Maule: tilings, Young and Tamari lattices under the same roof

IMSc, Chennai
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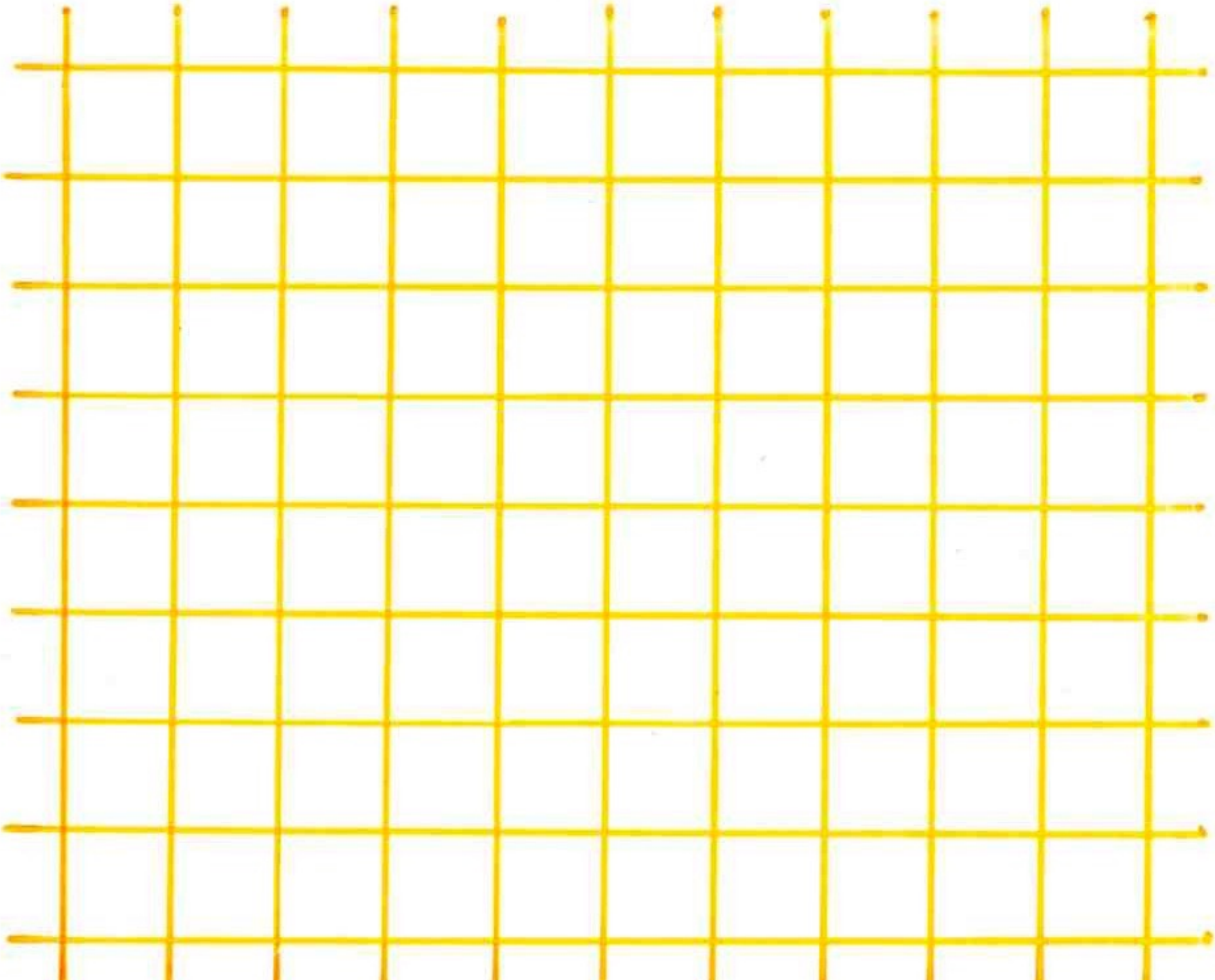
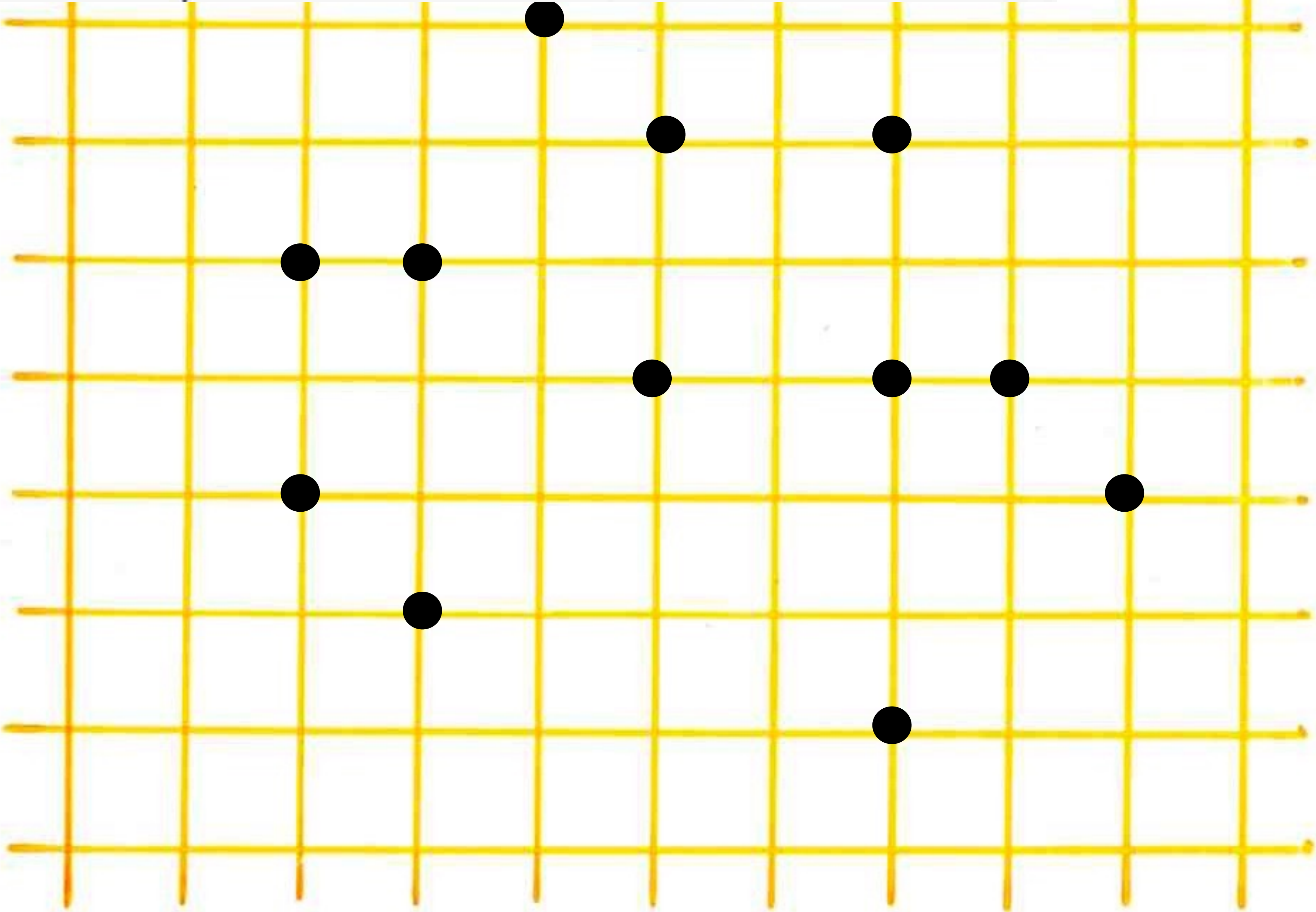
Maule



A cloud is a finite subset of the square lattice $\mathbb{Z} \times \mathbb{Z}$



X **cloud** is a finite subset of the square lattice $\mathbb{Z} \times \mathbb{Z}$

Definition

Γ - move

X cloud. Let $\alpha, \beta, \gamma \in X$ in Γ-position, that is

$$\begin{tikzcd} \alpha \arrow[r, dash, dashed] \arrow[d, dash, dashed] & \beta \\ \gamma & \end{tikzcd}$$

Suppose that all the vertices of the rectangle

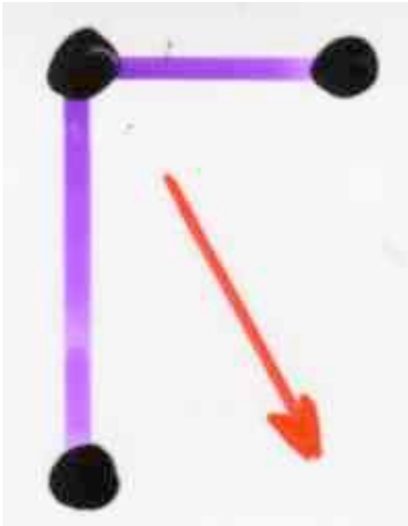
$$\begin{tikzcd} \alpha \arrow[r, dash, dashed] \arrow[d, dash, dashed] & \beta \arrow[d, dash, dashed] \\ \gamma \arrow[r, dash, dashed] & \delta \end{tikzcd}$$

, except α, β, γ, are empty (denoted $\times$)

Definition Γ- move

X, Y clouds $\quad Y = \Gamma(X)$

Γ - move



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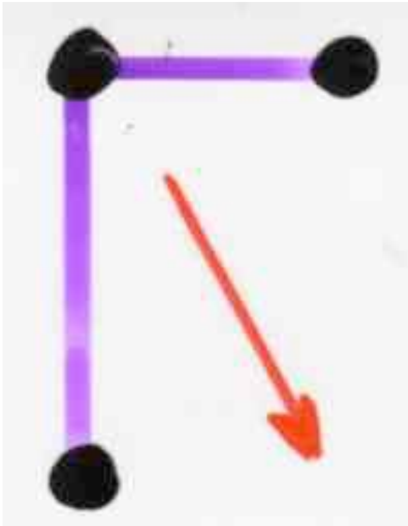
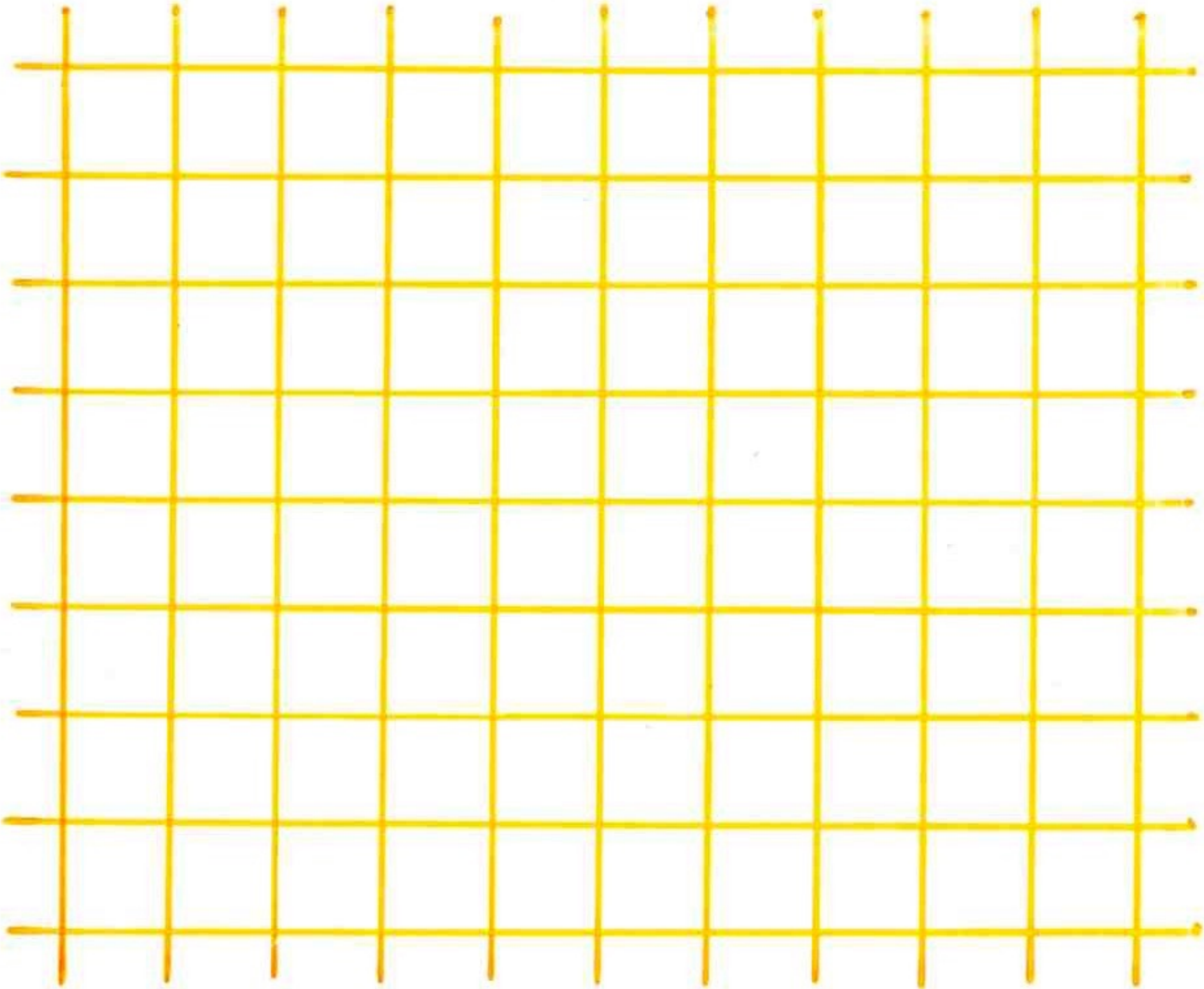
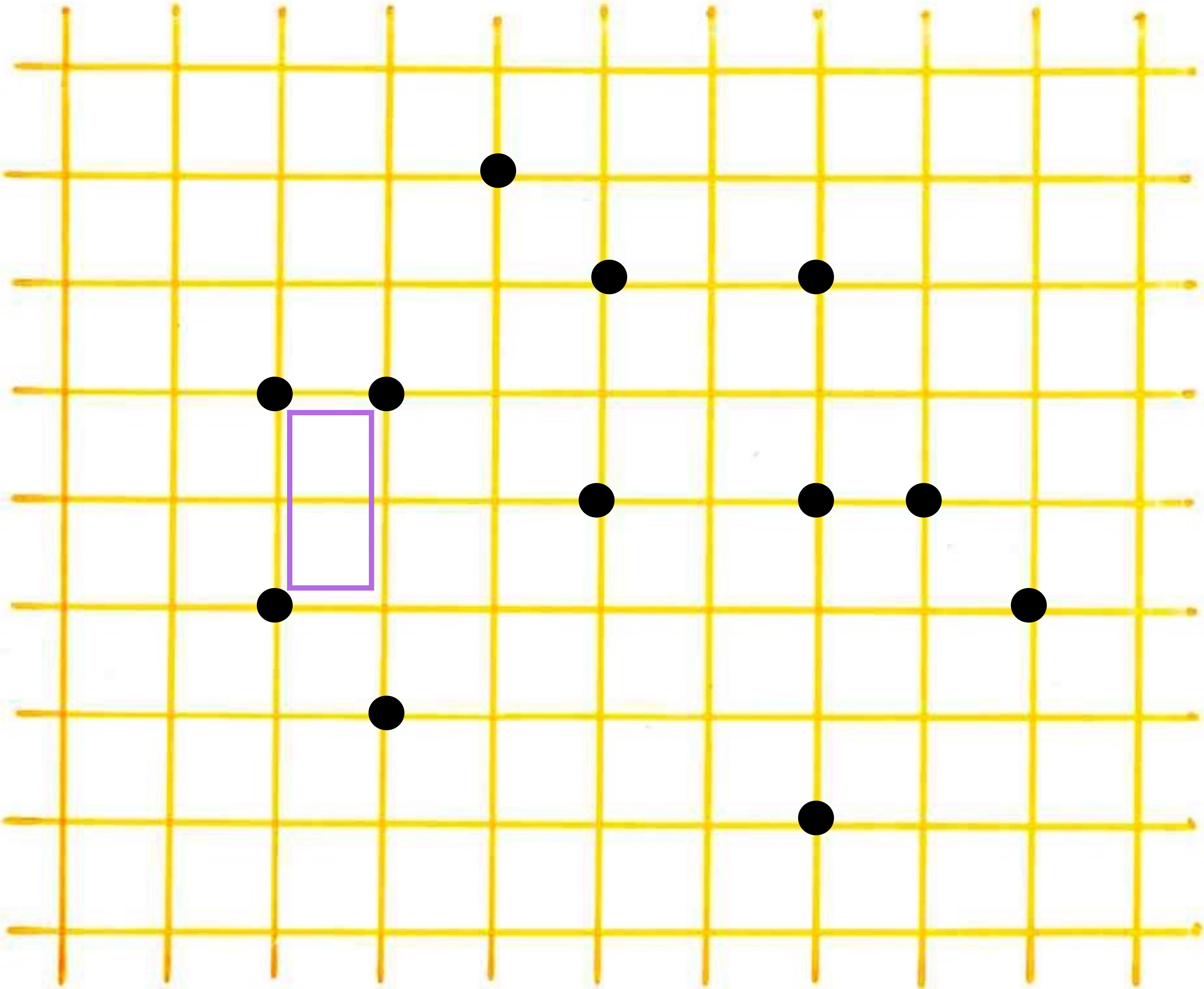


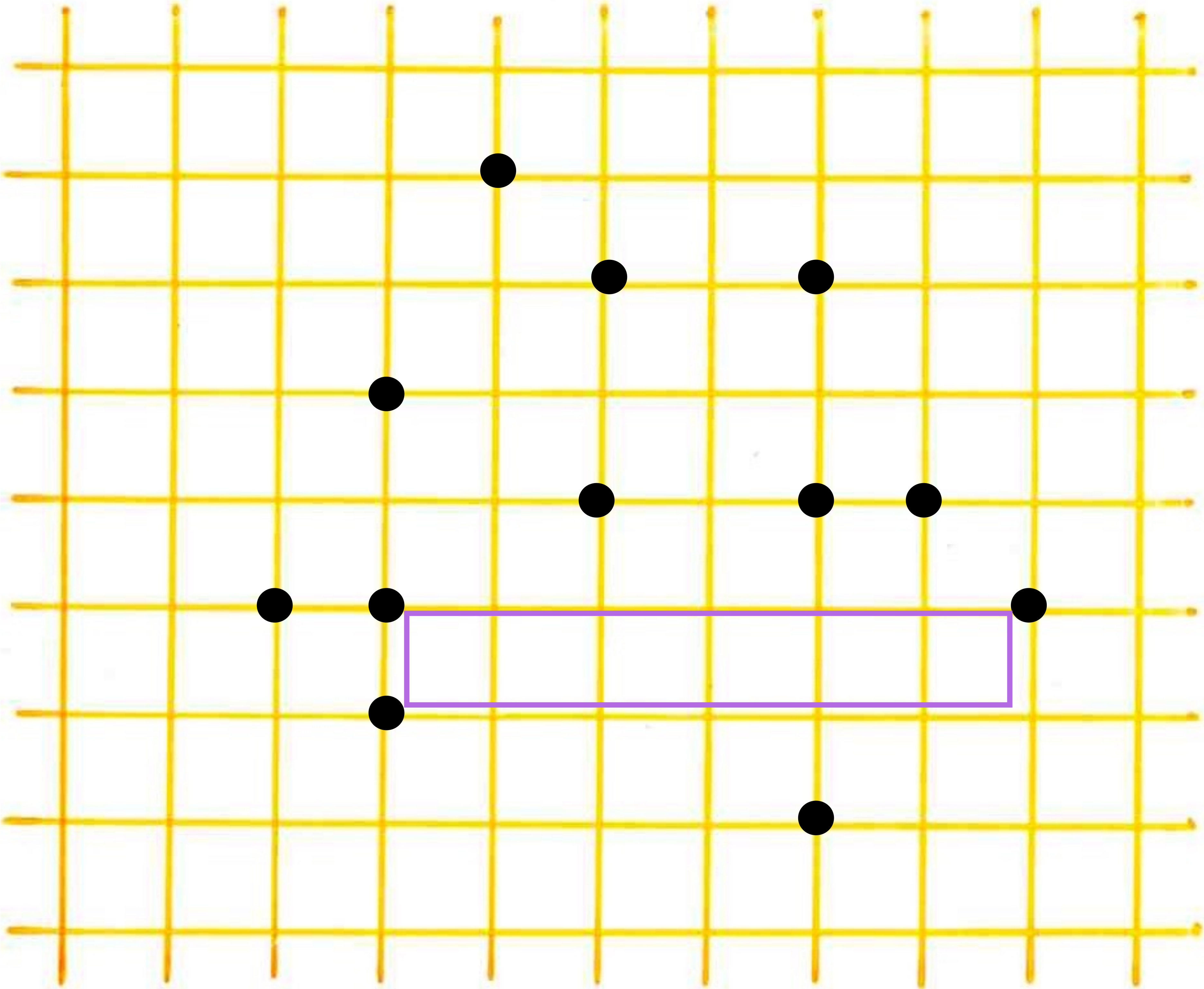
notation $X \xrightarrow{\Gamma} Y$

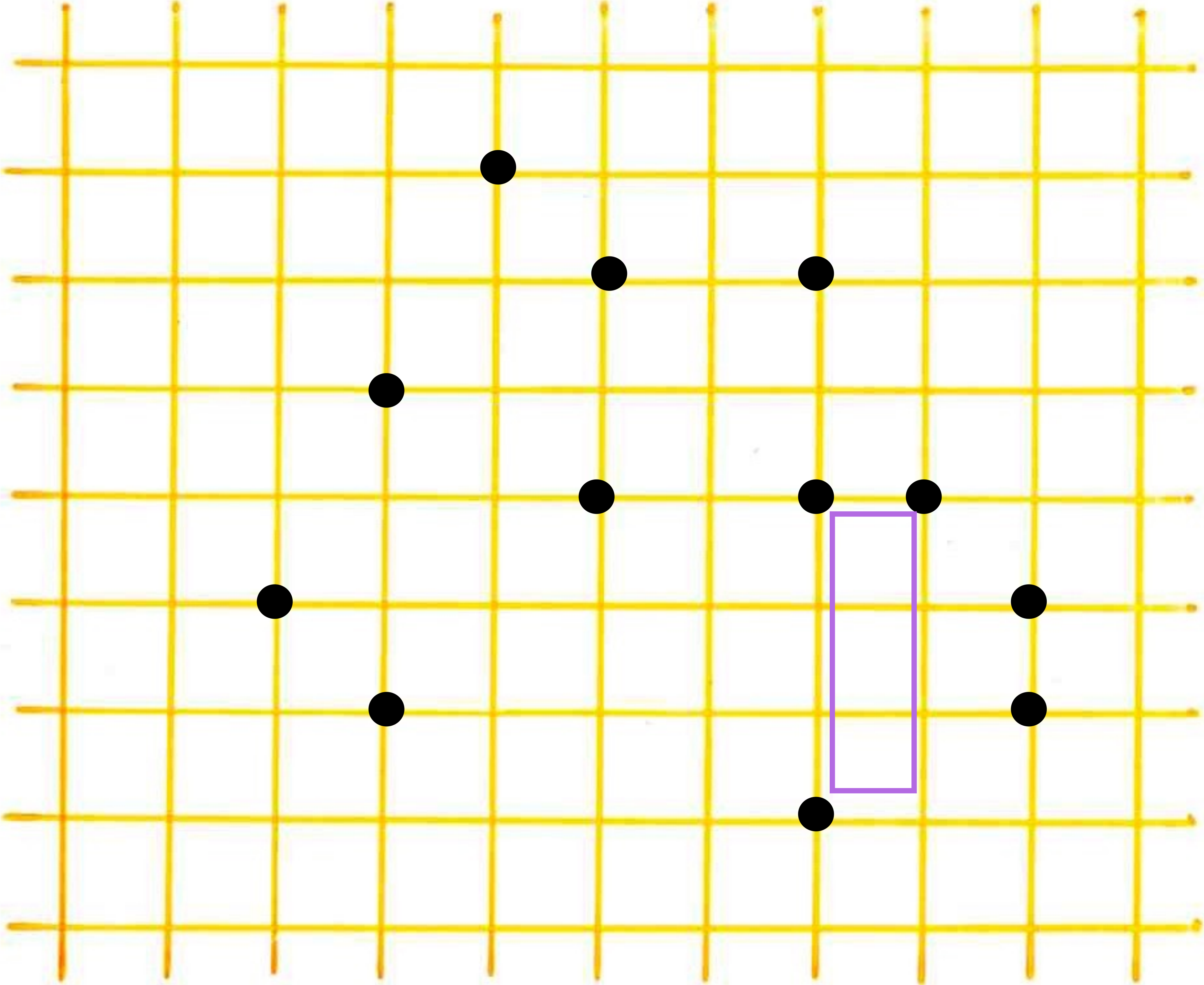
<u>Definition</u> The relation $X \xrightarrow{\Gamma^*} Y$ is the transitive closure of the relation $X \xrightarrow{\Gamma} Y$

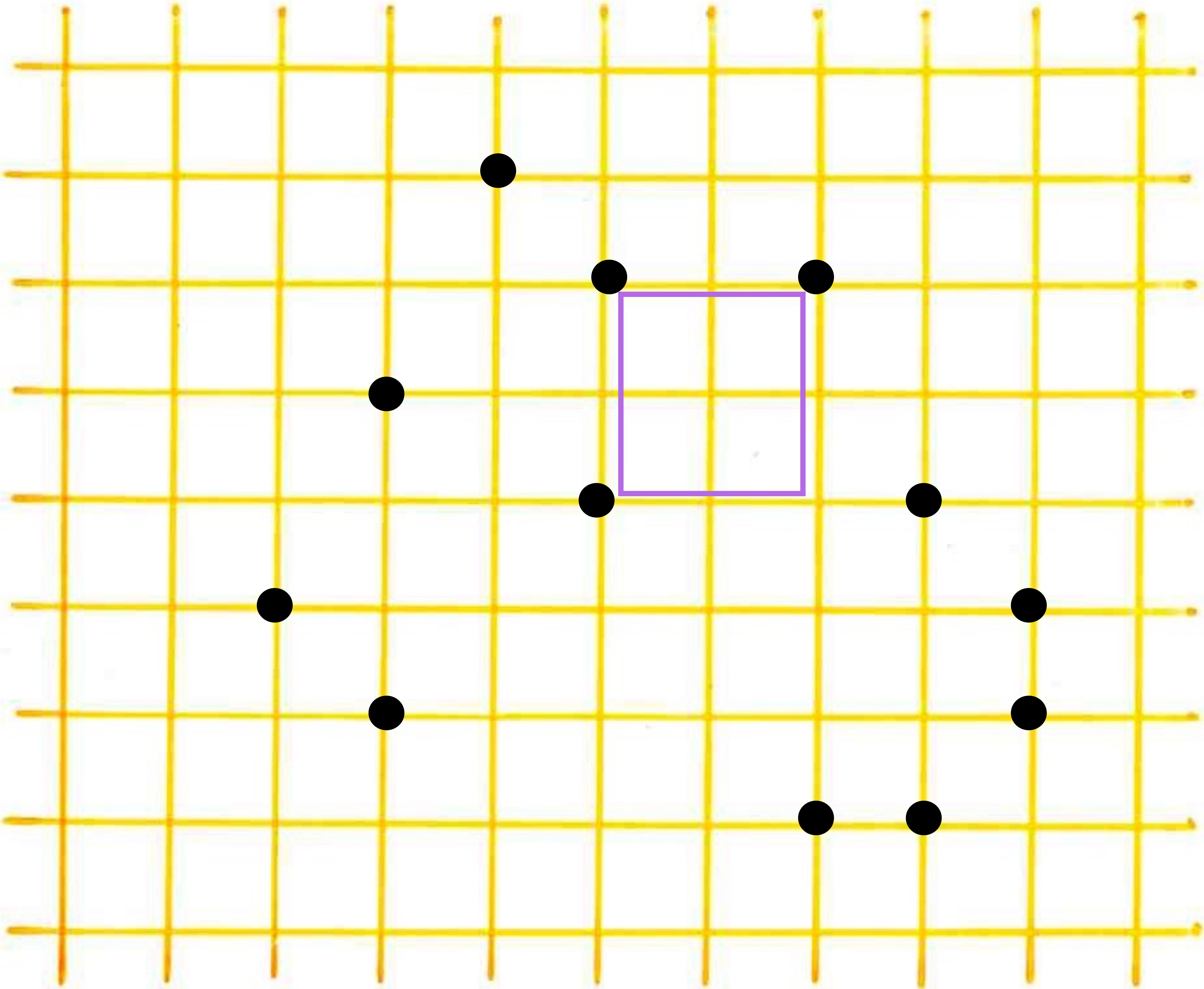
- $X \xrightarrow{\Gamma^*} Y$ is an order relation



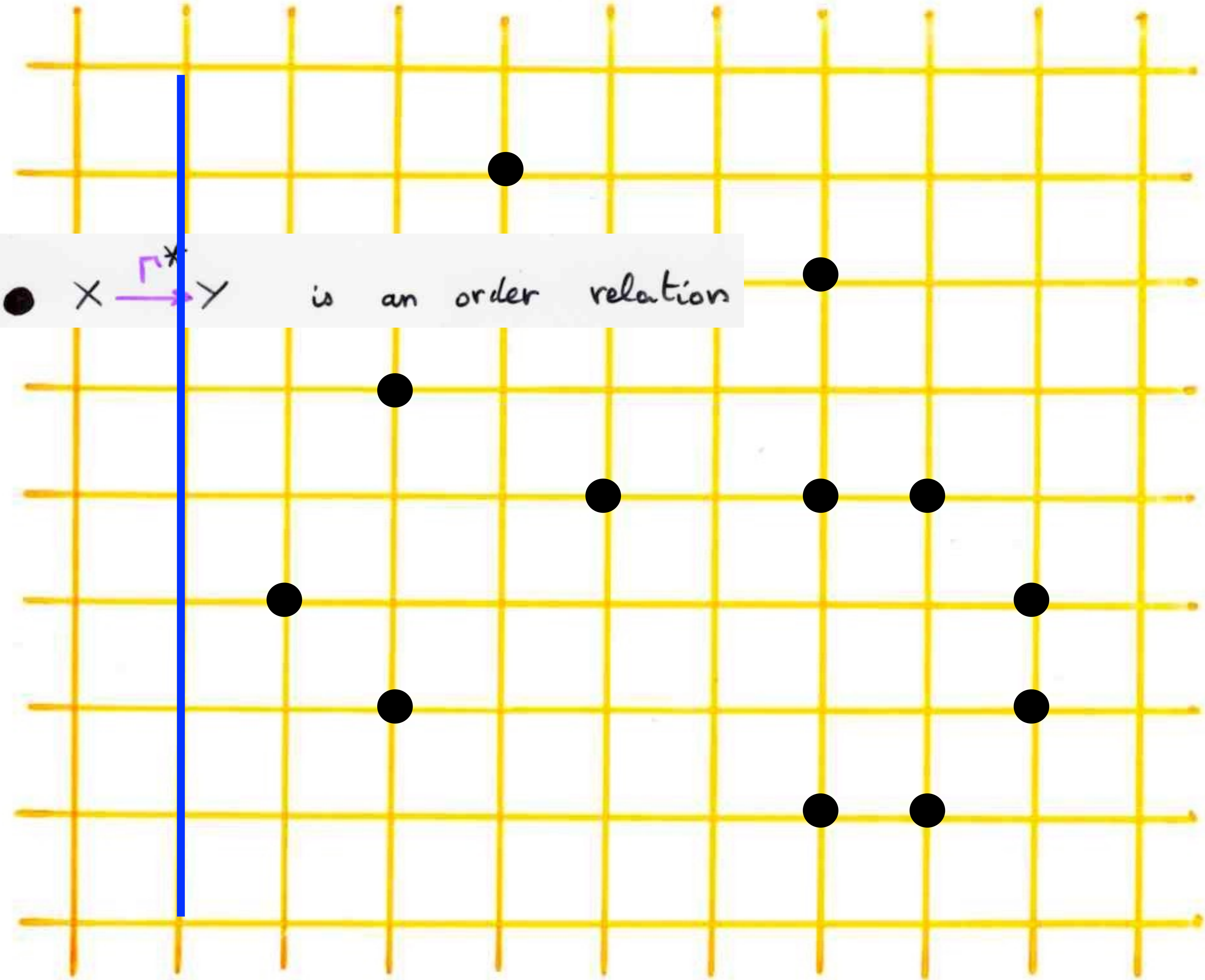




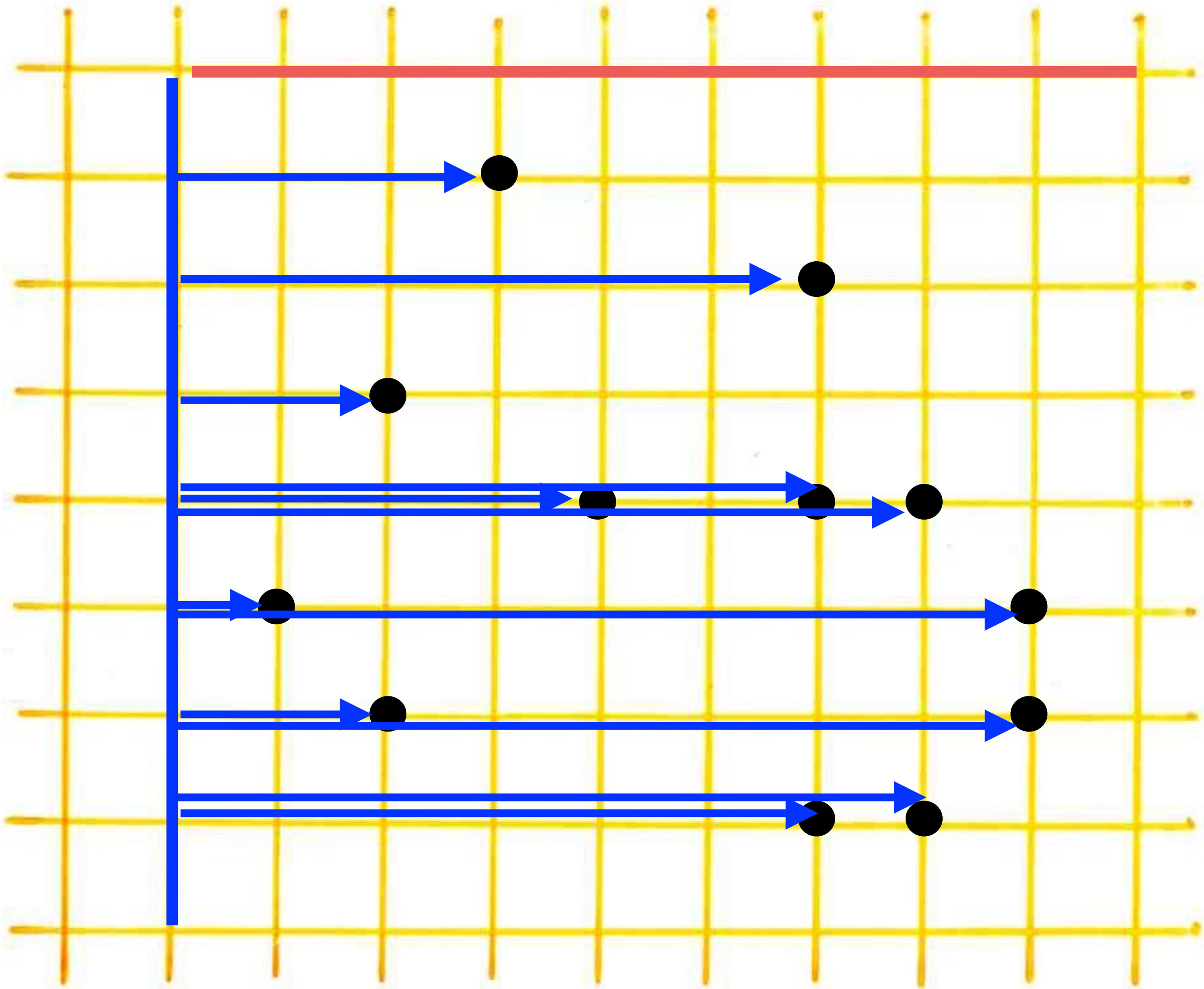




• $X \xrightarrow{\Gamma^*} Y$ is an order relation



- $X \xrightarrow{\Gamma^*} Y$ is an order relation



After a Γ-move, the sum of the distances of the points of the cloud to the blue vertical line will increase at least by one, and thus no cycles are possible.

Main definition The poset $\mathrm{Maule}(X)$ is the set of all clouds obtained from X by a succession of Γ-moves, (i.e. $X \xrightarrow{\Gamma^*} Y$) equipped with the order relation $Y \xrightarrow{\Gamma^*} Z$ for $Y, Z \in \mathrm{Maule}(X)$.

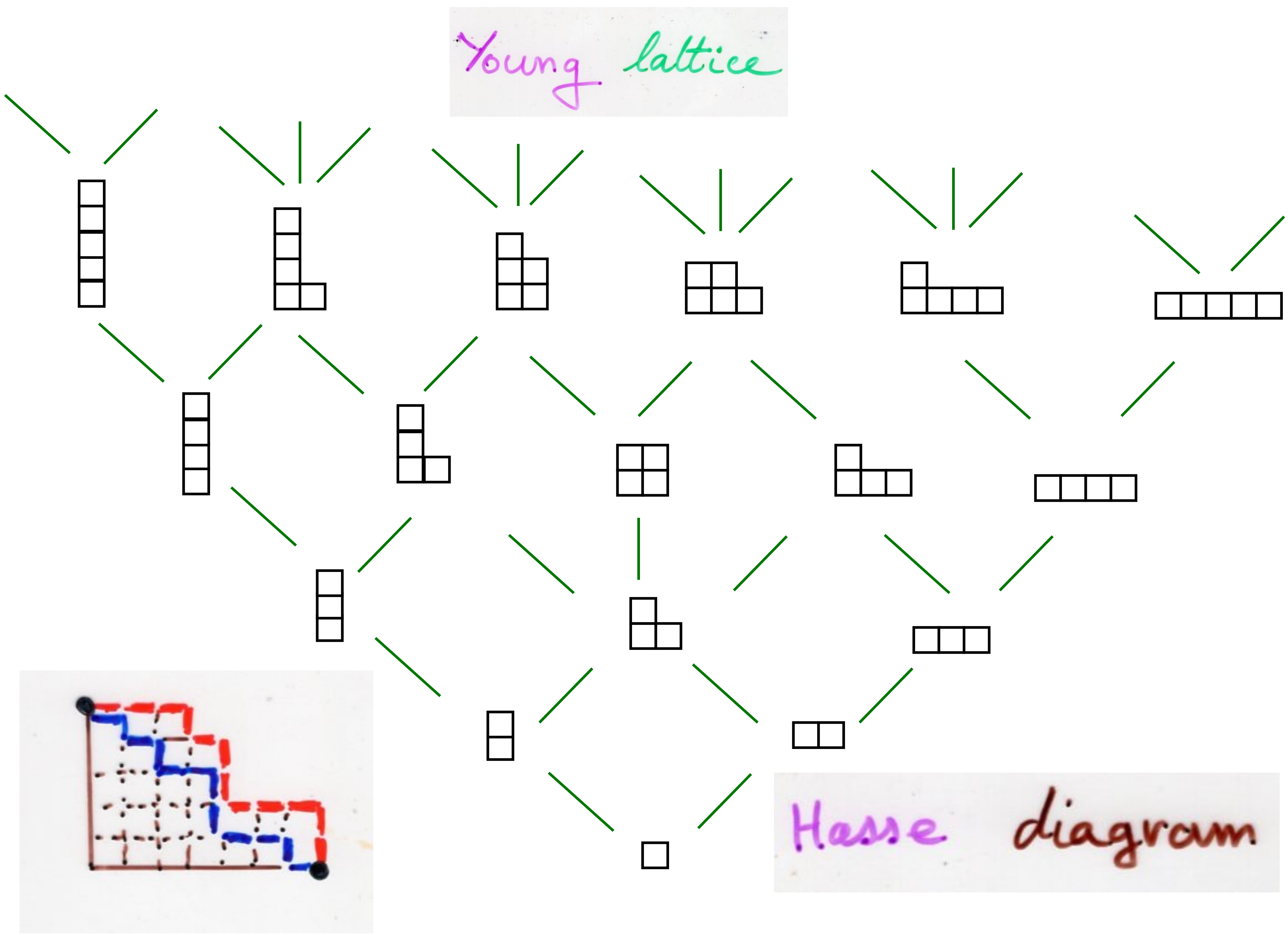
poset $\preccurlyeq$

partially ordered set

covering relation

$\alpha \preccurlyeq \beta$
no γ between
α and β

Hasse diagram

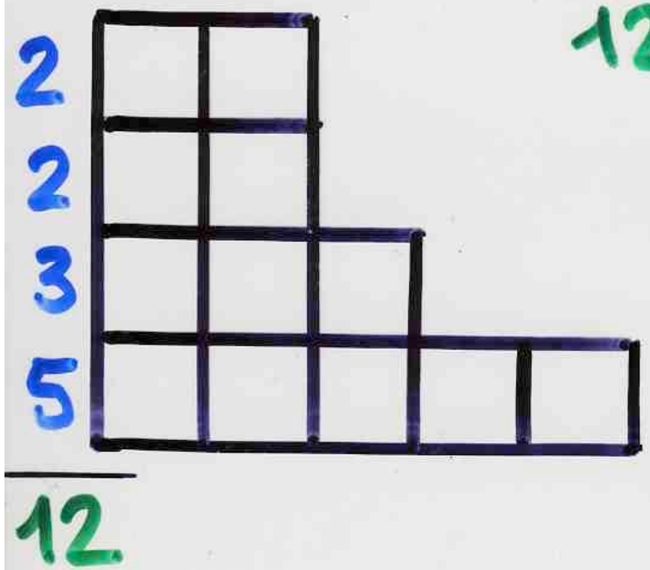

Young lattice
Hasse diagram

$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$$

partition of the integer n

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

λ_i part of the partition



$12 = n = 5 + 3 + 2 + 2$

Ferrers diagram (or Young diagram in physics)

Partition of n

lattice

every two elements
have a unique
least upper bound (join)

and a unique
greatest lower bound
(meet)

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- The relation $Y \xrightarrow{\Gamma} Z$ (for $Y, Z \in$ Maule(X)) is the covering relation of the poset Maule (X).

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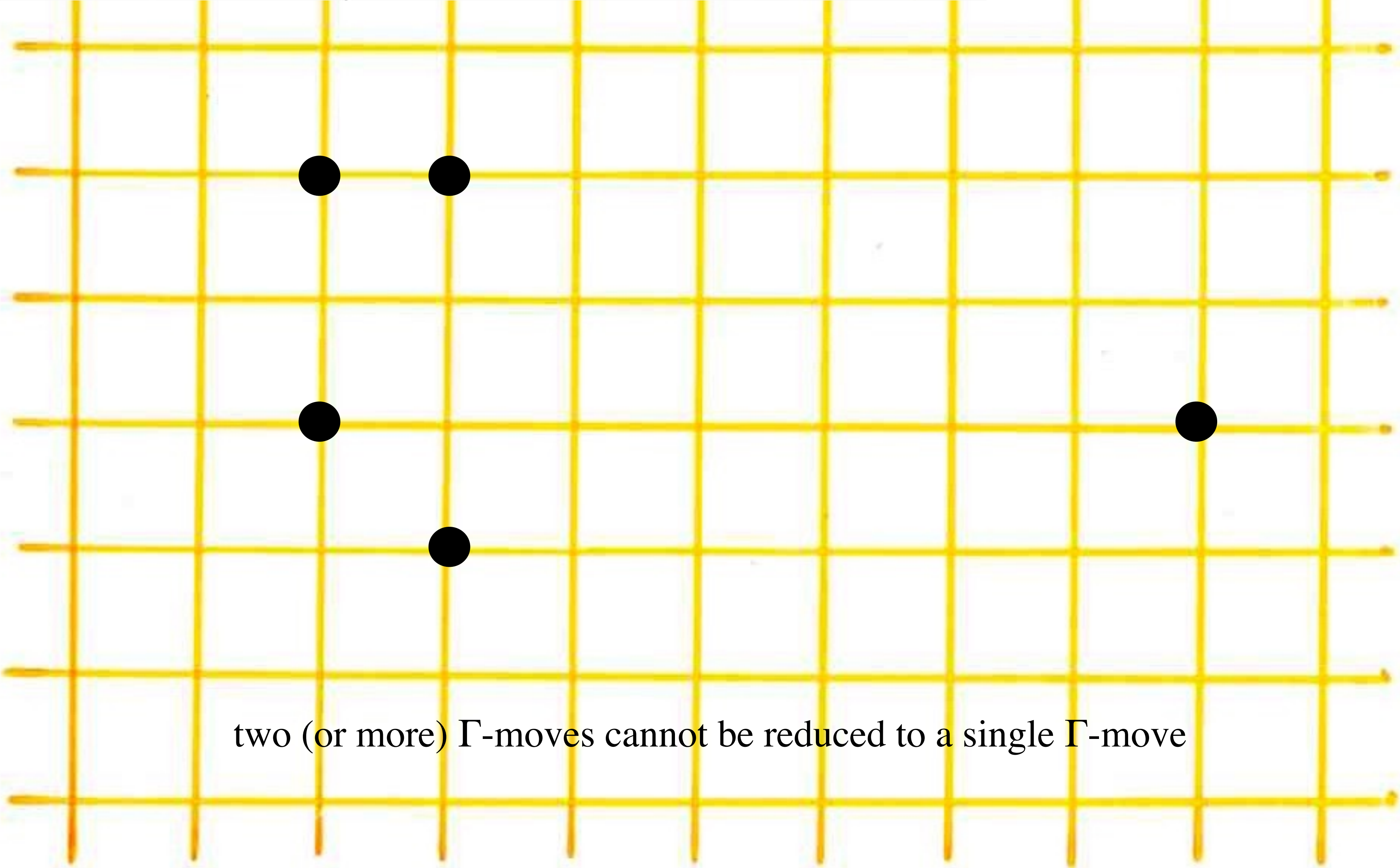

two (or more) Γ-moves cannot be reduced to a single Γ-move

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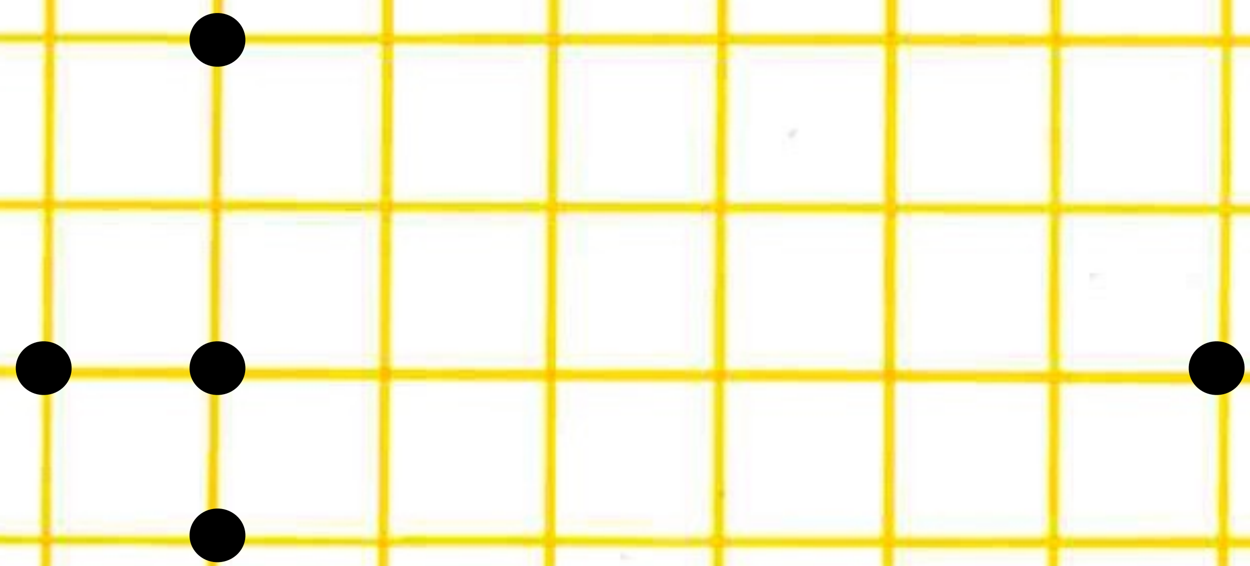

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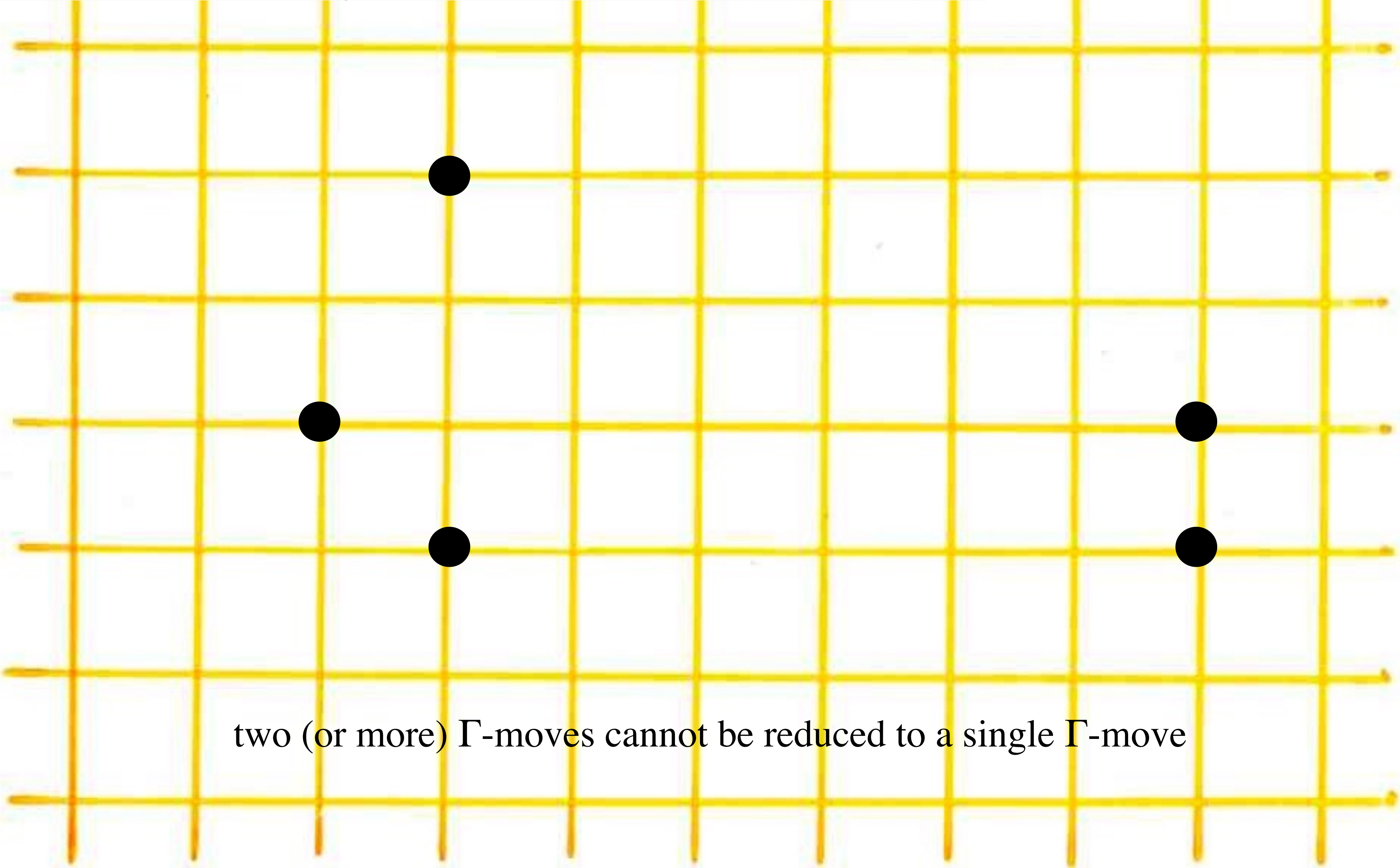

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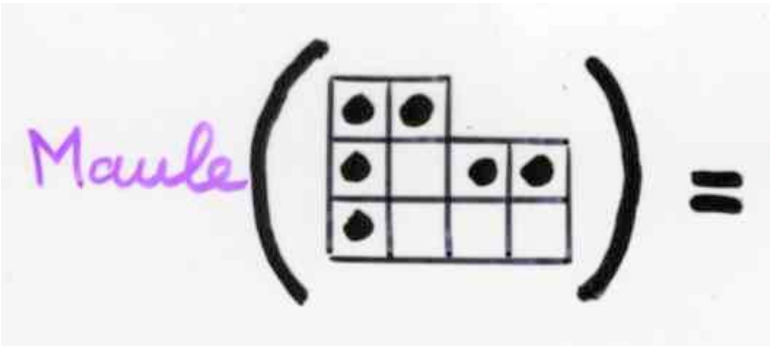
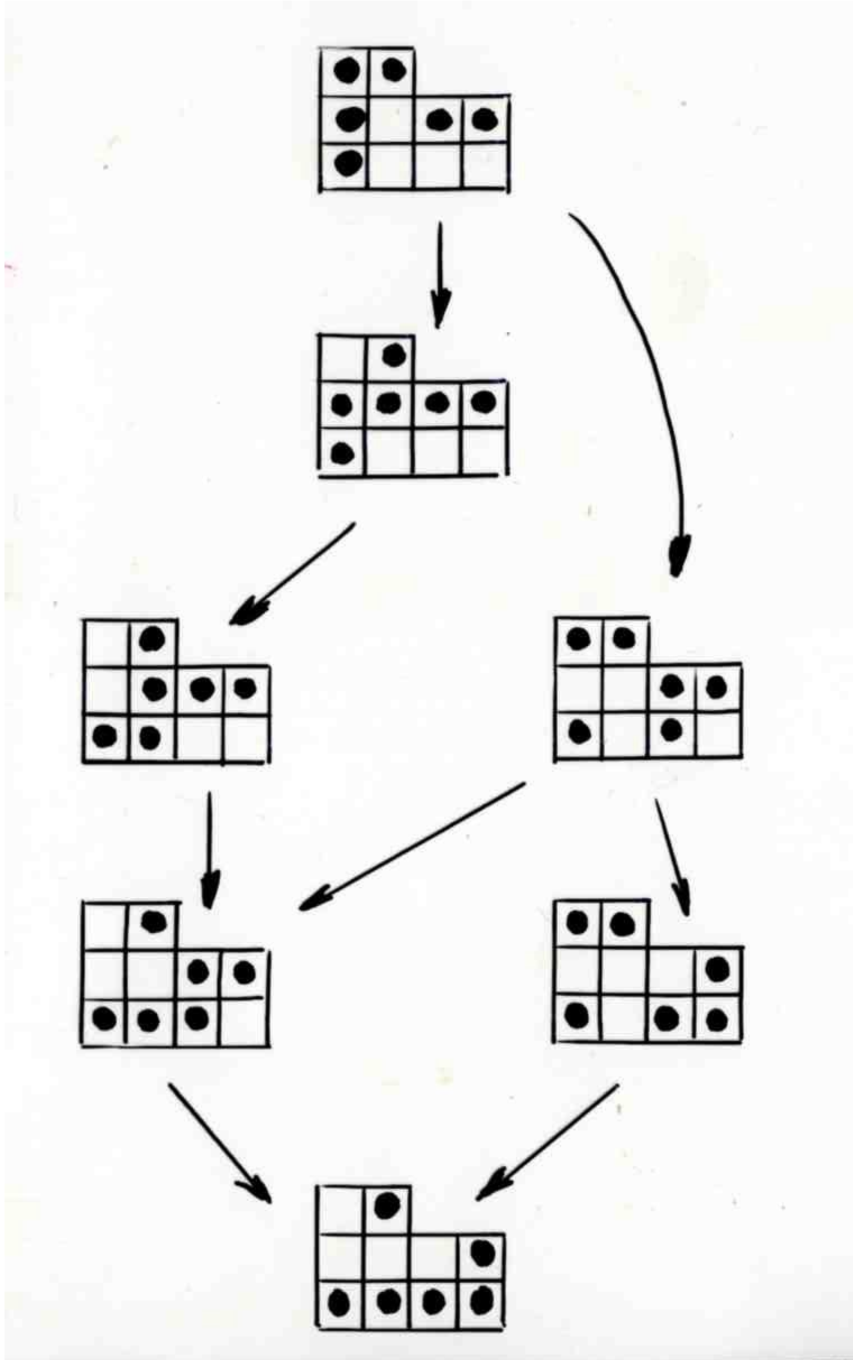
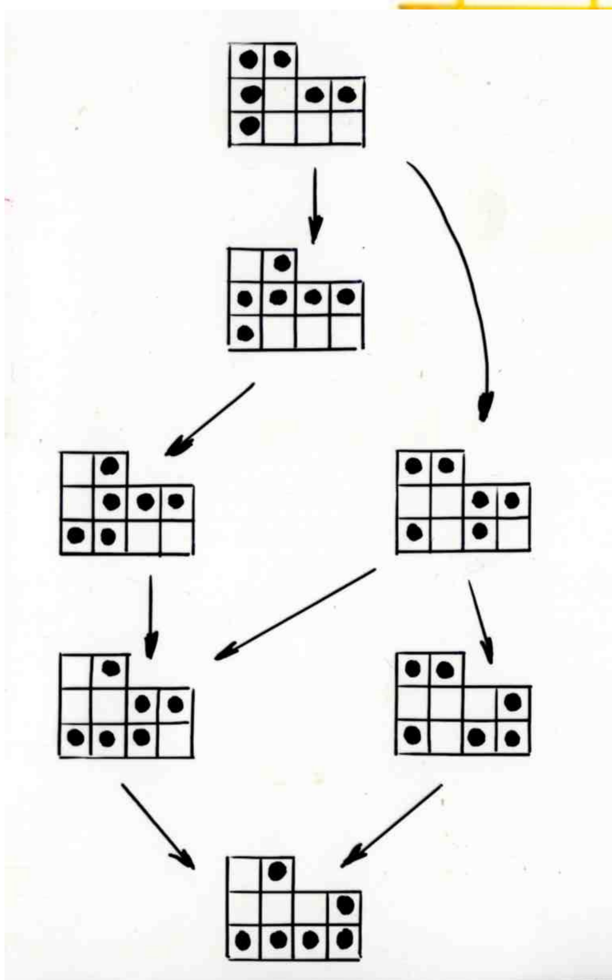
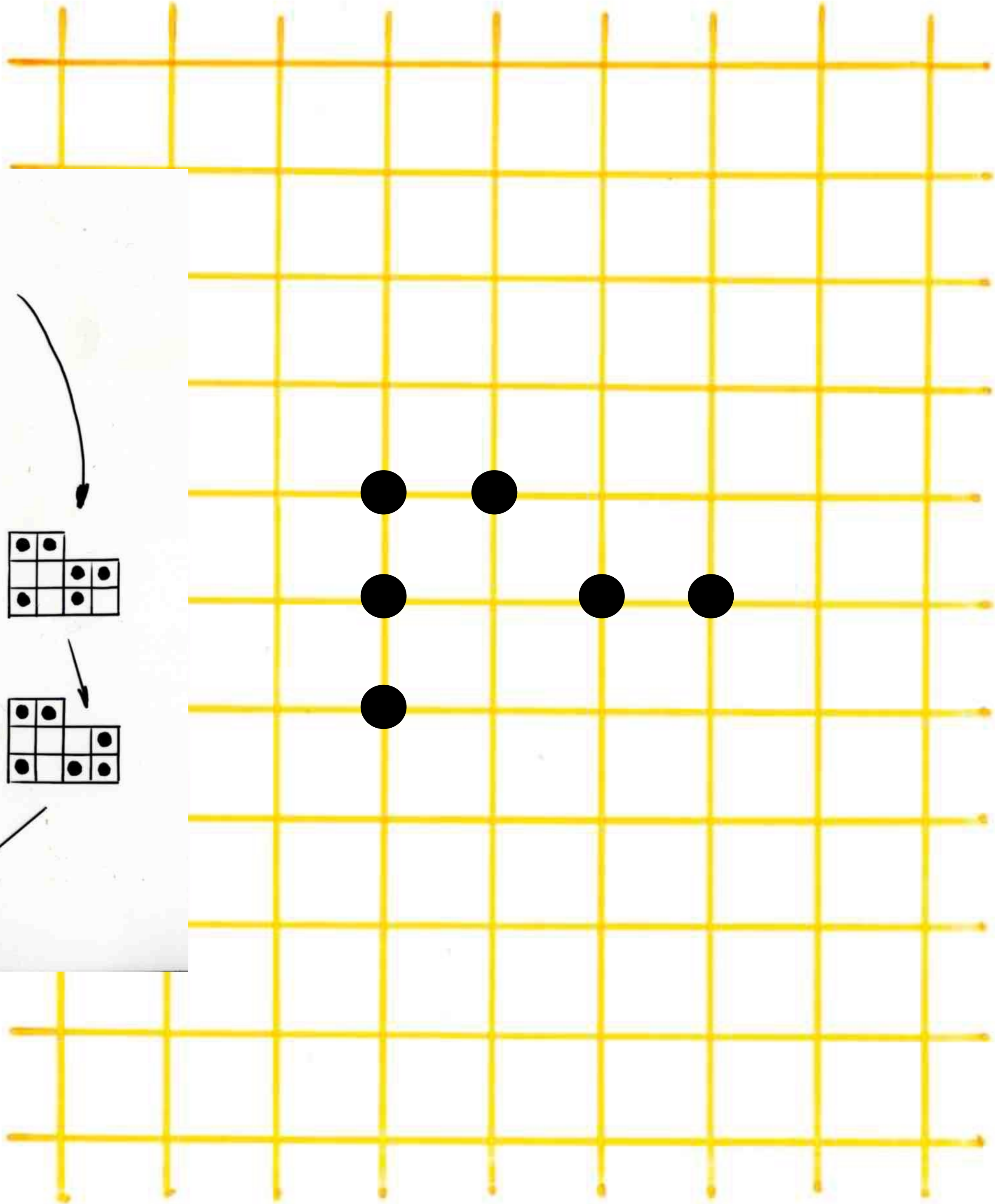
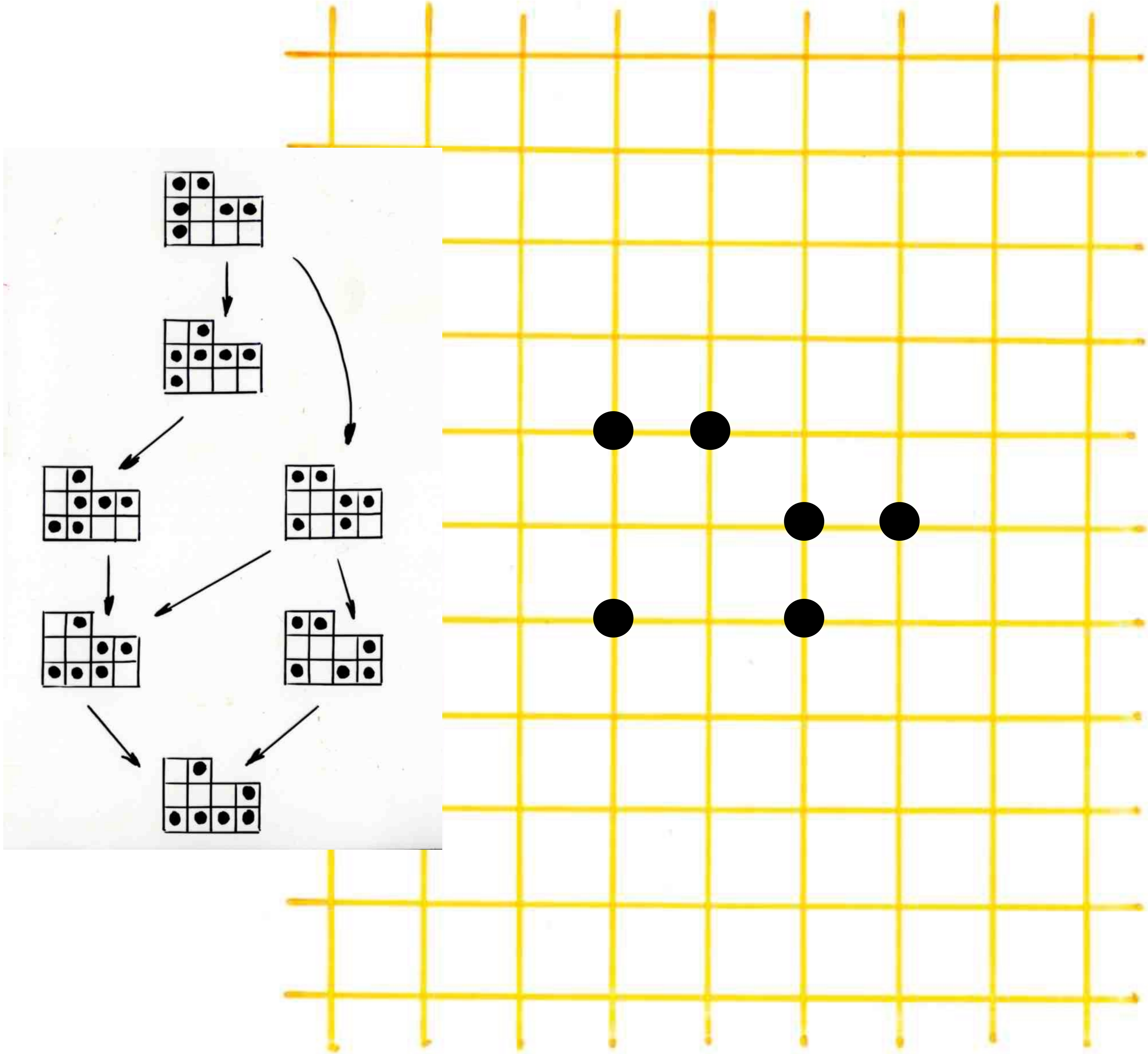
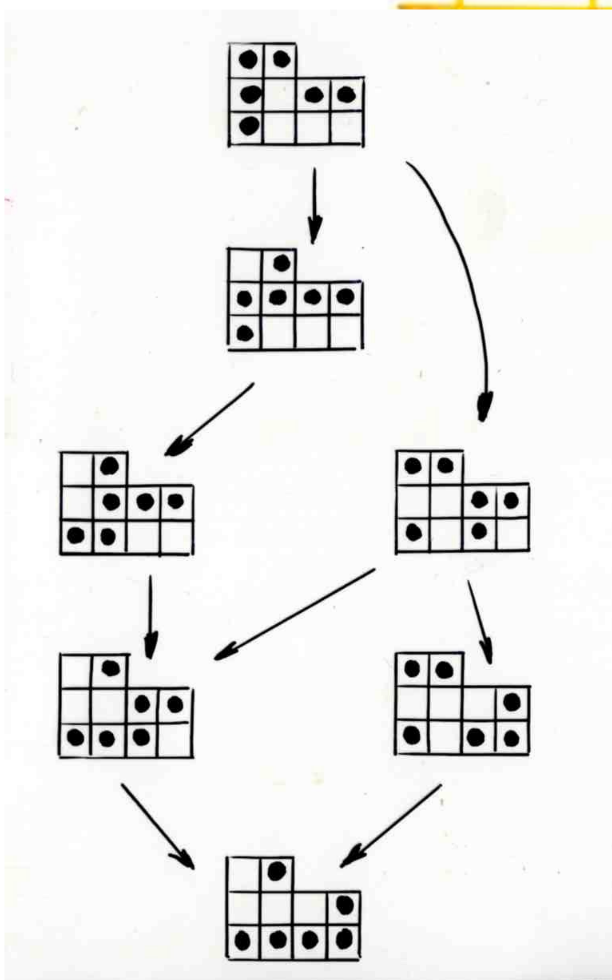
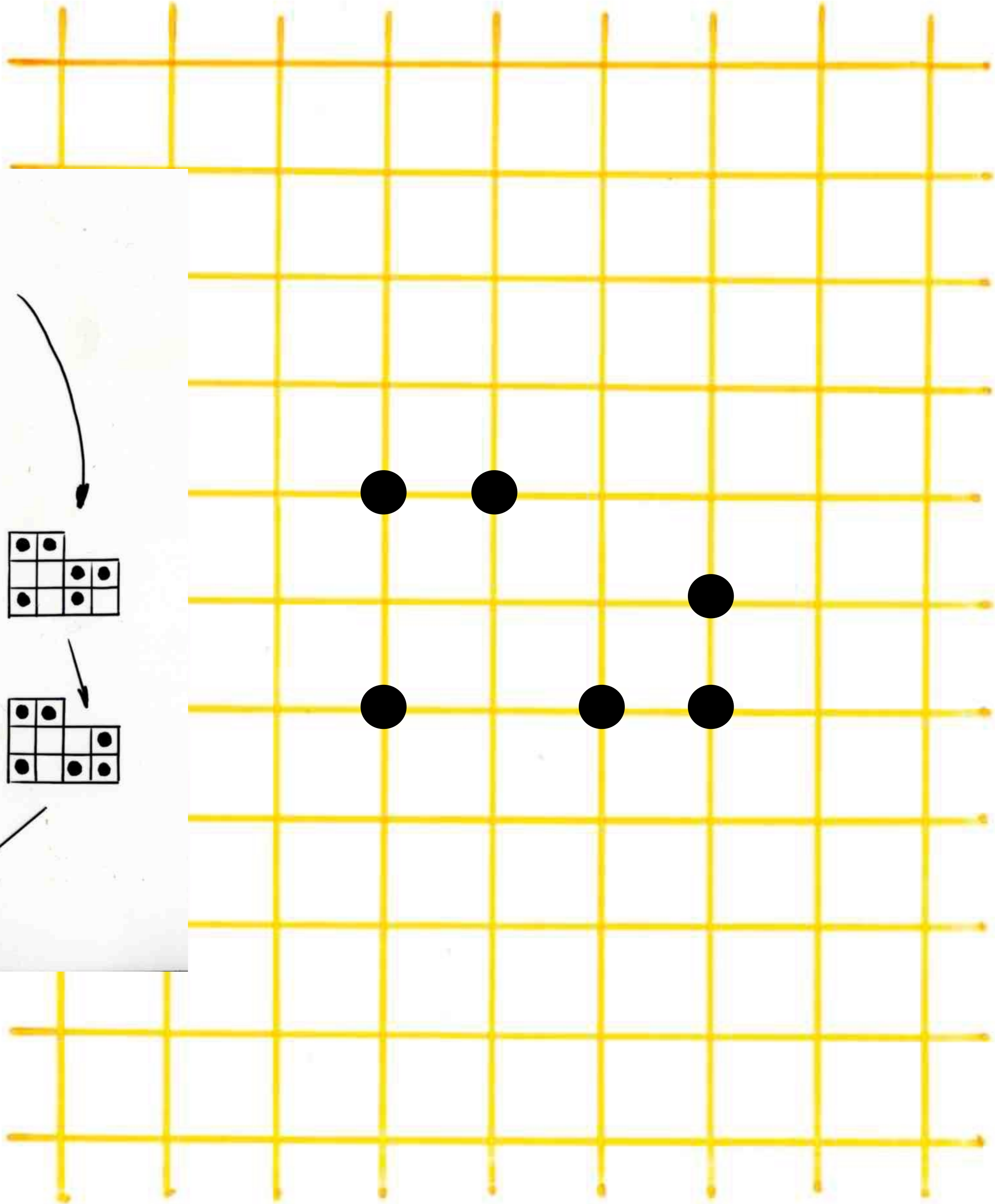

Maule

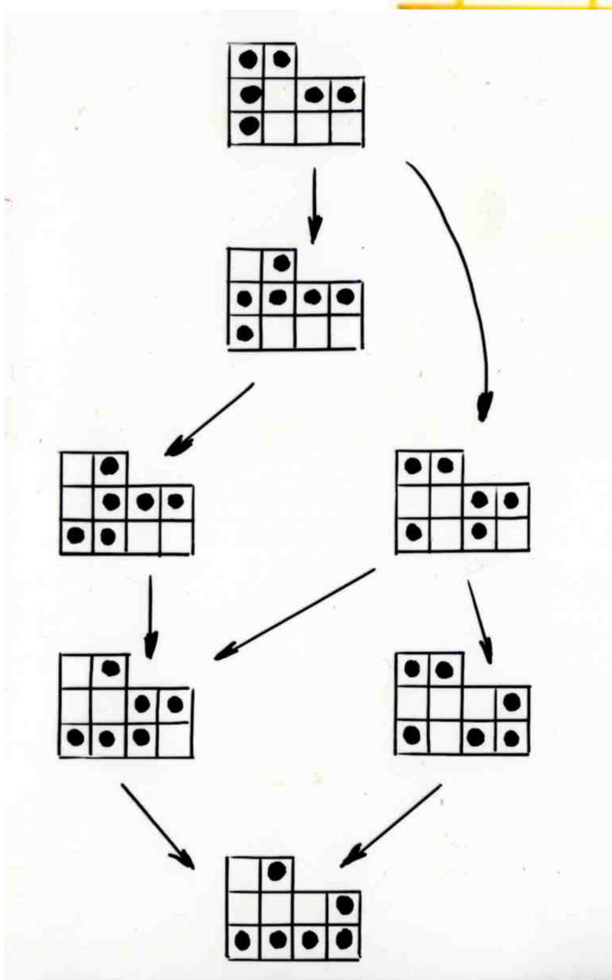


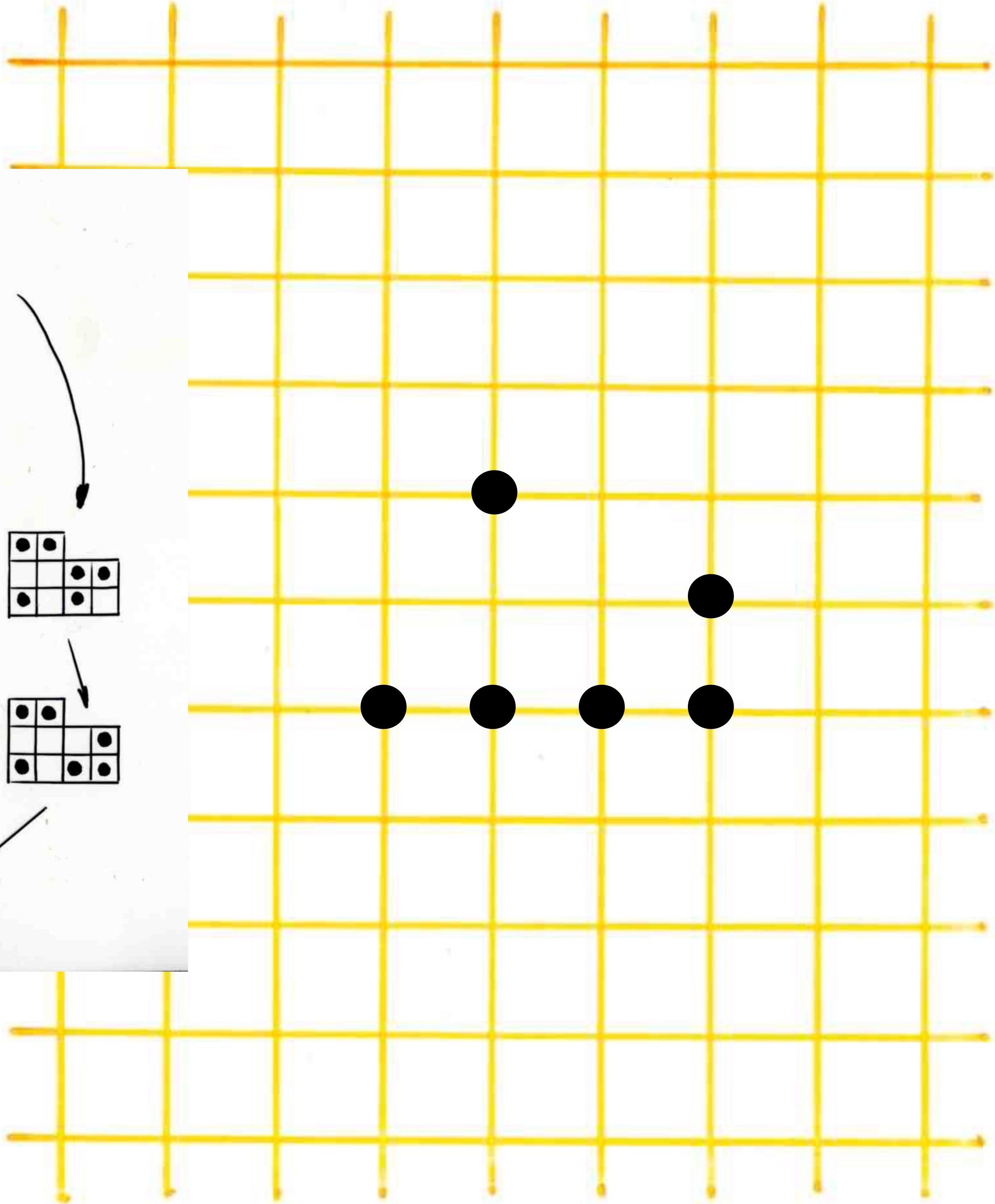










Remark Maule

- name of an area in Chile where this research was started, thanks to an invitation of Luc Lapointe (Talca Univ.)
- also the name of the river crossing this area

Mapuche name : pronouce Ma-ou-lé
signification: rainy

Luc Lapointe

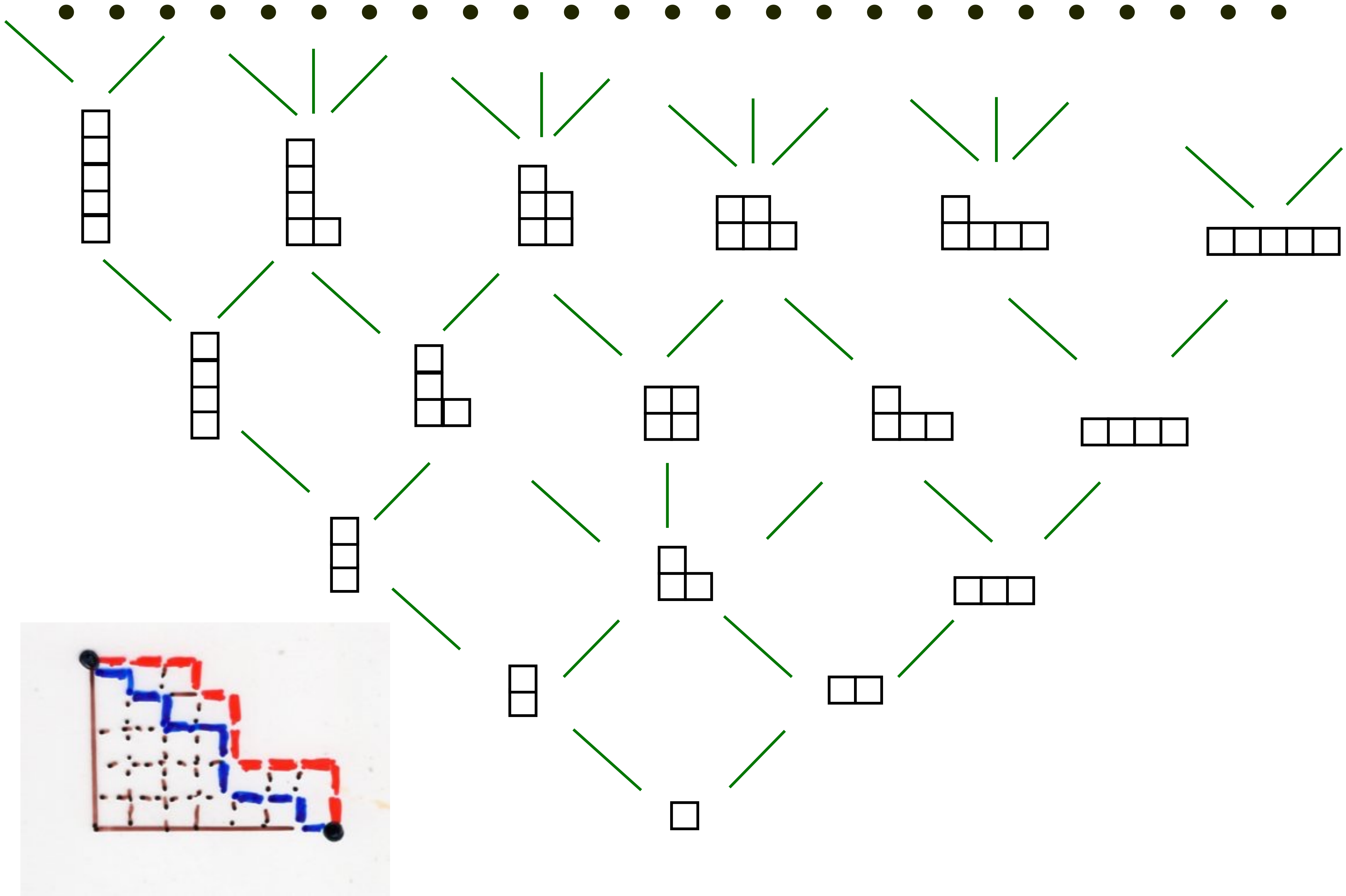
OCEANO PACIFICO
CORDILLERA DE LOS ANDES
Lago Rapel
Punta Verde
Lago Vichuquén
Lago Colbún
TALCA
Laguna del Maule

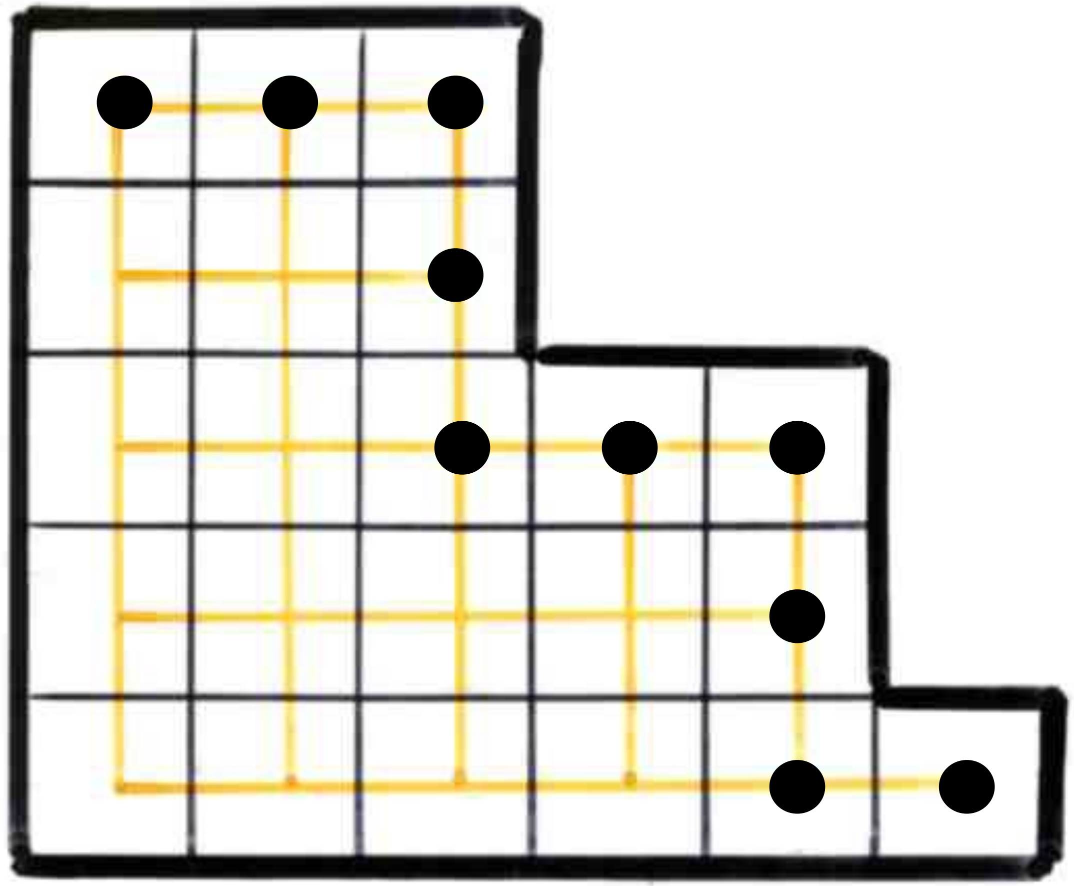
Maule valley

From Talca to Constitution

Young lattice

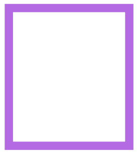
Young lattice

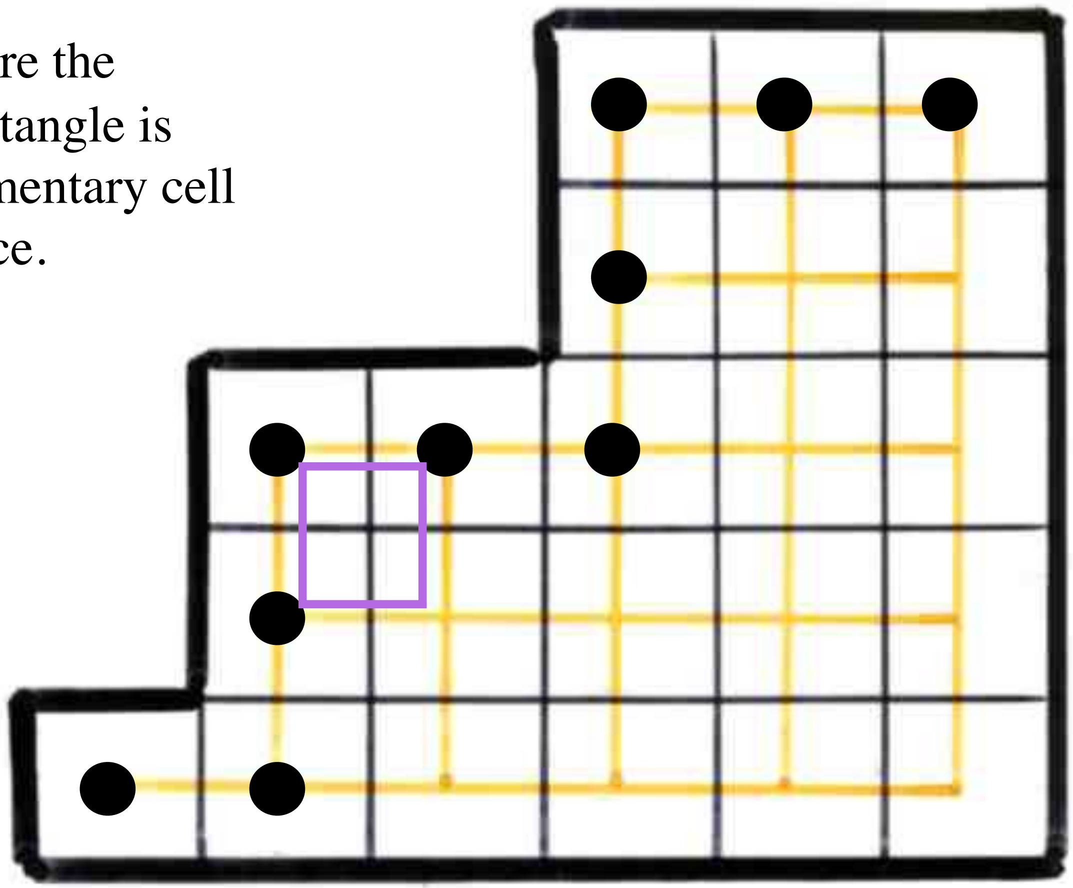




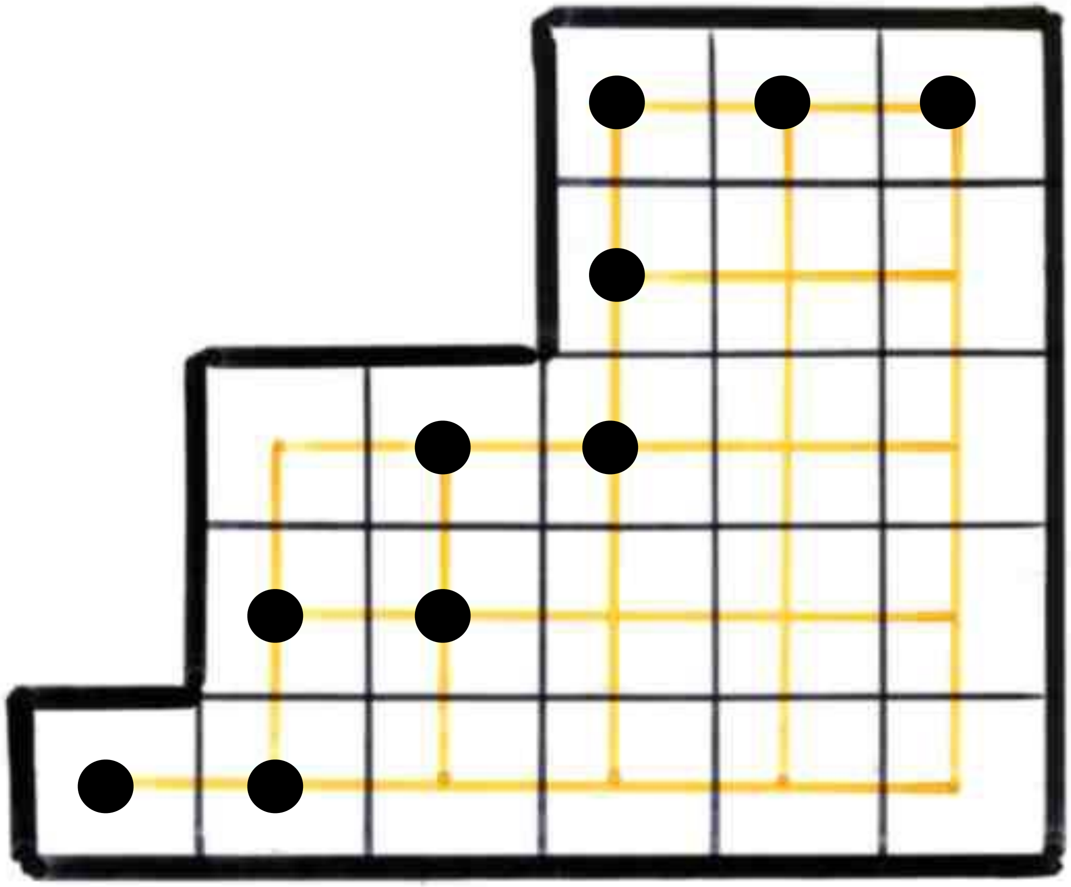
In the case of the maule generated by this cloud of points
Γ-moves are only elementary Γ-moves,

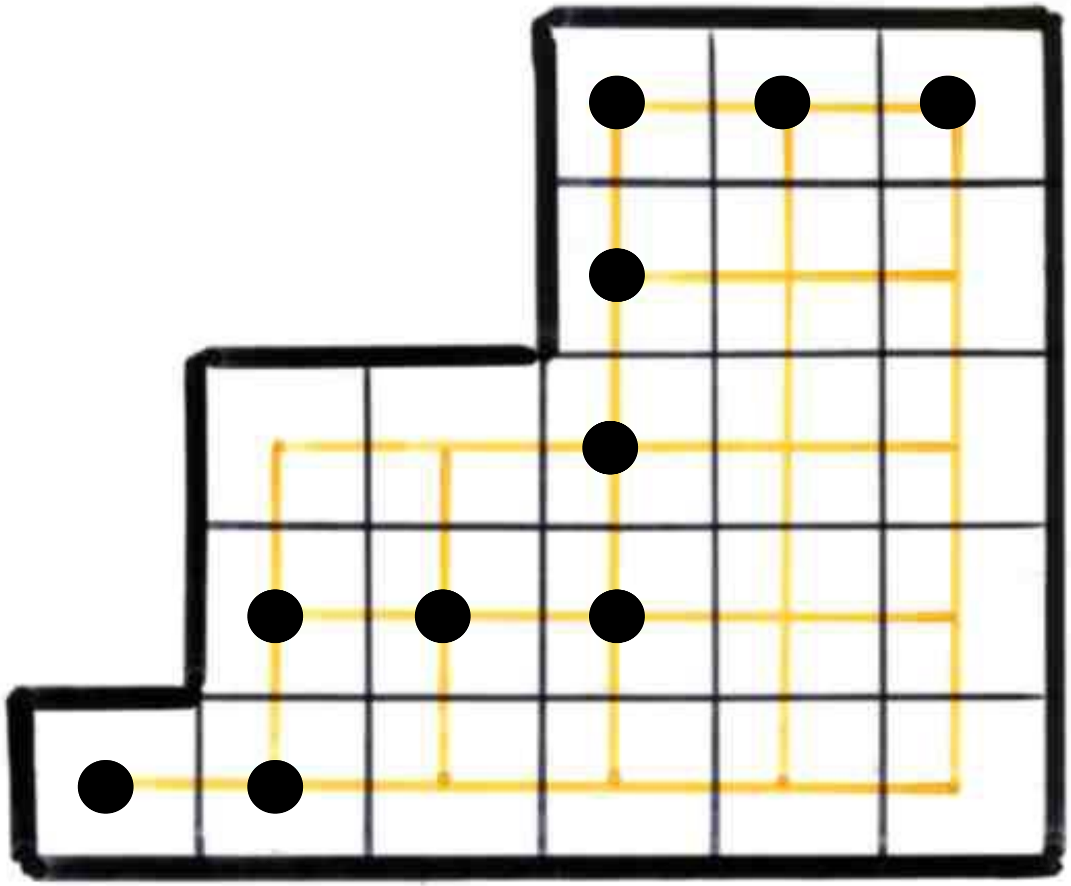
that is moves where the corresponding rectangle is reduced to an elementary cell of the square lattice.

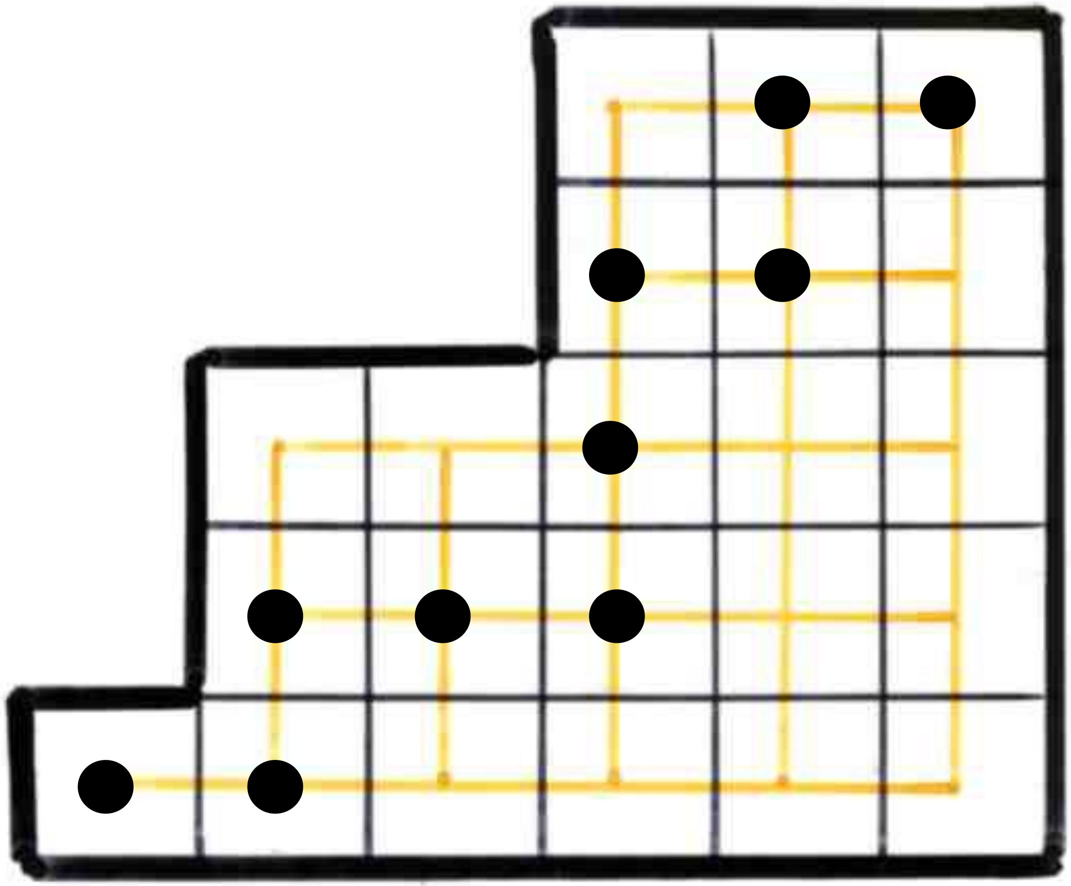


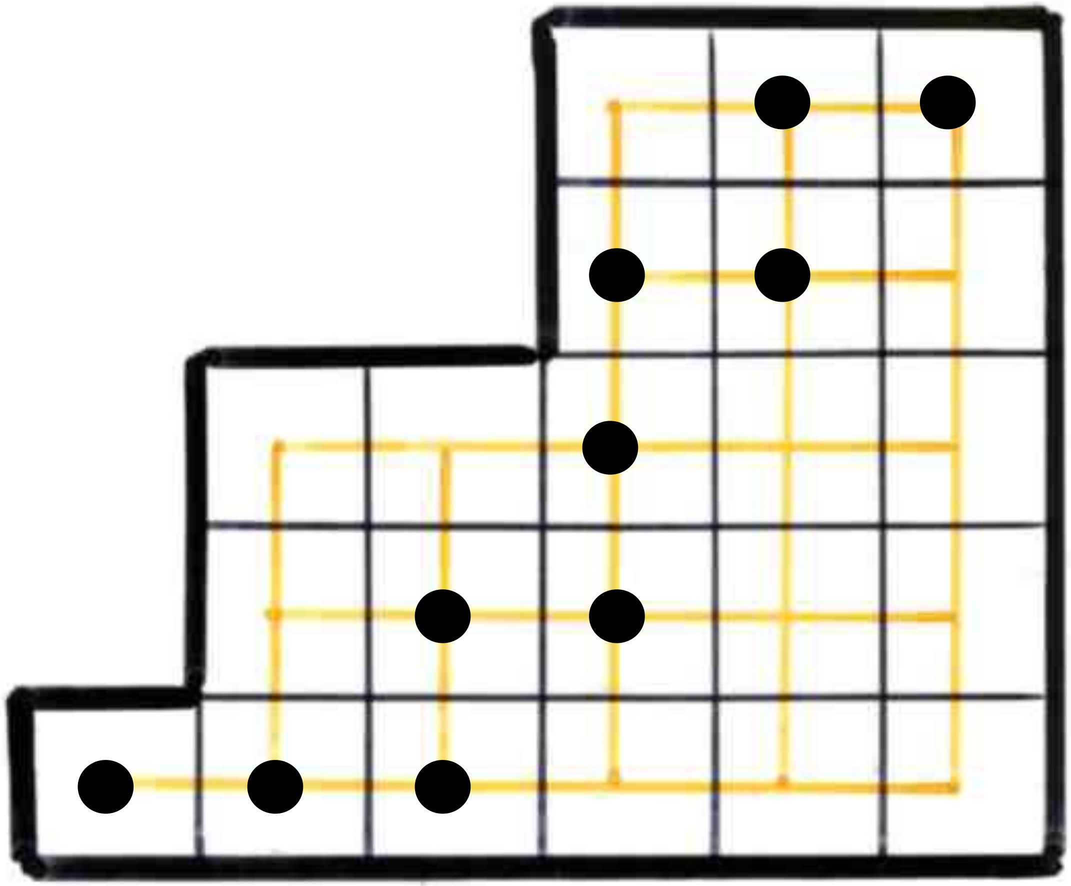


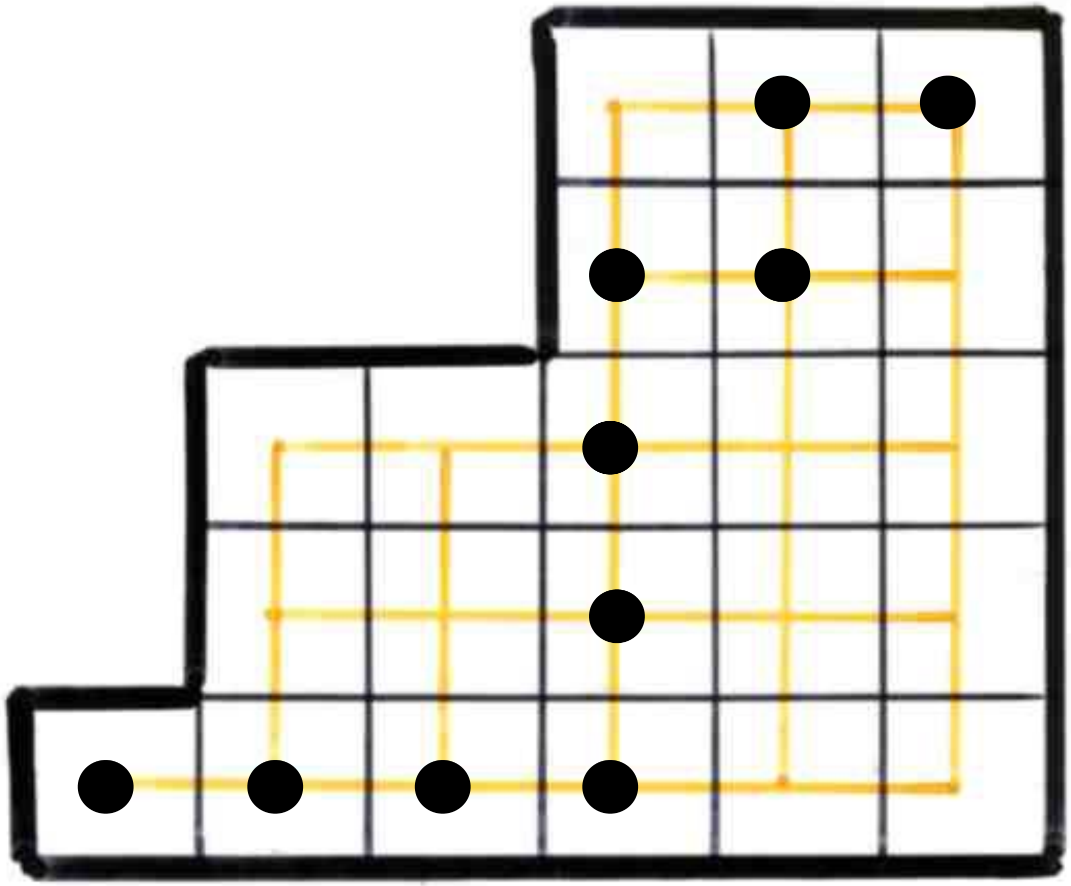
Such maules will be called simple maule.

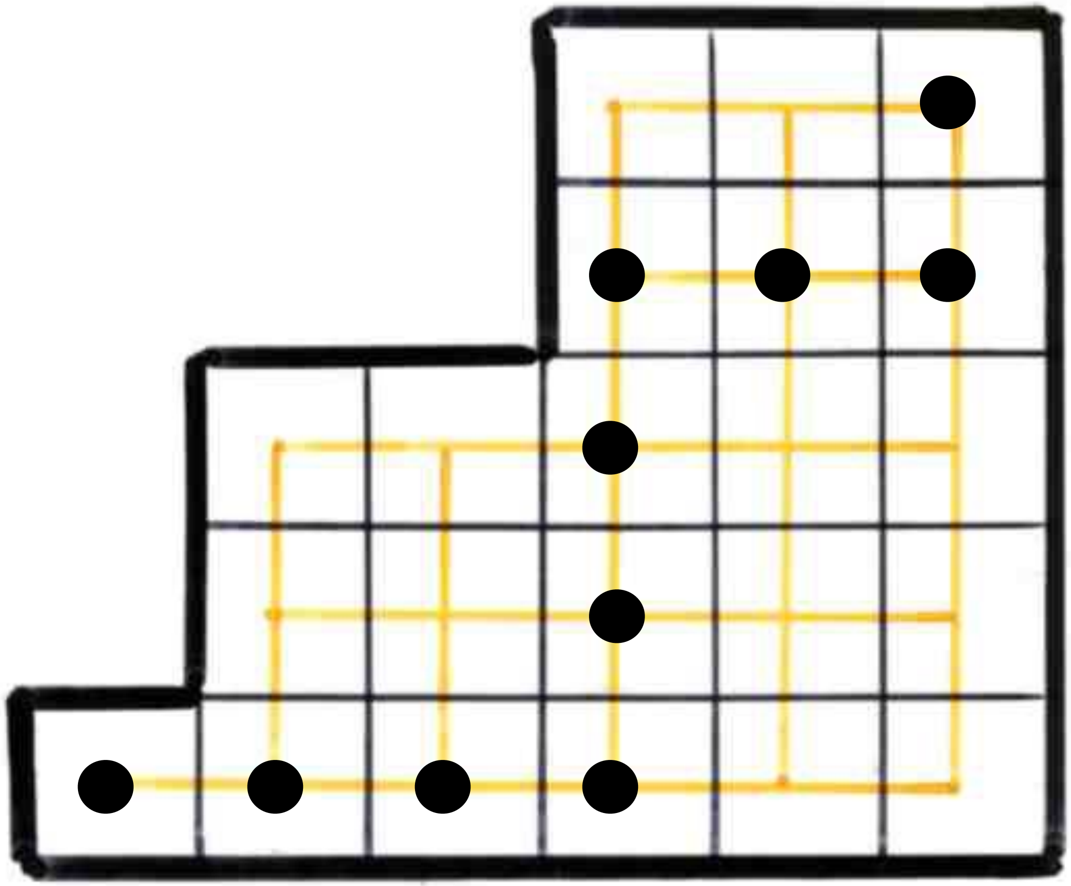


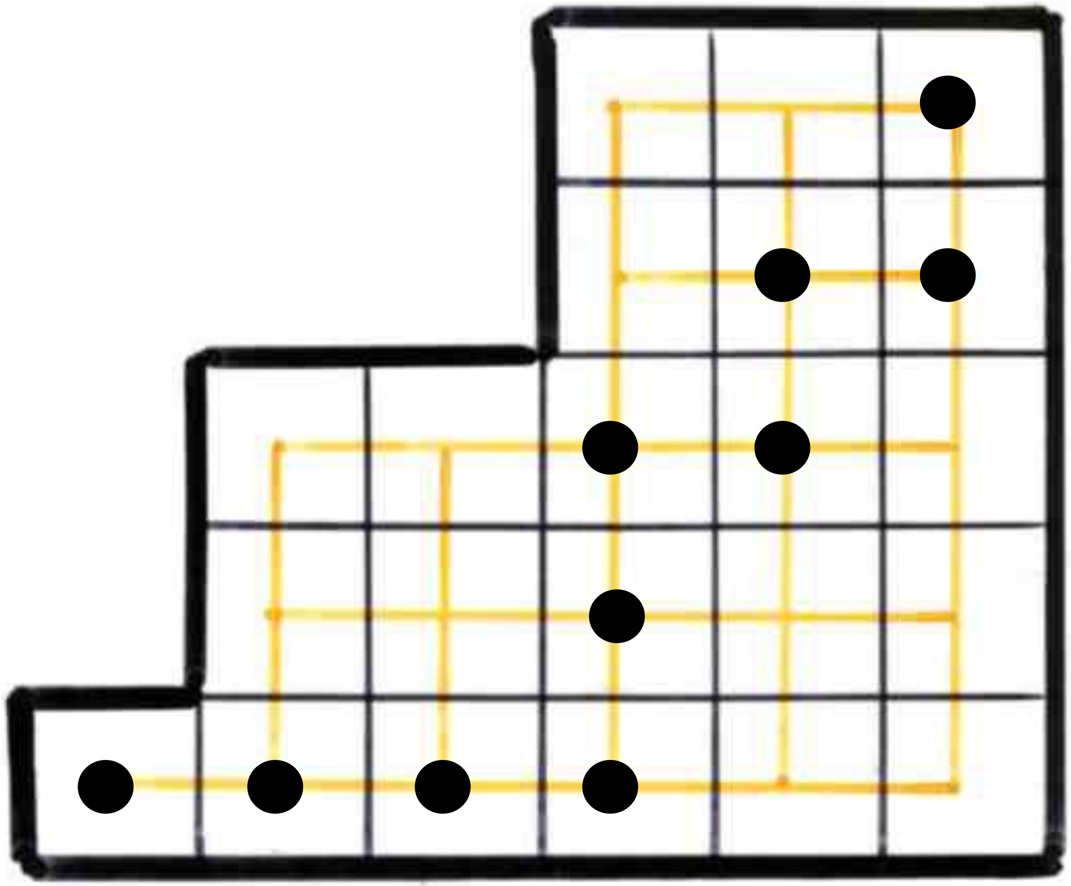


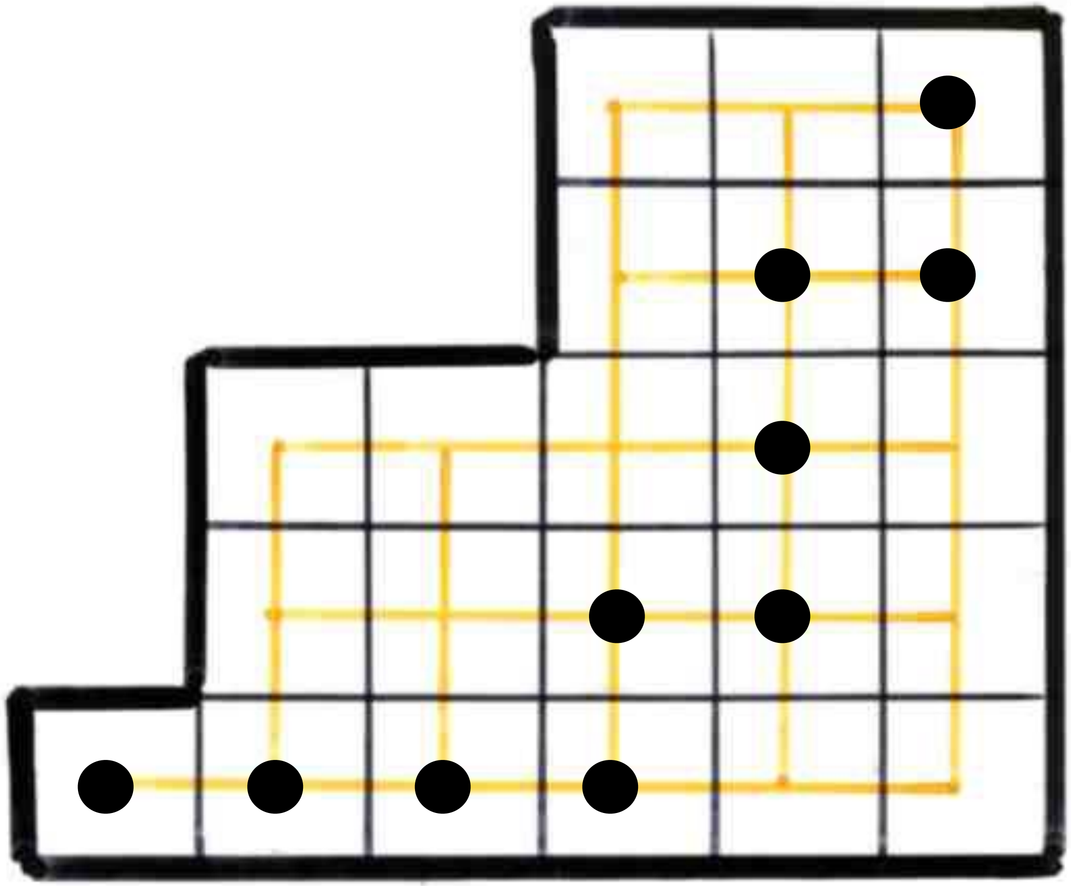


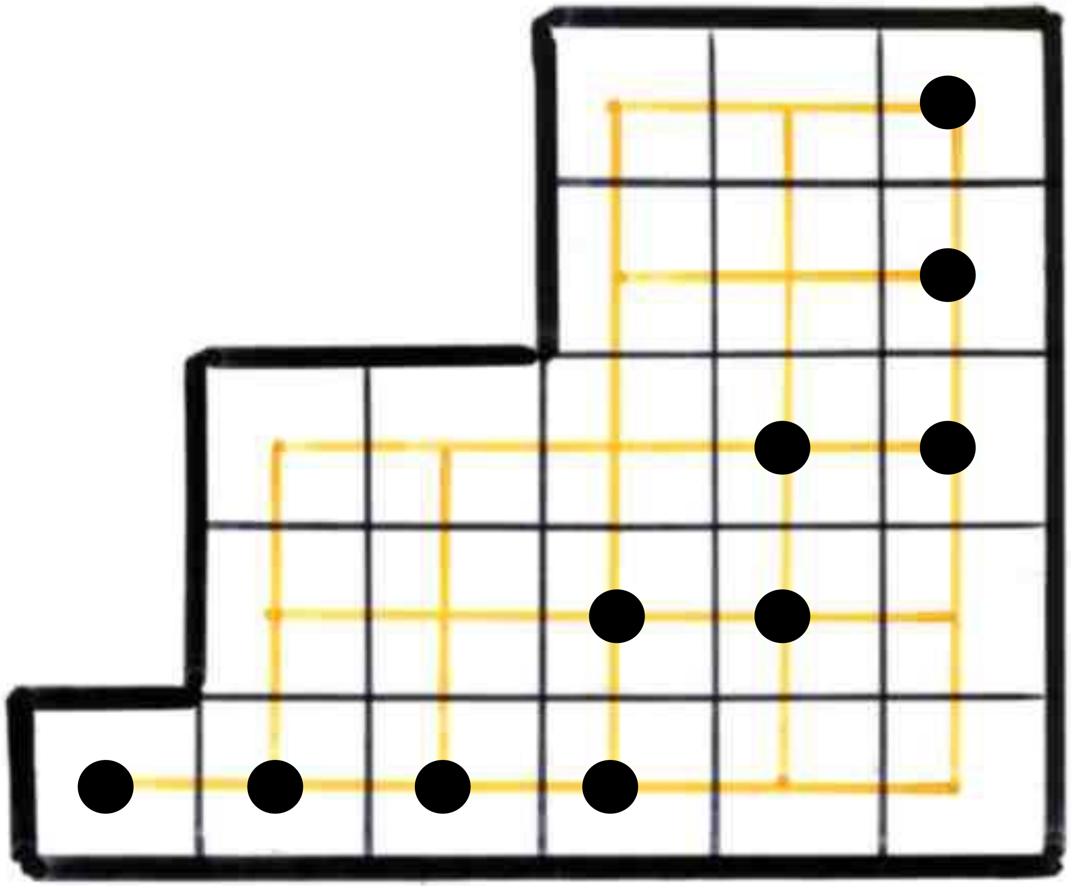


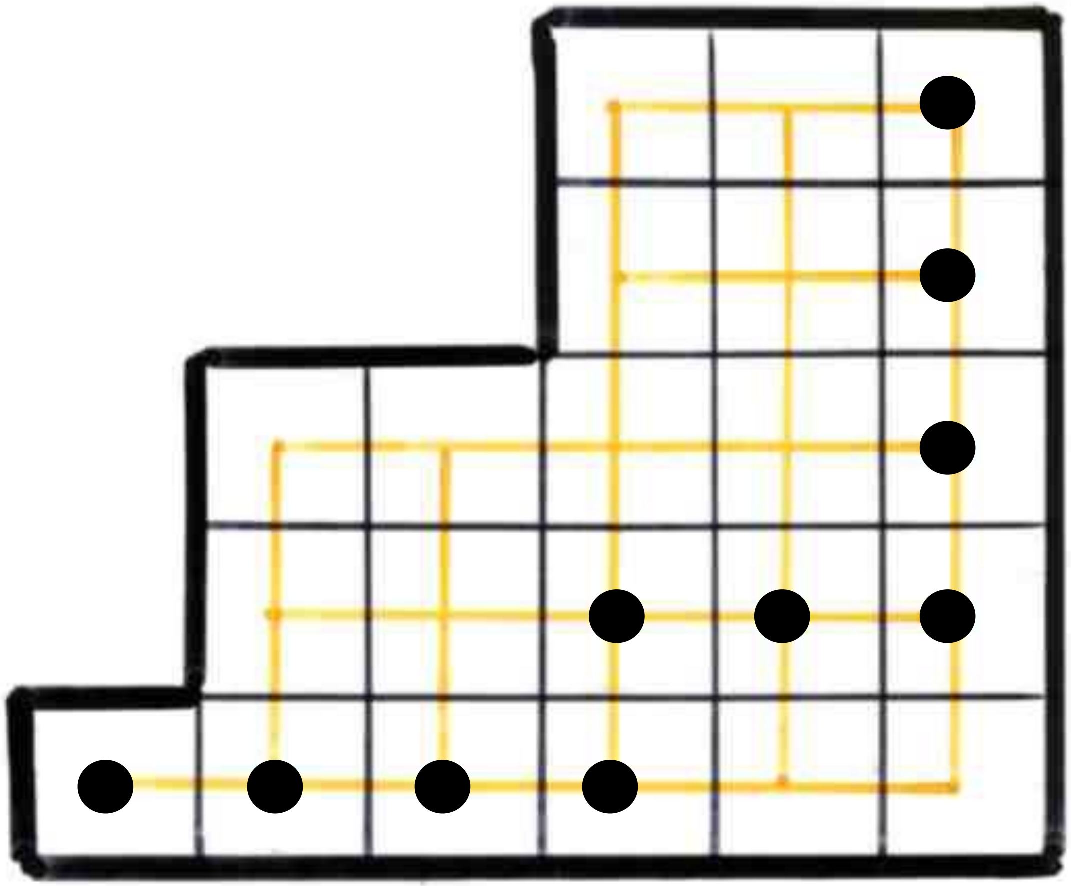


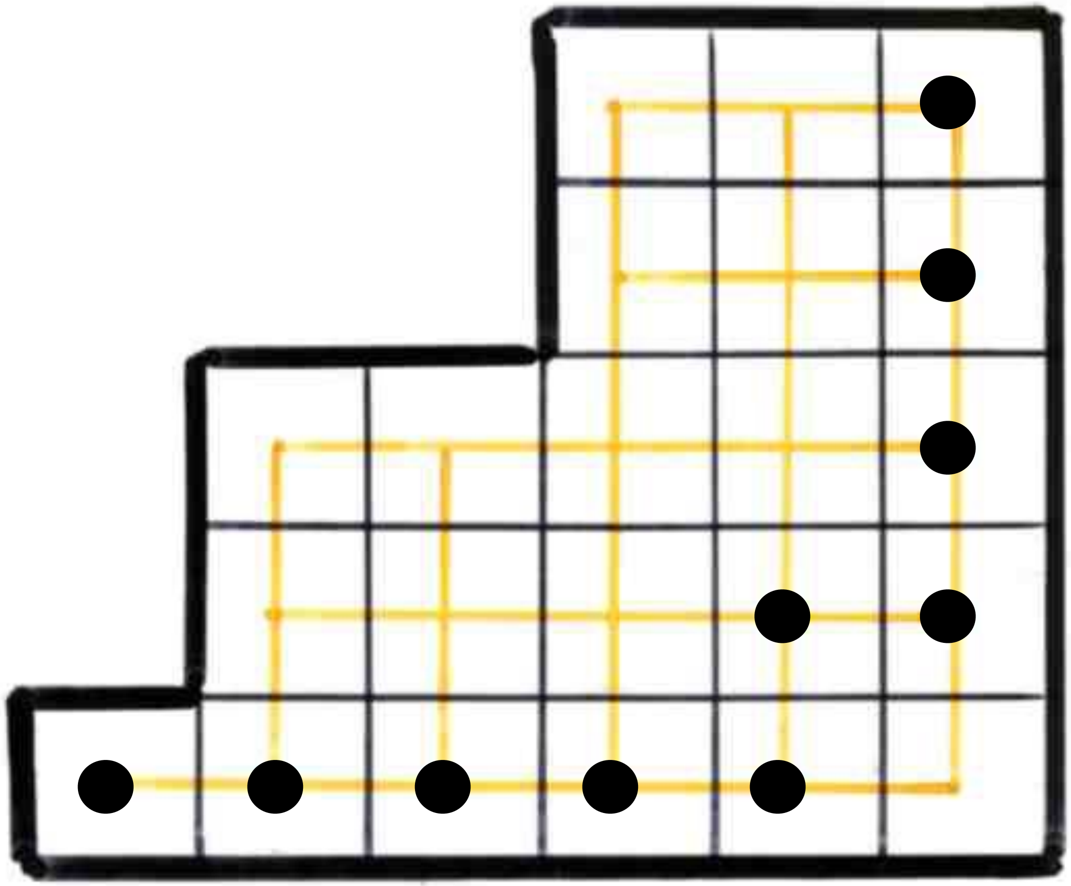


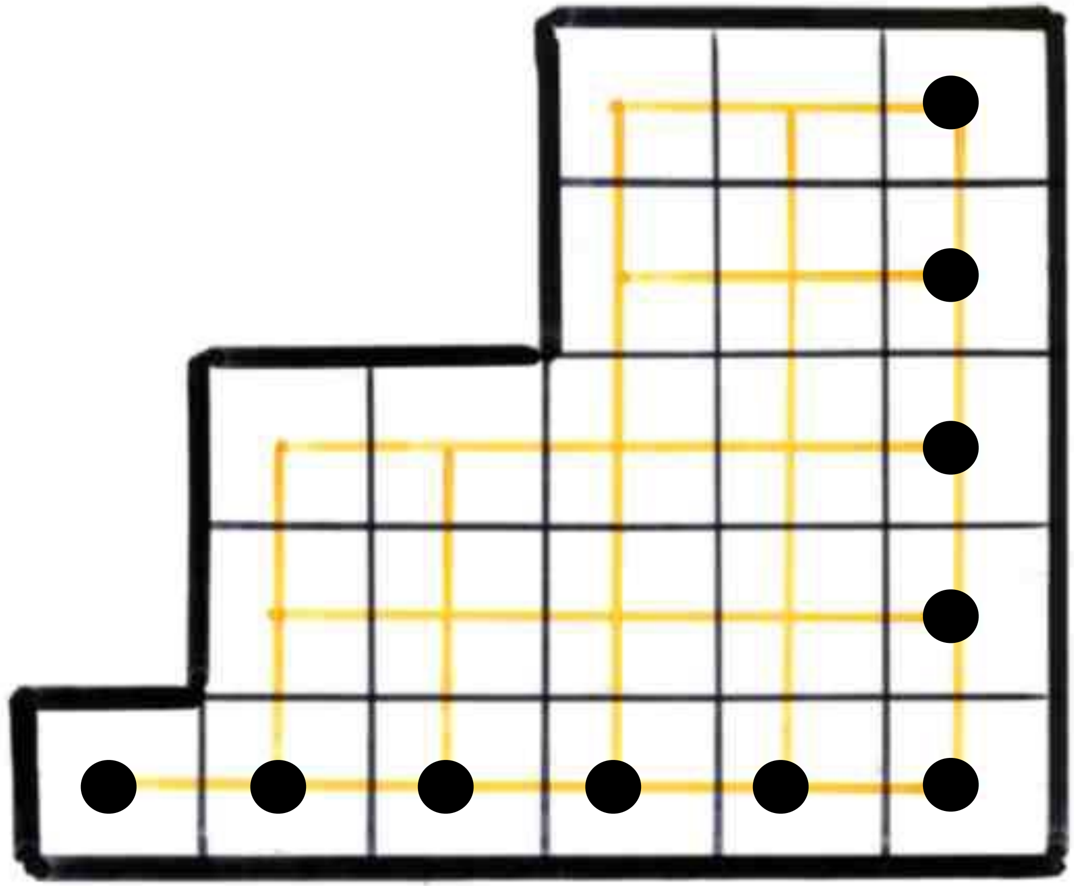


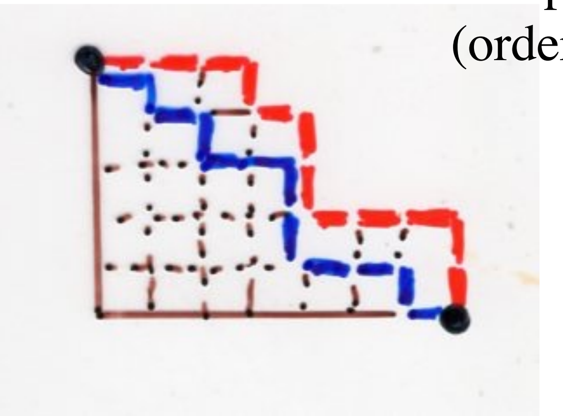






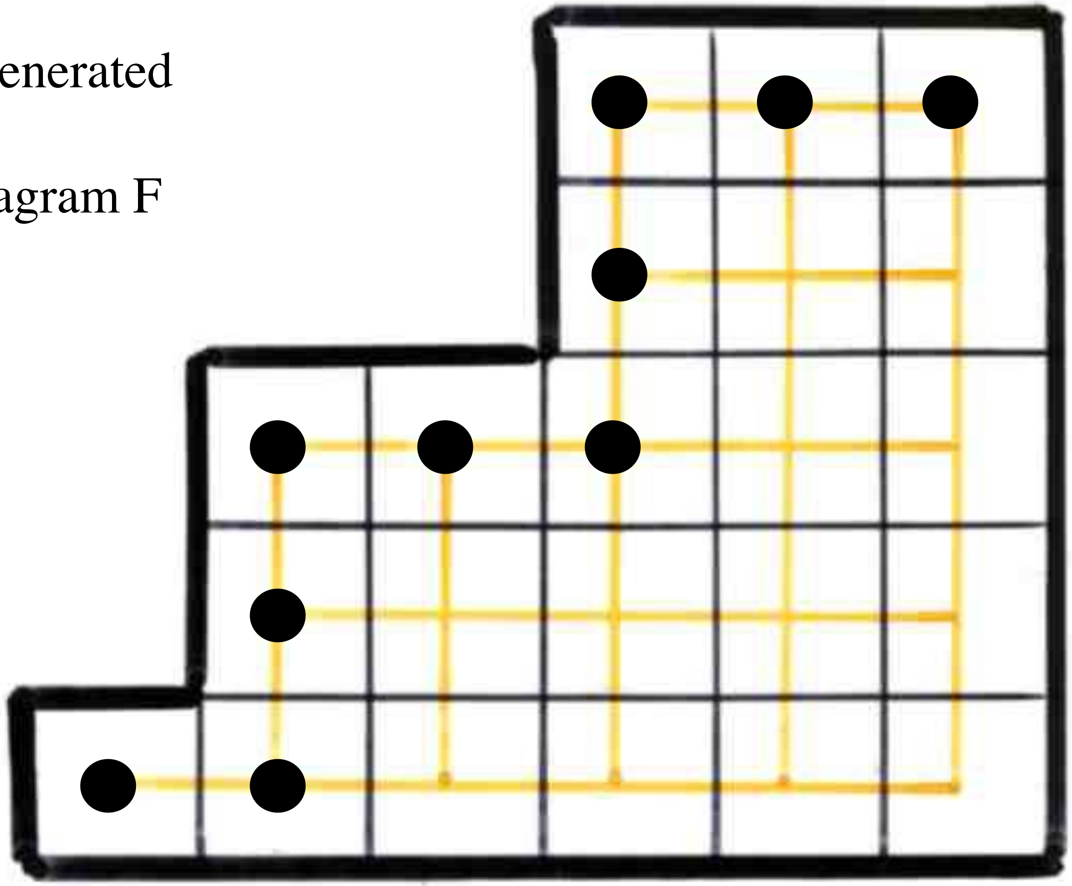




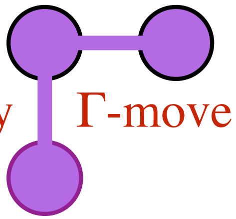


The poset of Ferrers diagrams included in a given Ferrers diagram F
(ordered by inclusion of diagrams)

is ismorphic to the maule generated
by the following cloud
associated to the Ferrers diagram F



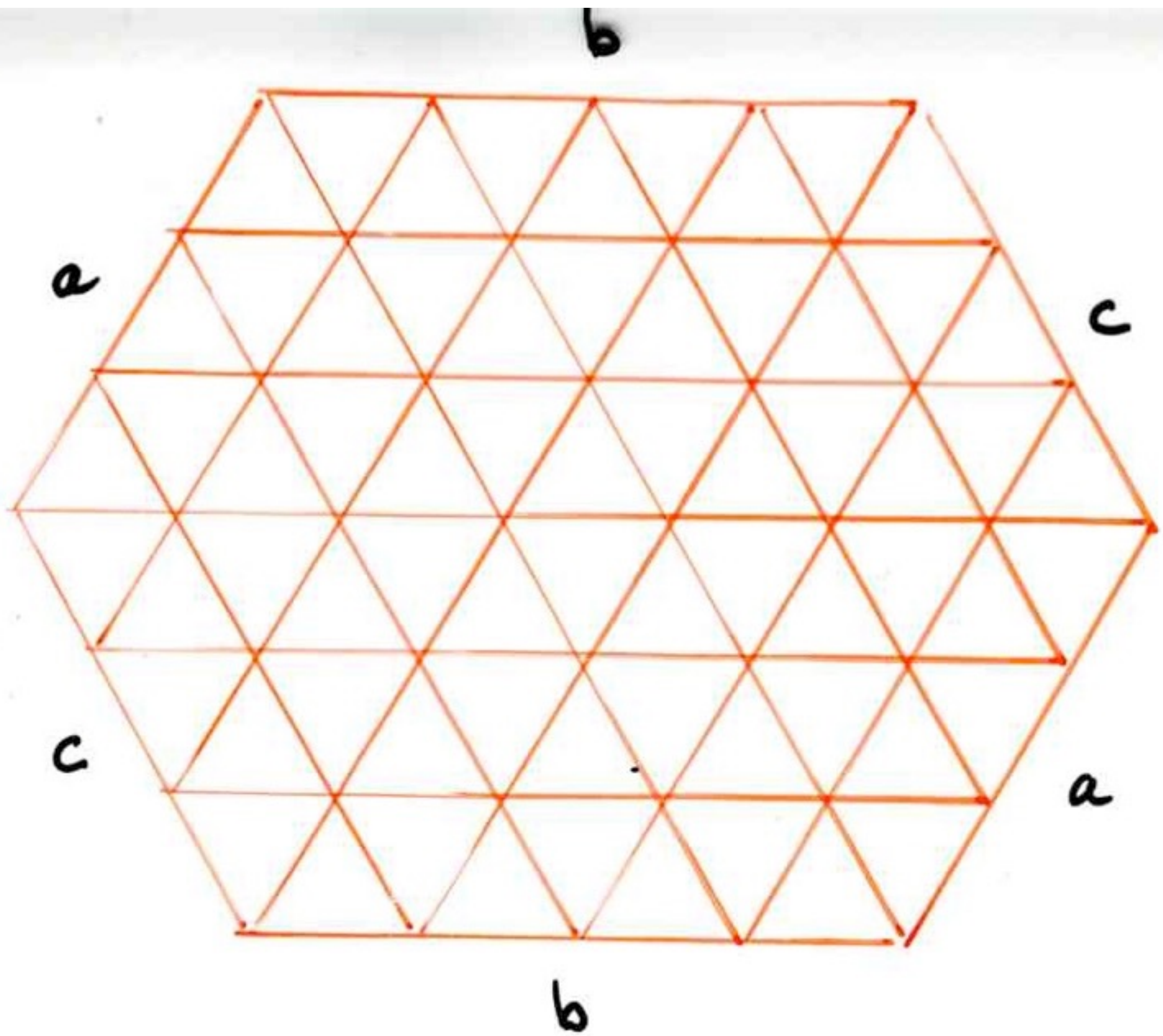
Maules having only

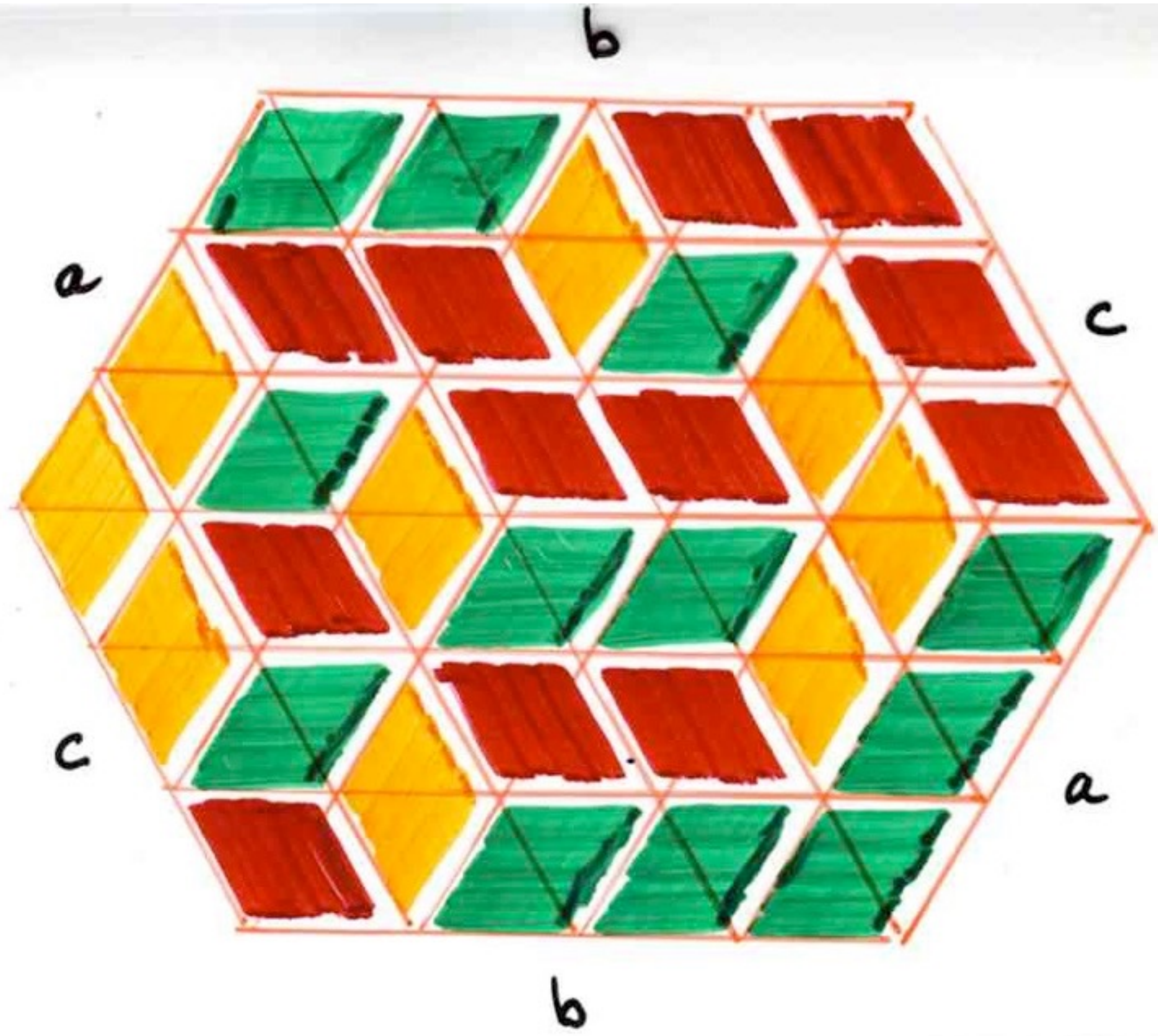


elementary Γ-move

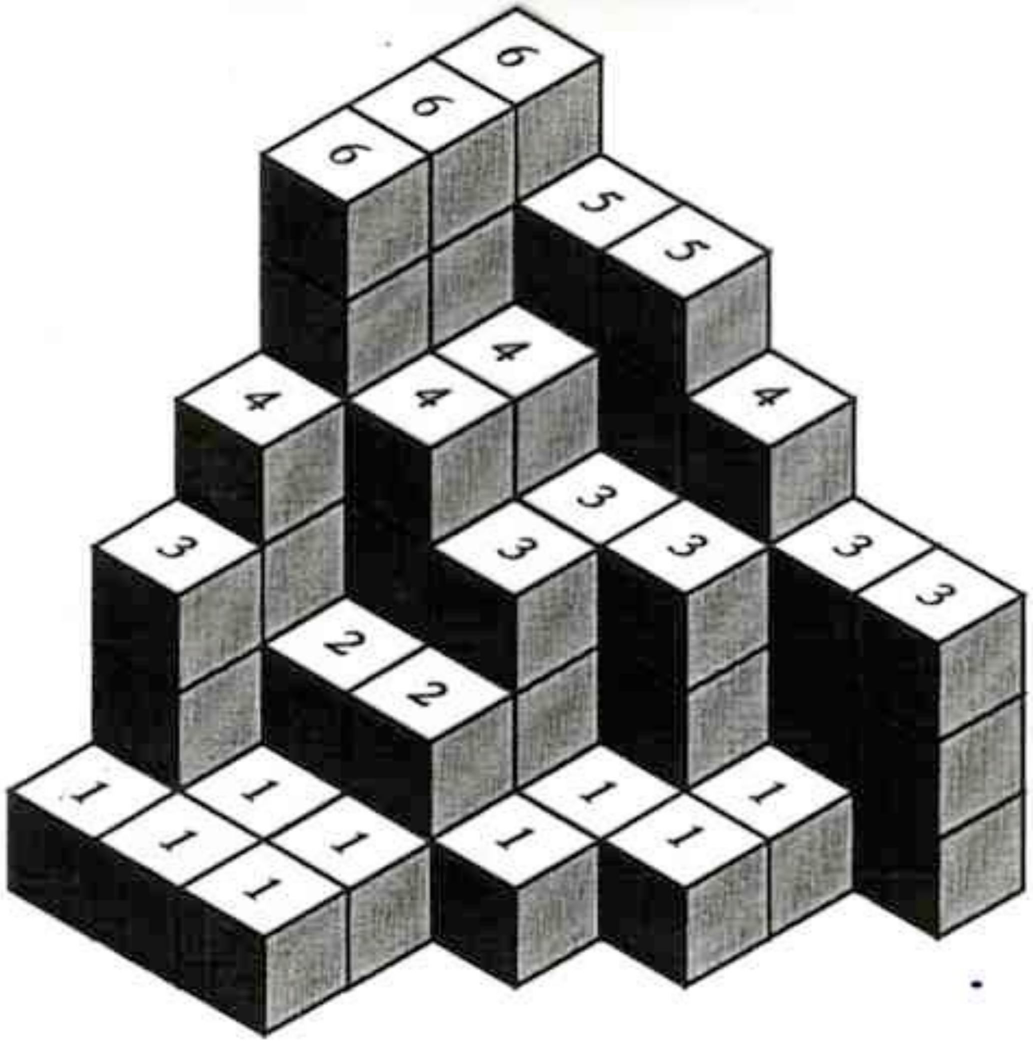
are called simple maule.

Tilings lattice


b
a
c
c
a
b



b
a
c
c
a
b



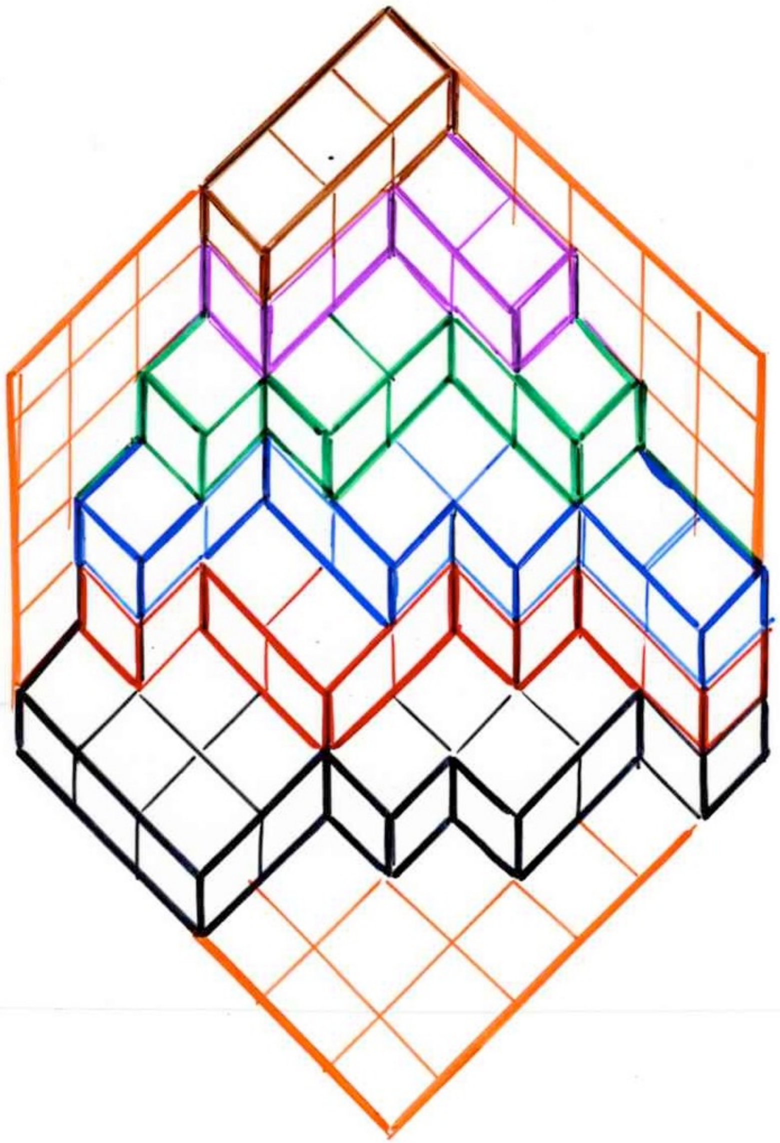
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

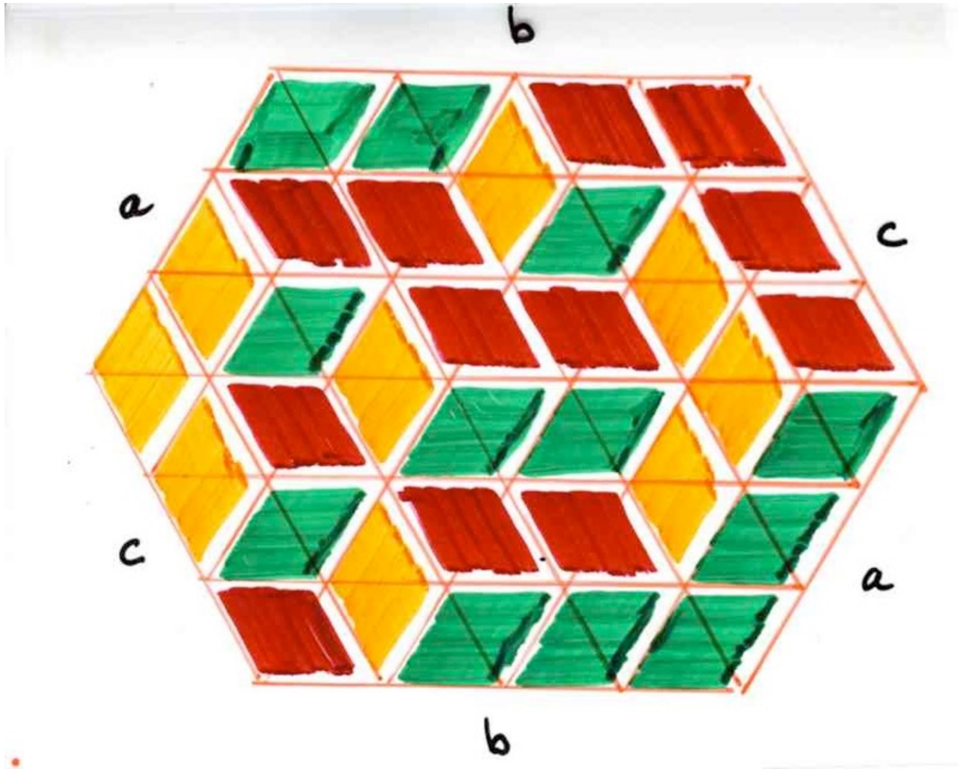
plane partitions

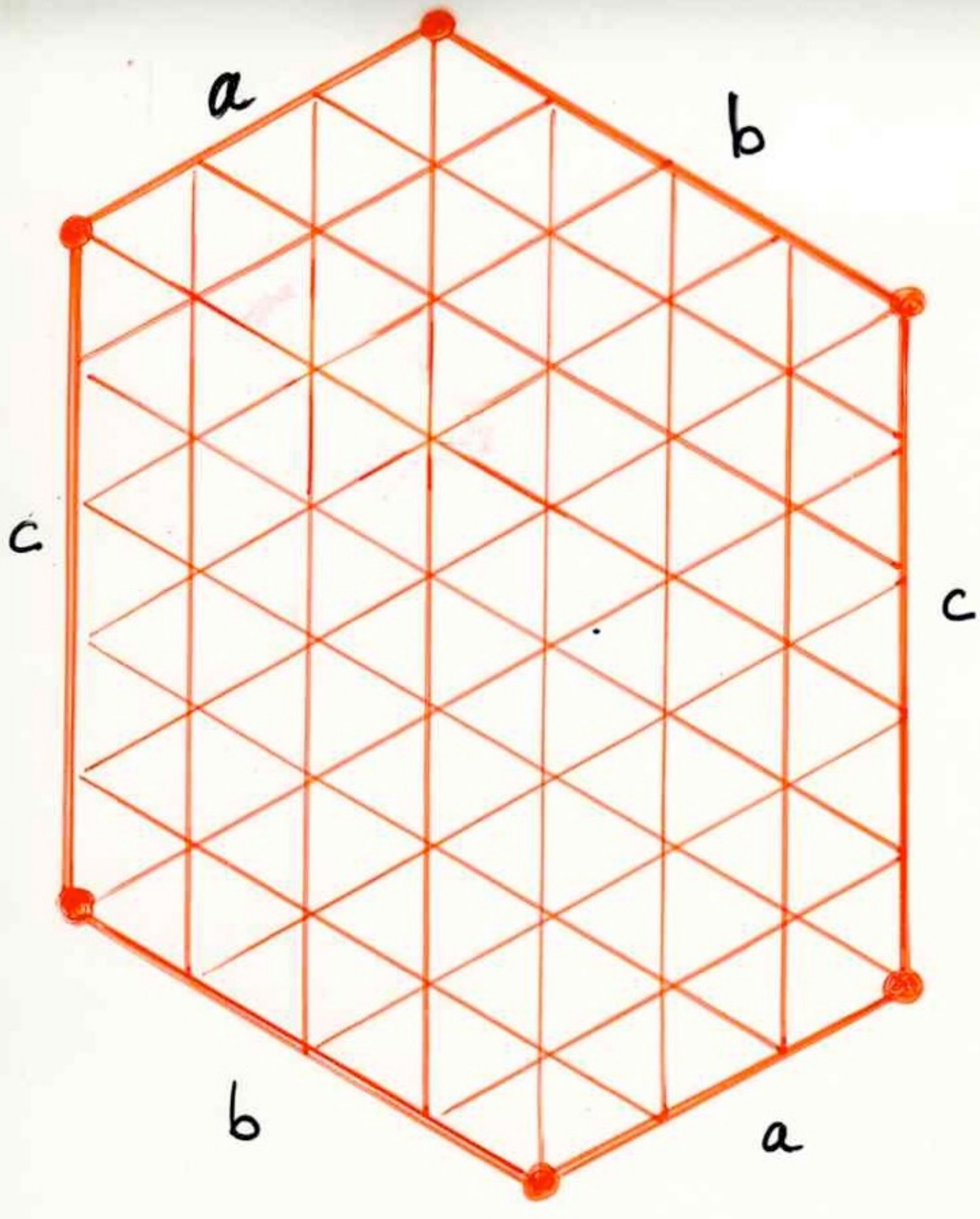
3D Ferrers diagrams

in a box $\mathcal{B}(a,b,c)$

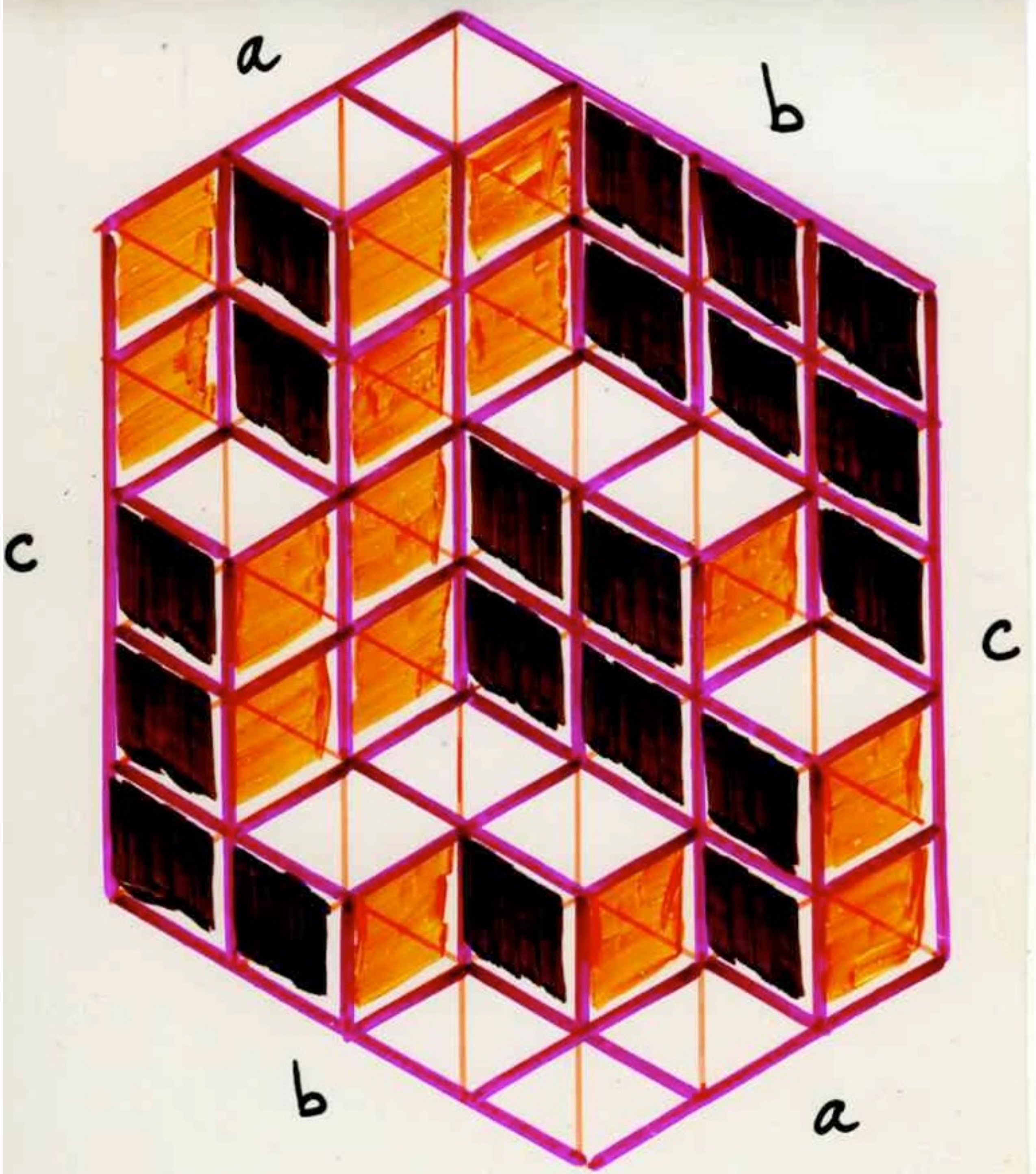
plane partitions
in a box (a, b, c)





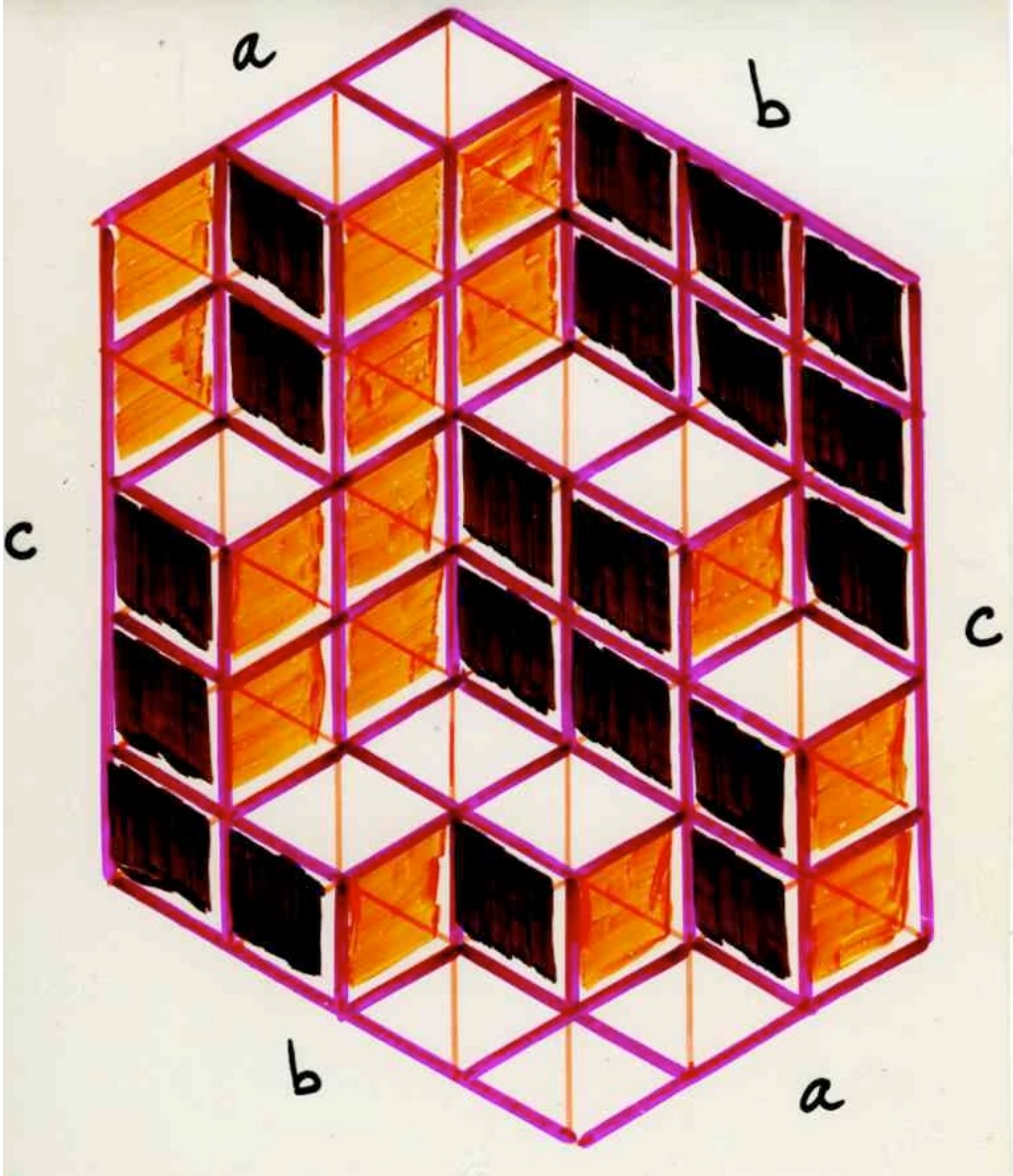

a
b
c
c
b
a

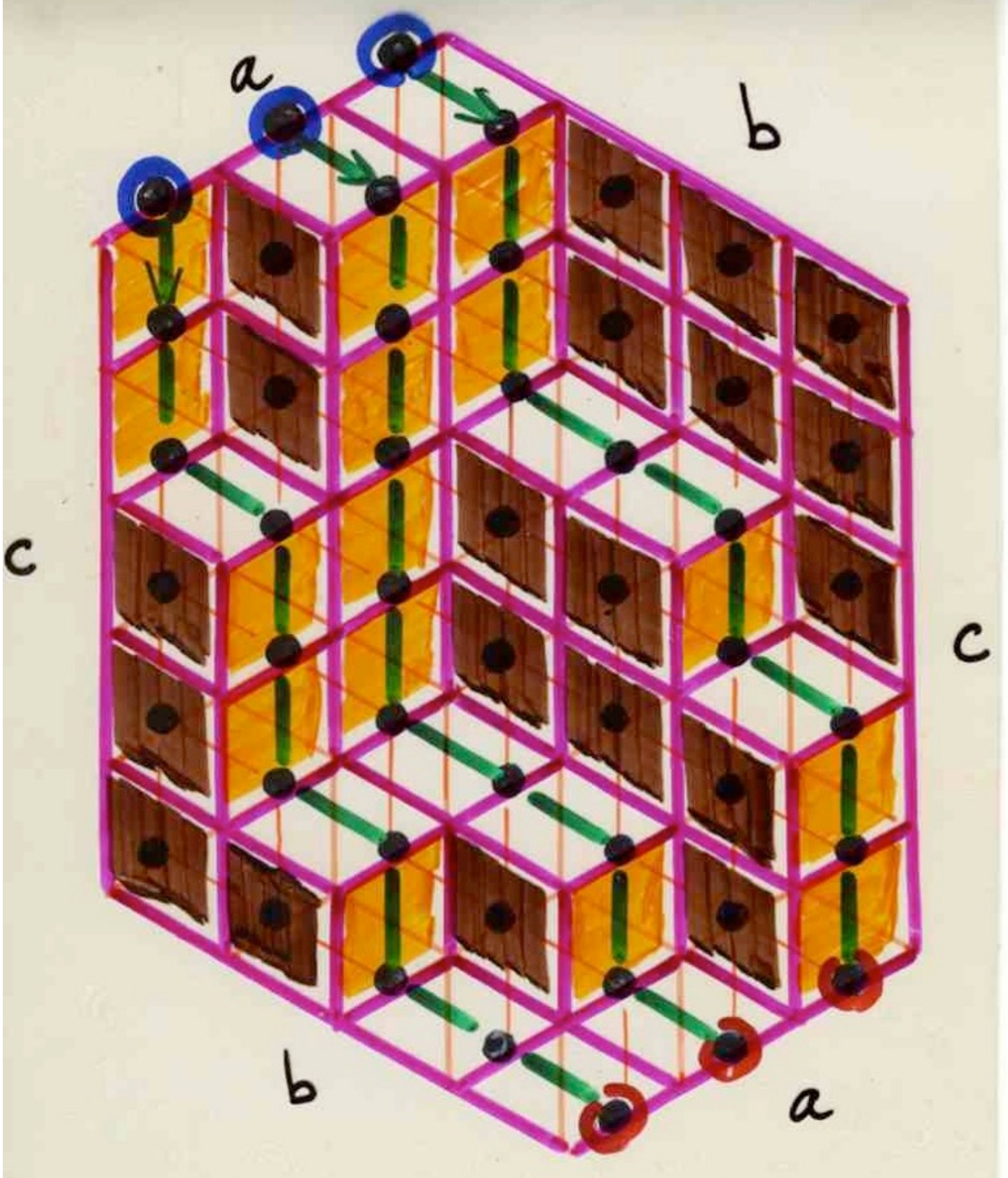
The poset of plane partitions included in a given box of size (a, b, c) (ordered by inclusion of 3D diagrams),

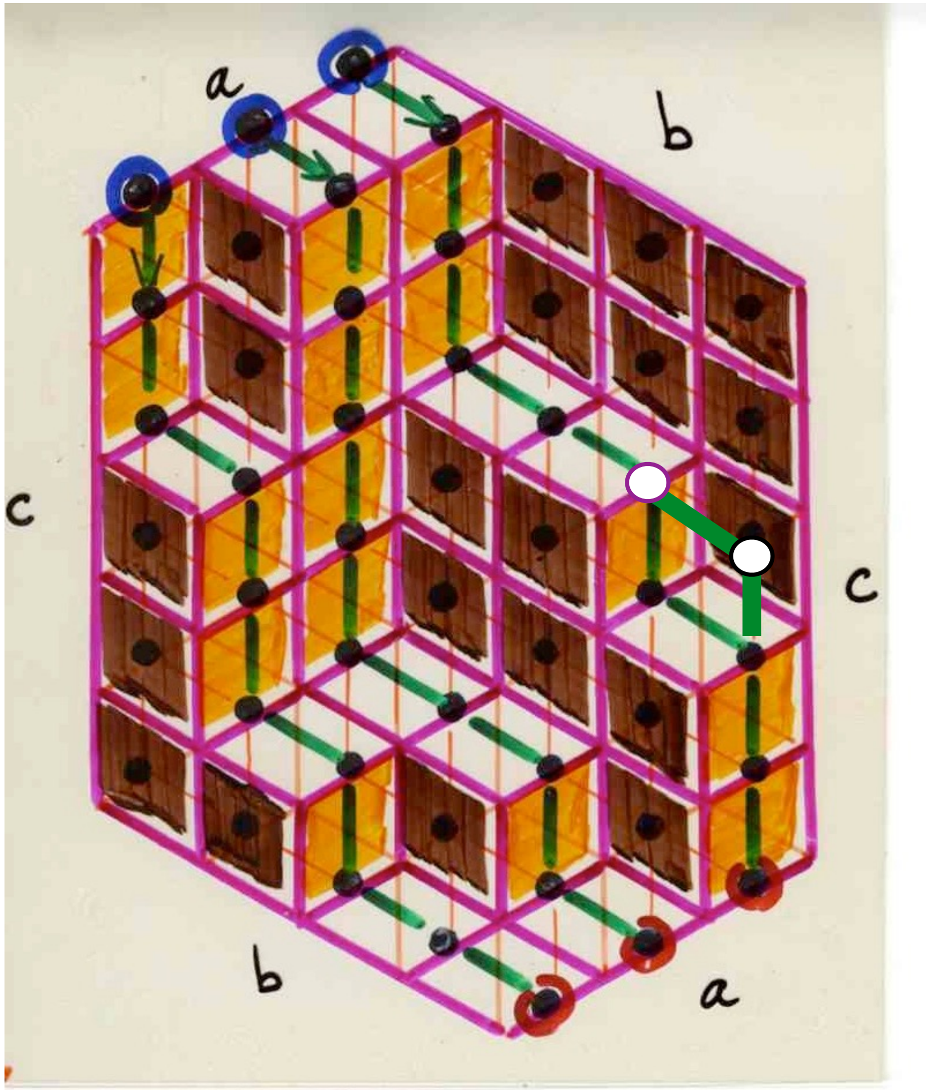


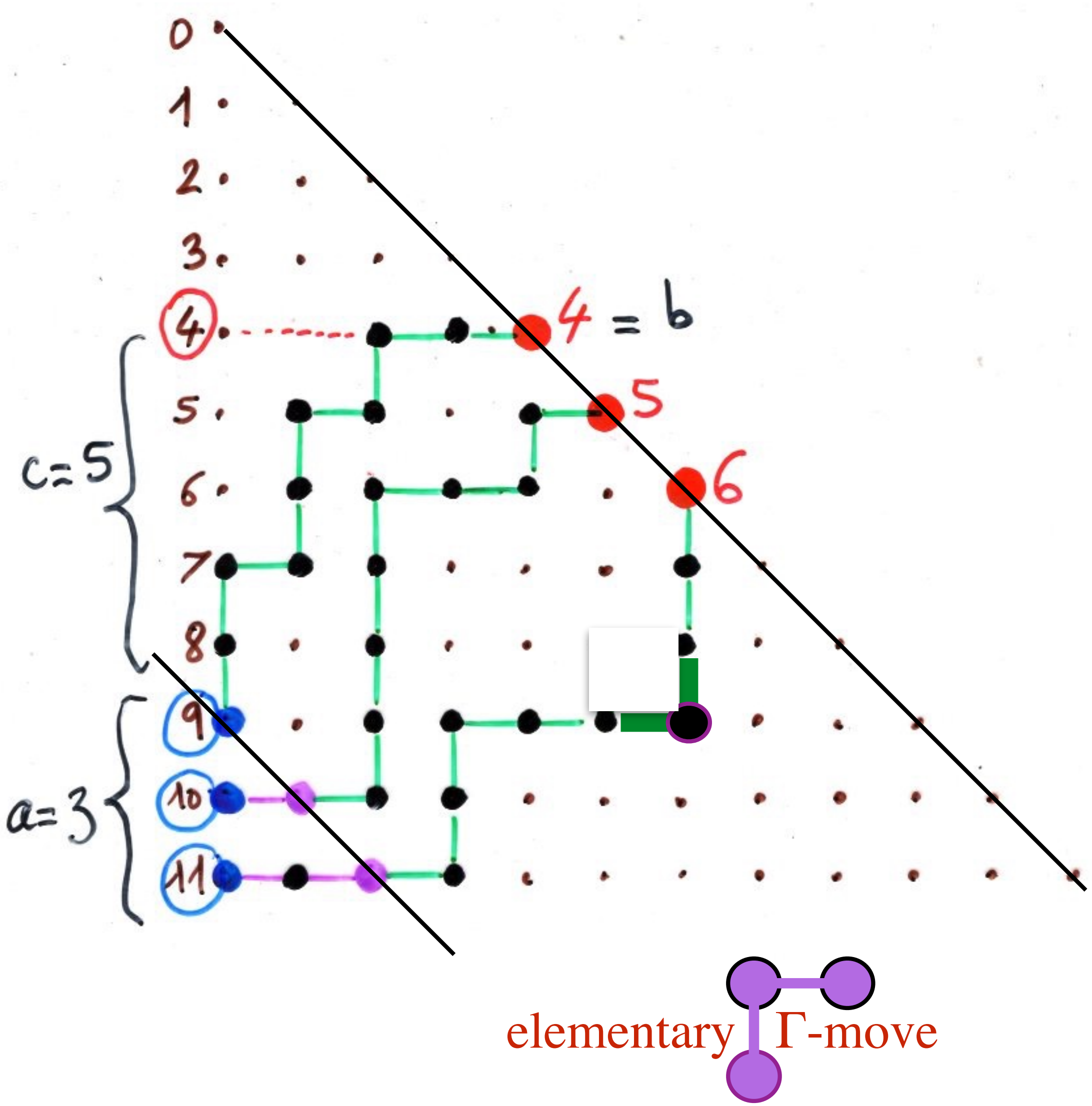
$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

MacMahon famous formula for the number of plane partitions included in a box (a, b, c) can be proved using a coding of the plane partitions with configuration of non-crossing paths

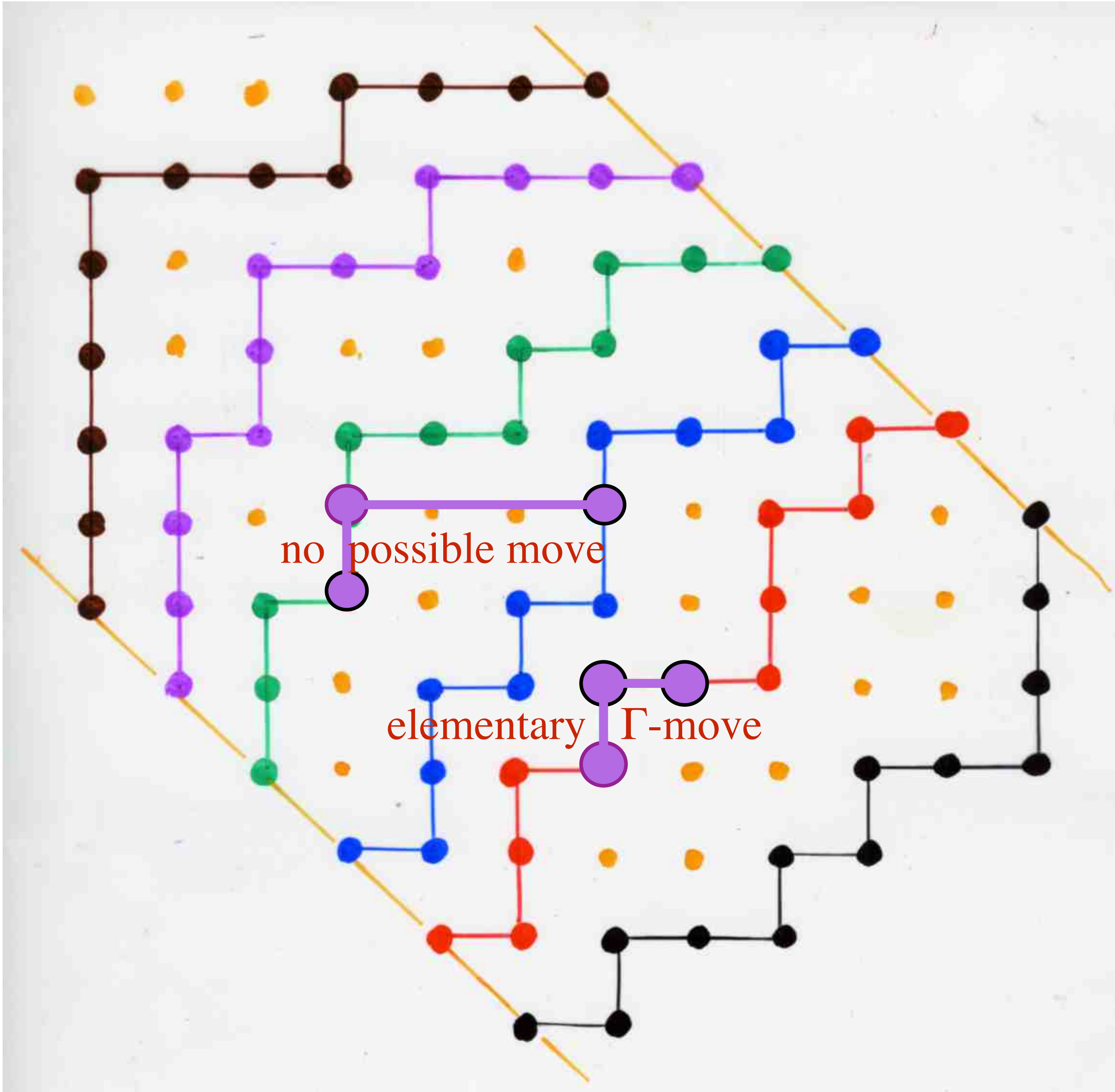

a
b
c
c
b
a


a
b
c
c
b
a

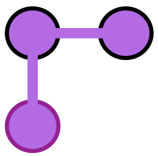

a
b
c
c
b
a


0
1
2
3
4
4 = b
5
5
6
6
7
8
9
10
11
c=5
a=3
elementary Γ-move

the associated cloud of points are all the vertices of all the paths



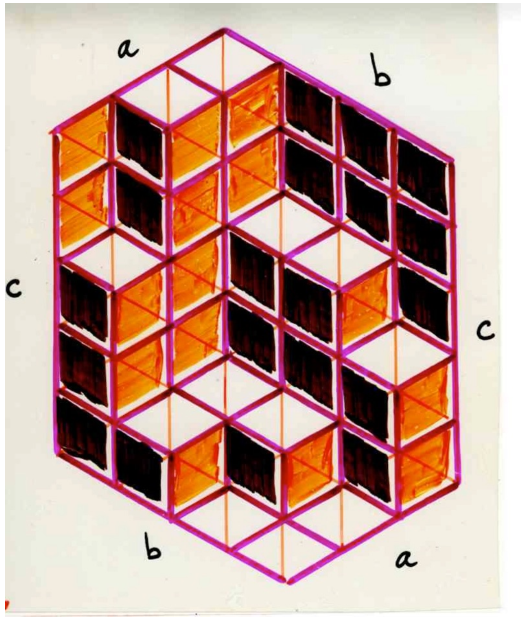
As in the case of Young lattice, Γ-moves are only elementary Γ-moves,



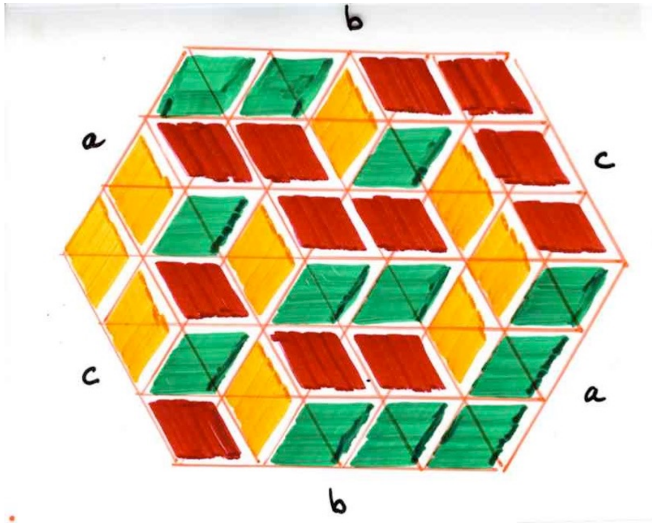
that is moves where the corresponding rectangle is reduced to an elementary cell of the square lattice.

Such maules are called simple maule.

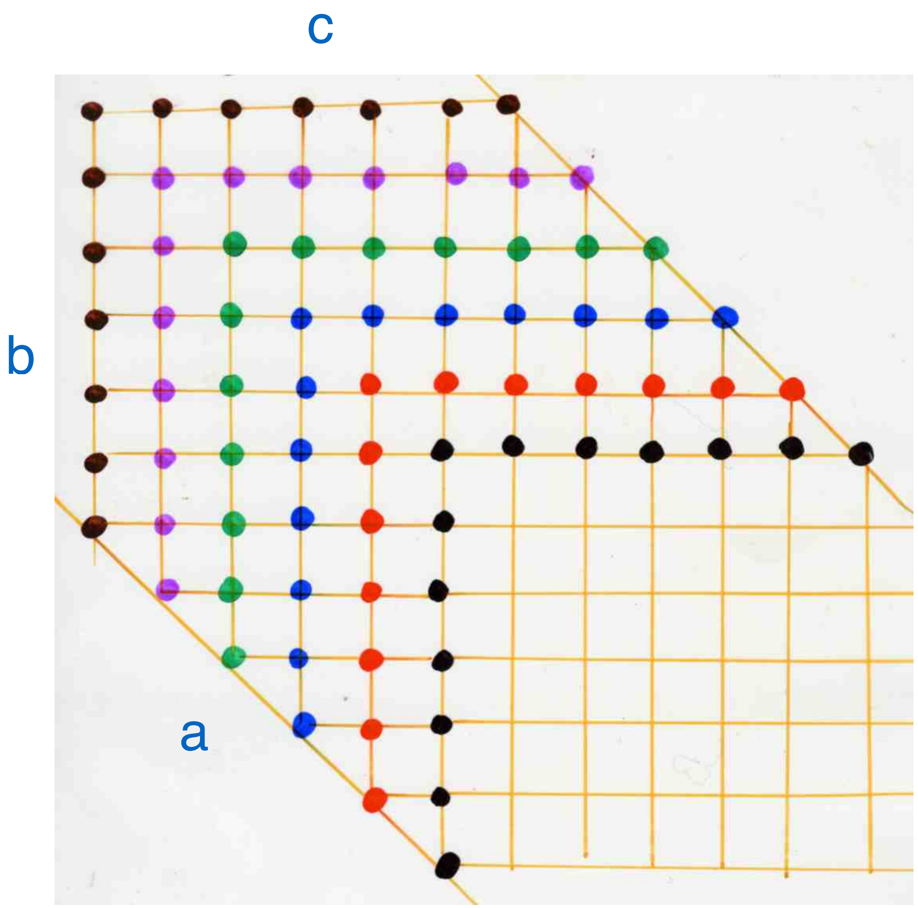
The poset of plane partitions included in a given box of size (a, b, c) (ordered by inclusion of 3D diagrams),



equivalently tilings of an hexagon of size (a,b,c),

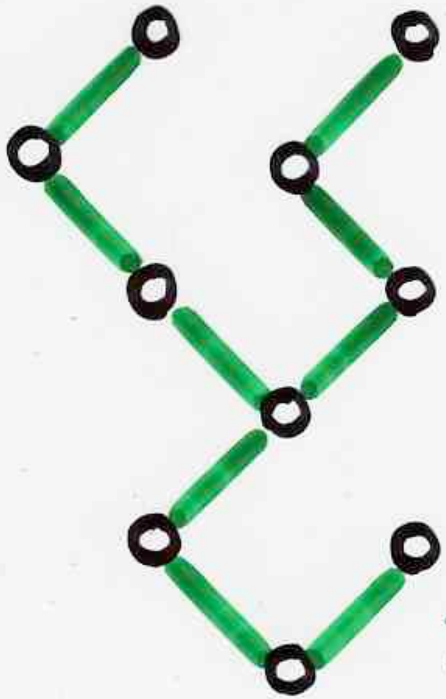


is isomorphic to the simple maule generated by the following cloud of points associated to the triple (a, b, c).



Tamari lattice

definition



B

$$\begin{cases} B = (L, r, R) \\ \text{or} \\ B = \emptyset \end{cases}$$

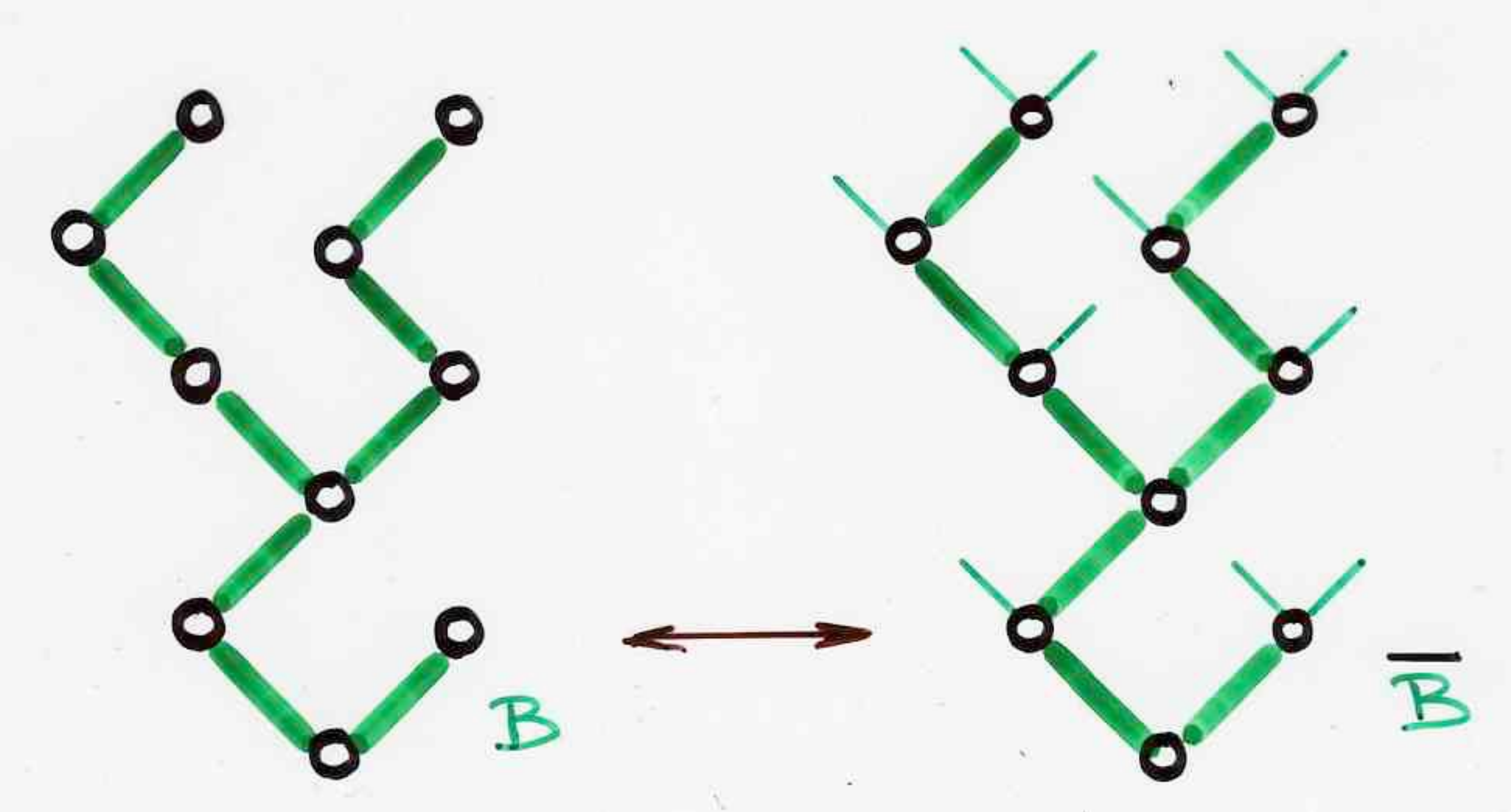
L, R binary trees

r root

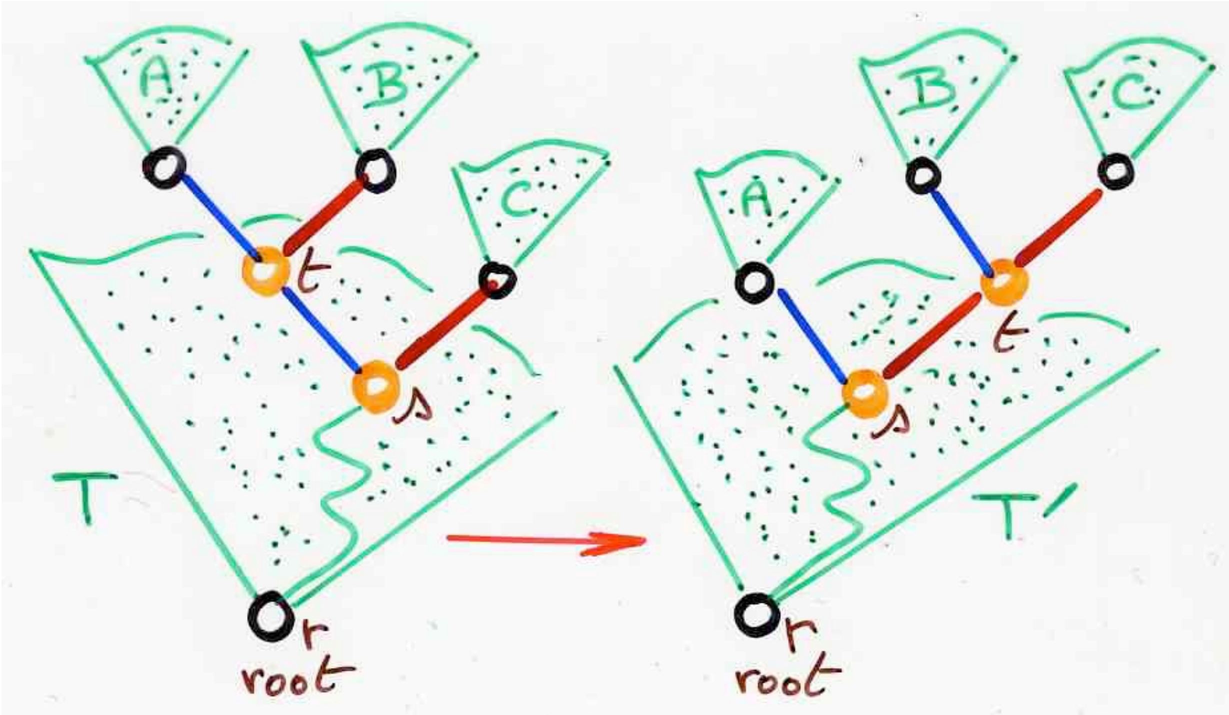
a binary tree B

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



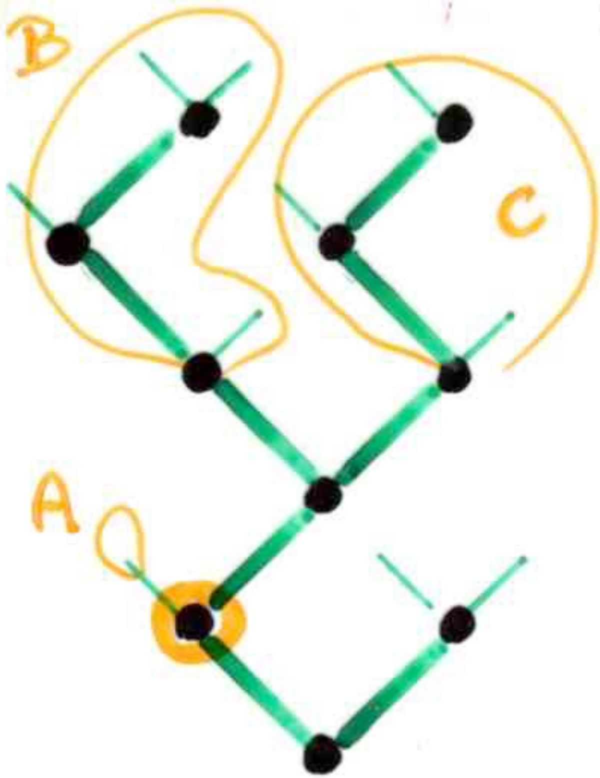
a binary tree B
and its associated complete binary tree $\overline{B}$
(full)

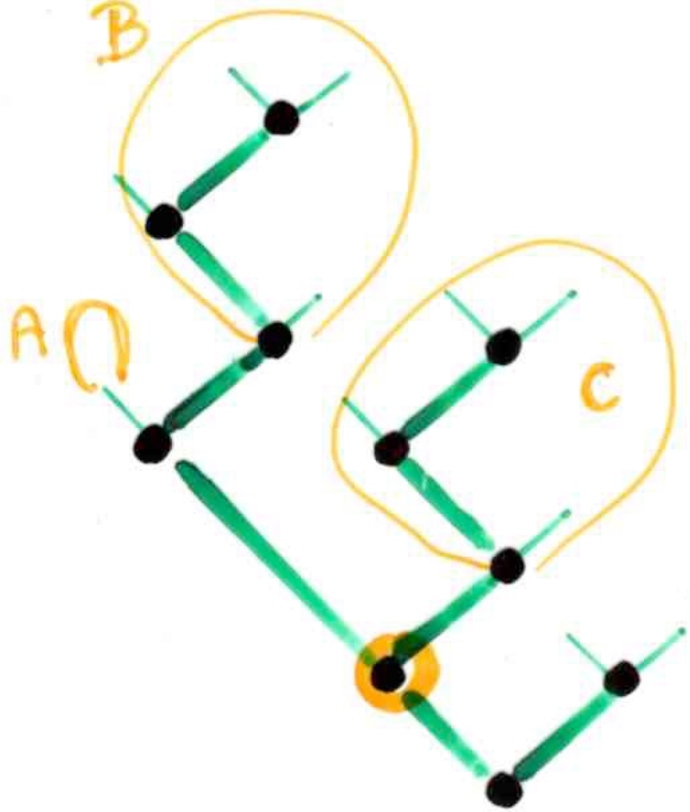


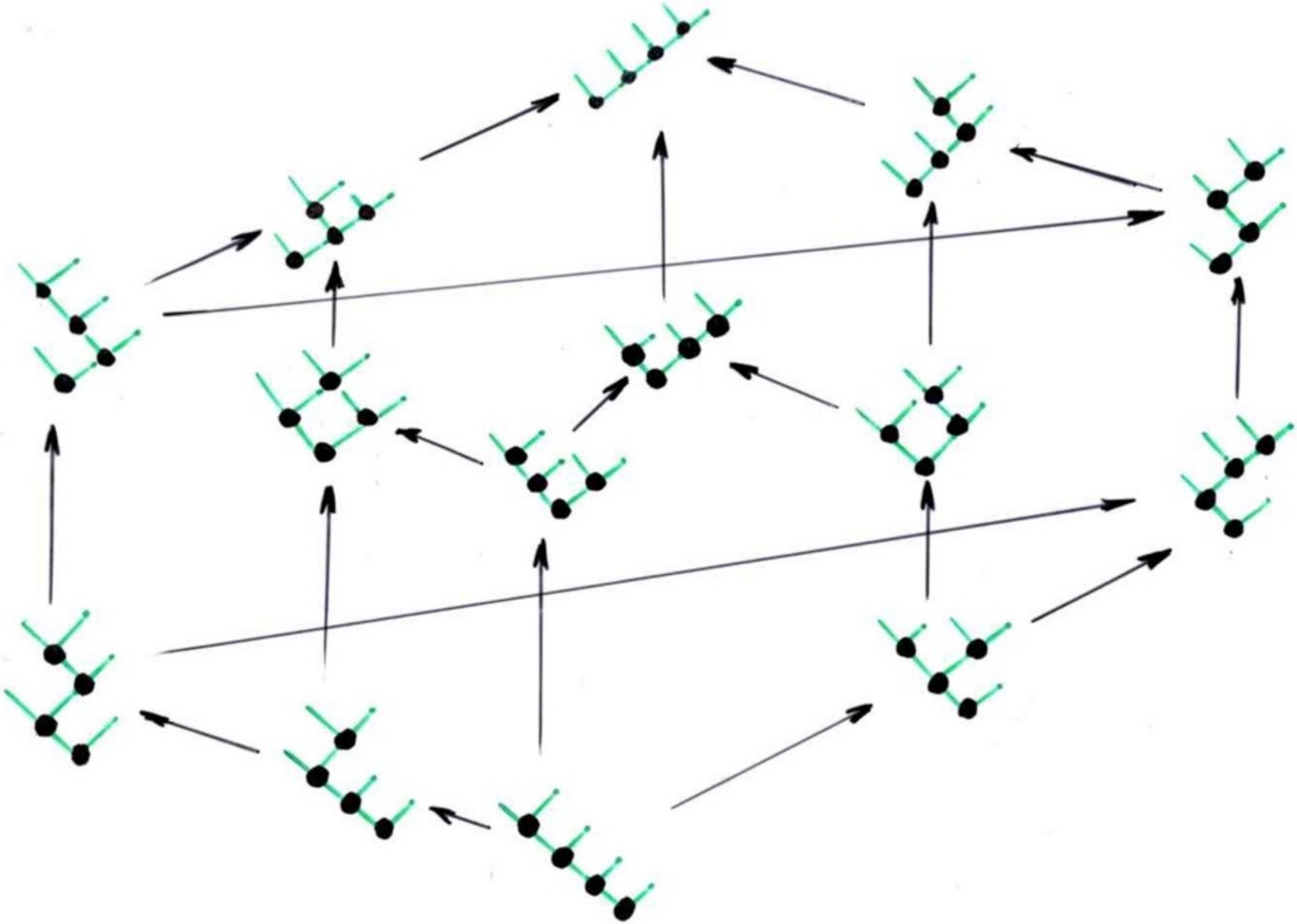
Tamari lattice

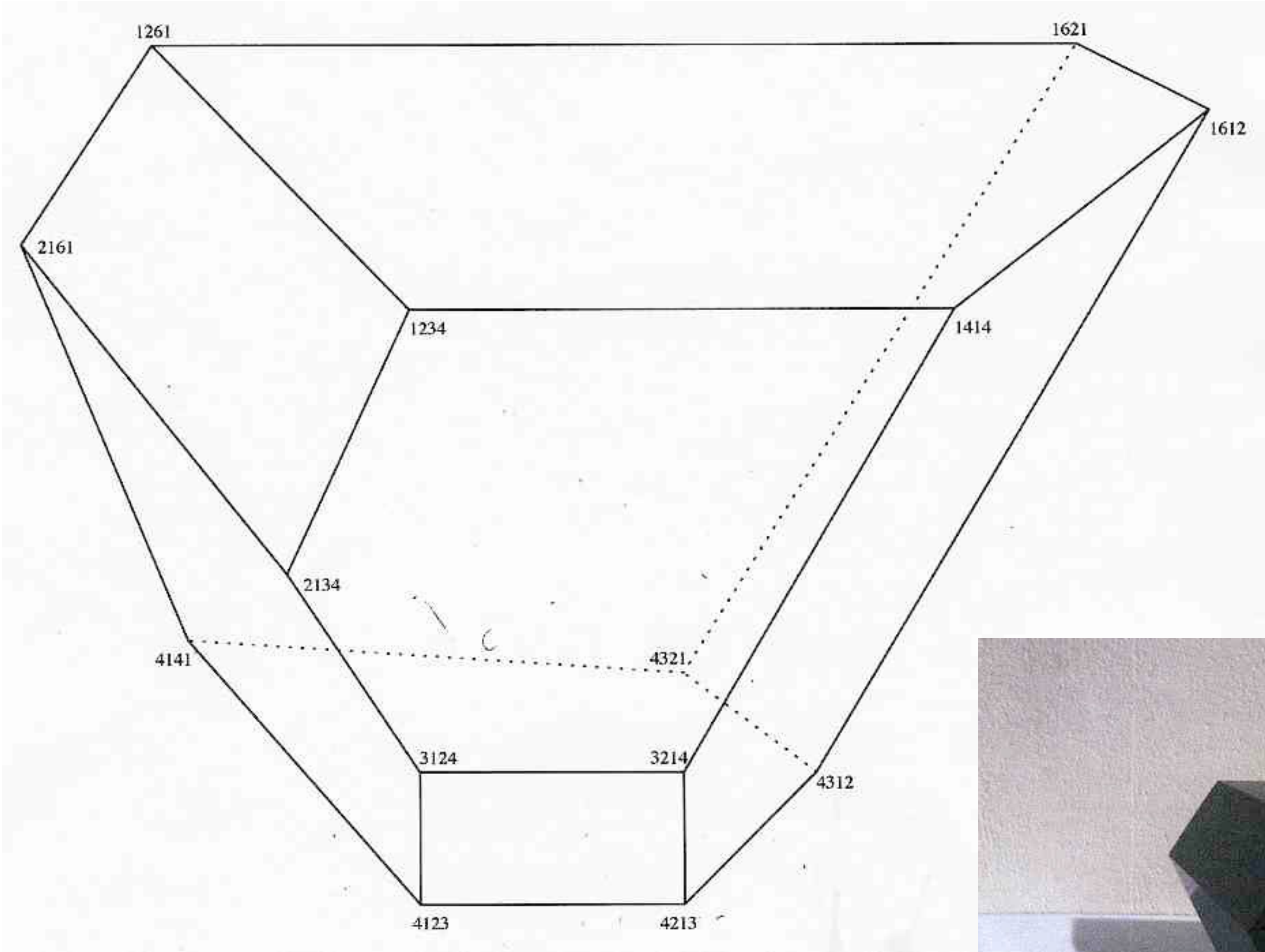
Rotation in a binary tree:
the covering relation in the
Tamari lattice

Dov Tamari (1951) thèse Sorbone
"Monoides preórdonnés et chaînes de Malcev"


B
C
A


B
A
C

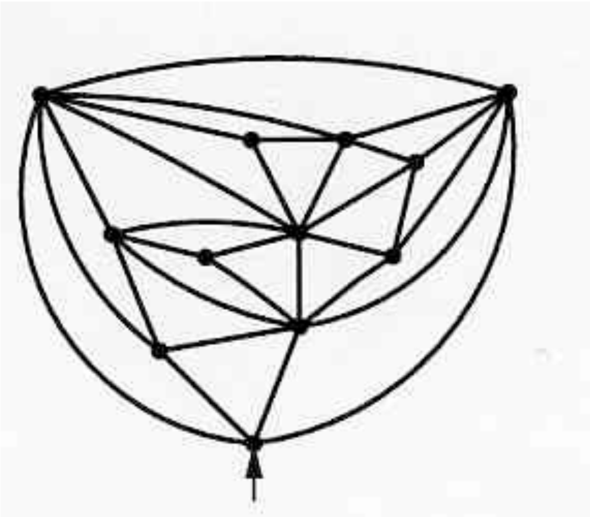




associahedron

number of intervals in Tamari(n)

$$\frac{2\,(4n+1)!}{(n+1)!\,(3n+2)!}$$ Chapoton (2006)

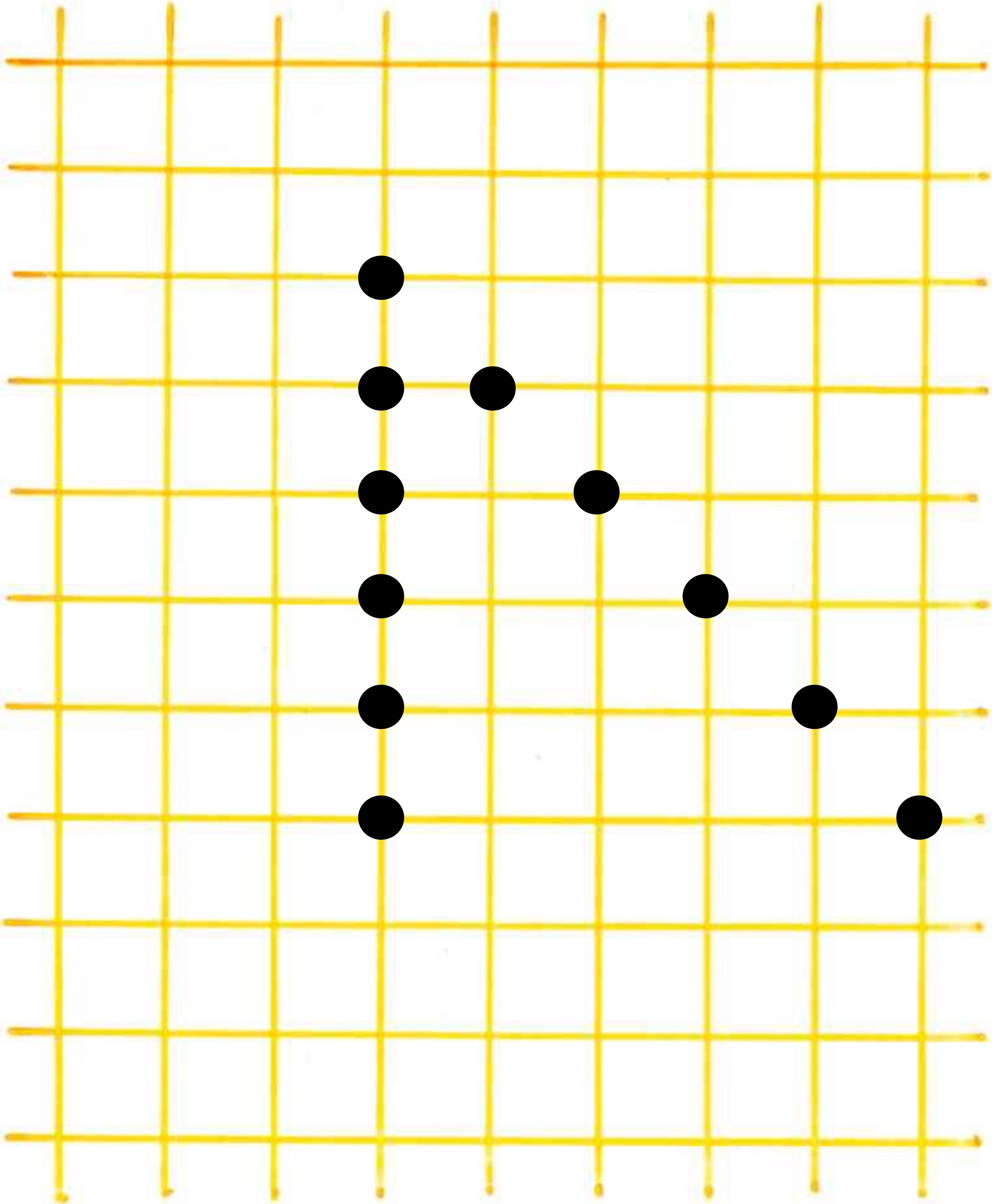


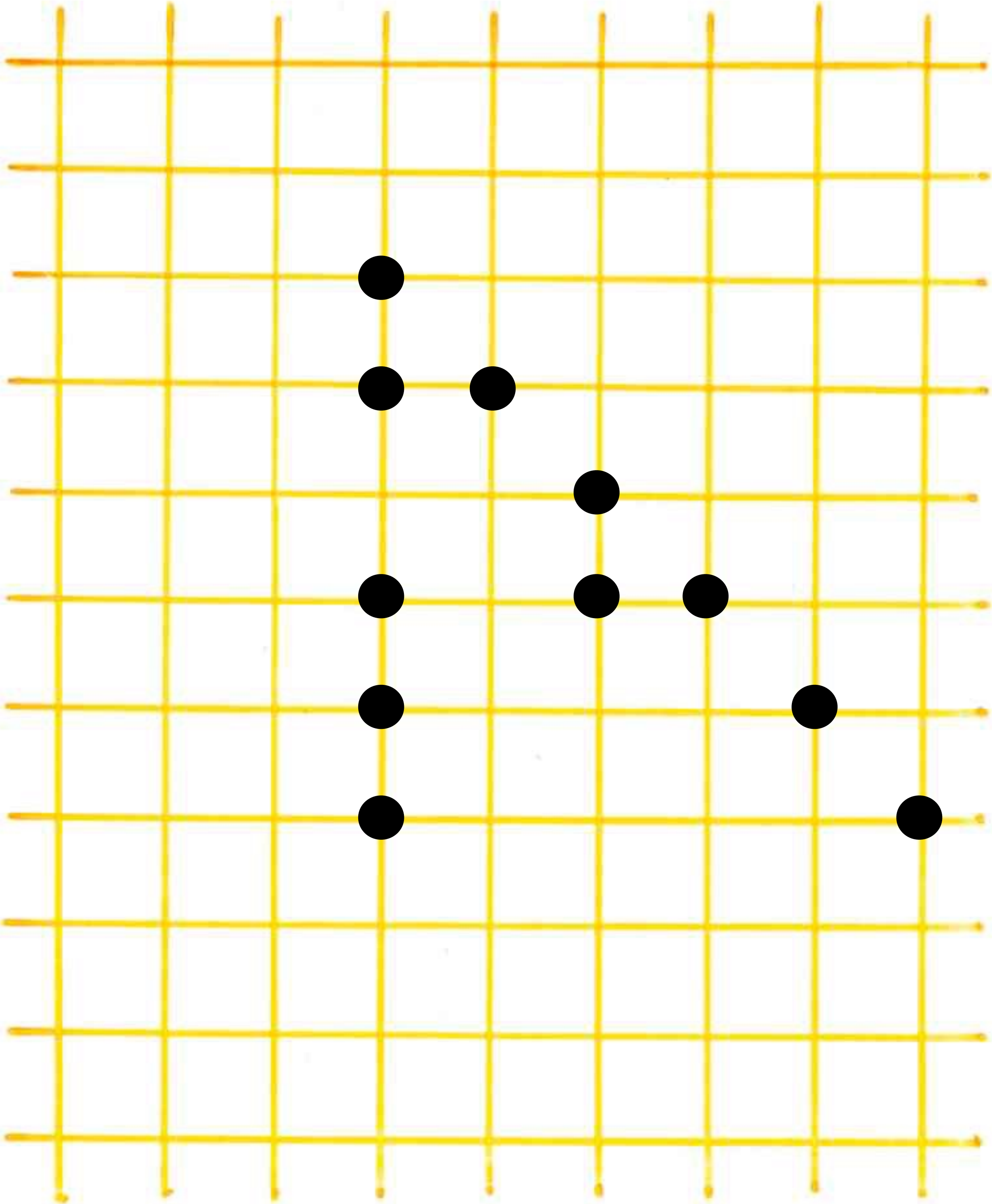
number of triangulations (i.e. maximal planar graphs)

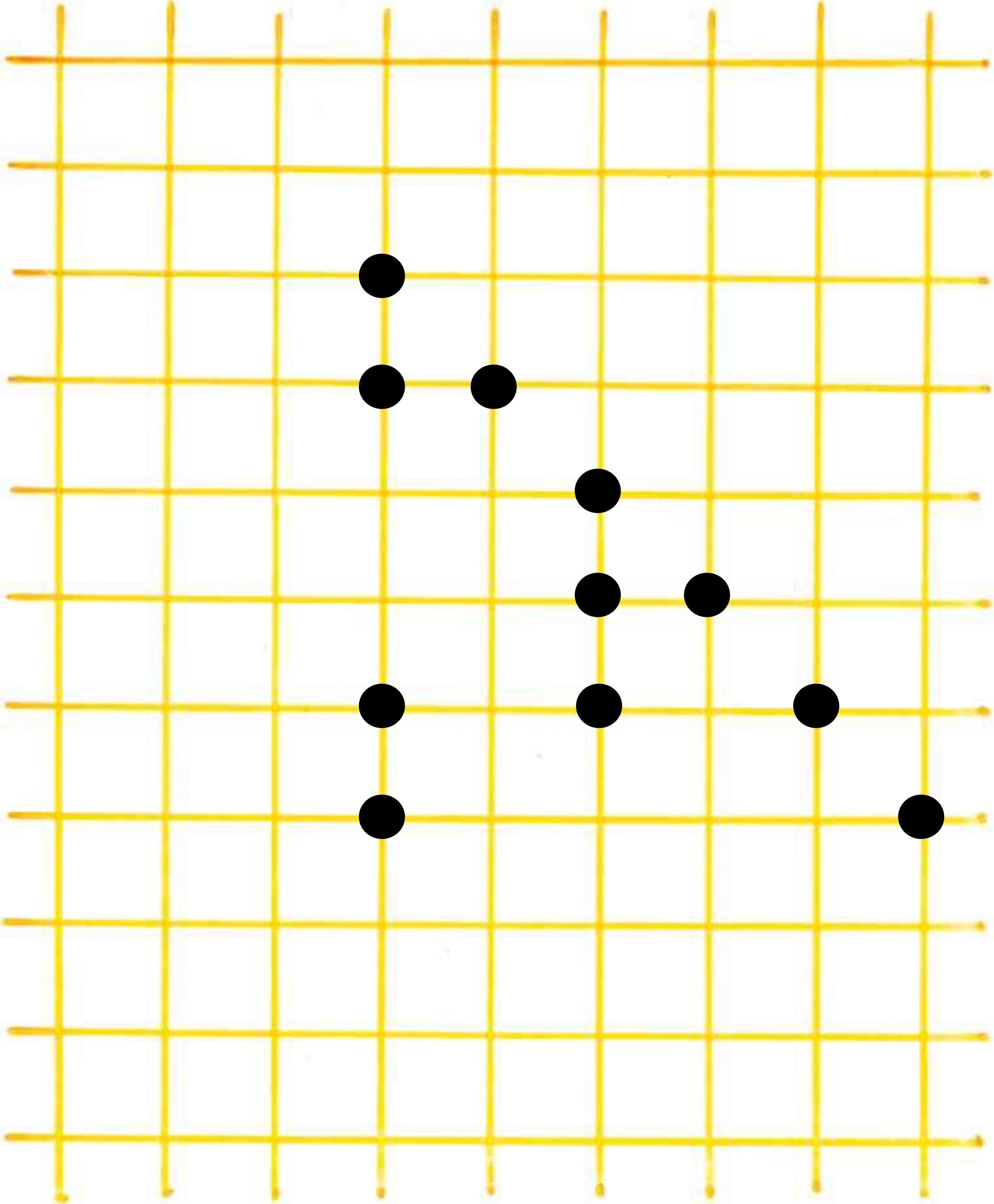
Bernardi, Bonichon (2007)

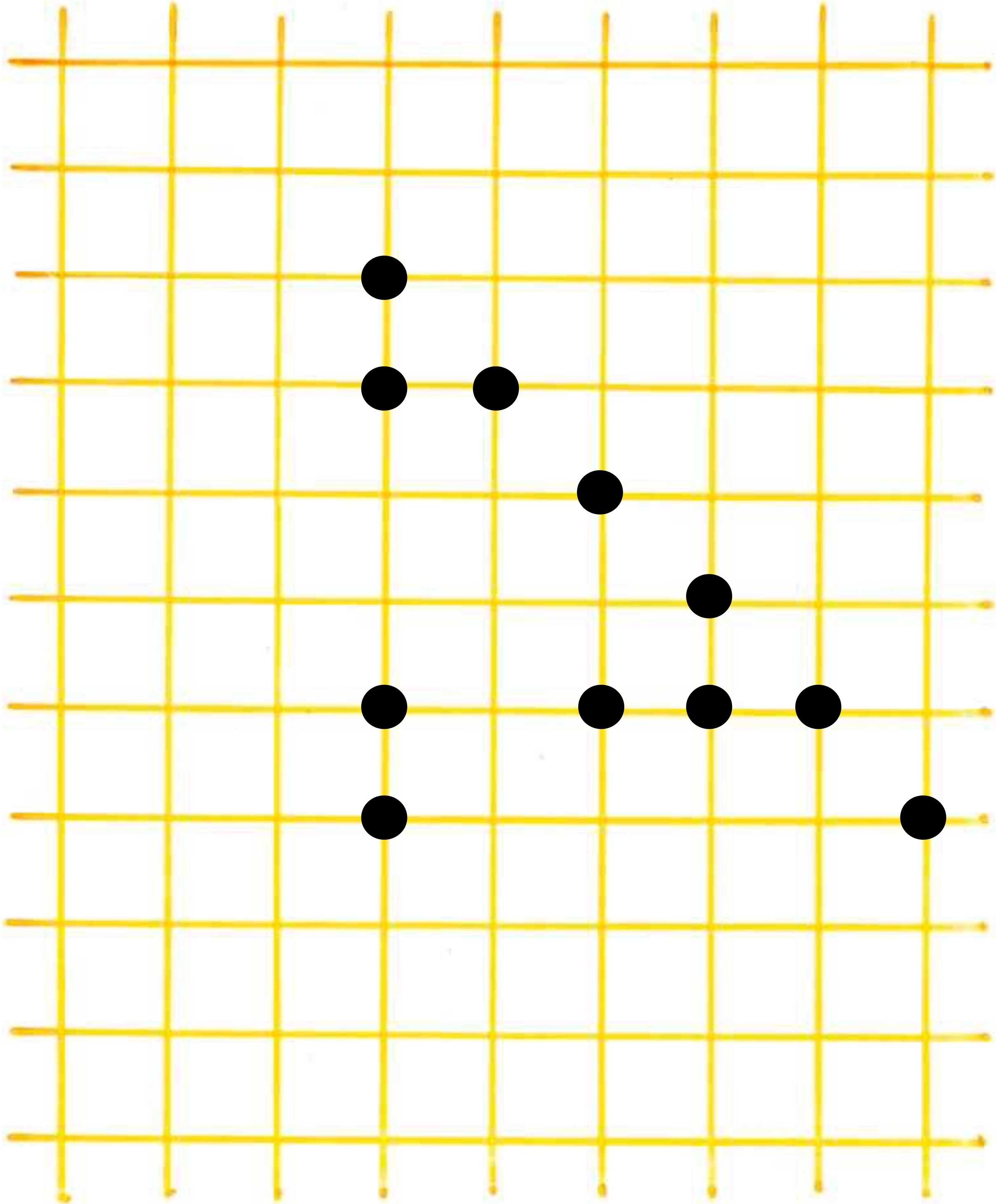
Tamari lattice

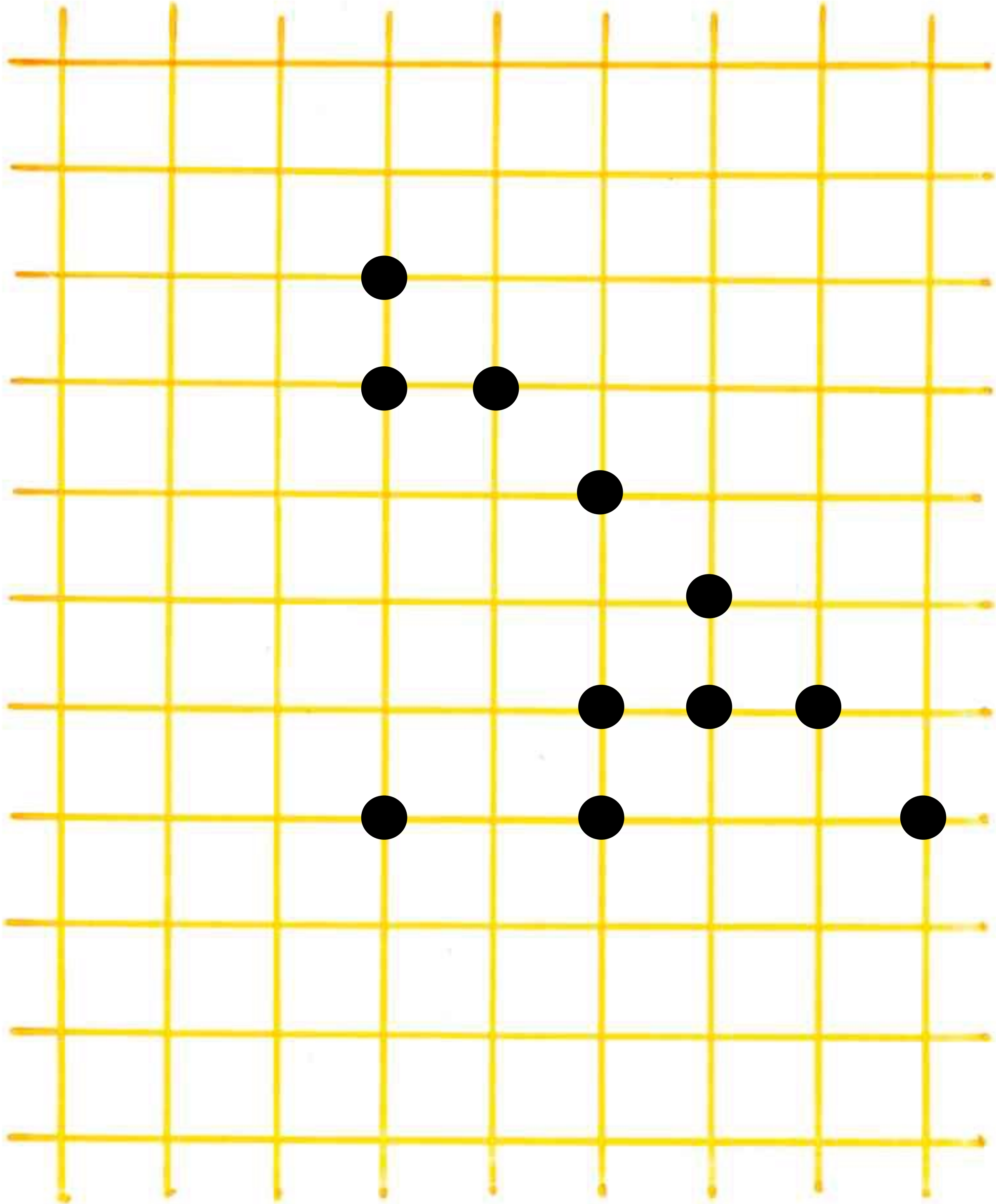
as a maule

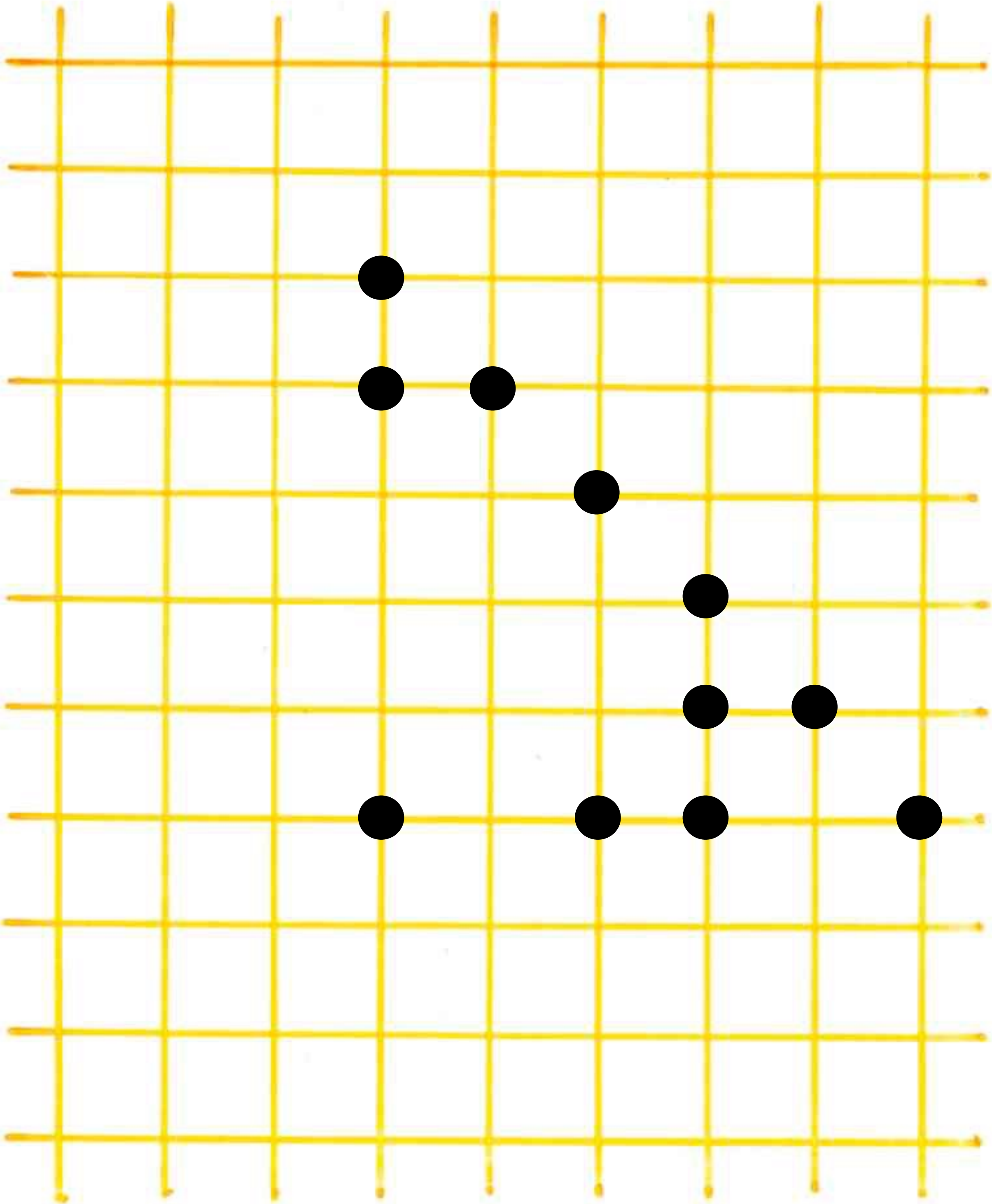


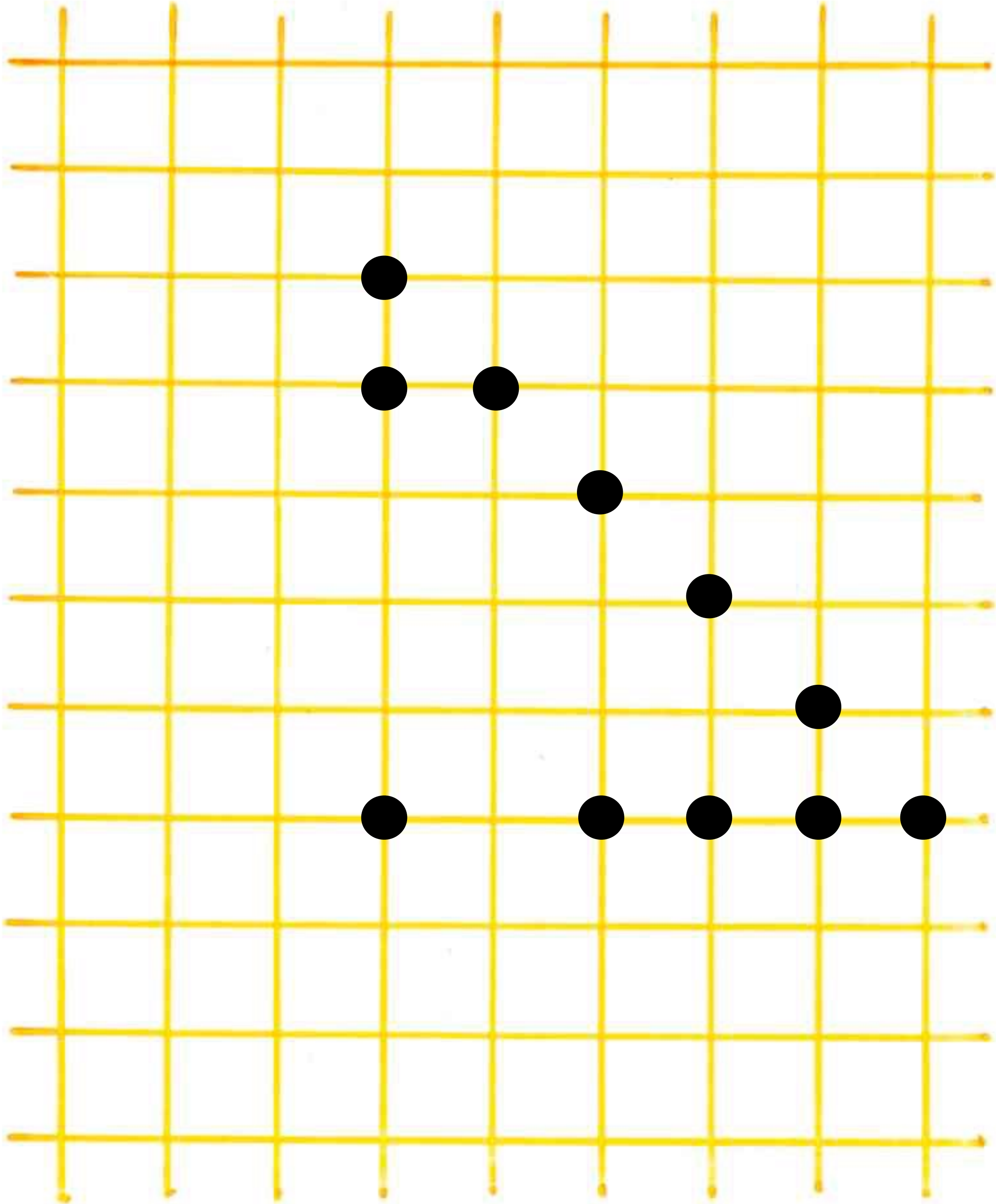


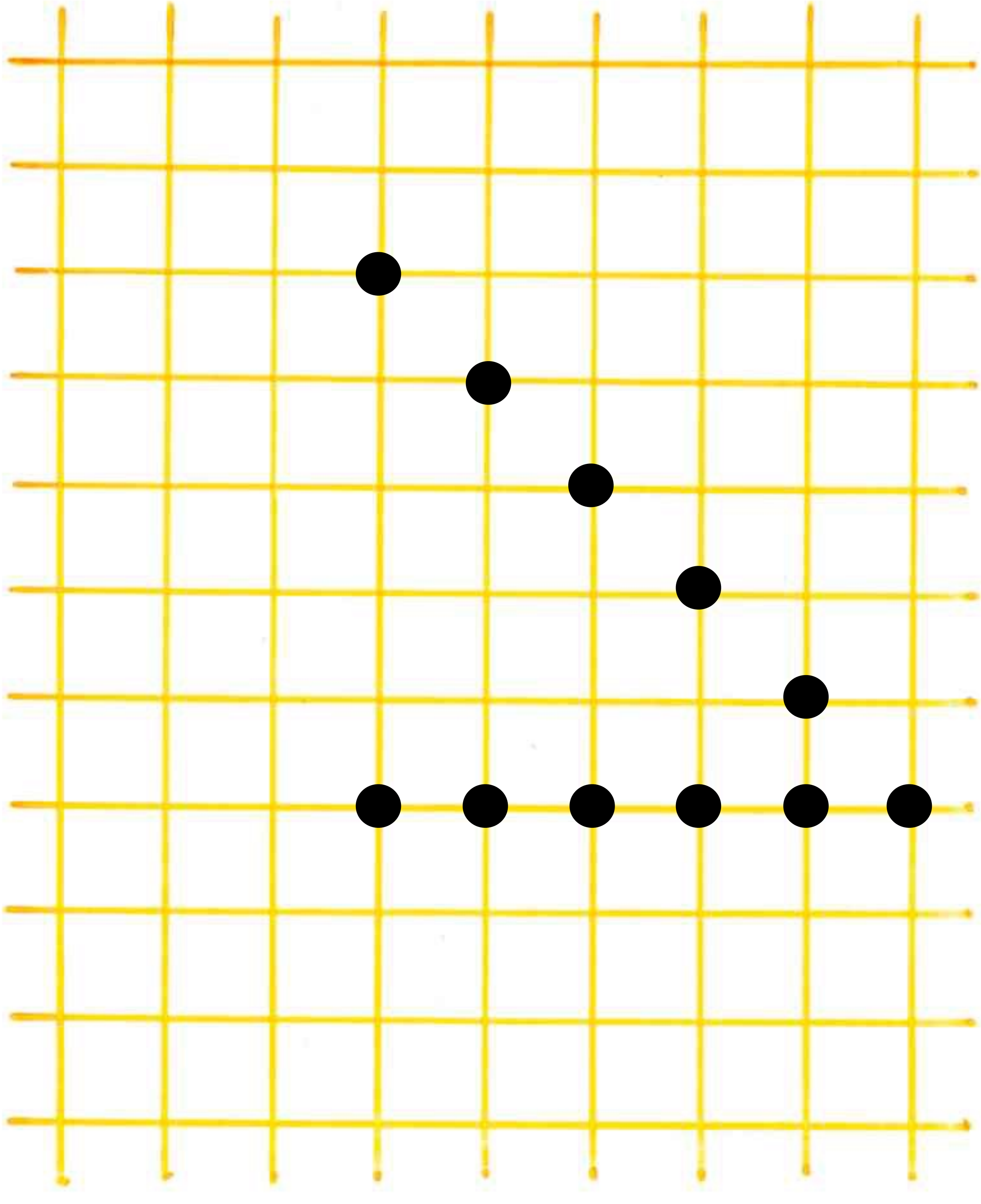




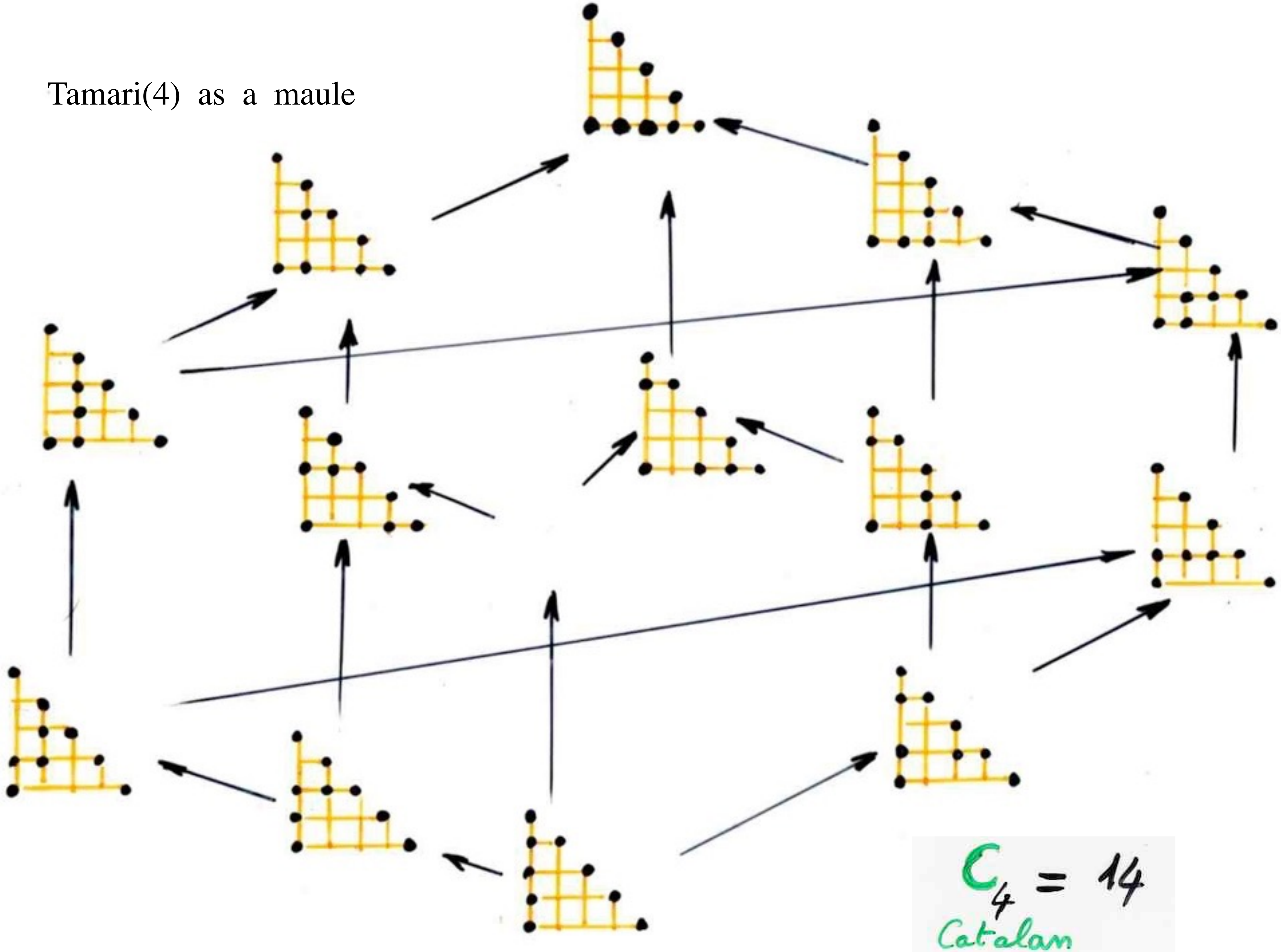




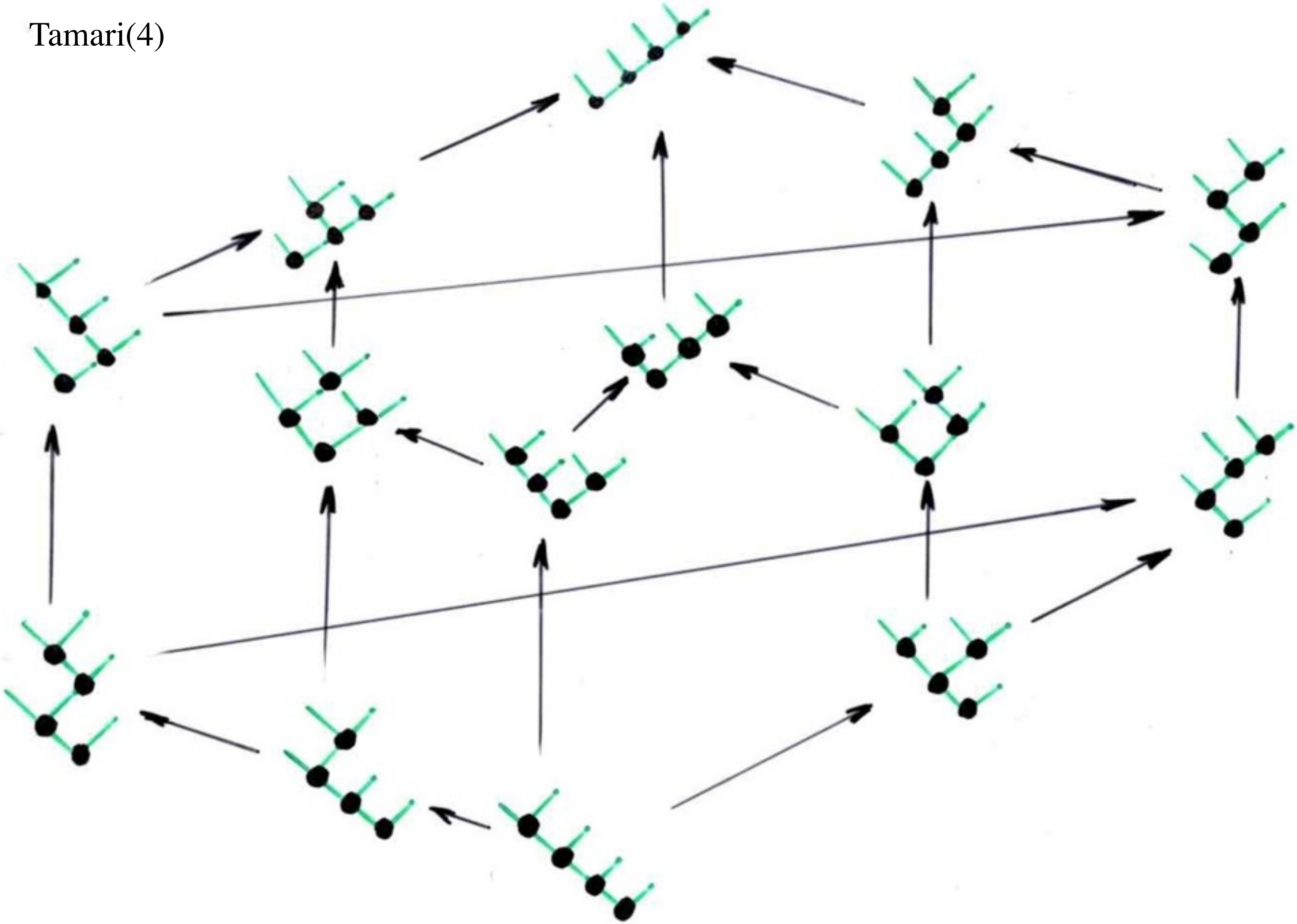




Tamari(4) as a maule



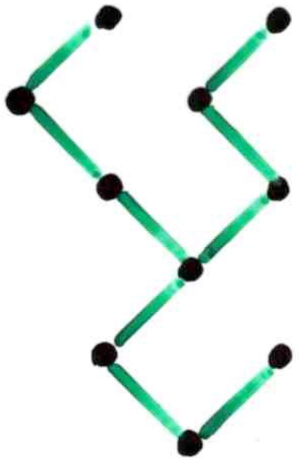
Tamari(4)

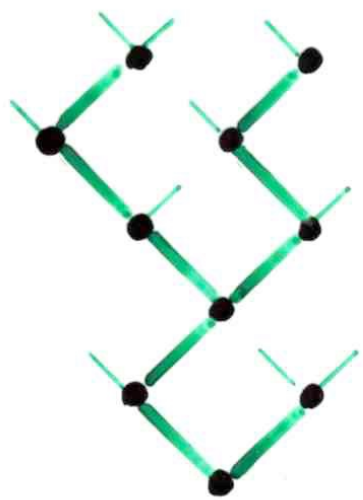


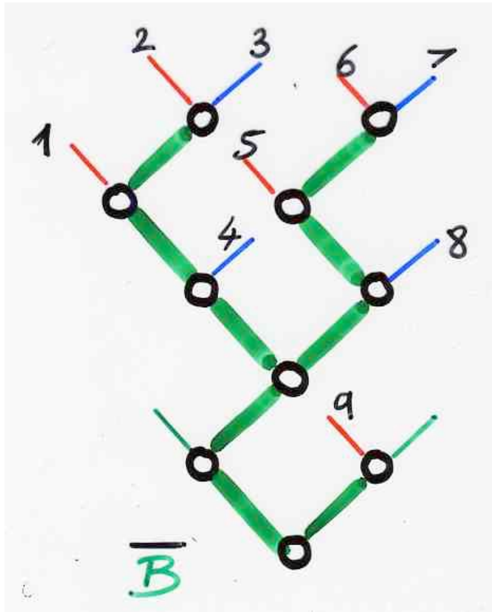
Proposition

$$\mathrm{Tamari}(n) = \mathrm{Maule}\begin{pmatrix} \bullet & & & & & & \\ \bullet & \bullet & & & n+1 & & \\ \bullet & & \bullet & & & & \\ \vdots & & & \bullet & & & \\ \vdots & & & & \bullet & & \\ \vdots & & & & & \ddots & \\ \bullet & & & & & & \bullet \\ \bullet & & & & & & & \bullet \end{pmatrix}$$

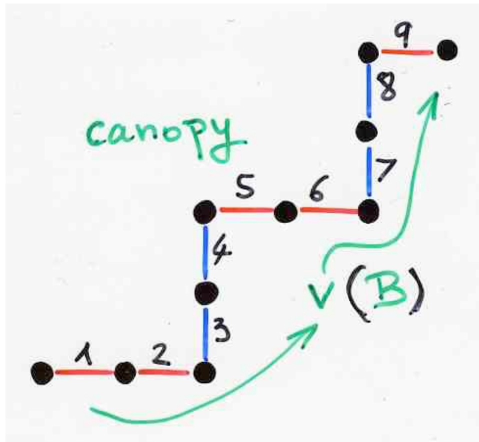
canopy of a binary tree



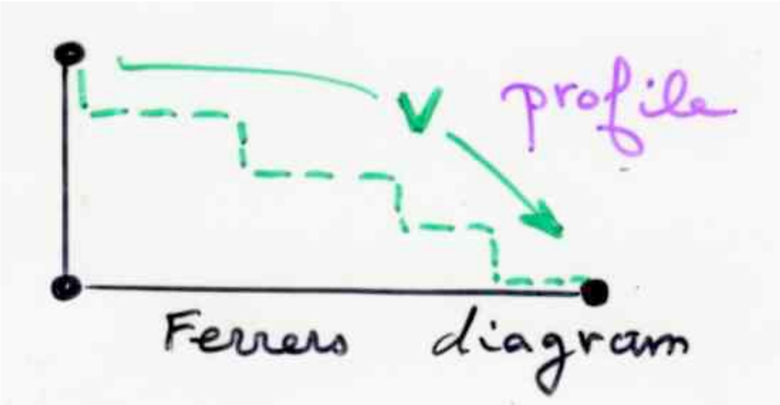


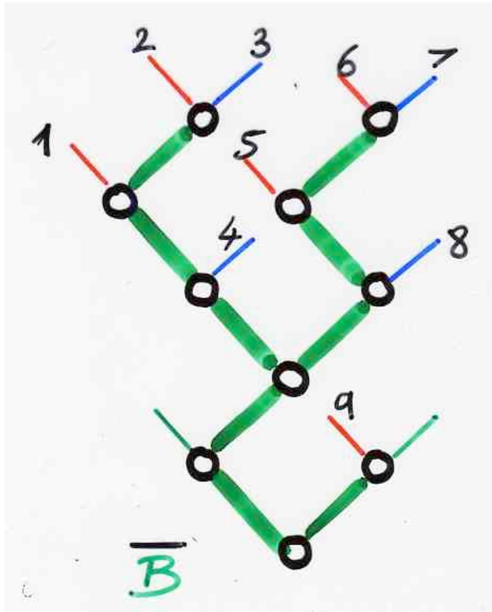


Loday, Ronco (1998) (2012)



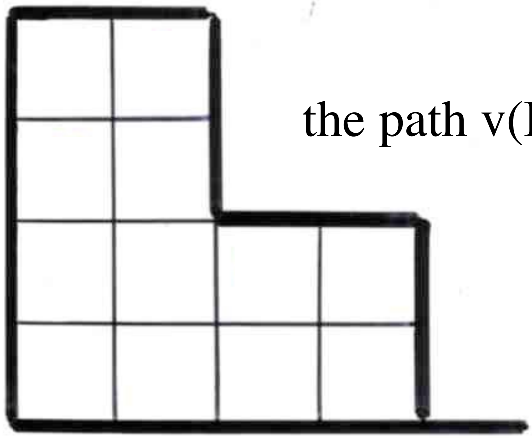
The external edges (except the first and last) of the extended binary tree are ordered from left to right (symmetric order). According to the fact the edge is left (red) or right (blue), this gives a word of length (n-1) in 2 letters, which can be seen as a path with elementary steps East (red) and North (blue).



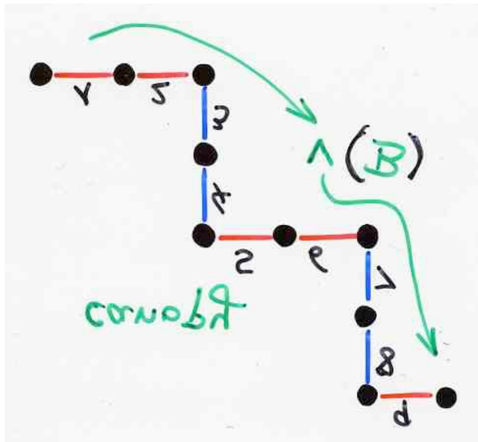


With the french notation for Ferrers diagrams, we will need to see the canopy as a path v(B) with elementary steps East and South, which define a Ferrers diagram F(B) (with possibly empty row or column). The path v, called the **profile** of F(B) is its North-East border.

F(B)



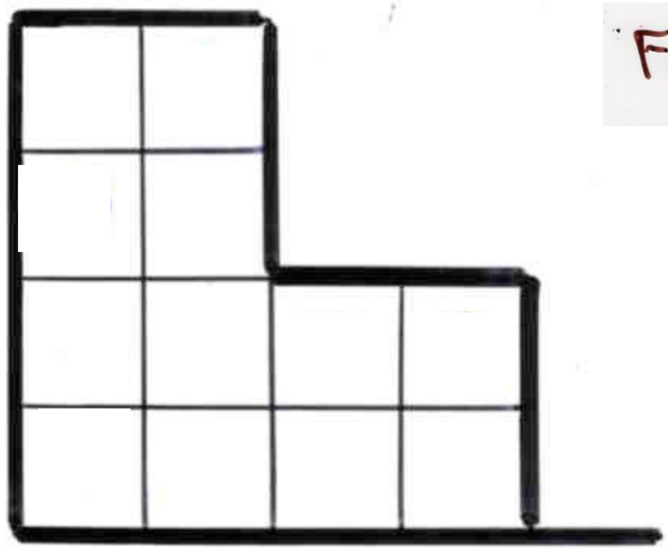
the path v(B)



alternative tableaux

alternative tableau

Definition



Ferrers diagram **F**

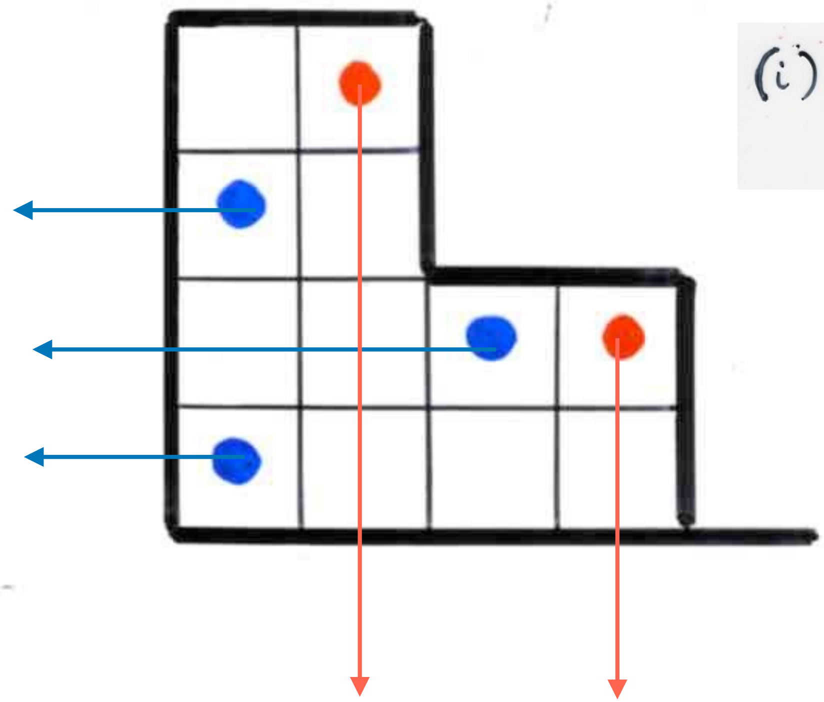
with possibly empty rows or columns

size of F

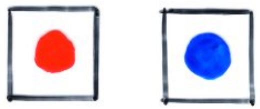
$\mathbf{n} = (\text{number of rows}) + (\text{number of columns})$

alternative tableau

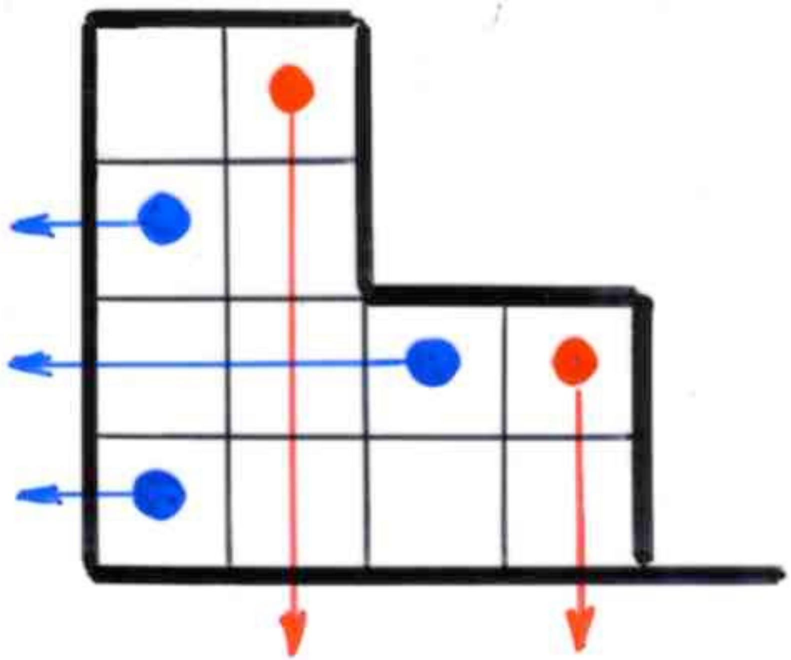
Definition



(i) some cells are coloured red or blue



(ii) • no coloured cell at the left of a blue cell
• no coloured cell below a red cell

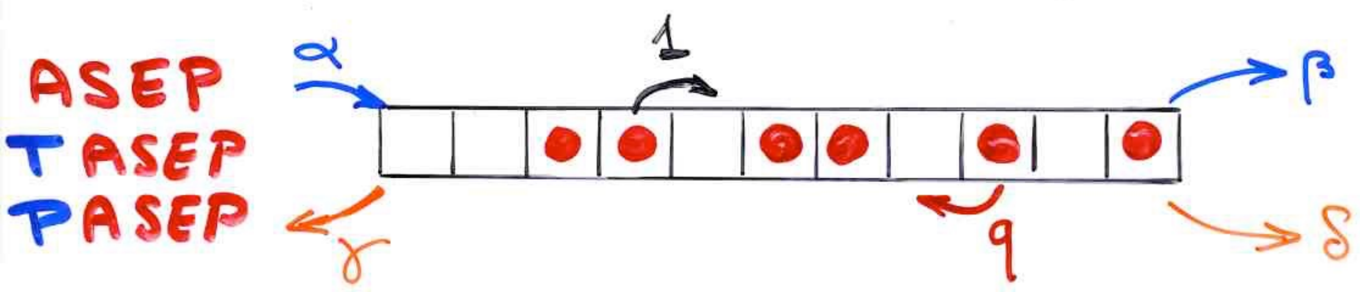


Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

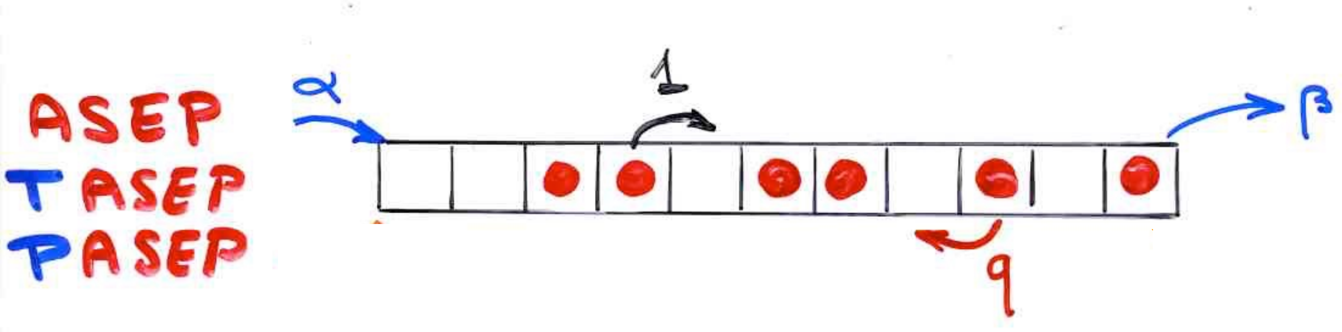
The general PASEP model in physics with its 3 parameters.
(partially asymmetric exclusion model)

toy model in the physics of
dynamical systems far from equilibrium

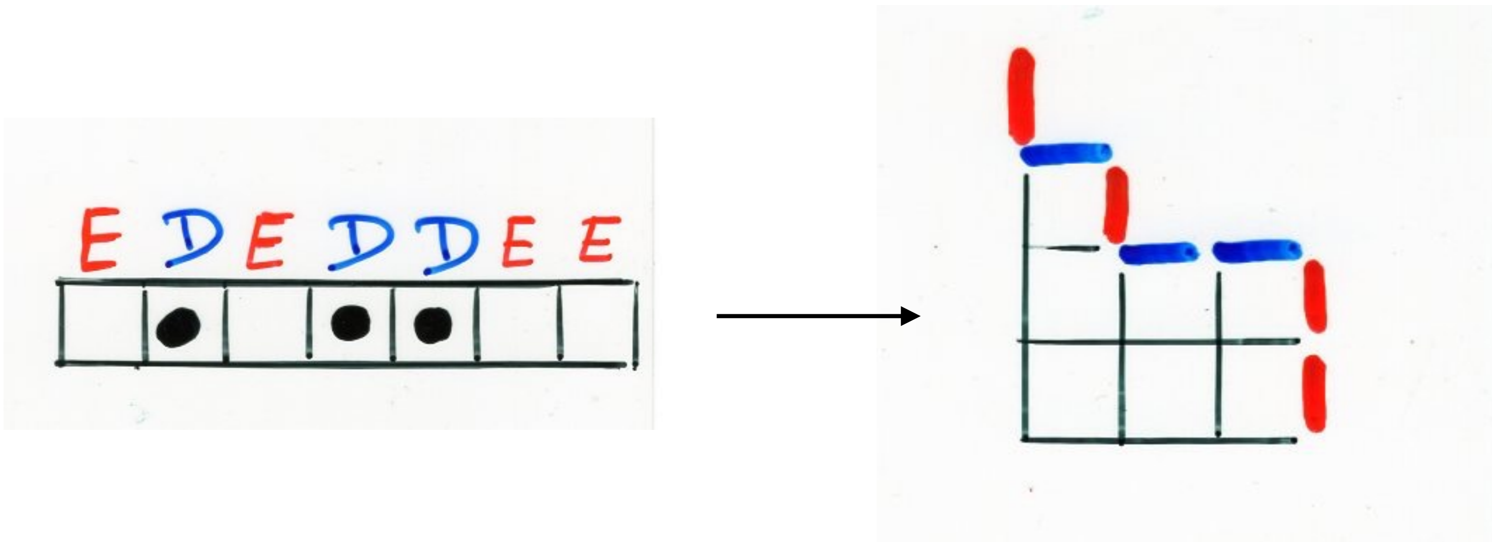


There is at most one particle per cell. Particles are moving one step forward (with probability one) and backward with probability q. The parameters α, β are probabilities for a particle to get in or out of the strip.

Alternating tableaux give an interpretation of the stationary probabilities for the PASEP model with 3 parameters α, β and q. Catalan alternative tableaux correspond to the TASEP (totally asymmetric exclusion model) where $q=0$.



computation of the "stationary probabilities"

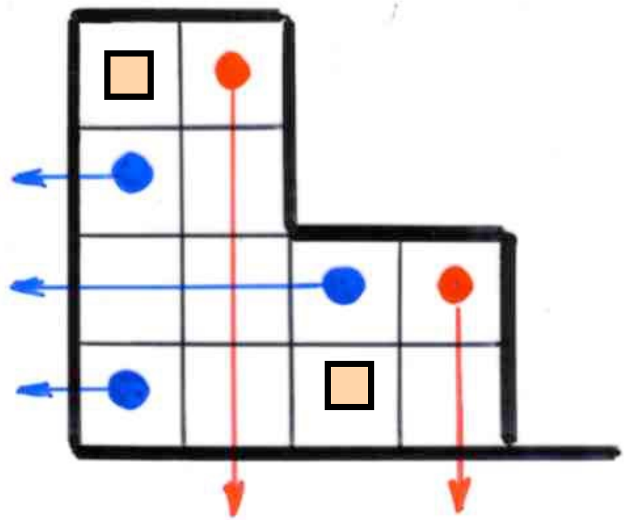


Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

<u>Corollary</u>. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\mathrm{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$



$k(T)$ = nb of cells ☐

$i(T)$ = nb of rows without ●

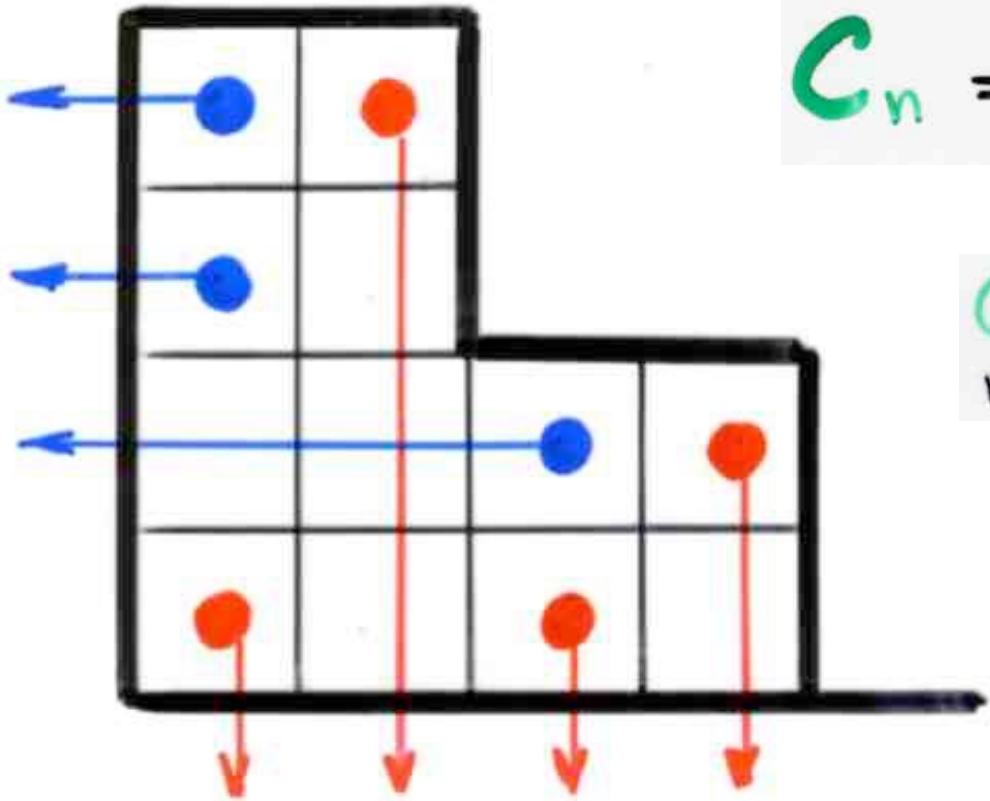
$j(T)$ = nb of columns without ●

Catalan alternative tableaux

Def Catalan alternative tableau T

alt. tab. without cells $\square$

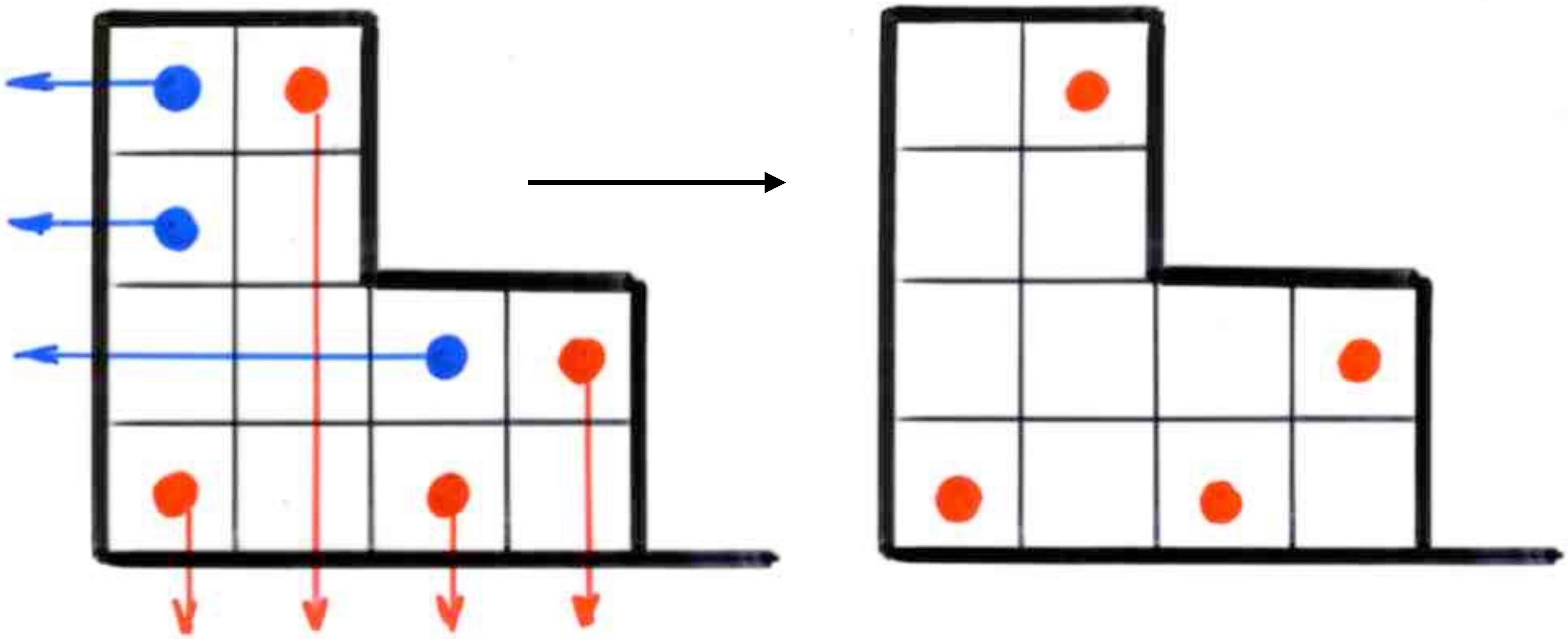
i.e: every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

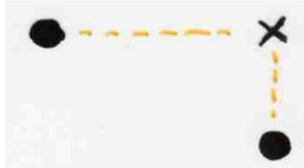
Characterisation of alternative Catalan tableaux

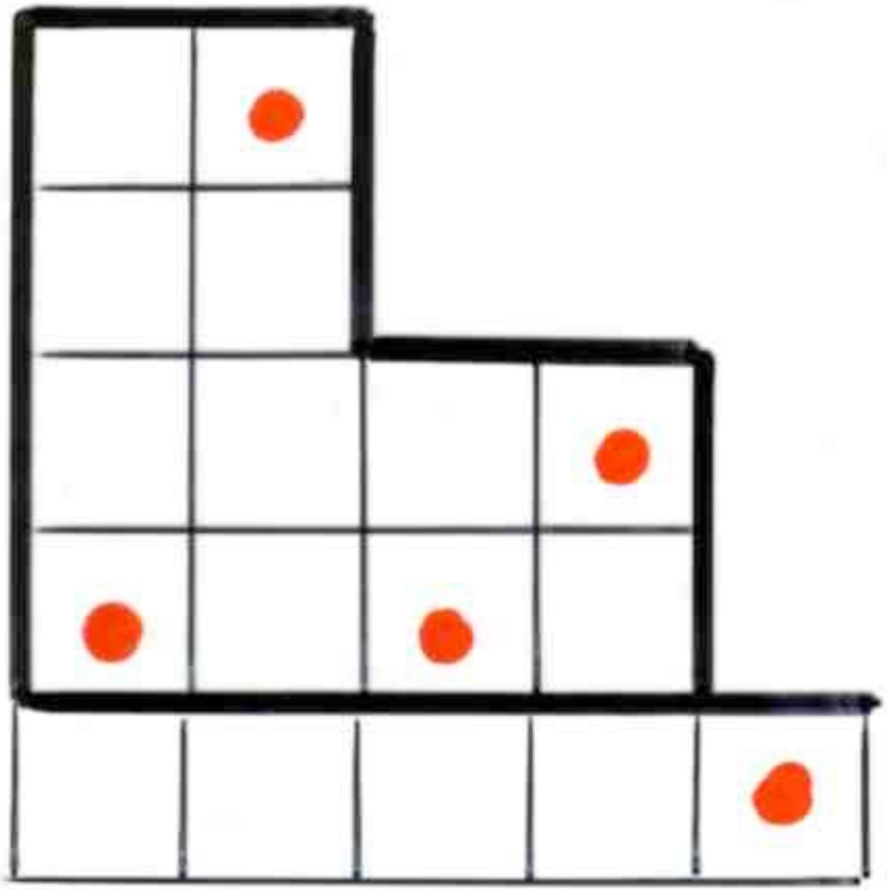


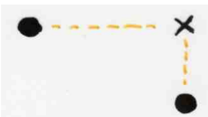
taking only the red points of a Catalan alternative tableau

one can reconstruct the original tableau from the knowledge of the red part

the augmented red part to the Catalan alternative tableau:
adding a red point in the new first row for each empty column of the red tableau

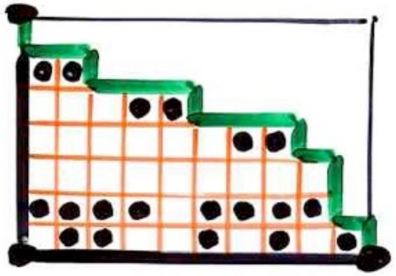
the original tableau is a Catalan alternative tableau if and only if the pattern  is forbidden



Such tableaux are the so-called « Catalan permutation tableaux », that is a tableau where the pattern  is forbidden and where in each column there is one and only one (red) point)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) $1 \cdots\cdots 0$ forbidden
$\quad\quad\quad\ \ \vdots$
$\quad\quad\quad\ \ 1$

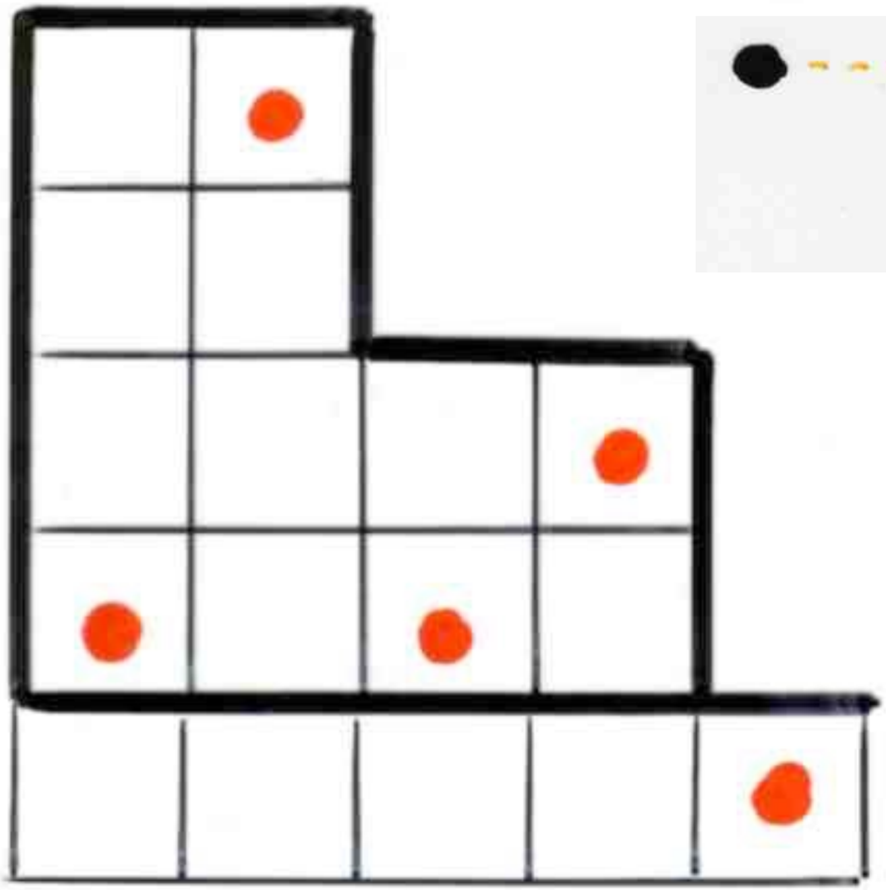
permutation tableau

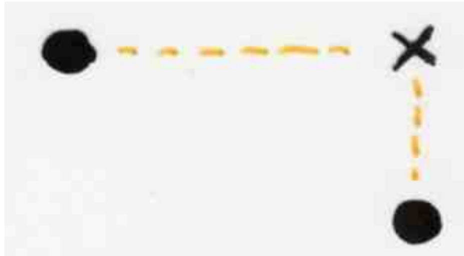
A. Postnikov (2001, ...)

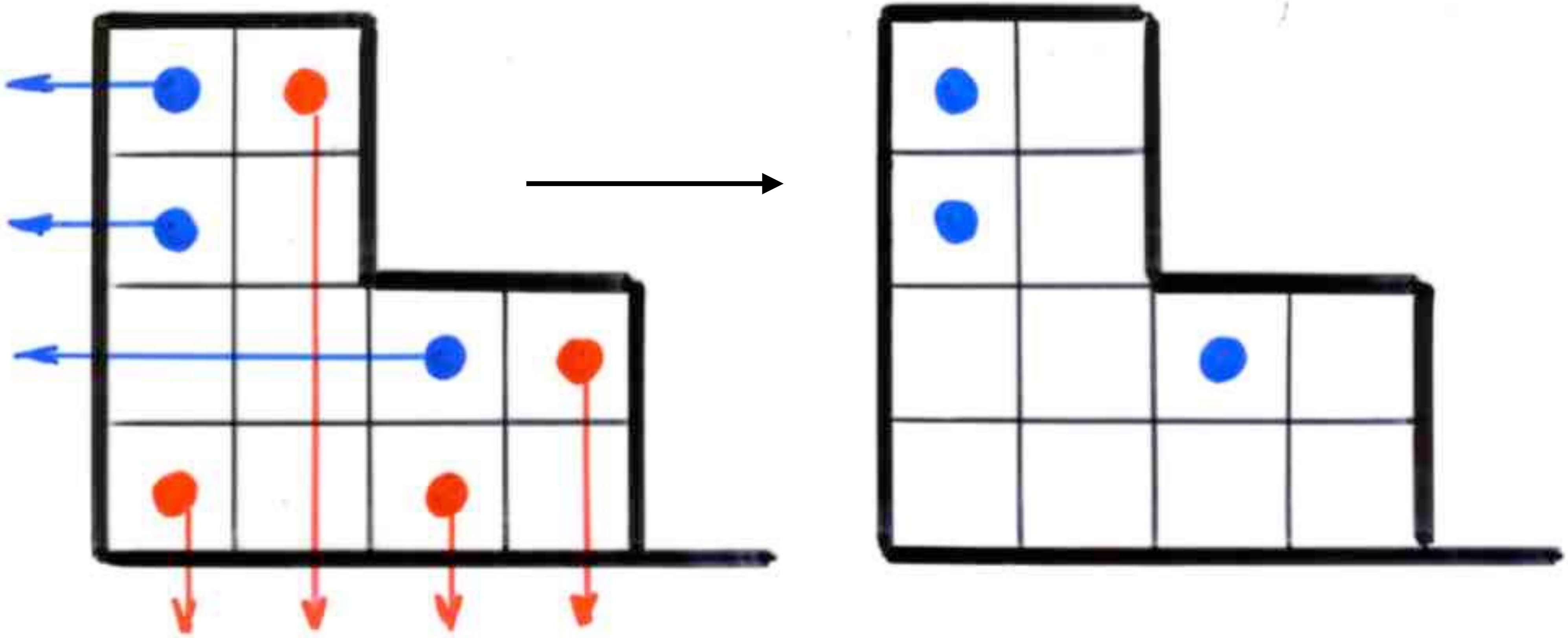
totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

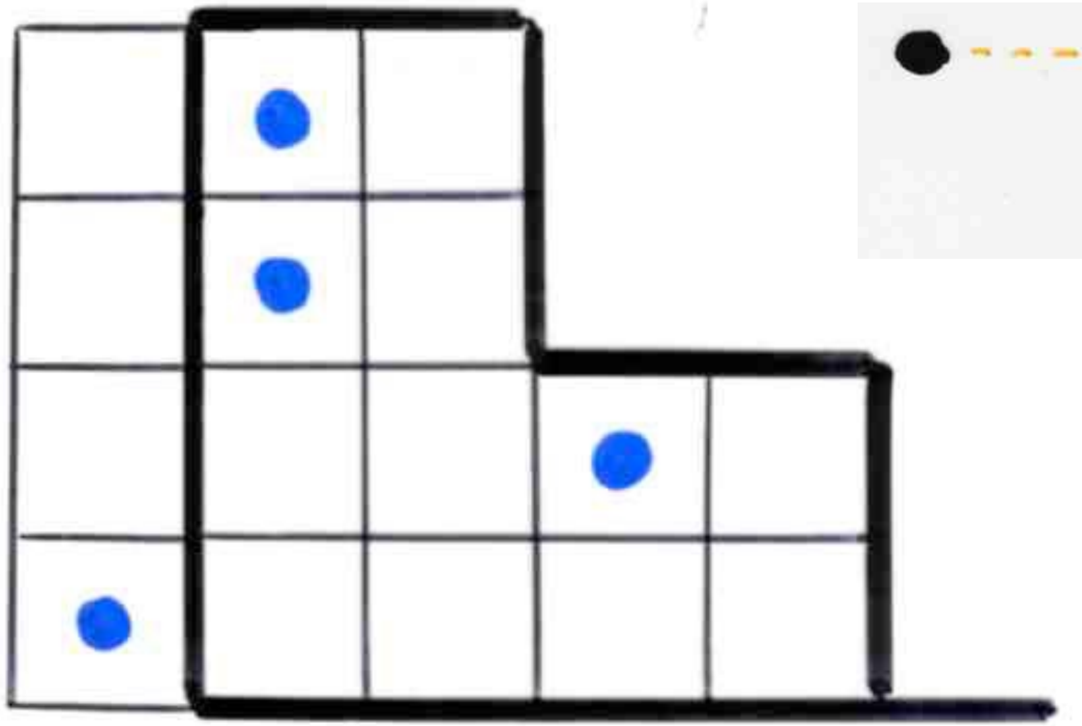
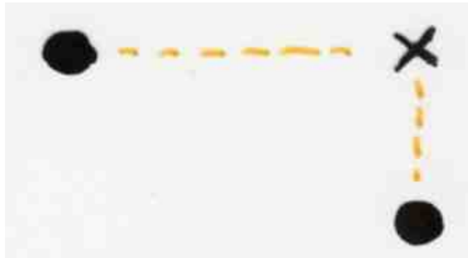
Catalan
permutation
tableaux



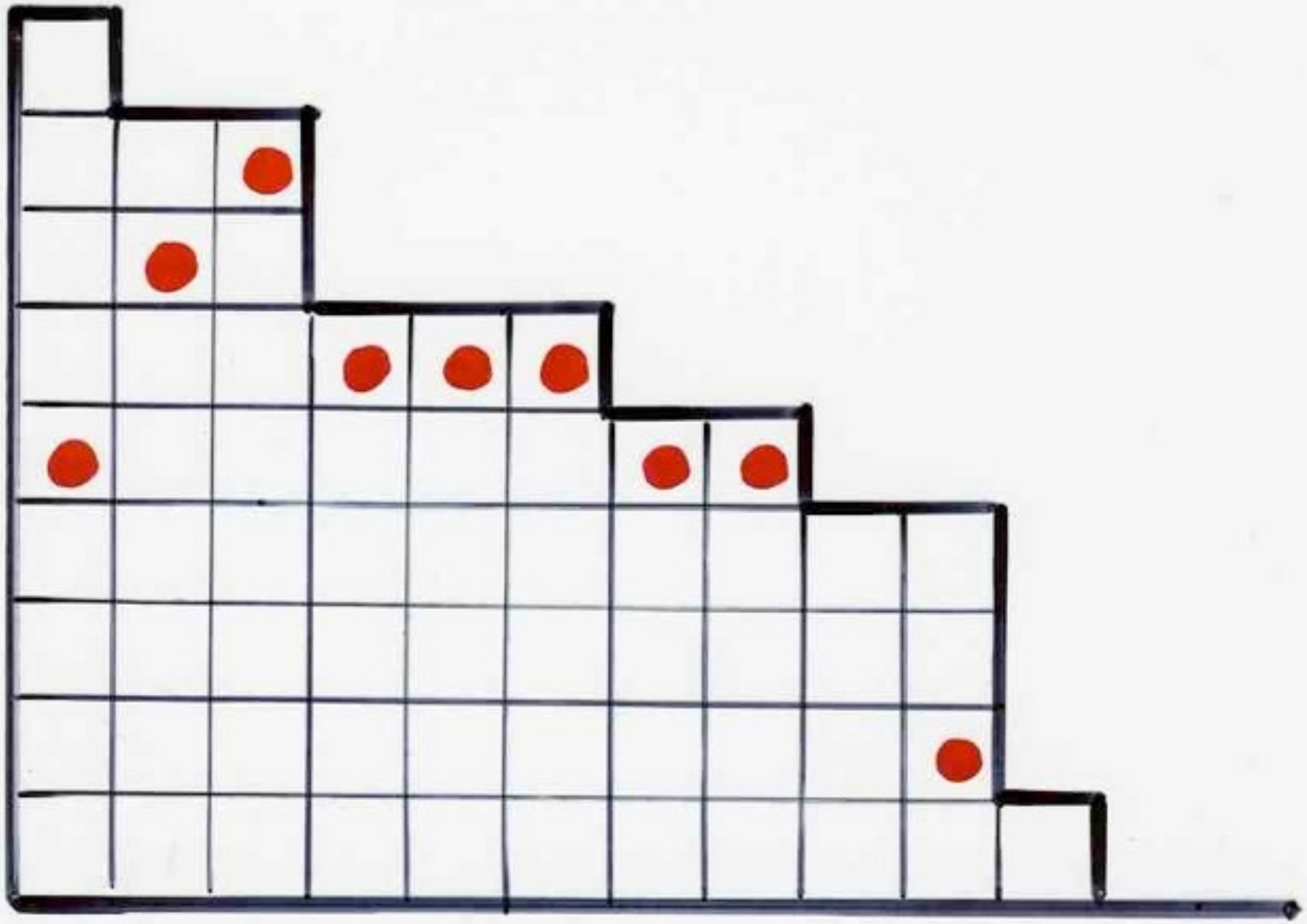


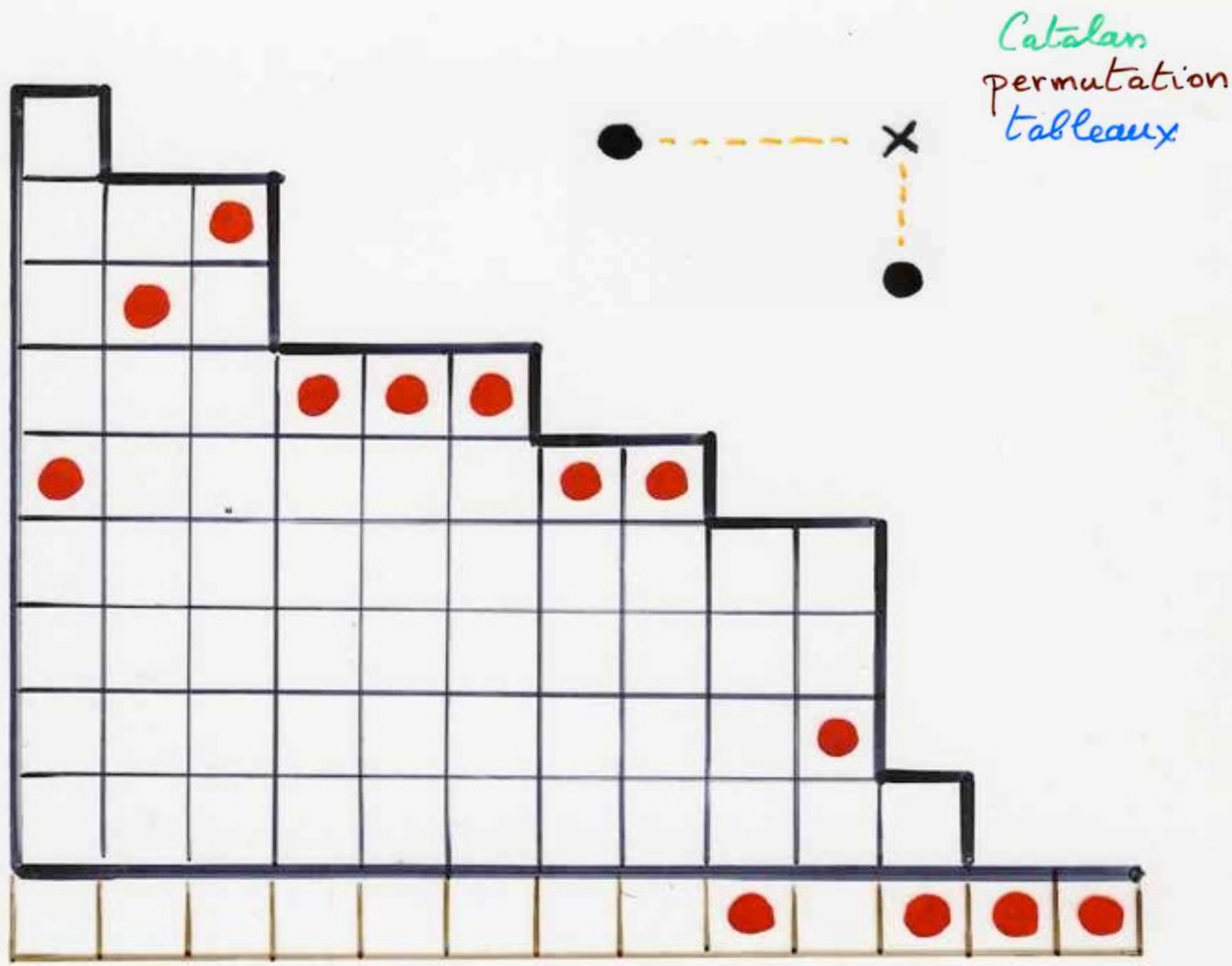


same with the blue points

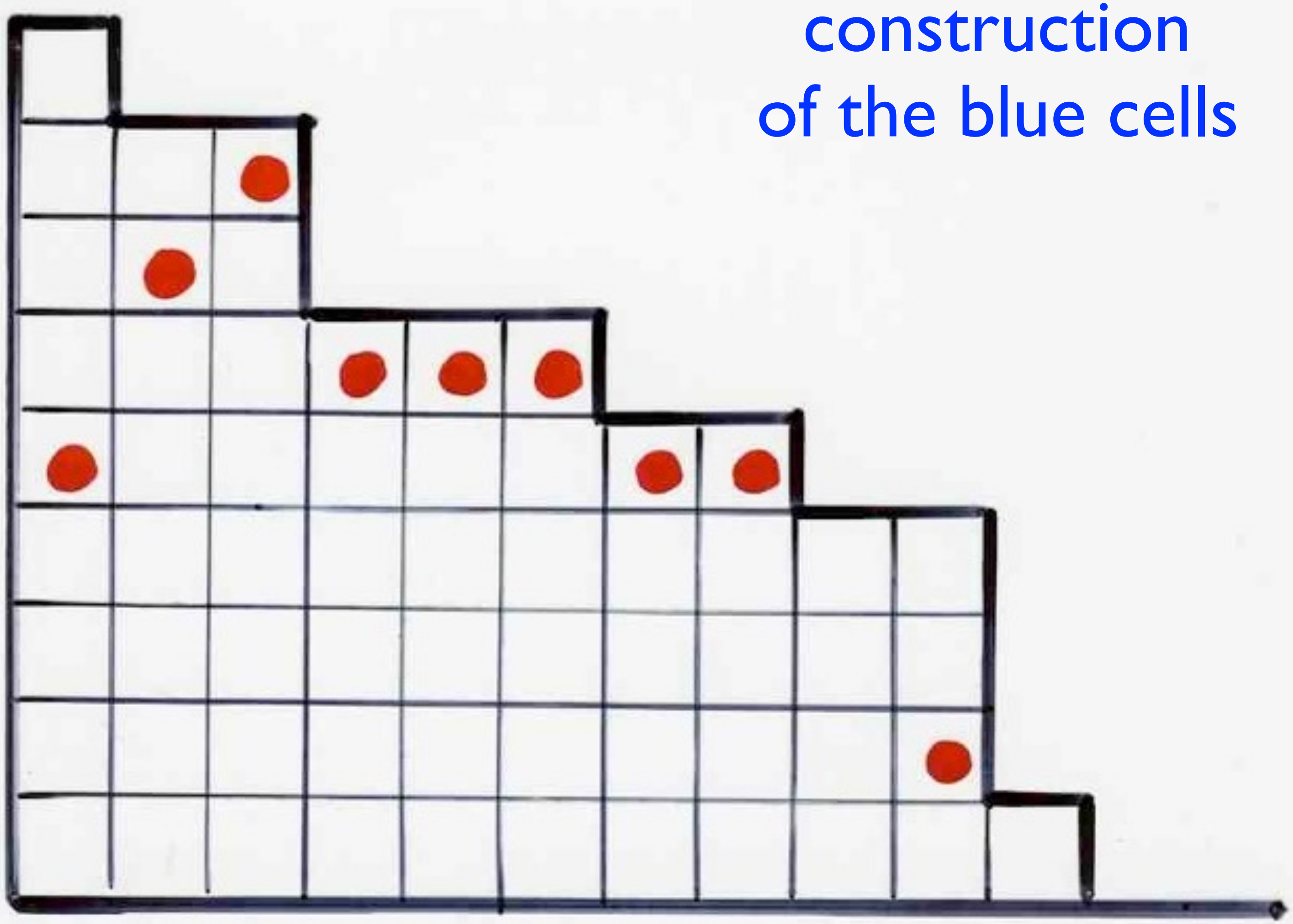



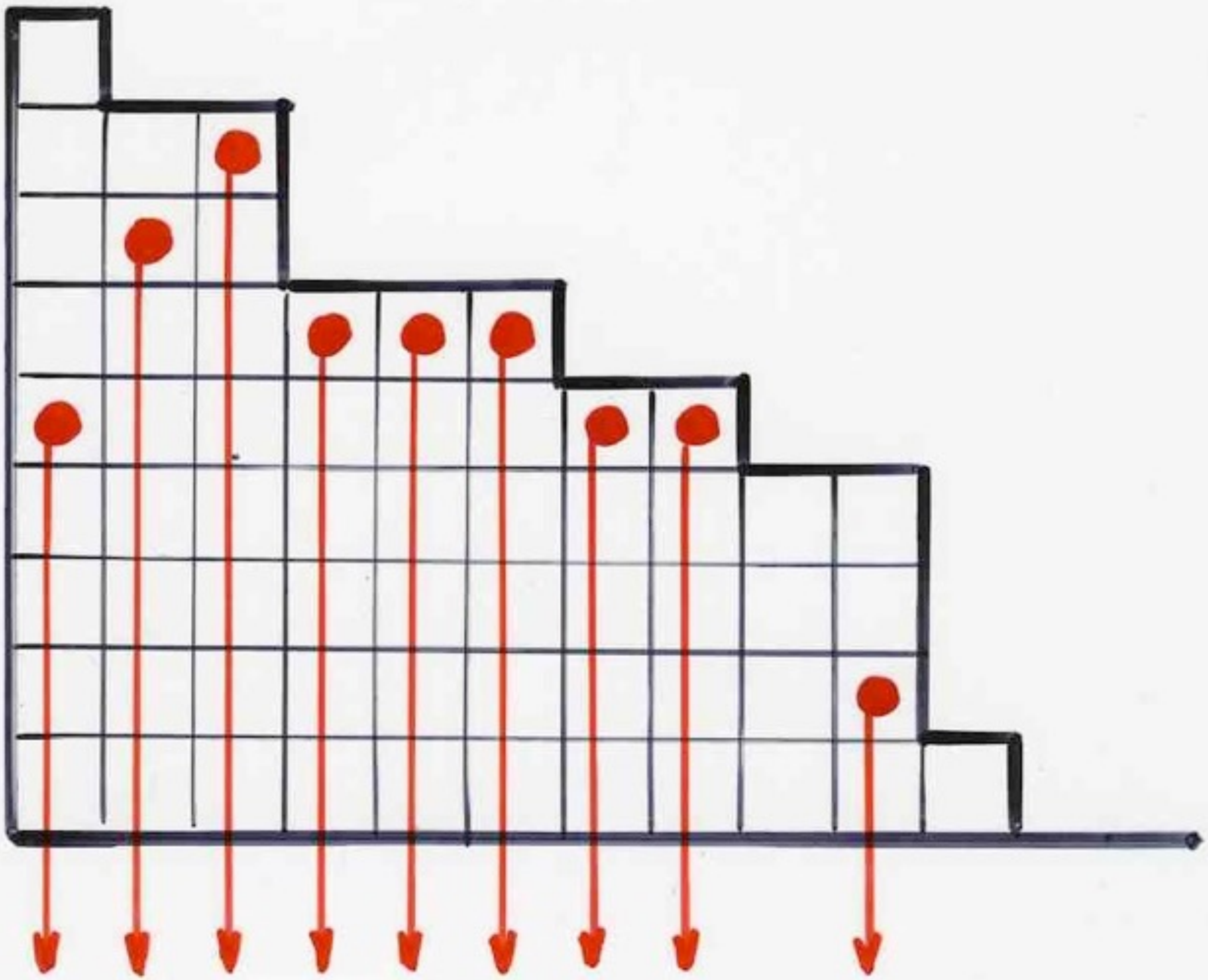
example

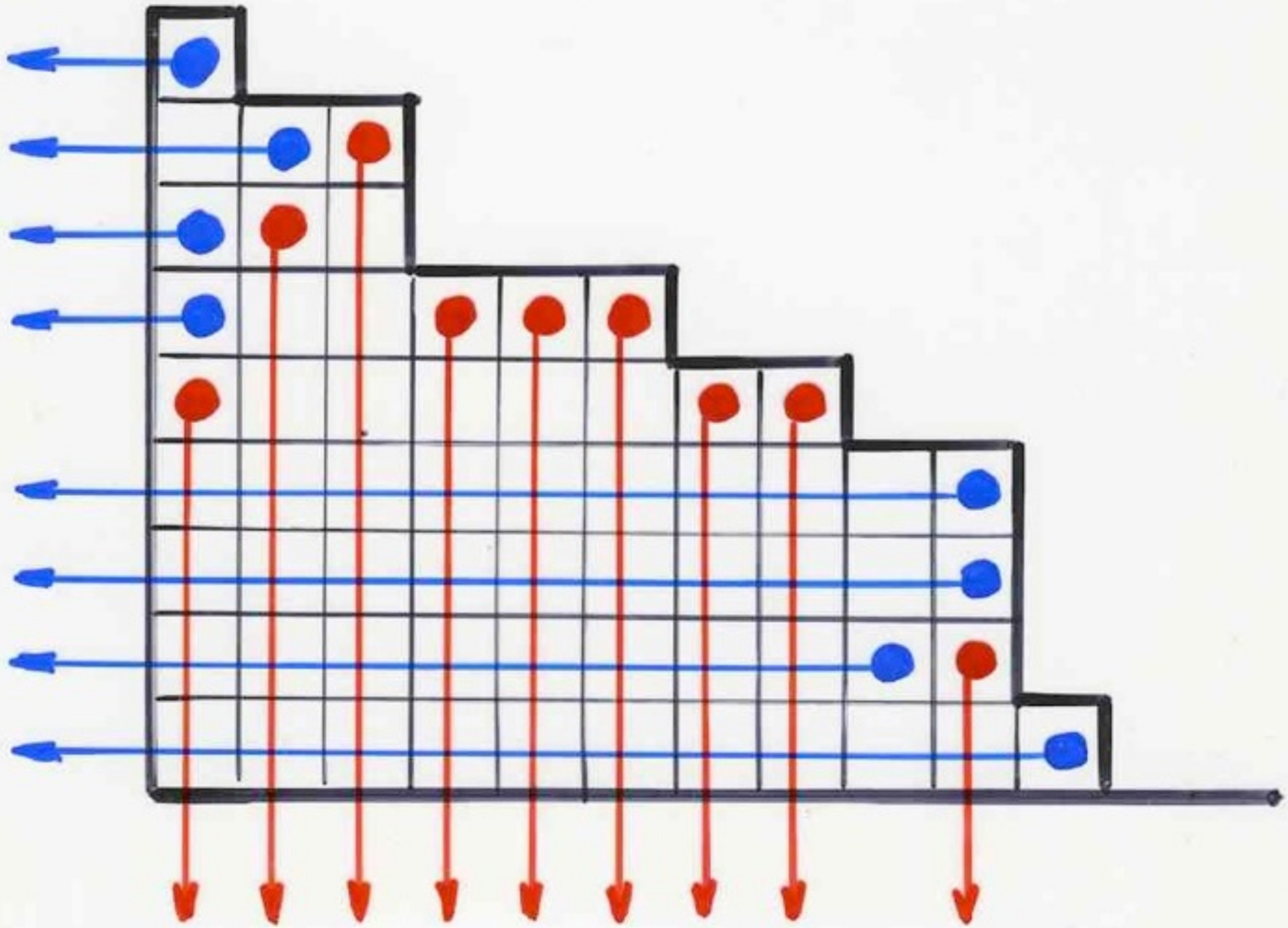


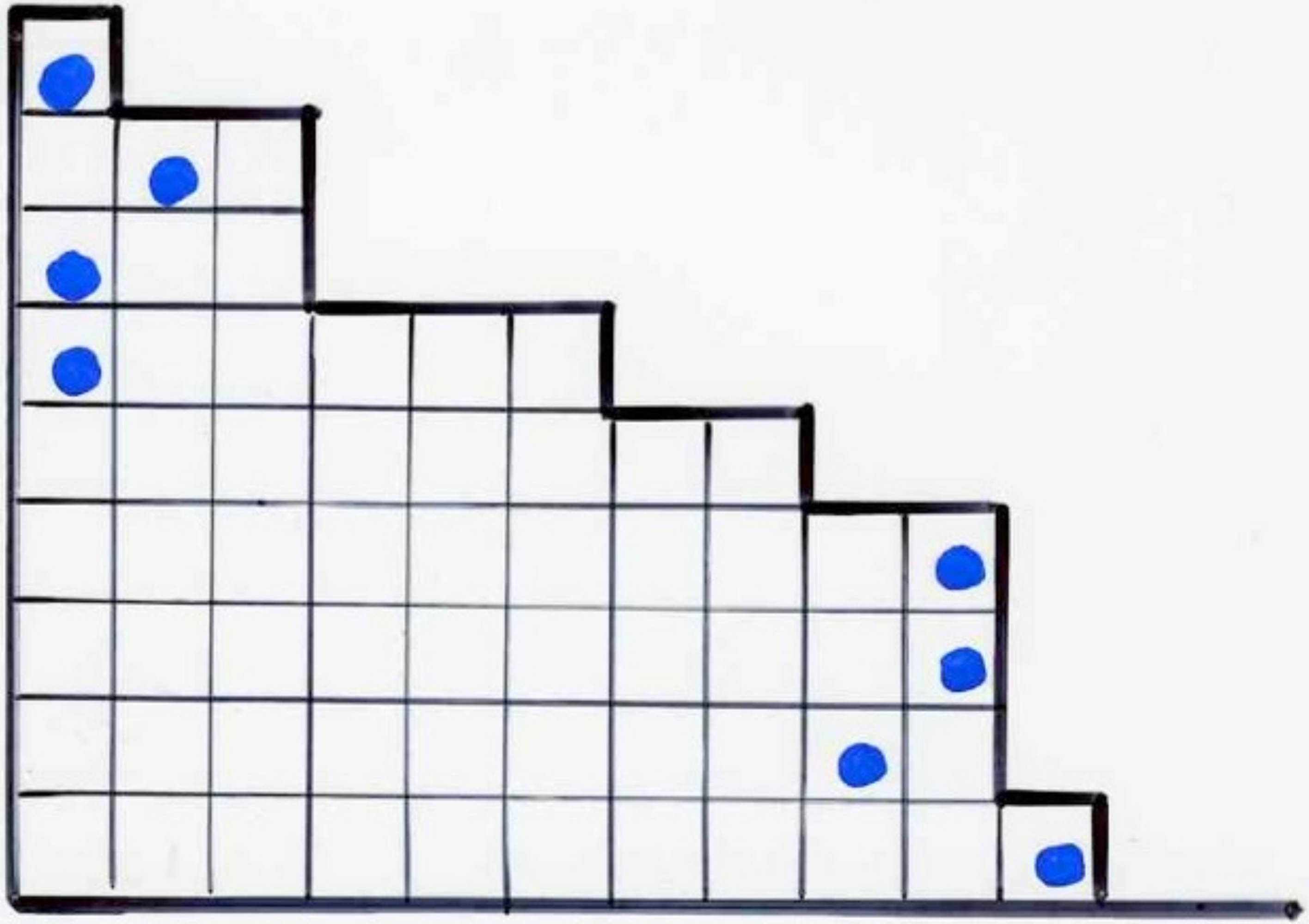

Catalan
permutation
tableaux

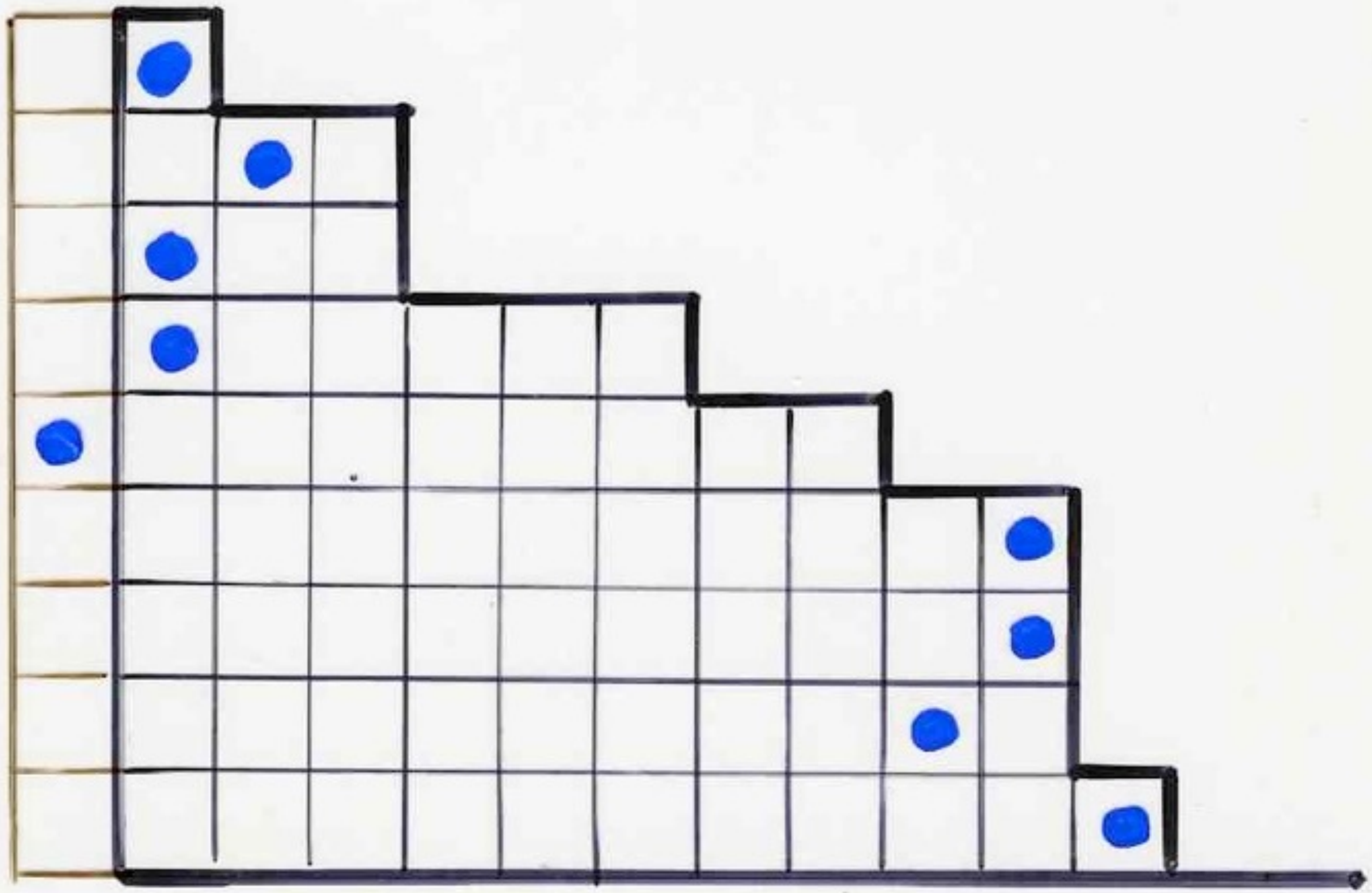
construction
of the blue cells

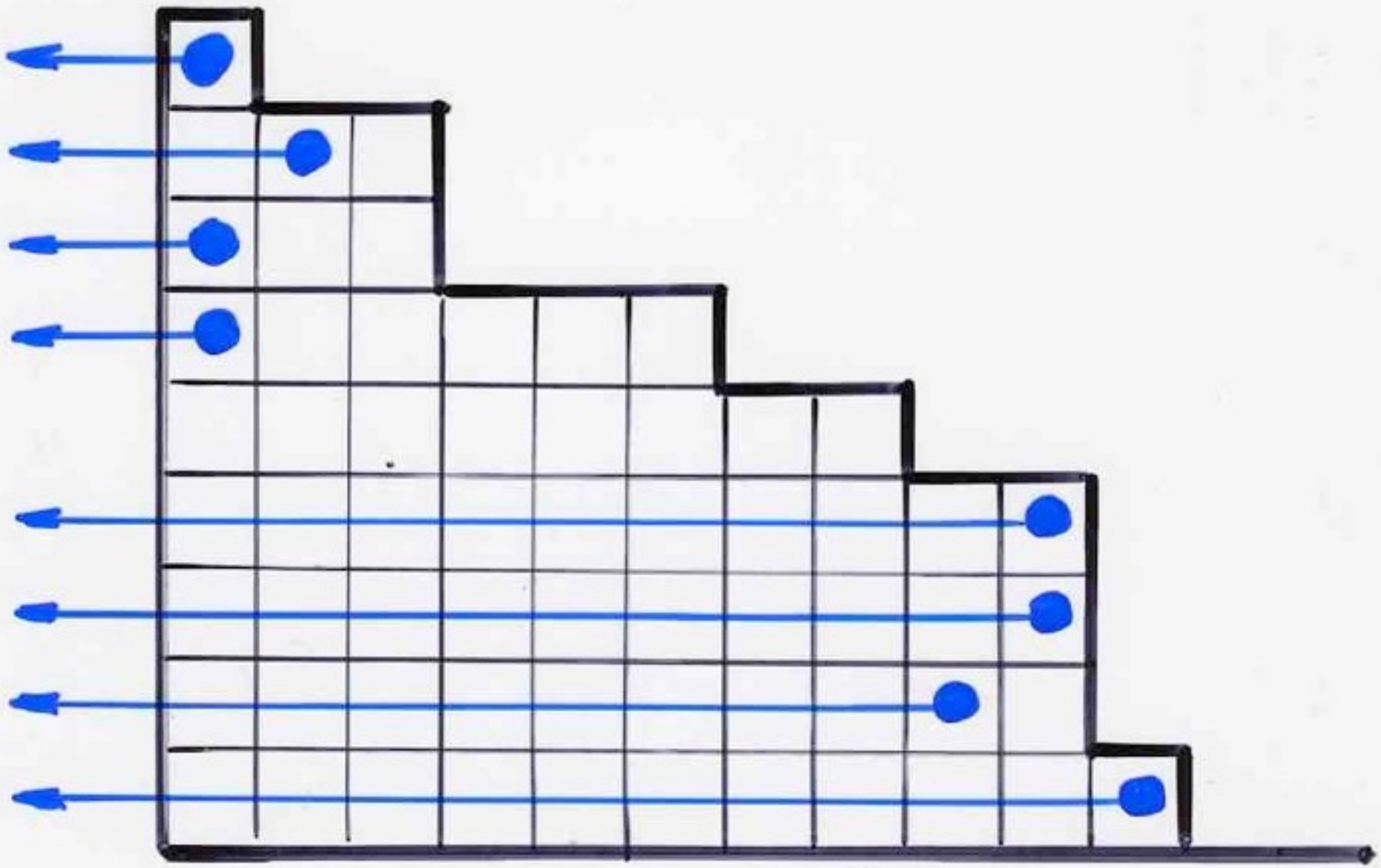


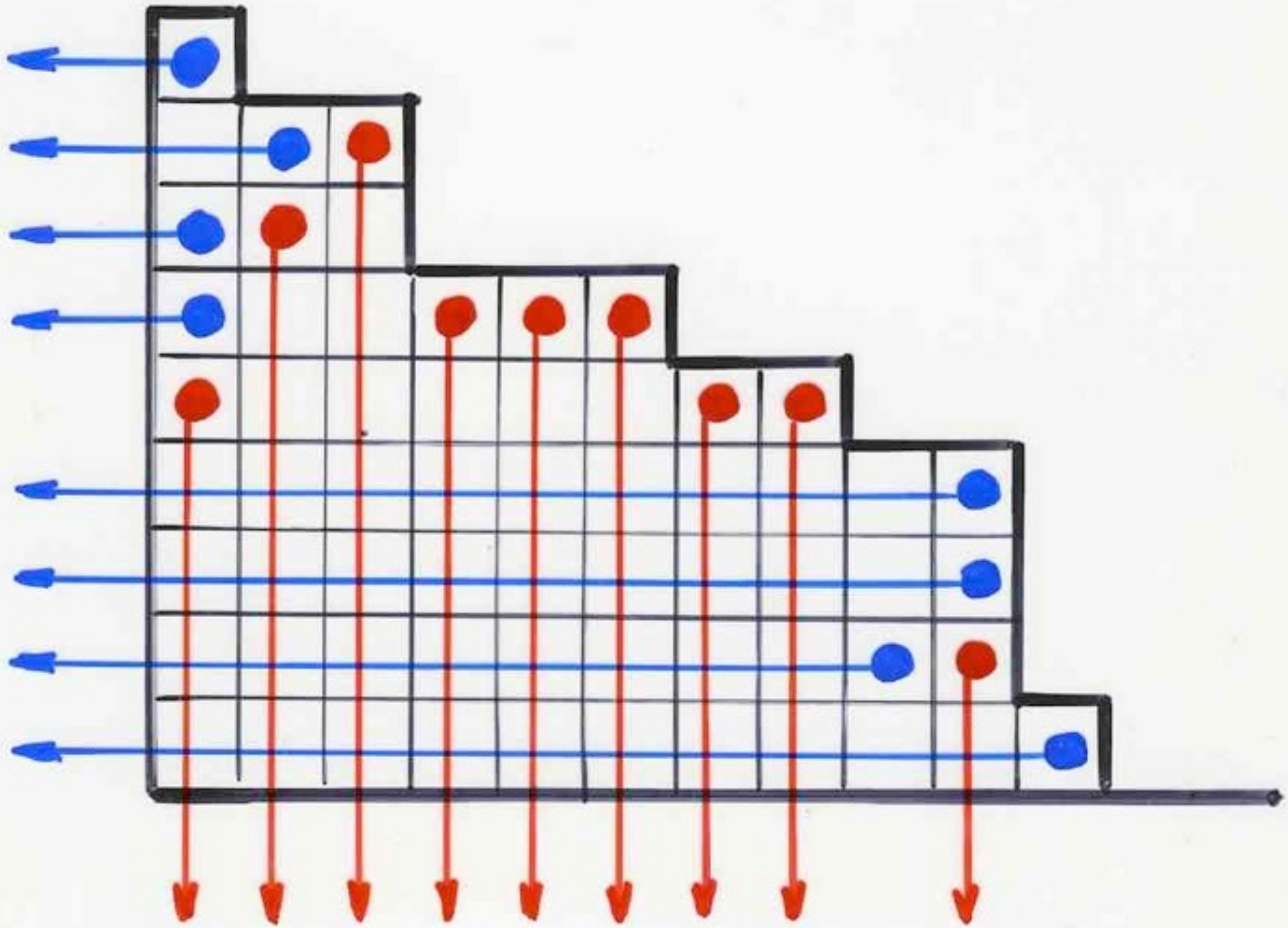








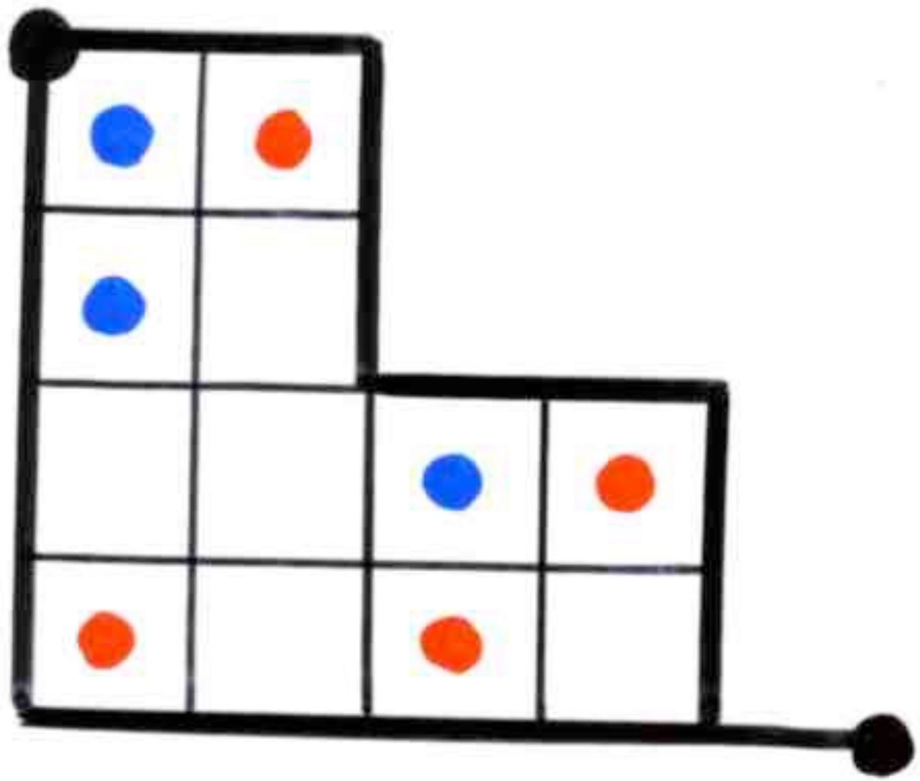




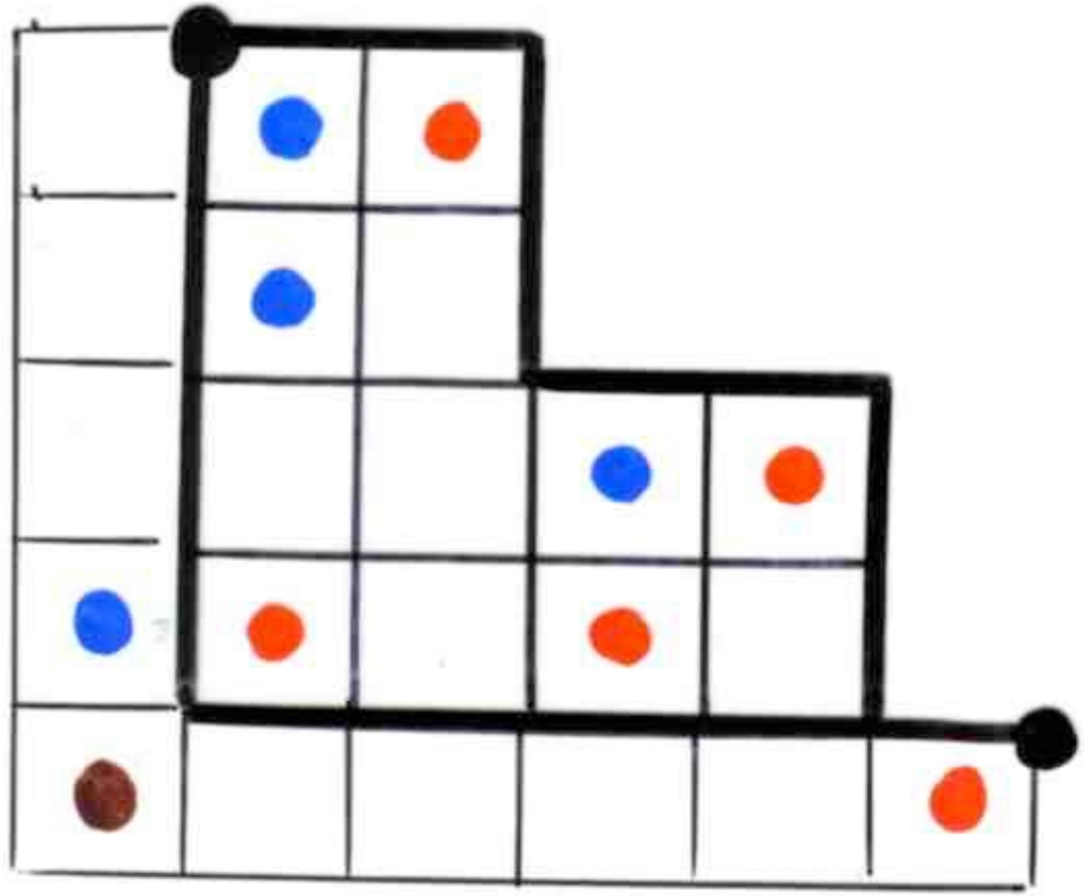
bijection

Catalan alternative tableaux

binary trees

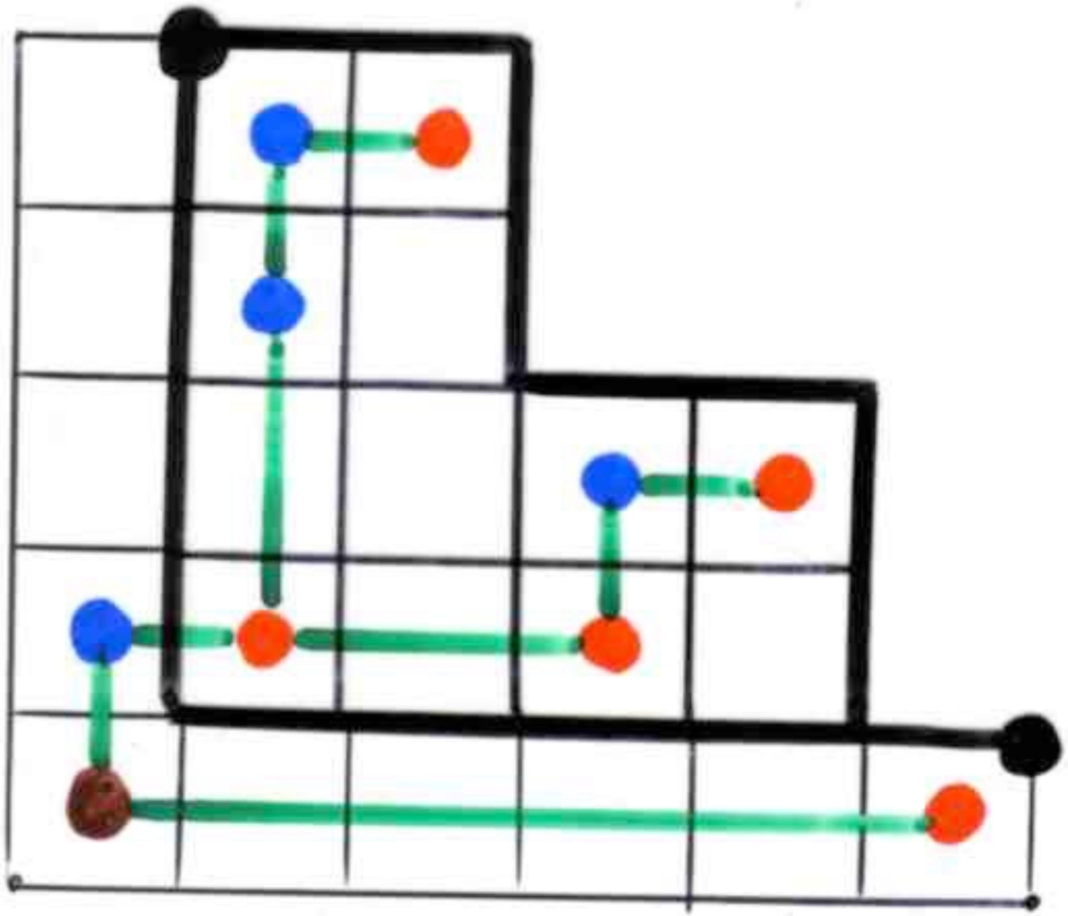


a Catalan alternative tableau

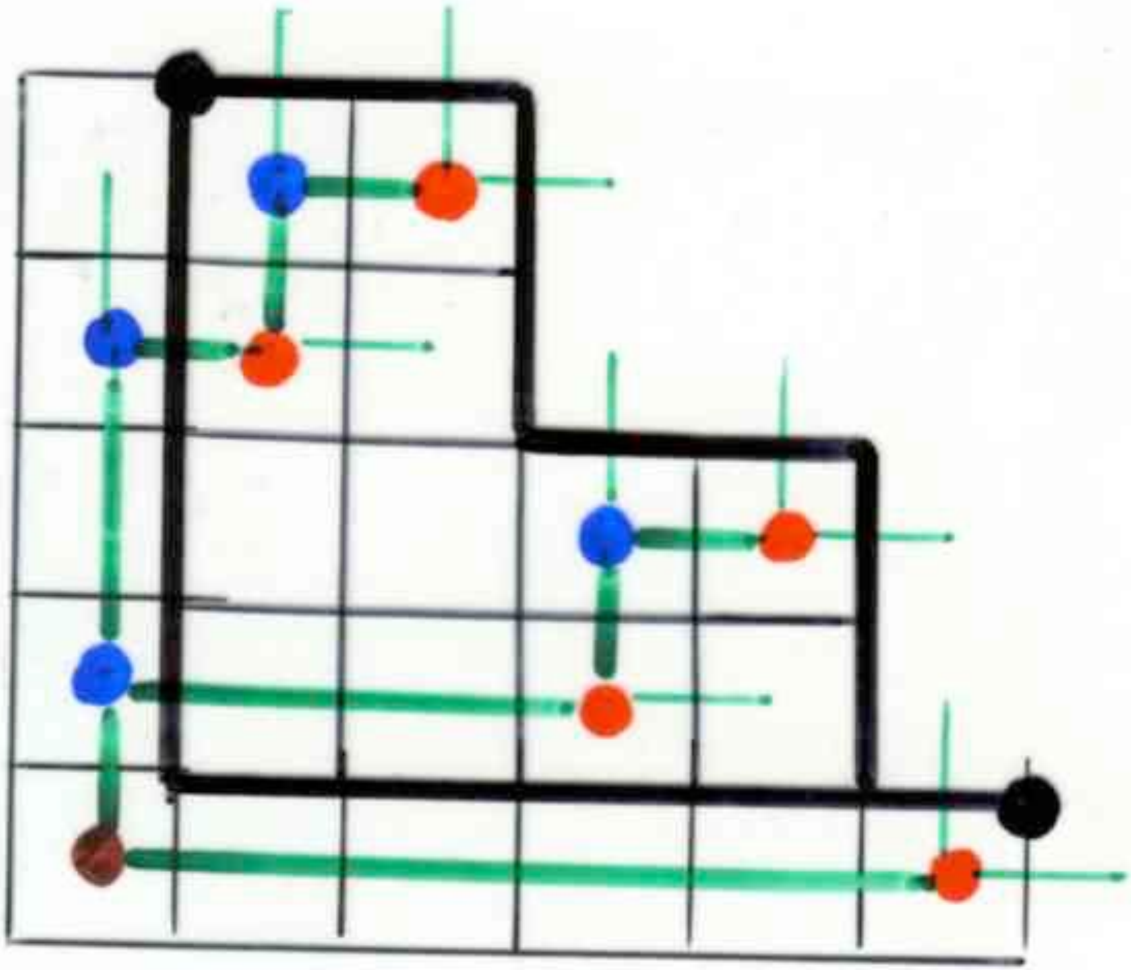


the extended Catalan alternative tableau

for each blue point add a vertical (green) edge below the point
for each red point add an horizontal (green) edge at the left of he point

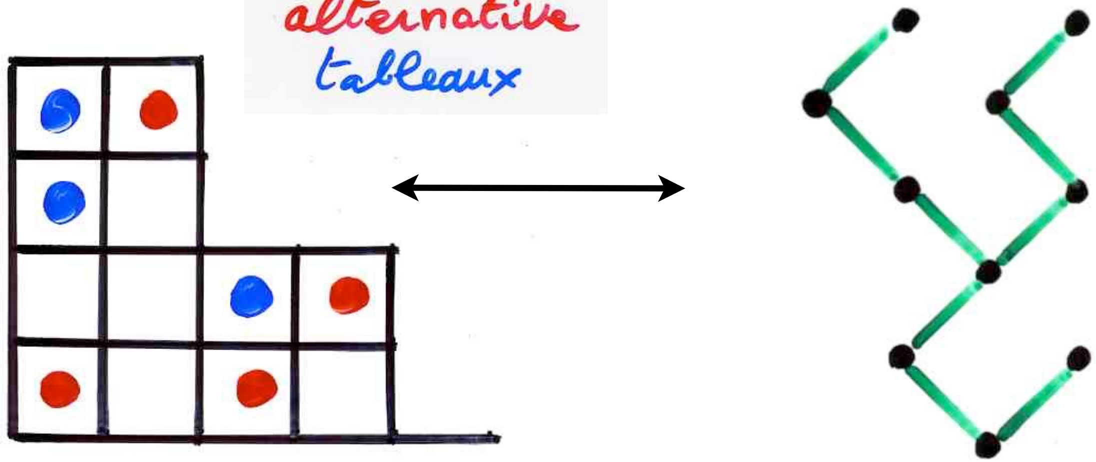


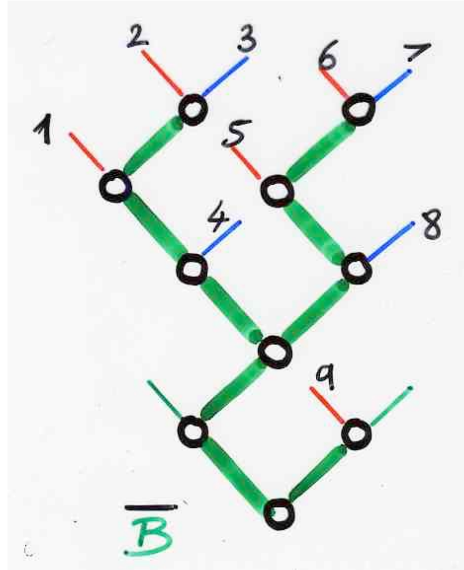
one get a binary tree



the associated extended (also called complete) binary tree

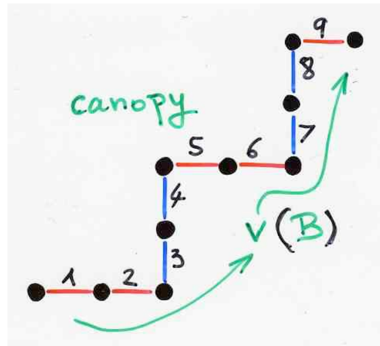
Catalan alternative tableaux

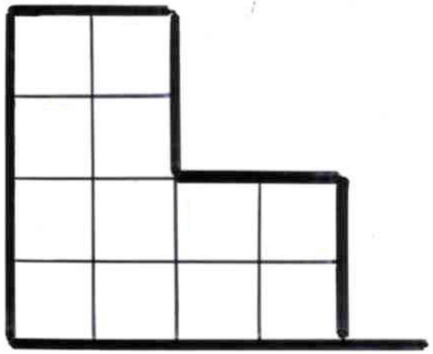


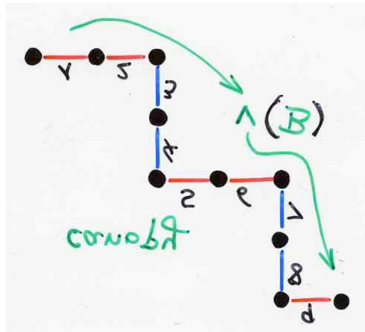


Proposition

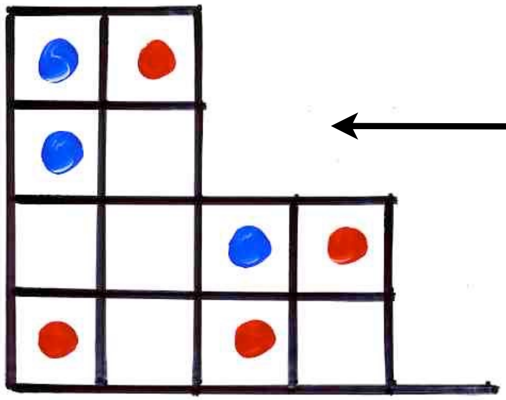
The map defined above is a bijection between alternative tableaux with profile v and binary trees with canopy v

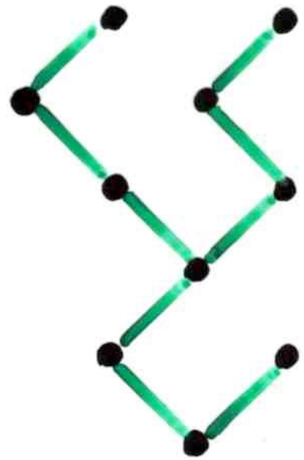


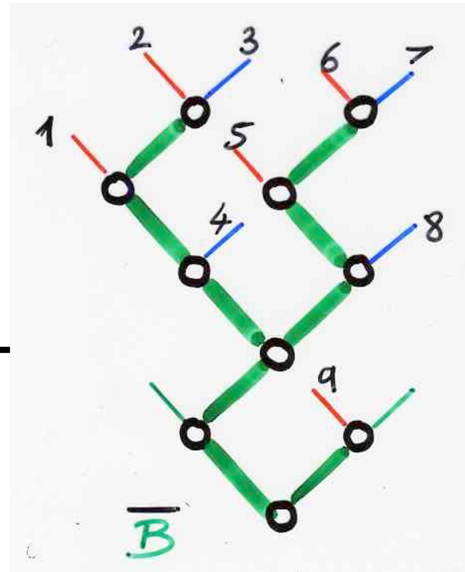


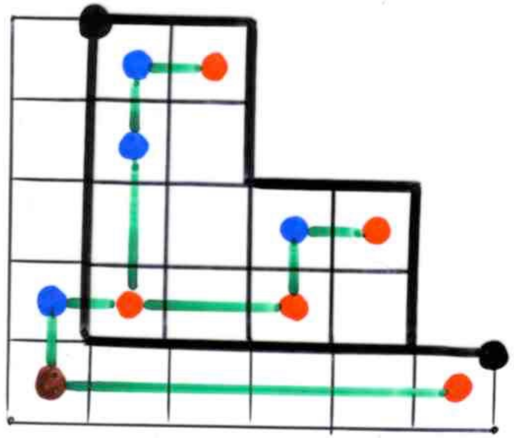


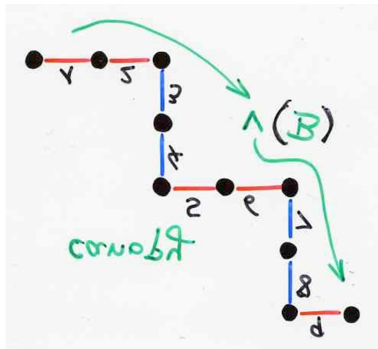
Catalan
alternative
tableaux

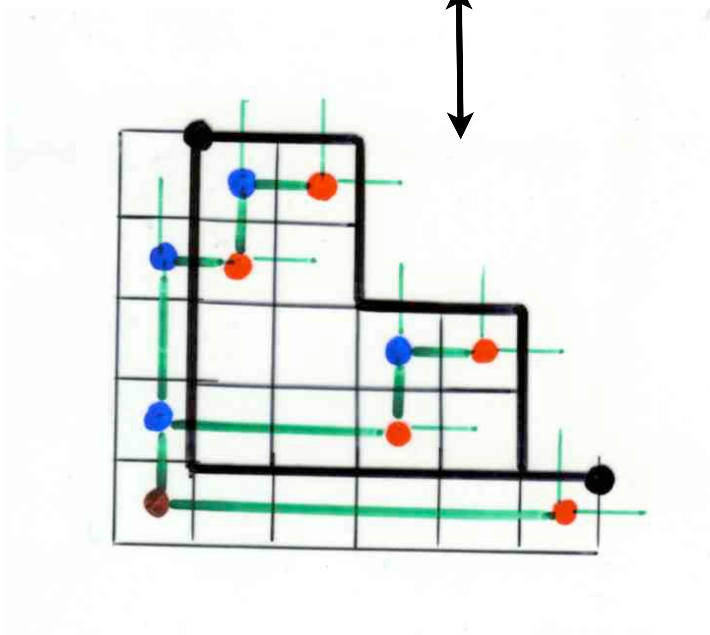








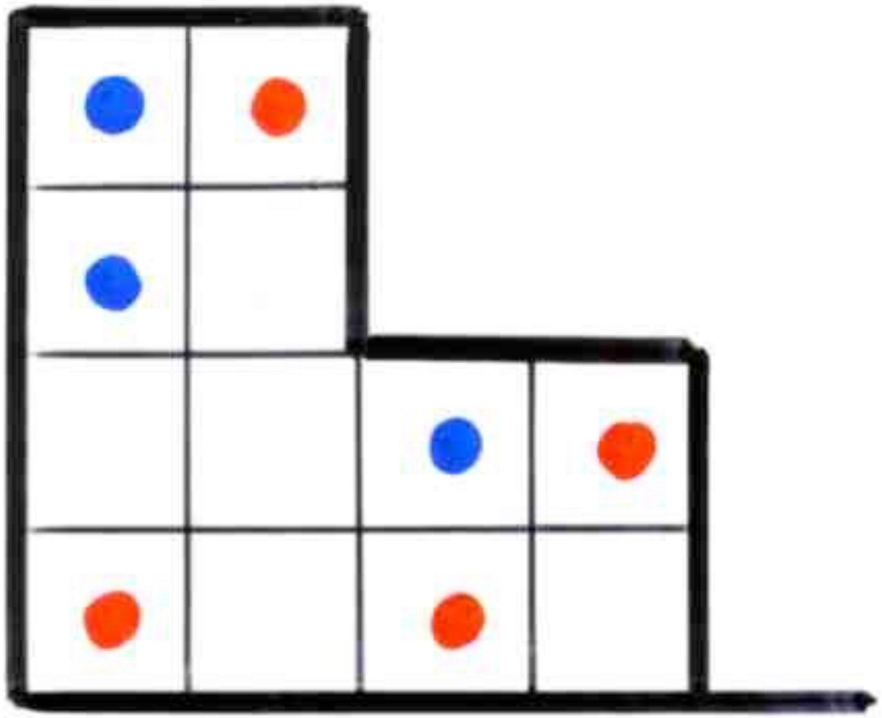




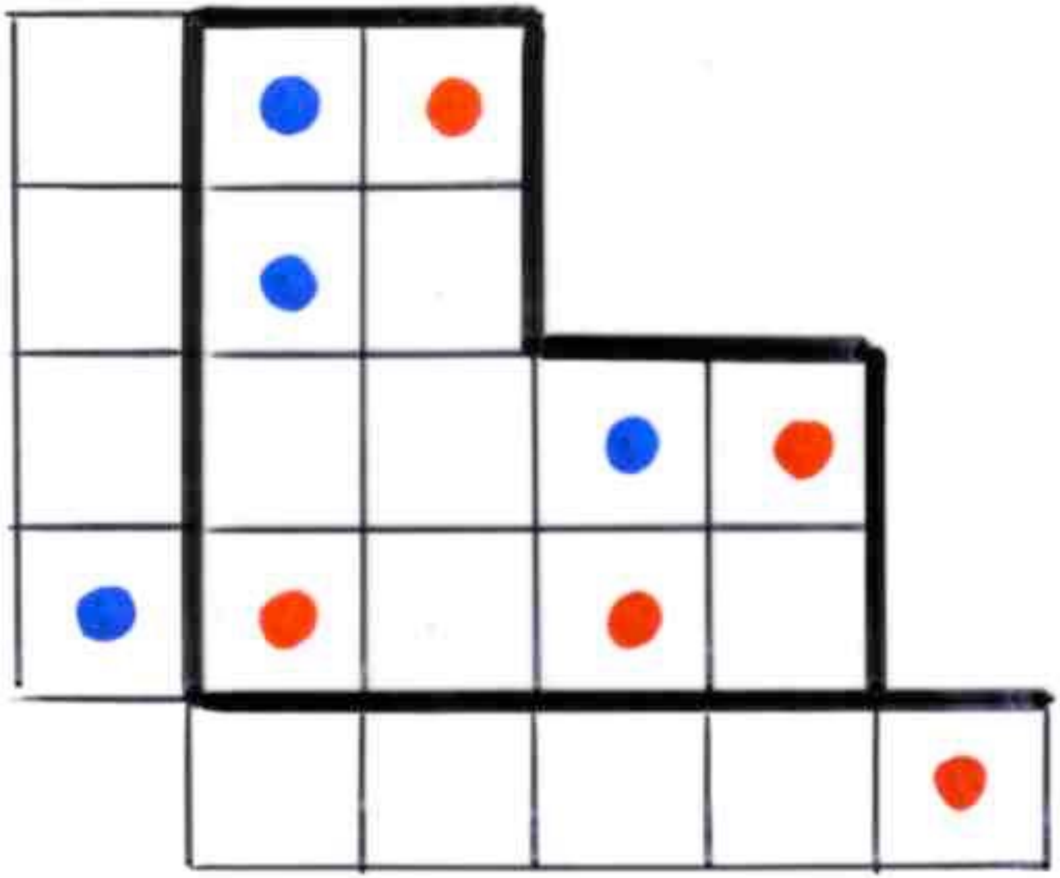
Catalan
alternative
tableaux

complete
binary
trees

Binary
trees

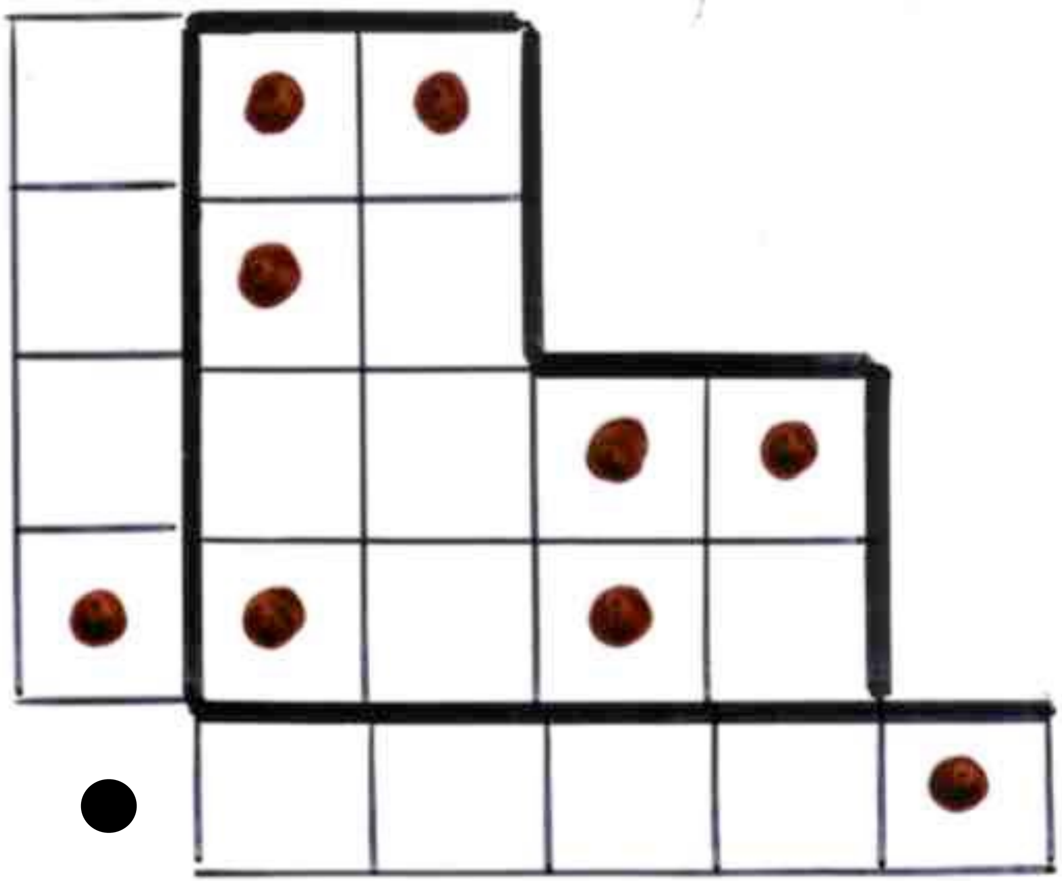


back to the original Catalan alternative tableau

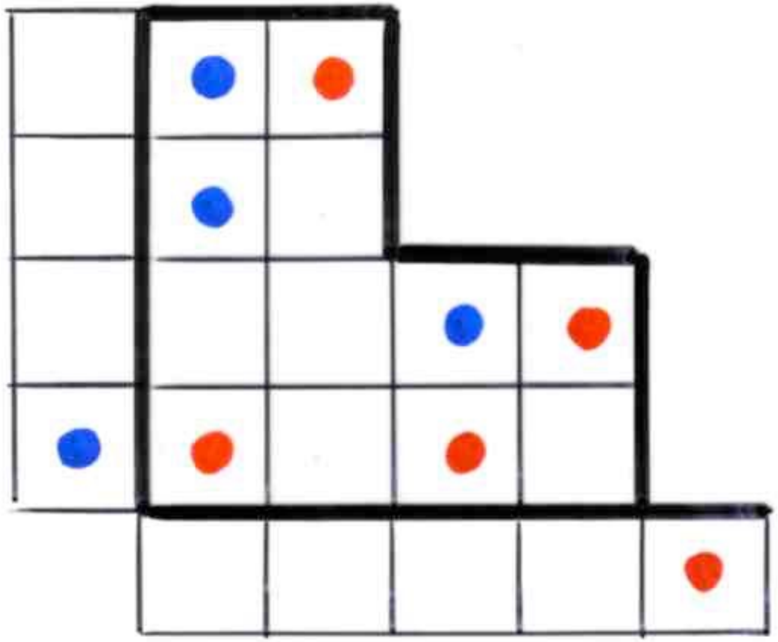


the augmented Catalan alternative tableau

Catalan
tree-like
tableaux

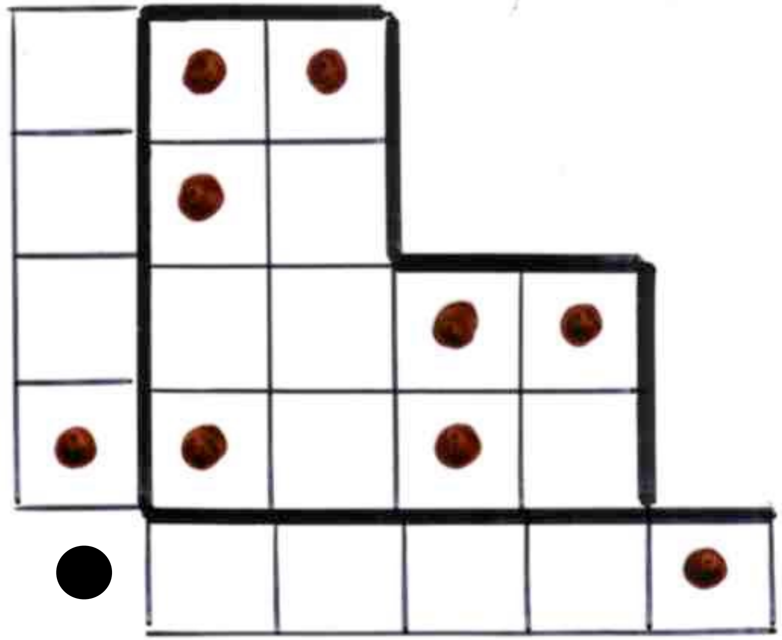


If one forgets the colors of the augmented Catalan alternative tableau, one can reconstruct the original tableau. Adding a point in the SW corner, one get a Catalan tree-like tableau. (see references in part II and slide 109, part II)



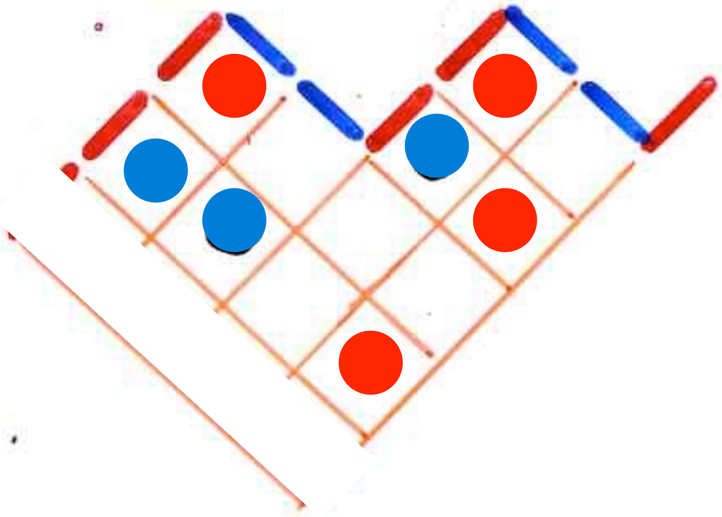
Catalan
alternative
tableaux

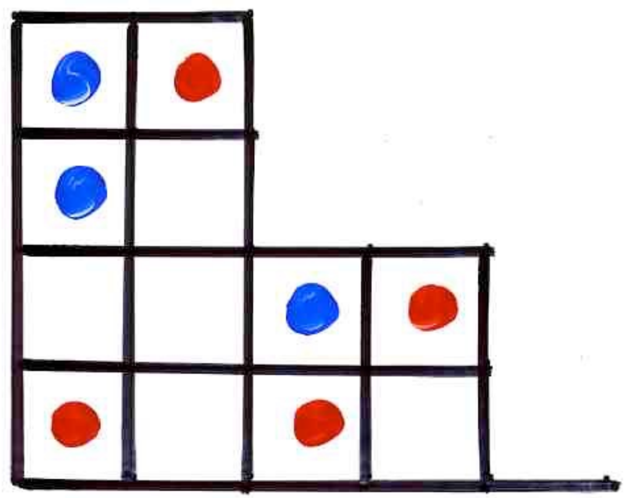
$\longleftrightarrow$

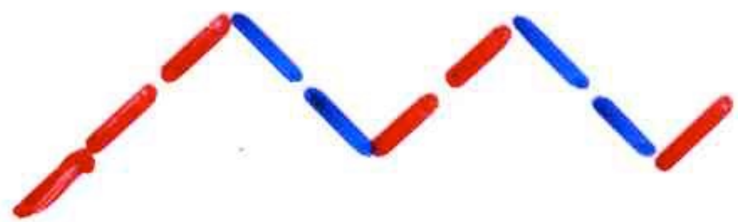


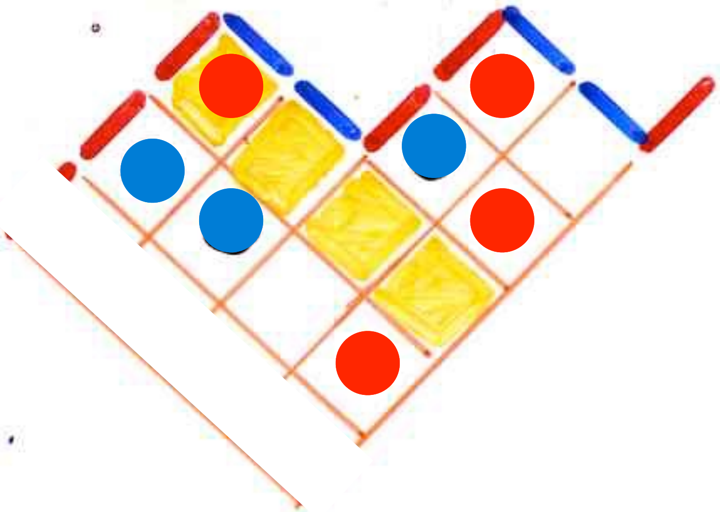
Catalan
tree-like
tableaux

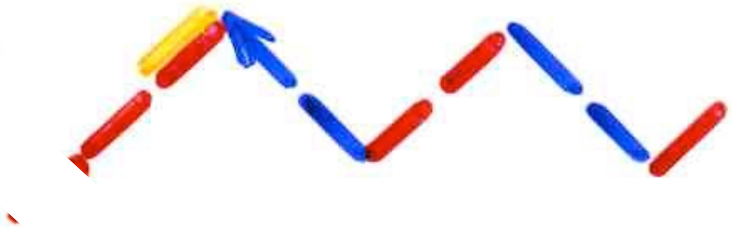
2nd bijection
Catalan alternative tableaux
binary trees

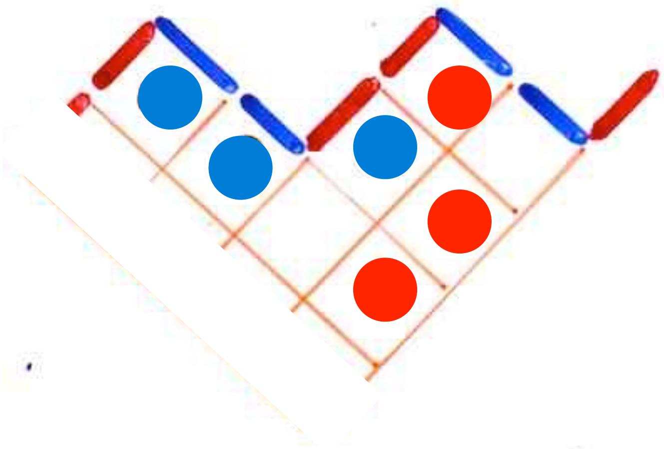


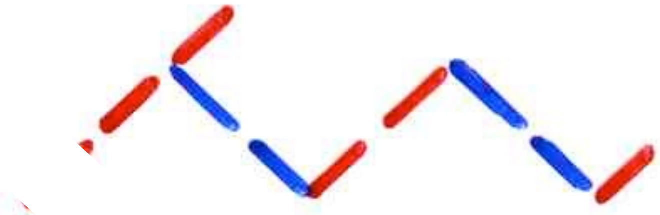


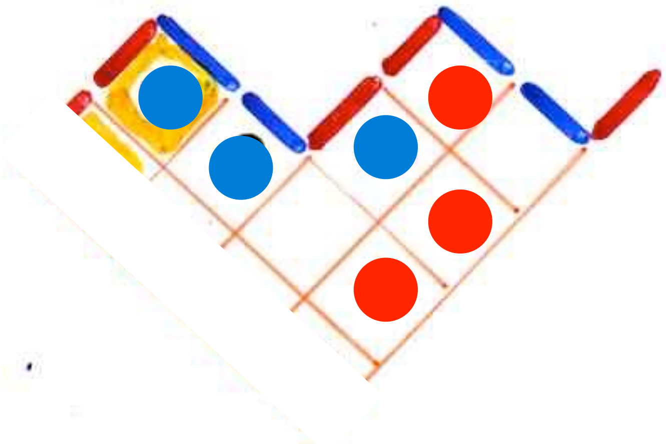


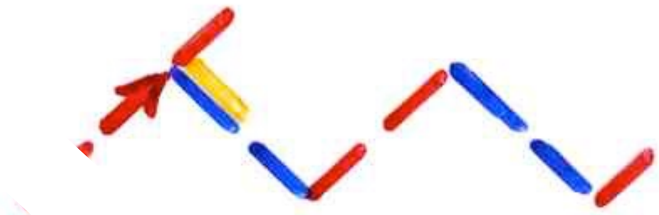


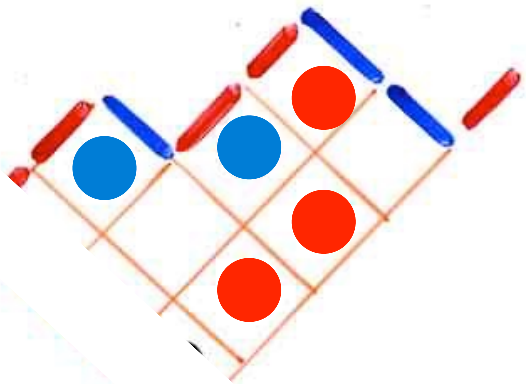


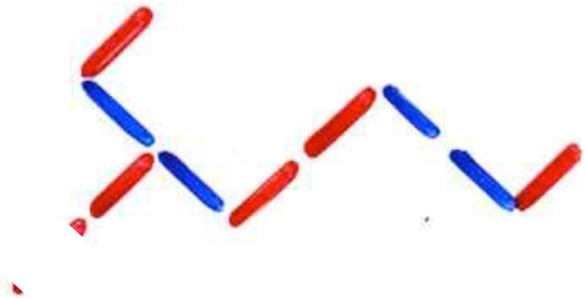


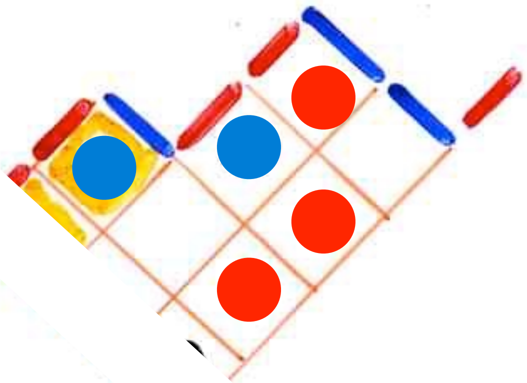


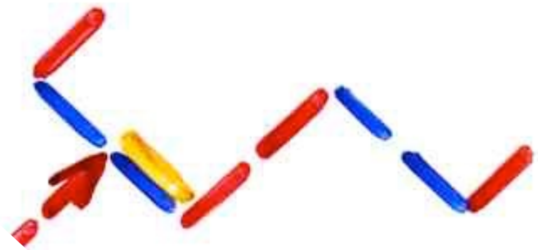


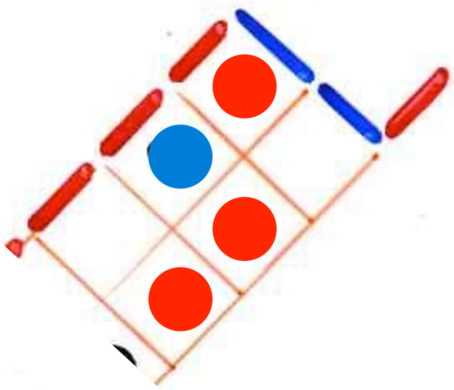


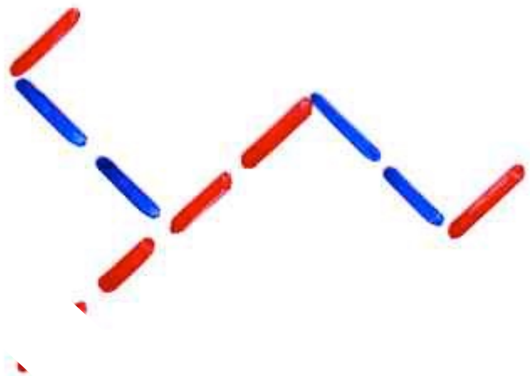


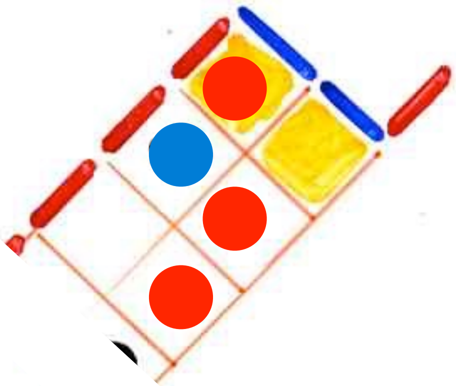


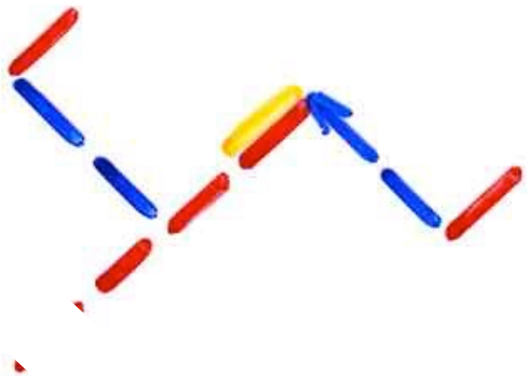


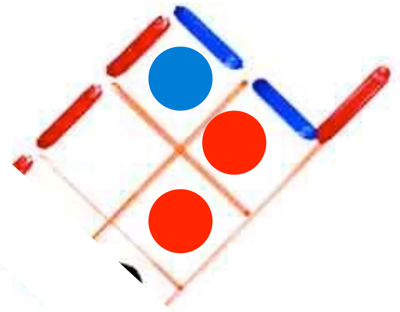


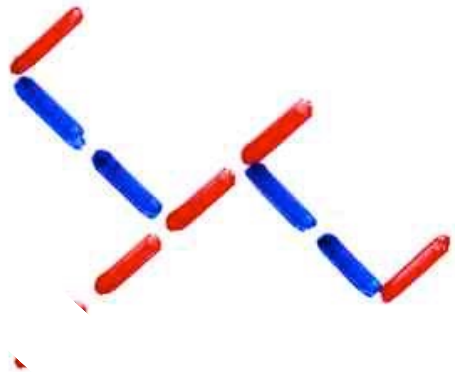


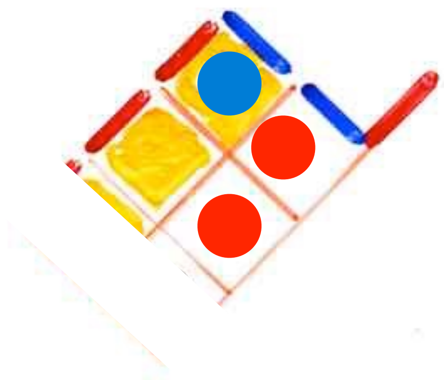


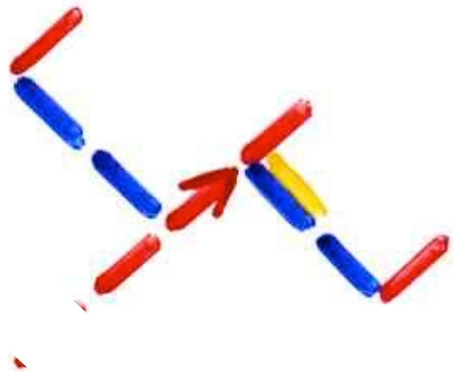


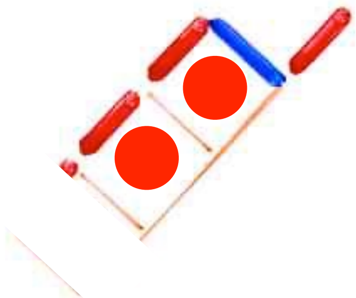


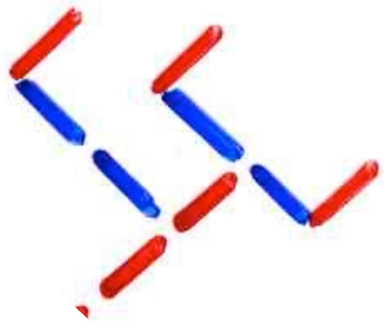


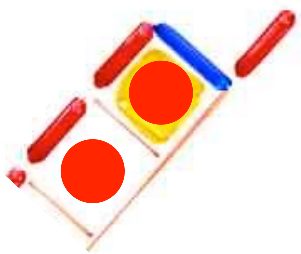


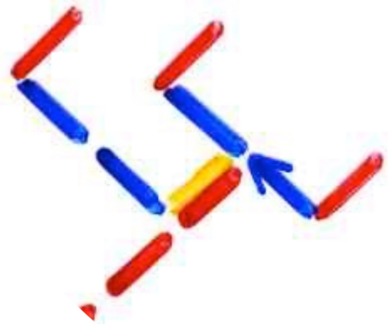


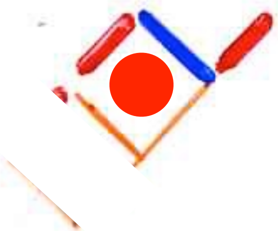


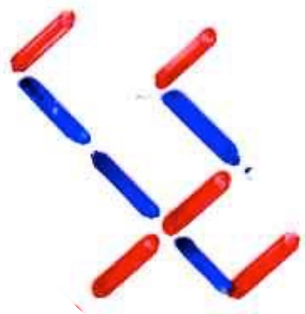


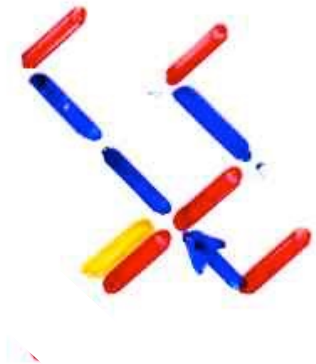






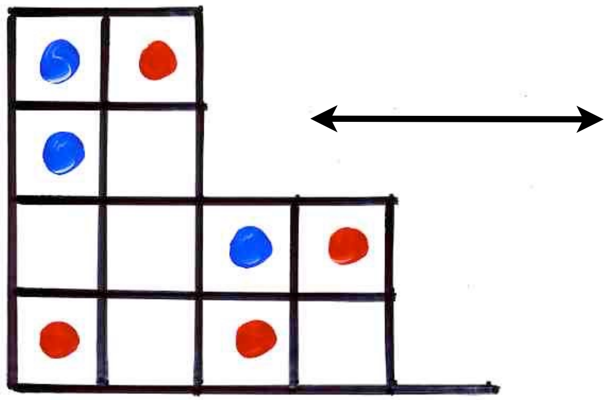


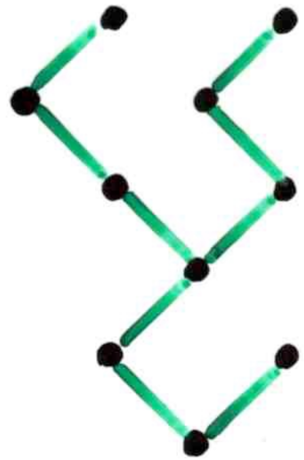


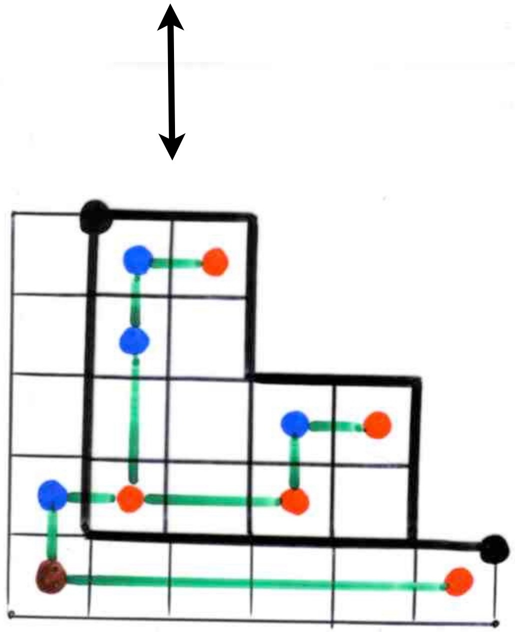


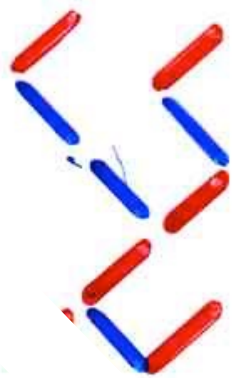
This algorithm based on a kind of « jeu de taquin » on « tableaux and trees » is reversible. One get a bijection between Catalan alternative tableaux and binary trees, which is the same as the one described on slide 115.

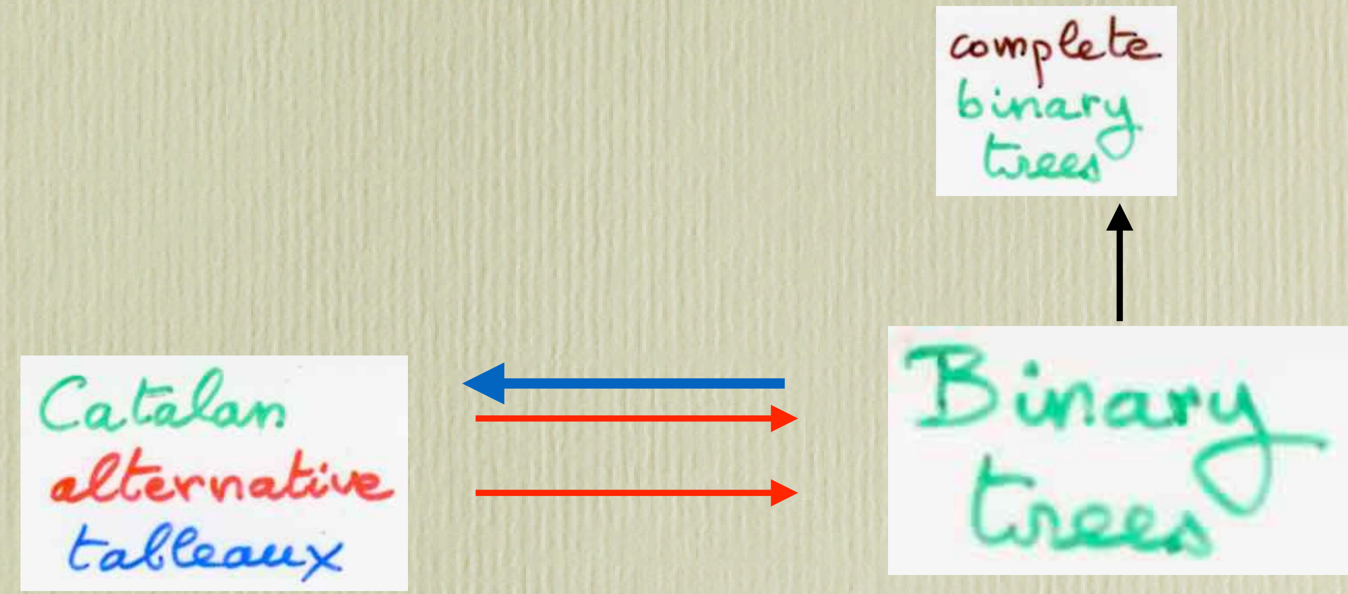
Catalan alternative tableaux



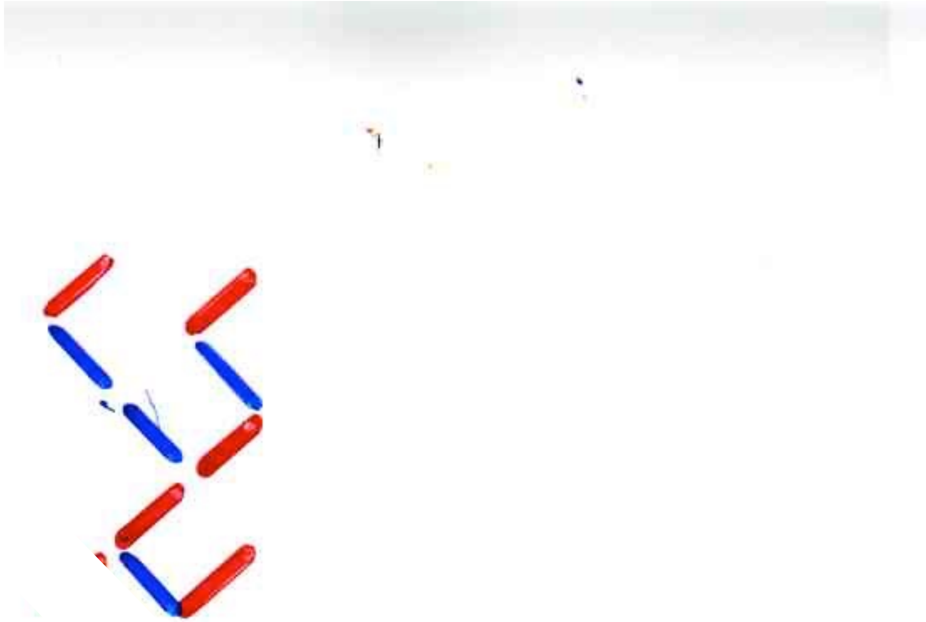


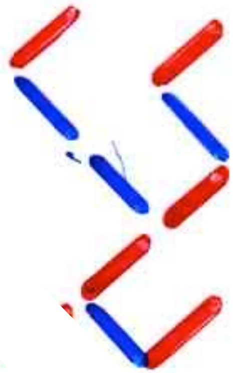


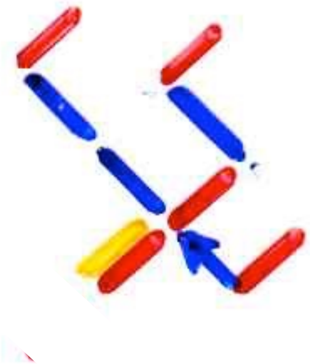


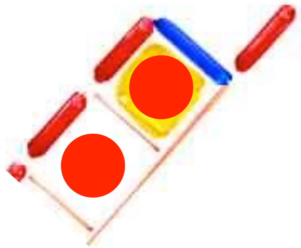

complete binary trees
Catalan alternative tableaux
Binary trees

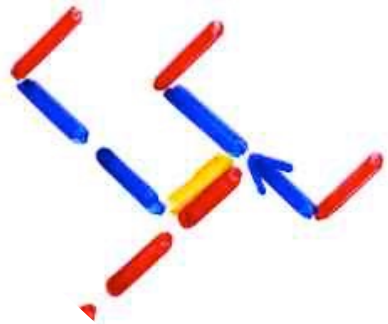
reverse bijection

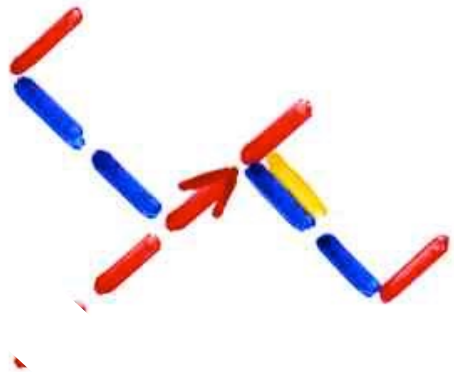


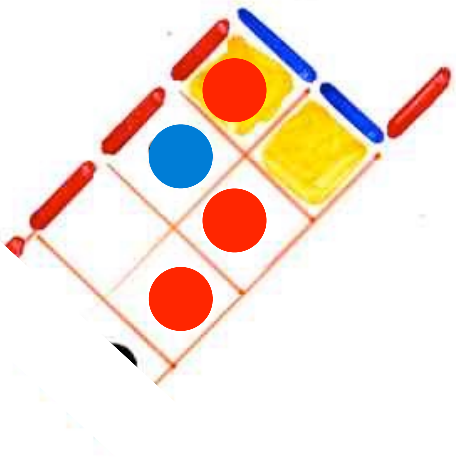


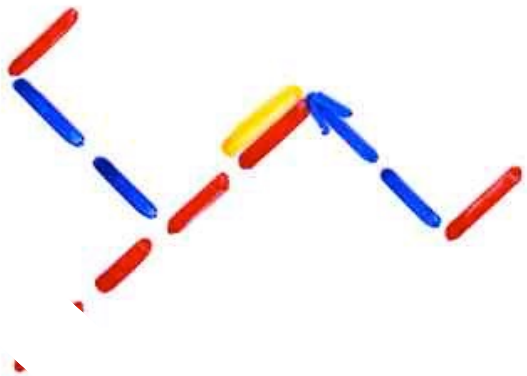


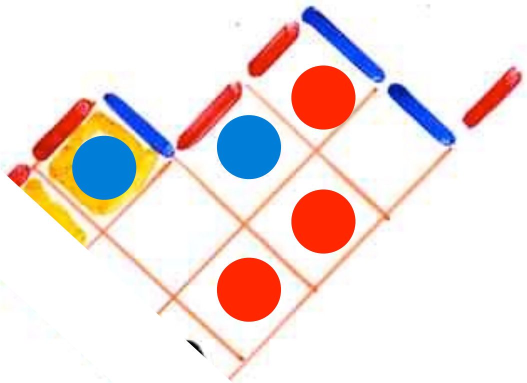


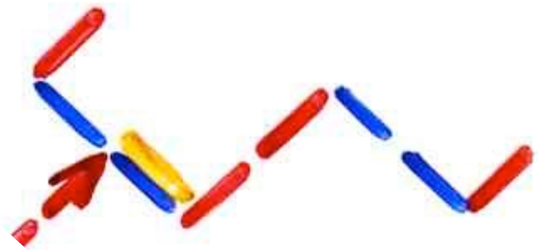


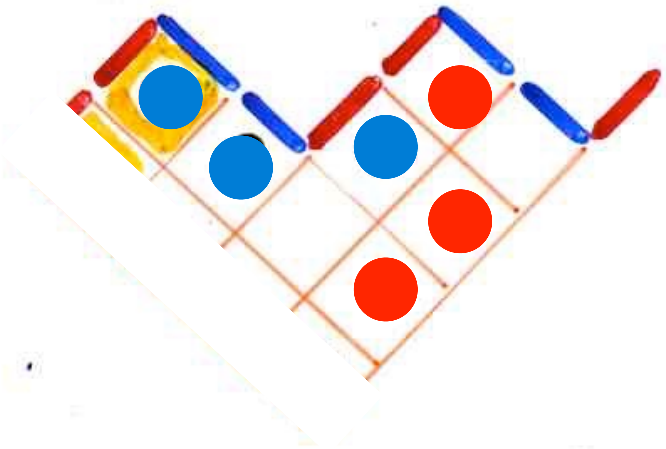


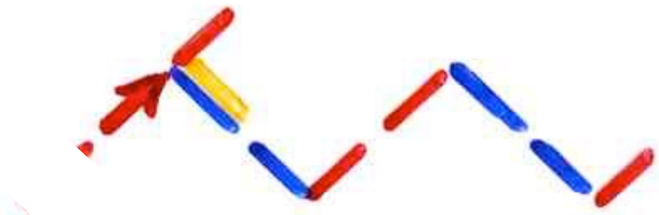


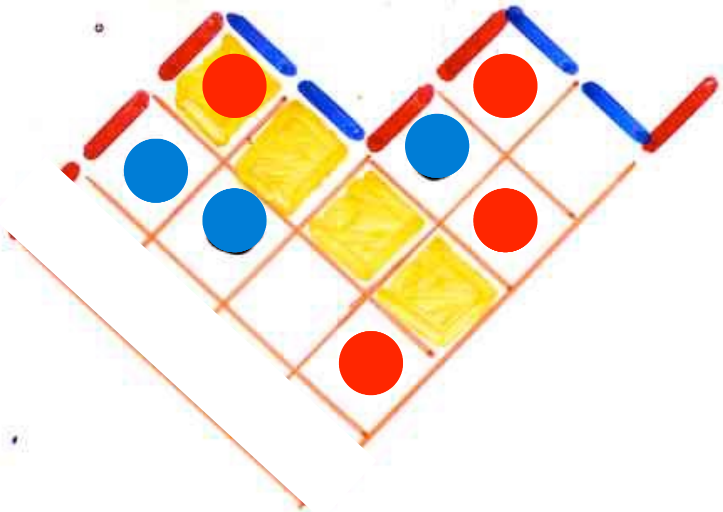


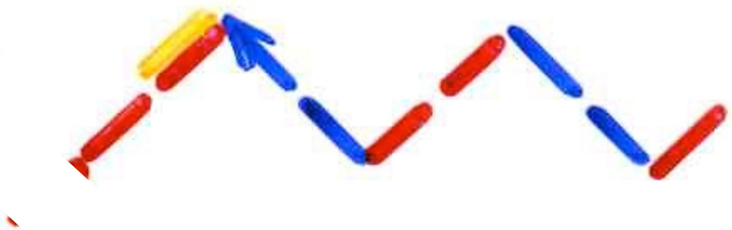








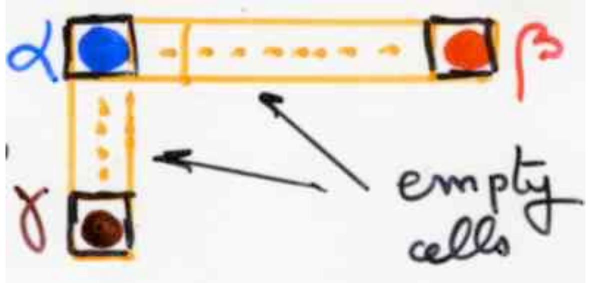




Tamari and alternative tableaux

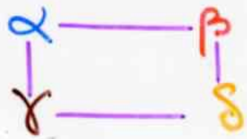
Main Lemma

In a Catalan alternative tableau let α, β, γ be 3 colored cells in a Γ position (α is necessarily blue and γ red)



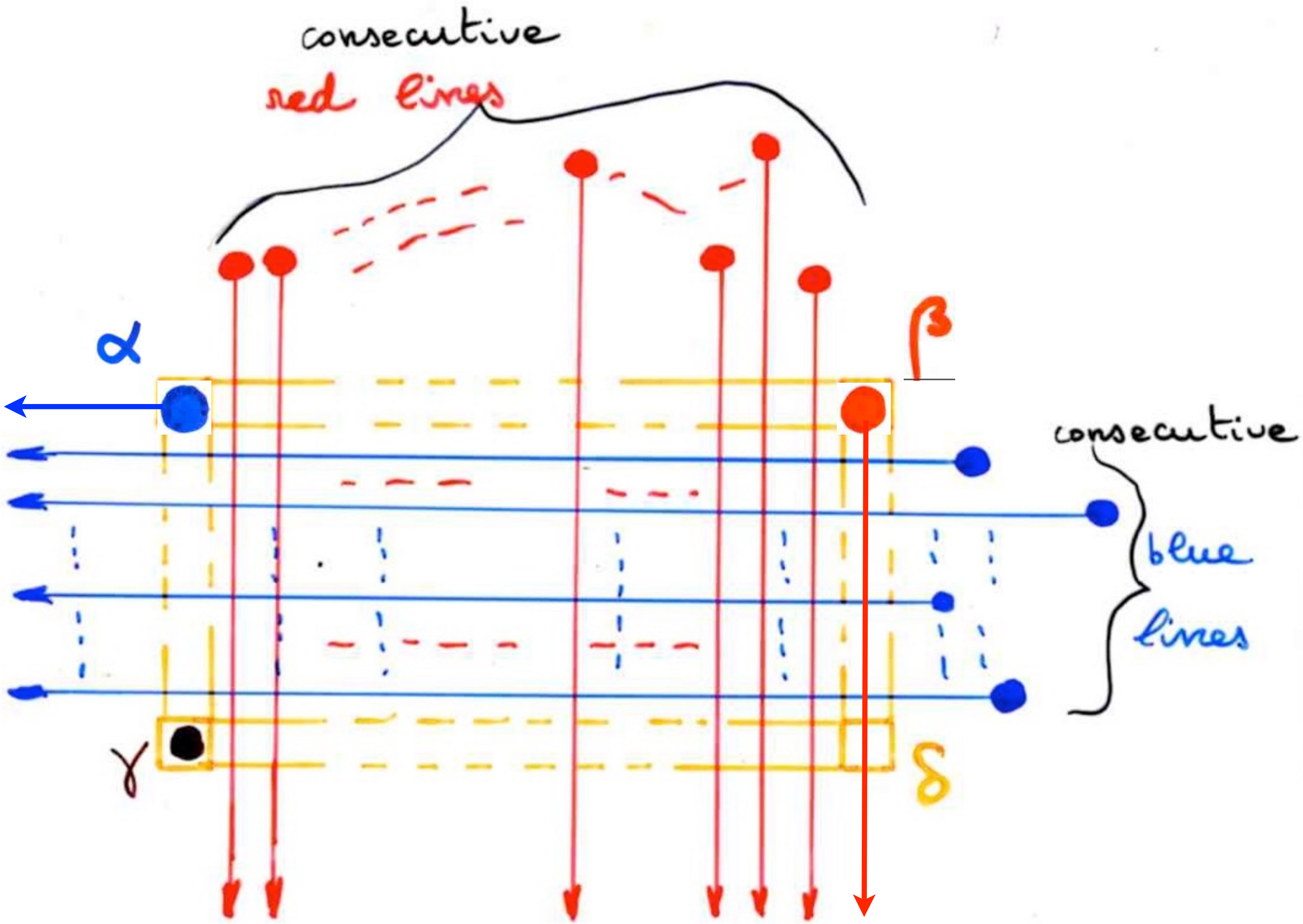
such that there is no colored cell between α and β and between α and γ.

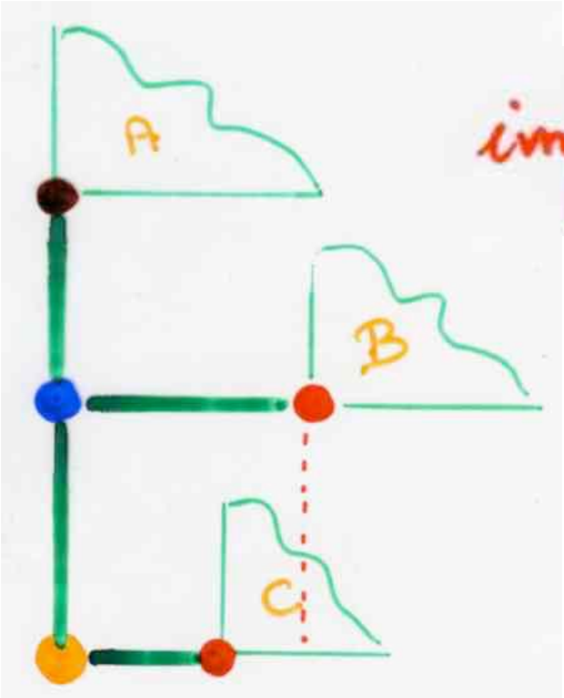
Then the cells of the whole rectangle



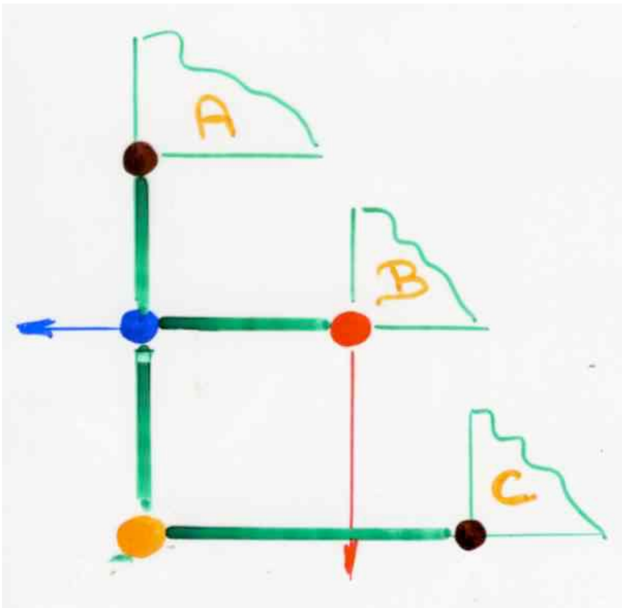
are empty (except α, β, γ)

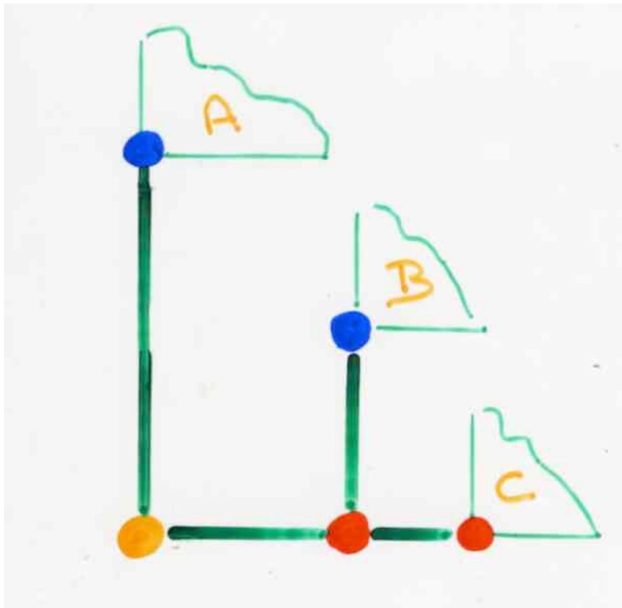
Moreover we have the following configuration of blue cells and lines, with red cells and lines :

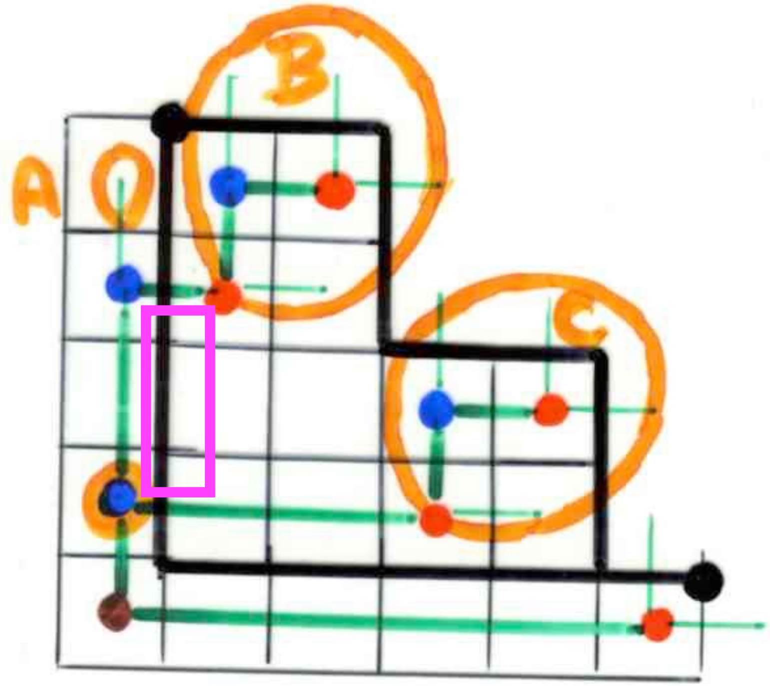

consecutive
red lines
α
β
consecutive
blue
lines
γ
δ

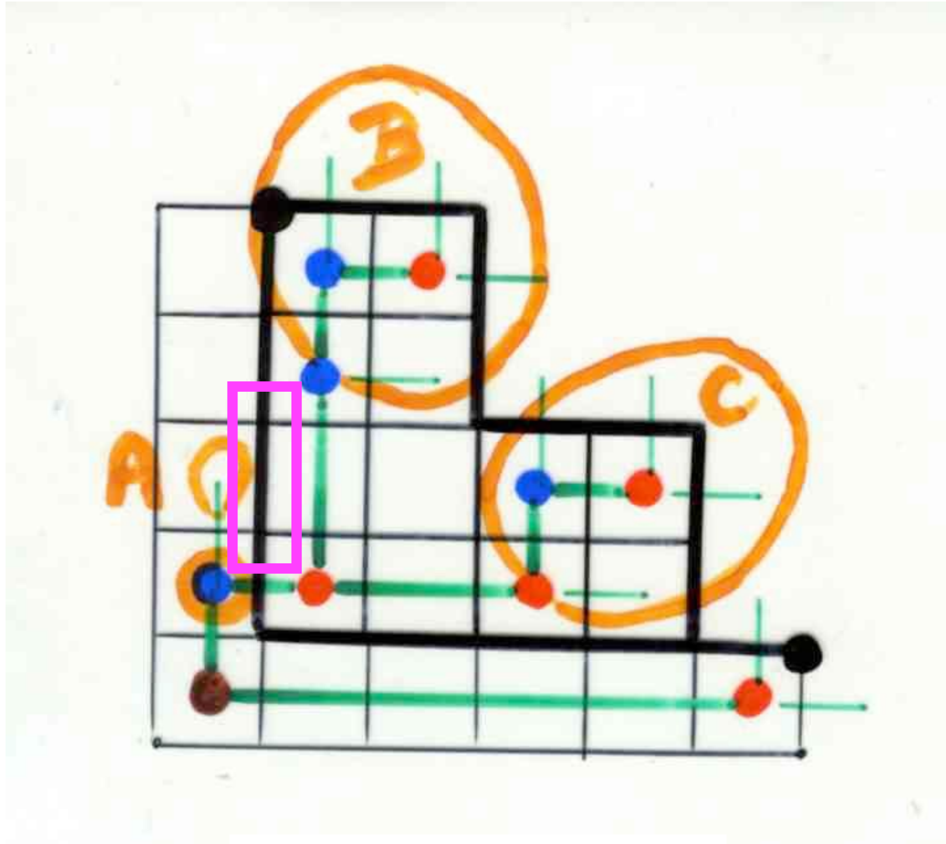


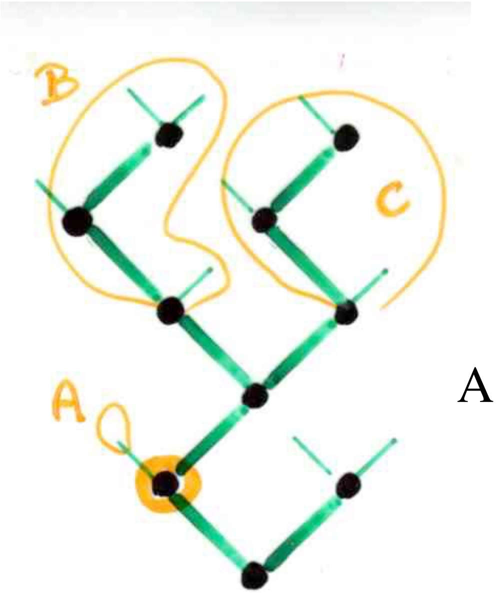
impossible !

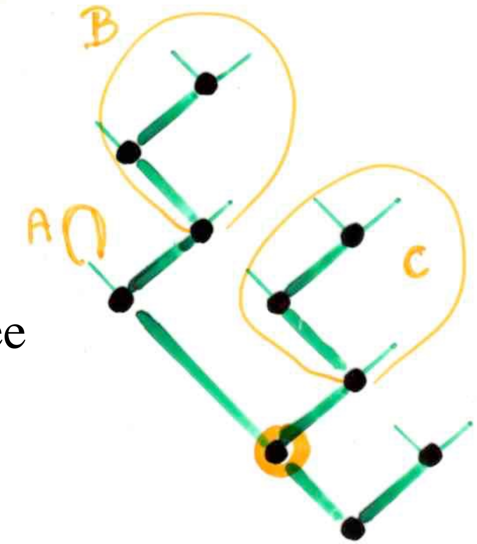




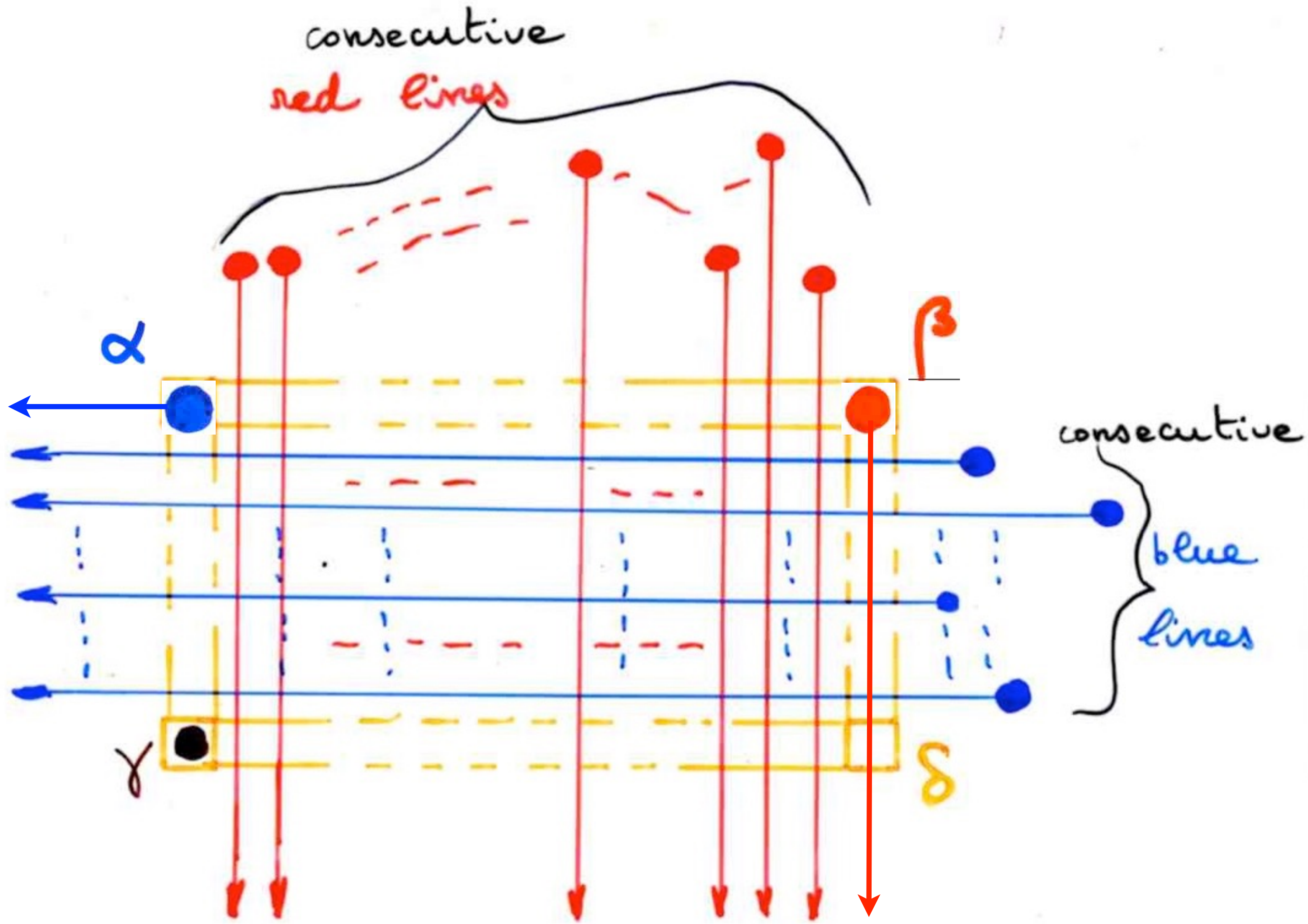


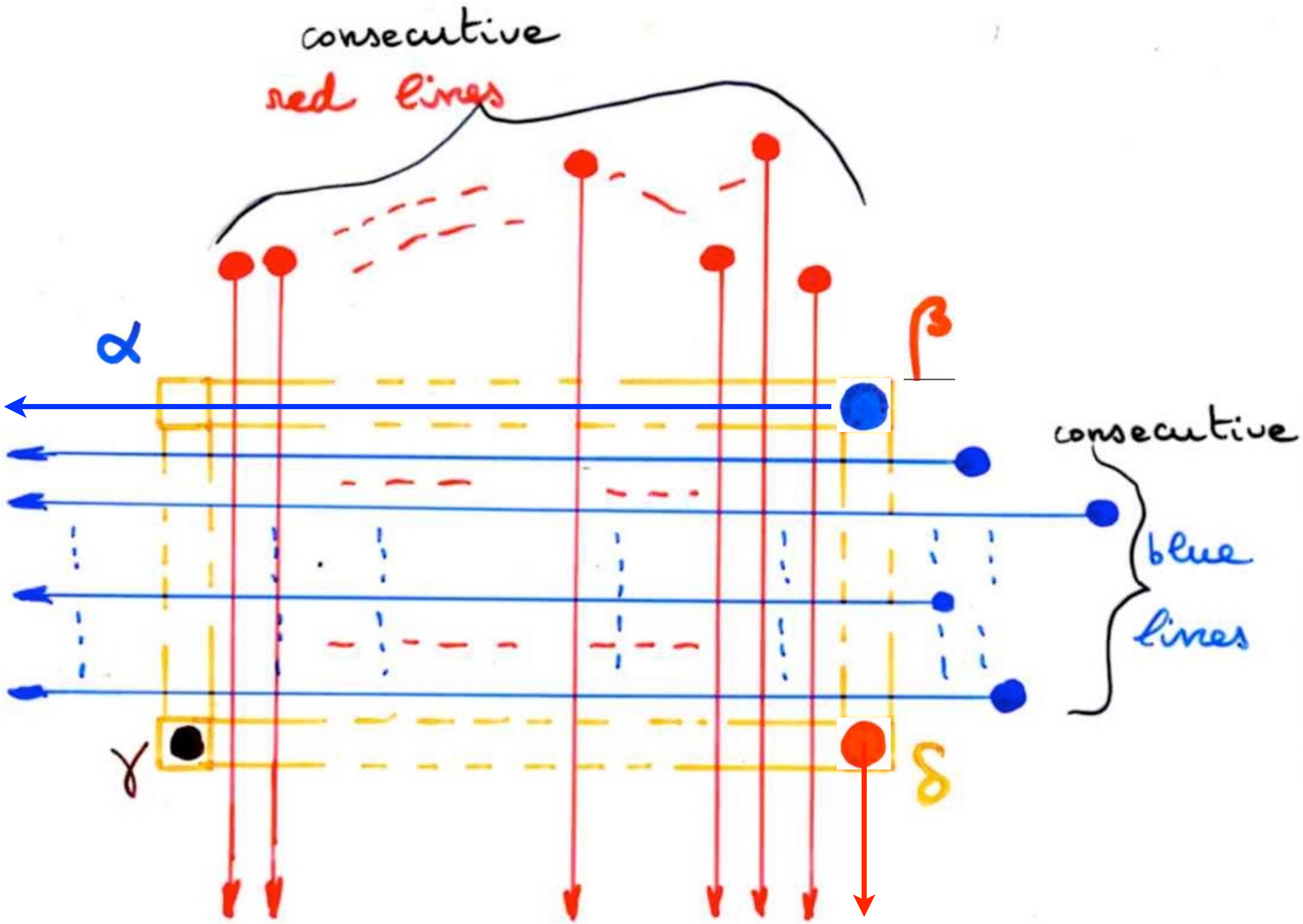






A rotation in the binary tree
corresponds exactly
to a certain Γ-move
in the associated
Catalan alternative tableau.

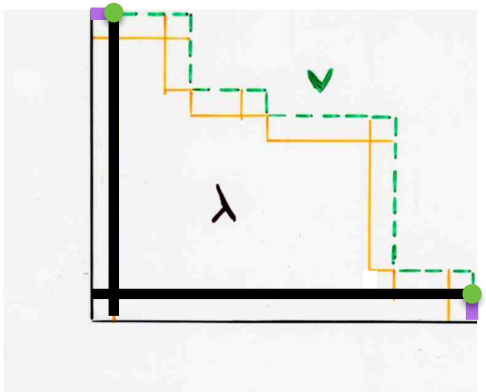

consecutive
red lines
α
β
consecutive
blue
lines
γ
δ


consecutive
red lines
α
β
consecutive
blue
lines
γ
δ

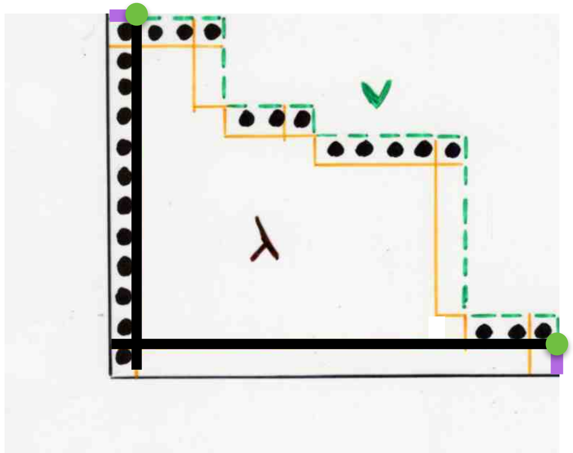
The main theorem

Main theorem

Ferrers diagram λ
with profile v

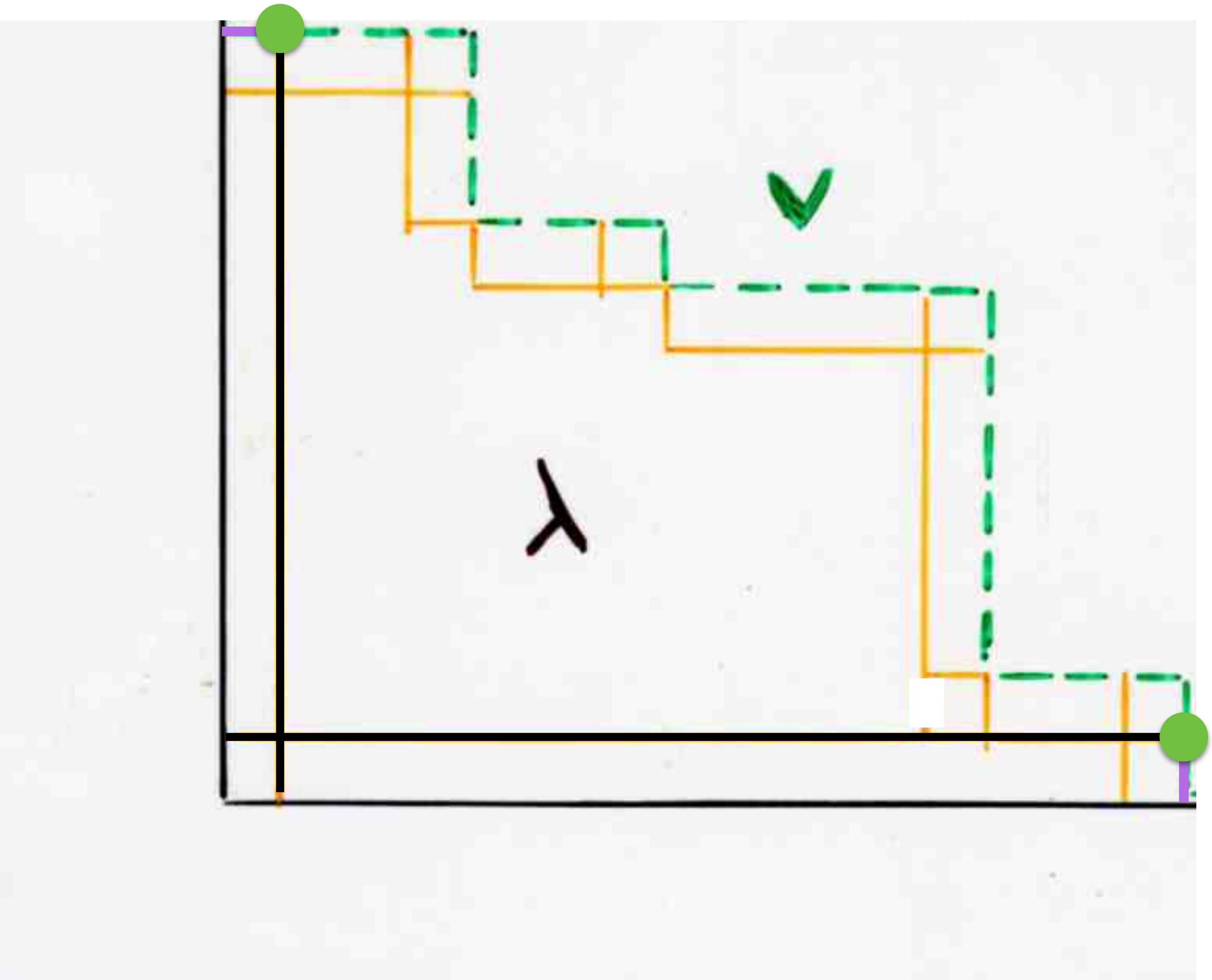


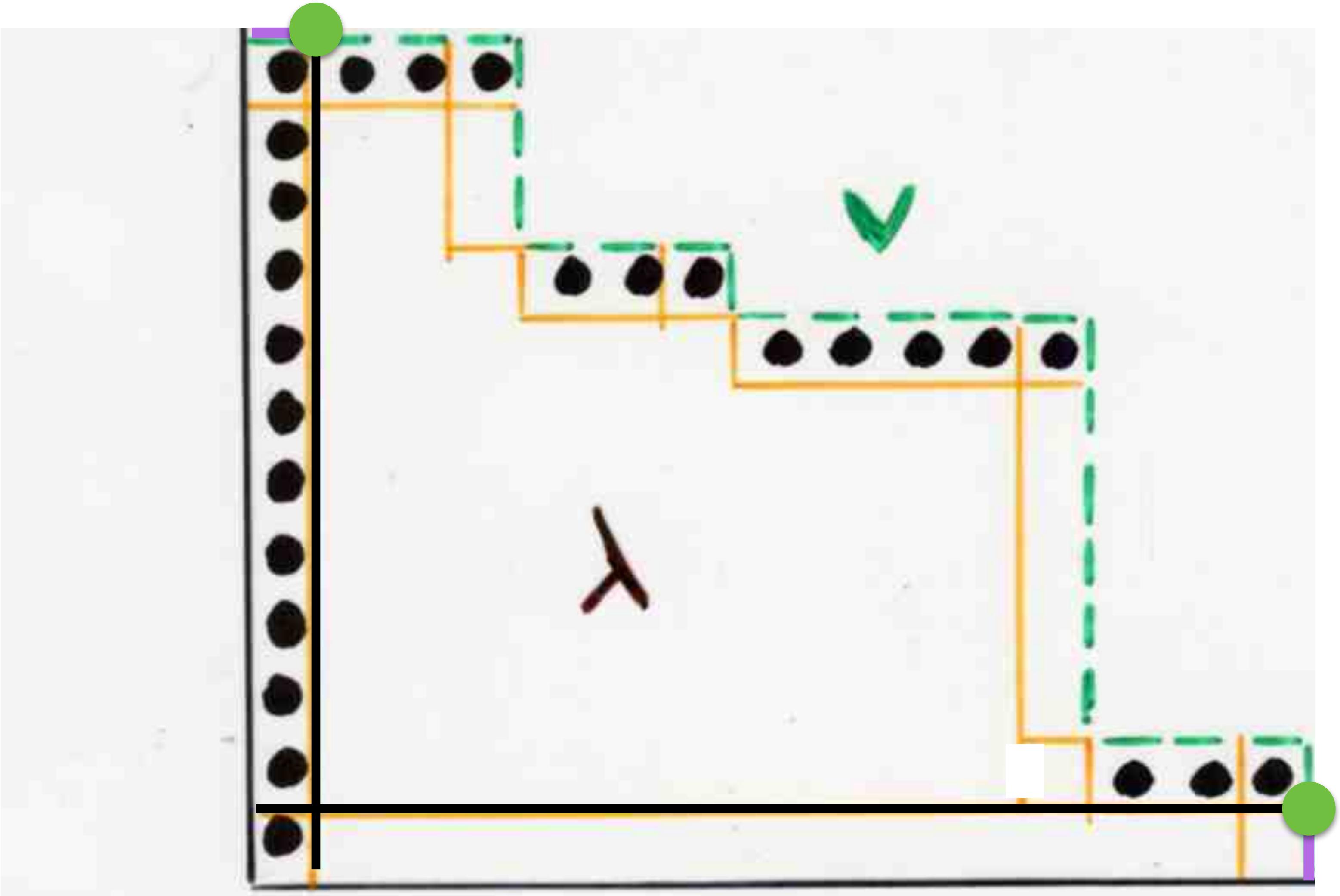
Let $X(\lambda) = X(v)$ be the cloud



The set of binary trees having a given canopy v is an interval of the Tamari lattice.

$$\mathrm{Int}(v) = \mathrm{Maule}(X(v))$$


ν
λ

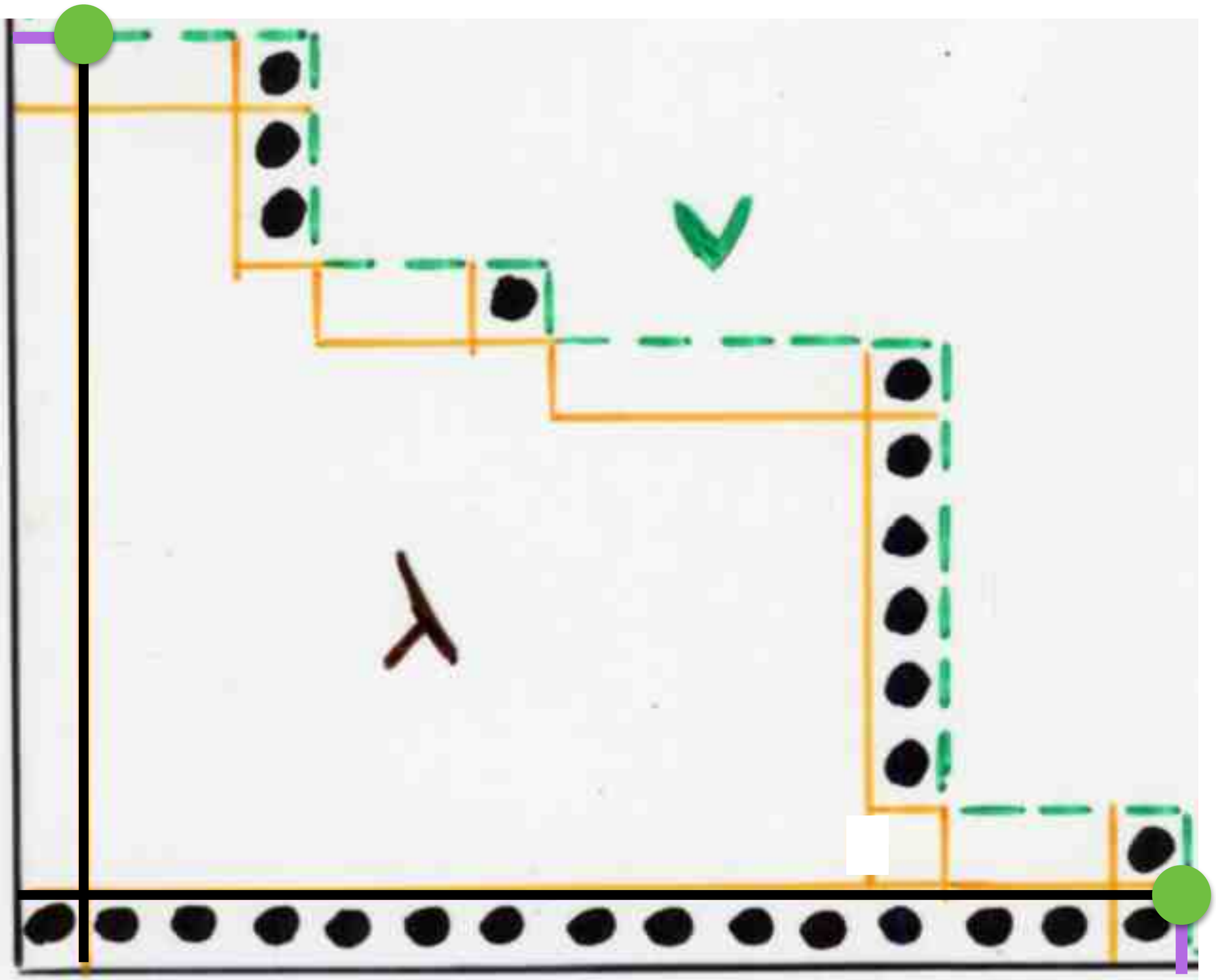


$X(v)$

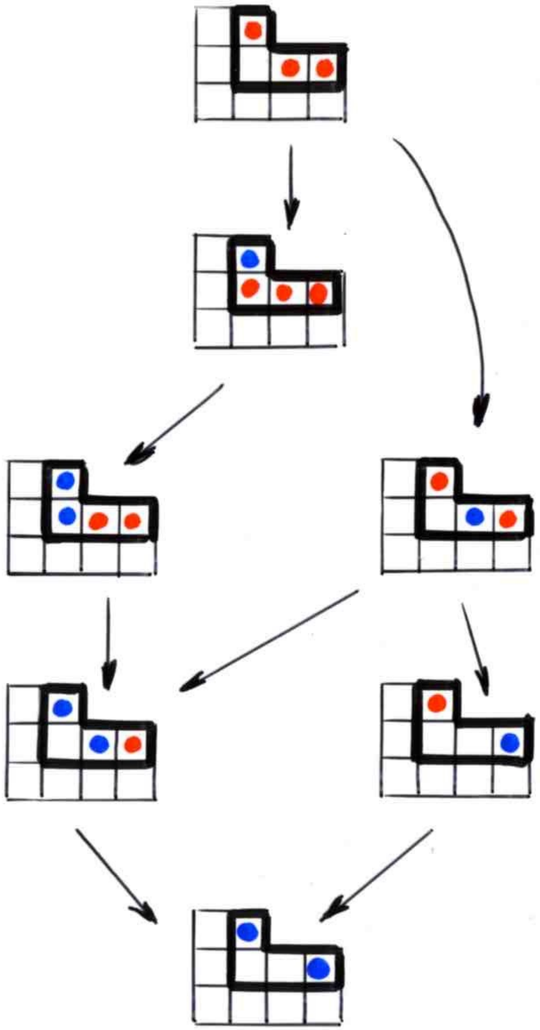
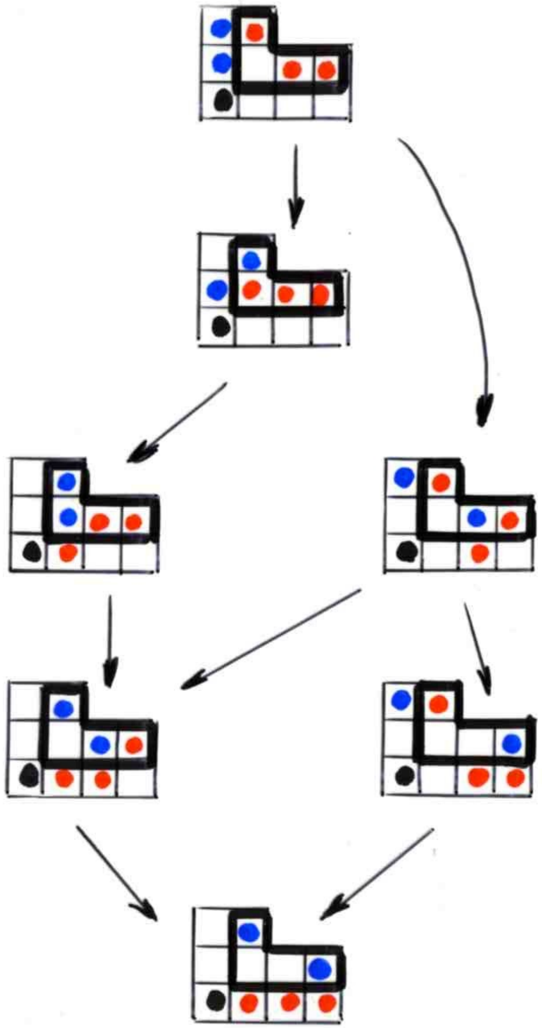
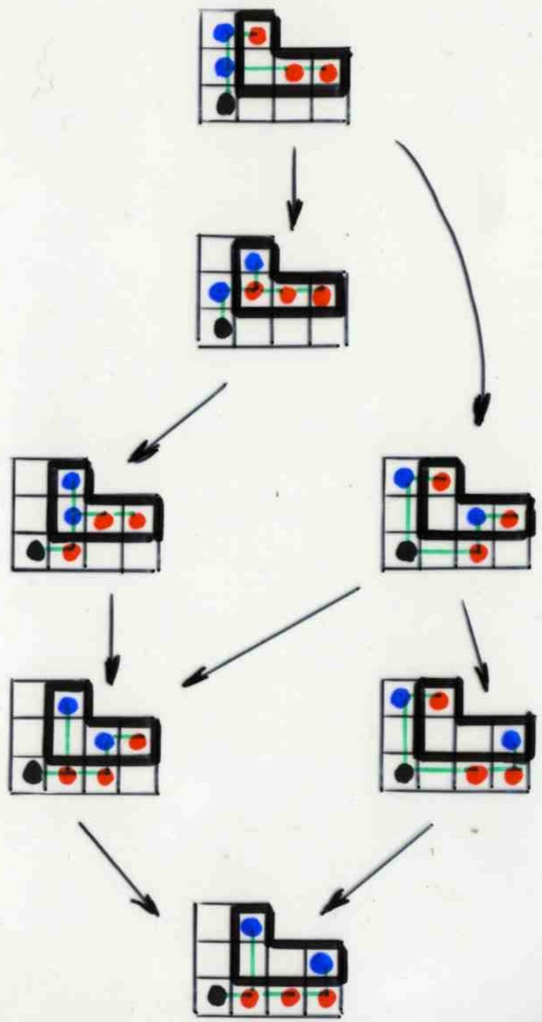
$X(\lambda)$

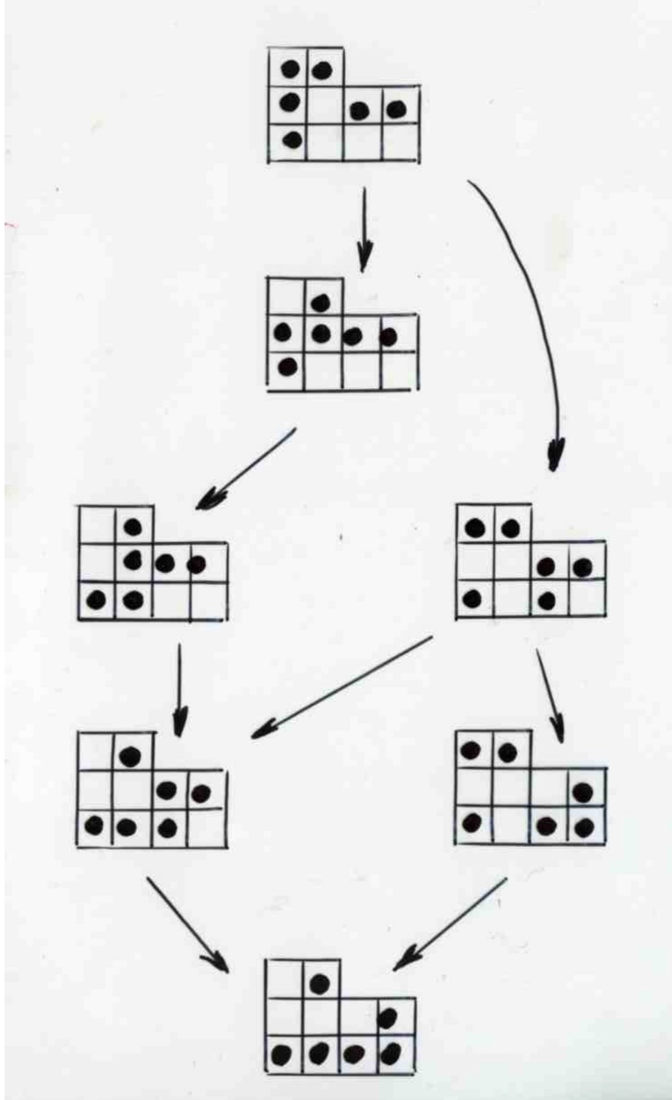
minimum element of the maule

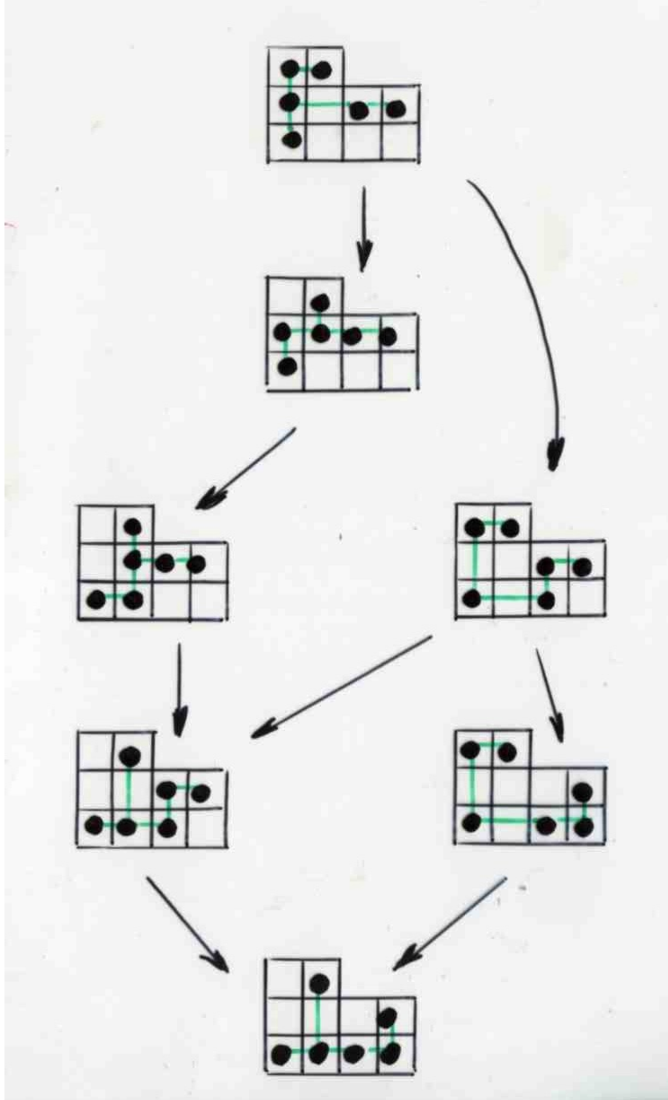
Maule$(X(v))$



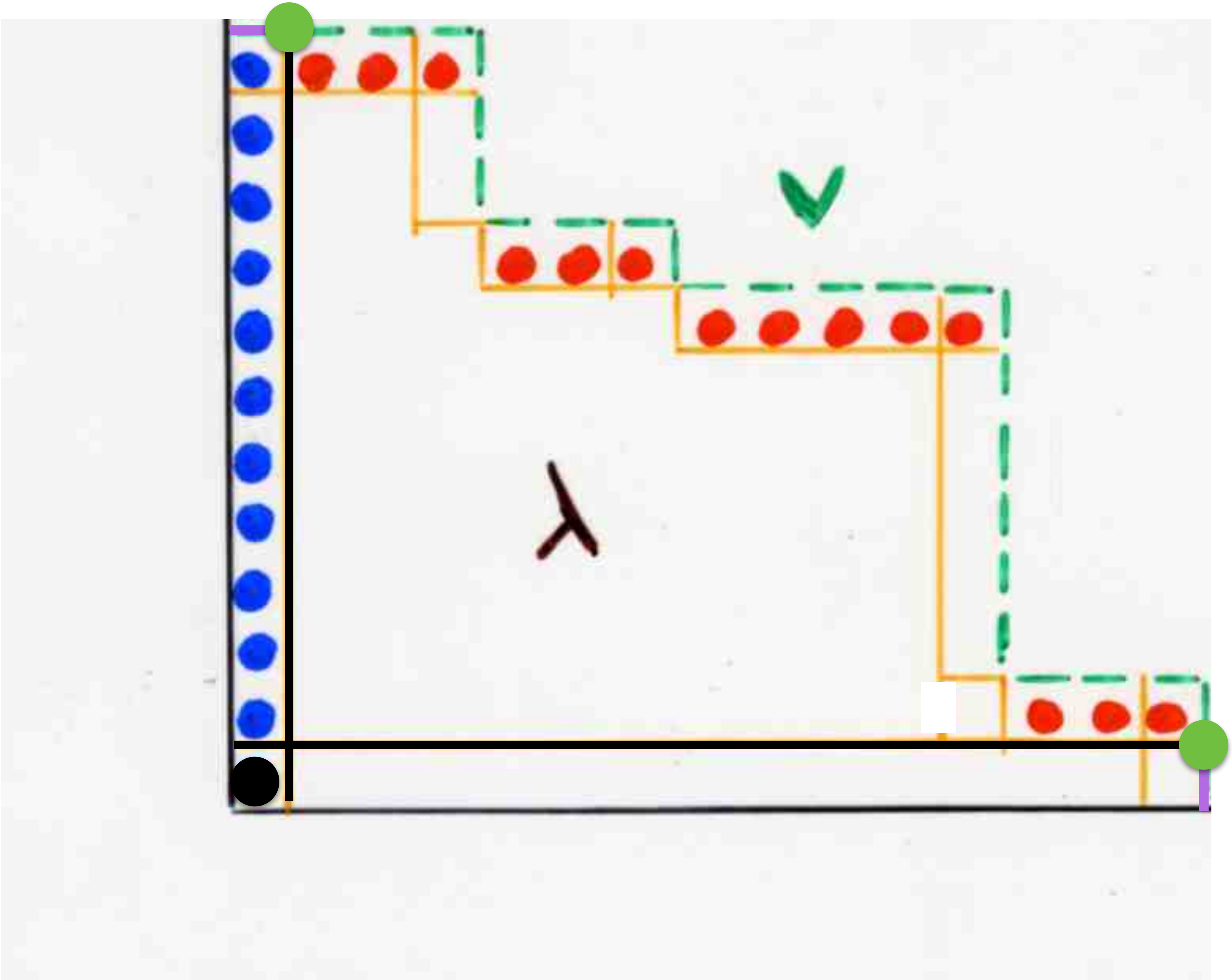
maximum element of the maule Maule(X(v))





an example



ν
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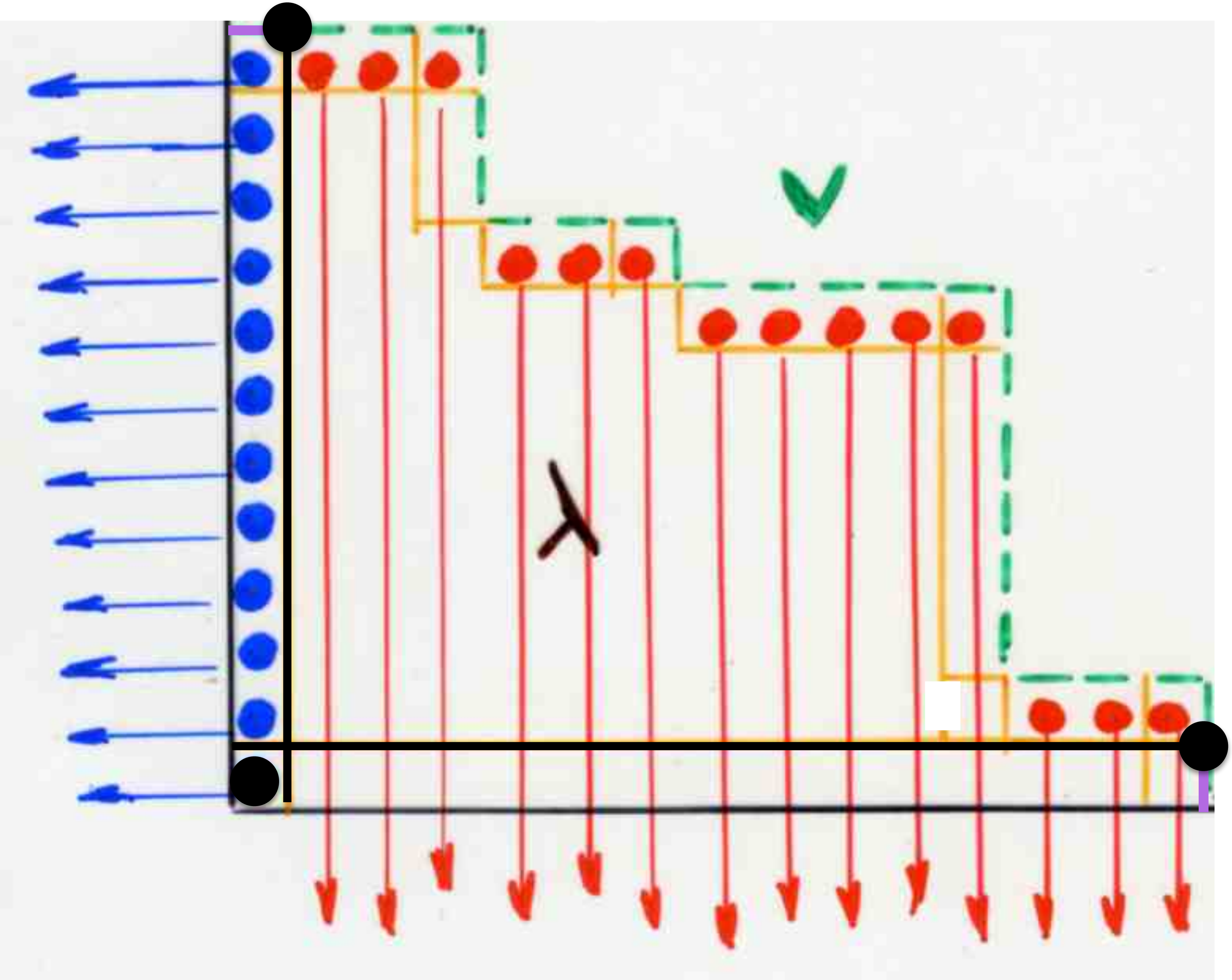
ν
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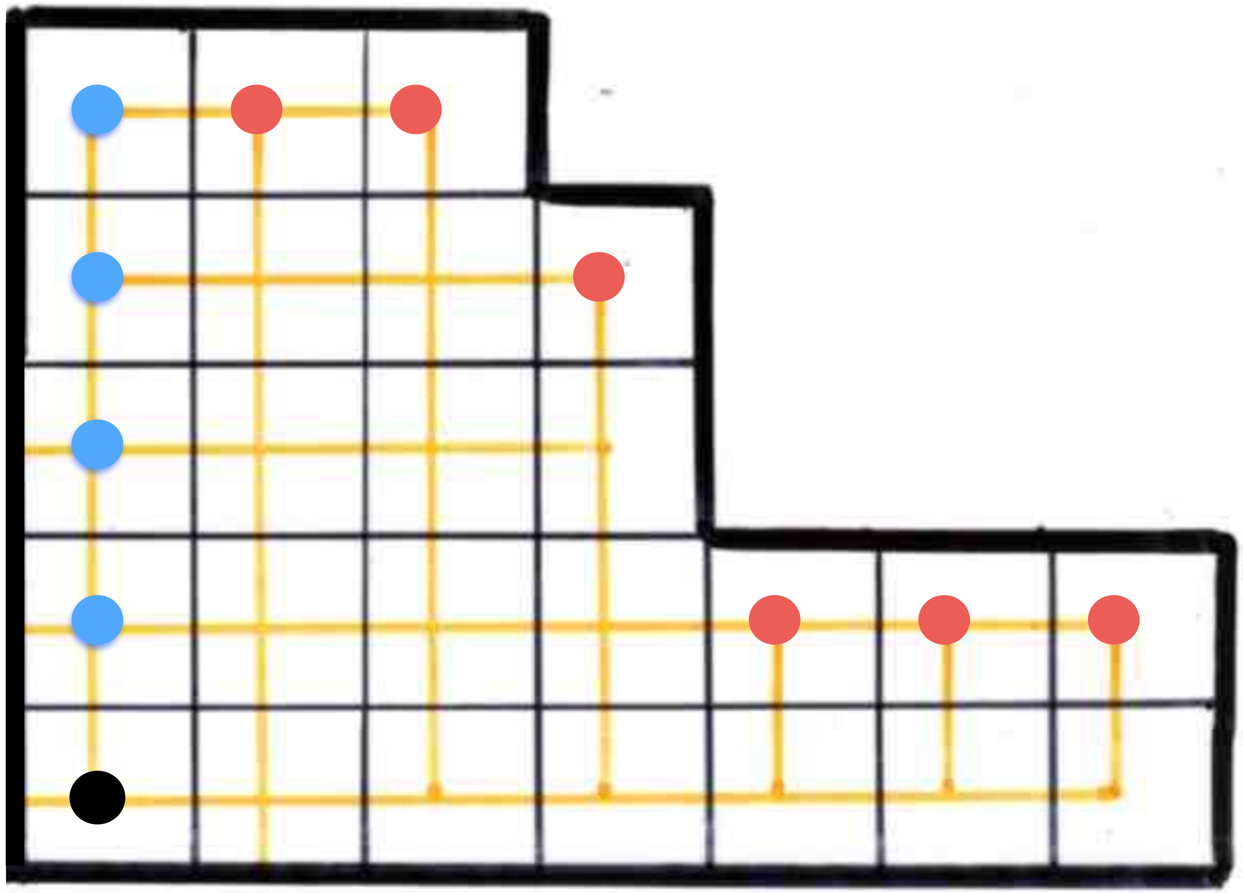
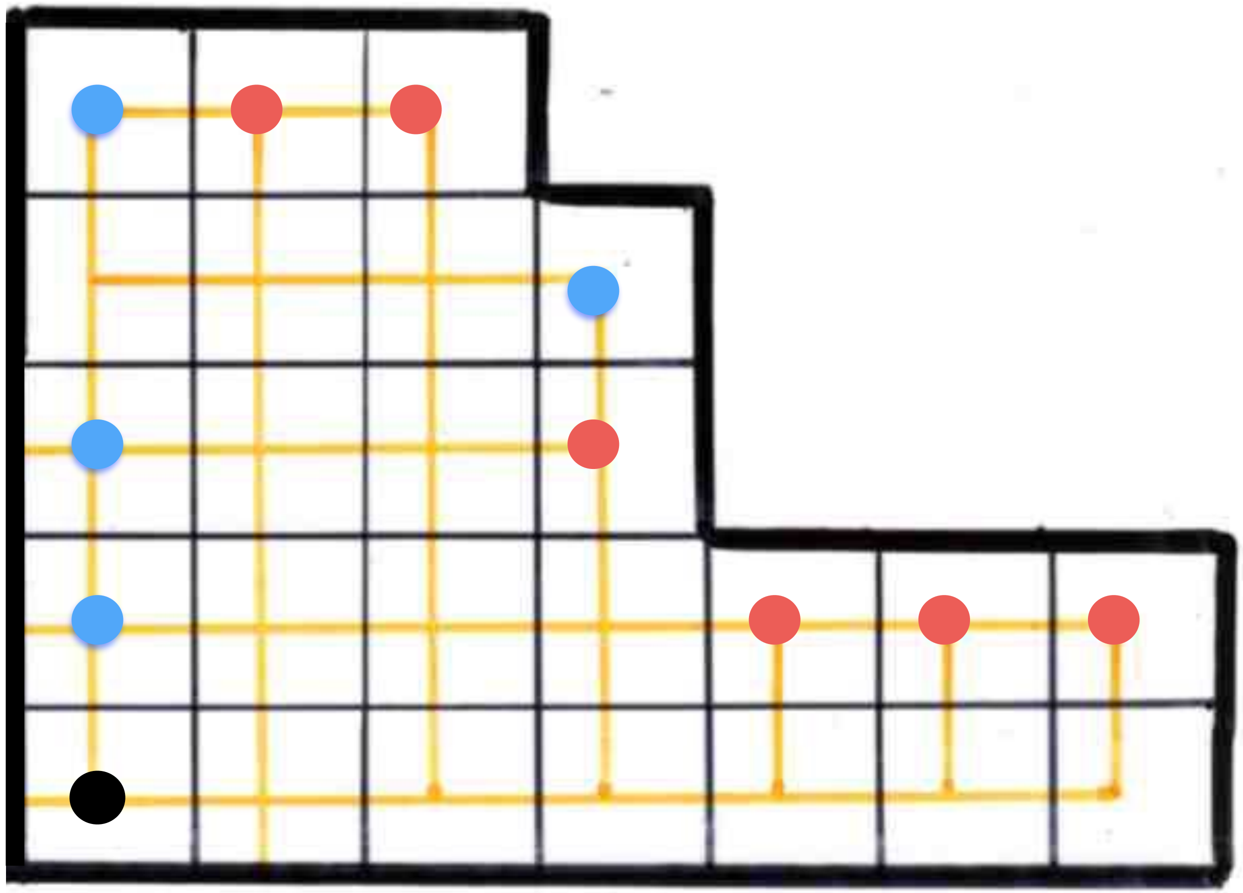
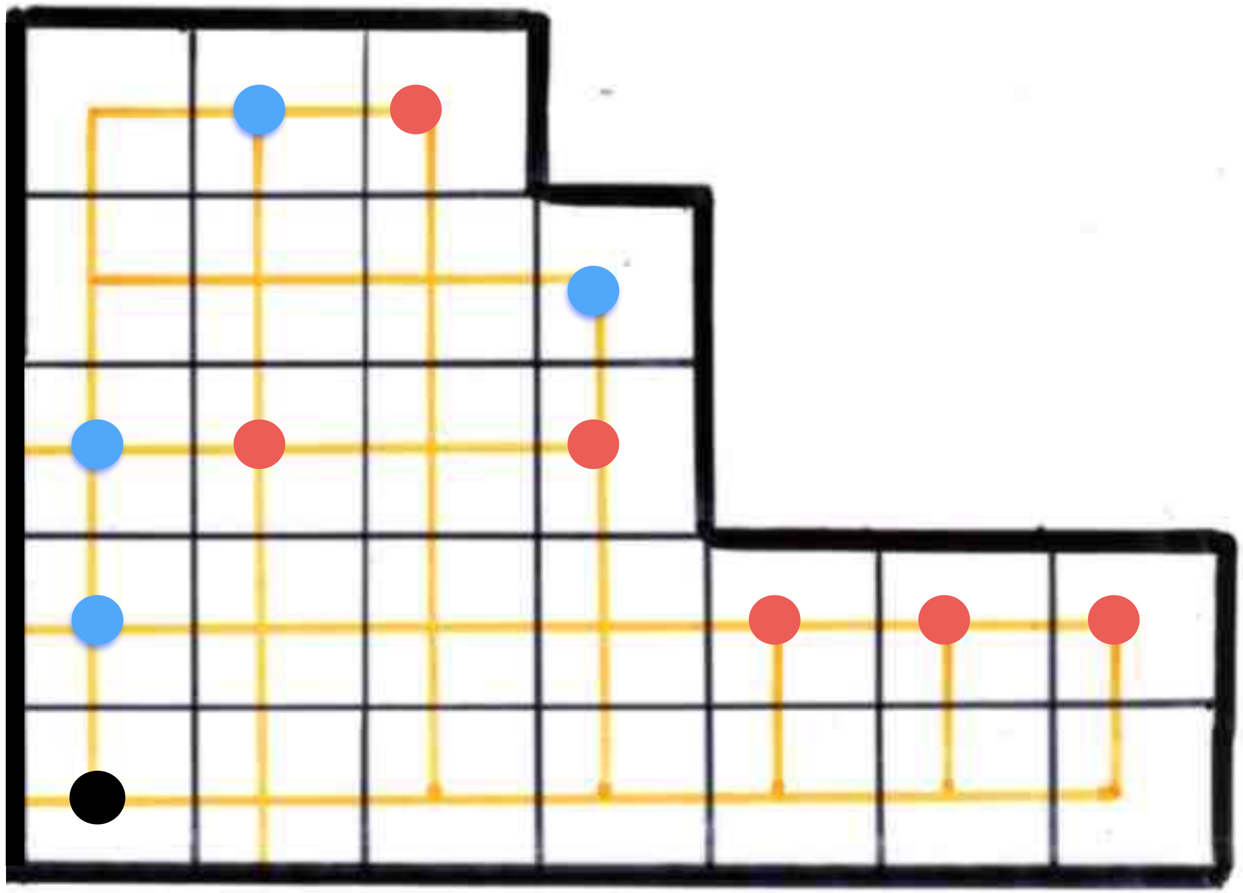
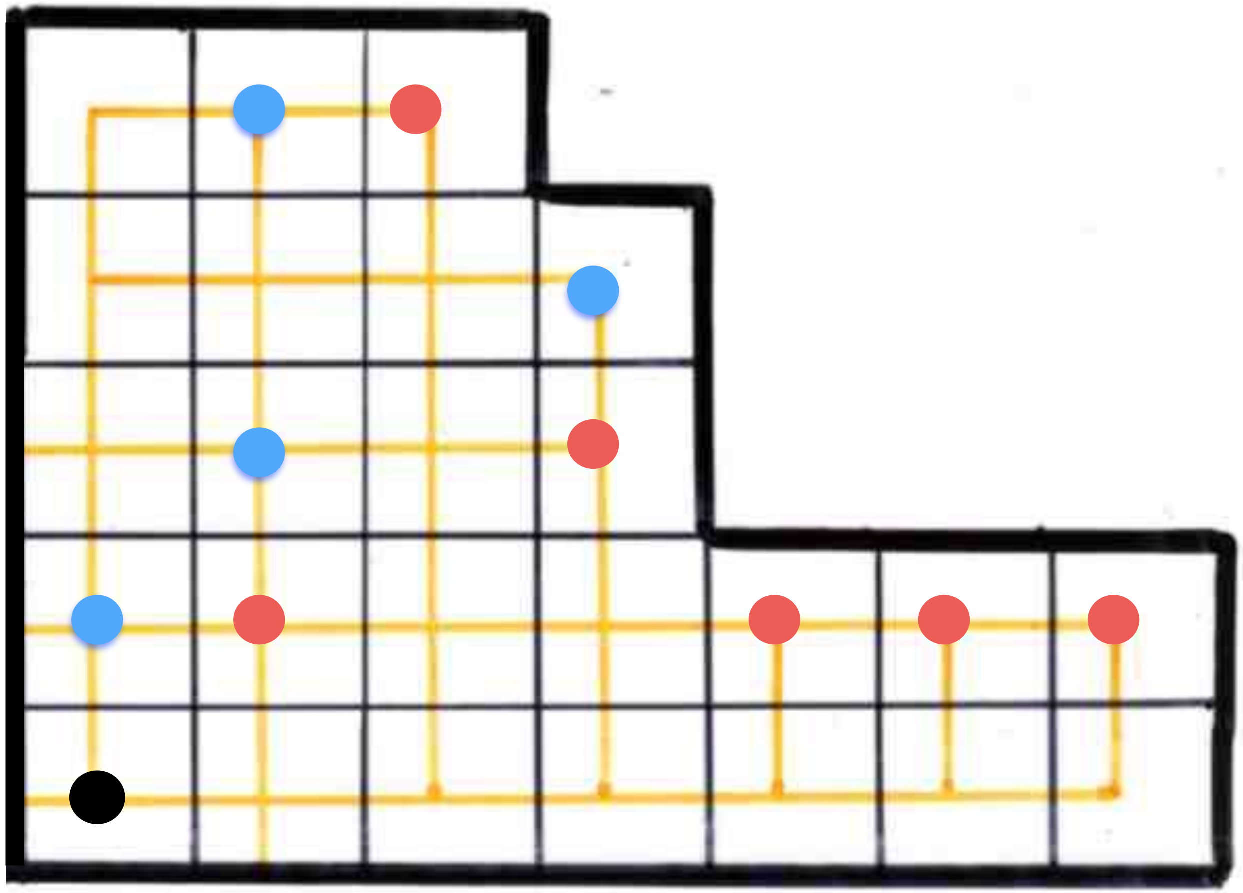
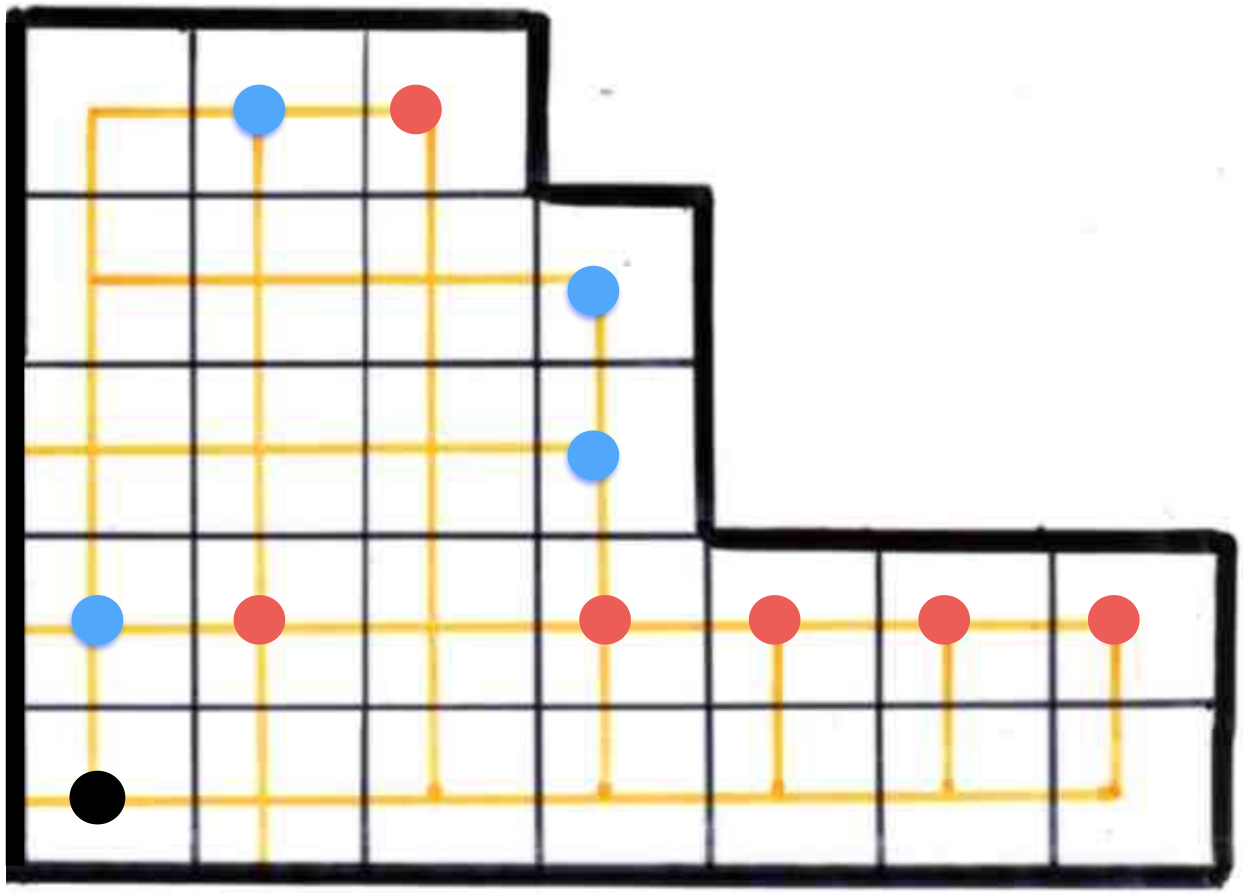
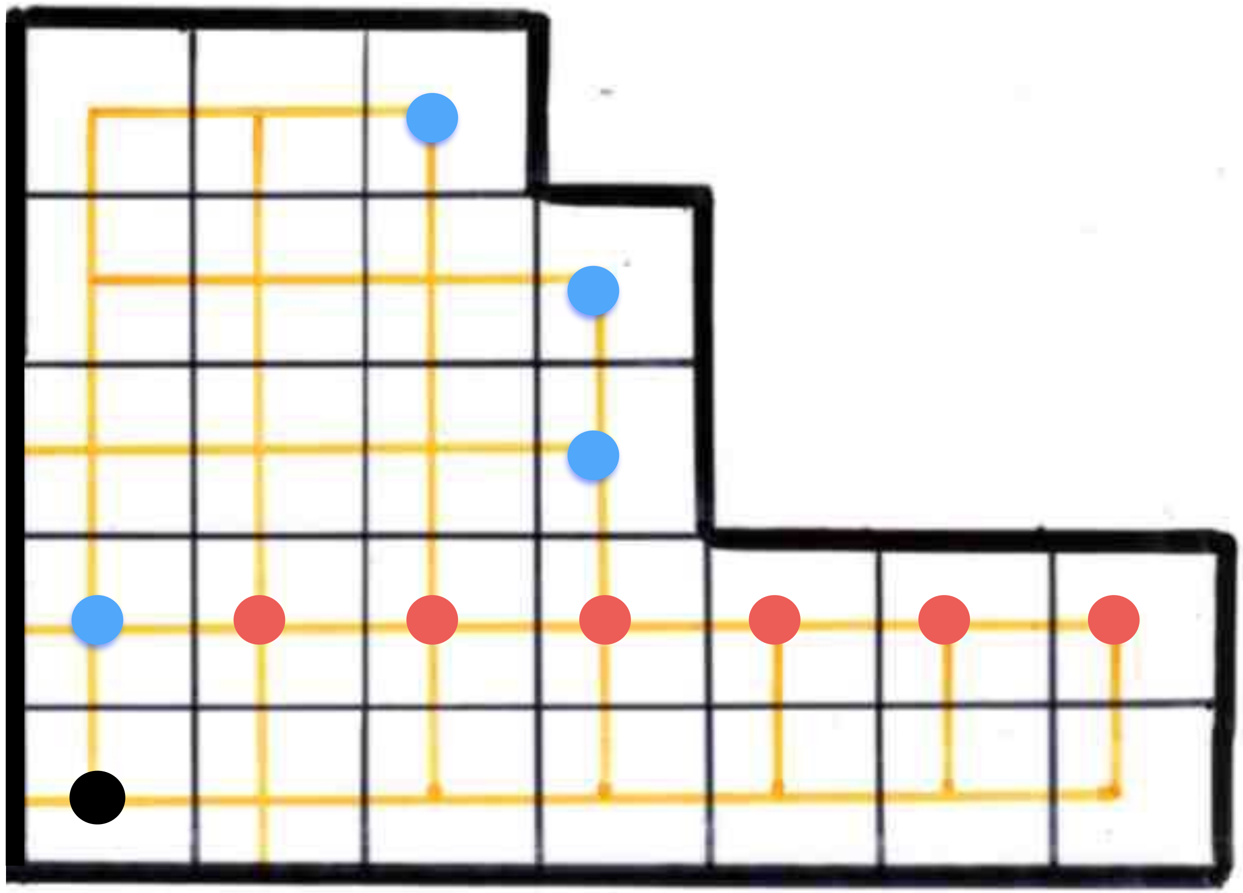


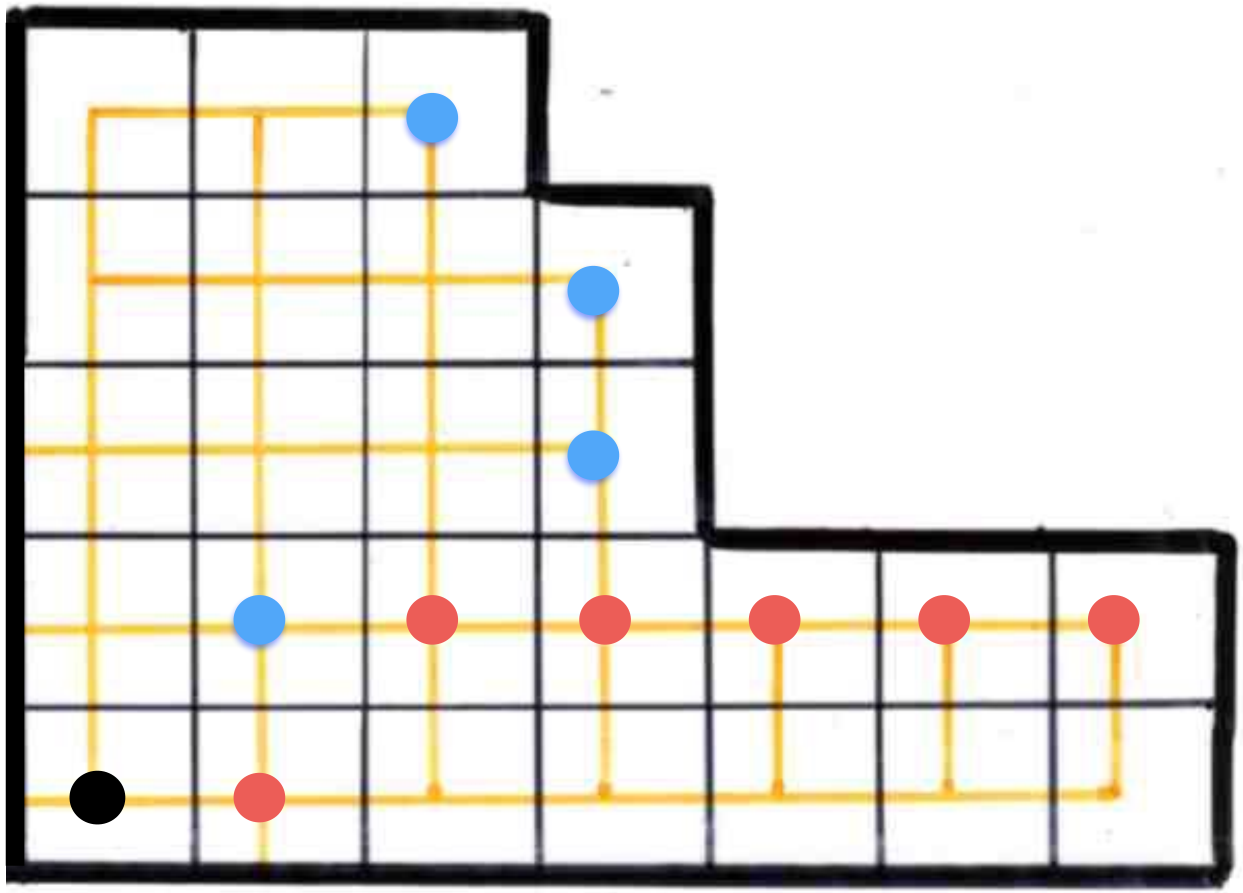


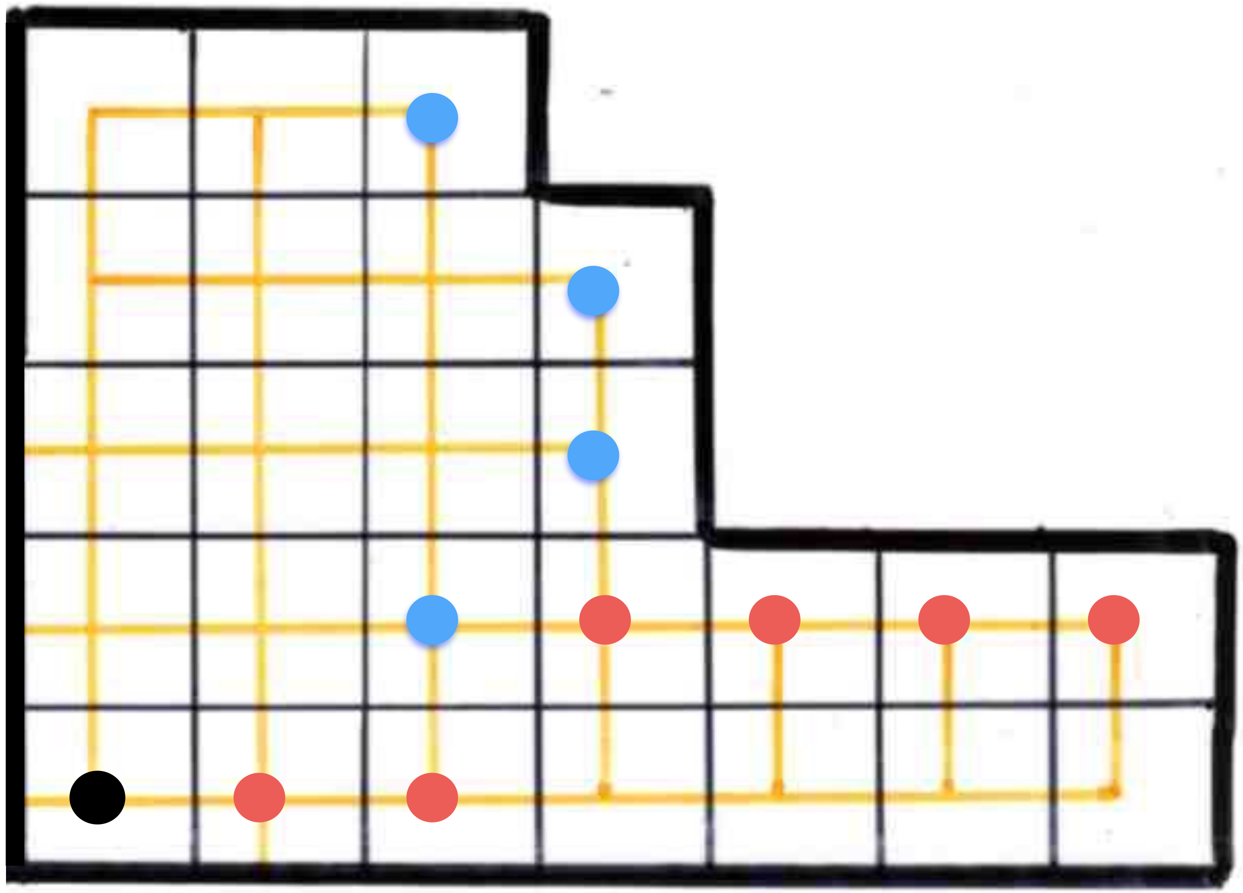


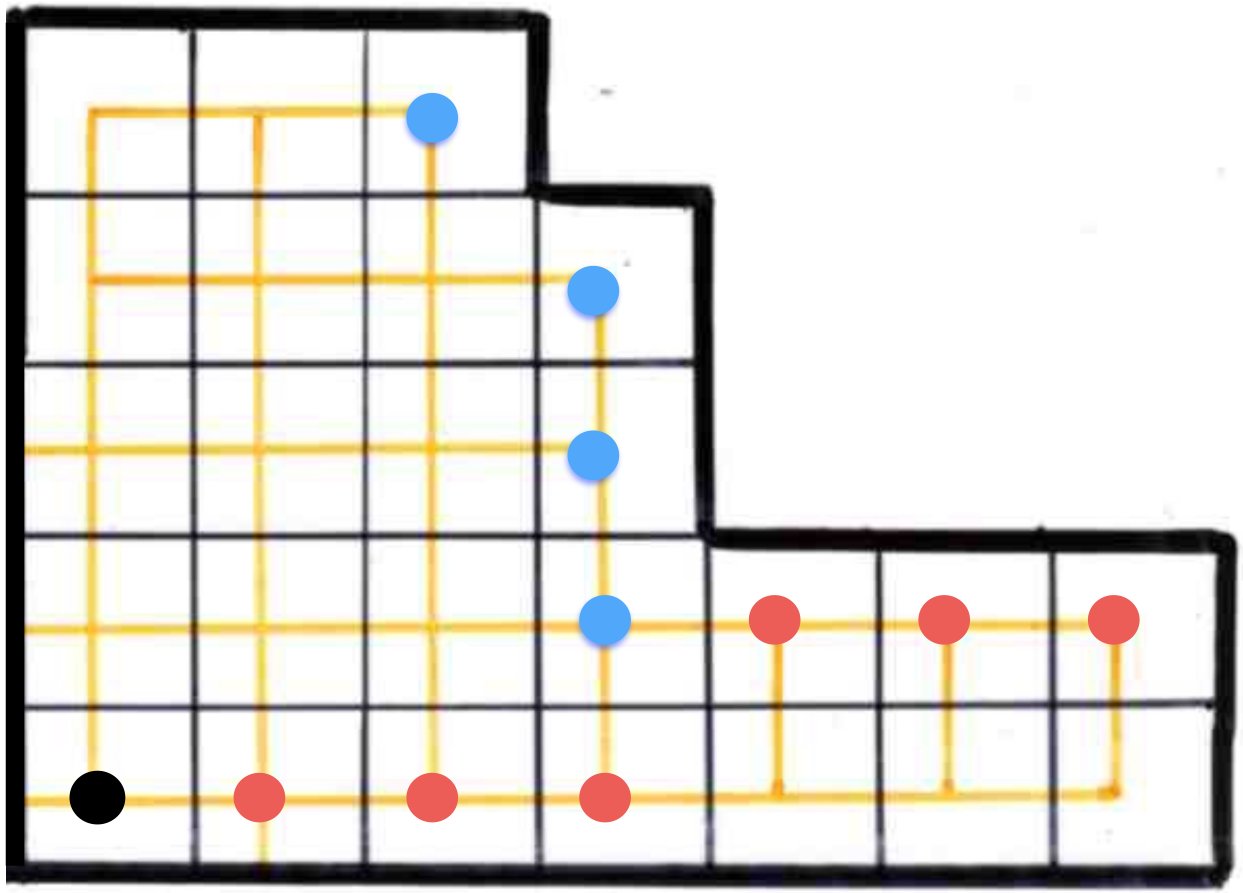


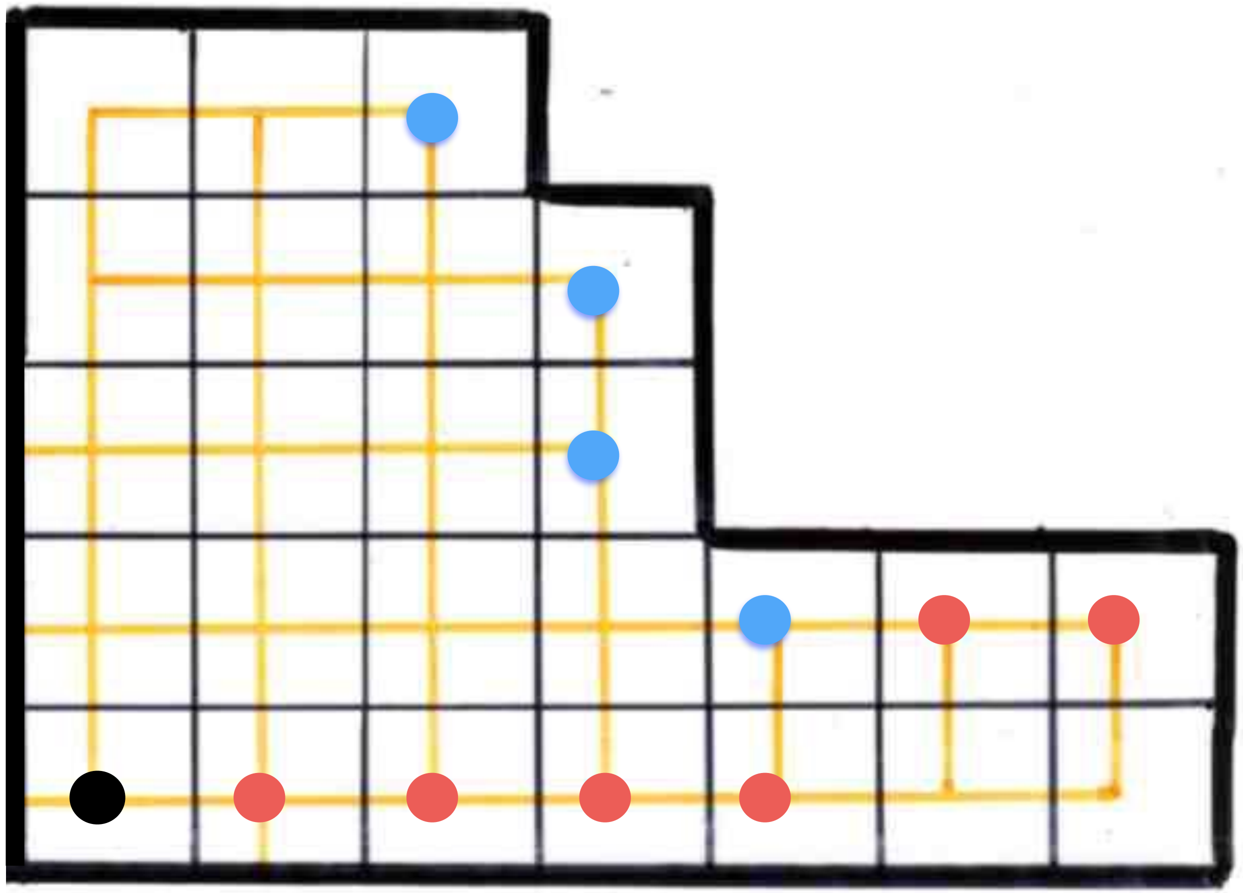


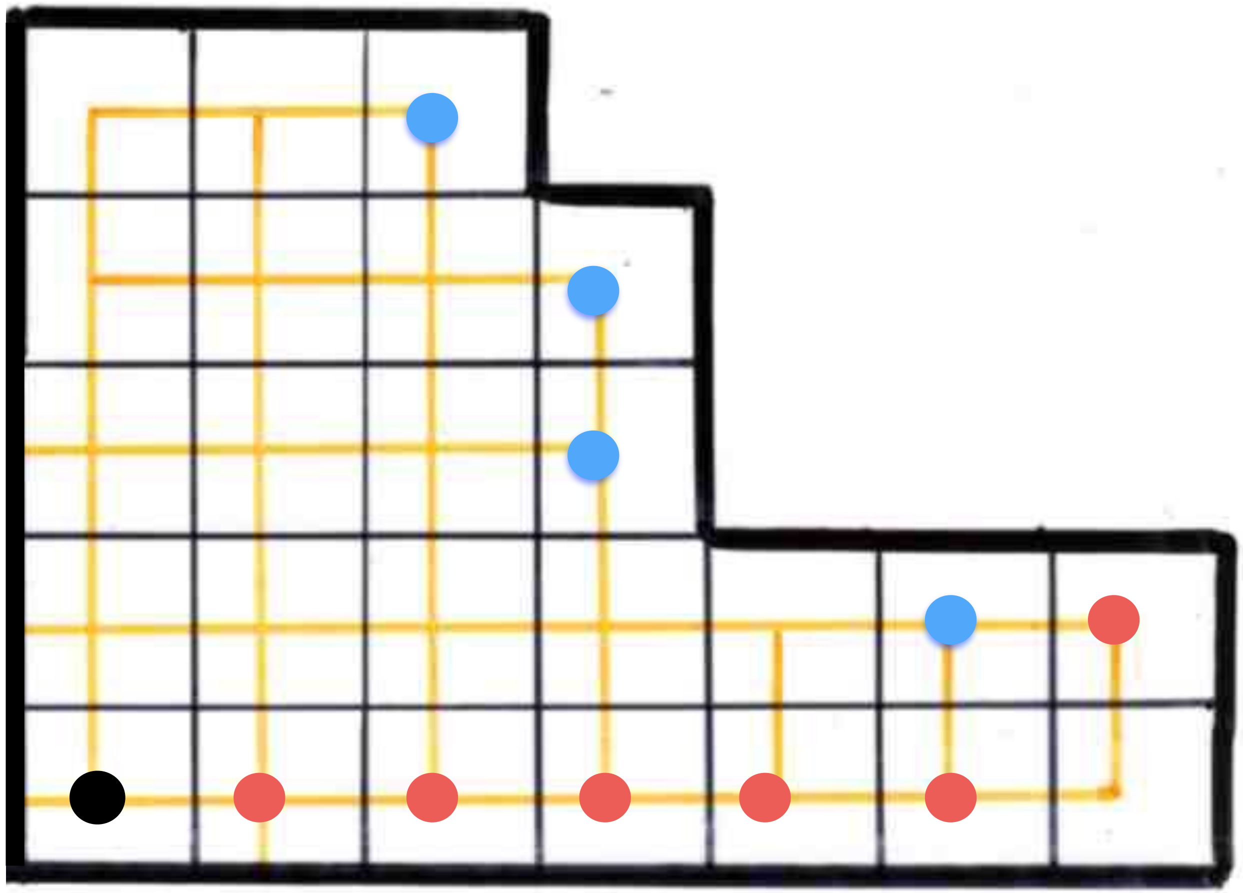


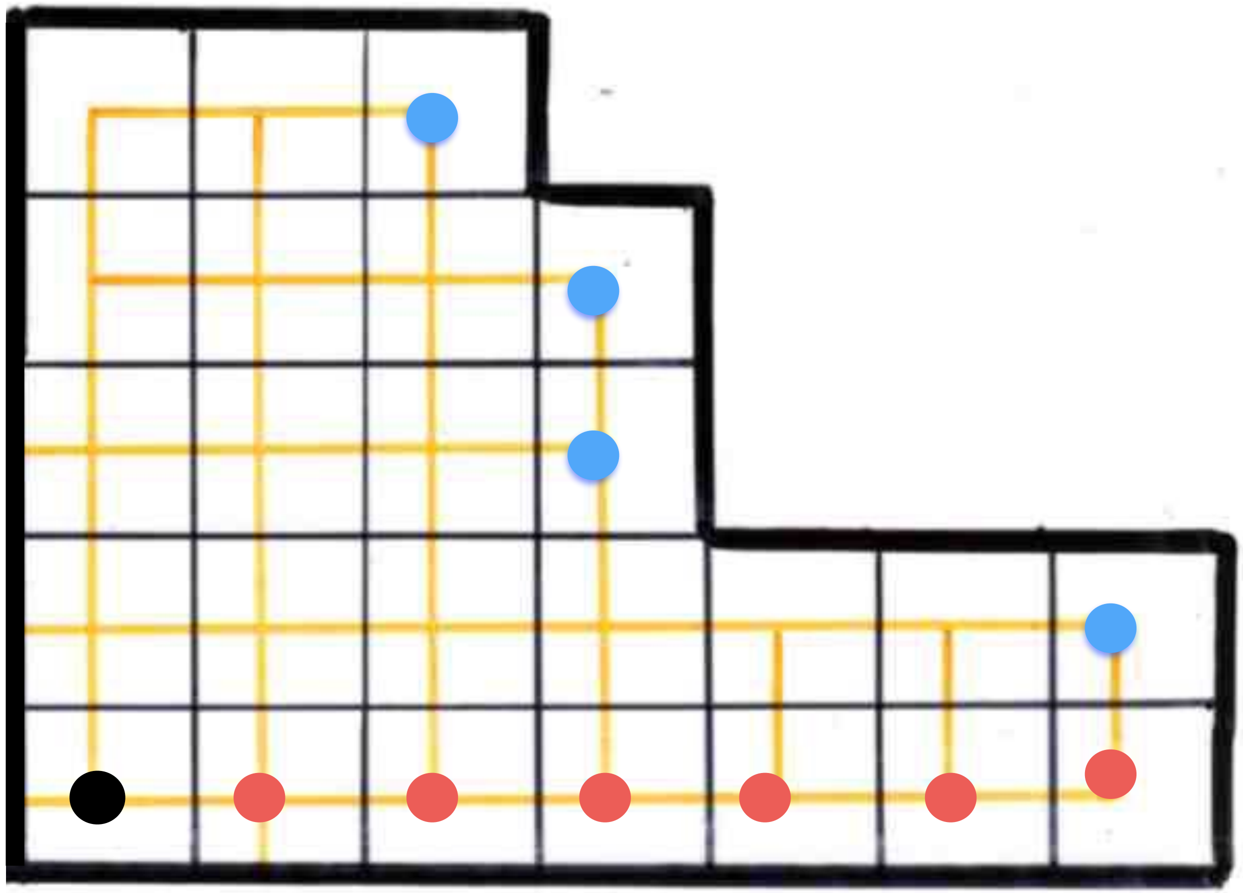


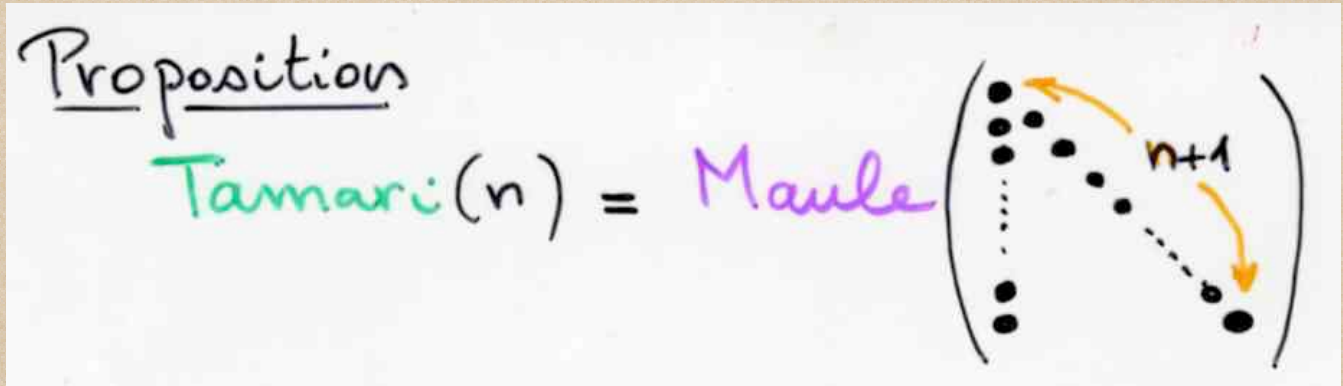








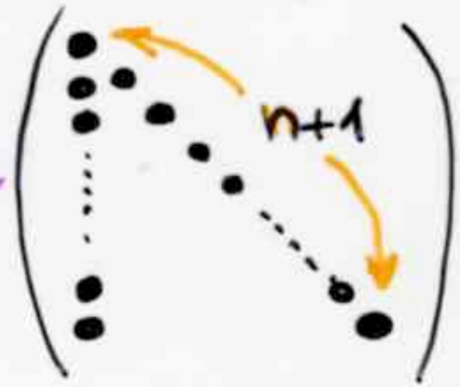


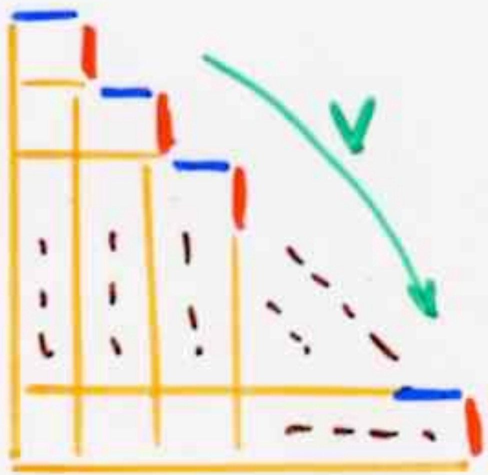


end of the proof

Proposition

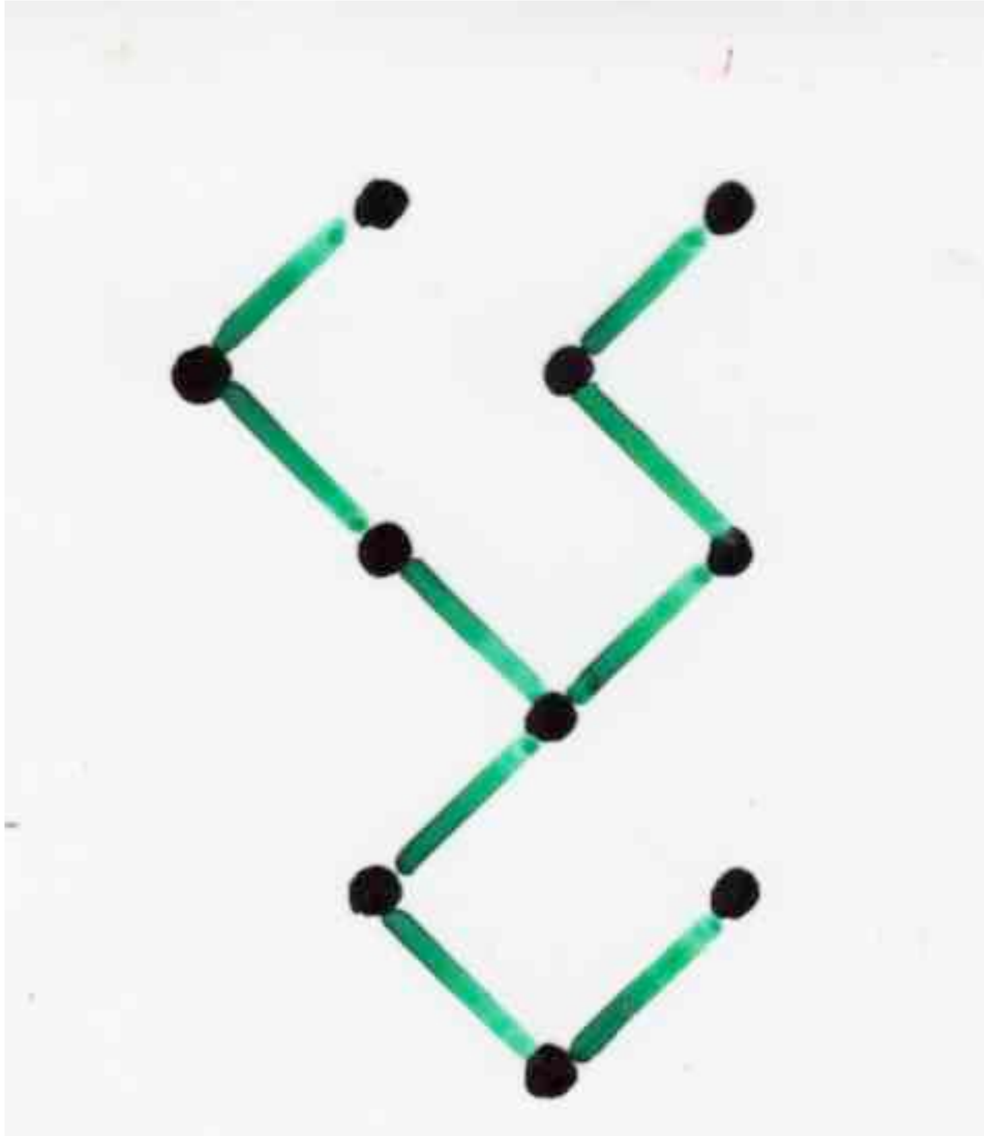
Tamari(n) = Maule

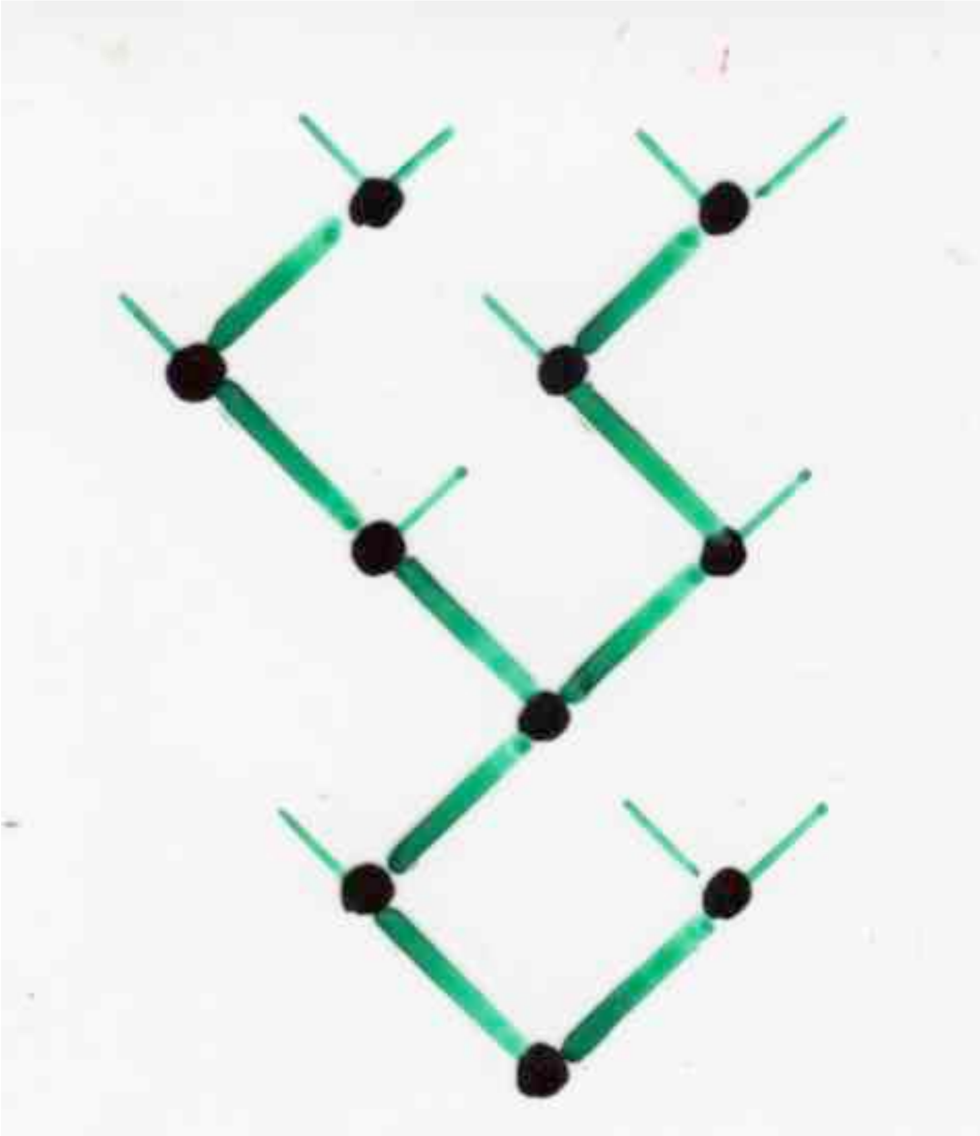


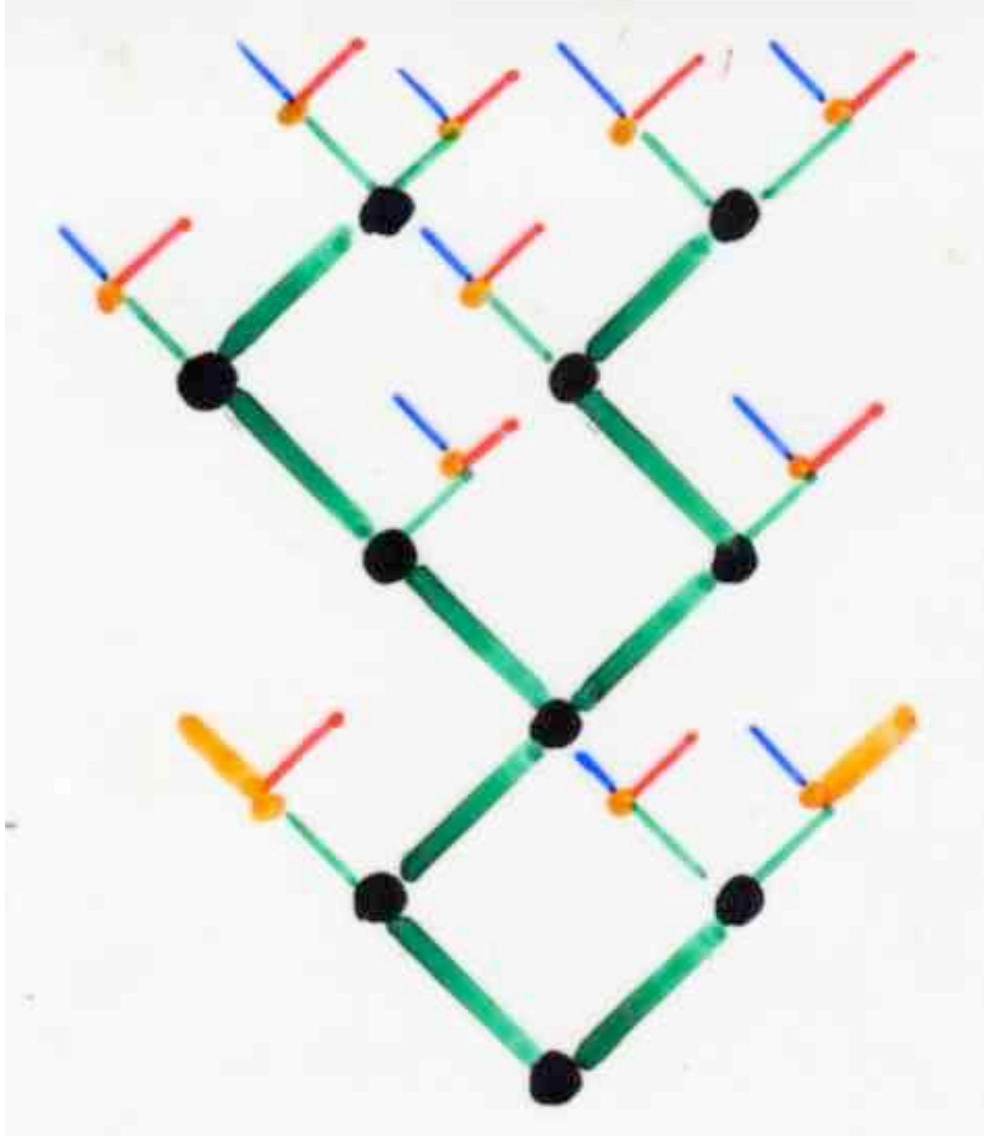


staircase
Catalan
alternative
tableaux

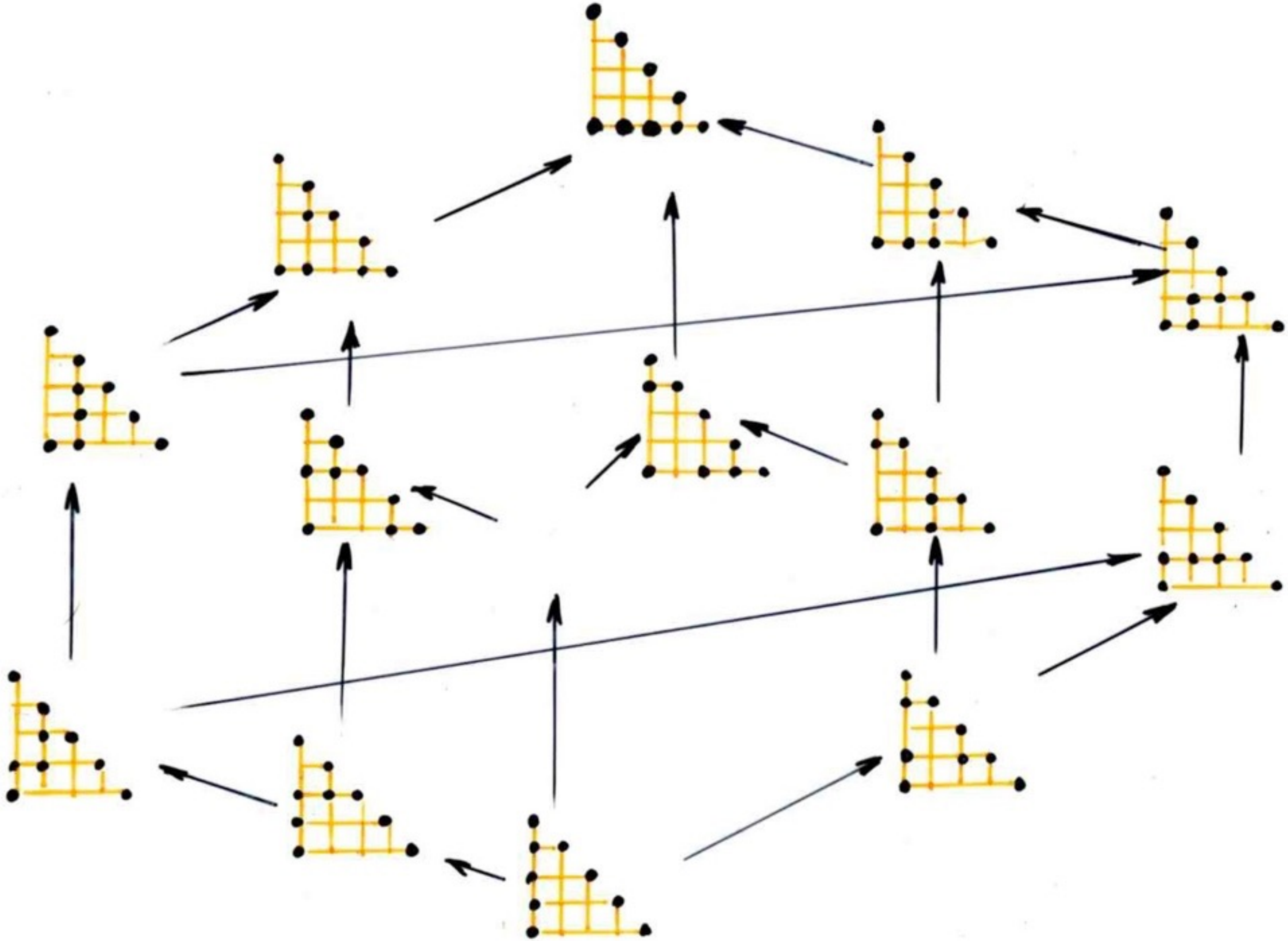
alternating
canopy

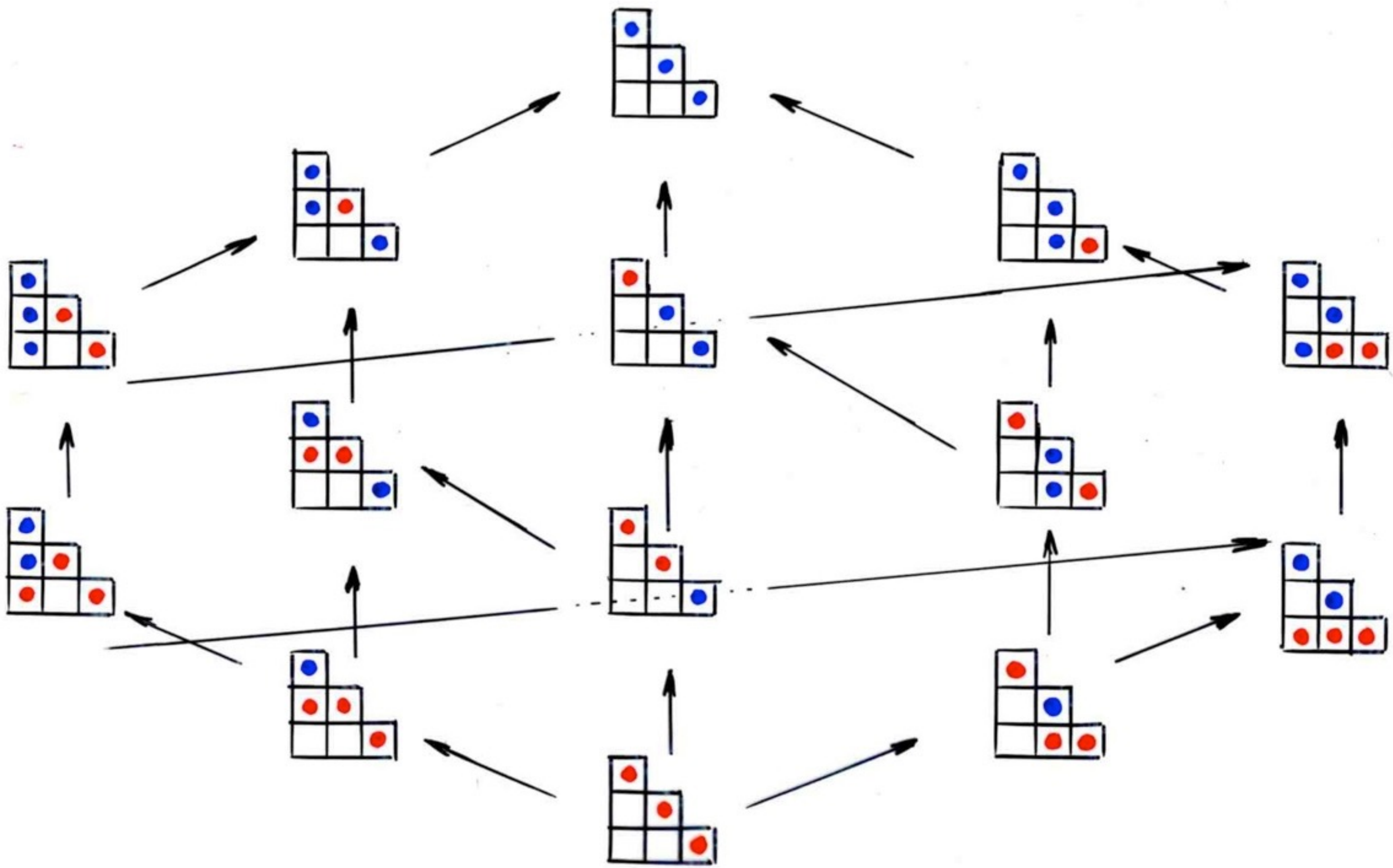


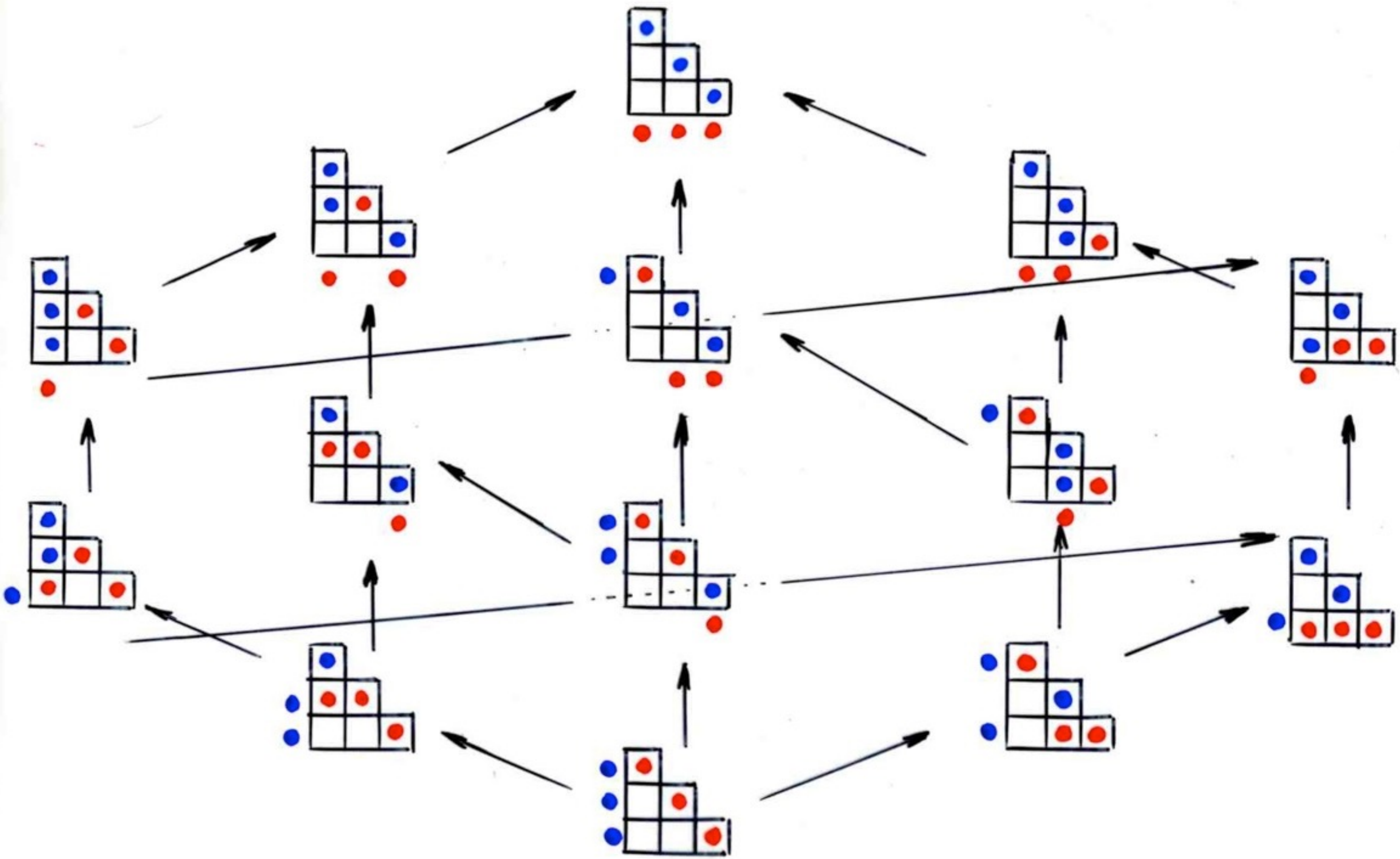


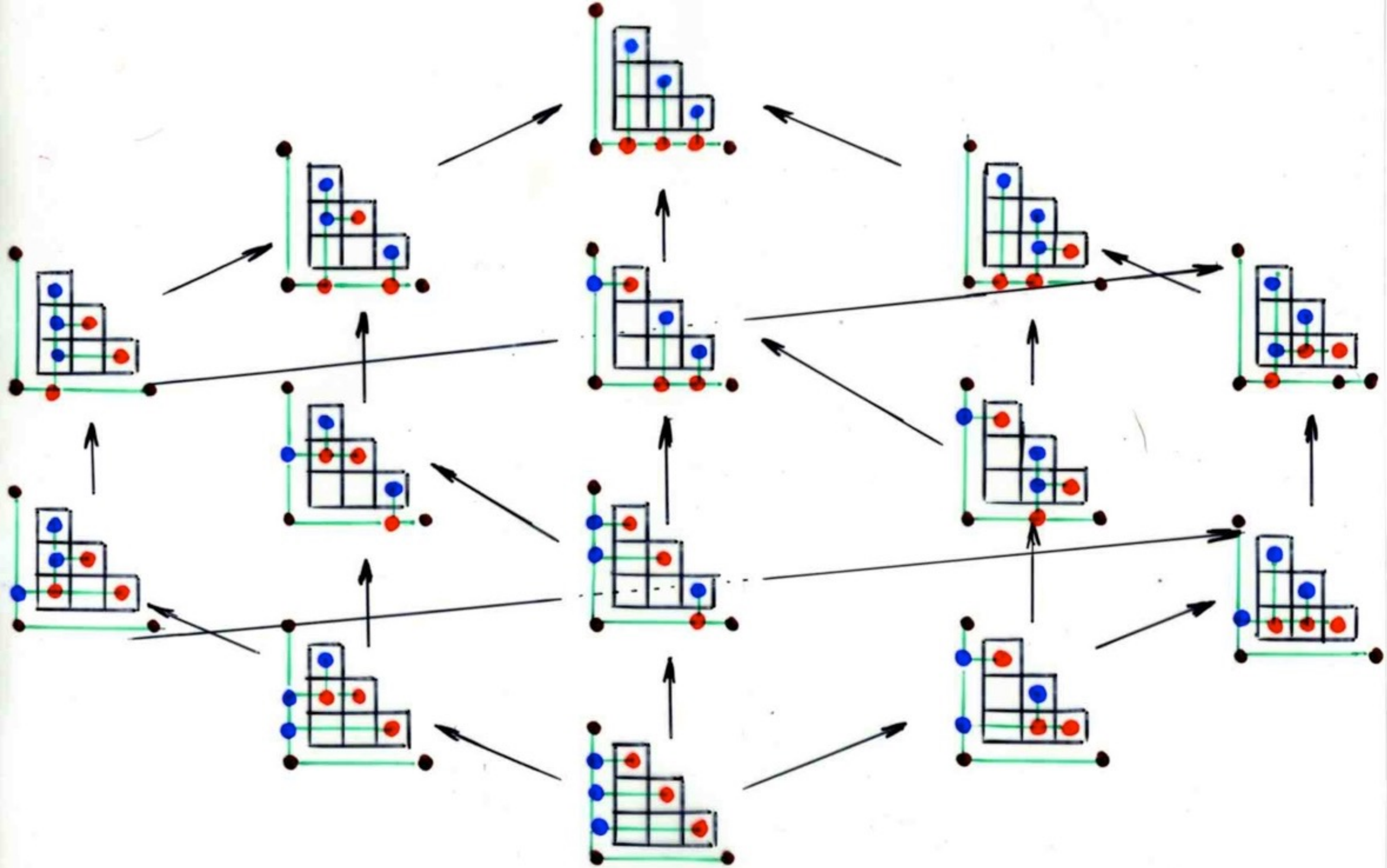


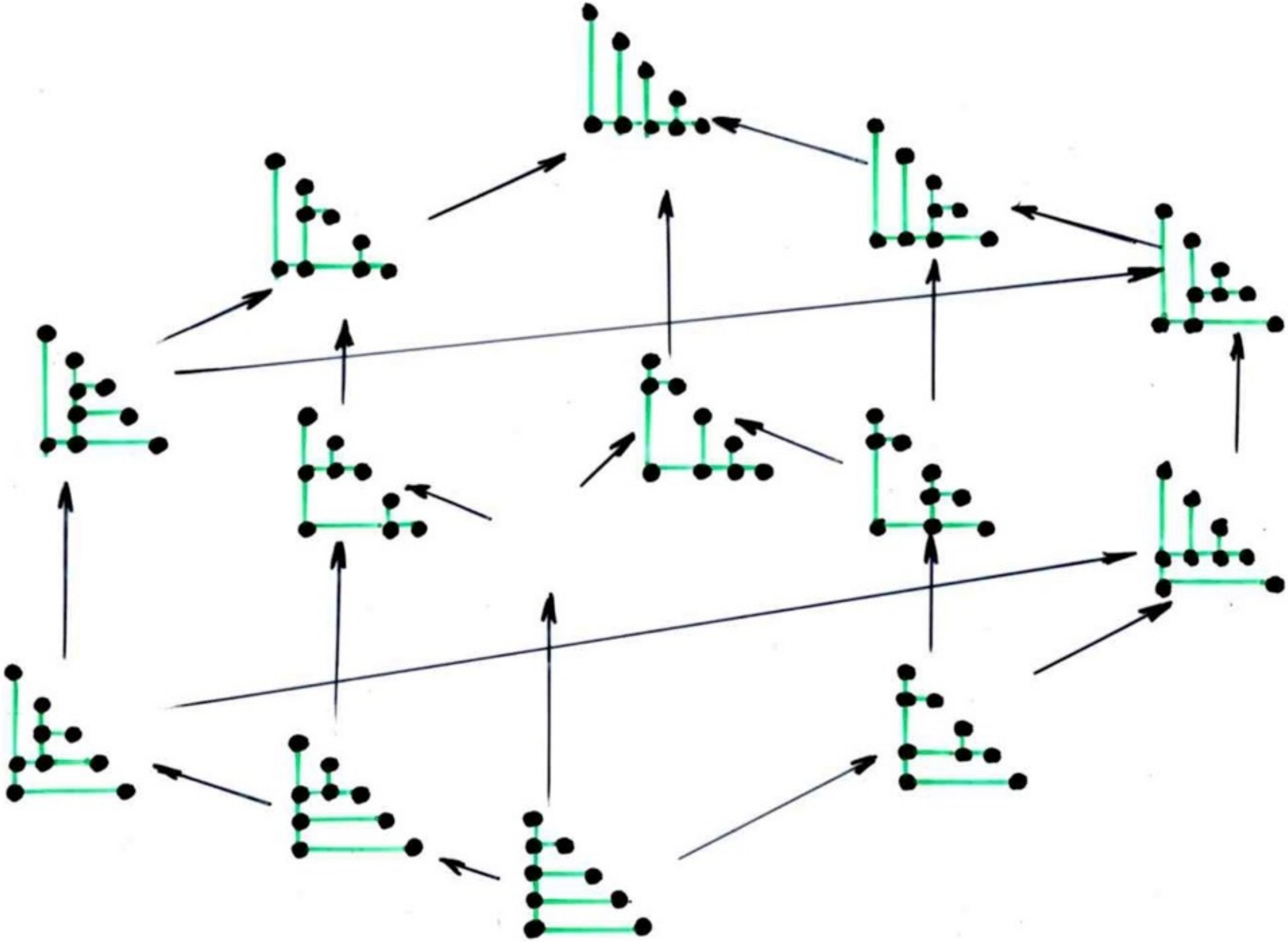
alternating
canopy

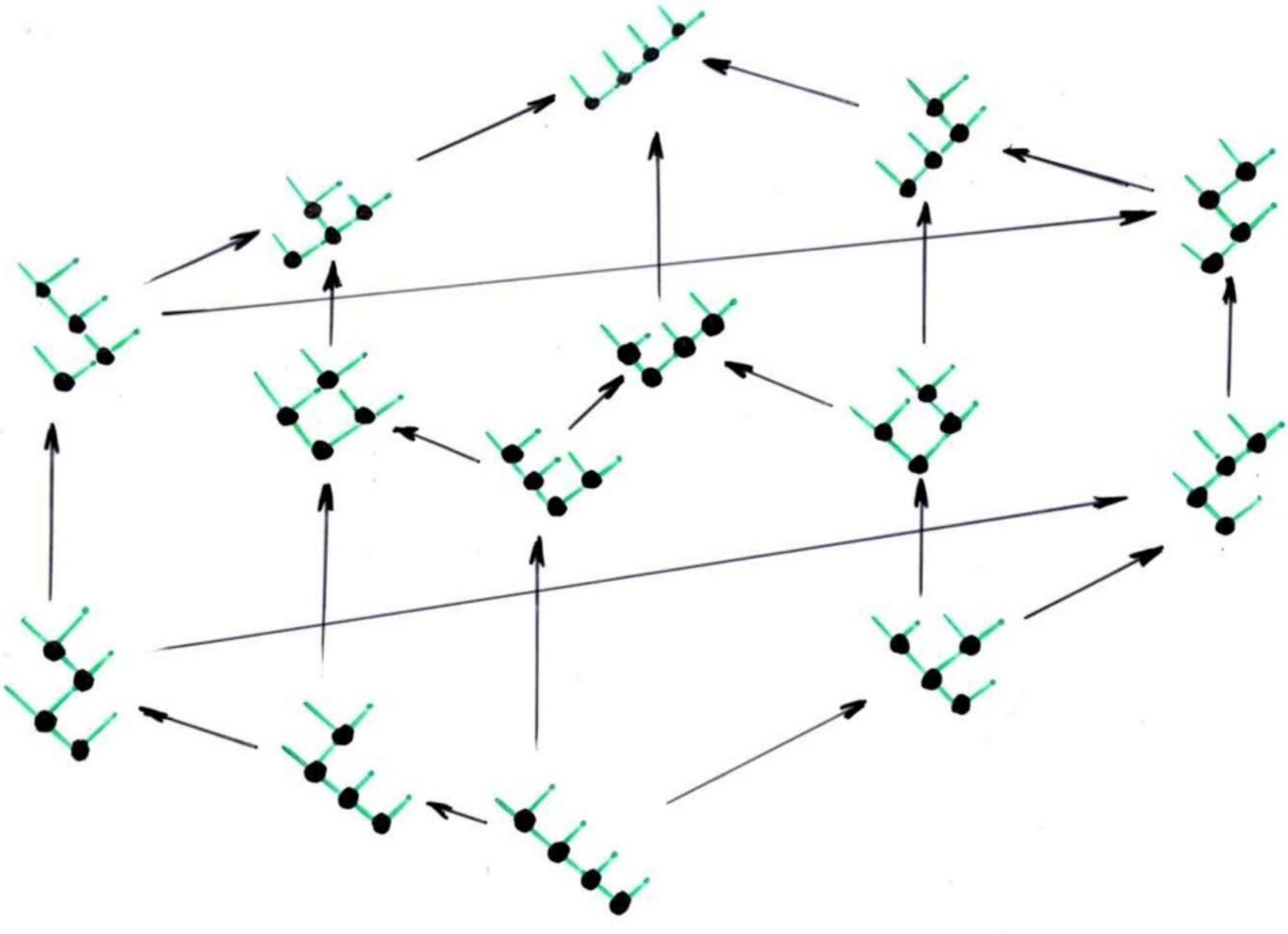








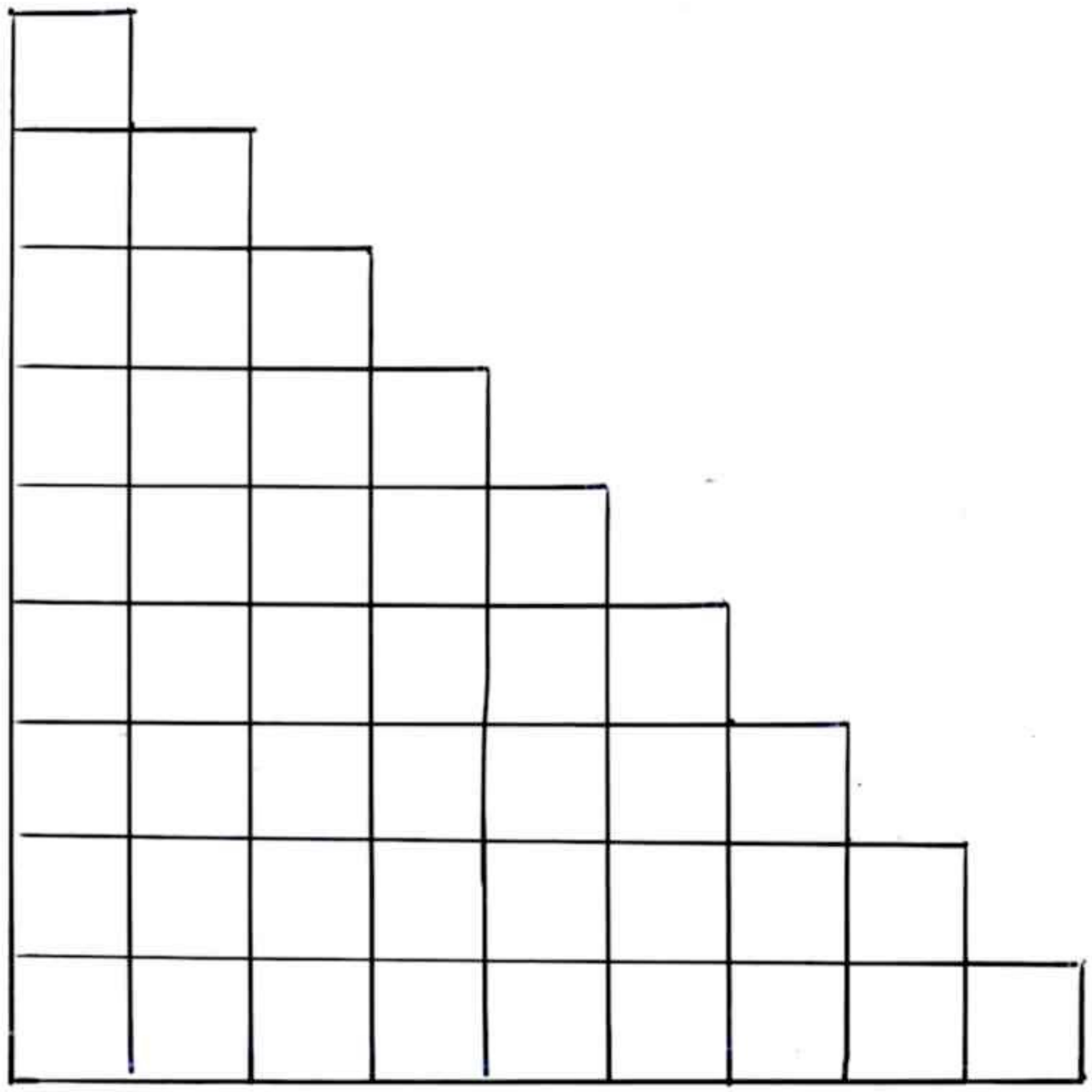


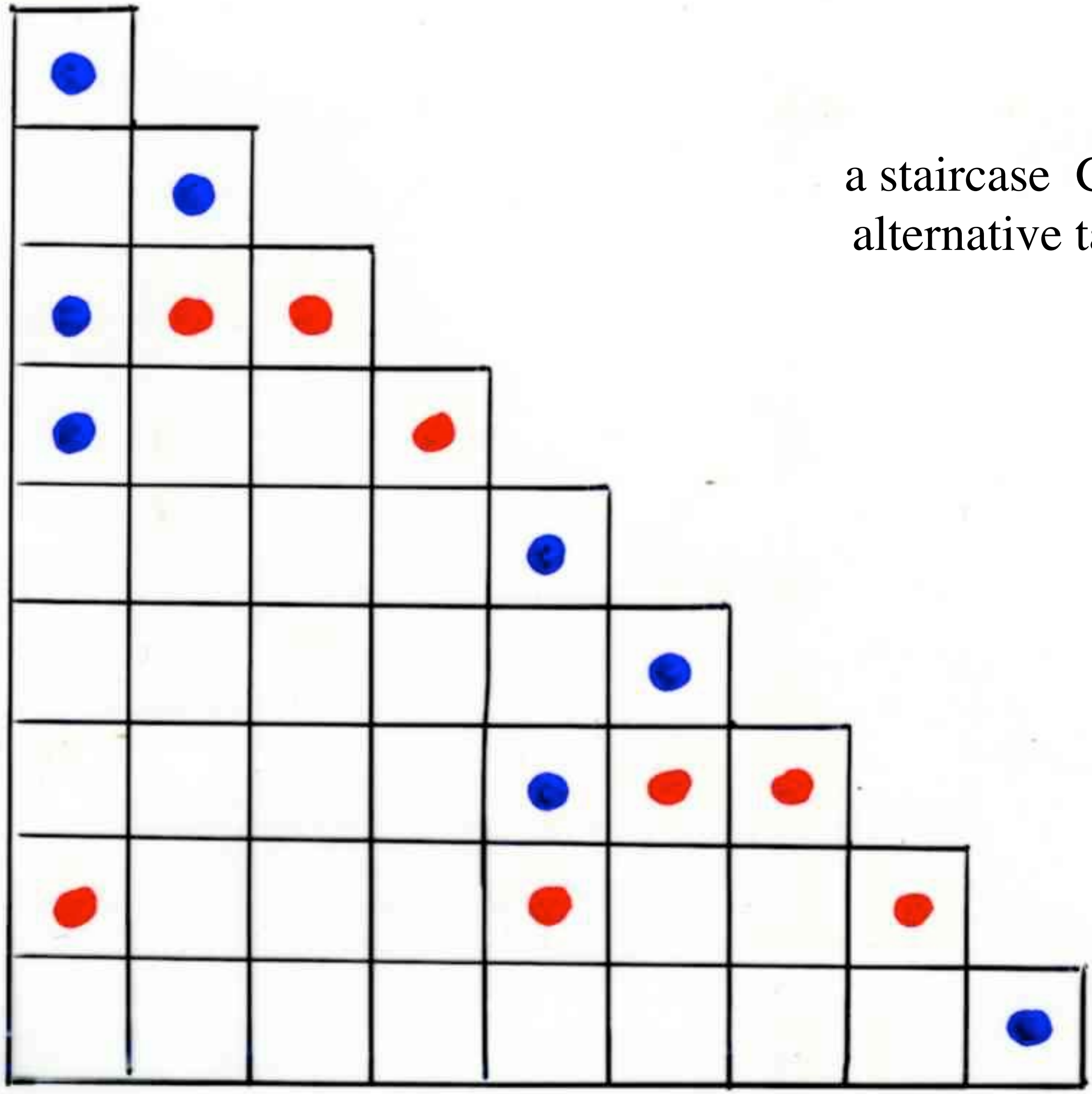


bijection

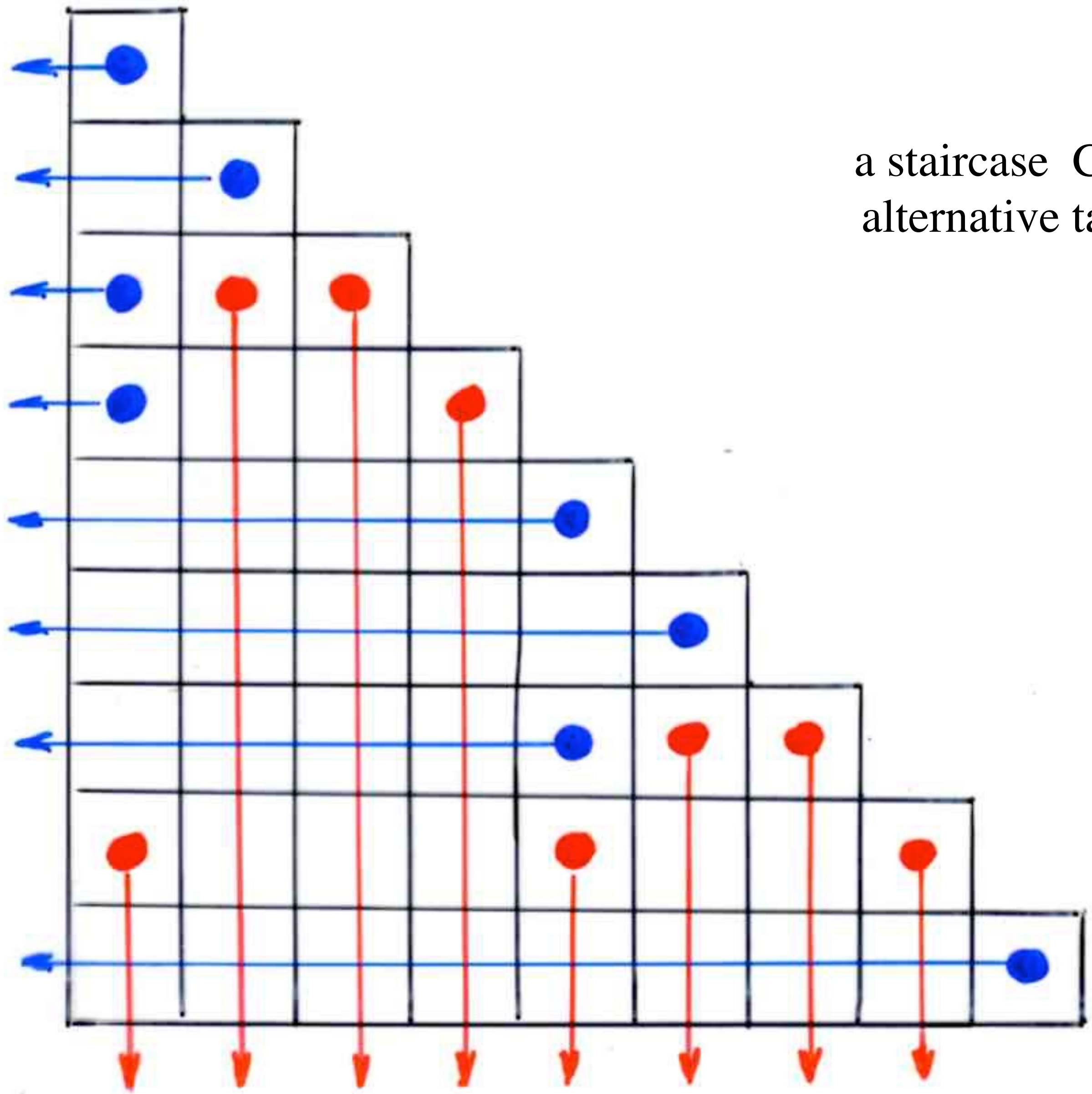
Catalan alternative tableaux

staircase Catalan alternative tableaux

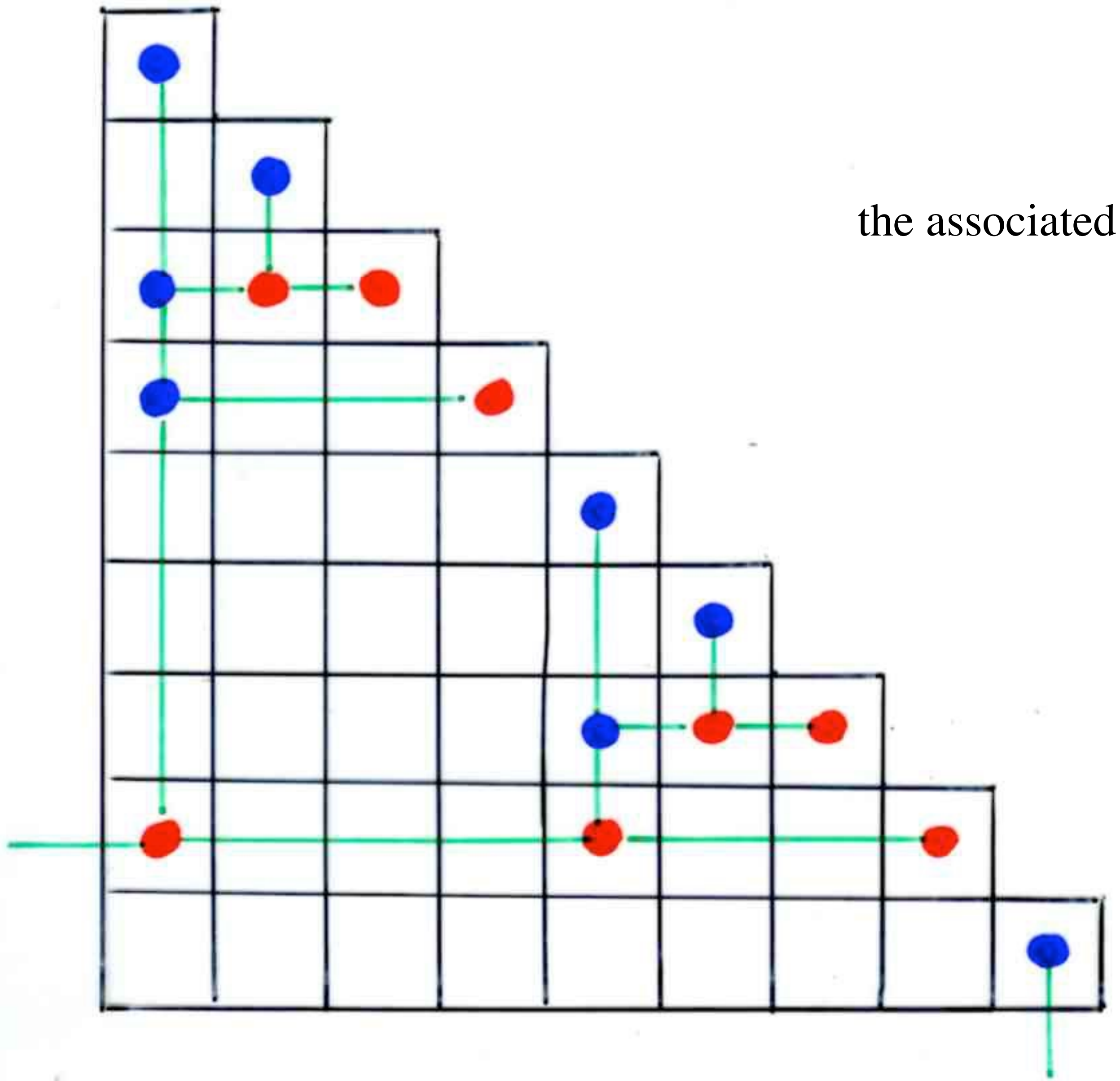




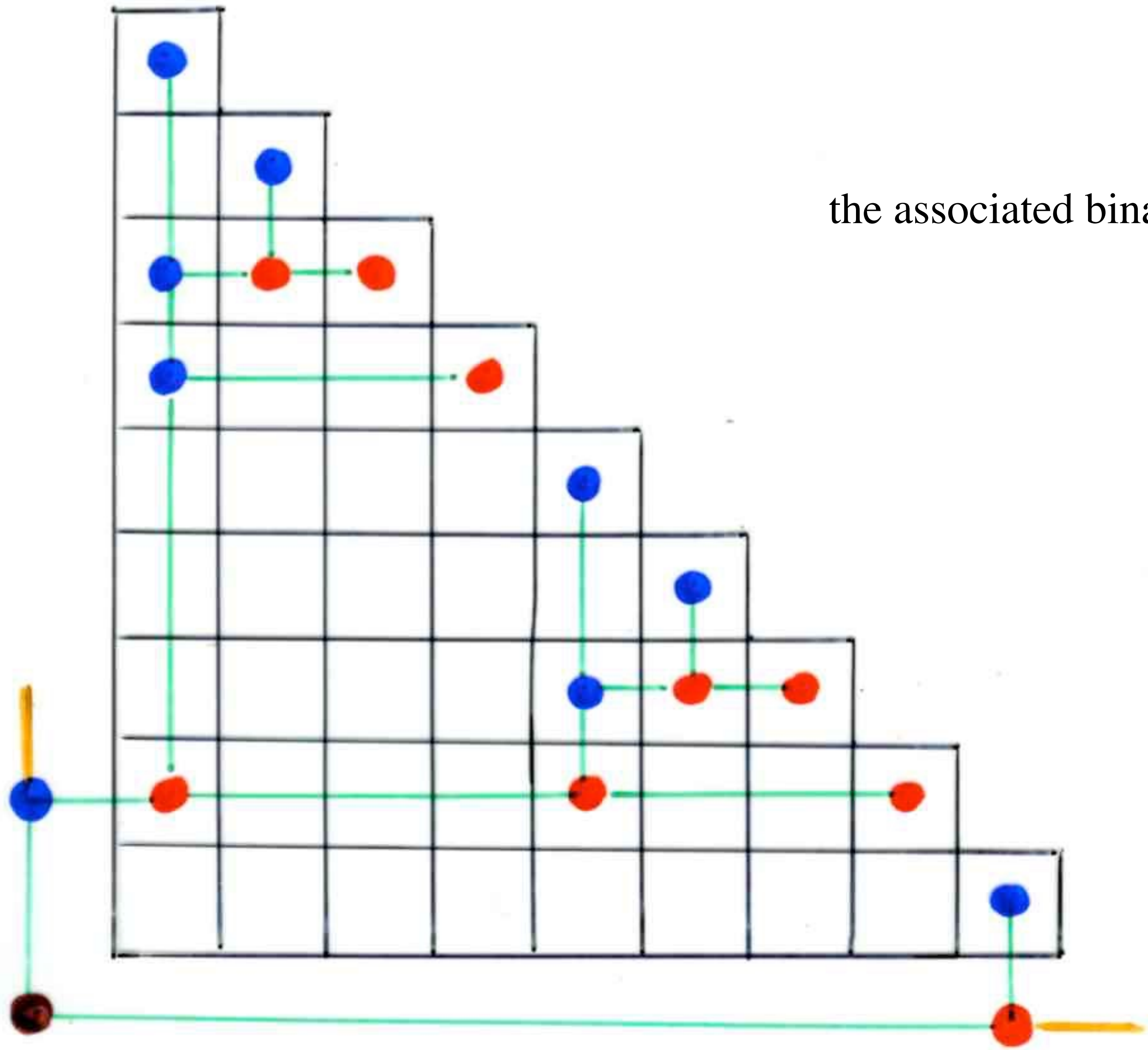
a staircase Catalan
alternative tableau



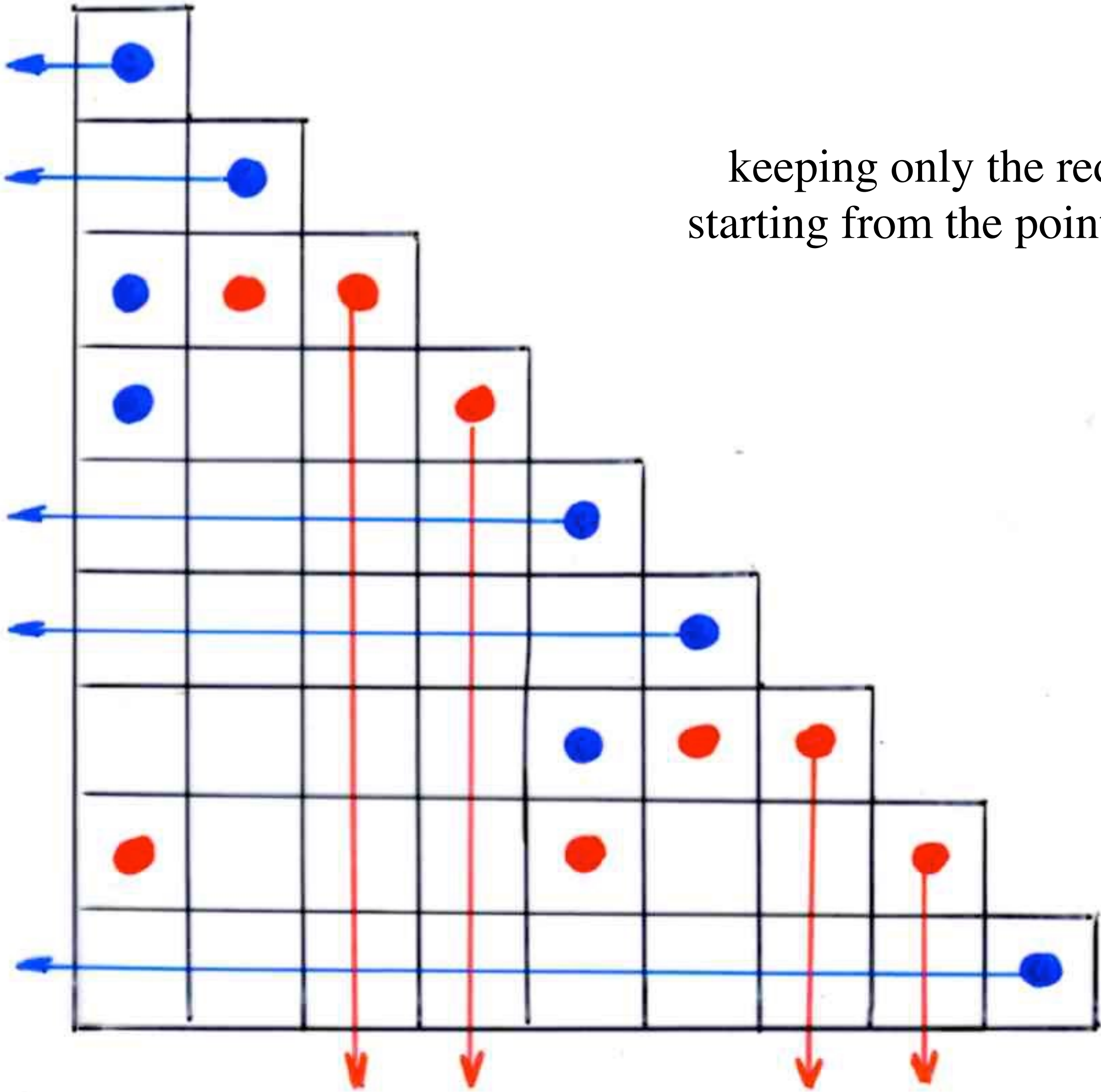
a staircase Catalan
alternative tableau



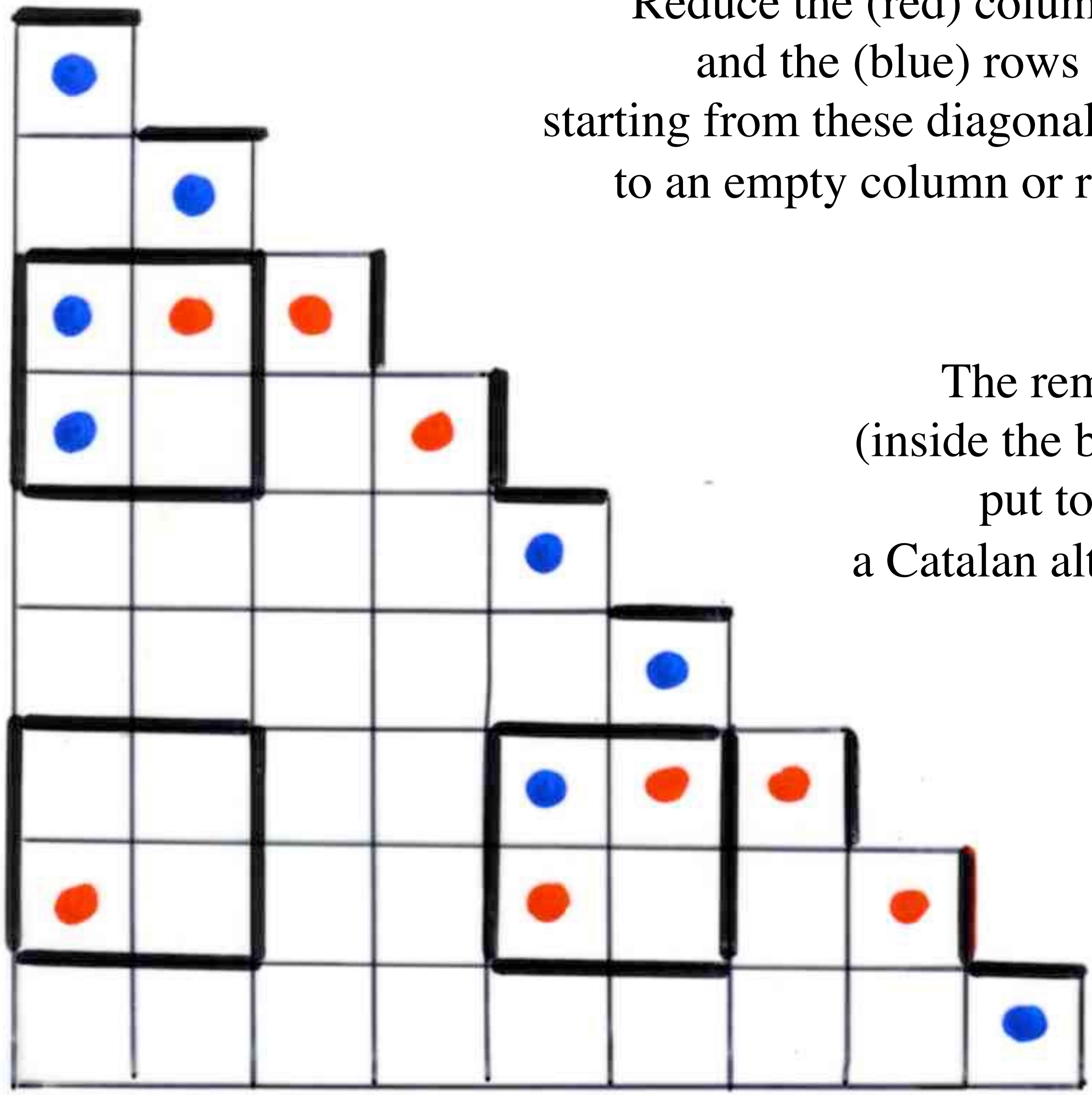
the associated binary tree



the associated binary tree

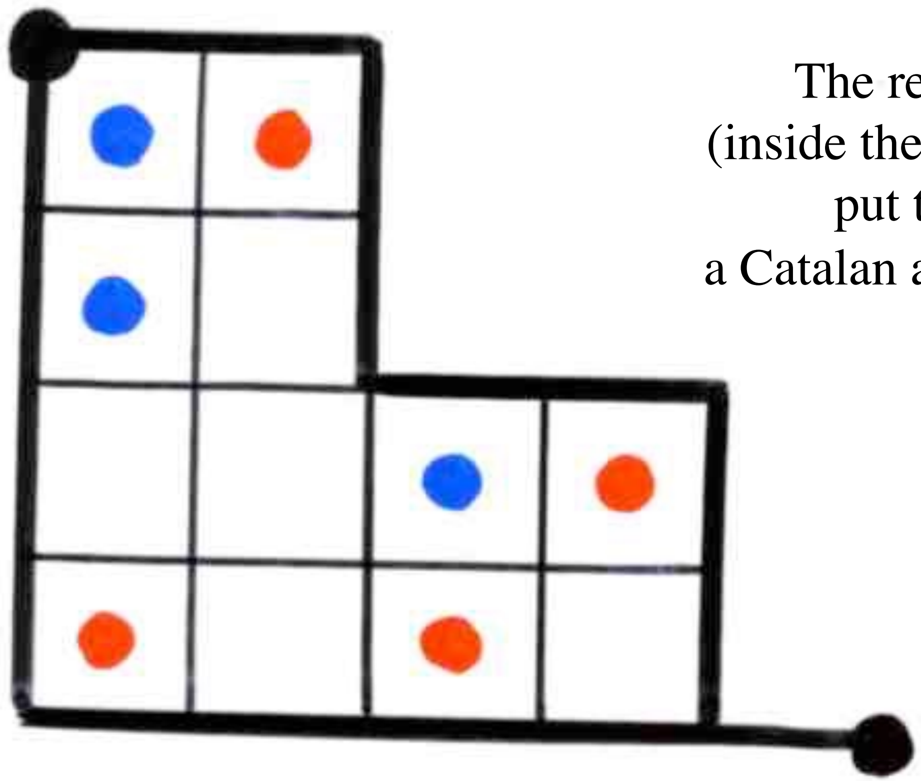


keeping only the red and blue lines
starting from the points on the diagonal

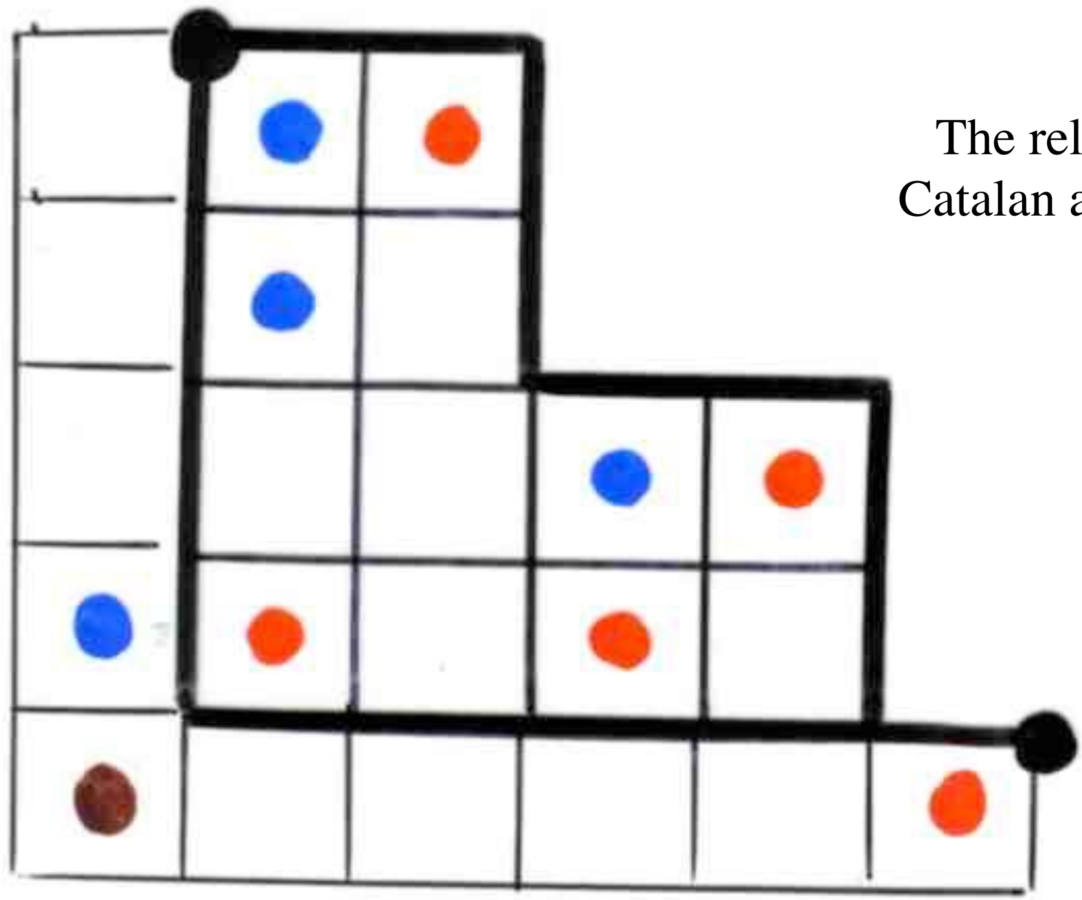


Reduce the (red) columns
and the (blue) rows
starting from these diagonal points
to an empty column or row.

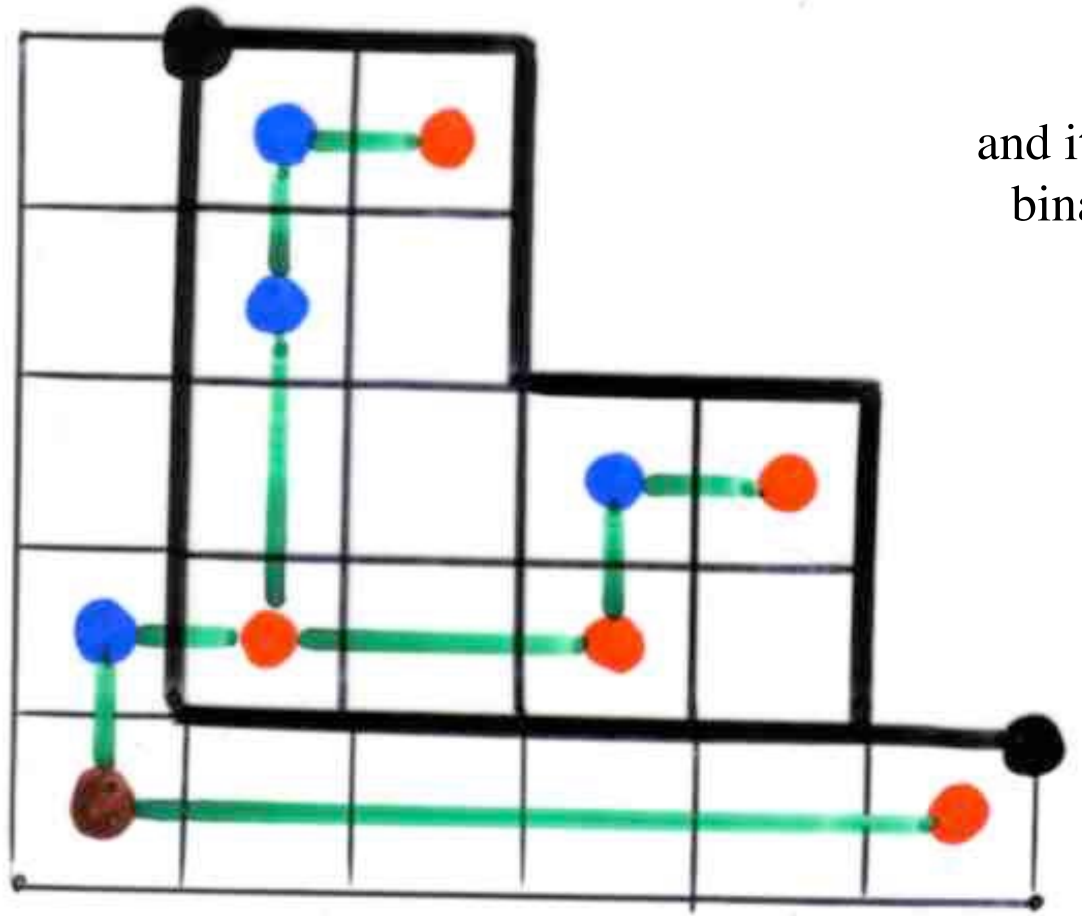
The remaining parts
(inside the black rectangles),
put together give
a Catalan alternative tableau.



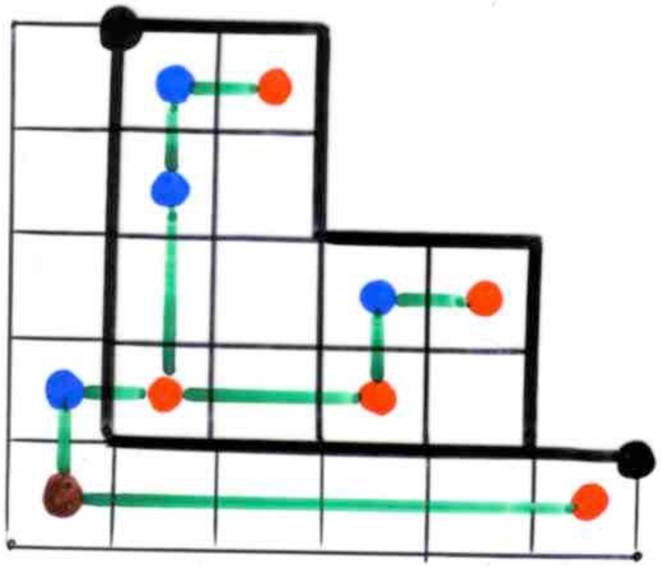
The remaining parts
(inside the black rectangles),
put together give
a Catalan alternative tableau.



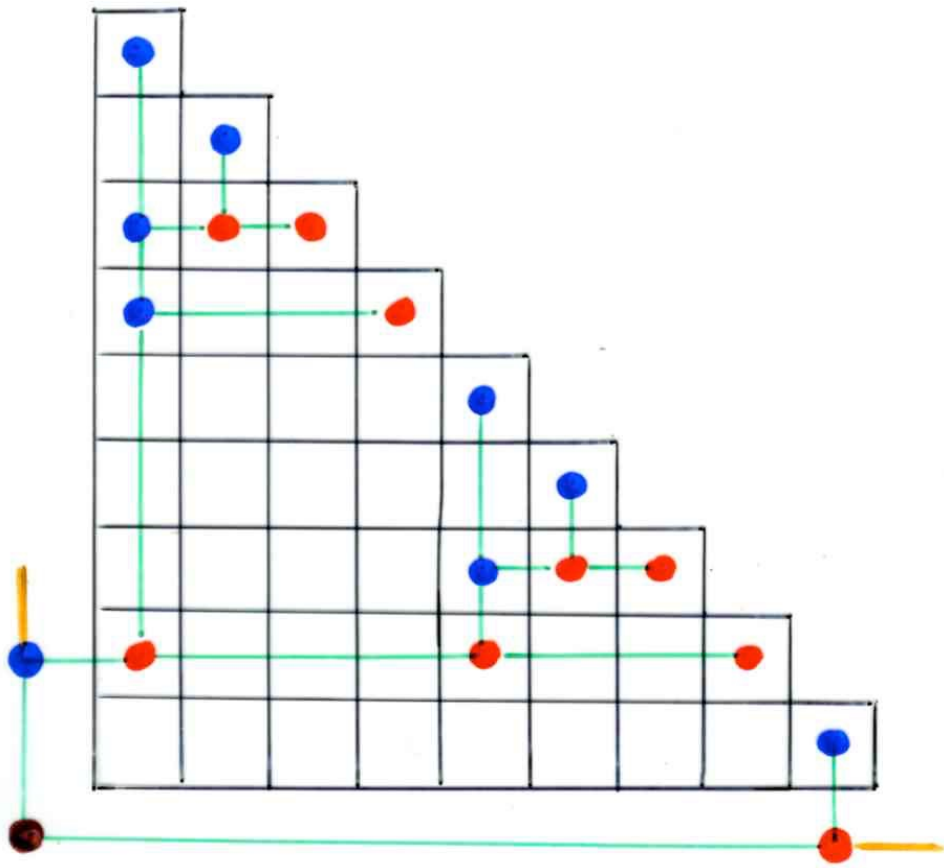
The related augmented
Catalan alternative tableau



and its associated
binary tree B.

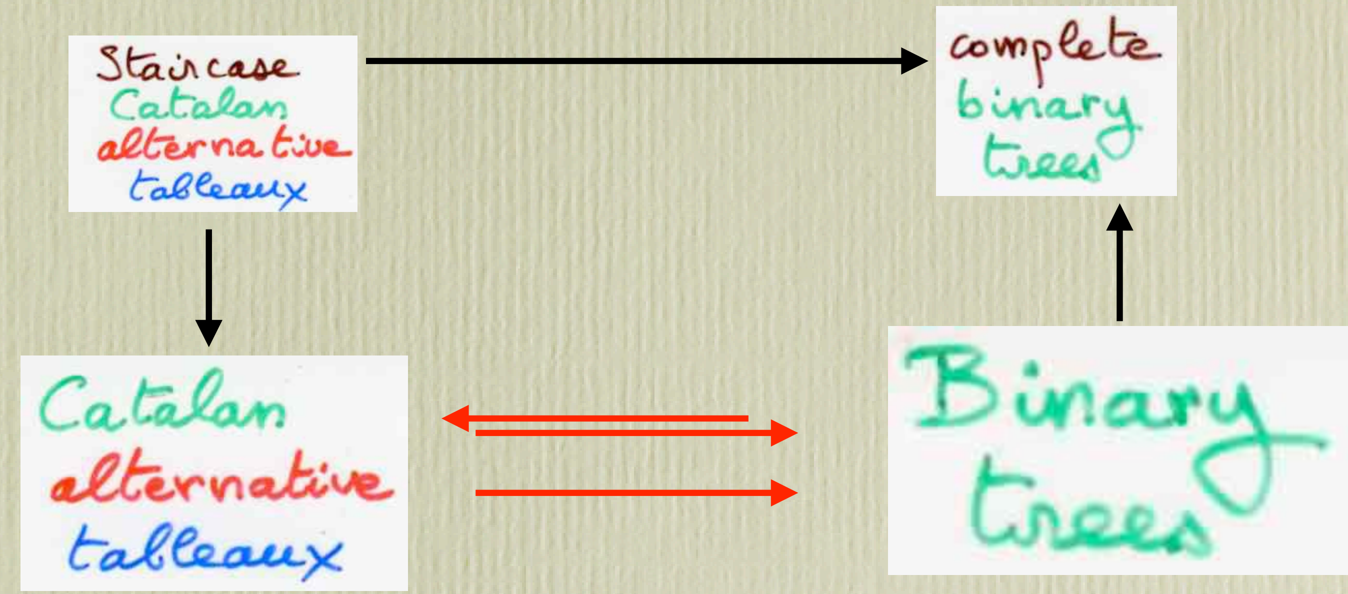


the associated
binary tree B.



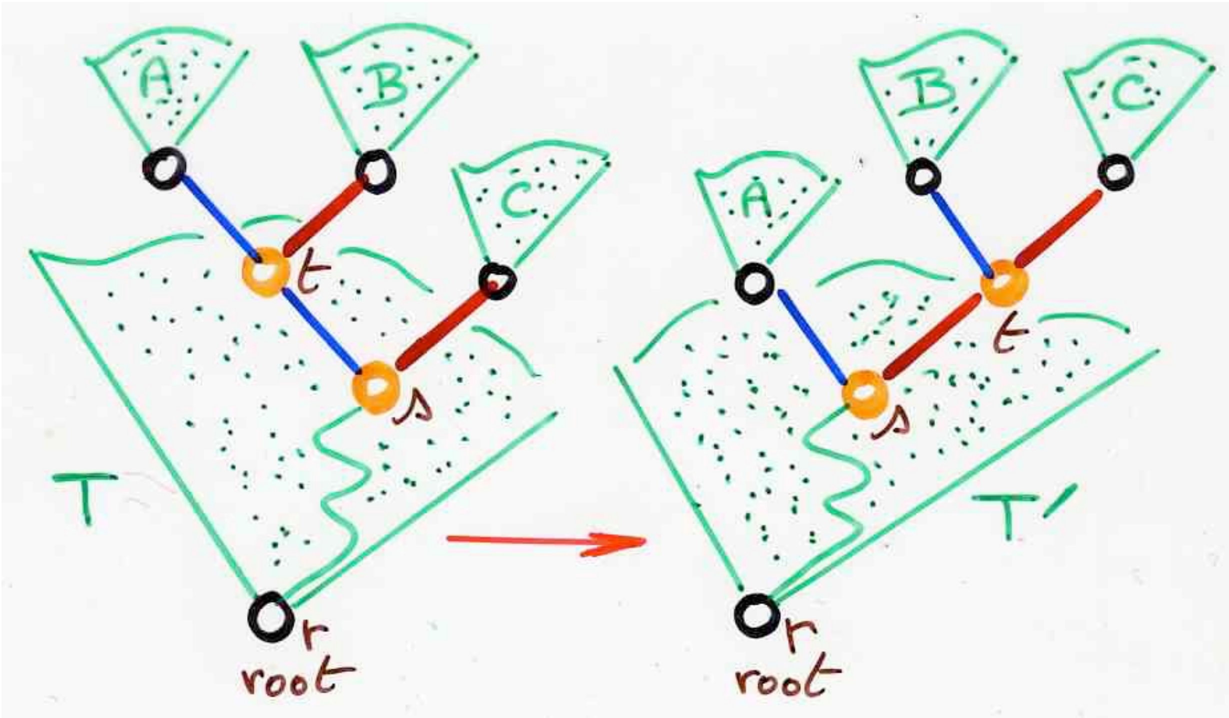
the binary tree associated to the staircase
Catalan alternative tableau
is the extension of the binary tree B

The canopy of B is the word in blue and red obtained by following
downward the diagonal of the staircase Catalan alternative tableau.



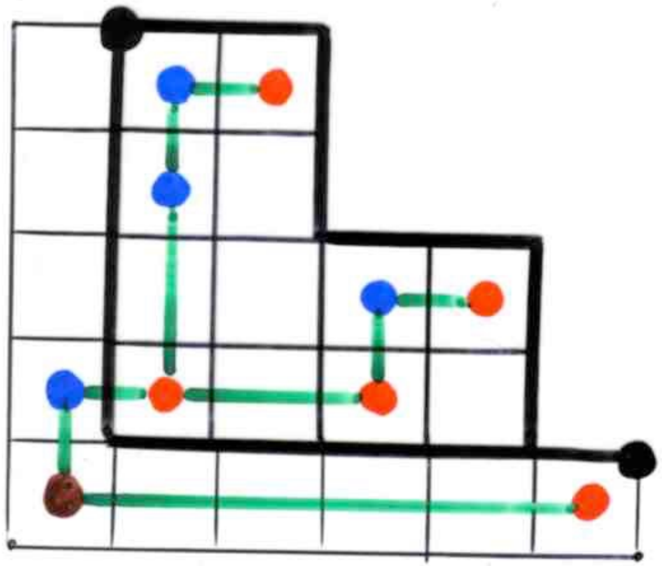
Staircase
Catalan
alternative
tableaux
complete
binary
trees
Catalan
alternative
tableaux
Binary
trees

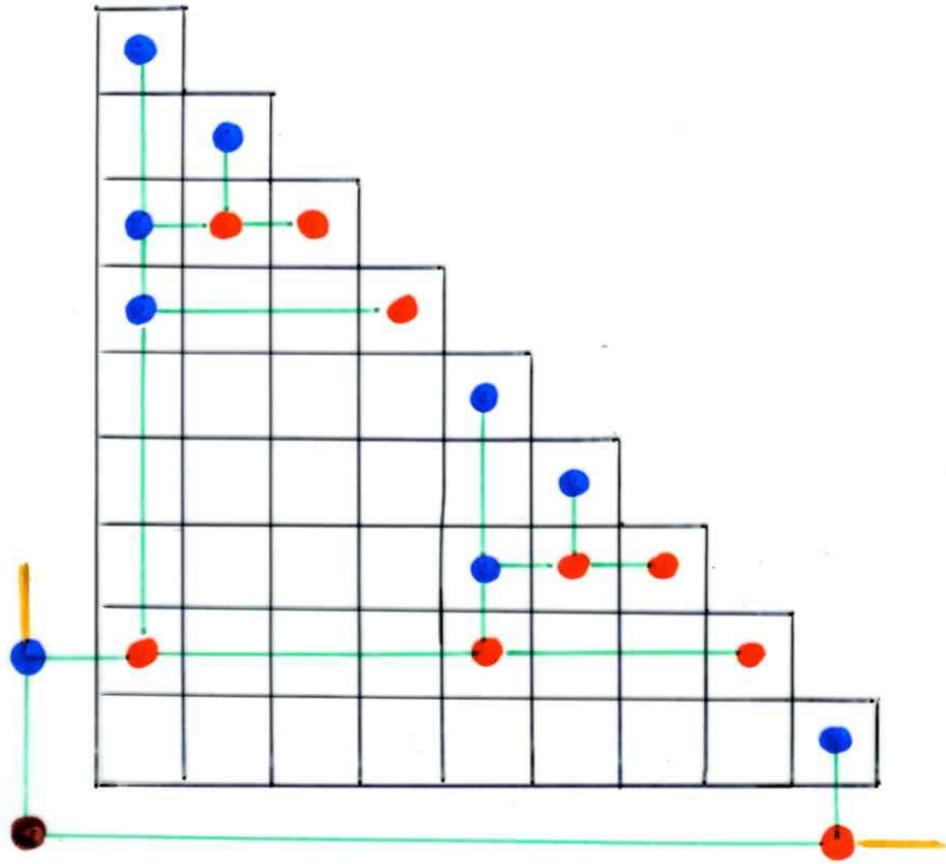
Canopy and rotation in binary trees

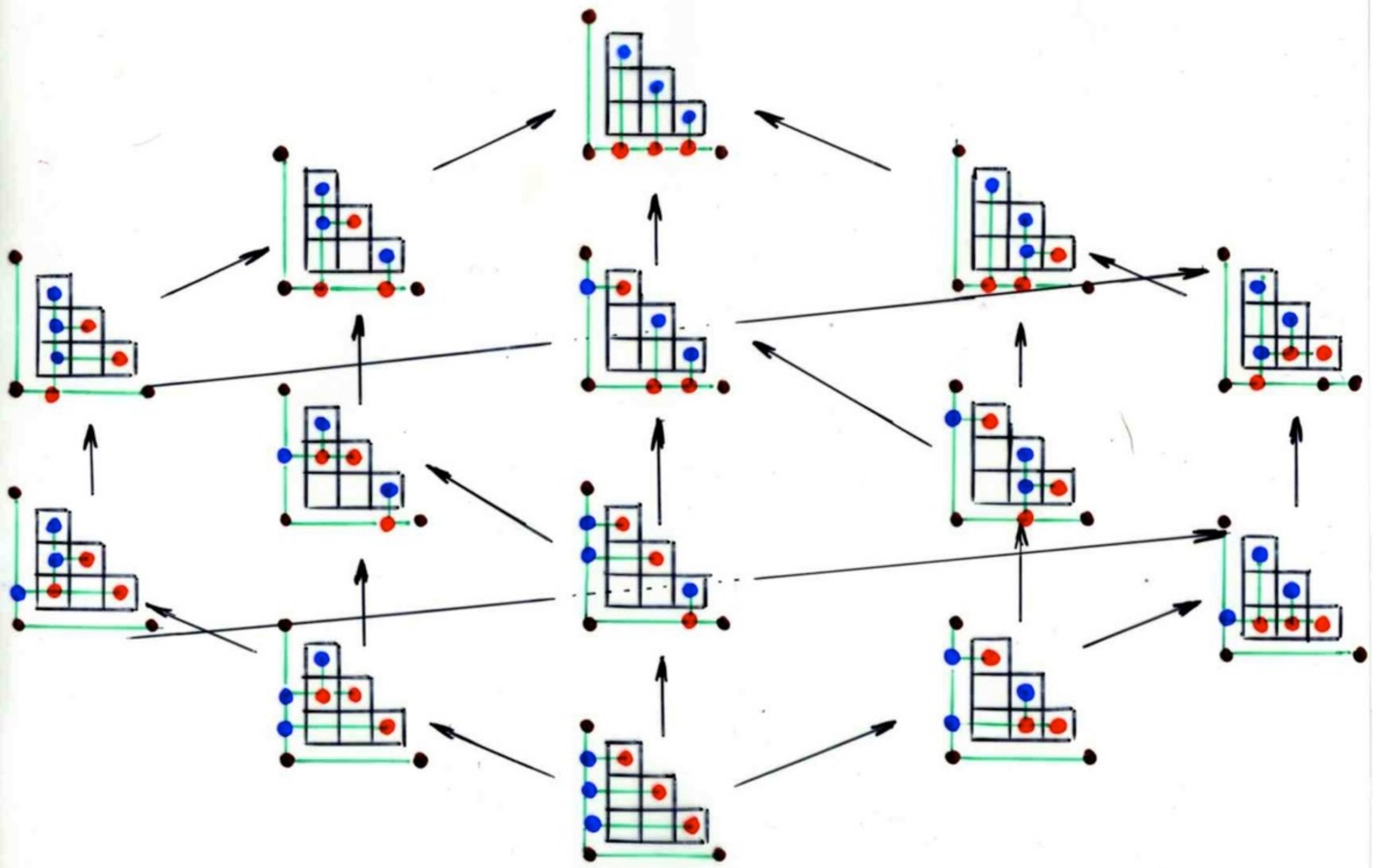


In the rotation the canopy of T is invariant if and only if the binary subtree B is not reduced to a single vertex. If B is reduced to a single vertex, the canopy of T' is deduced from the canopy of T by changing one edge to the right (red) into an edge to the right (blue).

In the associated Catalan staircase alternating tableau (see slide 153), this corresponds to a Γ-move where the rectangle is touching the diagonal.

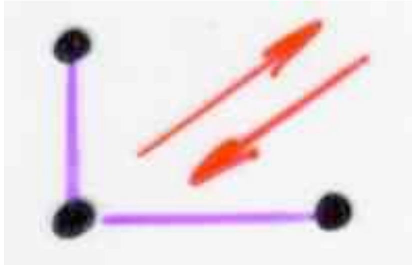






comments and remarks

Lam, Williams (2008)
total positivity for cominuscule Grassmannians



┘- move

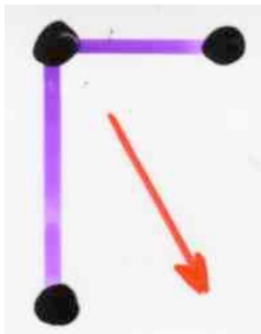
┐- move

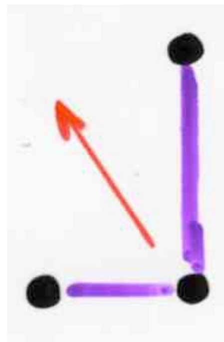
L- move

Karp, Williams, Zhang (2017)
decompositions of amplituhedra
m=4 scattering amplitudes in $\mathcal{N}=4$ supersymmetric Yang-Mills theory

┘- move

Γ- move

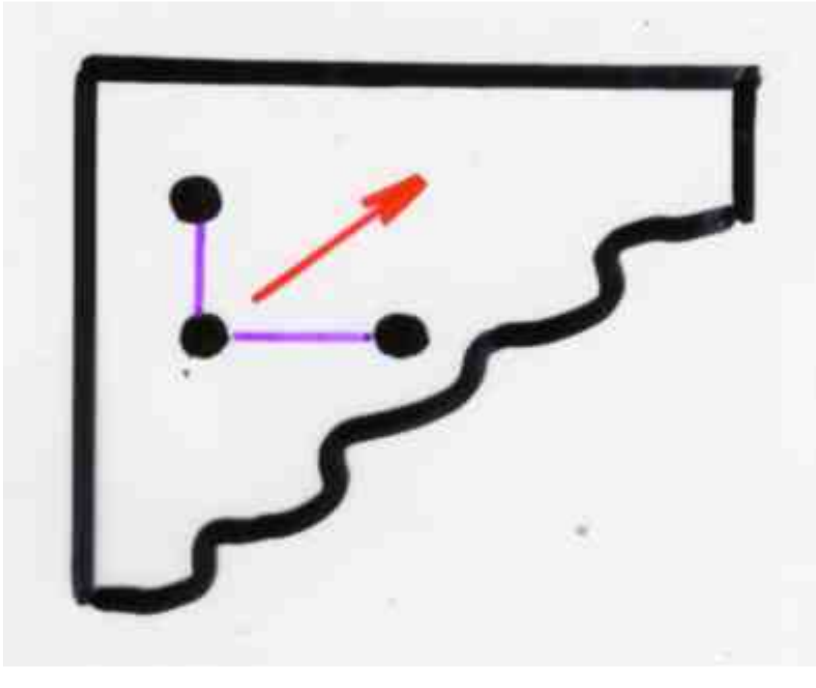




N. Bergeron, S. Billey (2010)
RC-graphs and Schubert polynomials

M. Rubey (2010)
Maximal 0-1 fillings of moon polyominoes
with restricted chain length and RC-graphs

chute move



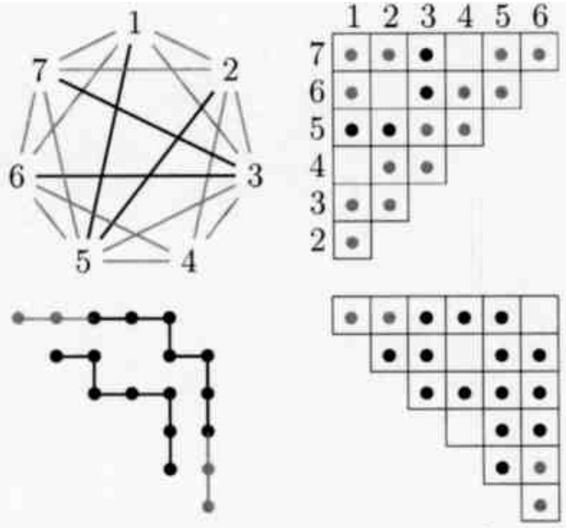
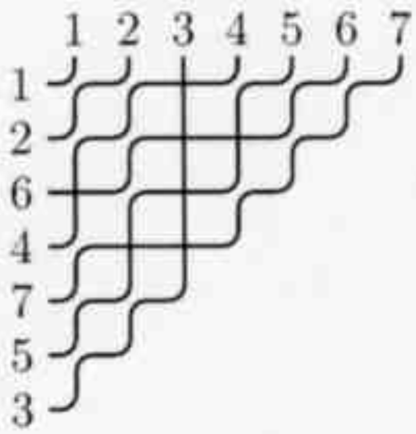


FIGURE 2. a 2-triangulation with corresponding filling of the staircase λ_0 and a fan of two Dyck paths with corresponding filling of the reverse staircase λ_0^{rev}.

M.Rubey (2010)



	1	2	3	4	5	6	7
1	•	•	+	•	•	•	•
2	•	•	+	+	•	•	
6	+	•	+	•	•		
4	•	+	+	•			
7	•	•	•				
5	•	•					
3	•						

	1	2	3	4	5	6	7
1	•	•	3	•	•	•	•
2	•	•	4	5	•	•	
6	3	•	5	•	•		
4	•	5	6	•			
7	•	•	•				
5	•	•					
3	•						

FIGURE 3. the reduced pipe dream associated to the reduced factorisation $s_3 s_5 s_4 s_5 s_3 s_6 s_5$ of $1, 2, 6, 4, 7, 5, 3$.

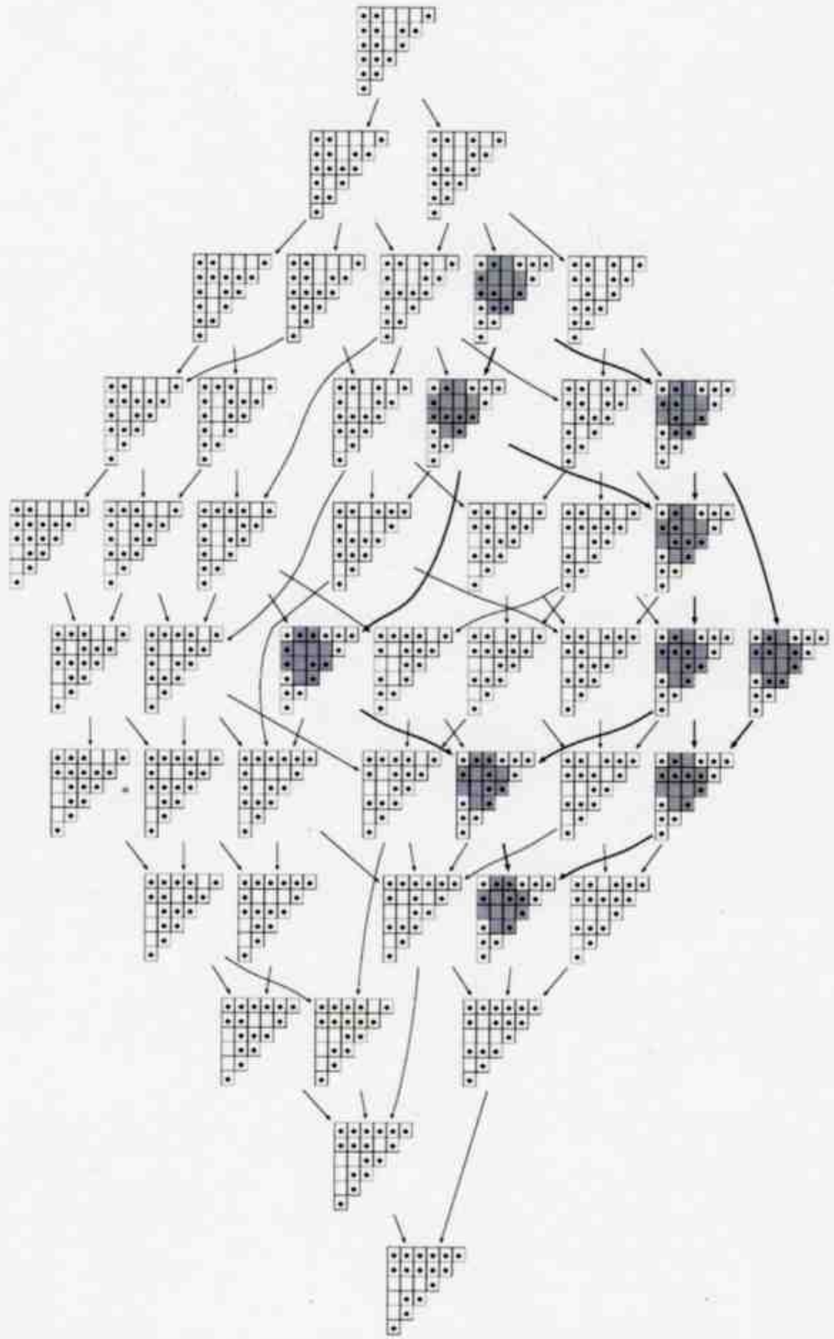


FIGURE 4. the poset of reduced pipe dreams for the permutation 1, 2, 6, 4, 5, 3. The interval of 0-1-fillings with $k = 1$ of the moon polyomino is emphasised.

M.Rubey (2010)

conjecture:
this maule is a lattice

number of maximal chains in Tamari(n) ?

Nelson (2016) Ph.D.

maximal chain in a poset

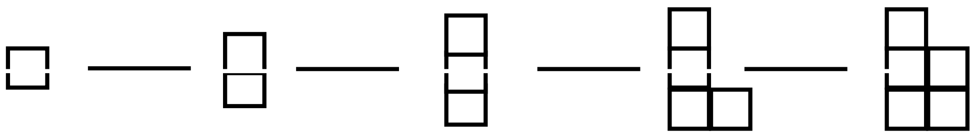
$\alpha_1 \lessdot \alpha_2 \lessdot \cdots \lessdot \alpha_k$

each α_{i+1} is covering α_k

bijection

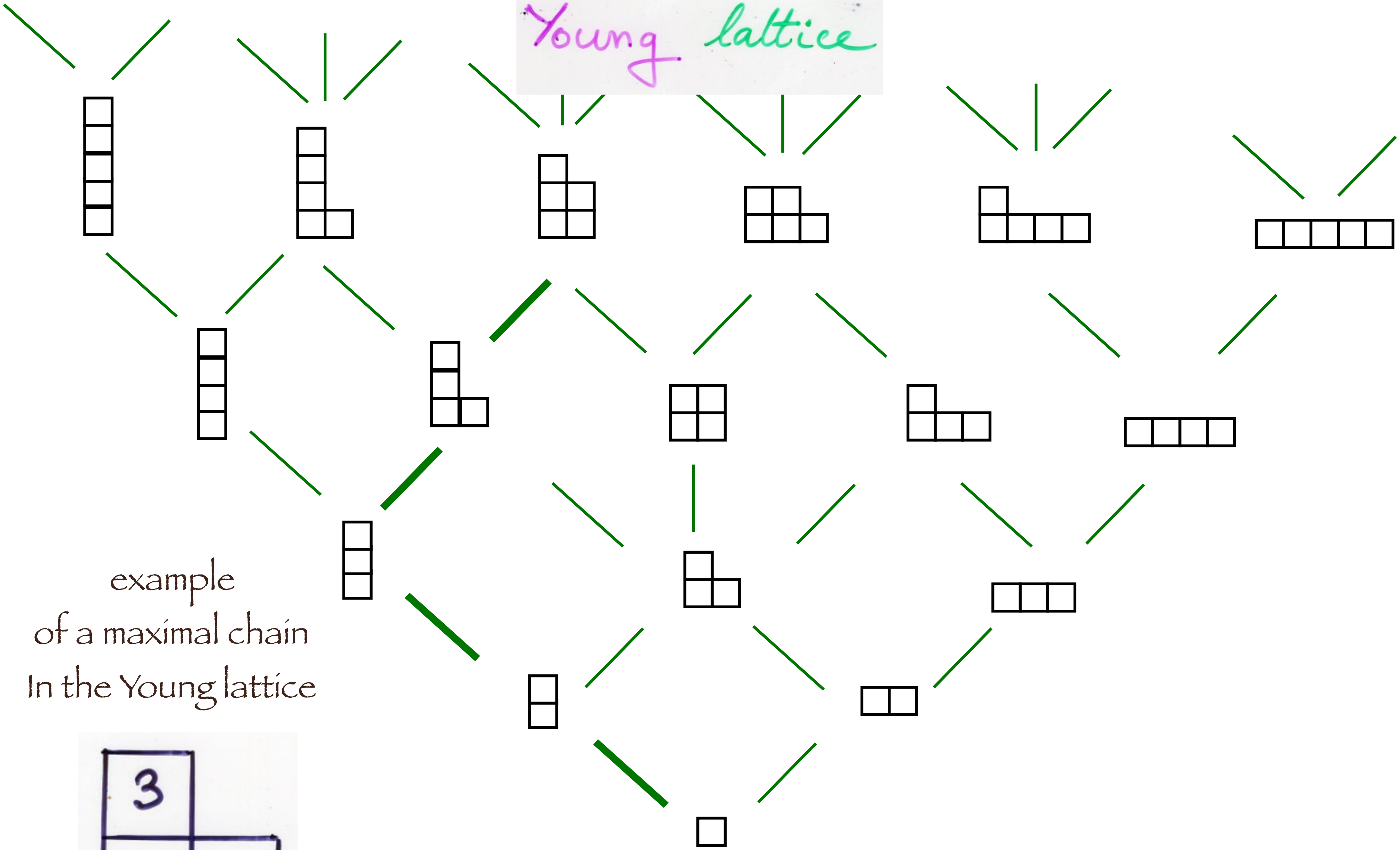
maximal chain in the Young lattice

$\alpha_1 = \emptyset \lessdot \cdots \lessdot \alpha_k = \lambda$ $\longleftrightarrow$ and Young tableau with shape λ

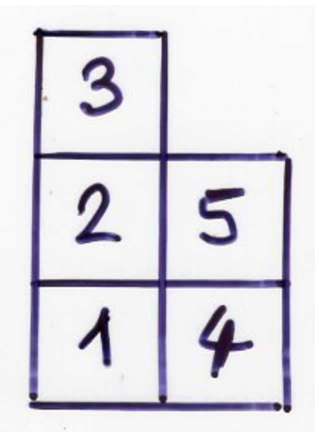


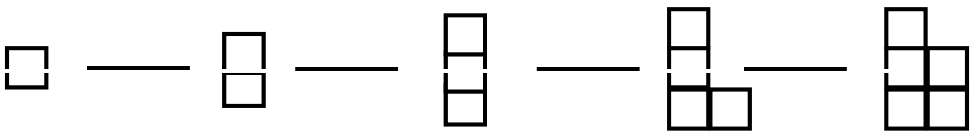
3	
2	5
1	4

Young lattice



example
of a maximal chain
In the Young lattice



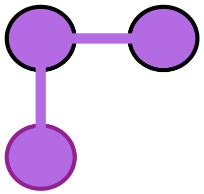


number of maximal chains in Tamari(n) ?

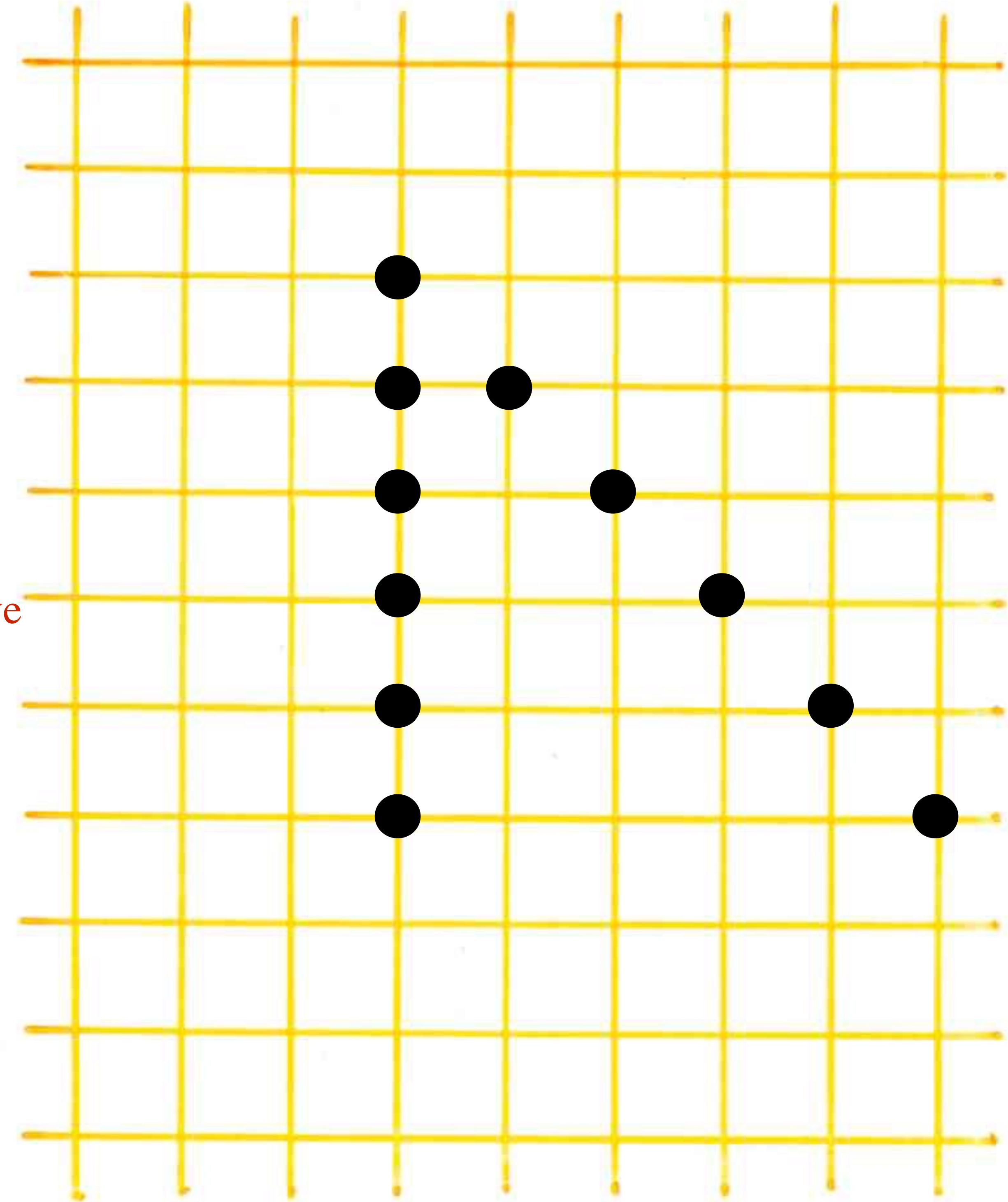
Nelson (2016) Ph.D.

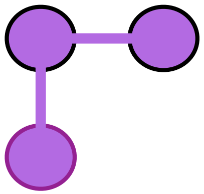
number of chains with maximum length

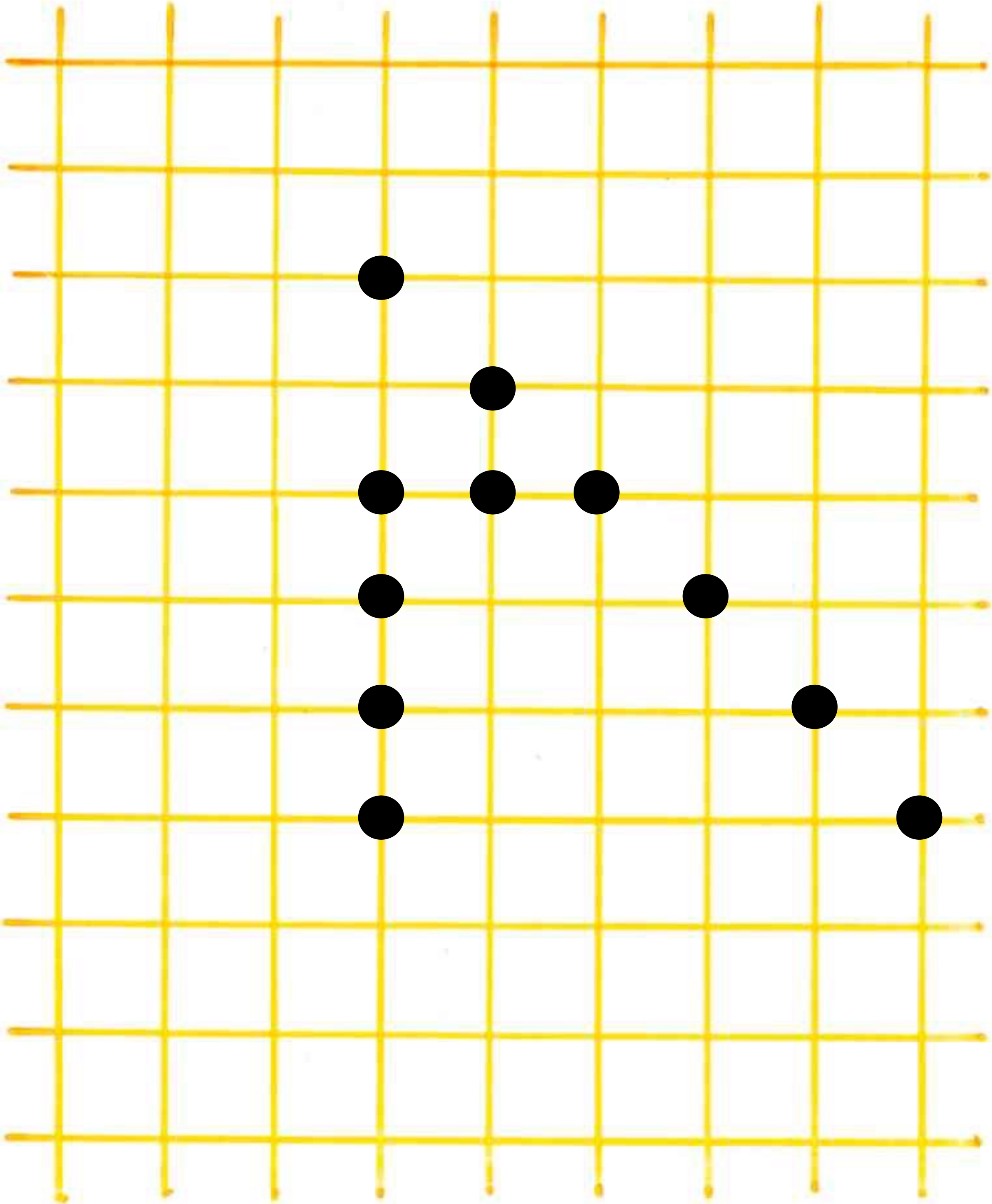
Fishel, Nelson (2014)
bijection with standard shifted tableaux of staircase shape

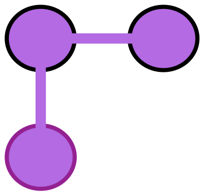


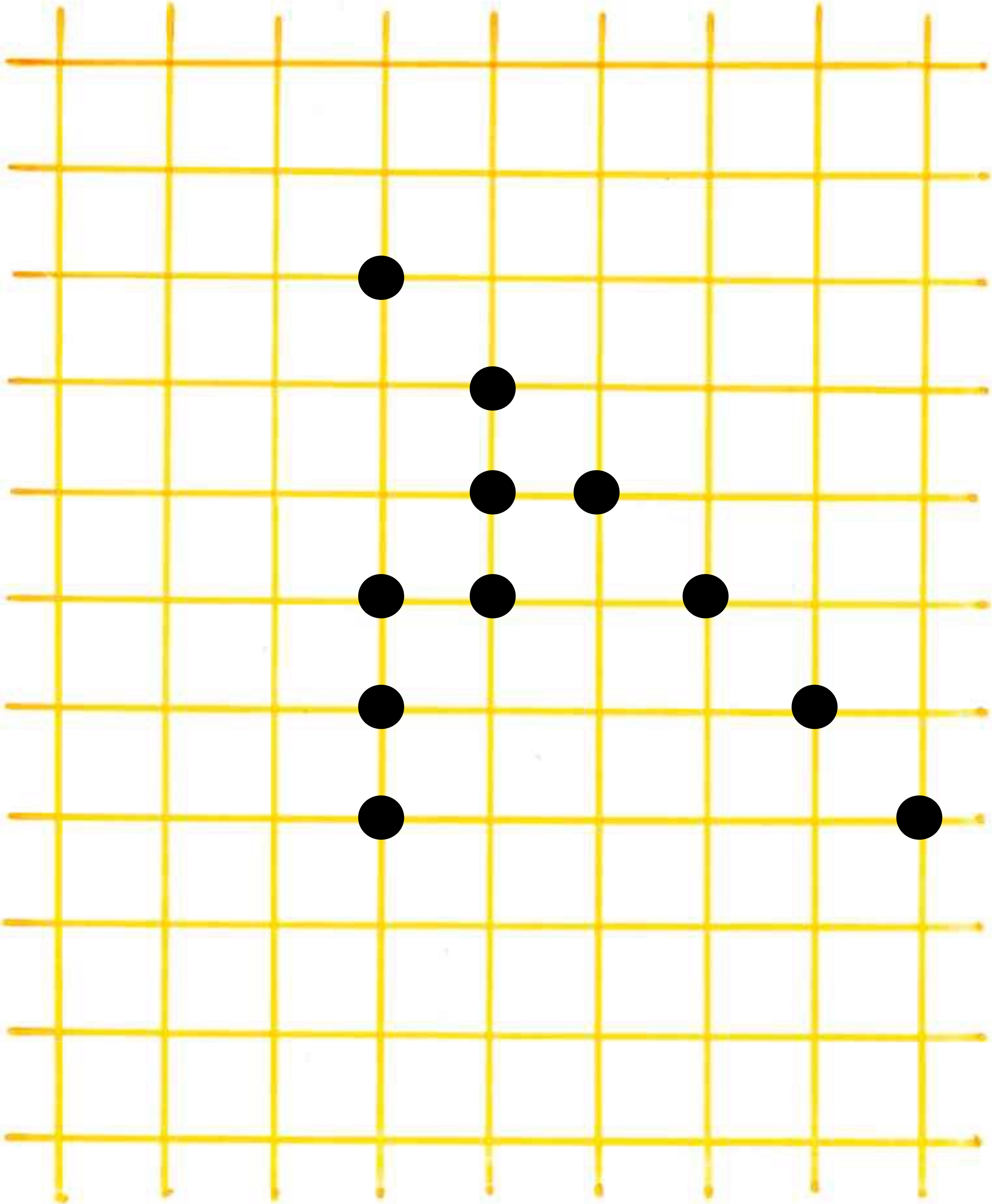
elementary Γ-move

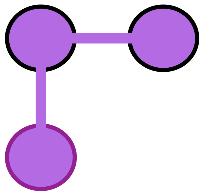


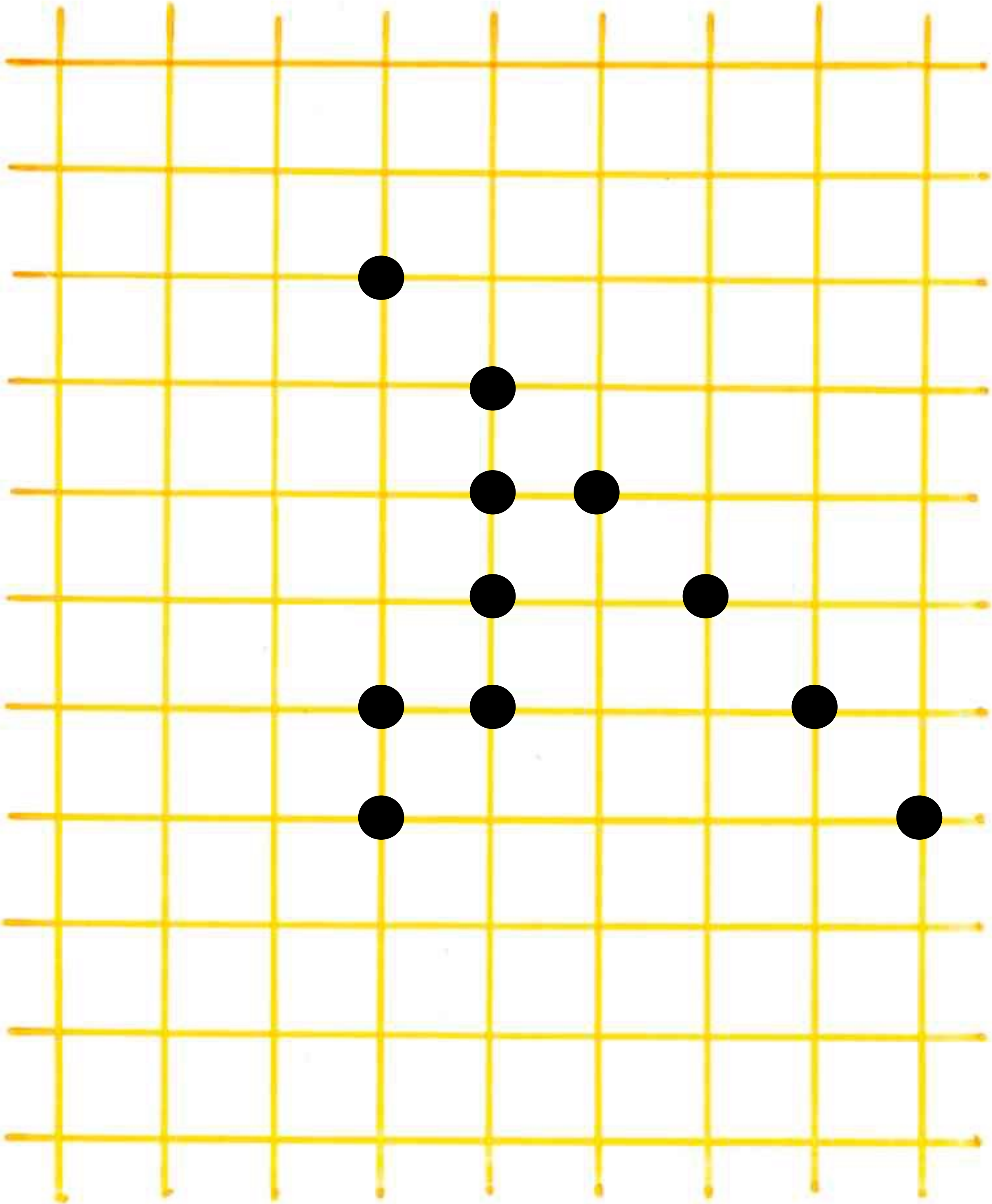


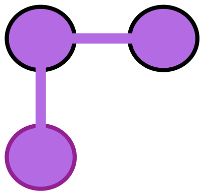


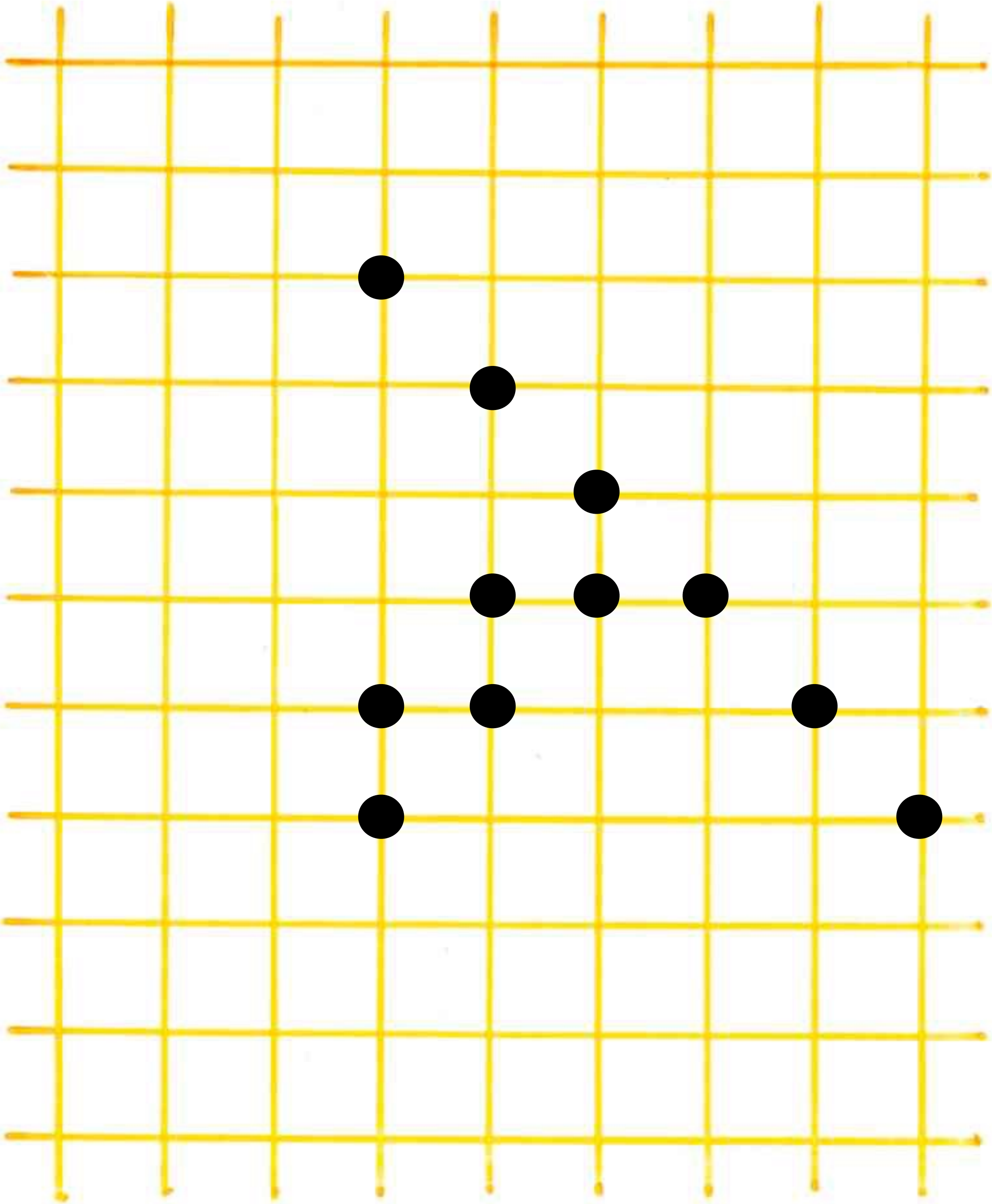


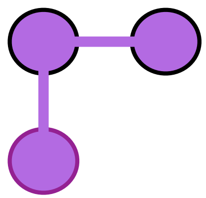


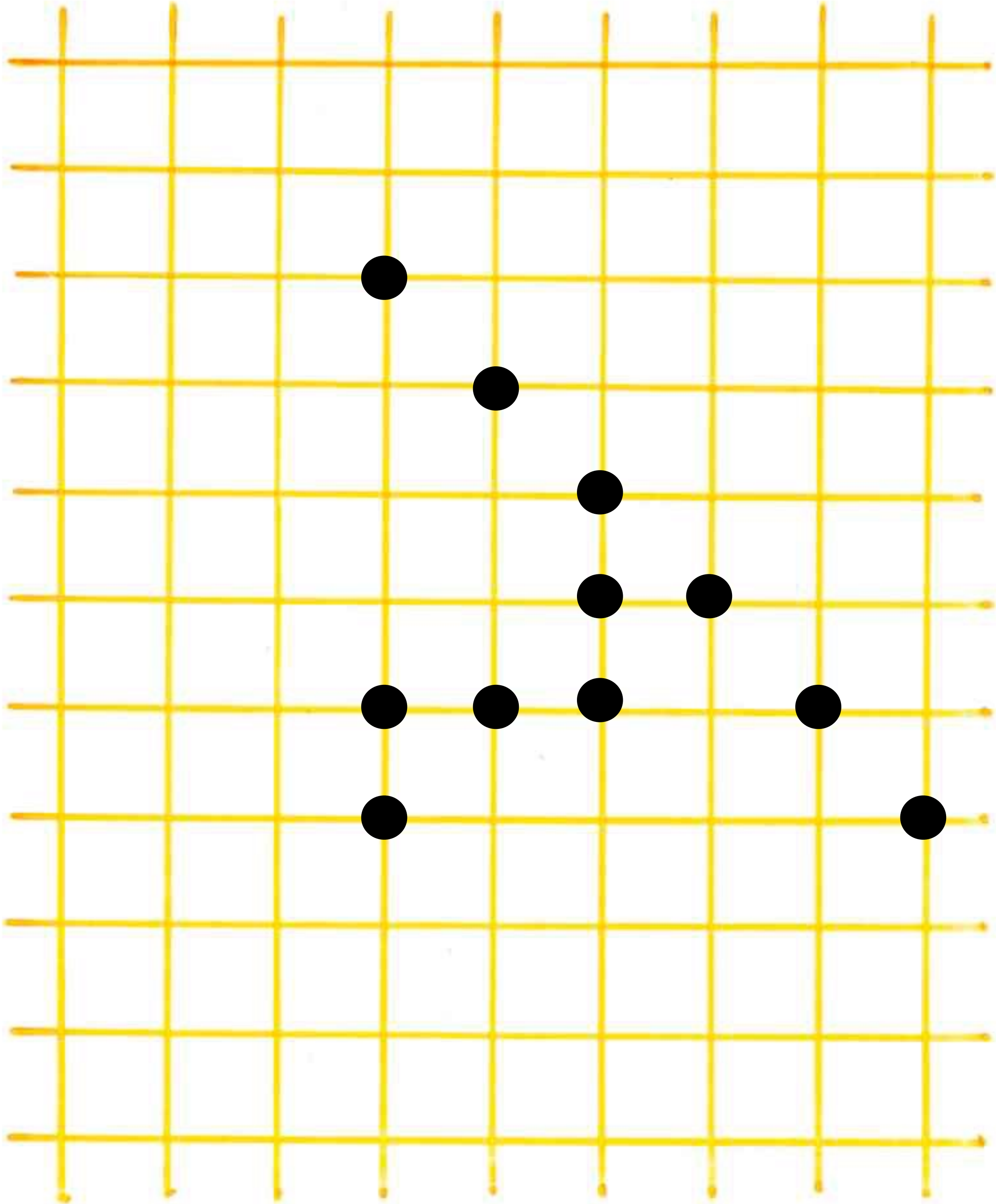


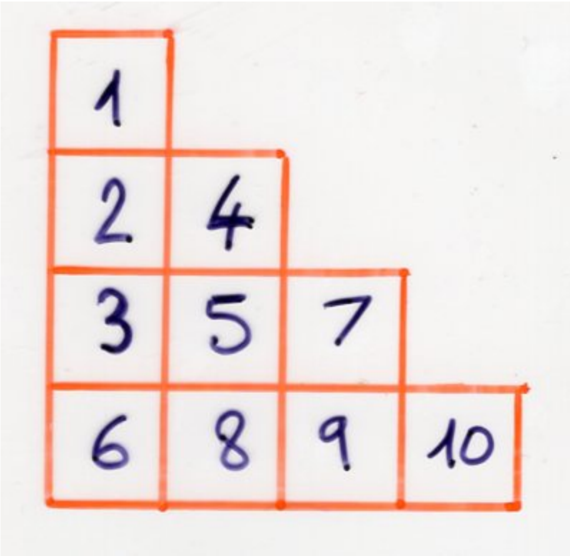


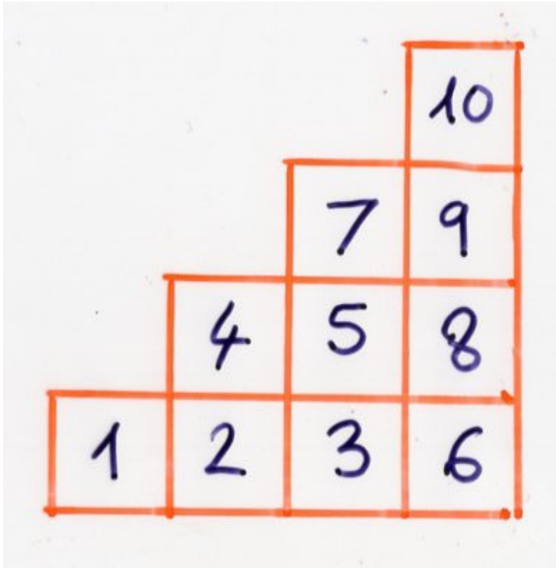












Fishel, Nelson (2014)
bijection with standard shifted tableaux
of staircase shape

slides on the website of SLC 79,
Bertinoro, 10-13 September 2017

Séminaire
Lotharingien de
Combinatoire

thank you!