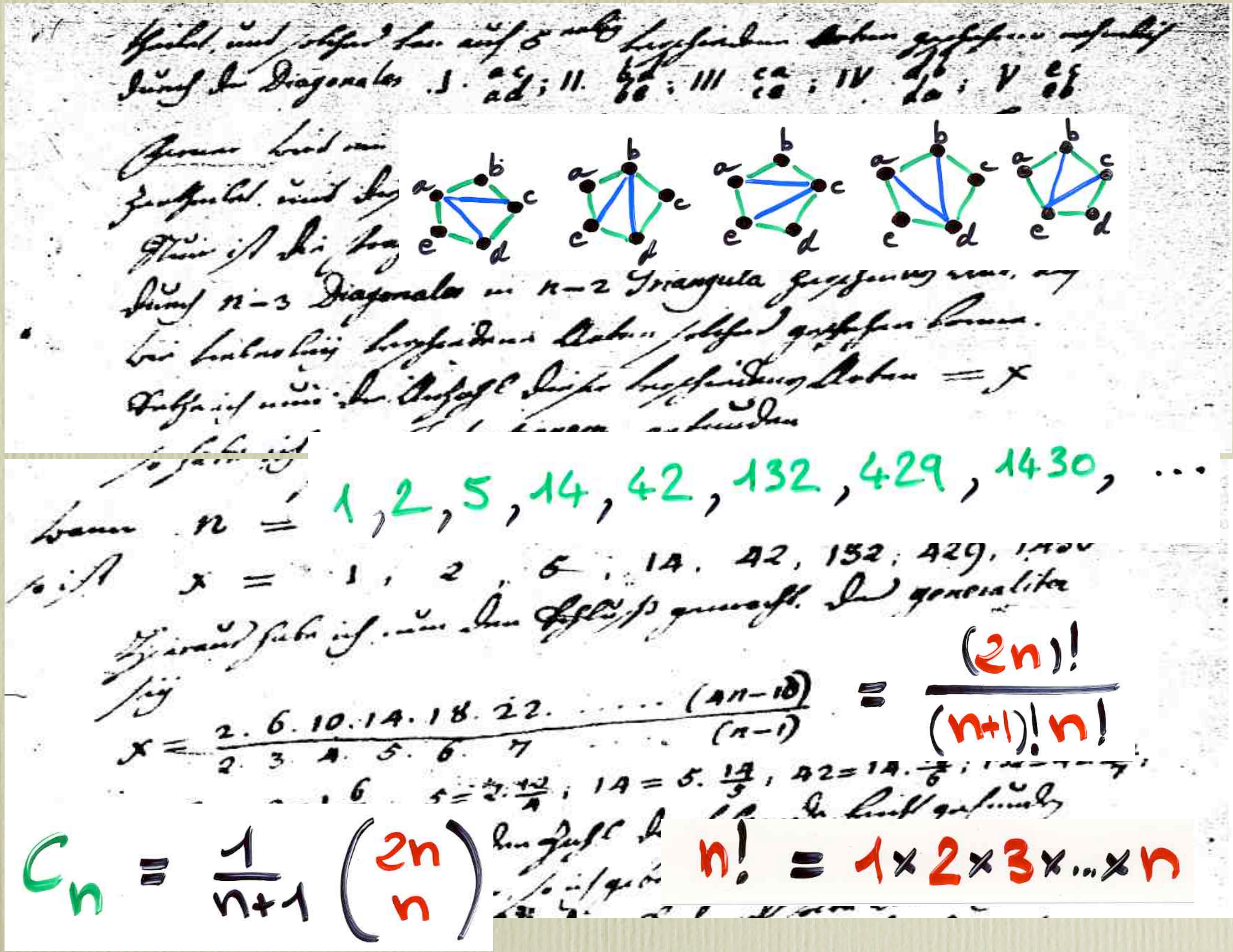
also wann $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4$.

Ew. [illegible] meinige [illegible] sich zu Ew. Hochwohlgeb. [illegible] und beständigen Gewogenheit gehorsamst empfehlen, und ich habe die Ehre mit der schuldigsten Hochachtung lebenslang zu beharren

Ew. Hochwohlgebohrnen

Berlin d. 4. Sept 1751.

gehorsamster Diener
L. Euler



Diagonalen I. ac ad ; II. bd be ; III. ca ce ; IV. db da ; V. ec eb
a b c d e
n − 3 Diagonalen in n − 2 Triangula
= x
1 , 2 , 5 , 14 , 42 , 132 , 429 , 1430 , ...
n =
x = 1 , 2 , 5 , 14 , 42 , 132 , 429 , 1430
generaliter
$x = \frac{2 . 6 . 10 . 14 . 18 . 22 . \ldots (4n-10)}{2 . 3 . 4 . 5 . 6 . 7 . \ldots (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$
$5 = 2 . \frac{10}{4}$; $14 = 5 . \frac{14}{5}$; $42 = 14 . \frac{18}{6}$
$C_n = \frac{1}{n+1}\binom{2n}{n}$
$n! = 1 \times 2 \times 3 \times \ldots \times n$

$$\frac{1-2a-\sqrt{1-4a^2}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$a = \frac{1}{4}$$

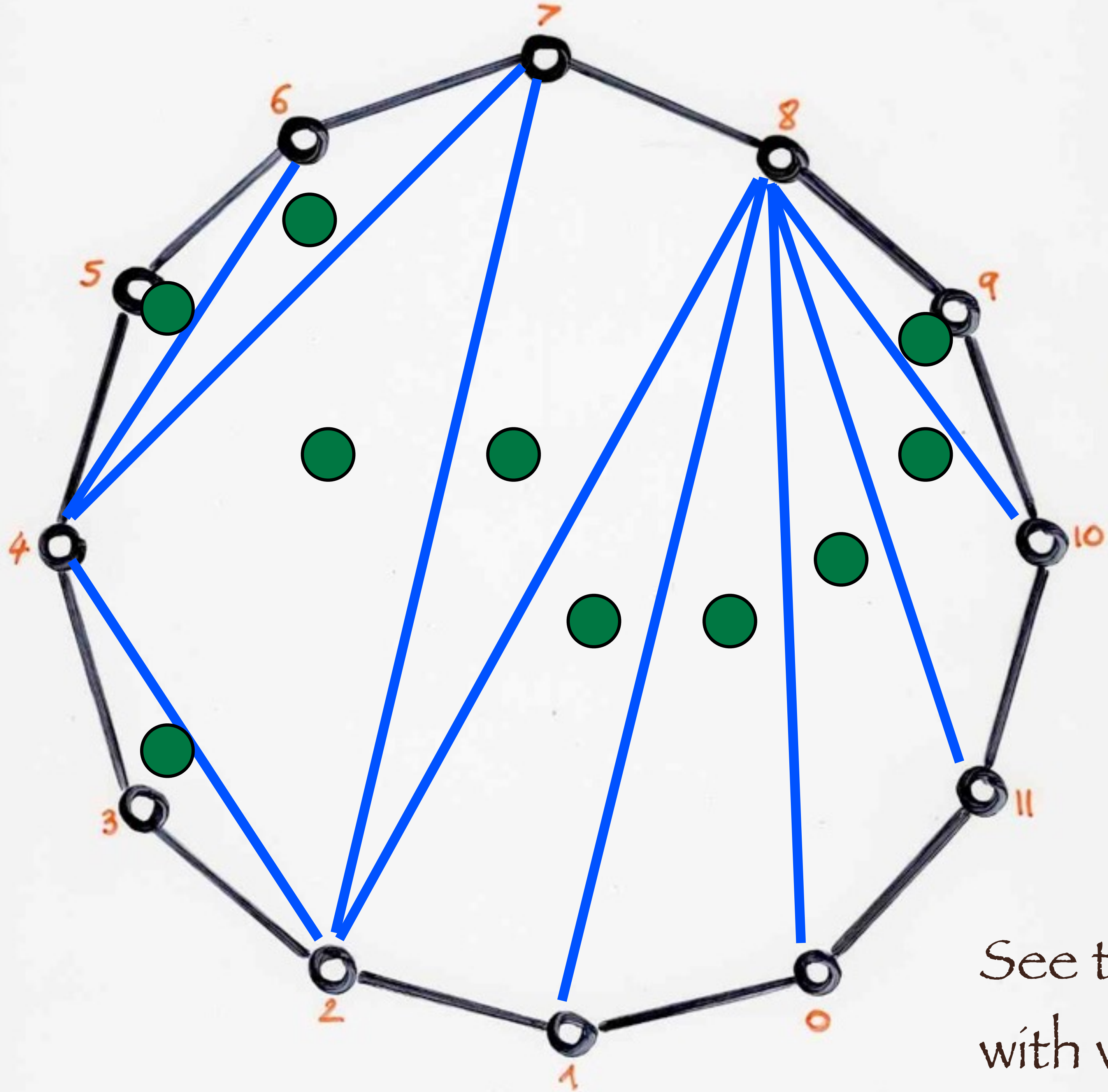
$$1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc} = 4.$$

Berlin d. 4. Sept 1751.

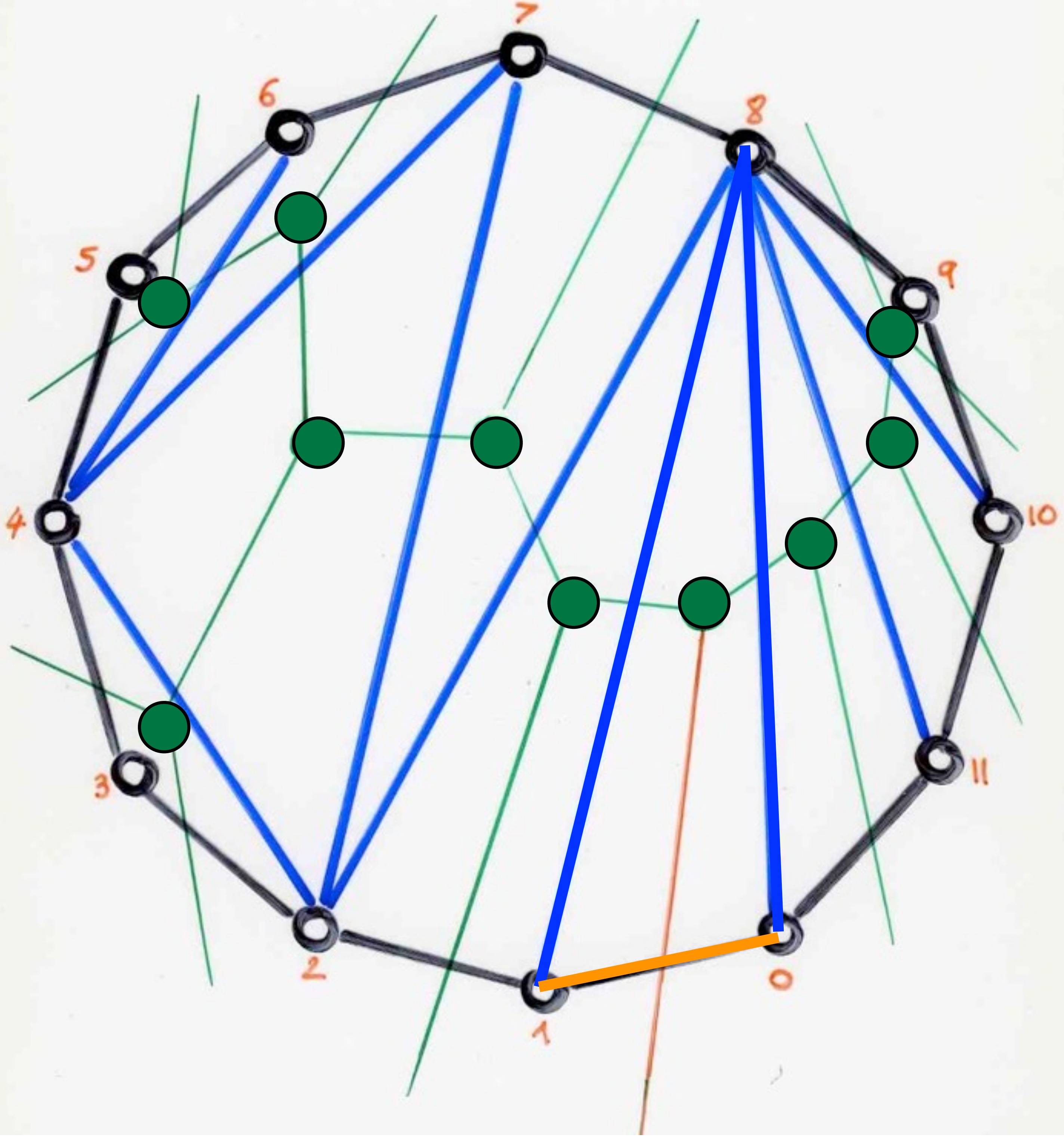
L. Euler

4 Sept 1751
Berlin

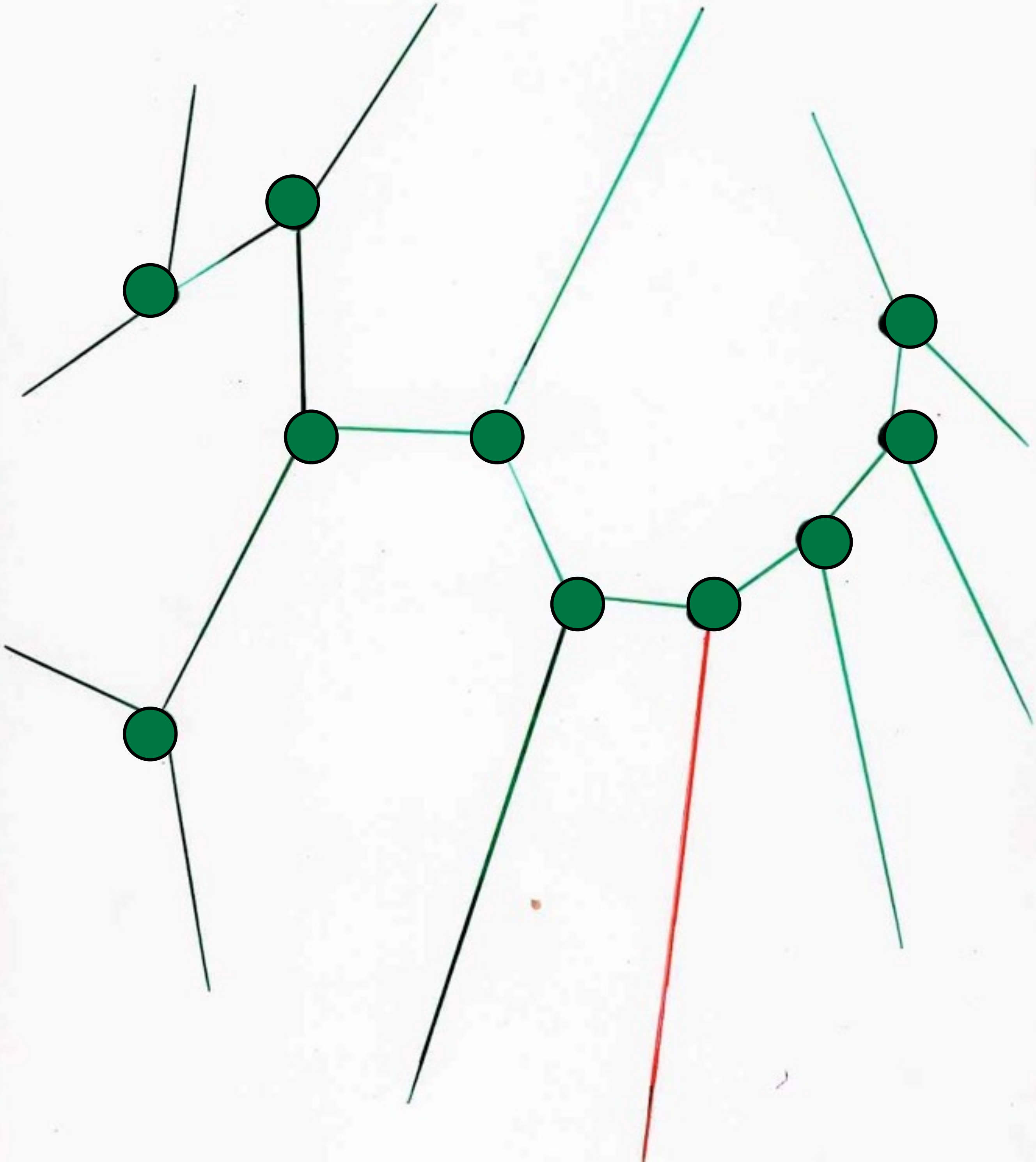
From triangulations to binary trees …

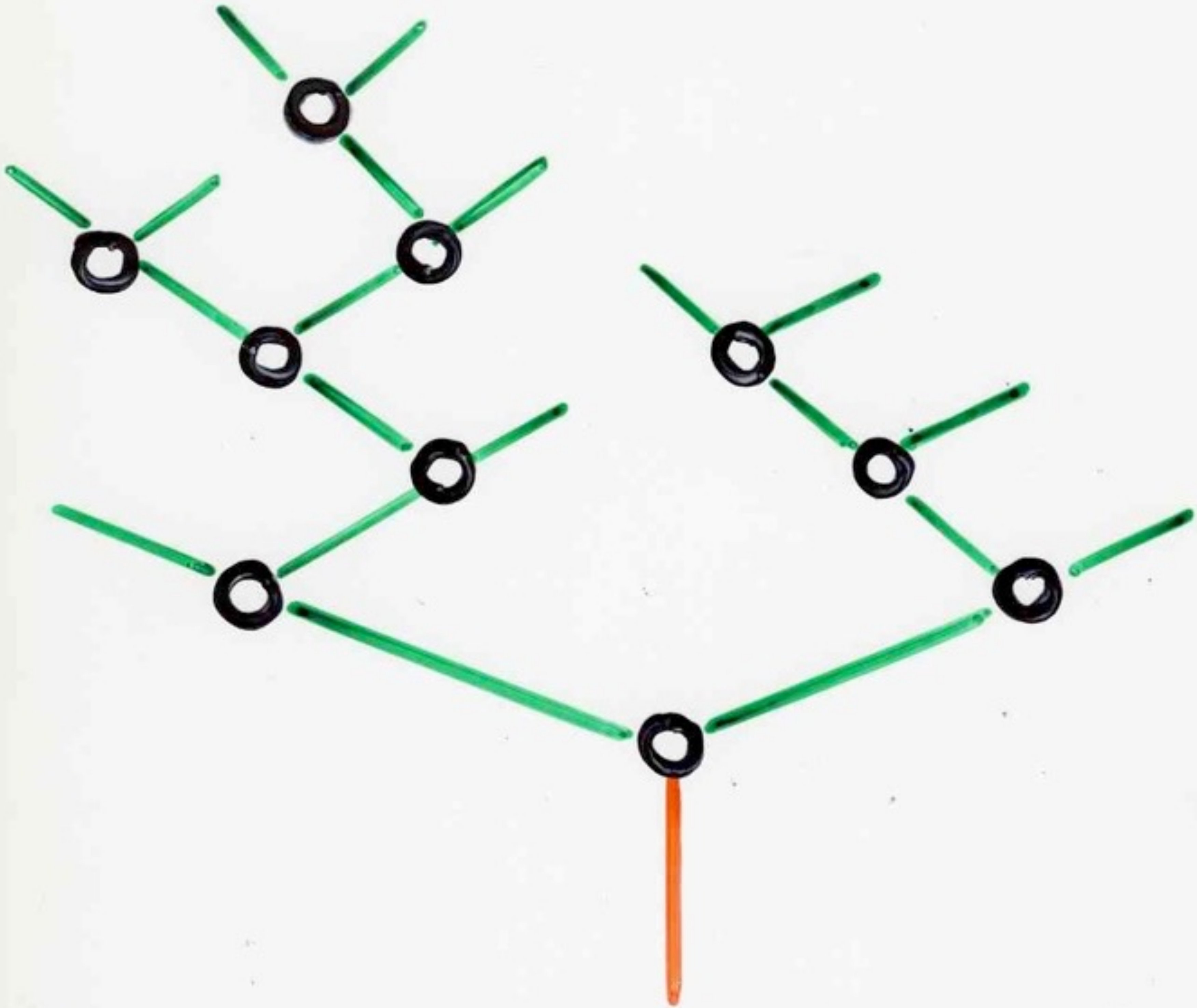


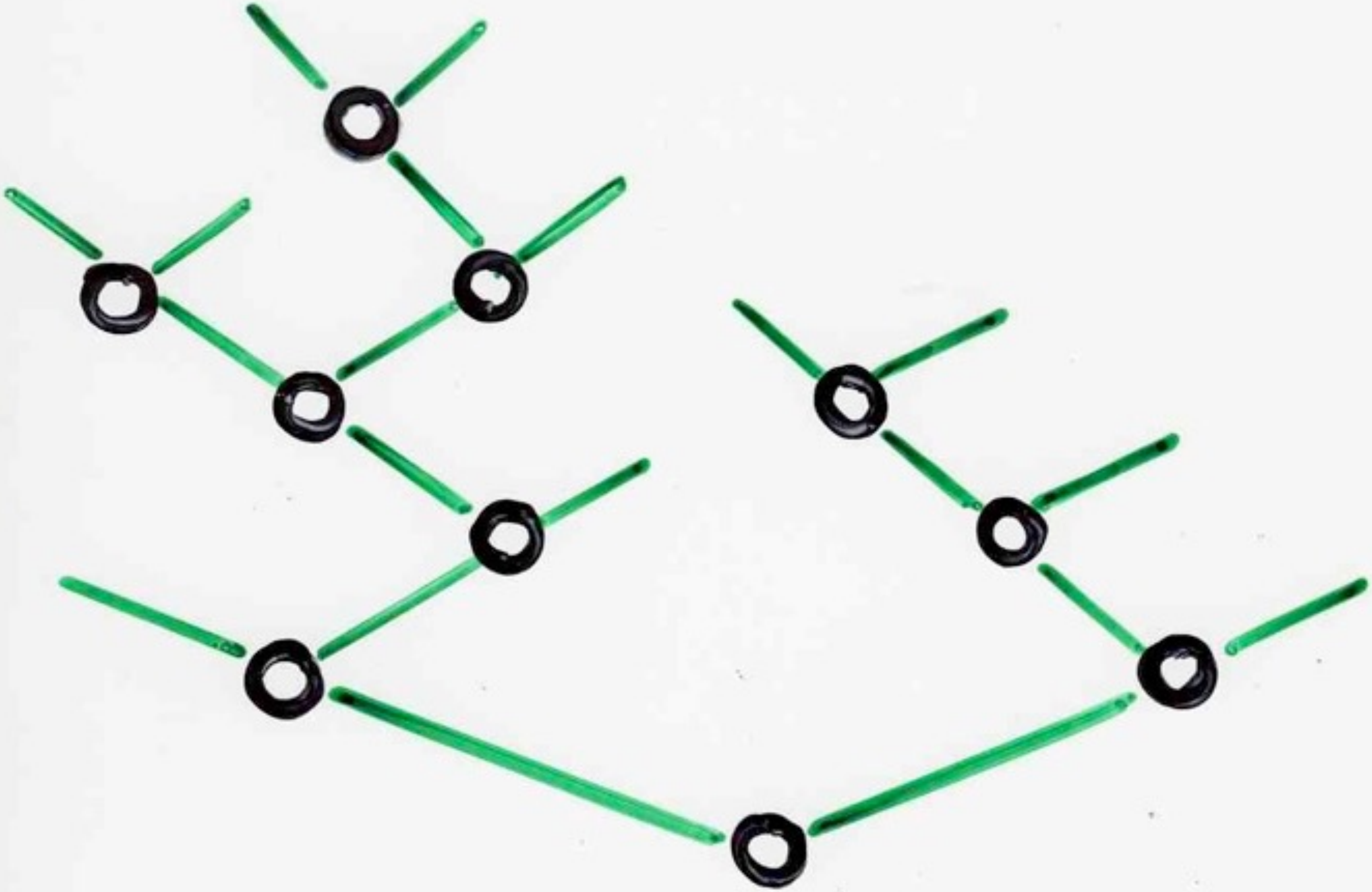
See the video
with violinists



7
6
8
5
9
4
10
3
11
2
0
1







violins:

Mariette Freudentheil

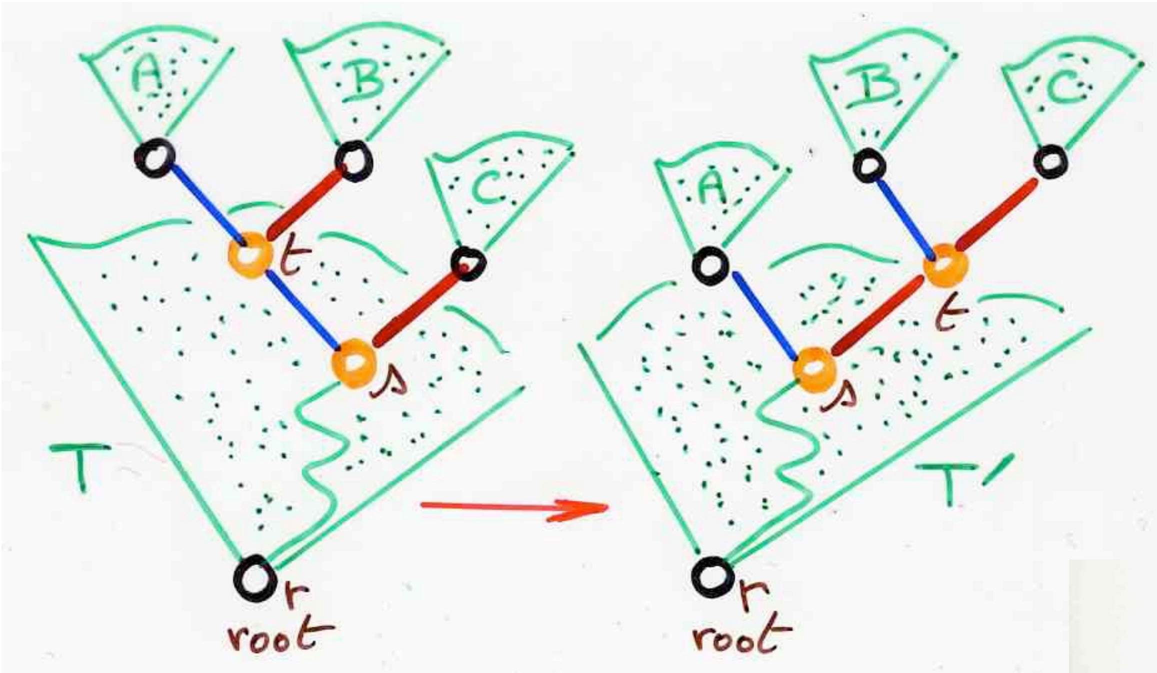
Gérard H.E. Duchamp

Association
Cont'Science:

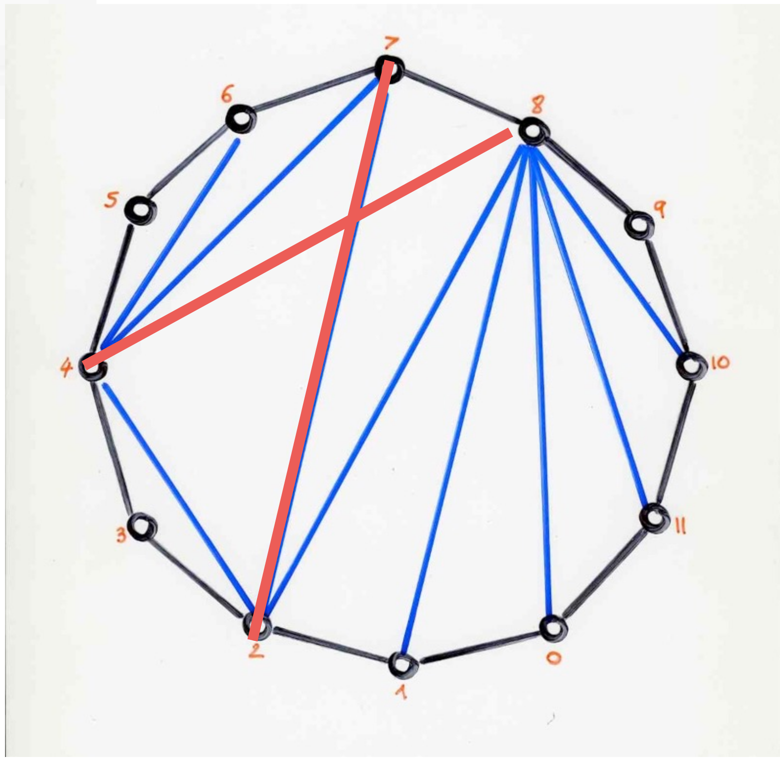
G. Duchamp
M. Pig Lagos
X. Viennot

Atelier audiovisuel
Université Bordeaux 1
Yves Descubes
Franck Marmisse

Tamari lattice with triangulations



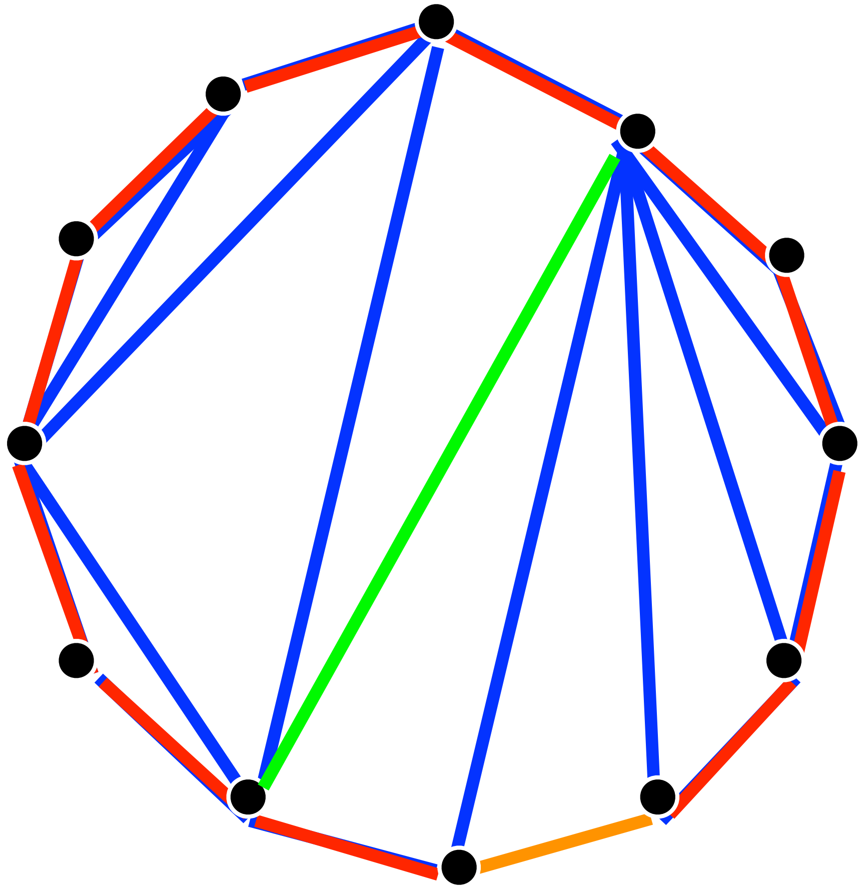
Rotation in a binary tree:
the covering relation in the
Tamari lattice



order relation

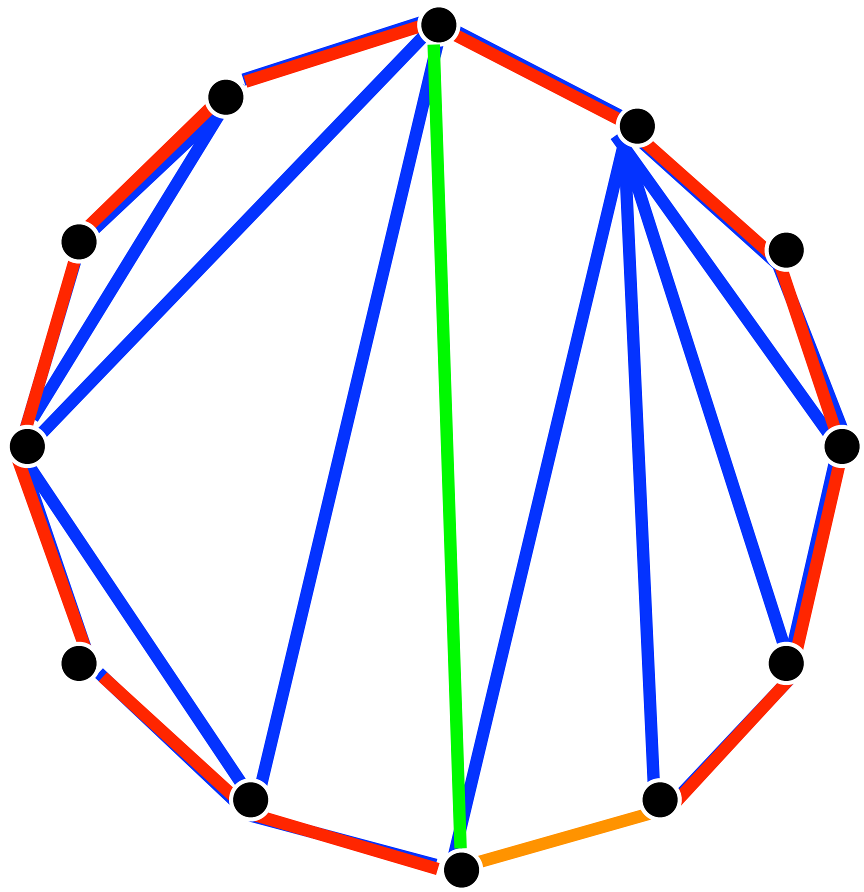
Tamari lattice

Triangulations
(of a regular polygon)

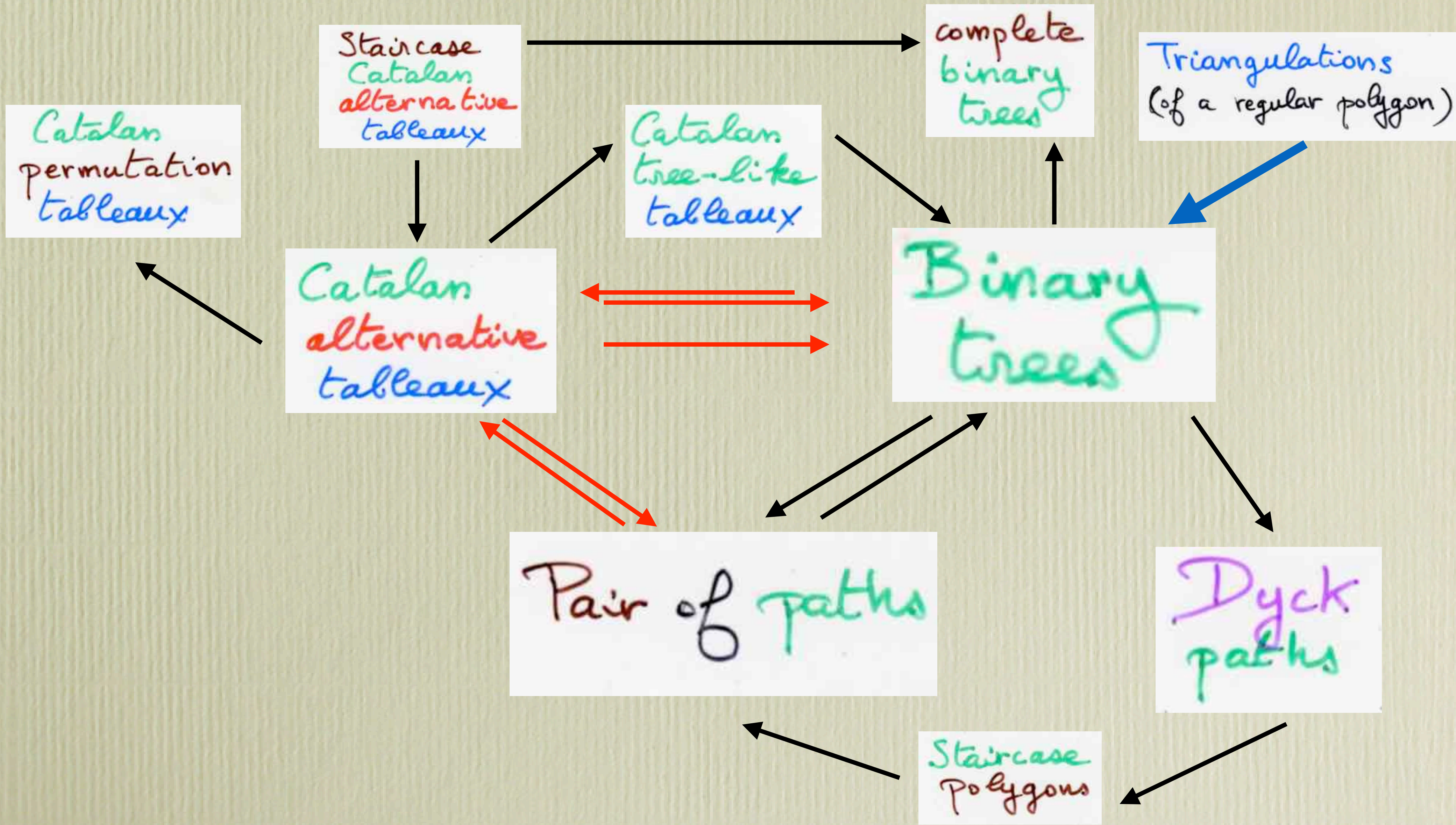


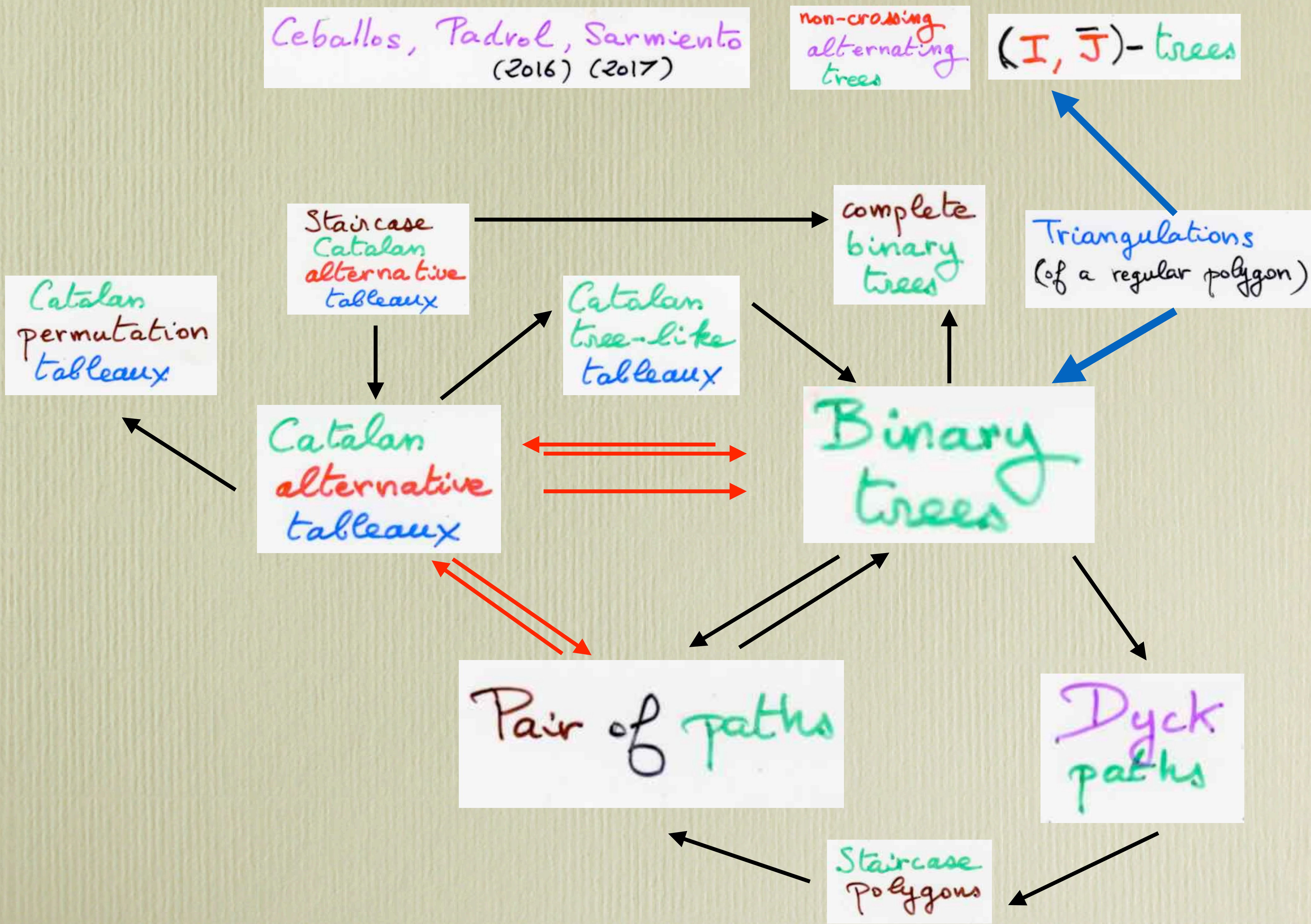
a flip in a triangulation defining the Tamari lattice

Triangulations
(of a regular polygon)



a flip in a triangulation defining the Tamari lattice


Staircase Catalan alternative tableaux
complete binary trees
Triangulations (of a regular polygon)
Catalan permutation tableaux
Catalan tree-like tableaux
Catalan alternative tableaux
Binary trees
Pair of paths
Dyck paths
Staircase polygons


Ceballos, Padrol, Sarmiento (2016) (2017)
non-crossing alternating trees
(I, J)- trees
Staircase Catalan alternative tableaux
complete binary trees
Triangulations (of a regular polygon)
Catalan permutation tableaux
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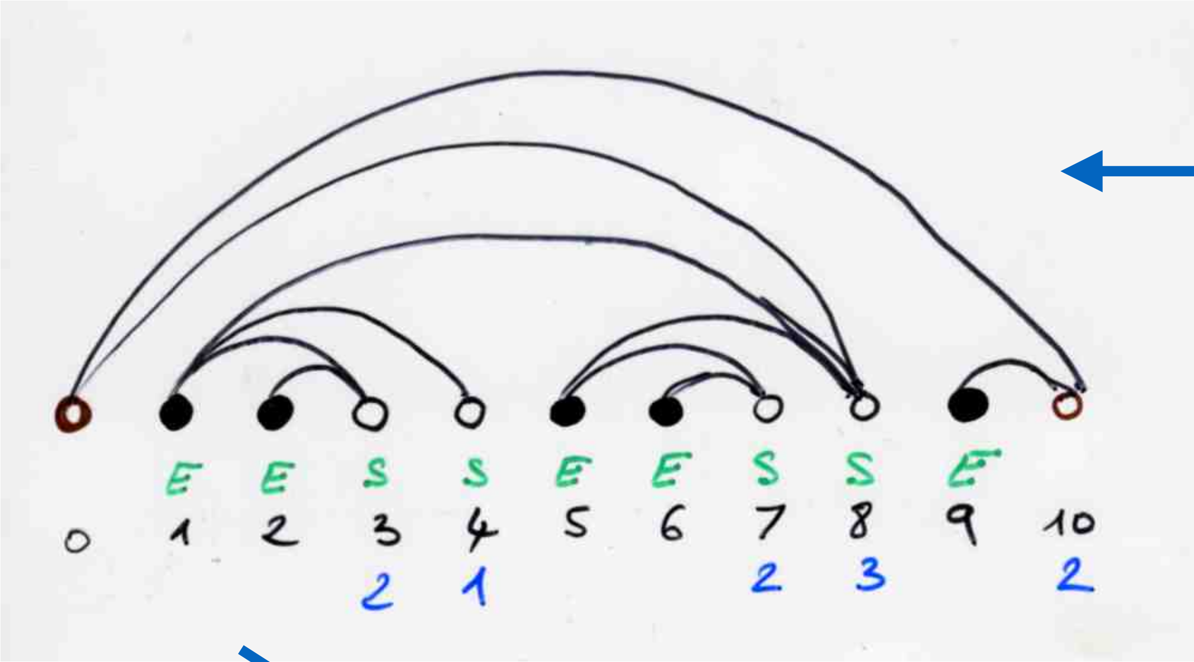
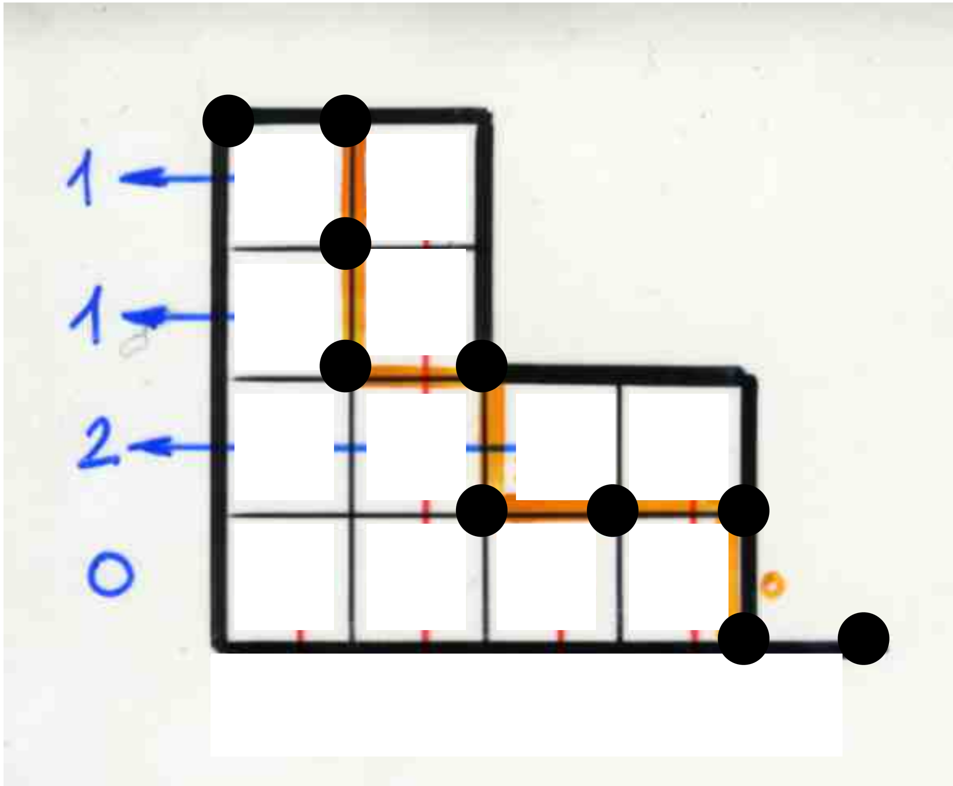


Triangulations
(of a regular polygon)

$(I, \bar{J})$-trees

Pair of paths



Ceballos, Padrol, Sarmiento
(2016) (2017)

non-crossing alternating trees

$(I, \bar{J})$-trees

ν-trees

Triangulations
(of a regular polygon)

Catalan tree-like tableaux

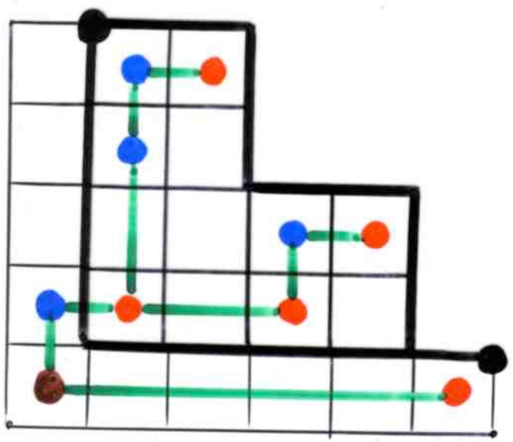
Pair of paths

Ceballos, Padrol, Sarmiento
(2016) (2017)

v-tree introduced by the 3 authors are the same as the binary tree underlying an alternative tableau, or equivalently a tree-like tableau

v-trees

geometry of v-Tamari lattice



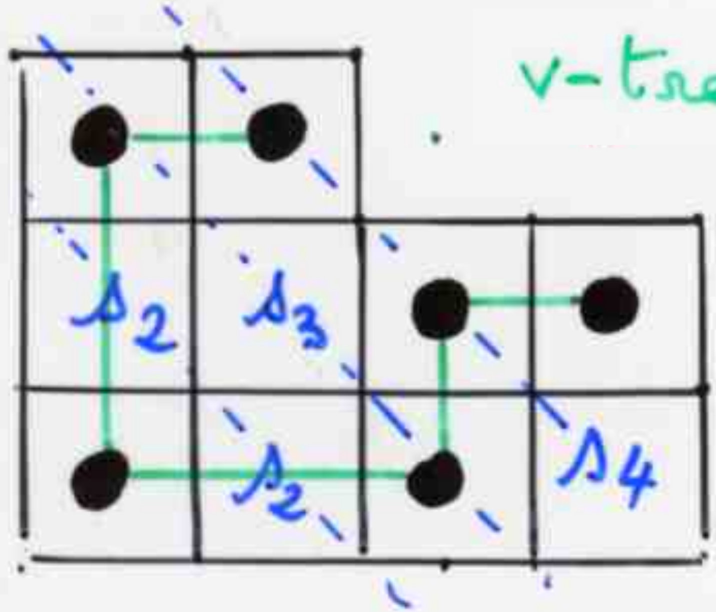
geometric realization of the Hasse diagram of the poset as the graph of a polyhedral complex

Catalan tree-like tableaux

Fuss-Catalan case (m-Tamari) m integer

→ subdivision of an associahedron induced by a tropical hyperplane arrangement

Ceballos, Padrol, Sarmiento
(2016) (2017)

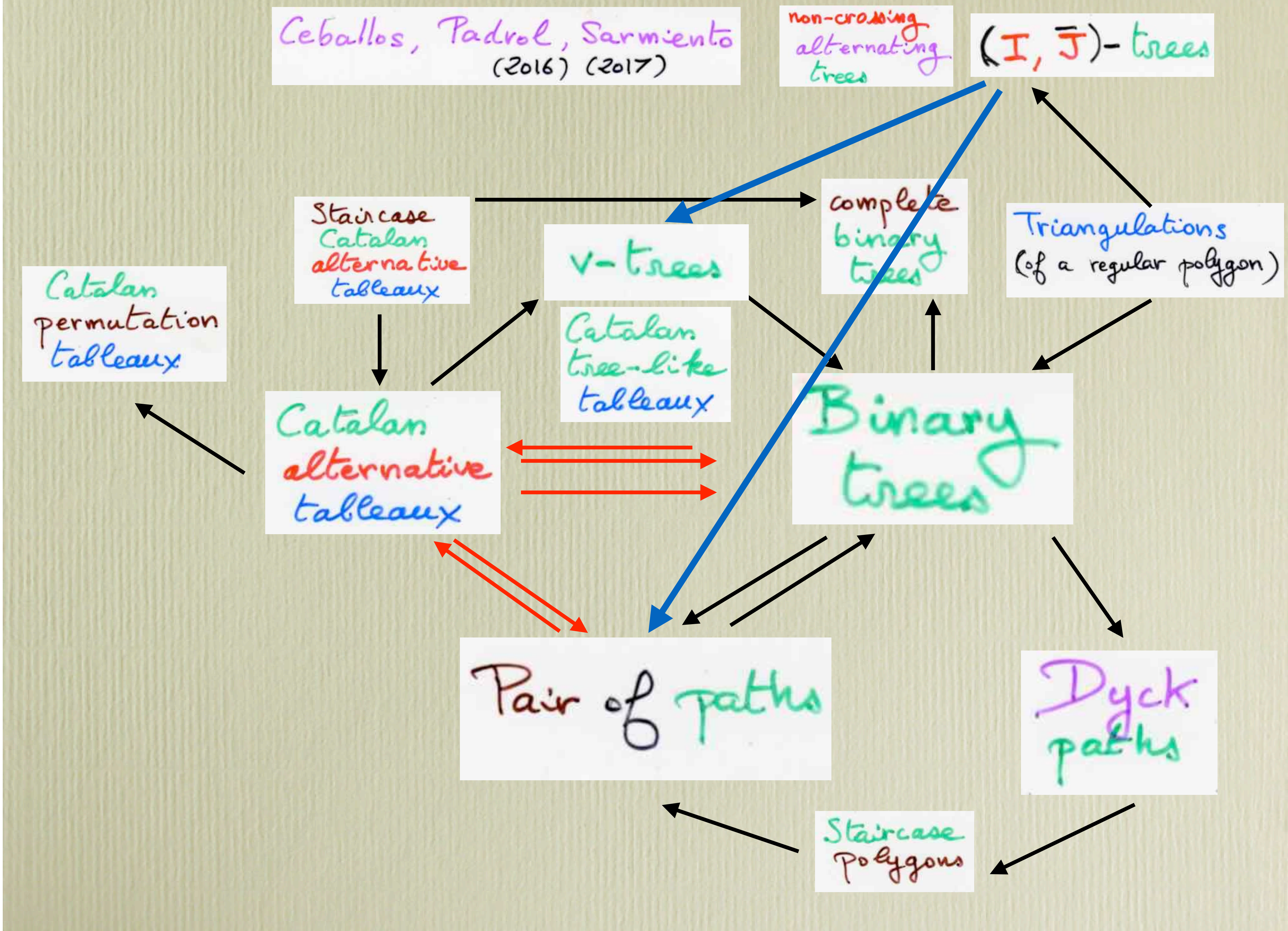


v-trees

subword complex

v-Tamari lattice : dual of a well chosen

$s_2 s_3 s_2 s_4 = [1, 4, 3, 5, 2, 6]$



Ceballos, Padrol, Sarmiento (2016) (2017)
non-crossing alternating trees
(I, J)- trees
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v-trees
complete binary trees
Triangulations (of a regular polygon)
Catalan permutation tableaux
Catalan tree-like tableaux
Catalan alternative tableaux
Binary trees
Pair of paths
Dyck paths
Staircase polygons

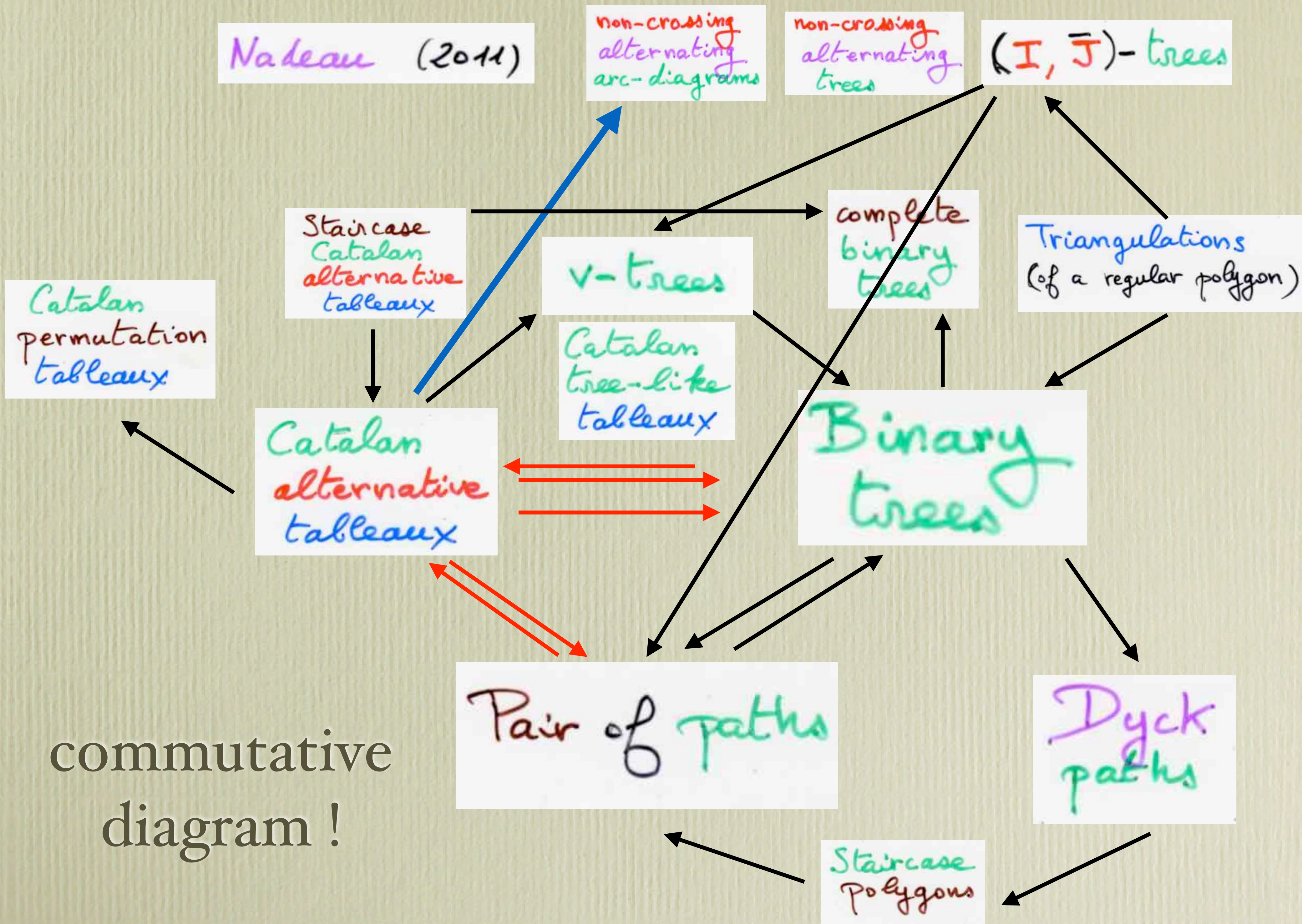
a festival of
commutative diagrams !

Nadeau (2011)

non-crossing
alternating
arc-diagrams = $(I, \bar{J})$-trees

more with ...

Catalan
alternative
tableaux

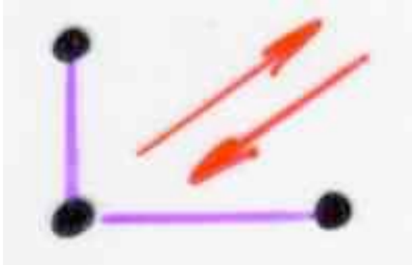


commutative diagram !

comments, remarks, references

Lam, Williams (2008)
total positivity for cominuscule Grassmannians

┘- move



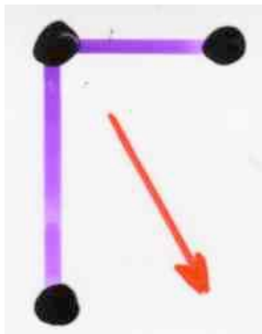
┐- move

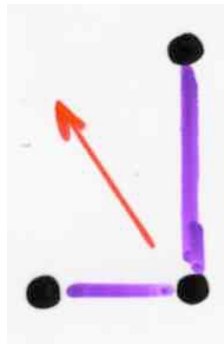
L- move

Karp, Williams, Zhang (2017)
decompositions of amplituhedra
$m=4$ scattering amplitudes in $\mathcal{N}=4$ supersymmetric Yang-Mills theory

┘- move

Γ- move

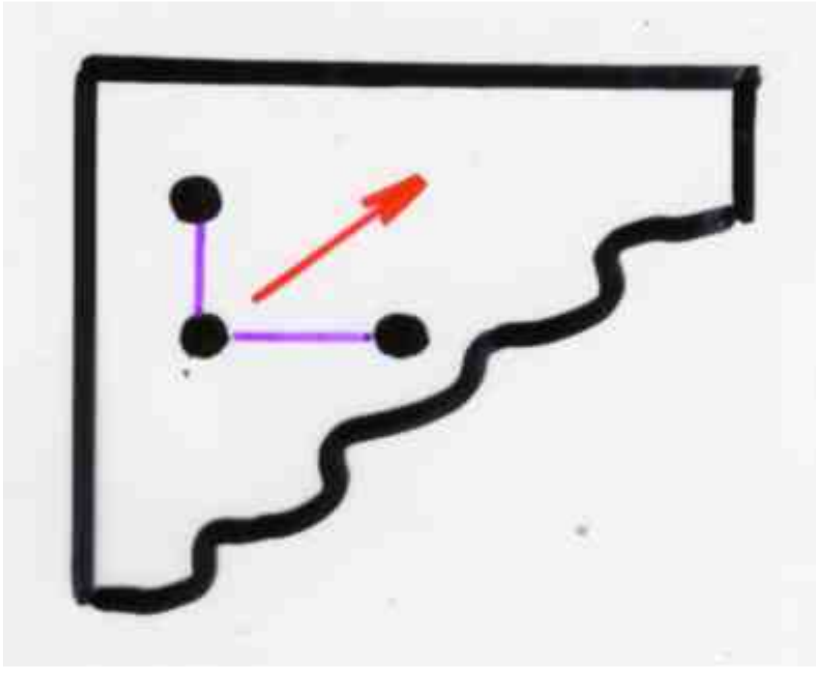


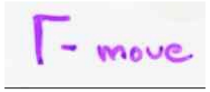


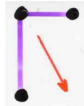
N. Bergeron, S. Billey (2010)
RC-graphs and Schubert polynomials

M. Rubey (2010)
Maximal 0-1 fillings of moon polyominoes with restricted chain length and RC-graphs

chute move

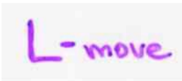


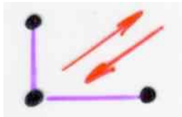
other references using what I call « Γ-move » 

 are:

N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment Math. 2 (1993), n°4, 257-269 available from http://projecteuclid.org/getRecord?id=euclid.em/1048516036.
(Γ-moves in the case of rectangle with 2 rows)

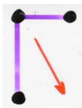
T. Lam and L. Williams, total positivity for cominuscule Grassmannians, New-York J. math., 14: 53-99, 2008, arXiv: 0710.2932 [math.CO]

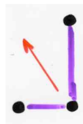
in here fact 

 Ferrers diagrams are in french notations

M. Rubey, Maximal 0-1-fillings of moon polyominoes with restricted chain lengths and RC-graphs, arXiv: 1009.3919v4 [math.CO] ((Γ-moves called « chutes »)

S. Karp, L. Williams, Y. Zhang, Decompositions of amplituhedra, ArXiv: 1708.09525 [math.CO]
here Γ-moves are

 and « Le-move » 



number of maximal chains in Tamari(n) ?

number of chains with maximum length

Nelson (2016) Ph.D.

Fishel, Nelson (2014)
bijection with standard shifted tableaux of staircase shape

This bijection is an immediate consequence of fact that the classical Tamari lattice is a maule: maximal chains with maximum length correspond to Γ-moves which are elementary, that is the corresponding rectangle is reduced to a cell of the square lattice. This property extends to Tamari(v) and the extension mixing Young and Tamari (slides 55-68, part II)

references:

S.Fishel and L.Nelson, Chains of maximum length in the Tamari lattice, Proc. Amer. math Soc. 142 (10):3343-3353, 2014

L.Nelson, Toward the enumeration of maximal chains in the Tamari lattices, Ph.D. Arizona sSate University, August 2016

L.Nelson, A recursion on maximal chains in the Tamari lattices, arXiv: 1709.02987 [math;CO]

alternative tableaux
and avatars

combinatorial structures

hypercube — associahedron — permutahedron

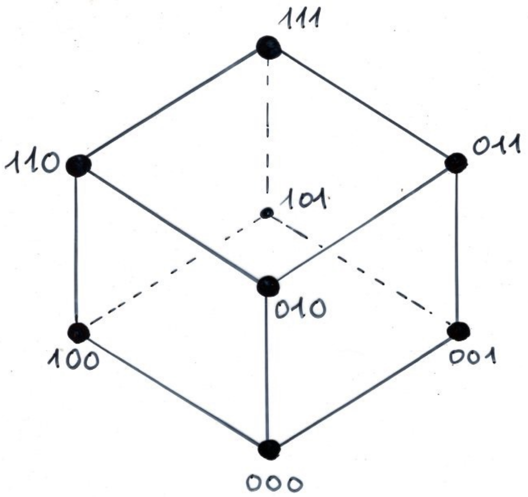
Boolean lattice inclusion $\longleftrightarrow$ Tamari order $\longleftrightarrow$ weak Bruhat order

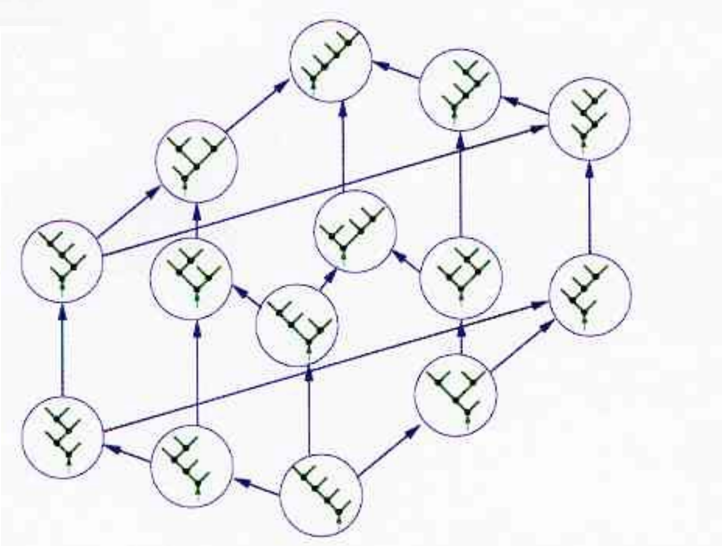
dim 2^{n-1} $\quad C_n \quad$ $n!$

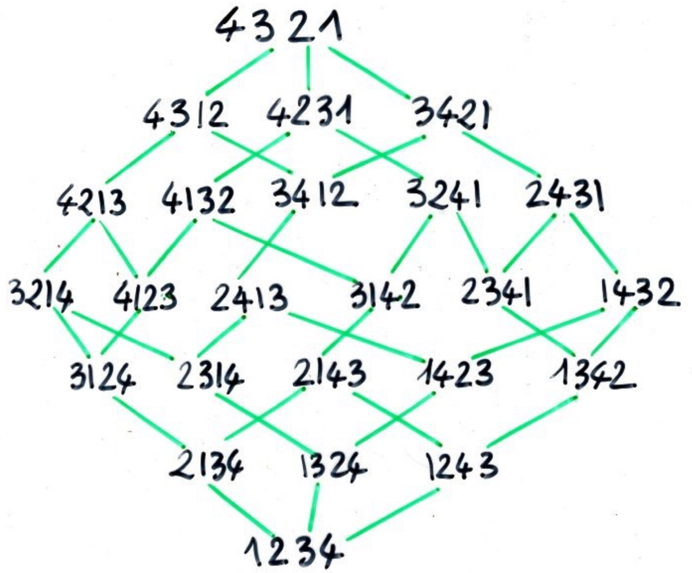
Catalan

algebraic structures

Hopf algebra







Some references for alternative tableaux and its avatars (enumerated by n!):

permutations tableaux: A. Postnikov, Total positivity, Grassmannians and networks, arXiv math/0609764, 2006

alternative tableaux, X.V. ("video-preprint") talk at Newton Institute, 23 April 2008, slides and video at https://sms.cam.ac.uk/media/1004

P. Nadeau, "On the structure of alternative tableaux", JCTA, Volume 118, Issue 5, July 2011, p1638-1660 or ArXiv 0908.4050,

P. Nadeau introduced a class of "alternative trees" in bijection with alternative tableaux, and a subclass of "non-crossing alternative trees" in bijection with Catalan alternative tableaux, objects which are the same as "(I,J_) trees ".

staircase tableaux: S. Corteel and L. Williams, Duke Math J. 159 (2011), 385--415 , arXiv math/0910.1858, 2009

tree-like tableaux, J.C. Aval, A. Boussicault and P. Nadeau (FPSAC2011, Reikjavik) and Electronic Journal of Combinatorics, Volume 20, Issue 4 (2013), P34

more with permutations tableaux:

S. Corteel, A simple bijection between permutations tableaux and permutations, arXiv: math/0609700

S. Corteel and P. Nadeau, Bijections for permutation tableaux, Europ. J. of Combinatorics, 2007

S. Corteel and L.K. Williams, Tableaux combinatorics for the asymmetric exclusion process, Adv in Apl Maths, to appear, arXiv:math/0602109

E. Steingrimsson and L. Williams Permutation tableaux and permutation patterns, J. Combinatorial Th. A., 114 (2007) 211-234. arXiv:math.CO/0507149

For the four subclasses enumerated by Catalan numbers see:

X.V., FPSAC 2007, Tianjiin : Chine (2007) or arXiv math/ 0905.3081 (bijection Catalan permutation tableaux -- pair of paths (u,v))

J.C. Aval and X.V., (about Catalan alternative tableaux and Loday-Ronco Hopf algebra of trees) SLC, 63 (2010) B63h or arXiv math 0912.0798

here we have rewritten the above bijection Catalan permutation tableaux -- pair (u,v) as a bijection Catalan alternative tableaux -- pair of paths (u,v).

the bijection Catalan alternative tableaux -- Catalan tree-like tableaux can be easily found as a special case of the bijection between alternative tableaux -- tree-like tableaux, see for example:
tree-like tableaux, J.C. Aval, A.Boussicault and P.Nadeau (FPSAC2011, Reikjavik) and Electronic Journal of Combinatorics, Volume 20, Issue 4 (2013), P34

more material about permutations tableaux, alternative, tree-like and staircase tableaux in:

The cellular ansatz: bijective combinatorics and quadratic algebra
Course given par X.V. at IMSc, Chennai, January-March 2018
website: https://www.imsc.res.in/~viennot/bjc-course.html# Part III
or http://www.viennot.org/bjc-course.html Part III
(with links to slides and videos)

the paper introducing the lattice Tamari(v) is:
P.-L. Préville-Ratelle and X.V., « An extension of Tamari lattices », Transactions AMS, 369 (2017) 5219-5239
note: curiously the title in the Transactions « The enumeration of generalised Tamara intervals » is wrong (!). This is the title of the paper [13] quoted in our paper.
An extended abstract of the paper can be found in the Proceeding of the FPSAC'2015, Daejon, South Korea, DMTCS proc. FPSAC'15, 2015, 133-144

The work of C.Ceballos, A.Padrol and C.Sarmiento we very briefly mentioned in slides 79-90 (part II) can be found in:
C.Ceballos, A.Padrol and C.Sarmiento, Geometry of v-Tamari in types A and B, ArXiv: 1611.09794 [math.CO] (47 pages). To be published in Transactions of the A.M.S.
and in the slides of a talk at the 78th SLC devoted to the 60th birthday of Jean-Yves Thibon http://www.mat.univie.ac.at/~slc/ see « preface » with the talk of Cesar Ceballos « v-Tamari lattices via subwords complex »

v-trees introduced by the 3 authors are the same as the binary tree underlying an alternative tableau, or equivalently a tree-like tableau

Catalan
permutation
tableaux

Catalan
tree-like
tableaux

Catalan
alternative
tableaux

n!

permutations

permutation tableaux

tree-like tableaux

Aval, Boussicault, Nadeau (2013)

alternative tableaux

X.V. (2008)

n!

permutations

permutation tableaux

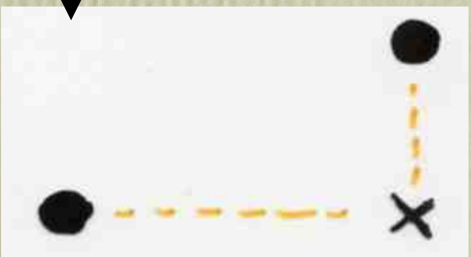
tree-like tableaux

Aval, Boussicault, Nadeau (2013)

alternative tableaux

X.V. (2008)

Steingrímsson, Williams (2007)

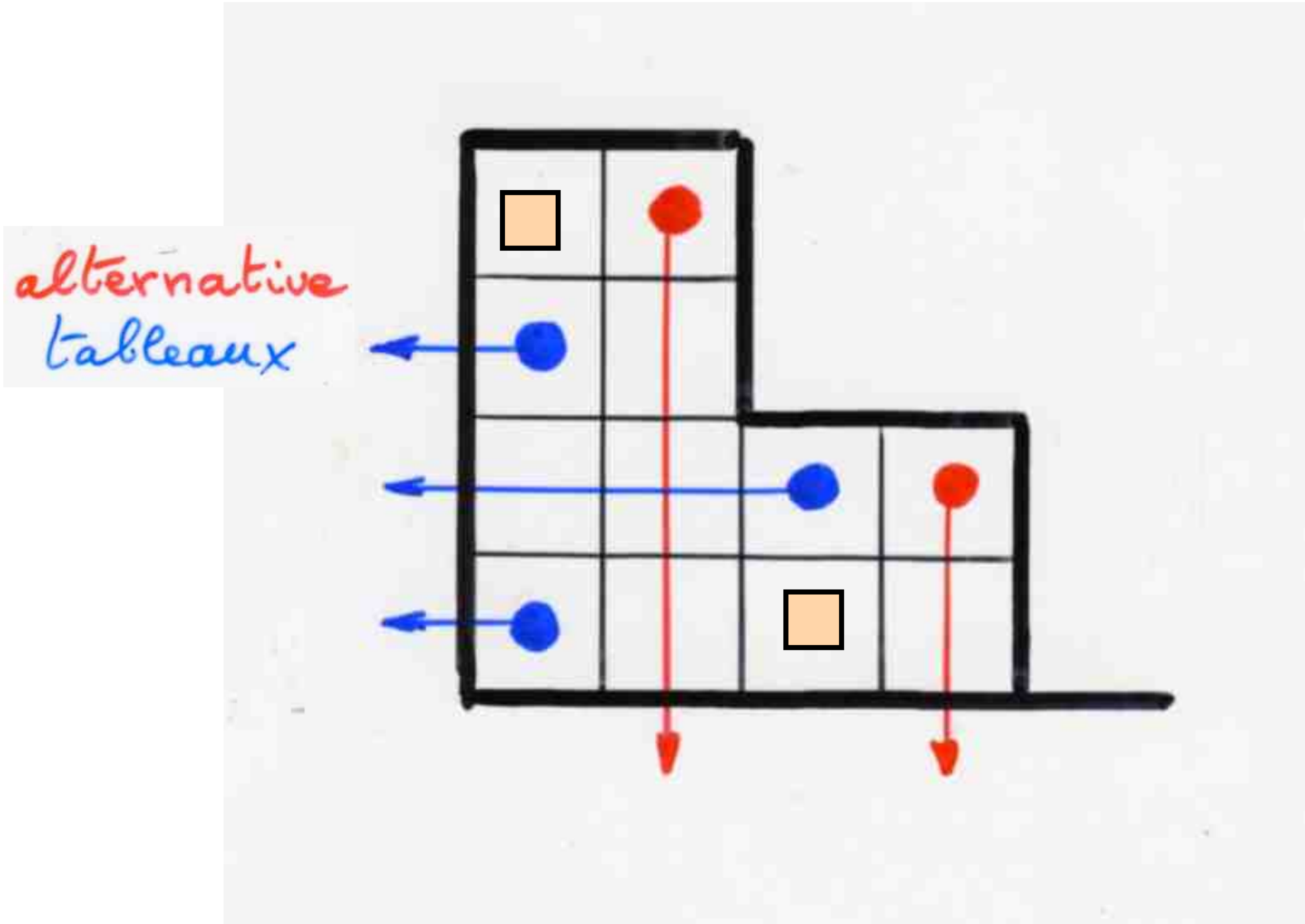


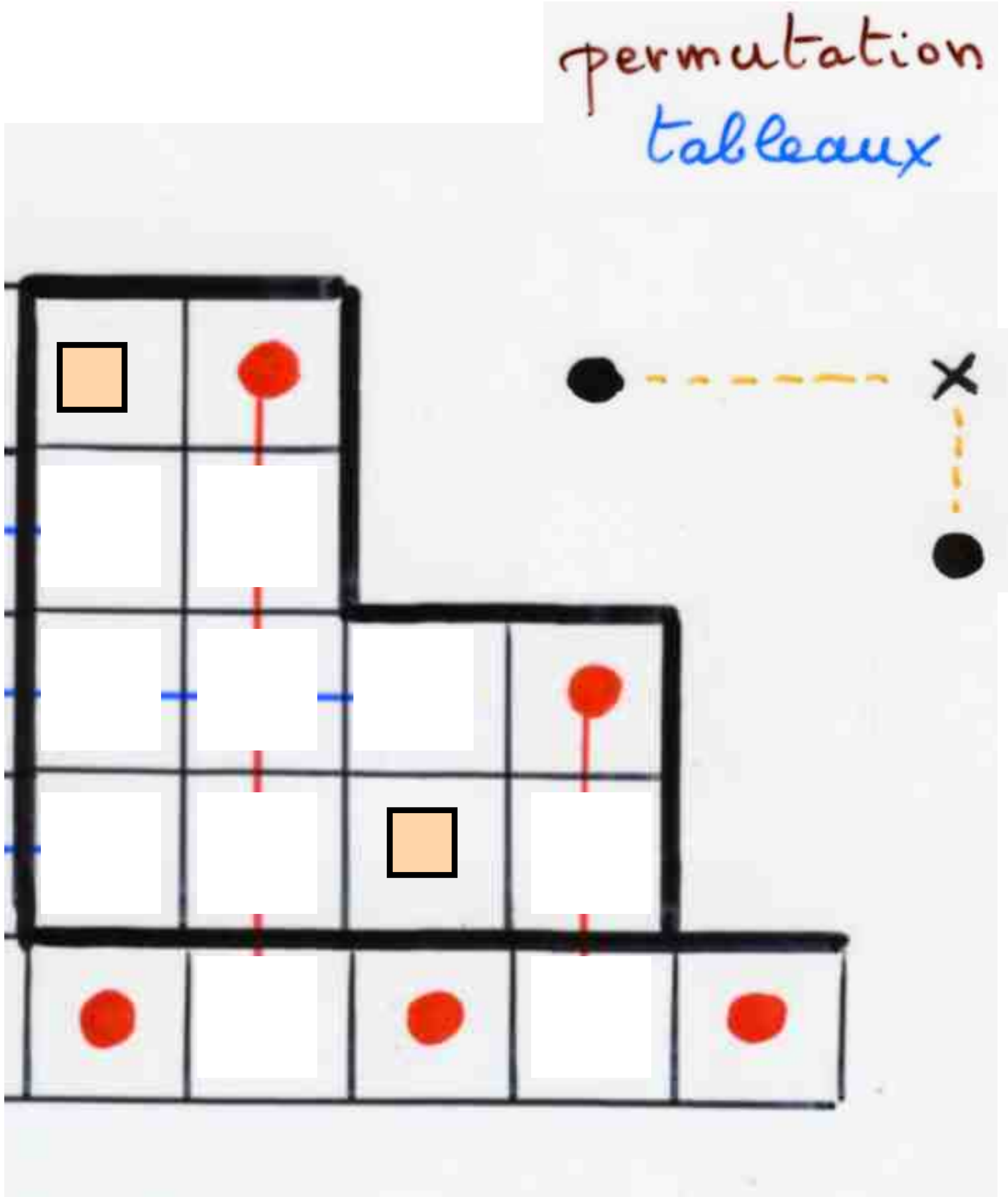
⅃-diagrams

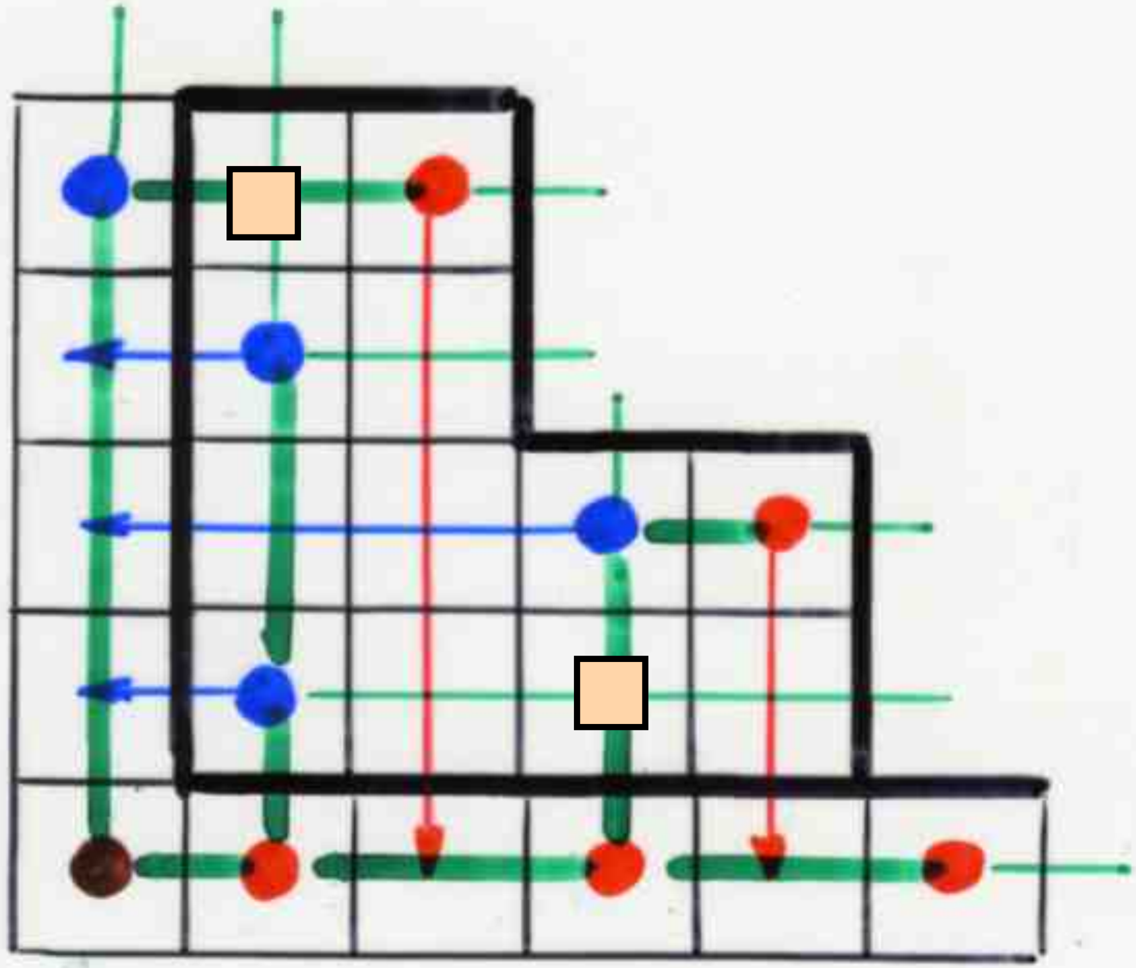
decorated permutations

Postnikov (2006)

totally non-negative part of the type A Grassmannian


alternative
tableaux


permutation
tableaux



The Adela bijection

demultiplication
In the PASEP algebra

PASEP algebra

$$Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

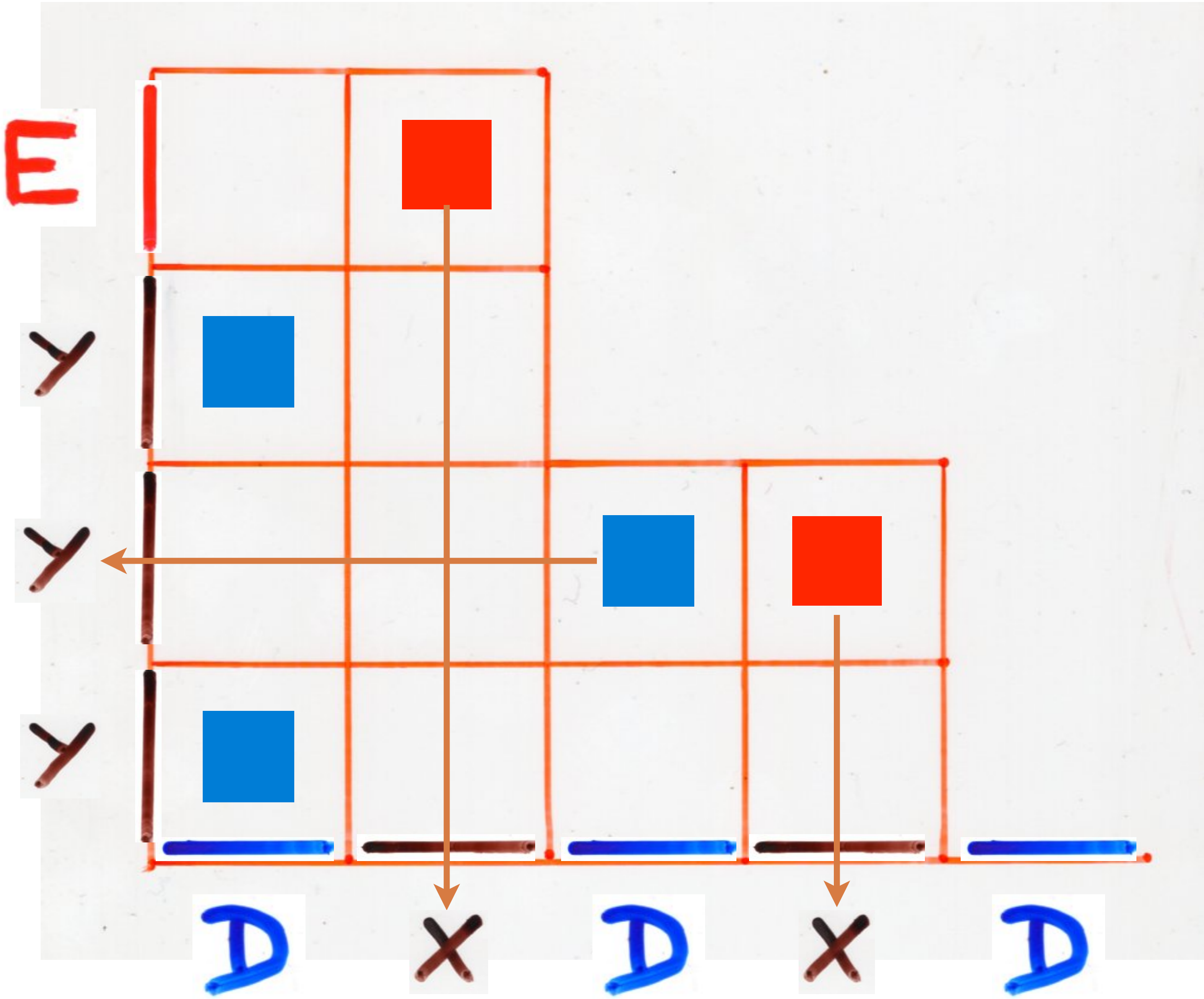
see Ch 2c, p3-8
duplication of equations in
quadratic algebrea
Ch 2c, p9-15
duplication in the PASEP algebra

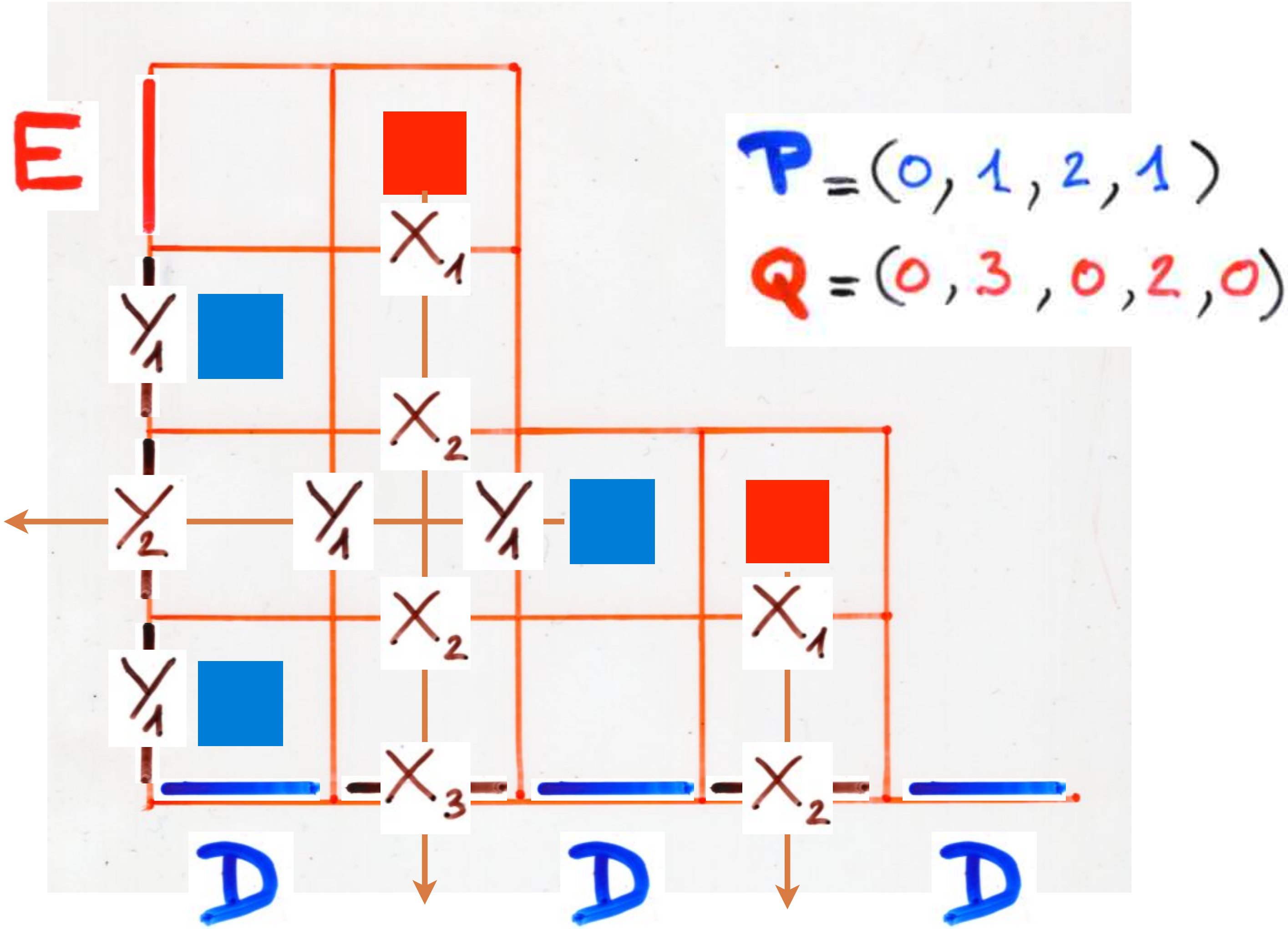
BJC3 ,Ch 2c, the bijective course, IMSc, Chennai, 2018
website: https://www.imsc.res.in/~viennot/bjc-course.html# Part III

$$DE = ED + EX_1 + Y_1 D$$

$$\begin{cases} X_1 E = E X_2 \\ \cdots \\ X_i E = E X_{i+1} \\ \cdots \end{cases} \qquad \begin{cases} D Y_1 = Y_2 D \\ \cdots \\ D Y_i = Y_{i+1} D \\ \cdots \end{cases}$$

$$X_i Y_j = Y_j X_i$$


E
Y
Y
Y
D
X
D
X
D



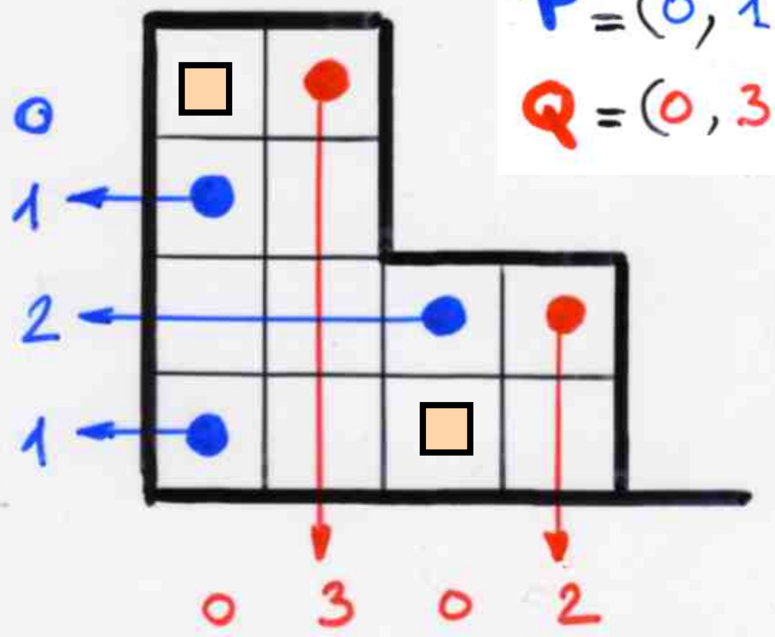
E
P = (0, 1, 2, 1)
Q = (0, 3, 0, 2, 0)
X_1
X_2
X_2
X_3
X_1
X_2
Y_1
Y_2
Y_1
Y_1
Y_1
D
D
D

Adela bijection

Adela(T) = (P, Q)

$P(T) = (a_1, a_2, \dots, a_k)$

$Q(T) = (b_1, \dots, b_\ell)$



$P = (0, 1, 2, 1)$

$Q = (0, 3, 0, 2, 0)$

$a_i = \begin{cases} 0 & \text{if no } \bullet \text{ in row } i \\ 1 + \text{number of cells} & \text{in row } i \end{cases}$

$b_j = \begin{cases} 0 & \text{if no } \bullet \text{ in the } j^{th} \text{ column} \\ 1 + \text{number of cells} & \text{in the } j^{th} \text{ column} \end{cases}$

the Adela bijection

Adela(T) = (P, Q)

The map T $\longrightarrow$ (P, Q) is a bijection between alternative tableaux and some pairs (P, Q) of vectors of integers.

This fact can be proved using the « **cellular ansatz** » methodology described in the course:
The cellular ansatz: bijective combinatorics and quadratic algebra
Course given par X.V. at IMSc, Chennai, January-March 2018
website: https://www.imsc.res.in/~viennot/bjc-course.html# Part III
or http://www.viennot.org/bjc-course.html Part III. (with links to slides and videos)

The cellular ansatz methodology associate certain combinatorial objects to some quadratic algebra, together with a systematic way to construct some bijections analogue to the RSK bijection between permutations and pair of Young tableaux. In the case of the so-called PASEP algebra defined by generators E, D and the relation DE = ED+E+D, we get the alternative tableaux enumerated by n!.
In the case of the Weyl-Heisenberg algebra defined by UD = DU+Id, we get the permutations.

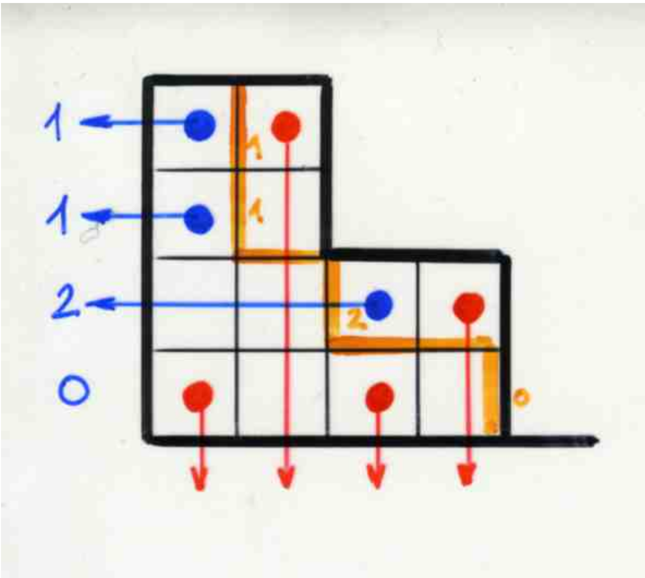
Then we define a methodology called « demultiplication » of equations (see Ch2b and Ch2c of this course given at IMSc 2018), which gives the RSK bijection in the case of the algebra UD = DU+Id, and the above Adela bijection in the case of the PASEP algebra.

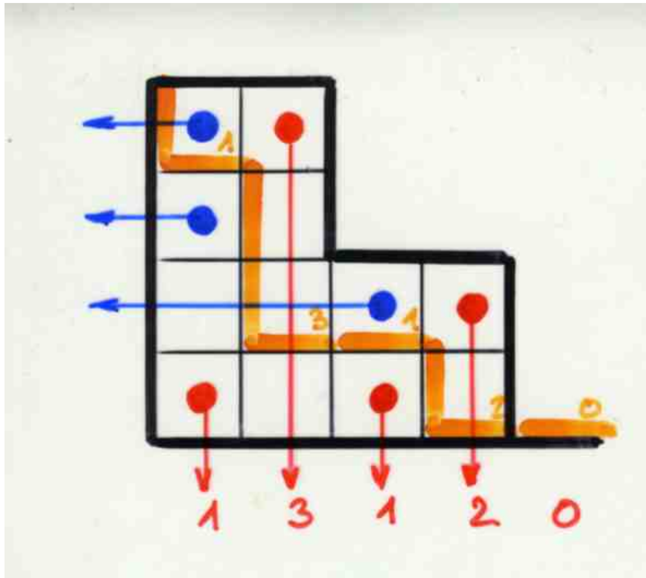
the Adela duality

$$\mathrm{Adela}(T) = (P, Q)$$

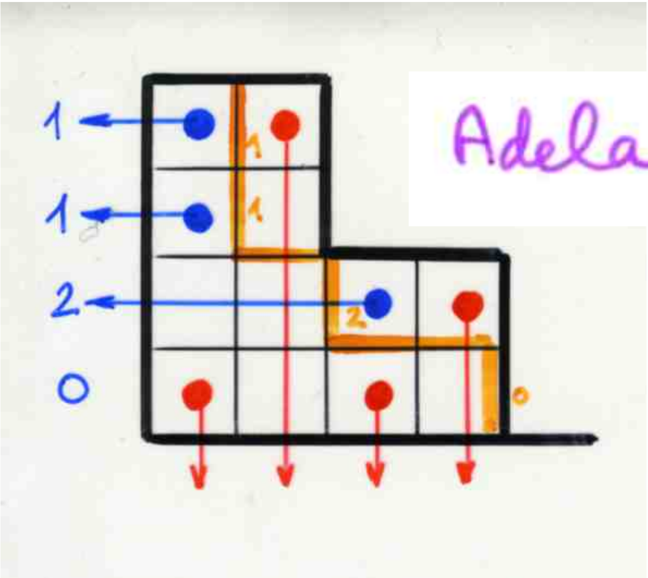
In the case of Catalan alternative tableaux, the column vector Q is determined by the row vector P and in that case the Adela bijection is reduced to the bijection $T \longrightarrow P$ described in this talk (slide 100).

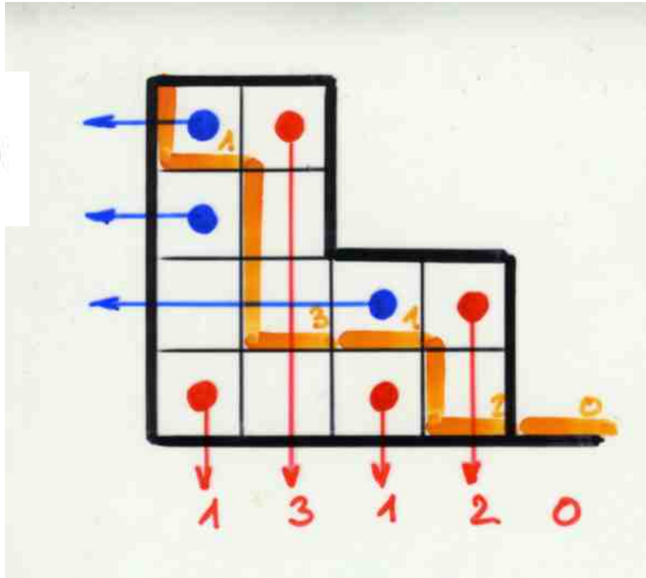
In that case I call the map exchanging $P \longrightarrow Q$ « the Adela duality » (see next slide). This is equivalent to the duality described on slides 64-65 (theorem 2).





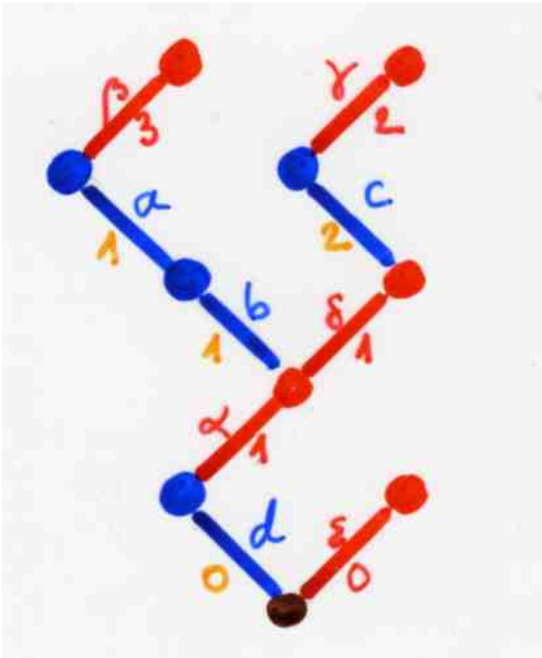
Adela(T) = (P, Q)





the Catalan case

Adela duality



a	b	c	d
1	1	2	0

α	β	γ	δ	ε
1	3	1	2	0

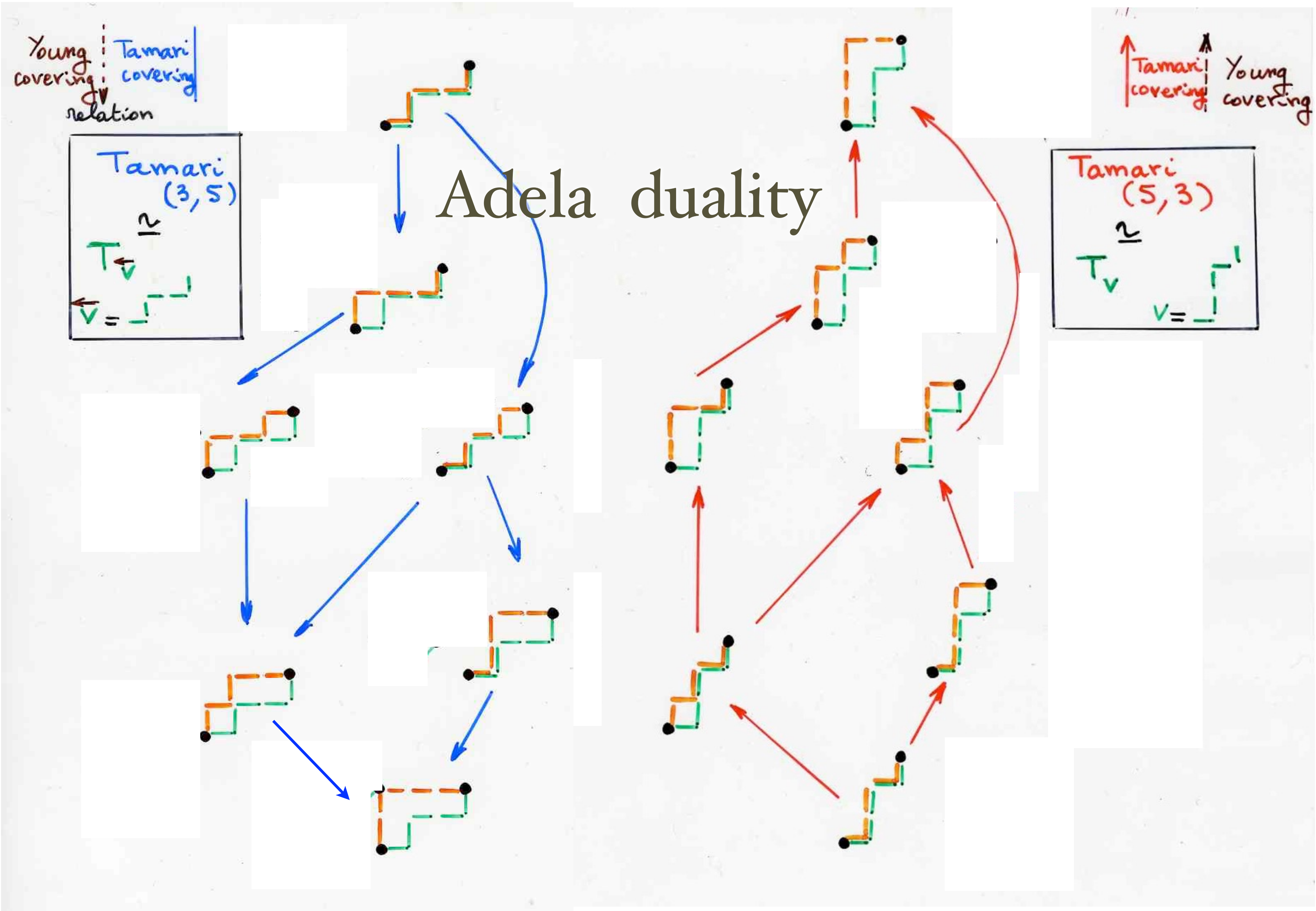
right height of left edges

left height of right edges

inorder

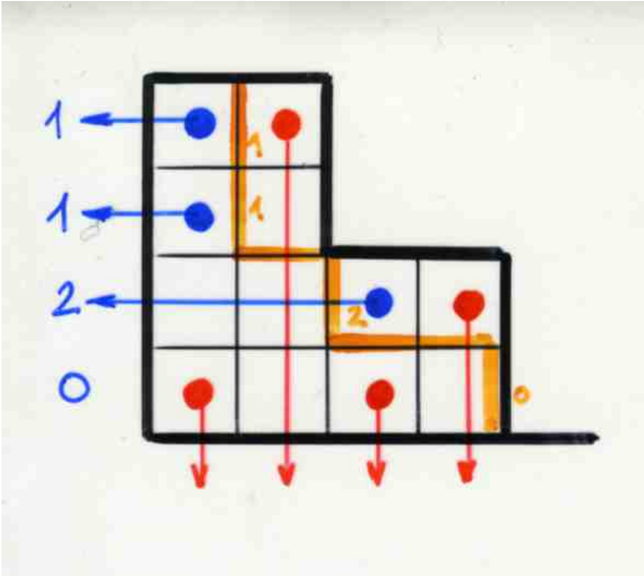
(= symmetric order)

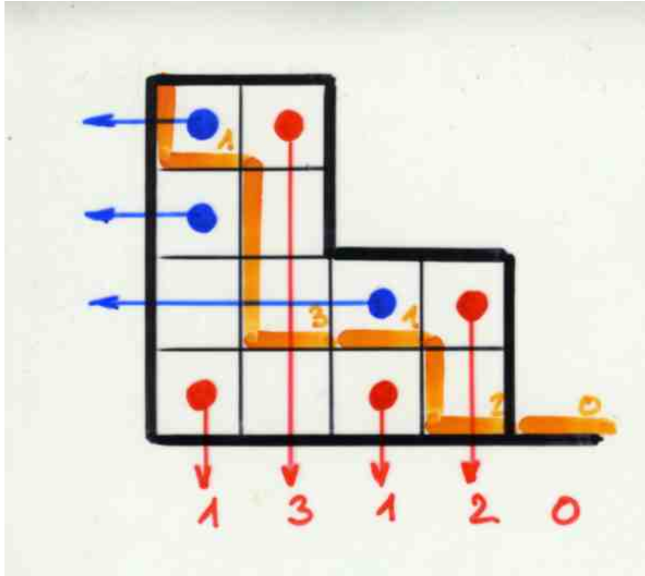
Duality $T_v \leftrightarrow T_{\overleftarrow{v}}$



Adela duality

Adela(T) = (P , Q)





Catalan alternative tableaux ⟷ Pair of paths

see Ch4, this course BJC3

The "Adela duality"

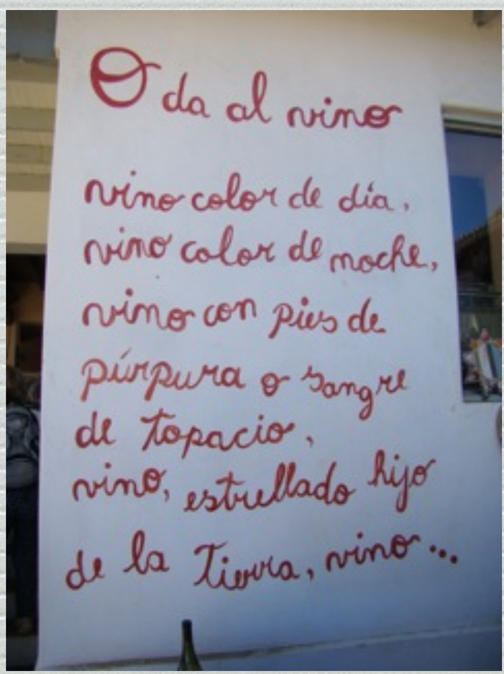
P (T) ⟷ Q (T)

Why Adela bijection ?

Isla Negra
Pablo Neruda

The names «Adela bijection» and «Adela duality» is in honour of my friend Adela where part of this research was done in her house in Isla Negra, Chile, inspiring place where Pablo Neruda spent many years in his house in front of the Pacific Ocean.

Isla Negra
Pablo Neruda



slides on the website of SLC 79,
Bertinoro, 10-13 September 2017

Séminaire
Lotharingien de
Combinatoire

also on www.viennot.org

Thank you !

Link to the video of this maths seminar on:

http://www.viennot.org

https://www.imsc.res.in/~viennot