Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 2
Moments and histories

Ch 2b

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Reminding Ch2a

bijection

Laguerre histories $\longrightarrow$ permutations

description with words

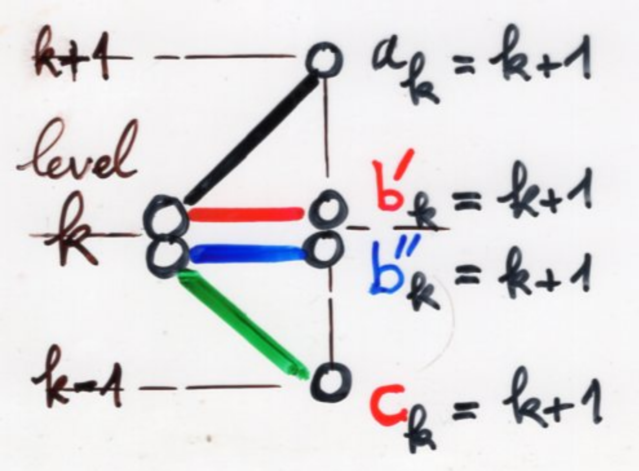
Laguerre history

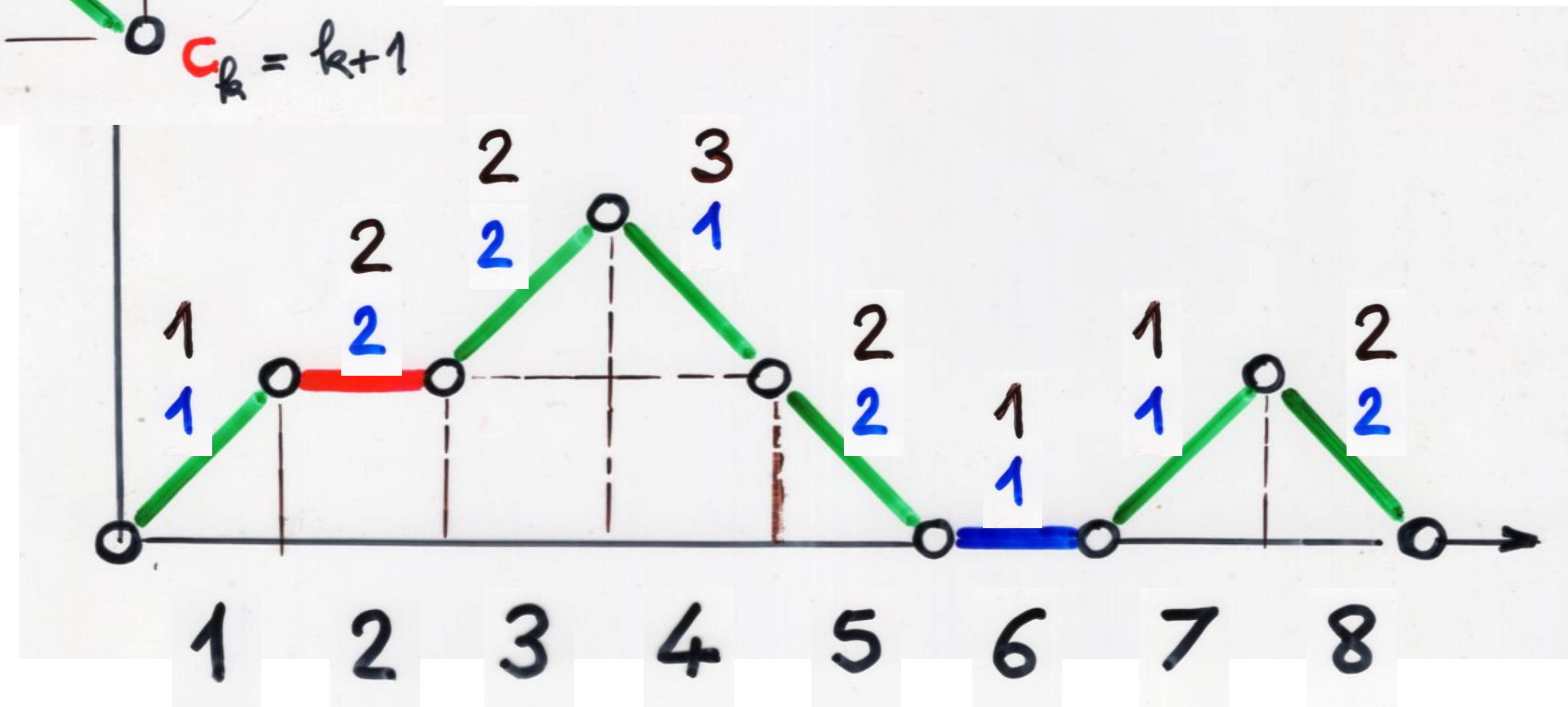
$$h = (\omega_c, p)$$

choice function

$$p = (p_1, \dots, p_n) \qquad 1 \leqslant p_i \leqslant v(\omega_i)$$

number of possibilities





bijection

$$h = (\omega_c ; \underbrace{(p_1, \ldots, p_n)}_{p})$$
$|\omega| = n$

$\longleftrightarrow$

permutations
$\sigma \in \mathfrak{S}_{n+1}$

Laguerre histories

$(n+1)!$

$|h| = |\omega|$
length of the history

J. Françon, X.V. (1979)

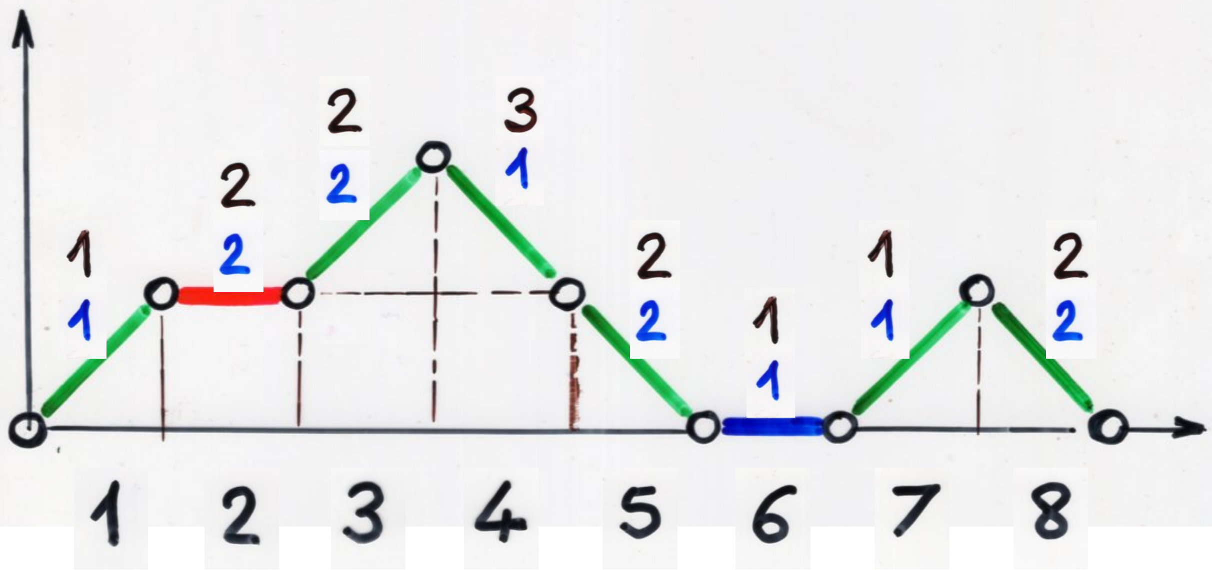
Laguerre history $h = (\omega_c, p)$

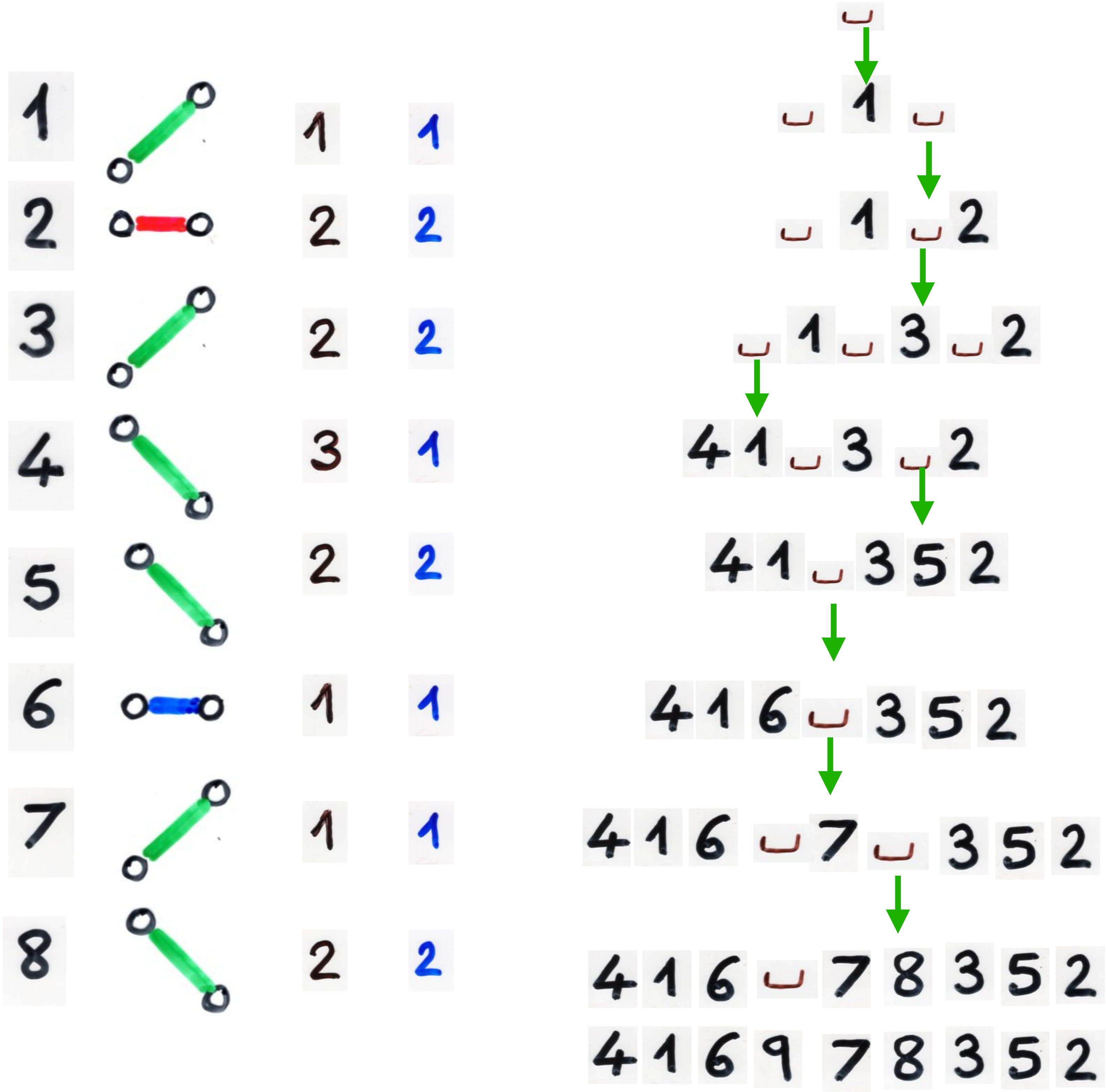
choice function $p = (p_1, \dots, p_n)$

$$1 \leq p_i \leq v(\omega_i)$$

$$\omega = (\omega_1 \cdots \omega_n)$$

1	1	1
2	2	2
3	2	2
4	3	1
5	2	2
6	1	1
7	1	1
8	2	2





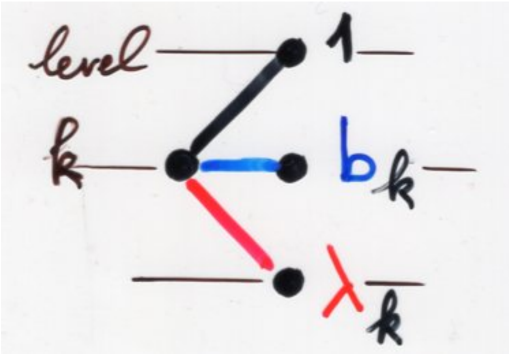
1 1 1
2 2 2
3 2 2
4 3 1
5 2 2
6 1 1
7 1 1
8 2 2
␣
␣ 1 ␣
␣ 1 ␣ 2
␣ 1 ␣ 3 ␣ 2
4 1 ␣ 3 ␣ 2
4 1 ␣ 3 5 2
4 1 6 ␣ 3 5 2
4 1 6 ␣ 7 ␣ 3 5 2
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

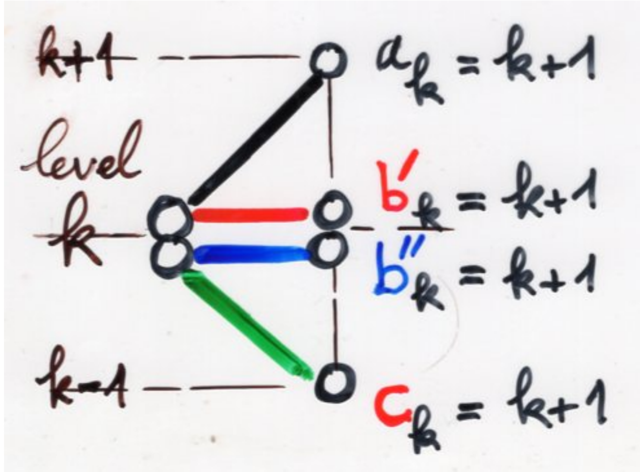
Laguerre polynomials

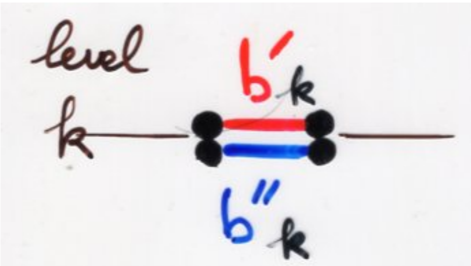
$L^{(1)}_n(x)$

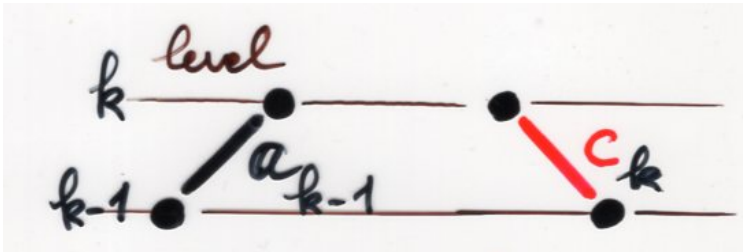
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

moments $\mu_n = (n+1)!$









$$b_k = b'_k + b''_k$$

$$a_{k-1}\, c_k = \lambda_k$$

weigthed Laguerre histories

Laguerre $L_n^{(\alpha)}$

(monic)

$\lambda_k = k(k+\alpha)$

$b_k = 2k+\alpha+1$

$$\sum_{\substack{|\omega|=n \\ \text{Motzkin} \\ \text{path}}} v(\omega) = \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin} \\ \text{path}}} v^*(\omega) = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

Laguerre $L_n^{(\alpha)}(x)$

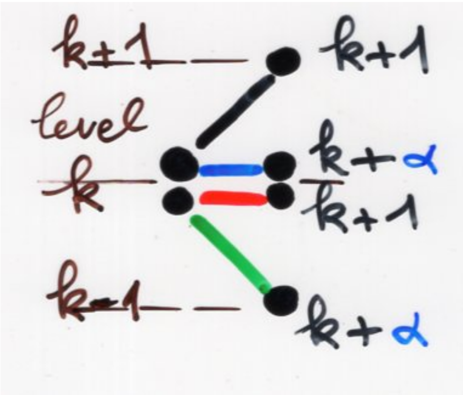
$b_k = 2k + \alpha + 1$

$b_k = b'_k + b''_k$

$\lambda_k = k(k+\alpha)$

$a_{k-1} c_k = \lambda_k$

$v^*(\omega)$



$a_k = k+1$

$b''_k = k + \alpha$ $(k \geq 0)$

$b'_k = k+1$

$c_k = k + \alpha$

$(k \geq 1)$

Laguerre polynomials

$L_n^{(\alpha)}(x)$

weight (α)

$v_\alpha(h)$

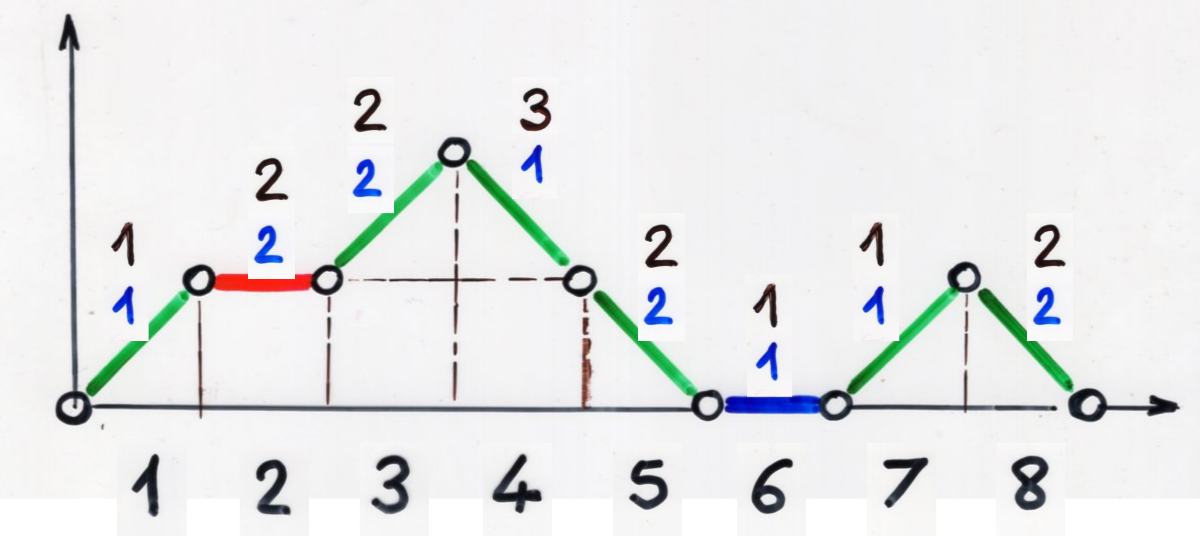
weighted Laguerre histories

$h = (\omega_c ; \underbrace{(p_1, \ldots, p_n)}_{P})$

$|\omega| = n$

$\omega_c = \omega_1 \cdots \omega_n$

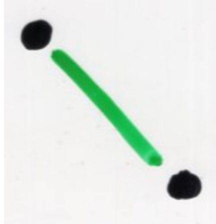
Laguerre histories



put a weight α for each choice

$p_i = 1$

with $\omega_i = \begin{cases} \text{blue East step} \\ \text{or South-East step} \end{cases}$



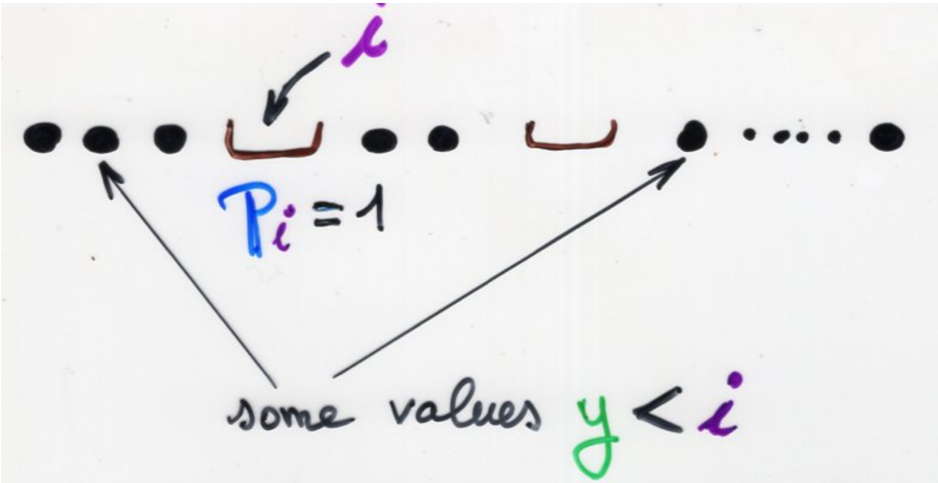
<u>Lemma</u>

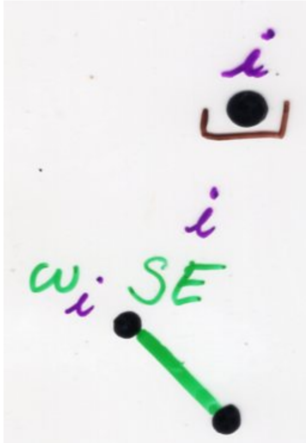
this equivalent to say that the element i is a lr-max element of the permutation σ (exept $i=n+1$)

left to right maximum element

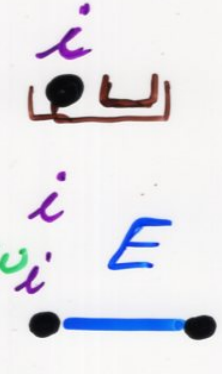
(lr-max)

insertion of i in first free position (= open) $\sqcup$





or



i will be a lr-max element of σ

if ω_i = NE or E

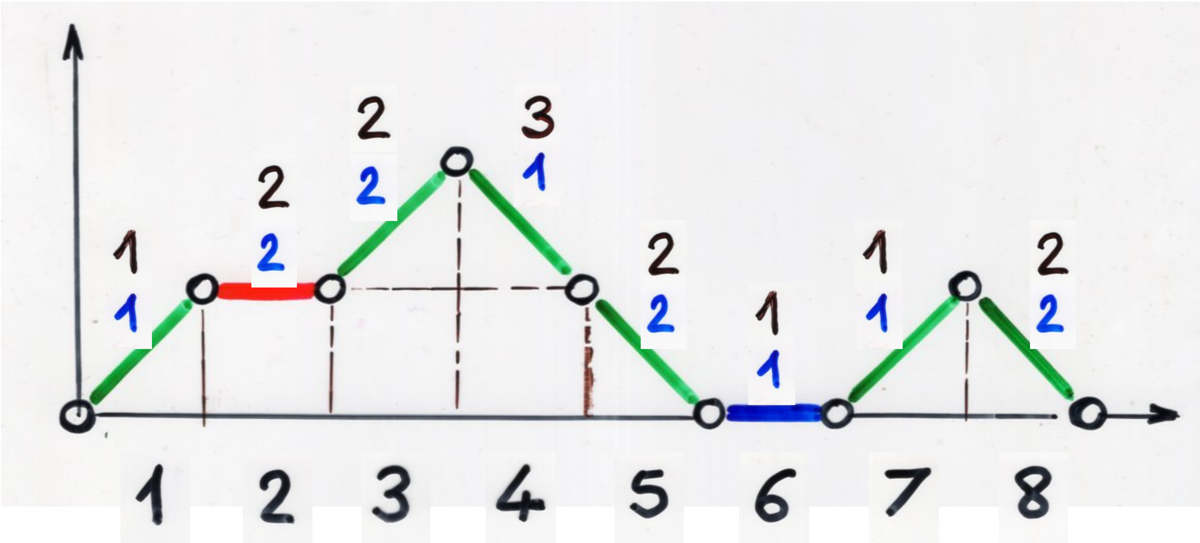
i will not be a lr-max element of σ (for any value of P_i

example

$\sigma = 4\ 1\ 6\ \boxed{9}\ 7\ 8\ 3\ 5\ 2$

lr-max

max



$i = 4$ $\omega_4 =$  ; $P_4 = 1$

$i = 6$ $\omega_6 =$  ; $P_6 = 1$

Corollary

(monic)

The moments of the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ are

$$\mu_n = (\alpha+1)(\alpha+2)\cdots(\alpha+n)$$

$$(\alpha)_n$$

$\alpha = 1 \quad \mu_n = (n+1)!$

$\alpha = 0 \quad \mu_n = n!$

Definition $h = (\omega_c ; P) \to \sigma \in \mathfrak{S}_{n+1}$

an element x is called *initial*

iff $P_x = 1$

Remark

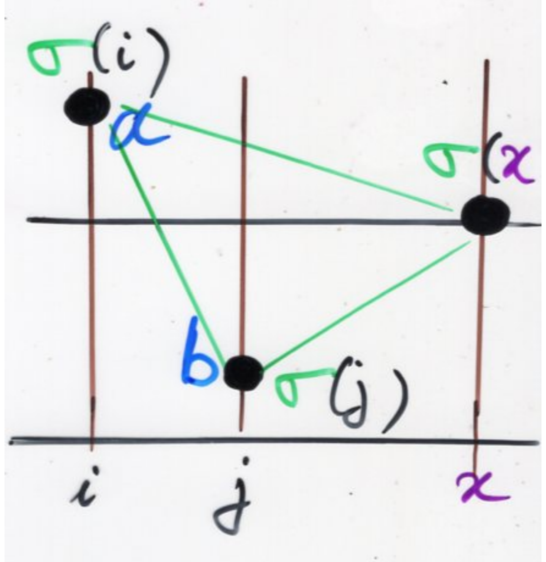
$x \in [1, n]$ is *initial* $\iff$

there no elements a, b in σ

such that $a > \sigma(x) > b$

with $a = \sigma(i)$, $b = \sigma(j)$, $i < j < x$

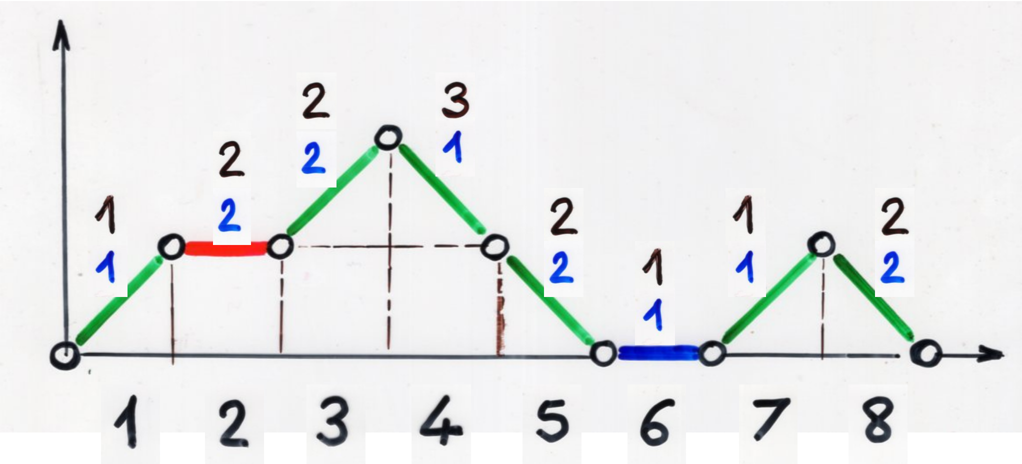
and $P_x = 1$



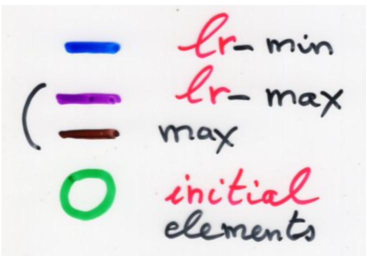
$\begin{cases} lr\text{-min} \\ lr\text{-max} \end{cases}$ elements of σ are *initial* elements

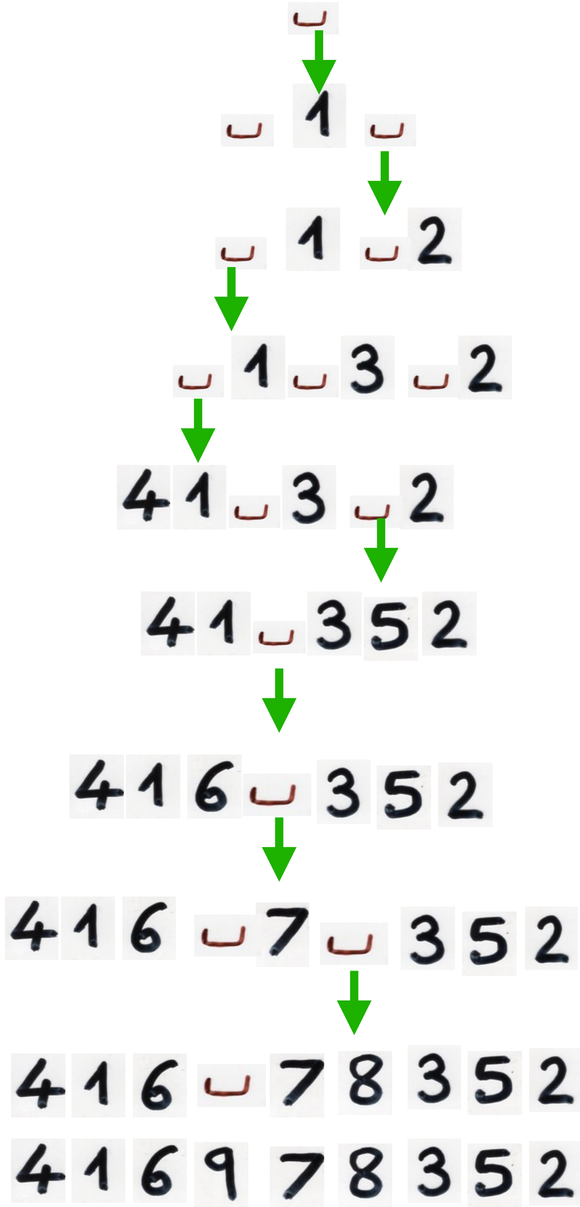
exercise $h = (\omega_c, P) \to \sigma \in \mathfrak{S}$

give a characterization of lr-min elements of σ.



$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$

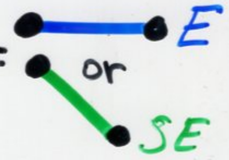




Restricted Laguerre histories

Definition restricted Laguerre history

$h = (\omega_c \,;\, \mathbf{p}) \qquad \mathbf{p} = (p_1, \ldots, p_n)$

such that $p_i > 1$ for step $\omega_i =$ 

In other words, during the insertion process $h \rightarrow \sigma$ the first open position $\sqcup$ is always kept at the beginning (of the sequence of values $1, 2, \ldots$ and $\sqcup$)

restricted Laguerre histories

$\sigma(1) = (n+1)$

$\mu_n = n!$

$\beta = \alpha + 1$

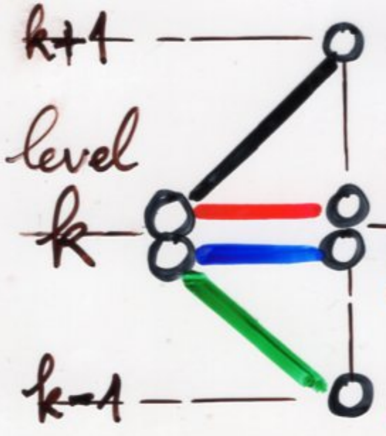
for a restricted Laguerre history, put a weight β for each choice $p_i = 1$ with $\omega_i =$  or 

this is equivalent to say that the element i is a lr-min element of the corresponding permutation σ.

Corollary moments of $L_n^{(\beta)}(x)$

$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$

Laguerre histories



$a_k = k+1$

$b'_k = k+1$ $(k \geq 0)$

$b''_k = k+1$

$c_k = k+1$ $(k \geq 1)$

$b_k = (2k+2)$

$\lambda_k = k(k+1)$

$$\begin{cases} \lambda_k = a_{k-1} c_k \\ b_k = b'_k + b''_k \end{cases}$$

$L_n^{(1)}(x)$

moments

$\mu_n = (n+1)!$

restricted Laguerre histories

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases}$$

$(k \geq 0)$

$(k \geq 1)$

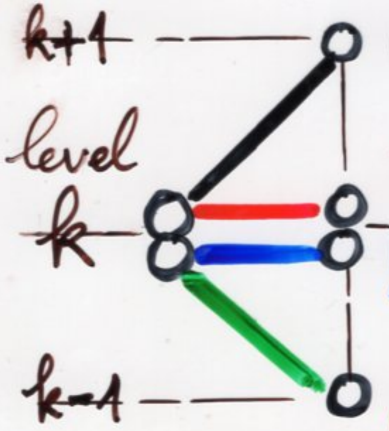
$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

$L_n^{(0)}(x)$

$\mu_n = n!$

Laguerre histories

restricted Laguerre histories



$$a_k = k+1$$
$$b'_k = k+1$$
$$b''_k = k+\alpha$$
$$c_k = k+\alpha$$

$$\beta = \alpha + 1$$

$$\begin{cases} a_k = k+\beta \\ b'_k = k+\beta \end{cases} \quad (k \geq 0)$$
$$\begin{cases} b''_k = k \\ c_k = k \end{cases} \quad (k \geq 1)$$

$$\begin{cases} \lambda_k = a_{k-1} c_k \\ b_k = b'_k + b''_k \end{cases}$$

$$\begin{cases} b_k = 2k+\alpha+1 \\ \lambda_k = k(k+\alpha) \end{cases}$$

$$\begin{cases} b_k = 2k+\beta \\ \lambda_k = (k-1+\beta)k \end{cases}$$

$$\mu_n = (\alpha+1)\cdots(\alpha+n)$$

$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$

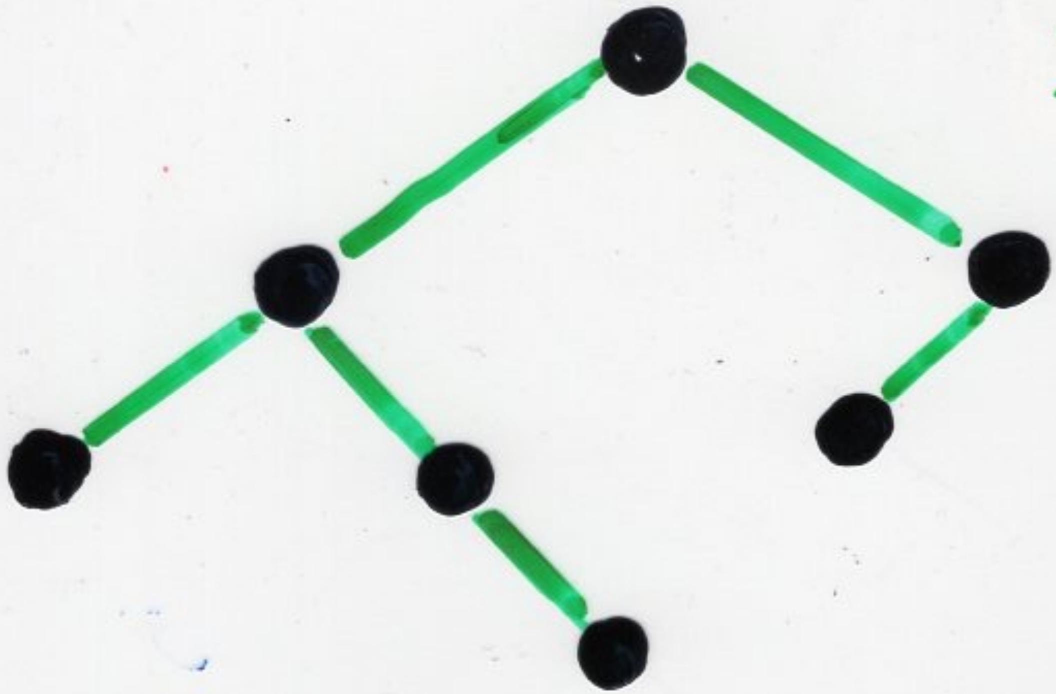
bijection

Laguerre histories $\longrightarrow$ permutations

description with binary trees

Binary trees

binary tree



$B = \emptyset$

or

$B = \langle L, r, R \rangle$

left subtree, root, right subtree

C_n = number of binary trees having n internal vertices

(or $n+1$ leaves = external vertices)

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

binary trees

n vertices

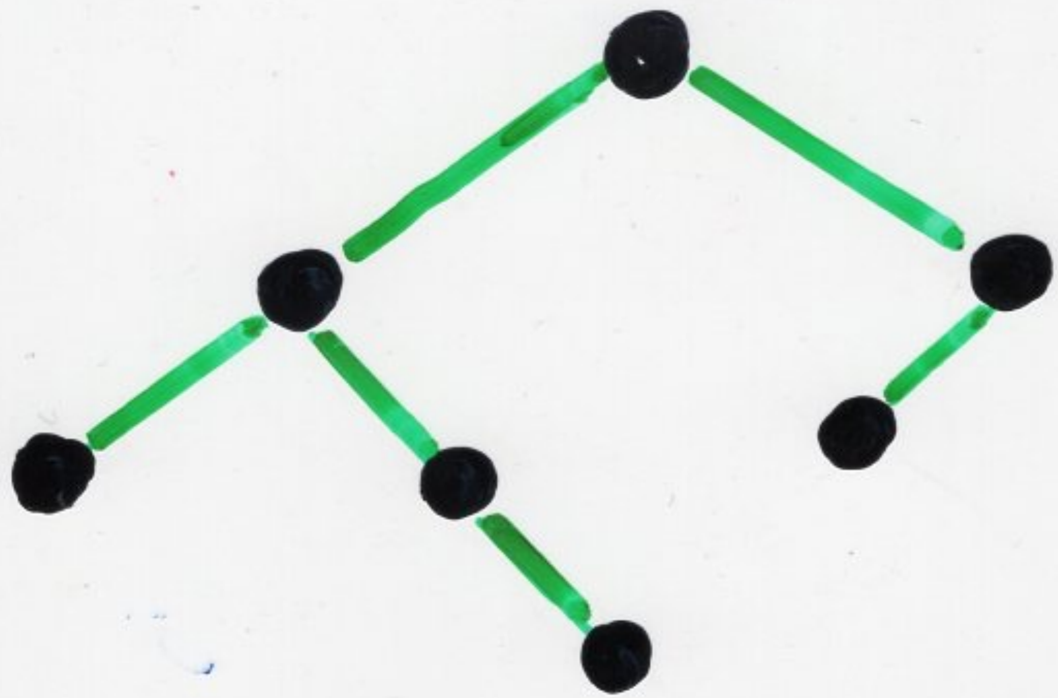
$\updownarrow$ bijection

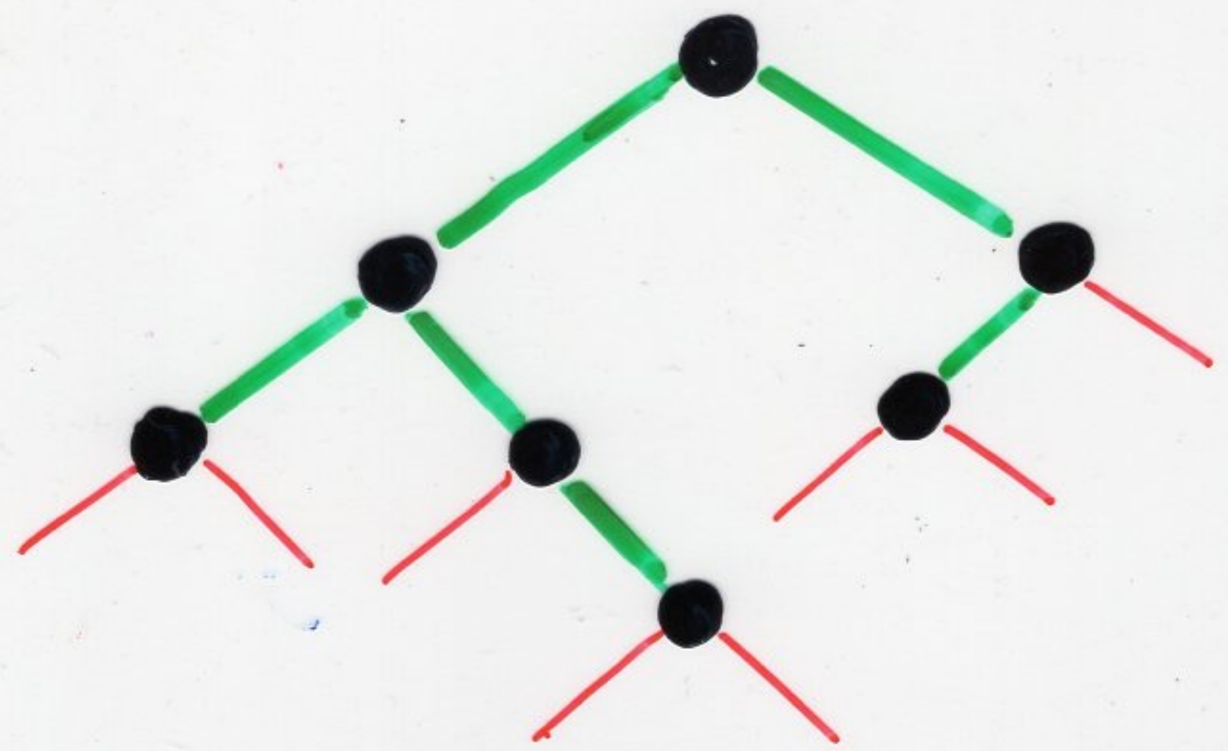
complete

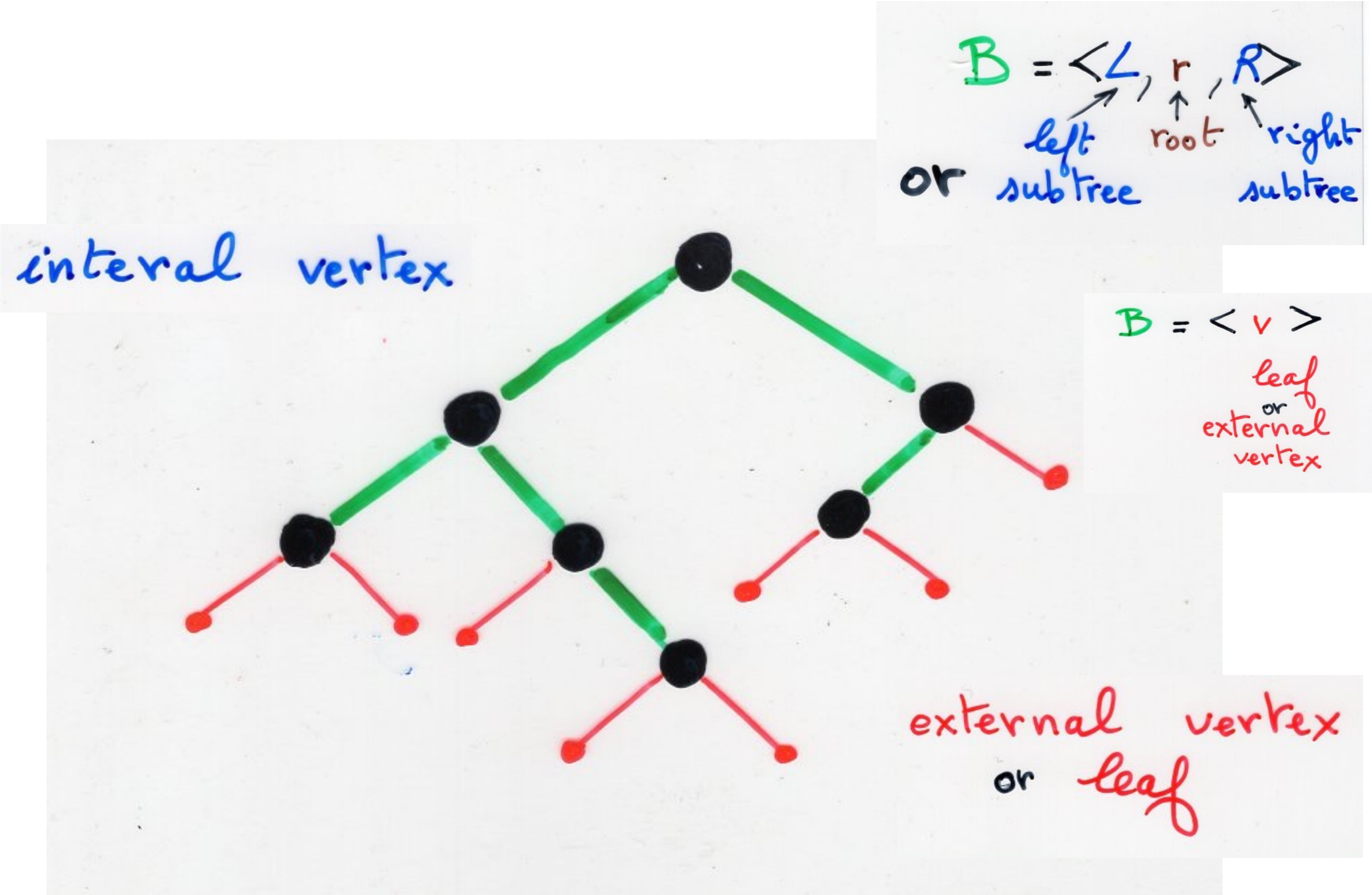
binary trees

$(2n+1)$ vertices

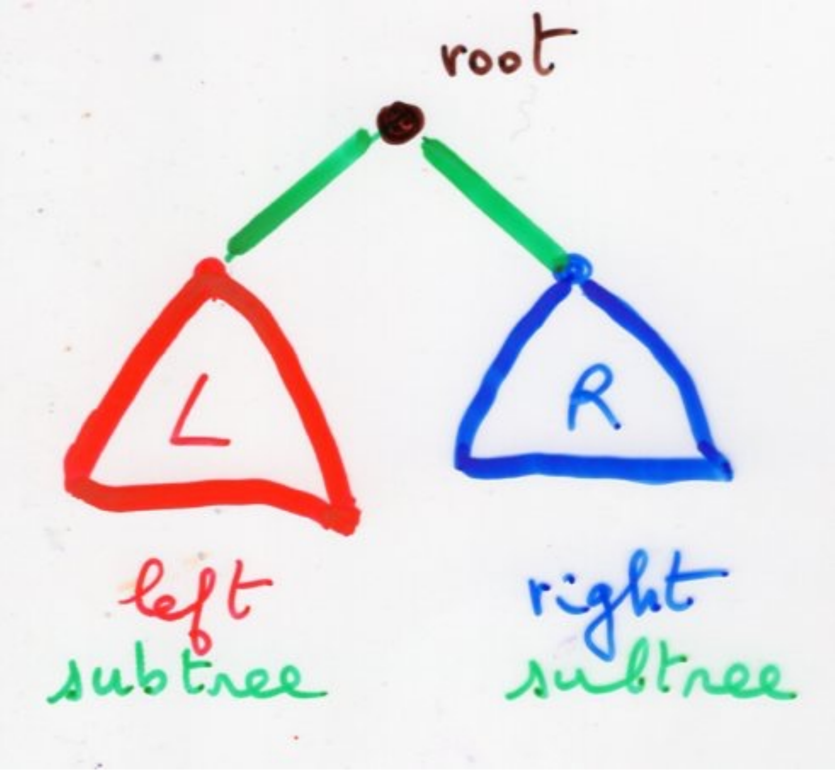
$\left\{\begin{array}{ll} n & \text{internal} \\ n+1 & \text{external} \end{array}\right.$ vertices

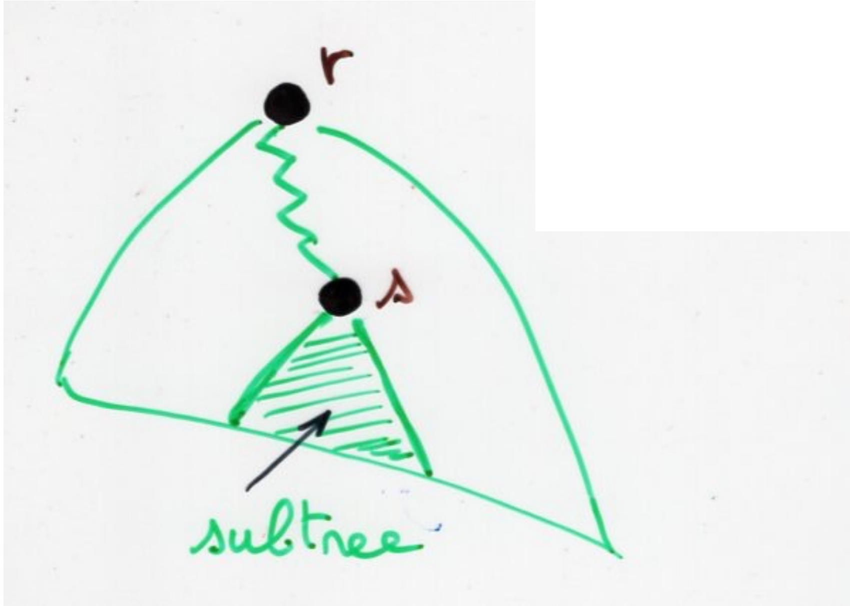






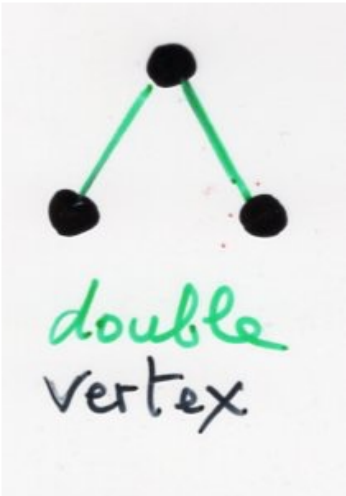
B = <L, r, R>
left subtree
root
right subtree
or
interal vertex
B = < v >
leaf
or
external vertex
external vertex
or leaf


root
L
R
left
subtree
right
subtree


r
s
subtree


left
son
daughter

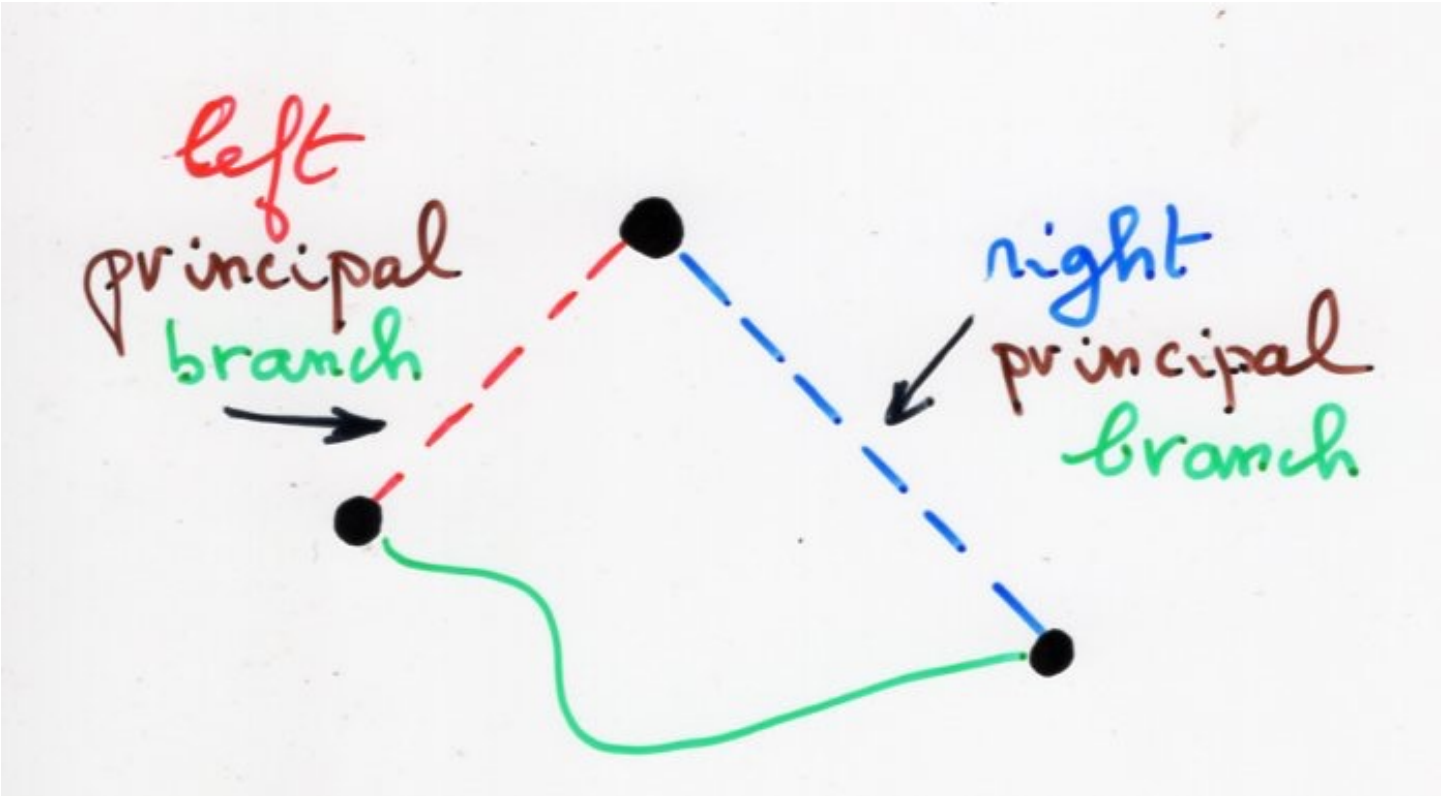

right
son
daughter


double
vertex


right
simple
vertex


left
simple
vertex


leaf



left
principal
branch
right
principal
branch

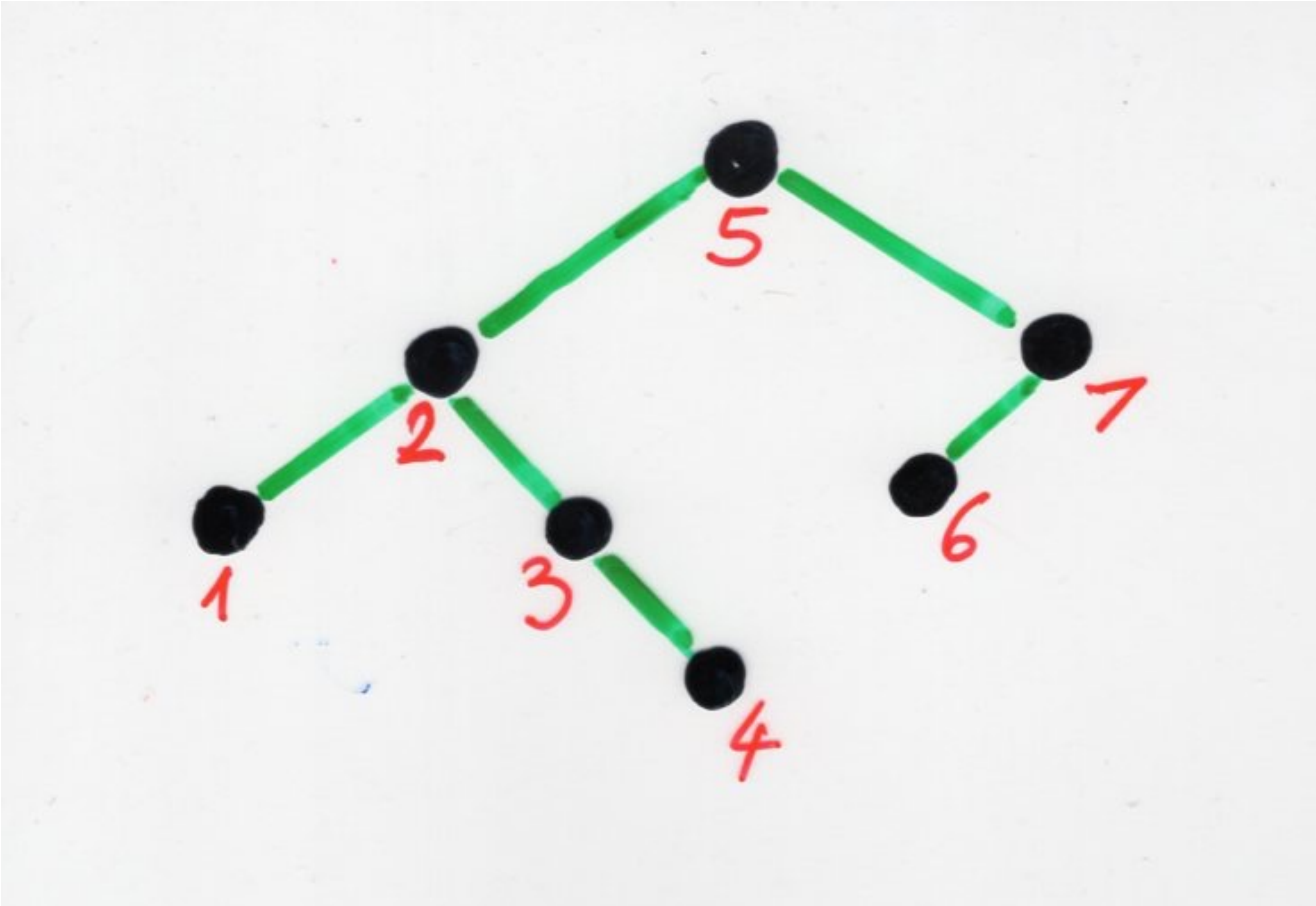
inorder

(symmetric order)

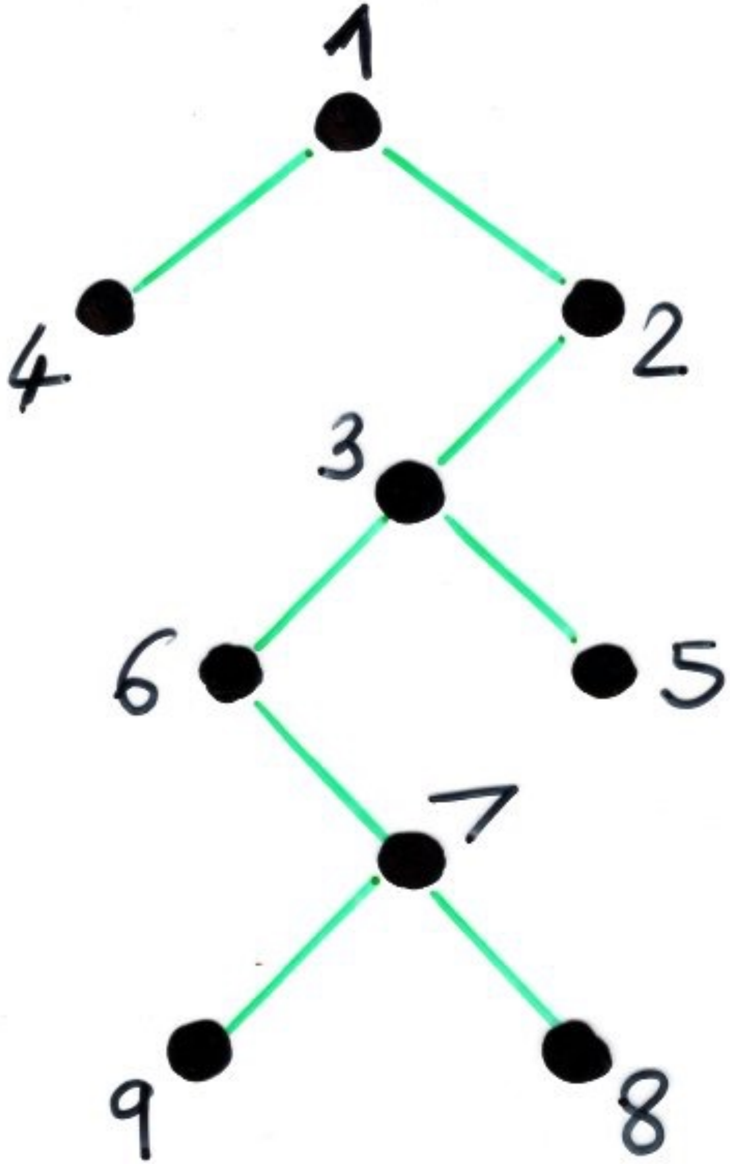
visit the left subtree

visit the root

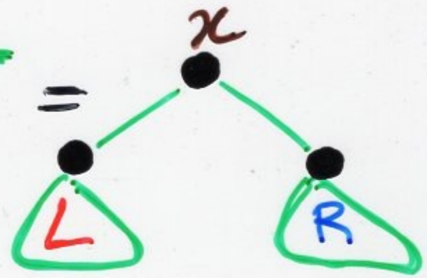
visit the right subtree


5
2
7
1
3
6
4

Increasing binary trees



inorder

$T =$ 

$$\pi(T) = \underbrace{\pi(L)}\, x\, \underbrace{\pi(R)}$$

projection of $T \in \mathcal{C}_n$

$$\pi(T) = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

w word of $\{1,2,\dots,n\}^*$
with all letters distinct

free monoid generated
by the "alphabet" $\{1,2,\dots,n\}$

<u>Definition</u>

$\delta(w)$ "déployé" of w

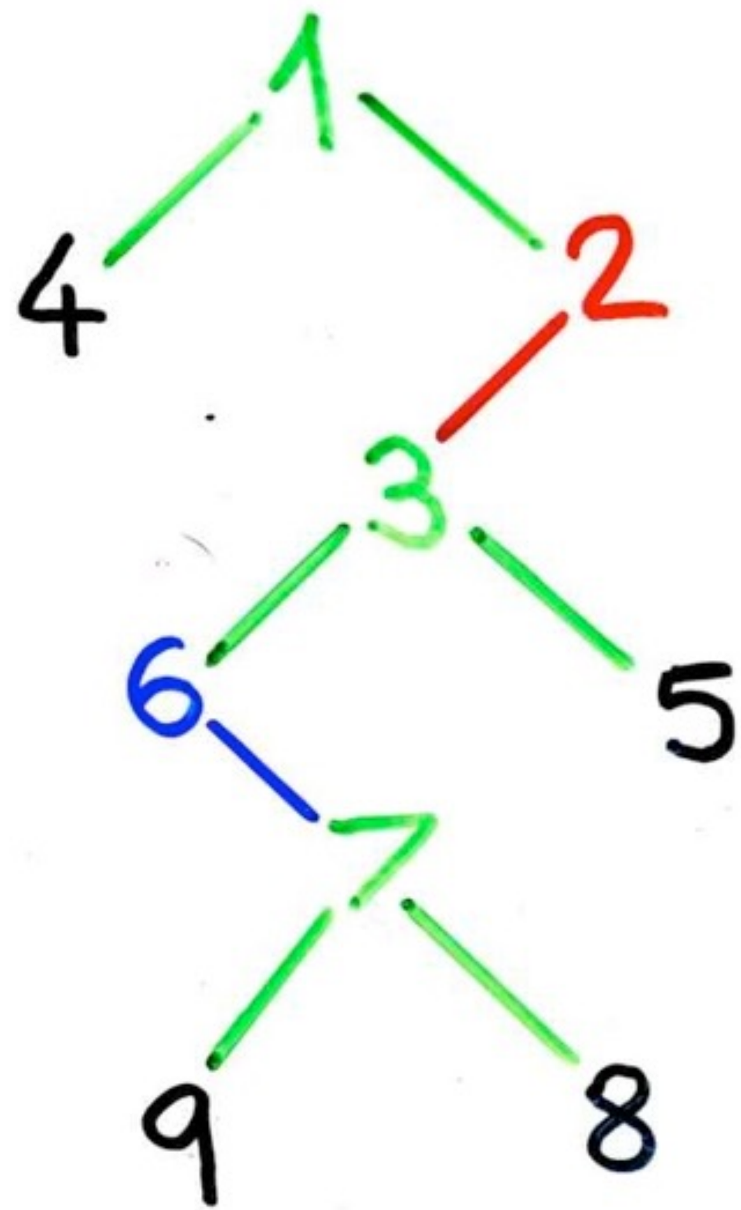
$$\begin{cases} \delta(e) = \emptyset & (e \text{ empty word}) \\ \delta(w) = (\delta(u), m, \delta(v)) \end{cases}$$

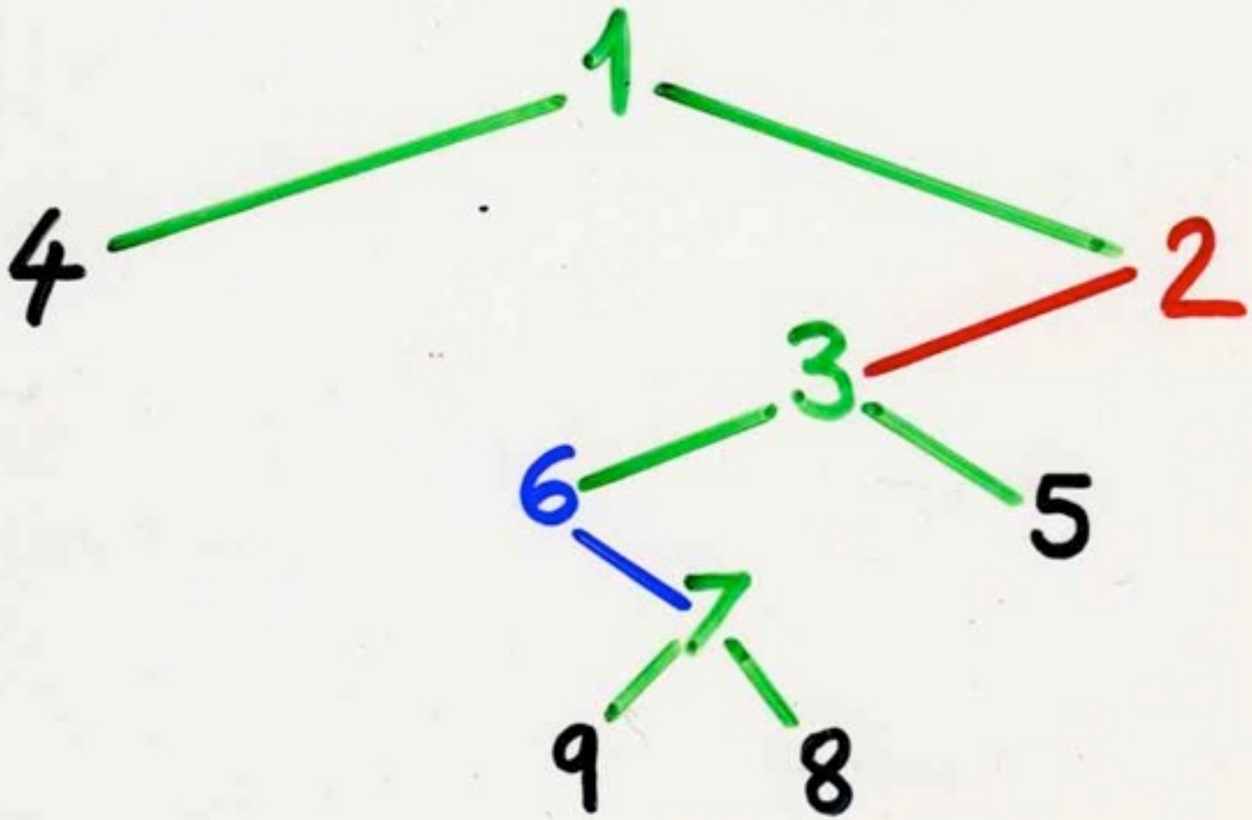
$w = umv$ where m is the
minimum letter of w

<u>Proposition</u>

$$\mathfrak{S}_n \underset{\delta}{\overset{\overline{\pi}}{\rightleftarrows}} \mathcal{E}_n$$

$\overline{\pi}$ and δ are bijections
and $\delta = \overline{\pi}^{-1}$


1
4
2
3
6
5
7
9
8


1
4
2
3
6
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7
9
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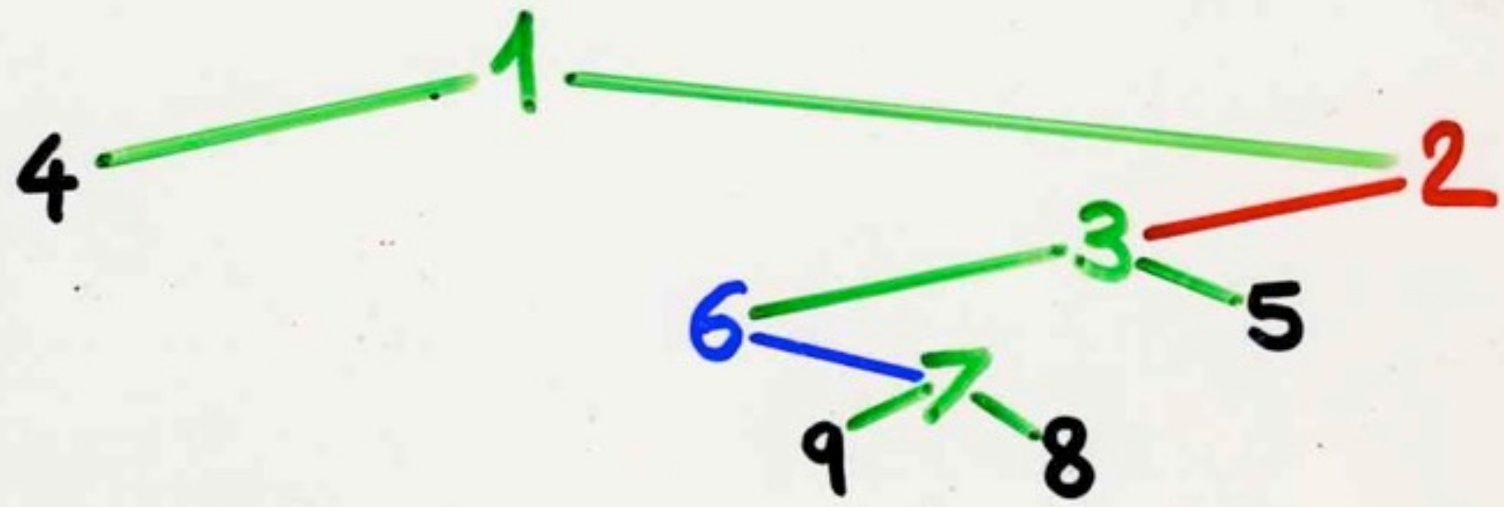

1
4
2
3
5
6
7
9
8


4
1
6
2
3
5
9
7
8

4 1 6 9 7 8 3 5 2

Definition x-factorization

$\sigma \in \mathfrak{S}_n$, $x \in [1,n]$ $\sigma = u\,\lambda(x)\,x\,\rho(x)\,v$

$$\begin{cases} \bullet \text{ the letters of } \lambda(x) \text{ and } \rho(x) \text{ are } > x \\ \bullet \text{ the lengths } |\lambda(x)| \text{ and } |\rho(x)| \text{ are maximum} \end{cases}$$

Lemma $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{T}_n$, $x \in [1,n]$

$u\,\lambda(x)\,x\,\rho(x)\,v$ x-factorization

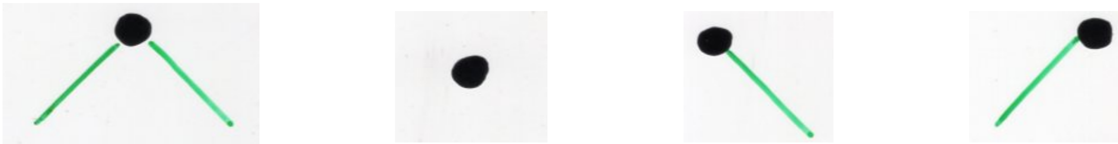
the the left (resp. right) subtree of the vertex x in the tree $\delta(\sigma)$ is:

$\delta(\lambda(x))$ (resp. $\delta(\rho(x))$)

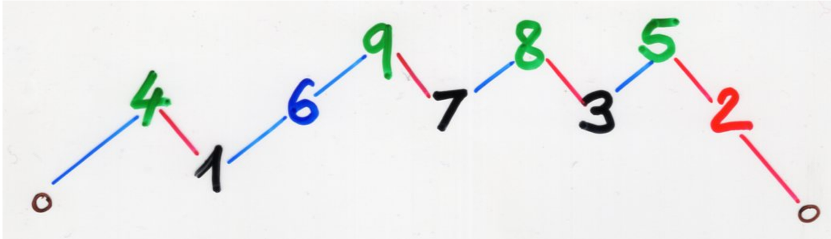
Corollary

in σ, x is :

	valley	peak	double rise	double descent
iff in $\delta(\sigma)$ x is :	double vertex	leaf	right simple	left simple

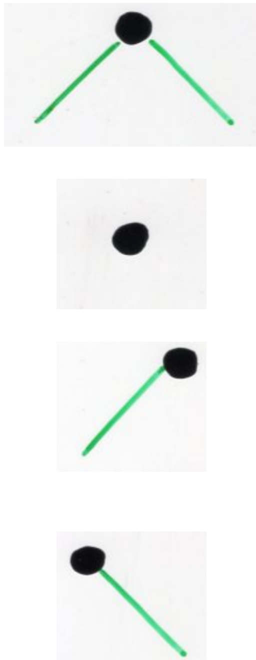


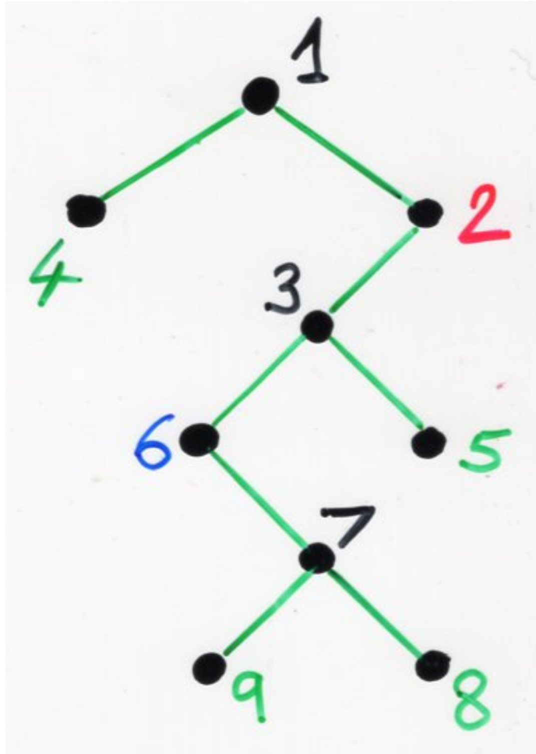
permutation σ



$$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$$

Valleys	1, 3, 7
peaks	4, 5, 8, 9
double descents	2
double rise	6





increasing binary tree

$LR\text{-}\min(\sigma)$ = set of lr-min elements of σ

$RL\text{-}\min(\sigma)$ = set of rl-min elements of σ

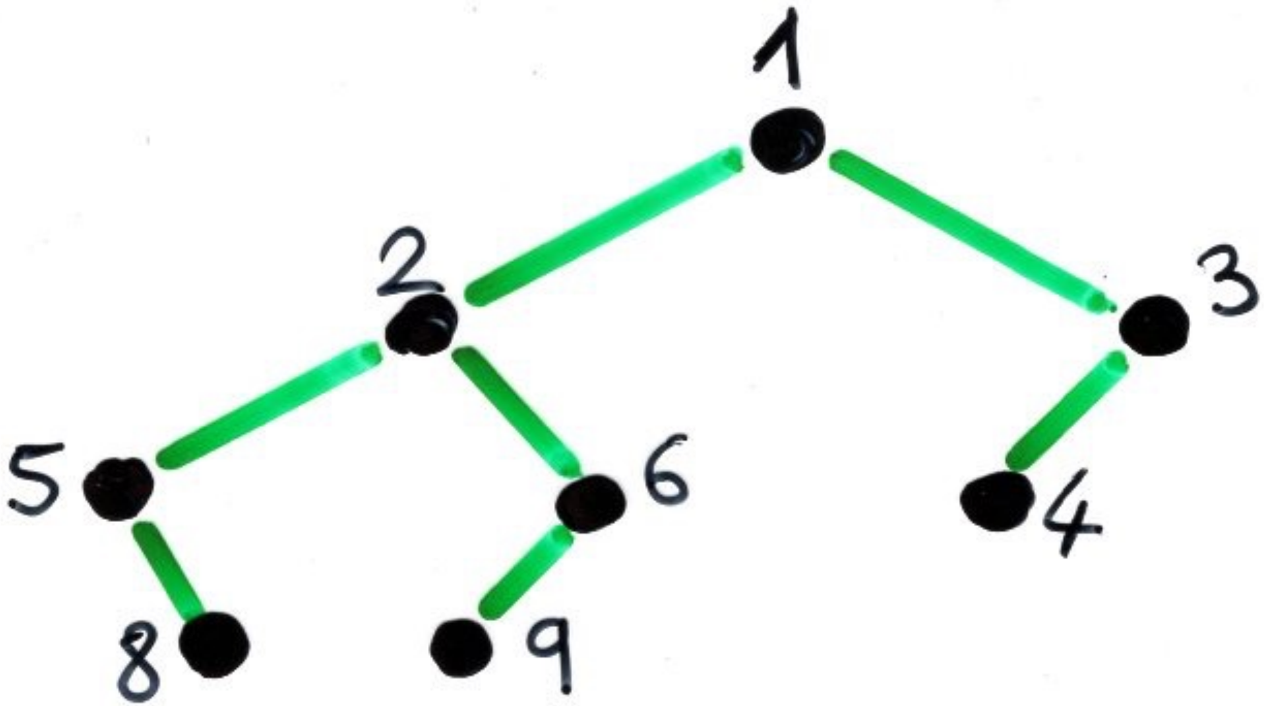
$LB(T)$ = left branch of $T \in \mathcal{T}_n$

$RB(T)$ = right branch of $T \in \mathcal{T}_n$

<u>Proposition</u> $\sigma \in \mathfrak{S}_n$, $T = \delta(\sigma) \in \mathcal{T}_n$

$$LR\text{-}\min(\sigma) = LB(T)$$

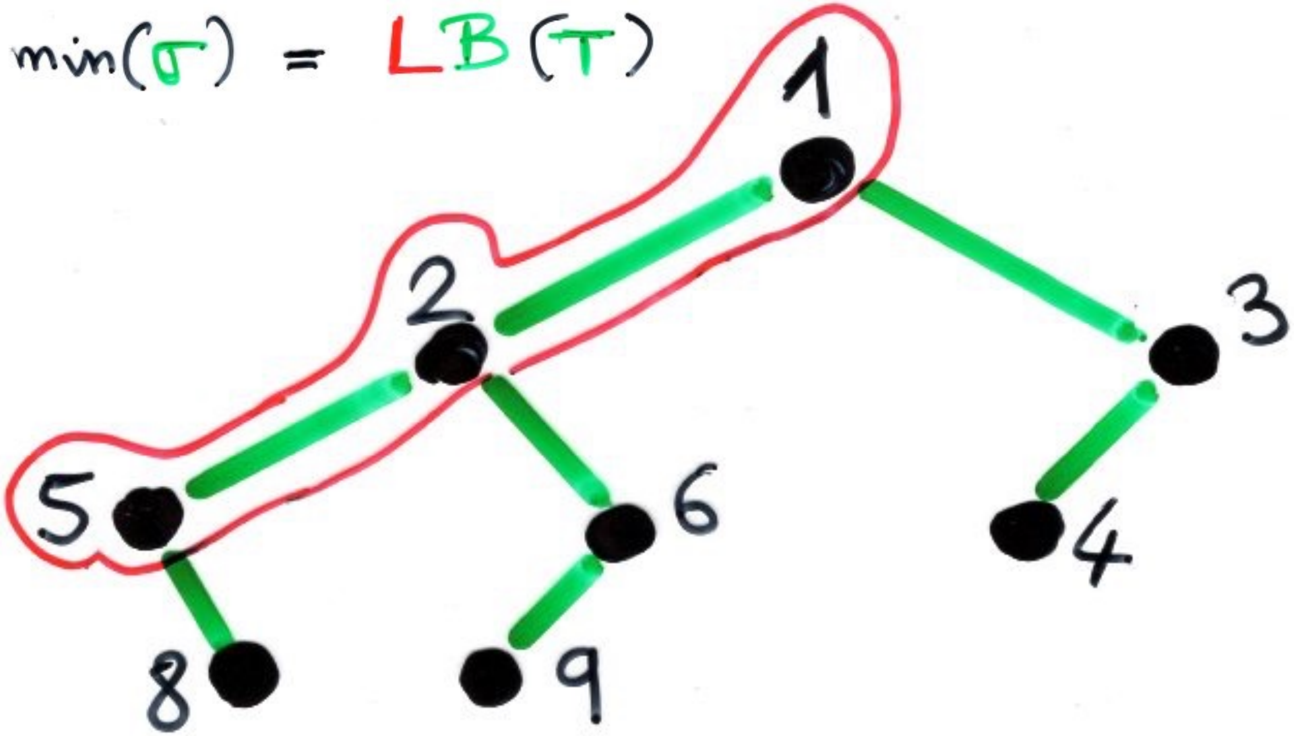
$$RL\text{-}\min(\sigma) = RB(T)$$



$$\pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$$

$$T = \delta(\sigma)$$

$LR\text{-}\min(\sigma) = LB(T)$

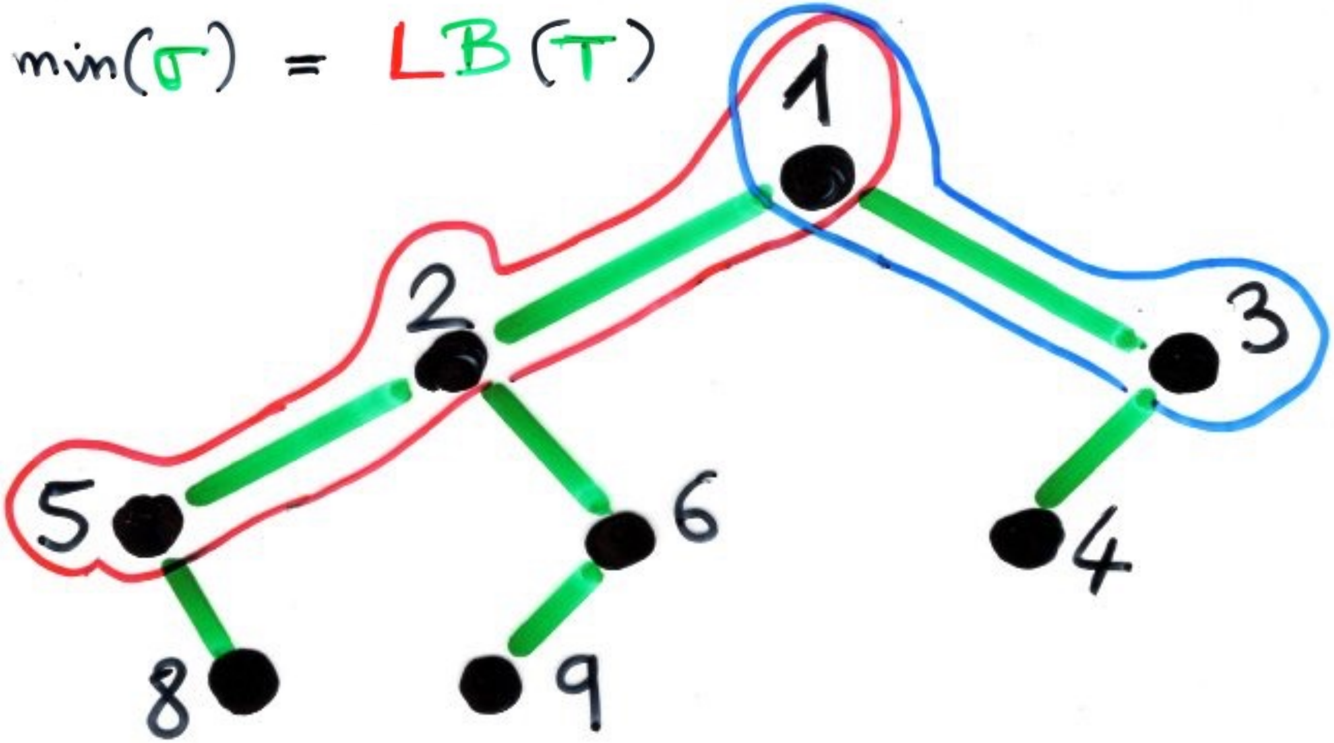


$$\Pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$$

$$T = \delta(\sigma)$$

$RL\text{-}\min(\sigma) = RB(T)$

$LR\text{-}\min(\sigma) = LB(T)$



$\Pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$

$T = \delta(\sigma)$

bijection

Laguerre histories $\longrightarrow$ permutations

description with binary trees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2 2 2
3 2 2
4 3 1
5 2 2
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7 1 1
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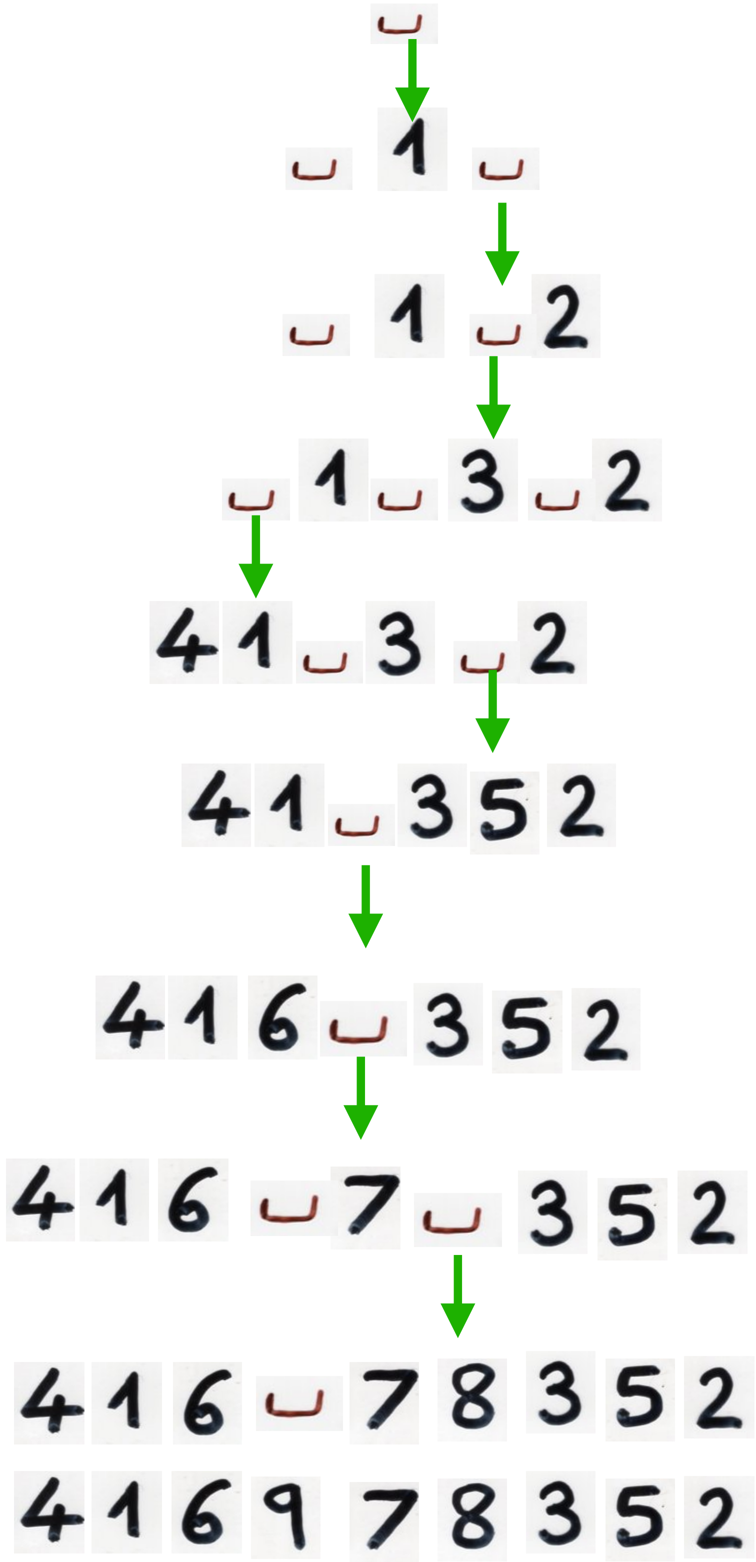
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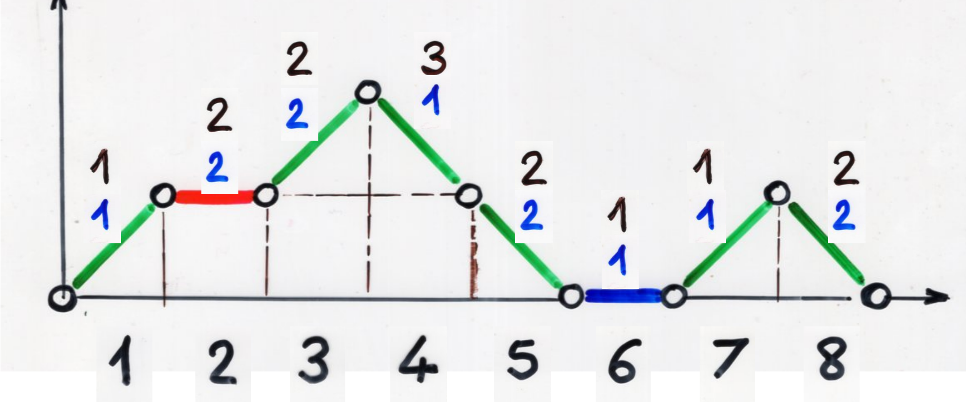
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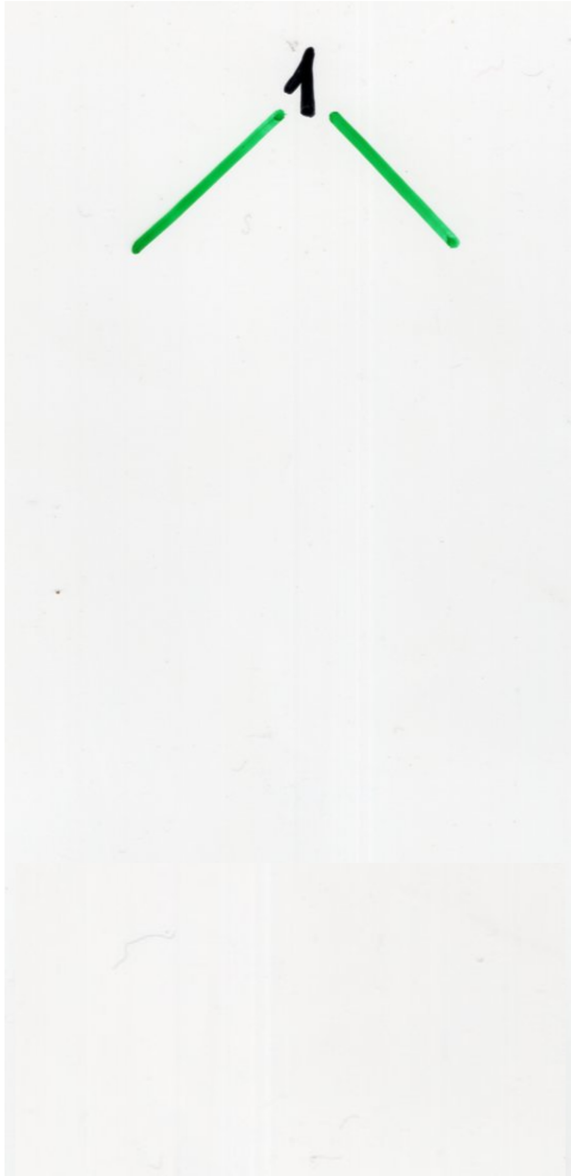
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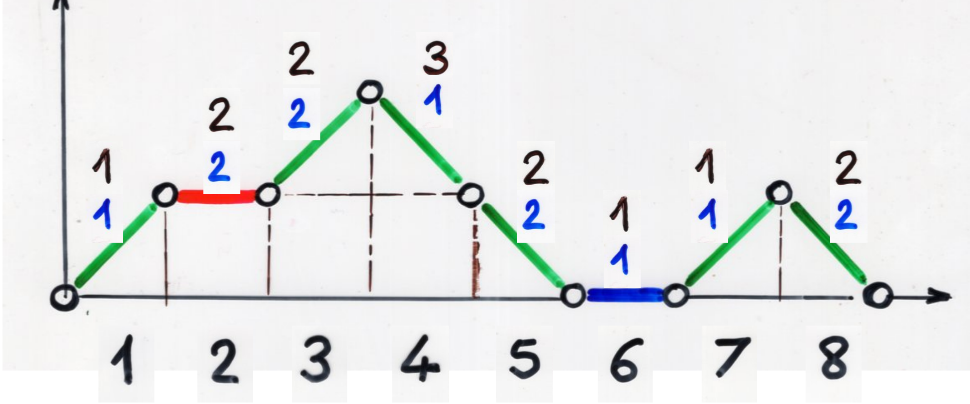
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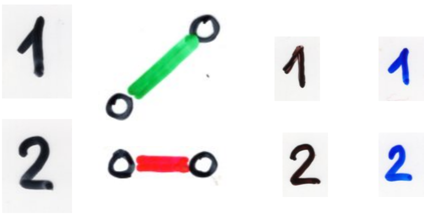
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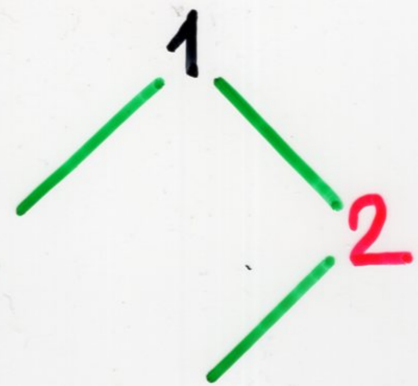
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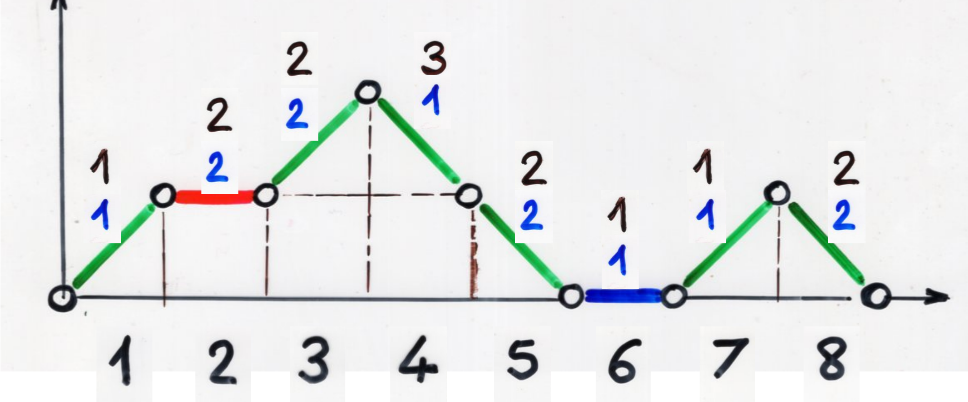
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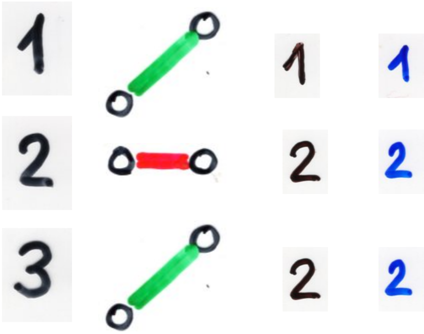
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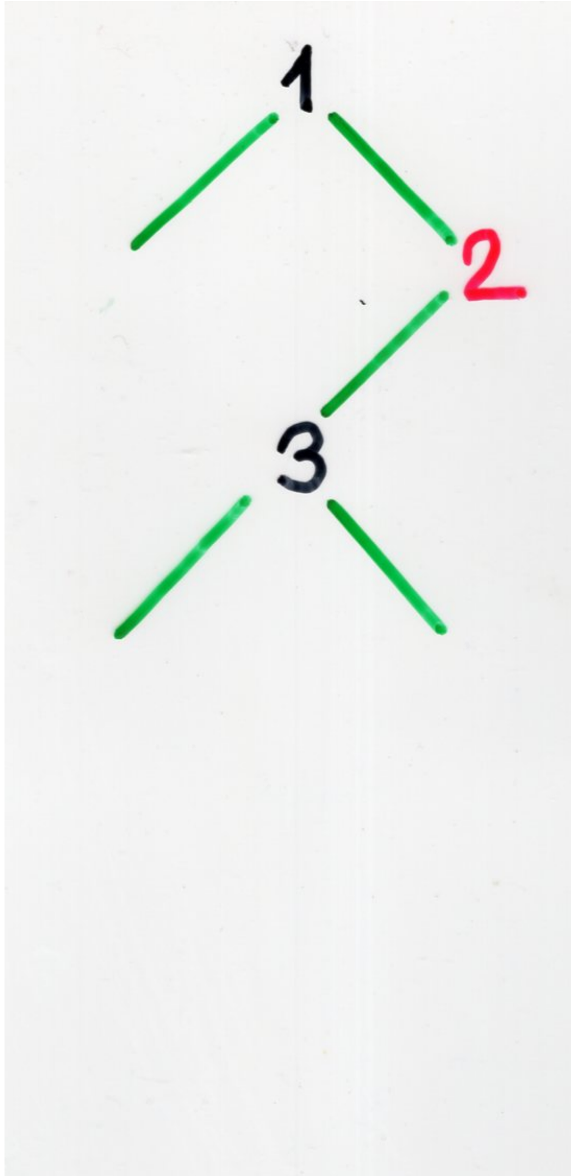
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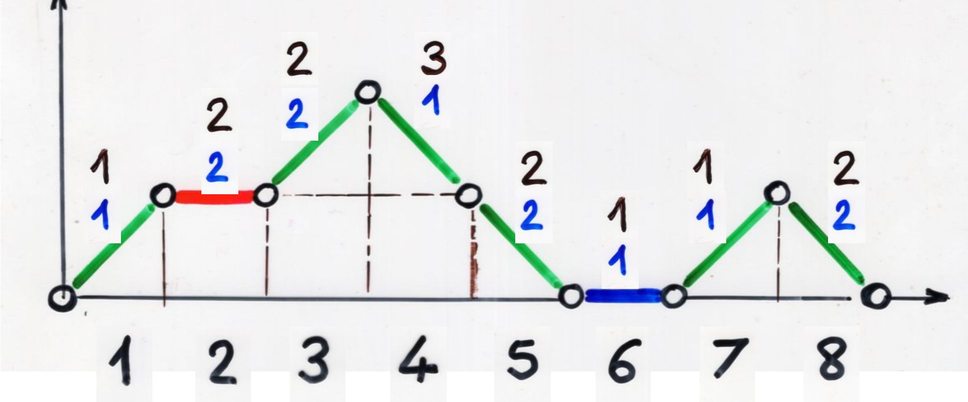
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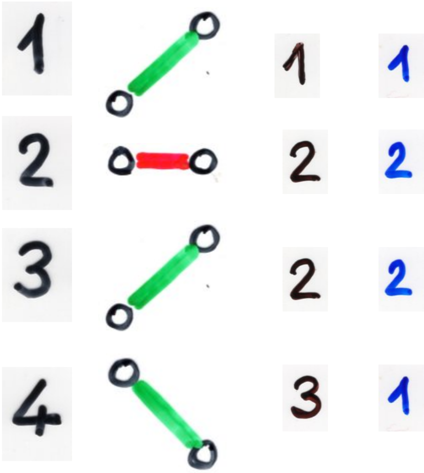
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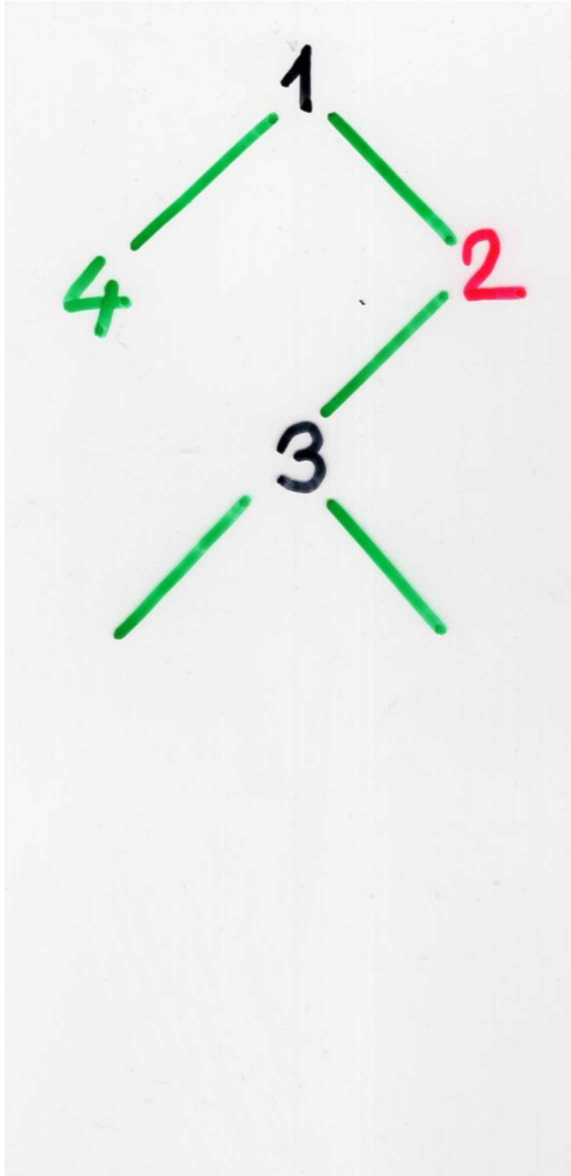
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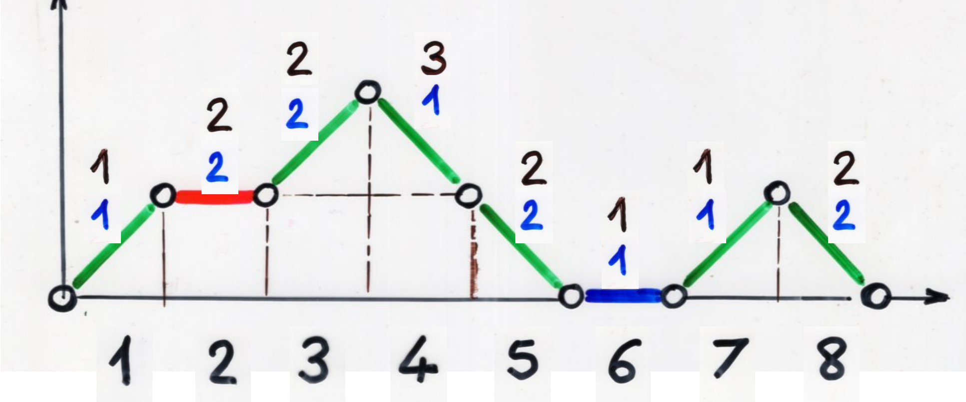
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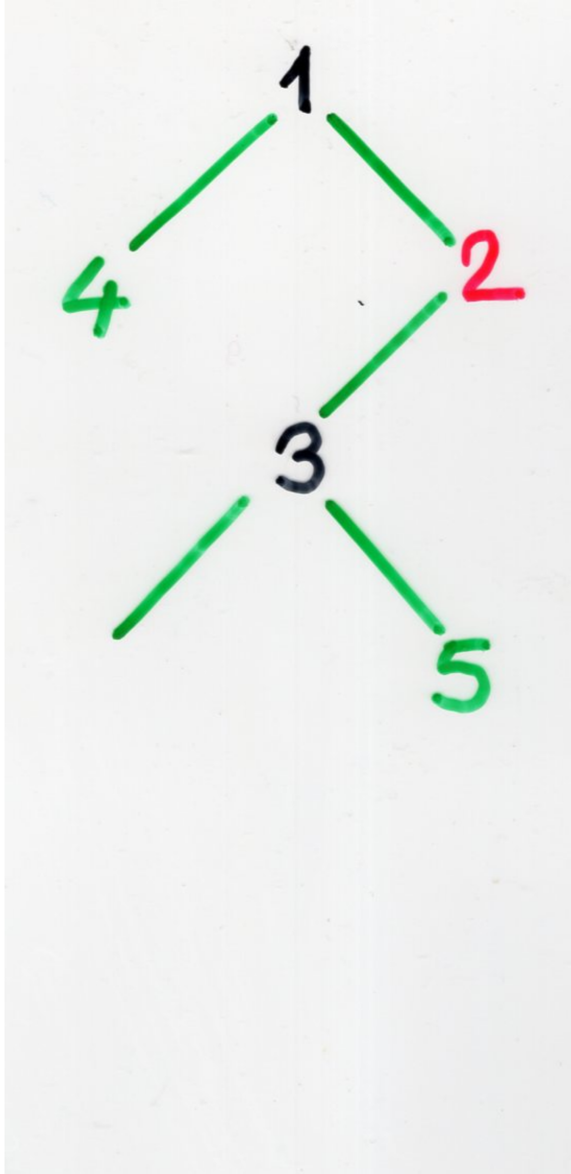
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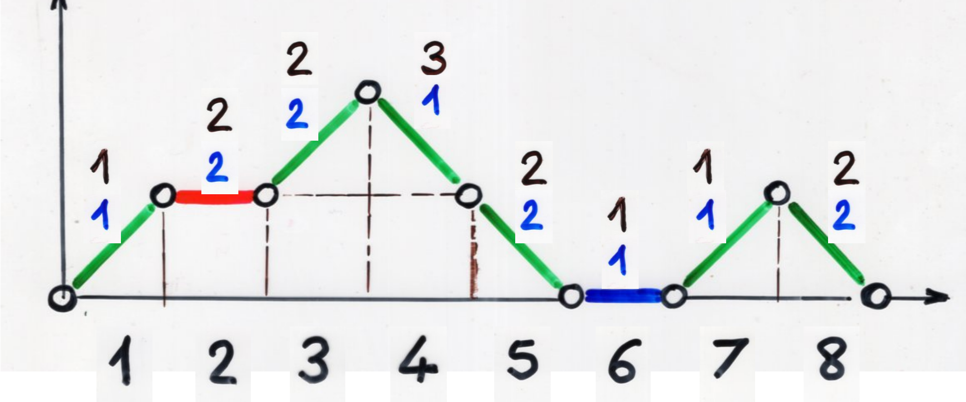
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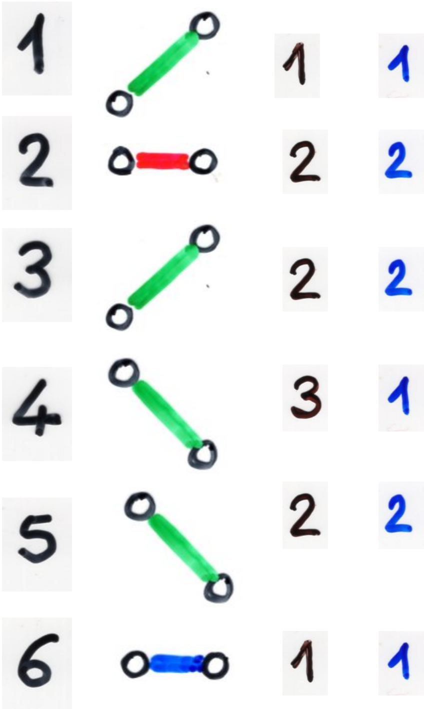
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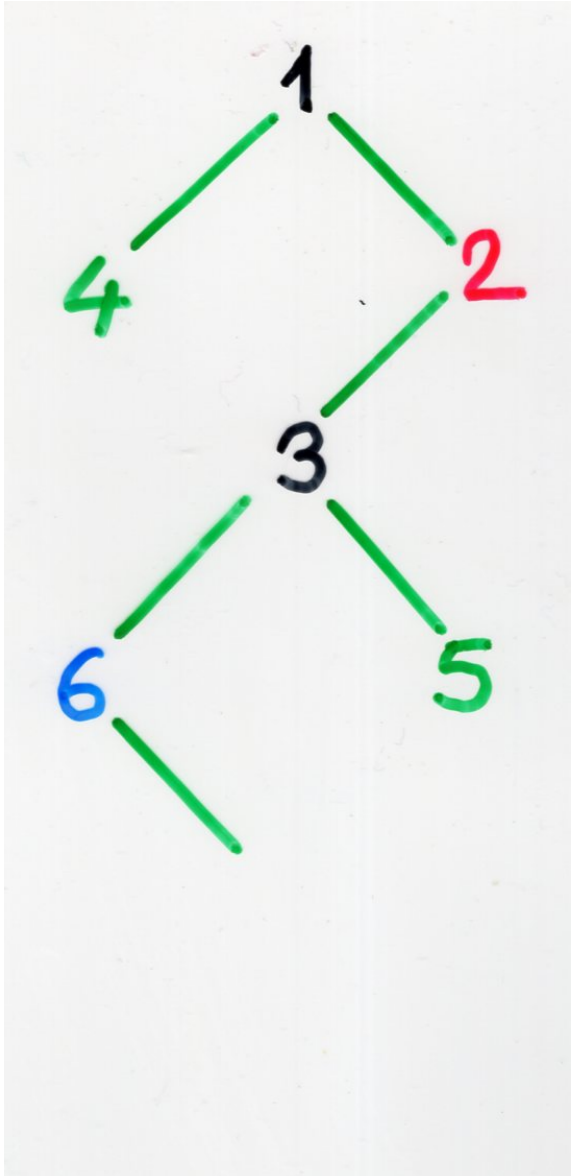
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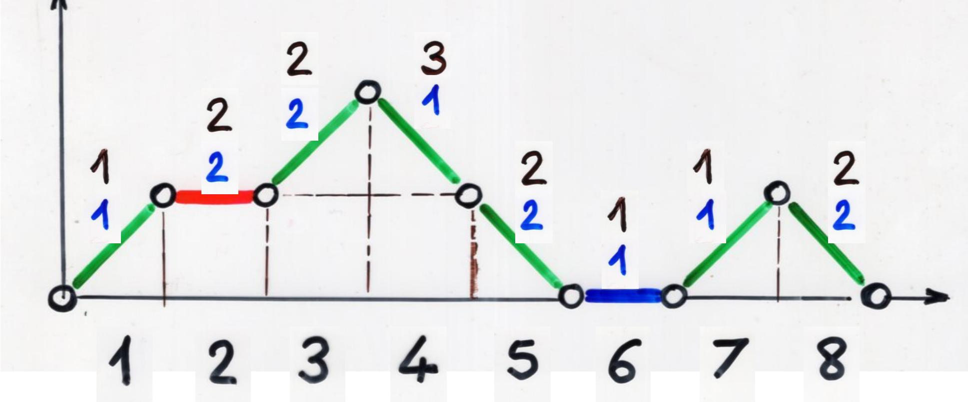
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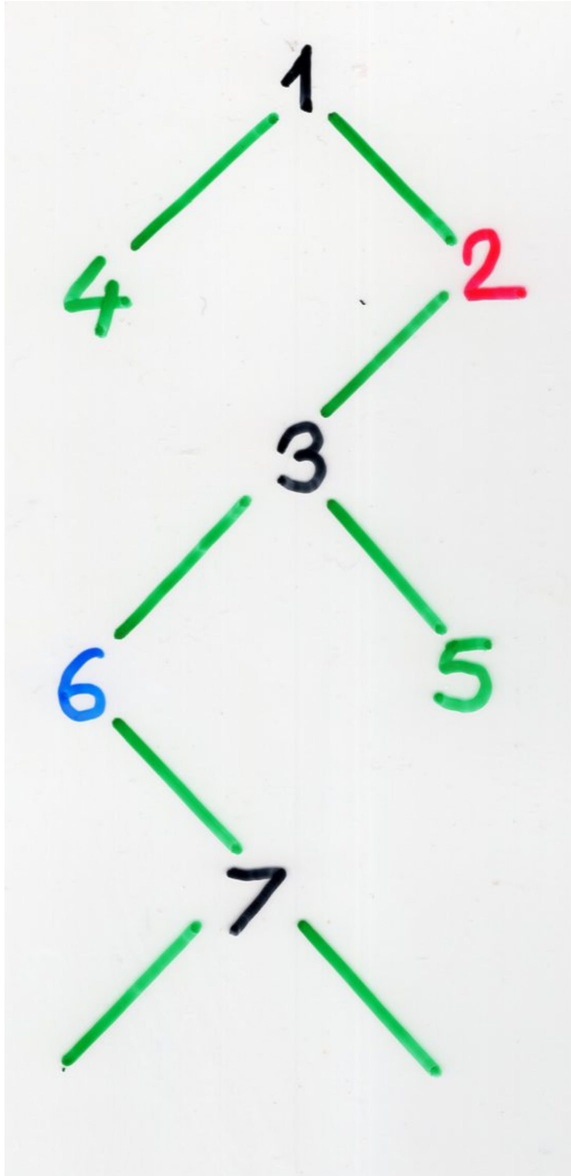
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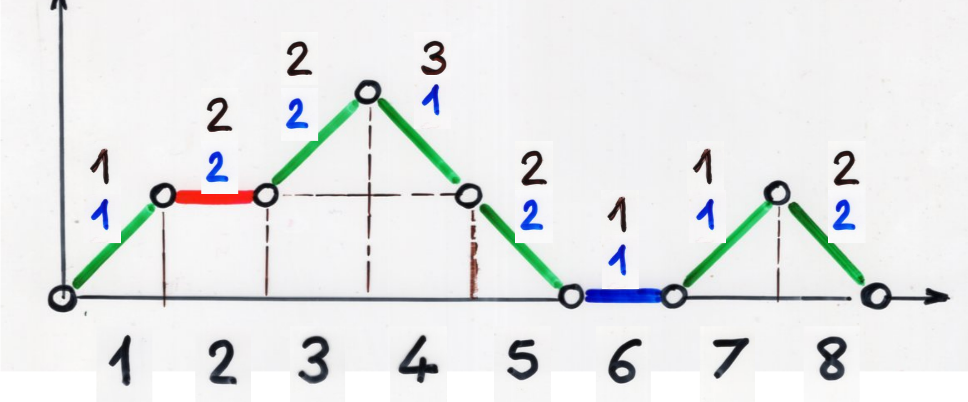

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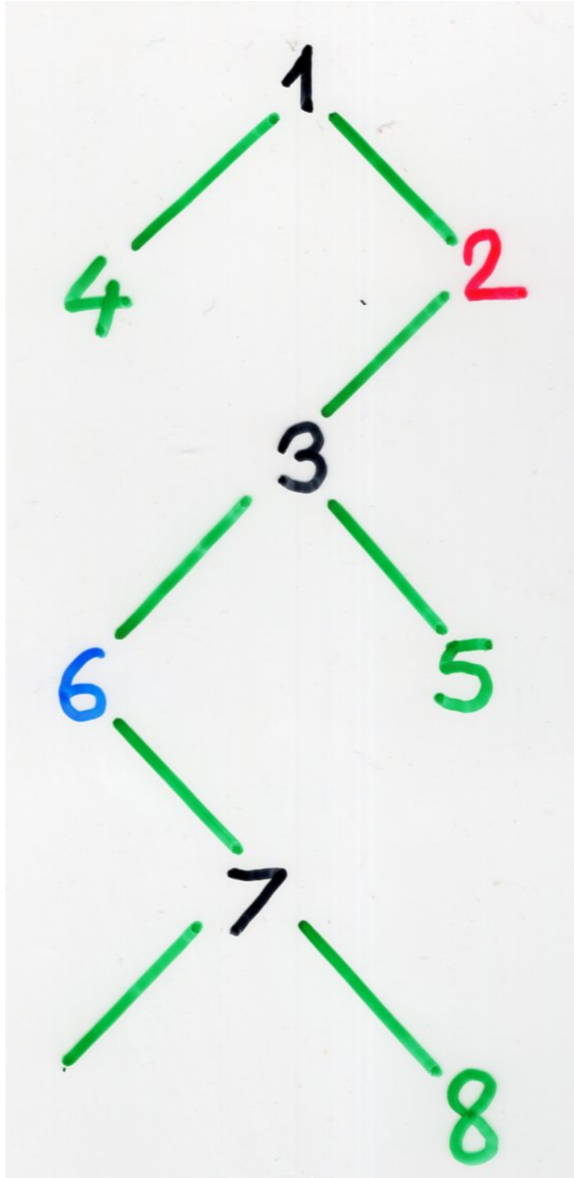

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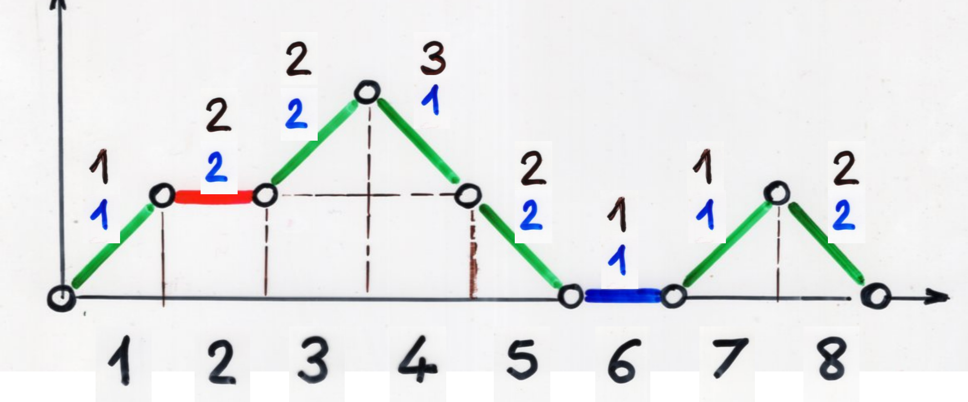
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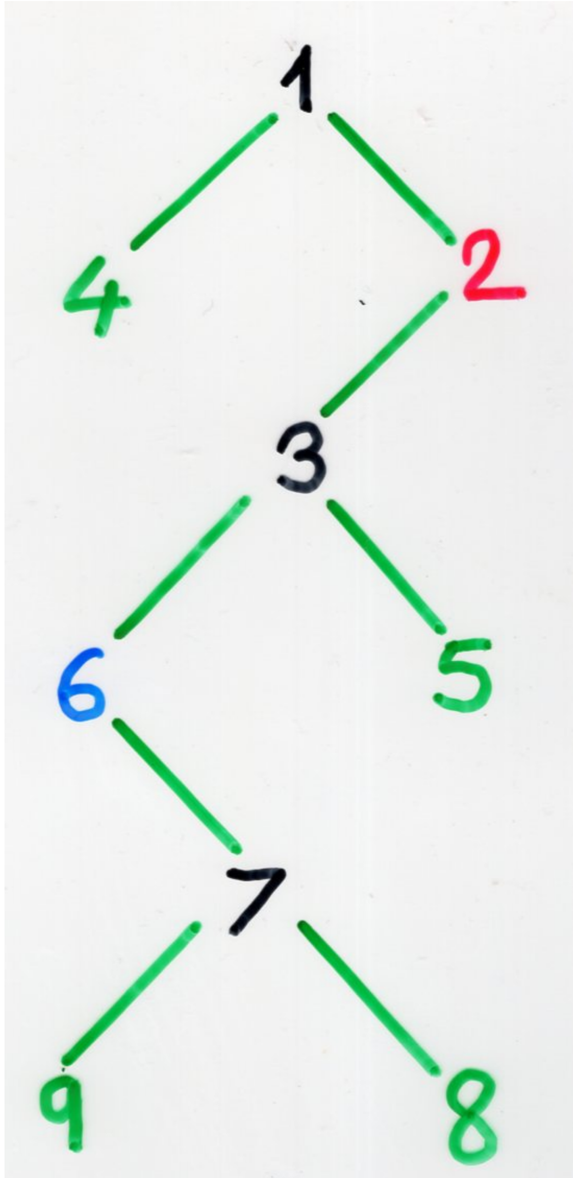
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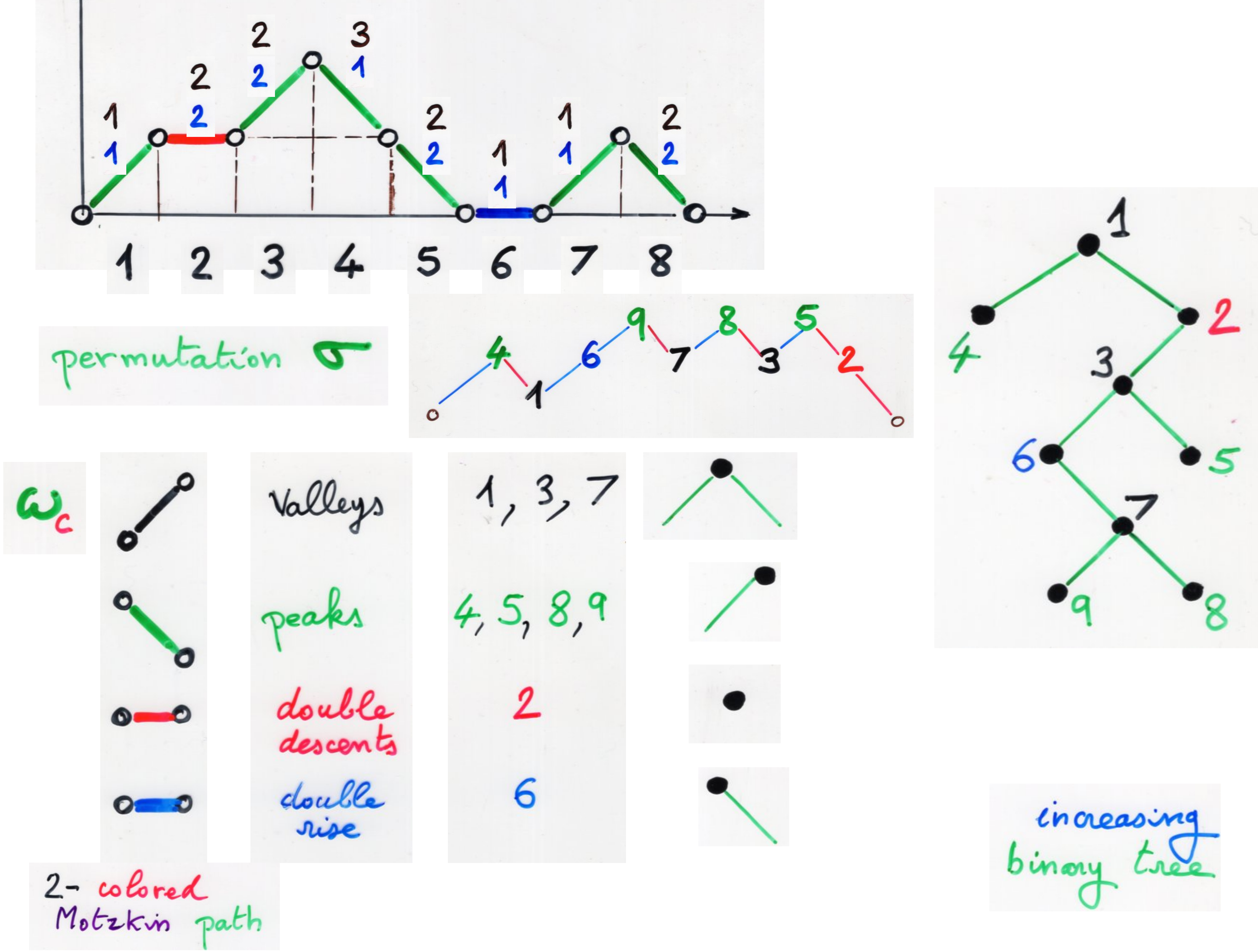
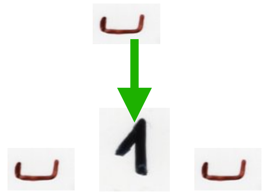


1 2 3 4 5 6 7 8
permutation σ
4 1 6 9 7 8 3 5 2
ω c
Valleys
1, 3, 7
peaks
4, 5, 8, 9
double descents
2
double rise
6
2- colored Motzkin path
increasing binary tree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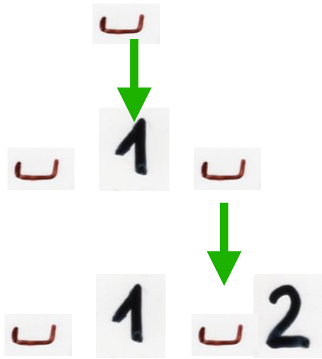
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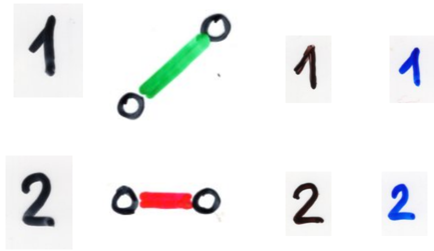
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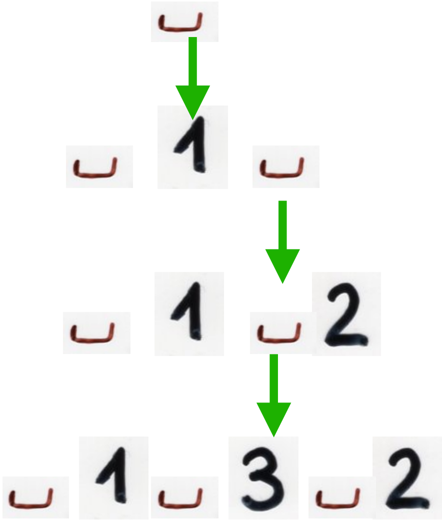
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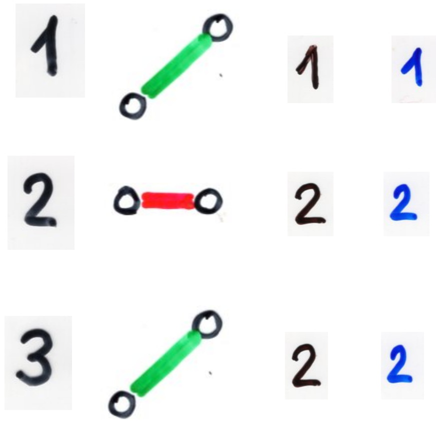
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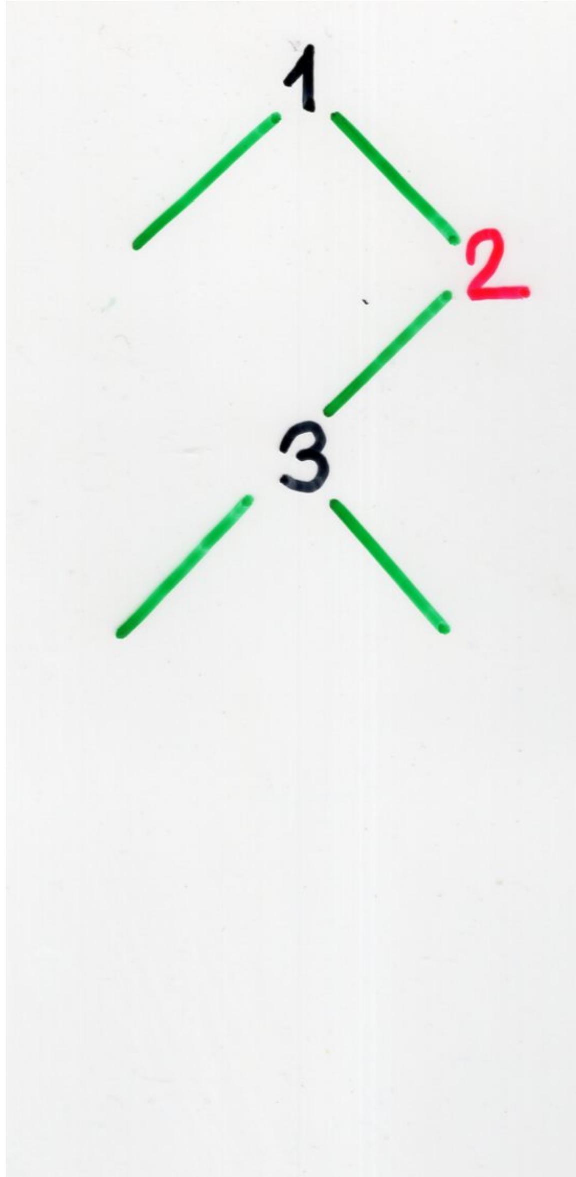
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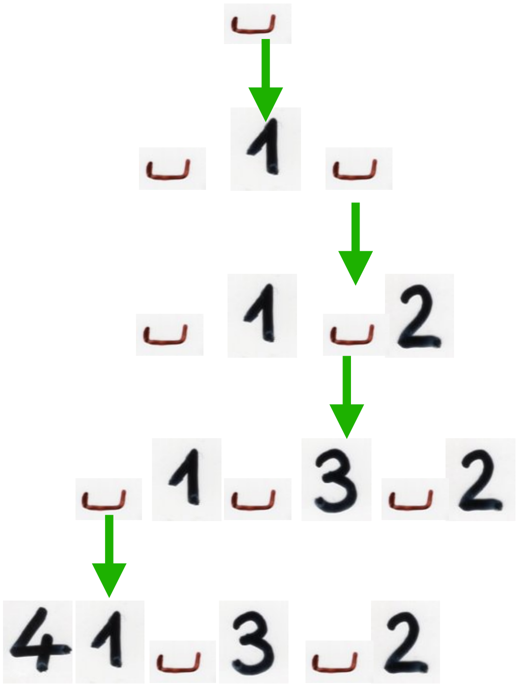
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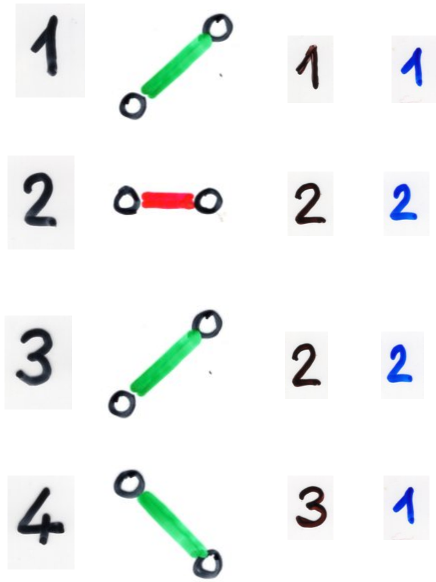
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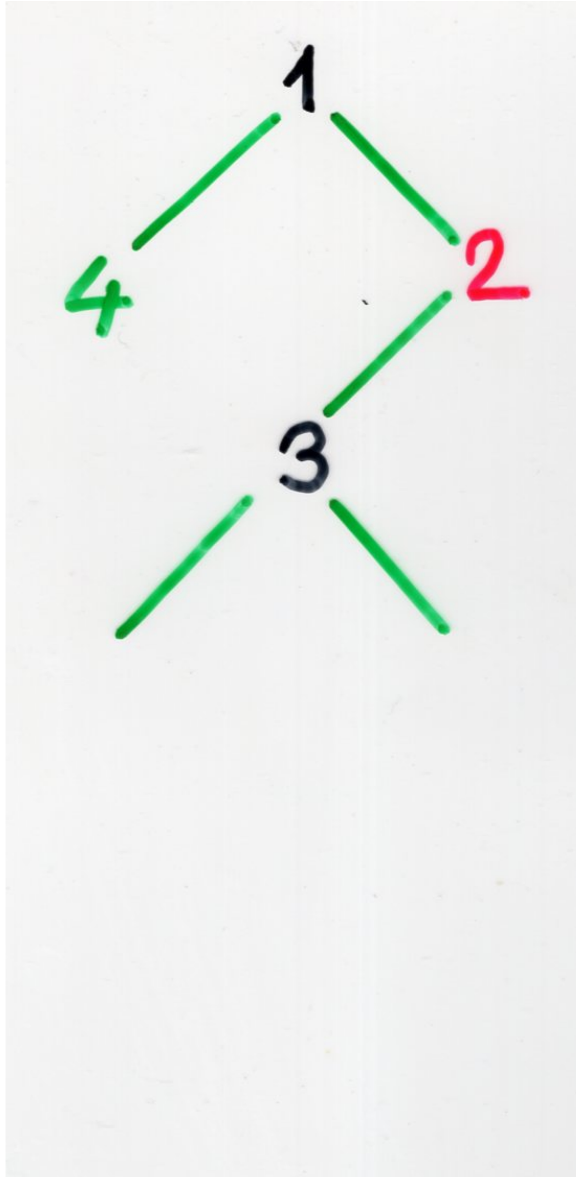
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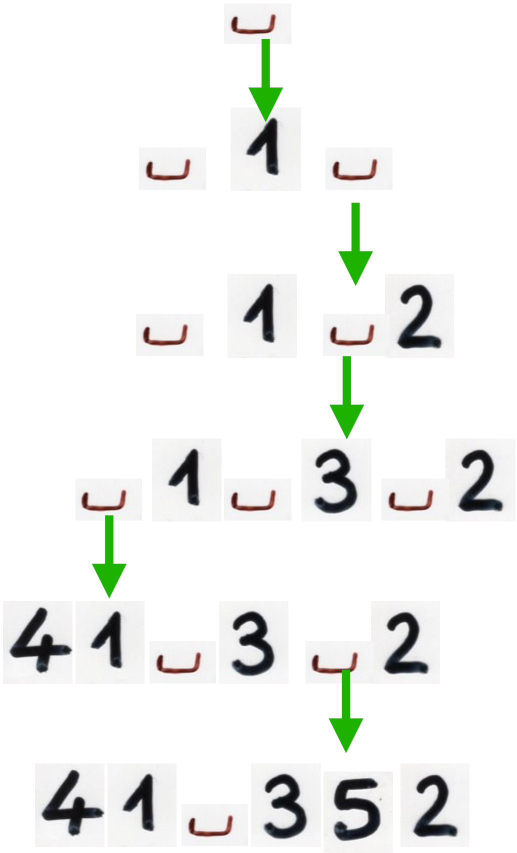
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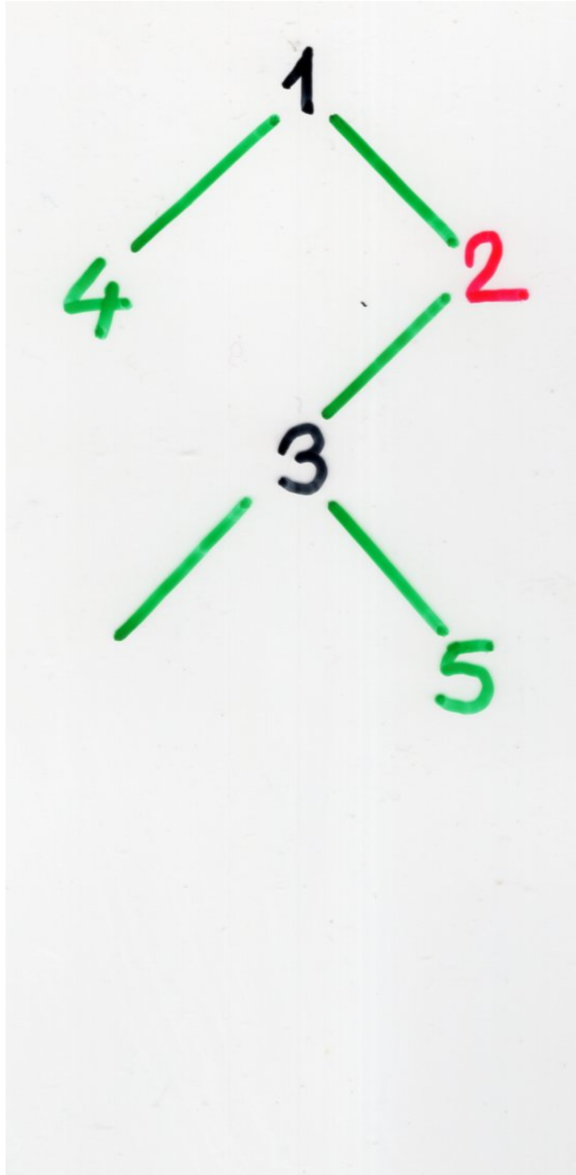
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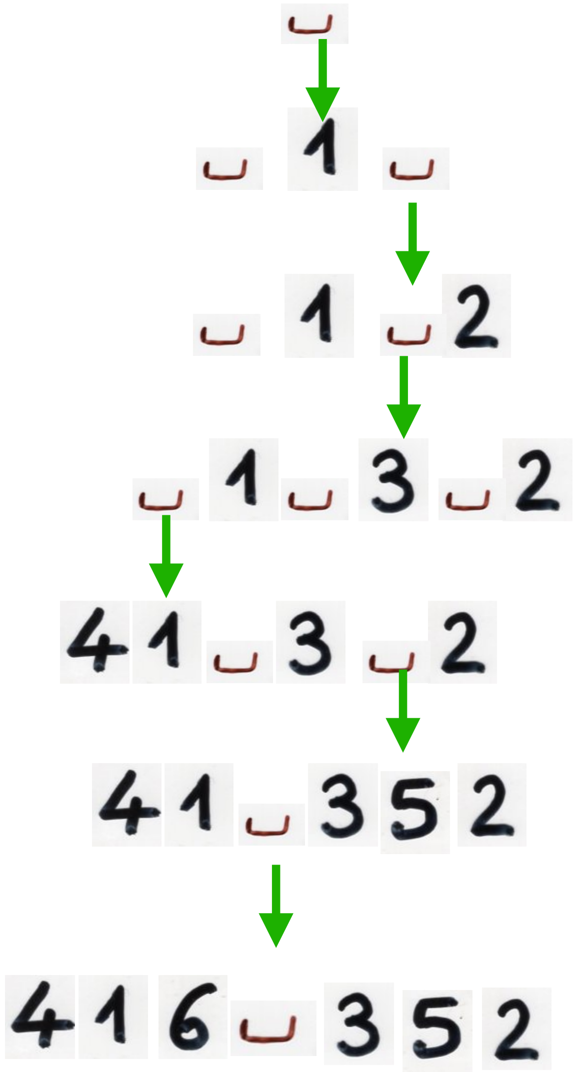
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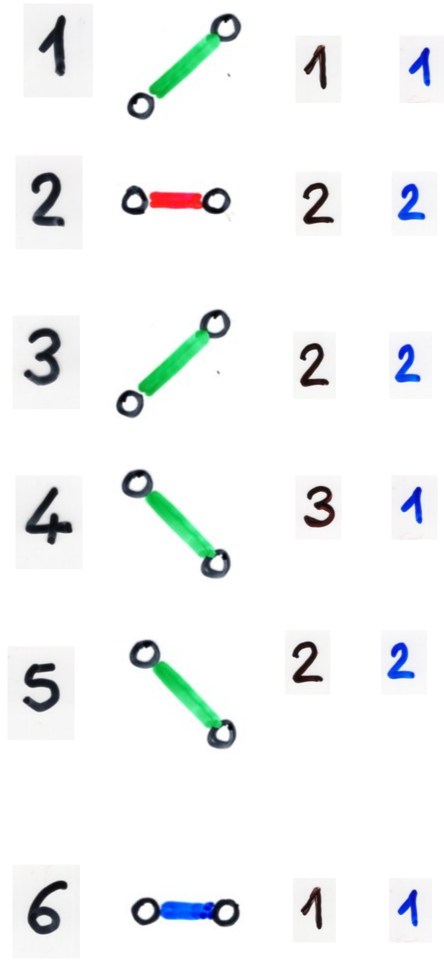
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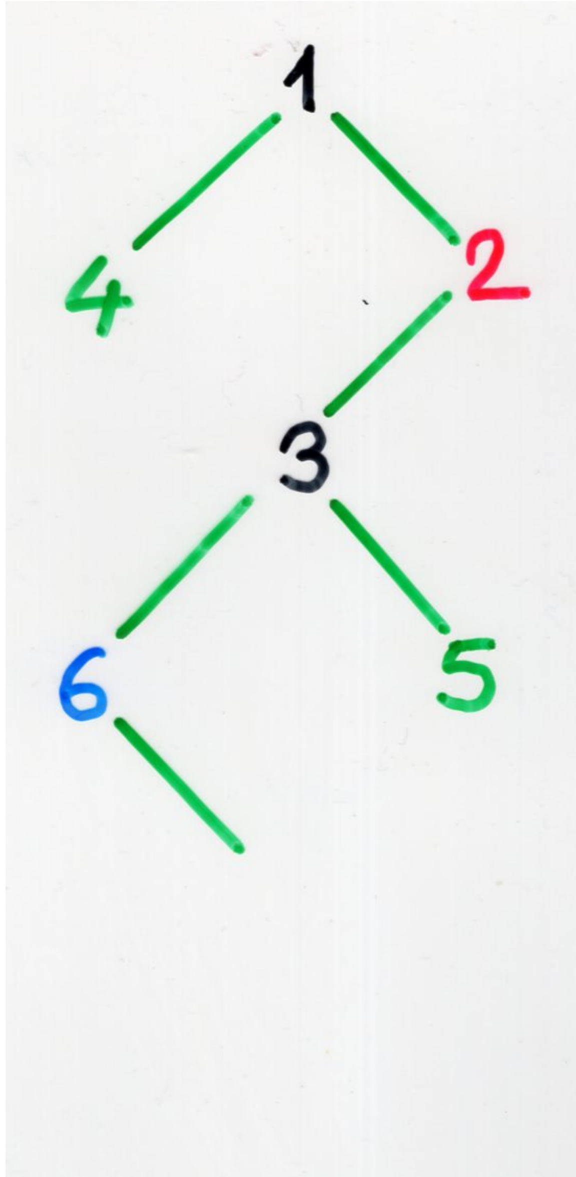
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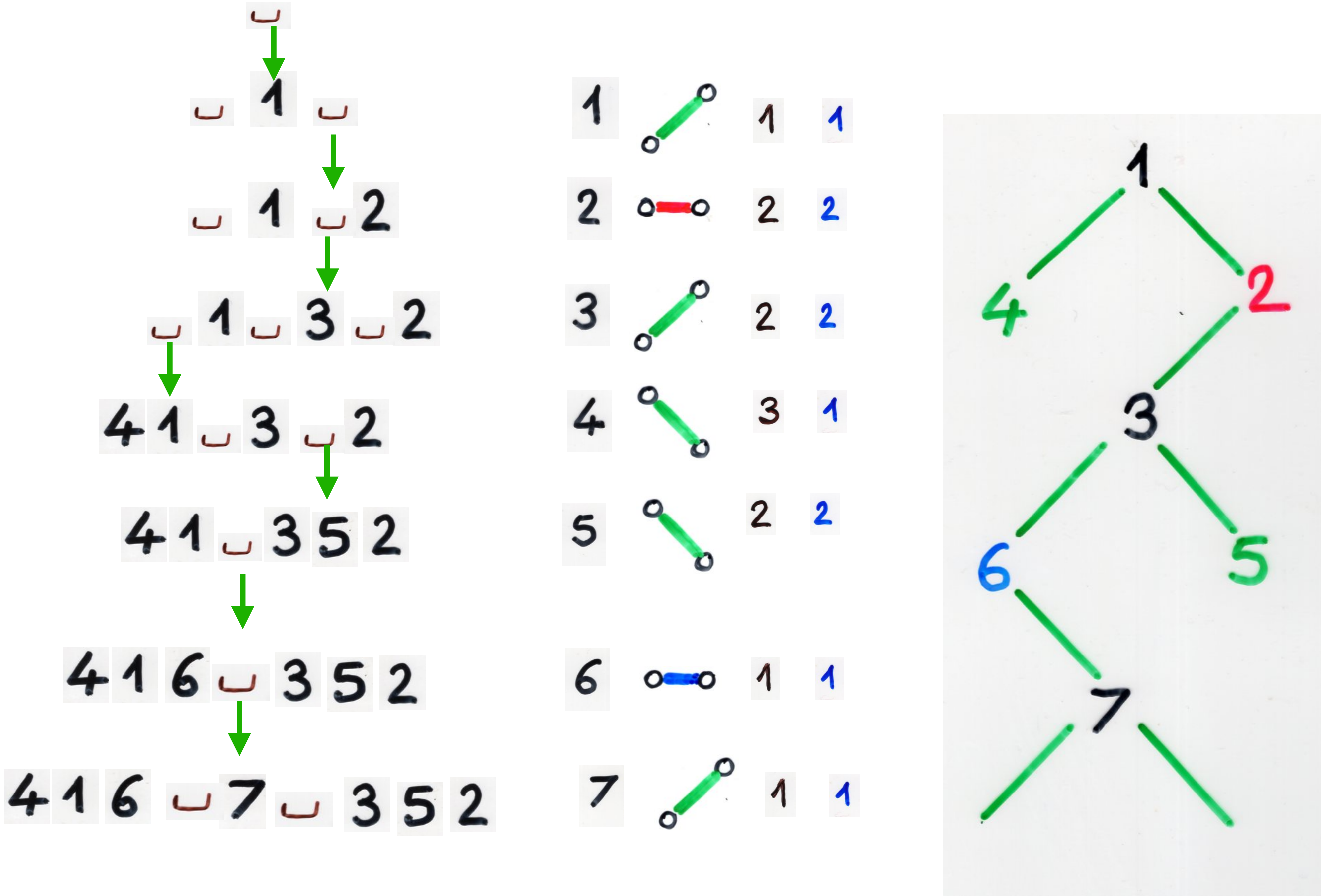

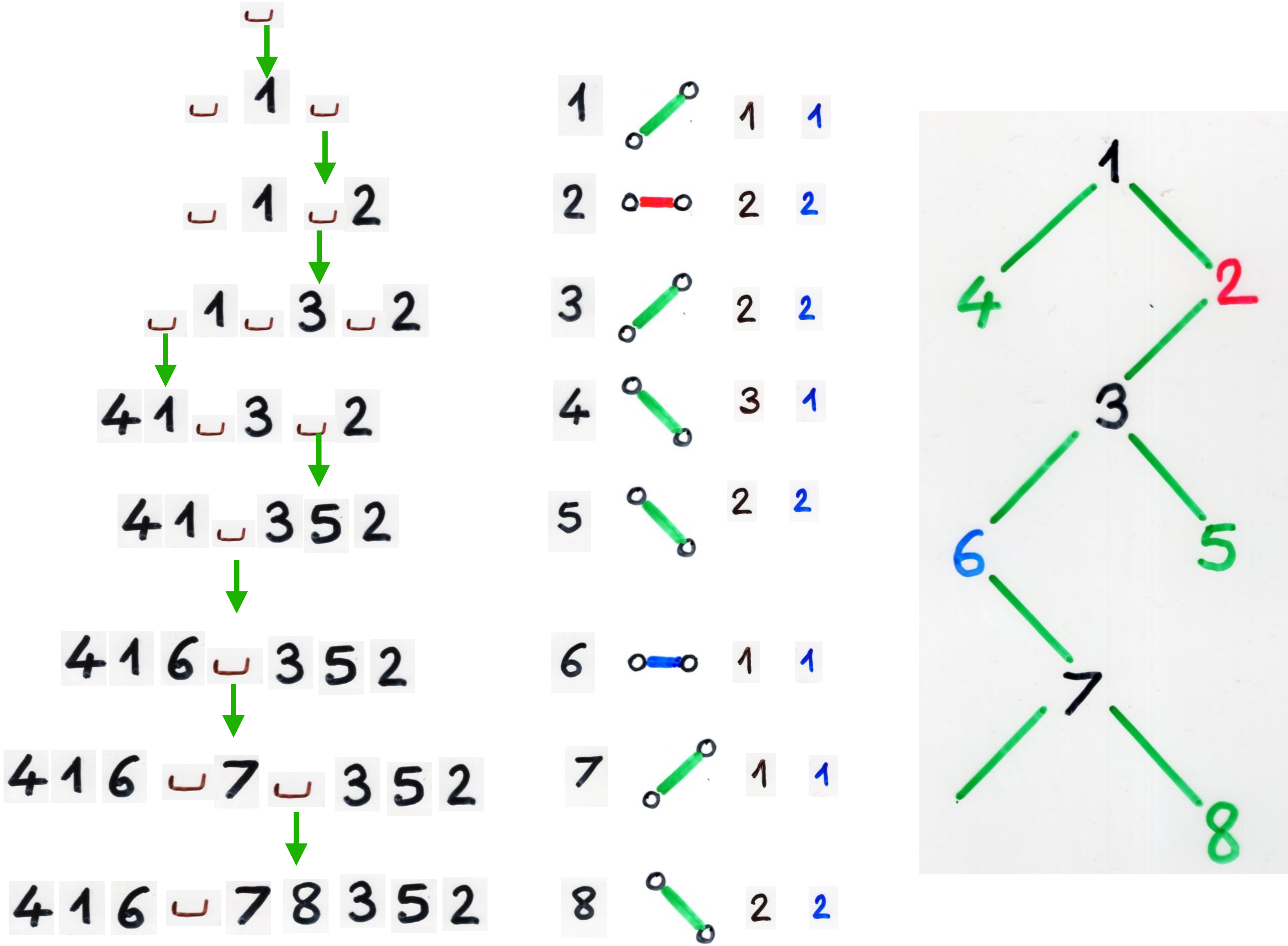
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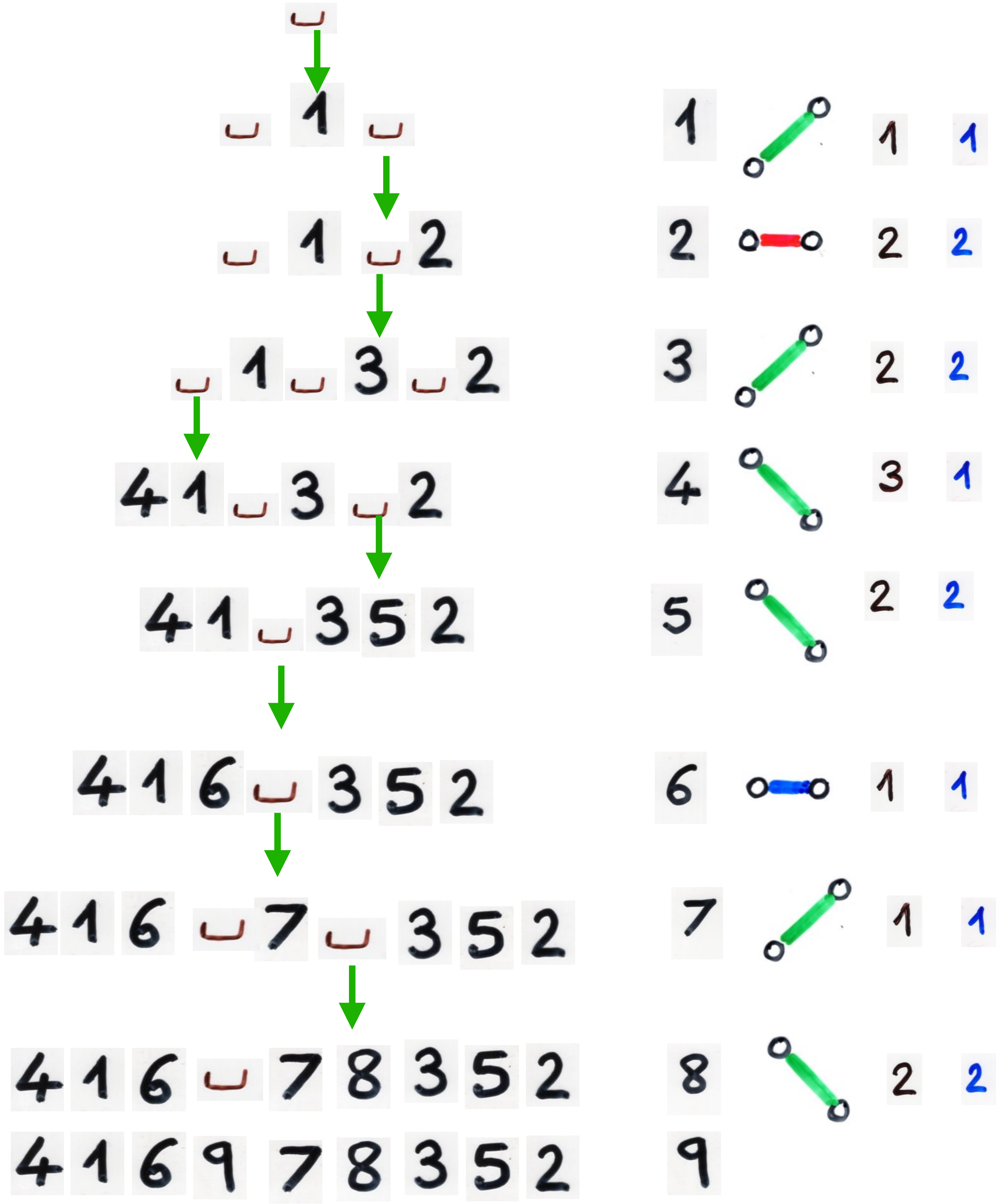
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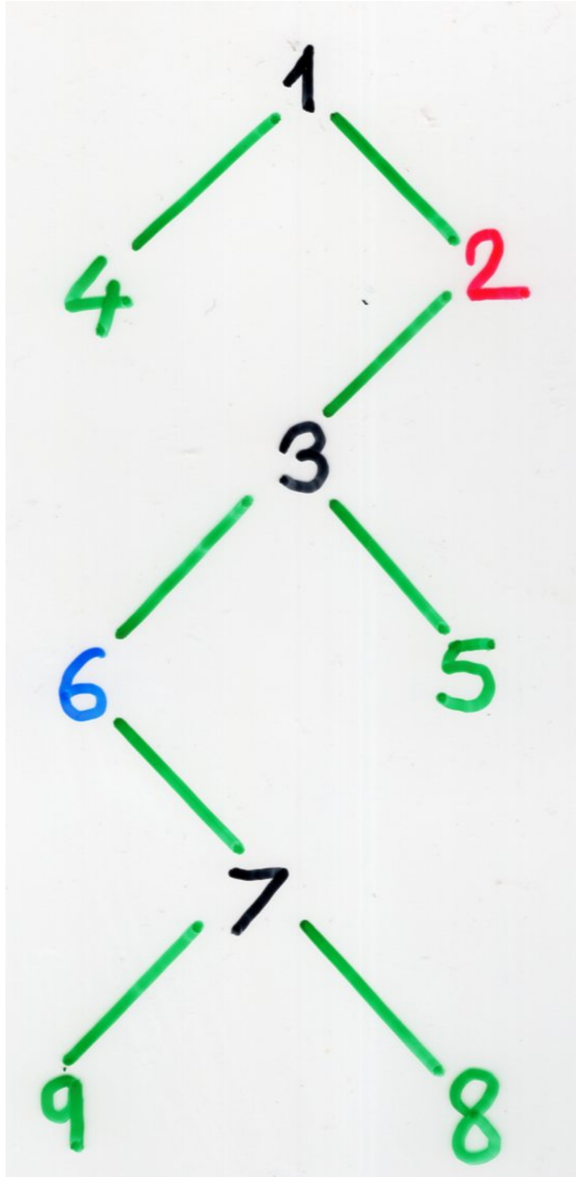
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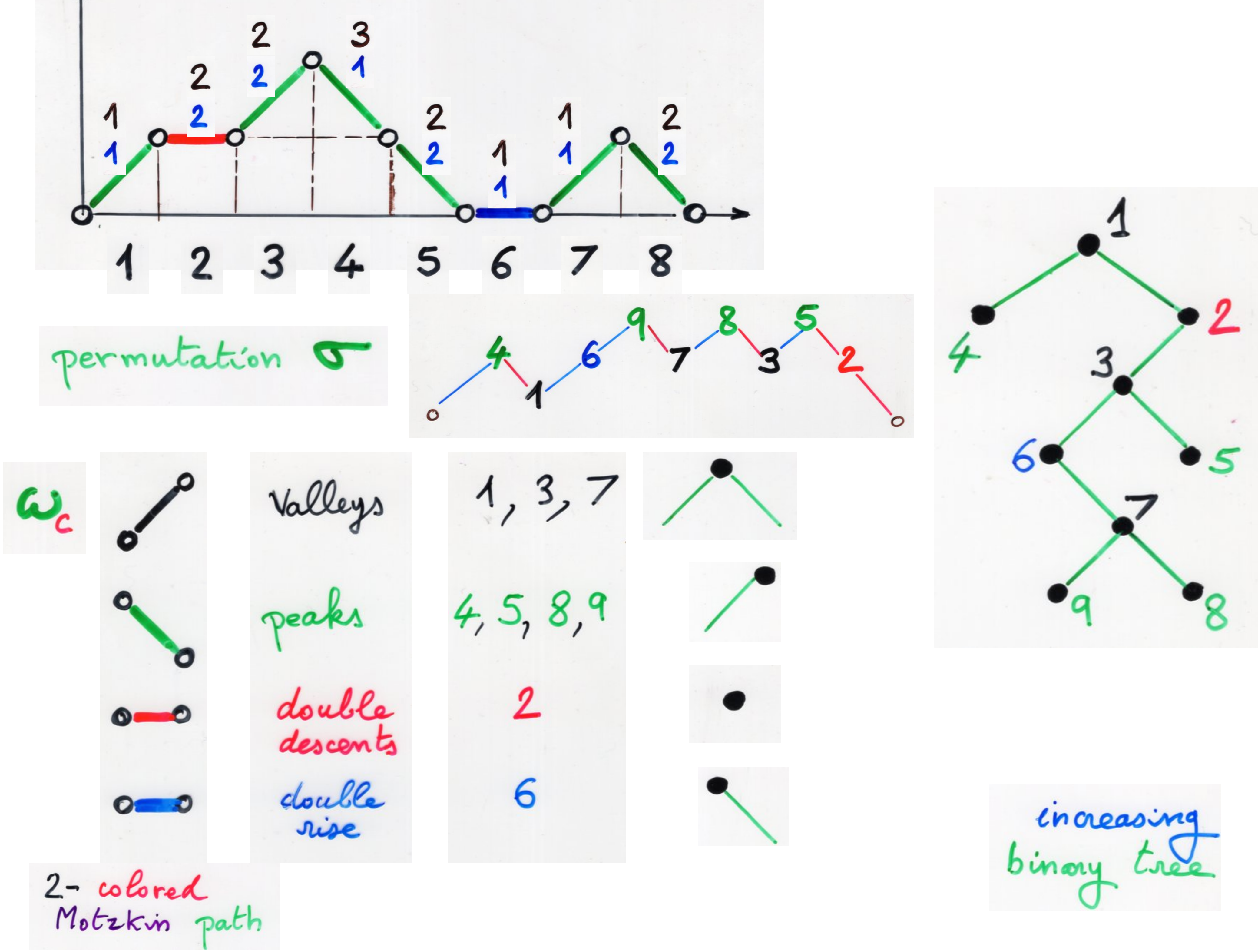


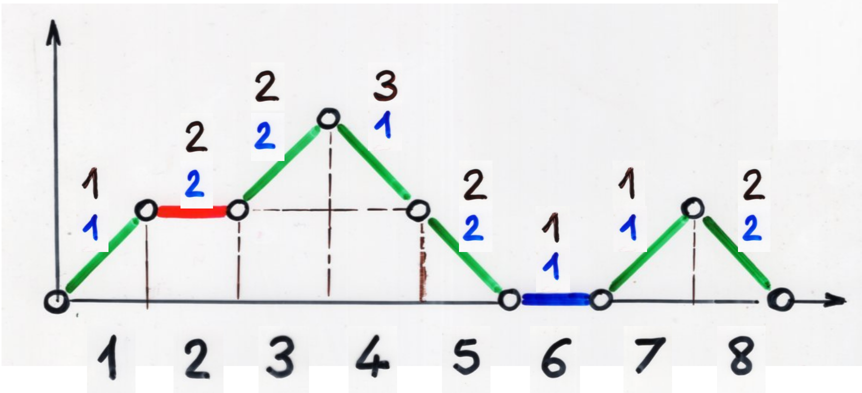
1 2 3 4 5 6 7 8
permutation σ
4 1 6 9 7 8 3 5 2
ω_c
Valleys 1, 3, 7
peaks 4, 5, 8, 9
double descents 2
double rise 6
2- colored Motzkin path
increasing binary tree

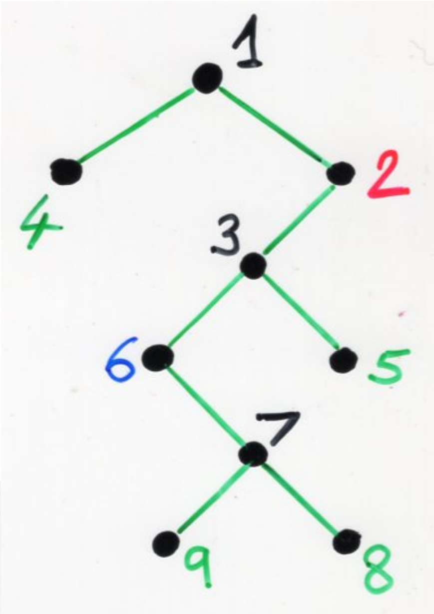
$$\mathcal{L}_n \xrightarrow{\varphi} \mathcal{T}_{n+1} \xrightarrow{\Pi} \mathfrak{S}_{n+1}$$

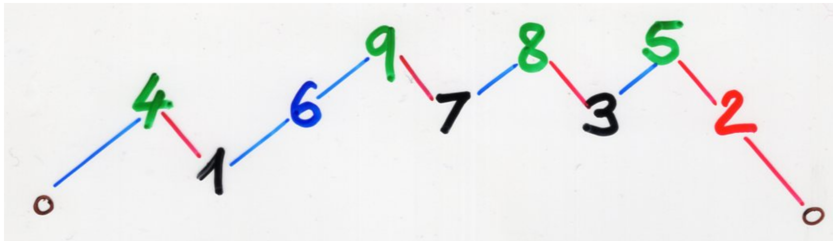
Laguerre histories — increasing binary trees — permutations

$$h = (\omega_c \,;\, (p_1, \ldots, p_n))$$

2-colored Motzkin path — choice function







Orthogonal Sheffer polynomials

Sheffer polynomials

$$\sum_{n \geqslant 0} P_n(x) \frac{t^n}{n!} = g(t) \exp\left(x f(t)\right)$$

Rota
umbral calculus

delta operator Q

$$D x^n = n x^{(n-1)}$$

Meixner (1934)

$\{P_n(x)\}_{n\geqslant 0}$ orthogonal polynomials

are Sheffer polynomials

$\Longleftrightarrow$

positive-definite OPS

Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak+b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a,b,c,d \in \mathbb{R} \\ c \geqslant 0, \quad c+d > 0 \end{cases}$

positive-definite OPS

Sheffer type $\Leftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$

(1) $a=0$, $c=0$ $b=0$ Hermite polynomials $H_n(x)$

(2) $a\neq 0$, $a^2-4c=0$ Laguerre polynomials $L_n^{(\alpha)}(x)$

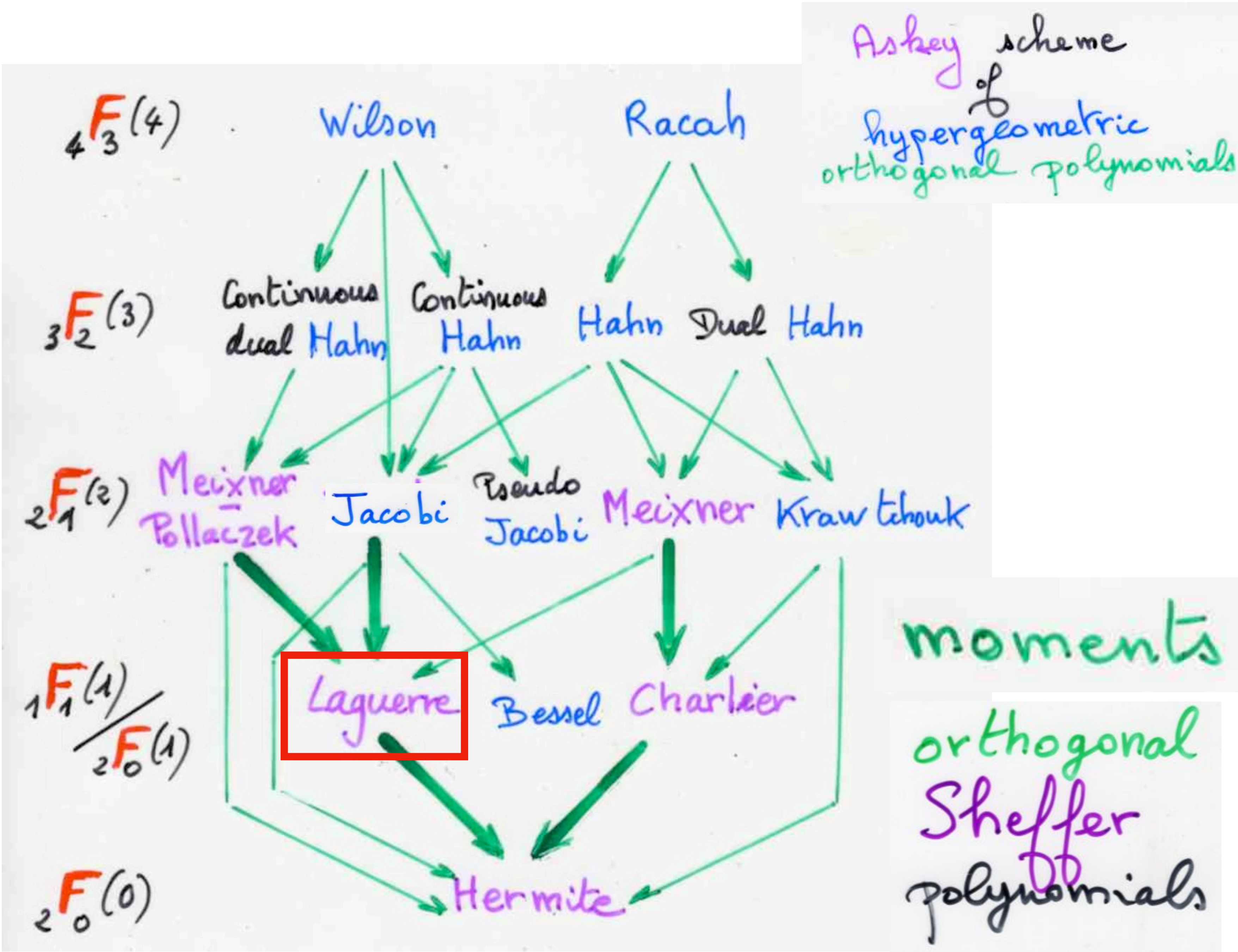
(3) $a\neq 0$, $c=0$ Charlier polynomials $C_n^{(a)}(x)$

(4) $a^2-4c>0$ Meixner polynomials $M_n^{(\beta,c)}(x)$

(5) $a^2-4c<0$ Meixner-Pollaczek polynomials $C_n^{(a)}(x)$

Hermite	$\nu(x^n) =$	$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-x^2/2} dx$
Laguerre	$\psi(x^n) =$	$\frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^n x^{\alpha} e^{-x} dx$
Charlier	$\mu(x^n) =$	$e^{-a} \sum_{x=0}^{\infty} x^n \frac{a^x}{x!}$
Meixner	$\rho(x^n) =$	$(1-c)^{\beta} \sum_{x=0}^{\infty} x^n \frac{c^x (\beta)_x}{x!}$
Meixner-Pollaczeck	$\varphi(x^n) =$	$\frac{1}{\int_{-\infty}^{+\infty} w(x) dx} \int_{-\infty}^{+\infty} x^n w(x) dx$

$$w(x) = \left[\Gamma(\eta/2)\right]^{-2} \left|\Gamma((\eta+ix)/2)\right|^2 \exp(-x \arctan \delta)$$



Askey scheme of hypergeometric orthogonal polynomials
$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner-Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
$_1F_1(1)$ / $_2F_0(1)$
Laguerre
Bessel
Charlier
$_2F_0(0)$
Hermite
moments
orthogonal
Sheffer
polynomials

Charlier histories

Charlier polynomials

$$C_n^{(a)}(x) = \sum_{0 \leq k \leq n} \binom{n}{k}\binom{x}{k} k! \, (-a)^{n-k}$$

$$\sum_{n \geqslant 0} C_n^{(a)}(x) \frac{t^n}{n!} = e^{-at}(1+t)^x$$

$$\int_0^\infty C_m^{(a)}(x)\, C_n^{(a)}(x)\, d\psi^{(a)}(x) = a^n\, n!\, \delta_{mn}$$

$\psi^{(a)}$ jumps $d\psi^{(a)}(x) = \dfrac{e^{-a} a^x}{x!}$

Charlier polynomials

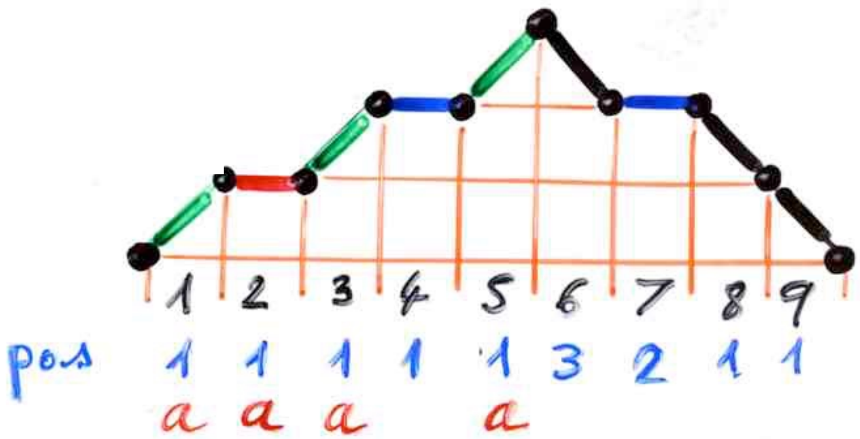
$$\begin{cases} \lambda_k = ak & (k \geq 1) \\ b_k = k + a & (k \geq 0) \end{cases}$$

$$\mu_n = \sum_{1 \leq k \leq n} S(n,k)\, a^k$$

moments

Stirling numbers (2nd kind)

Charlier histories



$$\begin{cases} b'_k = a \\ b''_k = k \end{cases}$$

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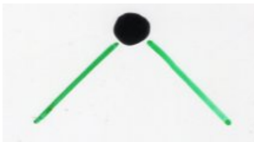

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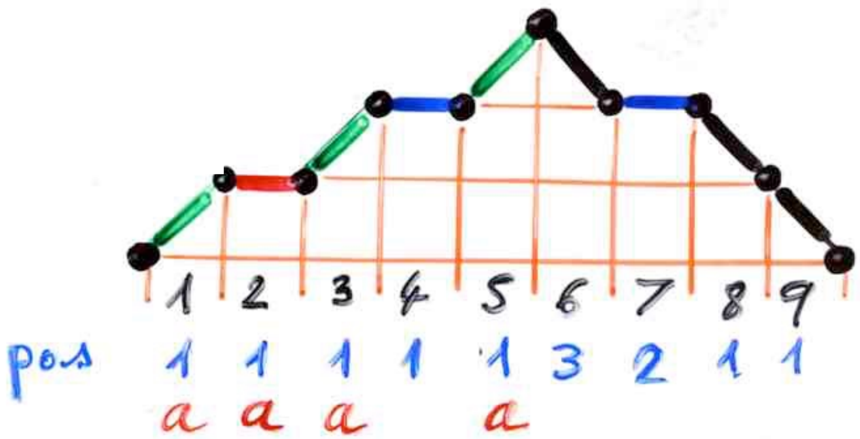

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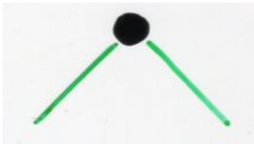

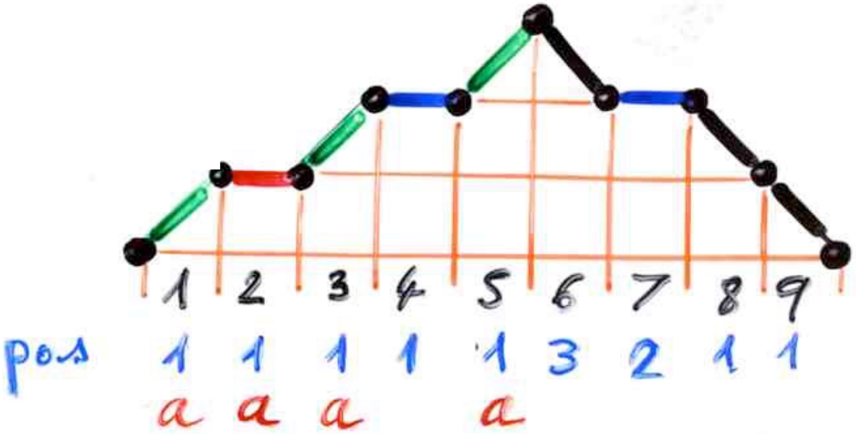

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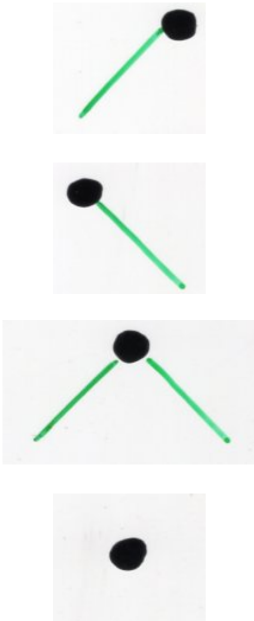

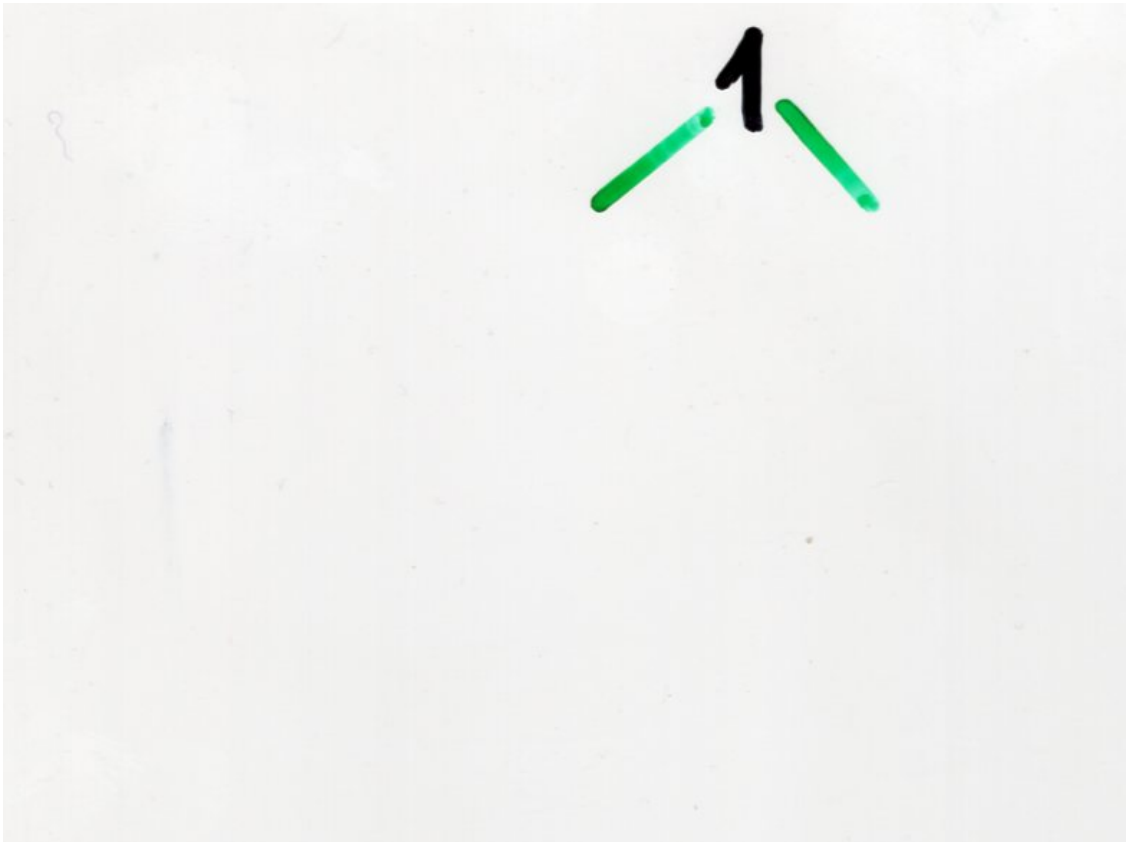

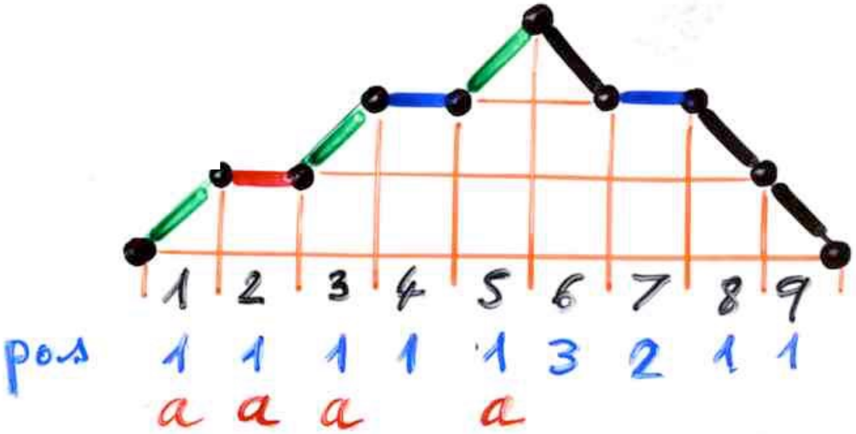

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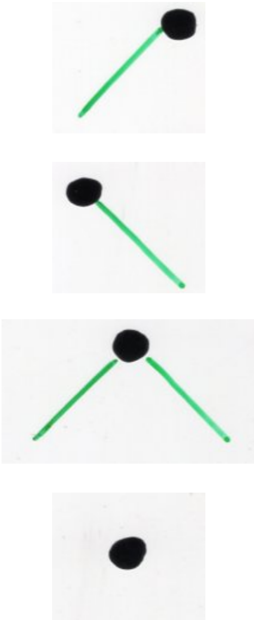
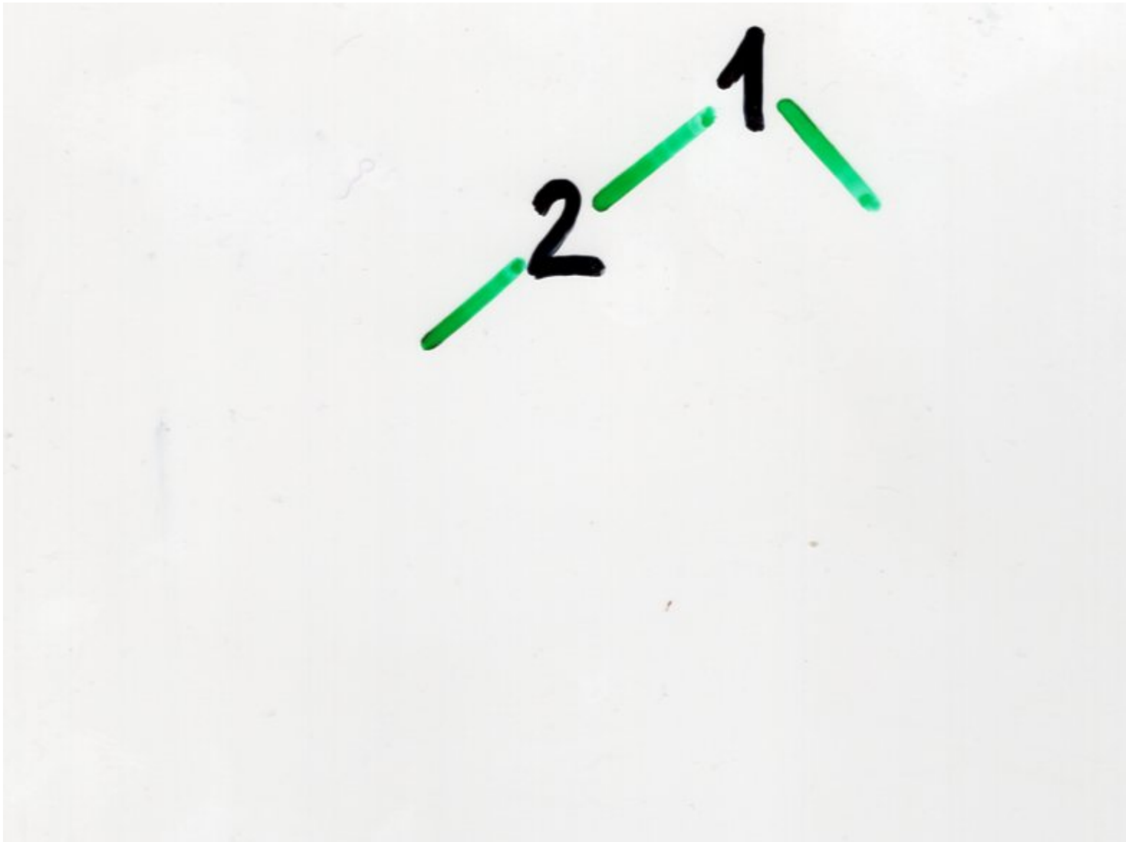
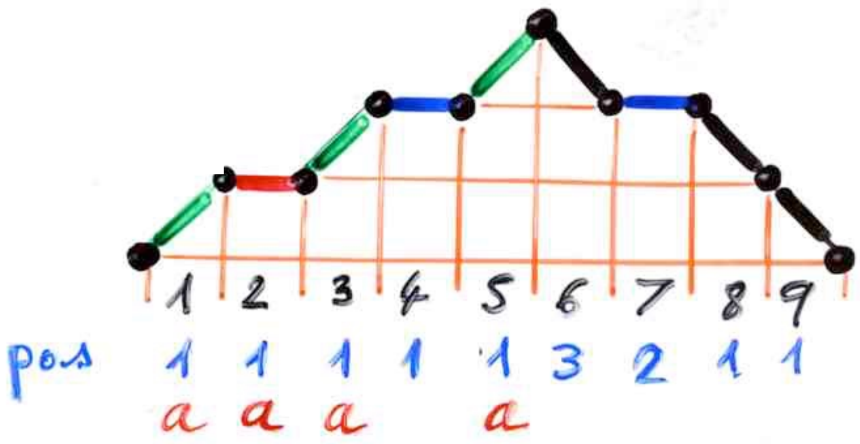




Charlier histories



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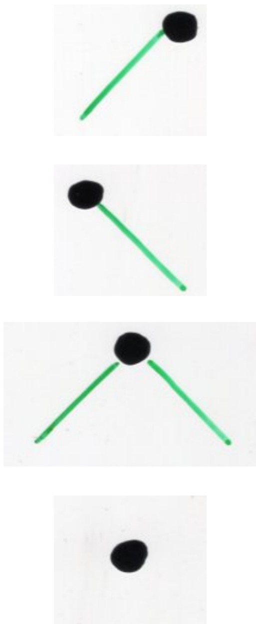
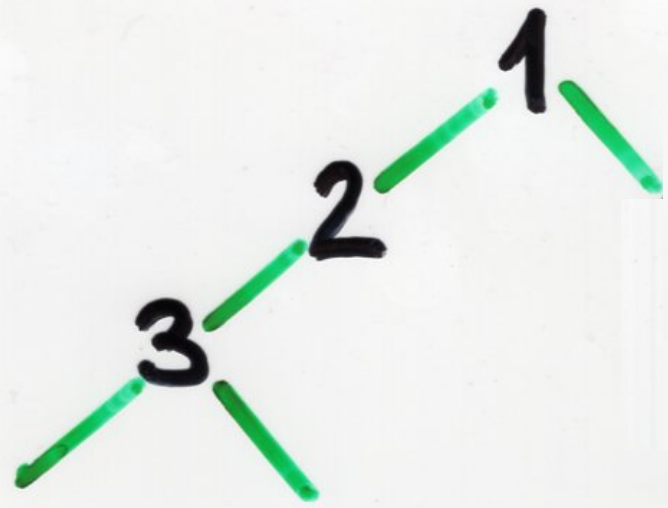
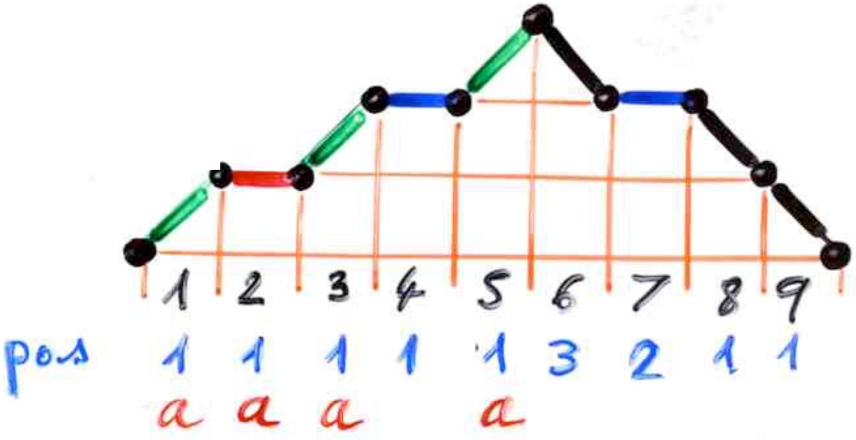




Charlier histories



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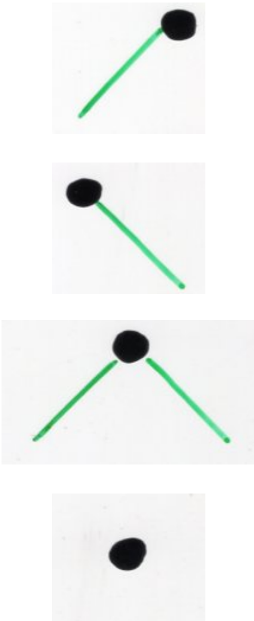
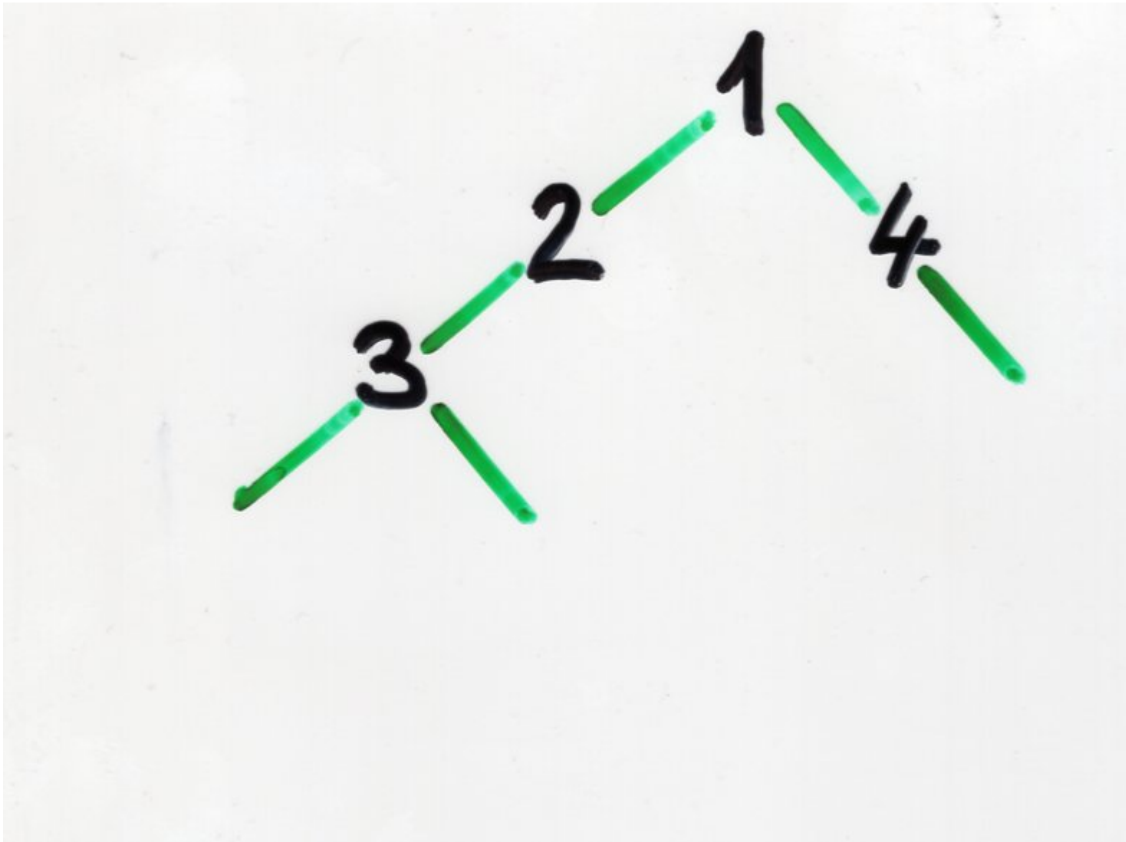
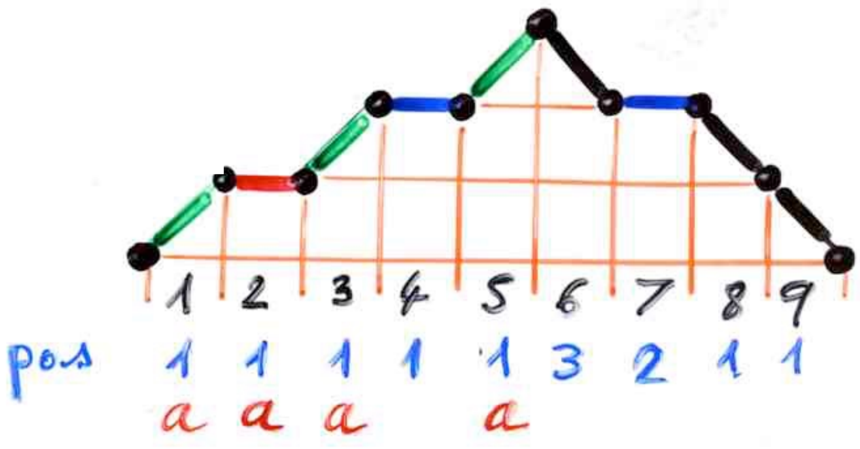




Charlier histories



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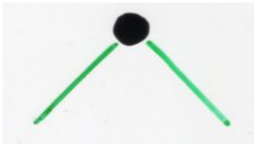

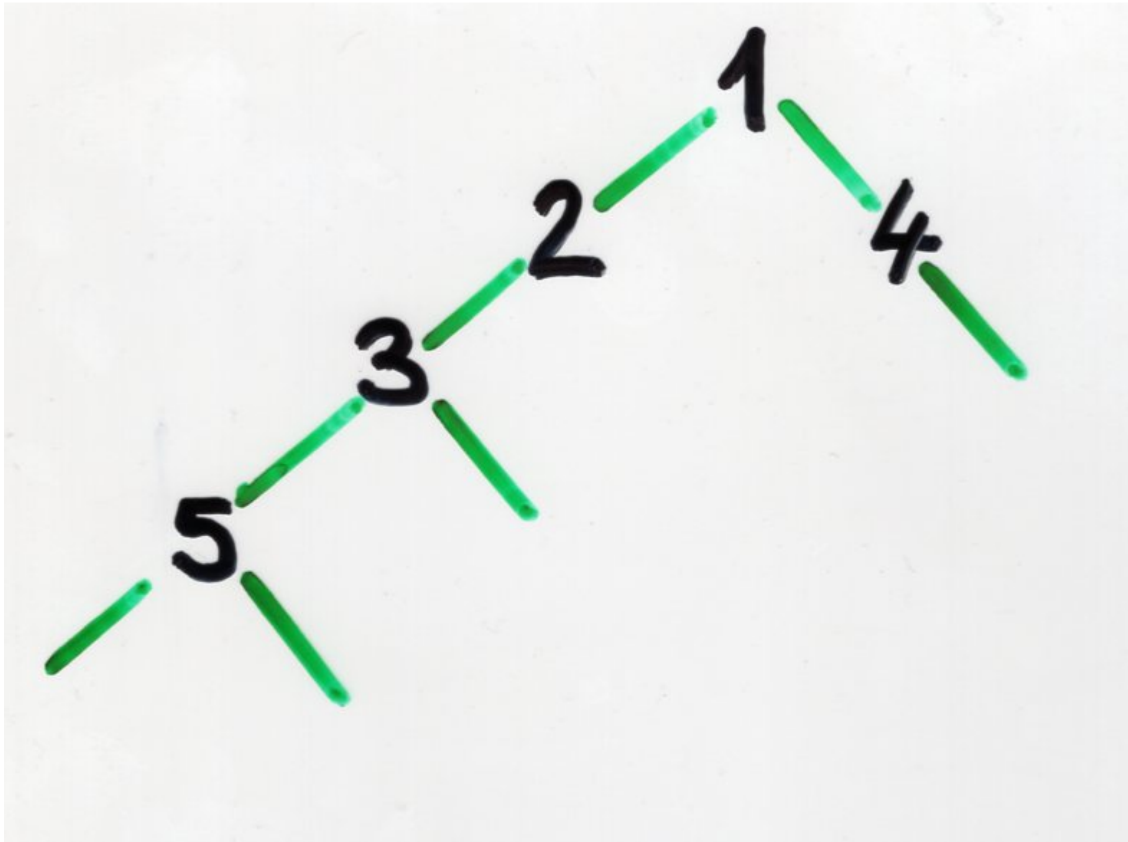

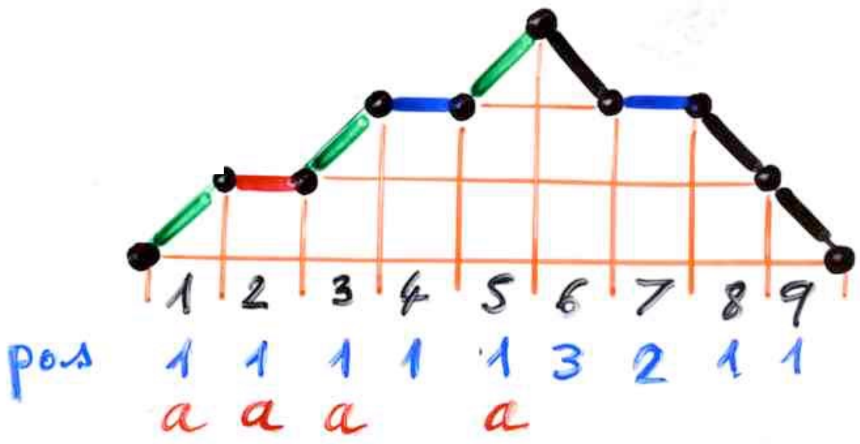

Charlier histories

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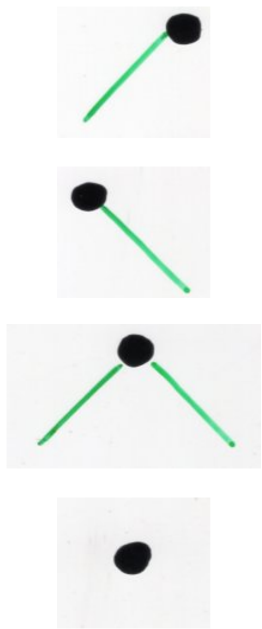

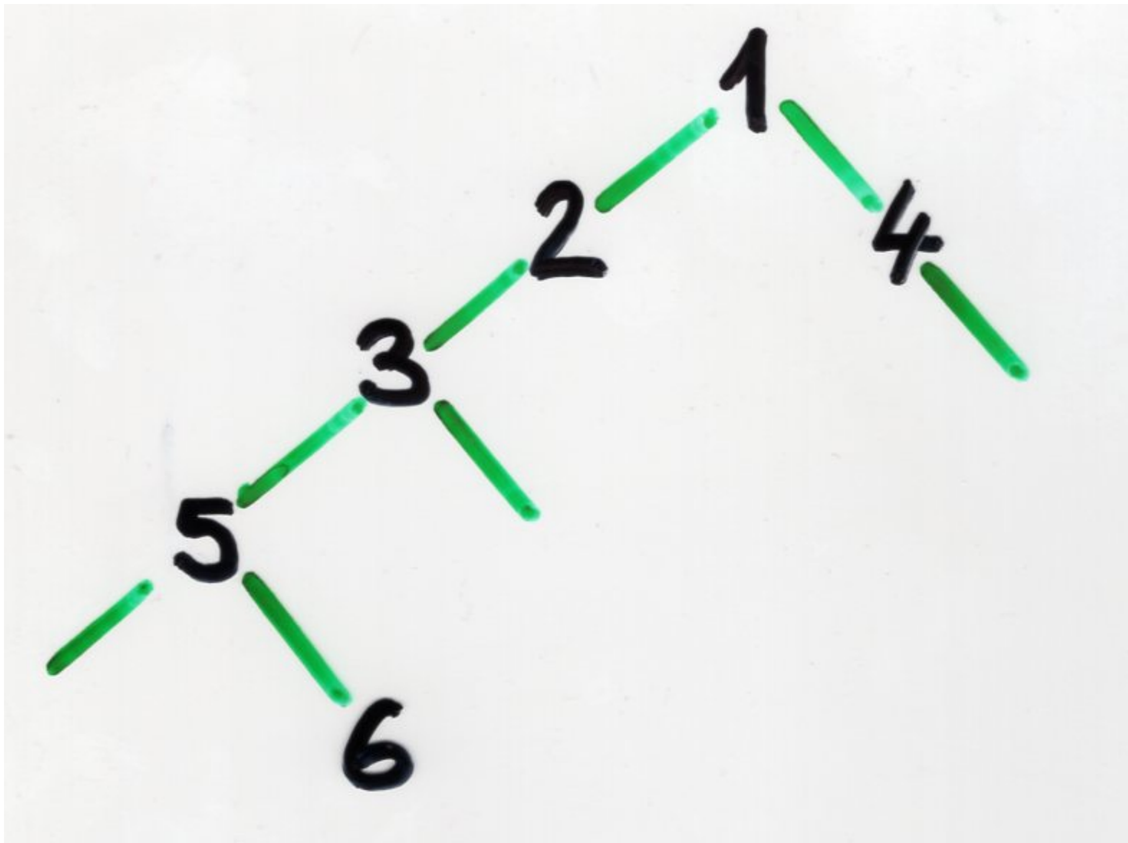

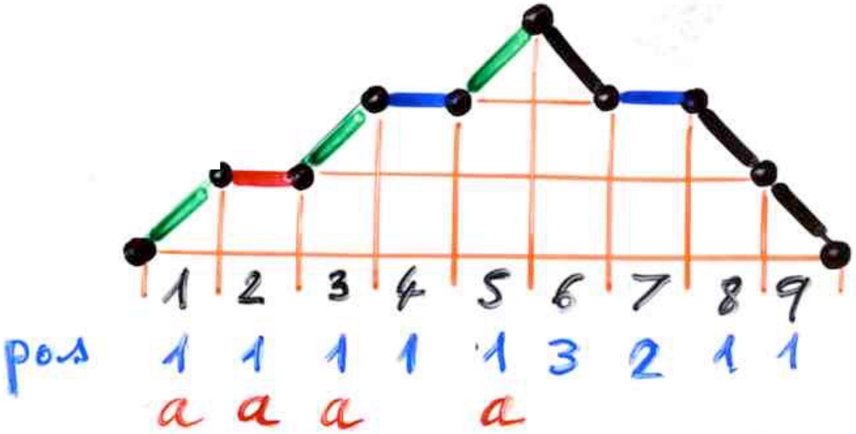

Charlier histories

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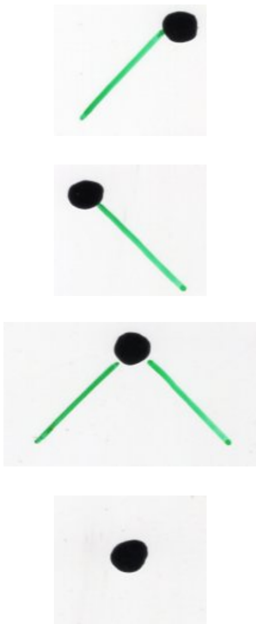

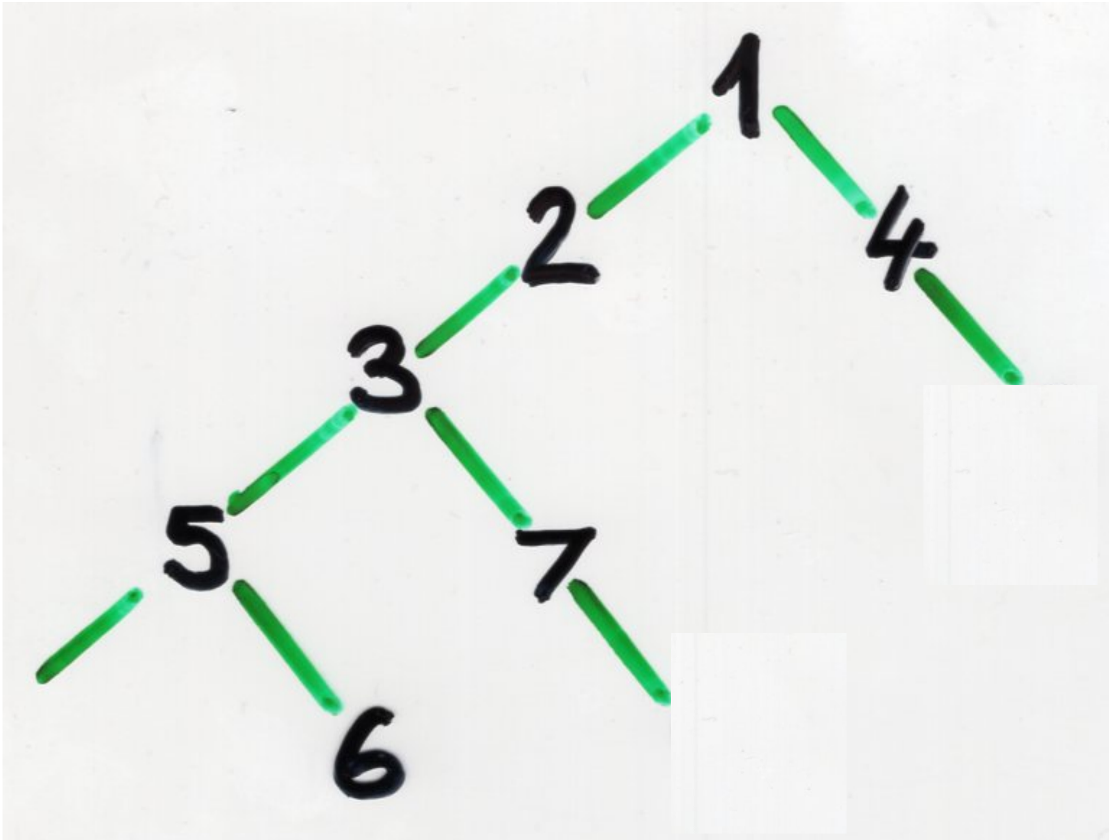

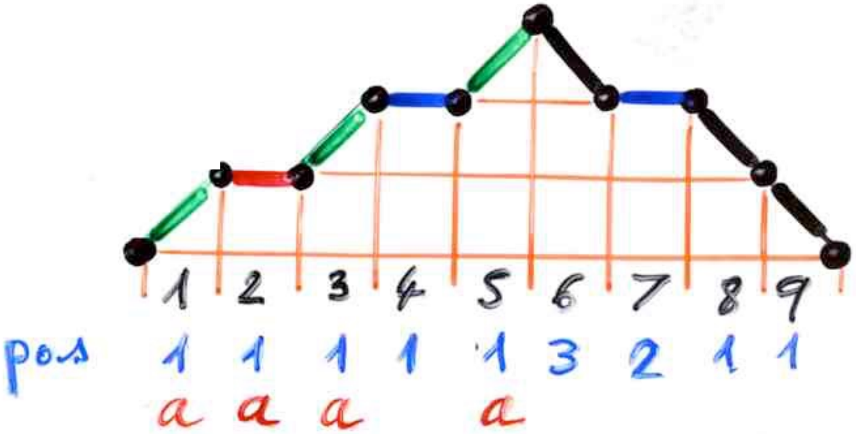

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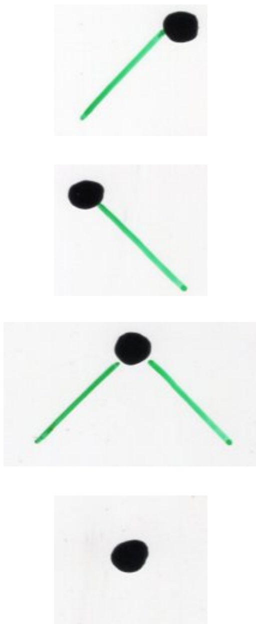
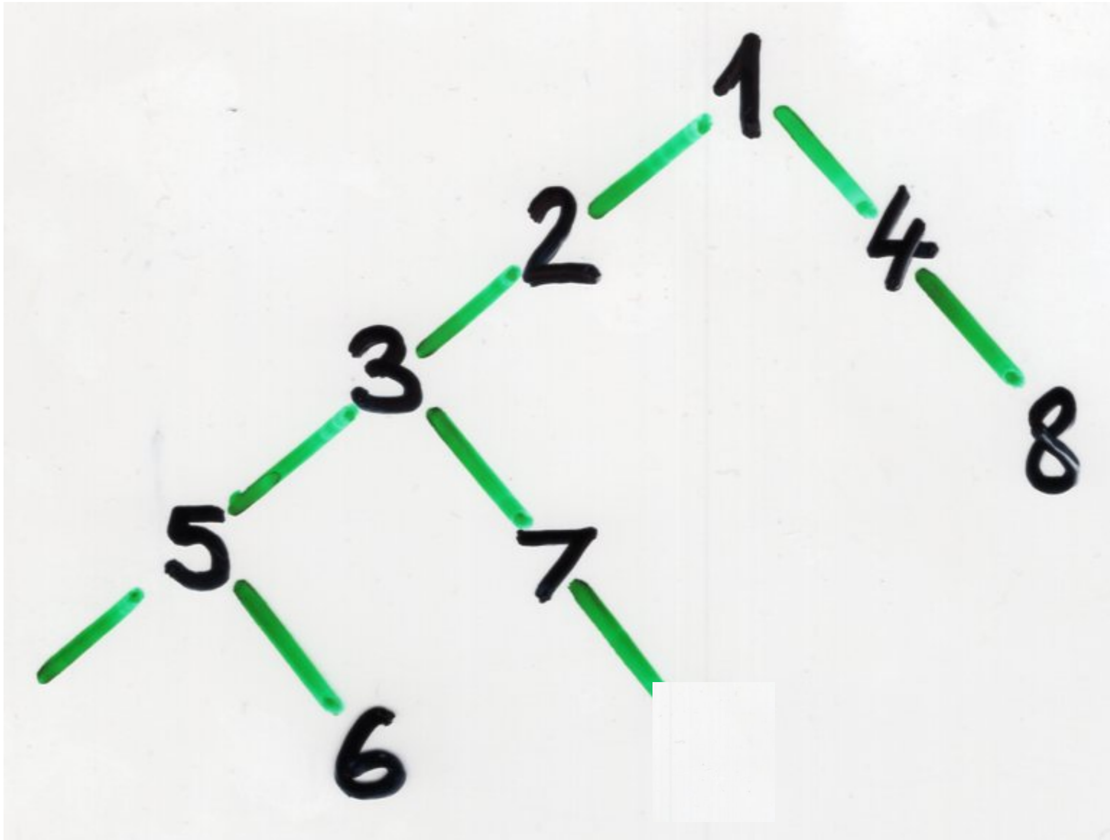
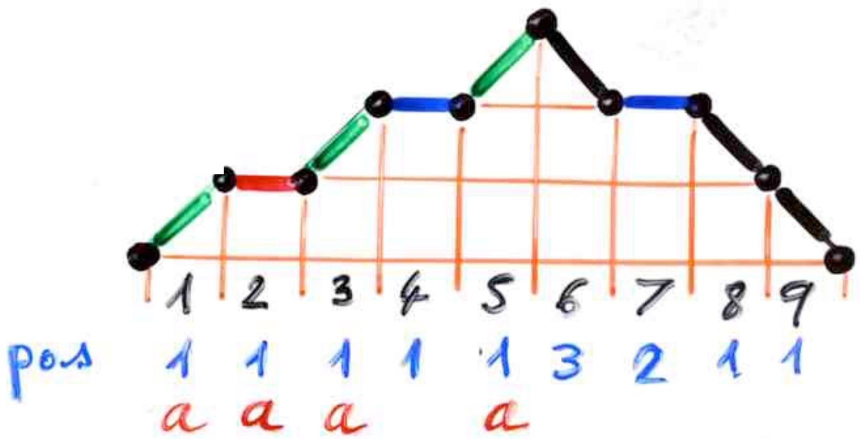




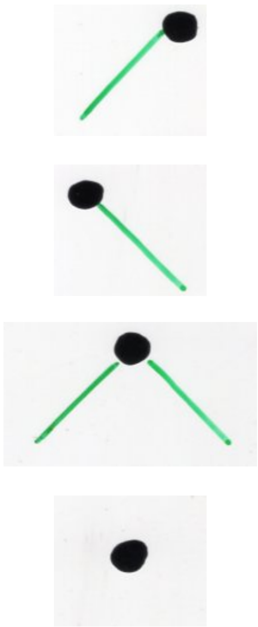


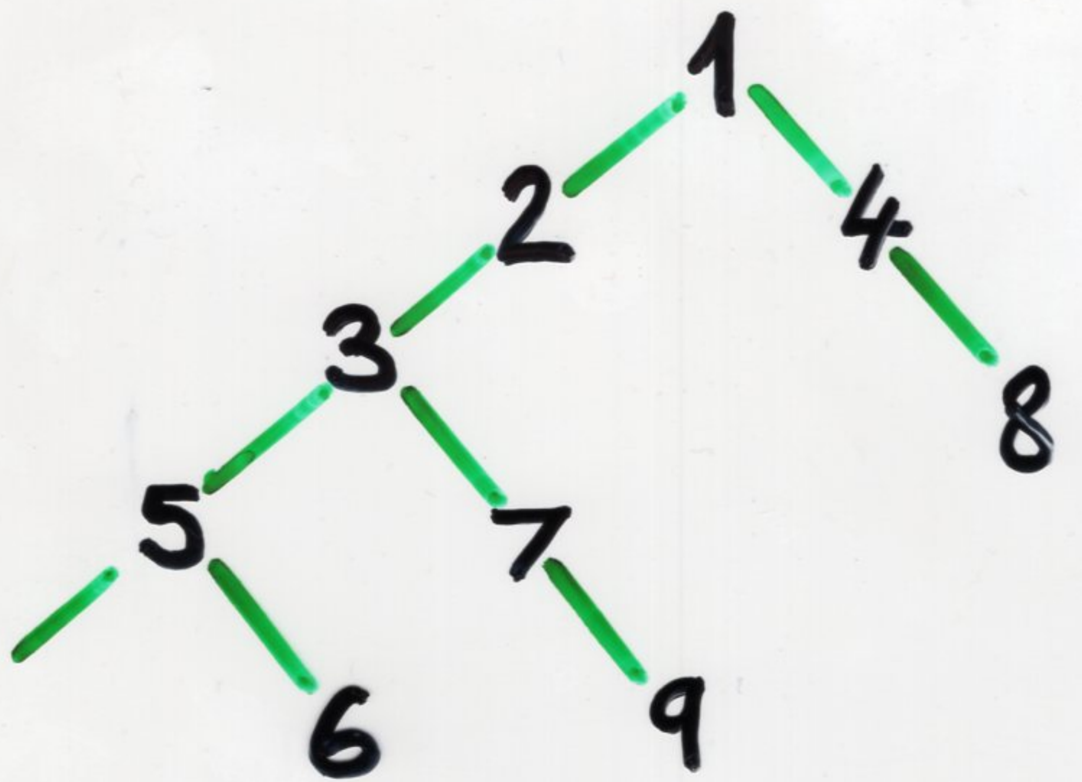
Charlier histories

$$\underset{\text{moments}}{\mu_n} = \sum_{1 \le k \le n} S(n,k)\, a^k$$

$$\begin{cases} b'_k = a \\ b''_k = k \end{cases}$$

$$\begin{cases} a_k = a \\ c_k = k \end{cases}$$

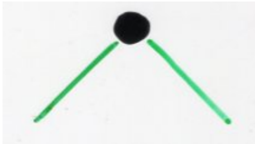


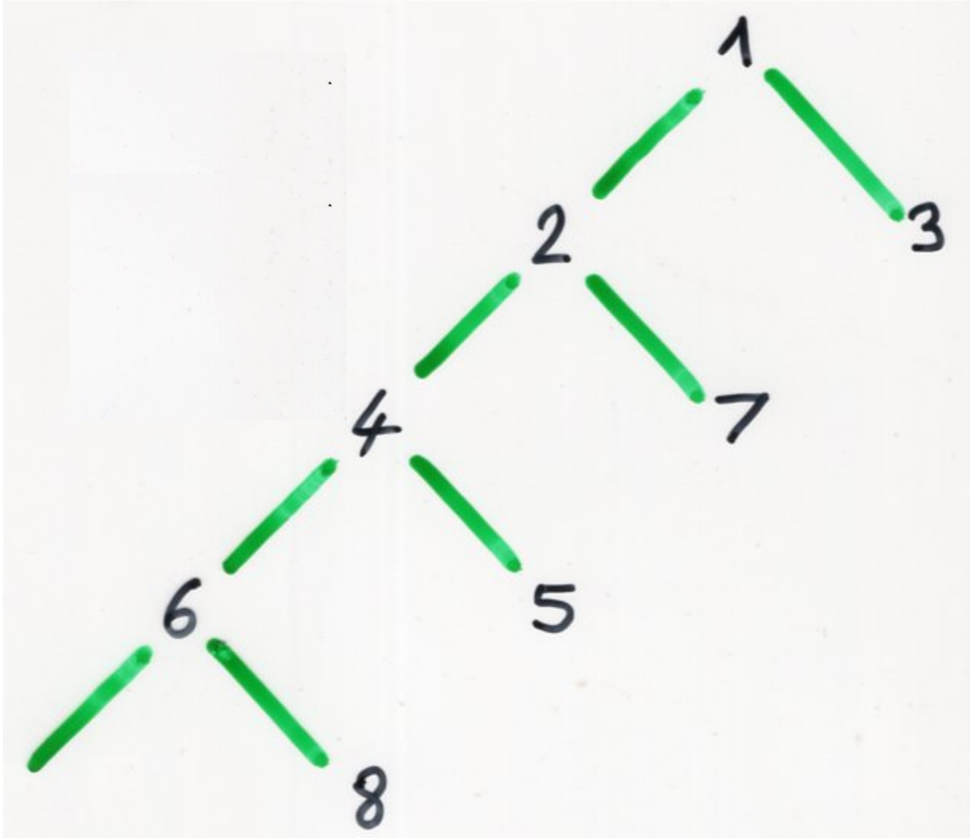


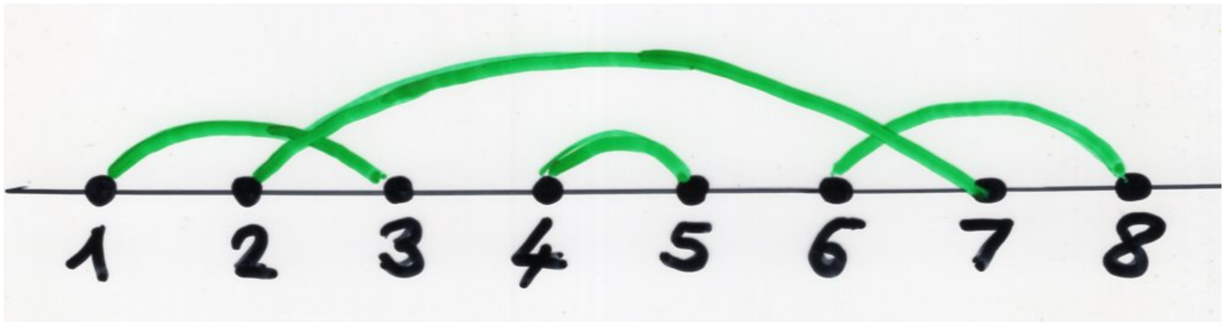
Hermite histories

$b_k = 0$

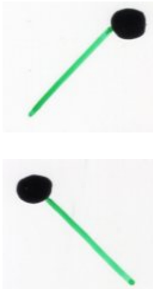
$$\begin{cases} a_k = 1 \\ c_k = k \end{cases}$$



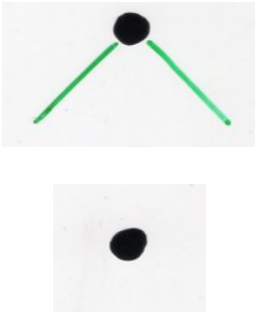


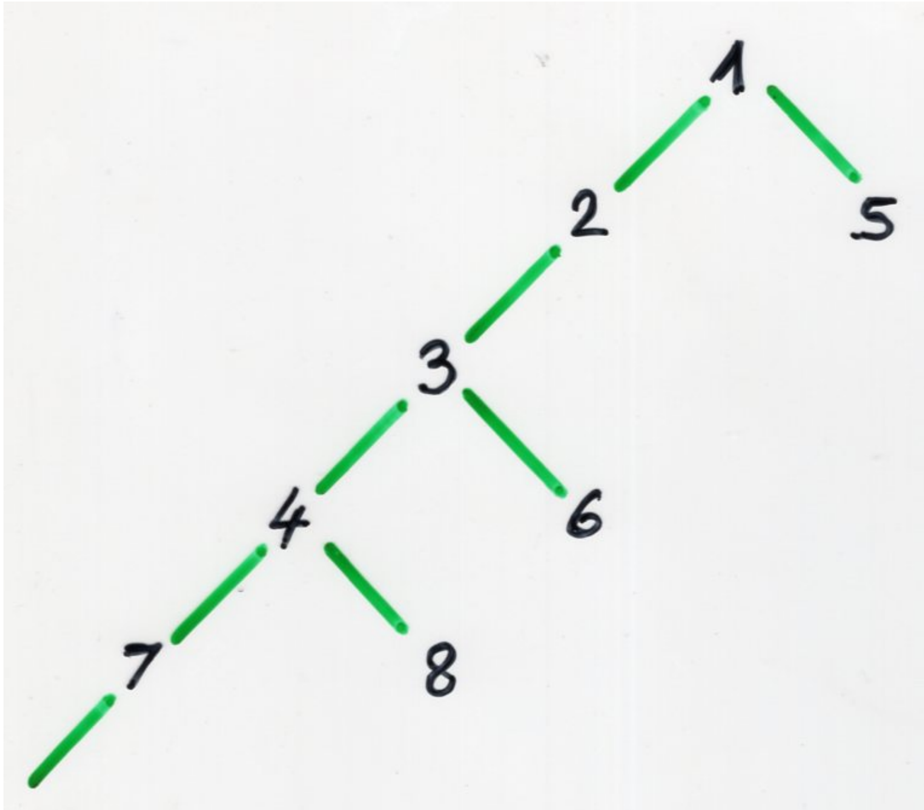


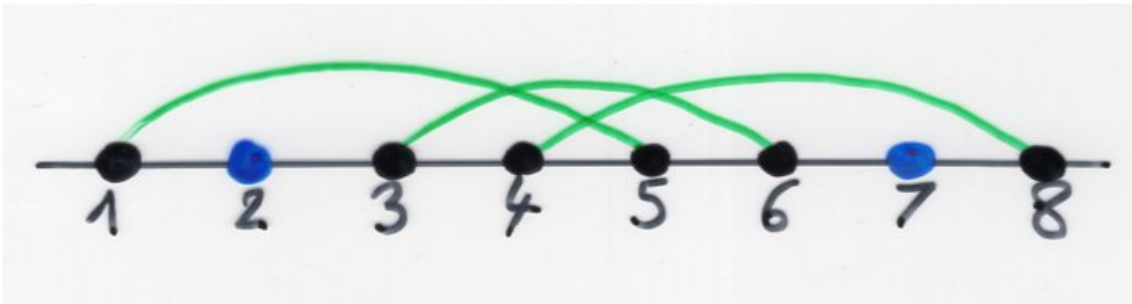
$$\begin{cases} b'_k = a \\ b''_k = 0 \end{cases}$$



$$\begin{cases} a_k = 1 \\ c_k = k \end{cases}$$







Moments of Meixner polynomials

Meixner

$$M_n(x;\beta,c) = (-1)^n n! \sum_{0\le k\le n} \binom{x}{k}\binom{-x-\beta}{n-k} c^{-k}$$

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta}$$

$$\sum_{k\geq 0} M_m(x_k;\beta,c) M_n(x_k;\beta,c)\frac{c^{-k}(\beta)_k}{k!} = \left(\frac{c}{c-1}\right)^{\beta} c^{-n}\, n!\,(\beta)_n\,\delta_{mn}$$

$$x_k = -k-\beta \quad (k\geq 0)$$

$$\tilde{M}_n(x;\beta,c) = (\beta)_n \left(\frac{c}{c-1}\right)^n M_n(x;\beta,c)$$

$$\tilde{b}_k = \frac{(1+c)k + \beta c}{(1-c)}$$

$$\tilde{\lambda}_k = \frac{ck(k+\beta-1)}{(1-c)^2}$$

$$b_k = (1+c)k + \beta c$$

$$\lambda_k = ck(k+\beta-1)$$

$$\tilde{v}(\omega) = \frac{1}{(1-c)^n} v(\omega)$$

$$|\omega| = n$$

$$b_k = (1+c)k + \beta c$$

$$\lambda_k = c k (k+\beta-1)$$

$$b'_k = c(k+\beta)$$

$$a_k = c(k+\beta)$$

$$b''_k = k$$

$$c_k = k$$

$$\sum_{|\omega|=n} v(\omega) = \sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} c^{d(\sigma)}$$

Motzkin path

$$\sum_{\substack{|\omega|=n \\ \text{Motzkin path}}} v(\omega) = \sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} c^{d(\sigma)}$$

$$\mu_n = \sum_{\substack{|\omega|=n \\ \text{Motzkin path}}} \tilde{v}(\omega)$$

$$\mu_n = \frac{1}{(1-c)^n} \sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} c^{d(\sigma)}$$

$\beta = 1$ $$\sum_{\sigma \in \mathfrak{S}_n} c^{d(\sigma)} = A_n(c)$$

Eulerian polynomials $A_n(x)$

$\beta = 1$

$$\sum_{\sigma \in \mathfrak{S}_n} c^{d(\sigma)} = A_n(c)$$

Eulerian polynomials

$A_n(x)$

$$\frac{A_n(c)}{(1-c)^n} = (1-c) \sum_{k \geq 0} k^n c^k$$

$$\mu_n(\beta, c) = (1-c)^{\beta} \sum_{k \geq 0} k^n c^k \frac{(\beta)_k}{k!}$$

Meixner

$$\mu_n(\beta, c) = (1-c)^\beta \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$$

$$\sum_{k\geq 0} M_m(x_k;\beta,c) M_n(x_k;\beta,c) \frac{c^{-k}(\beta)_k}{k!} = \left(\frac{c}{c-1}\right)^\beta c^{-n}\, n!\, (\beta)_n\, \delta_{mn}$$

$$x_k = -k-\beta \quad (k\geq 0)$$

$$\rho(x^n) = (1-c)^\beta \sum_{x=0}^{\infty} x^n \frac{c^x (\beta)_x}{x!}$$

$\beta = 1, \; c = \frac{1}{2}$

$$\begin{cases} \tilde{b}_k = 3k+1 \\ \tilde{\lambda}_k = 2k^2 \end{cases}$$

$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} 2^{d(\sigma)}$ = number of ordered partitions of $\{1, 2, \dots, n\}$

<u>exercise</u> direct proof by constructing a bijection between ordered partitions and some histories associated to weighted colored Motzkin paths with weight $\tilde{b}_k = 3k+1$, $\tilde{\lambda} = 2k^2$

$c = \frac{1}{2}$ $\quad \tilde{b}_k = 3k + \beta$

$\tilde{\lambda}_k = 2k(k+\beta-1)$

parameter β : number of blocks ?

Moments of Meixner-Pollaczek polynomials

Meixner–Pollaczeck

$$\sum_{n\geqslant 0} P_n(x;\eta,\delta)\frac{t^n}{n!} = \left[(1+\delta t)^2+t^2\right]^{-\eta/2}\exp\left[x\arctan\left(\frac{t}{1+\delta t}\right)\right]$$

$$\delta\in\mathbb{R},\ \eta>0$$

$$\int_{-\infty}^{\infty} P_m(x;\eta,\delta)P_n(x;\eta,\delta)\,w(x)\,dx = (\delta^2+1)^n\, n!\,(\eta)_n\int_{-\infty}^{\infty} w(x)\,dx\;\delta_{mn}$$

$$w(x;\eta,\delta) = \left[\Gamma(\eta/2)\right]^{-2}\left|\Gamma\left(\frac{\eta+ix}{2}\right)\right|^2\exp(-x\arctan\delta)$$

$$\begin{cases} b_k = (2k+\eta)\delta \\ \lambda_k = (1+\delta^2)\,k\,(k+\eta-1) \end{cases}$$

$$\begin{cases} b'_k = \delta(k+\eta) \\ b''_k = \delta k \end{cases}$$

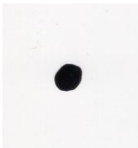
double descent

double rise

$$b_k = b'_k + b''_k$$

$$\begin{cases} a_k = (1+\delta^2)(k+\eta) \\ c_k = k \end{cases}$$

valley



$$\lambda_k = a_{k-1}\,c_k$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \, \delta^{dr(\sigma)+dd(\sigma)} (1+\delta^2)^{v(\sigma)}$$

$s(\sigma)$ = number of lr-min elements of σ

$dr(\sigma)$ = number of double rises of σ

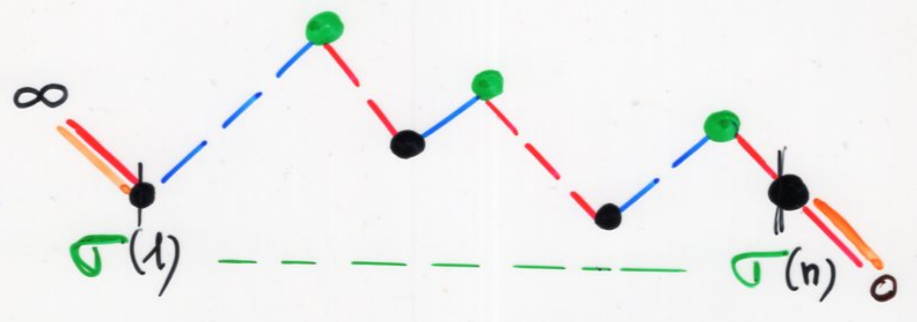
$dd(\sigma)$ = number of double descents of σ

$v(\sigma)$ = number of valleys of σ

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \delta^{dr(\sigma)+dd(\sigma)} (1+\delta^2)^{v(\sigma)}$$

$$= \delta^n \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{v(\sigma)}$$

$$n = dr(\sigma) + dd(\sigma) + 2\, v(\sigma)$$



Meixner-Pollaczeck

$$\varphi(x^n) = \frac{1}{\int_{-\infty}^{+\infty} w(x)\,dx} \int_{-\infty}^{+\infty} x^n w(x)\,dx$$

$$w(x) = \left[\Gamma(\eta/2)\right]^{-2} \left|\Gamma((\eta+ix)/2)\right|^2 \exp(-x \arctan \delta)$$

$$\mu_n = \delta^n \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \left(1 + \frac{1}{\delta^2}\right)^{v(\sigma)}$$

special case

$\delta = 0, \ \eta = 1$

$$\begin{cases} b_k = 0 \\ \lambda_k = k^2 \end{cases}$$

$\mu_{2n} = E_{2n}$

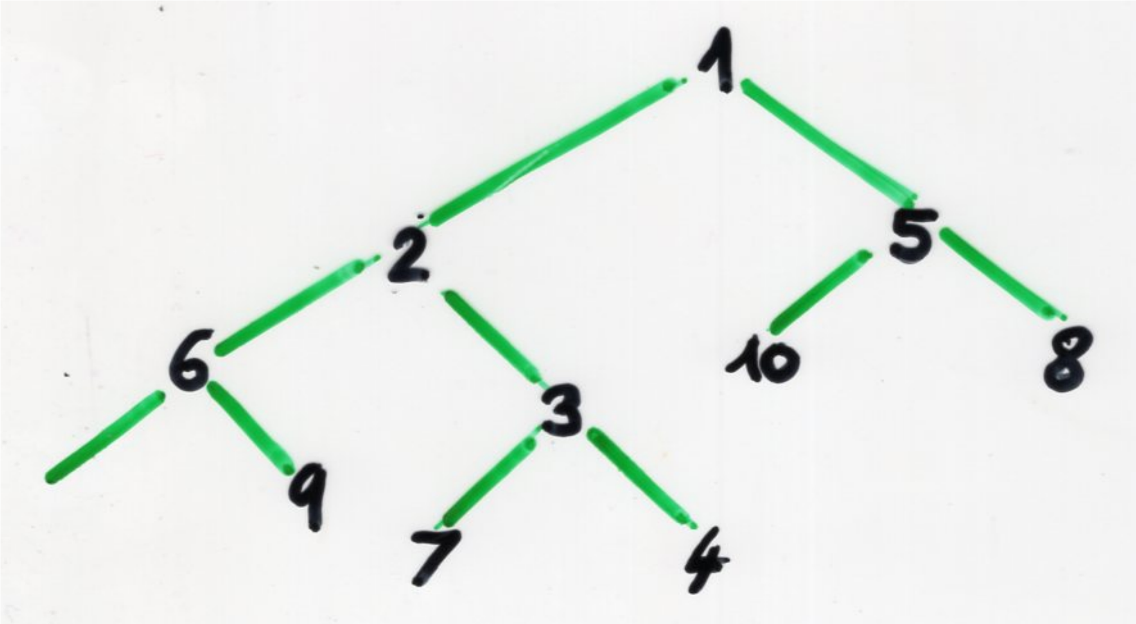
$\mu_{2n+1} = 0$

number of alternating permutations on $\{1, \ldots 2n\}$

Euler numbers

$$\sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!} = \frac{1}{\cos t}$$

$\sec t$



$$\sigma = 6\;9\;2\;7\;3\;4\;1\;(10)\;5\;8$$

Moments of the five Sheffer orthogonal polynomials

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$			$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{(a)}(x)$			$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta, c; x)$			$\sum_{\sigma\in\mathfrak{S}_n} \frac{\beta^{s(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Meixner Pollaczek $P_n(\delta,\eta;x)$			$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1}=0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\displaystyle\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\dfrac{(1+c)k+\beta c}{(1-c)}$	$\dfrac{ck(k-1+\beta)}{(1-c)^2}$	$\displaystyle\sum_{\sigma\in\mathfrak{S}_n} \frac{\beta^{s(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\displaystyle\delta^n\sum_{\sigma\in\mathfrak{S}_n}\eta^{s(\sigma)}\left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n =$ $(\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\dots\times(2n-1)$ $\mu_{2n+1}=0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\frac{(1+c)k+\beta c}{(1-c)}$	$\frac{ck(k-1+\beta)}{(1-c)^2}$	$=(1-c)^{\beta}\sum_{k\geq 0} k^n c^n \frac{(\beta)_k}{k!}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n\sum_{\sigma\in\mathfrak{S}_n}\eta^{s(\sigma)}\left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
$M_n(\beta, c; x)$ (Kreweras) $\beta=1,\ c=\frac{1}{2}$	$3k+1$	$2k^2$	number of ordered partitions on $[1,n]$
$P_n(\delta, \eta; x)$ $\delta=0,\ \eta=1$	0	k^2	$\mu_{2n} = E_{2n}$ secant number (Euler)

Moments of the general formal Sheffer orthogonal polynomials

positive-definite OPS

$$\text{Sheffer type} \Longleftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$$

$$\text{with } \begin{cases} a, b, c, d \in \mathbb{R} \\ c \geq 0, \quad c+d > 0 \end{cases}$$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma\in\mathfrak{S}_n} a^{v(\sigma)}\, b^{p(\sigma)}\, c^{dr(\sigma)}\, d^{dd(\sigma)}\, \alpha^{f(\sigma)}\, \beta^{s(\sigma)}$$

$s(\sigma)$ = number of lr-min elements of σ

$f(\sigma)$ = number of lr-min elements which are a descent of σ

$dr(\sigma)$ = number of double rises of σ

$dd(\sigma)$ = number of double descents of σ

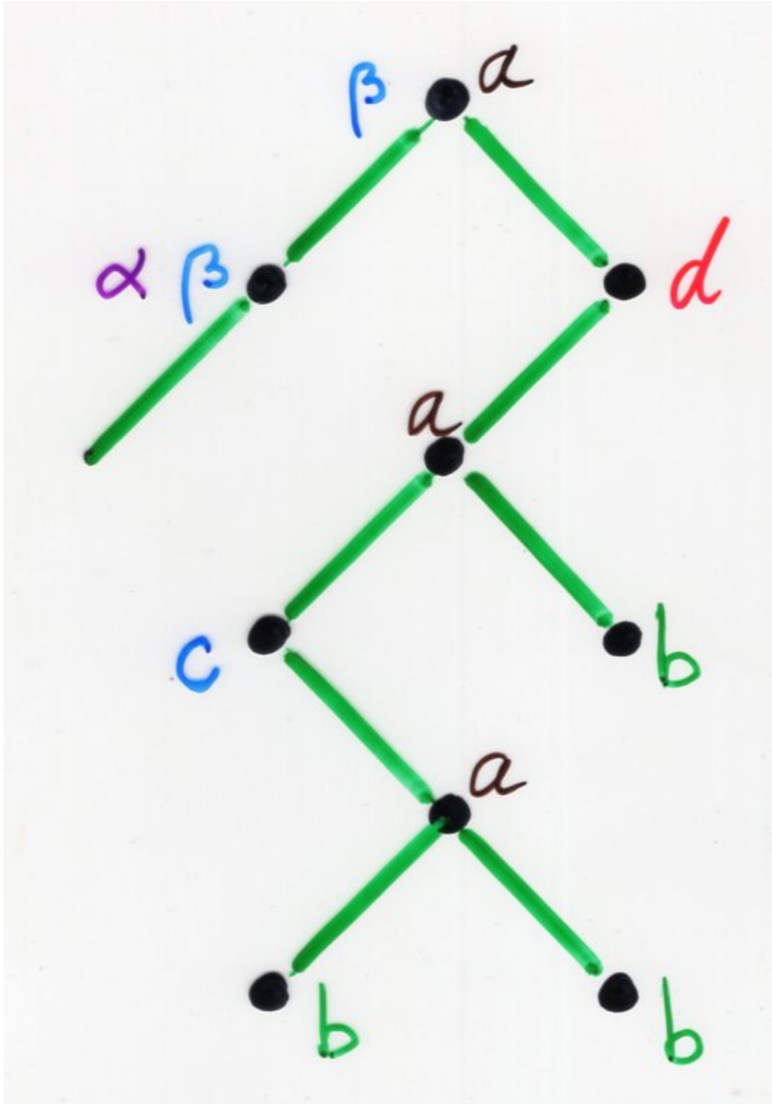
$v(\sigma)$ = number of valleys of σ

$p(\sigma)$ = number of peaks of σ

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

$$\begin{cases} b'_k = \alpha\beta + kd \\ b''_k = kc \end{cases} \qquad \begin{cases} a_k = (k+\beta)a \\ c_k = kb \end{cases}$$


β
a
α β
d
a
c
b
a
b
b