Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 4

Expanding a power series into continued fraction

Chapter 4b

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Chapter 4

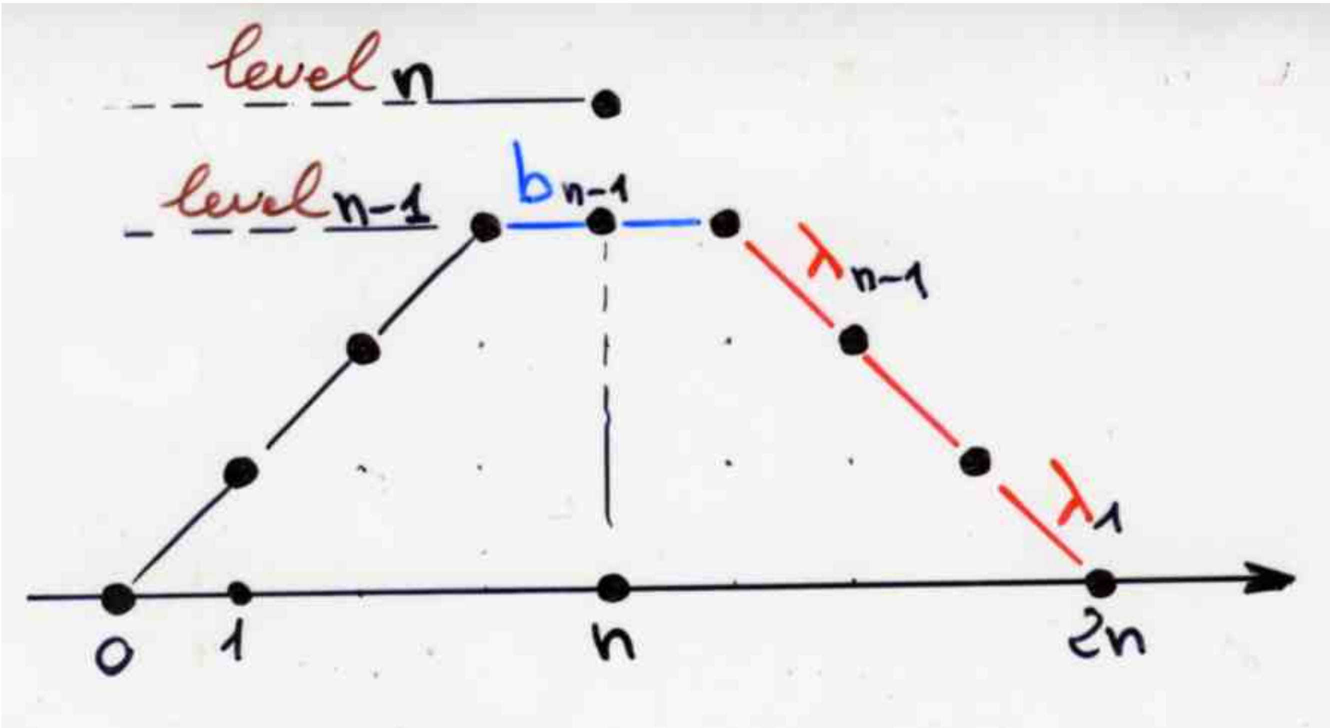
equivalently:

computing the coefficients

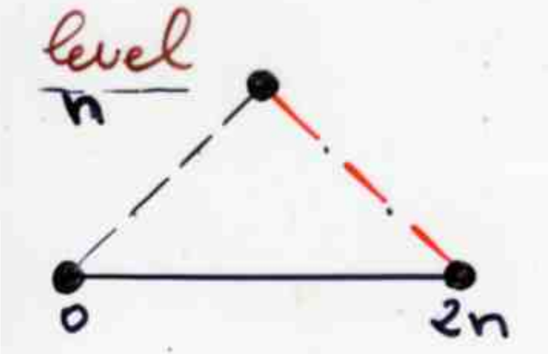
$$\lambda_k \qquad b_k$$

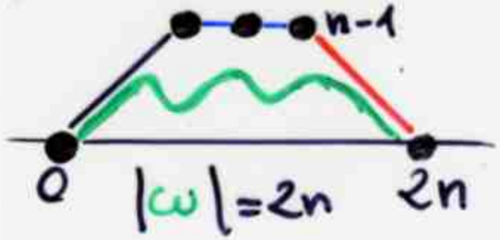
of the 3-terms linear recurrence knowing
the moments of the orthogonal polynomials

Reminding Ch 4a

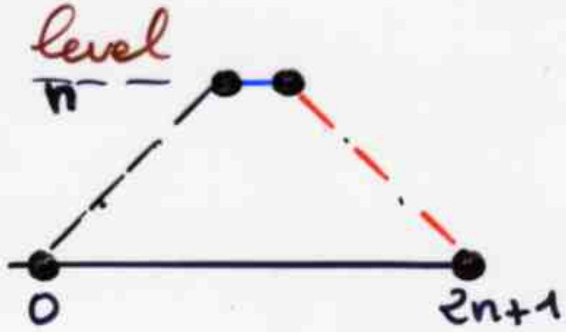


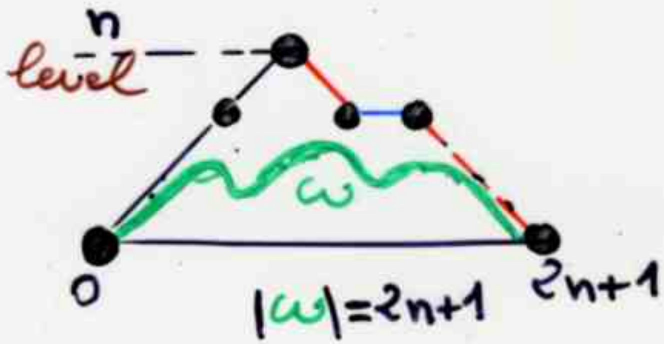
$$\mu_{2n} = \lambda_1 \cdots \lambda_n + \sum_{\omega \text{ . Motzkin path}} v(\omega)$$





$$\mu_{2n+1} = \lambda_1 \; \lambda_n b_n + \sum_{\omega \text{ Motzkin path}} v(\omega)$$





Hankel determinant

any minor of

$$\begin{array}{c|cccccc} & & & & & & j \\ \hline & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \\ & \mu_1 & \mu_2 & \mu_3 & \cdots & & \\ & \mu_2 & \mu_3 & \cdots & & & \\ & \mu_3 & \vdots & & & & \\ i & & & & & & \mu_{i+j} \end{array}$$

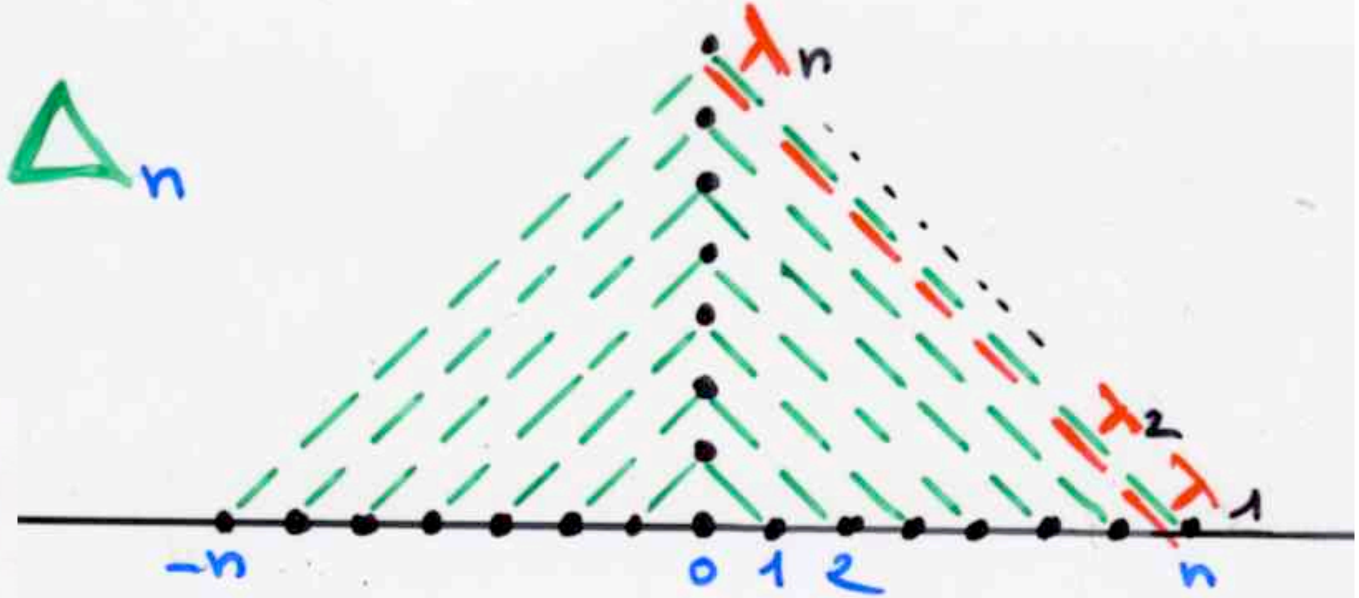
$$\mathbf{H}\begin{pmatrix} \alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k \end{pmatrix}$$

$$0 \leq \alpha_1 < \cdots < \alpha_k$$

$$0 \leq \beta_1 < \cdots < \beta_k$$

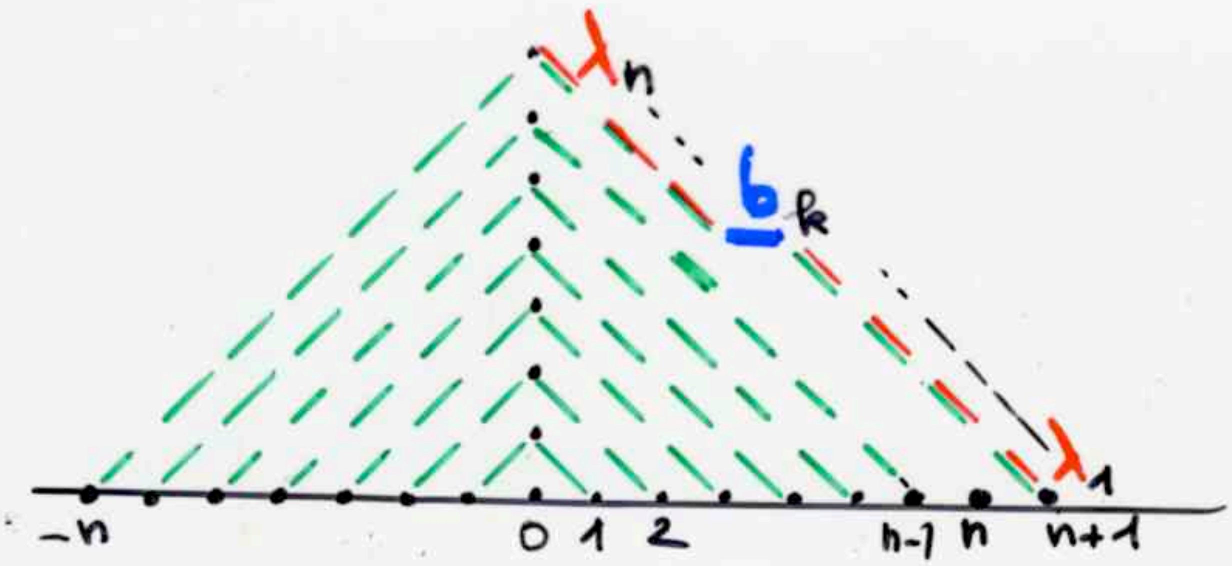
$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\Delta_n = H\begin{pmatrix} 0, & 1, & \ldots, & n \\ 0, & 1, & \ldots, & n \end{pmatrix}$$



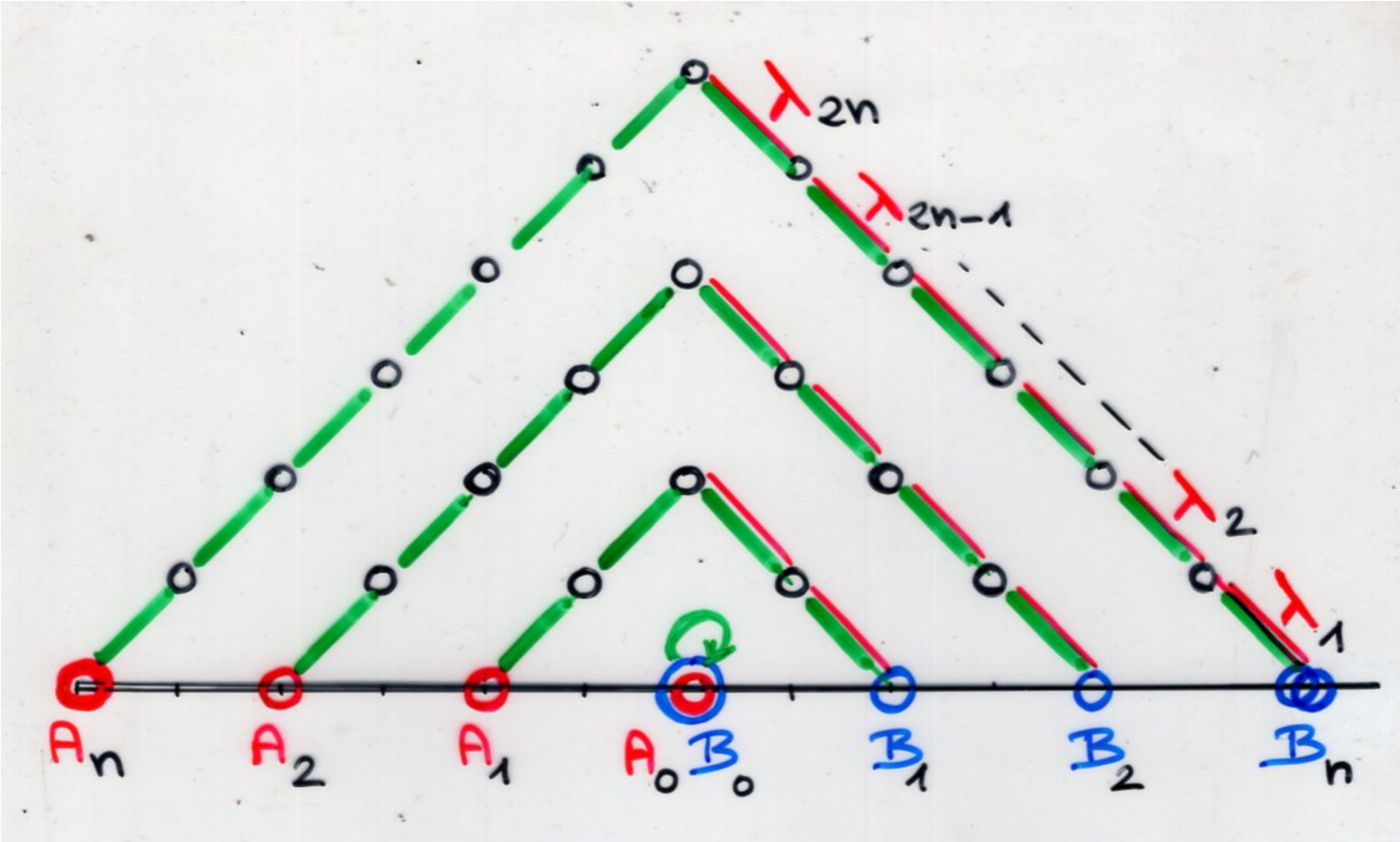
$$\lambda_n = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$$

$$\chi_n = H\begin{pmatrix} 0, 1, \dots, n-1; n \\ 0, 1, \dots, n-1; n+1 \end{pmatrix}$$

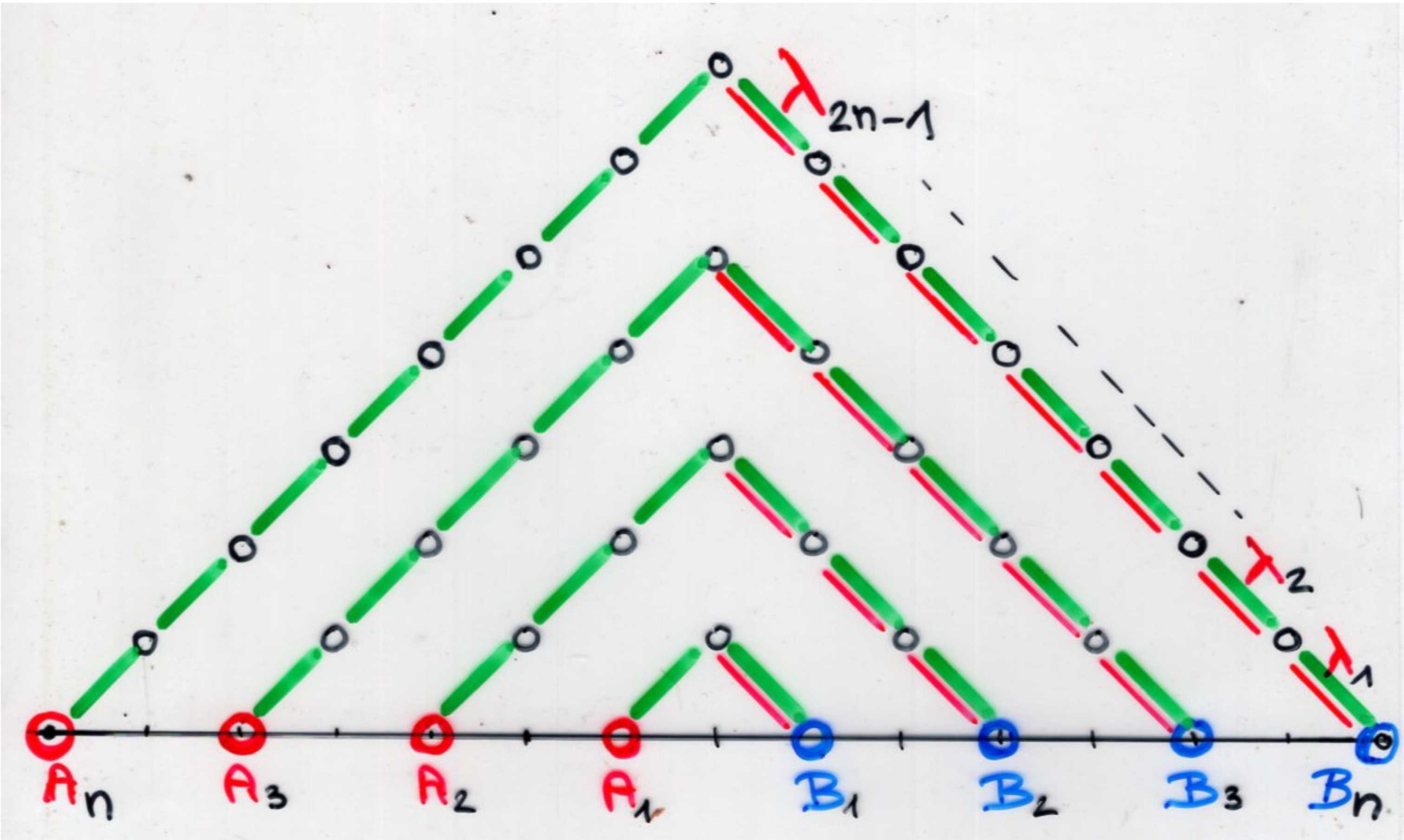


$$b_n = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}$$

$$\Delta_n^{(0)}(\nu) = H_\nu\begin{pmatrix} 0, 1, \dots, n \\ 0, 1, \dots, n \end{pmatrix}$$



$$\Delta_n^{(1)}(\nu) = H_\nu\begin{pmatrix}1, \dots, n\\ 1, \dots, n\end{pmatrix}$$



$$\lambda_{2n} = \frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} : \frac{\Delta_n^{(1)}(\nu)}{\Delta_{n-1}^{(1)}(\nu)} \quad (n \geqslant 1)$$

$$\lambda_{2n+1} = \frac{\Delta_{n+1}^{(1)}(\nu)}{\Delta_n^{(1)}(\nu)} : \frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} \quad (n \geqslant 0)$$

Ramanujan’s algorithm

Ramanujan algorithm

Notebook Chapter 12, entry 17

Write $\dfrac{1}{1+} \dfrac{a_1 x}{1+} \dfrac{a_2 x}{1+} \dfrac{a_3 x}{1+\cdots} = \sum_{k=0}^{\infty} A_k (-x)^k,$

where $A_0 = 1$.

Let $P_n = a_1 a_2 \cdots a_{n-1} (a_1 + a_2 + \cdots + a_n), \quad n \geqslant 1$

Then

$$P_1 = A_1$$
$$P_2 = A_2$$
$$P_3 = A_3 - a_1 A_2$$
$$P_4 = A_4 - (a_1 + a_2) A_3$$
$$P_5 = A_5 - (a_1 + a_2 + a_3) A_4 + a_1 a_3 A_3$$
$$P_6 = A_6 - (a_1 + a_2 + a_3 + a_4) A_5 + (a_1 a_3 + a_2 a_4 + a_1 a_4) A_4$$

In general, for $n \geqslant 1$

$$P_n = \sum_{0 \leq k < \frac{n}{2}} (-1)^k \varphi_k(n) A_{n-k}$$

where $\varphi_0(n) \equiv 1$ and $\varphi_r(n)$, $r \geqslant 1$, is defined recursively by

$$\varphi_r(n+1) - \varphi_r(n) = a_{n-1} \varphi_{r-1}(n-1)$$

bijective proof

with the notations of the course :

$a_k = \lambda_k$ $(k \geq 1)$ $A_n = \mu_n$ $(n \geq 0)$

The continued fraction is the Stieltjès continued fraction

$S(t;\lambda)$

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cfrac{\cdots}{1-\cfrac{\lambda_k t}{\cdots}}}}}$$

$$\sum_{n \geq 0} \mu_n t^n = S(t;\lambda)$$

$$\mu_n = \sum_{|\omega| = 2n} v(\omega)$$

Dyck paths

v

related to $\{\lambda_k\}$

$$\varphi_r(n) = \sum_{\alpha} v(\alpha)$$

α pavage of $[0, n-2]$ with r dimers

$$d(\alpha) = r$$

Ramanujan's theorem can be restated as:

$$\sum_{0 \leq k < \frac{n}{2}} \left(\sum_{(\alpha, \omega)} (-1)^k \, v(\alpha) \, v(\omega) \right) =$$

$$\begin{cases} \bullet \ \alpha \text{ pavage of } [0, n-2] \\ \qquad \text{with } k = d(\alpha) \text{ dimers} \\ \bullet \ \omega \text{ Dyck path } \ |\omega| = 2n-2k \end{cases}$$

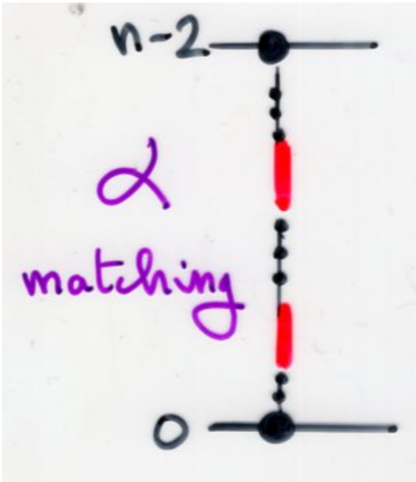
$$= \lambda_1 \cdots \lambda_{n-1} (\lambda_1 + \cdots + \lambda_n)$$

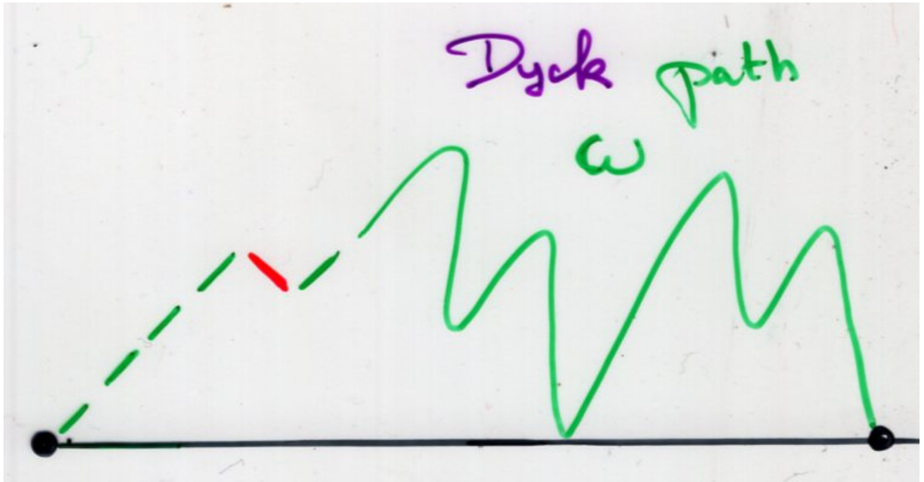
here pavages are only with dimers, that is are in fact matchings (of $[0, n-2]$).

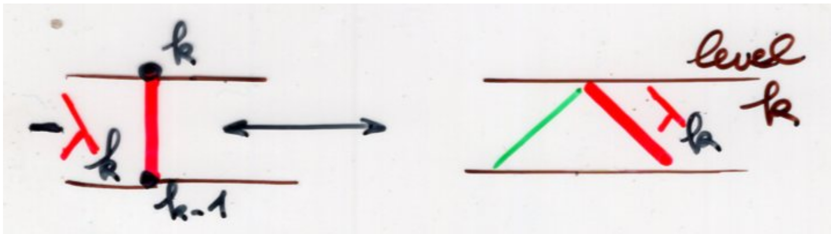
sign-reversing
weight preserving involution

$$(\alpha, \omega) \xrightarrow{\Phi} (\alpha', \omega')$$

same involution as in Ch1c, 26-27
(different "border" conditions)







$h(\alpha)$ = smallest index i of $[0, n-2]$
"occupied" by a dimer
(if $\alpha \neq \emptyset$)

$h(\omega)$ = level of the starting point of the first elementary step SE (always exist)

$$\begin{cases} (i) & h(\alpha) \leq h(\omega) \\ (ii) & h(\alpha) > h(\omega) \end{cases}$$

(i) $h(\alpha) \leq h(\omega)$ and $\alpha \neq \emptyset$

delete from the pavage α the leftmost piece, i.e. the dimer $(i, i+1)$ if $i = h(\alpha)$

incorporate  in the path ω as $(i+1, i+2)$ steps

equivalently the level of the first vertex of  is i

$$(\alpha, \omega) \xrightarrow{\Phi} (\alpha', \omega')$$

$$h(\omega') = h(\alpha)$$
$$h(\alpha') > h(\alpha)$$

we are in cas (ii)

the weight is preserved:
$$v(\alpha)\,v(\omega) = v(\alpha')\,v(\omega')$$

sign-reversing

(ii) $h(\alpha) > h(\omega)$ and $h(\omega) \leq (n-2)$

delete from the path ω the $(i, i+1)^{th}$ steps 

and add the dimer $(i-1, i)$ to α

$$(\alpha, \omega) \xrightarrow{\Phi} (\alpha', \omega')$$

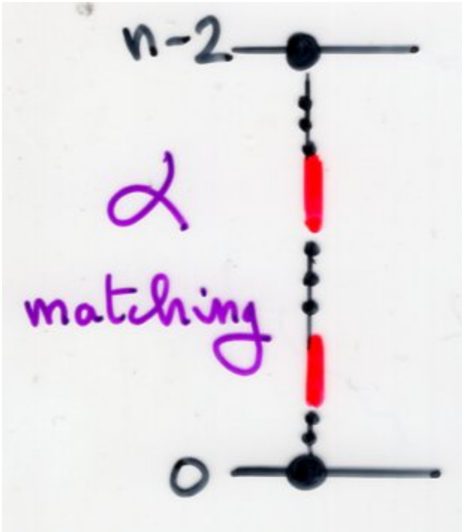
$h(\omega') = h(\alpha) + 1$
$h(\alpha') > h(\alpha) + 1$

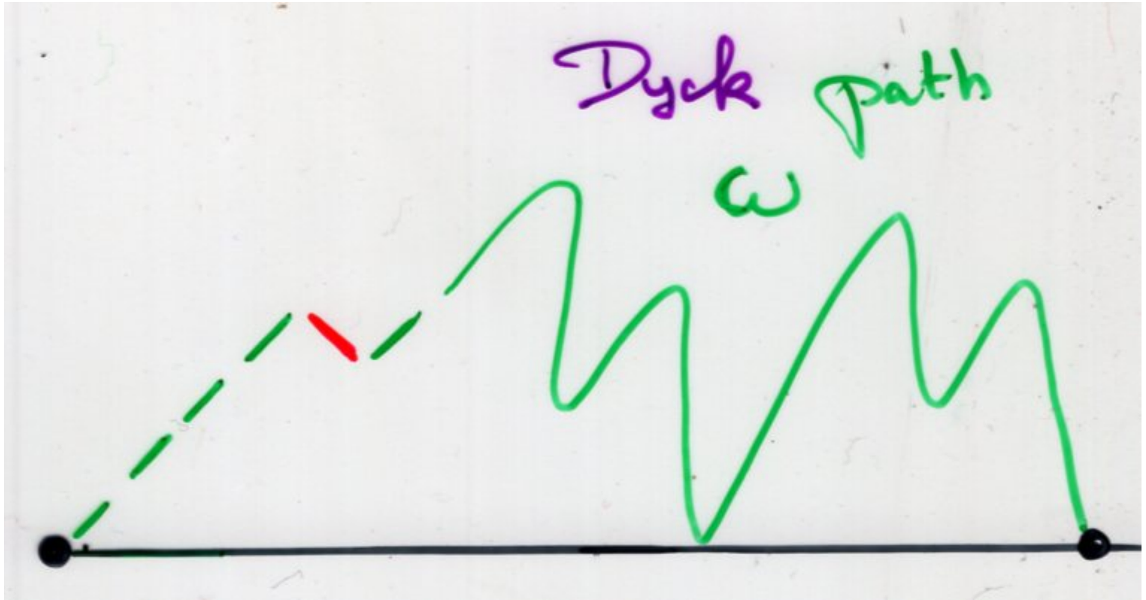
we are in cas (i)

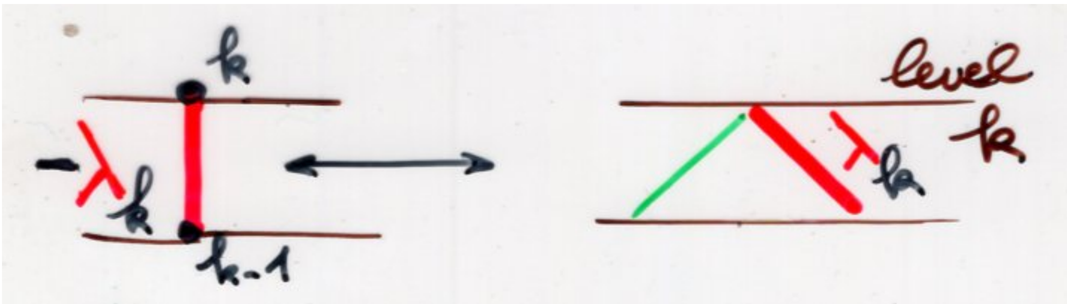
the weight is preserved:
$v(\alpha)v(\omega) = v(\alpha')v(\omega')$

sign-reversing

$$(\alpha, \omega) \xrightarrow{\Phi} (\alpha', \omega')$$







Φ is not defined for the pairs

$\alpha = \emptyset$, ω with length $|\omega| = 2n$

and the first $(n-1)$ steps are $/$

NE

thus : $\omega = \omega' \omega''$

ω'' is of the following type:

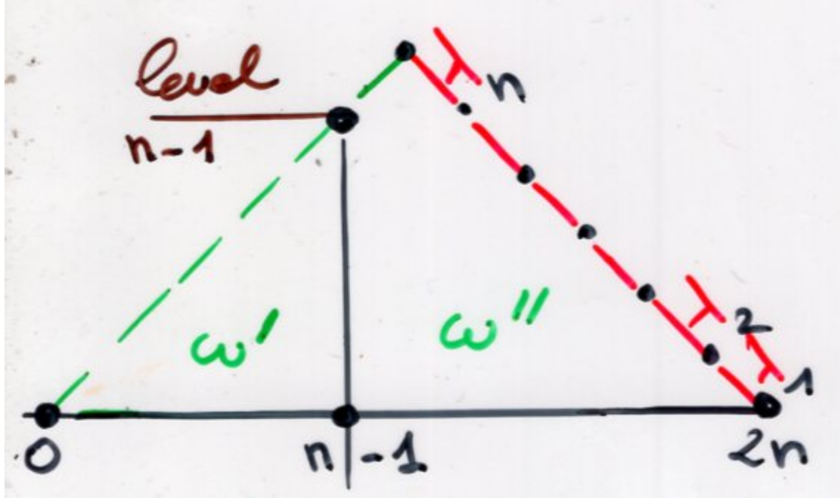
$$\begin{cases} \omega' = (/)^{n-1} \\ \omega'' \quad \text{"Dyck path"} \end{cases}$$

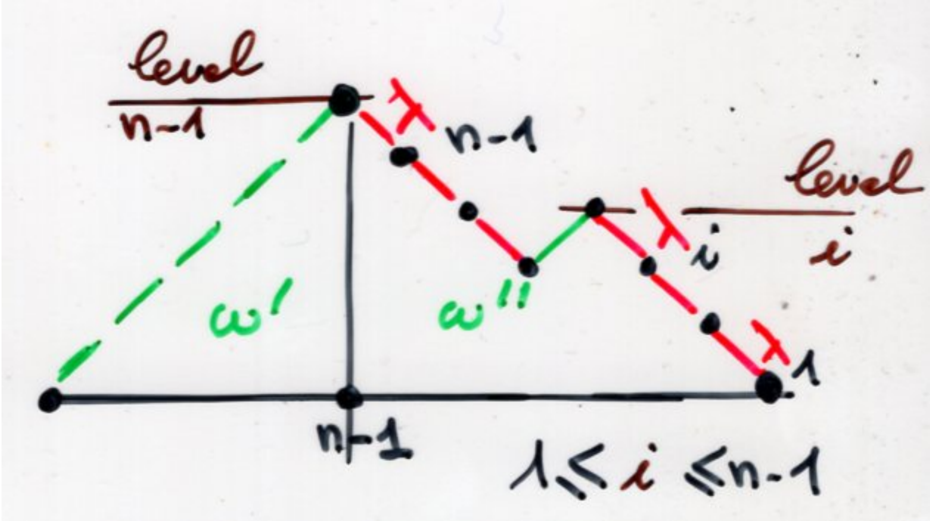
$(n-1)$ level

0 level

$|\omega''| = n+1$

ω'' is of the following type:





total weight: $= \lambda_1 \cdots \lambda_n + \sum_{1 \leq i \leq n-1} \lambda_i (\lambda_1 \cdots \lambda_{n-1})$

$$\sum_{0 \leq k < \frac{n}{2}} \left(\sum_{(\alpha, \omega)} (-1)^k v(\alpha) v(\omega) \right) = \sum_{|\omega| = 2n} v(\omega)$$

Dyck paths

$$\begin{cases} \bullet\ \alpha \text{ pavage of } [0, n-2] \text{ with } k = d(\alpha) \text{ dimers} \\ \bullet\ \omega \text{ Dyck path } |\omega| = 2n - 2k \end{cases}$$

$$\omega = \omega' \omega''$$

$$= \lambda_1 \cdots \lambda_n + \sum_{1 \leq i \leq n-1} \lambda_i (\lambda_1 \cdots \lambda_{n-1})$$

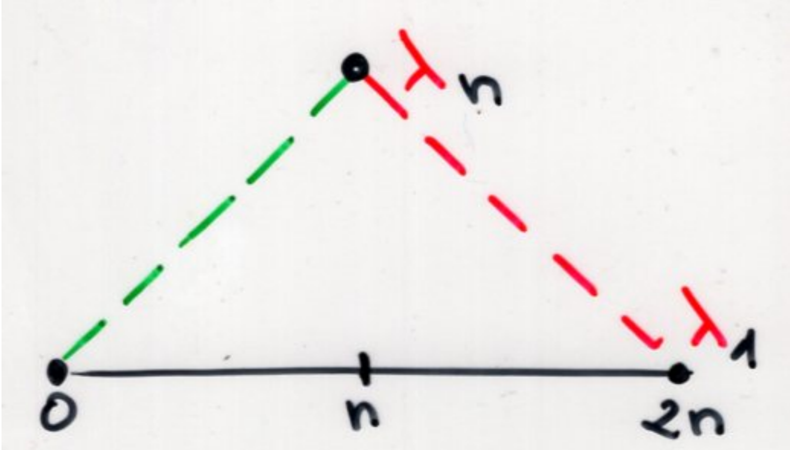
end of the proof $\square$

If we change the definition of $\varphi_r(n)$
by taking $\varphi_r(n+1) = \overline{\varphi}_r(n)$

i.e. $$\overline{\varphi}_r(n) = \sum_{\substack{\alpha \\ \text{pavages of } [0,n-1] \\ \text{with } r \text{ dimers}}} v(\alpha)$$

same formula, same involution
but now $\underline{\Phi}$ is not defined for the
pair (α, ω), $\alpha = \emptyset$, ω Dyck path
$|\omega| = 2n$, the first n steps are NE

i.e.

$\omega =$ 

$= \lambda_1 \dots \lambda_n$

other proofs

Berndt, Lamphere, Wilson (1985)

Goulden, Jackson (1984)

Andrews

by induction
à "sieving process"
(analogue to "inclusion-exclusion")

$\mathbf{b} = \{b_k\}_{k\geq 0}$ $\quad$ $\boldsymbol{\lambda} = \{\lambda_k\}_{k\geq 1}$ $\quad$ $J(t; \mathbf{b}, \boldsymbol{\lambda})$

Goulden, Jackson (1984)

$$P_k^*(t)\, f(t) - \delta P_{k-1}^*(t) = \lambda_1 \cdots \lambda_k\, t^k f_k(t)$$

$$f_k(t) = t^k J(t)\, J^{[1]}(t) \cdots J^{[k]}(t)$$

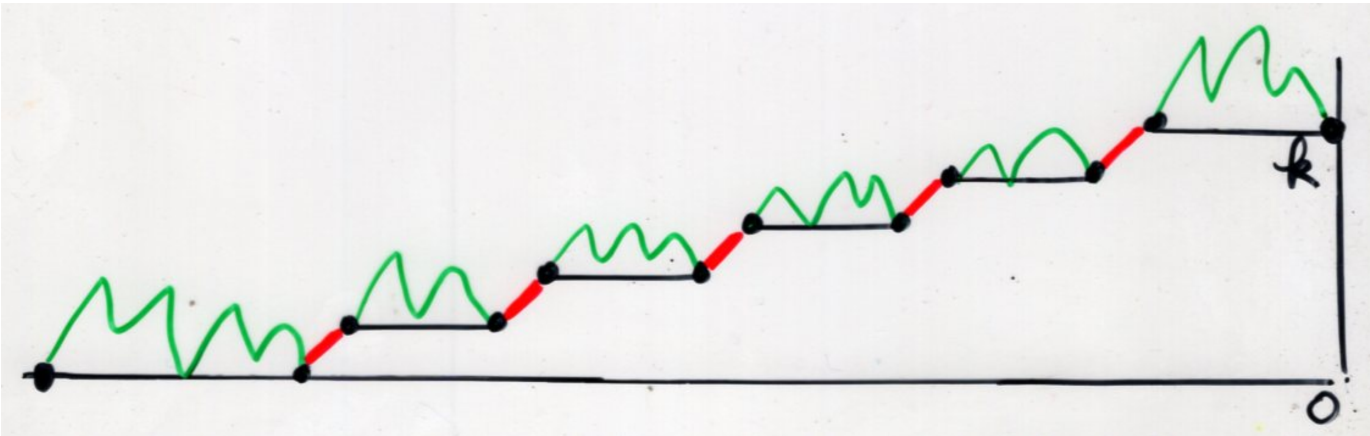
$$J^{[k]}(t) = \cfrac{1}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{1 - b_{k+1} t - \cfrac{\lambda_{k+2} t^2}{\cdots}}}$$

in fact

$$f_k(t) = \sum_{\substack{\omega \\ \text{Motzkin path} \\ 0 \to k}} v(\omega)\, t^{|\omega|}$$

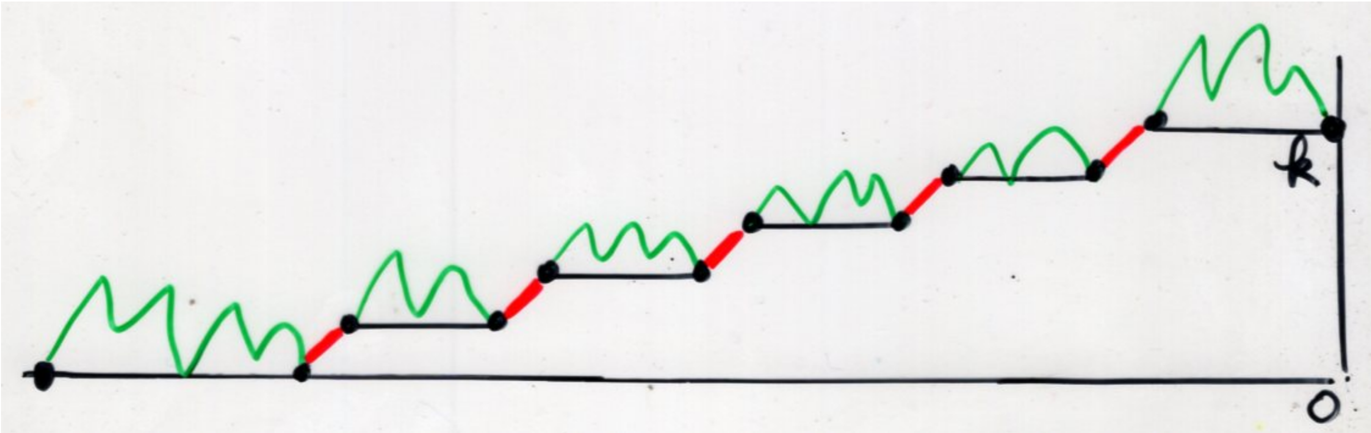
$$f_0(t) = f(t)$$

$$= J(t)$$



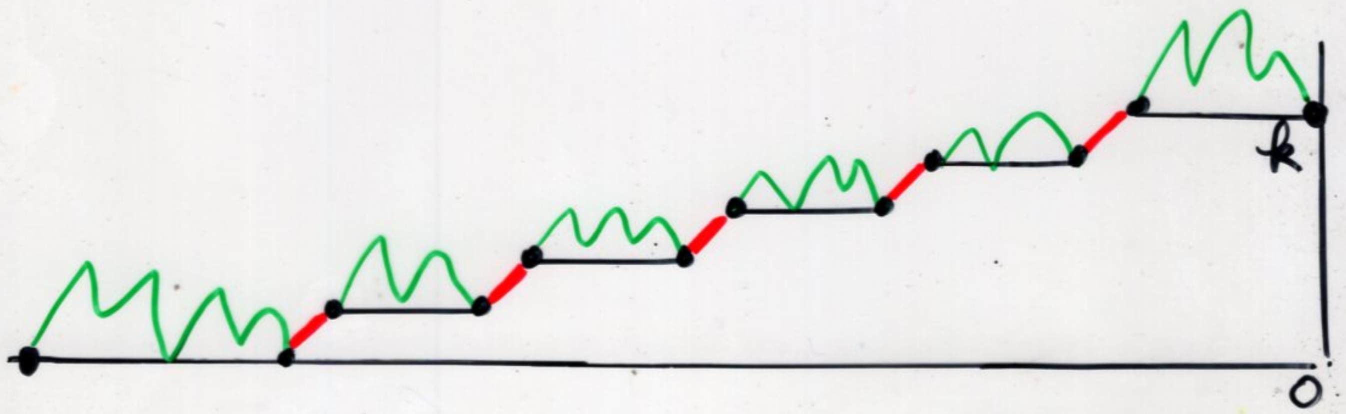
$$f_k(t) = \sum_{n \geq k} \mu_{n,k} t^n$$

coefficient of the vertical polynomials
$V_n(x)$, inverse polynomials of $T_n(x)$
(→ Ch 1d, 26-36)



$$f_k(t) = t^k \, J(t) \, J^{[1]}(t) \cdots J^{[k]}(t)$$

$$J^{[k]}(t) = \cfrac{1}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{1 - b_{k+1} t - \cfrac{\lambda_{k+2} t^2}{\cdots}}}$$



$$P_k^*(t) = \sum_{\substack{\alpha \\ \text{pavage of } [0,n-1]}} (-1)^{|\alpha|} v(\alpha)\, t^{m(\alpha)+2d(\alpha)}$$

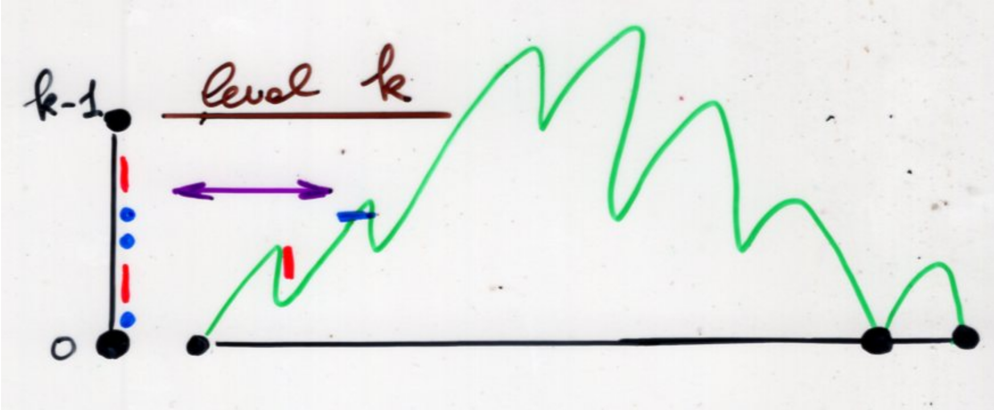
$$|\alpha| = m(\alpha) + d(\alpha)$$

$m(\alpha)$ = number of monomers of the pavage α

$d(\alpha)$ = number of dimers of the pavage α

$$P^*_k(t)\, f(t) = \sum_{(\alpha,\omega)} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)\, t^{m(\alpha)+2d(\alpha)+|\omega|}$$

$$\begin{cases} \alpha & \text{pavage of } [0, k-1] \\ \omega & \text{Motzkin path } 0 \rightsquigarrow 0 \end{cases}$$



sign-reversing
weight preserving involution

same involution as in Ch1c, 26-27
(different "border" conditions)

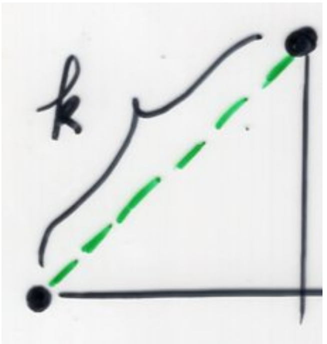
$$P_k^{*}(t)\, f(t) = \sum_{(\alpha,\omega)} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)\, t^{m(\alpha)+2d(\alpha)+|\omega|}$$

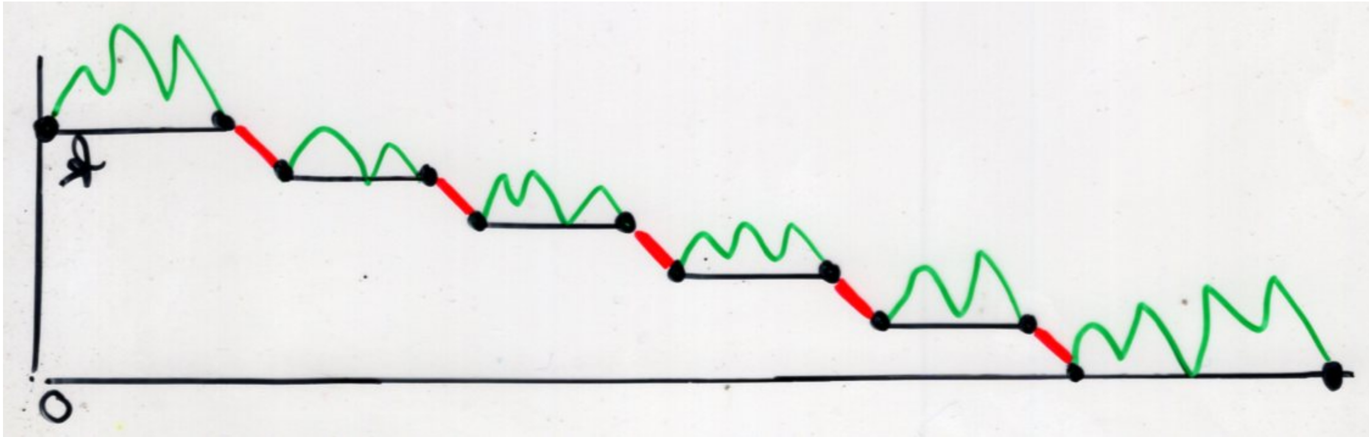
- α empty pavage
- ω Motzkin path $0 \rightsquigarrow 0$
  such that the first k steps are NE

or

- α pavage of $[0, k-1]$ such that
  0 is an isolated point
- ω empty path

- α empty pavage
- ω Motzkin path $0 \rightsquigarrow 0$
  such that the first k steps are NE



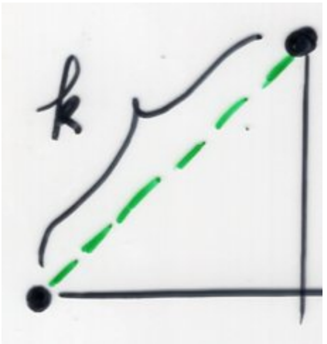


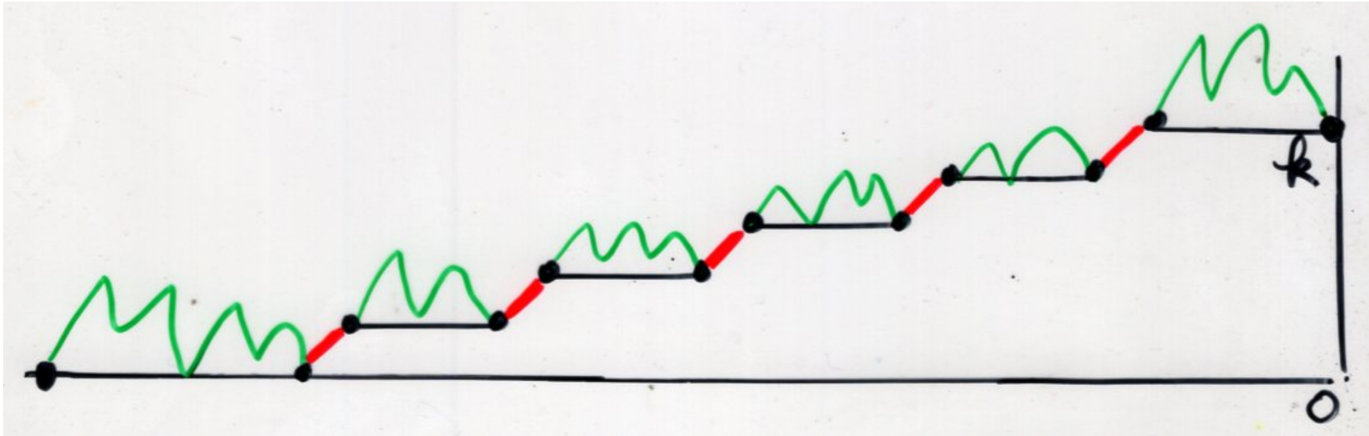
$$t^k$$

$$\sum_{\substack{\omega \\ \text{Motzkin path}}} v(\omega)\, t^{|\omega|}$$

$k \rightsquigarrow 0$

- α empty pavage
- ω Motzkin path $0 \rightsquigarrow 0$ such that the first k steps are NE





$$t^k \qquad \lambda_1 \cdots \lambda_k f_k(t)$$

$$f_k(t) = \sum_{\substack{\omega \\ \text{Motzkin path} \\ 0 \rightsquigarrow k}} v(\omega)\, t^{|\omega|}$$

$$P^{*}_{k}(t)\, f(t) = \sum_{(\alpha,\omega)} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)\, t^{m(\alpha)+2d(\alpha)+|\omega|}$$

or

- α parage of $[0, k-1]$ such that 0 is an isolated point
- ω . empty path

$$\delta P^{*}_{k-1}(t)$$

$$P^{*}_{k}(t)\, f(t) = t^{k}\, \lambda_1 \cdots \lambda_k\, f_k(t) + \delta\, P^{*}_{k-1}(t)$$

- α empty passage
- ω Motzkin path $0 \rightsquigarrow 0$ such that the first k steps are NE

- α passage of $[0, k-1]$ such that 0 is an isolated point
- ω empty path

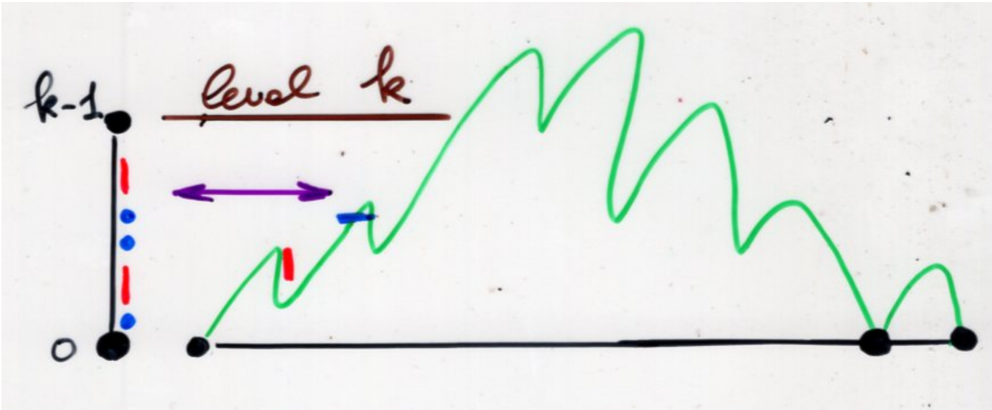
$$P^{*}_{k}(t)\, f(t) - \delta\, P^{*}_{k-1}(t) = t^{k}\, \lambda_1 \cdots \lambda_k\, f_k(t)$$

same "essence" of the involution

sign-reversing, weight preserving

(with some variations

and different "border conditions"



- Ramanujan's formula
  (Notebook, entry 17, Ch. 12)
- The "main theorem" Ch 1
  $\Rightarrow$ Favard's theorem
- Convergents of continued fractions

- Goulden, Jackson formula

$$P_k^*(t)\,f(t) - \delta\, P_{k-1}^*(t) = t^k \lambda_1 \cdots \lambda_k\, f_k(t)$$

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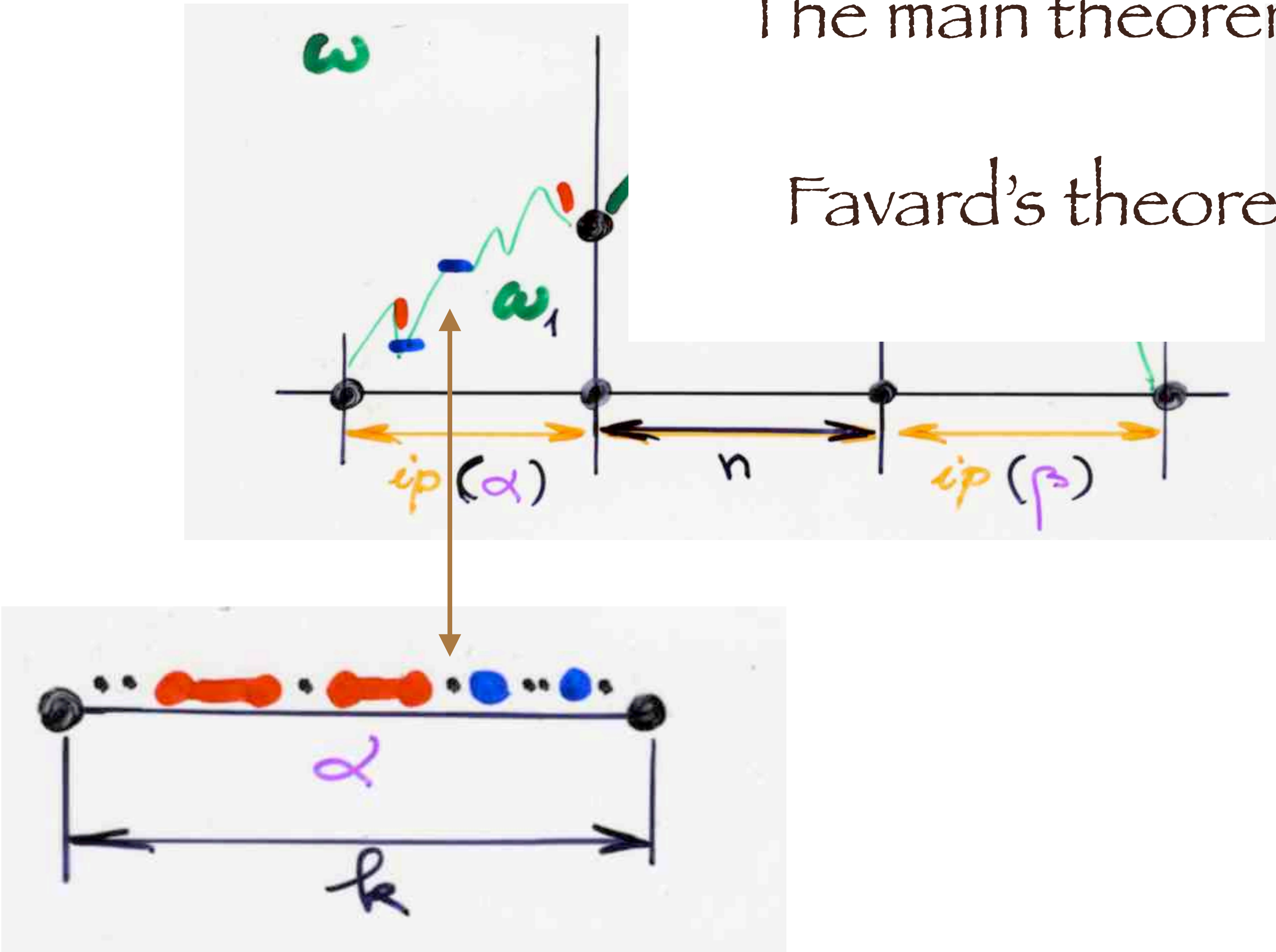
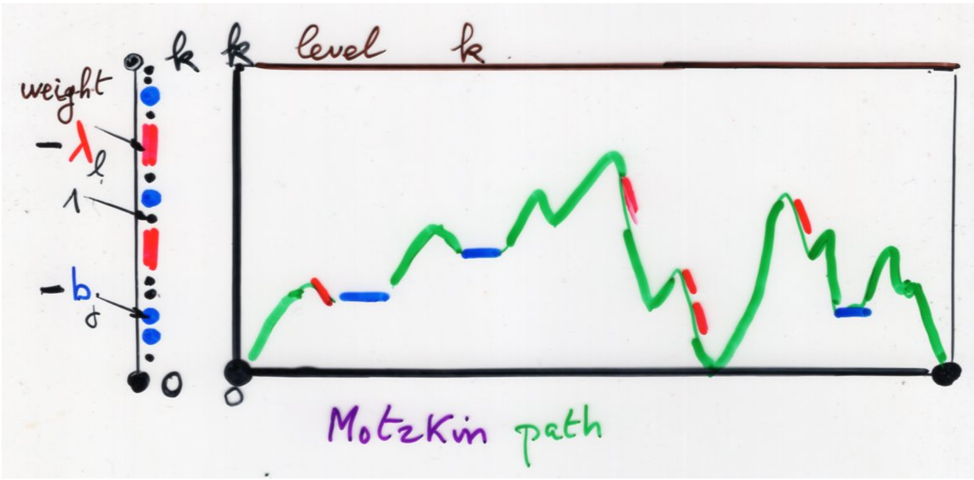
The main theorem

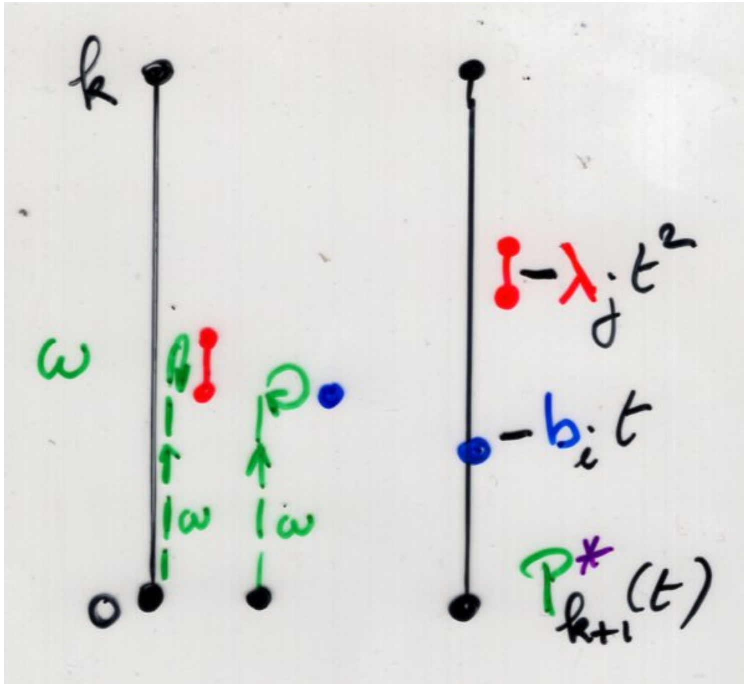
Favard's theorem



$$(\omega, \alpha, \beta) \in E_{n,k,\ell} \setminus L_{n,k,\ell}$$

$$P^*_{k+1}(t)\left[\sum_{\substack{\omega \\ \text{Motzkin path} \\ \text{height} \leq k}} v(\omega)\, t^{|\omega|}\right] = \delta P^*_k(t)$$





The quotient-difference algorithm

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$\mu_0 = 1$

$$\underbrace{\hspace{4em}}_{S(t;\lambda)}$$

Stieltjes continued fraction

$$\sum_{n\geq 0} \nu_n t^n = S(t;\lambda)$$

Stieltjes continued fraction

$$\{\nu_n\}_{n\geq 0} \longrightarrow \lambda = \{\lambda_k\}_{k\geq 1}$$

equivalently $P_{k+1}(x) = xP_k(x) - \lambda_k P_{k-1}(x)$

$$\begin{cases} \mu_{2n} = \nu_n \\ \mu_{2n+1} = 0 \end{cases}$$

$$\sum_{n\geq 0} \nu_n t^n = S(t;\lambda)$$

Stieltjès continued fraction $\{\nu_n\}_{n\geq 0}$ (moments)

$$\sum_{n\geq 0} \mu_{2n} t^{2n} = S(t^2;\lambda)$$
$$= J(t;0,\lambda)$$

$$\begin{cases} \mu_{2n} = \nu_n \\ \mu_{2n+1} = 0 \end{cases}$$

$\cdot\, b_k = 0$
$(k \geq 0)$

the ring $\mathbb{K}$ is a field

quotient-difference algorithm

qd-algorithm

Steifel (1958)

Rutishauser (1957)

Henrici (1958, 1974)

Gragg (1972)

continued fractions

orthogonal polynomials

→ Padé approximants

applied mathematics

numerical analysis
theoretical physics

Sogo (1993)

application of the qd-algorithm
to the solution of the Toda molecule equation

" Excited states of Calogero-Sutherland-
Moser model - classification by Young
diagrams "

numerical analysis
theoretical physics

→ Padé approximants

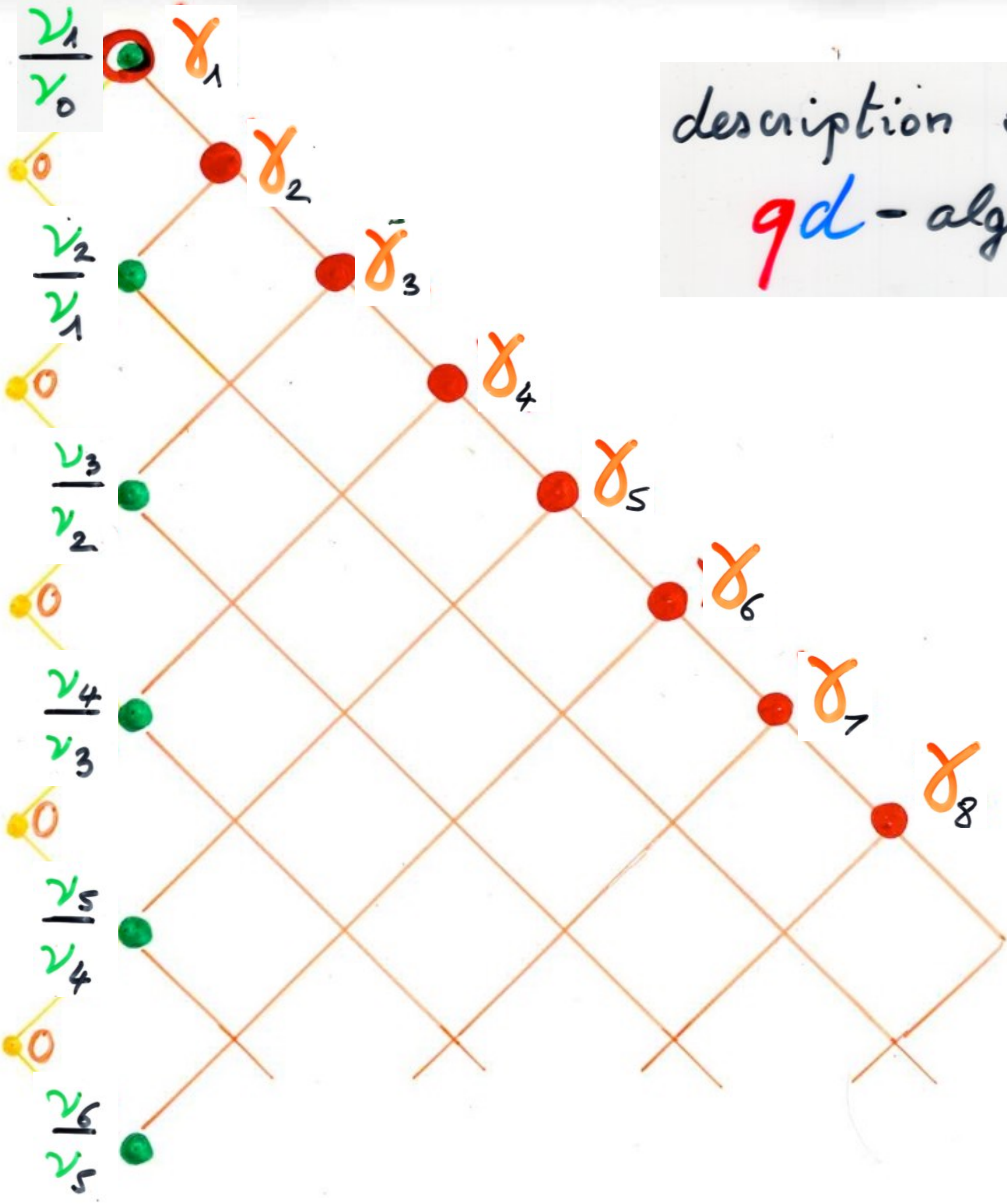
books:

Baker (1975)
Baker, Grave-Morris (1981)
(vols 1, 2)
Brezinski (1980)
Gilewicz (1978)

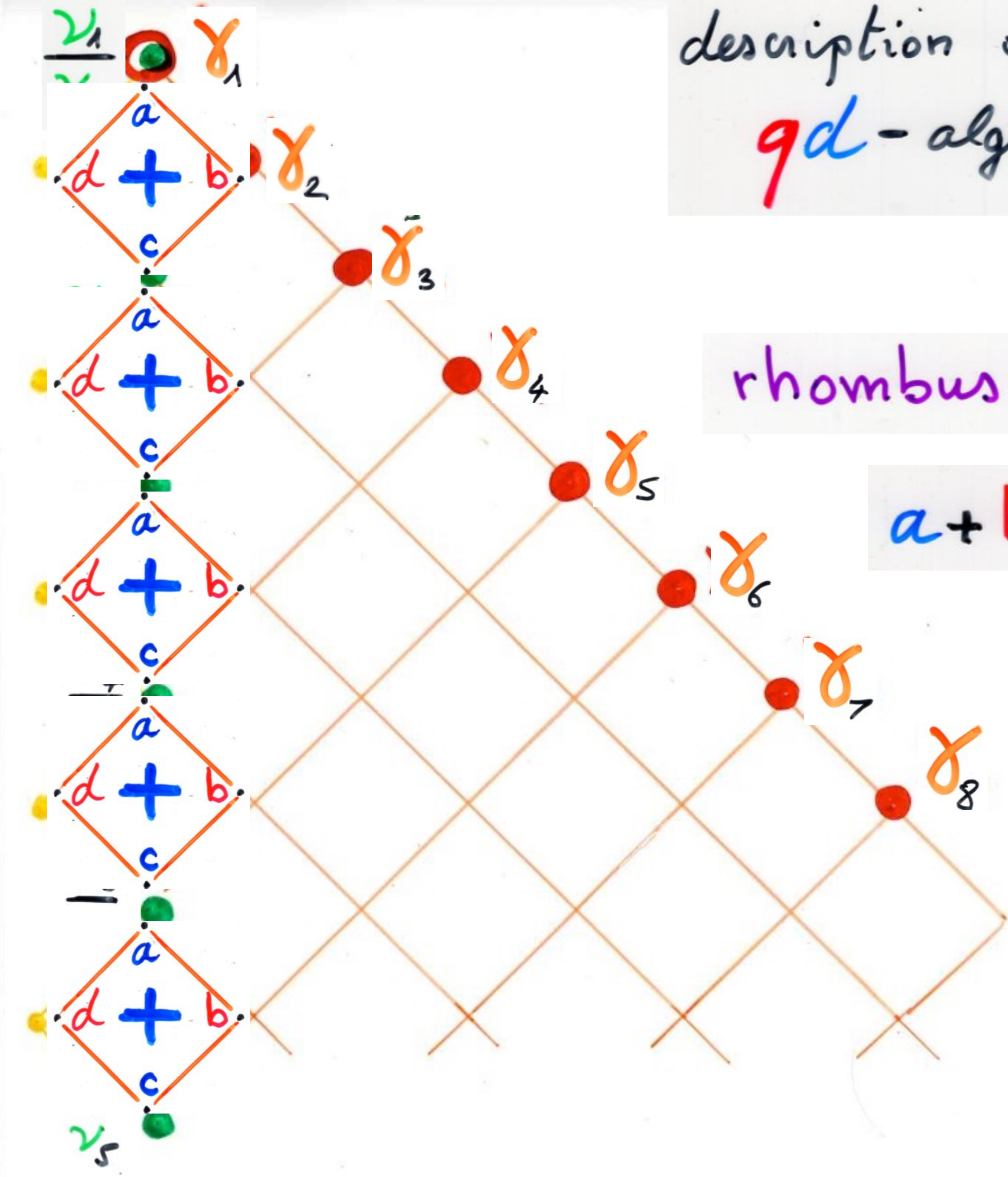
combinatorial interpretation

Roblet (1994)

description of the qd-algorithm



description of the qd-algorithm

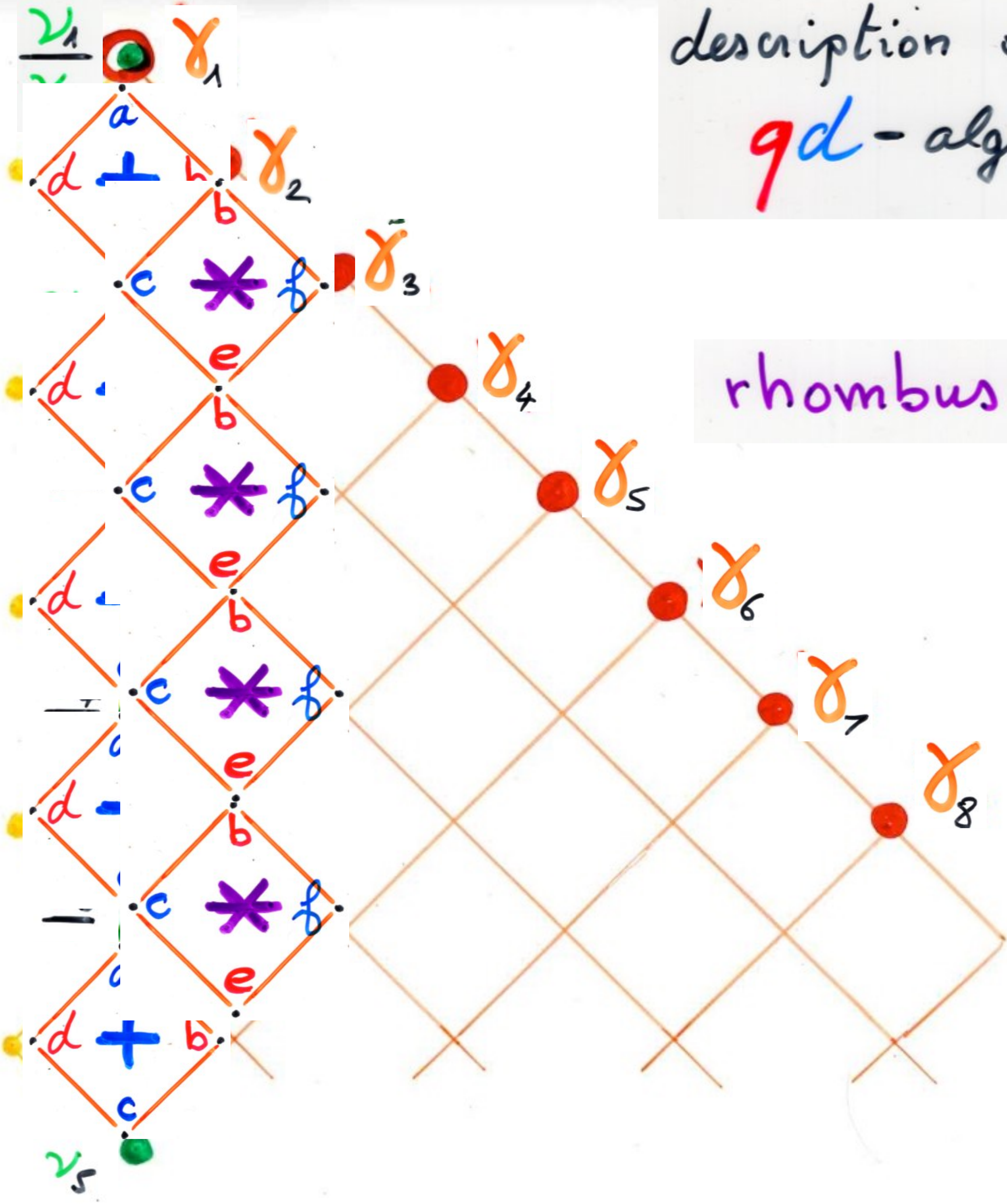


rhombus rules

$$a + b = c + d$$

description of the

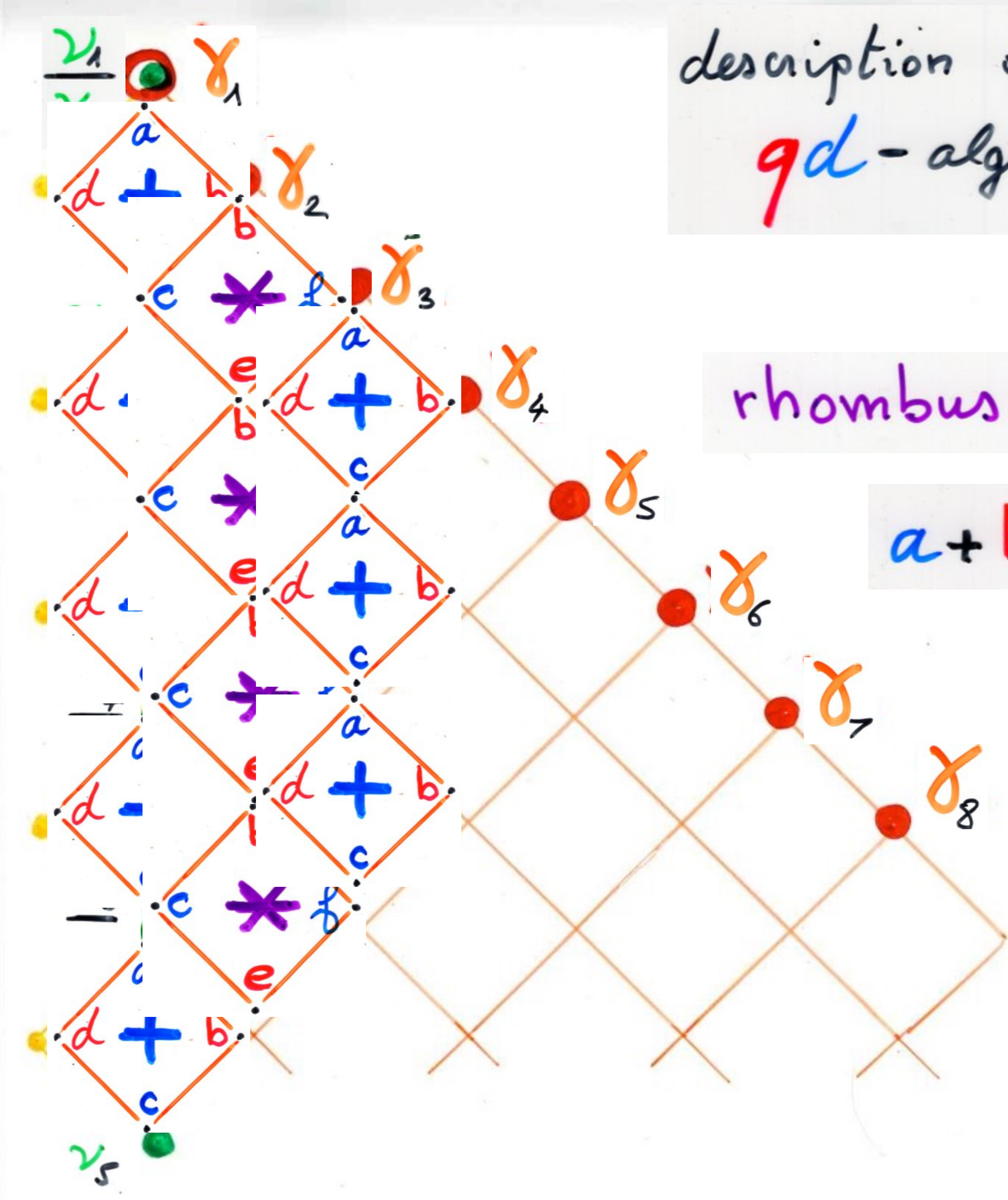
qd - algorithm



rhombus rules

$bf = ec$

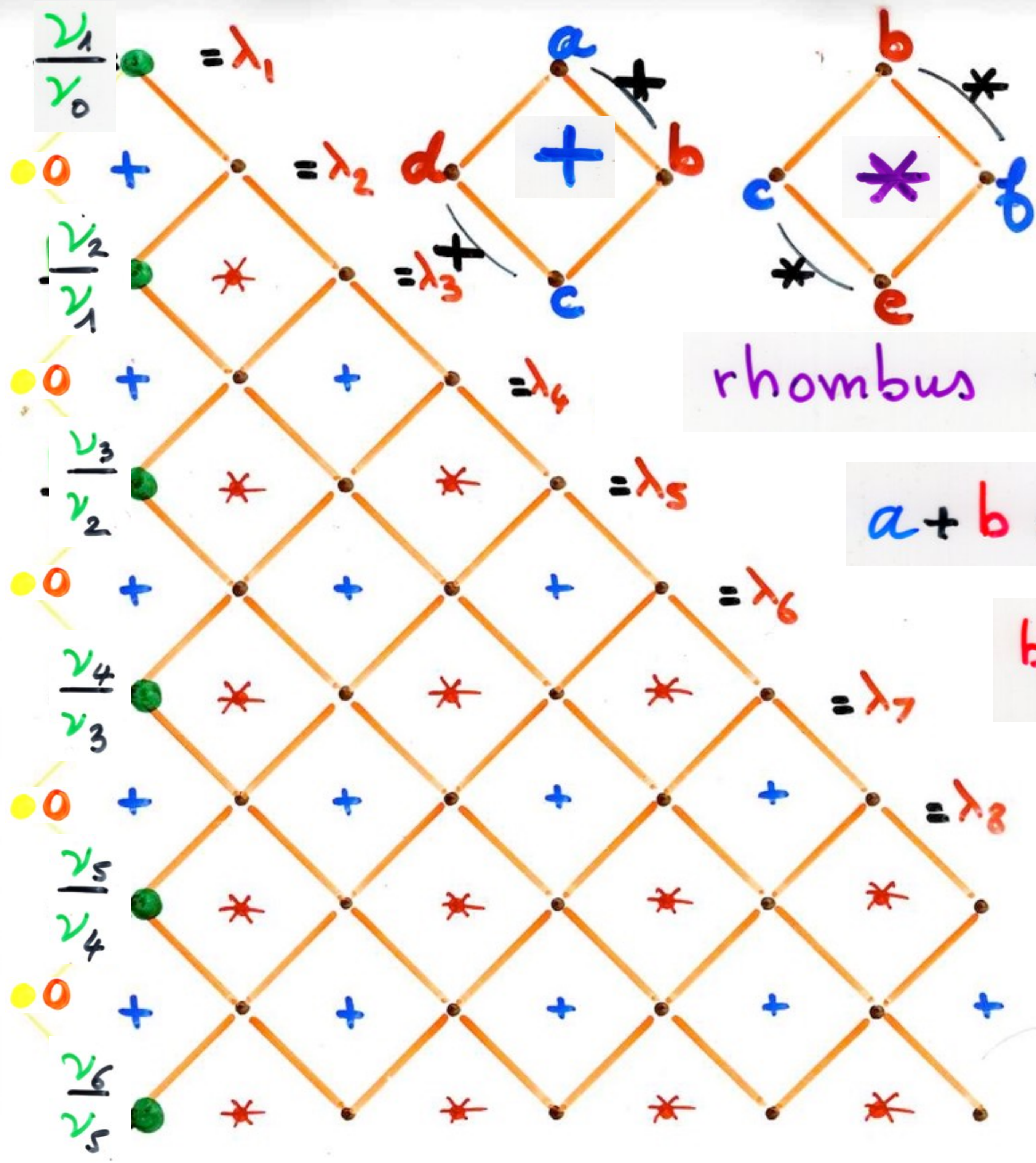
description of the qd-algorithm



rhombus rules

$$a + b = c + d$$

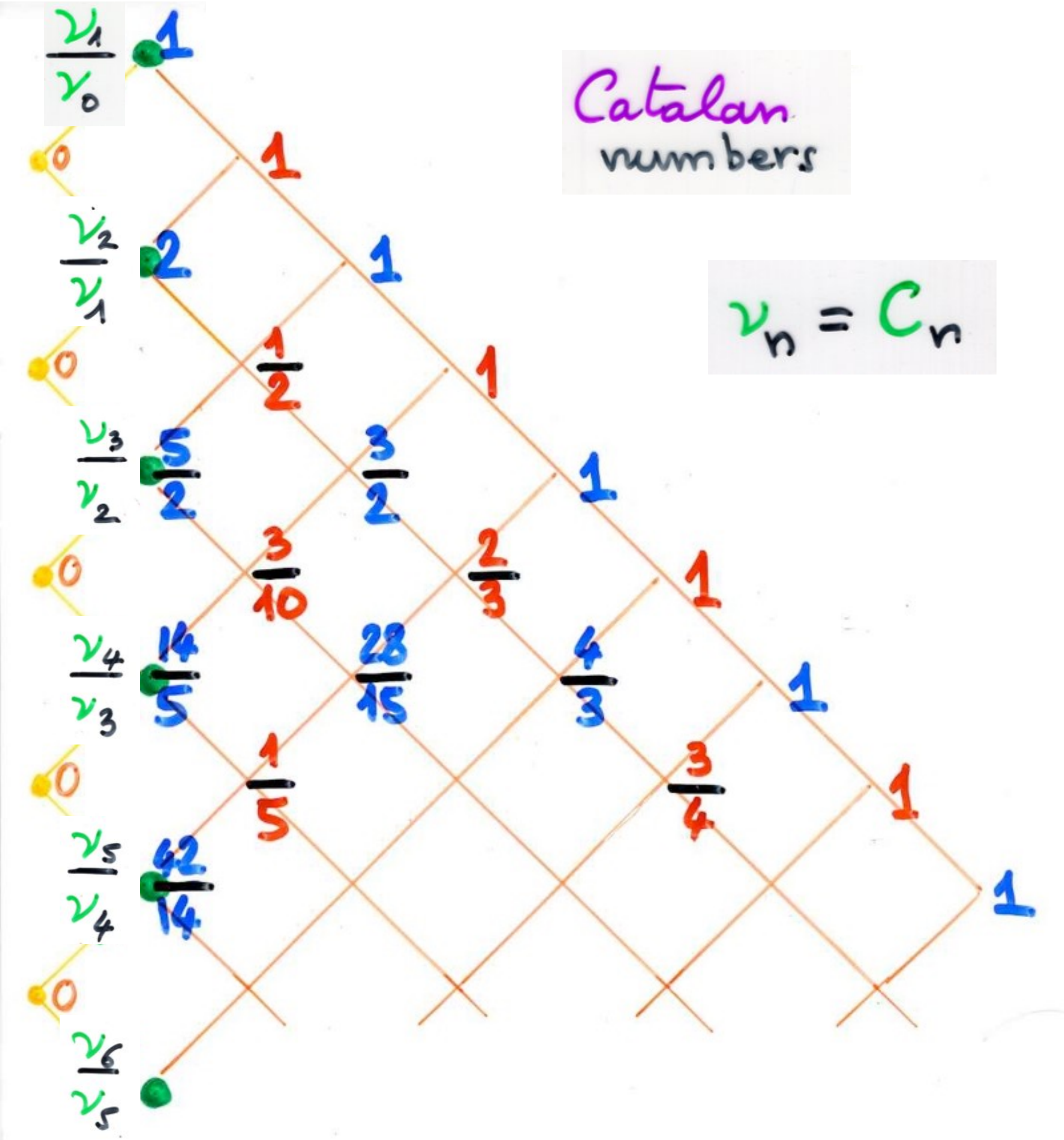
$$b f = e c$$



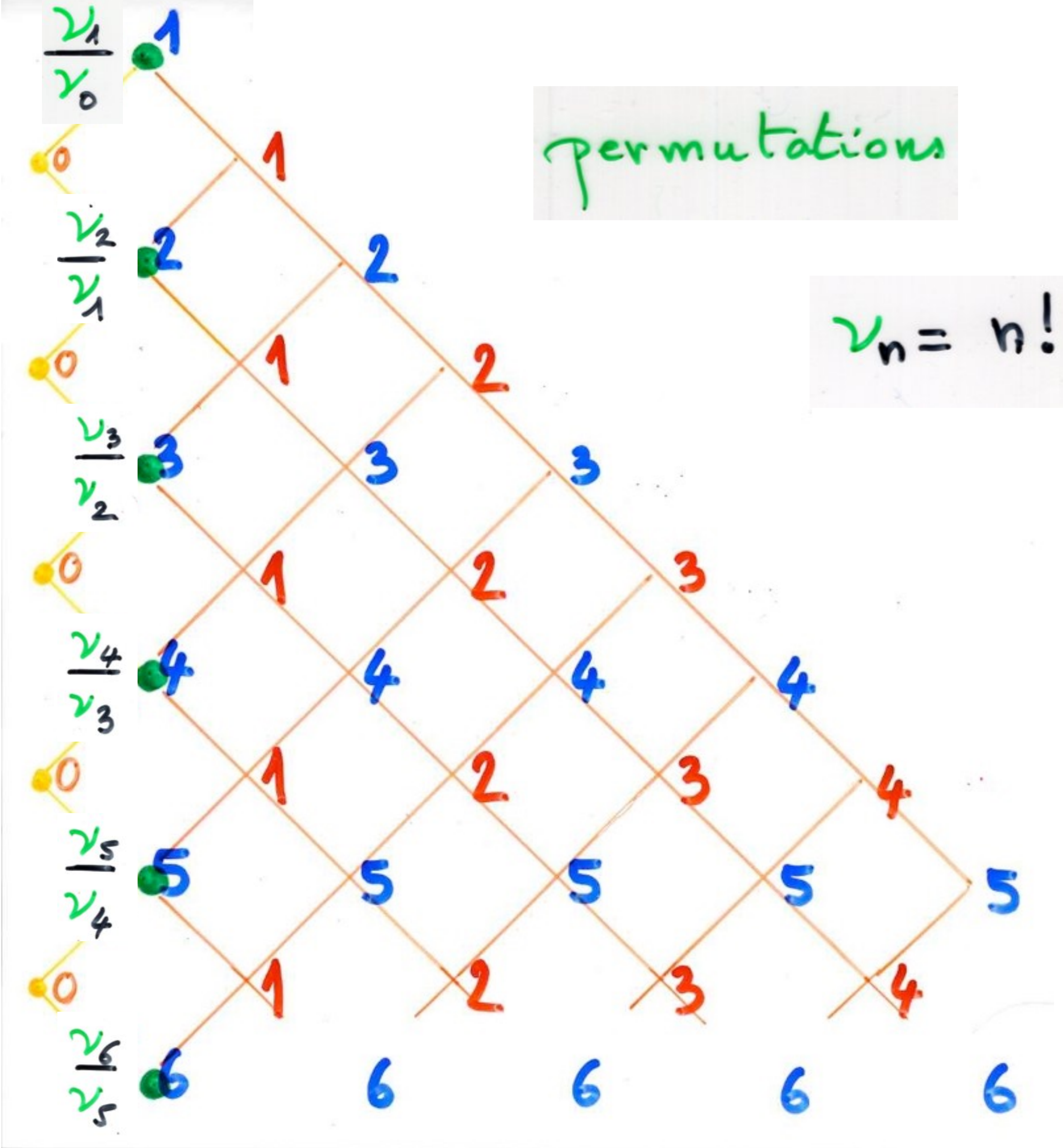
rhombus rules

$a + b = c + d$

$bf = ec$

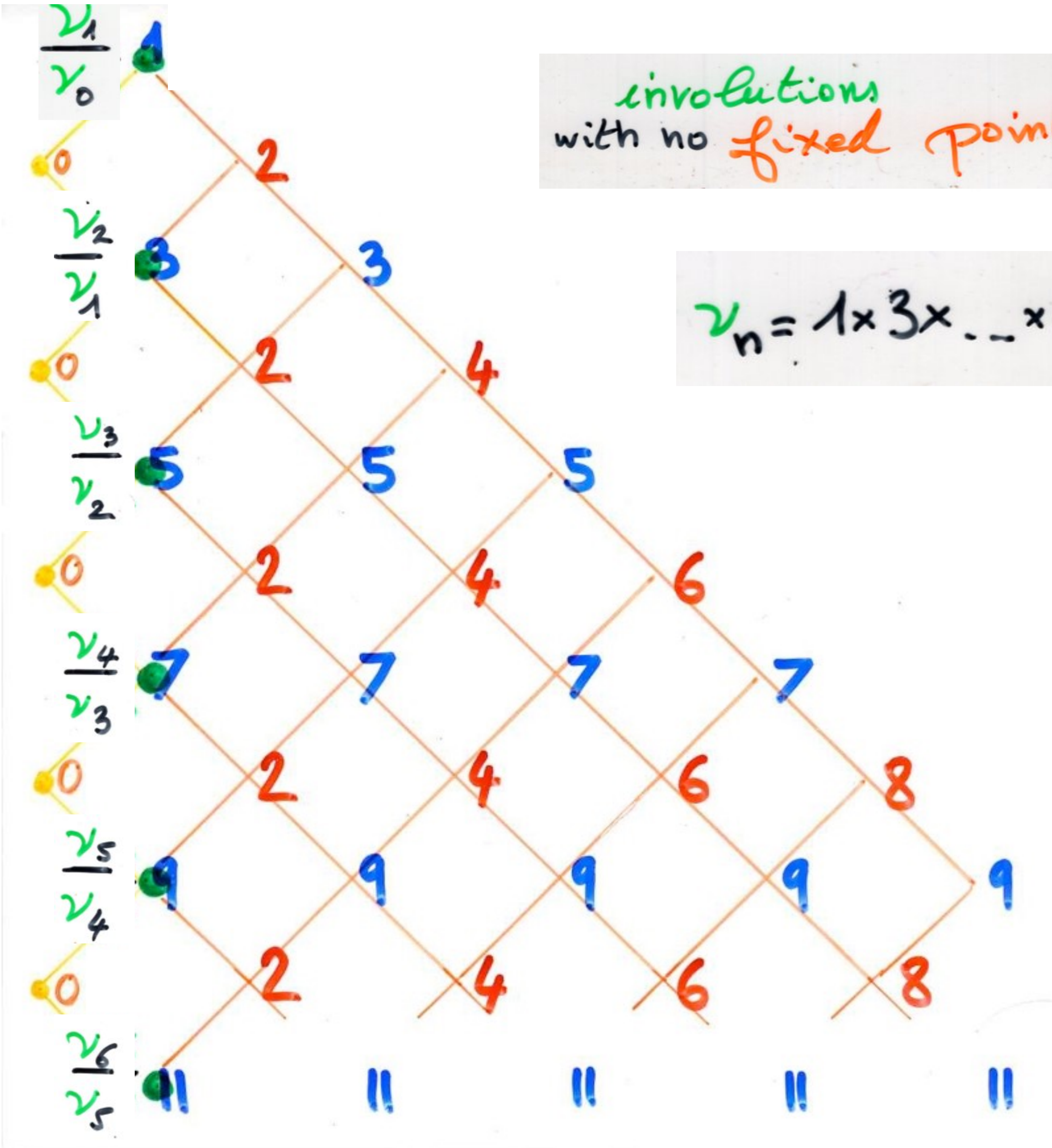


Catalan numbers
$\nu_n = C_n$
$\frac{\nu_1}{\nu_0}$
$\frac{\nu_2}{\nu_1}$
$\frac{\nu_3}{\nu_2}$
$\frac{\nu_4}{\nu_3}$
$\frac{\nu_5}{\nu_4}$
$\frac{\nu_6}{\nu_5}$
0
0
0
0
0
1
1
1
1
1
1
1
1
1
1
1
2
$\frac{1}{2}$
$\frac{5}{2}$
$\frac{3}{2}$
$\frac{3}{10}$
$\frac{2}{3}$
$\frac{14}{5}$
$\frac{28}{15}$
$\frac{4}{3}$
$\frac{1}{5}$
$\frac{3}{4}$
$\frac{42}{14}$


permutations
$\nu_n = n!$
$\frac{\nu_1}{\nu_0}$
$\frac{\nu_2}{\nu_1}$
$\frac{\nu_3}{\nu_2}$
$\frac{\nu_4}{\nu_3}$
$\frac{\nu_5}{\nu_4}$
$\frac{\nu_6}{\nu_5}$
0
0
0
0
0
1
1
2 2
1 2
3 3 3
1 2 3
4 4 4 4
1 2 3 4
5 5 5 5 5
1 2 3 4
6 6 6 6 6

involutions with no fixed points

$\nu_n = 1 \times 3 \times \dots \times (2n-1)$



The qd-transform

$S(t;\gamma)$ $\gamma = \{\gamma_k\}_{k\geq 1}$

Definition the qd-transform

$\gamma = \{\gamma_k\}_{k\geq 1} \longrightarrow \gamma' = \{\gamma'_k\}_{k\geq 1}$

$\gamma_k, \gamma'_k \in \mathbb{K}$ $(k \geq 1)$

$\gamma' = qd(\gamma)$

$$S(t;\gamma) = 1 + \gamma_1 t\, S(t;\gamma')$$

(Hermite polynomials
or involutions with no fixed points)

4 examples

$\gamma = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$

$\downarrow$

$\gamma' = 3, 2, 5, 4, 7, 6, 9, 8, 11, 10, \ldots$

- $\gamma_k = k$, then $\gamma'_k = k$ if k even
  $\gamma'_k = k+2$ if k odd

(Catalan numbers)

$$\gamma = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots$$
$$\downarrow$$
$$\gamma' = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, \frac{3}{4}, \frac{5}{4}, \frac{4}{5}, \frac{6}{5}, \frac{5}{6}, \dots$$

- $\gamma_k = 1$, then $\gamma'_{2k} = \dfrac{k}{k+1}$, $\gamma'_{2k-1} = \dfrac{k+1}{k}$

(Euler's continued fraction for $n!$)

$$\gamma = 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$$
$$\downarrow$$
$$\gamma' = 2, 1, 3, 2, 4, 3, 5, 4, 6, 5, \ldots$$

- $\gamma_k = \left\lceil \frac{k}{2} \right\rceil$, $\begin{cases} \gamma'_k = \frac{k}{2} & k \text{ even} \\ \gamma'_k = 1 + \left\lceil \frac{k}{2} \right\rceil & k \text{ odd} \end{cases}$

  $k \geq 1$

$$\bullet \begin{cases} \gamma_k = 1, & k \text{ odd} \\ \gamma_k = 2, & k \text{ even} \end{cases} \quad \text{(small Schröder numbers)}$$

$$\gamma = 1, 2, 1, 2, 1, 2, 1, 2, \ldots$$

$$\downarrow$$

$$\gamma' = 3, \frac{2}{3}, \frac{7}{3}, \frac{6}{7}, \frac{15}{7}, \frac{14}{15}, \frac{31}{15}, \frac{30}{31}, \frac{63}{31}, \frac{62}{63}$$

$$\gamma'_{2k} = \frac{2^{k+1}-2}{2^{k+1}-1}, \quad \gamma'_{2k-1} = \frac{2^{k+1}-1}{2^{k}-1} \quad (k \geq 1)$$

Proposition

Let $\gamma = \{\gamma_k\}_{k\geq 1}$, $\gamma' = \{\gamma'_k\}_{k\geq 1}$ be two sequences of $\mathbb{K}$

$\gamma' = q\alpha(\gamma)$ iff for every $k \geq 0$

$$\begin{cases} \gamma_{2k+1} + \gamma_{2k+2} = \gamma'_{2k} + \gamma'_{2k+1} \\ \gamma_{2k}\,\gamma_{2k+1} = \gamma'_{2k-1}\,\gamma'_{2k} \end{cases}$$

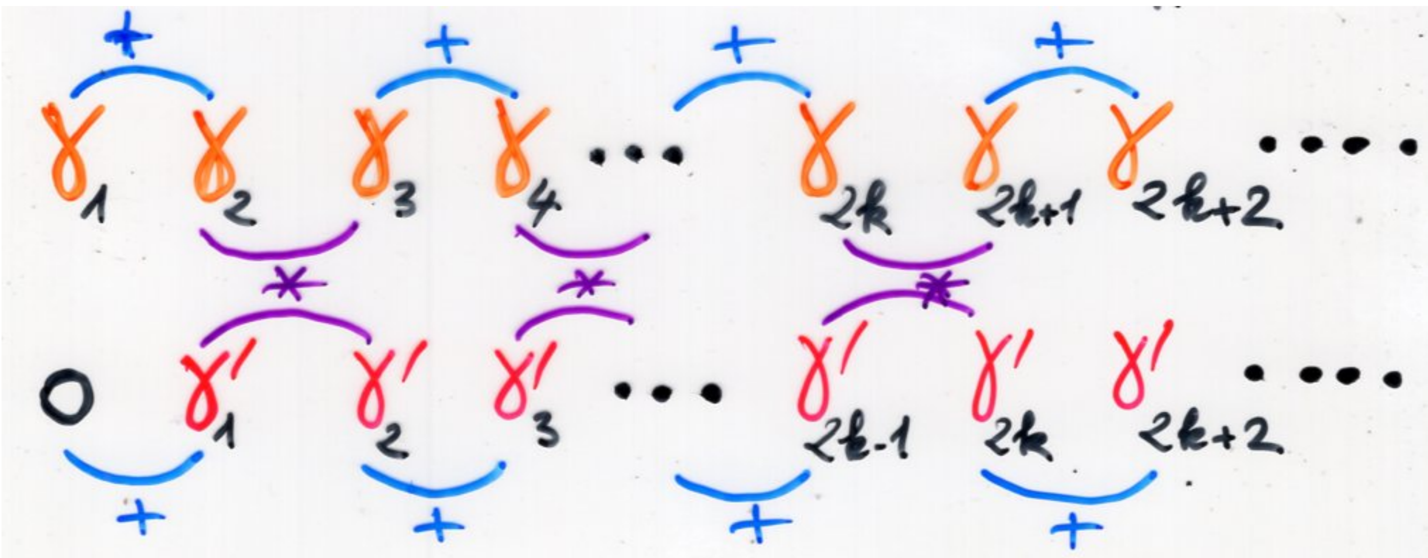
$+$ sum

$*$ product

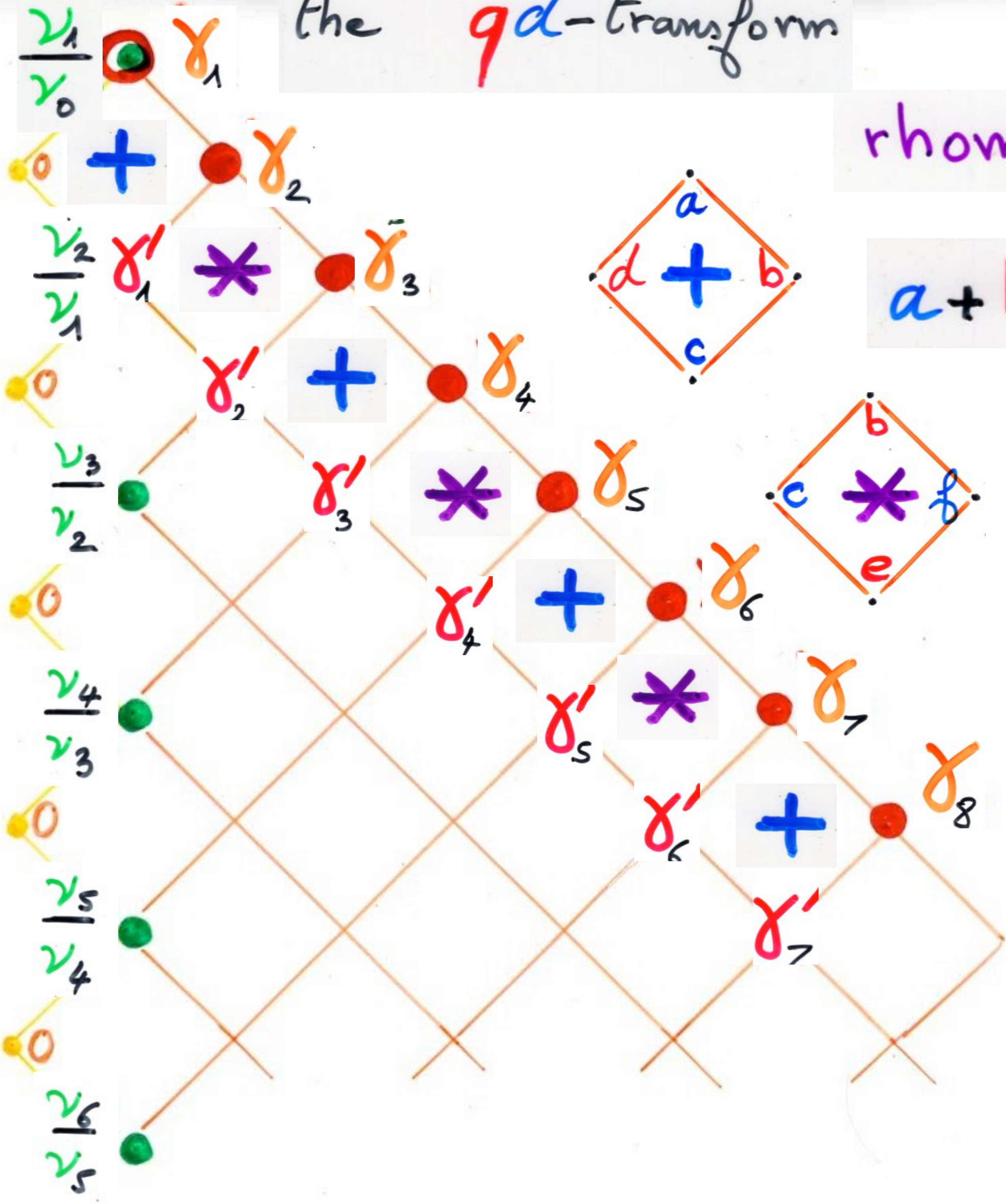
the qd-transform

computation of $\{\gamma'_k\}$ from $\{\gamma_k\}$

as soon as $\gamma'_k \neq 0$



the qd-transform



rhombus rules

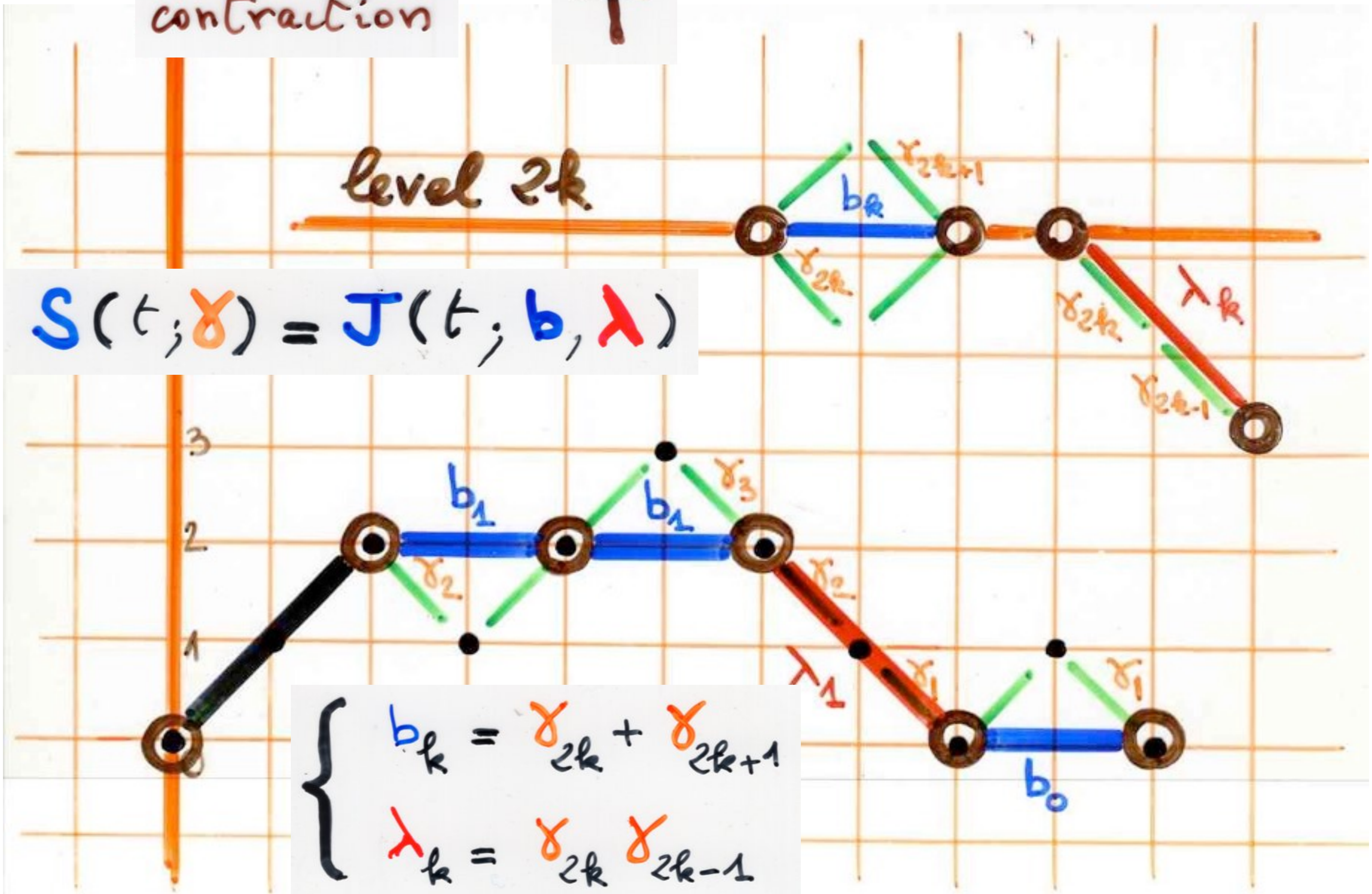
$a+b=c+d$

$bf=ec$

proof:

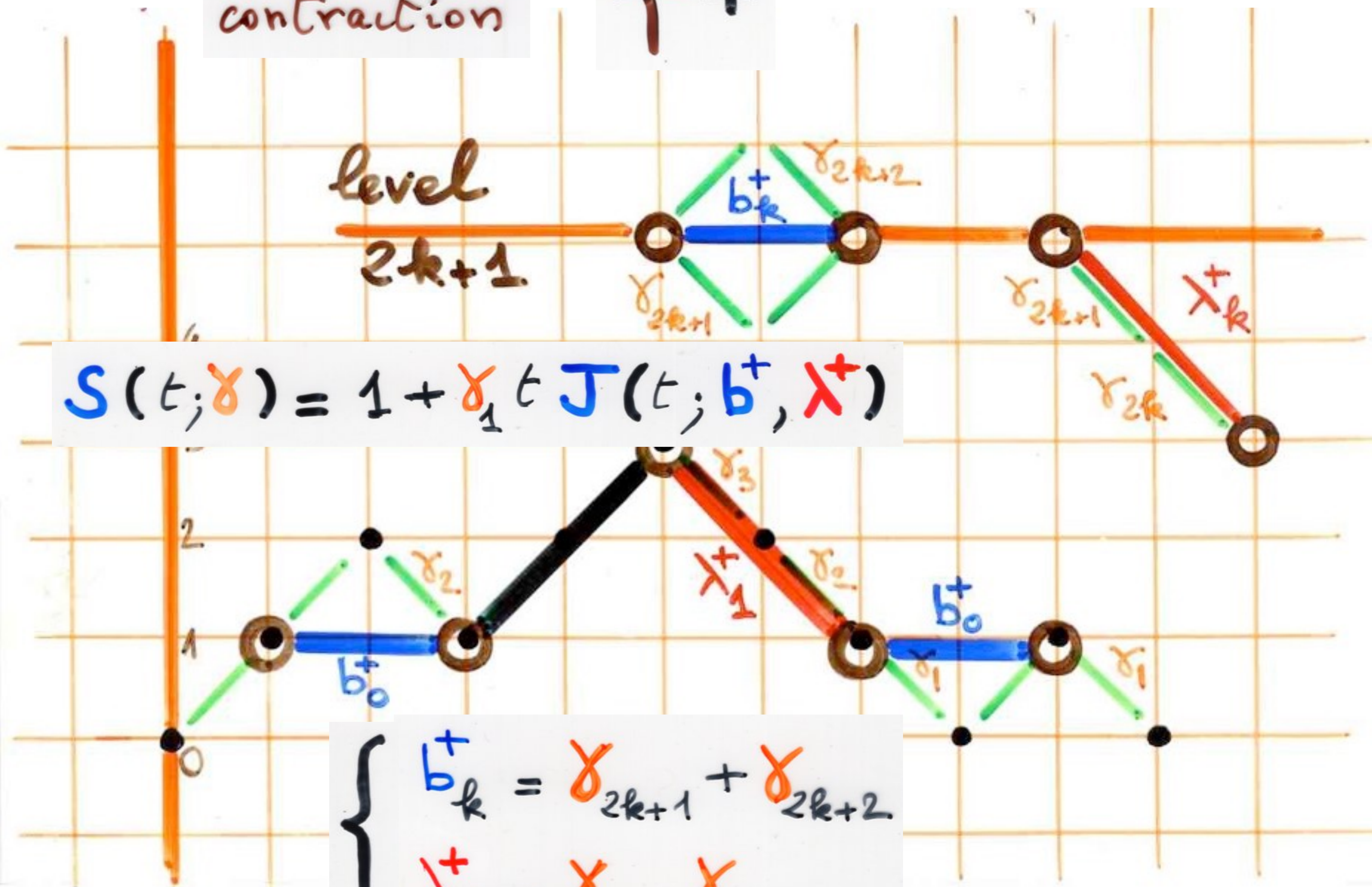
contraction T

$$S(t;\gamma) = J(t;b,\lambda)$$



$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$

contraction T^+



$$S(t;\gamma) = 1 + \gamma_1 t\, J(t; b^+, \lambda^+)$$

$$\begin{cases} b^+_k = \gamma_{2k+1} + \gamma_{2k+2} \\ \lambda^+_k = \gamma_{2k+1}\gamma_{2k} \end{cases}$$

$$S(t;\gamma) = 1 + \gamma_1 t\, J(t;\, b^+,\, \lambda^+)$$

contraction T

contraction T^+ $\quad S(t;\, \gamma')$

$qd(\gamma)$

$$\begin{cases} b_k = \gamma_{2k} + \gamma_{2k+1} \\ \lambda_k = \gamma_{2k}\,\gamma_{2k-1} \end{cases}$$

$$\begin{cases} b^+_k = \gamma_{2k+1} + \gamma_{2k+2} \\ \lambda^+_k = \gamma_{2k+1}\,\gamma_{2k} \end{cases}$$

$$\begin{cases} \gamma_{2k+1} + \gamma_{2k+2} = \gamma'_{2k} + \gamma'_{2k+1} \\ \gamma_{2k}\,\gamma_{2k+1} = \gamma'_{2k-1}\,\gamma'_{2k} \end{cases}$$

$+$ sum

$*$ product

$$\nu_n = \sum_{\substack{|\omega|=2n \\ \text{Dyck paths}}} v_\gamma(\omega)$$

$$\gamma = \{\gamma_k\}_{k\geq 1}$$

$$\nu_n = \nu_1 \sum_{\substack{|\omega|=2n-2 \\ \text{Dyck paths}}} v_{qd(\gamma)}(\omega)$$

"compression"
of weighted Dyck paths

ex :
$$C_n = \sum_{|\omega| = 2n-2} v_{qd(1,1,\ldots\ldots)}(\omega)$$

(Catalan numbers)

$$\gamma = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$
$$\downarrow$$
$$\gamma' = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, \frac{3}{4}, \frac{5}{4}, \frac{4}{5}, \frac{6}{5}, \frac{5}{6}, \ldots$$

- $\gamma_k = 1$, then $\gamma'_{2k} = \dfrac{k}{k+1}$, $\gamma'_{2k-1} = \dfrac{k+1}{k}$

Combinatorial proof for the quotient-difference algorithm

The qd-algorithm

$$\gamma^{(0)} = \gamma = \{\gamma_k\}_{k\geq 1},$$

$$S(t;\gamma) = \sum_{n\geq 0} \gamma_n t^n$$

define $\gamma^{(m)} = \{\gamma_k^{(m)}\}_{k\geq 1}$

$$= qd^{(m)}(\gamma)$$

$$S(t;\gamma) =$$

$$1 + \gamma_1^{(0)} t\, S(t;\gamma^{(1)})$$

$$1 + \gamma_1^{(0)} t + \gamma_1^{(0)} \gamma_1^{(1)} t^2 S(t;\gamma^{(2)})$$

$$\cdots\cdots\cdots\cdots$$

$$1 + \gamma_1^{(0)} t + \gamma_1^{(0)} \gamma_1^{(1)} t^2 + \cdots + \underbrace{\gamma_1^{(0)} \gamma_1^{(1)} \cdots \gamma_1^{(n-1)}}_{\nu_n} t^n S(t;\gamma^{(n)})$$

$$\gamma_1^{(n)} = \frac{\nu_{n+1}}{\nu_n} \qquad (n \geqslant 0)$$

$$\begin{cases} \gamma_{2k+1}^{(n)} + \gamma_{2k+2}^{(n)} = \gamma_{2k}^{(n+1)} + \gamma_{2k+1}^{(n+1)} & + \text{ sum} \\ \gamma_{2k}^{(n)}\,\gamma_{2k+1}^{(n)} = \gamma_{2k-1}^{(n+1)}\,\gamma_{2k}^{(n+1)} & * \text{ product} \end{cases}$$

Rhombus rules

change of notations : (and of colour!)

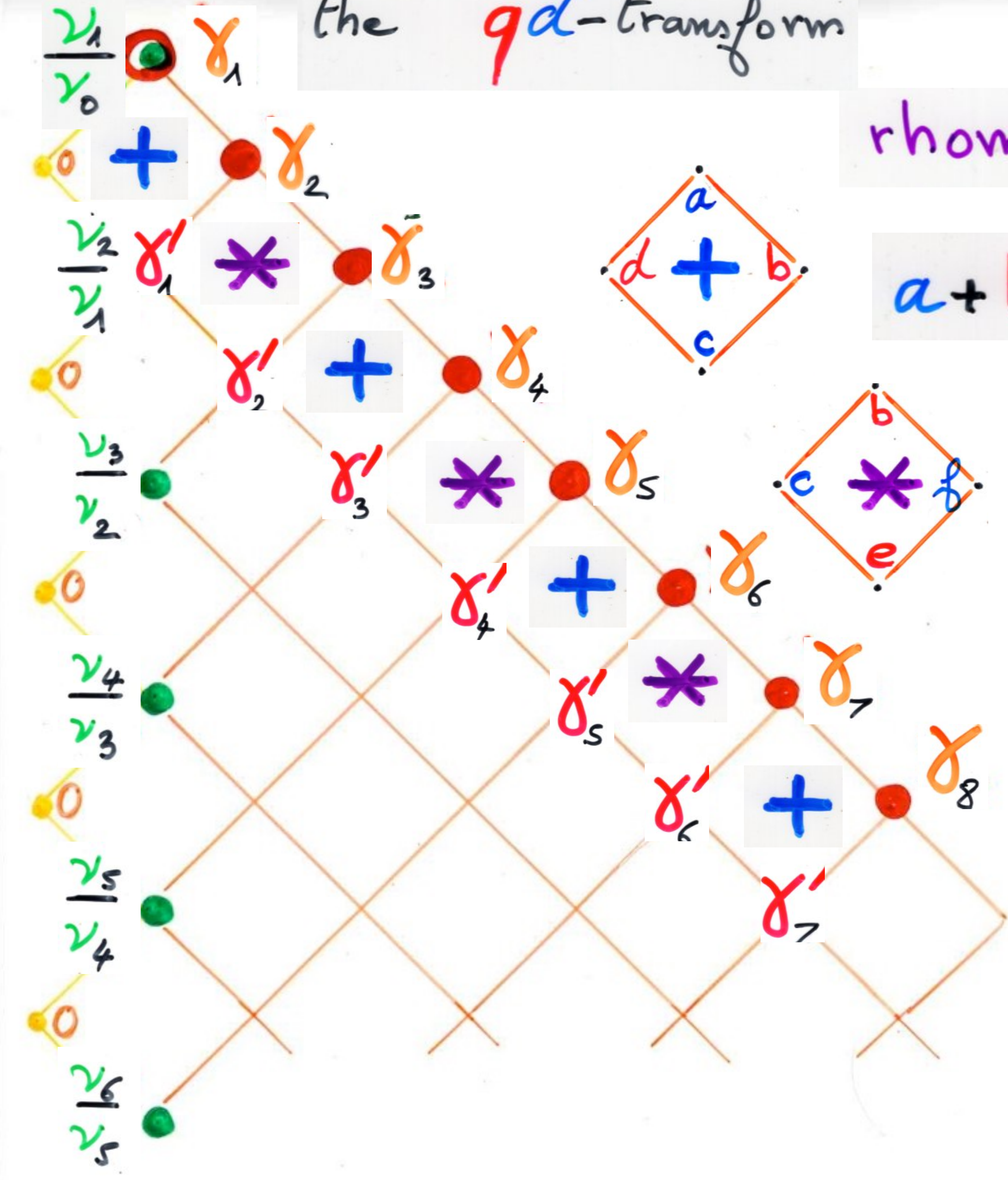
$$\gamma_{2k}^{(n)} = e_k^{(n)} \quad , \quad \gamma_{2k+1}^{(n)} = q_{k+1}^{(n)}$$

$$\begin{cases} q_{k+1}^{(n)} + e_{k+1}^{(n)} = q_{k+1}^{(n+1)} + e_k^{(n+1)} \\ e_k^{(n)} q_{k+1}^{(n)} = e_k^{(n+1)} q_k^{(n+1)} \end{cases}$$

with initial conditions:

$$e_0^{(n)} = 0 , \quad q_1^{(n)} = \frac{\nu_{n+1}}{\nu_n} , \quad \text{for every } n \geq 0$$

the qd-transform

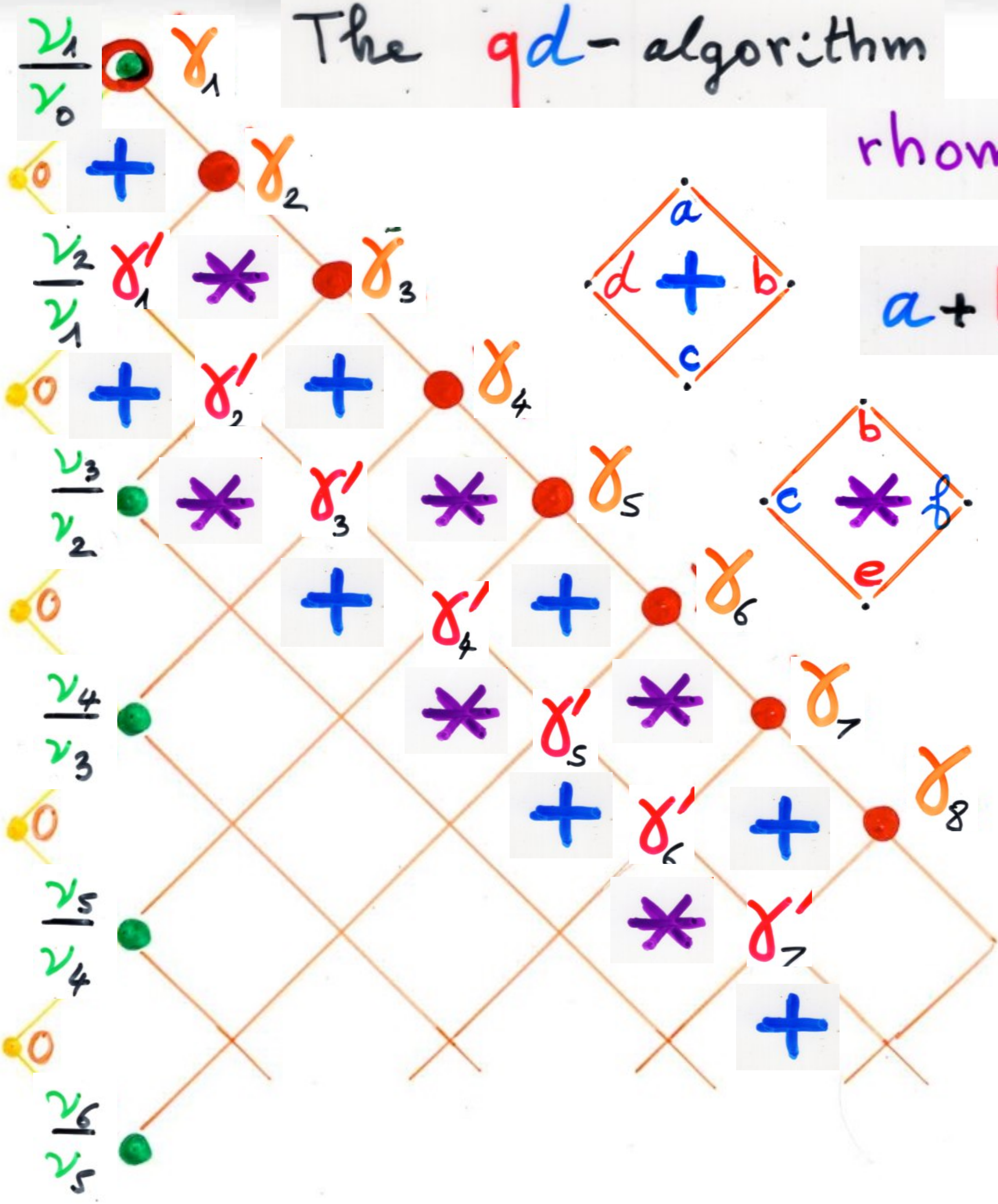


rhombus rules

$$a + b = c + d$$

$$b f = e c$$

The qd-algorithm



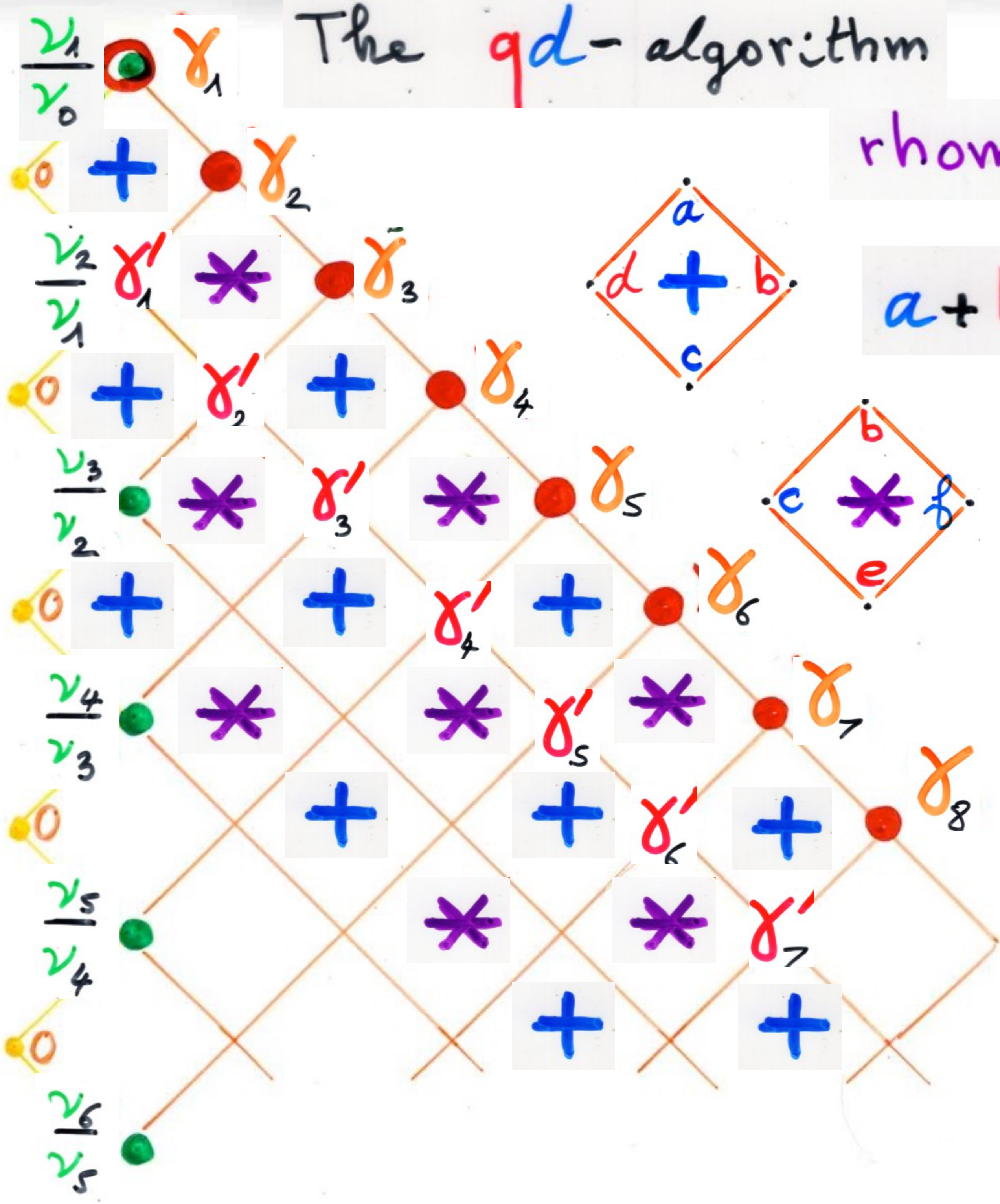
rhombus rules

$$a + b = c + d$$

$$b f = e c$$

The qd-algorithm

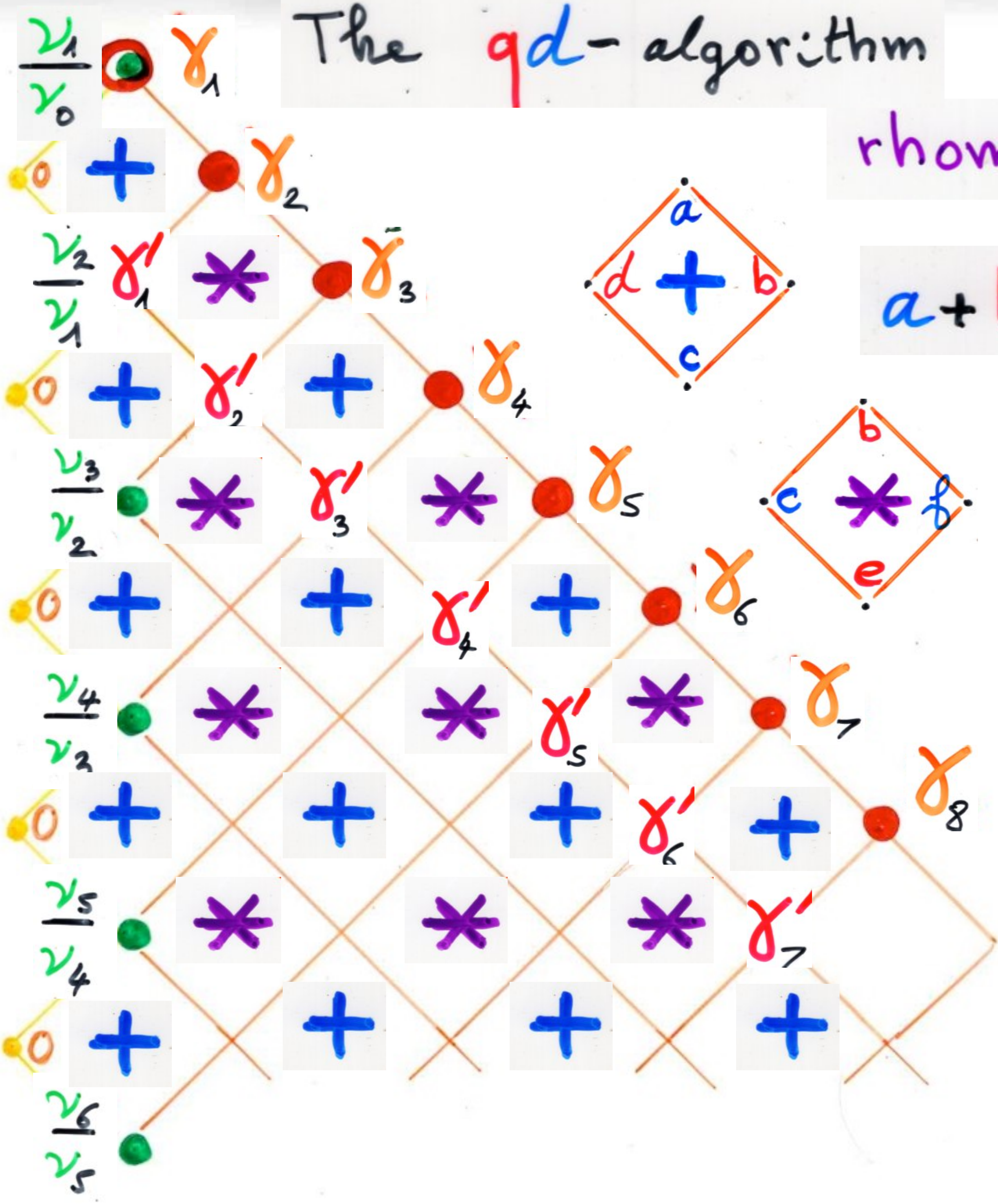
rhombus rules



$$a+b=c+d$$

$$bf=ec$$

The qd-algorithm

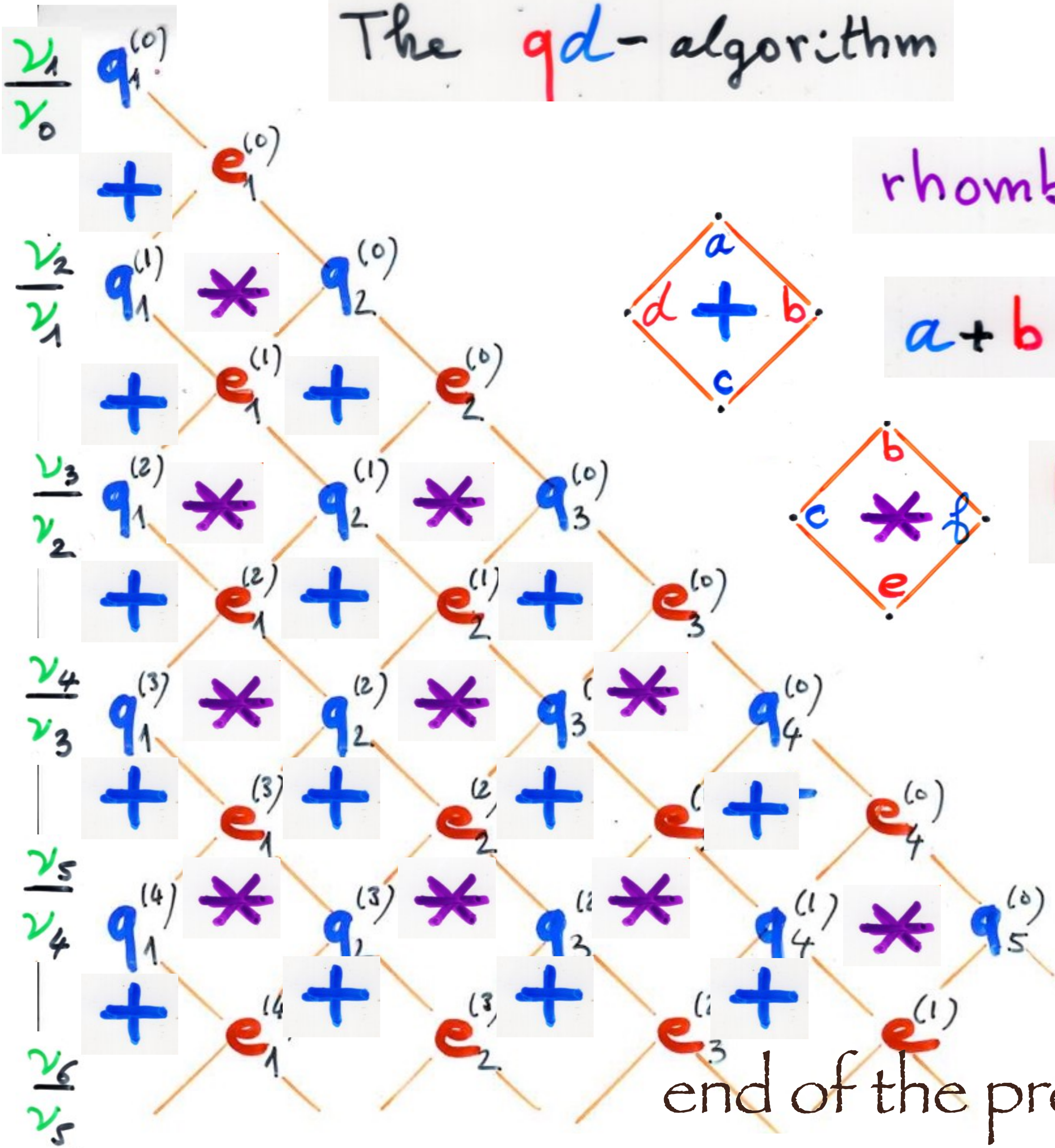


rhombus rules

$$a+b=c+d$$

$$bf=ec$$

The qd-algorithm



rhombus rules

$$a + b = c + d$$

$$bf = ec$$

end of the proof $\square$

Expression with Hankel determinants

$$\nu_{n+i} = \nu_n \sum_{|\omega|=2i} v_{\gamma(n)}(\omega)$$

Dyck paths

"compression"
of weighted Dyck paths

$$|\omega| = 2n+2p \longrightarrow |\omega'| = 2p$$

$$\nu^{(n)} = \{\nu_i^{(n)}\}_{i\geq 0} \qquad \nu_i^{(n)} = \nu_{n+i} \qquad i,n \geq 0$$

$$\overline{\nu}^{(n)} = \{\overline{\nu}_i^{(n)}\}_{i\geq 0} \qquad \overline{\nu}_i^{(n)} = \frac{\nu_{n+i}}{\nu_n}$$

$$\nu = \{\nu_n\}_{n\geq 0}$$

$$H_k^{(n)}(\nu) \qquad H_k^{(n)} = H\begin{pmatrix} n, & n+1, & \dots, & n+k-1 \\ n, & n, & \dots, & n \end{pmatrix}$$

$$H_k^{(n)}(\nu) = H_k^{(0)}(\nu^{(n)})$$

$$H_k^{(n+1)}(\nu) = H_k^{(1)}(\nu^{(n)})$$

$$H_k^{(n)}(\nu) = (\nu_n)^k H_k^{(0)}(\overline{\nu}^{(n)})$$

$$H_k^{(n+1)}(\nu) = (\nu_n)^k H_k^{(1)}(\overline{\nu}^{(n)})$$

$$\nu_{n+i} = \nu_n \sum_{|\omega|=2i} v_{\gamma^{(n)}}(\omega)$$

Dyck paths

$$\bar{\nu}_i^{(n)} = \frac{\nu_{n+i}}{\nu_n}$$

$$S(t;\gamma^{(n)}) = \sum_{i\geq 0} \bar{\nu}_i^{(n)} t^i$$

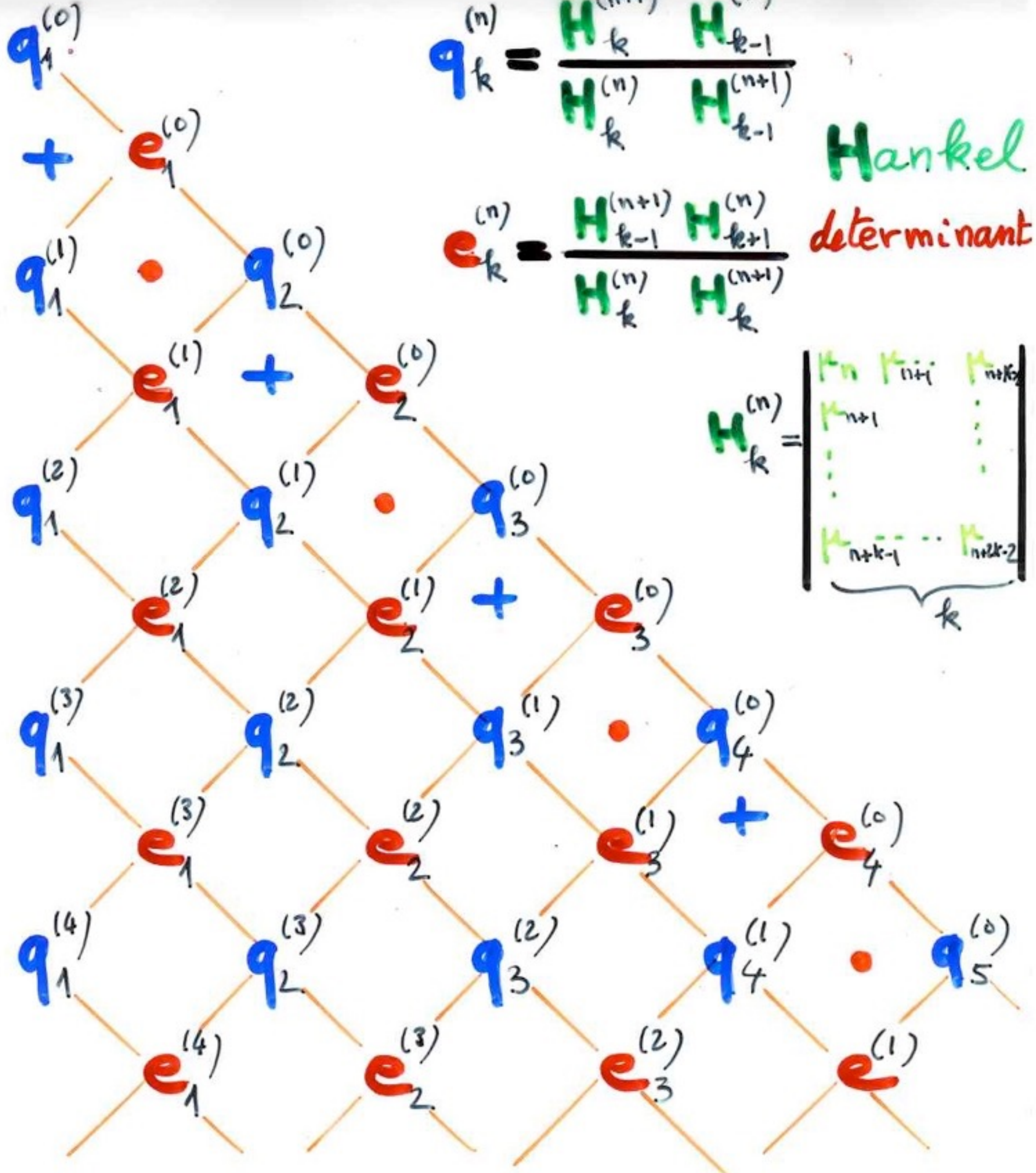
a kind of "compression"
of non-crossing Dyck paths

$$\frac{H_k^{(n)}(\nu)}{H_{k-1}^{(n)}(\nu)} = \nu_n \gamma_1^{(n)} \gamma_2^{(n)} \cdots \gamma_{2k-2}^{(n)}$$

$$\frac{H_k^{(n+1)}(\nu)}{H_{k-1}^{(n+1)}(\nu)} = \nu_n \gamma_1^{(n)} \gamma_2^{(n)} \cdots \gamma_{2k-1}^{(n)}$$

$$q_k^{(n)} = \frac{H_k^{(n+1)}\, H_{k-1}^{(n)}}{H_{k-1}^{(n+1)}\, H_k^{(n)}}$$

$$e_k^{(n)} = \frac{H_{(k+1)}^{(n)}\, H_{k-1}^{(n+1)}}{H_k^{(n)}\, H_k^{(n+1)}}$$



$$q_k^{(n)} = \frac{H_k^{(n+1)} \; H_{k-1}^{(n)}}{H_k^{(n)} \; H_{k-1}^{(n+1)}}$$

$$e_k^{(n)} = \frac{H_{k-1}^{(n+1)} \; H_{k+1}^{(n)}}{H_k^{(n)} \; H_k^{(n+1)}}$$

Hankel determinant

$$H_k^{(n)} = \underbrace{\begin{vmatrix} \mu_n & \mu_{n+1} & \cdots & \mu_{n+k-1} \\ \mu_{n+1} & & & \vdots \\ \vdots & & & \vdots \\ \mu_{n+k-1} & \cdots & & \mu_{n+2k-2} \end{vmatrix}}_{k}$$

Corollary

Starting from the sequence $\{\nu_n\}_{n\geq 0}$ the **qd**-algorithm can be performed iff $H_k^{(n)}(\nu) \neq 0$ for every $n, k \geq 0$

Corollary

The **qd**-transform of sequence $\gamma = \{\gamma_k\}_{k\geq 1}$ exists

iff $H_k^{(1)}(\nu) \neq 0$ and $H_k^{(2)}(\nu) \neq 0$ for every $k \geq 0$

Complexity of the algorithms

Direct algorithm with Motzkin paths

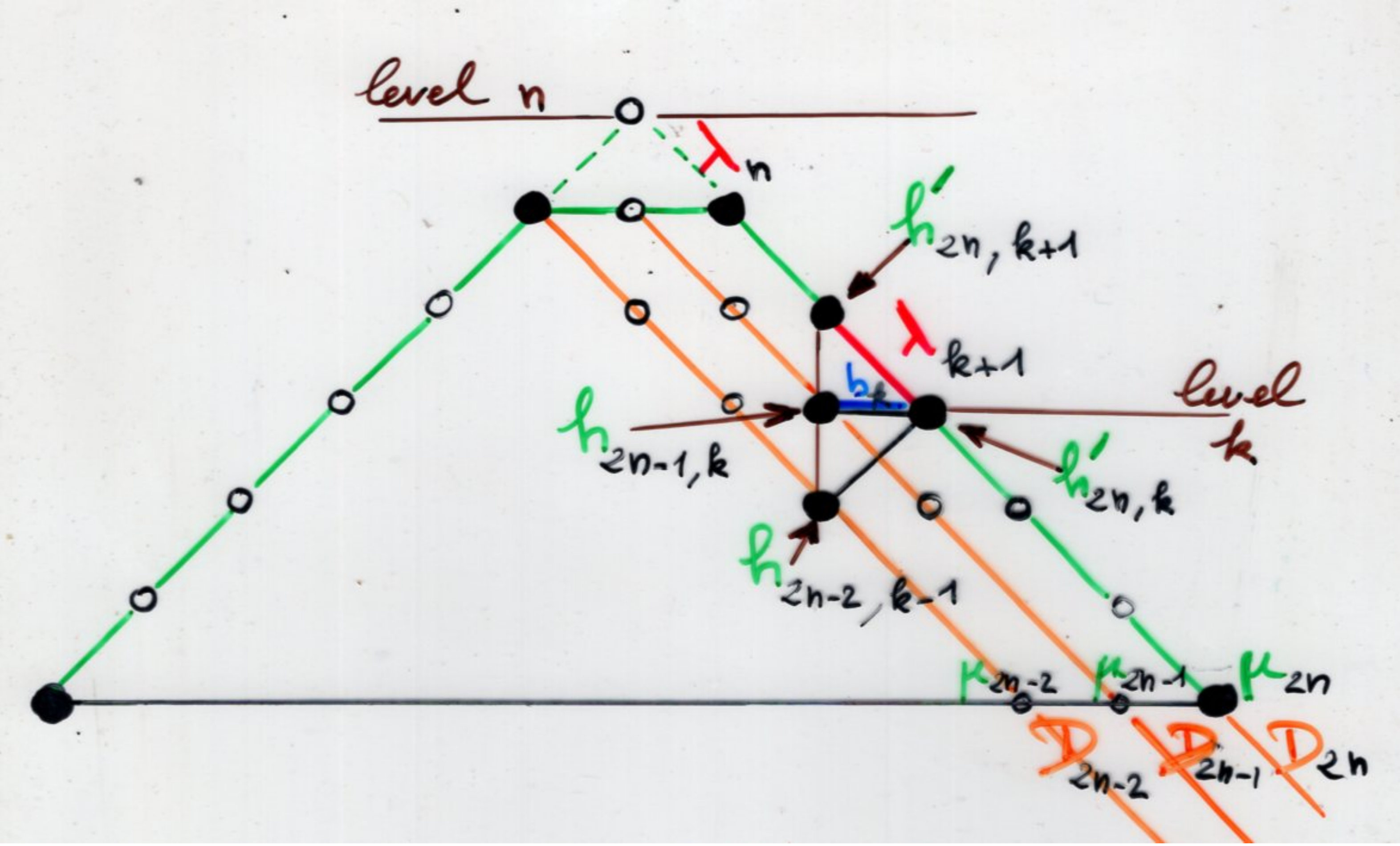
number of divisions:

- direct paths algorithm: linear n
- Ramanujan formula: linear n
- qd-algorithm : quadratic n^2

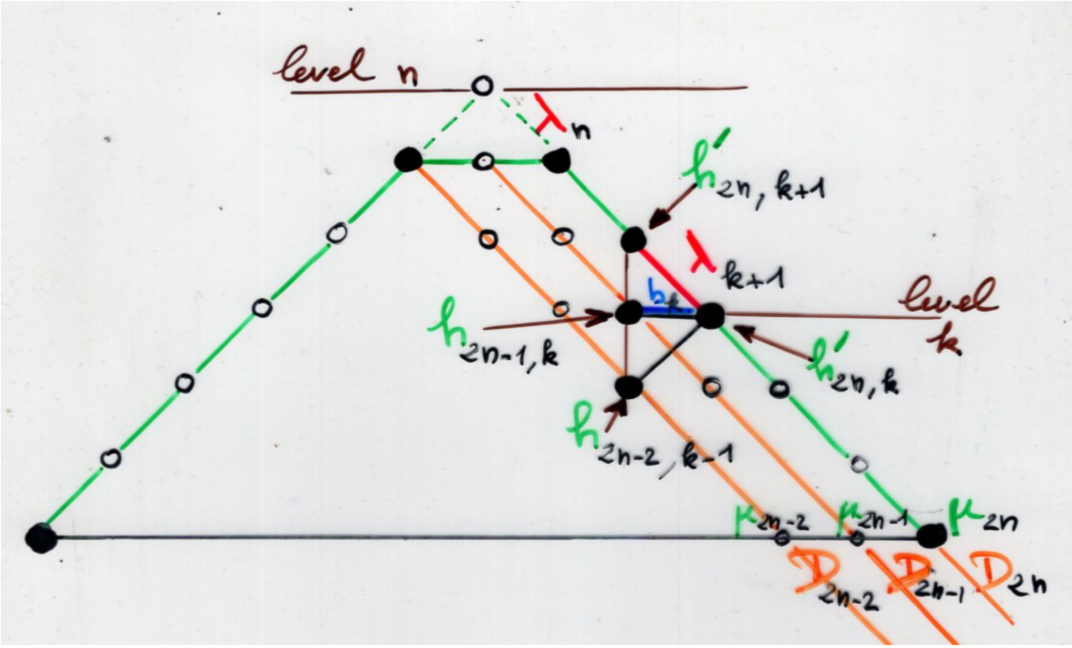
$$P_n = \sum_{0 \leq k < \frac{n}{2}} (-1)^k \varphi_k(n) A_{n-k}$$

$$\varphi_r(n+1) - \varphi_r(n) = a_{n-1} \varphi_{r-1}(n-1)$$

$$P_n = a_1 a_2 \cdots a_{n-1} (a_1 + a_2 + \cdots + a_n), \quad n \geq 1$$

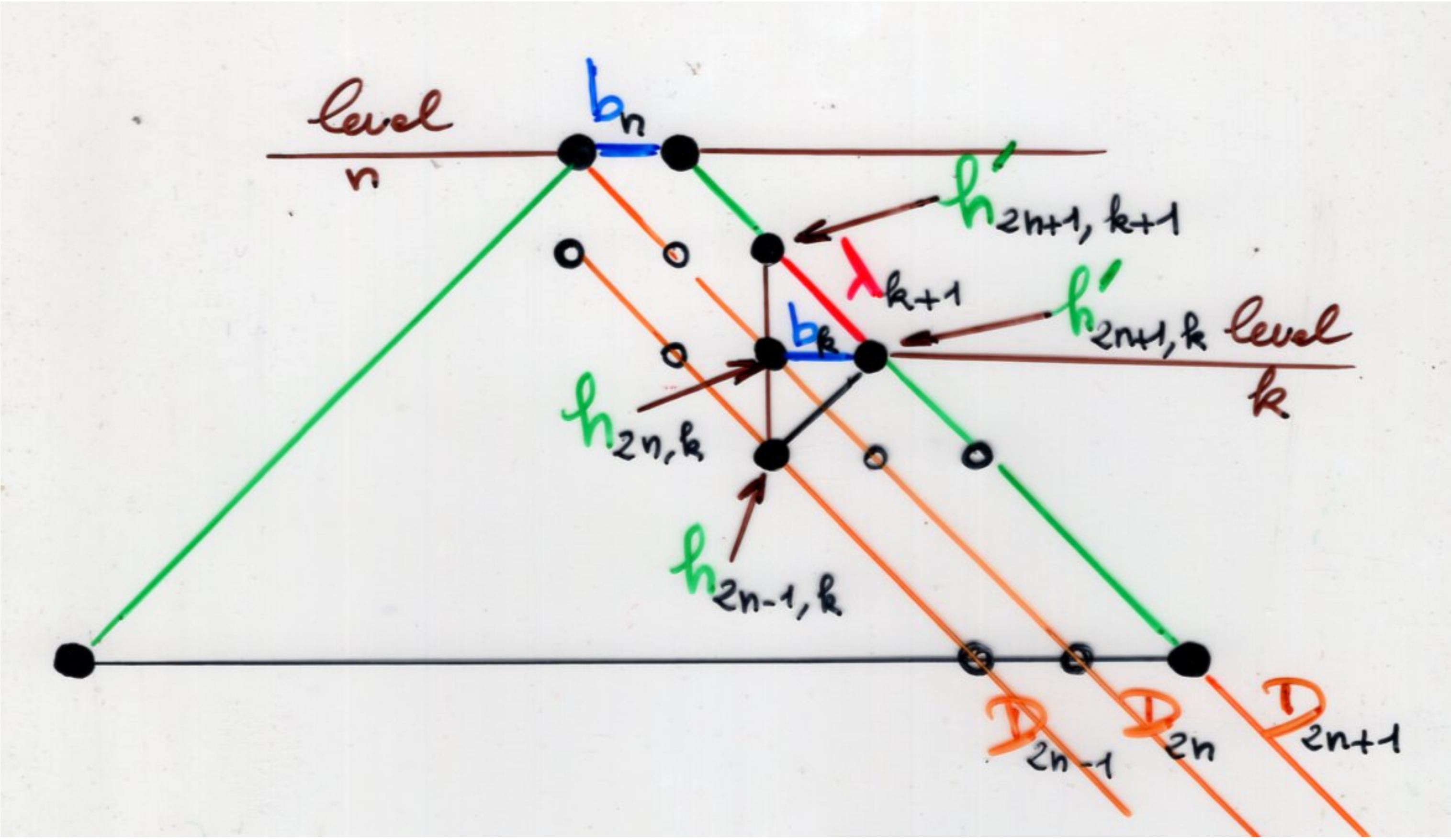


level n
λ_n
$h'_{2n,k+1}$
λ_{k+1}
b_k
level k
$h_{2n-1,k}$
$h'_{2n,k}$
$h_{2n-2,k-1}$
μ_{2n-2}
μ_{2n-1}
μ_{2n}
D_{2n-2}
D_{2n-1}
D_{2n}

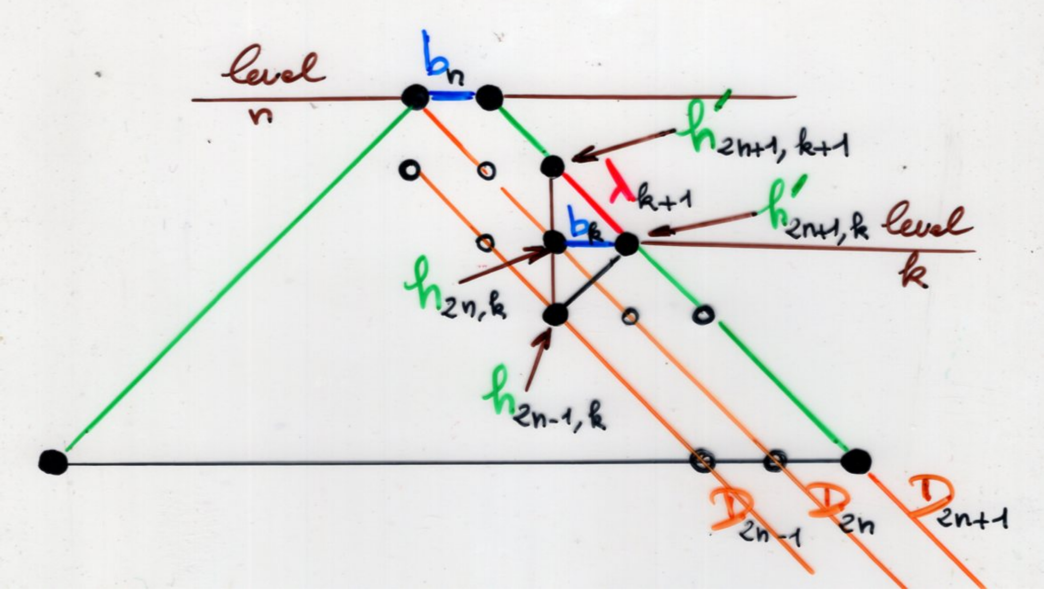


$$\begin{cases} h'_{2n,n-1} = b_{n-1}\, h_{2n-1,n-1} + h_{2n-2,n-2} \\ h'_{2n,k} = \lambda_{k+1}\, h'_{2n,k+1} + b_k\, h_{2n-1,k} + h_{2n-2,k-1} \quad (\text{for } k=n-2,\dots,1) \\ h'_{2n,0} = \lambda_1\, h'_{2n,1} + b_0\, \mu_{2n-1} \\ \mu_{2n} = h'_{2n,0} + (\lambda_1 \cdots \lambda_{n-1})\, \lambda_n \end{cases}$$

$$h_{2n,k} = h'_{2n,k} + (\lambda_{k+1} \cdots \lambda_n)$$



level
n
b_n
$h'_{2n+1,k+1}$
λ_{k+1}
$h'_{2n+1,k}$
level
k
b_k
$h_{2n,k}$
$h_{2n-1,k}$
D_{2n-1}
D_{2n}
D_{2n+1}



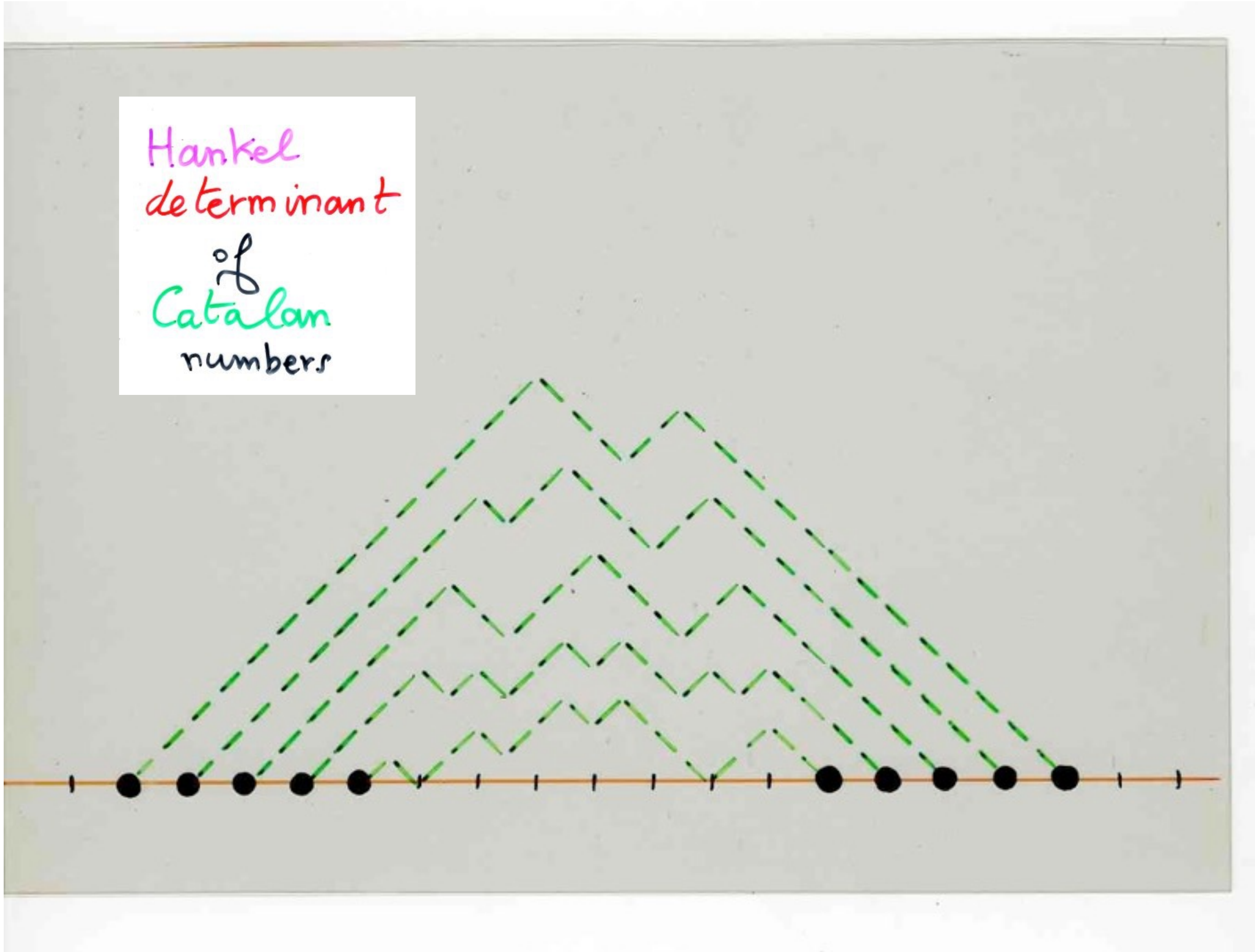
$$\begin{cases} h'_{2n+1,n} = h_{2n-1,n-1} \\ \text{------------} \\ h'_{2n+1,k} = \lambda_{k+1} h'_{2n+1,k+1} + b_k h_{2n,k} + h_{2n-1,k-1} \\ \text{------} \quad (\text{for } k = n-1, \dots, 1) \\ h'_{2n+1,0} = \lambda_1 h'_{2n+1,1} + b_0 \mu_{2n} \\ \mu_{2n+1} = h'_{2n+1,0} + (\lambda_1 \cdots \lambda_n) b_n \end{cases}$$

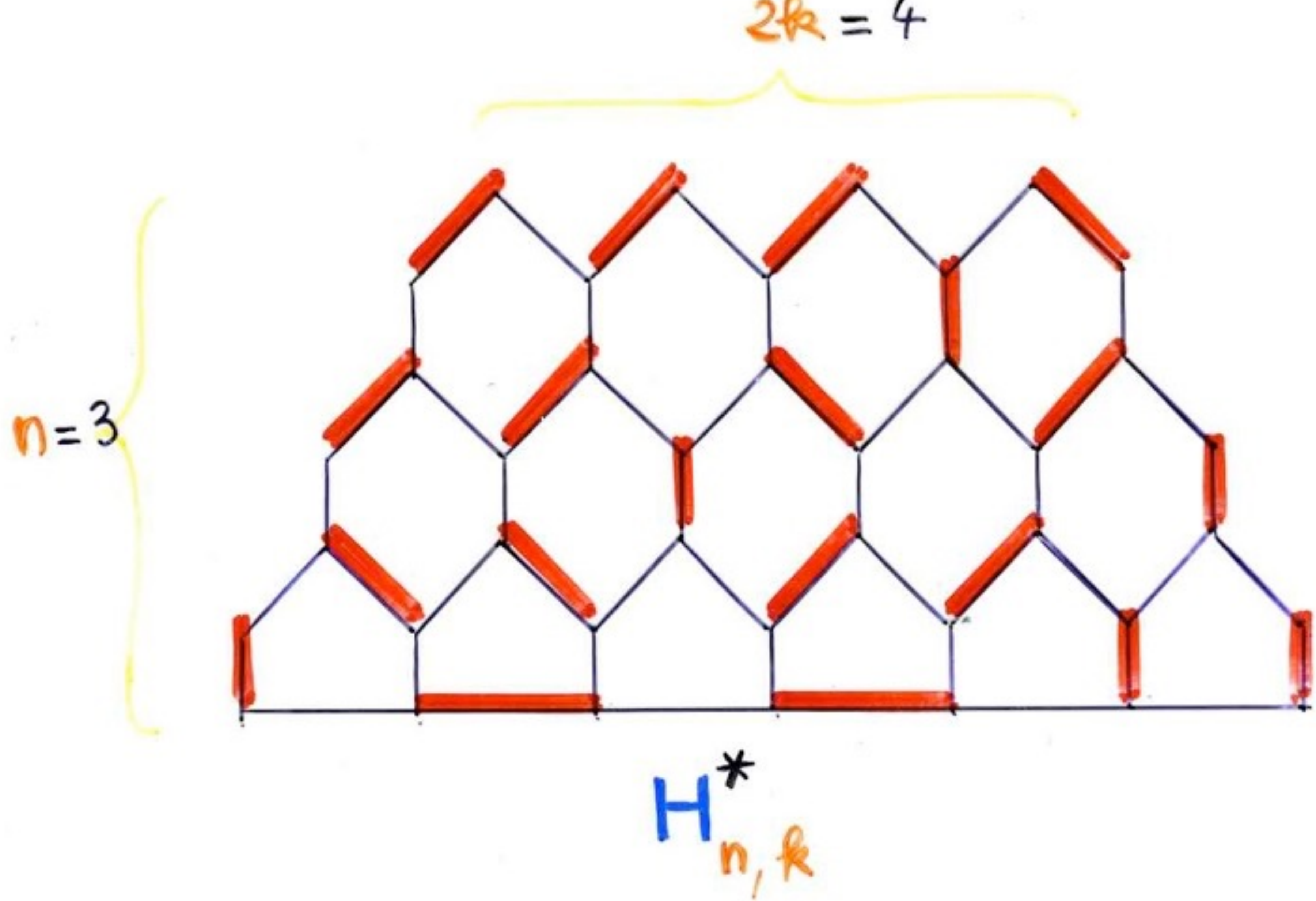
$$h_{2n+1,k} = h'_{2n+1,k} + (\lambda_{k+1} \cdots \lambda_n) b_n \quad (1 \le k \le n-1)$$
$$h_{2n+1,n} = h'_{2n+1,n} + b_n$$

Combinatorial applications

$$\begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+k-1} \\ C_{n+1} & \cdots & & \vdots \\ \vdots & & & \vdots \\ C_{n+k-1} & \cdots & \cdots & C_{n+2k-2} \end{vmatrix} = \prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$$

Hankel
determinant
of
Catalan
numbers

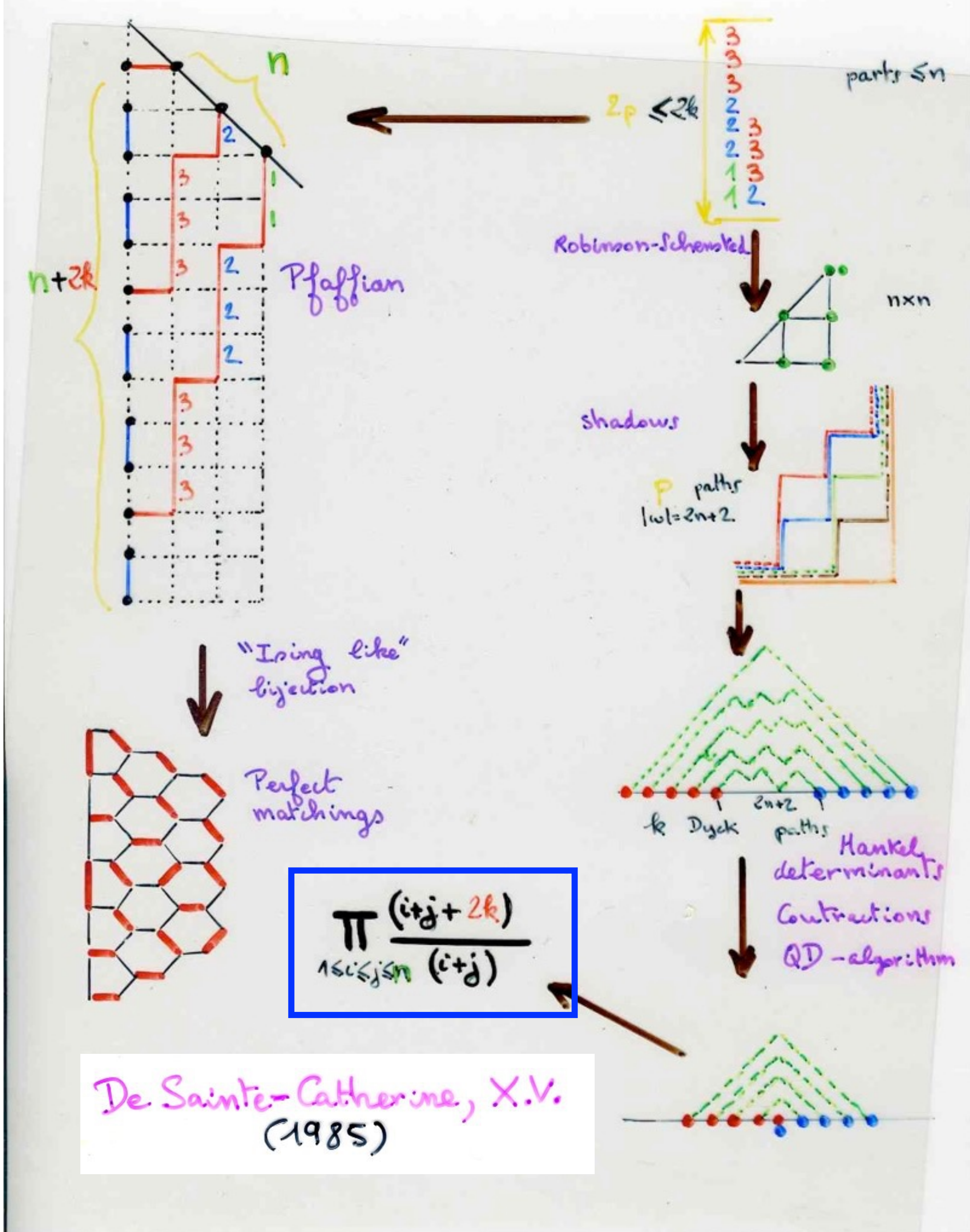

Hankel
determinant
of
Catalan
numbers



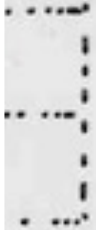
2k = 4
n = 3
$H^*_{n,k}$

number of perfect matchings of $H^*_{n,k}$

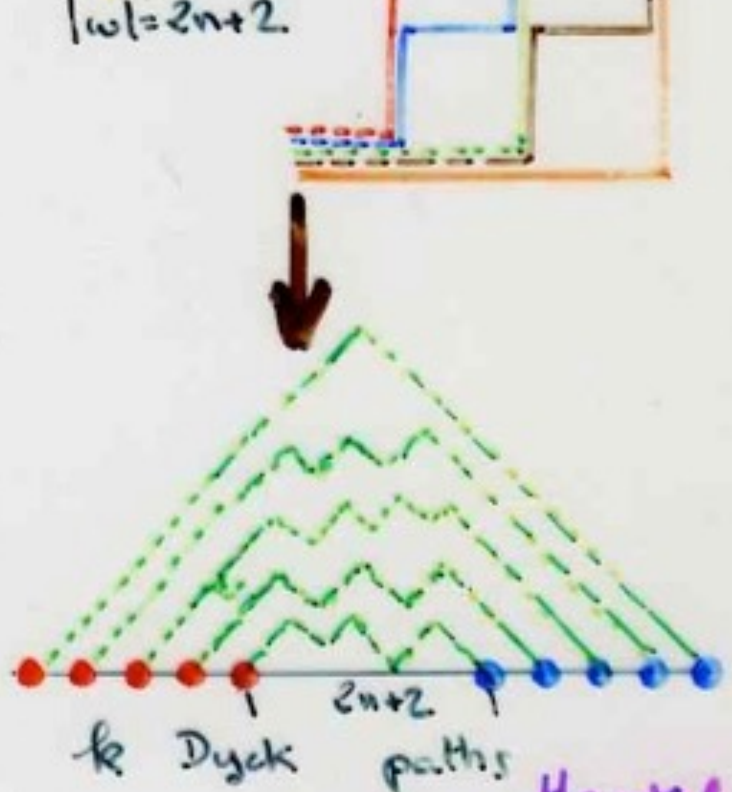
$$= \prod_{1 \leq i \leq j \leq n} \frac{(i+j+2k)}{(i+j)}$$


parts ≤ n
2p ≤ 2k
n
n+2k
Pfaffian
Robinson-Schensted
n×n
shadows
p paths
|ω|=2n+2
k Dyck paths
2n+2
Hankel determinants
Contractions
QD-algorithm
"Ising like" bijection
Perfect matchings
$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$
De Sainte-Catherine, X.V.
(1985)

2022



"Ising like" bijection



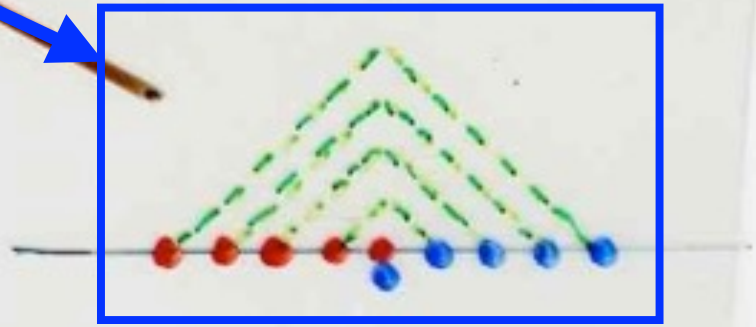
Perfect matchings

Hankel determinants

Contractions

QD-algorithm

$$\boxed{\prod_{1\leq i\leq j\leq n} \frac{(i+j+2k)}{(i+j)}}$$



2021

Proposition

qd-table for Catalan

$$q_k^{(n)} = \frac{(2n+2k-1)(2n+2k)}{(n+2k-1)(n+2k)}$$

$$e_k^{(n)} = \frac{2k(2k+1)}{(n+2k)(n+2k+1)}$$

Proof. We just have to check that these numbers satisfy the rhombus rules (18). We have successively

$$e_k^{(n)} q_{k+1}^{(n)} = \frac{2k(2k+1)(2n+2k+1)(2n+2k+2)}{(n+2k)(n+2k+1)(n+2k+1)(n+2k+2)} = e_k^{(n+1)} q_k^{(n+1)}.$$

$$\begin{aligned} q_{k+1}^{(n)} + e_{k+1}^{(n)} &= \frac{(2n+2k+1)(2n+2k+2)}{(n+2k+1)(n+2k+2)} + \frac{(2k+2)(2k+3)}{(n+2k+2)(n+2k+3)}, \\ &= \frac{4n^3 + n^2(16k+18) + n(24k^2+52k+26) + 2(2k+1)(2k+2)(2k+3)}{(n+2k+1)(n+2k+2)(n+2k+3)}, \\ &= \frac{(2n+2k+3)(2n+2k+4)}{(n+2k+2)(n+2k+3)} + \frac{2k(2k+1)}{(n+2k+1)(n+2k+2)}, \\ &= q_{k+1}^{(n+1)} + e_k^{(n+1)}. \square \end{aligned}$$

Corollary 11. *The number of non-crossing configurations of k Dyck paths $\eta = (w_1, w_2, \cdots, w_k)$ such that for i, $1 \le i \le k$, w_i goes from the point $(-2i+2, 0)$ to the point $(2n+2i-2, 0)$ is*

$$d_{n,k} = \prod_{1 \le i < j < n} \frac{i+j+2k}{i+j}.$$

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Proof. From the above considerations, this number is the $k \times k$ Hankel determinant (for $\mu_n = C_n$)

$$H_k^{(n)} = (C_n)^k (q_1^{(n)} e_1^{(n)})^{k-1} (q_2^{(n)} e_2^{(n)})^{k-2} \cdots (q_{k-1}^{(n)} e_{k-1}^{(n)}), \tag{27}$$

with $q_k^{(n)}$ and $e_k^{(n)}$ defined by (26). We have successively

$$C_n q_1^{(n)} e_1^{(n)} q_2^{(n)} e_2^{(n)} \cdots q_{k-1}^{(n)} e_{k-1}^{(n)} = \frac{(2k-1)!(2n+2k-2)!}{(n+2k-1)!(n+2k-2)},$$

$$d_{n,k}/d_{n,k-1} = \frac{(2k+n)(2k+n+1)\cdots(2k+2n-2)}{2k(2k+1)\cdots(2k+n-2)},$$

$$d_{n,k}/d_{n,k-1} = H_k^{(n)}/H_{k-1}^{(n)}, \ (n \geq 1, k \geq 2). \tag{28}$$

With $d_{n,1} = C_n = H_1^{(n)}$ $(n \geq 1)$. We deduce $H_k^{(n)} = d_{n,k}$. □

The formula of corollary 11 reappeared in Physics in the context of directed polymers with watermelons topoly in the presence of a wall. Extensions are given in Guttmann, Krattenthaler, Viennot [13]. This work follows Guttmann

$$H_k^{(n)} = (C_n)^k \left(q_1^{(n)} e_1^{(n)}\right)^{k-1} \left(q_2^{(n)} e_2^{(n)}\right)^{k-2} \cdots \left(q_{k-1}^{(n)} e_{k-1}^{(n)}\right)$$

$$= \prod_{1 \leq i \leq j < n} \frac{i+j+2k}{i+j}$$

The idea of compression of paths and configurations of non-crossing Dyck paths

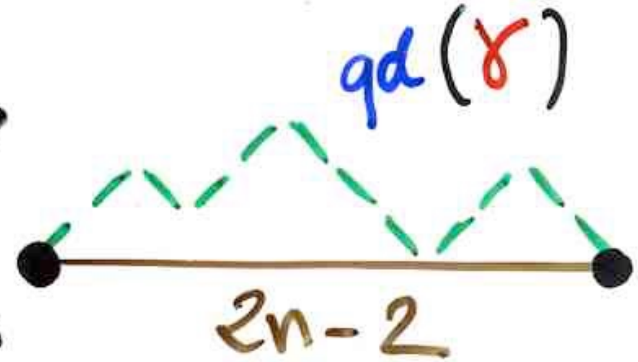
Catalan

$$C_n =$$

$$\frac{1}{n+1}\binom{2n}{n}$$

Catalan

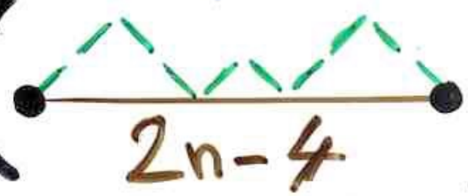
$$C_n = \gamma_1^{(0)} \sum_{\omega} v\Big(\quad \Big)$$



$$\frac{1}{n+1}\binom{2n}{n}$$

Catalan

$qd^{(2)}(\gamma)$

$C_n = \gamma_1^{(0)} \, \gamma_1^{(1)} \sum_{\omega} v\Big($  $\Big)$

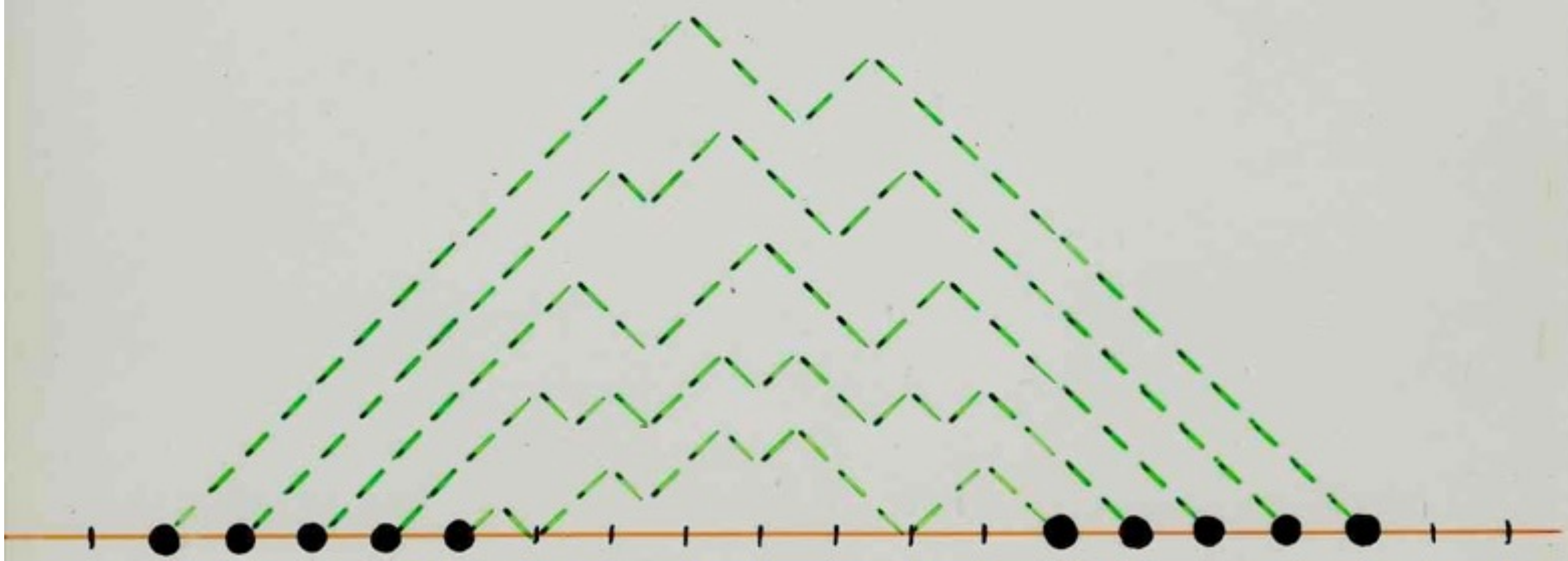
$$\frac{1}{n+1}\binom{2n}{n}$$

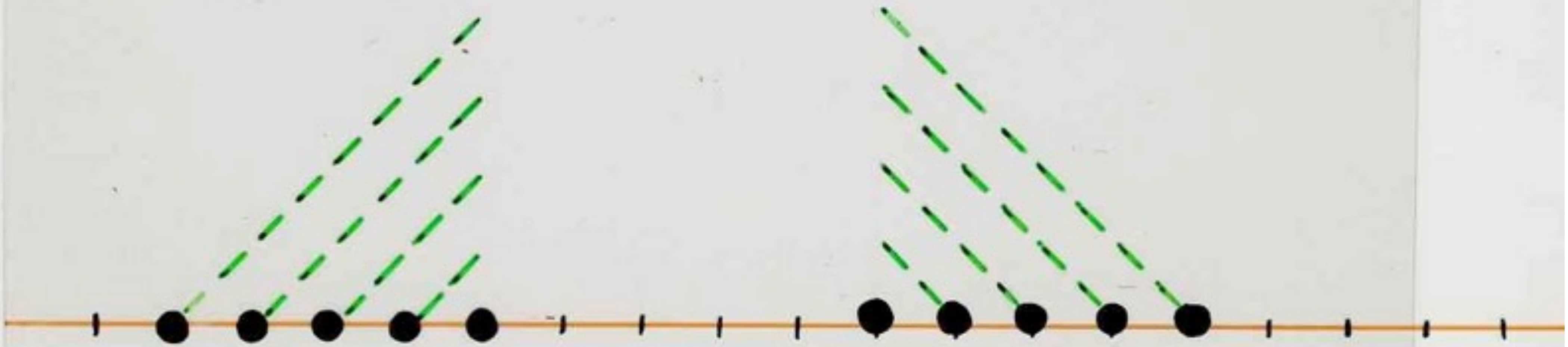
Catalan

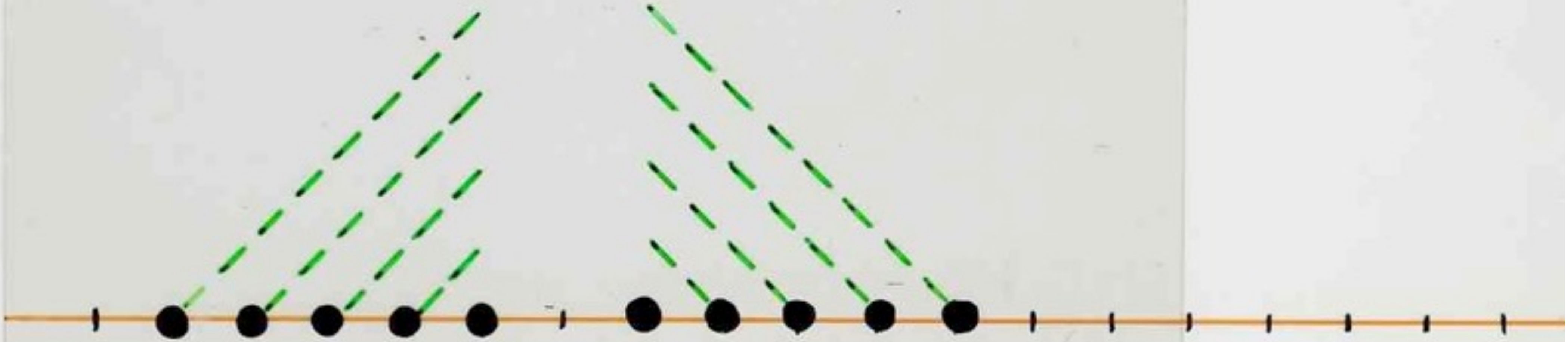
$$\mathrm{qd}^{(n-1)}(\gamma)$$

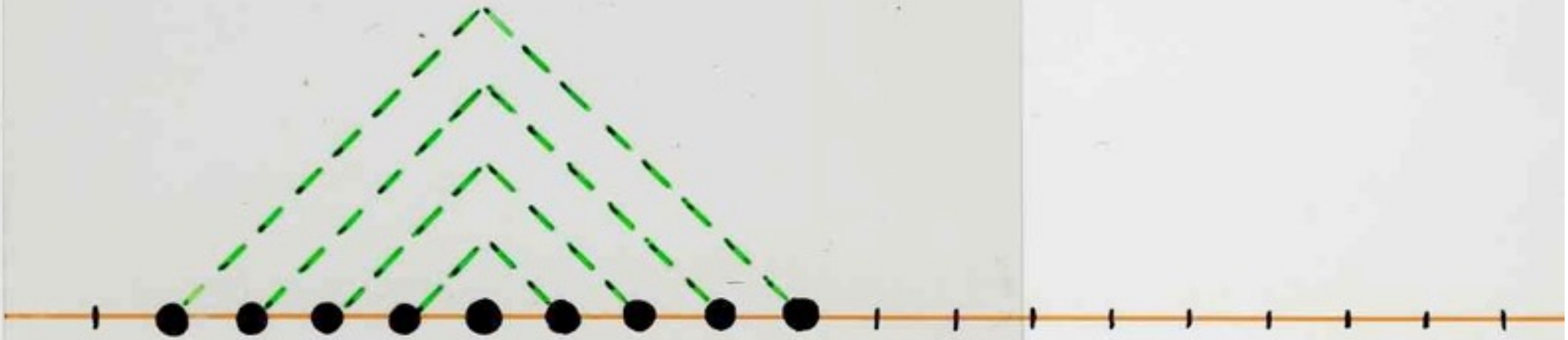
$$C_n = \gamma_1^{(0)}\,\gamma_1^{(1)} \cdots \gamma_1^{(n-1)} \vee (\bullet)$$

$$\frac{1}{n+1}\binom{2n}{n}$$



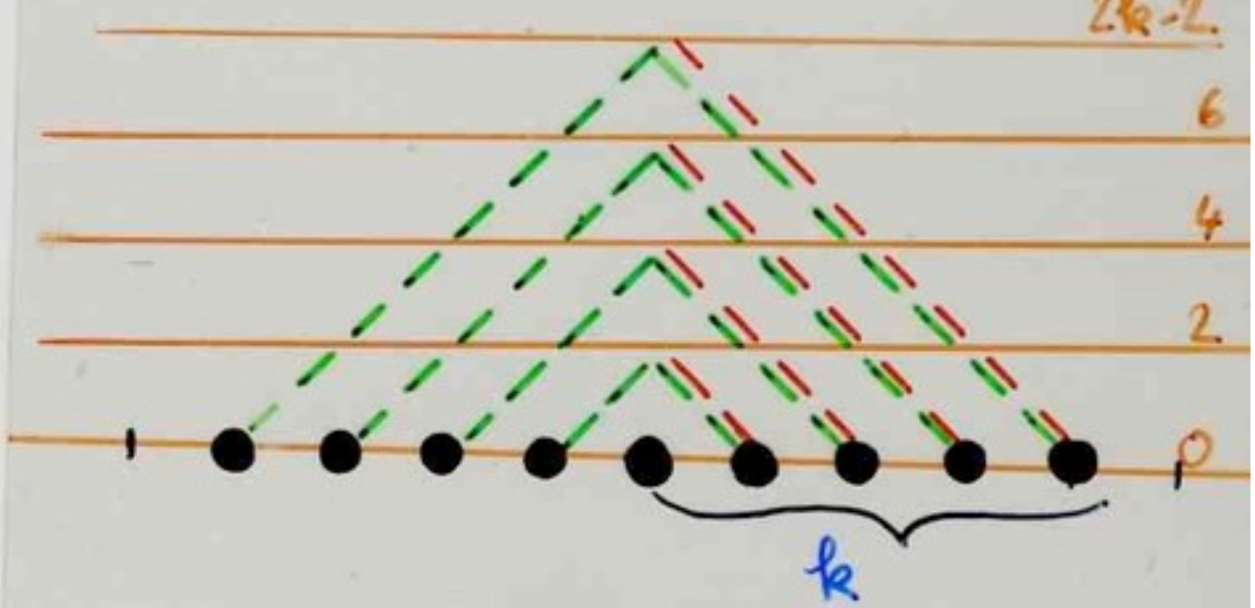






$$\Delta_{n,k} = \left(\gamma_1^{(n)}\gamma_2^{(n)}\right)^{k-1}\left(\gamma_3^{(n)}\gamma_4^{(n)}\right)^{k-2} \cdots\cdots \left(\gamma_{2k-1}^{(n)}\ \gamma_{2k-2}^{(n)}\right)$$

k paths



Research problem

number of perfect matchings of $H^*_{n,k}$

$$= \prod_{1 \leq i \leq j < n} \frac{i+j+2k}{i+j}$$

$$H_k^{(n)} = (C_n)^k \left(q_1^{(n)} e_1^{(n)}\right)^{k-1} \left(q_2^{(n)} e_2^{(n)}\right)^{k-2} \cdots \left(q_{k-1}^{(n)} e_{k-1}^{(n)}\right)$$

$$\frac{(\text{product})}{(\text{product})} = \prod_{i,j} \frac{a_{i,j}}{b_{i,j}}$$

many formulae of the type

$$a_n = \frac{(\text{product})}{(\text{product})}$$

$$a_{(\text{something})} = \frac{(\text{product})}{(\text{product})}$$

$$(\text{determinant})$$

- hook-length formula
  Young tableaux
- ASM alternating sign matrices
- TSSCPP
- ....

bijective proof ?

$$a_{(\text{something})} = \frac{(\text{product})}{(\text{product})}$$

$$(\text{product})\, a_{(\text{something})} = (\text{product})$$

Research idea

look at

$$a_{(\text{something})}$$

as a product of rational numbers

$$= \frac{a_{(\,)}}{b_{(\,)}} \times \cdots \times \frac{a_{(\,)}}{b_{(\,)}}$$

coming from some kind of "compression" of configurations of non-crossing configurations of paths related to some determinants