Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 6
q-analogues
of some orthogonal polynomials

Ch6a

IMSc, Chennai
March 4, 2019

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

mirror website
www.imsc.res.in/~viennot

q-analogue

n! binomial coefficients

q-analogue

$$[i]_q = 1 + q + \dots + q^{i-1}$$

$$= \frac{1-q^i}{1-q}$$

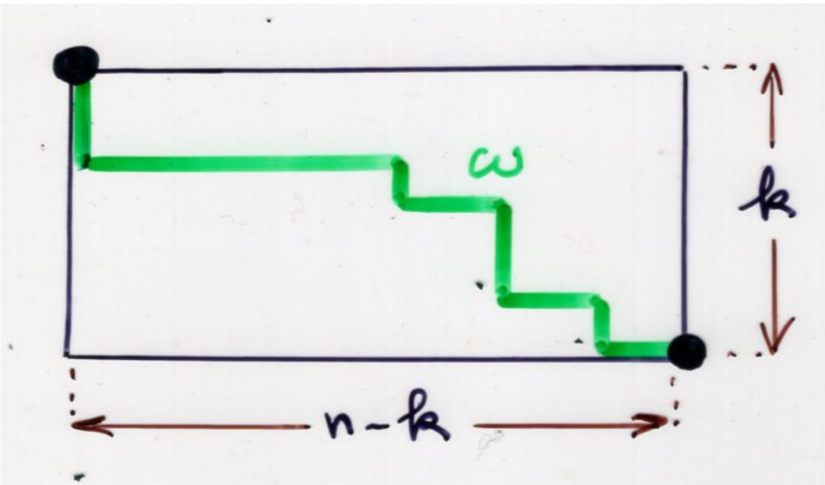
$$[n!]_q = [1]_q \times [2]_q \times \dots \times [n]_q$$

$$= \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n!]_q}{[k!]_q\,[n-k]_q}$$

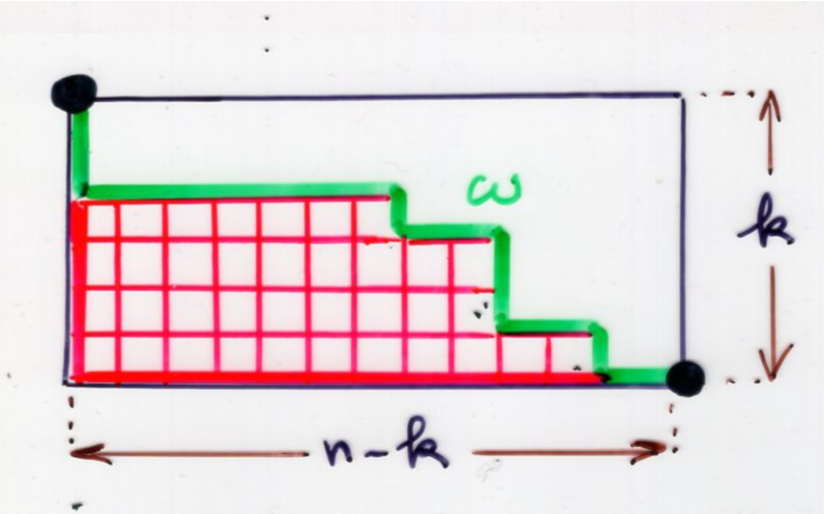
$$= \frac{(1-q)(1-q^2)\ \cdots\ (1-q^n)}{(1-q)\cdots(1-q^k)\ (1-q)\cdots(1-q^{n-k})}$$



$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n!]_q}{[k!]_q\,[n-k]_q}$$

$$= \frac{(1-q)(1-q^2)\ \cdots\ (1-q^n)}{(1-q)\cdots(1-q^k)\ (1-q)\cdots(1-q^{n-k})}$$



$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\omega} q^{\text{area}(\omega)}$$

Definition

sub-excedante functions

$$f : [1, n] \longrightarrow [0, n-1]$$

pour tout $1 \leqslant i \leqslant n$, $0 \leqslant f(i) < i$

$$|\mathcal{F}_n| = n!$$

$\mathcal{F}_n$ set of sub-excedante functions

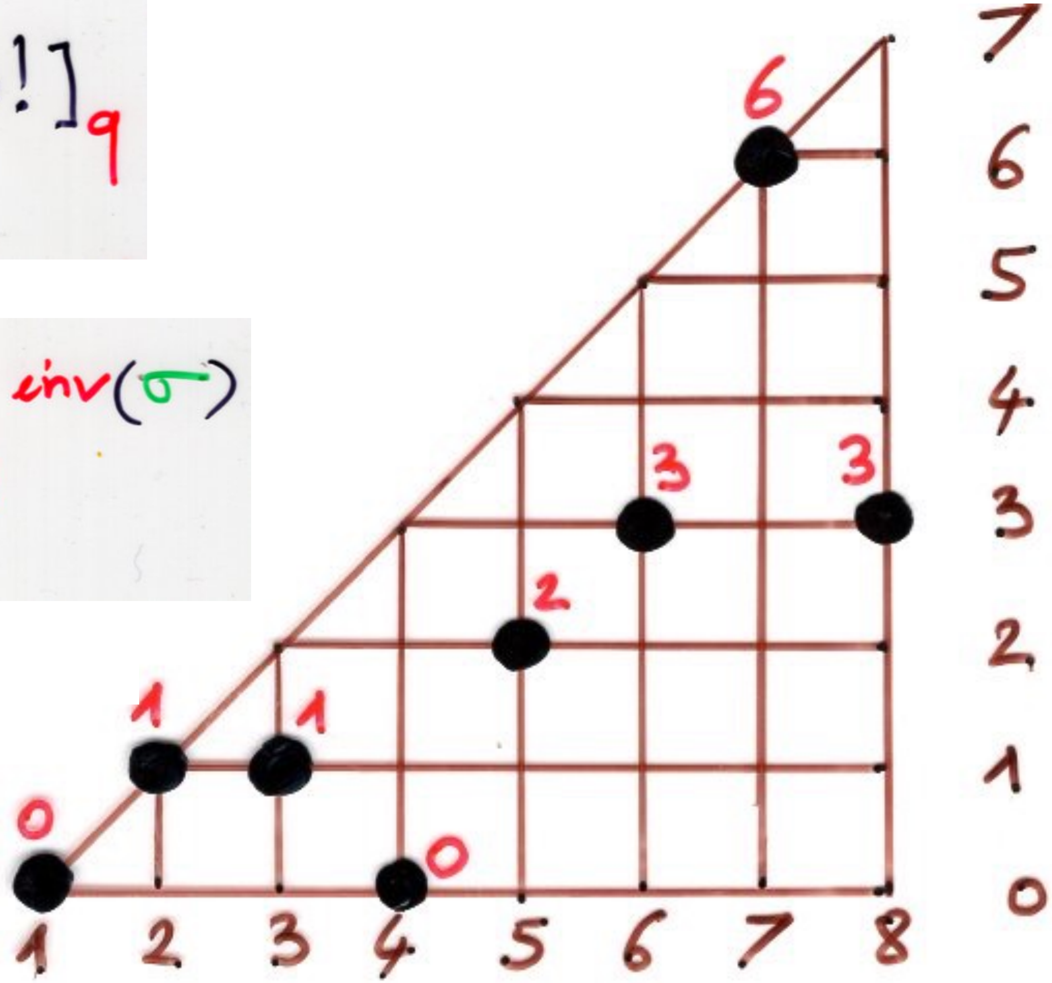
$$\mathrm{sum}(f) = \sum_{i=1}^{n} f(i)$$

$$\sum_{f \in \mathcal{F}_n} q^{\mathrm{sum}(f)} = [n!]_q$$

$$\mathrm{sum}(f) = \sum_{i=1}^{n} f(i)$$

$$\sum_{f \in \mathcal{F}_n} q^{\mathrm{sum}(f)} = [n!]_q$$

$$= \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$



$\sigma \in \mathfrak{S}_n \longrightarrow f \in \mathcal{F}_n$

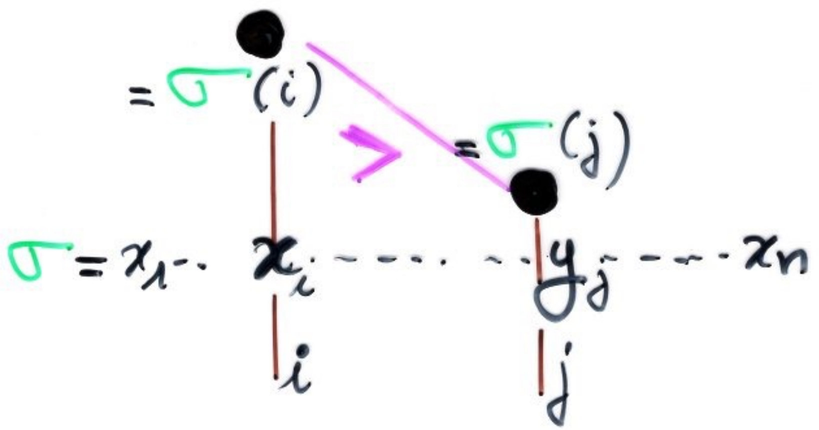
Inversion table

$\sigma = \dfrac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

$1 \leq x \leq n \qquad x = \sigma(i)$

$f(x) =$ number of j, $i < j \leq n$ with $\sigma(j) < \sigma(i)$



inversion of σ

$(i, j) \qquad 1 \leq i < j \leq n$

$\sigma(i) > \sigma(j)$

$inv(\sigma) =$ number of inversions

number of inversions

inv(σ) = 19

0+0+1+3+1+3+3+3+5

$$q^{0+0+1+3+1+3+3+3+5}$$

$$q^{i}$$

␣

␣ 1 ␣ — 0

␣ 1 ␣ 2 ␣ — 0

␣ 1 ␣ 3 ␣ 2 ␣ — 1

␣ 4 ␣ 1 ␣ 3 ␣ 2 ␣ — 3

␣ 4 ␣ 1 ␣ 3 ␣ 5 ␣ 2 ␣ — 1

␣ 4 ␣ 1 ␣ 6 ␣ 3 ␣ 5 ␣ 2 ␣ — 3

␣ 4 ␣ 1 ␣ 6 ␣ 7 ␣ 3 ␣ 5 ␣ 2 ␣ — 3

␣ 4 ␣ 1 ␣ 6 ␣ 7 ␣ 8 ␣ 3 ␣ 5 ␣ 2 ␣ — 3

␣ 4 ␣ 1 ␣ 6 ␣ 9 ␣ 7 ␣ 8 ␣ 3 ␣ 5 ␣ 2 ␣ — 5

In this chapter

6 q-analogues
of
orthogonal polynomials

Hermite

Charlier

Laguerre

q-Hermite I
(continuous)

$$\lambda_k = [k]_q$$

q-Hermite II
discrete

$$\lambda_k = q^{k-1}[k]_q$$

"continuous version"

q-Charlier I

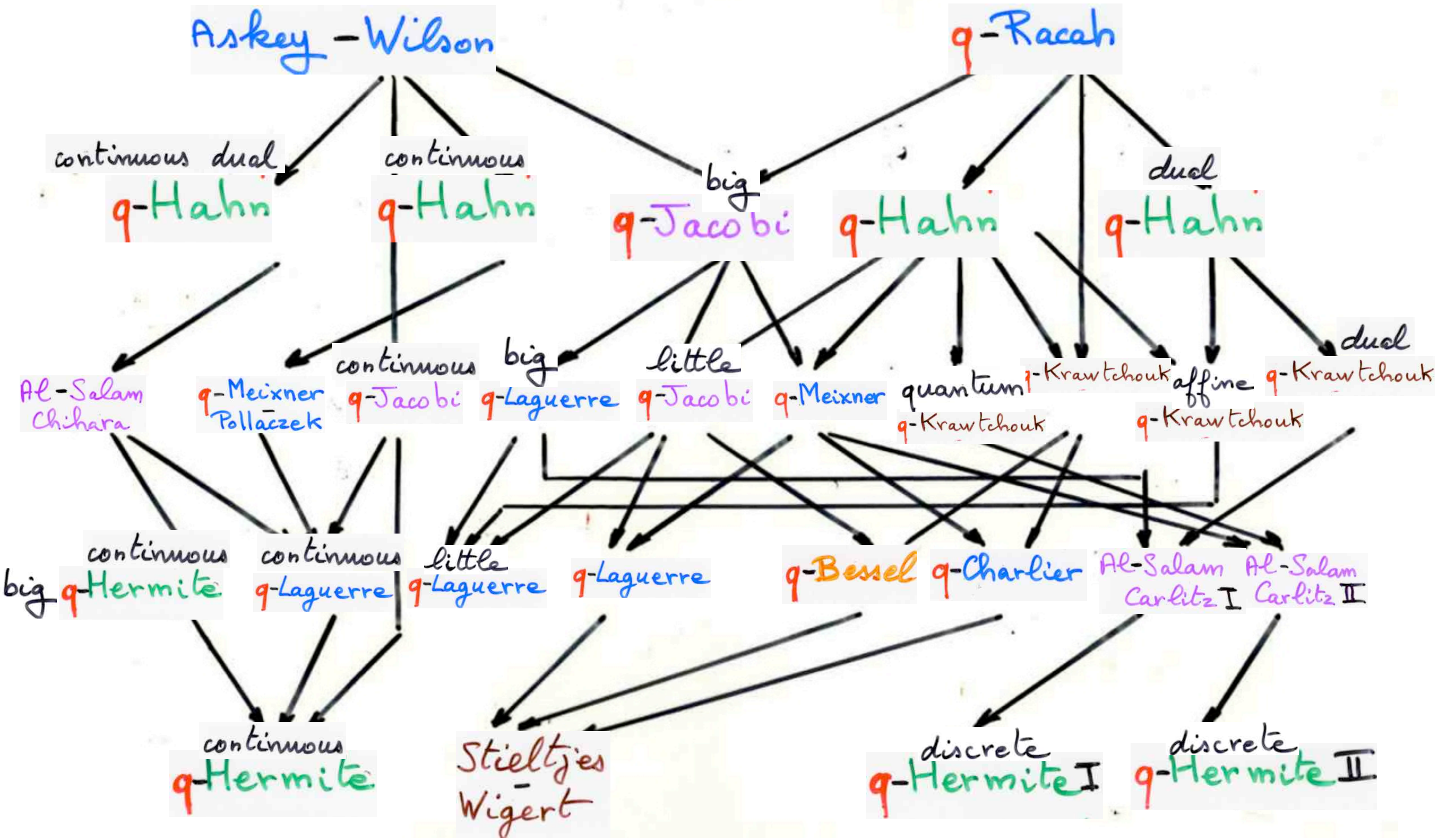
$$\begin{cases} b_k = a + [k]_q \\ \lambda_k = a[k]_q \end{cases}$$

discrete

q-Charlier II

$$\begin{cases} b_k = aq^k + [k]_q \\ \lambda_k = aq^{k-1}[k]_q \end{cases}$$

scheme of basic hypergeometric orthogonal polynomials



q-Hermite

continuous q-Hermite

(Hermite I)

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

Hermite polynomials

$$\cfrac{1}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{3t}{\cdots}}}}$$

moments

Hermite

polynomials

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

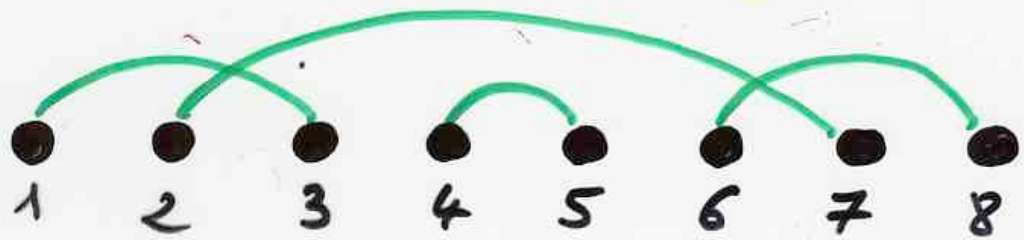
$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

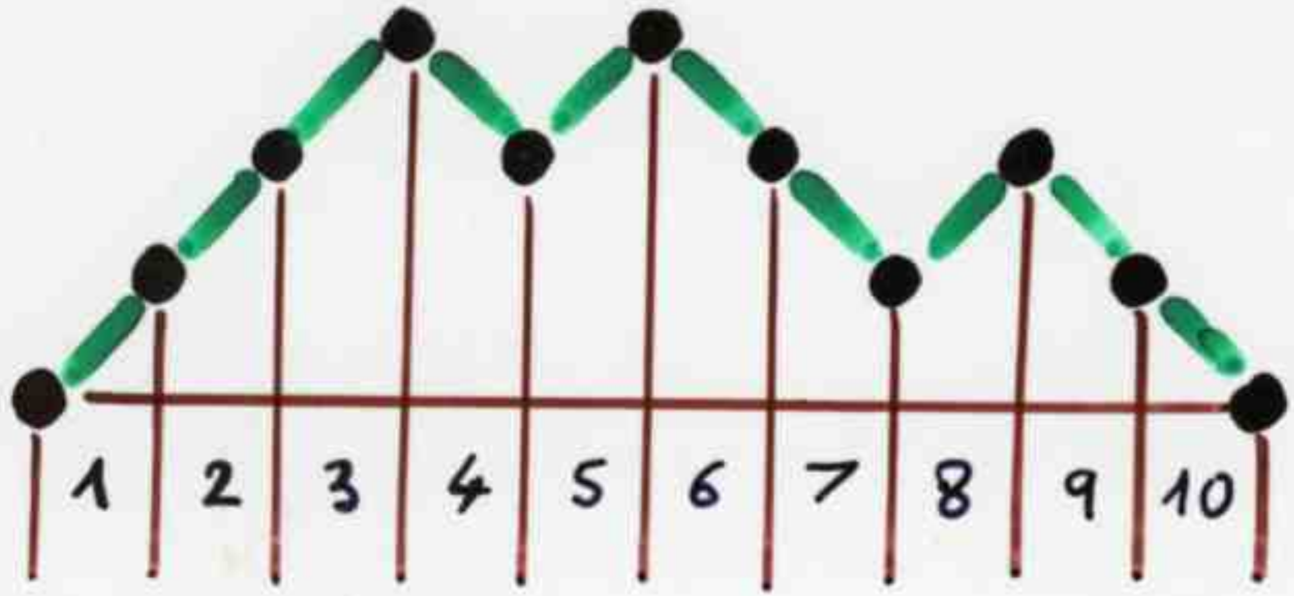
number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams

perfect matching

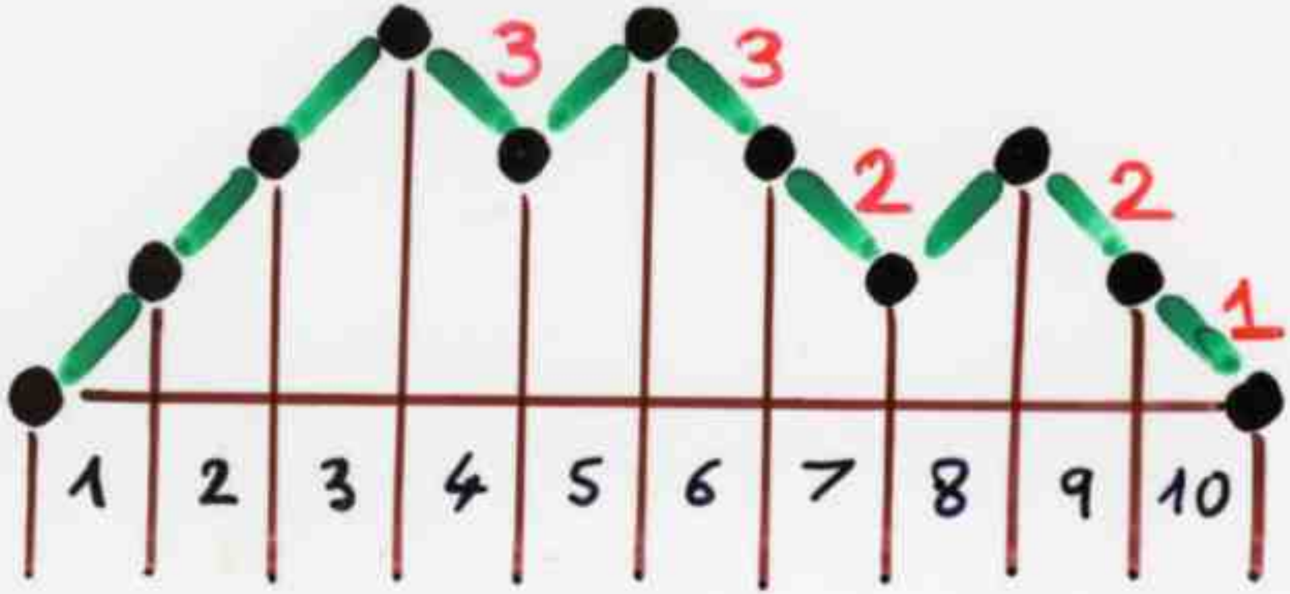


Hermite history



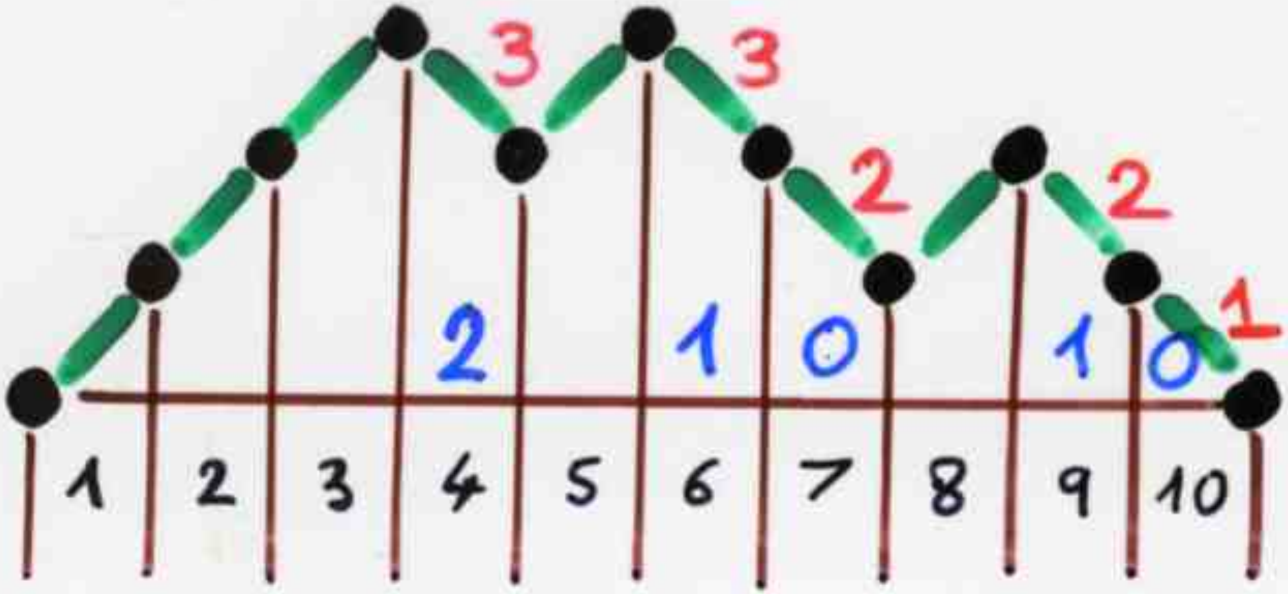
Hermite
history

$$\text{Hermite}\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

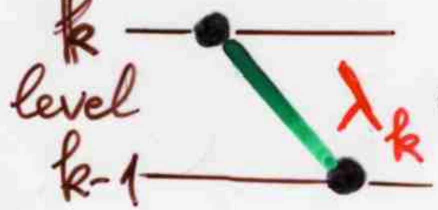


Hermite history

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$



$0 \leq i < \lambda_k = k$

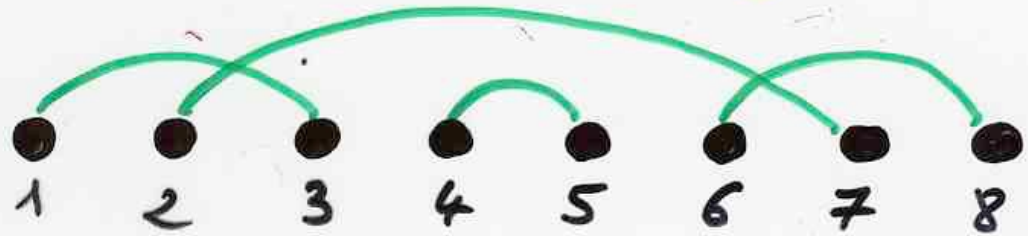


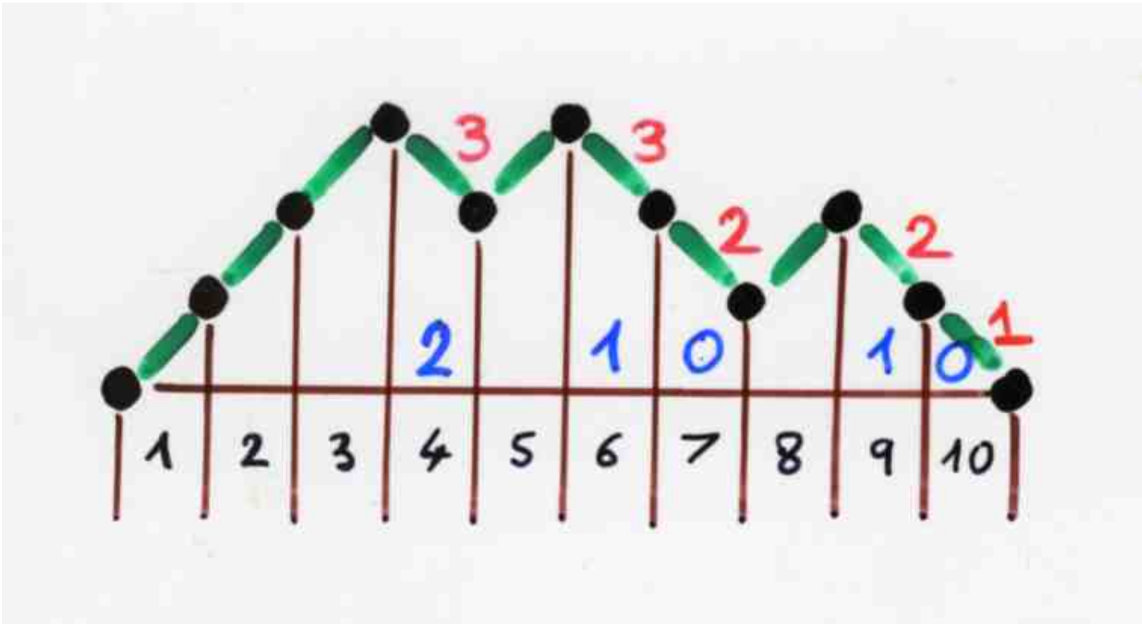
$\lambda_k = k$

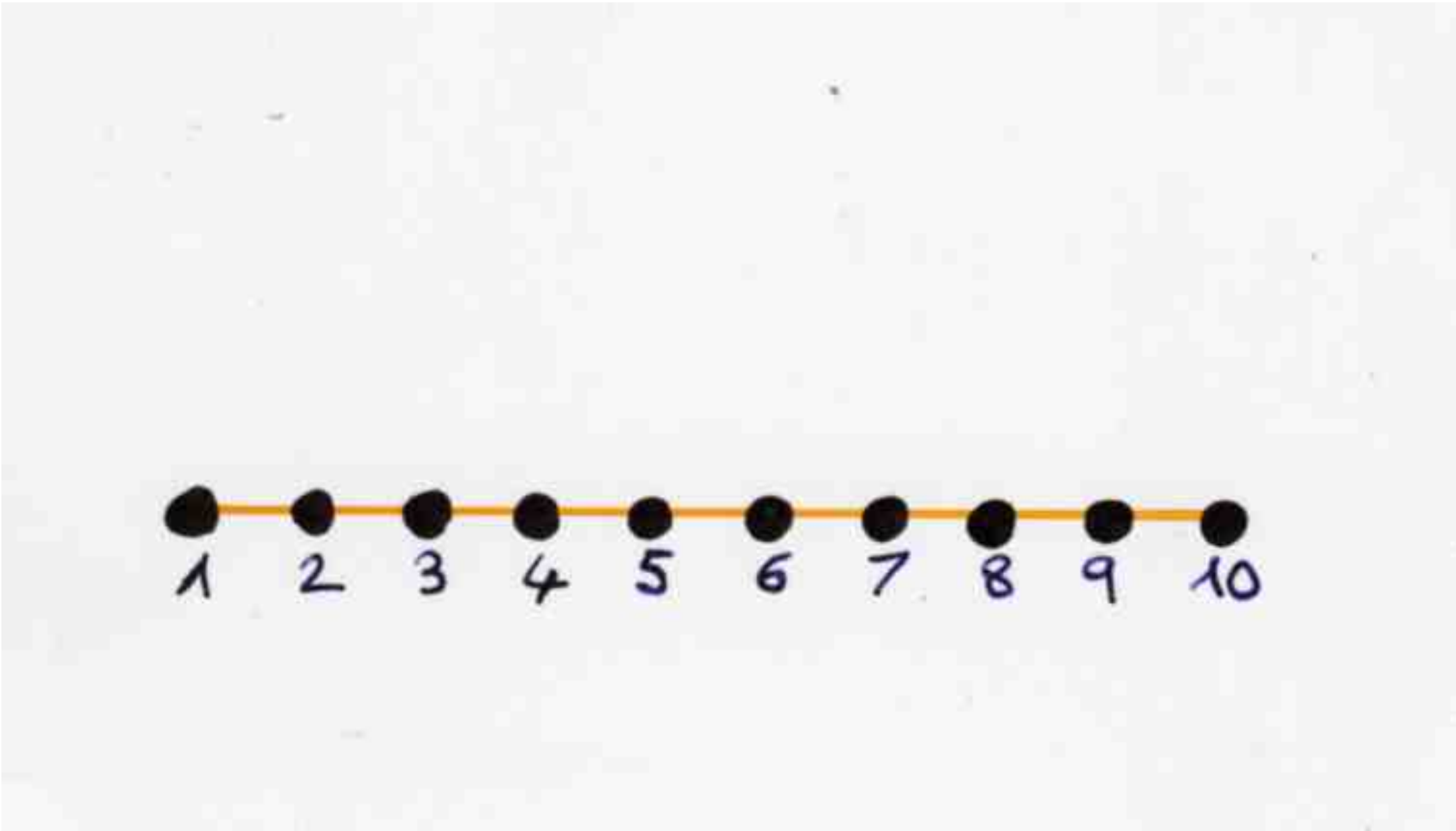
bijection

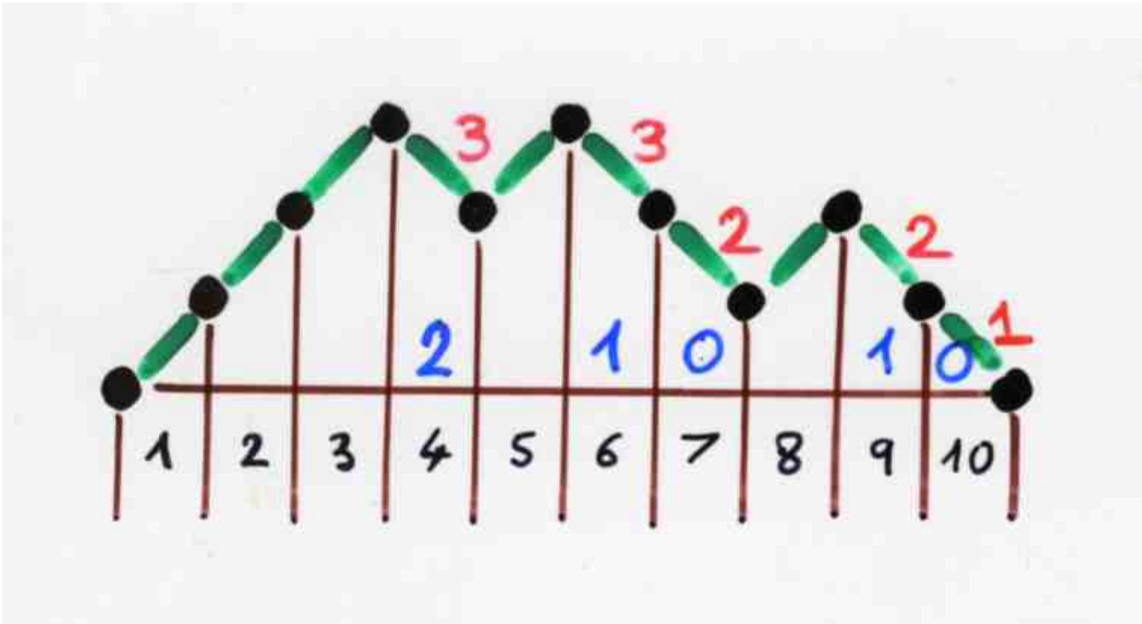
Hermite history $\longleftrightarrow$ chord diagrams

perfect matching

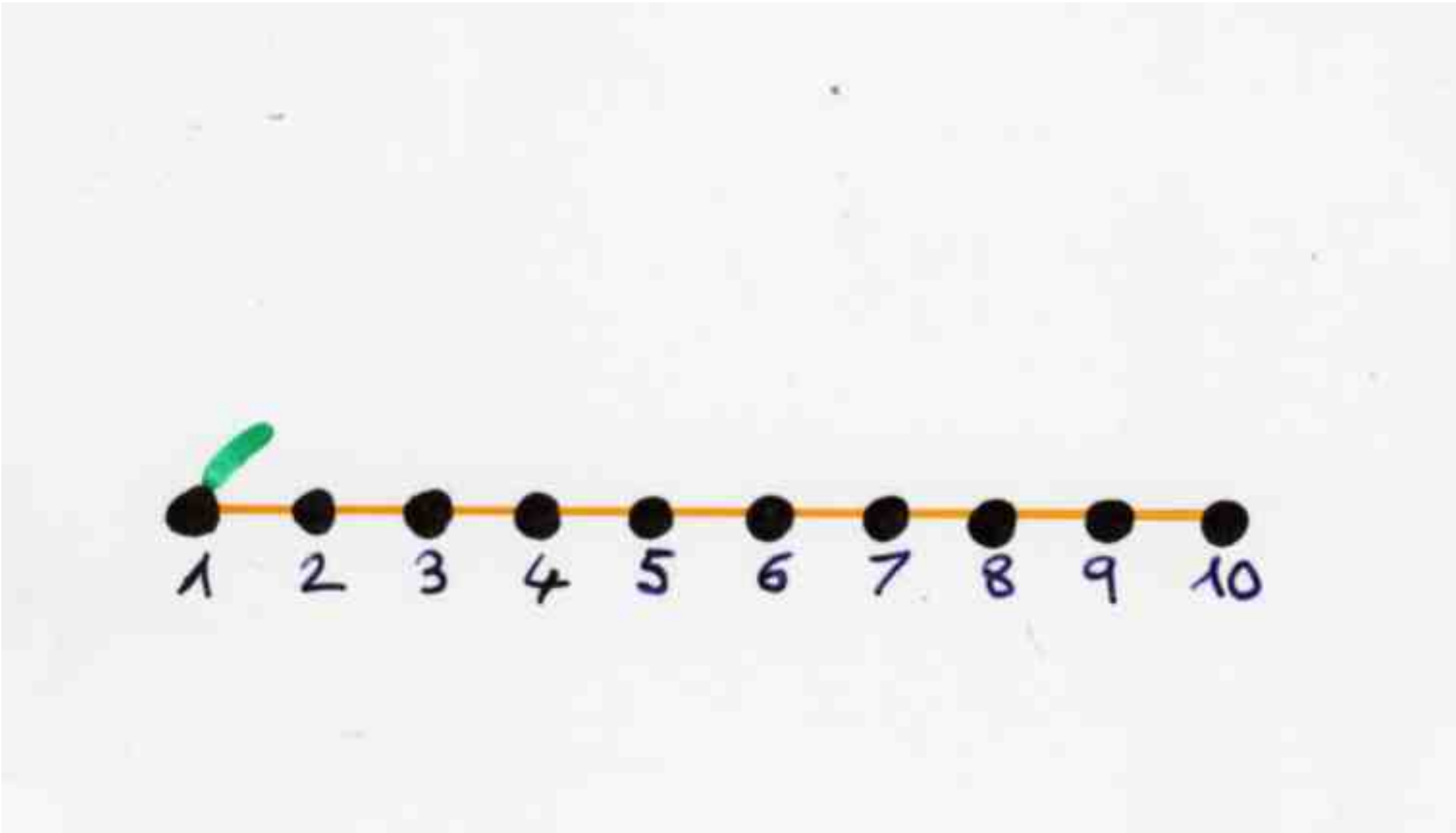



3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

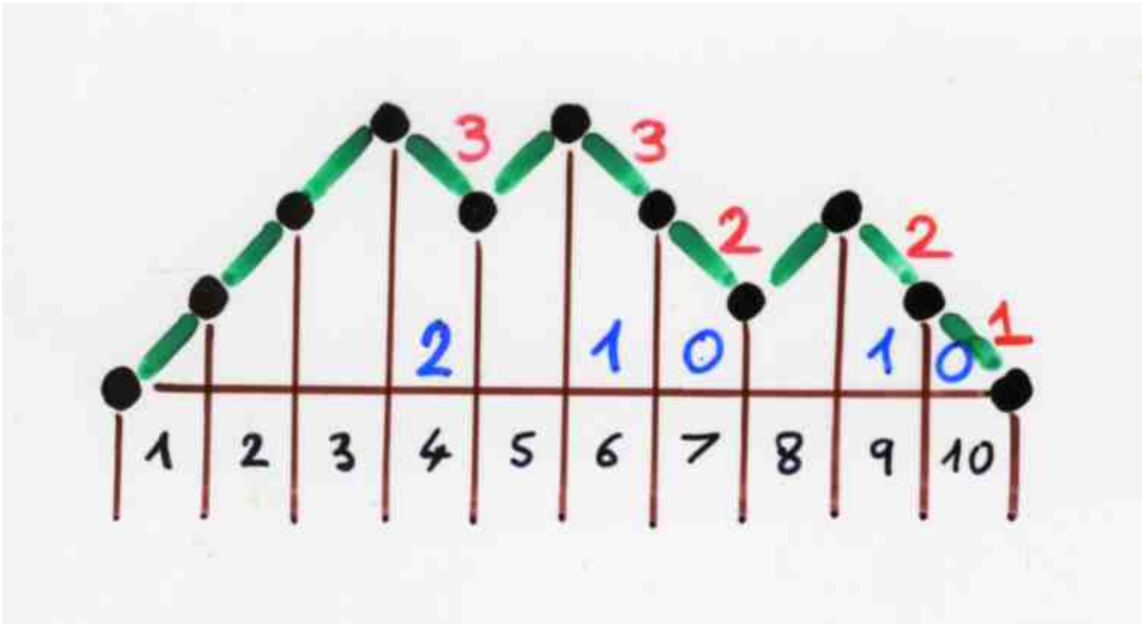

1 2 3 4 5 6 7 8 9 10

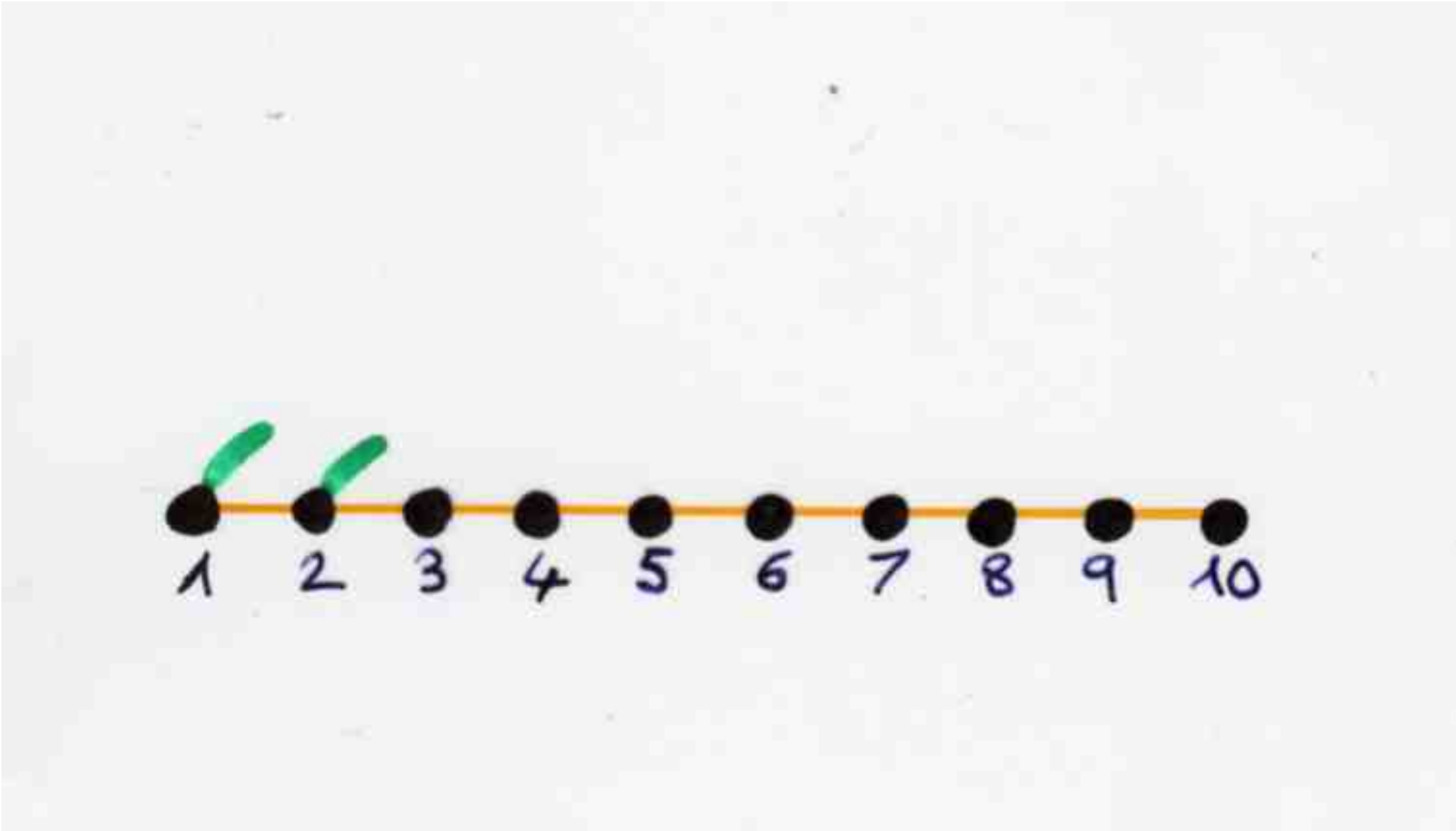


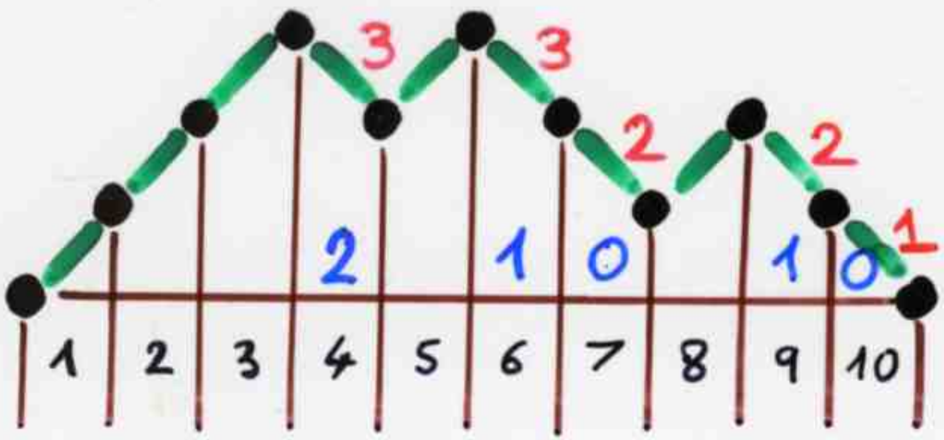
3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

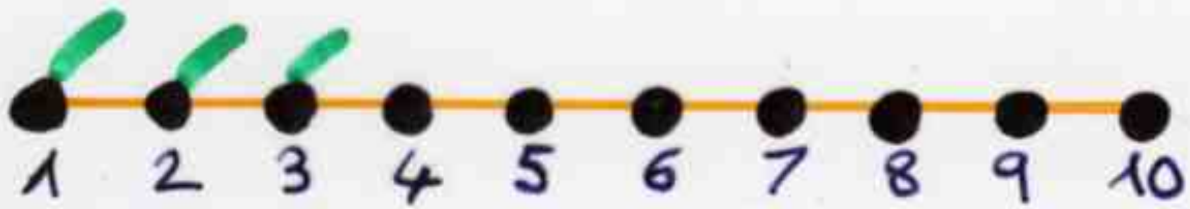


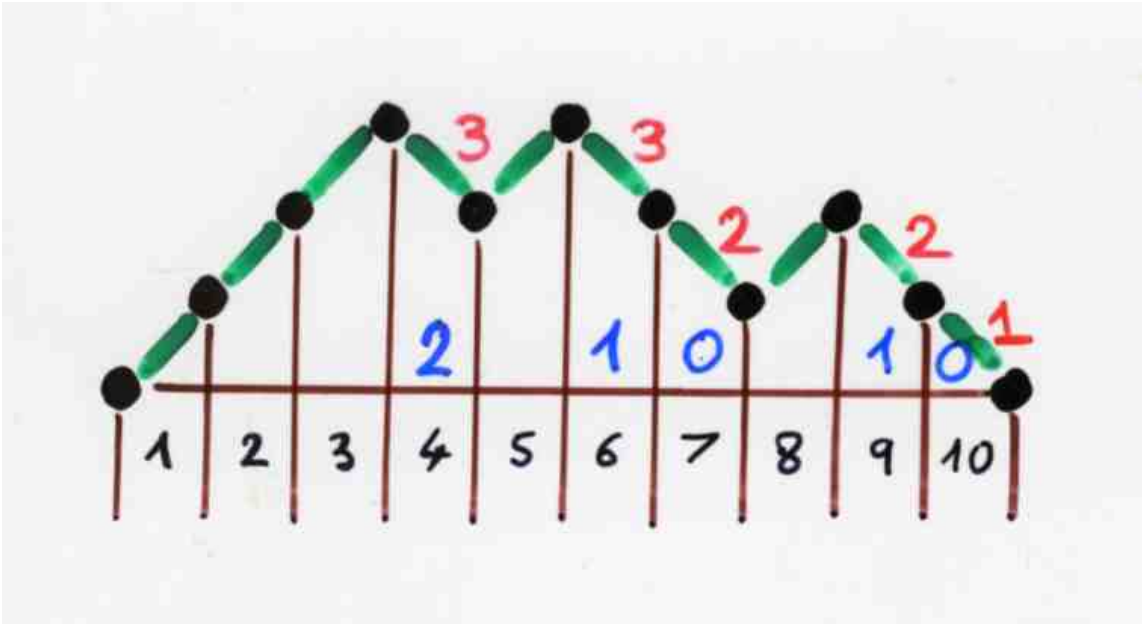
1
2
3
4
5
6
7
8
9
10

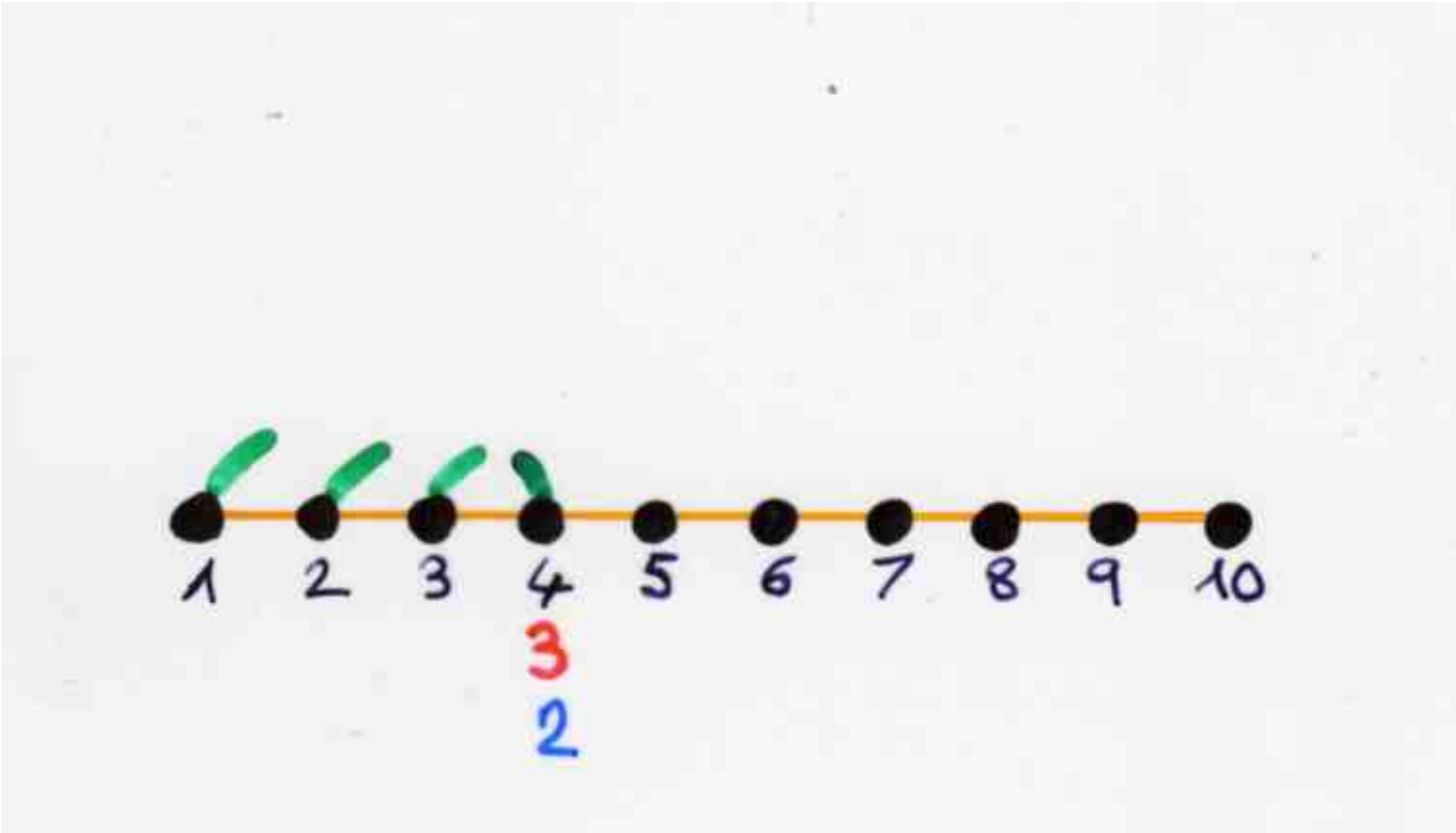

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

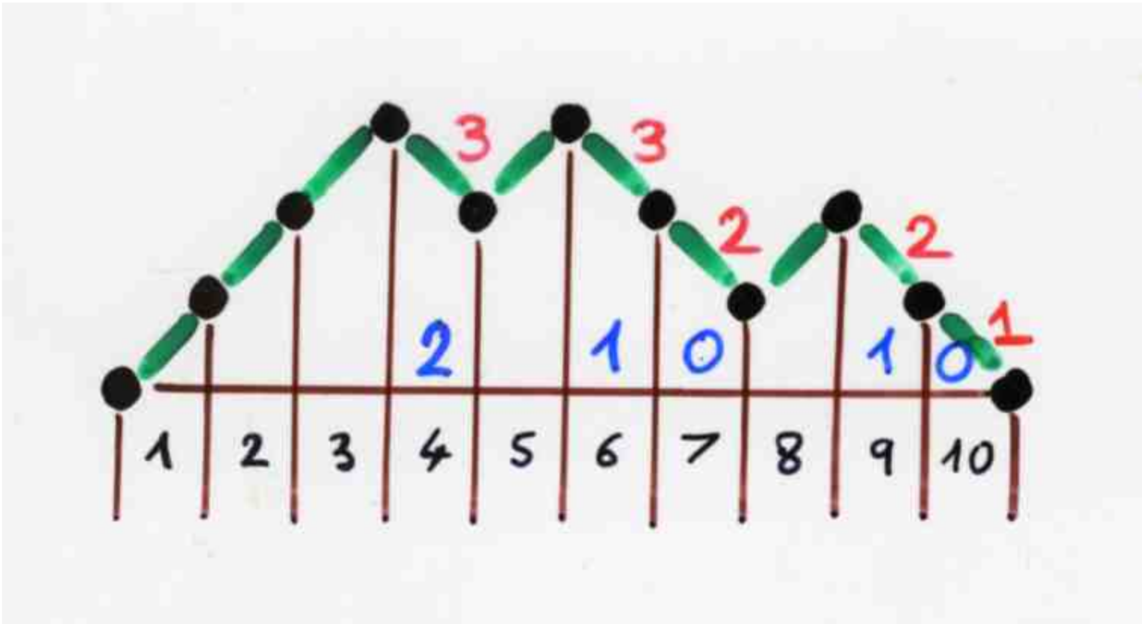

1 2 3 4 5 6 7 8 9 10

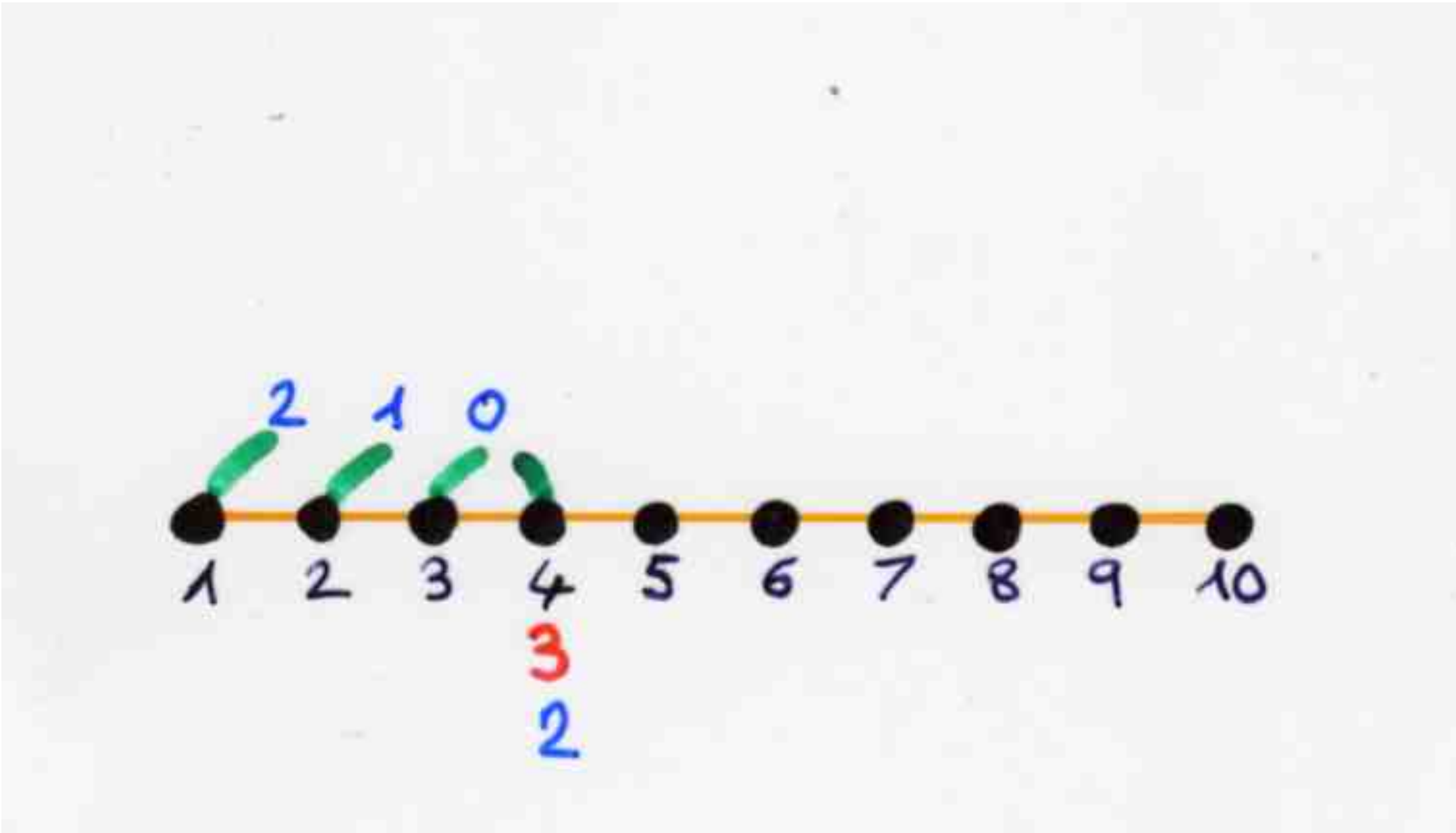

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

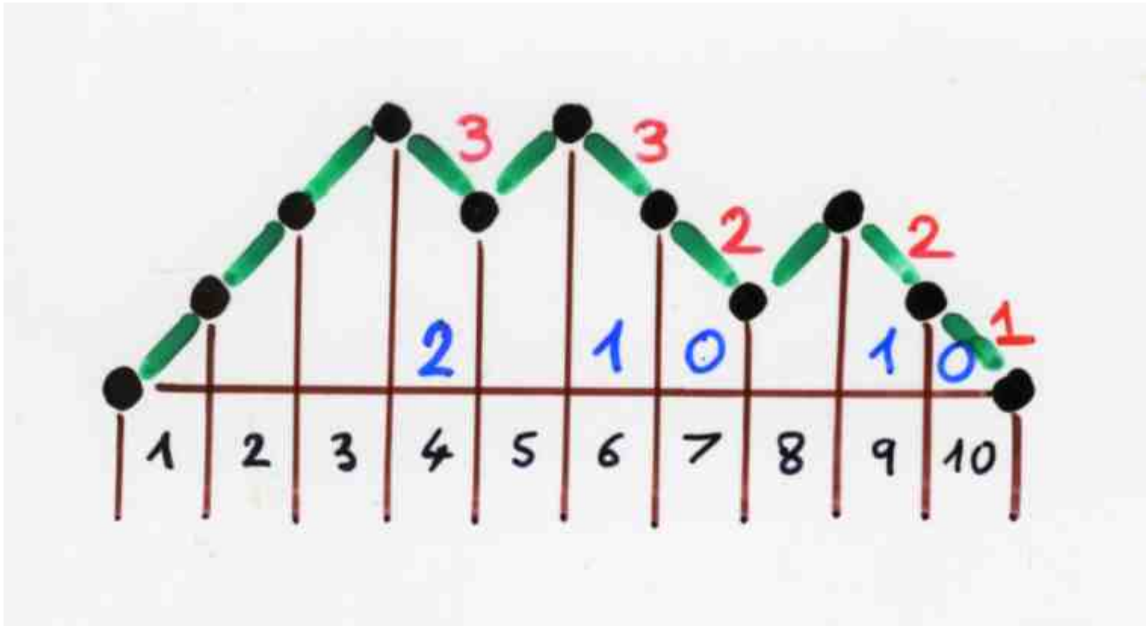

1 2 3 4 5 6 7 8 9 10


3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

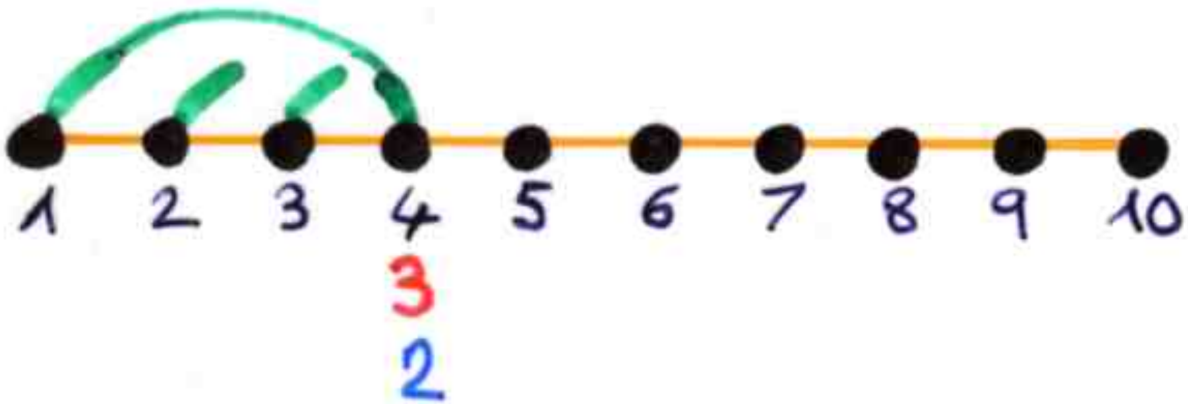

1 2 3 4 5 6 7 8 9 10
3
2


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10

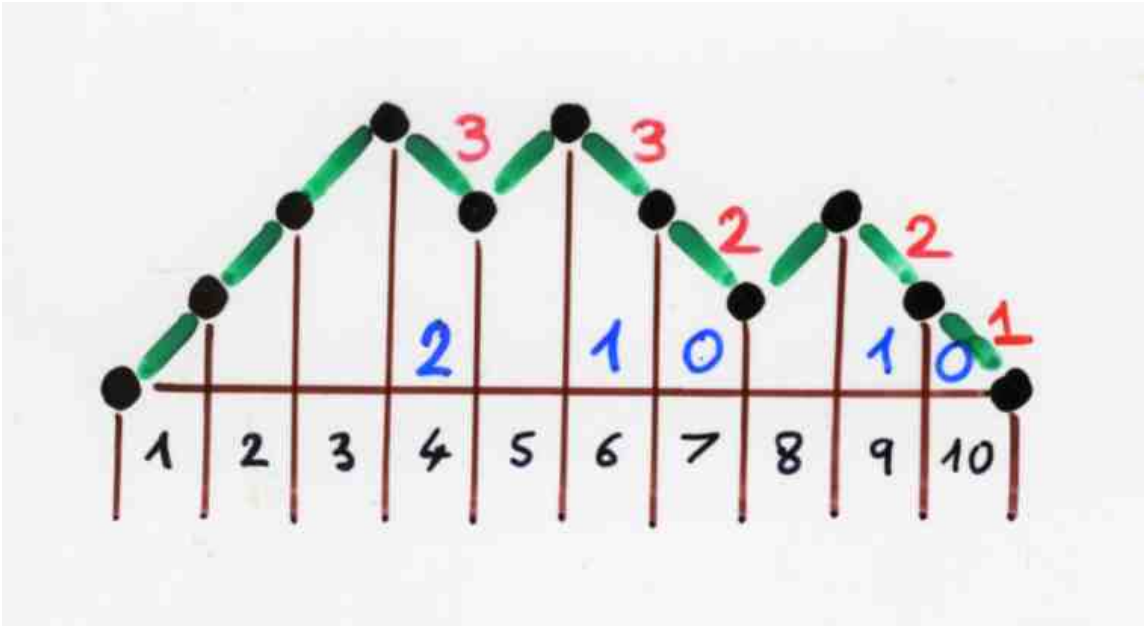

2 1 0
1 2 3 4 5 6 7 8 9 10
3
2

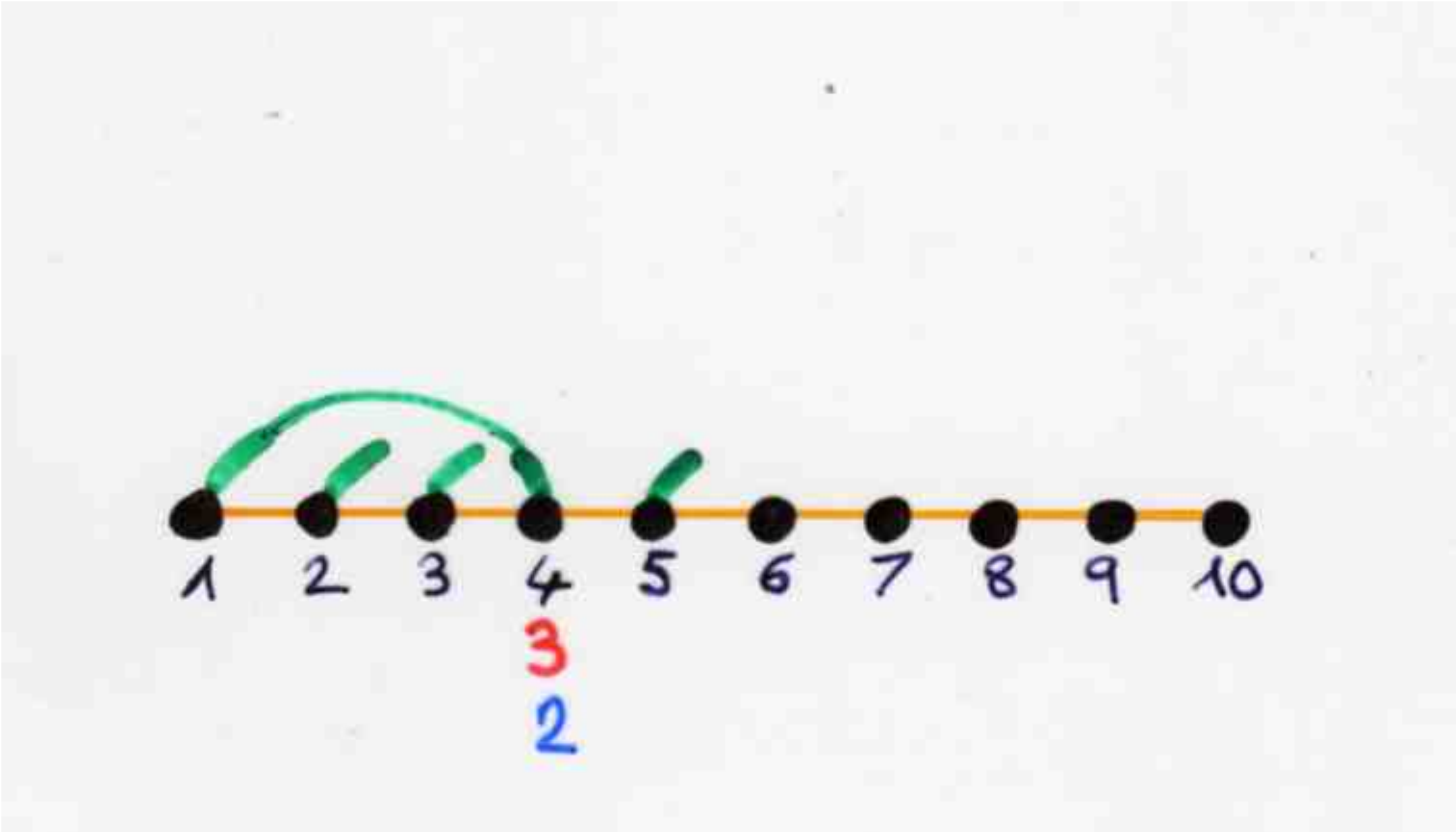


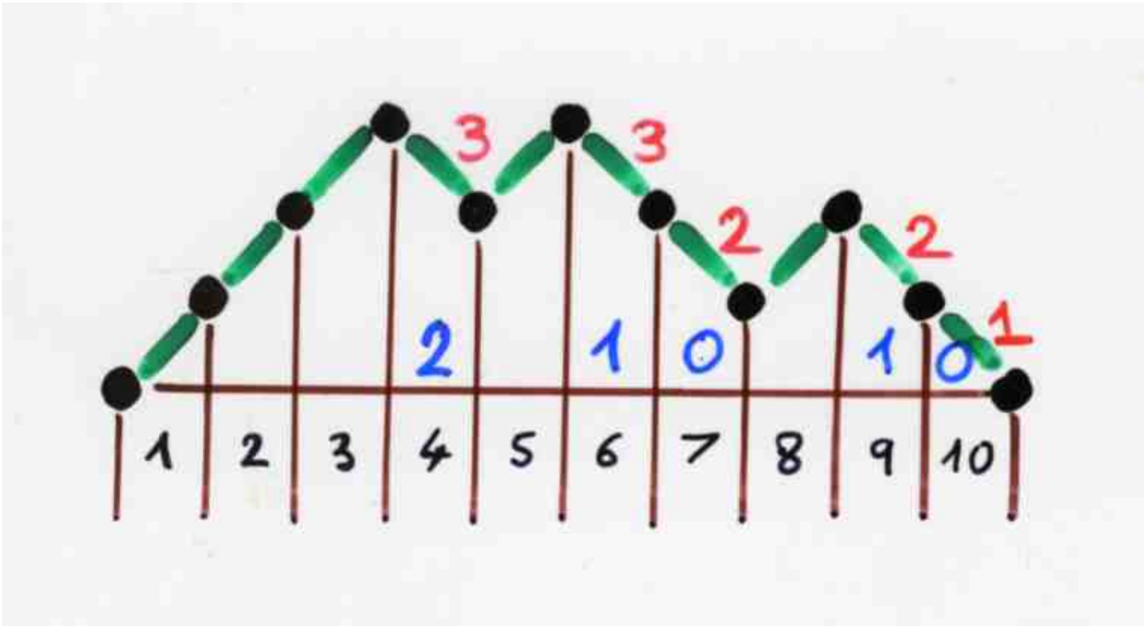
3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

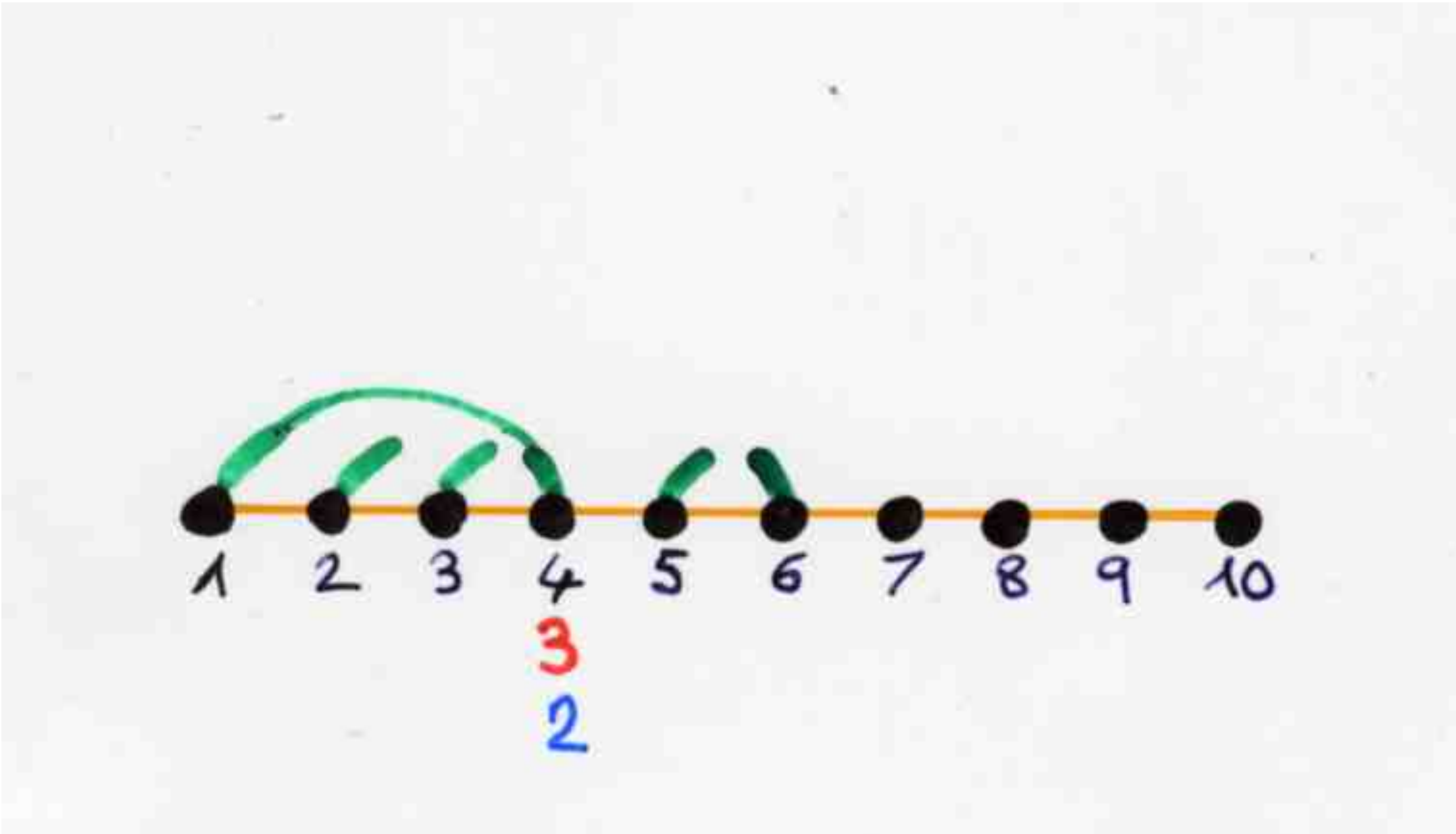


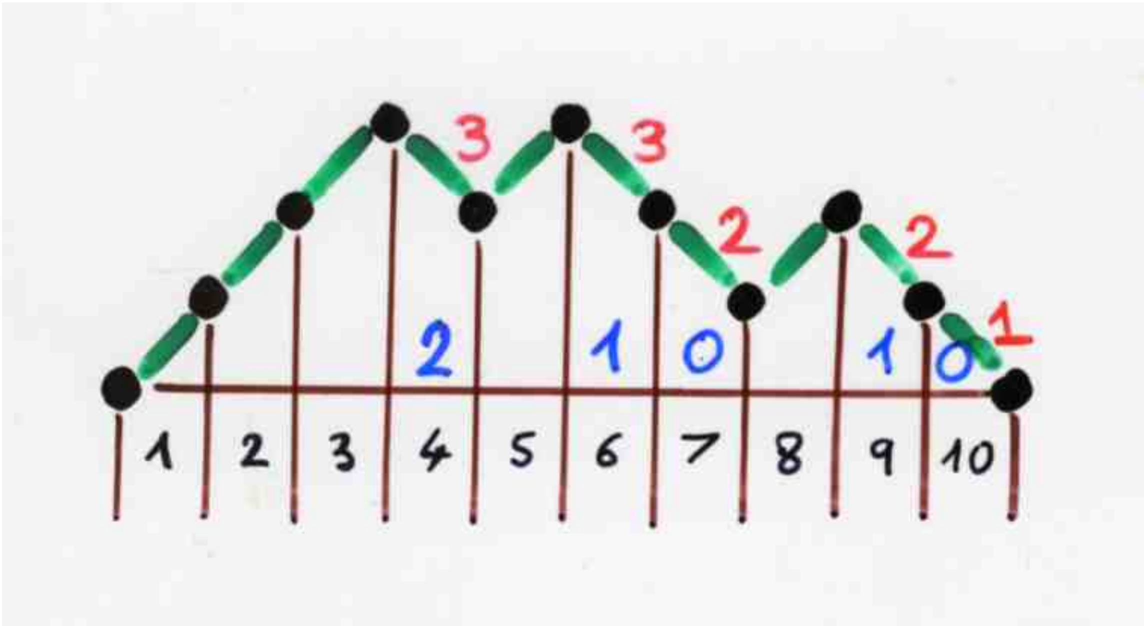
1 2 3 4 5 6 7 8 9 10
3
2

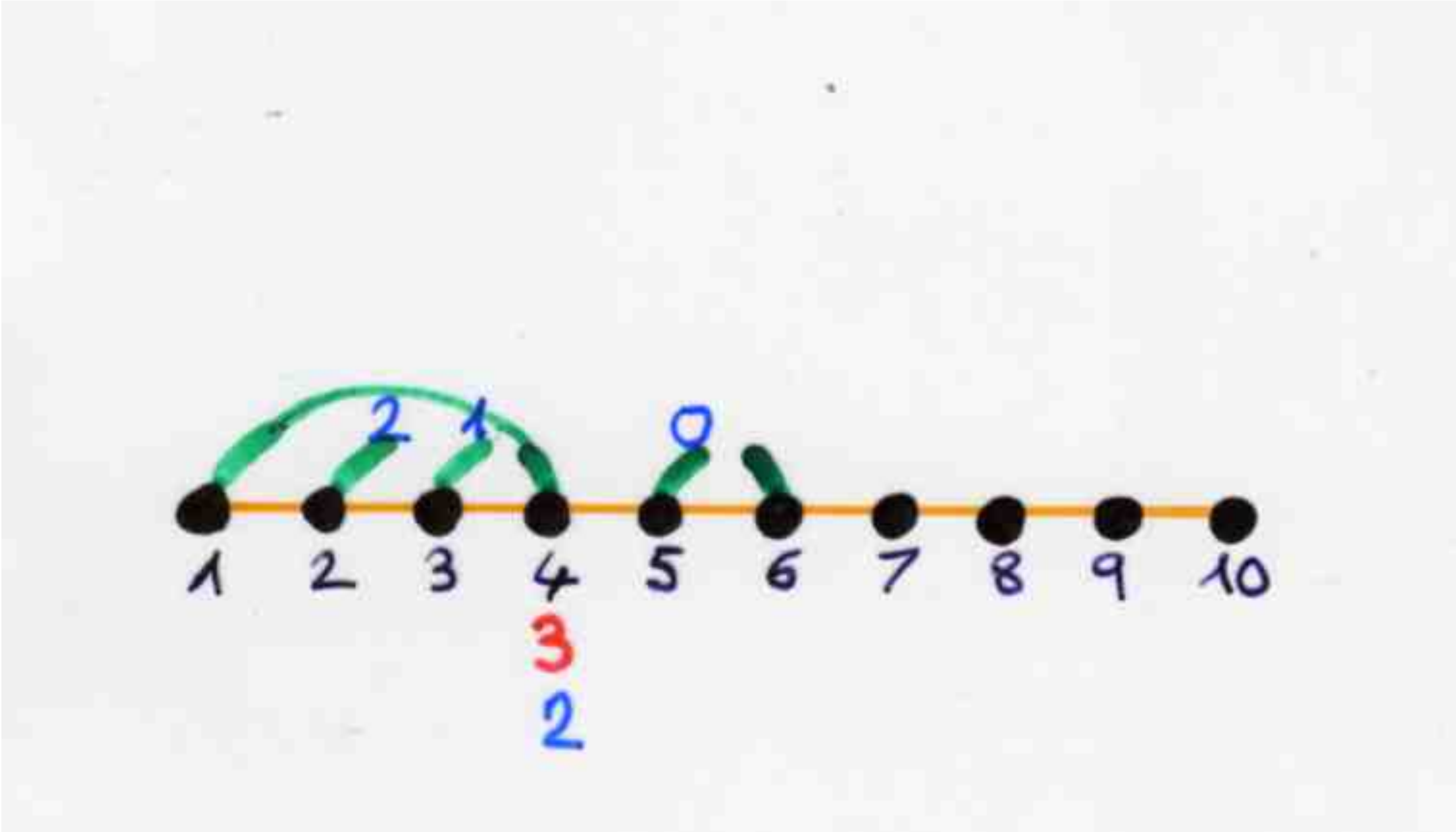

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

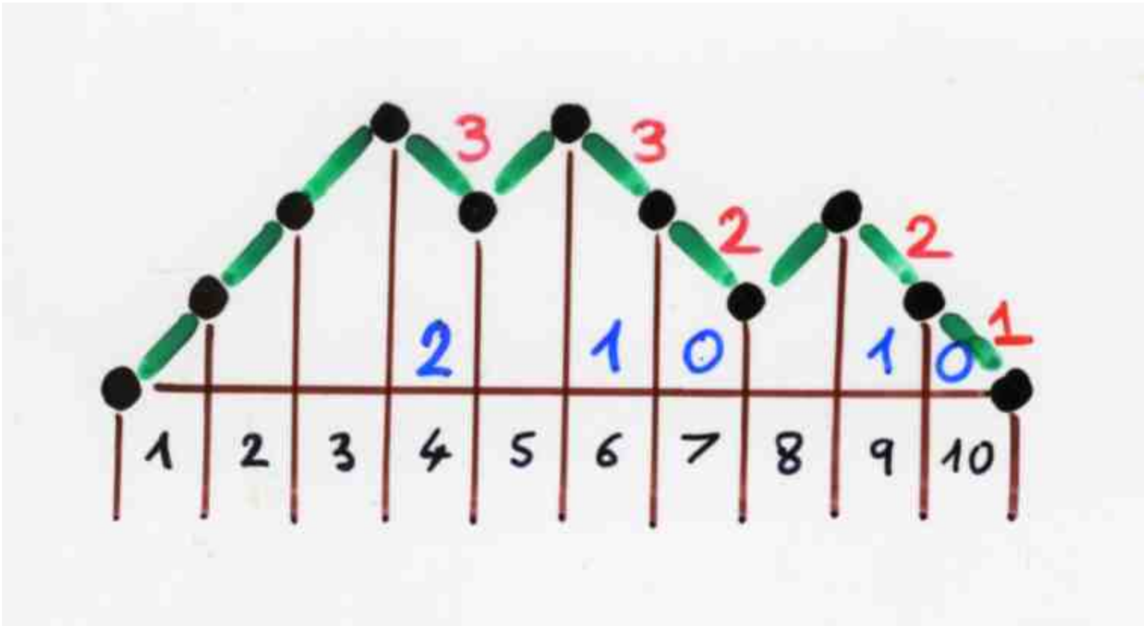

1
2
3
4
5
6
7
8
9
10
3
2

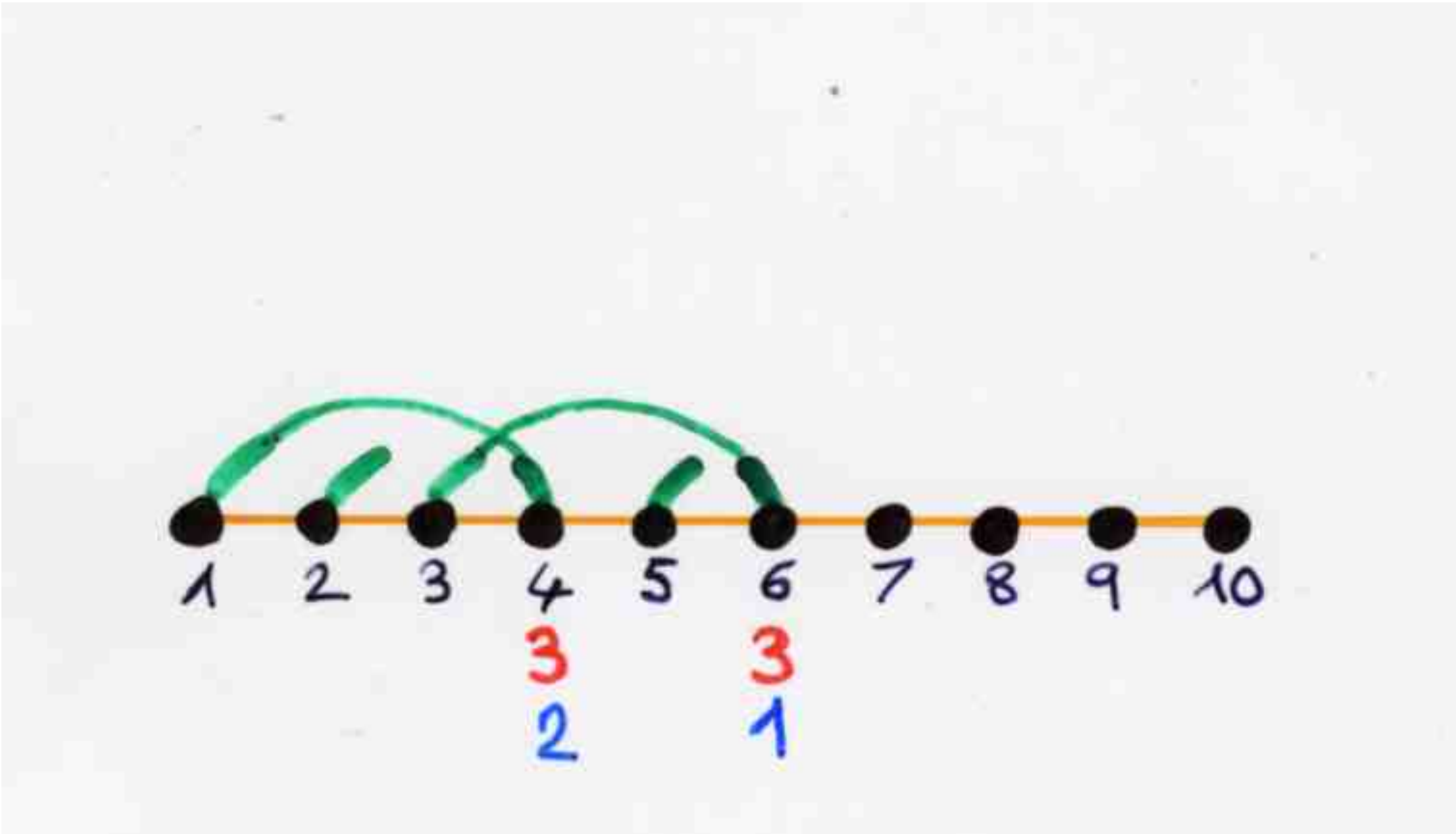

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

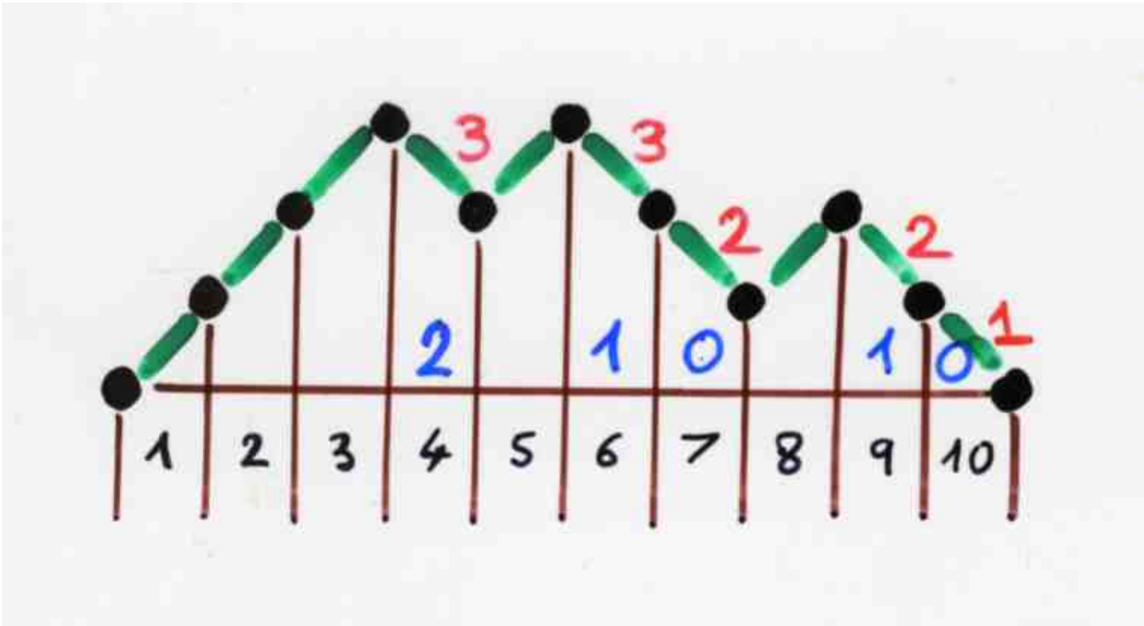

1 2 3 4 5 6 7 8 9 10
3
2

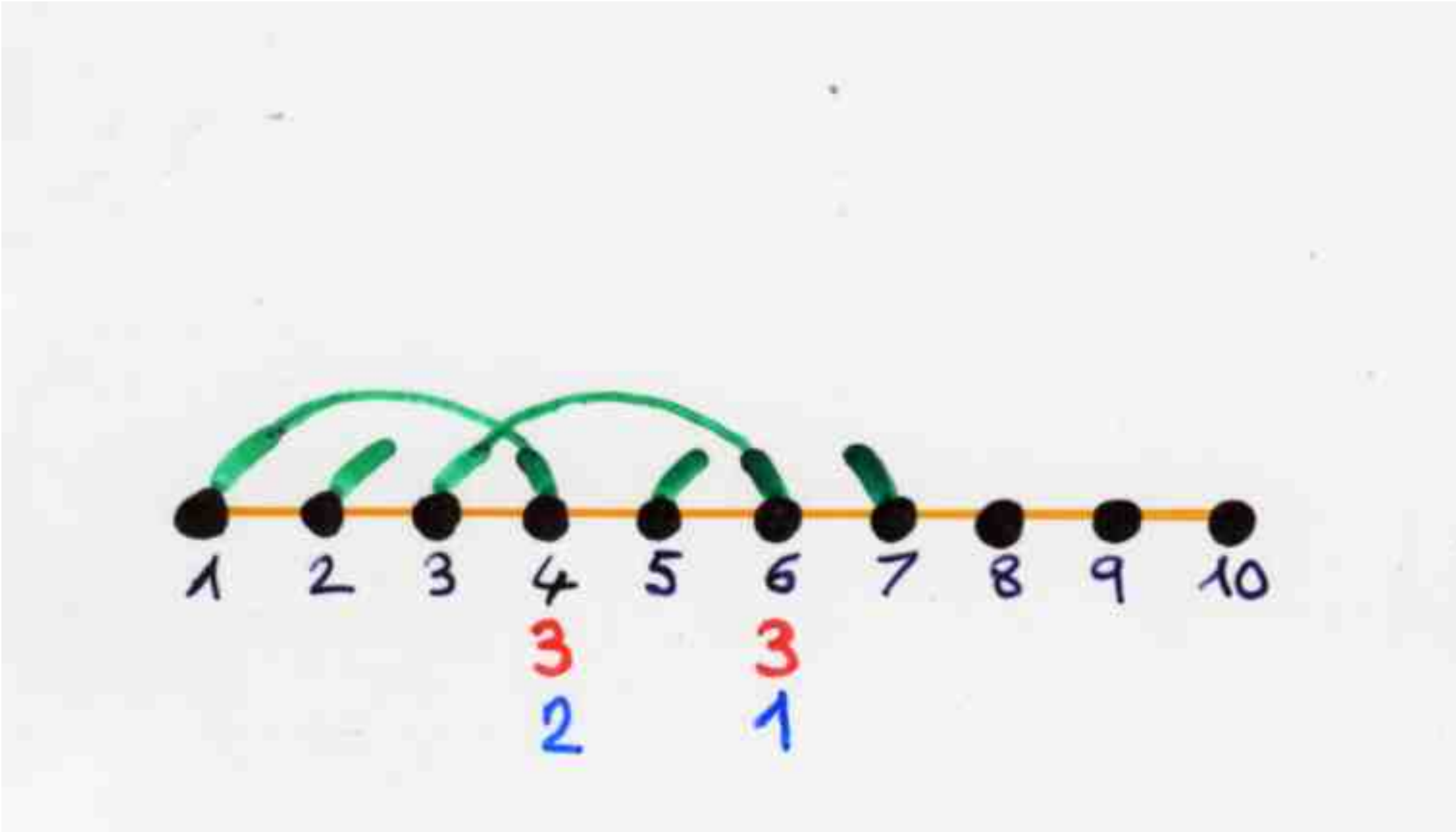

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

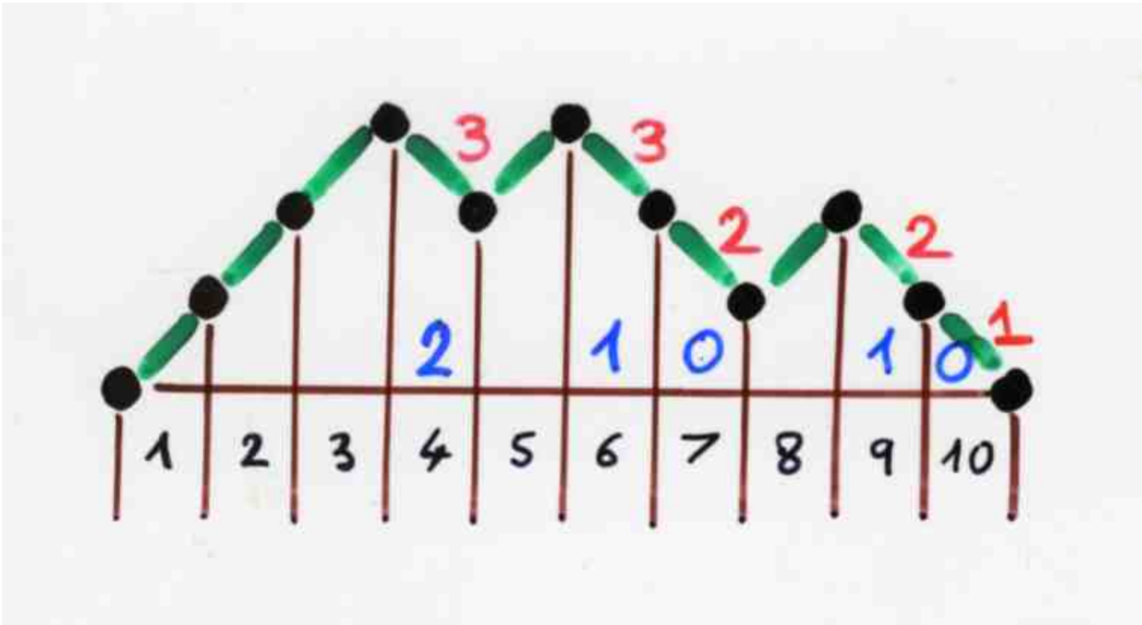

2
1
0
1 2 3 4 5 6 7 8 9 10
3
2

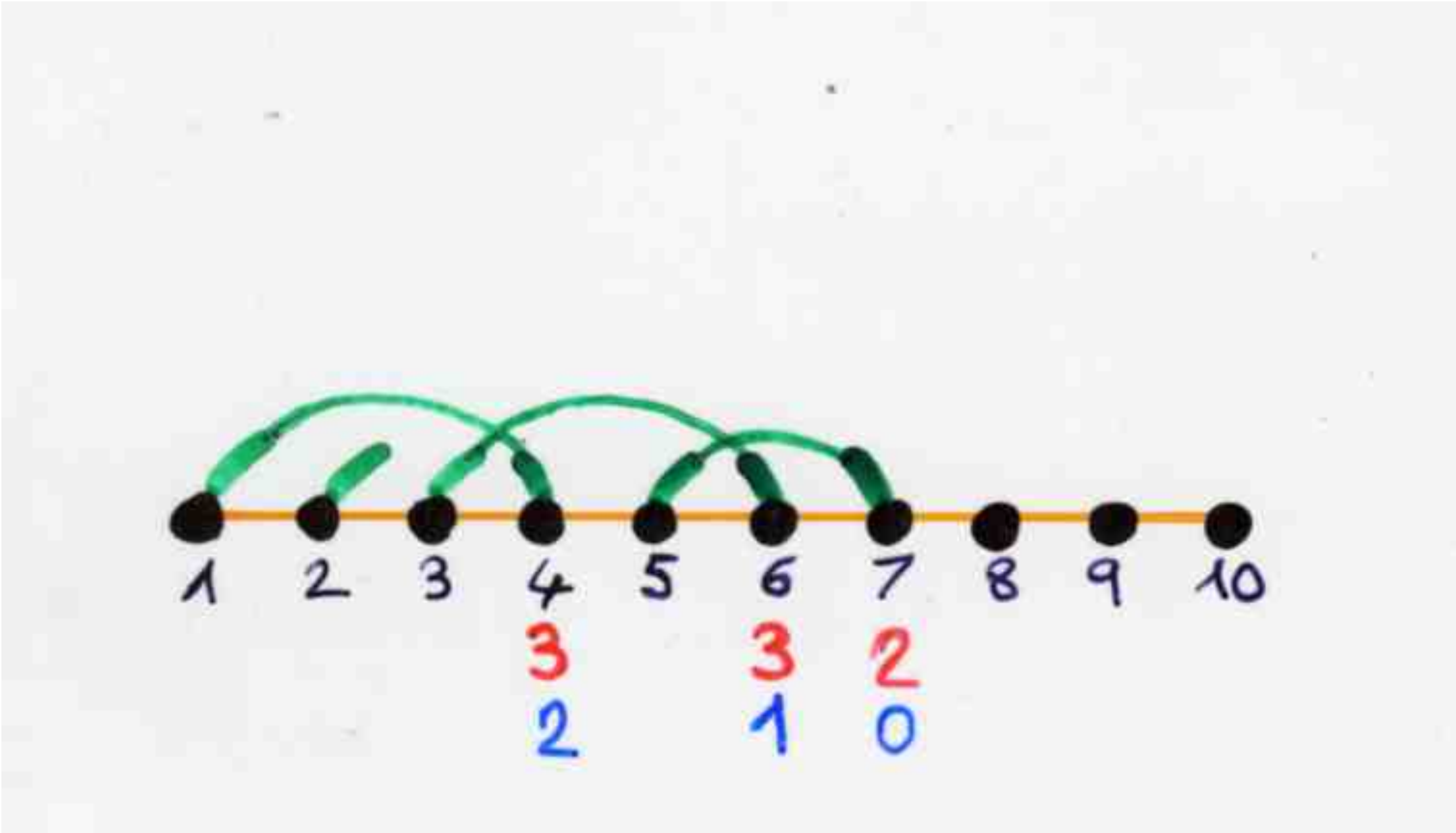

3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10

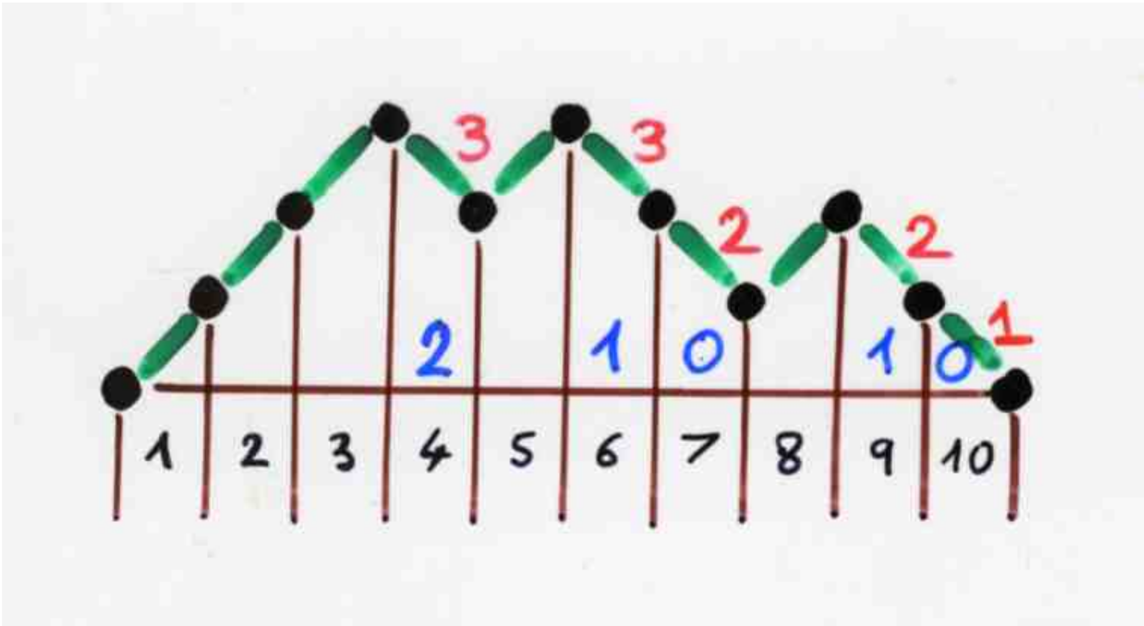

1 2 3 4 5 6 7 8 9 10
3 3
2 1

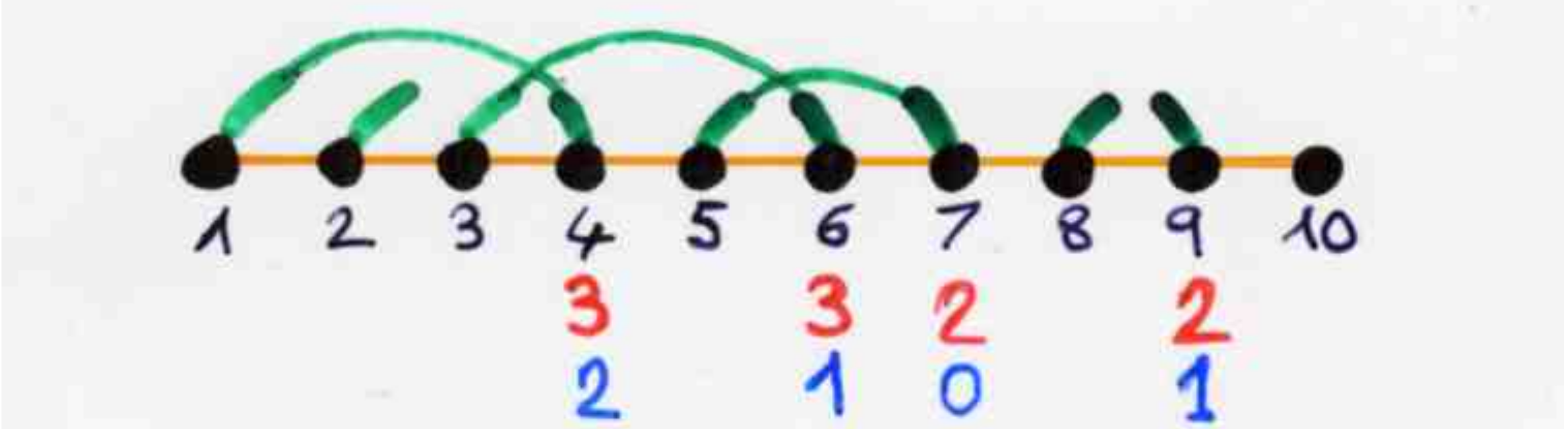

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

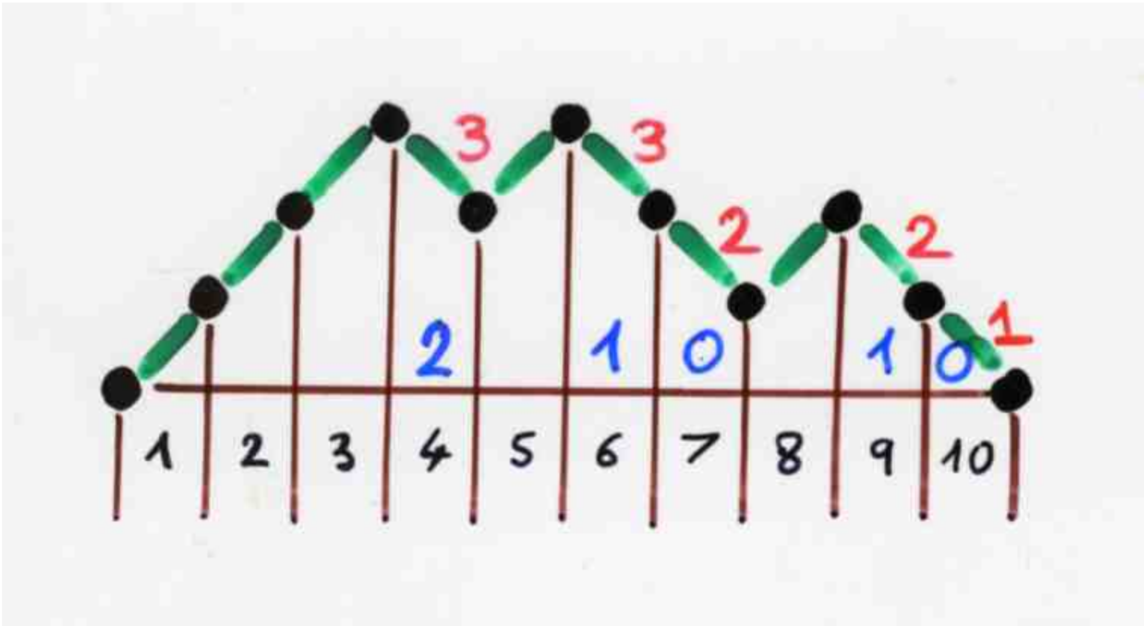

1 2 3 4 5 6 7 8 9 10
3
2
3
1

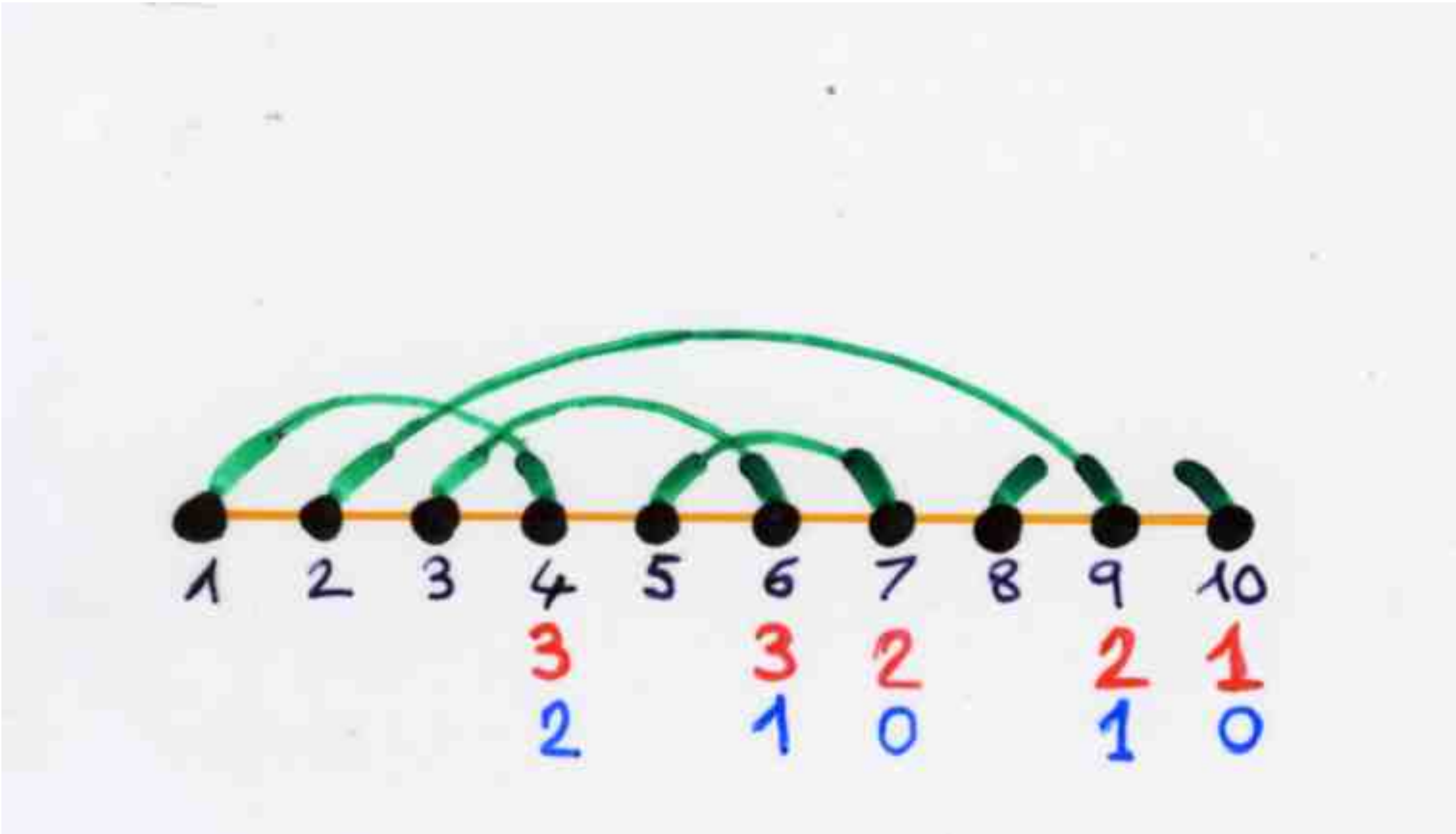

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

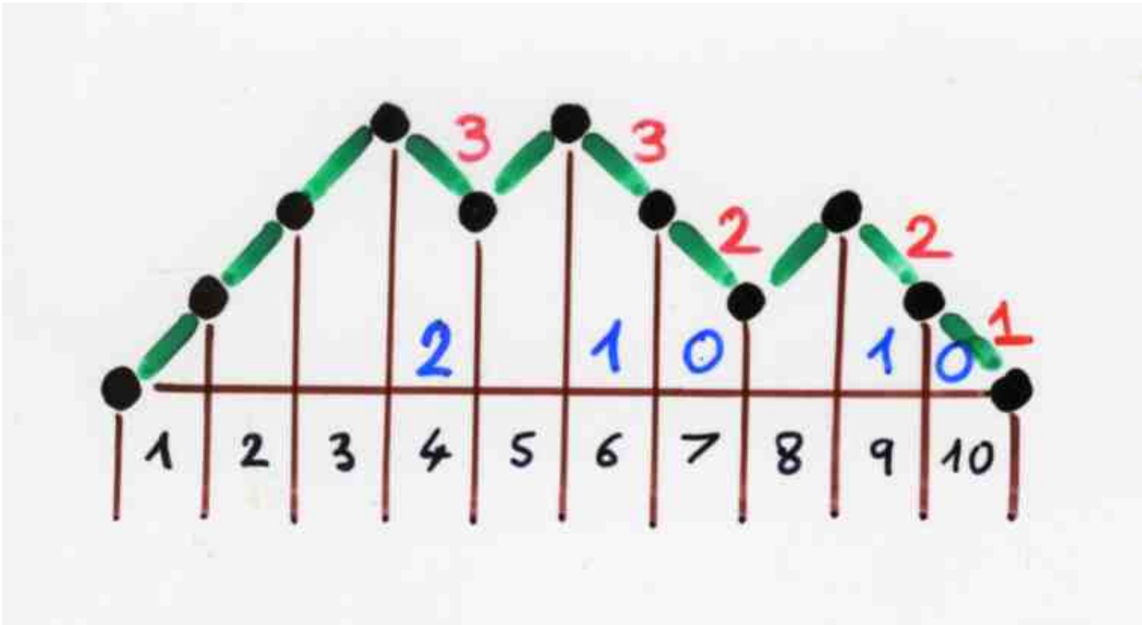

1 2 3 4 5 6 7 8 9 10
3 3 2
2 1 0


3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

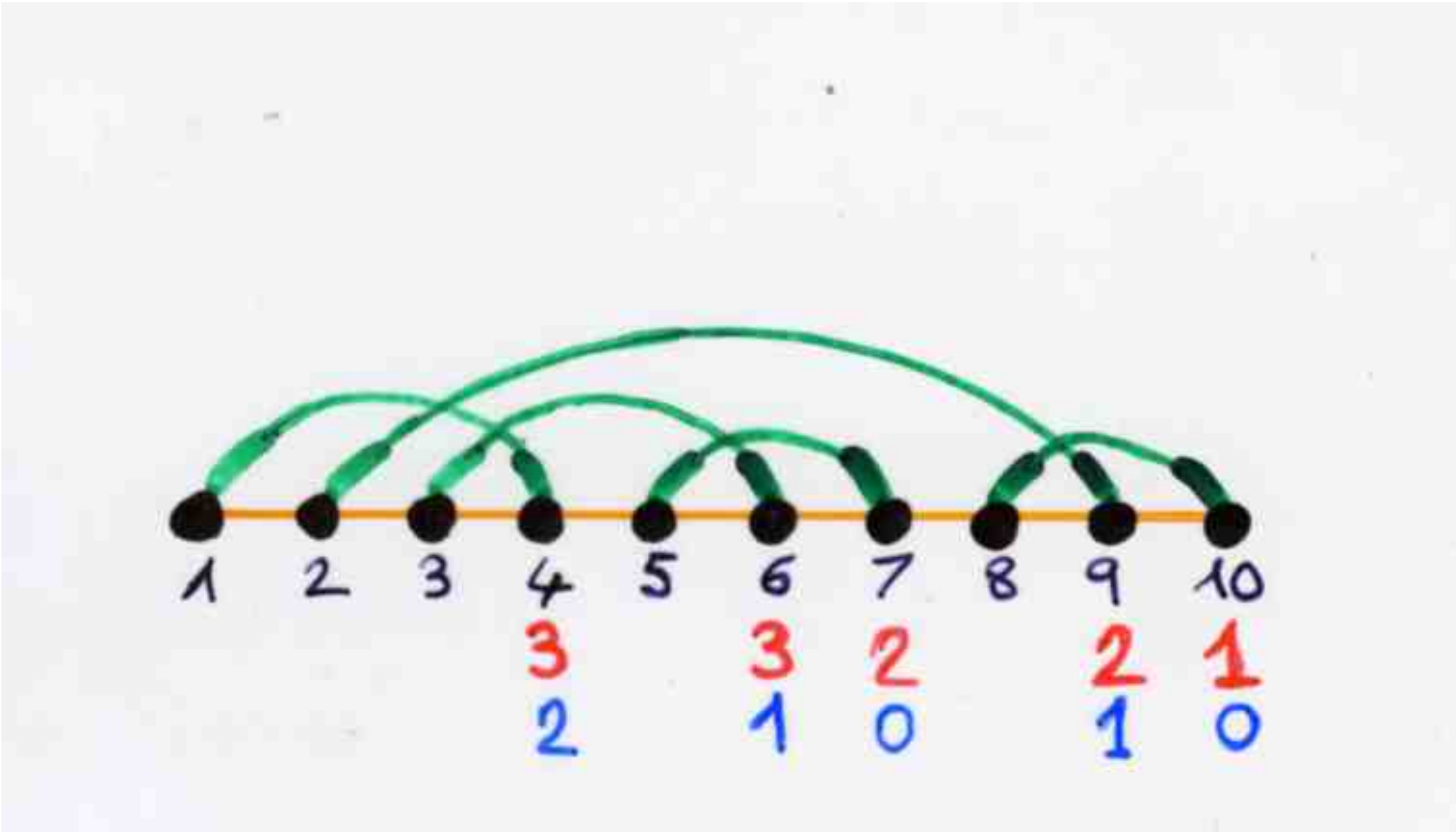

1 2 3 4 5 6 7 8 9 10
3 3 2 2
2 1 0 1


3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0



3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10



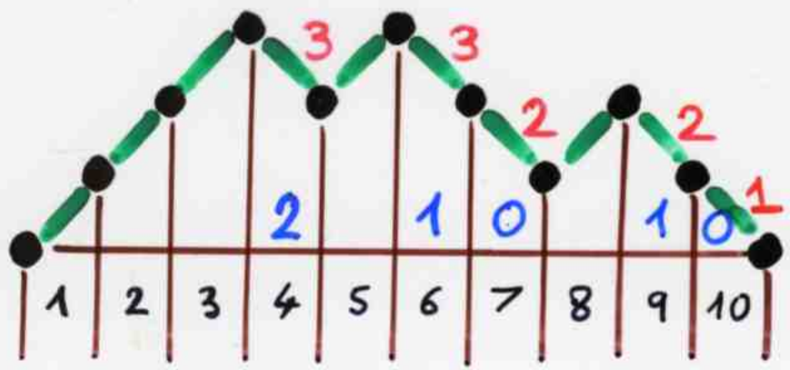
1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

q-analogue of Hermite histories

q-Hermite I
(continuous)

$\lambda_k = [k]_q$

$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$

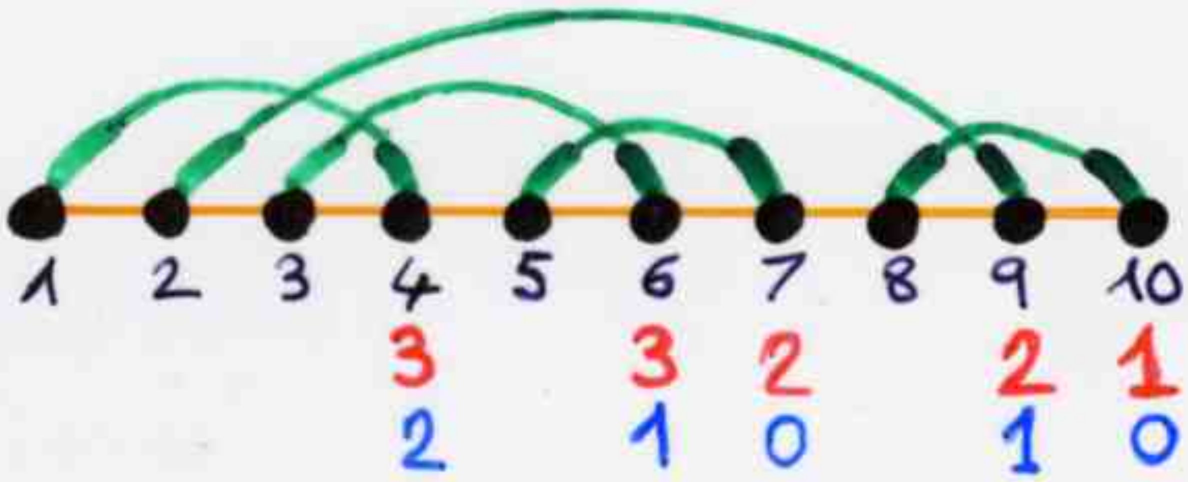


Hermite history related to ω

ω
Dyck path

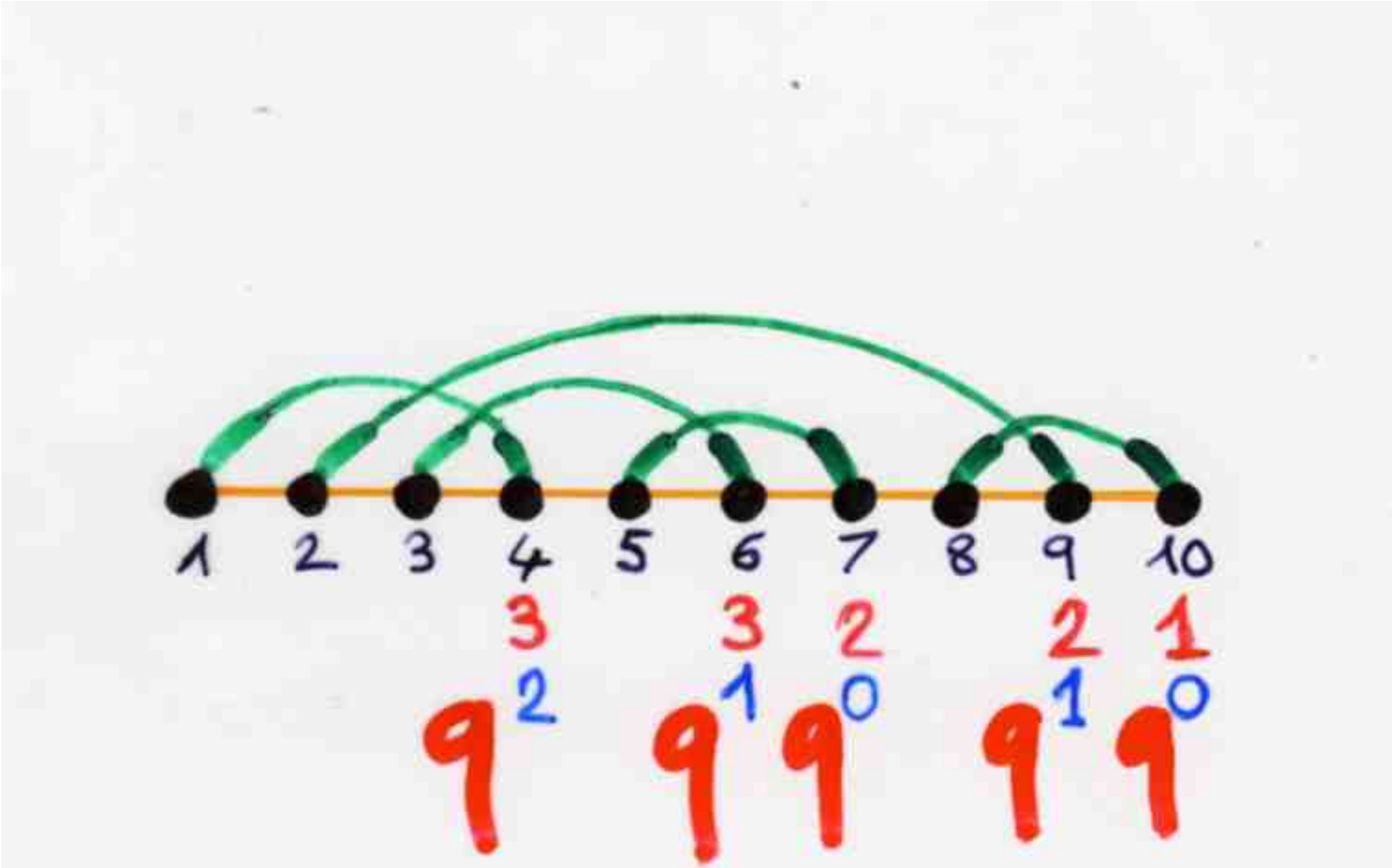
$v_q(h)$ $\quad q^{2+1+0+1+0} \quad = q^4$

q- weight



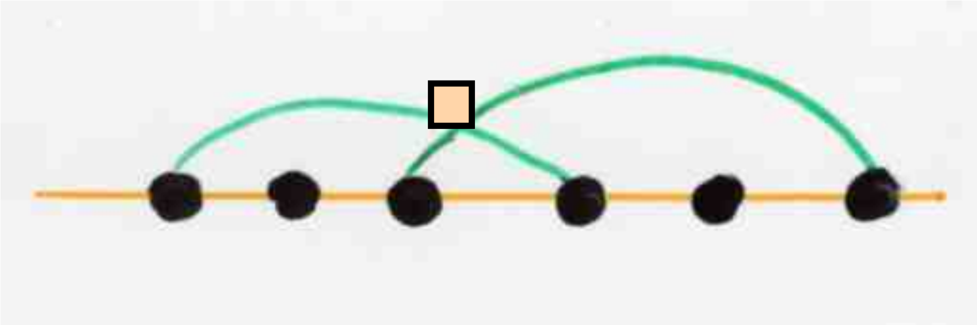
of an Hermite history

$v_q(h) = q^{\left(\sum_{i=1}^{n} p_i\right)}$



$$q^{2+1+0+1+0} = q^{4}$$

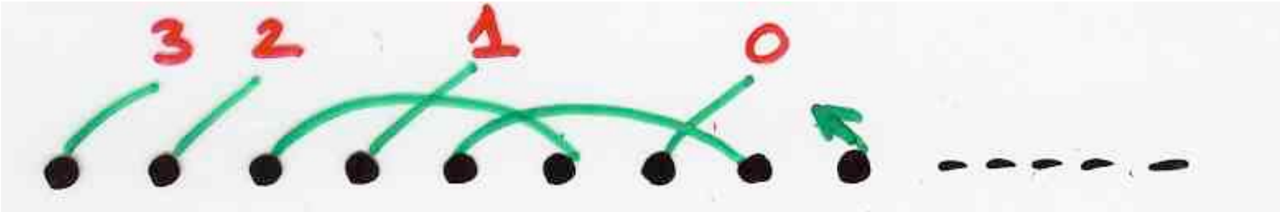
$$v_q(h)$$



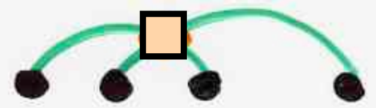
crossing

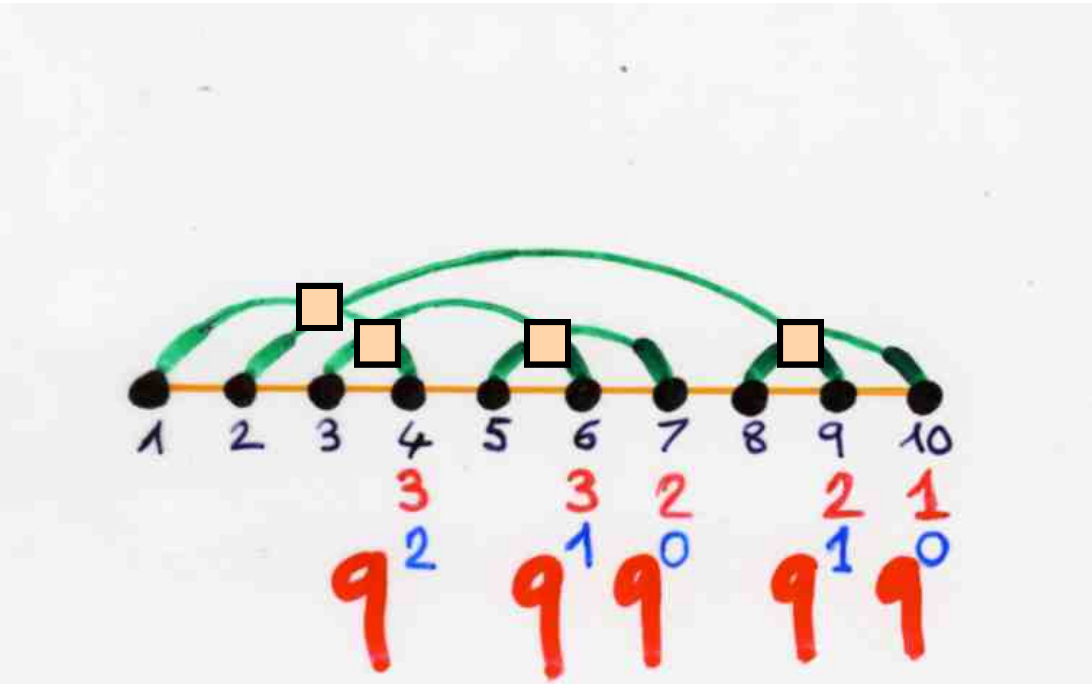
$$v_q(h) = cr(I)$$

number of crossings
of the chord diagram I



crossing





$$v_q(h)$$

$$q^{2+1+0+1+0} = q^4$$

continuous q-Hermite polynomials

q-Hermite I

$$\begin{cases} b_k = 0 \\ \lambda_k = [k]_q \end{cases}$$

$$\mu^{I}_{2n}(q) = \sum_{|h|=2n} v_q(h) = \sum_{h} q^{\mathrm{sum}(h)}$$

Hermite histories

Hermite histories

Proposition

$$\mu^{I}_{2n}(q) = \sum_{I} q^{\mathrm{cr}(I)}$$

chord diagrams on $[1, 2n]$

q-analog of
Hermite histories

(with nestings)

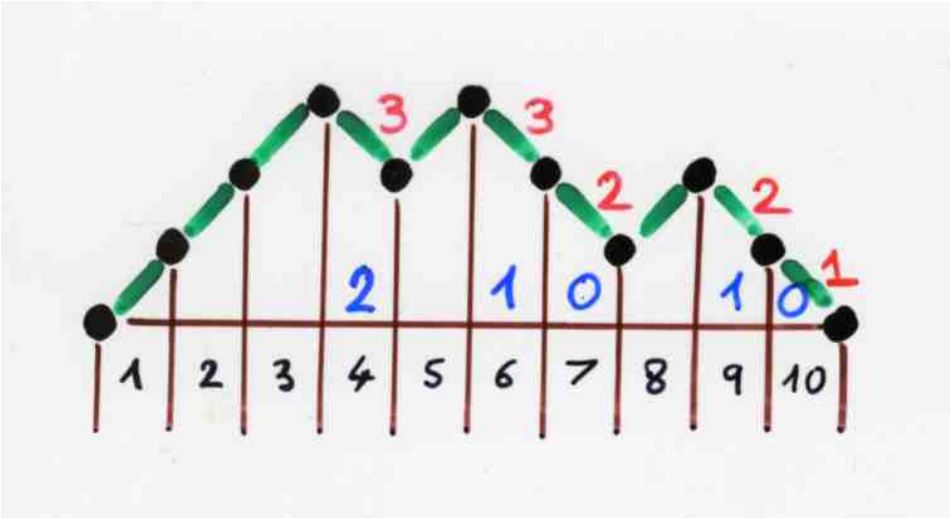
q-Hermite I
(continuous)

nesting 

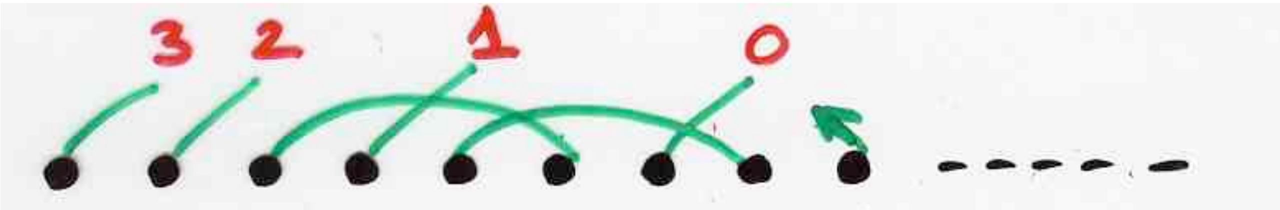
$$v_q(h) = \text{nest}(I)$$

number of nestings
of the chord diagram I

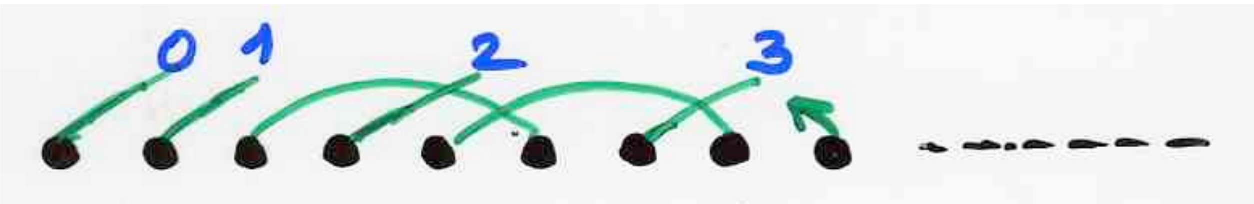
Hermite
history

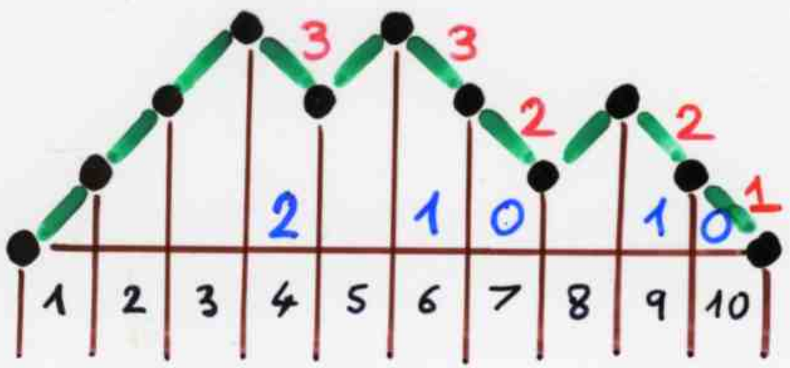


crossing



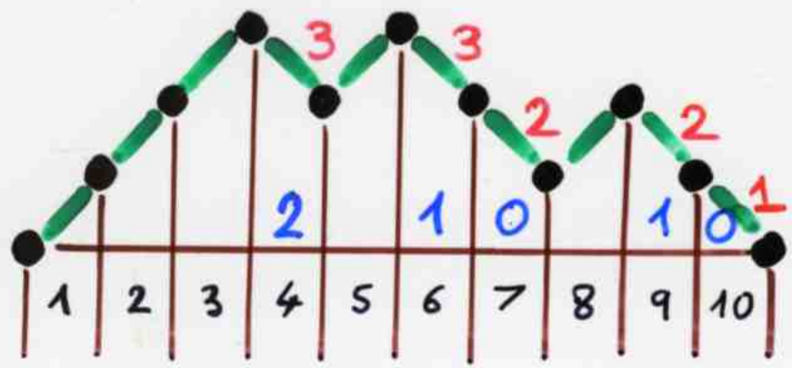
nesting

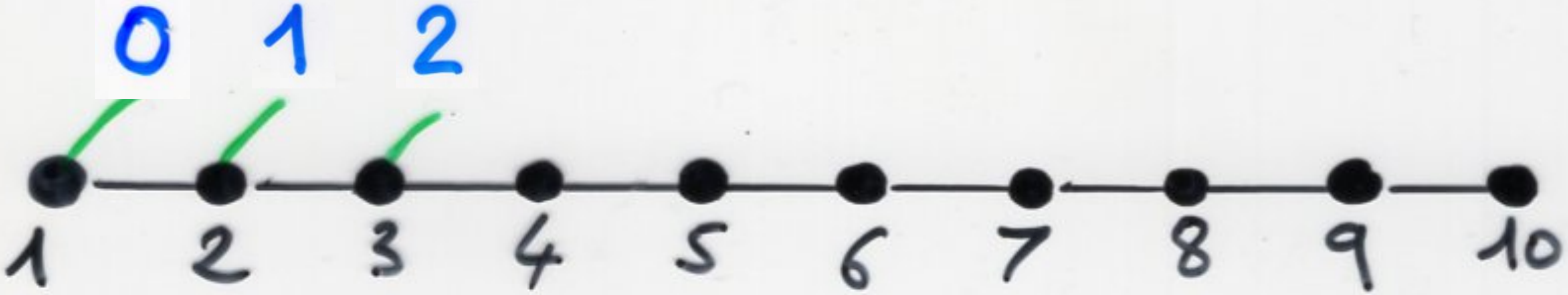


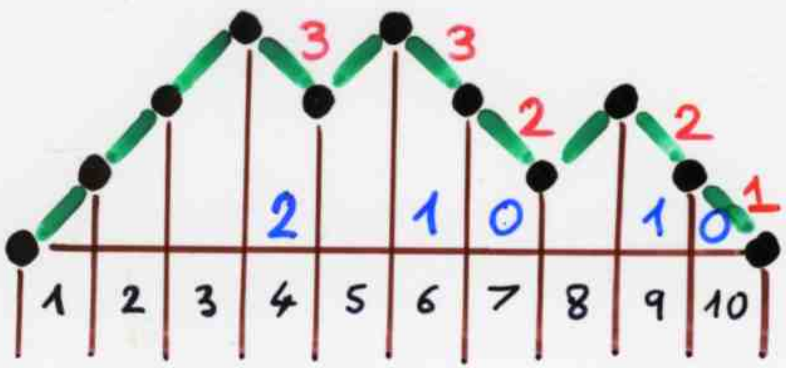

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

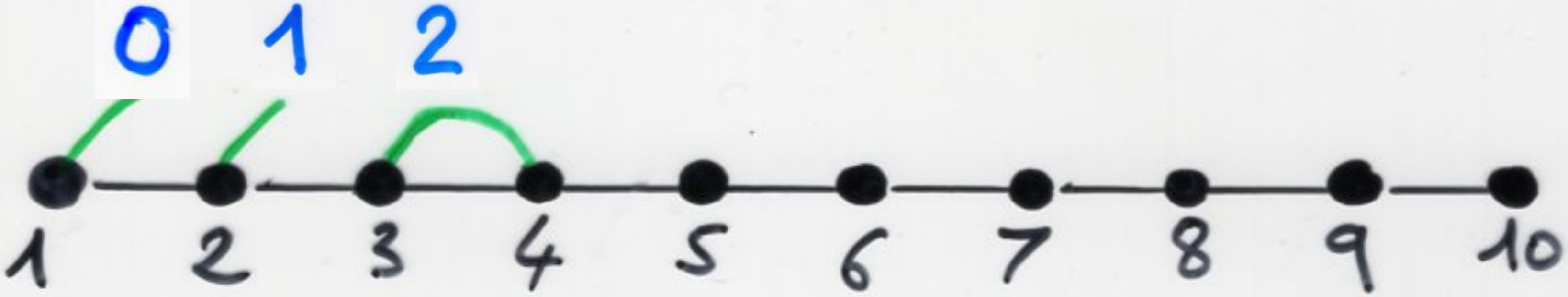

1 2 3 4 5 6 7 8 9 10

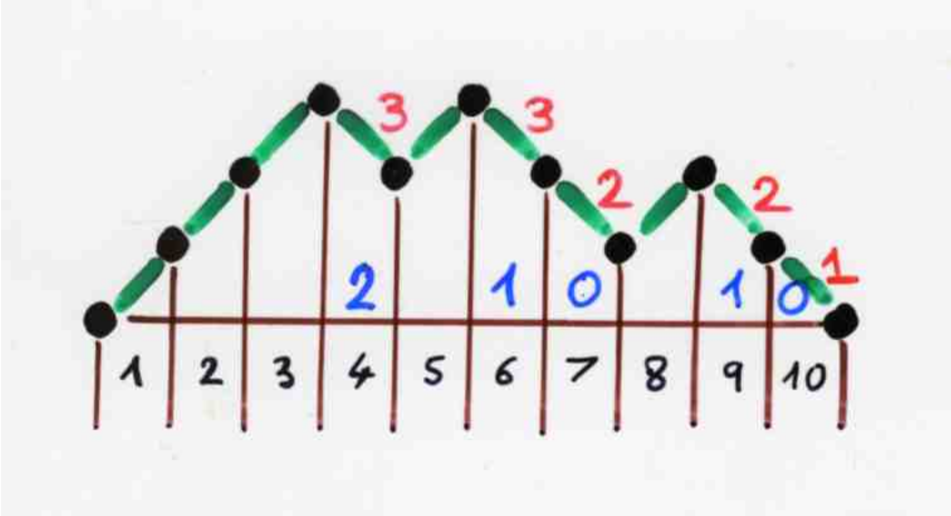

0 1 2 3

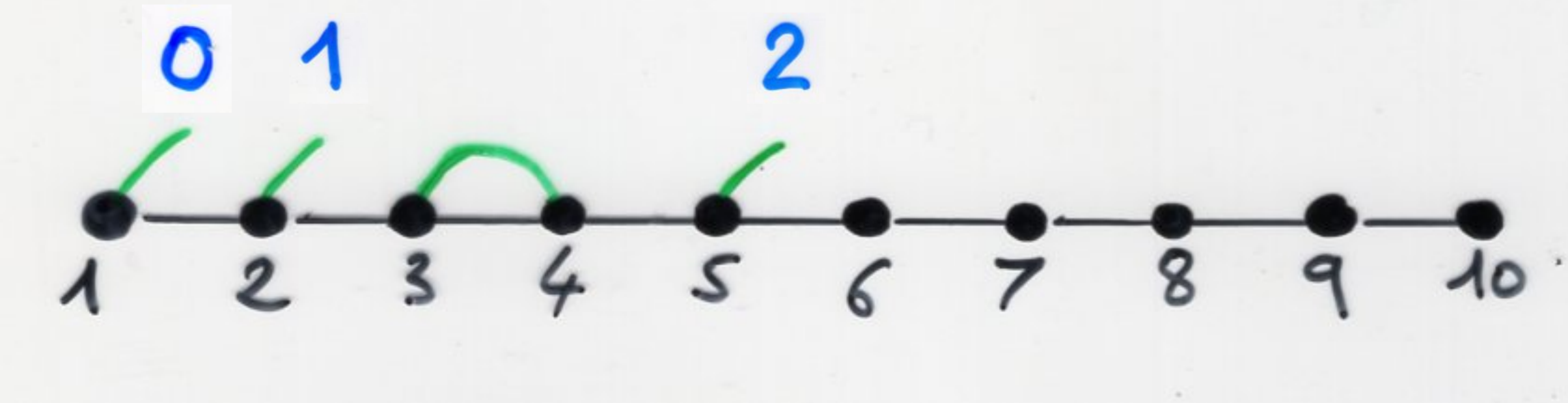

3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10

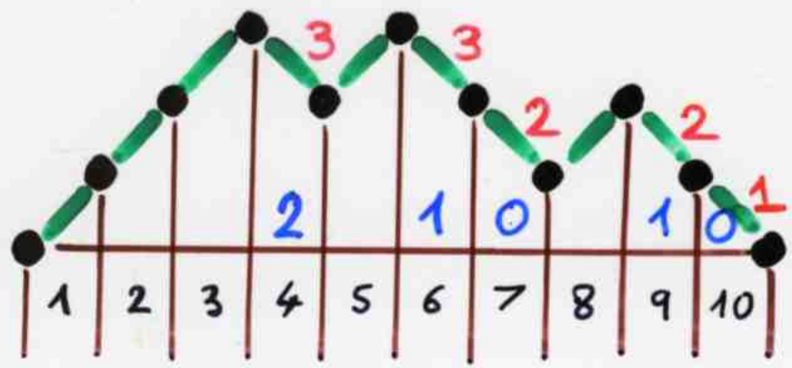

0 1 2
1 2 3 4 5 6 7 8 9 10


3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

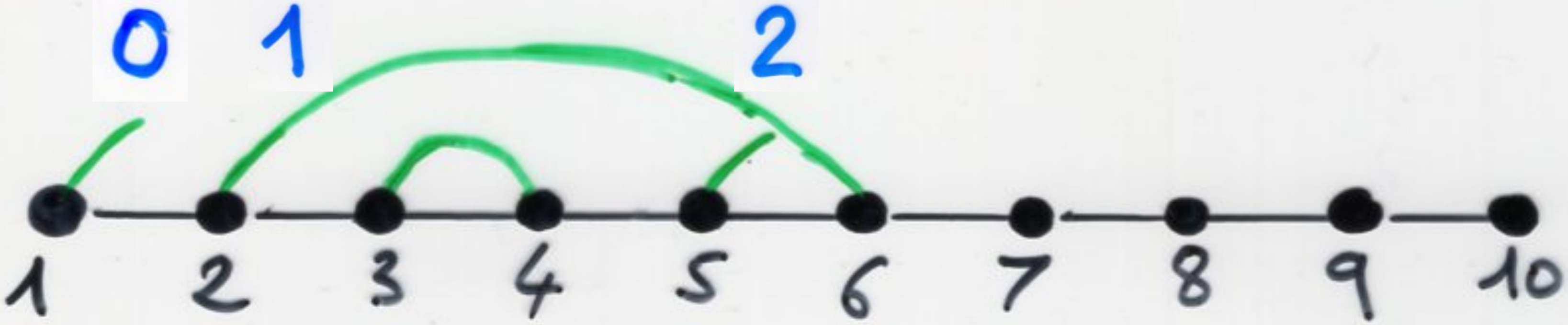

0
1
2
1
2
3
4
5
6
7
8
9
10


3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

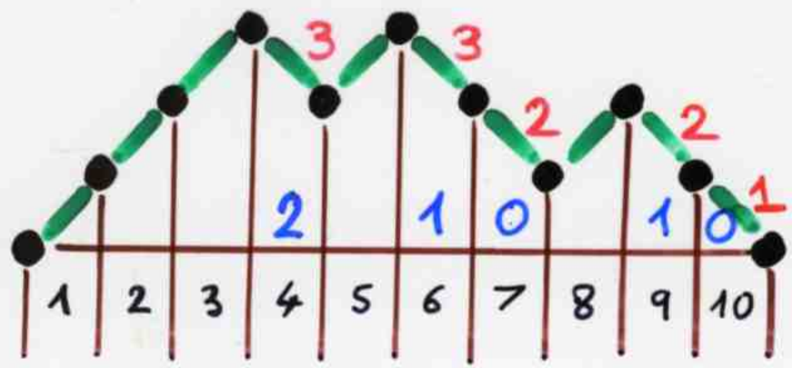

0
1
2
1
2
3
4
5
6
7
8
9
10

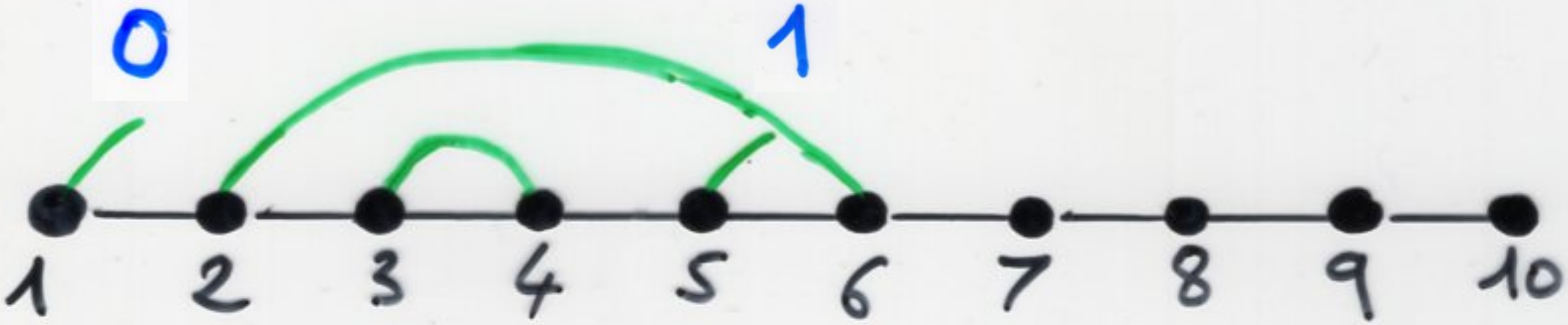


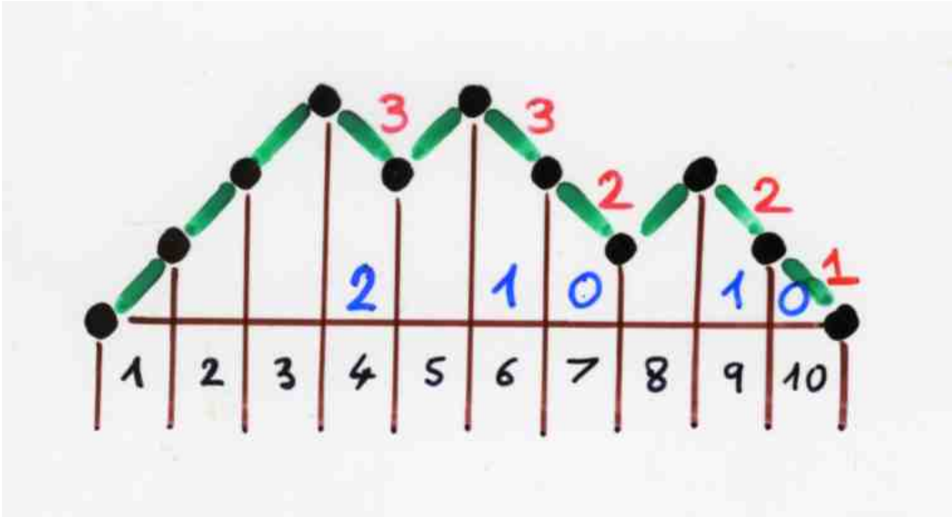
3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10

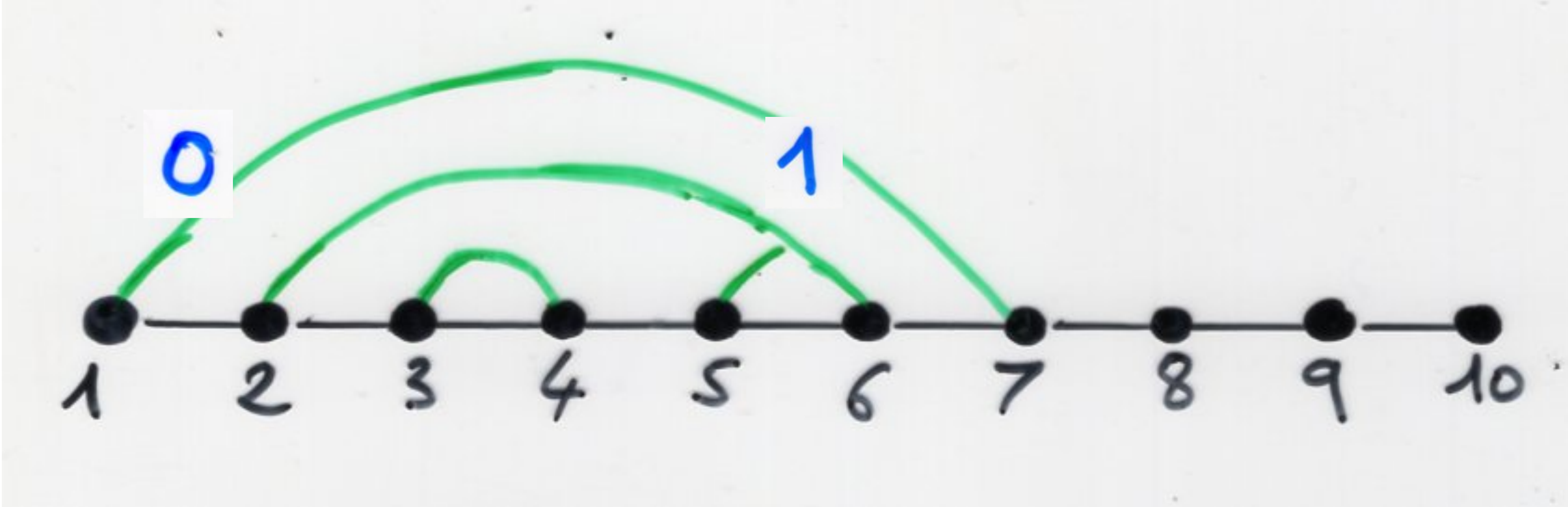


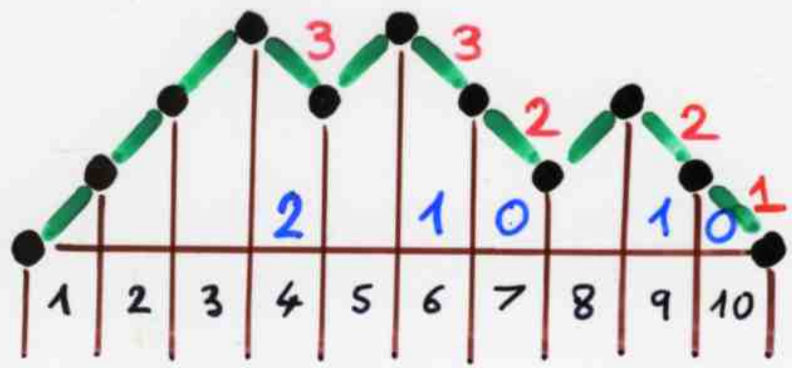
0 1 2
1 2 3 4 5 6 7 8 9 10

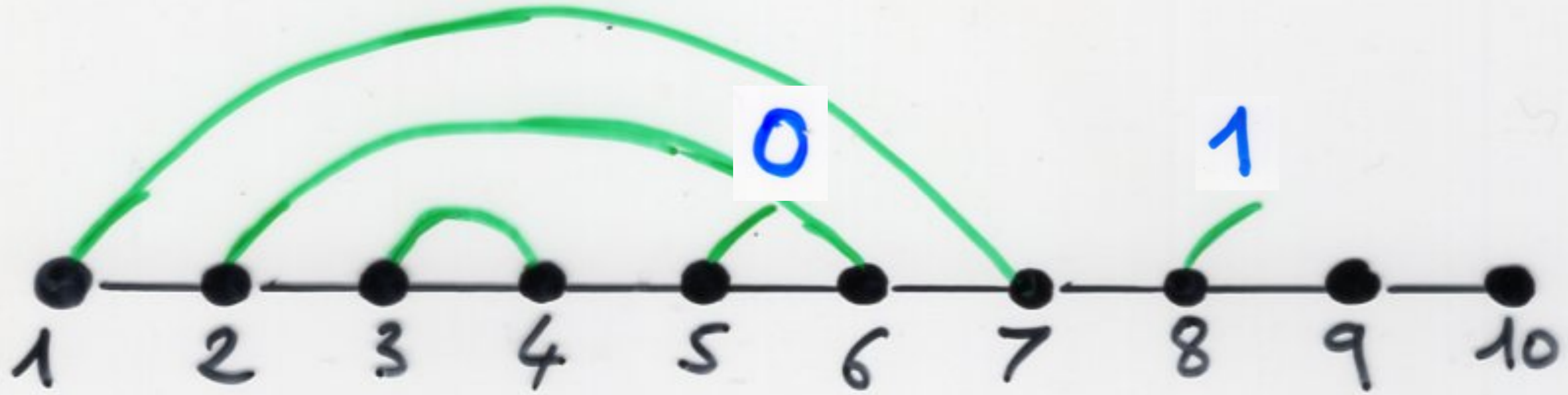

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

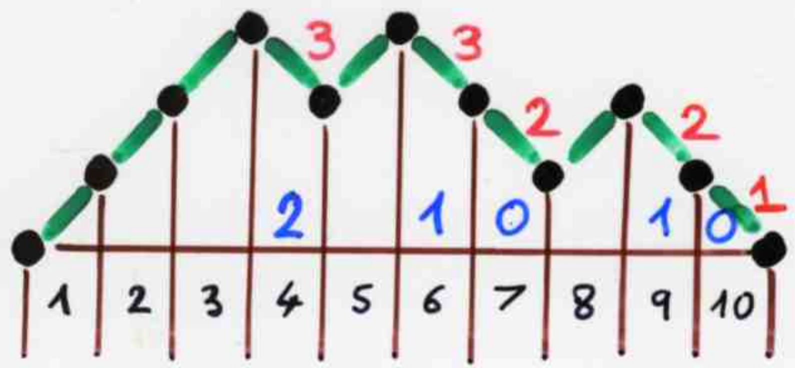

0
1
1 2 3 4 5 6 7 8 9 10

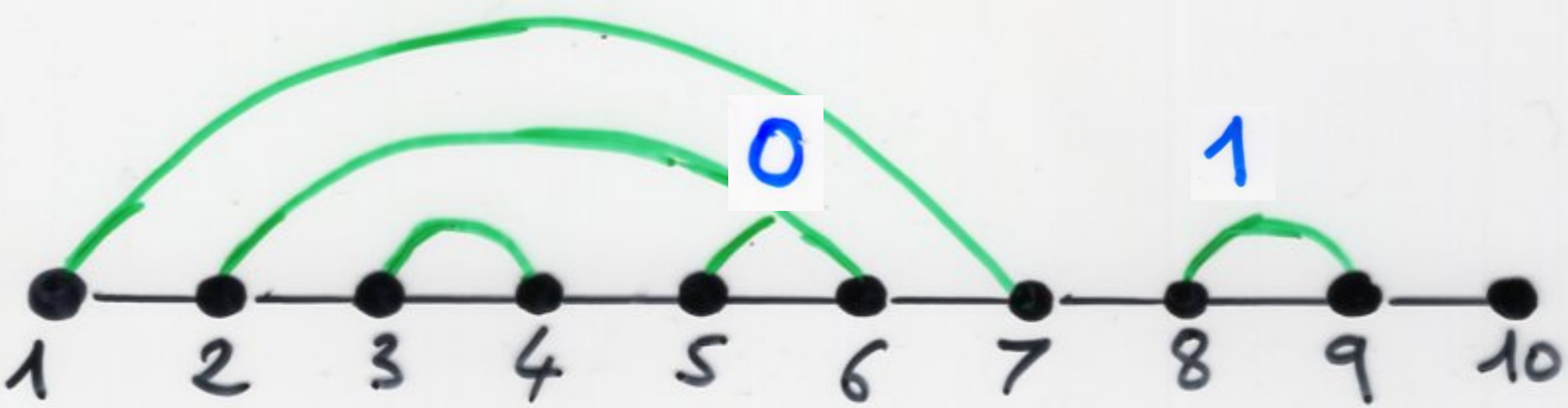

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

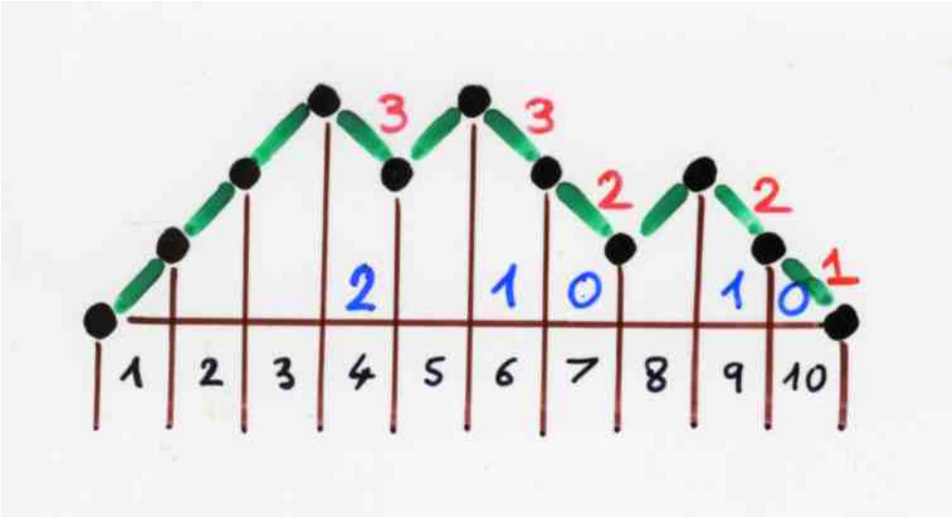

0
1
1 2 3 4 5 6 7 8 9 10

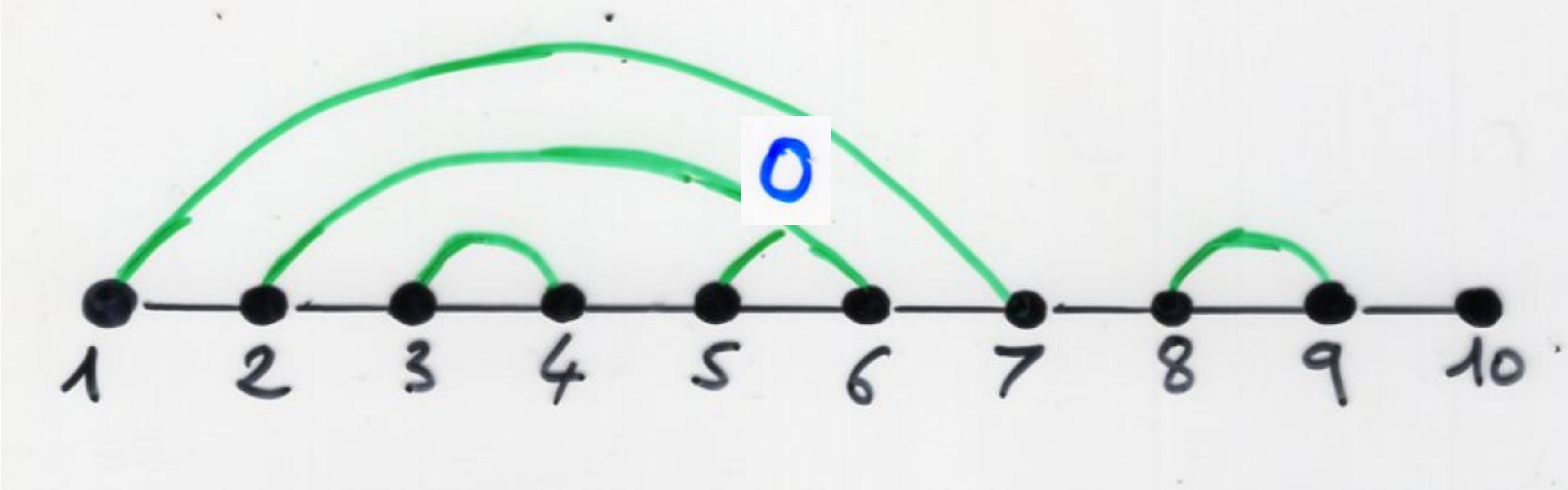

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

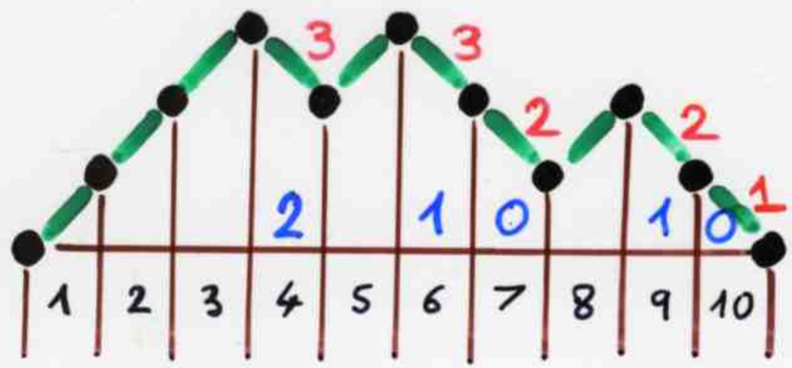

0
1
1 2 3 4 5 6 7 8 9 10


3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

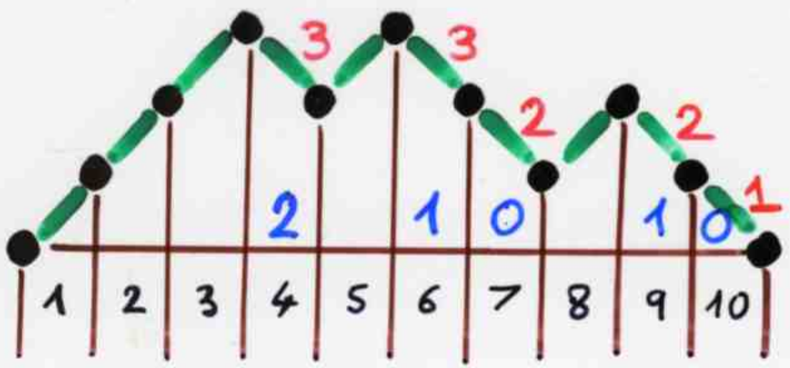

0
1
1
2
3
4
5
6
7
8
9
10

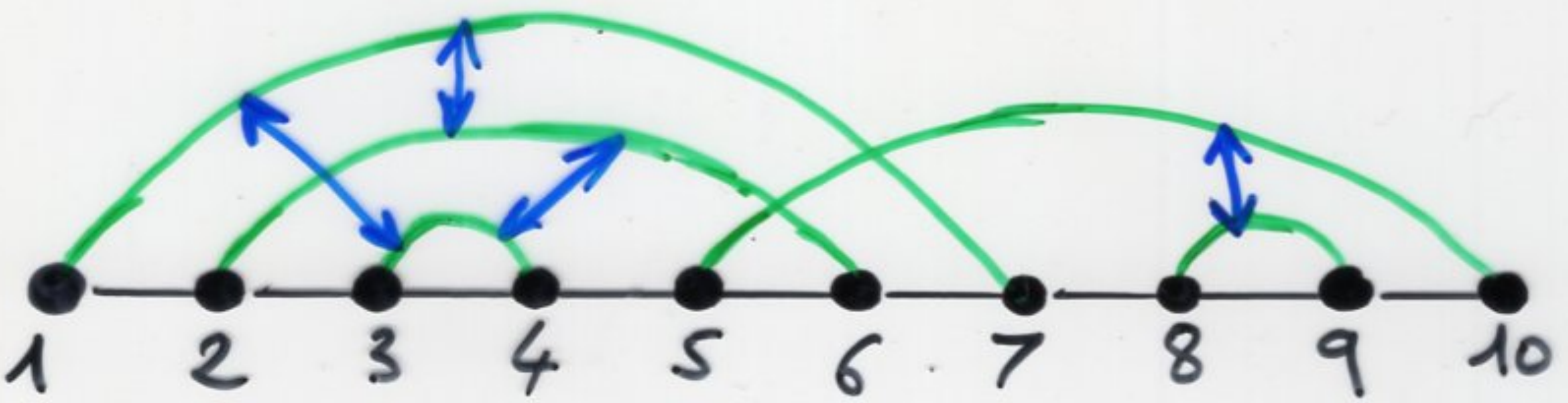

3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10


0
1 2 3 4 5 6 7 8 9 10


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10


3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10

moments of q-Hermite I polynomials

$$\mu_{2n}^{I}(q) = \sum_{\substack{I \\ \text{chord diagrams} \\ \text{on } [1,2n]}} q^{cr(I)}$$

$$= \sum_{\substack{I \\ \text{chord diagrams} \\ \text{on } [1,2n]}} q^{nest(I)}$$

$$\mu^{I}_{2n}(q) = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{\binom{k}{2}}$$

Touchard (1952)
Riordan (1975)
Read (1979)

Penaud (1995)
bijective proof

the philosophy of « histories »

and its q-analogues

S states

$s \xrightarrow{A} t$

operators

history

weight

$V_A(s,t)$ = number of possibilities to apply A

$H = h_1 h_2 \cdots h_n$

sequence of operators

initial state s_0

$P = (p_1, p_2, \dots, p_n)$

$s_i \xrightarrow{h_i} s_{i+1}$

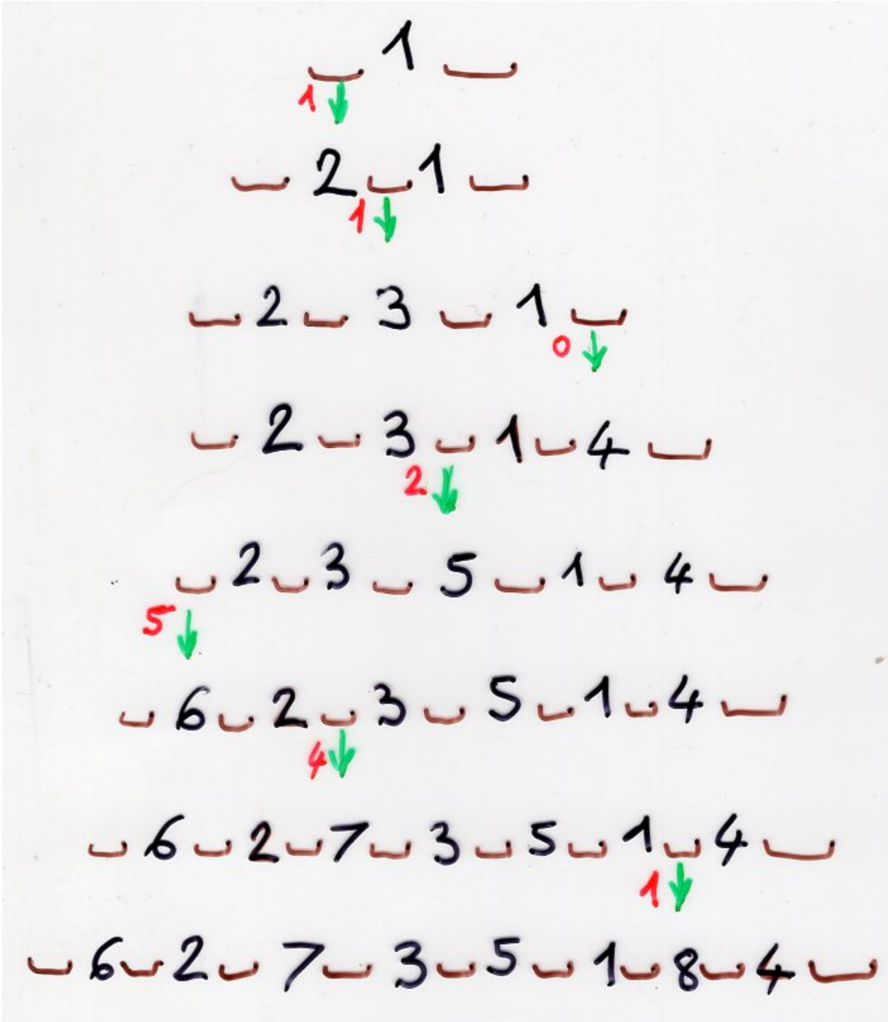
$0 \leq p_i < V_{h_i}(s_i, s_{i+1})$

q-weight

$V_q(H) = q^{\left(\sum_{i=1}^{n} p_i\right)}$

Inv

number of inversions



Maj

Major index

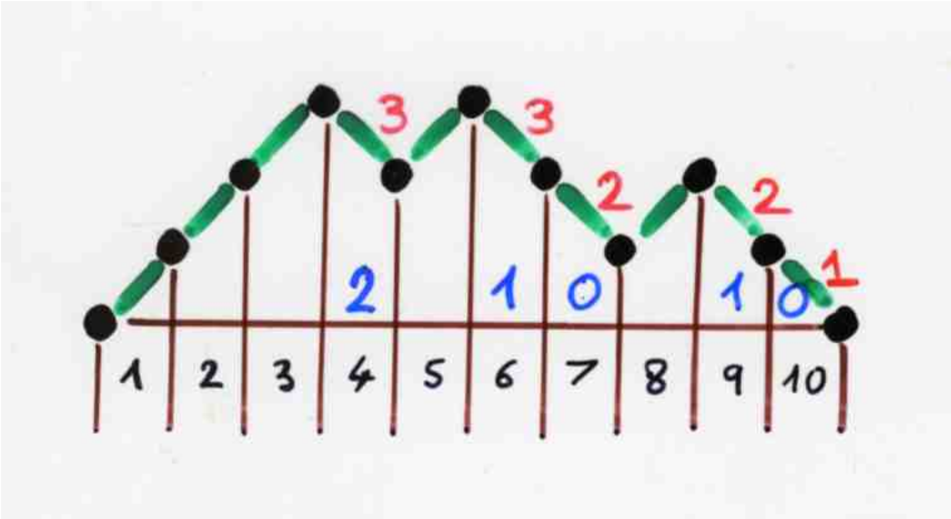
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

$$\mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma(i) > \sigma(i+1)}} i$$

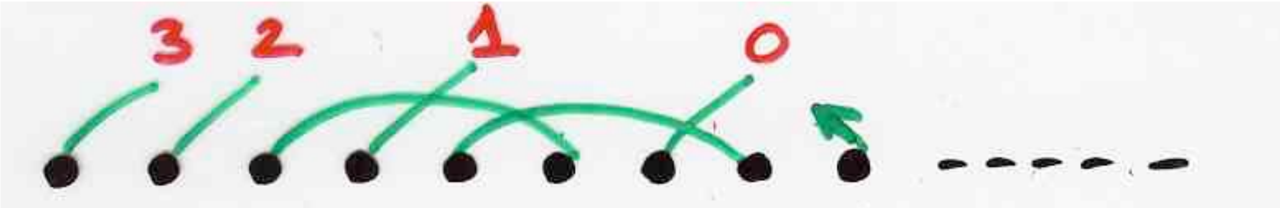
$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}$$

Mahonian
distribution

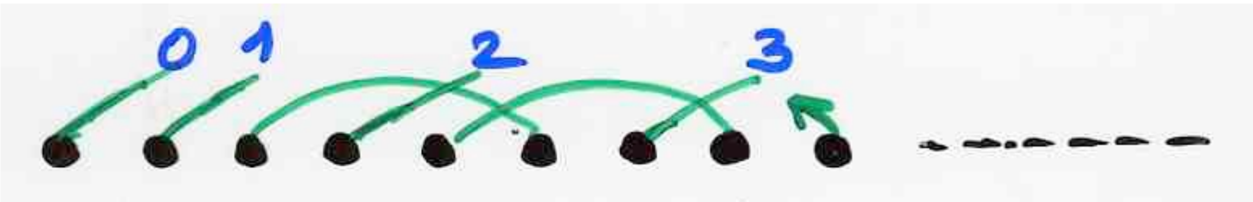
Hermite history

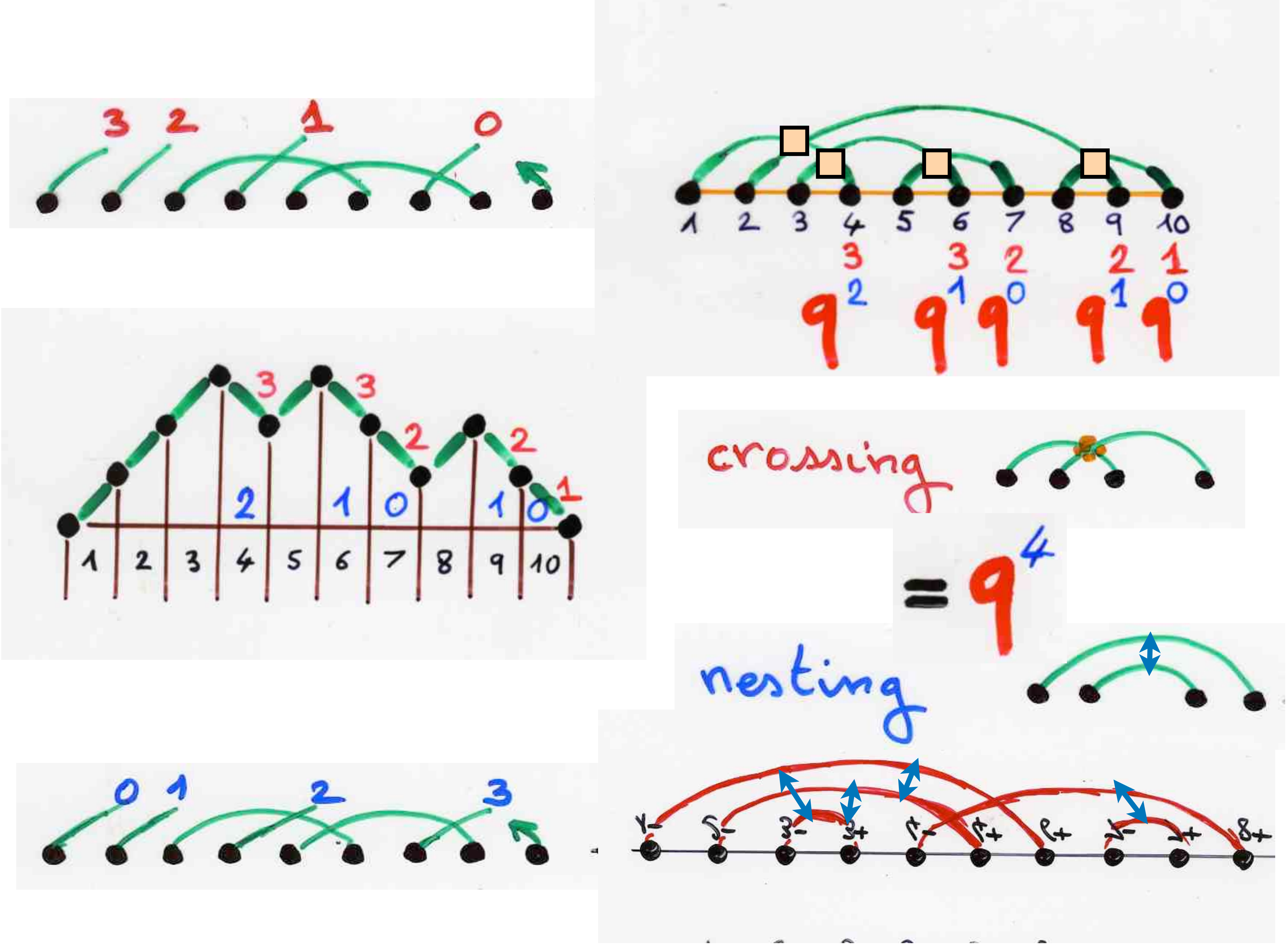


crossing 



nesting 

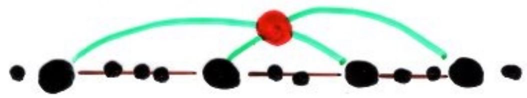




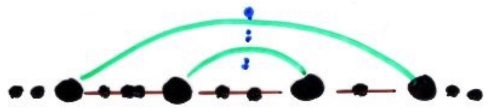
3 2 1 0
1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0
q q q q q
3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10
crossing
= q4
nesting
0 1 2 3

$$\sum_{\substack{\text{chord} \\ \text{diagrams } I \\ [1,2n]}} q^{cr(I)} = \sum_{\substack{I \\ \text{chord} \\ \text{diagrams} \\ [1,2n]}} q^{nest(I)} = \sum_{\substack{h \\ \text{Hermite} \\ \text{histories} \\ |h| = 2n}} q^{sum(h)}$$

$cr(I)$ = number of crossings



$nest(I)$ = number of nestings

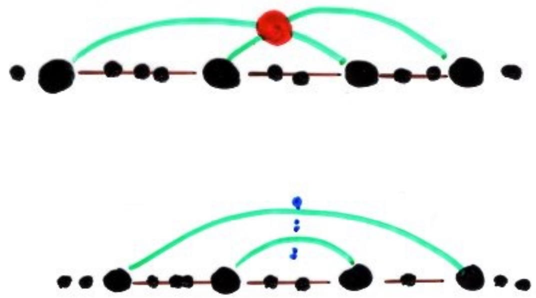


$$sum(h) = \sum_i (p_i - 1)$$

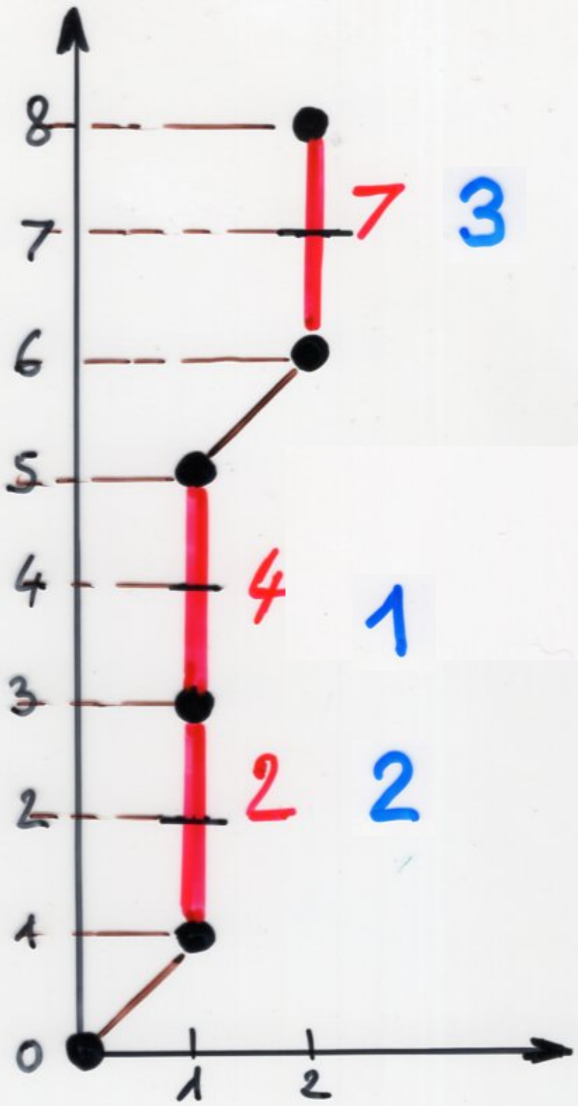
exercise

$$\sum_{\substack{I \\ \text{chord diagrams} \\ [1,2n]}} q^{cr(I)}\, t^{nest(I)}$$

(q,t)-polynomial
symmetric in q and t



q-analogue of Hermite polynomials

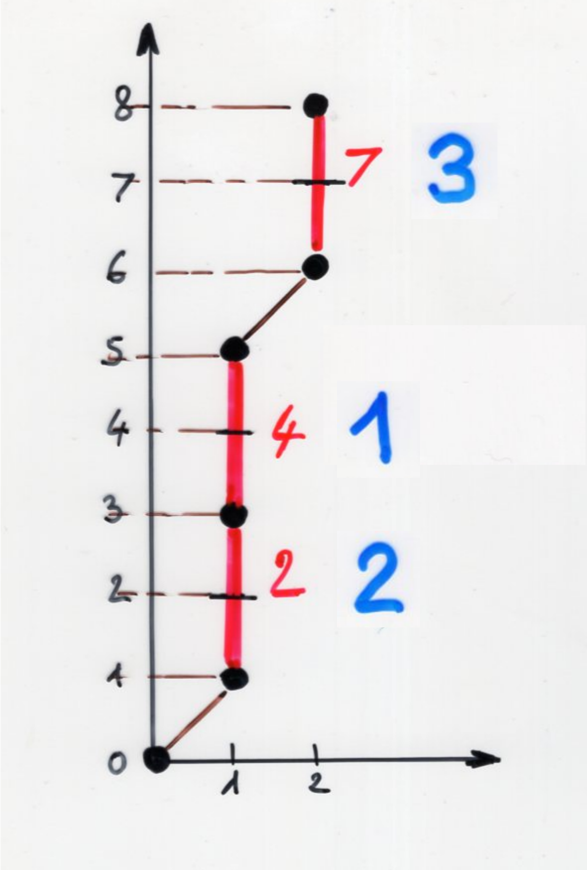


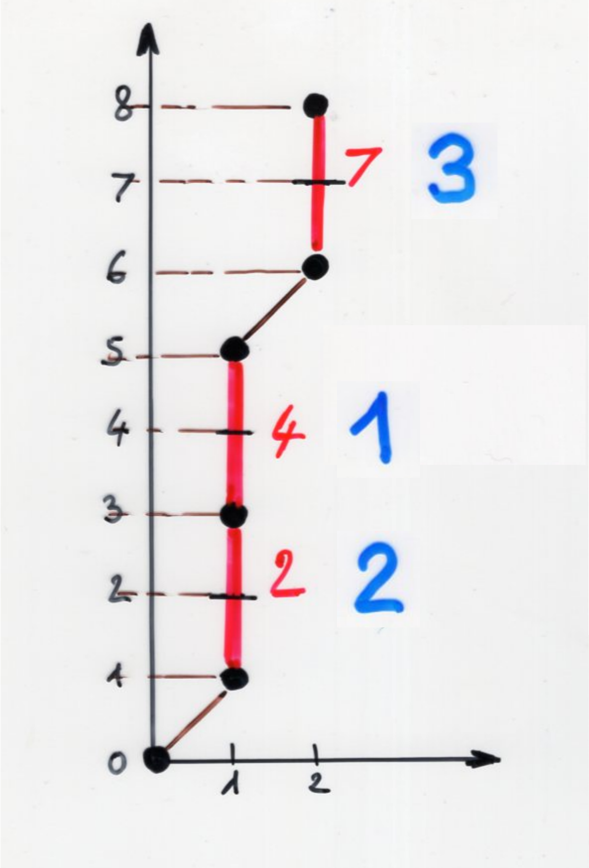
Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

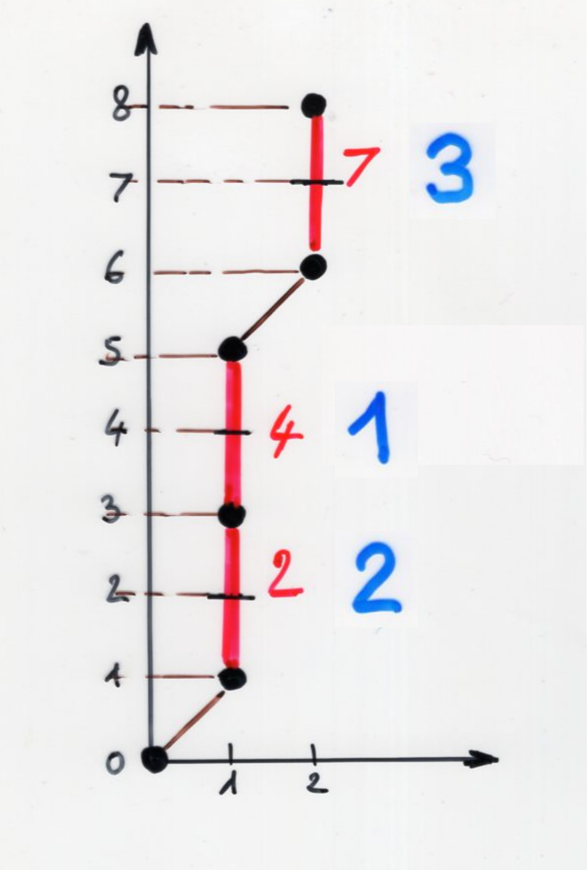
$H_n(x)$

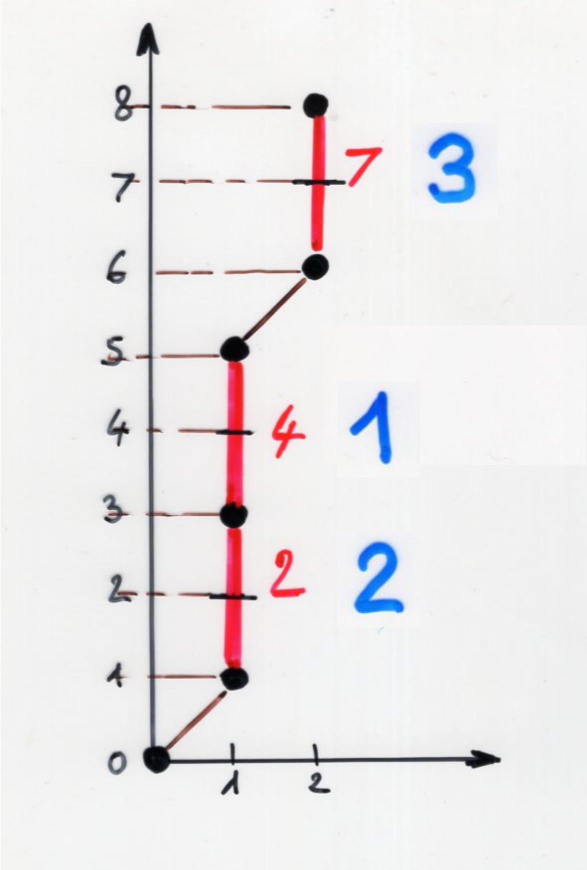
(combinatorial) Hermite polynomials

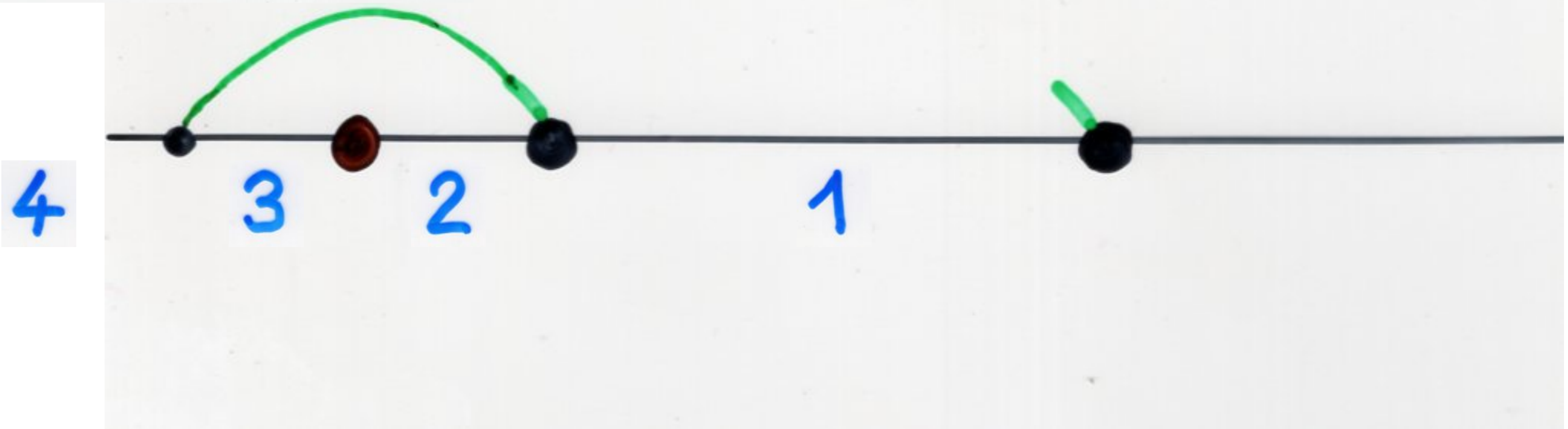

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

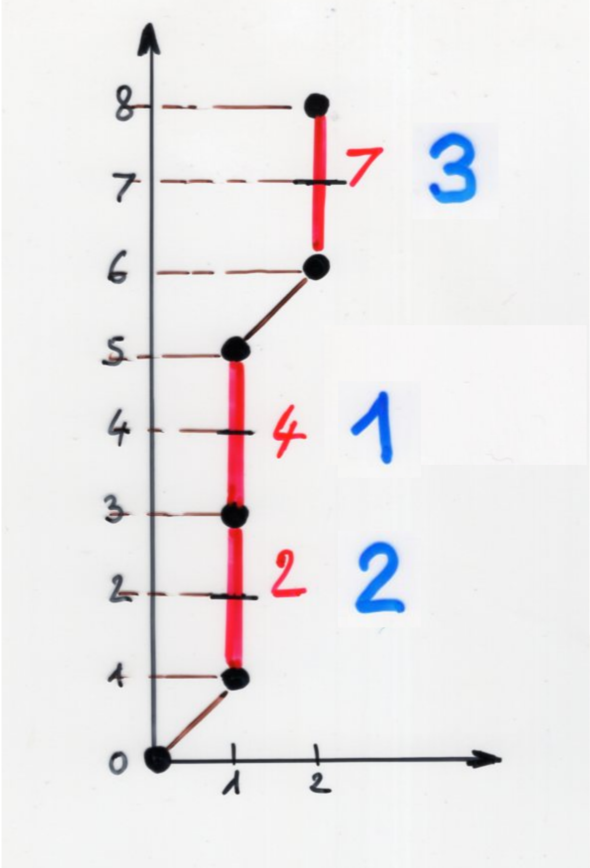

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

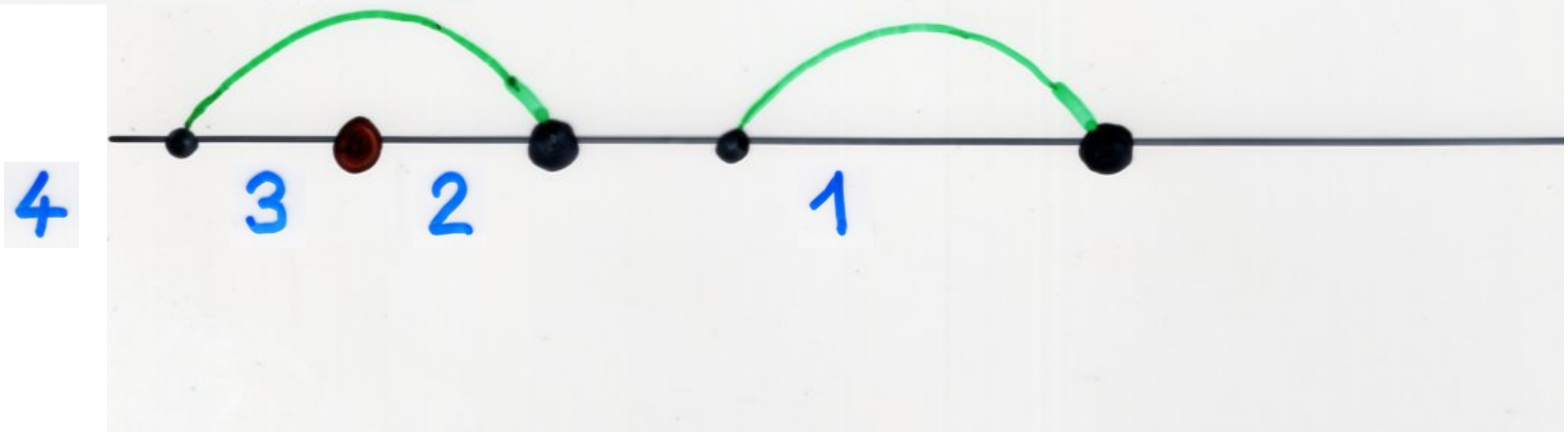

2
1

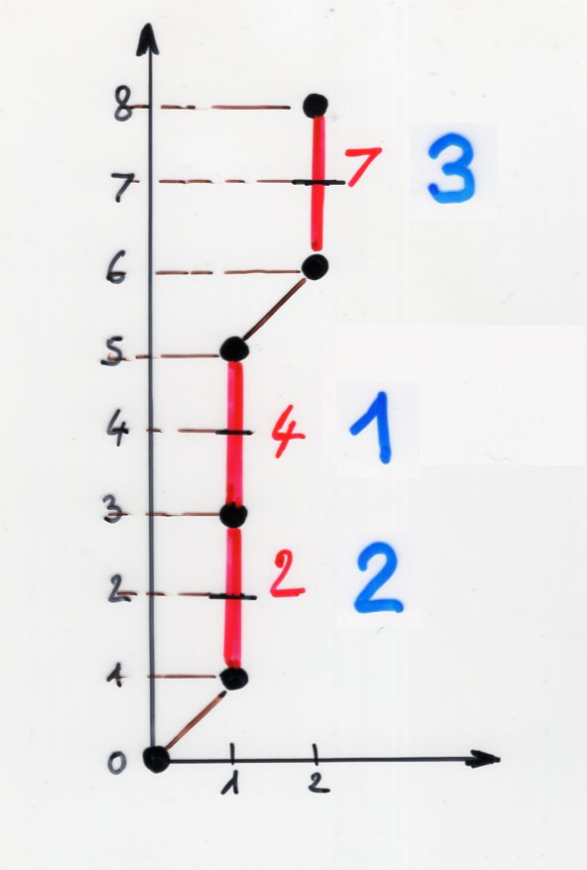

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

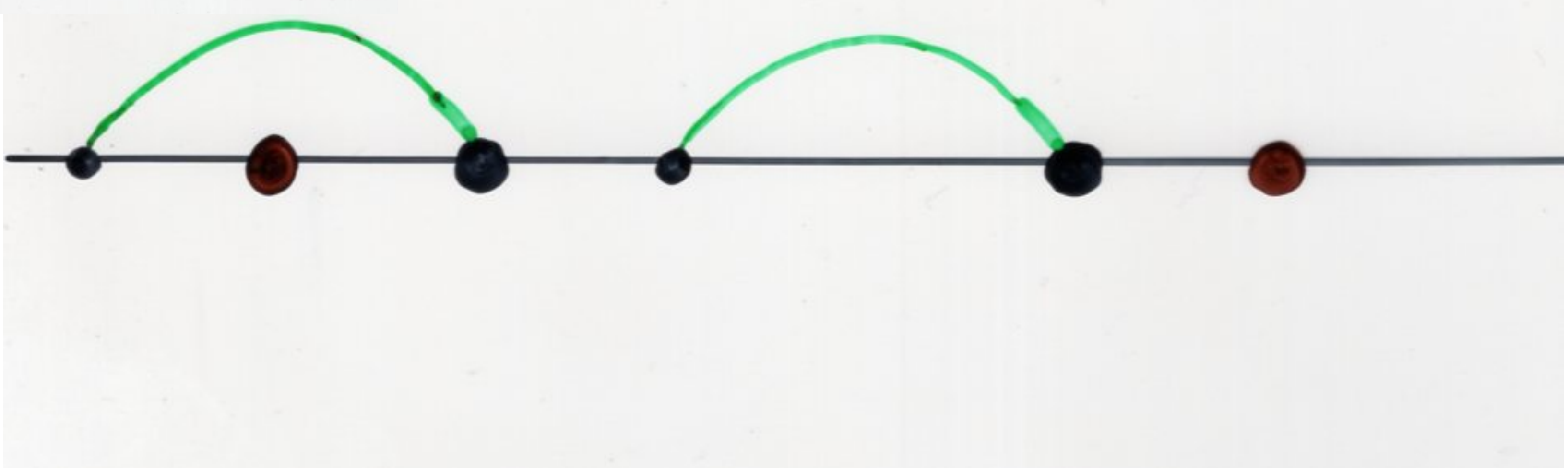

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

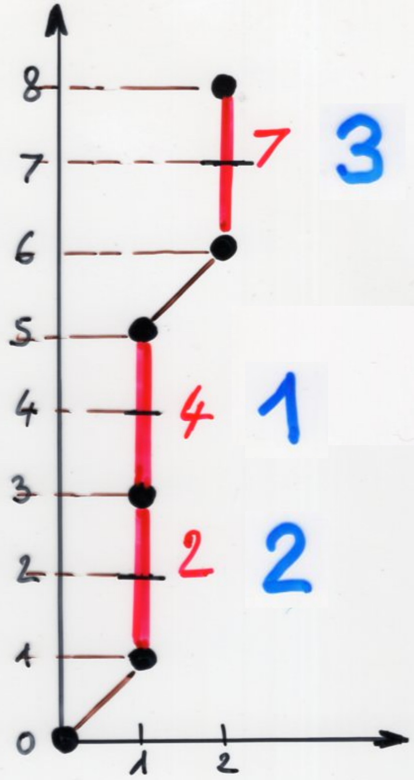

4
3
2
1

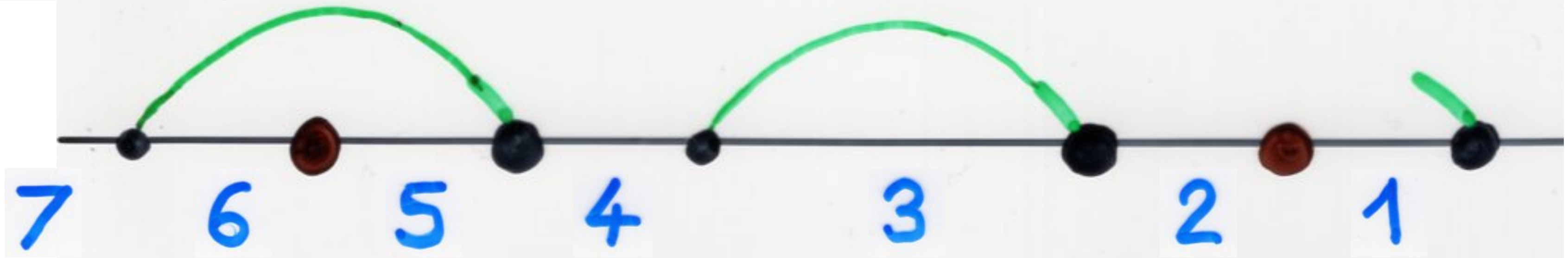

0
1
2
3
4
5
6
7
8
7
3
4
1
2
2

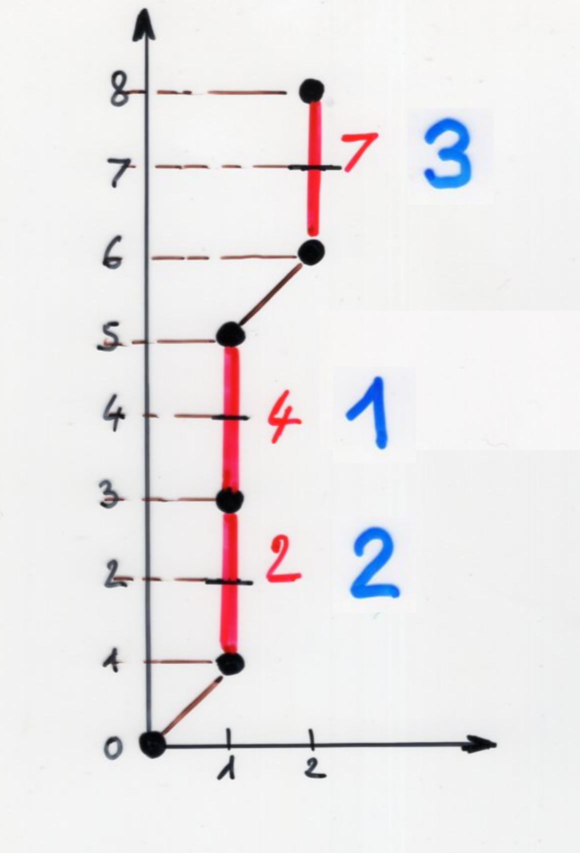

4
3
2
1


8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2




8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

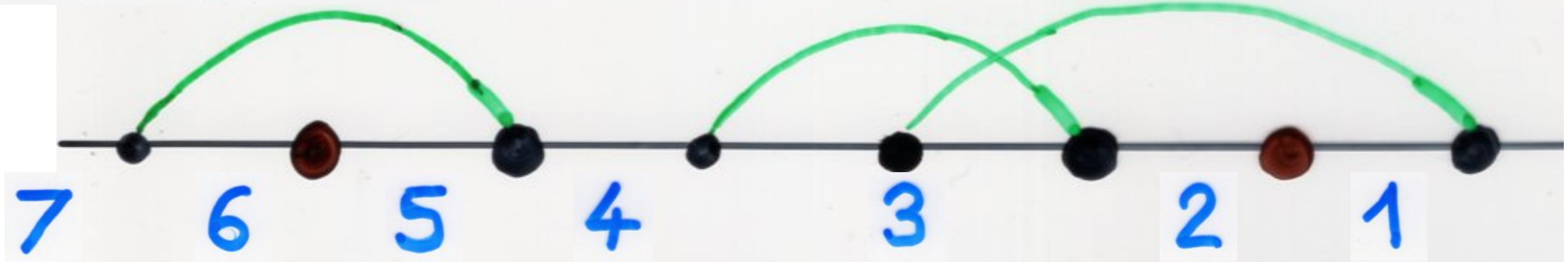

7
6
5
4
3
2
1



Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$H_n(x) = \sum_{0 \leq 2k < n} (-1)^k \frac{n!}{2^k\, k!\,(n-2k)!}\, x^{n-2k}$$



$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)}\, x^{fix(\sigma)}$$

q-analogue of Hermite polynomials

$H_k(x;q)$

$\lambda_k = [k]_q$

$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$

q-Hermite I (continuous)

α matching of $[1, n]$

$d(\alpha)$ = number of edges of α

$n - 2d(\alpha) = ip(\alpha)$

number of isolated points

σ involution

$d(\sigma)$ = number of cycles of length 2

$fix(\sigma)$ = number of fixed points

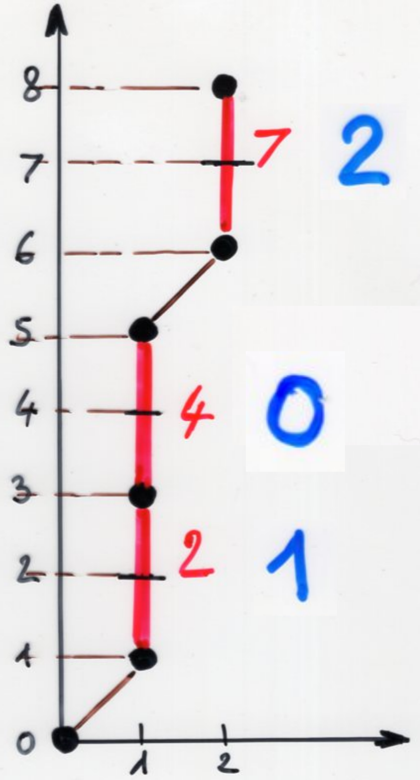
$$s(\sigma) = \sum_{\substack{e \\ \text{edges}}} s(e) \quad (\text{cycles. length } 2)$$

$e = \{i, j\}, \quad i < j$

$$s(e) = \left\{ \begin{array}{l} \text{number of indices } k, \\ \quad i < k < j, \quad \sigma(k) < j \end{array} \right\}$$

<u>Proposition</u>

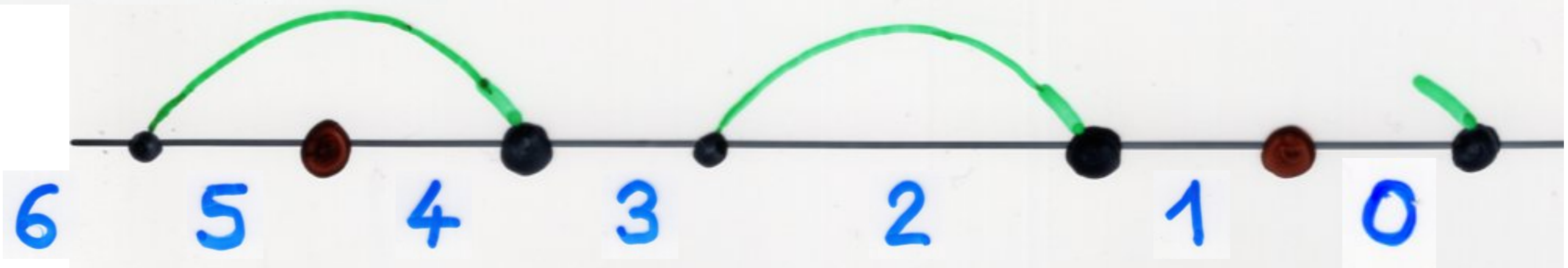
$$H_n(x;q) = \sum_{\substack{\alpha \\ \text{matching of } [1,n]}} (-1)^{d(\alpha)} x^{n-2d(\alpha)} q^{s(\alpha)}$$

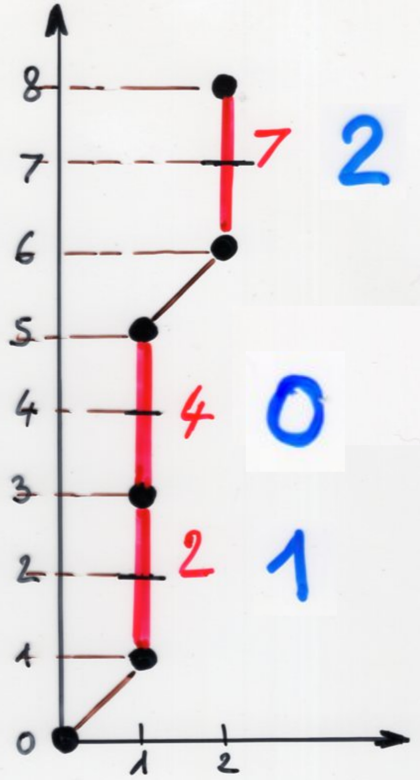


Hermite polynomials

$H_k(x;q)$

$\lambda_k = [k]_q$



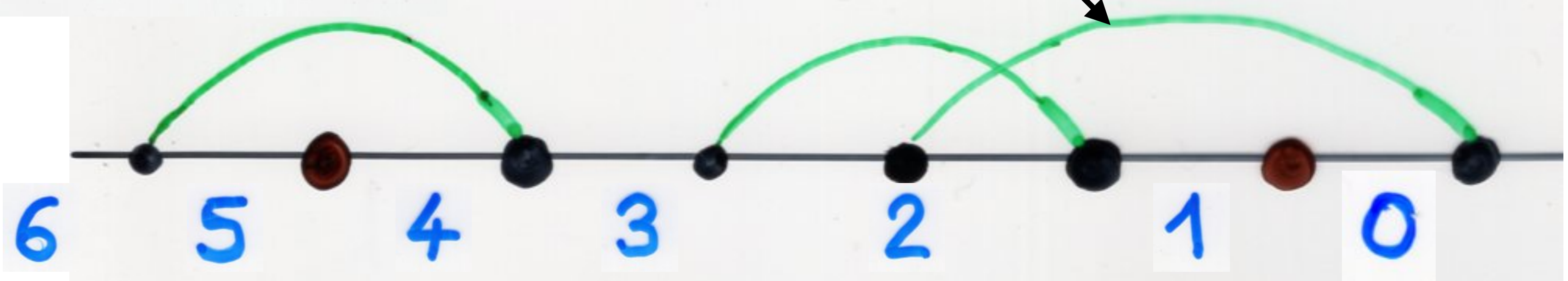


Hermite polynomials

$H_k(x;q)$

$\lambda_k = [k]_q$

$e = \{i, j\}, \quad i<j$

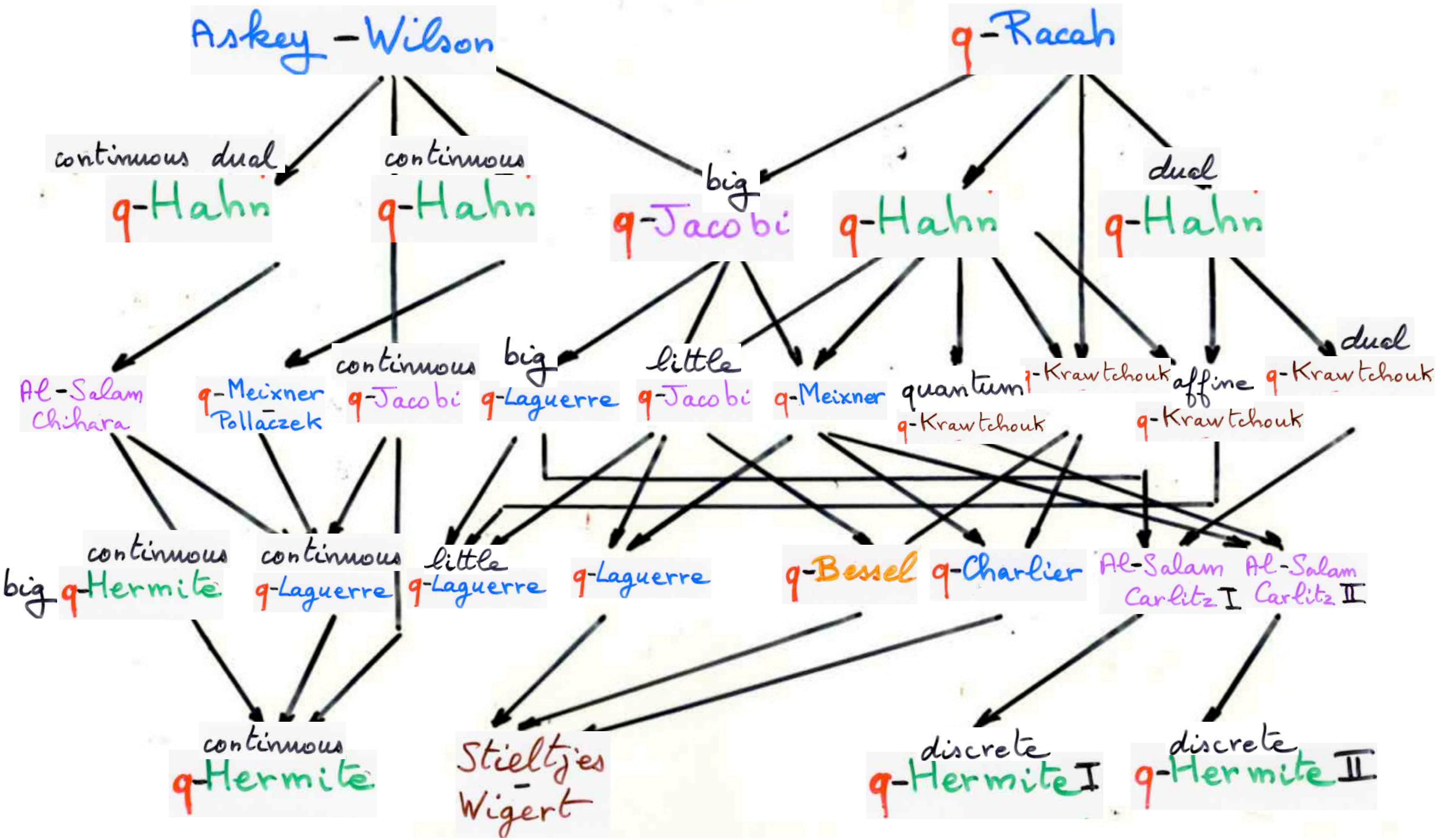


$$s(e) = \left\{ \begin{array}{l} \text{number of indices } k, \\ i<k<j, \quad \sigma(k)<j \end{array} \right\}$$

discrete q-Hermite

(Hermite II)

scheme of basic hypergeometric orthogonal polynomials



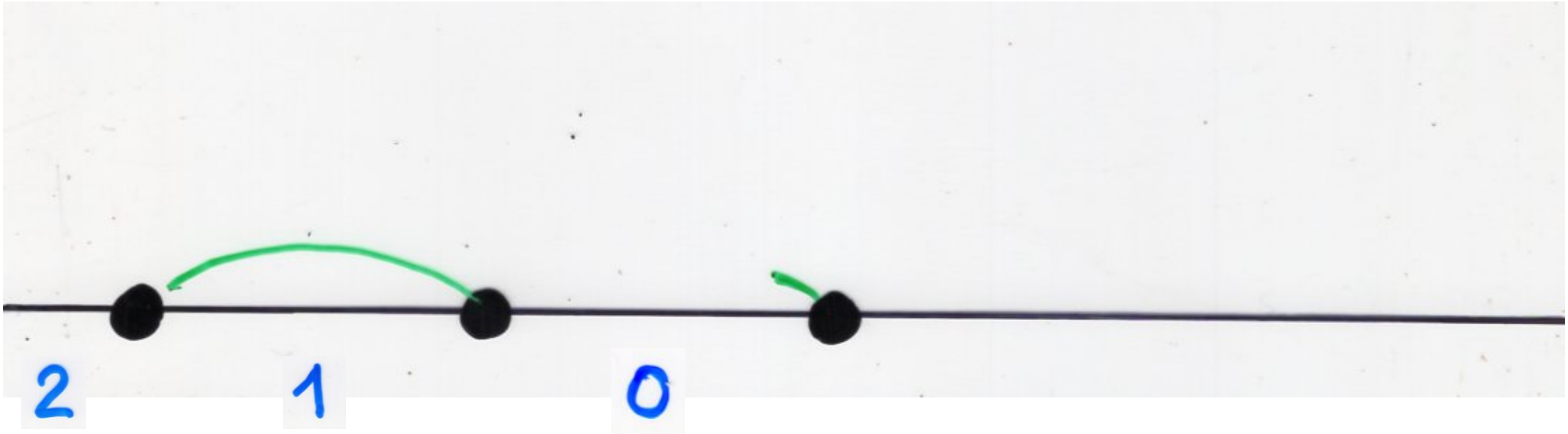
q-Hermite II
(discrete I)

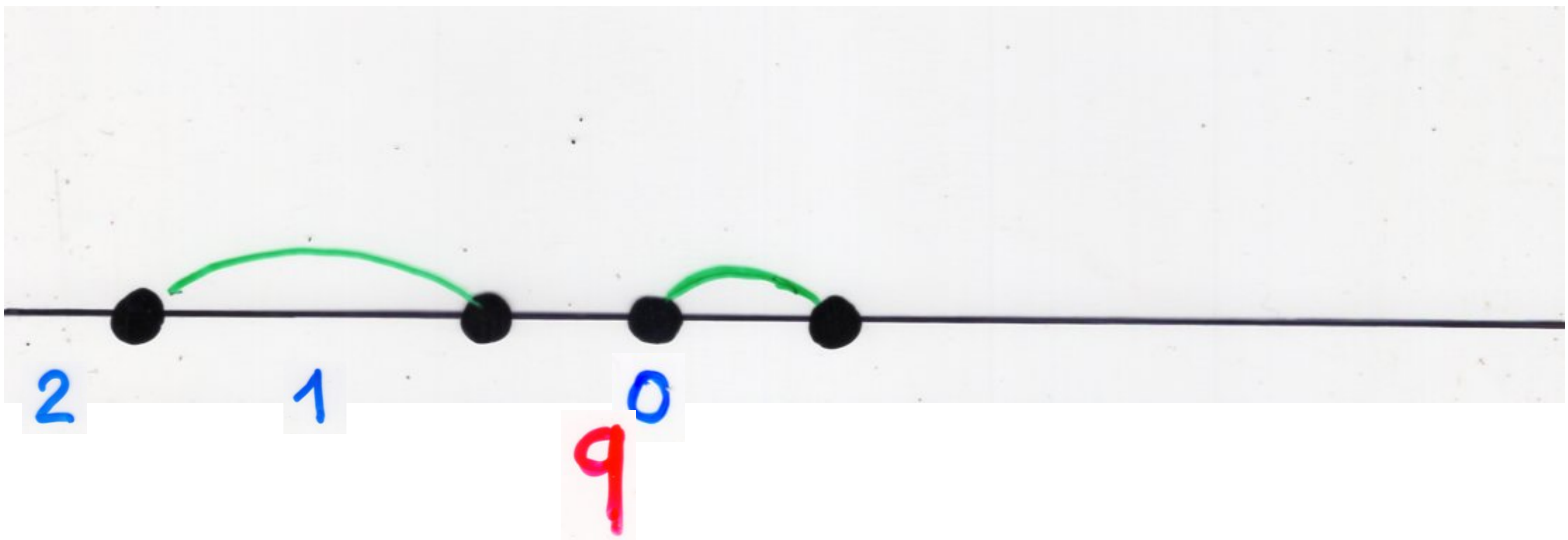
$$\lambda_k = q^{k-1}[k]_q$$

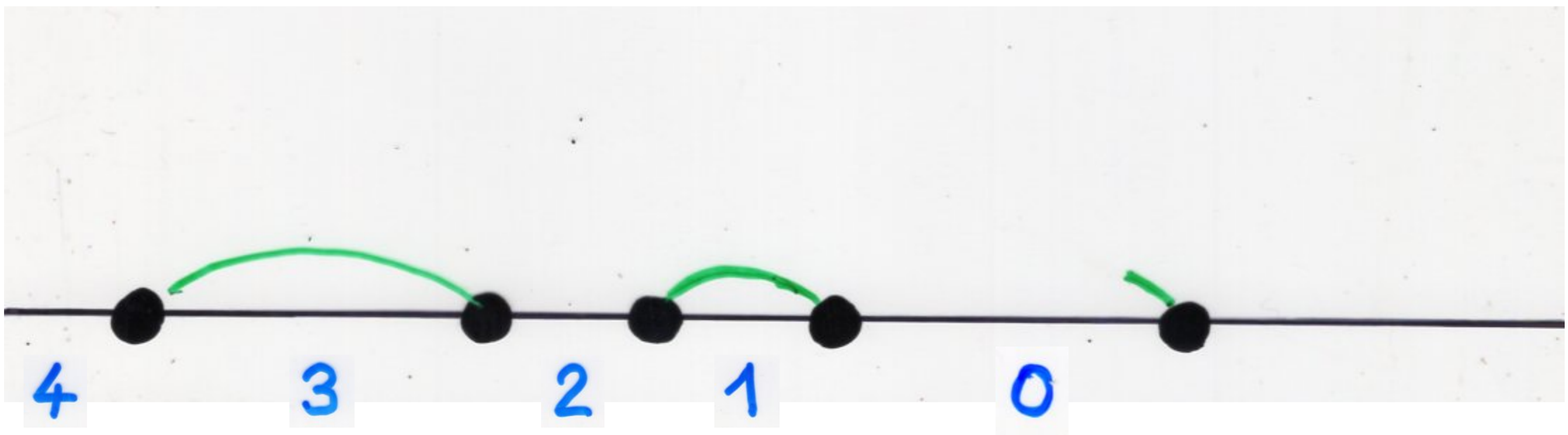
moments

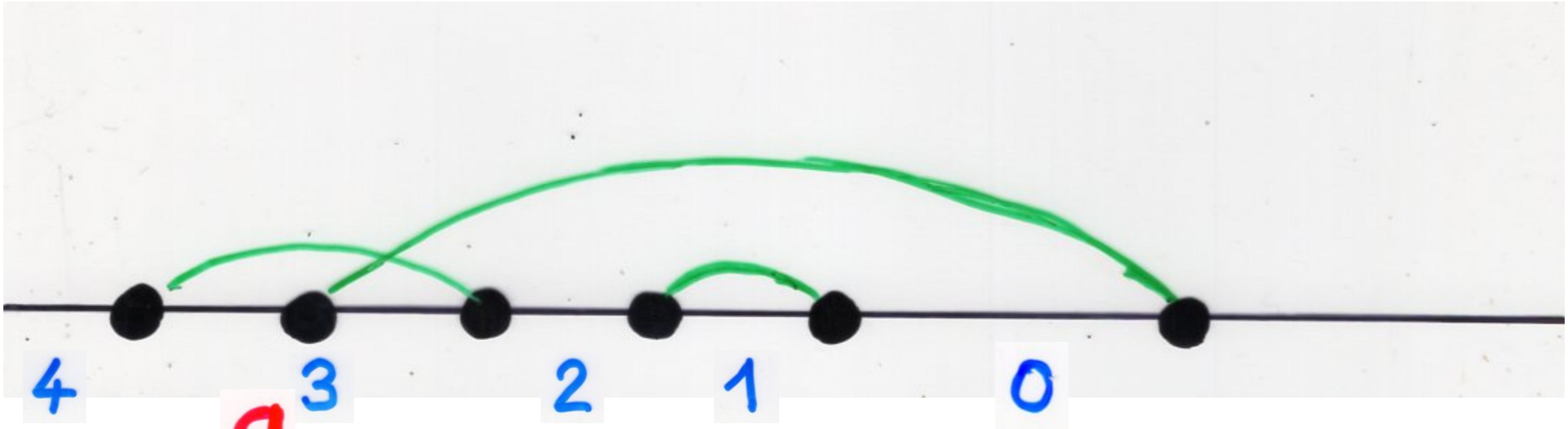
Proposition

$$\mu^{II}_{2n}(q) = [1]_q \cdot [3]_q \cdots [2n-1]_q$$

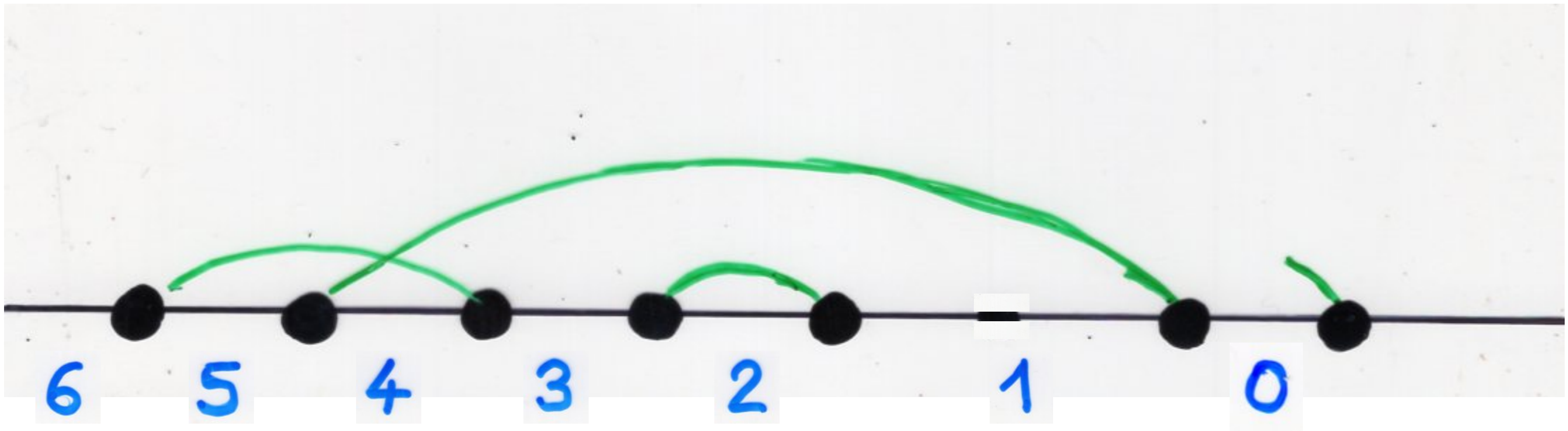

2
1
0

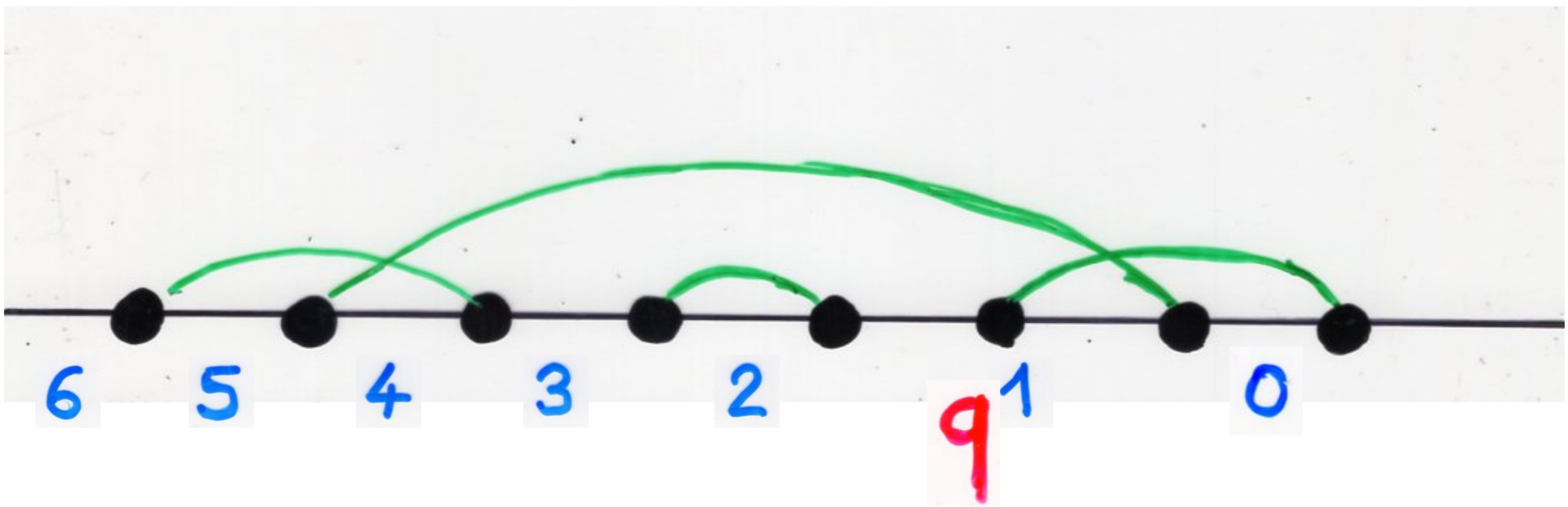

2
1
0
9


4
3
2
1
0

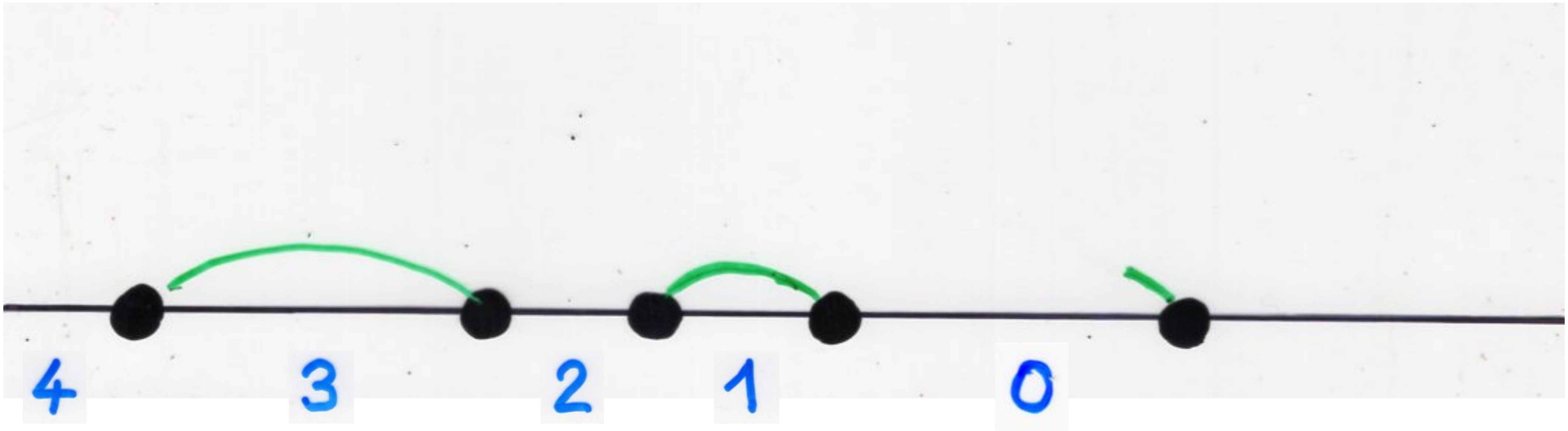


9

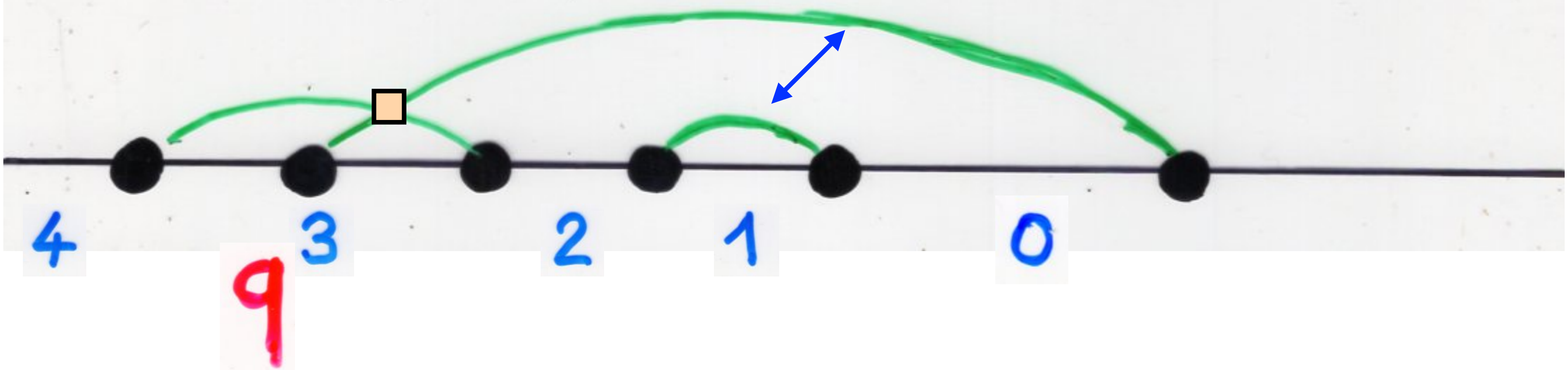

6
5
4
3
2
1
0

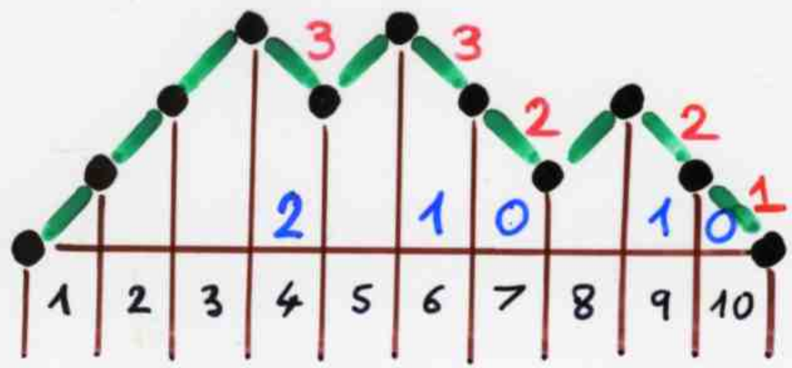


$$Inv(I) = q^0 q^3 q^1$$

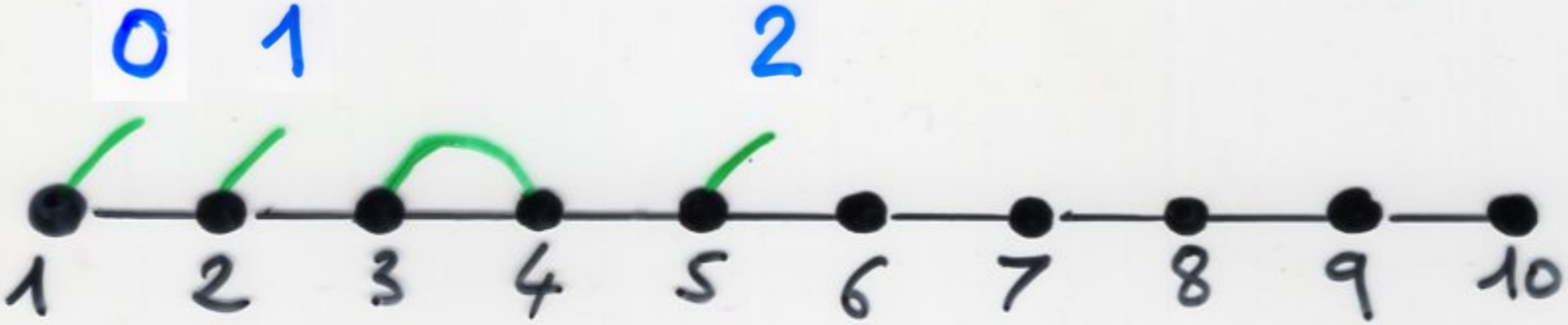

4
3
2
1
0

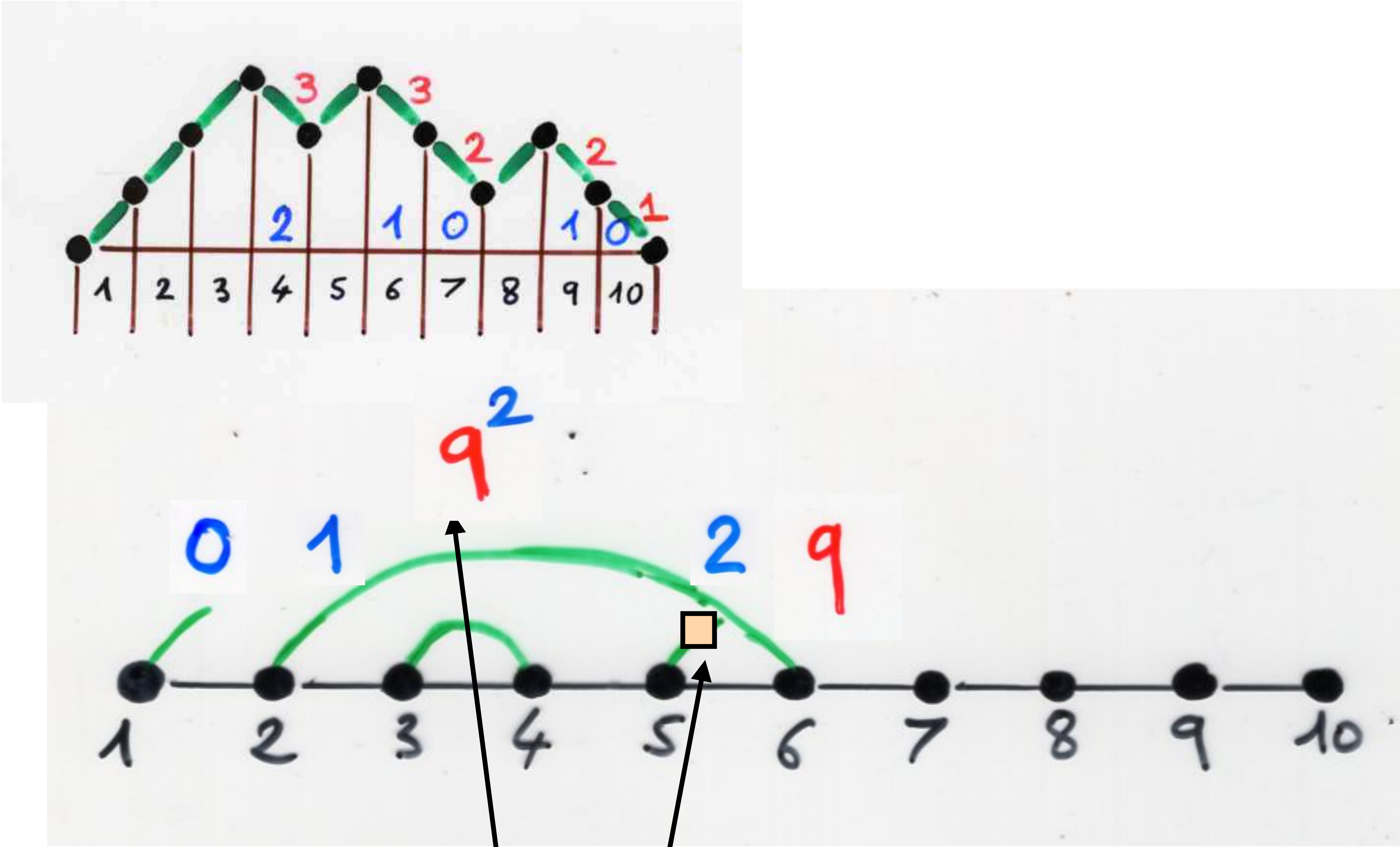
Lemma $\mathrm{Inv}(I) = \mathrm{cr}(I) + 2\,\mathrm{nest}(I)$



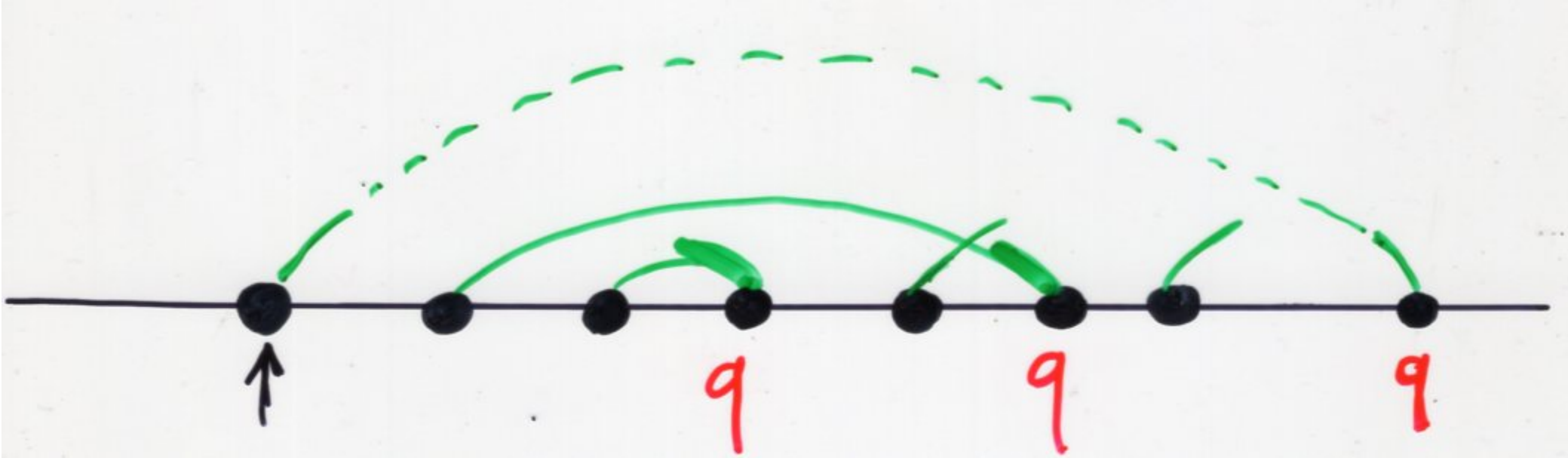


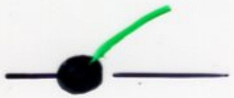
$$\lambda_k = q^{k-1}[k]_q$$

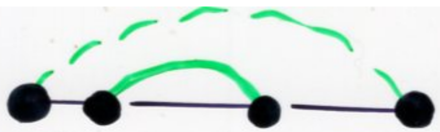




$$\lambda_k = q^{k-1}[k]_q$$

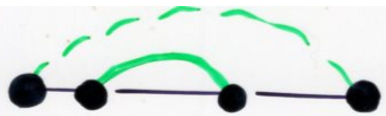


for each  q weight coming from the factor q^{k-1} in $\lambda_k = q^{k-1}[k]_q$

q number of  nesting

+ number of  crossing

$$\lambda_k = q^{k-1}[k]_q$$

q number of  nesting

$+$ number of  crossing

total $q^{nest(I) + 2cr(I)} = q^{Inv(I)}$

$$Inv(I) = cr(I) + 2\,nest(I)$$

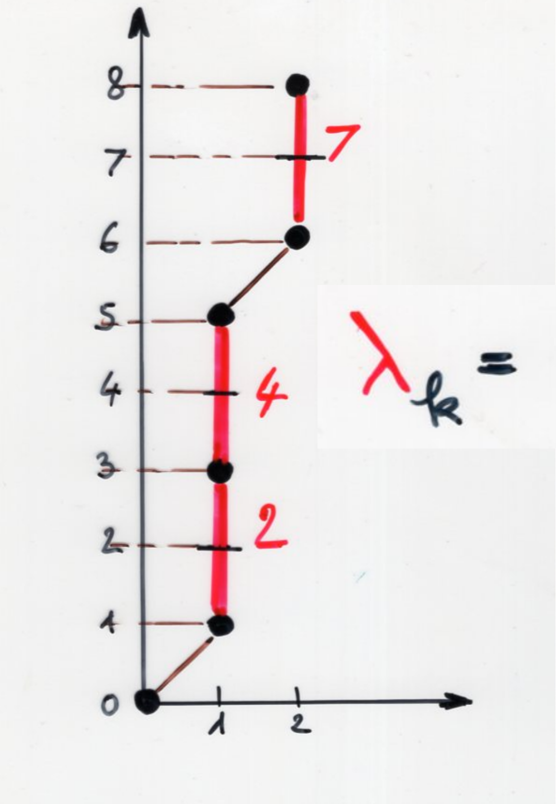
Proposition

q-Hermite II
(discrete I)

$$\lambda_k = q^{k-1} [k]_q$$

moments

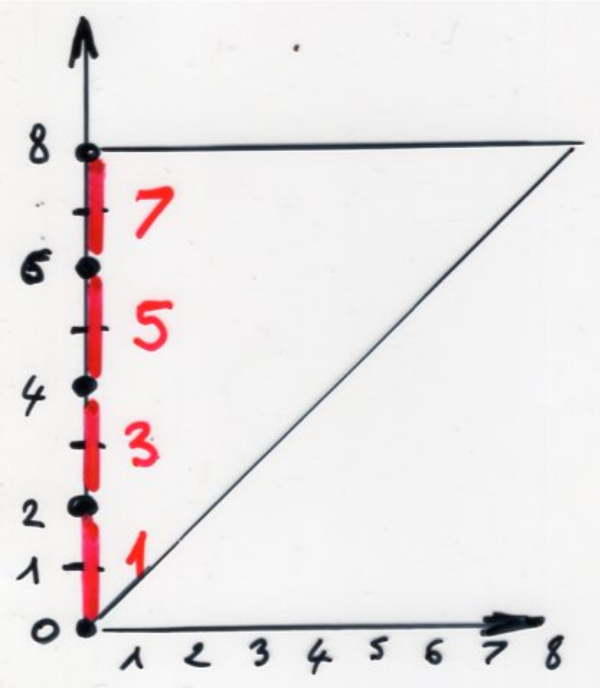
$$\mu^{II}_{2n}(q) = [1]_q \cdot [3]_q \cdots [2n-1]_q$$

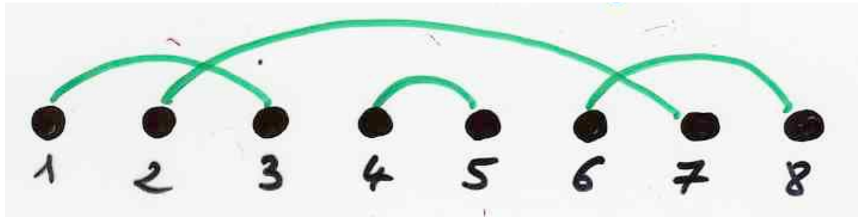


$\lambda_k = [k]_q$

$H_k(x;q)$

q-Hermite I
(continuous)





q-Hermite II
(discrete I)

$\lambda_k = q^{k-1}[k]_q$

$[1]_q \cdot [3]_q \cdots [2n-1]_q$

moments

Charlier

Charlier polynomials

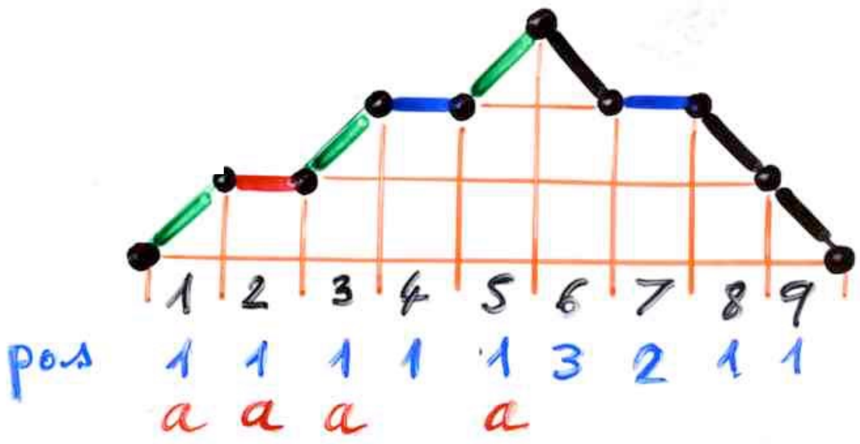
$$\begin{cases} \lambda_k = ak & (k \geq 1) \\ b_k = k + a & (k \geq 0) \end{cases}$$

$$\mu_n = \sum_{1 \leq k \leq n} S(n,k)\, a^k$$

moments

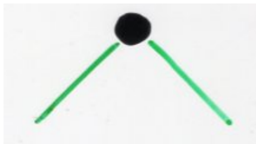
Stirling numbers (2nd kind)

Charlier histories

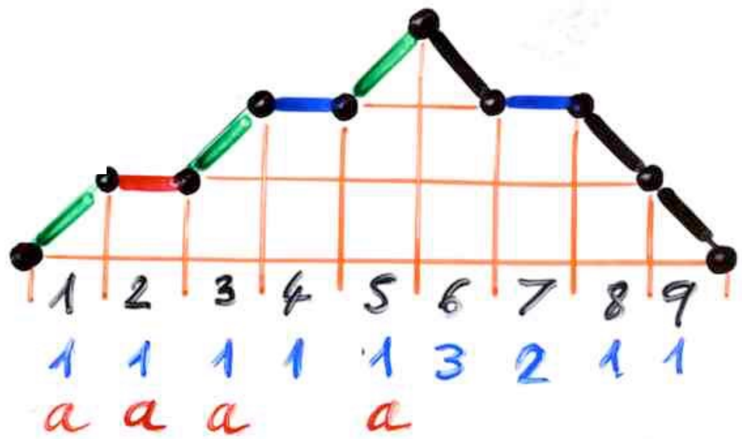


$$\begin{cases} b'_k = a \\ b''_k = k \end{cases}$$

$$\begin{cases} a_k = a \\ c_k = k \end{cases}$$



$$\begin{cases} \lambda_k = ak & (k \geq 1) \\ b_k = k + a & (k \geq 0) \end{cases}$$



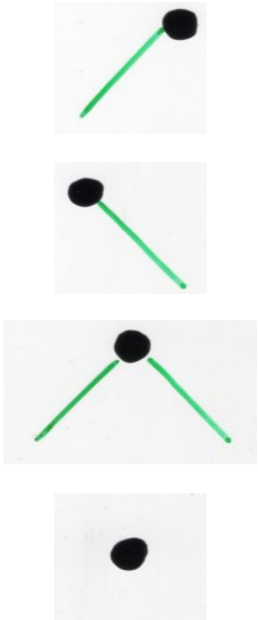
Charlier histories

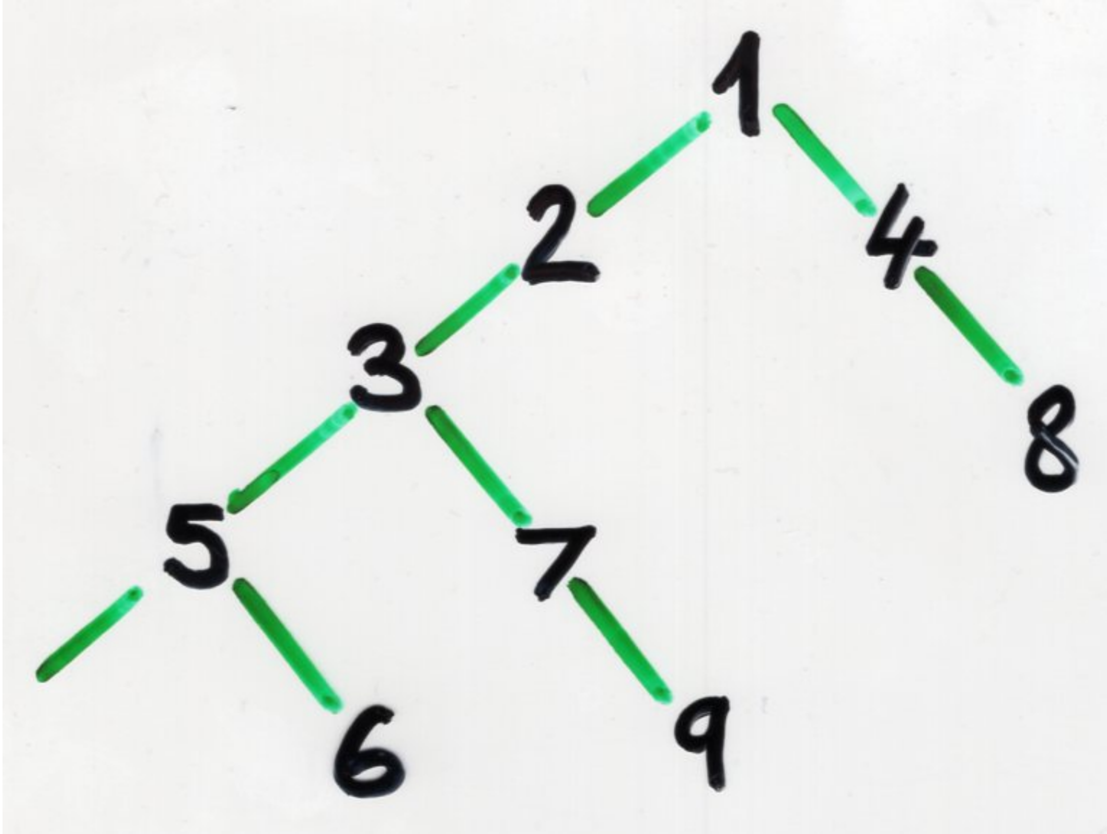
$$\mu_n = \sum_{1 \leq k \leq n} S(n,k)\, a^k$$

moments

$$\begin{cases} b'_k = a \\ b''_k = k \end{cases}$$

$$\begin{cases} a_k = a \\ c_k = k \end{cases}$$





q-Charlier II

discrete

de Médicis, Stanton, White (1995)

q-Charlier polynomials

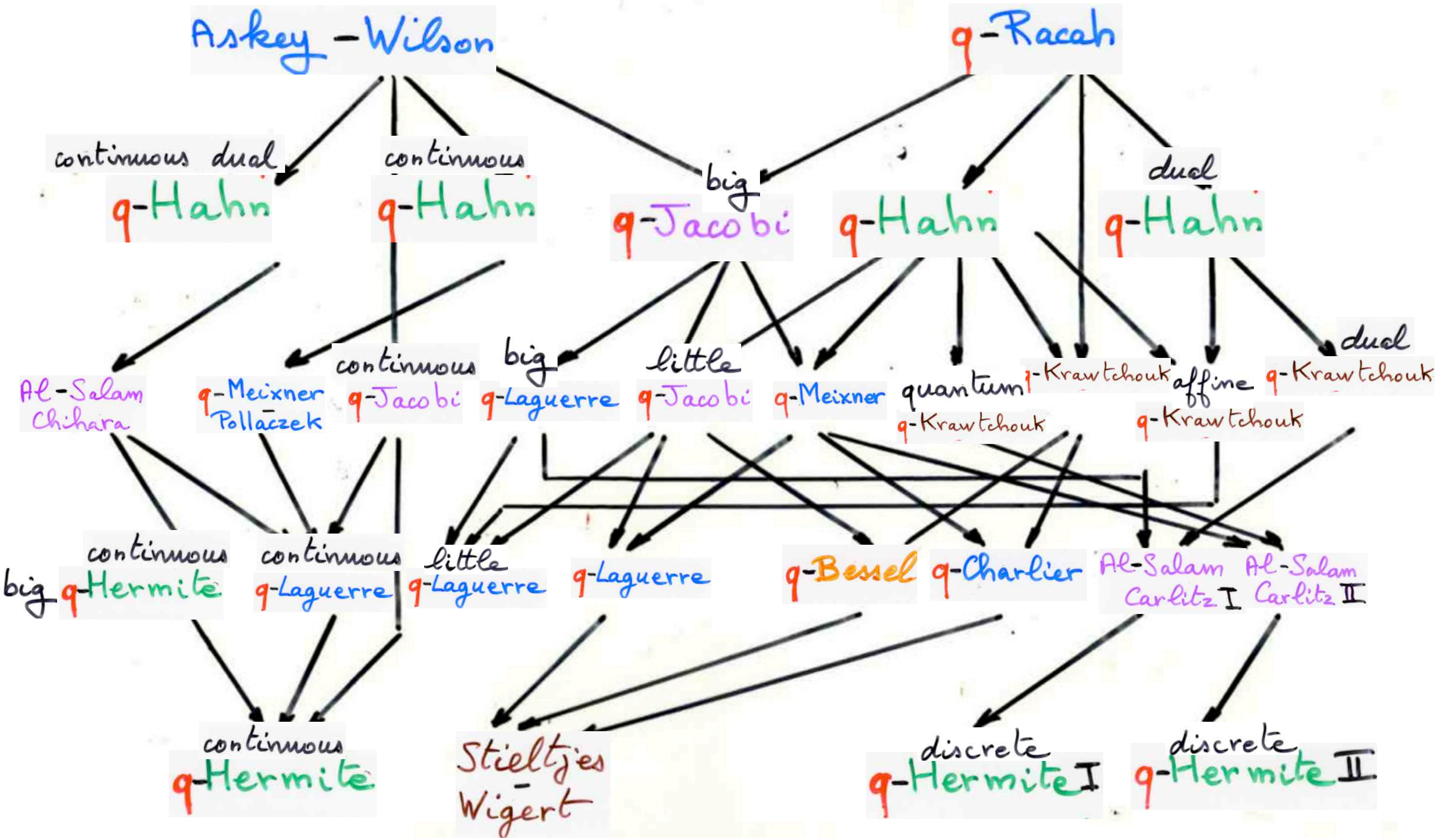
discrete q-Charlier II

$$\begin{cases} b_k = aq^k + [k]_q \\ \lambda_k = aq^{k-1}[k]_q \end{cases}$$

rescaled versions of

Al Salam - Carlitz polynomials

scheme of basic hypergeometric orthogonal polynomials



$$C_n(x,a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1} (x - [i]_q)$$

$$\prod_{i=0}^{k-1} (x - [i]_q) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k - cyc(\sigma)} q^{inv(\sigma)} \cdot x^{cyc(\sigma)}$$

$$inv(B) = \sum_{b \in B} (b-1)$$

sum over all subsets $B \subseteq [1,n]$

generating function for these subsets

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}$$

Proposition

$$C_n(x, a; q) = \sum_{B \subseteq [1,n]} \sum_{\sigma \in \mathfrak{S}_{n-|B|}} q^{inv(\sigma) + inv(B)} (-1)^{n - cyc(\sigma)} a^{|B|} x^{cyc(\sigma)}$$

combinatorial proof
3-terms recurrence relation

de Médicis (1993) (with an involution)

moments

$$\mu_n = \sum_{k=1}^{n} S_q(n,k)\, a^k$$

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q\, S_q(n-1,k)$$

$$S_q(0,k) = \delta_{0,k}$$

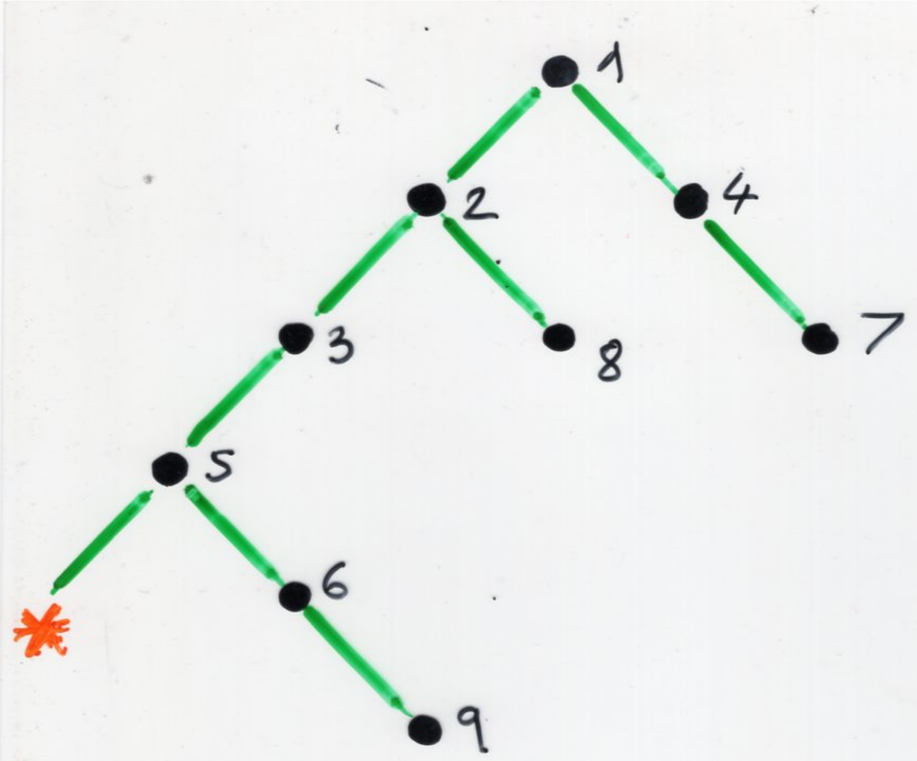
$$S_q(n,k) = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{n-k} \binom{n}{k+j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q (-1)^j$$

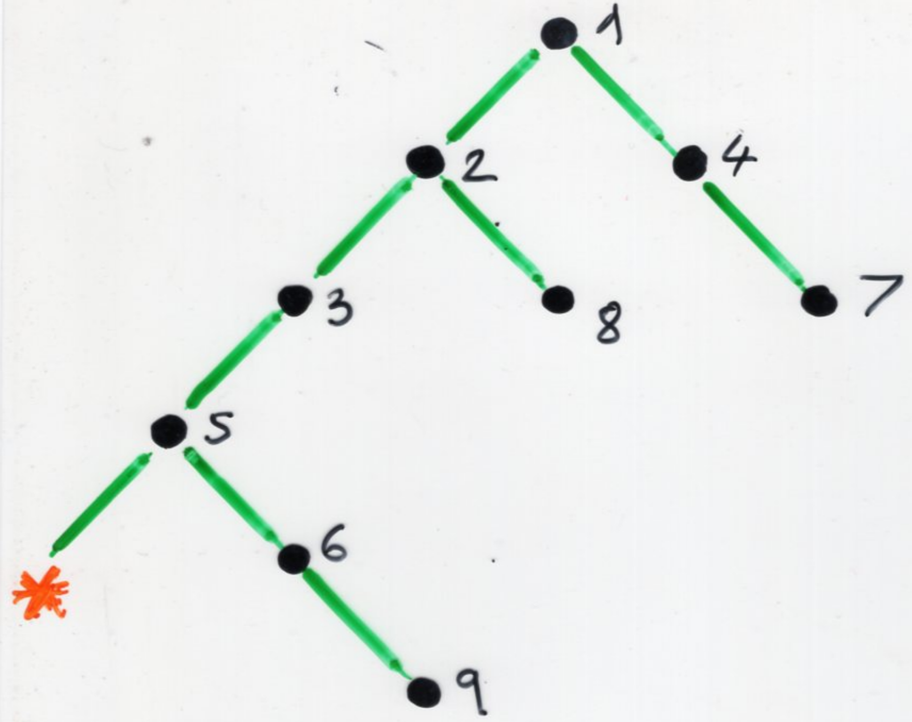
Gould (1961)

restricted growth functions
(RG functions)

set partition of $\{1, .., n\}$ π

blocks of π are ordered
by increasing minima





$w = w_1 \dots w_n$

w_i is the **block** where i is located

$$\pi = (1,4,7 \mid 2,8 \mid 3 \mid 5,6,9)$$

$w =$

1	2	3	4	5	6	7	8	9
1	2	3	1	4	4	1	2	4

Wachs, White (1991)

four natural statistics

$$\mathrm{rs}(\pi) = \mathrm{rs}(w)$$

$$= \sum_{i=1}^{n} \left(\begin{array}{l} \text{number of } j < w_i, \\ j \text{ appears to the right of } i \end{array} \right)$$

$$\pi = (1,4,7 \mid 2,8 \mid 3 \mid 5,6,9)$$

$$\mathrm{rs}(\pi) = 7$$

$$w = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 1 & 4 & 4 & 1 & 2 & 4 \end{array}$$

Wachs, White (1991)

four natural statistics

$$rs(\pi) = rs(w)$$

$$= \sum_{i=1}^{n} \left(\begin{array}{l} \text{number of } j < w_i, \\ j \text{ appears to the right of } i \end{array} \right)$$

$rs(\pi) = 7$

ls, lb

rs, rb

$<$; right

$>$; left

$lb(\pi) = 7$

$ls(\pi) = 13$

$rb(\pi) = 11$

same distribution (up to a constant) on the set $RG(n, k)$

restricted growth functions
length n, maximum k

generating function

$S_q(n, k)$ for rs and lb

$q^{\binom{k}{2}} S_q(n, k)$ for ls and rb

0-1 tableaux φ

Leroux (1990)

q-log concavity

Butler (1990)

$\mathrm{inv}(\varphi)$
lb

$\mathrm{nin}(\varphi)$
$ls - \binom{k}{2}$

σ permutation of $\{1,..,n\}$
k cycles

$\mathrm{inv}(\varphi) \longleftrightarrow \mathrm{inv}(\sigma)$

$s_q(n,k)$

q-Stirling numbers 1st kind

Proposition

$$\mu_n = \sum_{\pi \in \mathcal{P}(n)} a^{\mathrm{nbb}(\pi)} \, q^{\mathrm{rs}(\pi)}$$

$\mathrm{nbb}(\pi)$ = number of blocks of the (set) partition π

$\mathcal{P}(n)$ = set of all set partitions of $\{1, .., n\}$

Linearization coefficients
difficult formula

exact formula for

$$f_q\left(C_{n_1}(x,a;q)\,C_{n_2}(x,a;q)\,C_{n_3}(x,a;q)\right)$$

non-positive terms

5 weight-preserving
sign-reversing

involutions

"classical" q-Charlier polynomials

$$c_n(x;a;q) = {}_2\phi_1\left(q^{-n}, x; 0; q, -q^{n+1}/a\right)$$

monic polynomials

$$\begin{cases} b_k = aq^{-1-2k} + q^{-k} + aq^{-2k} - aq^{-k} \\ \lambda_k = -aq^{1-2k}(1-q^{-k})(1+aq^{-k}) \end{cases}$$

$$\mu_n = \prod_{i=1}^{n} (1+aq^{-i})$$

rescaled version

$$\hat{C}_n(z;a;q) = q^{-n^2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} \, q^{\binom{k+1}{2}} \prod_{i=0}^{k-1} \left(q^i z - [i]_q \right)$$

$$\begin{cases} b_k = q^{-k} [k]_q \left(1 + a(1-q) q^{-k}\right) + a q^{-1-2k} \\ \lambda_k = a q^{1-3k} [k]_q \left(1 + a(1-q) q^{-k}\right) \end{cases}$$

$$\mu_n = \sum_{j=1}^{n} q^{-\binom{j}{2} - n} S_{1/q}(n,j) \, a^j$$

Zeng (1995)

q-Charlier I

Kim, Stanton, Zeng (2006) 54th SLC

q-Charlier I "continuous version"

$$\begin{cases} b_k = a + [k]_q \\ \lambda_k = a\,[k]_q \end{cases}$$

rescaled version of
Anshelevich (2005) polynomials

q-Charlier I "continuous version"

$$\begin{cases} b_k = a + [k]_q \\ \lambda_k = a[k]_q \end{cases}$$

also:
rescaled version of
Al-Salam - Chihara polynomials

$$C_n(x, a; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (-a)^{n-k} \prod_{i=0}^{k-1} \left(x - [i]_q + a(q^{-i} - 1) \right)$$

Simion, Stanton (1996)

σ permutation on A (A, B)

σ fixed points cycles length > 1

for any $k \in A$

$C = (k_0, k_1, \ldots, k_s)$ k_s maximum element

$w(k) = 0$ if k is the maximum of its cycle

otherwise $k = k_j$ is on a cycle C

then $w(k) = k-1-\left|\{i : j<i<s,\ k_i < k_j\}\right|$

$$- \sum_{\text{cycles } Q,\ \max(Q)>k} (\text{number of points on } Q,\ <k)$$

$$w(B,\sigma) = \sum_{k\in A} w(k)$$

<u>Proposition</u>

$$C_n(x,a;q) = \sum_{(B,\sigma)} (-1)^{n-cyc(\sigma)}\, a^{|B|}\, x^{cyc(\sigma)}\, q^{w(B,\sigma)}$$

facts about measure Al-Salam – Chihara q-Hermite

$$\mu_n(a;q) = u^{-n} \sum_{m=0}^{n} \binom{n}{m} (-v)^{n-m} \theta_m$$

$$u = \frac{1}{2}\sqrt{\frac{1-q}{a}} \qquad v = \frac{a(1-q)+1}{2\sqrt{a(1-q)}}$$

$\theta_0 = 1$ odd values of $m \geqslant 1$

$$\theta_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{k} \sum_{\ell=0}^{\lfloor m/2 \rfloor - k} \frac{(-1)^{m-\ell} \left(a(1-q)\right)^{k+\ell} \left(1-q^{m-2k}\right)}{\left(2\sqrt{a(1-q)}\right)^m \left(1-q^{m-2k-\ell}\right)} \begin{bmatrix} m-2k-\ell \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}}$$

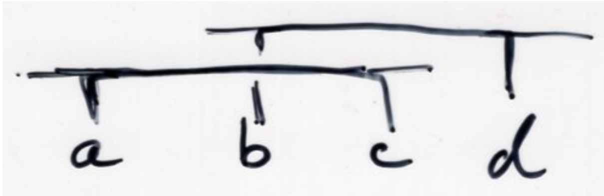
even values of $m \geqslant 1$

$$\theta_m = \sum_{k=0}^{\lfloor m/2 \rfloor - 1} \binom{m}{k} \sum_{\ell=0}^{\lfloor m/2 \rfloor - k} \frac{(-1)^{m-\ell} \left(a(1-q)\right)^{k+\ell} \left(1-q^{m-2k}\right)}{\left(2\sqrt{a(1-q)}\right)^m \left(1-q^{m-2k-\ell}\right)} \begin{bmatrix} m-2k-\ell \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}} + \frac{1}{2^m} \binom{m}{m/2}$$

Moments

π (set) partition of P_n

crossing



a, c ∈ same block
b, d ∈ same block

$a < b < c < d$

$e < f$, f follows e iff

$e, f \in$ same block, and no g in this block with $e < g < f$

restricted crossing

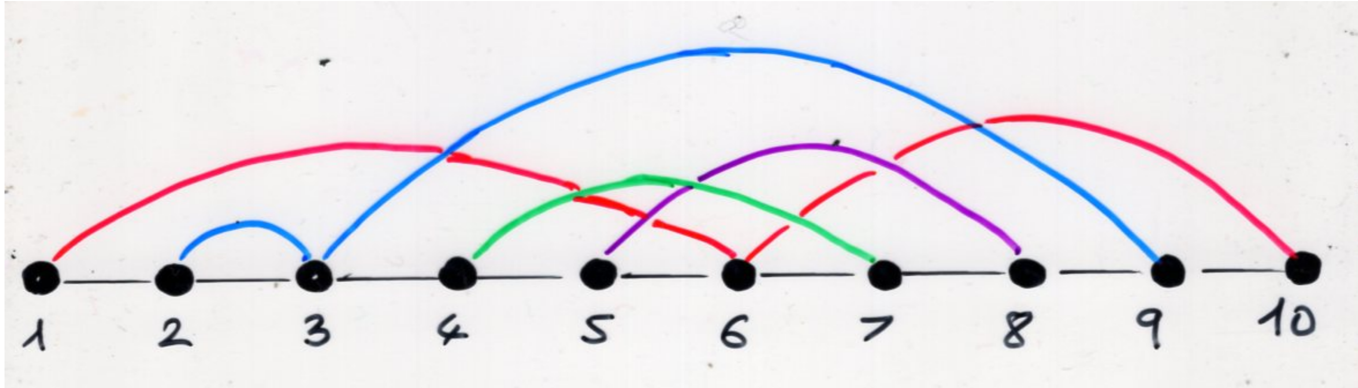
nesting (a, b, c, d)

crossing (a, b, c, d) such that
c follows a and d follows b

restricted nesting

$rc(\overline{\pi})$ number of restricted crossings

$rn(\overline{\pi})$ number of restricted nestings



$$\overline{\pi} = \{1, 6, 10\} / \{2, 3, 9\} / \{4, 7\} / \{5, 8\}$$

$rc(\overline{\pi}) = 7 \qquad rn(\overline{\pi}) = 3$

Proposition

$$\mu_n(a;q) = \sum_{\pi \in \mathcal{P}_n} a^{\mathrm{nbb}(\pi)} \, q^{\mathrm{rc}(\pi)}$$

$$\mu_n(a;q) = \sum_{\pi \in \mathcal{P}_n} a^{\mathrm{nbb}(\pi)} \, q^{\mathrm{rn}(\pi)}$$

in conclusion ...

"continuous version"

q-Charlier I

- complicated q-Stirling number associated to their moments
- complicated explicit formula
- most natural linearization formula

linearization coefficients with positive integers

discrete

q-Charlier II

- natural q-Stirling number associated to their moments
- simple explicit formula

Linearization coefficients difficult formula

non-positive terms

q-Hermite I
(continuous)

$$\lambda_k = [k]_q$$

q-Hermite II
discrete

$$\lambda_k = q^{k-1}[k]_q$$

"continuous version"

q-Charlier I

$$\begin{cases} b_k = a + [k]_q \\ \lambda_k = a[k]_q \end{cases}$$

discrete

q-Charlier II

$$\begin{cases} b_k = aq^k + [k]_q \\ \lambda_k = aq^{k-1}[k]_q \end{cases}$$

moments

q-Hermite I (continuous) $\lambda_k = [k]_q$

$$\mu^{I}_{2n}(q) = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{\binom{k}{2}}$$

q-Hermite II (discrete I) $\lambda_k = q^{k-1}[k]_q$

$$\mu^{II}_{2n}(q) = [1]_q \cdot [3]_q \cdots [2n-1]_q$$

"continuous version"

discrete

q-Laguerre I

q-Laguerre II

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

No class this thursday

Next class Monday 11 March