Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 6

Extensions, complements tableaux for the 2-PASEP algebra

IMSc, Chennai

15 March, 2018

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"The cellular ansatz" Ch 1, 2, 3, 4, 5

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

$UD = qDU + Id$

permutations

RSK

pairs of Young tableaux

towers placements

Physics

$DE = qED + E + D$

alternative tableaux

EXF

"Laguerre histories"

permutations

commutations

data structures "histories"

orthogonal polynomials

rewriting rules

ASM alternating sign matrices

tilings

?

planarization

"planar automata"

non-crossing paths

8-vertex model

quadratic algebra $\mathbf{Q}$

generators

$\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

for every $i \in I$, $j \in J$

$\mathcal{A} \cap \mathcal{B} = \emptyset$

$$B_j A_i = \sum_{k\ell} c_{ij}^{k\ell} A_k B_\ell$$

commutations

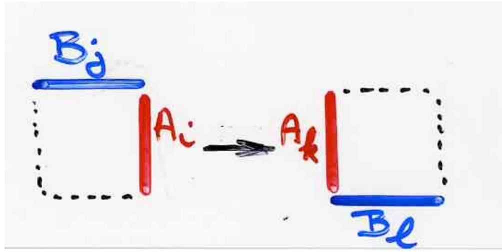
<u>Lemma</u> In $\mathbf{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

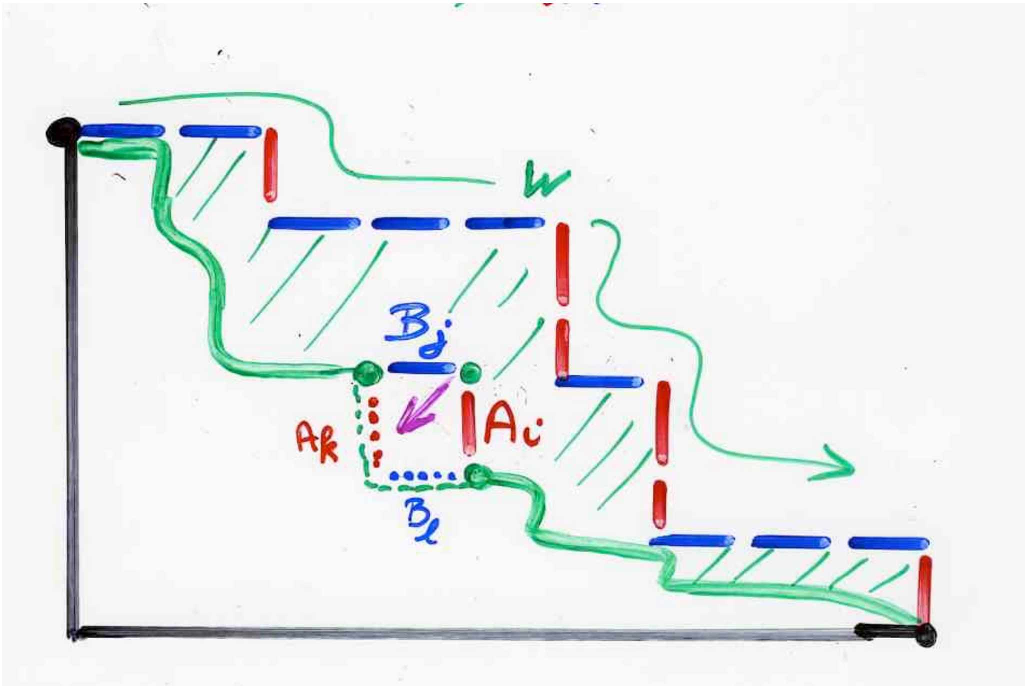
$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

normal ordering in physics

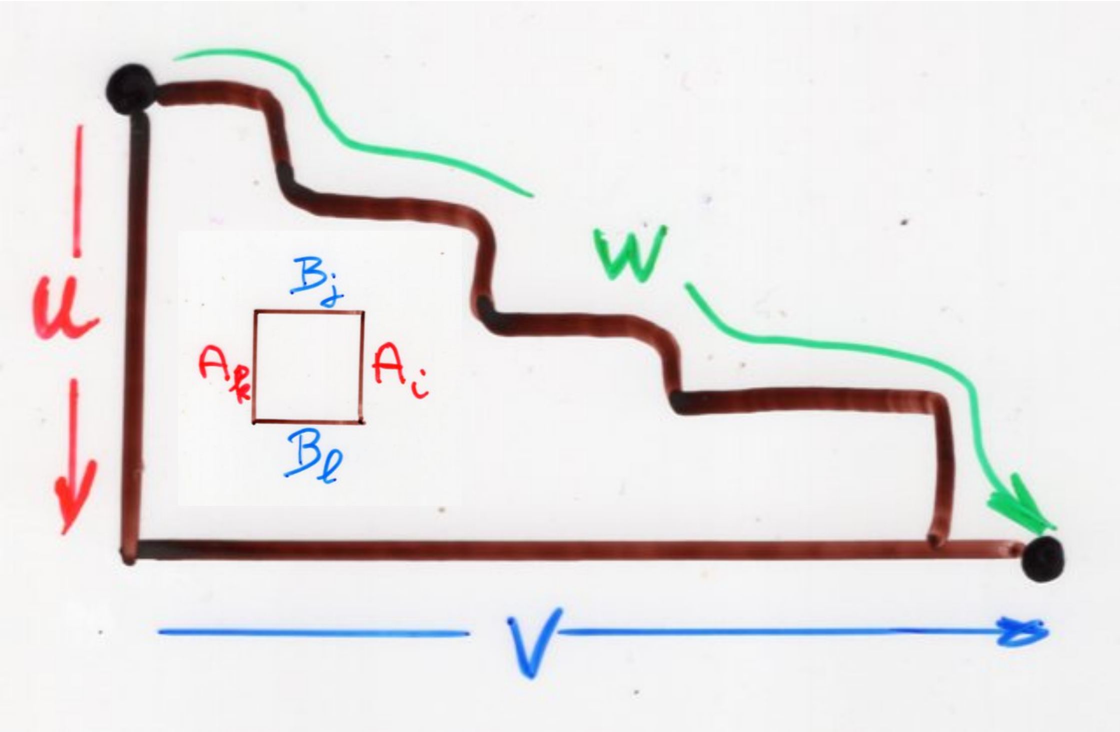
$B_j A_i \longrightarrow c^{kl}_{ij} \, A_k B_l$ rewriting rules

planarization of the rewriting rules





complete Q-tableau



$$c(u,v;w)=\sum_{T} wgt(T)$$

T complete Q-tableau

$uwb(T) = w$

$lwb(T) = uv$

L set of "labels"

$$\varphi : \left\{ \boxed{\begin{matrix} k & l \\ i & j \end{matrix}} \right\} = R \longrightarrow L$$

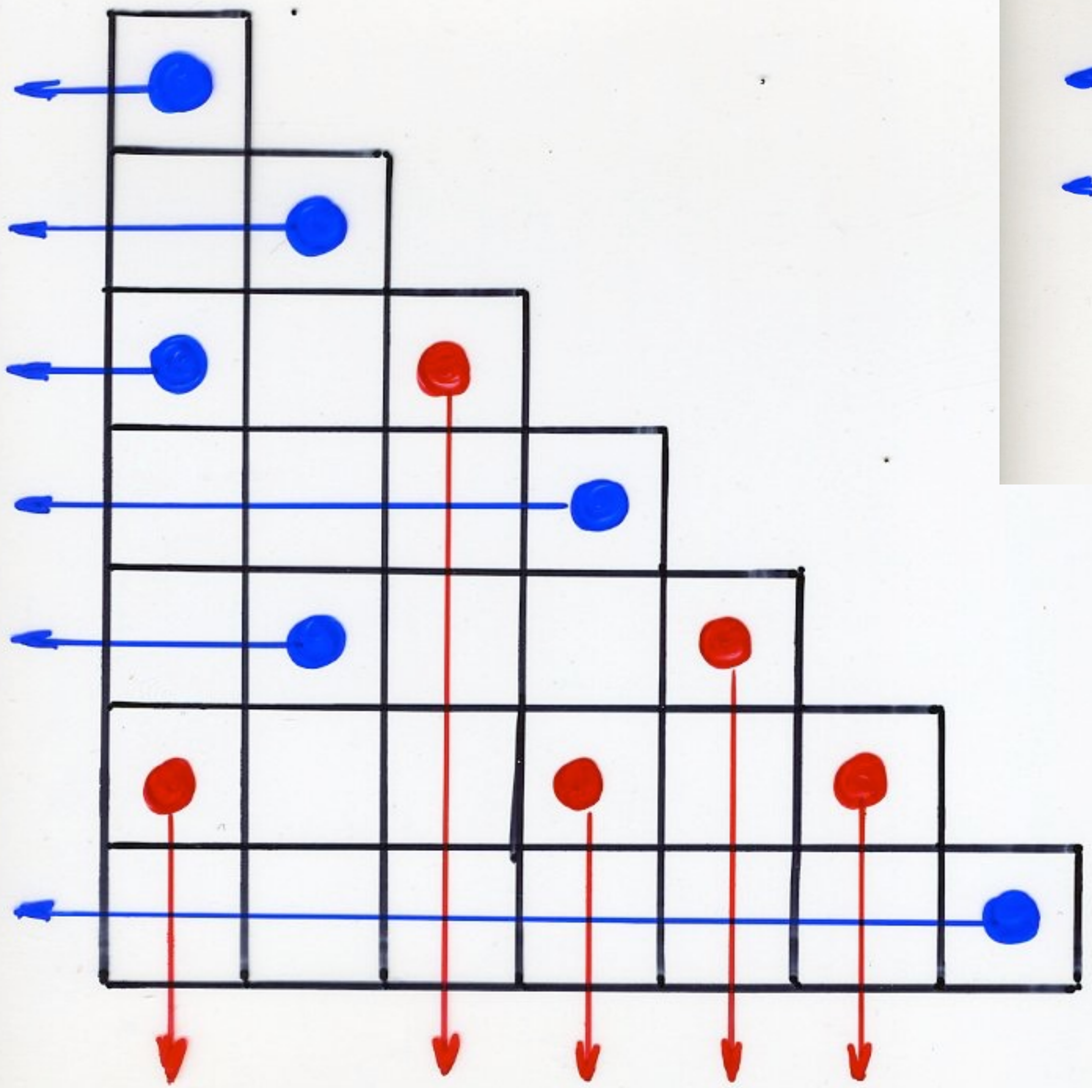
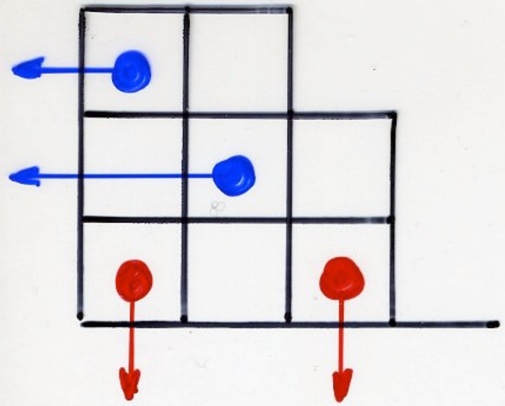
set of rewriting rules

$$B_j A_i \longrightarrow C^{kl}_{ij} A_k B_l$$

$(*)$

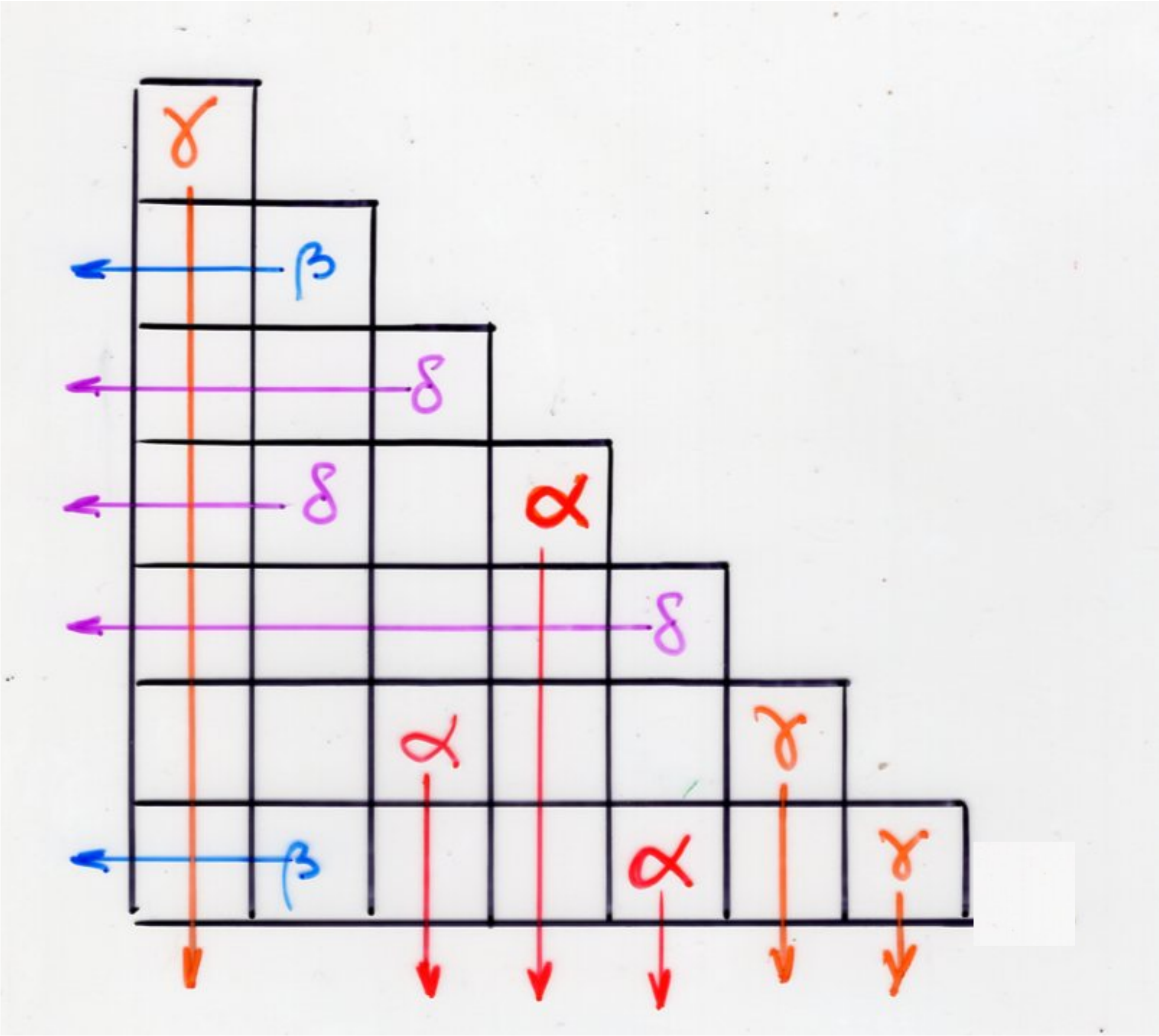
complete Q-tableaux $\longleftrightarrow$ Q-tableaux

complete Q-tableaux $\longleftrightarrow$ reverse Q-tableaux

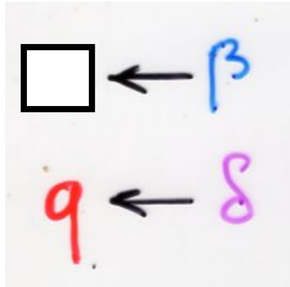



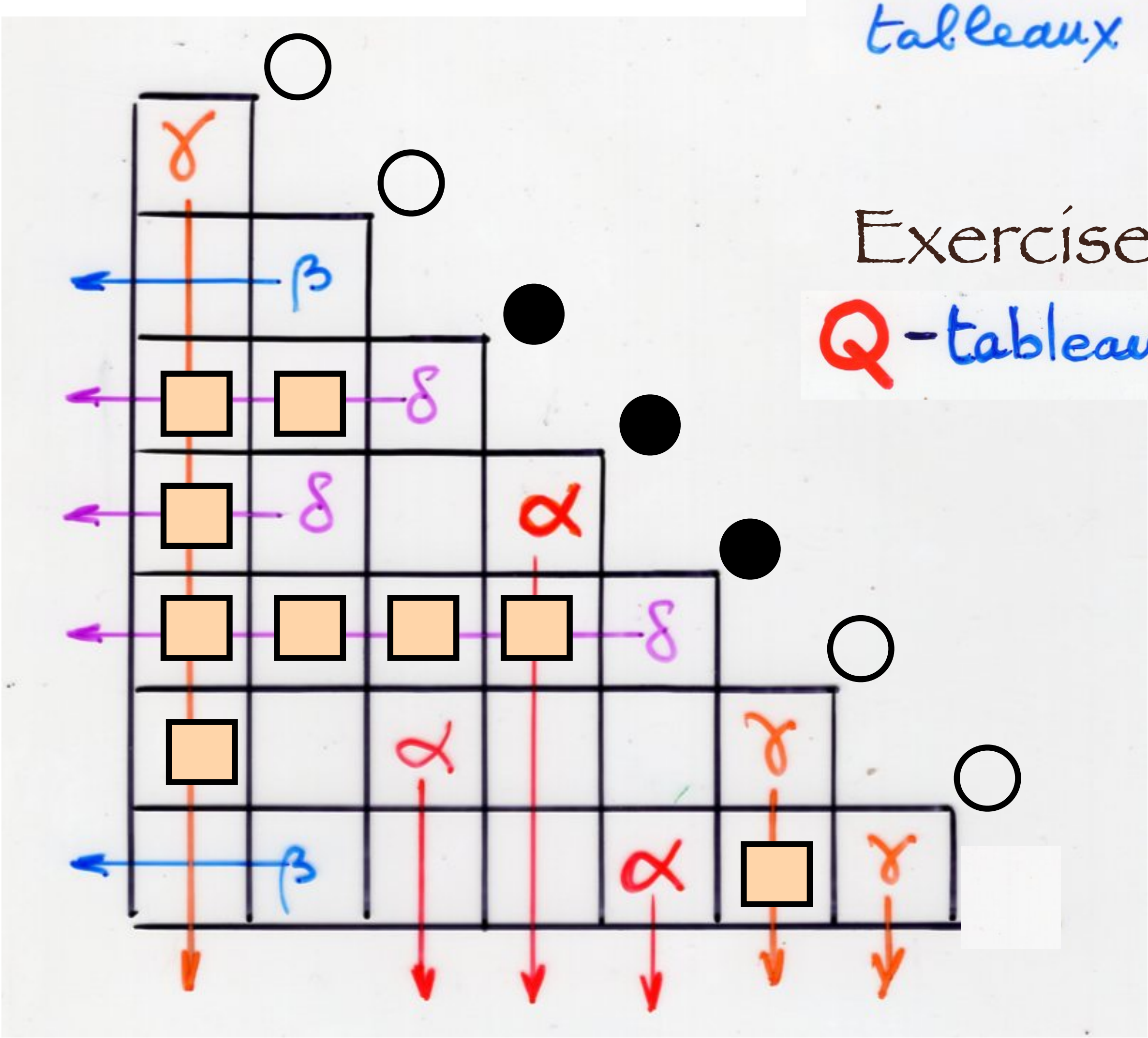
S. Corteel, L. Williams (2009)

staircase tableaux



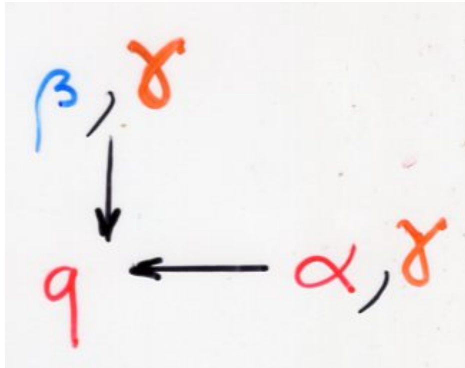
weight for staircase tableaux

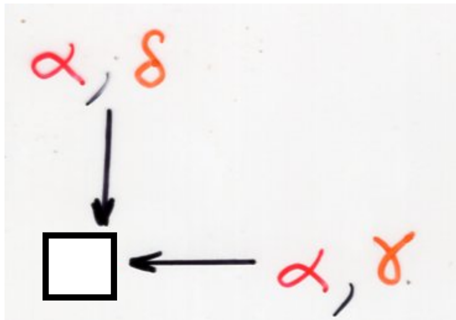




Exercise

Q-tableaux

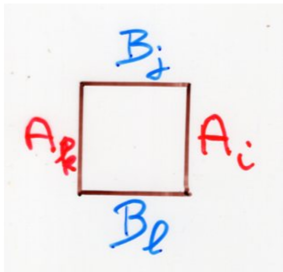




$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

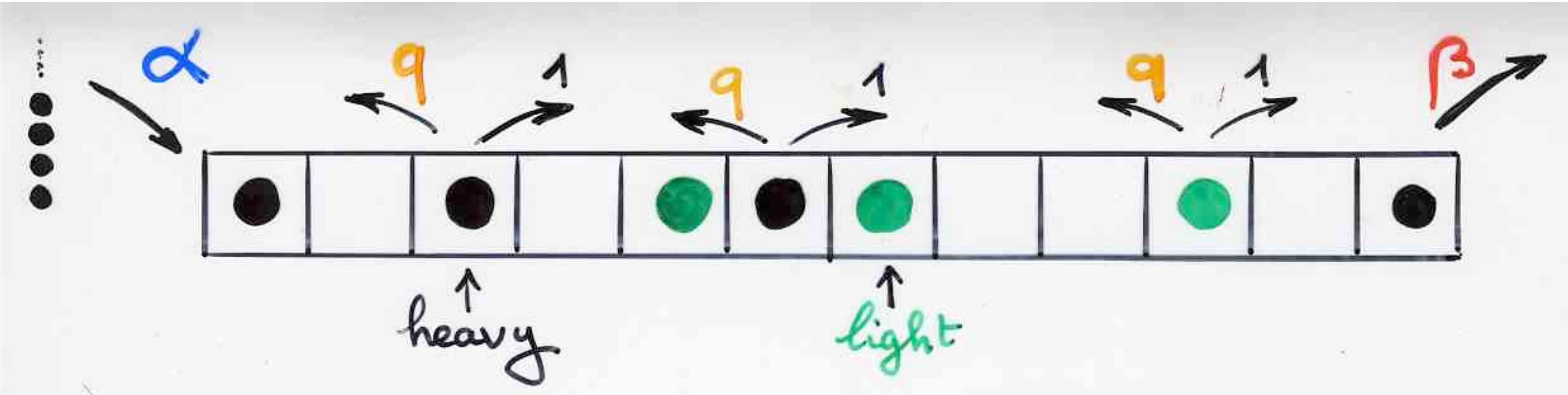
Extension of the cellular ansatz

$$B_j A_i = \sum_{k\ell} c_{ij}^{k\ell} A_k B_\ell$$



The 2-species PASEP

The 2-species PASEP



Matrix ansatz

for the 2-species PASEP

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n \qquad X_i \in \{\bullet, \bullet, \circ\}$

D, E, A matrices

W row vector $\qquad V$ column vector

$$\begin{cases} DE = qED + D + E \\ DA = qAD + A \\ AE = qEA + A \end{cases}$$

$$\langle W| E = \frac{1}{\alpha} \langle W|$$

$$D|V\rangle = \frac{1}{\beta} |V\rangle.$$

Matrix Ansatz (Uchiyama, 2008)

$X = X_1 \dots X_n \qquad X_i \in \{ \bullet, \bullet, \circ \}$

D, E, A matrices

W row vector $\qquad V$ column vector

$$\mathrm{Prob}(X) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\,\mathbb{1}_{(X_i=\bullet)} + A\,\mathbb{1}_{(X_i=\bullet)} + E\,\mathbb{1}_{(X_i=\circ)} | V \rangle$$

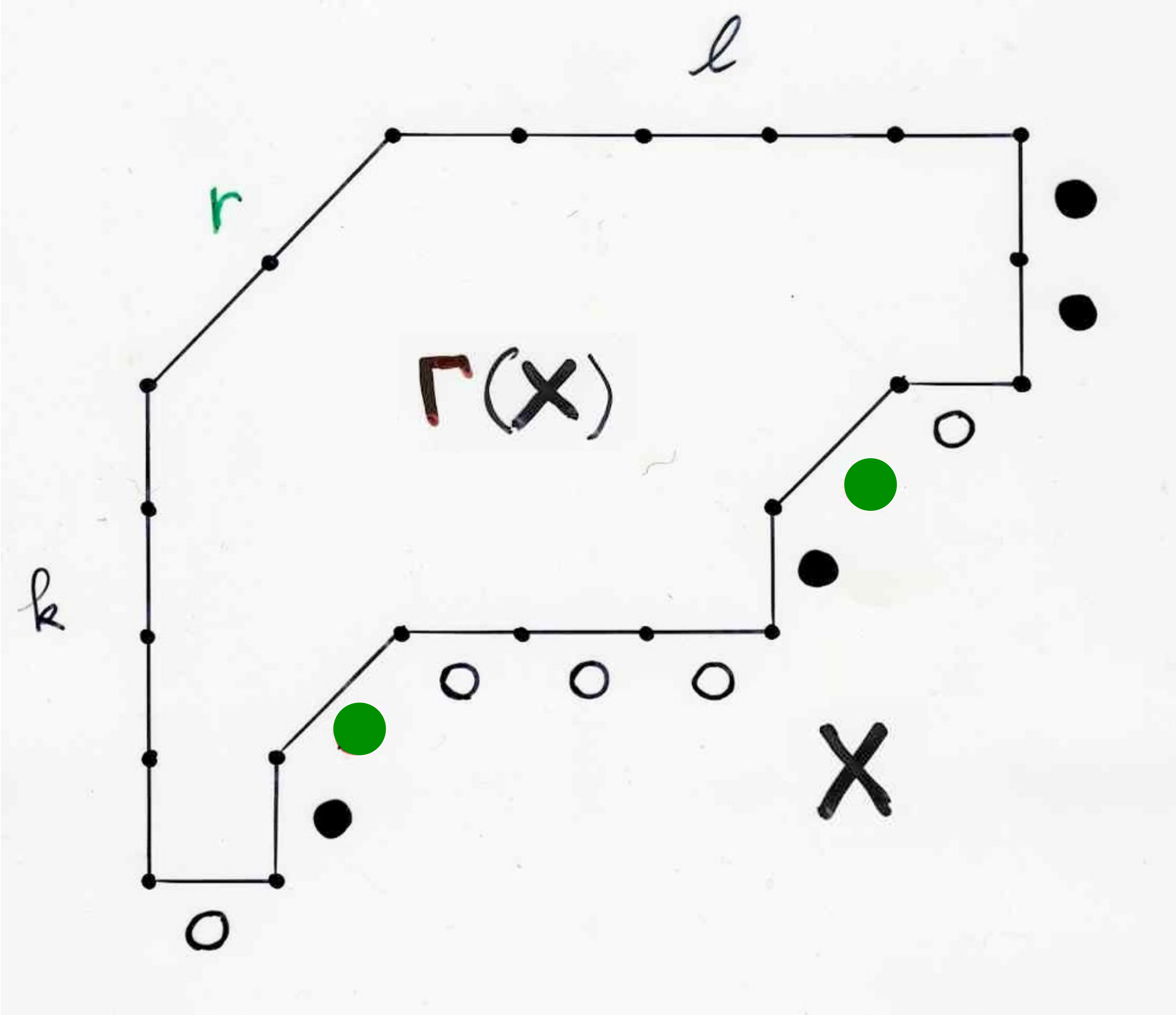
$$Z_{n,r} = \text{coeff. of } y^r \text{ in } \langle W | (D + yA + E)^n | V \rangle$$

Rhombic alternative tableaux

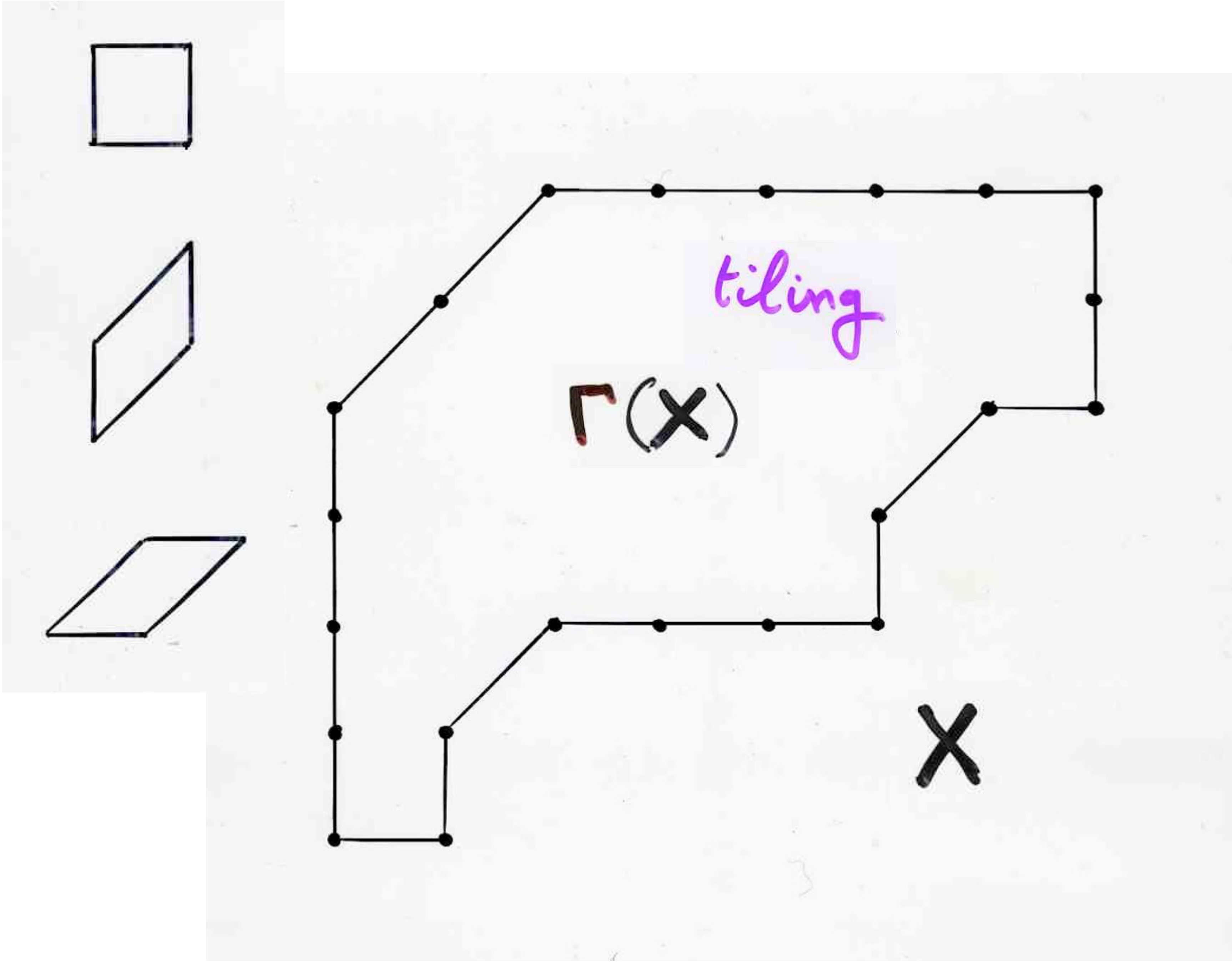
(RAT)

Rhombic alternative tableaux

O. Mandelshtam, X.V. (2015)



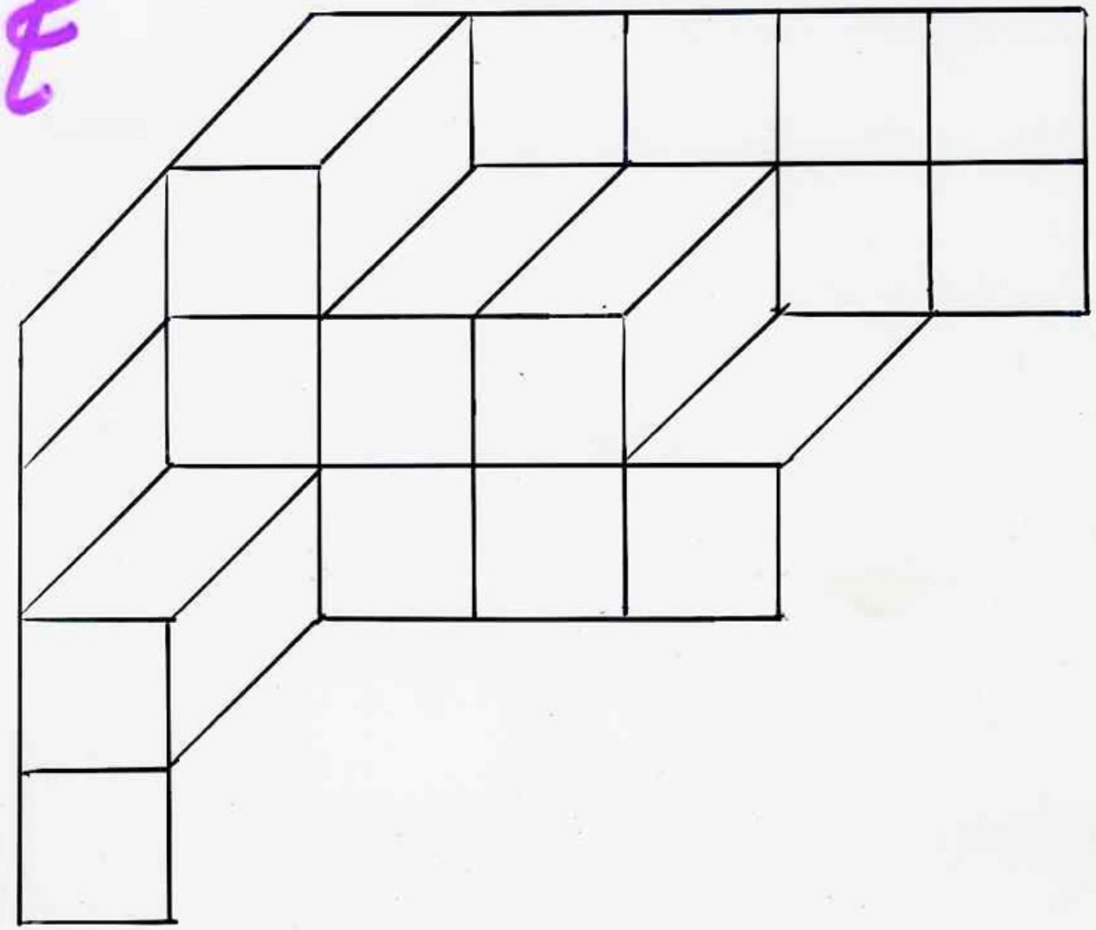
ℓ
r
Γ(X)
k
X



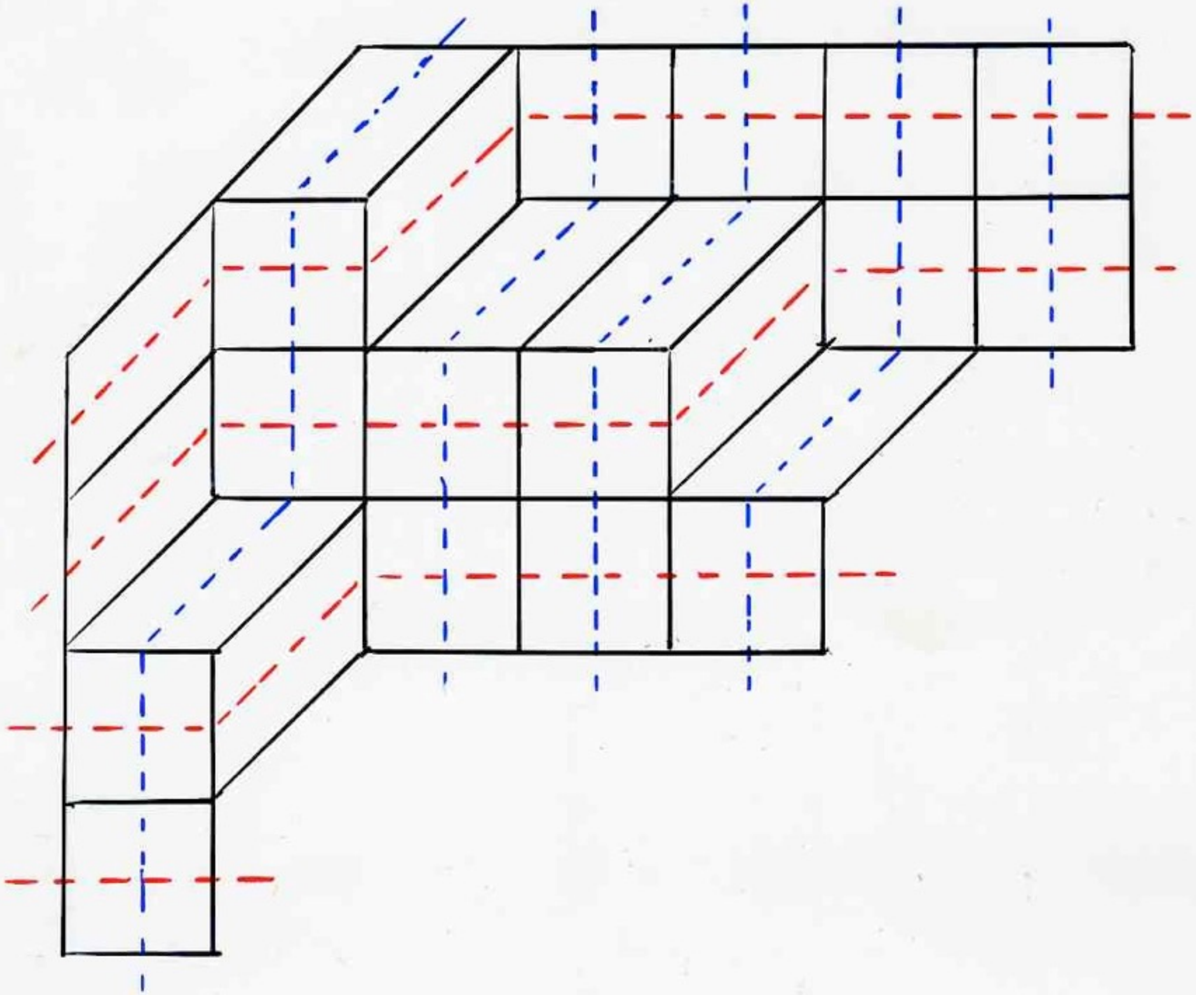
tiling
Γ(X)
X

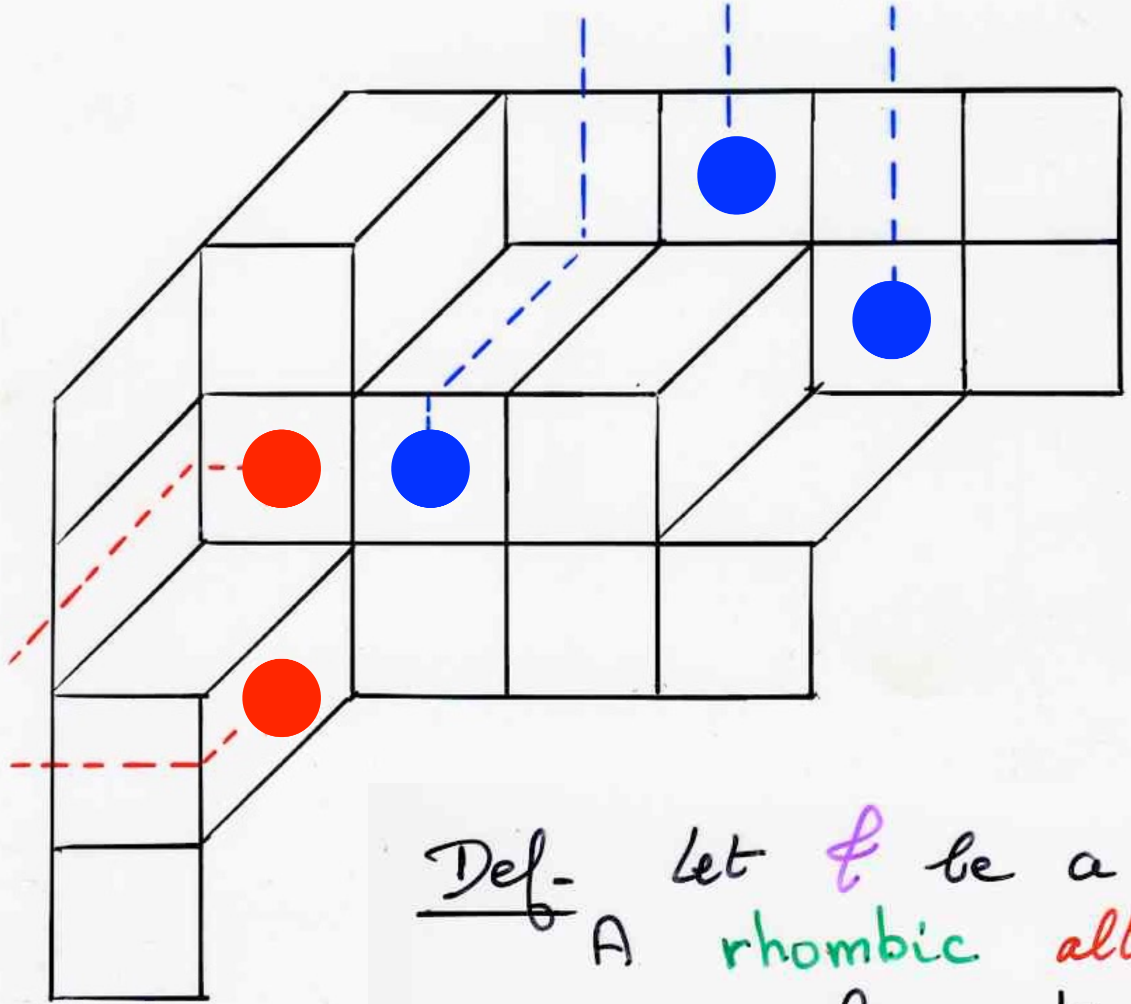
tiling

ℓ



west- strips
north-strips



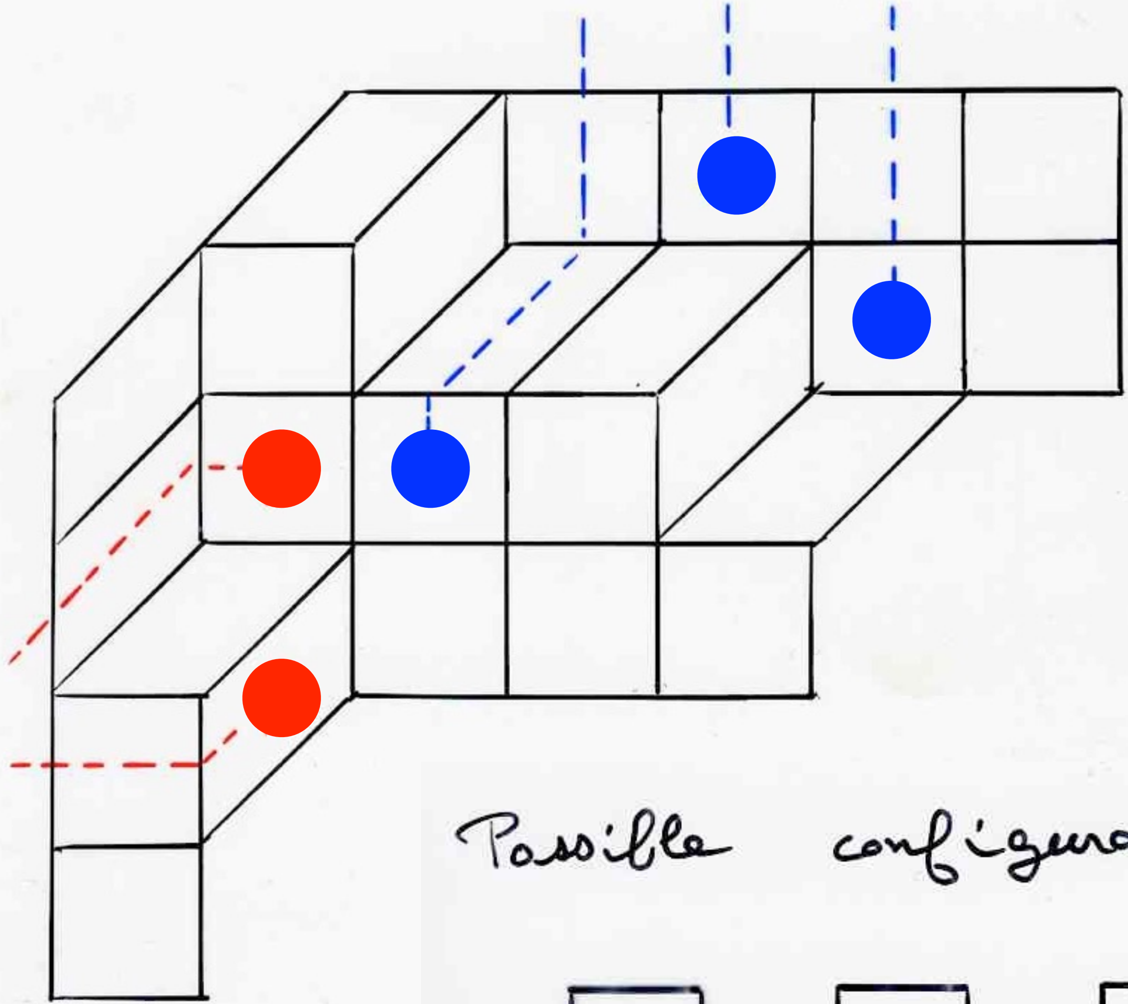


west-strips

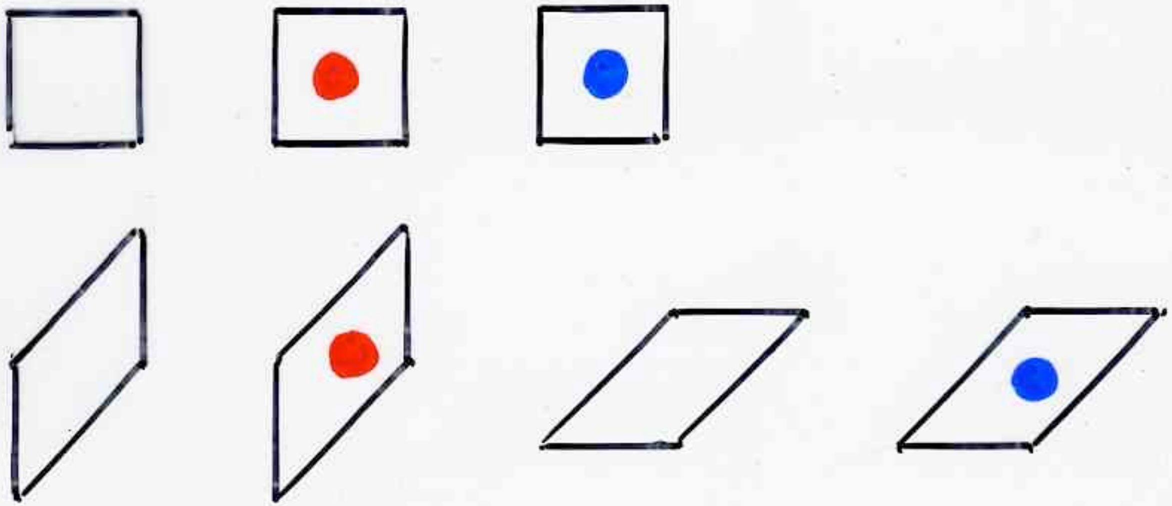
north-strips

Def- Let f be a tiling of $\Gamma(X)$. A rhombic alternative tableau T is a placement of ●, ● in the tiles such that:

- a ● is on a west-strip and any tile left of this ● is empty
- a ● is on a north-strip and any tile north of this ● is empty

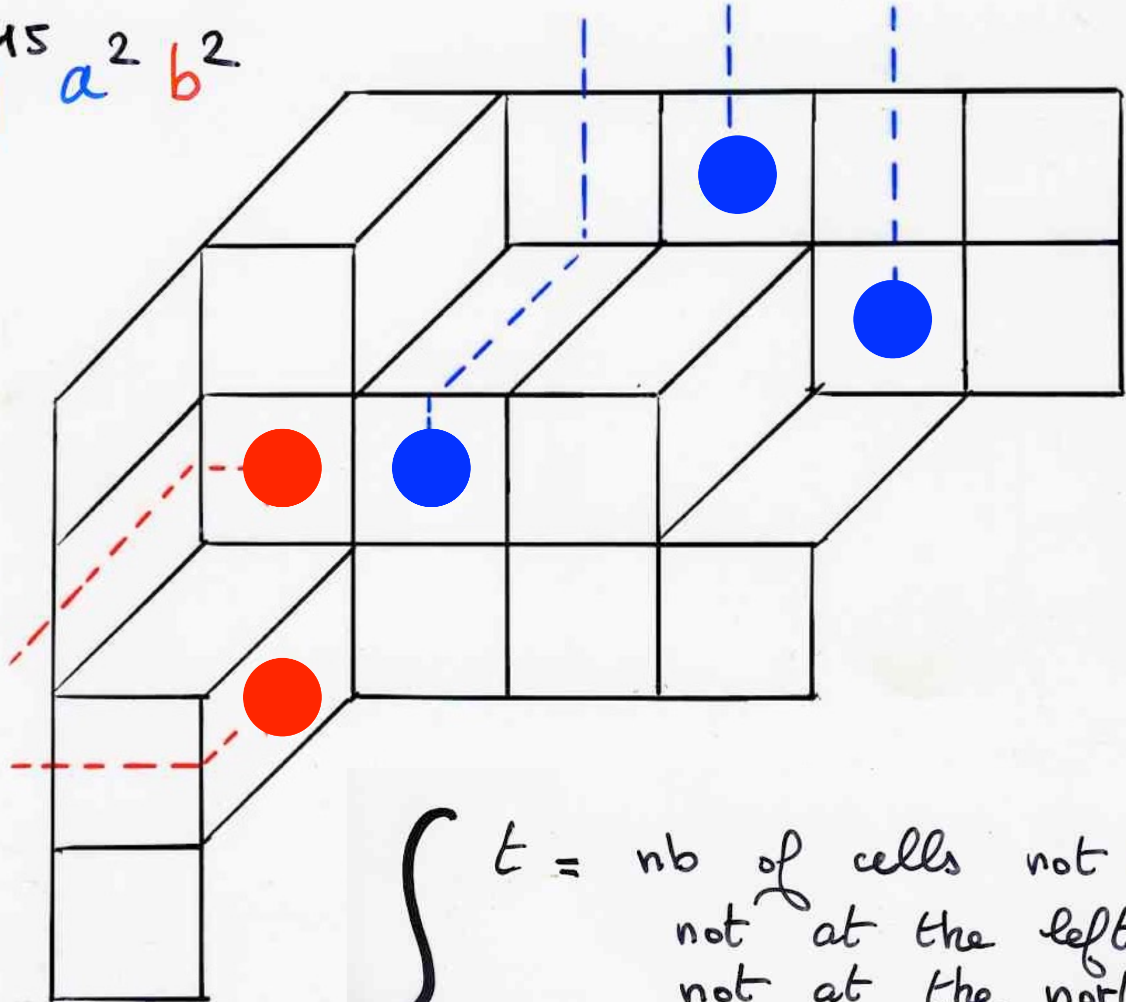


Possible configurations for a tile :



weight $\mathbf{wt}(T) = q^t a^i b^j$

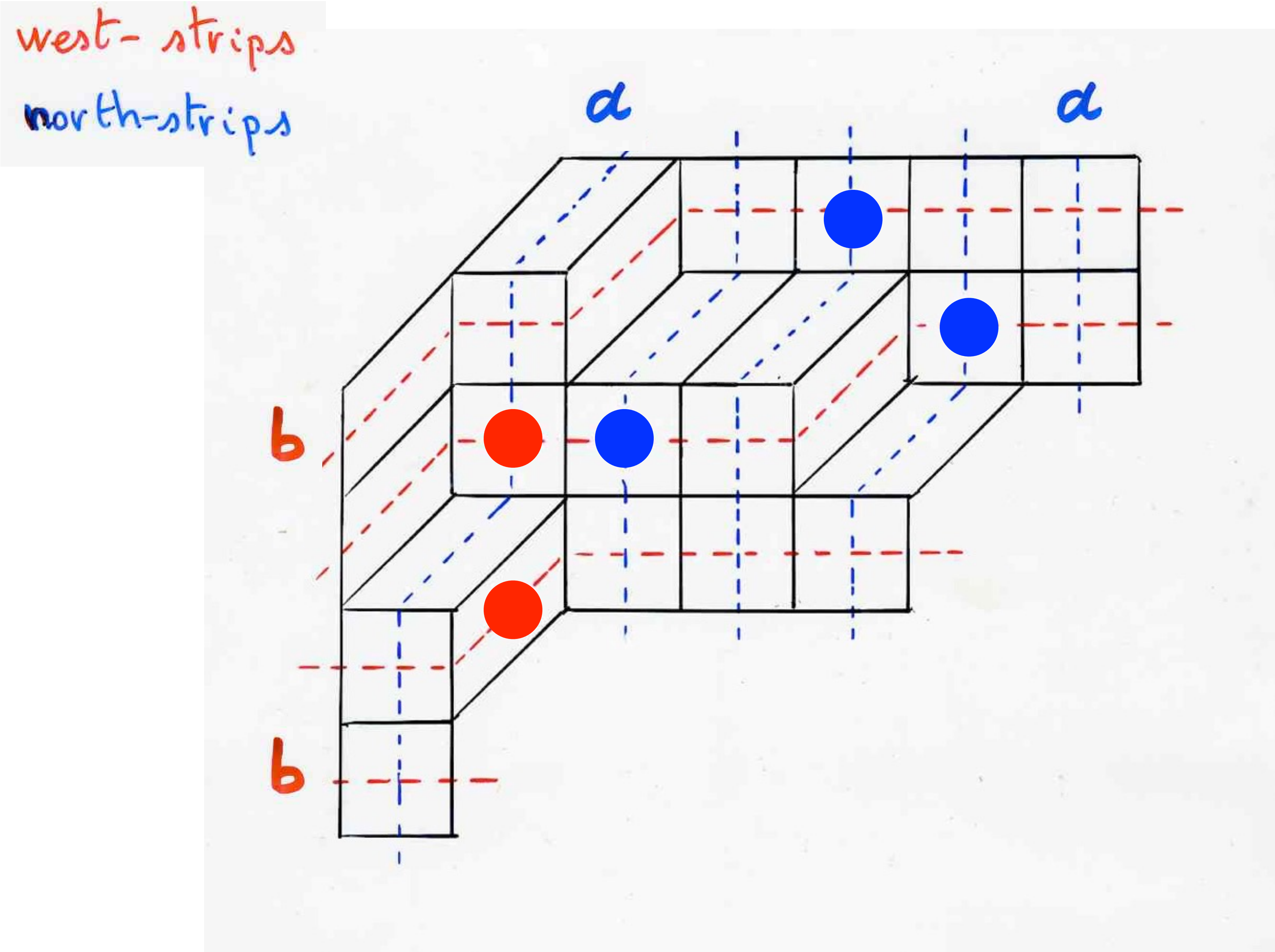
$wt(T) = q^{15} a^2 b^2$

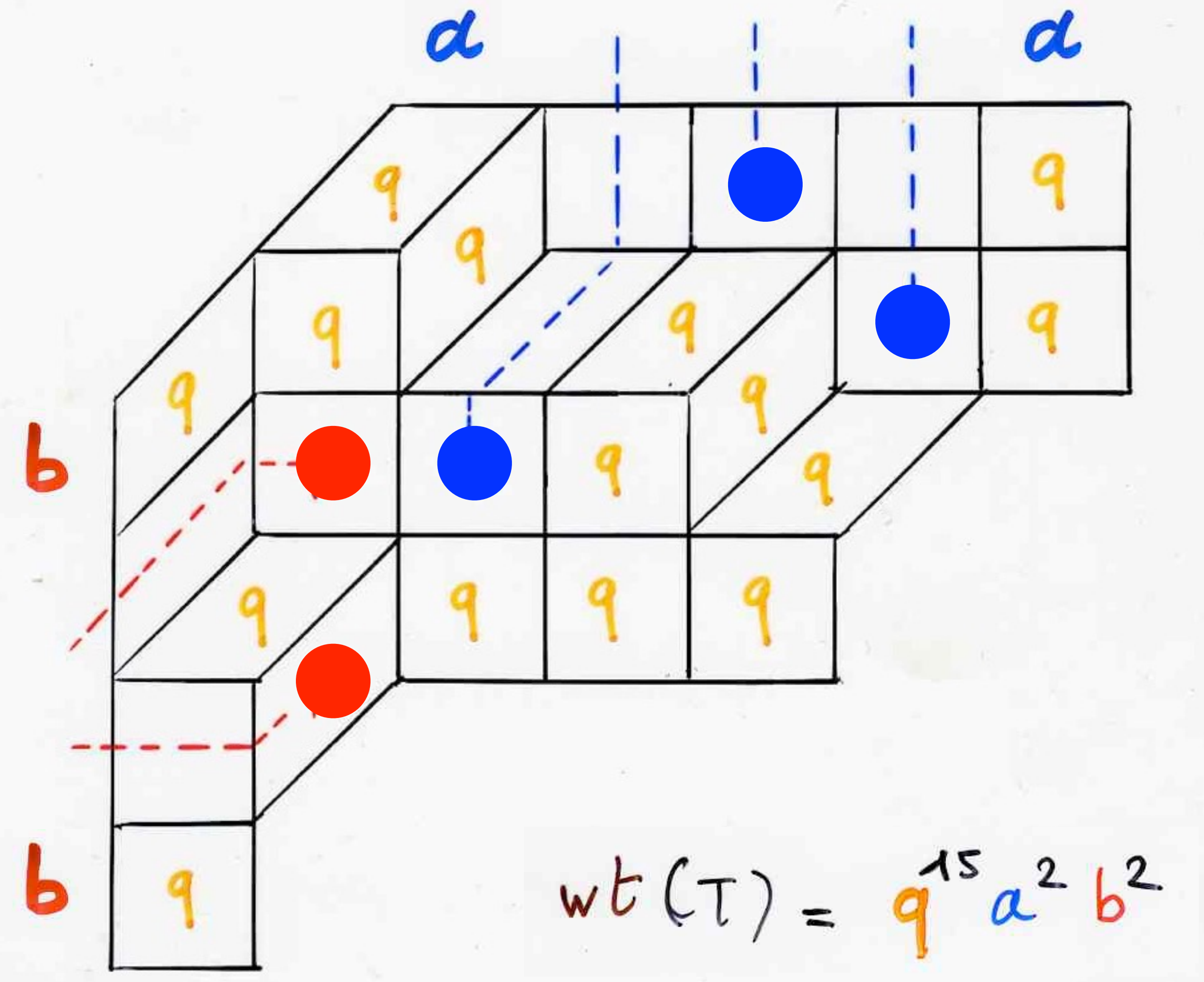


t = nb of cells not ●, not ●,
not at the left of a ●,
not at the north of a ●

i = nb of north-strips without a ●

j = nb of west-strips without a ●


west- strips
north-strips
a
a
b
b



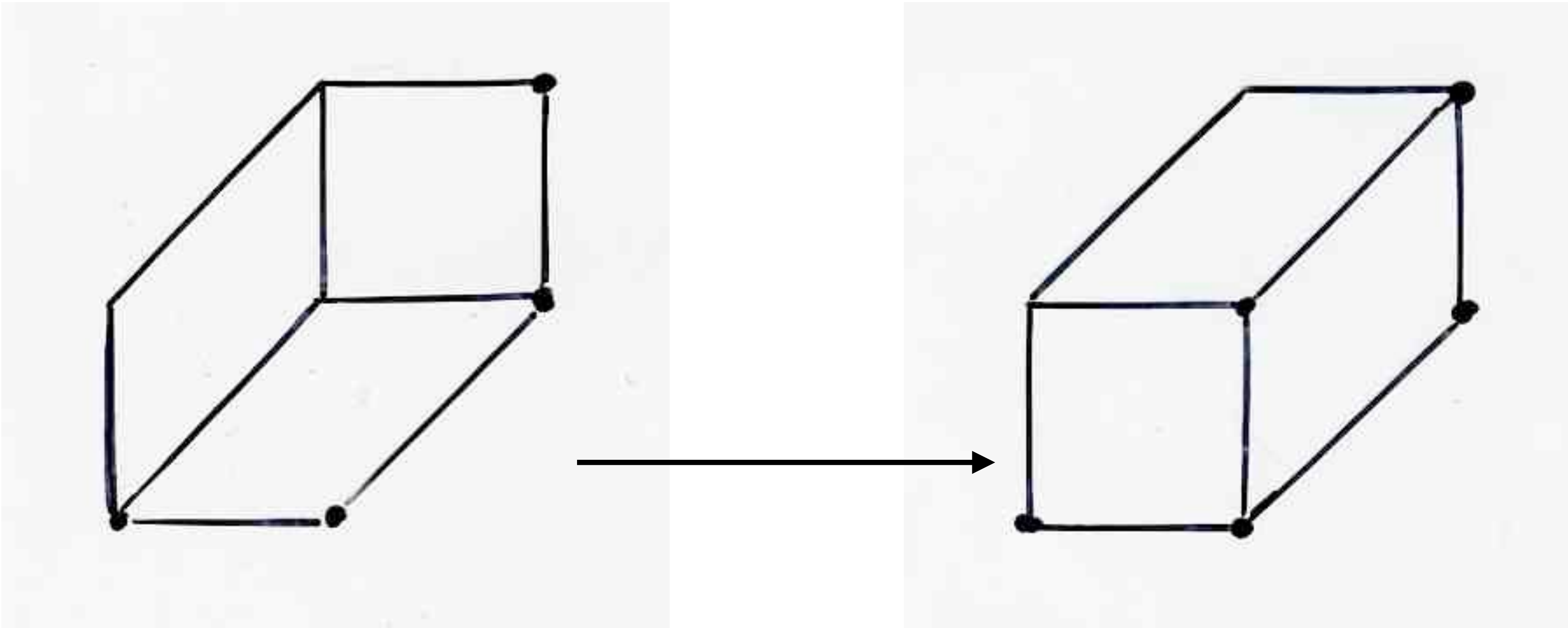
a
a
b
b
$wt(T) = q^{15} a^2 b^2$

$R(X, \mathfrak{t})$ set of rhombic alternative tableaux related to X, with the tiling $\mathfrak{t}$ of $\Gamma(X)$

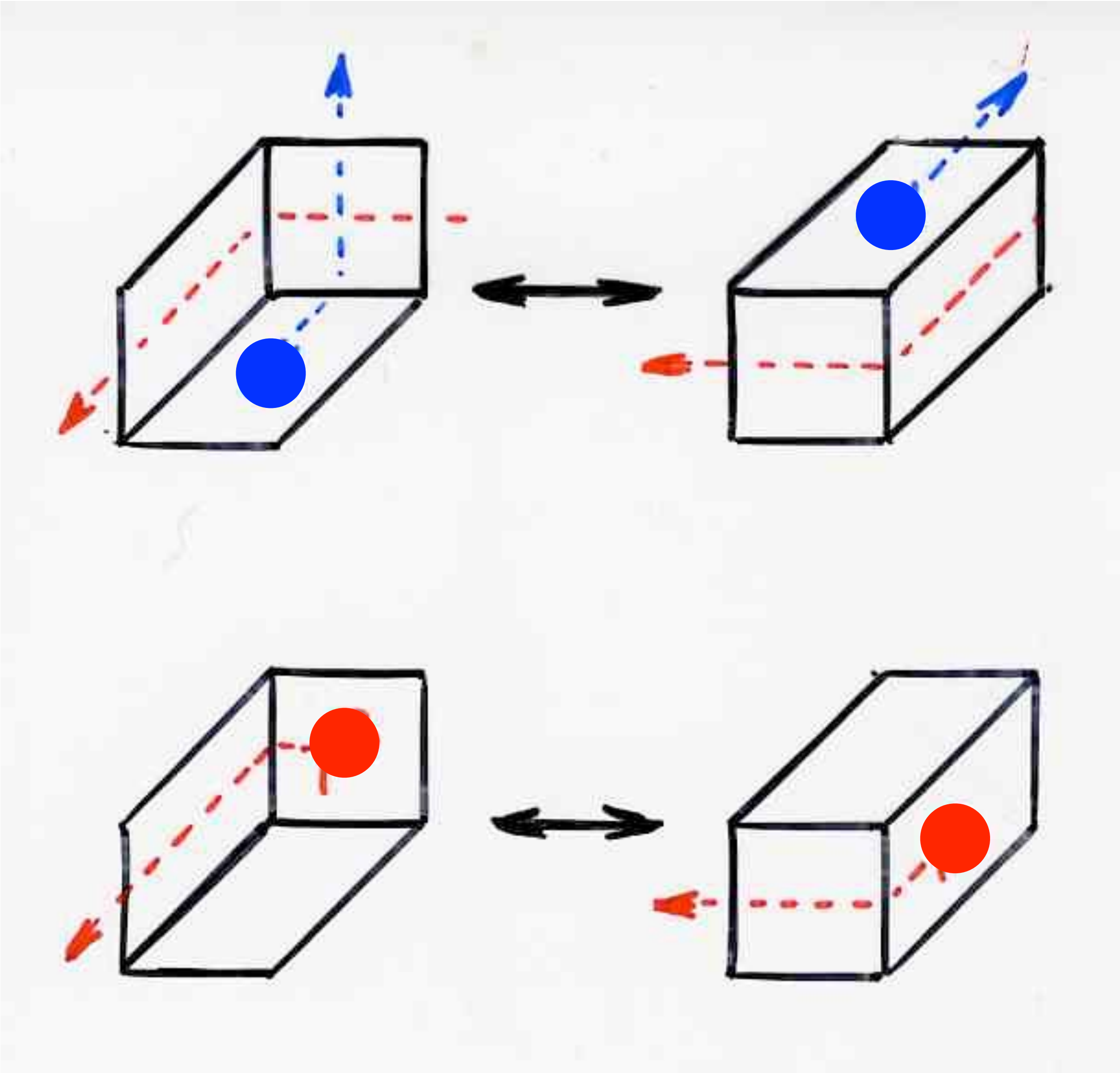
Prop X, $\Gamma(X)$ diagram

$\mathfrak{t}$, $\mathfrak{t}'$ tiling of $\Gamma(X)$

$$\sum_{T \in R(X, \mathfrak{t})} wt(T) = \sum_{T \in R(X, \mathfrak{t}')} wt(T)$$



a flip



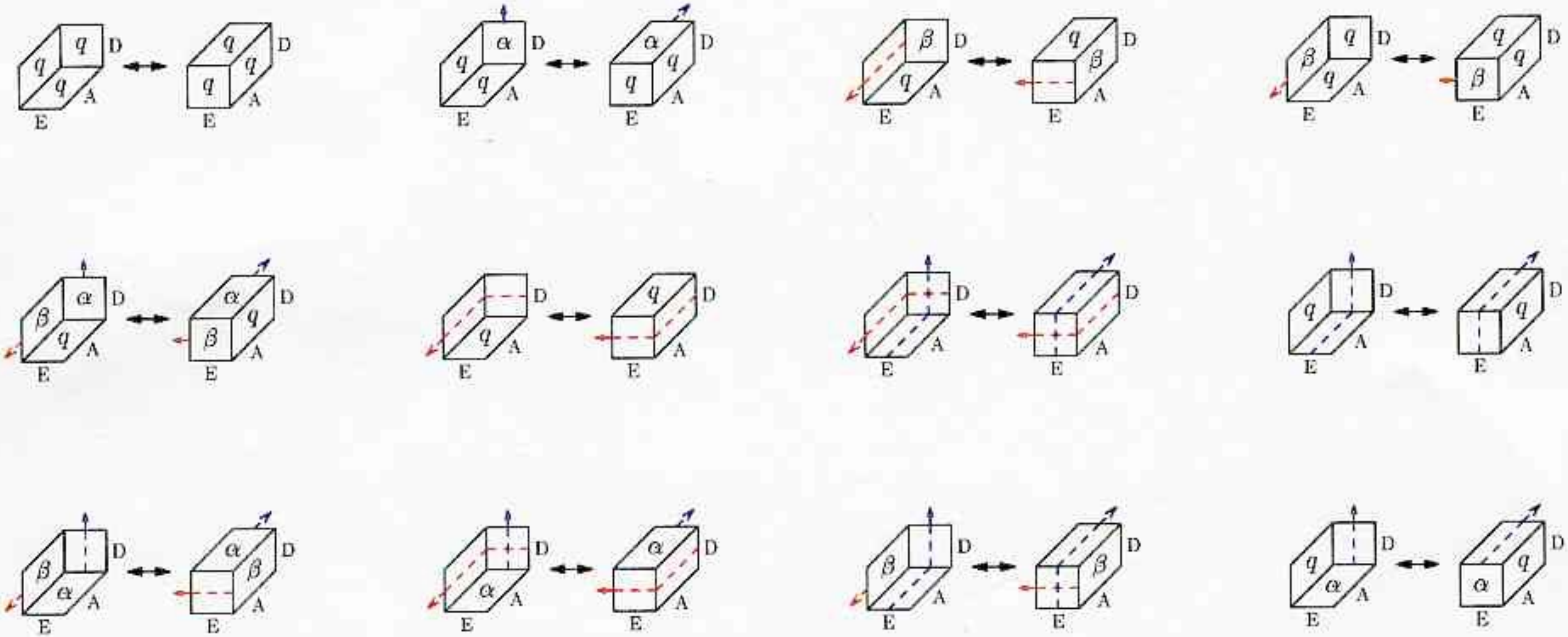


Figure 11: The involution ϕ from each possible filling of a minimal hexagon (left) to a maximal hexagon (right). The arrows imply compatibility requirements.

combinatorial interpretation
of
stationary probabilities

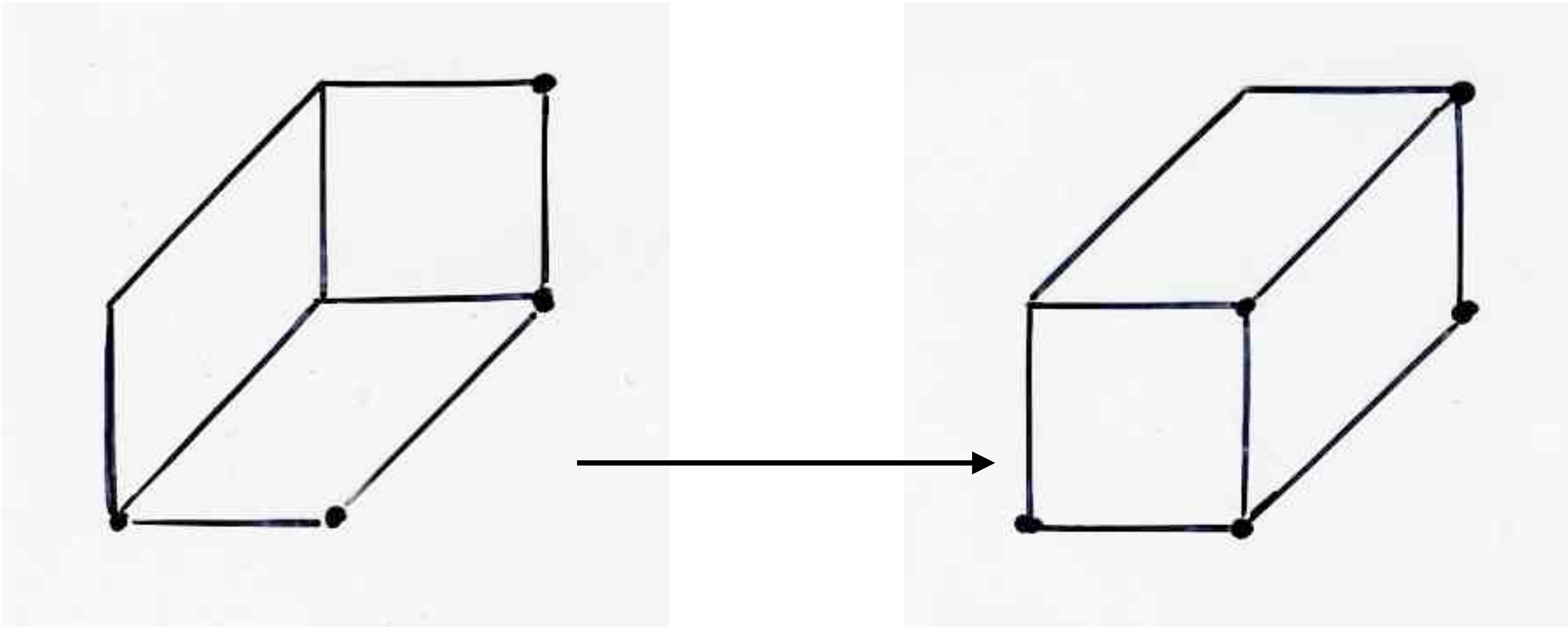
$$\mathrm{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X,\tau_X)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

$$a = \frac{1}{\alpha}$$

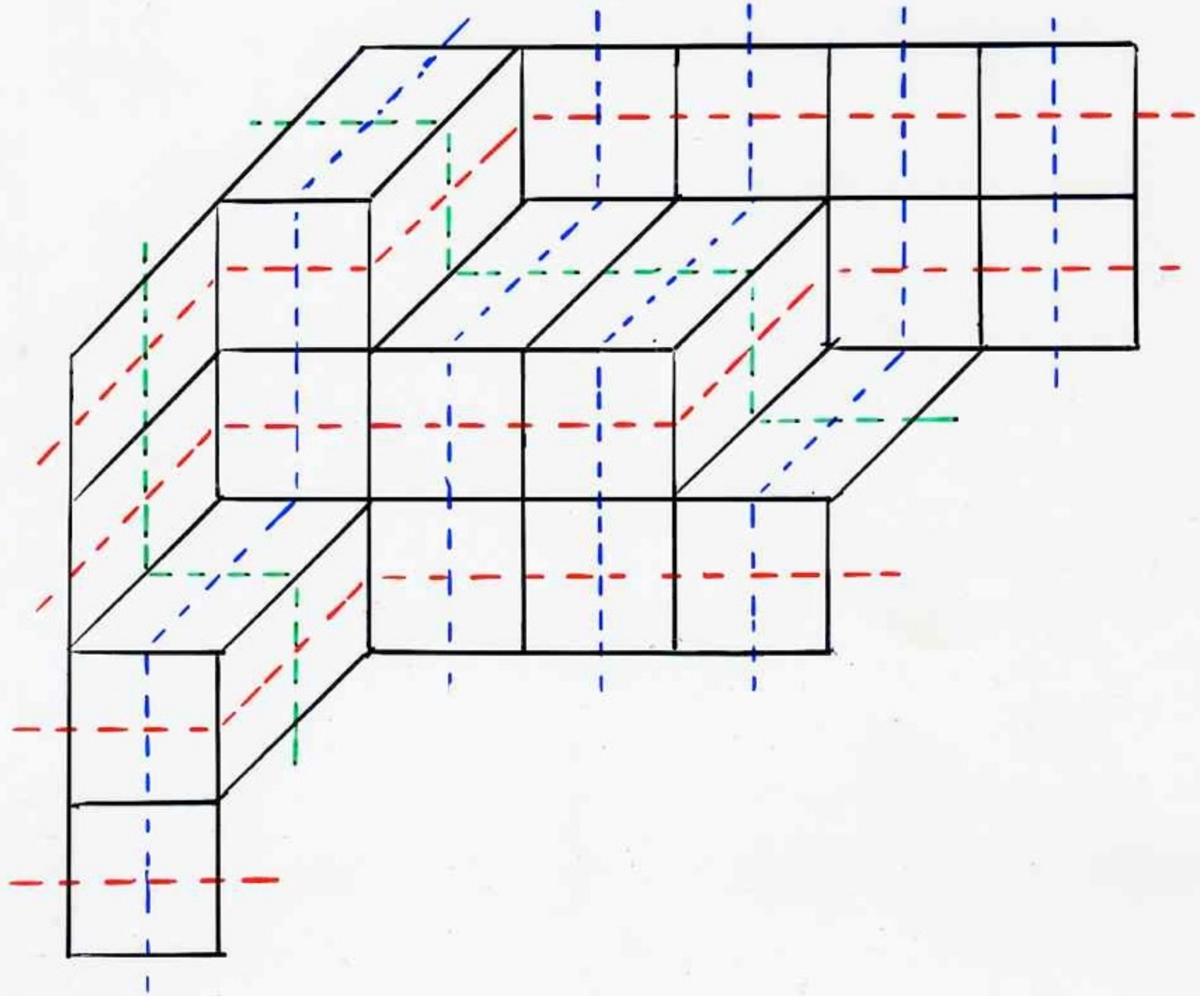
$$b = \frac{1}{\beta}$$

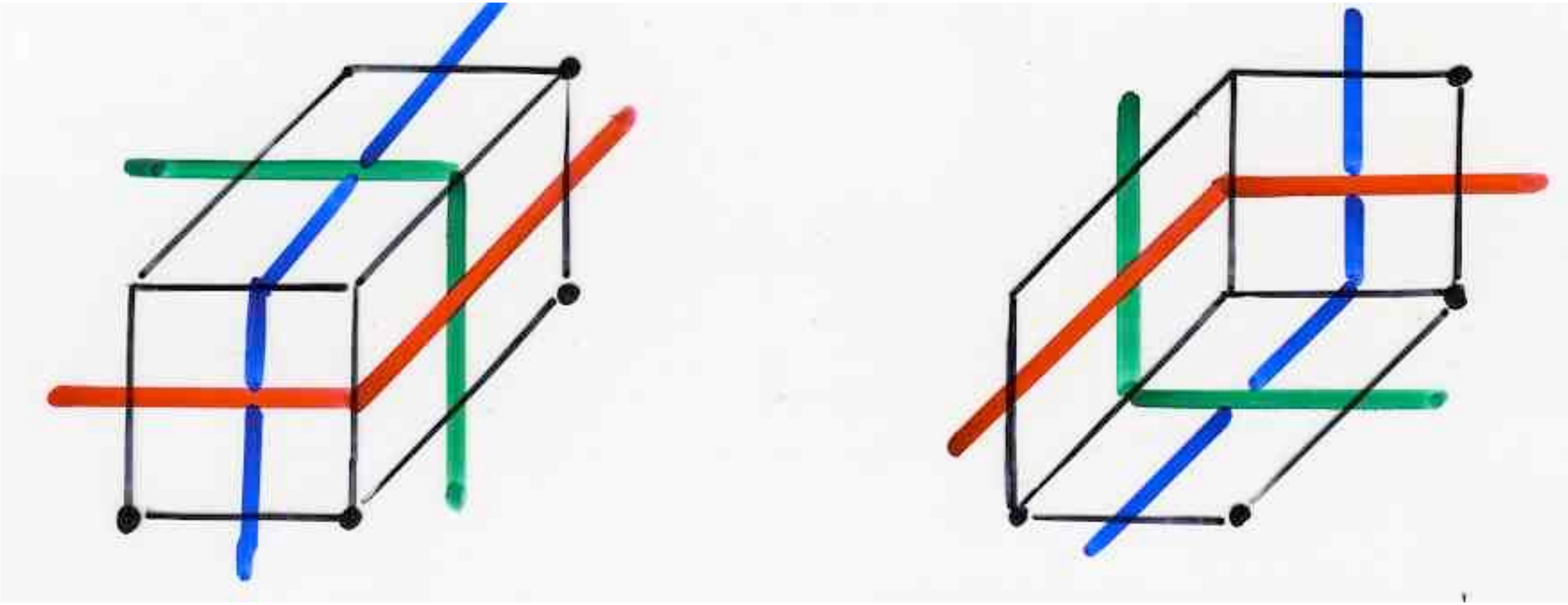
$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X,\tau_X)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

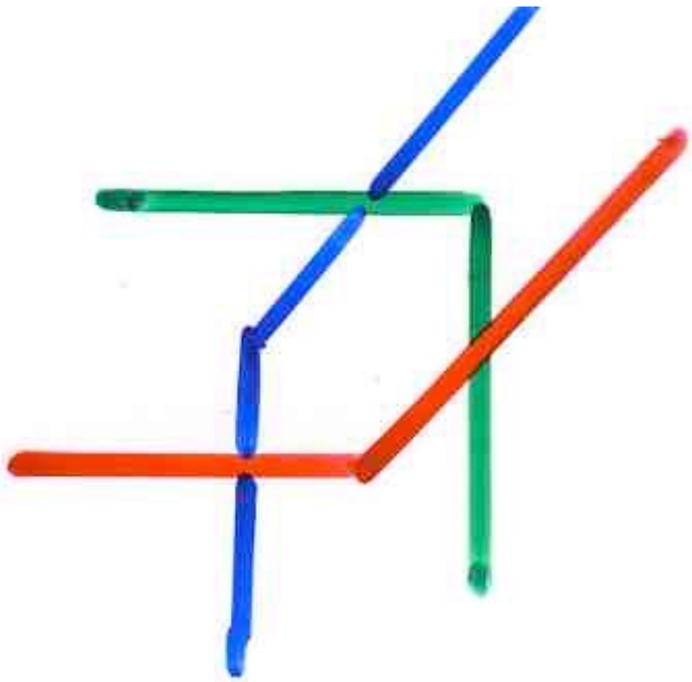
some remarks

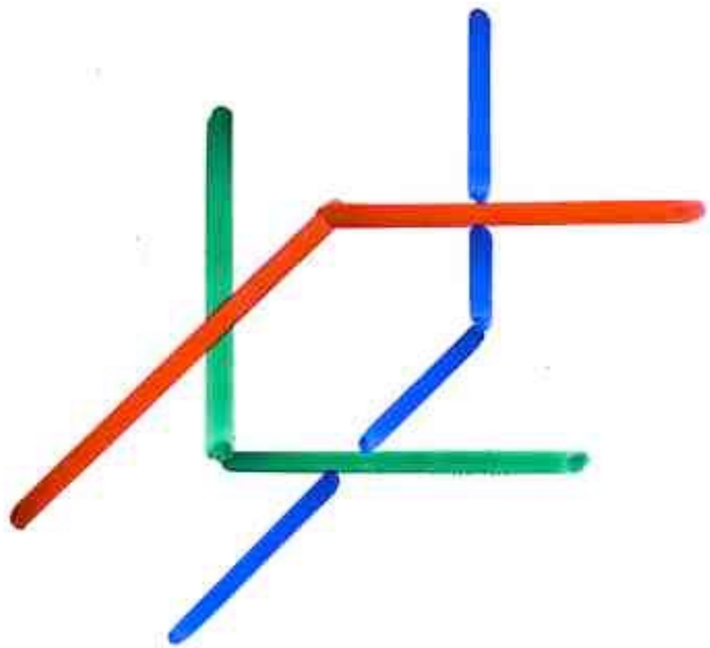


a flip



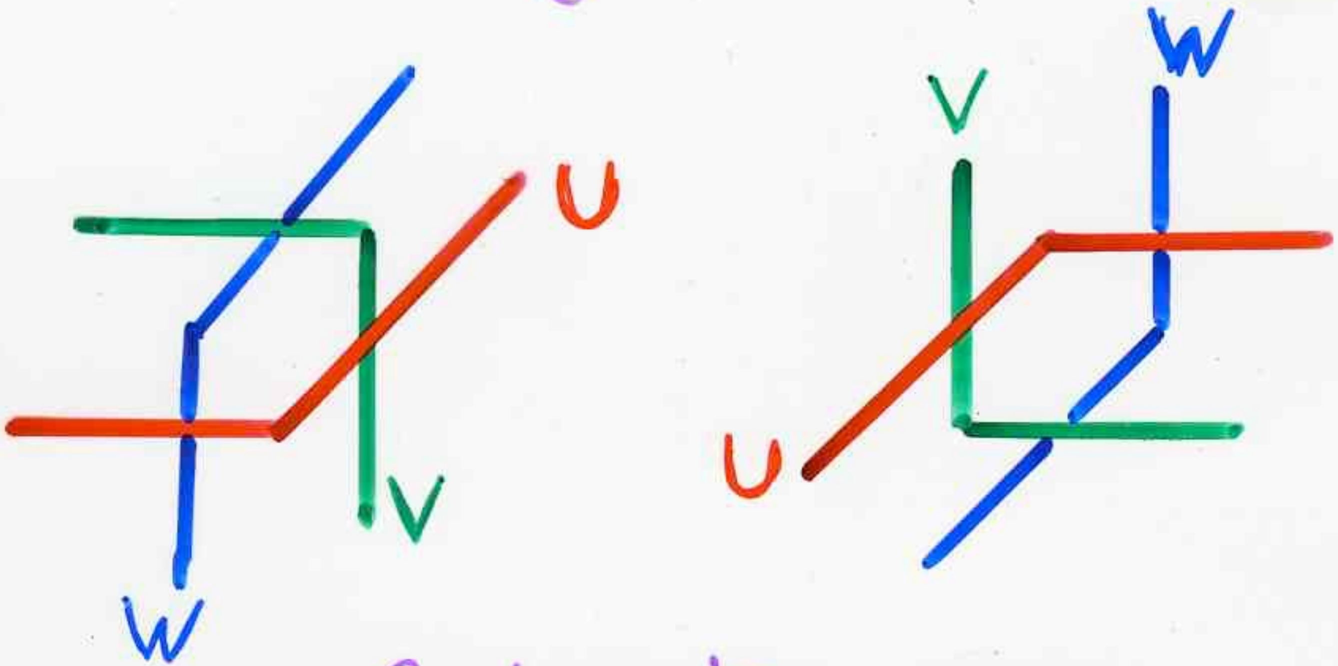






Yang-Baxter equation $UVW = WVU$

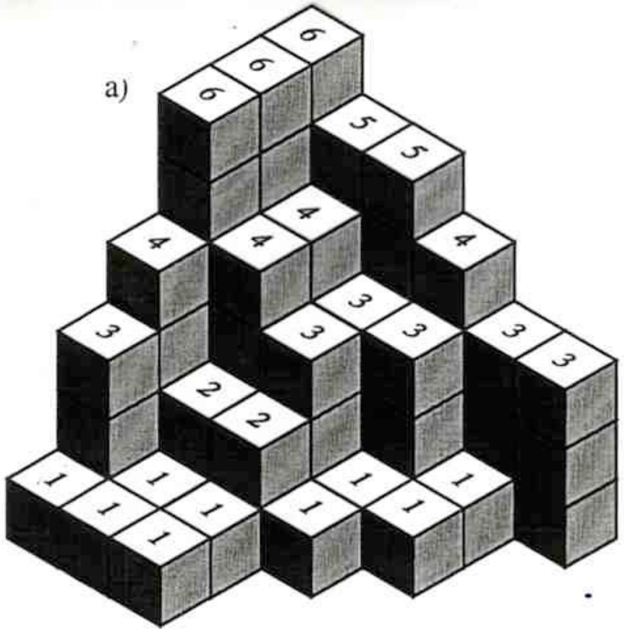
Yang-Baxter moves



Reidemeister moves

(knot theory)

a)

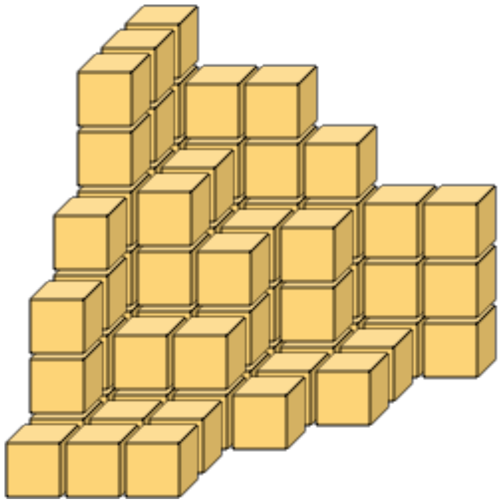


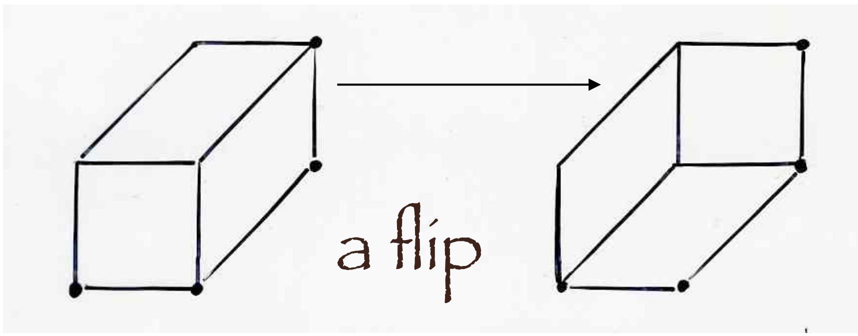
b)

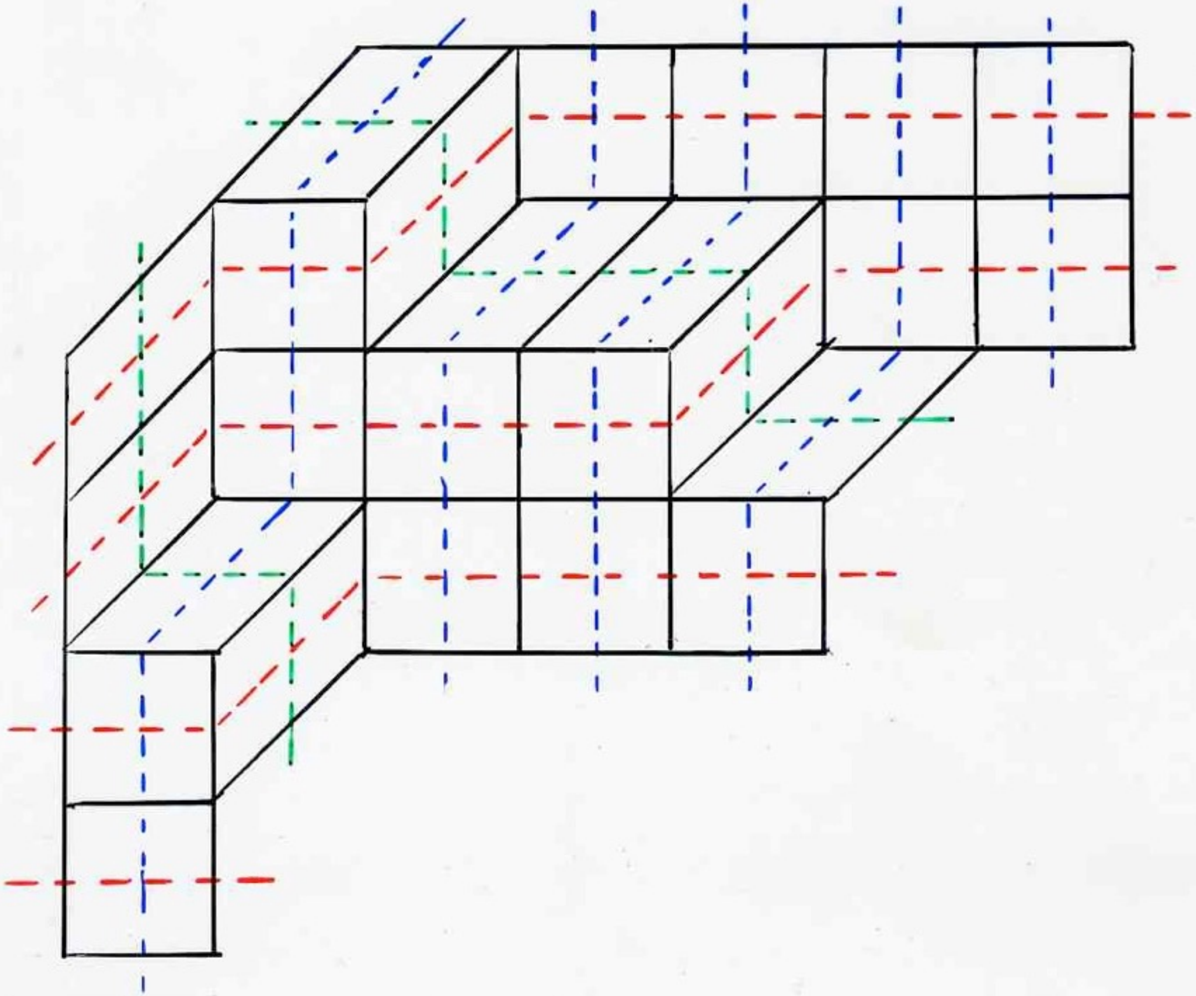
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

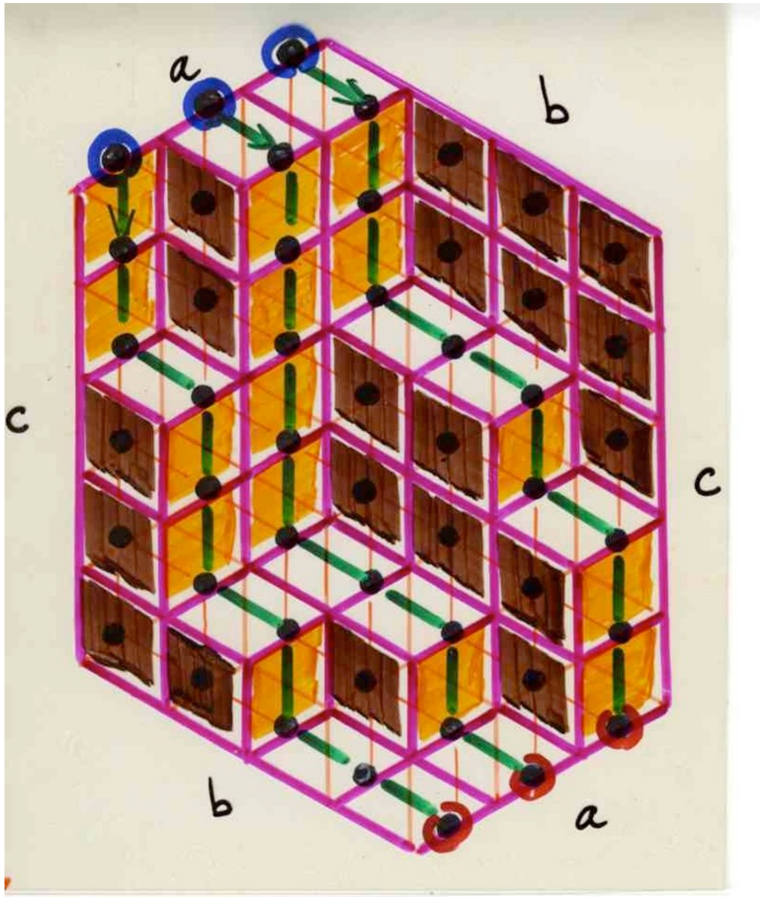
Covering relation of the poset of plane partitions

(see seminar 1 on Maules)









$$\prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{i+j+k-1}{i+j+k-2}$$

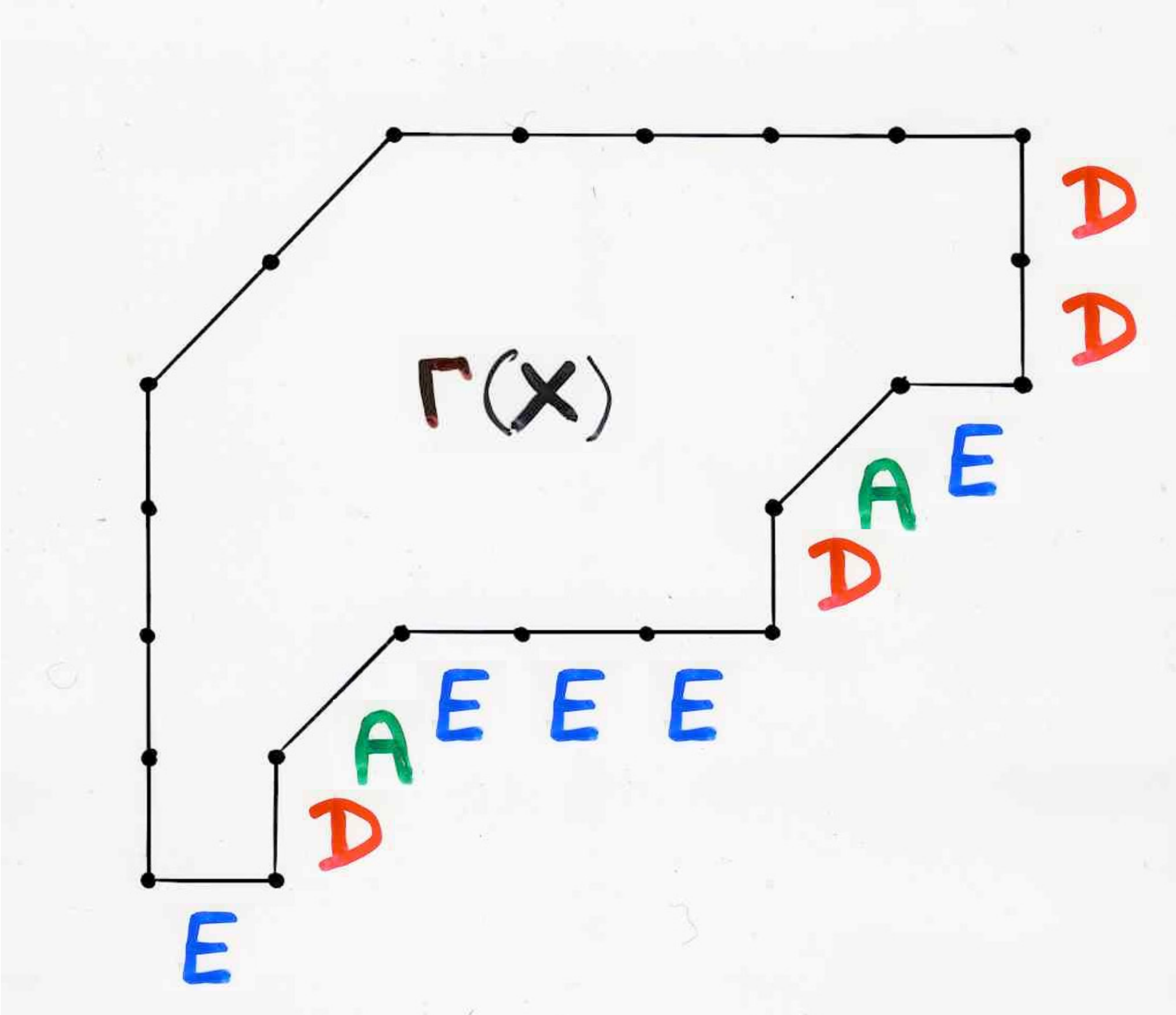
proof with a determinant

(LGV Lemma)

2-PASEP algebra

$$
\begin{cases}
DE = qED + D + E \\
DA = qAD + A \\
AE = qEA + A
\end{cases}
$$

$X =$ D D E A D E E E A D E



In the 2-PASEP algebra

Every word $X \in \{D, E, A\}^*$ can be expressed in a unique way

$$X = \sum_{T \in R(X, \mathcal{T})} q^{t} E^{i} A^{r} D^{j}$$

where:

$r = |X|_A$ (nb of A in the word X)

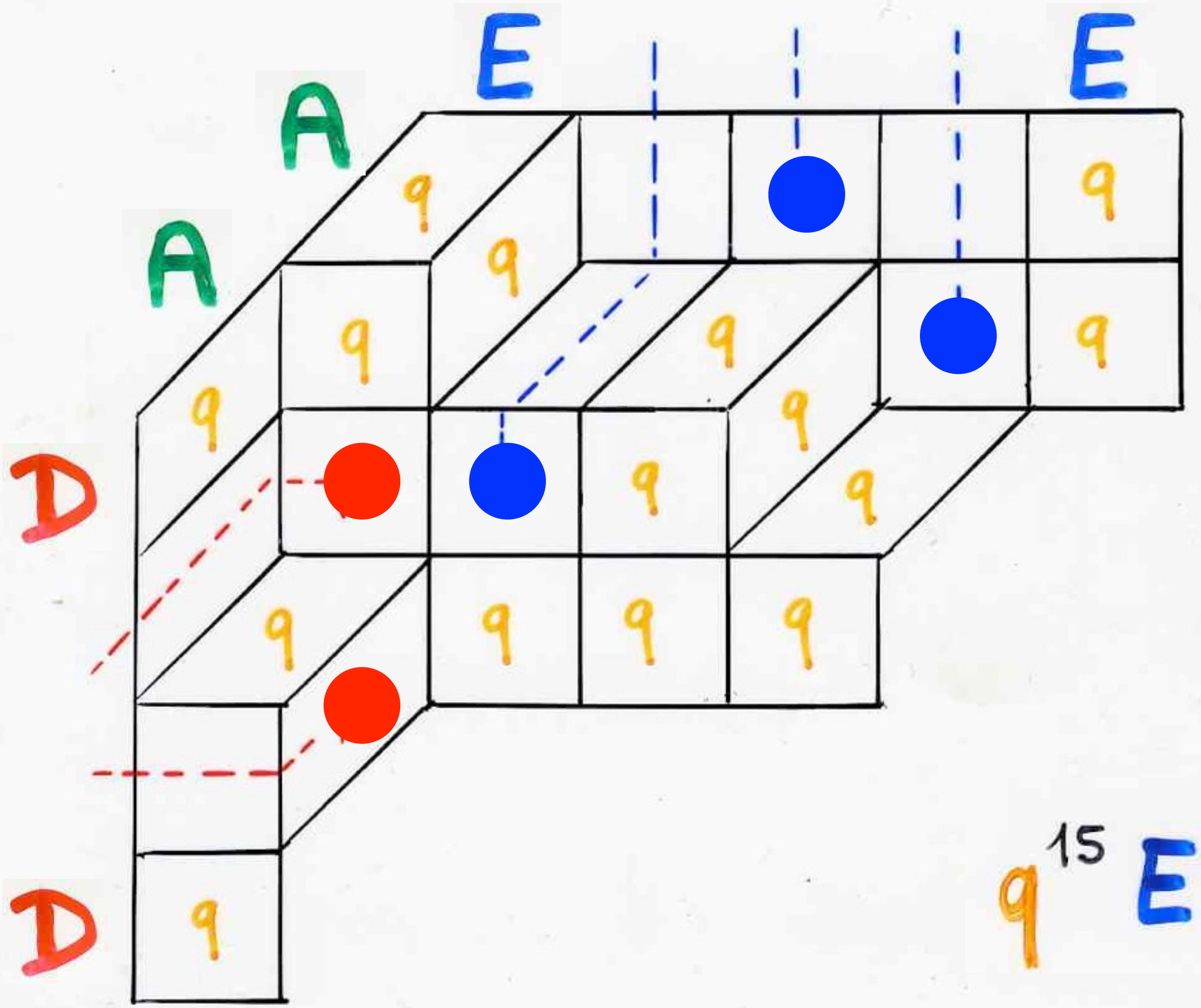
$\mathcal{T}$ a fixed tiling of $\Gamma(X)$

i = nb of free north-strips in T (=not containing an)

j = nb of free south-strips in T (=not containing a)

t = nb of cells labeled q in T

D D E A D E E E A D E



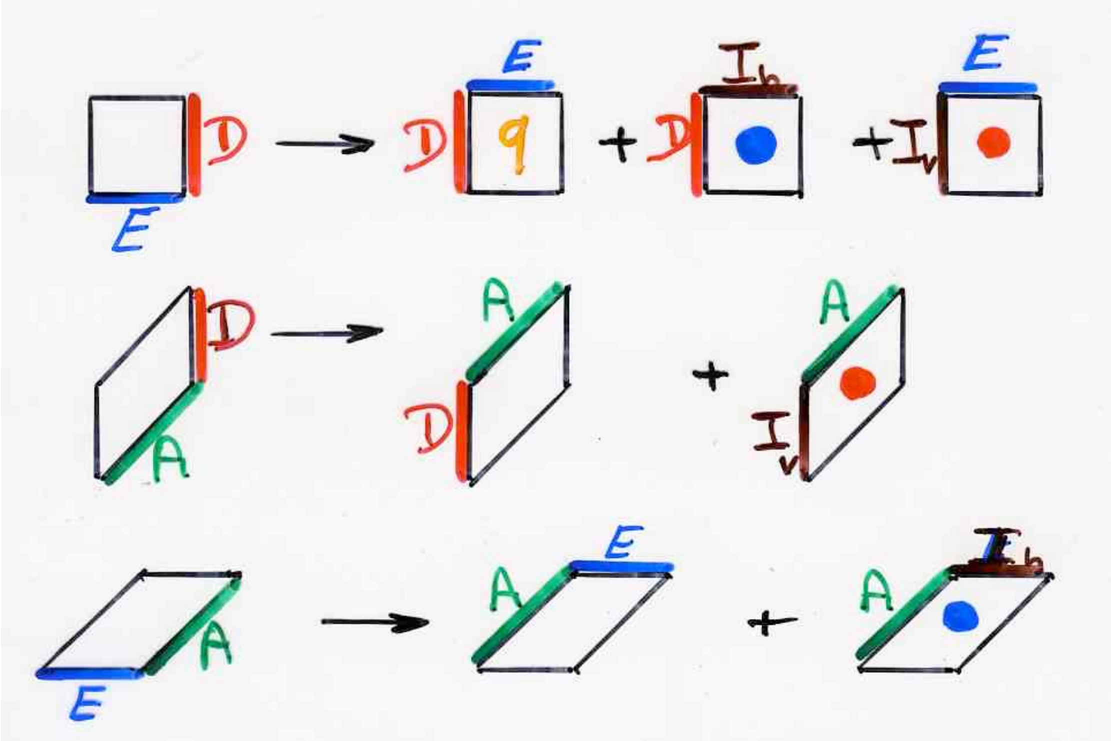
$q^{15} E^{2} A^{2} D^{2}$

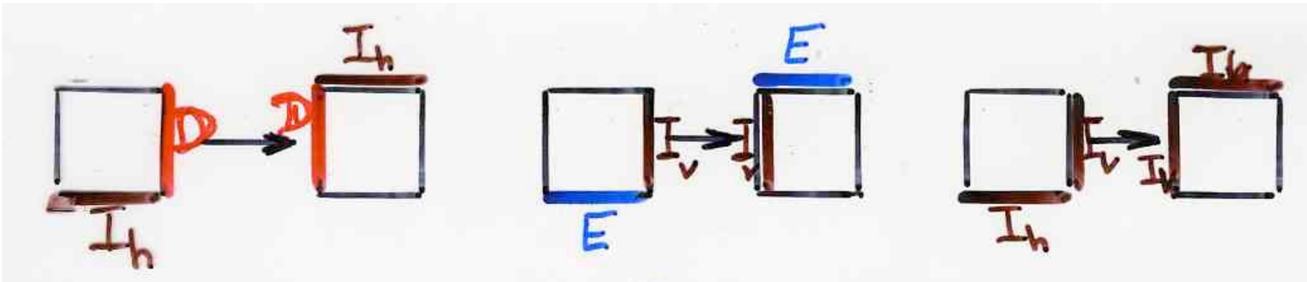
$$
\begin{cases}
DE = qED + D + E \\
DA = qAD + A \\
AE = qEA + A
\end{cases}
$$

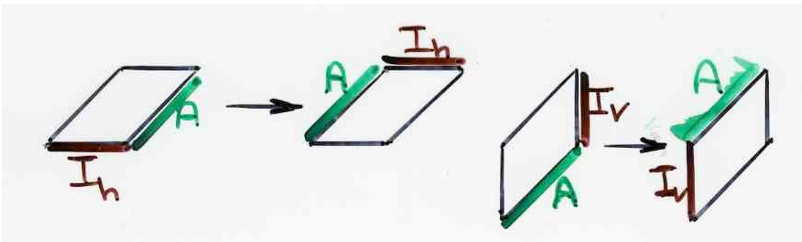
rewriting rules

$$
\begin{aligned}
DE &\rightarrow qED \text{ or } E \text{ or } D \\
DA &\rightarrow AD \text{ or } A \\
AE &\rightarrow EA \text{ or } A
\end{aligned}
$$

Planarisation of the rewriting rules


E
D
E
D
9
I_h
D
E
I_v
D
A
A
D
A
I_v
A
E
A
E
A
I_h

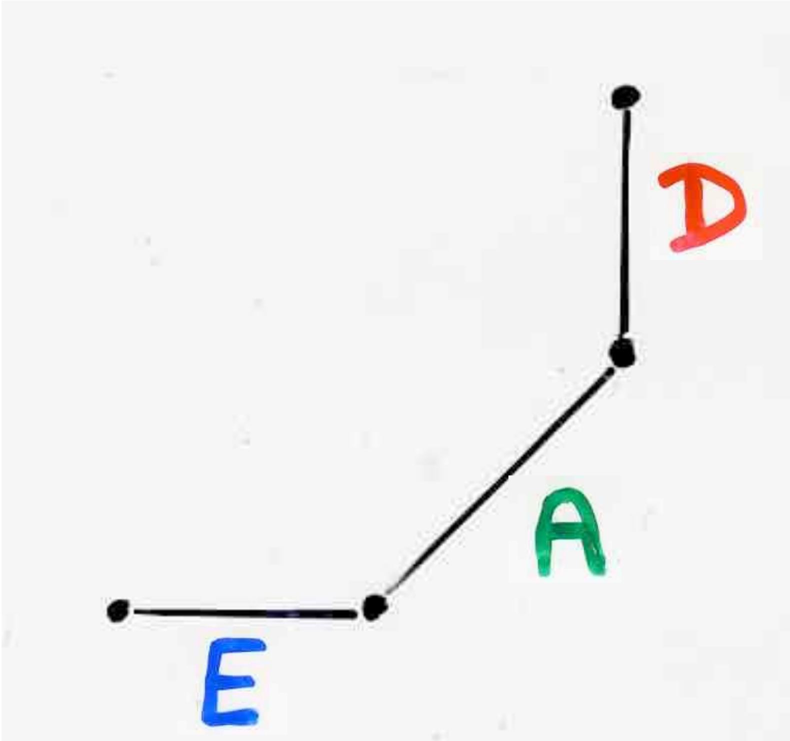

I_h
D
D
I_h
E
I_v
I_v
E
I_h
I_v
I_v
I_h


A
I_h
A
I_h
I_v
A
A
I_v

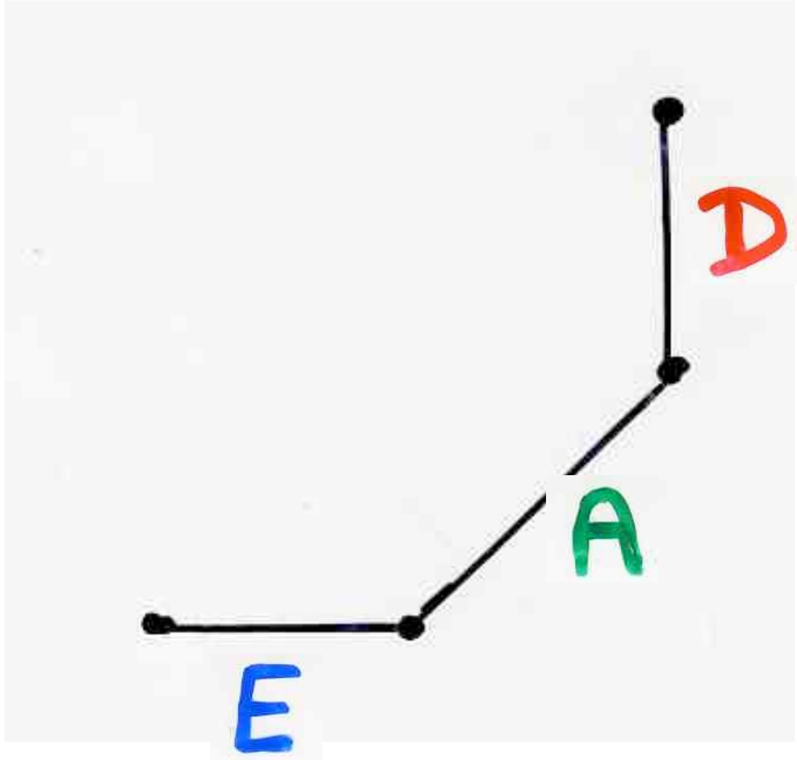
$$\begin{cases} DE = qED + I_v D + E I_v \\ DA = qAD + A I_v \\ AE = qEA + I_h A \end{cases}$$

$$\begin{cases} D I_h = I_h D \\ I_v E = E I_v \\ I_v I_h = I_h I_v \\ A I_h = I_h A \\ I_v A = A I_v \end{cases}$$

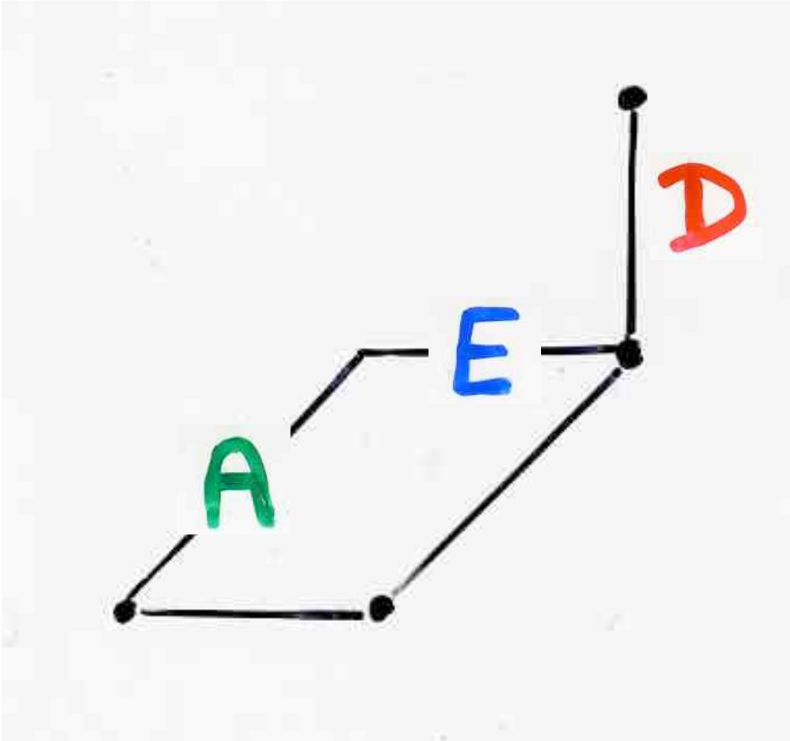
an example



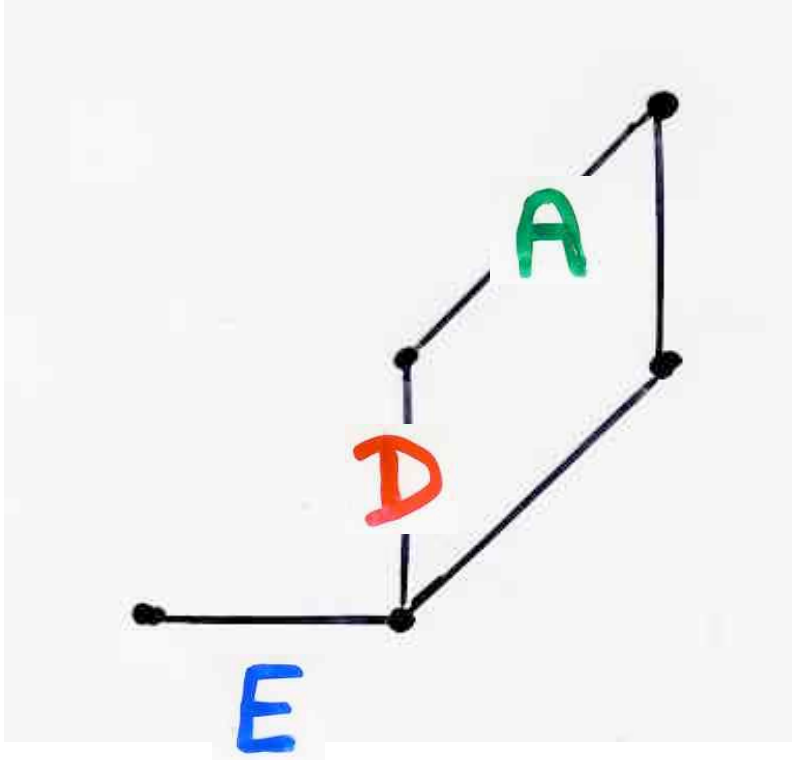
D A E



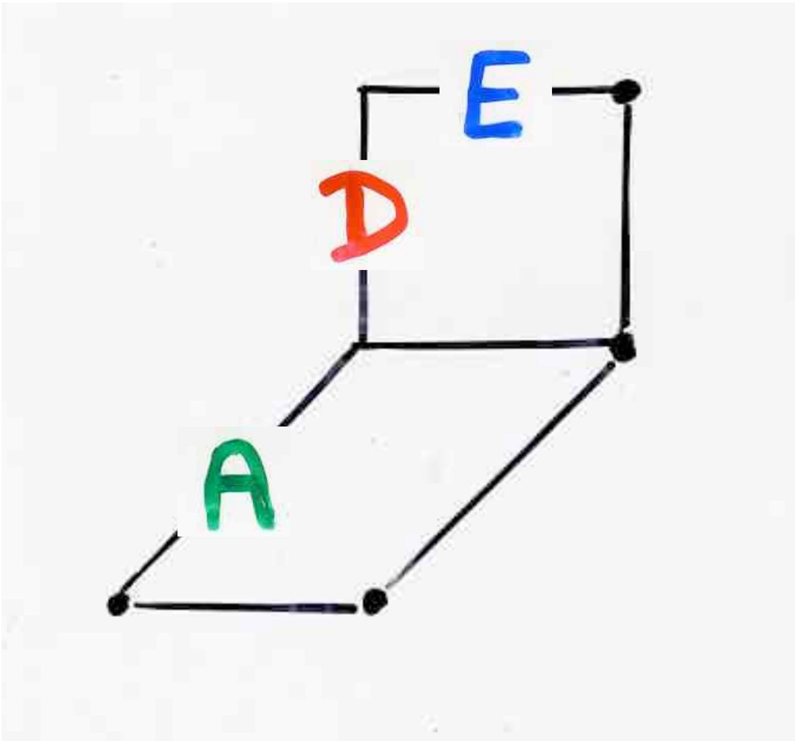
D A E

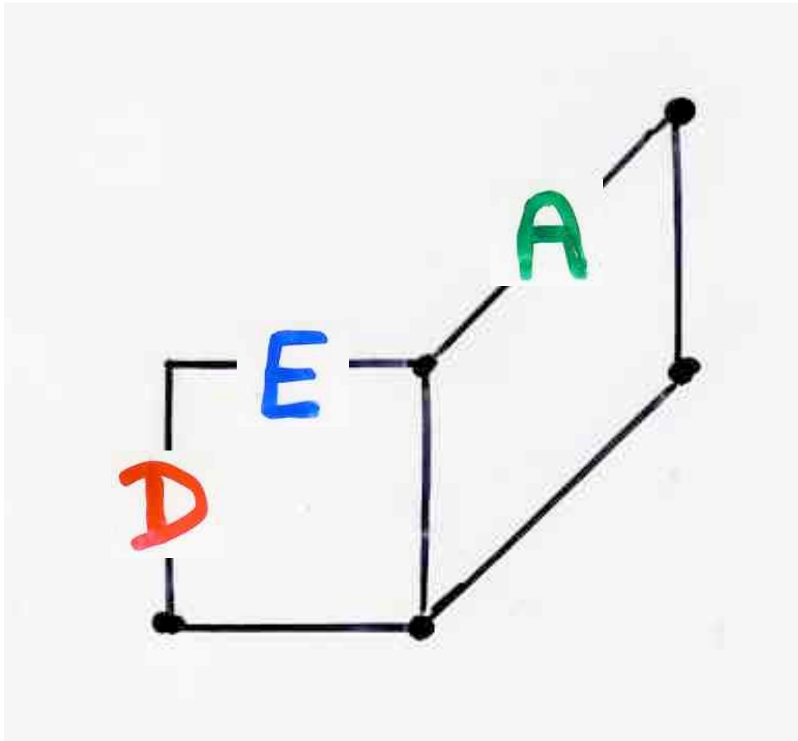


D A E
D E A



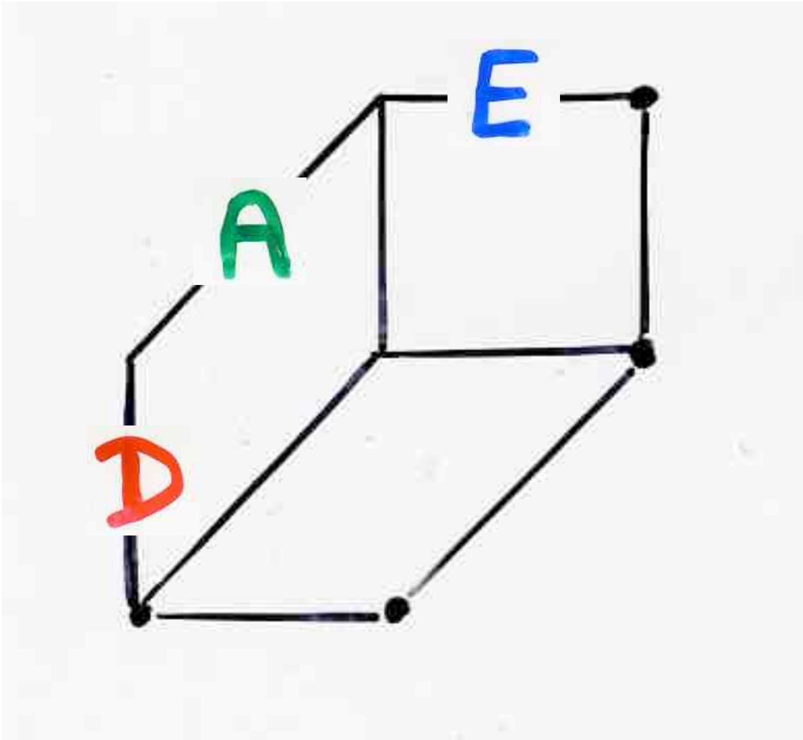
D A E
A D E

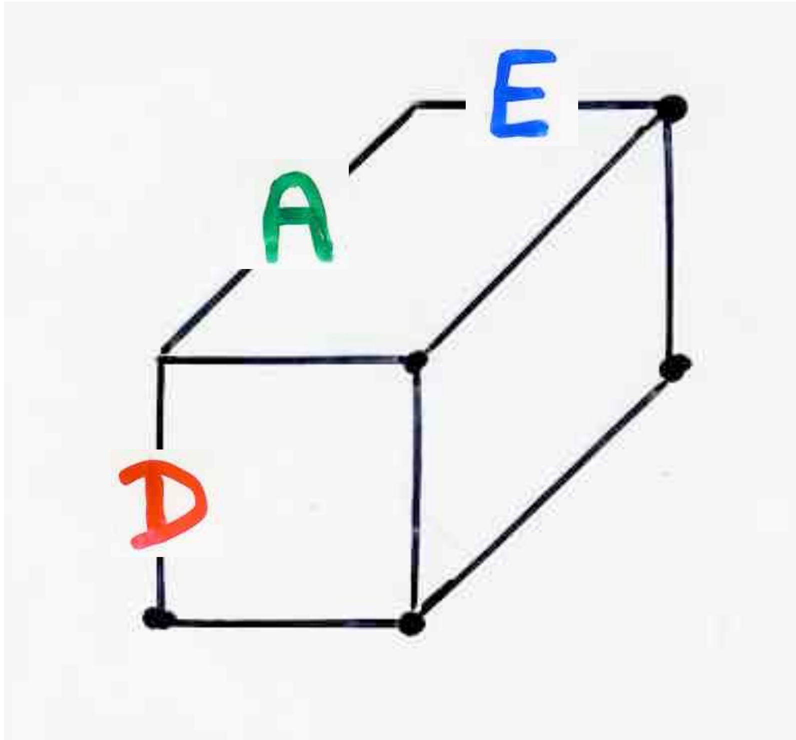




D	A	E
D	E	A
E	D	A

D	A	E
A	D	E
A	E	D

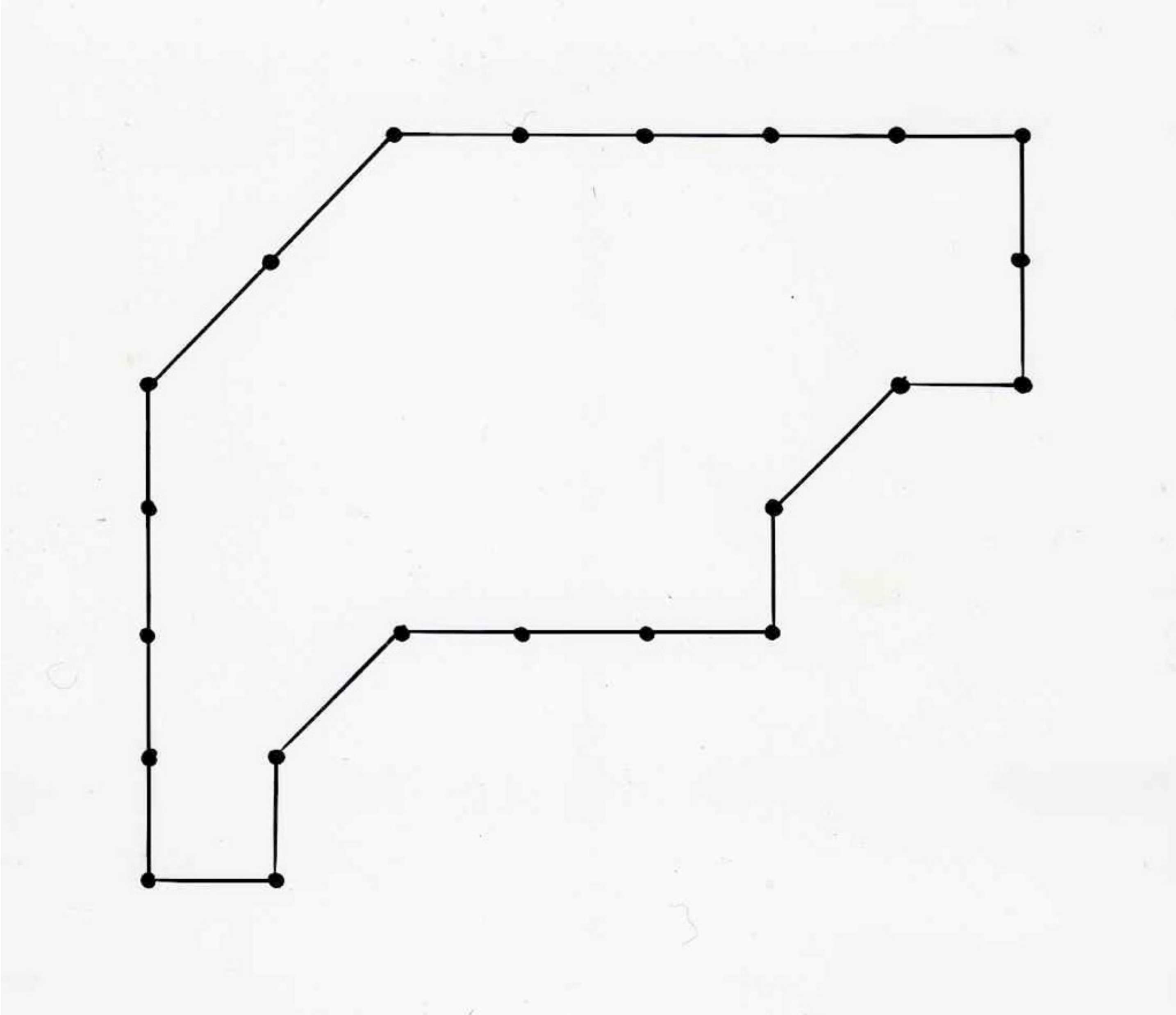




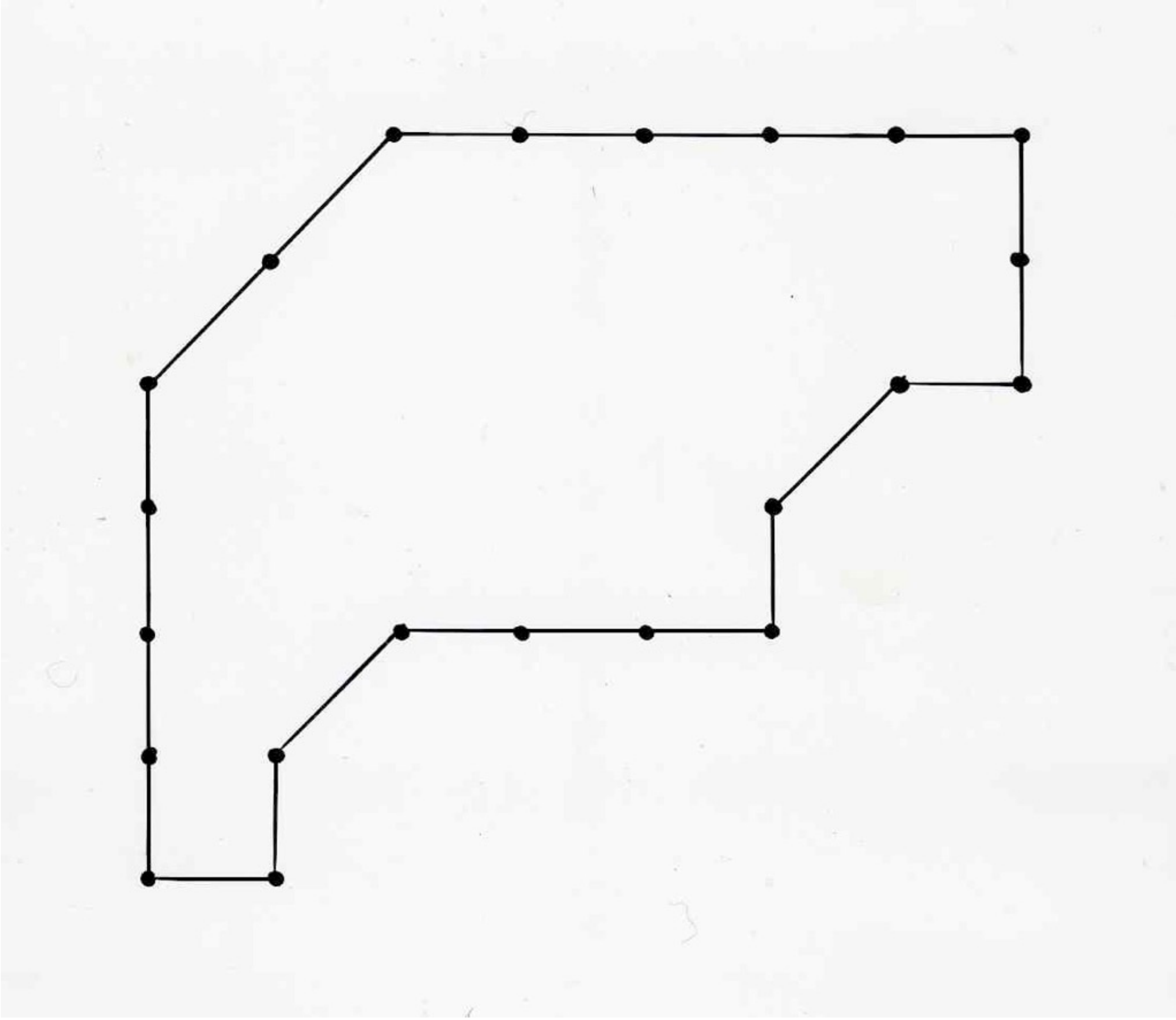
D	A	E
D	E	A
E	D	A
E	A	D

D	A	E
A	D	E
A	E	D
E	A	D

D D E A D E E E A D E



D D E A D E E E A D E



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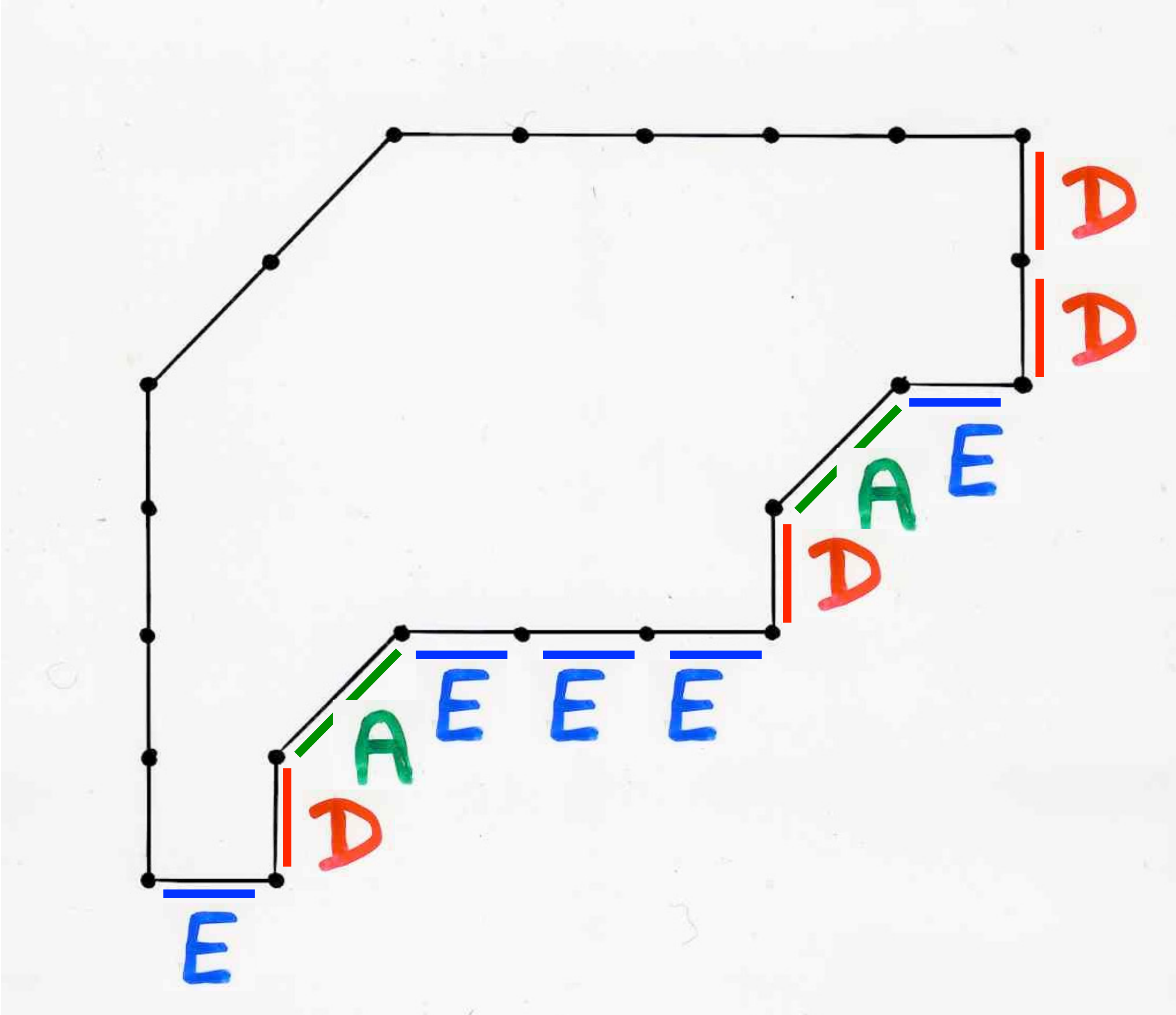

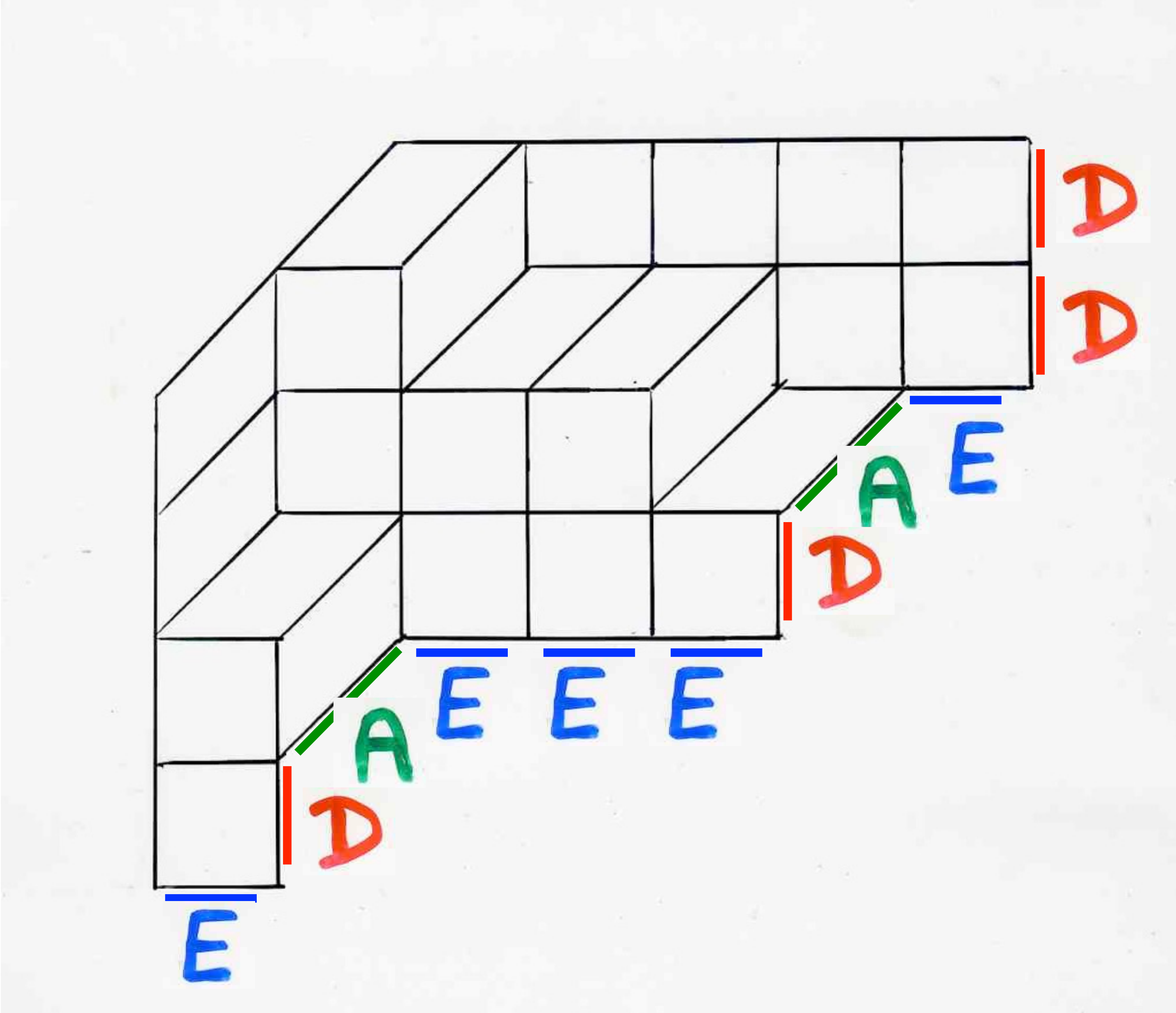

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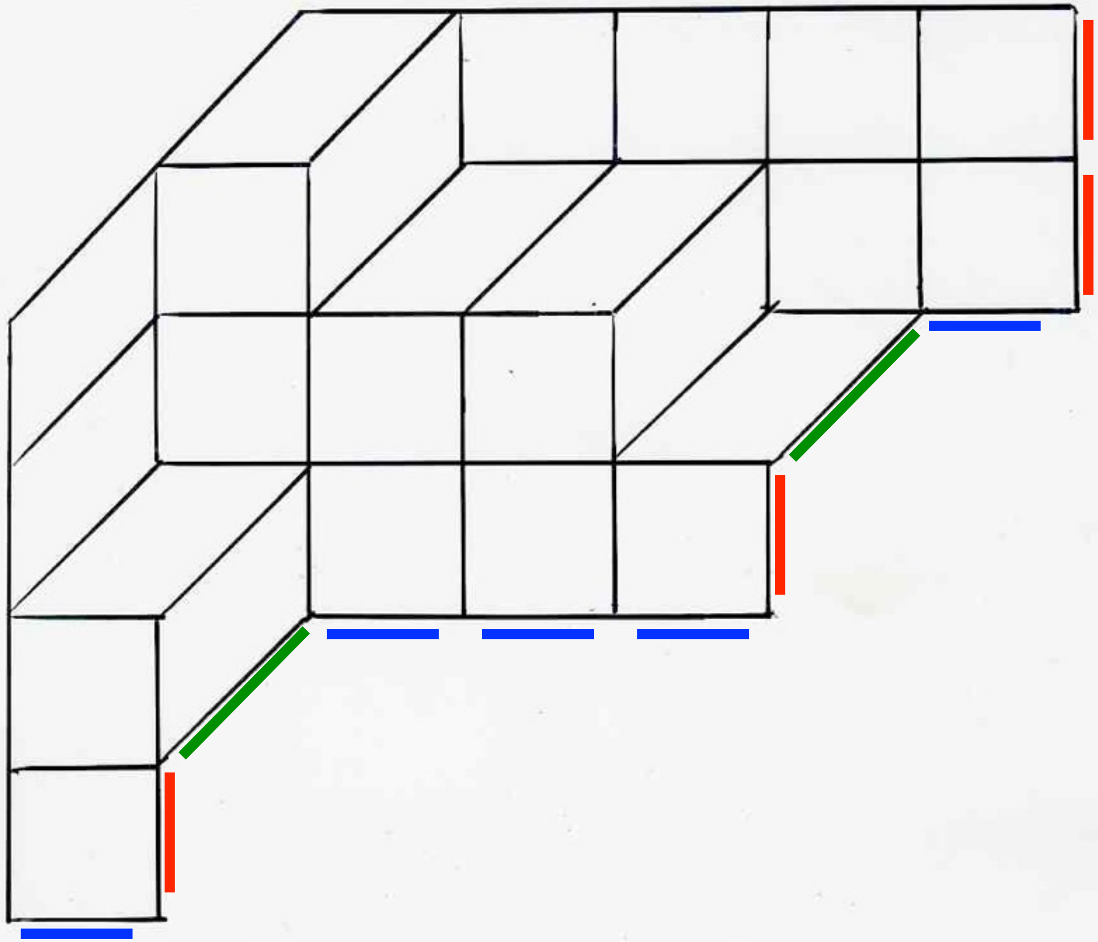

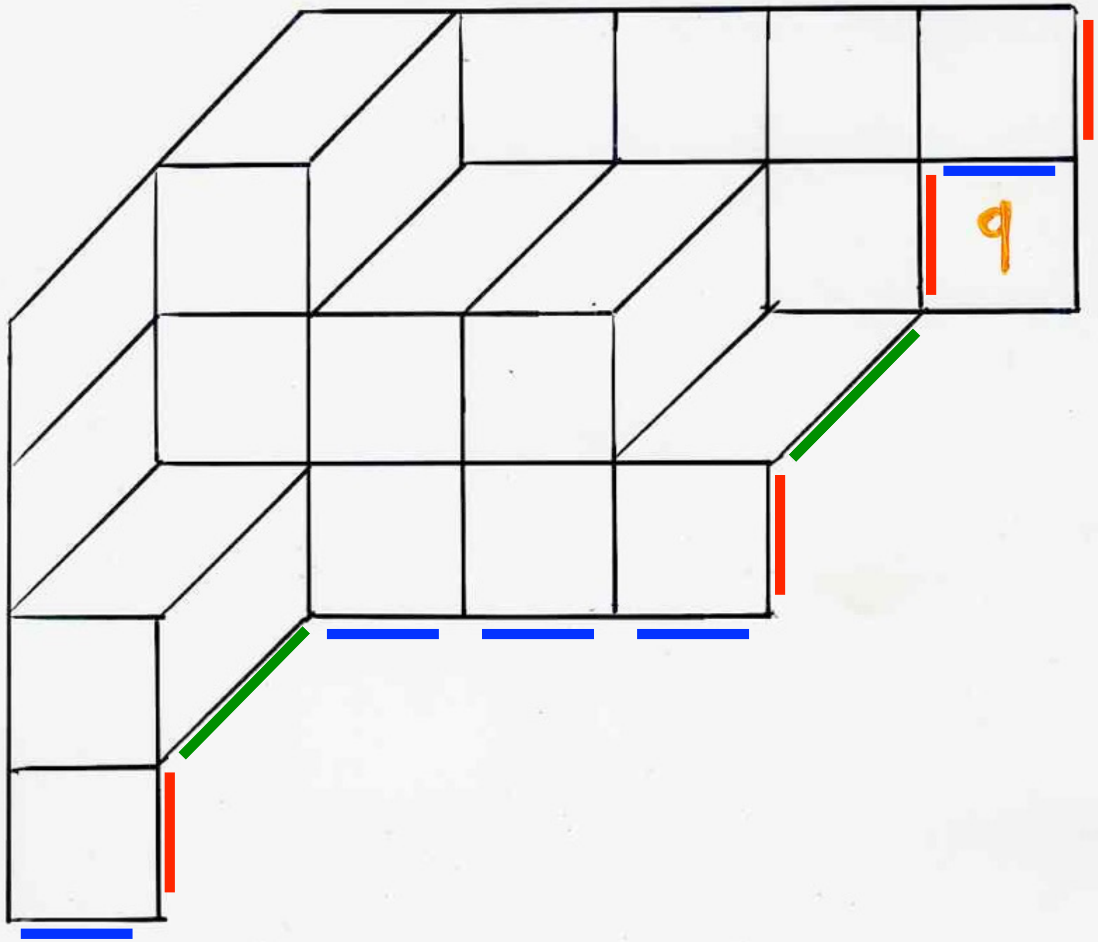
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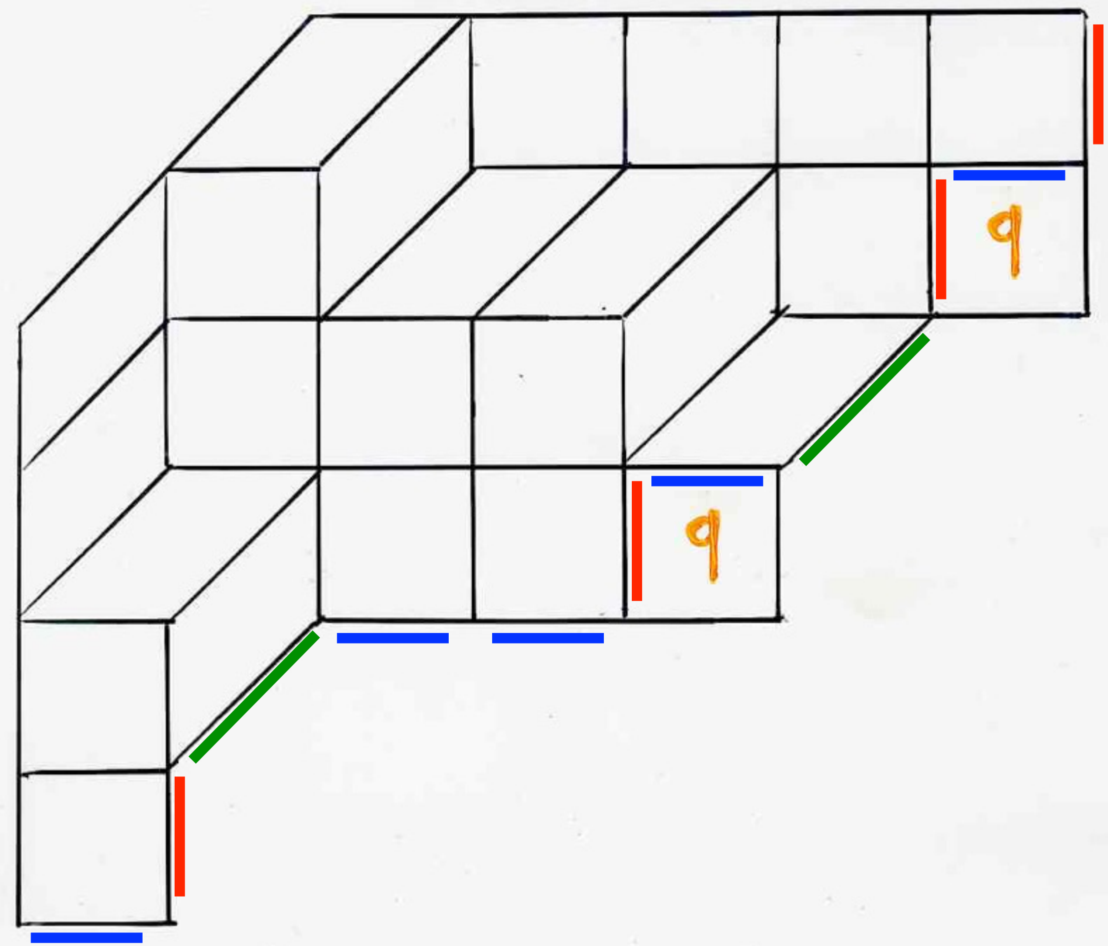
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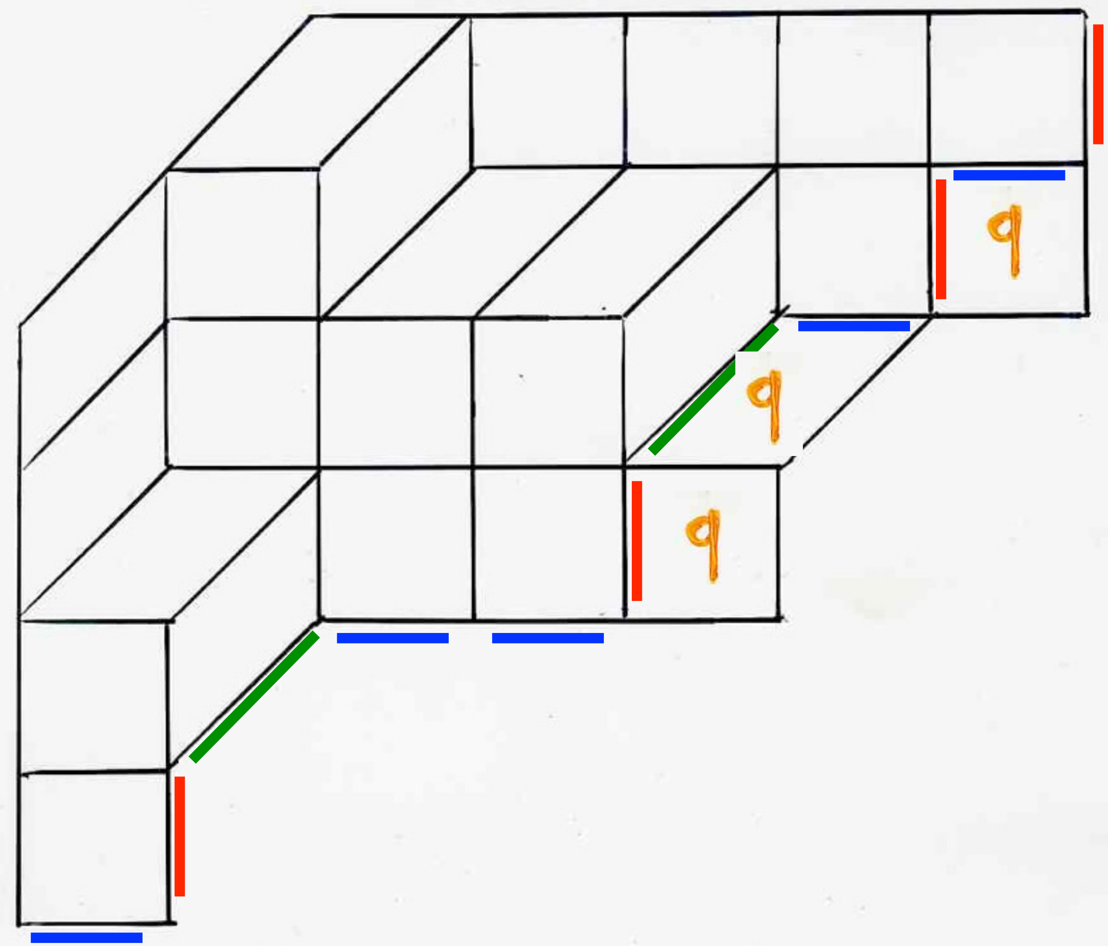
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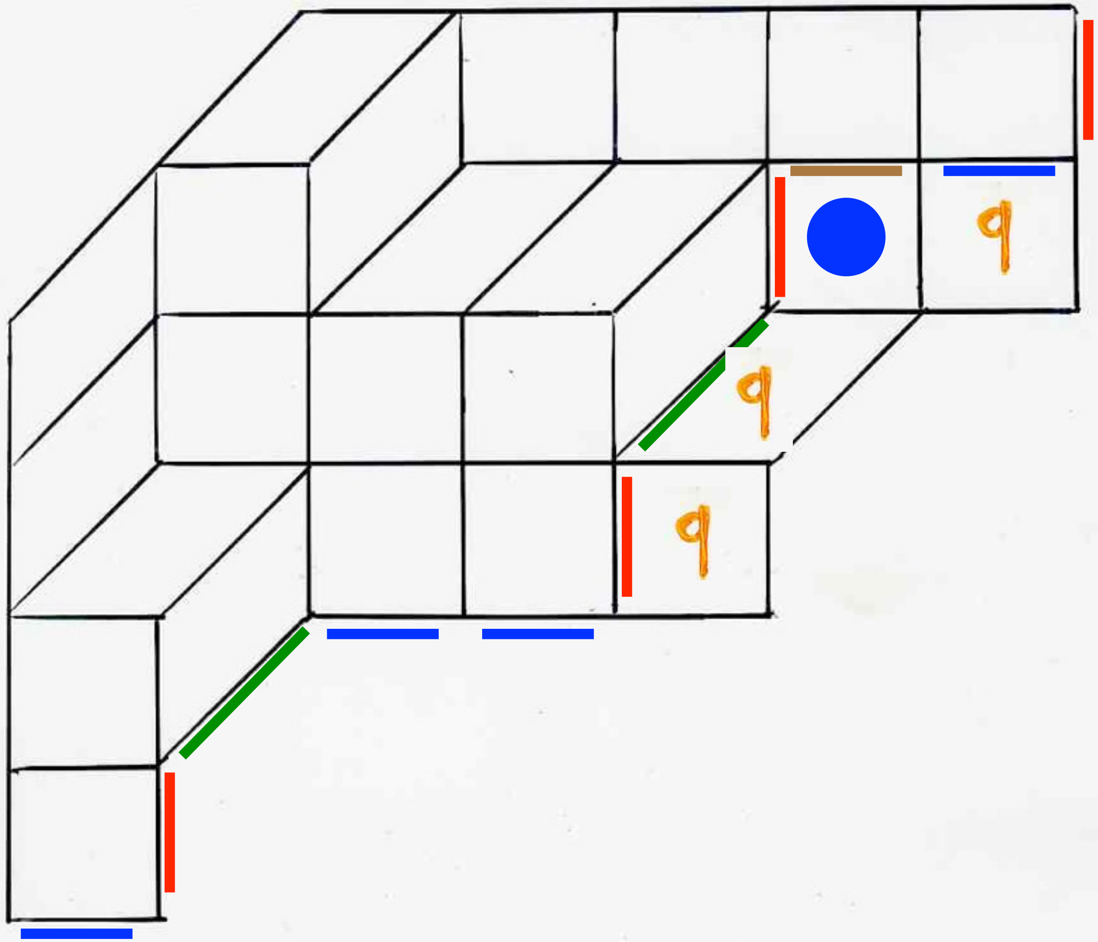
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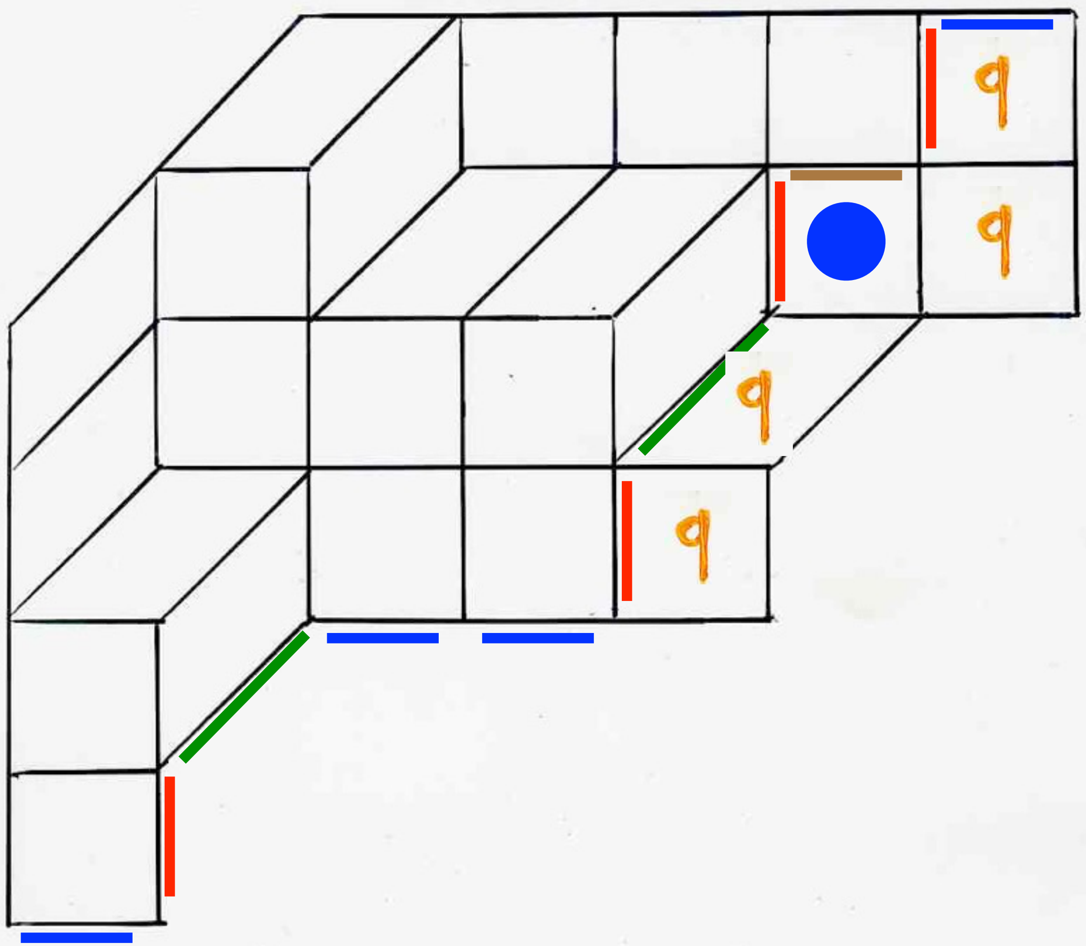
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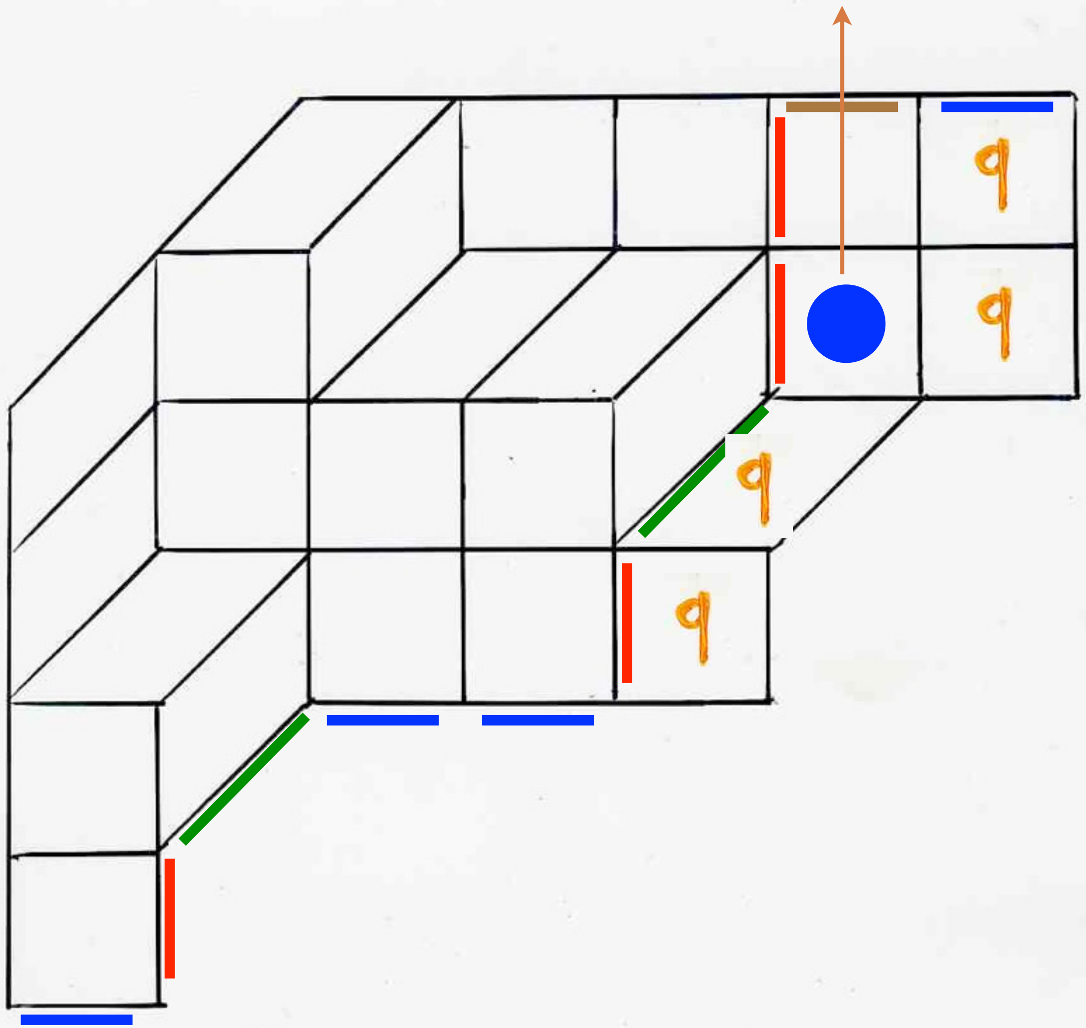
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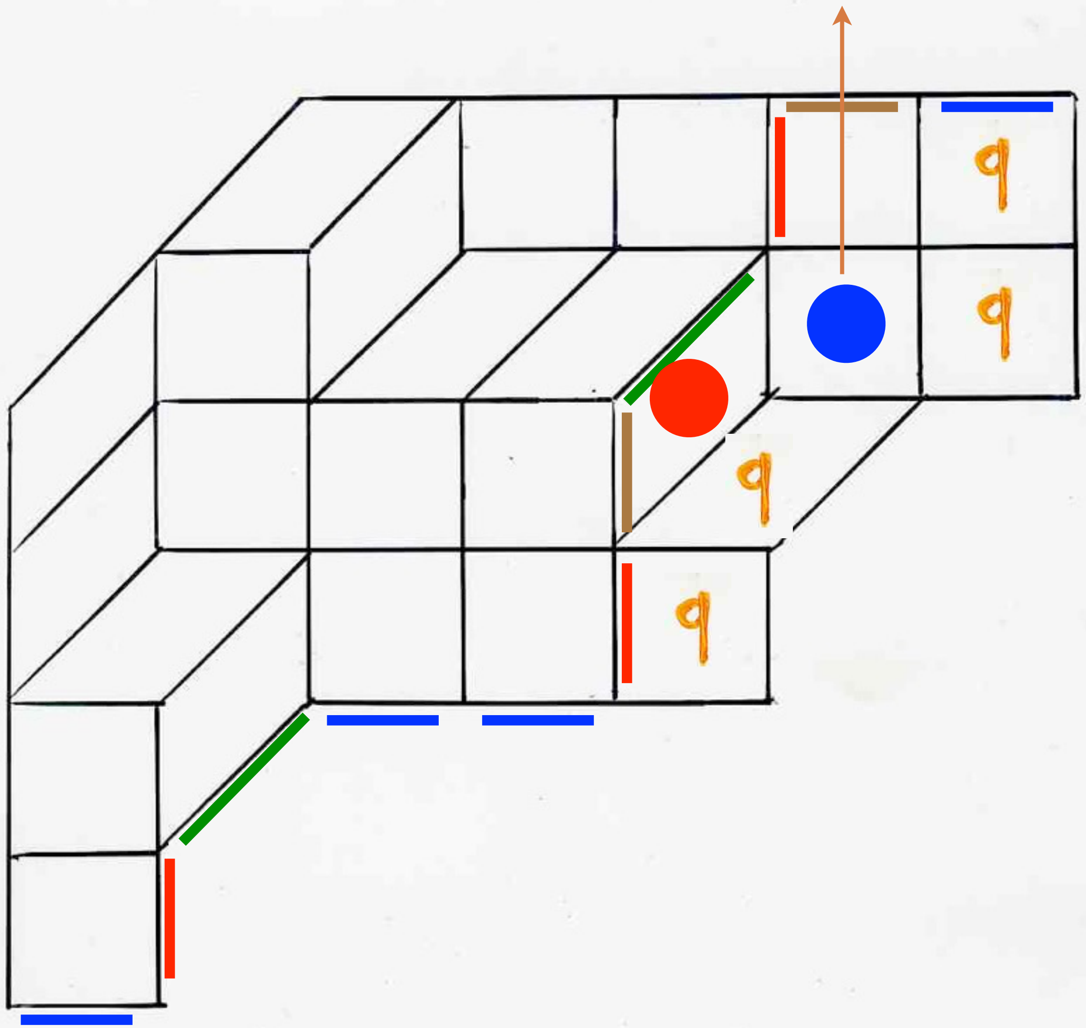
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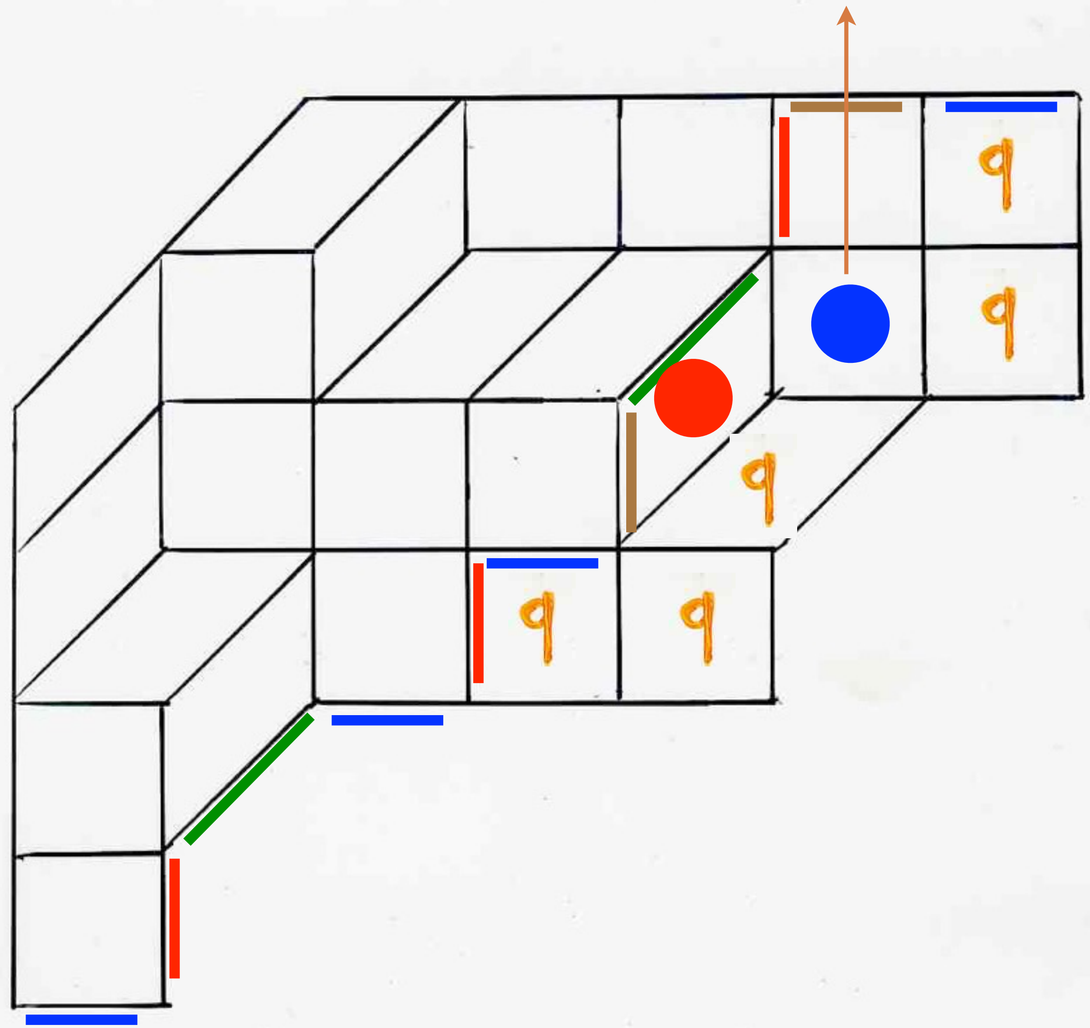

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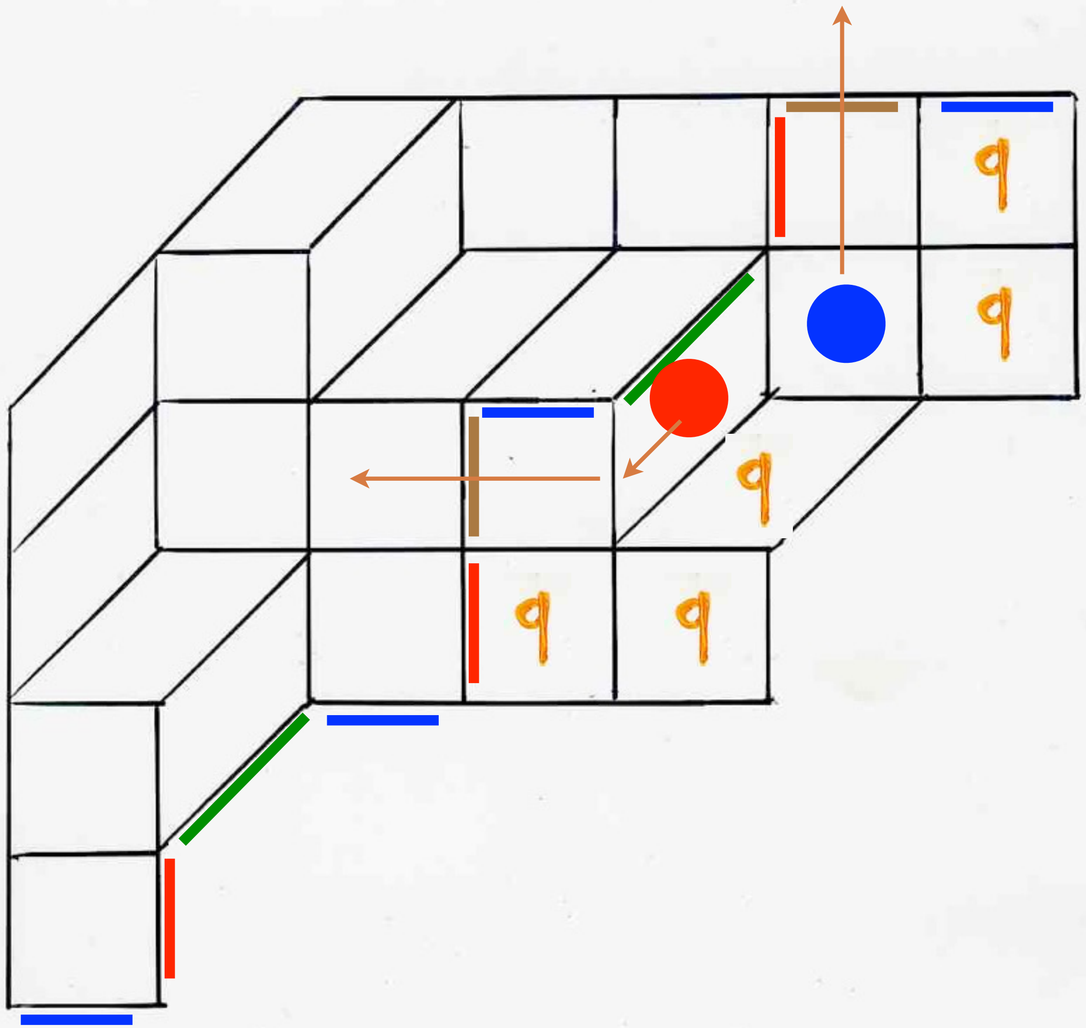

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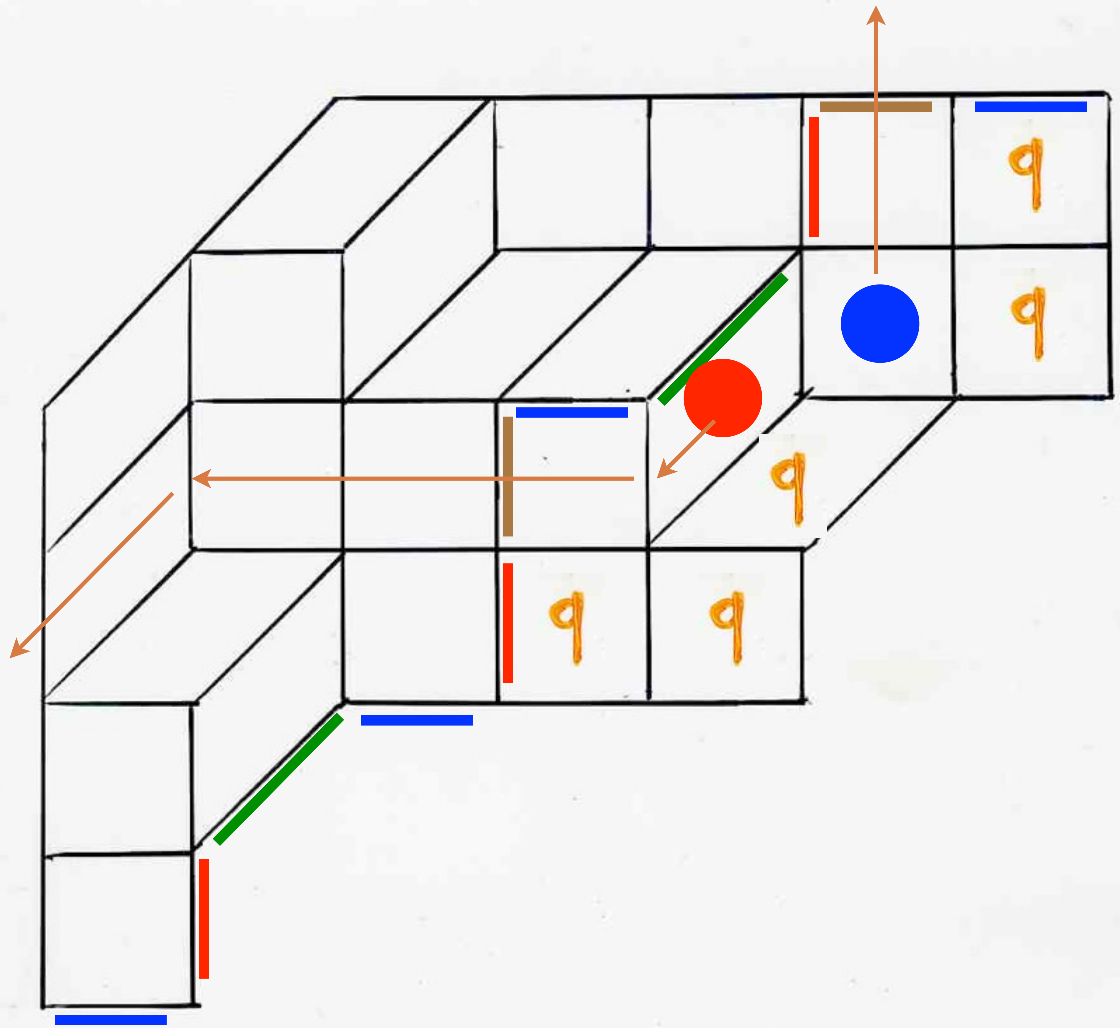

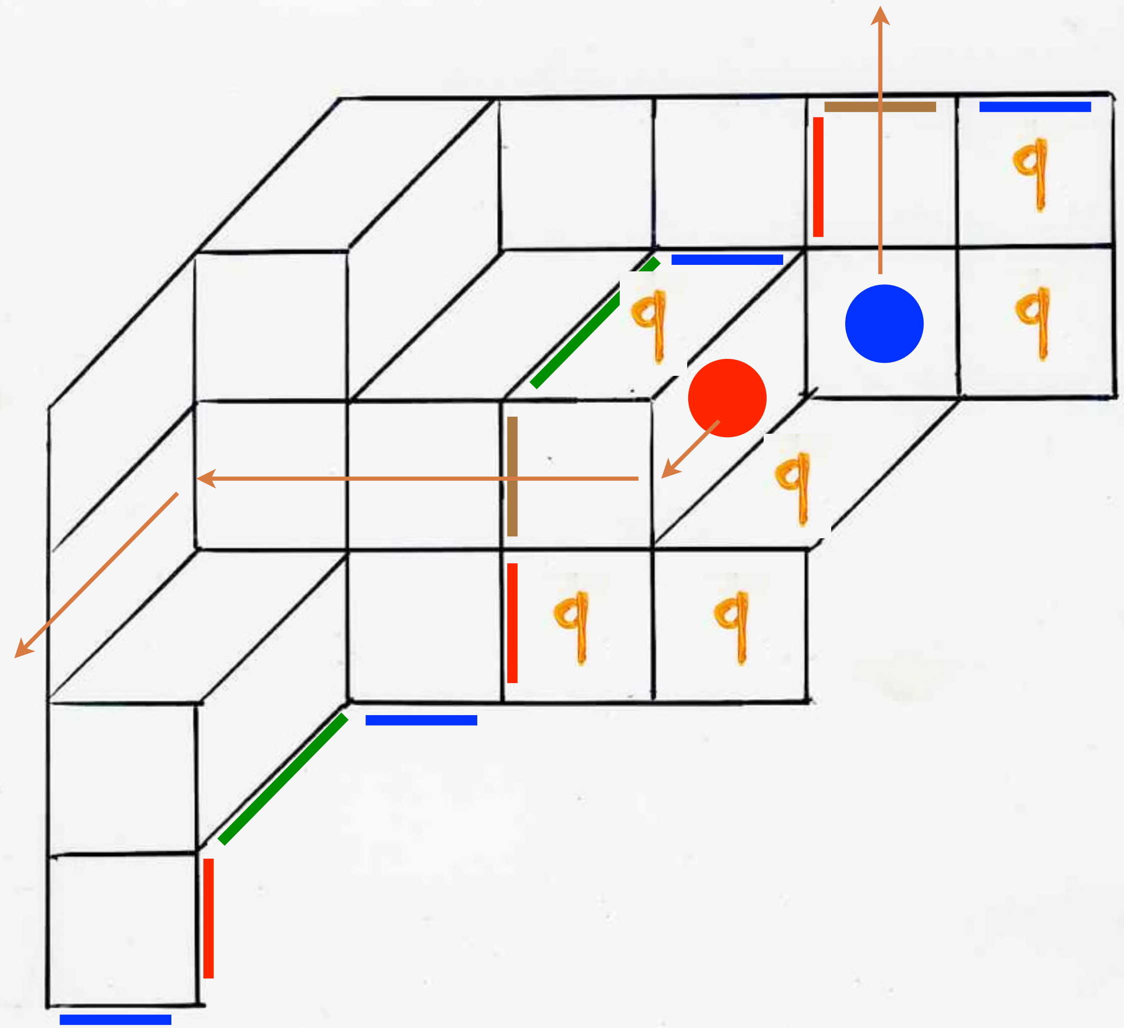
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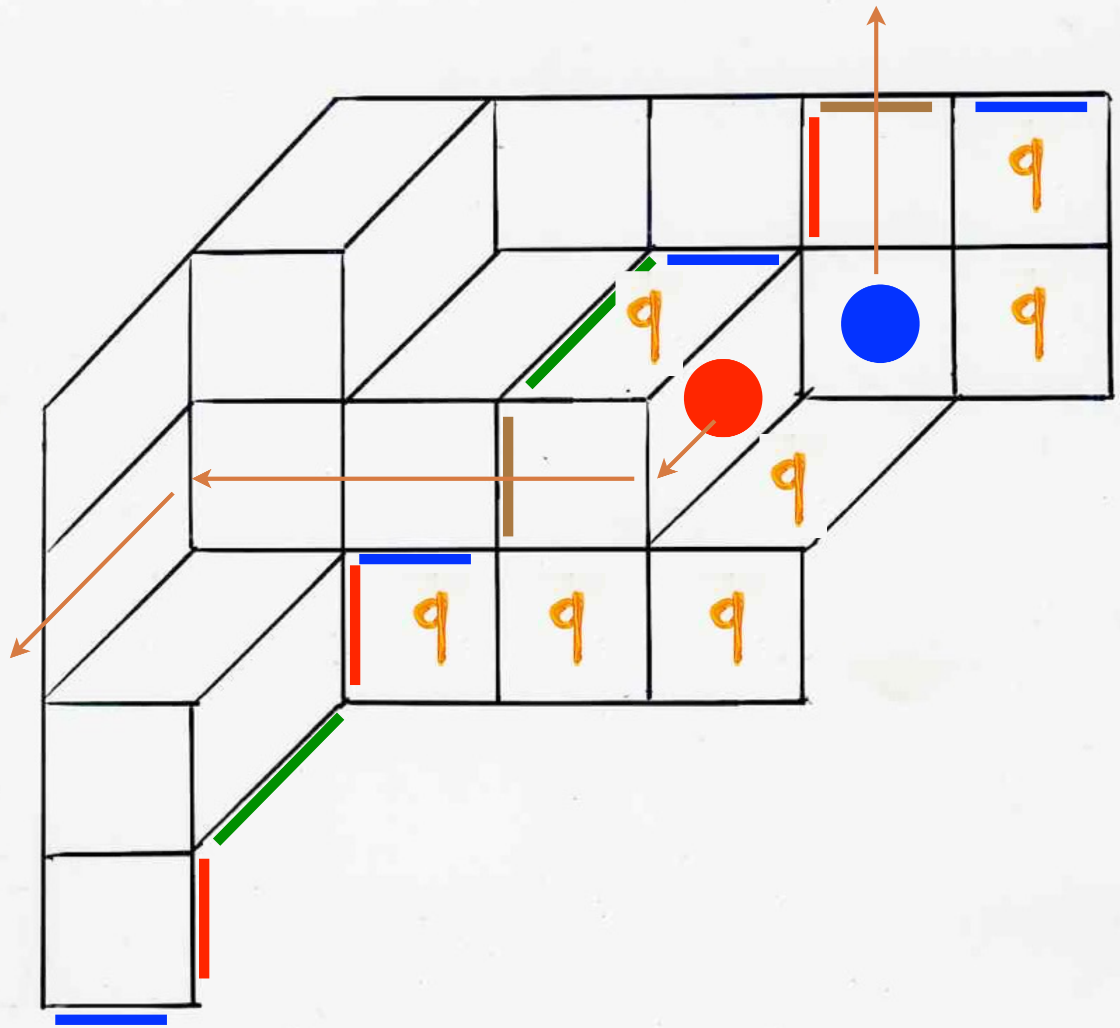
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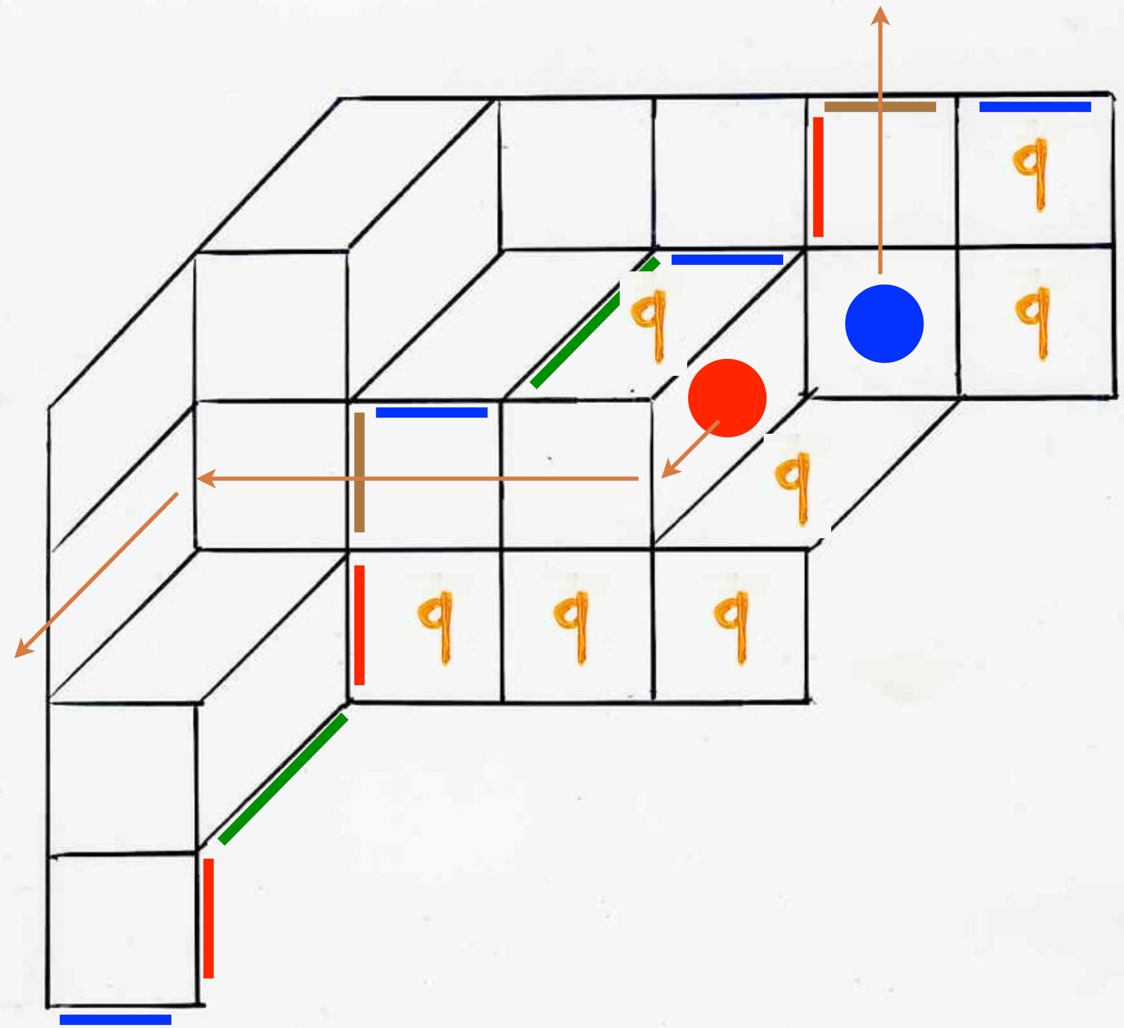
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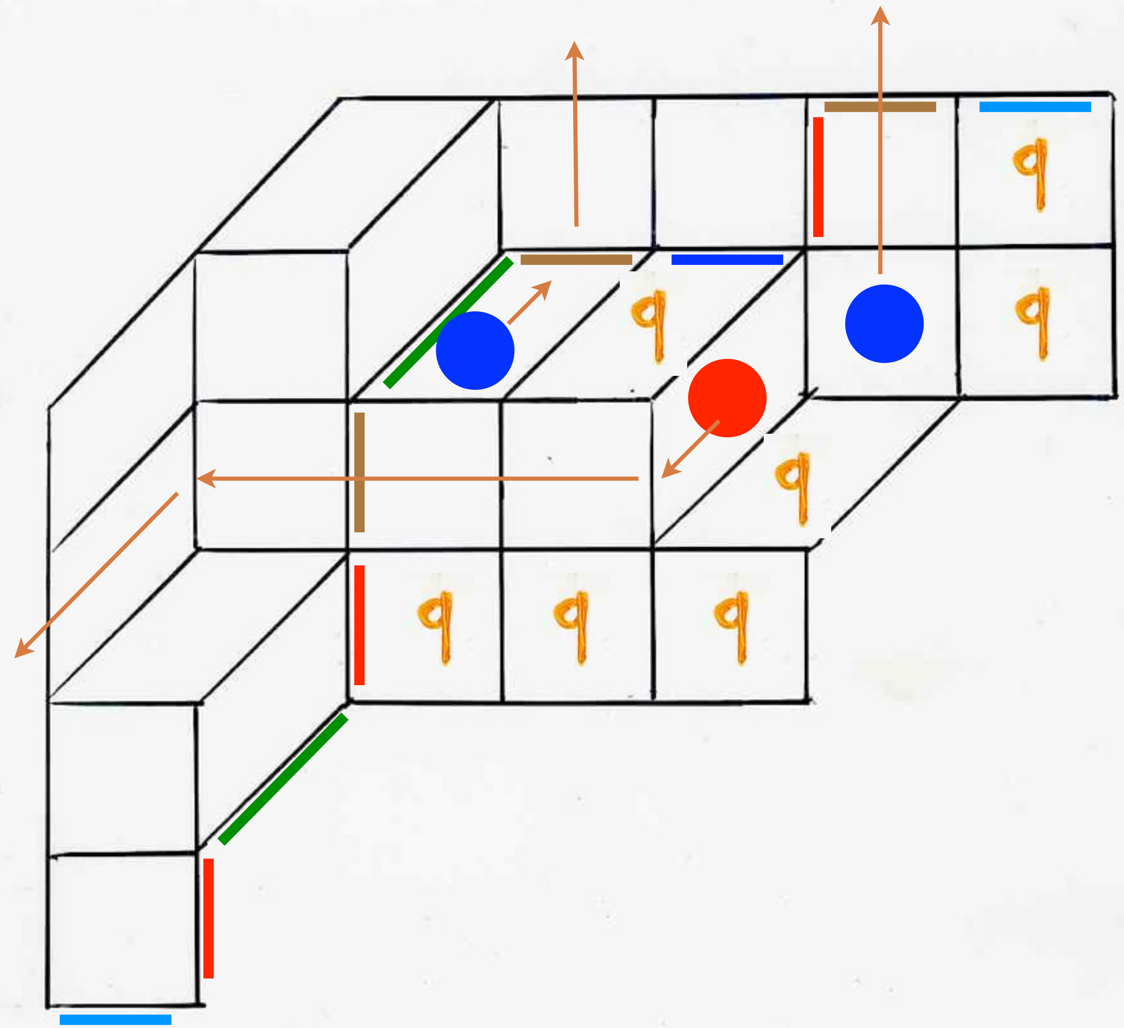
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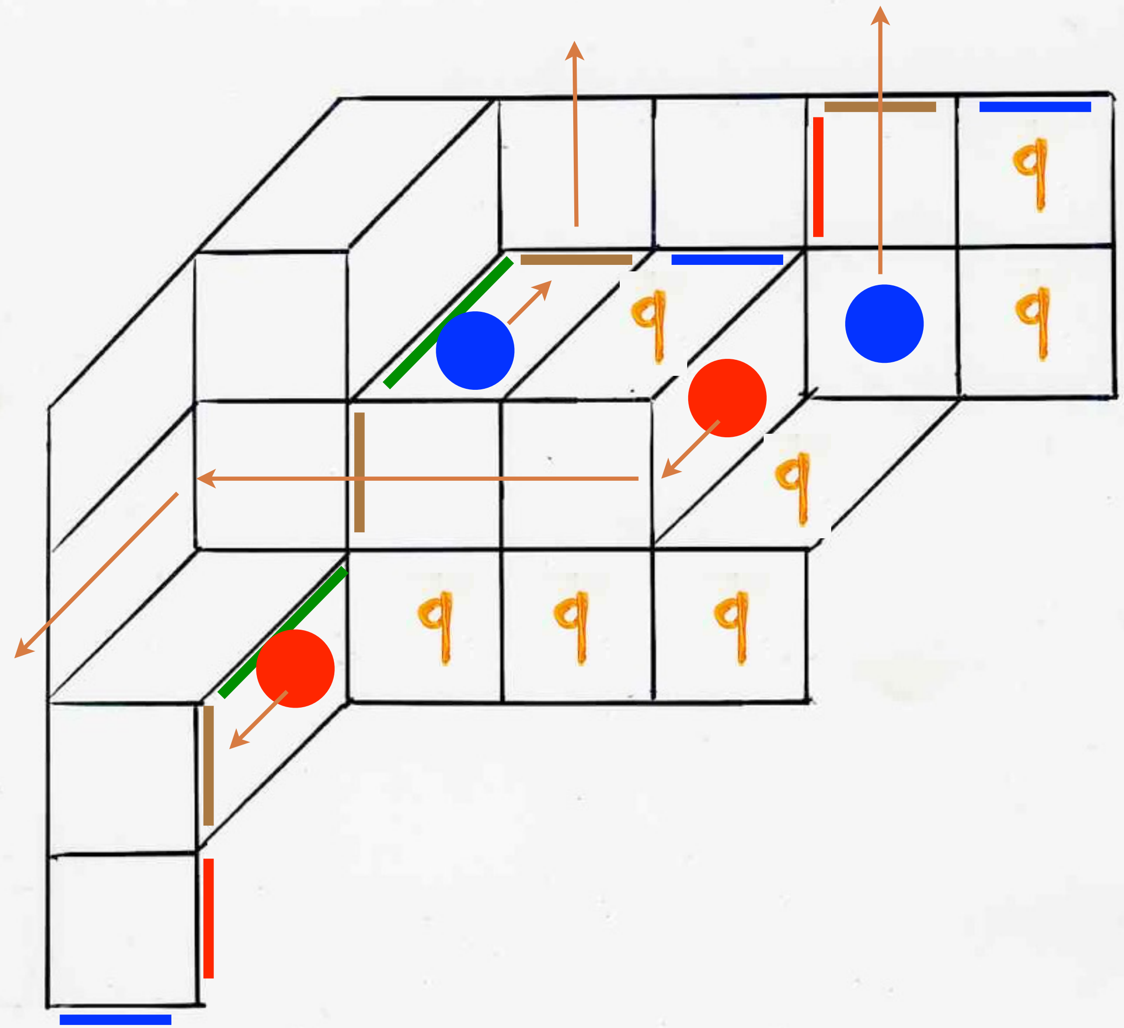

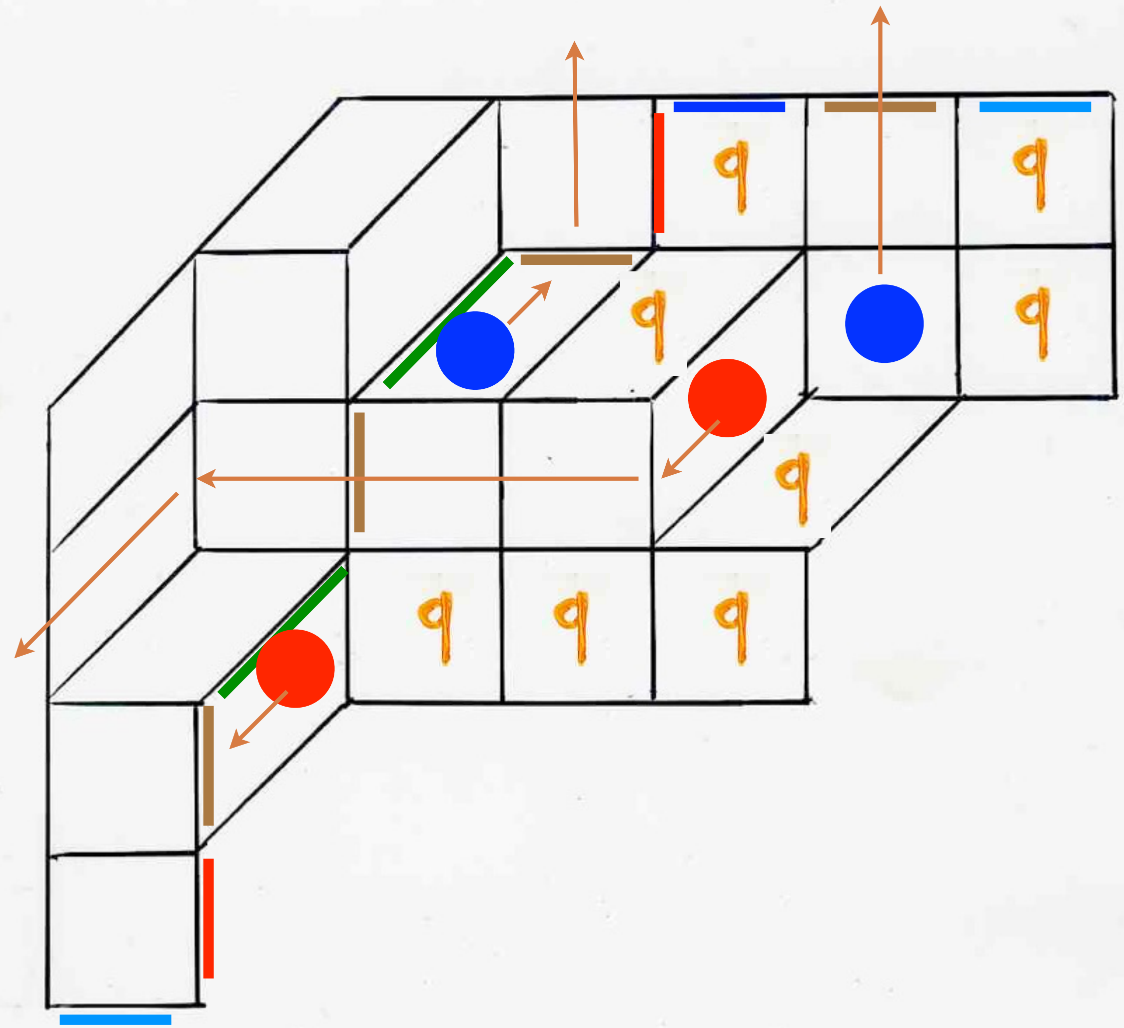

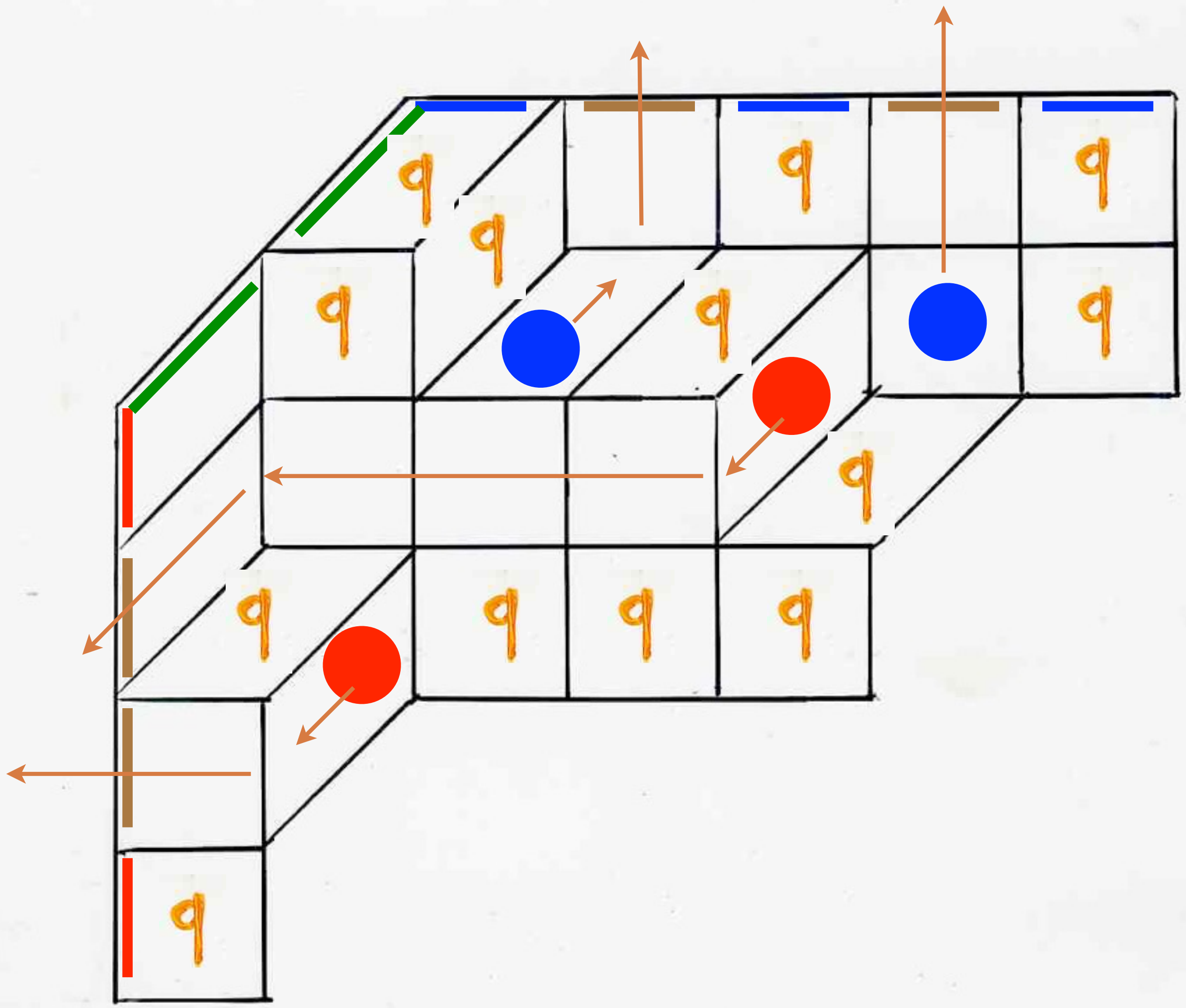

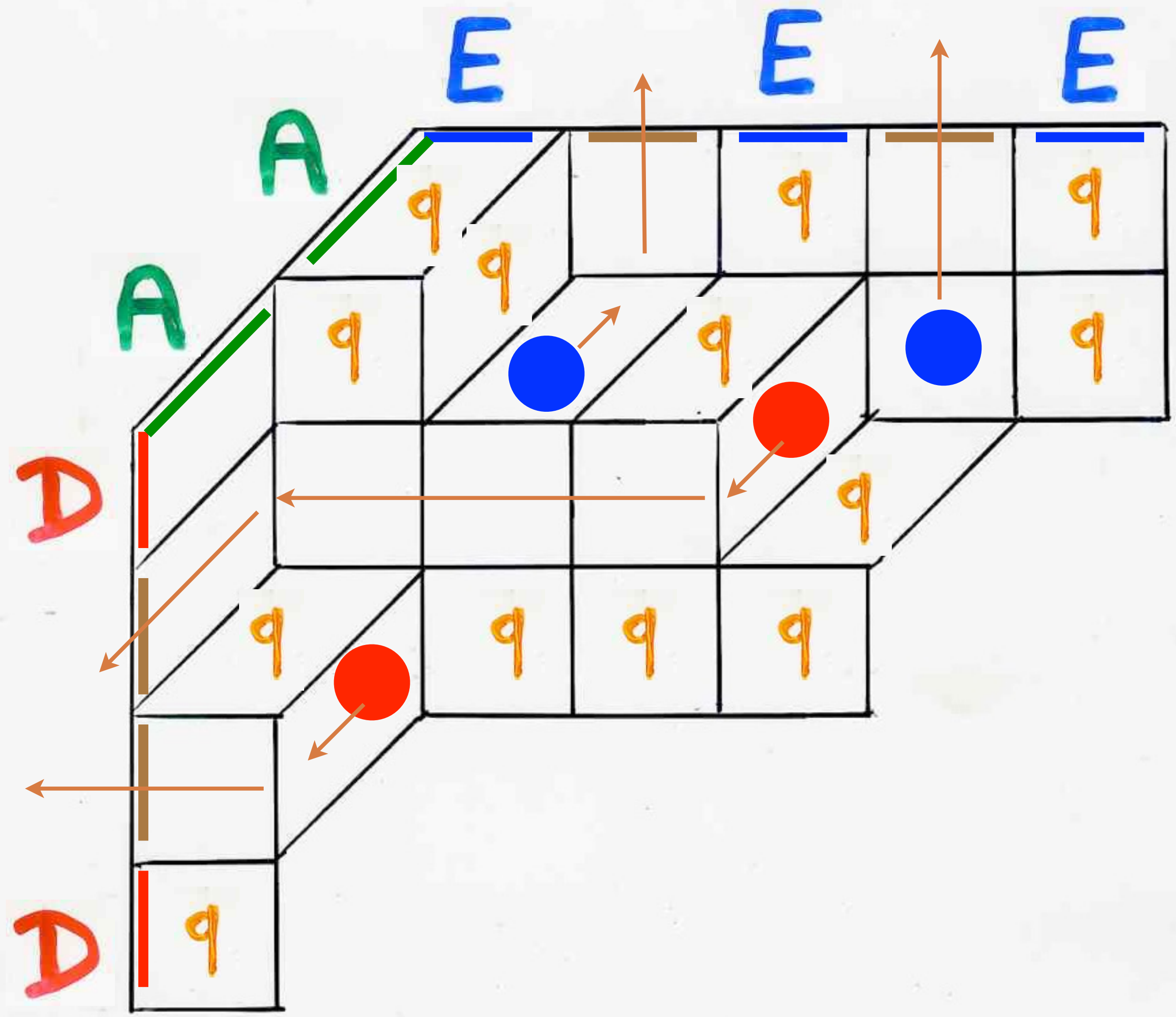
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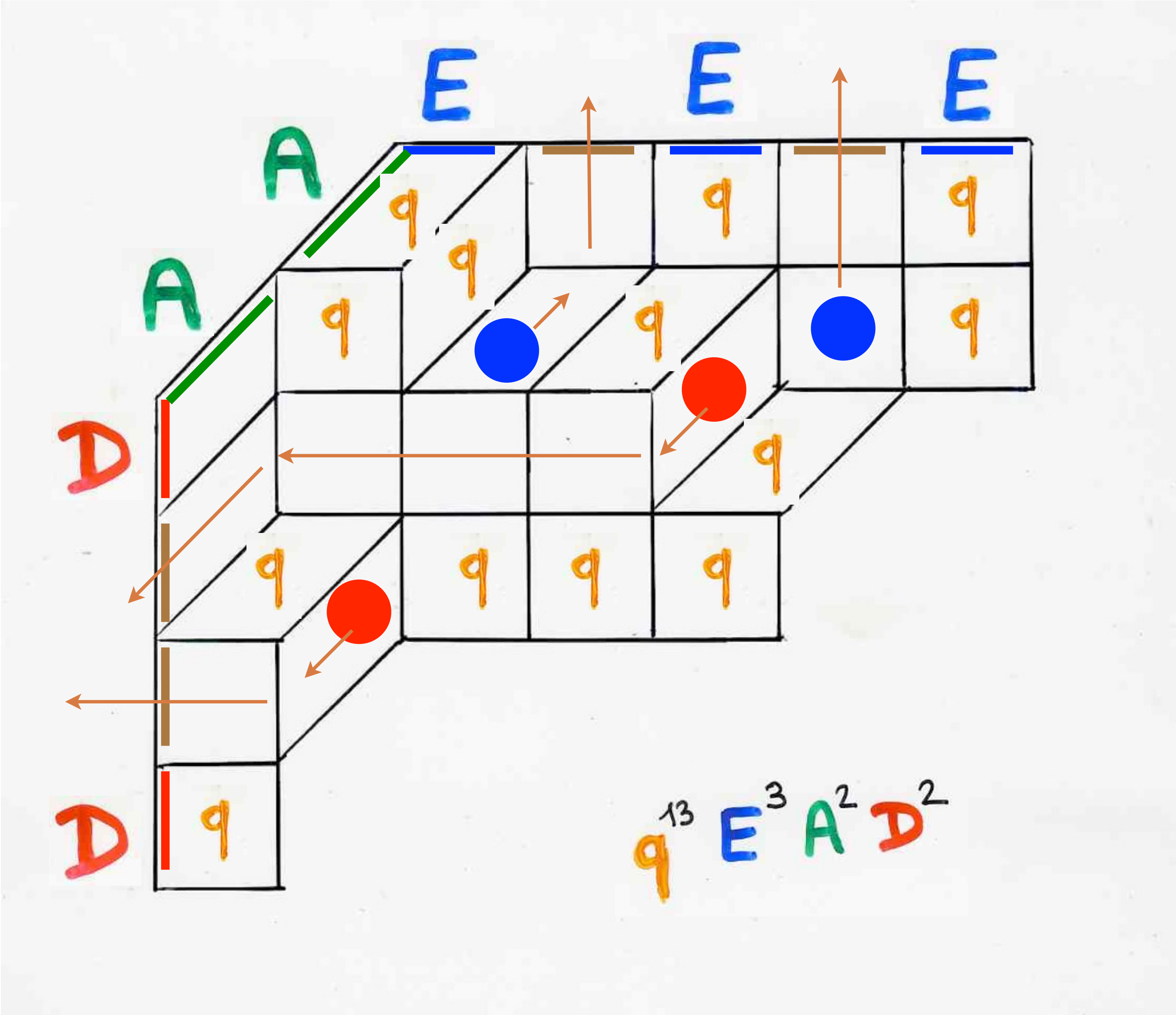


combinatorial interpretation
of
stationary probabilities
(proof)

$$\mathrm{Prob}(X) = \frac{1}{Z_{n,r}} \langle W | \prod_{i=1}^{n} D\,\mathbb{1}_{(x_i=\bullet)} + A\,\mathbb{1}_{(x_i=\bullet)} + E\,\mathbb{1}_{(x_i=\circ)} | V \rangle$$

$$\langle W | X | V \rangle = \sum_{T \in R(X,\tau)} q^{t} \left(\frac{1}{\alpha}\right)^{i} \left(\frac{1}{\beta}\right)^{j} \langle W | A^{r} | V \rangle$$

i = nb of free north-strips in T
(=not containing an $\bullet$)

j = nb of free south-strips in T
(=not containing a $\bullet$)

t = nb of cells labeled q in T

$$Z_{n,r} = \text{coeff. of } y^r \text{ in } \langle W | (D + yA + E)^n | V \rangle$$

$$Z_{n,r} = Z^*_{n,r} \langle W | A^r | V \rangle$$

$$Z^*_{n,r} = \sum_{X} \sum_{T \in R(X, T_X)} q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j$$

$$\mathrm{Prob}(X) = \frac{1}{Z^*_{n,r}} \sum_{T \in R(X, T_X)} \underbrace{q^t \left(\frac{1}{\alpha}\right)^i \left(\frac{1}{\beta}\right)^j}_{wt(T)}$$

enumeration
of
rhombic alternative tableaux

$$Z^*_{n,r}(\alpha=\beta=q=1) = \binom{n}{r}\frac{(n+1)!}{(r+1)!}$$

Lah numbers

nb of "assemblées" of permutations

$$\left\{\begin{matrix}[7,10,5,8] & & [9,2,11,6]\\ & [3,1,4] & \end{matrix}\right\}$$

$$\exp\left(\frac{xt}{1-t}\right)$$

$$Z^*_{n,r}(\alpha, \beta, q=1) = \binom{n}{r} \prod_{i=r}^{n-1} (\alpha^{-1} + \beta^{-1} + i)$$

extension of the "exchange-fusion algorithm"

O. Mandelshtam, X.V. (2016)

extension of Laguerre histories to (a subset of) B_n

signed permutations

q

S. Corteel, A. Nunge (2017)

«assemblées» and species

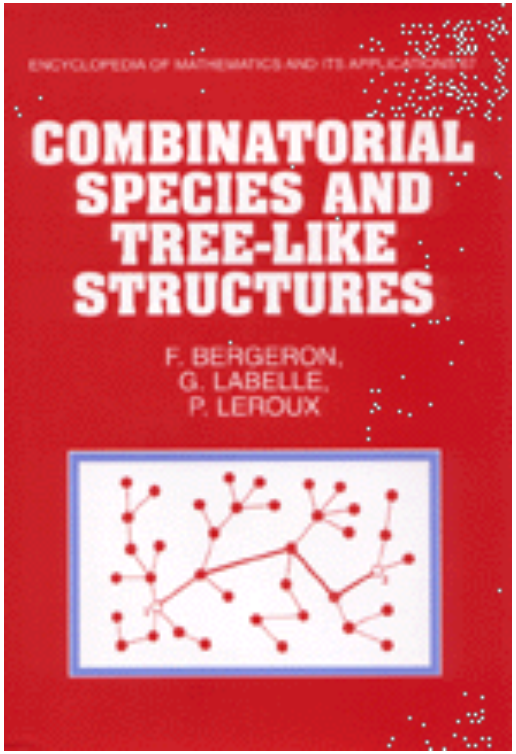
Reminding
(see BJC I, Ch3)

Combinatorial model
for exponential generating function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Species
(combinatorial)
structures

UQAM Montréal
Québec



A. Joyal
F. Bergeron
G. Labelle
P. Leroux
- - - - -

Encyclopedia of Maths.
and Applications
Cambridge Univ. Press
(1977)

Convention.

enumeration

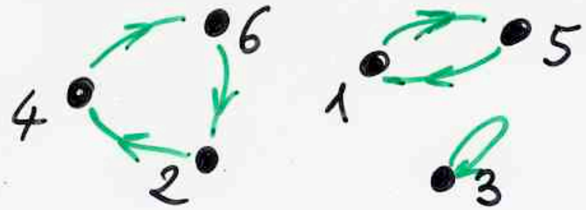
$$a_n = |F\{1,2,\dots,n\}|$$

Def. Generating function of the species F

$$F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

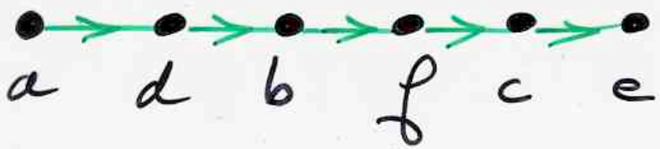
Examples

- **Permutations S** $\quad a_n = n! \quad S(t) = \frac{1}{1-t}$



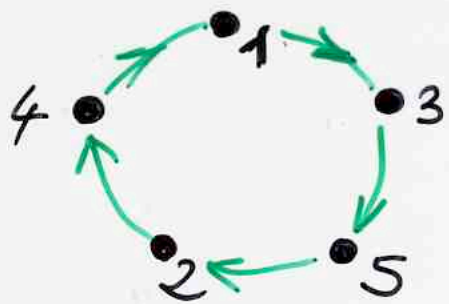
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}$$

- **Total order L** $\quad a_n = n! \quad L(t) = \frac{1}{1-t}$

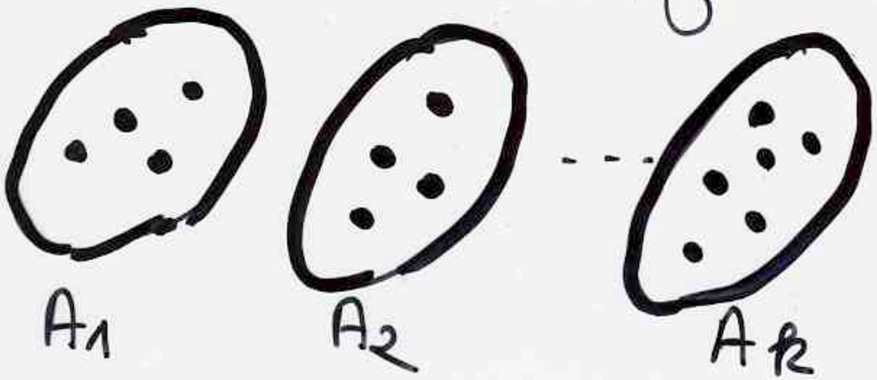


- **Cycle C** $\quad a_n = (n-1)!$

$$C(t) = \sum_{n \geq 1} \frac{t^n}{n} = \log (1-t)^{-1}$$



"**assemblée**" of $\mathbf{F}$-structures



partition of $\{1, 2, \ldots, n\}$

"assemblée" of $\mathbf{F}$-structures

A_1 A_2 A_k

partition of $\{1, 2, \ldots, n\}$

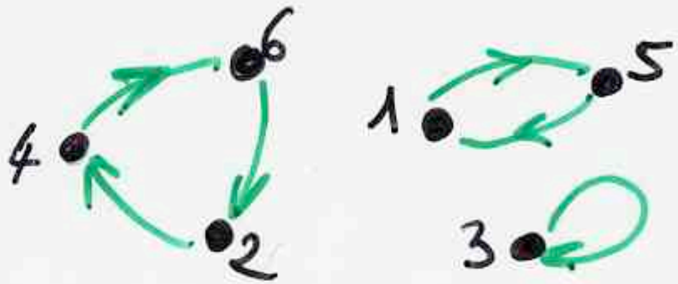
α_i $\mathbf{F}$-structure on A_i

$\mathbf{H} = \exp \mathbf{F}$

$$h(t) = \exp(f(t))$$

$$f(t) = \log(h(t))$$

Permutations $\mathbf{S} = \exp \mathbf{C}$ cycle



$$\sum_{n\geq 0} n!\,\frac{t^n}{n!} = \frac{1}{1-t}$$

$$\sum_{n\geq 1} (n-1)!\,\frac{t^n}{n!} = \sum_{n\geq 1} \frac{t^n}{n} = \log \frac{1}{1-t}$$

$$Z^{*}_{n,r}(\alpha=\beta=q=1) = \binom{n}{r} \frac{(n+1)!}{(r+1)!}$$

Lah numbers

nb of "assembleés" of permutations

$$\left\{ [7,10,5,8] \quad [3,1,4] \quad [9,2,11,6] \right\}$$

$$\exp\left(\frac{x\,t}{1-t}\right)$$

permutation

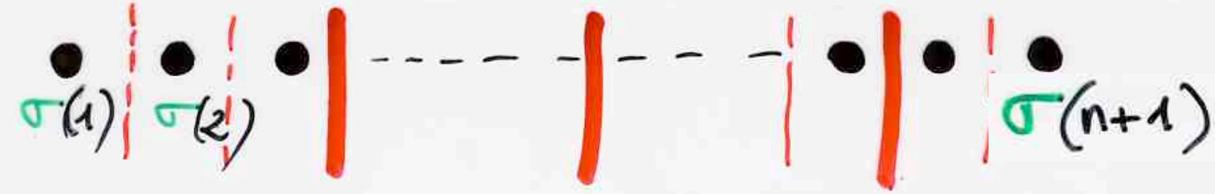
on $\{1, 2, \dots, n+1\}$

$(n+1)!$

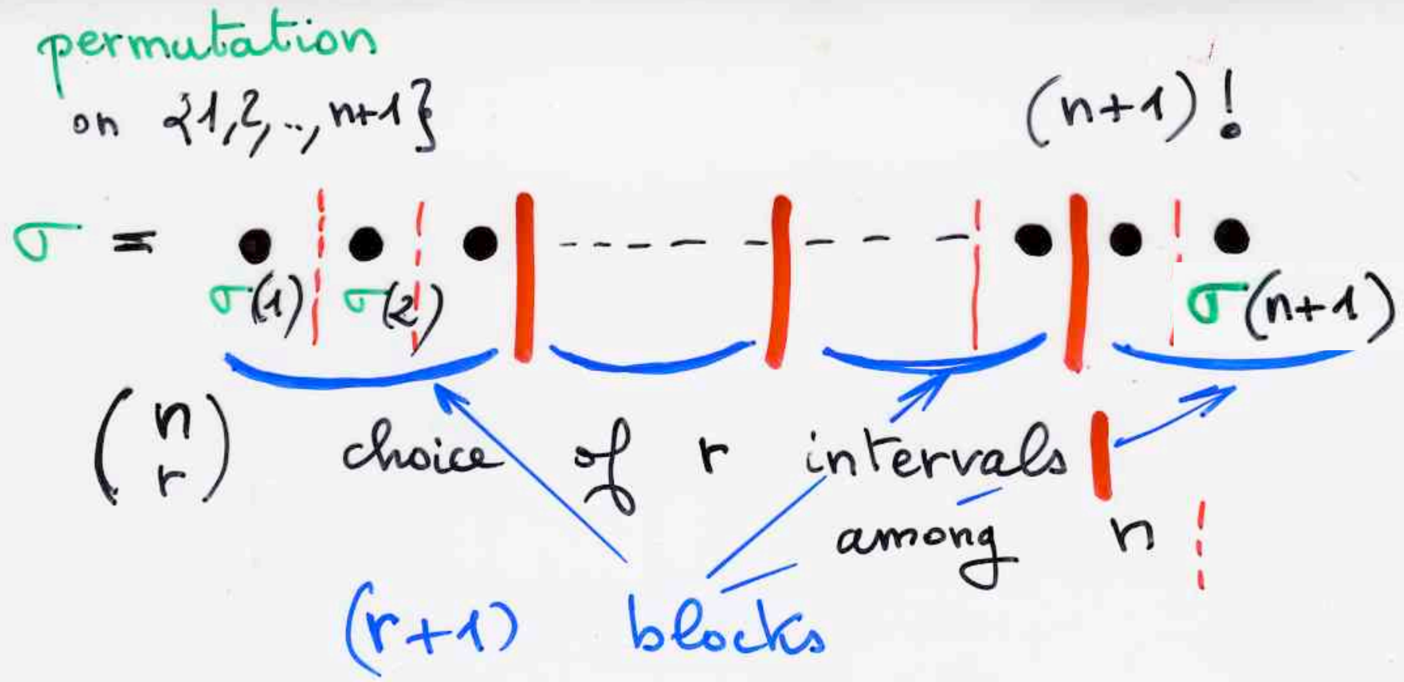
$$\sigma = \underset{\sigma(1)}{\bullet} \quad \underset{\sigma(2)}{\bullet} \quad \bullet \; \cdots\cdots\cdots \; \bullet \quad \bullet \quad \underset{\sigma(n+1)}{\bullet}$$

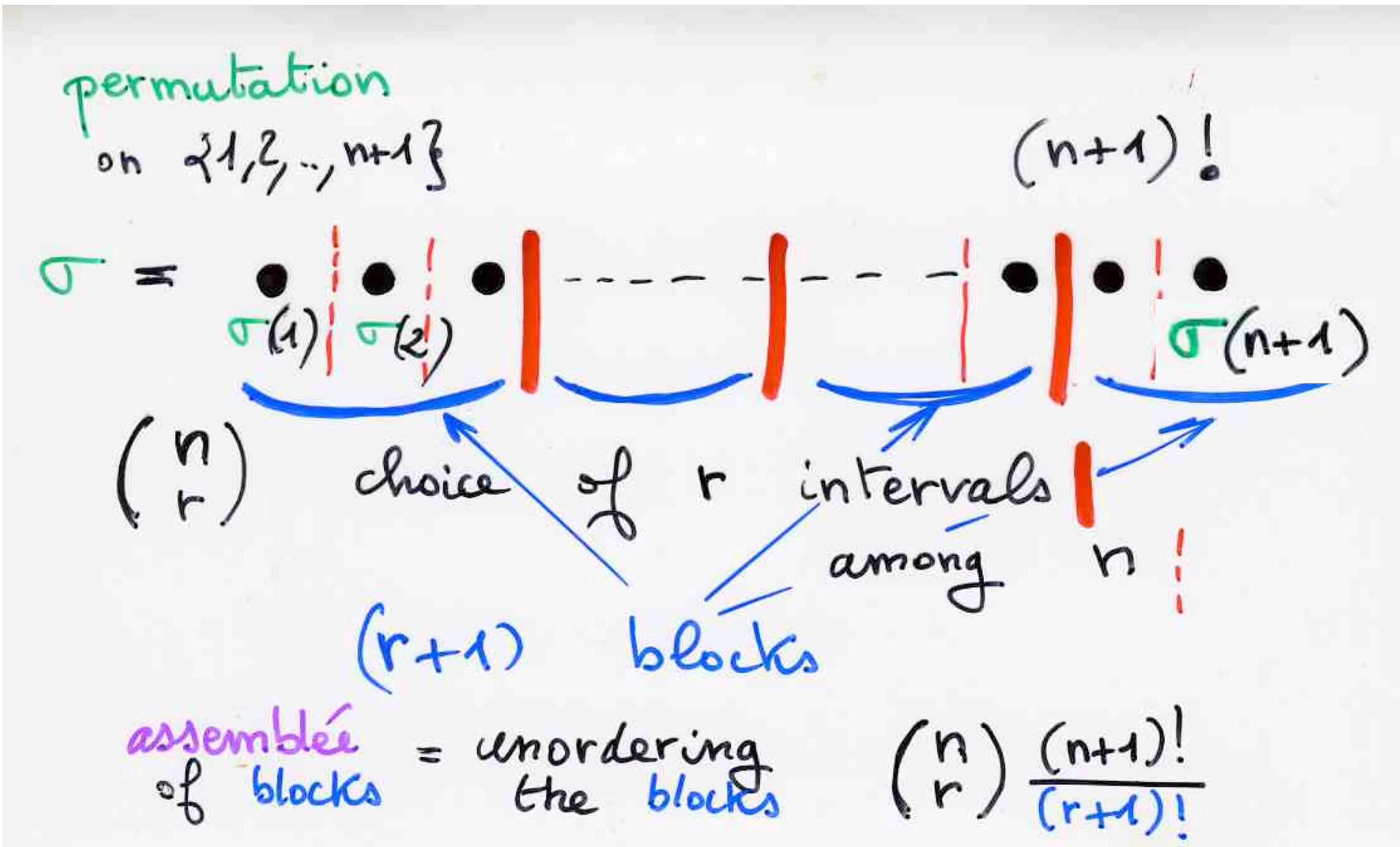
permutation on $\{1,2,\dots,n+1\}$

$(n+1)!$

$\sigma =$ 

$\binom{n}{r}$ choice of r intervals among n


permutation
on {1, 2, .., n+1}
(n+1) !
σ =
σ(1)
σ(2)
σ(n+1)
$\binom{n}{r}$
choice of r intervals among n
(r+1) blocks


permutation
on {1,2,..,n+1}
(n+1)!
σ =
σ(1) σ(2)
σ(n+1)
$\binom{n}{r}$ choice of r intervals among n
(r+1) blocks
assemblée of blocks = unordering the blocks
$\binom{n}{r} \frac{(n+1)!}{(r+1)!}$

from «assemblées» of permutations
to
rhombic alternative tableaux

$$\left\{ [2, 10, 12, 7] \quad [3, 11, 4] \quad [5, 9, 1, 8, 6] \right\}$$

"assemblée" of permutations

from an "assemblée" of permutations
→ σ permutation
concatenation of the blocks
such that their last elements
go decreasing

2 10 12 7 5 9 1 8 6 3 11 4

$$\left\{ [2, 10, 12, 7] \quad [3, 11, 4] \quad [5, 9, 1, 8, 6] \right\}$$

"assemblée" of permutations

$$2 \quad 10 \quad 12 \quad 7 \quad 5 \quad 9 \quad 1 \quad 8 \quad 6 \quad 3 \quad 11 \quad 4$$

$$\begin{cases} \text{increase} & \ldots\; x \;\text{-----}\; x+1 \;\text{--} \\ \text{decrease} & \ldots\; x+1 \;\text{---}\; x \;\text{----} \qquad (\max) \end{cases}$$

$$\left\{ \begin{matrix} [2, 10, 12, 7] & & [3, 11, 4] \\ & [5, 9, 1, 8, 6] & \end{matrix} \right\}$$

"assemblée" of permutations

$$2 \quad 10 \quad 12 \quad 7 \quad 5 \quad 9 \quad 1 \quad 8 \quad 6 \quad 3 \quad 11 \quad 4$$

$$\begin{cases} \text{increase} & \dots\; x \xrightarrow{\quad\quad} x+1 \;\; \text{- -} \\ \text{decrease} & \dots\; x+1 \xleftarrow{\quad} x \;\; \text{- - - -} \qquad (\text{max}) \end{cases}$$

$$\left\{ \begin{array}{ll} [2, 10, 12, 7] & [3, 11, 4] \\ & [5, 9, 1, 8, 6] \end{array} \right\}$$

"assemblée" of permutations

2 10 12 7 5 9 1 8 6 3 11 4

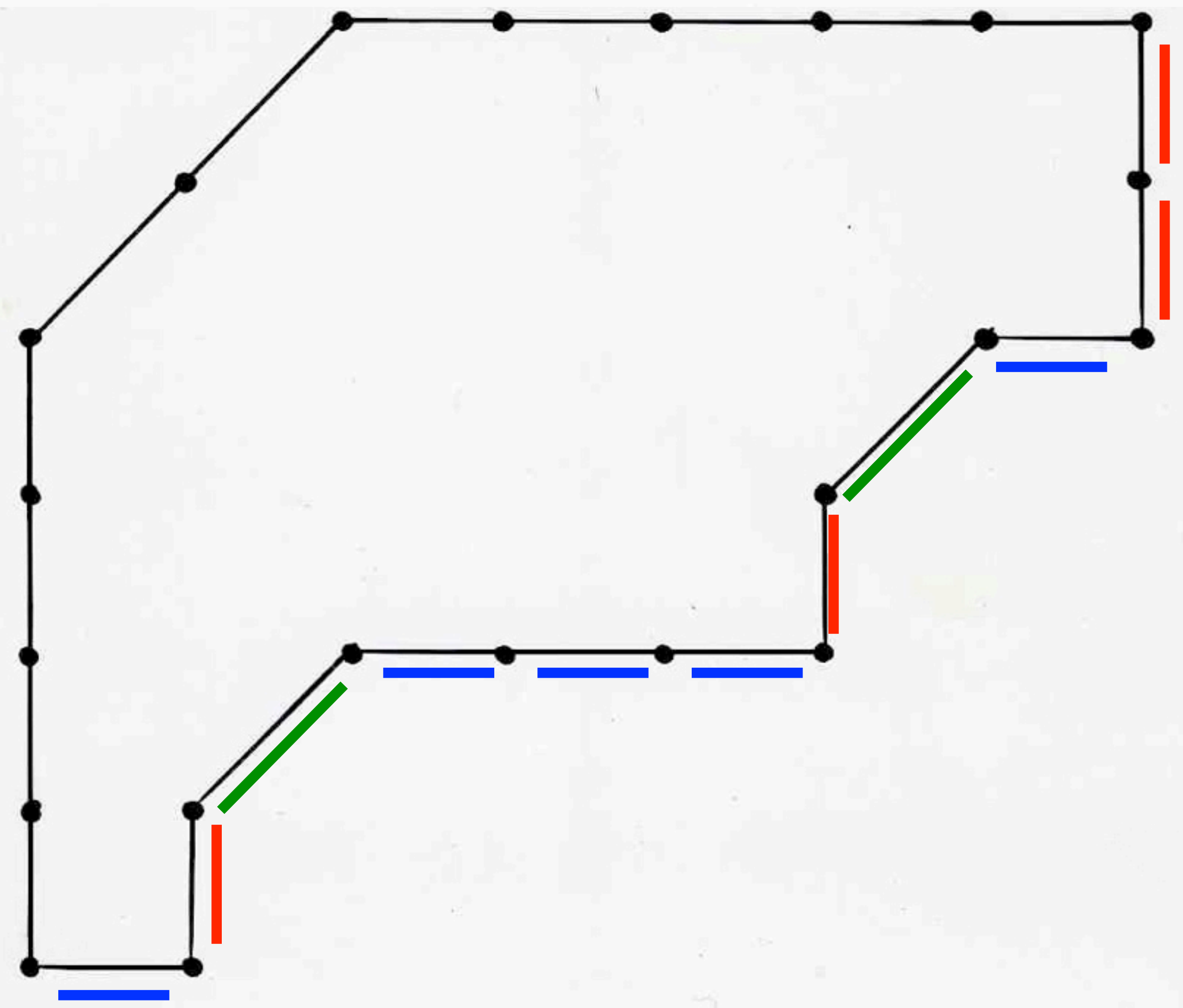
$$\left\{ \begin{array}{ll} \text{increase} & \ldots x \; \text{-----} \; x+1 \; \text{--} \\ \text{decrease} & \ldots x+1 \; \text{---} \; x \; \text{----} \qquad (\text{max}) \end{array} \right.$$

for an assemblée $\alpha \longrightarrow$ permutation σ

$\alpha \to \sigma \longrightarrow X$ word $\{\bullet, \bullet, \bullet\} \longrightarrow \Gamma(X)$ diagram

2 10 12 7 5 9 1 8 6 3 11 4

2 10 12 7 5 9 1 8 6 3 11 4



exchange-fusion algorithm

Def. A *label* (or *segment*) is a list (possibly empty) of consecutive integers

for two disjoint nonempty labels A and B,
B covers A (denoted $B \succ A$)
if for the smallest $i \in B$ and the largest $j \in A$, we have $i = j+1$

ex: $B = (\underset{i}{6}, 7, 8)$ $A = (4, \underset{j}{5})$

then $A \cup B$ is also a *label*

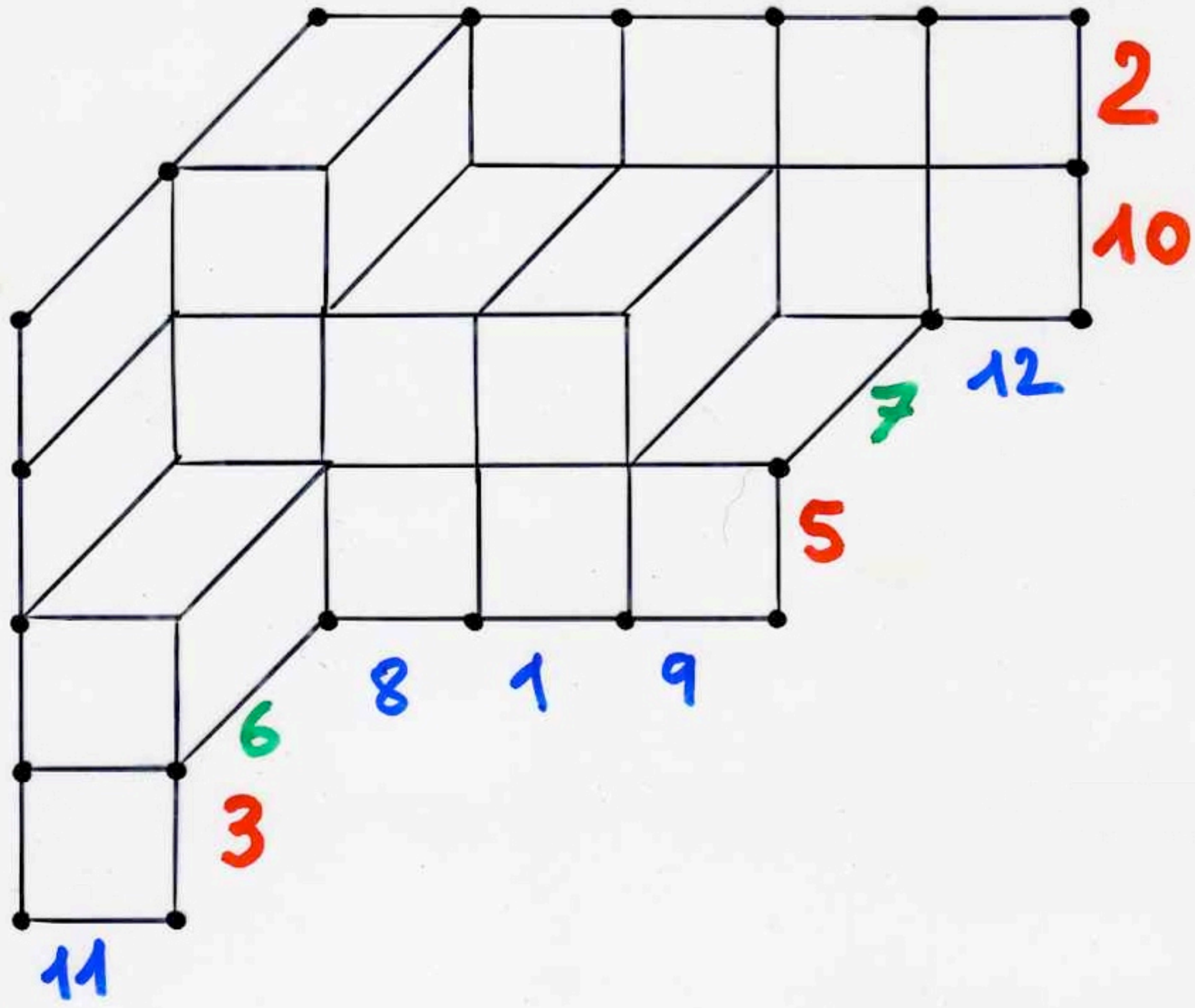
exchange-fusion algorithm

Initiation

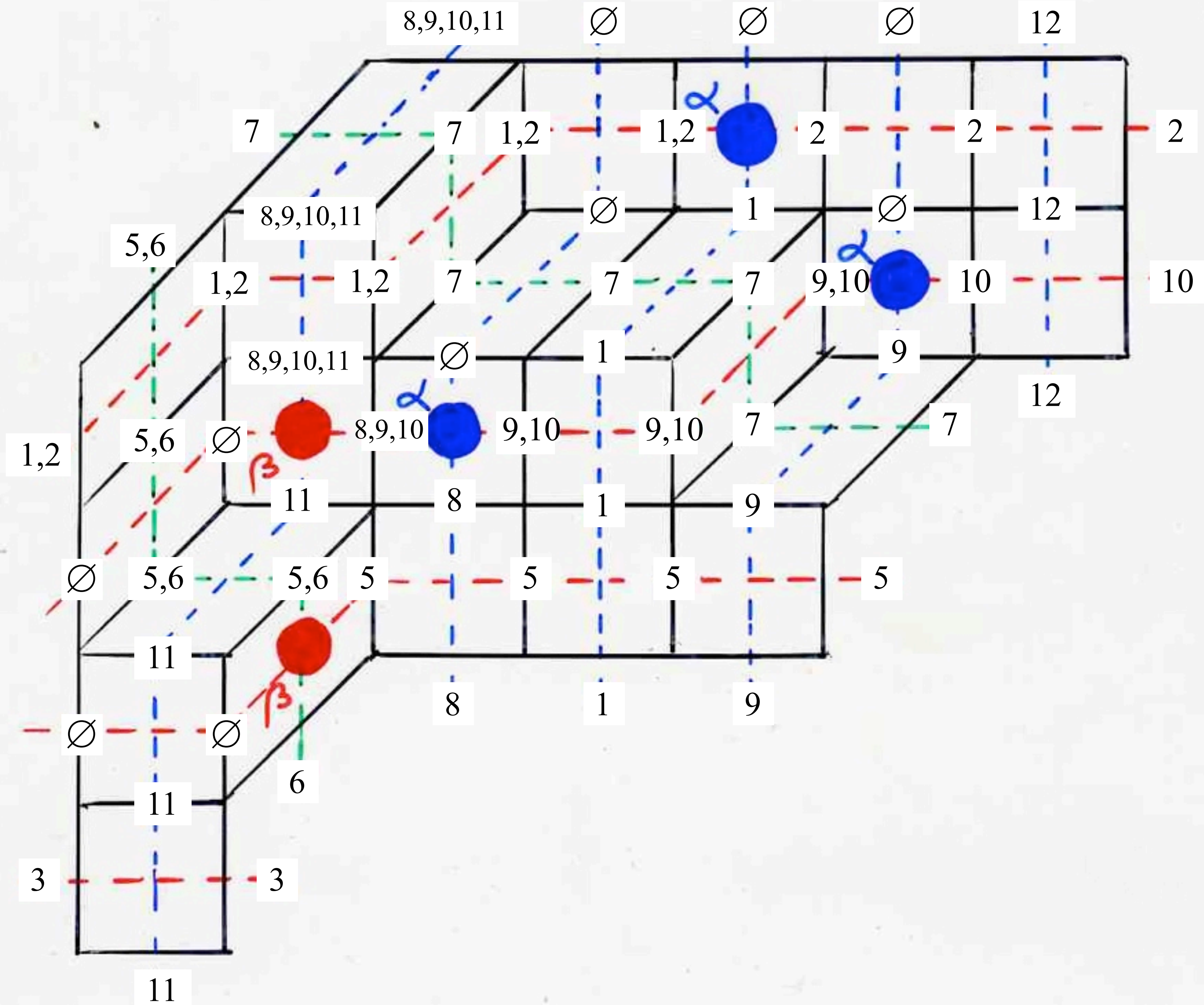
every SW edge of $\Gamma(X)$ receive a label reduced to the corresponding element of σ

choice of a tiling $\mathcal{E}$ of $\Gamma(X)$

2 10 12 7 5 9 1 8 6 3 11 4



(4)


8,9,10,11
∅
∅
∅
12
α
7
7
1,2
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2
2
2
8,9,10,11
∅
1
∅
12
5,6
α
1,2
1,2
7
7
7
9,10
10
10
8,9,10,11
∅
1
9
12
α
1,2
5,6
∅
8,9,10
9,10
9,10
7
7
β
11
8
1
9
∅
5,6
5,6
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8
1
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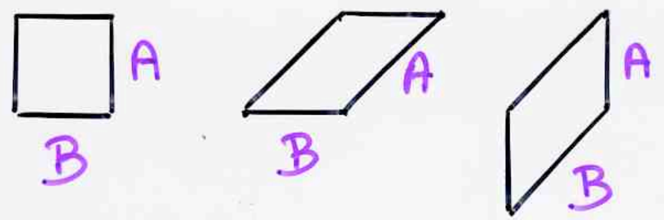
11

(4)

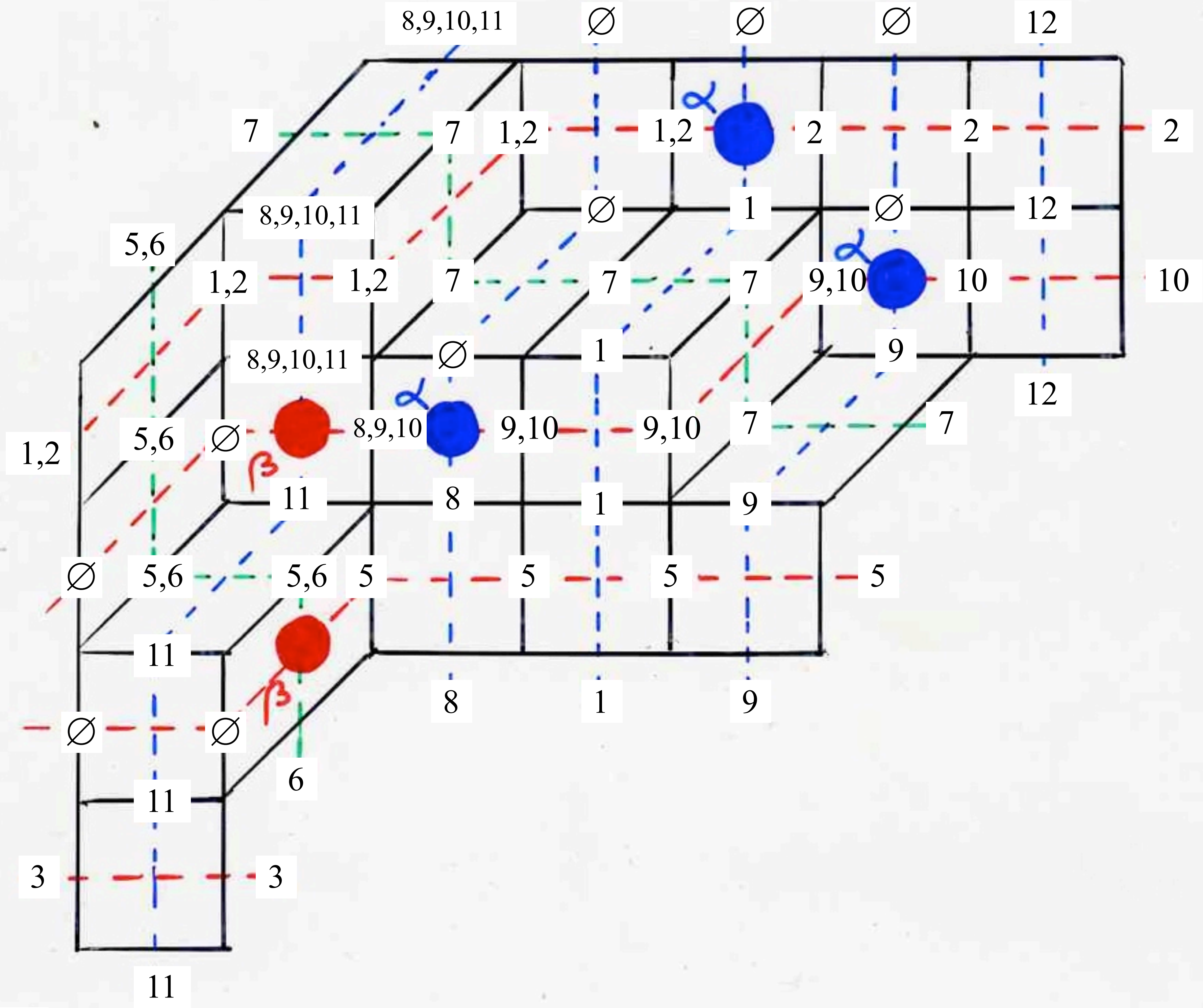
Let a tile with labels A and B



(i) • if $A \not\succ B$ and $B \not\succ A$, the labels "cross"

(ii) • if $A \succ B$ or $B \succ A$, $A \cup B$ is a label which follows the line of biggest label A or B

- except if the smallest label is on a green line, in that case the two labels "cross"
- when $A \cup B$ is a label, the other edge receives the label $\emptyset$


8,9,10,11
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12
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7
1,2
1,2
α
2
2
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8,9,10,11
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1
∅
12
5,6
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∅
1
9
12
α
1,2
5,6
∅
8,9,10
9,10
9,10
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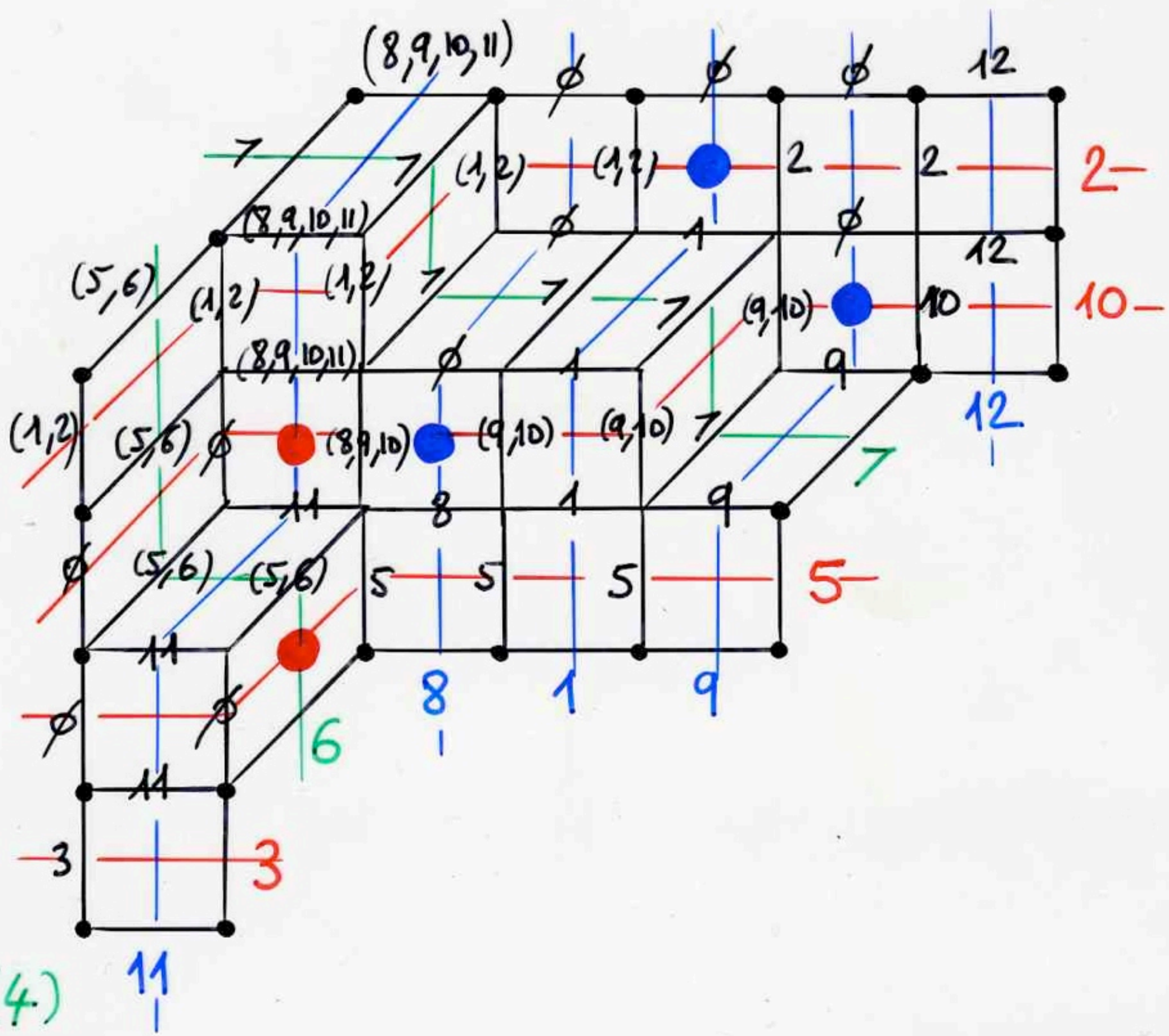
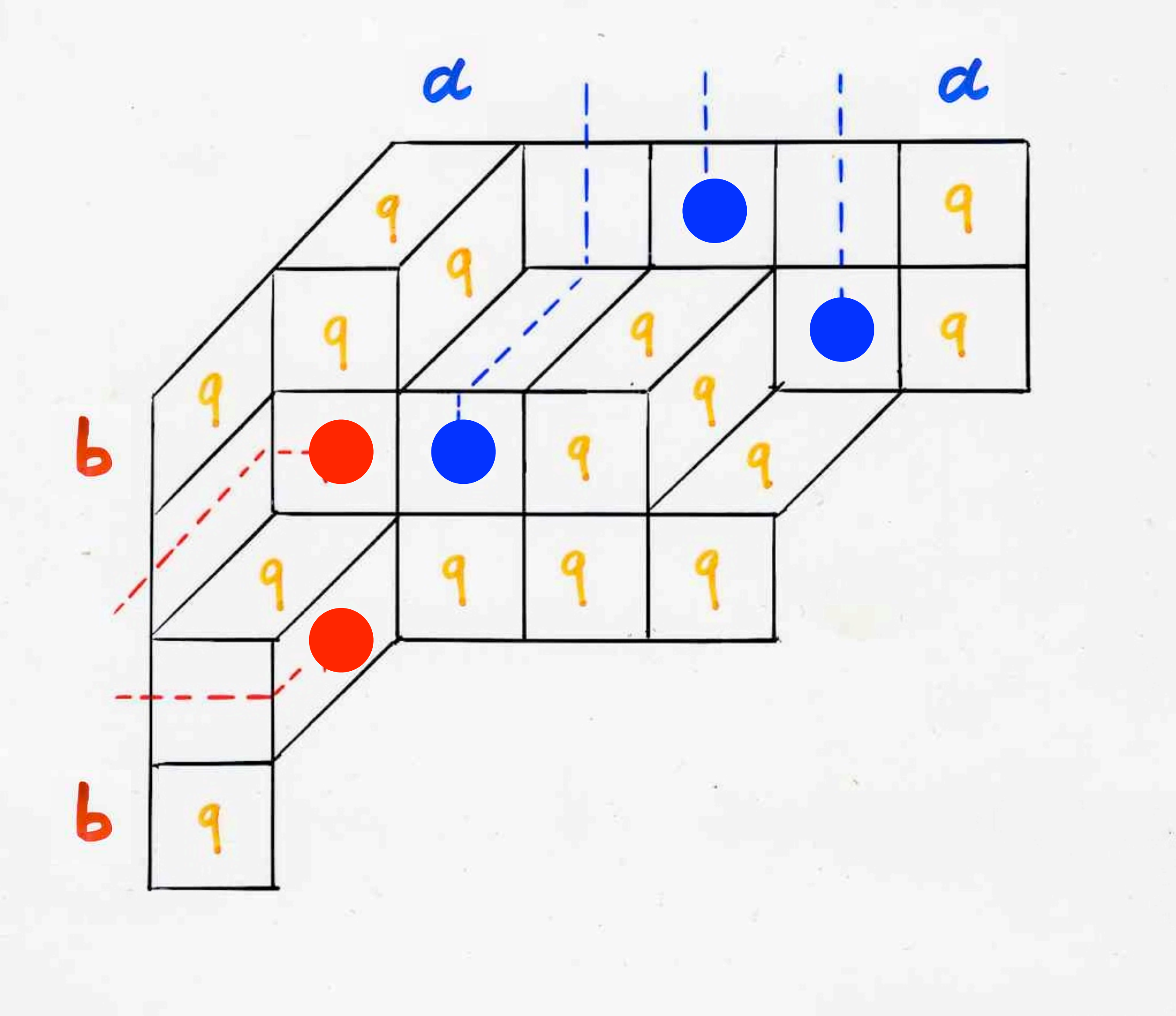
(4)

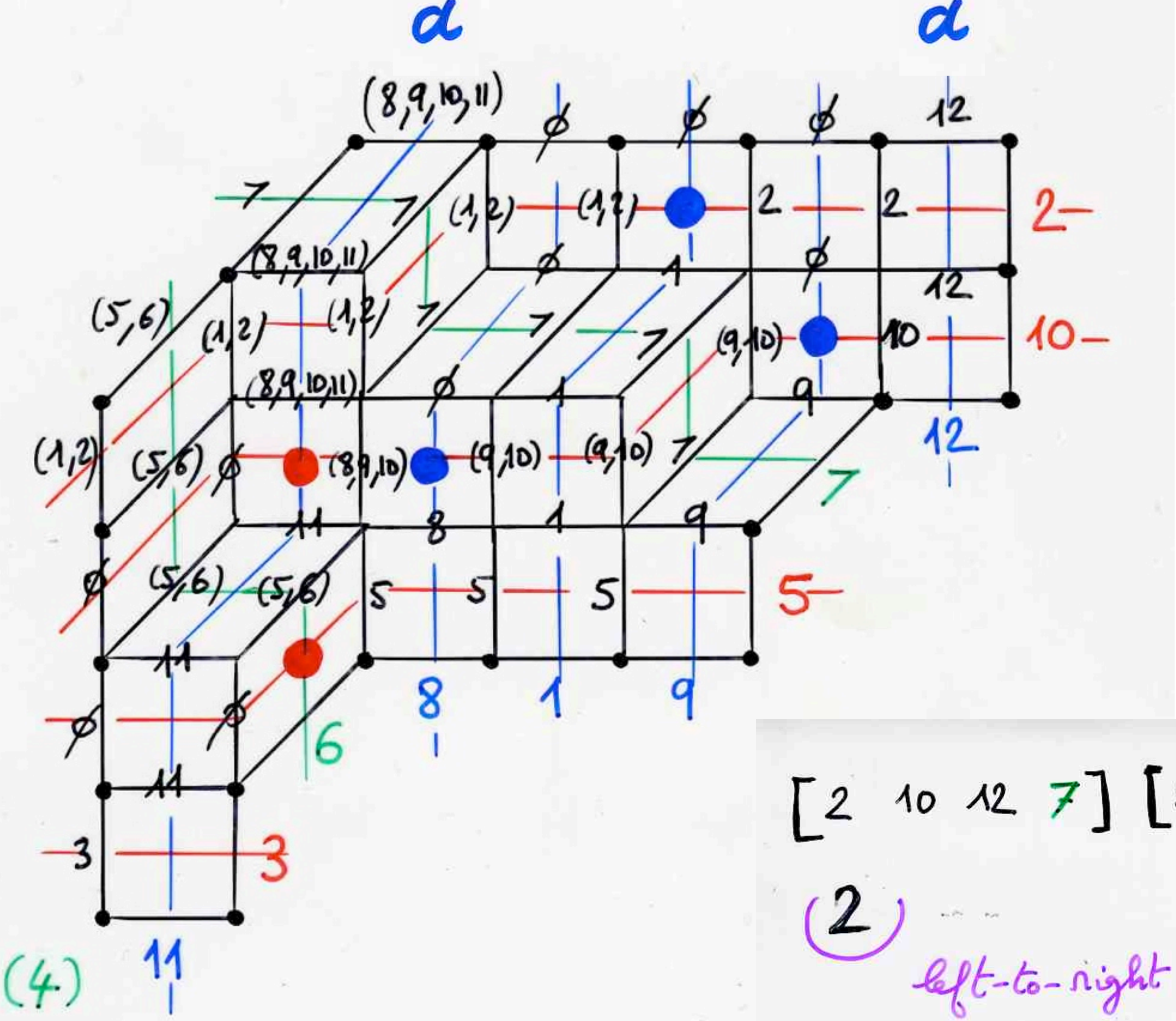


(4)

$$wt(T) = q^{15} a^{2} b^{2}$$



$$wt(T) = q^{15} a^2 b^2$$



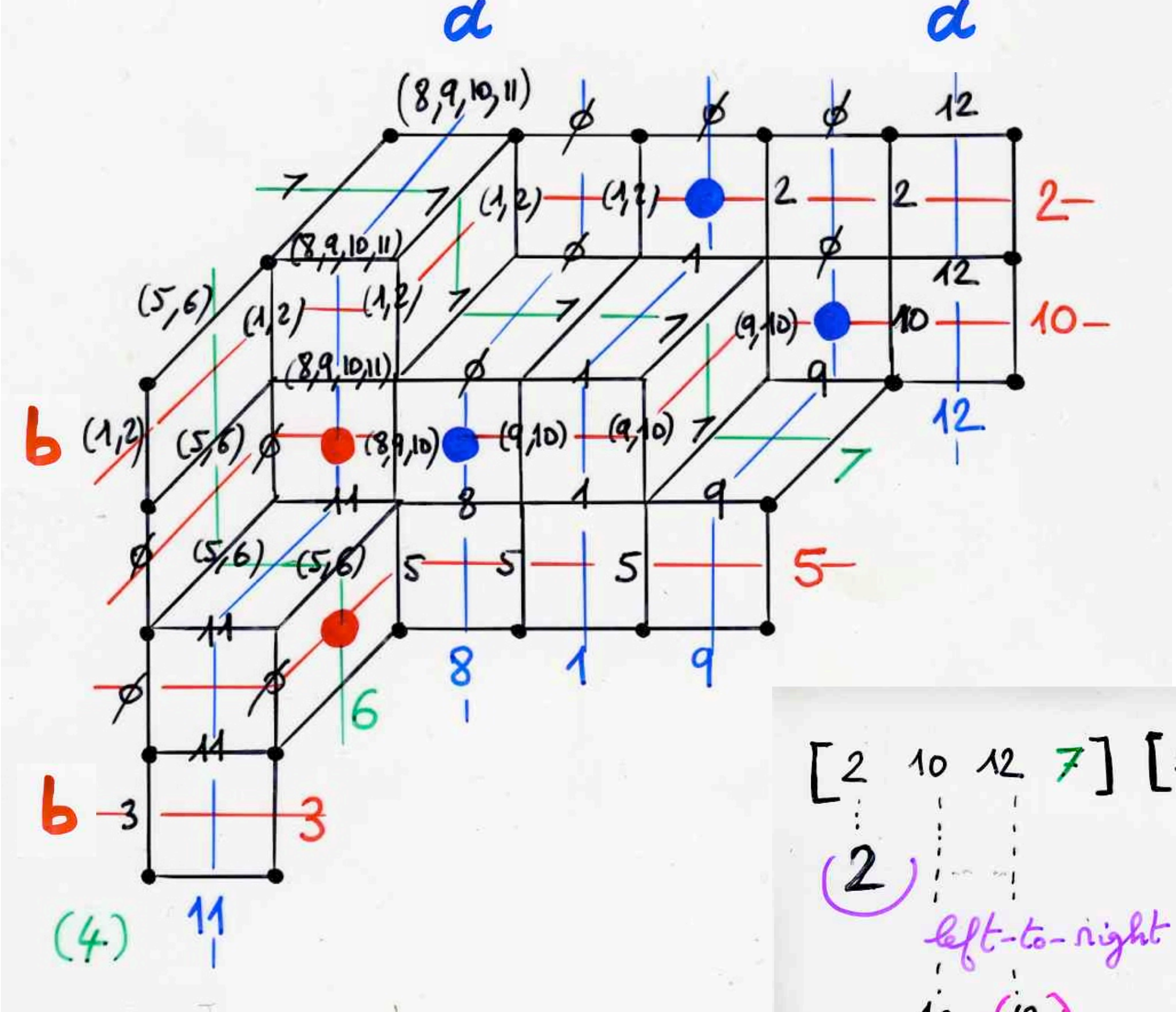
[2 10 12 7] [5 9 1 8 6] [3 11 4]

2 1 3

left-to-right max

d

$$wt(T) = q^{15} a^{2} b^{2}$$



[2 10 12 7] [5 9 1 8 6] [3 11 4]

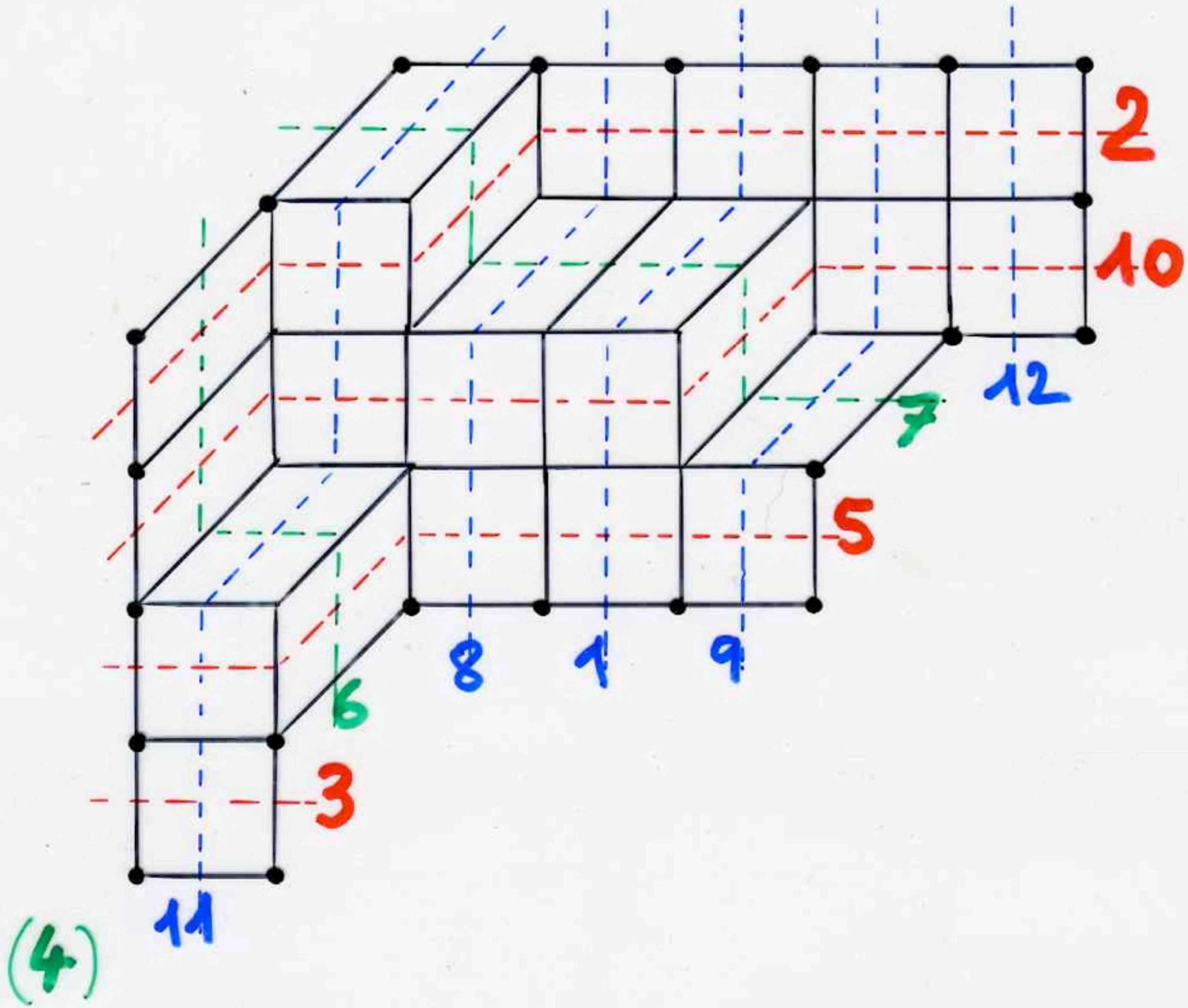
2 1 3 a

left-to-right max

10 12 9 8 11 b

right-to-left max

$$\mathrm{wt}(T) = q^{15} a^2 b^2$$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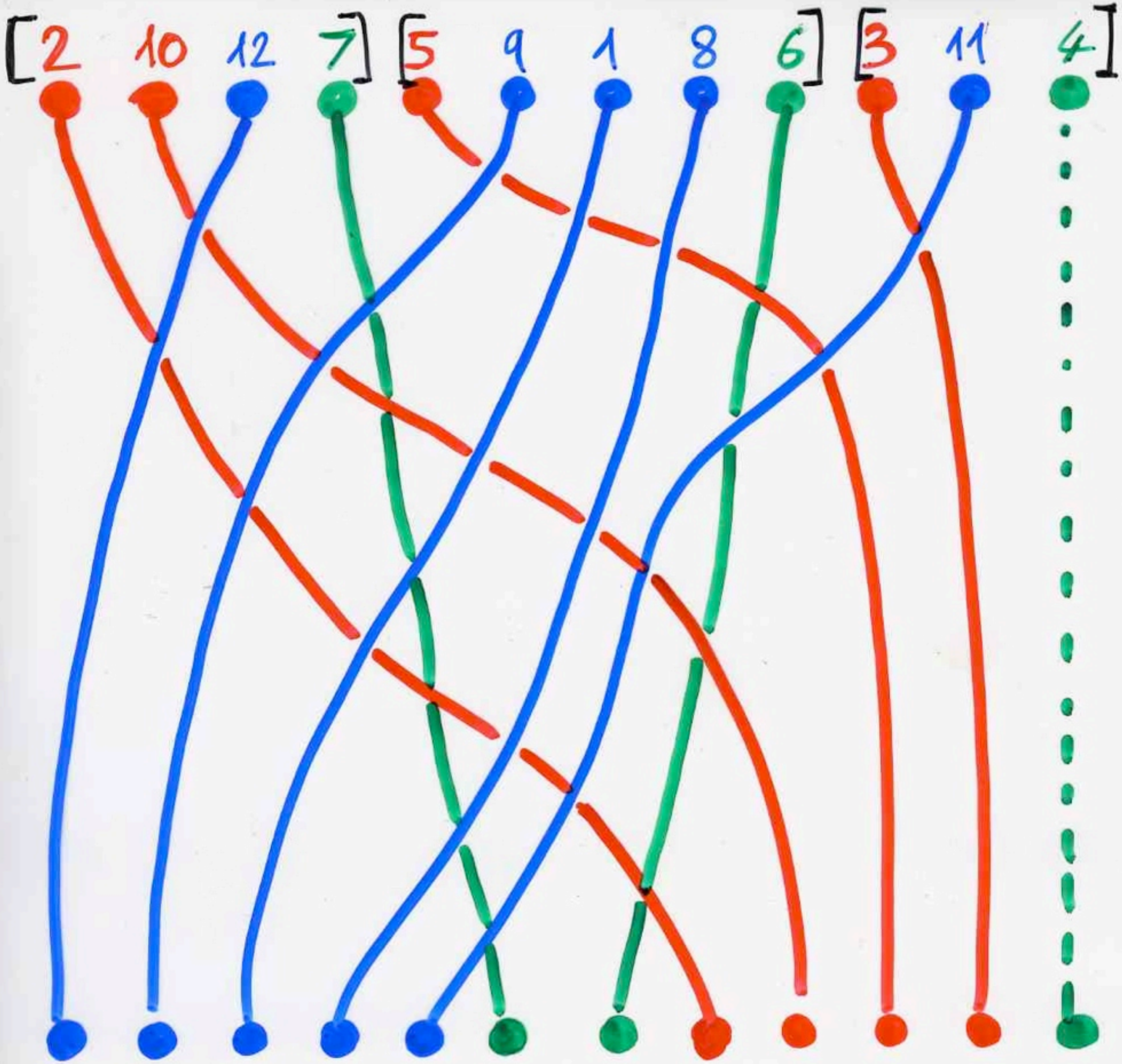
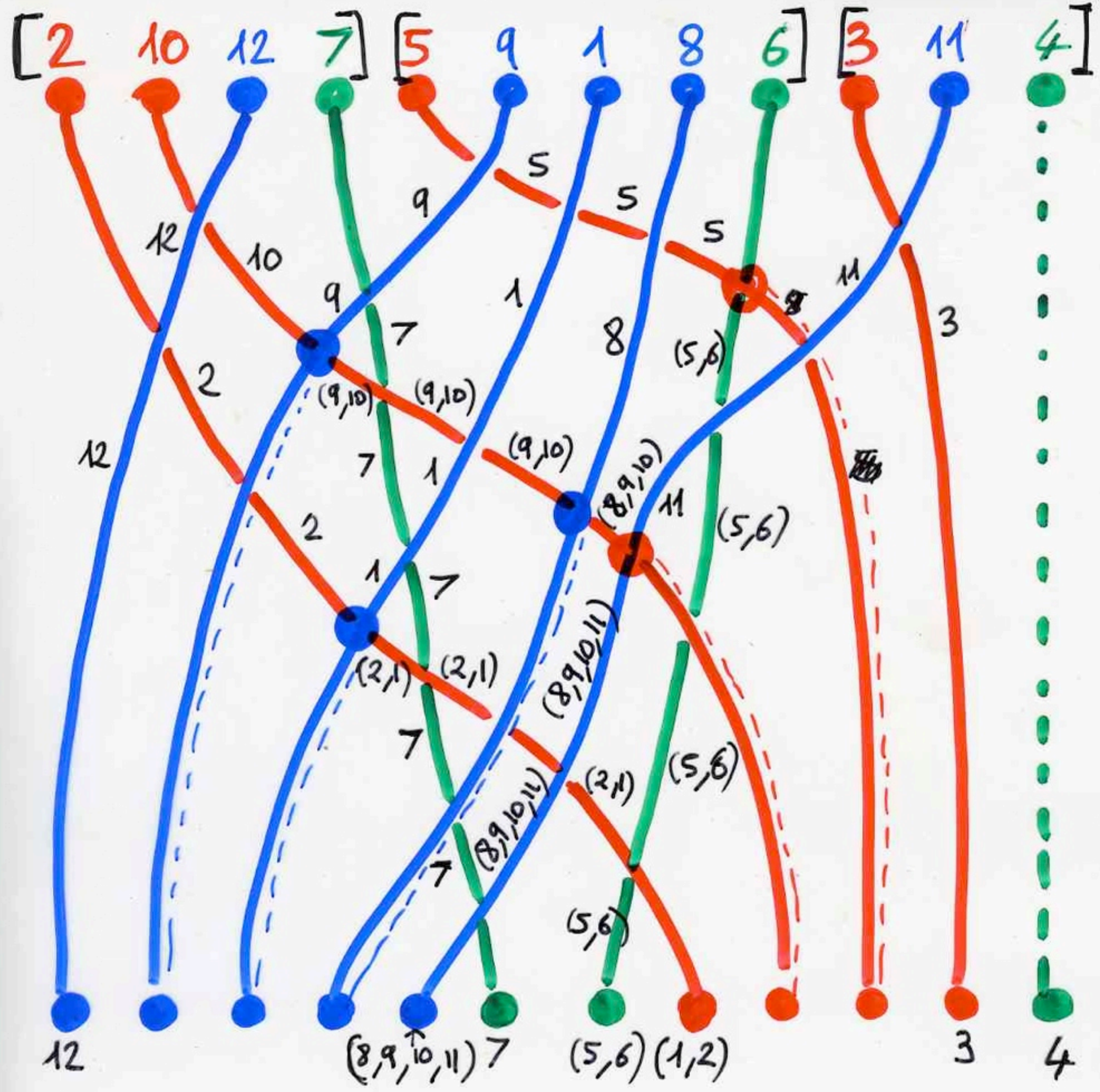
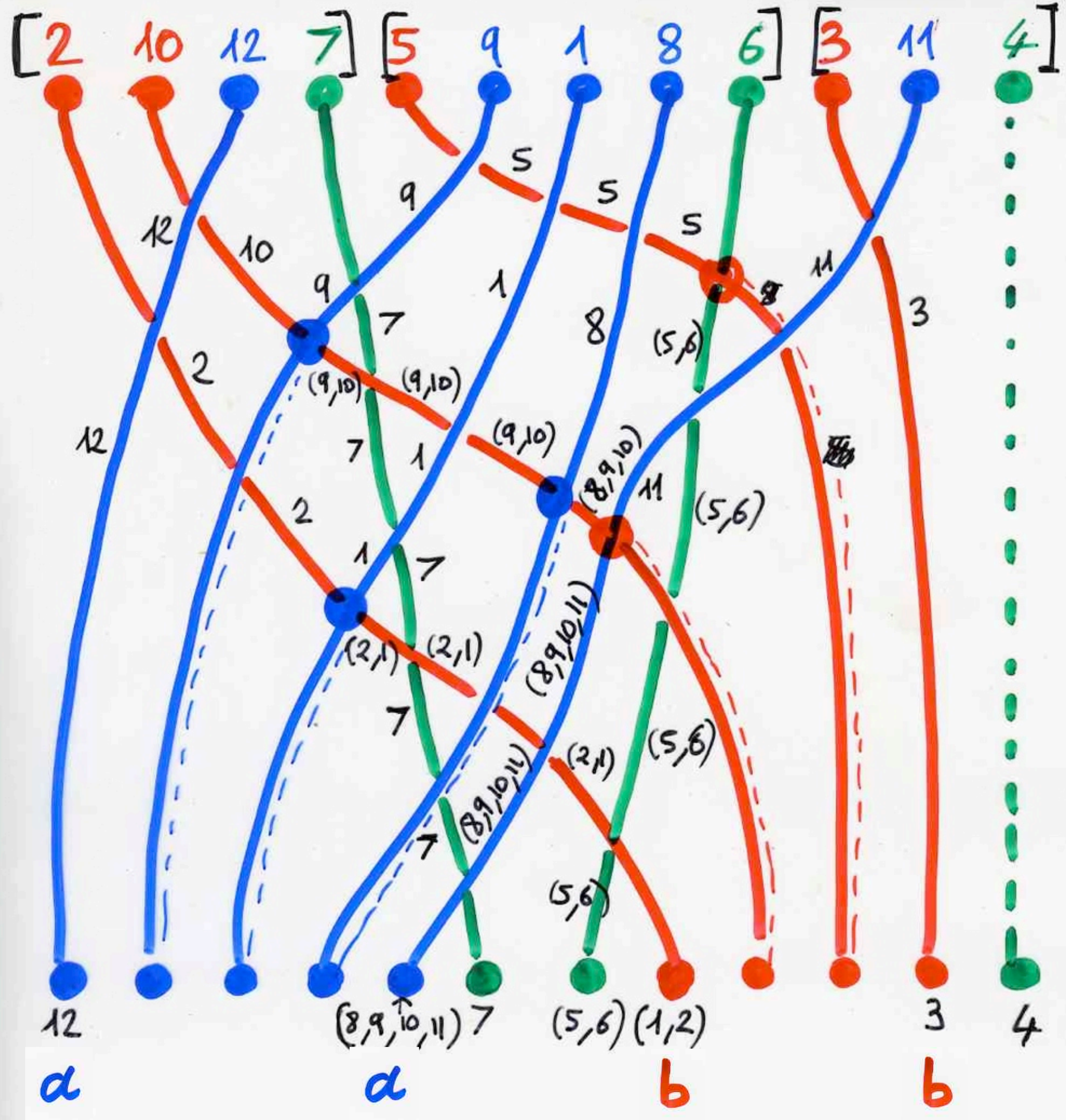
1
9
6
3
11
(4)


[2 10 12 7] [5 9 1 8 6] [3 11 4]



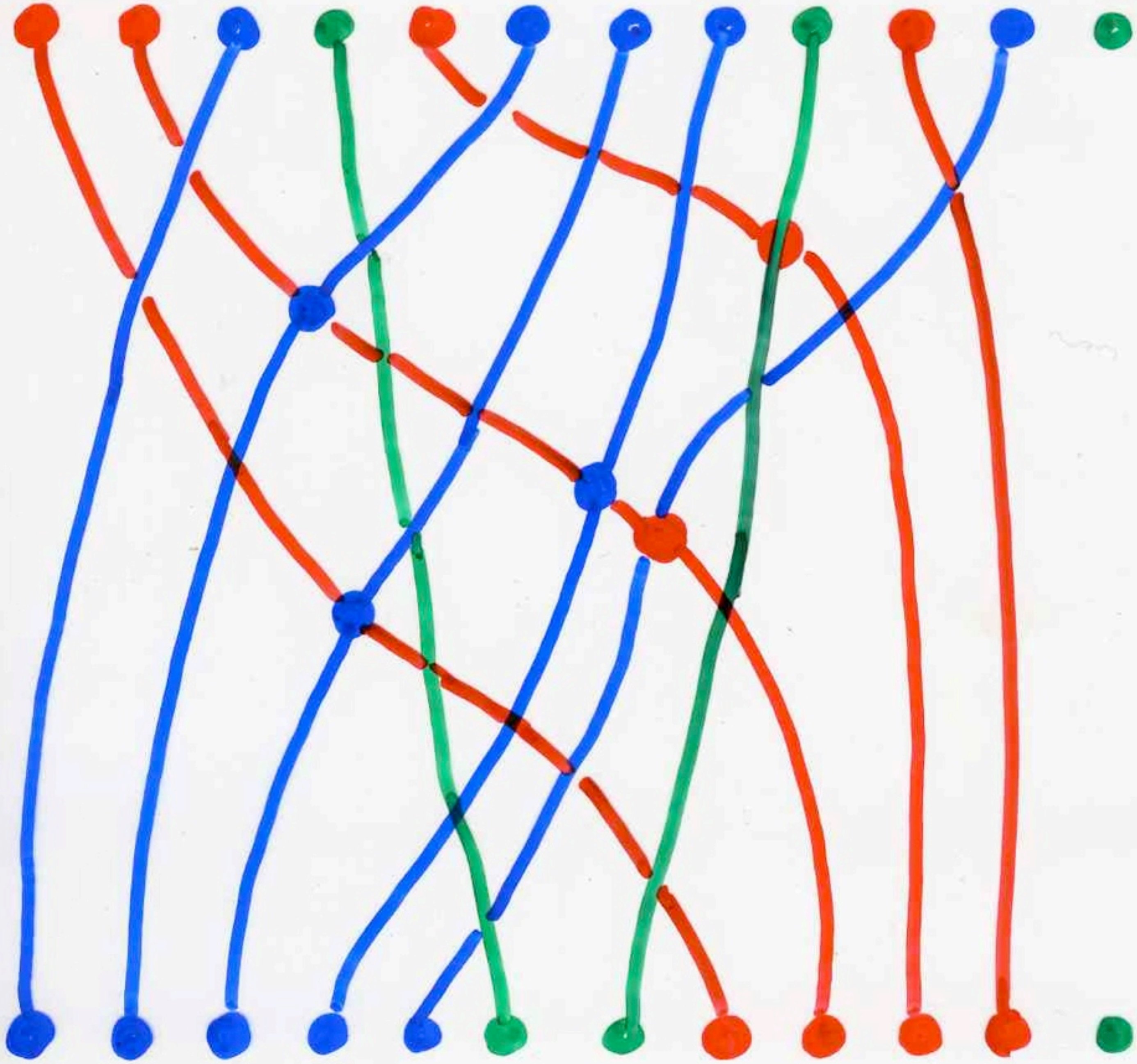
[2 10 12 7] [5 9 1 8 6] [3 11 4]
12
(8,9,10,11) 7
(5,6) (1,2)
3
4

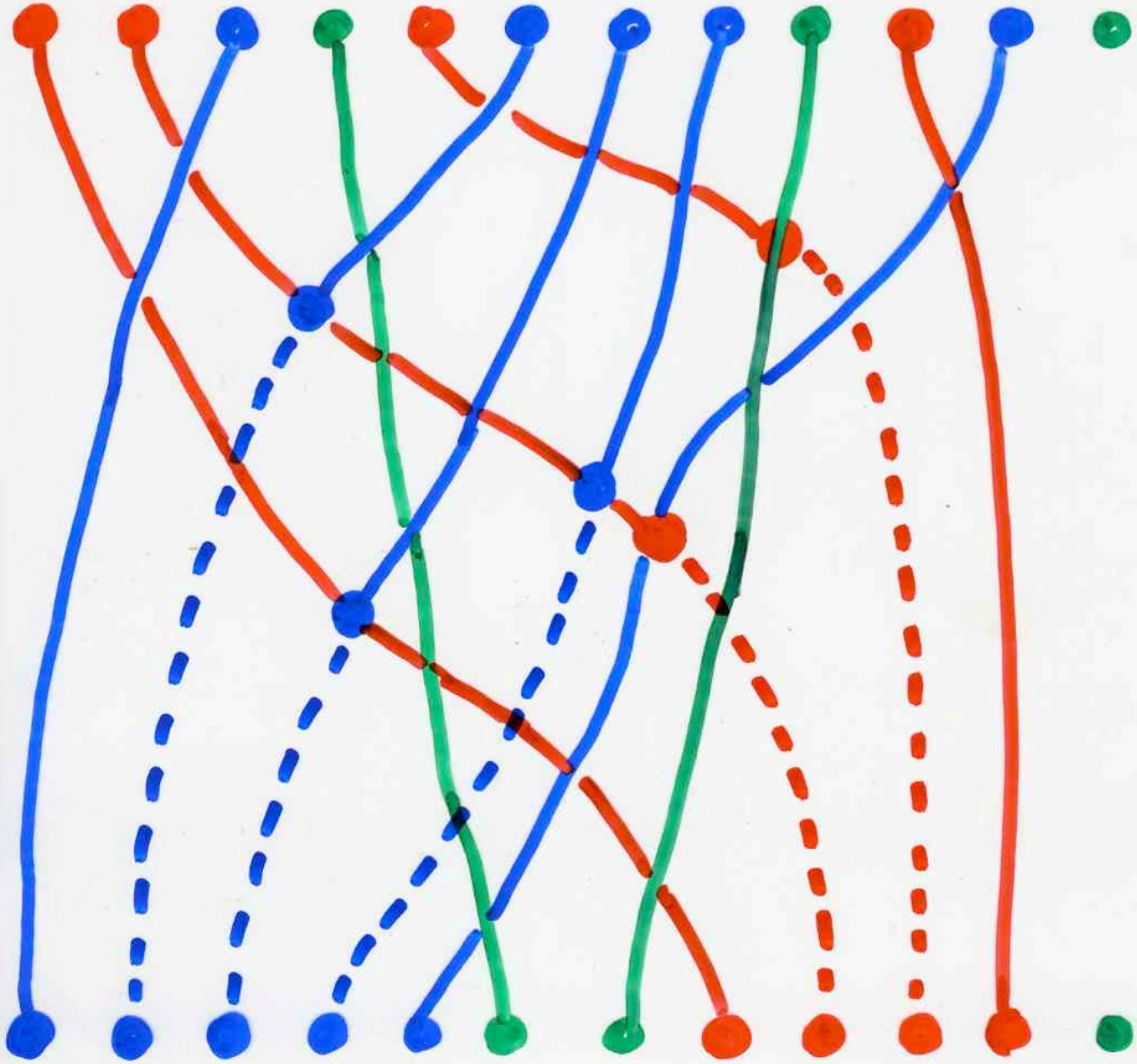

[2 10 12 7] [5 9 1 8 6] [3 11 4]
12
(8,9,10,11) 7
(5,6) (1,2)
3
4
a a b b

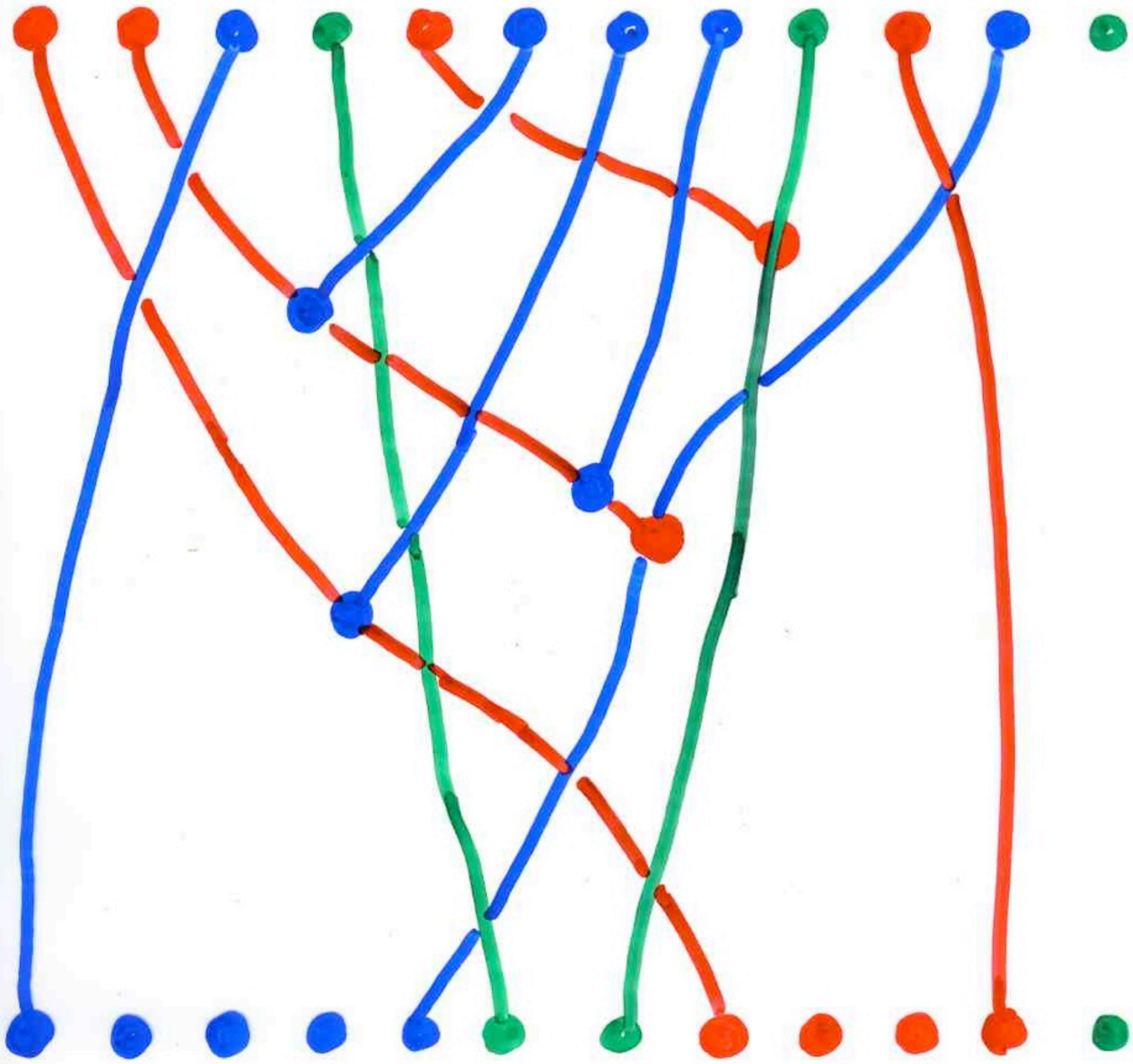
from rhombic alternative tableaux

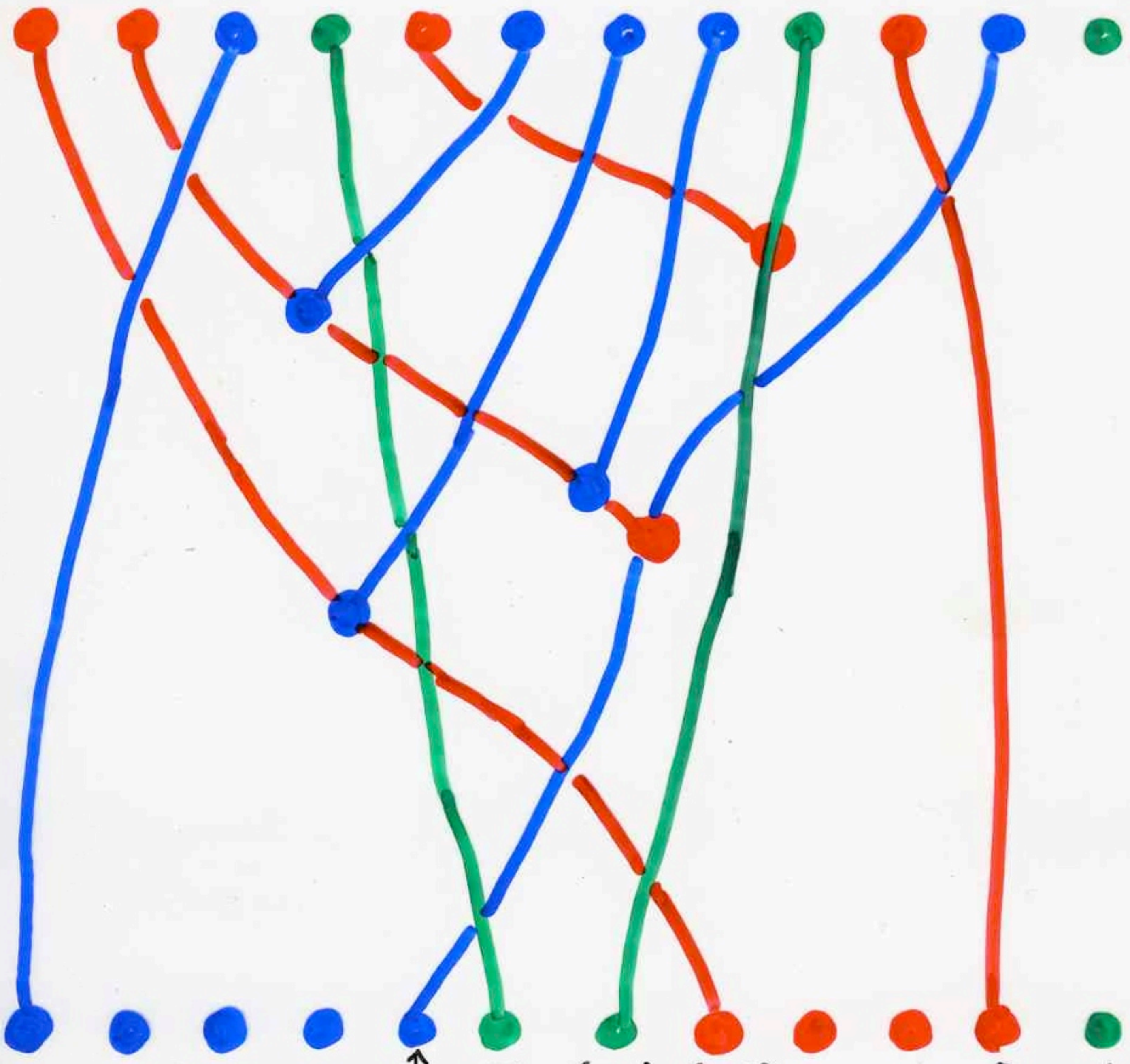
to

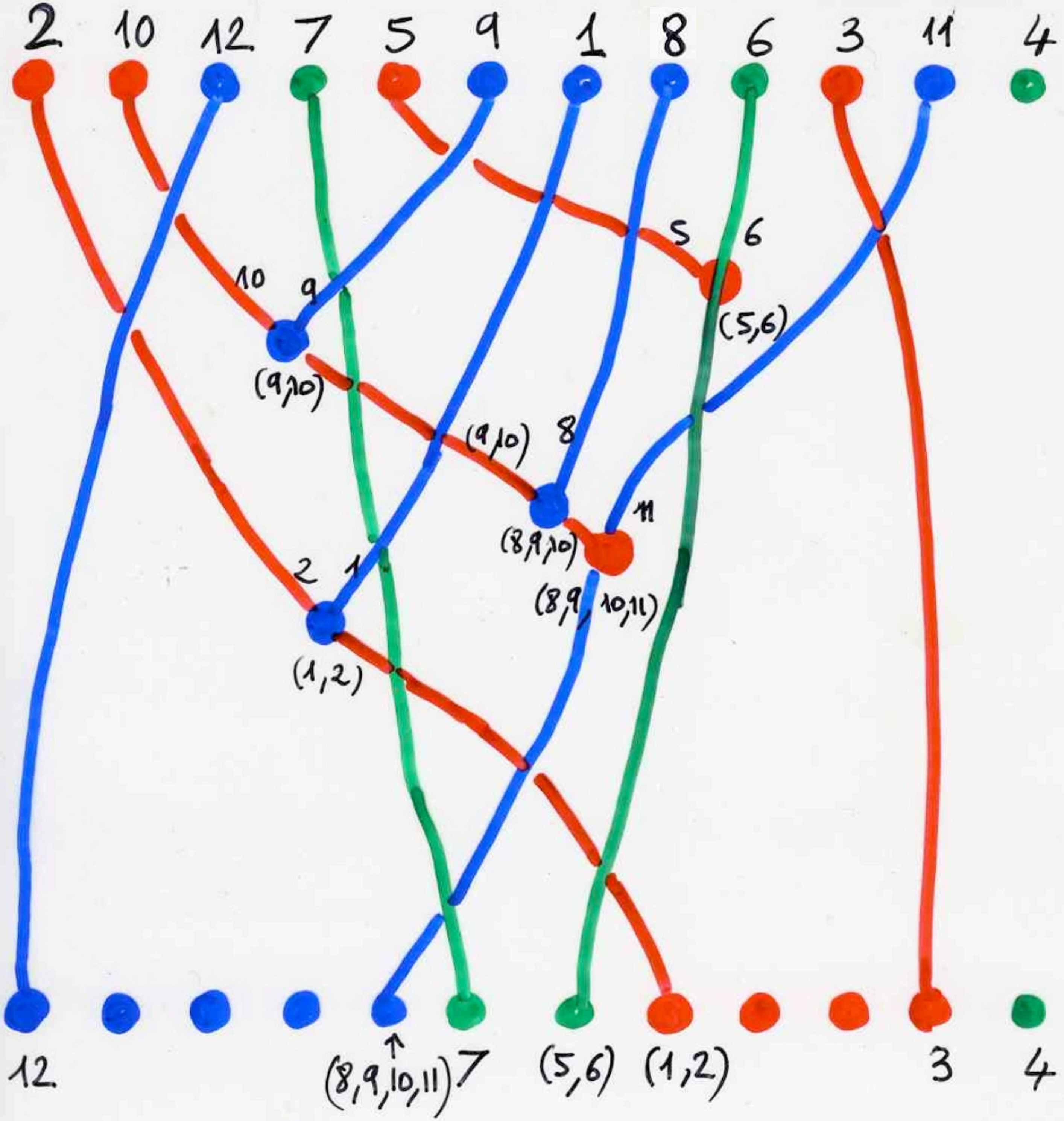
assemblée of permutations



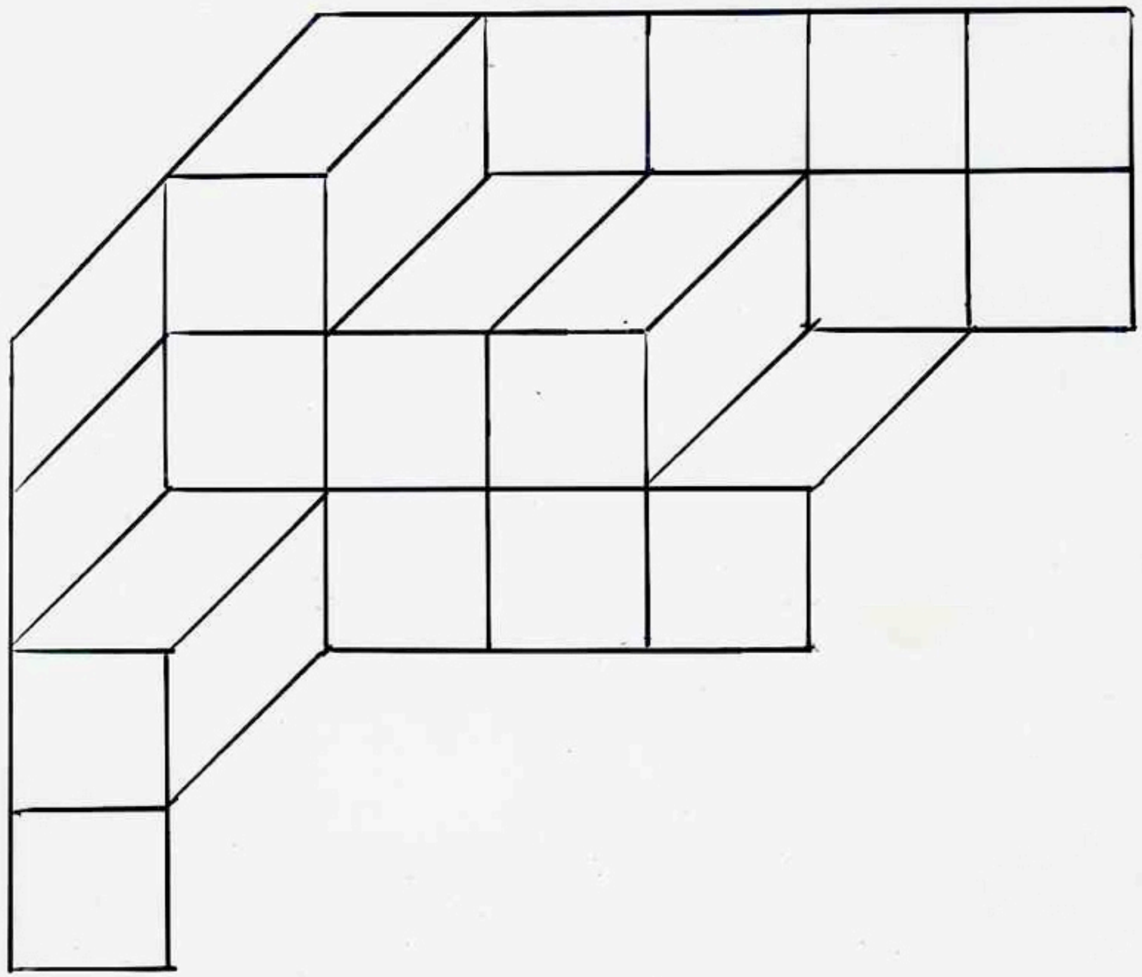


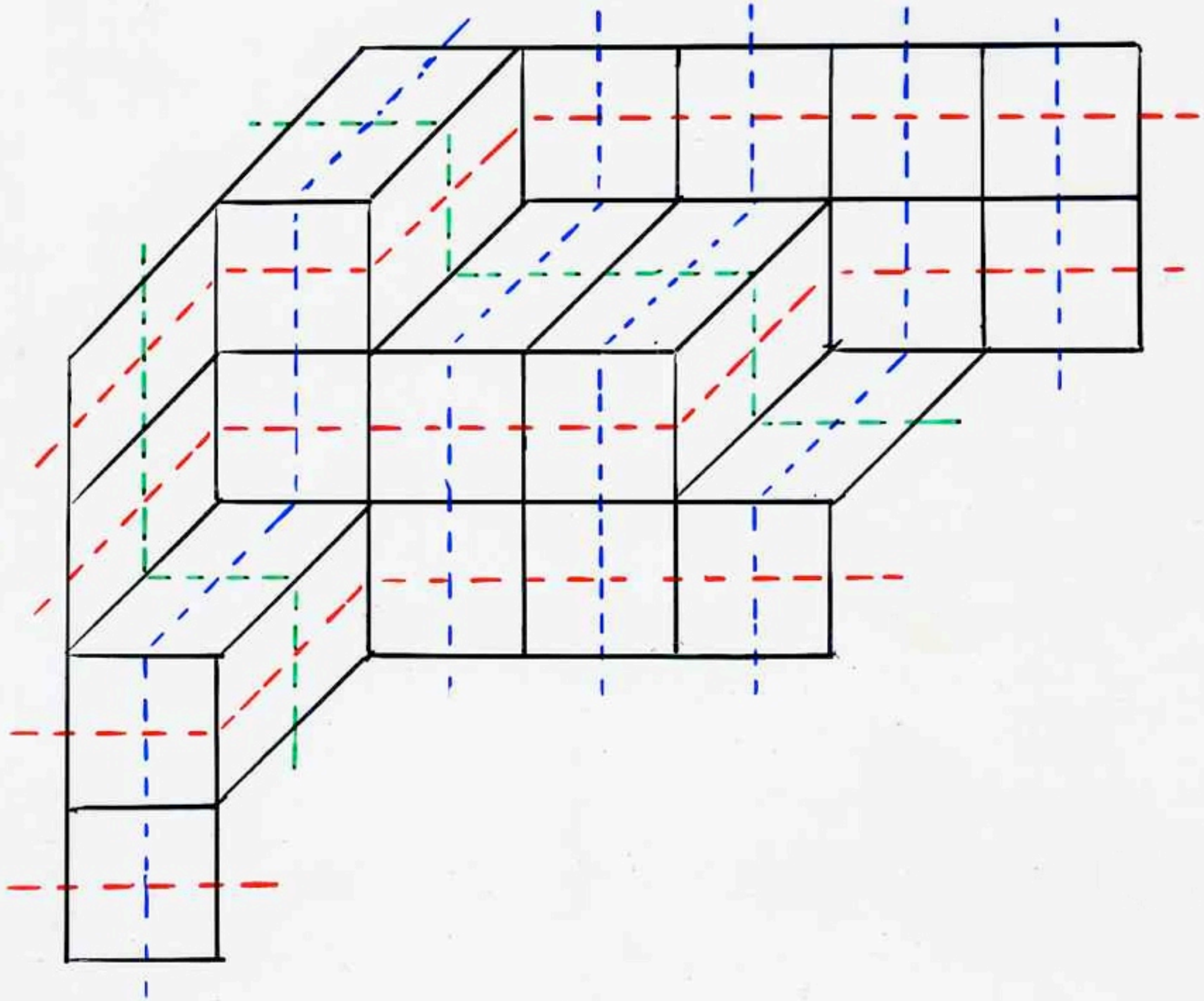


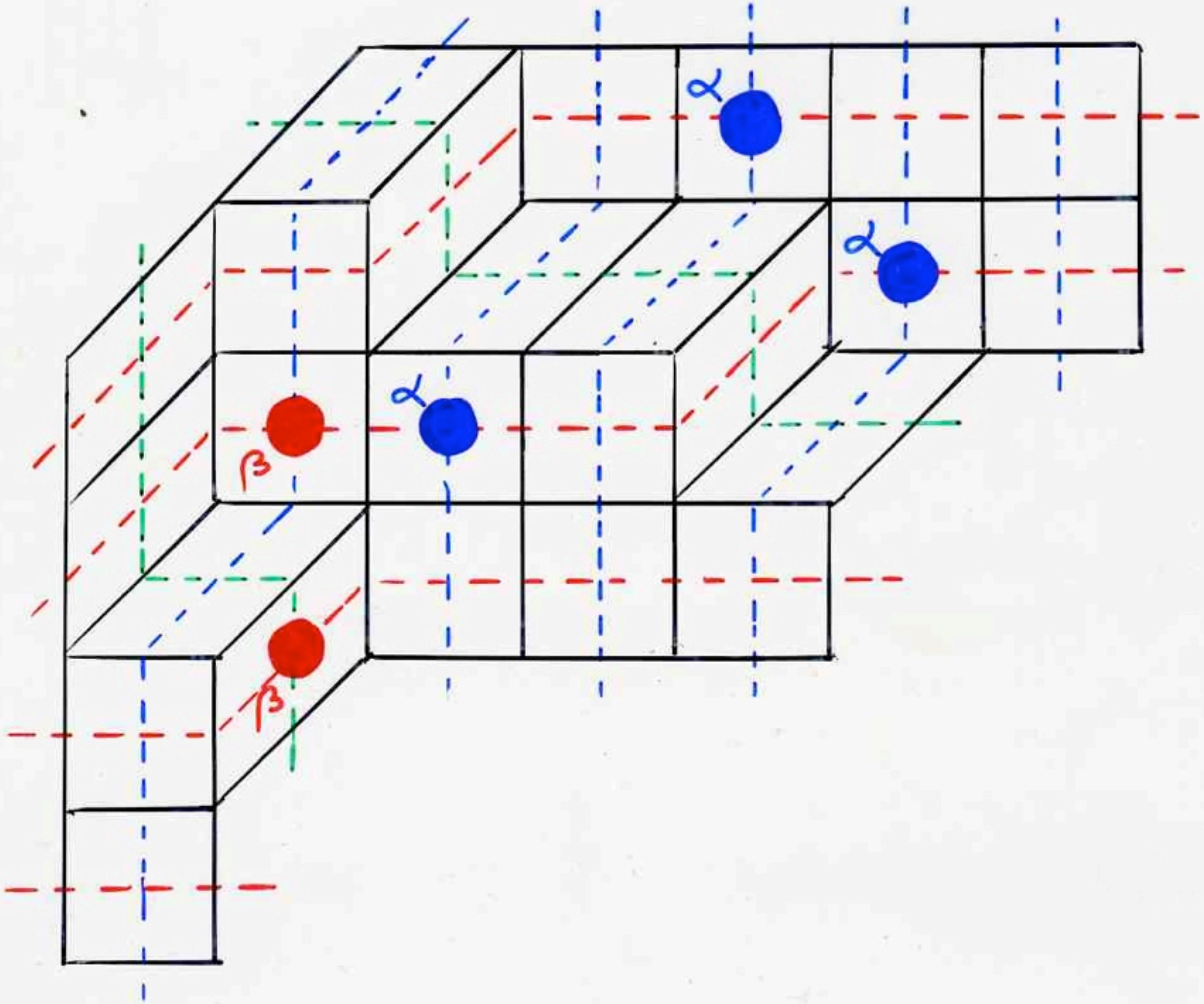

12
(8,9,10,11)
7
(5,6)
(1,2)
3
4



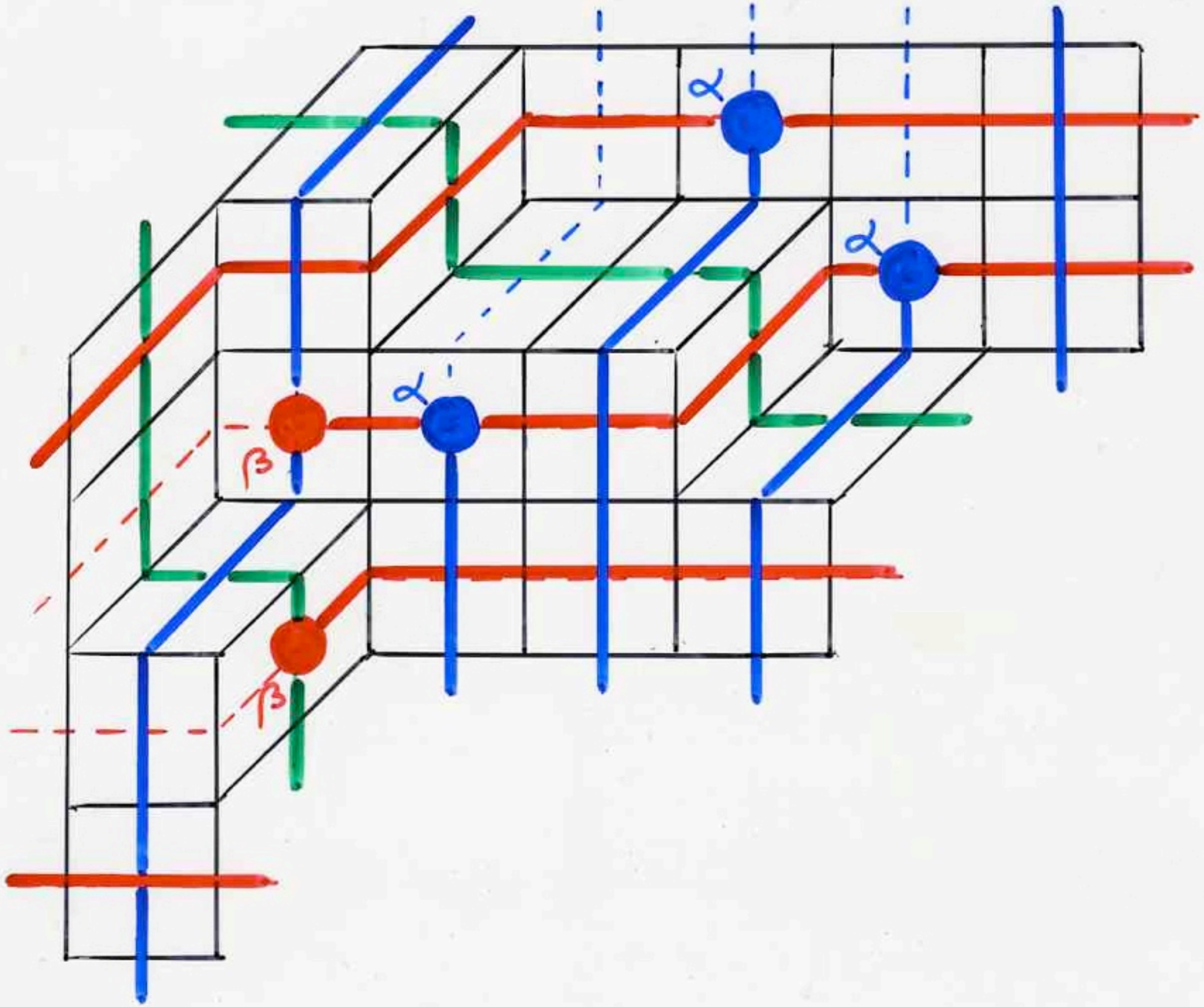
2 10 12 7 5 9 1 8 6 3 11 4
10
9
5
6
(5,6)
(9,10)
(9,10)
8
11
(8,9,10)
(8,9,10,11)
2
1
(1,2)
12
(8,9,10,11)
7
(5,6)
(1,2)
3
4

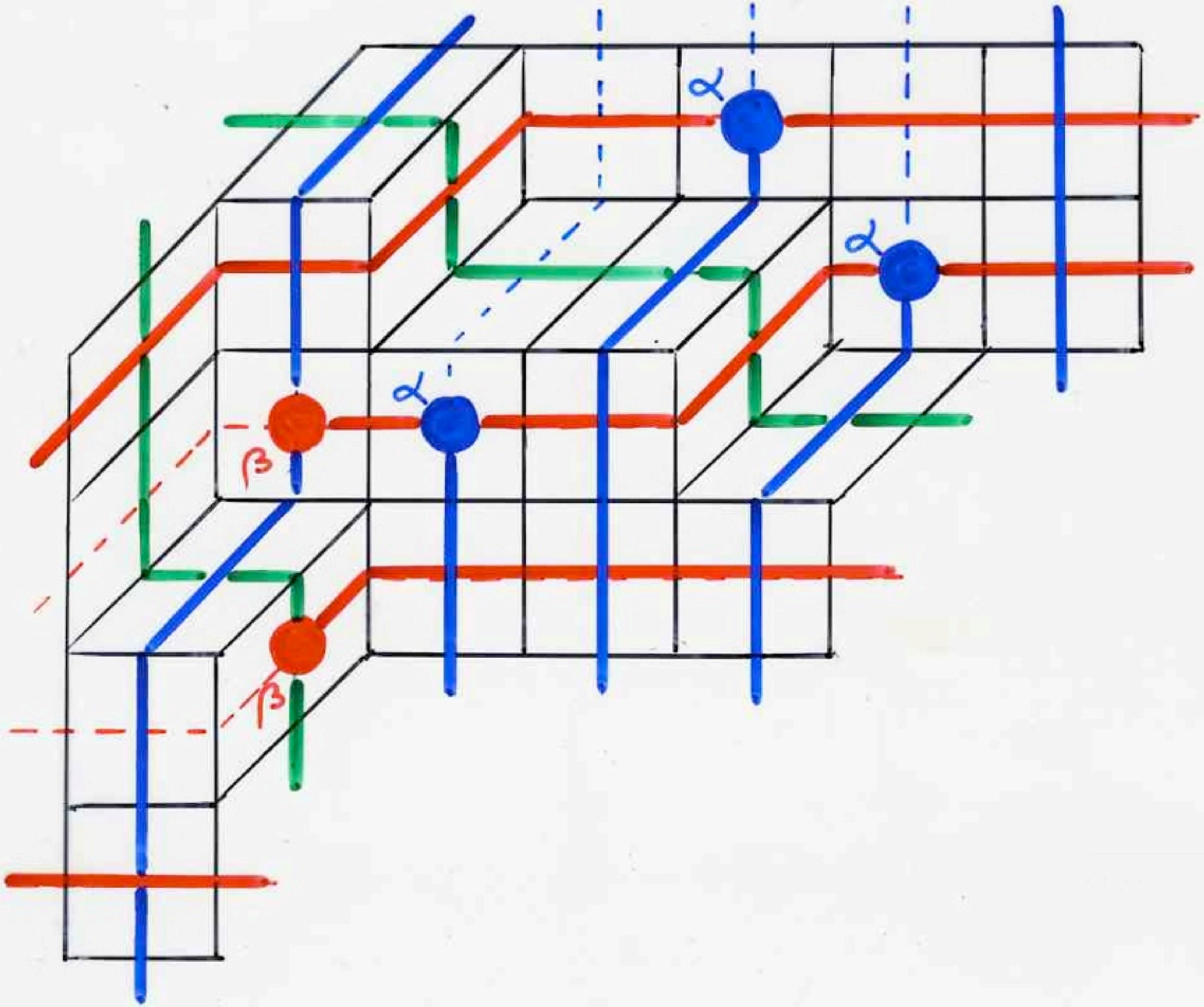




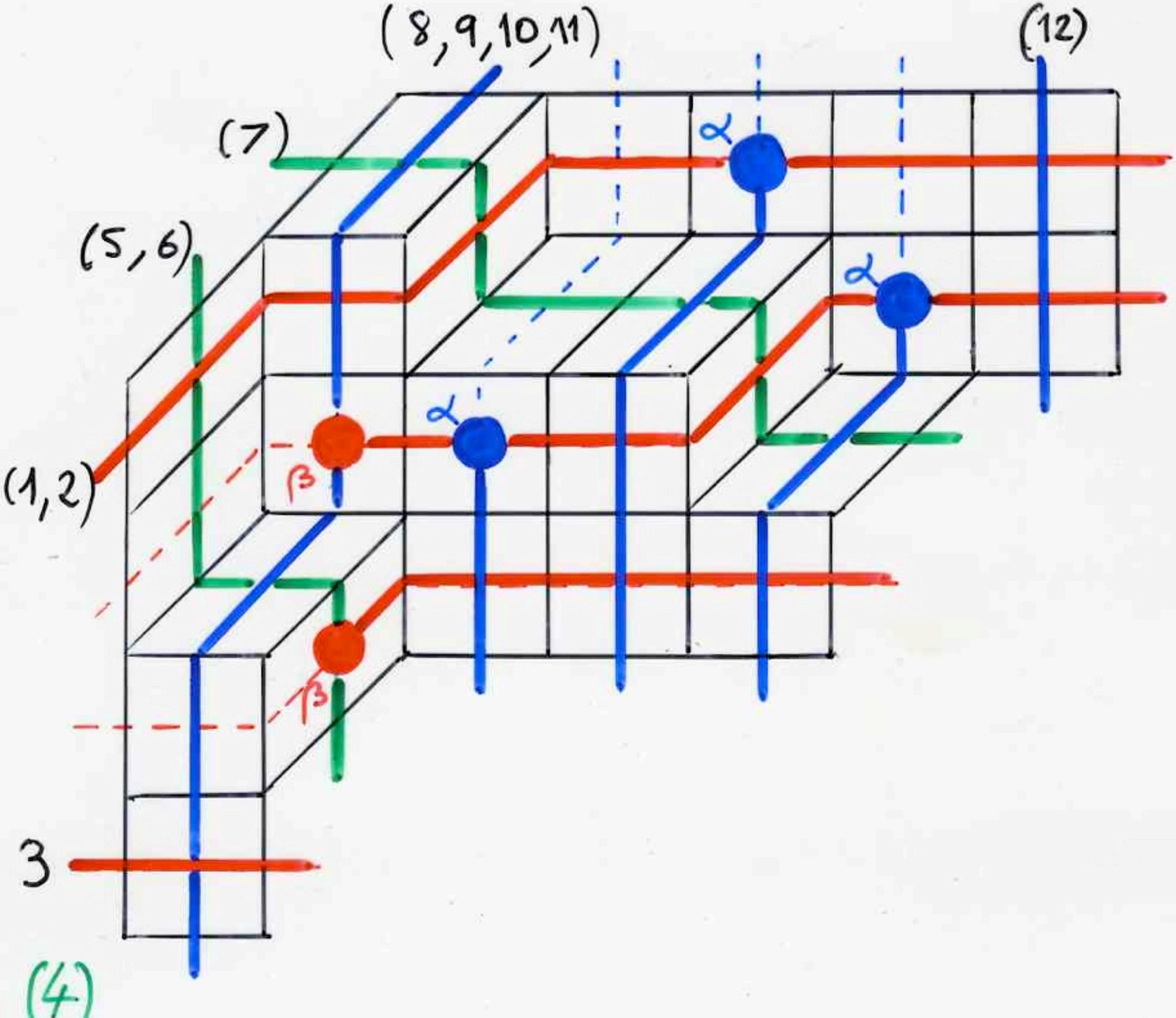


α
α
β
α
β

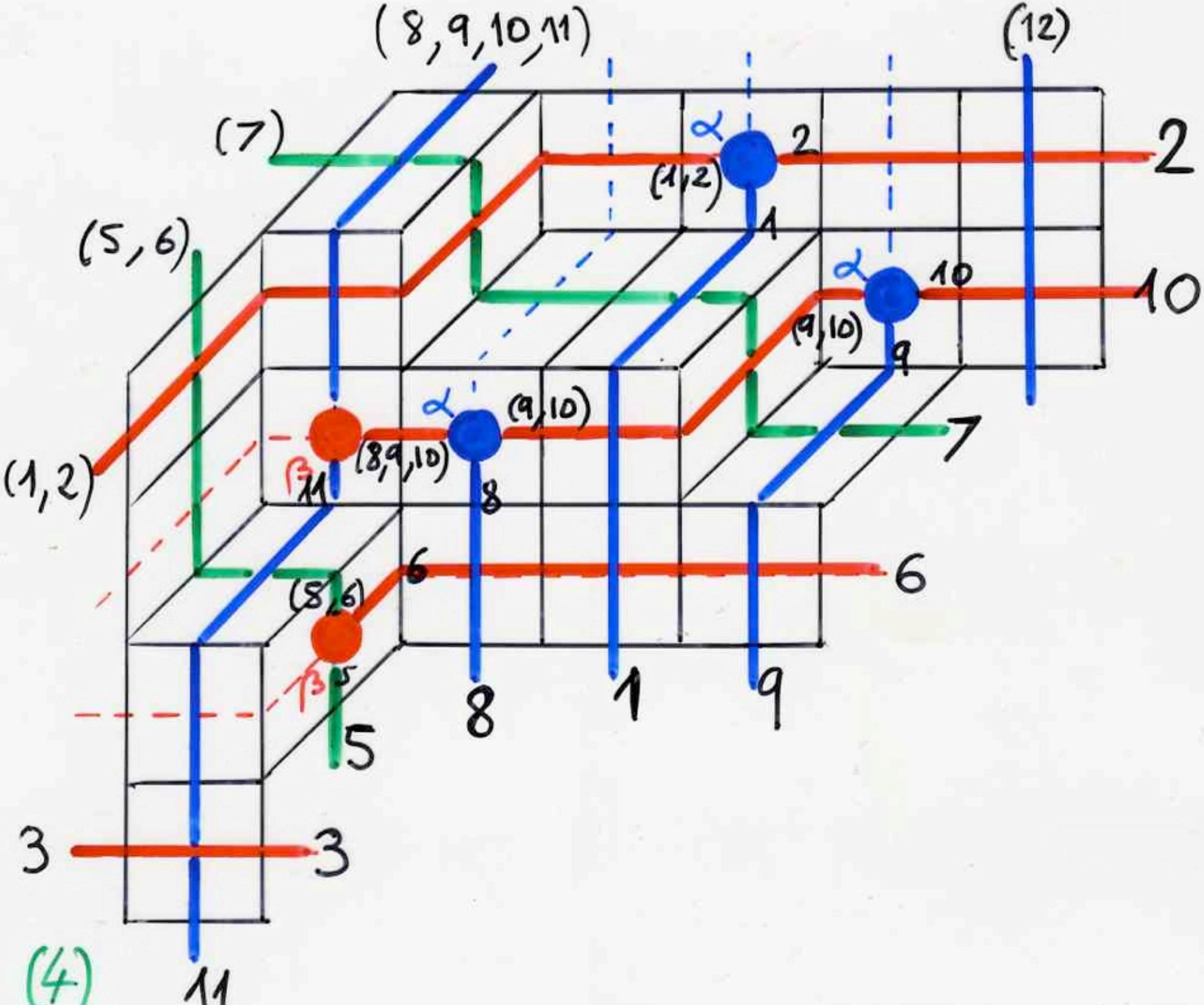

α
α
α
β
β



α
α
α
β
β



(8, 9, 10, 11)
(12)
(7)
(5, 6)
(1, 2)
3
(4)
α
α
α
α
β
β


(8,9,10,11)
(12)
(7)
(5,6)
(1,2)
2
1
10
(9,10)
9
(9,10)
(8,9,10)
11
8
7
6
(8,6)
5
α
β
1
9
3
3
(4)
11

bijection

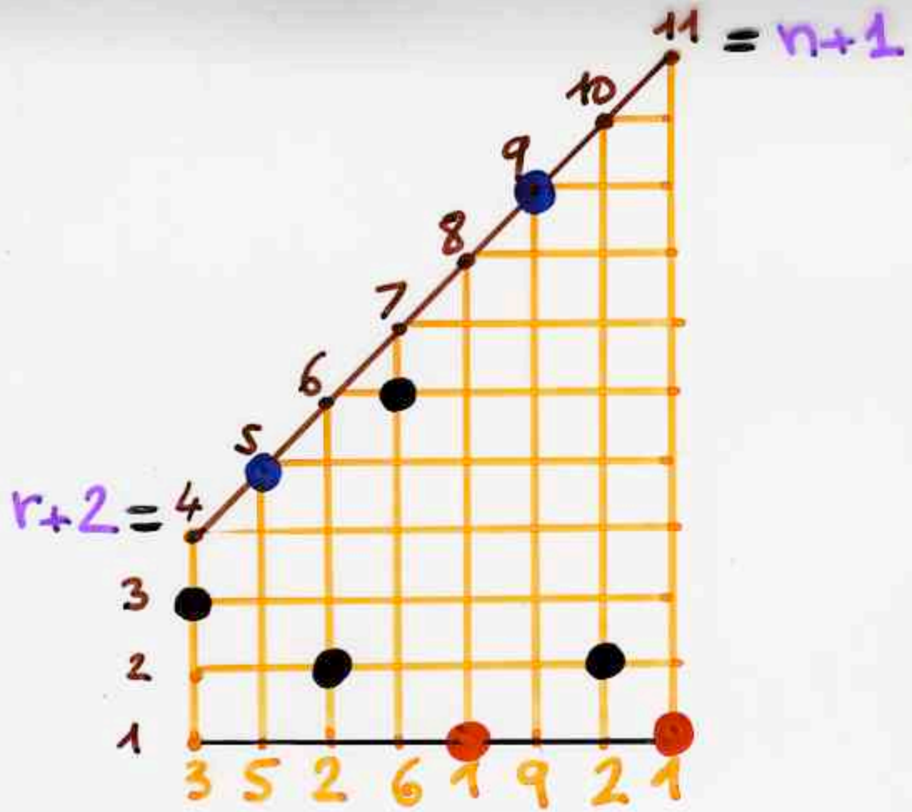
assemblée of permutations

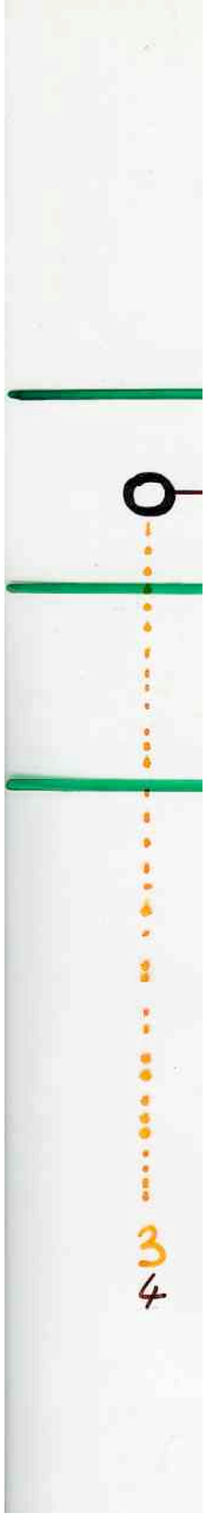
$\updownarrow$

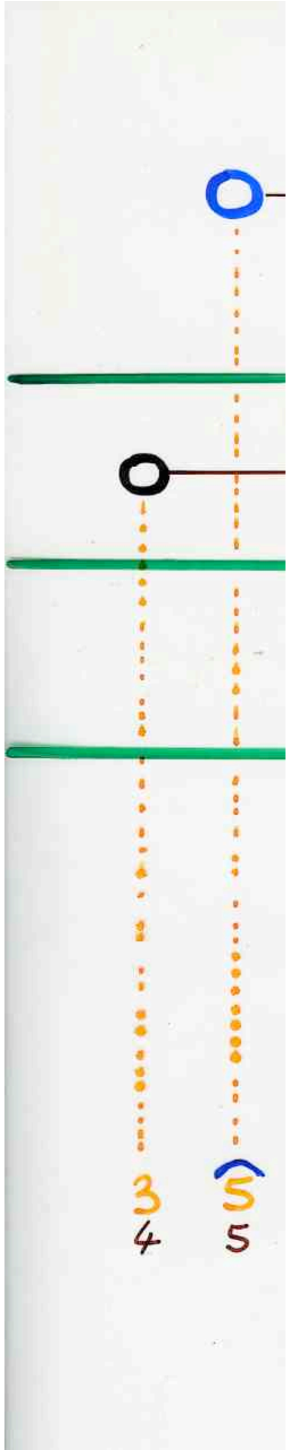
(subset of r elements among n) x
(r-truncated subexceedant functions)

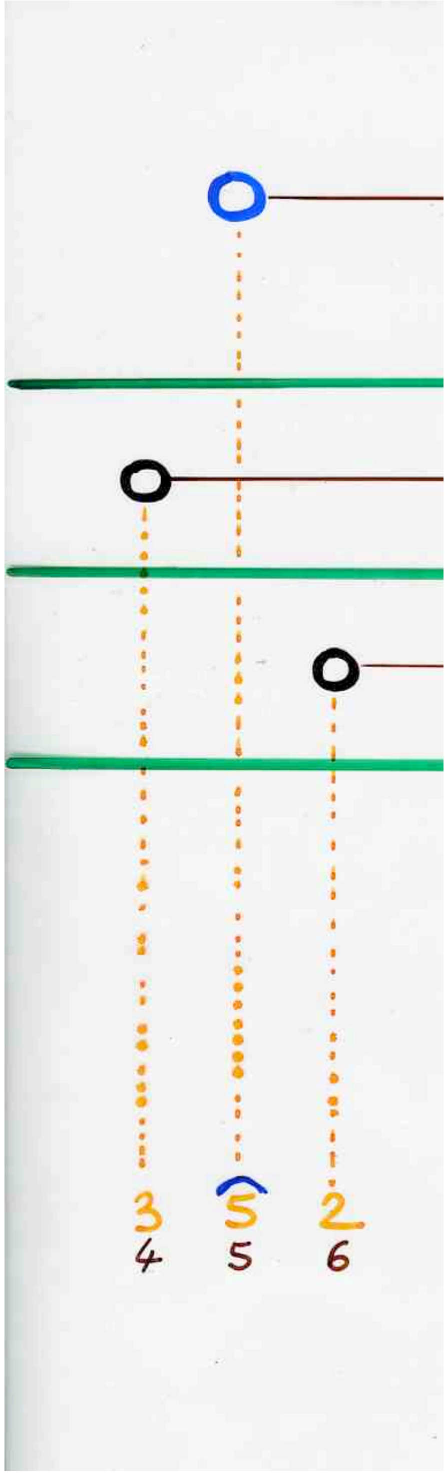
$$Z^{*}_{n,r}(\alpha, \beta, q=1) = \binom{n}{r} \prod_{i=r}^{n-1} \left(\alpha^{-1} + \beta^{-1} + i\right)$$

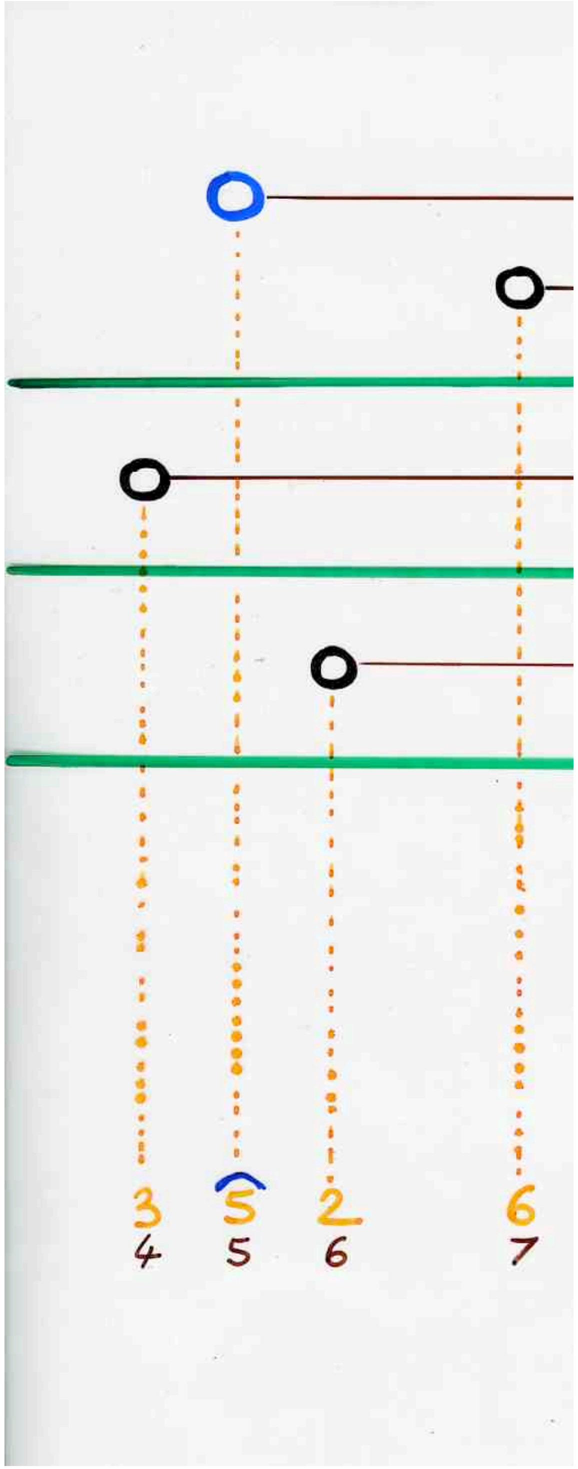
$$\binom{n}{r}$$

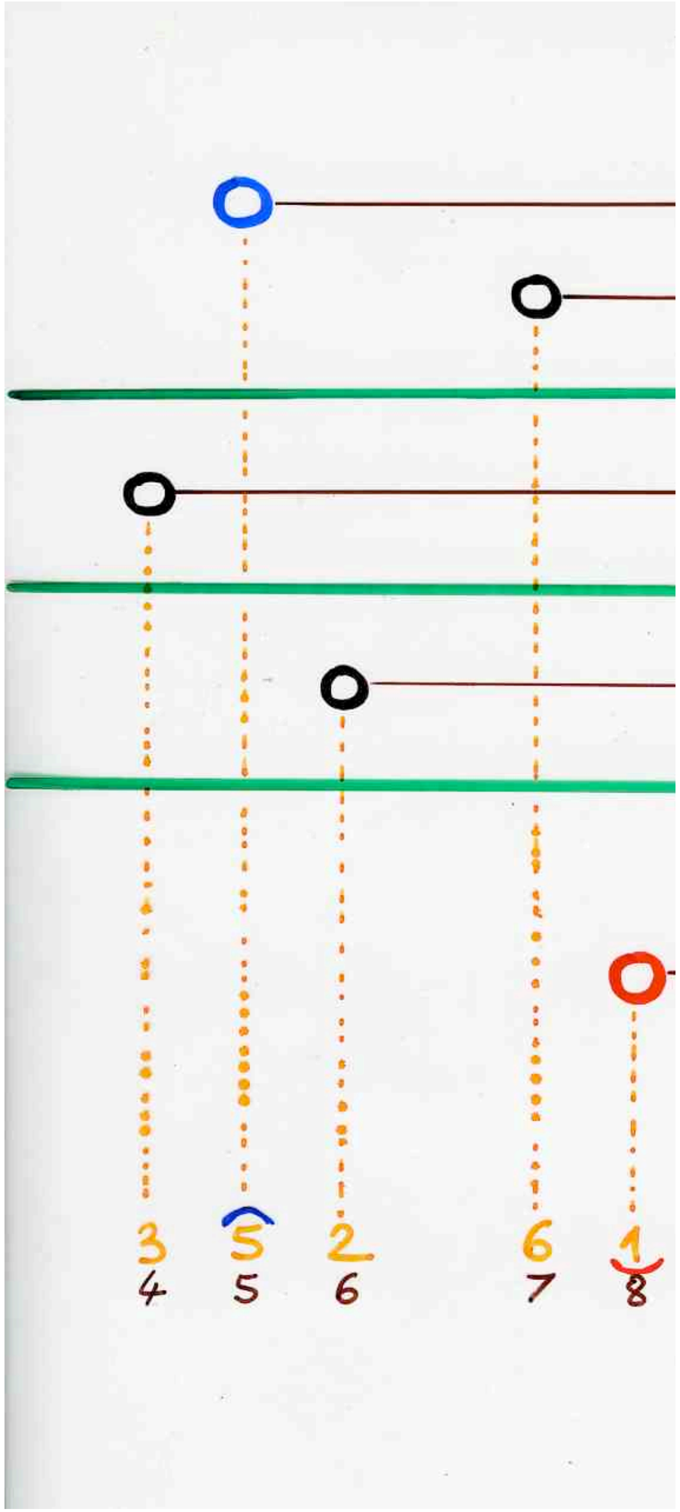


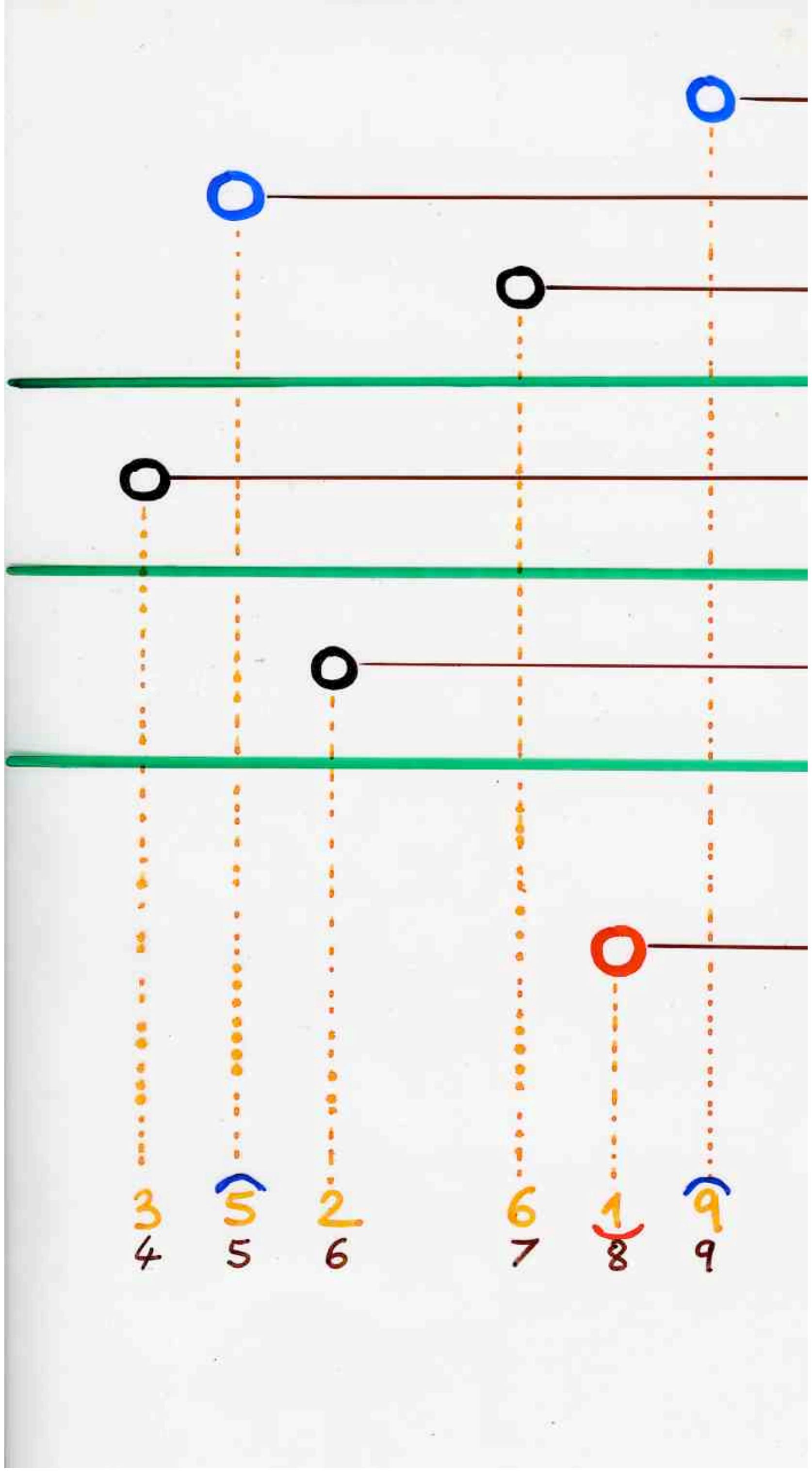

3
4

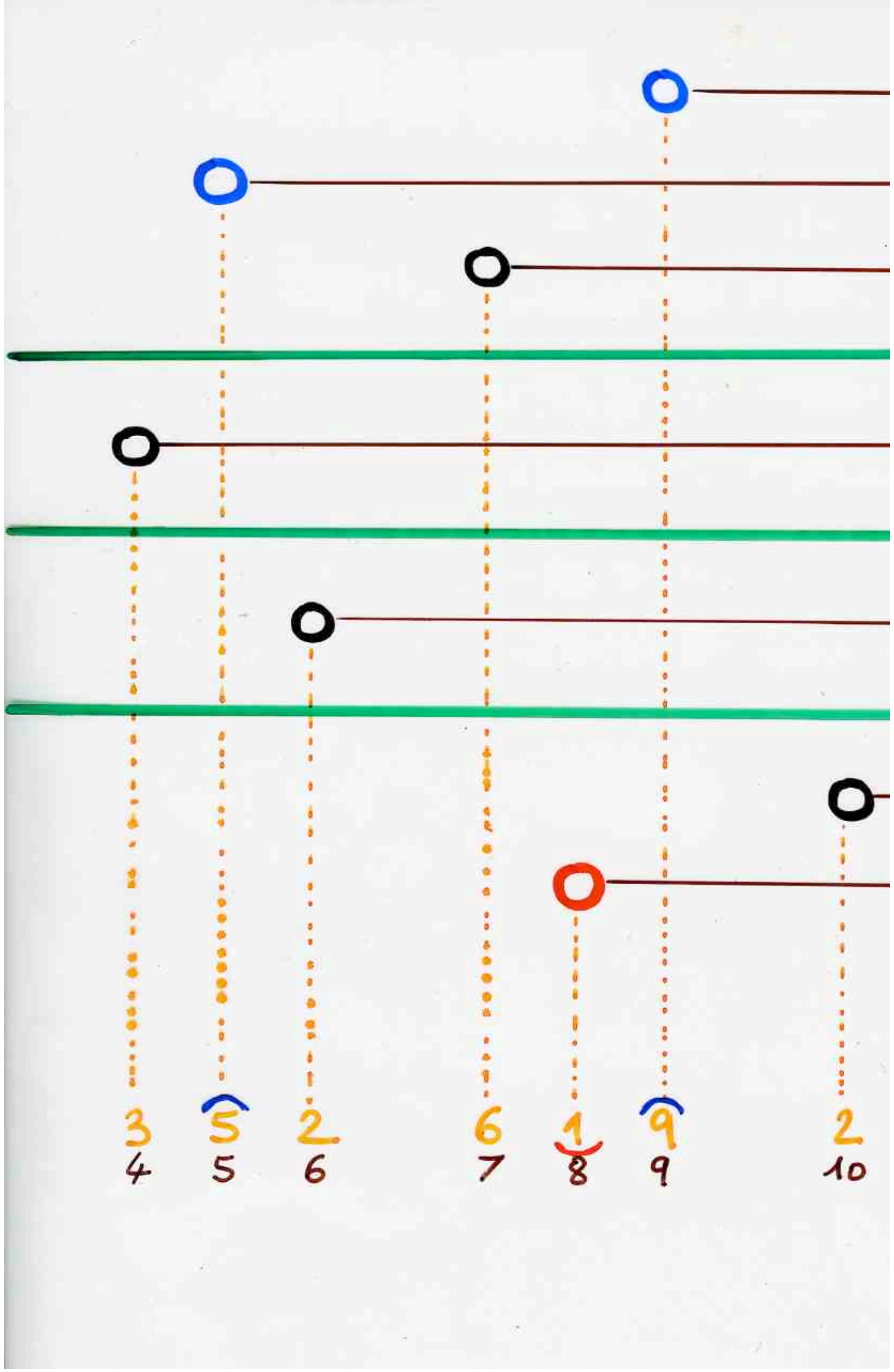

3 5
4 5

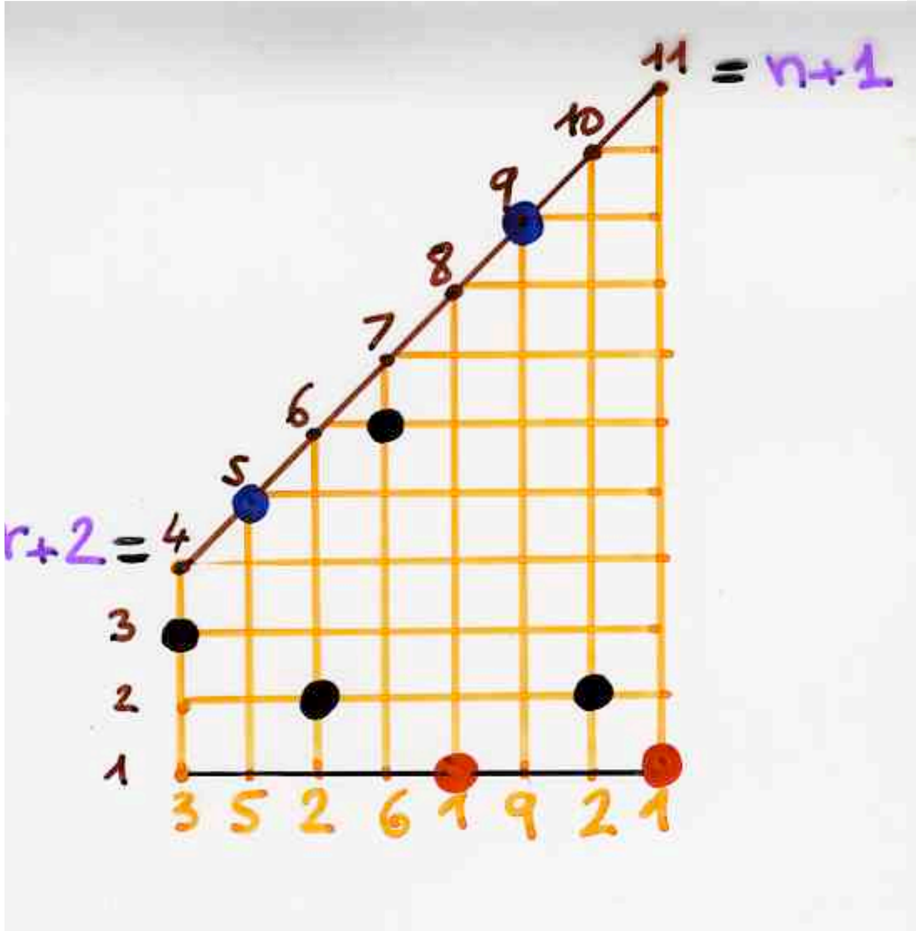

3 5 2
4 5 6


3 5 2 6
4 5 6 7

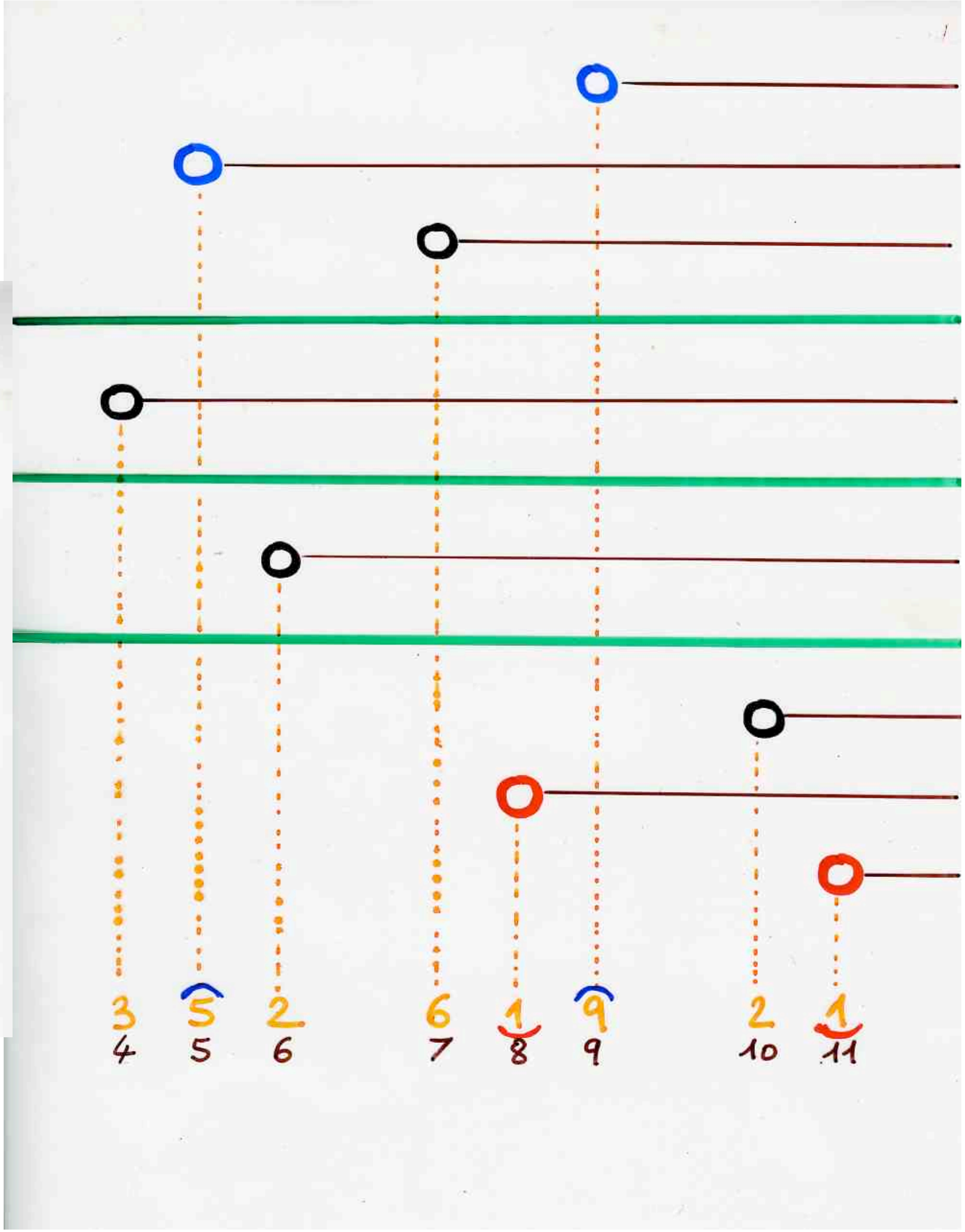

3 5 2 6 1
4 5 6 7 8


3 5 2 6 1 9
4 5 6 7 8 9


3 5 2 6 1 9 2
4 5 6 7 8 9 10

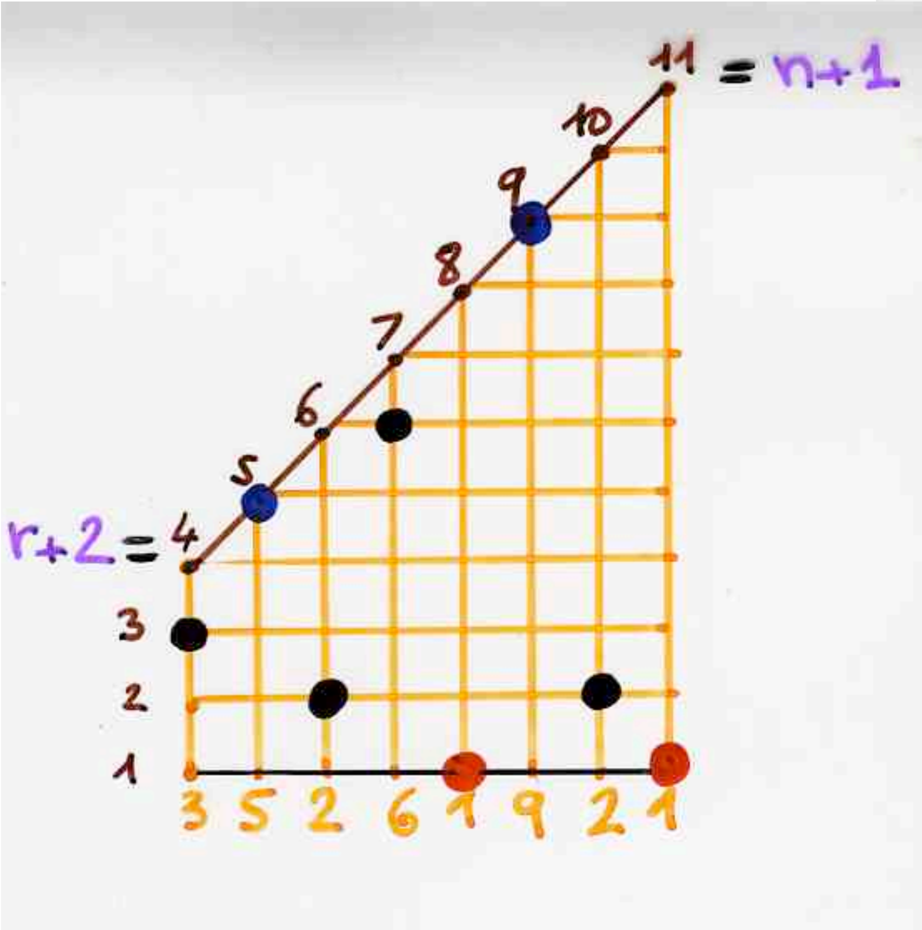


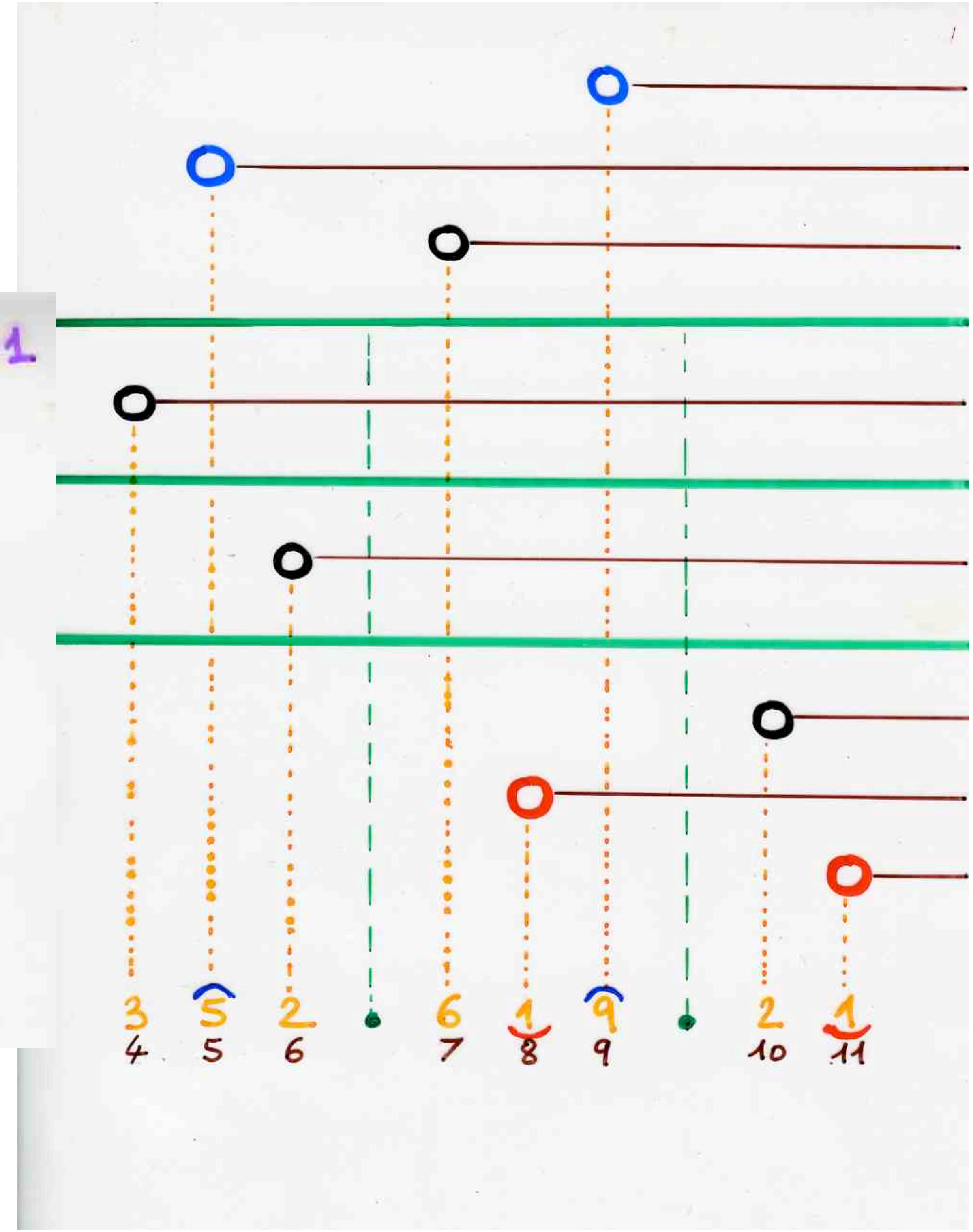
11 = n+1
10
9
8
7
6
5
r+2 = 4
3
2
1
3 5 2 6 1 9 2 1

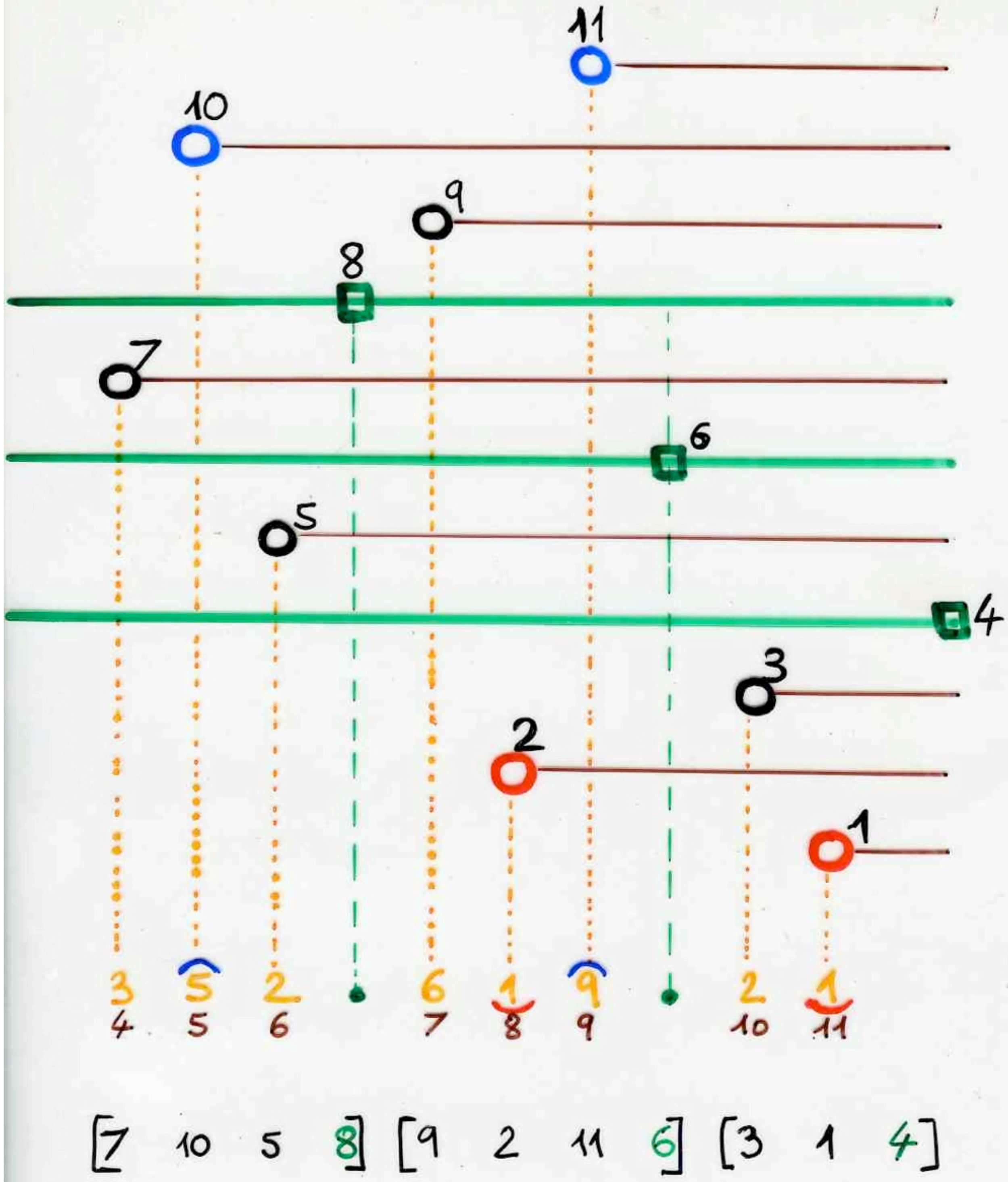


3 5 2 6 1 9 2 1
4 5 6 7 8 9 10 11

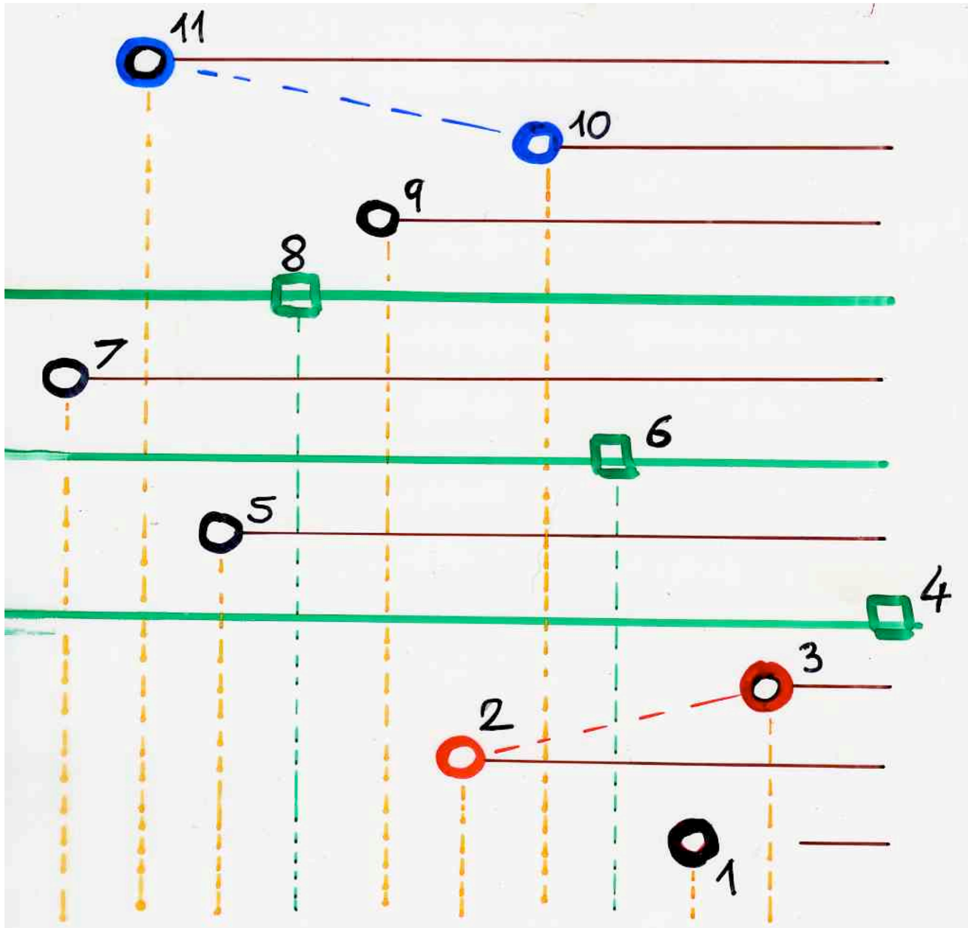
$$\binom{n}{r}\prod_{i=r}^{n-1}\left(\alpha^{-1}+\beta^{-1}+i\right)$$

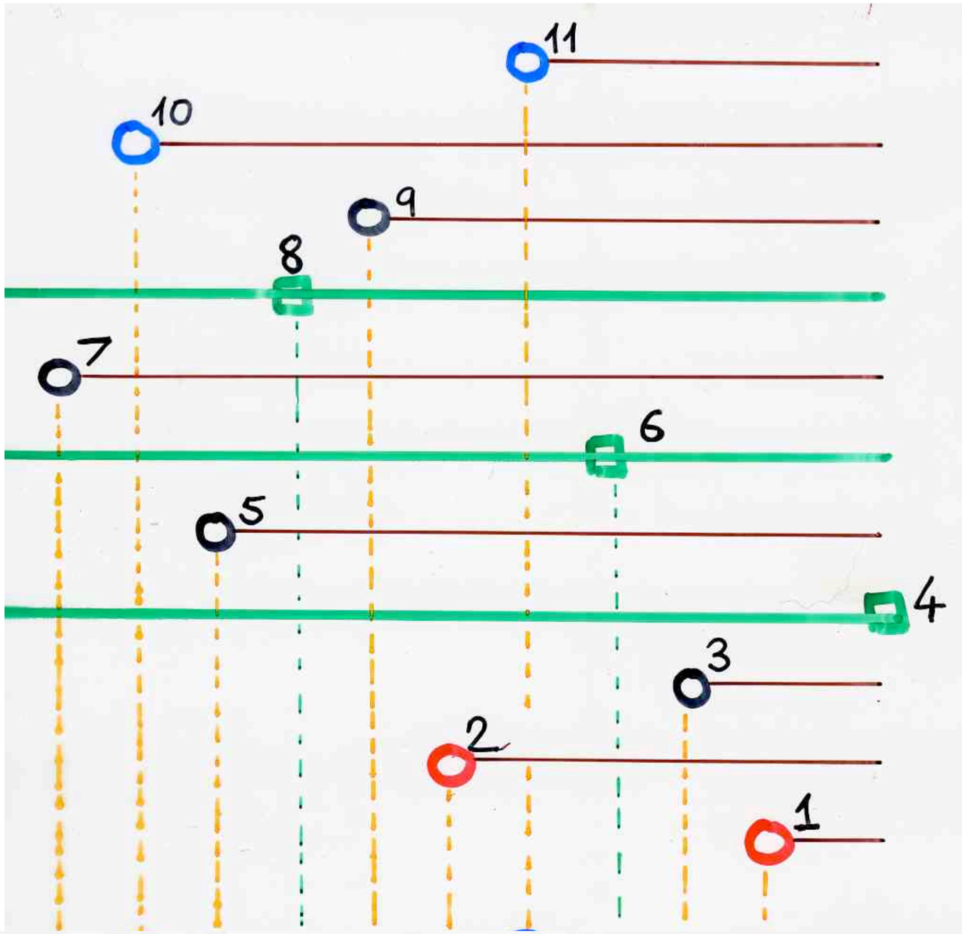






11
10
9
8
7
6
5
4
3
2
1
3 5 2 6 1 9 2 1
4 5 6 7 8 9 10 11
[7 10 5 8] [9 2 11 6] [3 1 4]





$[7 \quad 10 \quad 5 \quad 8] \qquad [9 \quad 2 \quad 11 \quad 6] \qquad [3 \quad 1 \quad 4]$

$[7 \quad 10 \quad 5 \quad 8] \qquad [9 \quad 2 \quad 11 \quad 6] \qquad [3 \quad 1 \quad 4]$

$u = \underline{2} \; 3 \; \underline{1} \qquad u^c = \bar{2} \; 1 \; \bar{3}$

$v = \bar{10} \; 9 \; \bar{11} \qquad \tilde{v} = \bar{11} \; 9 \; \bar{10}$

$[7 \quad \bar{11} \quad 5 \quad 8] \qquad [9 \quad \bar{2} \quad \bar{10} \quad 6] \qquad [1 \quad \bar{3} \quad 4]$

$$\binom{n}{r}(r+\bar{\alpha}+\bar{\beta})\cdots(n-1+\bar{\alpha}+\bar{\beta})$$

$$[7\ \ 10\ \ 5\ \ 8]\quad[9\ \ 2\ \ 11\ \ 6]\quad[3\ \ 1\ \ 4]$$

$$[7\ \ 10\ \ 5\ \ 8]\quad[9\ \ 2\ \ 11\ \ 6]\quad[3\ \ 1\ \ 4]$$

$$u = \underline{2}\ \ 3\ \ \underline{1} \qquad u^c = \bar{2}\ \ 1\ \ \bar{3}$$

$$v = \overline{10}\ \ 9\ \ \overline{11} \qquad \tilde{v} = \overline{11}\ \ 9\ \ \overline{10}$$

$$[7\ \ \overline{11}\ \ 5\ \ 8]\quad[9\ \ \bar{2}\ \ \overline{10}\ \ 6]\quad[1\ \ \bar{3}\ \ 4]$$

further enumerative results

$q = 0$ (Olya Mandelshtam)

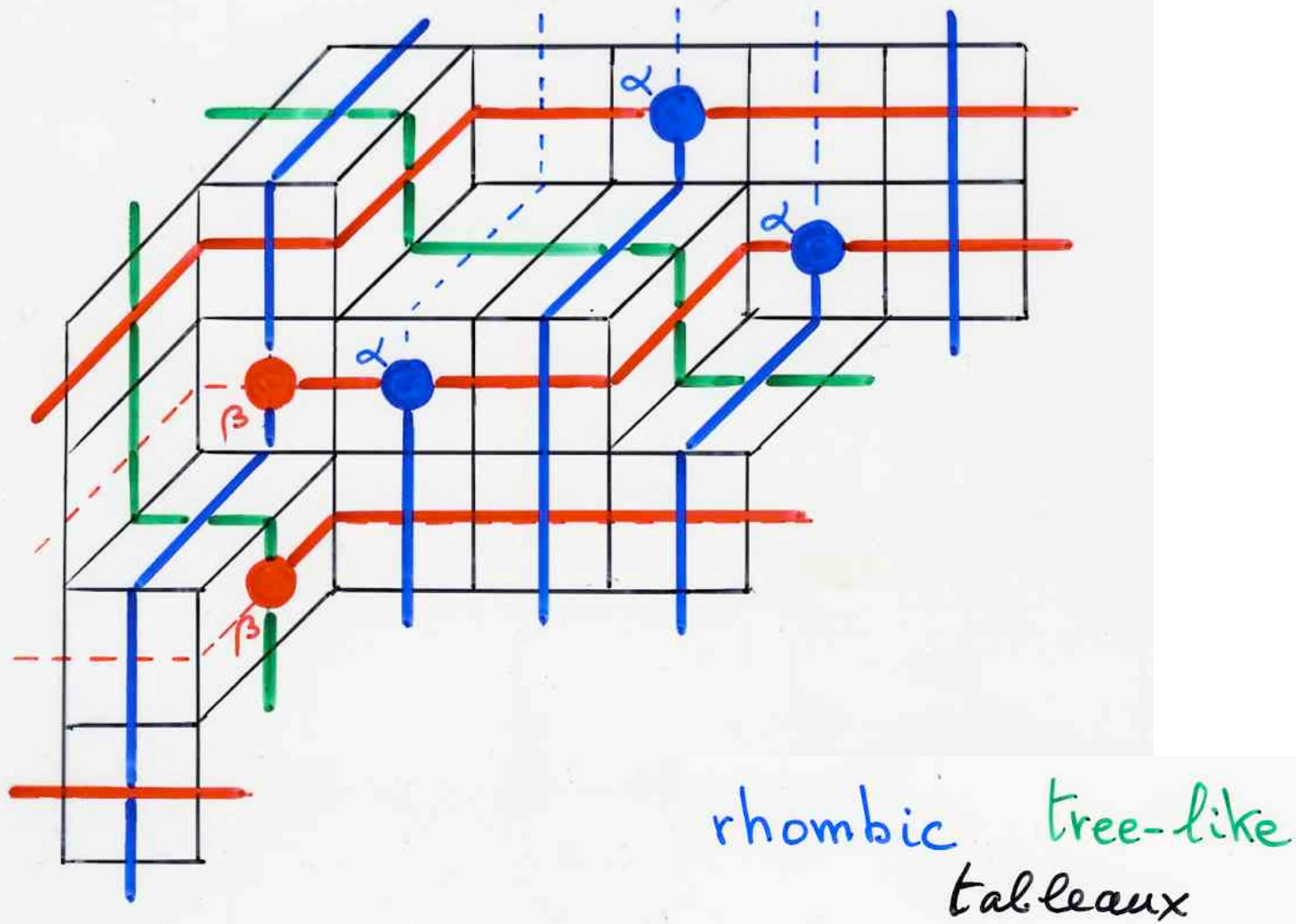
$$Z^{*}_{n,r}(\alpha, \beta, 0) = \sum_{p=1}^{n-r} \frac{2r+p}{2n-p} \binom{2n-p}{n+r} \frac{\bar{\alpha}^{p+1} - \bar{\beta}^{p+1}}{\bar{\alpha} - \bar{\beta}}$$

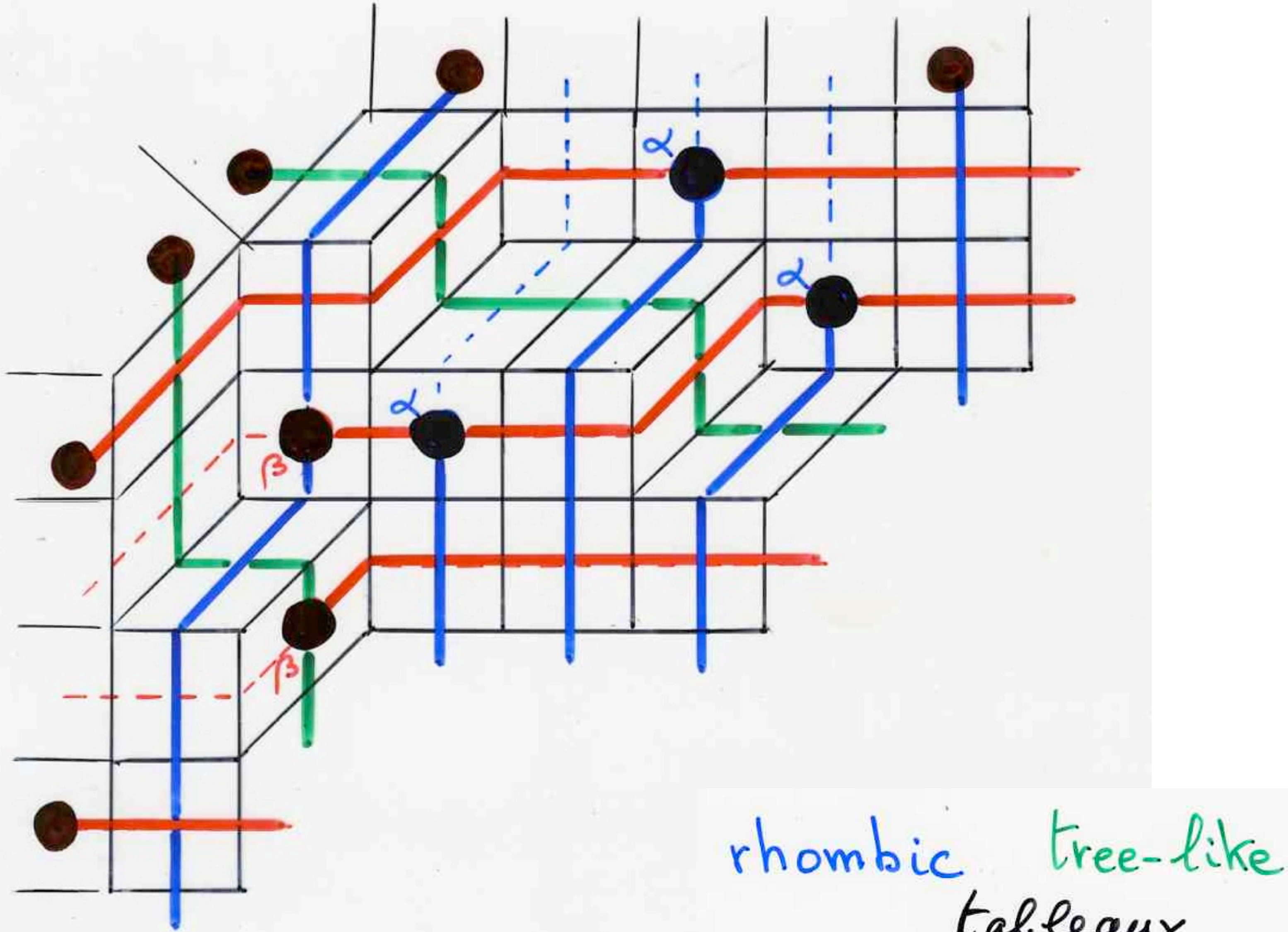
$$Z^{*}_{n,r}(1,1,0) = \frac{2(r+1)}{n+r+2} \binom{2n+1}{n-r}$$

nb of RAT size $n = r + k + \ell$
r A's, k D's, ℓ E's

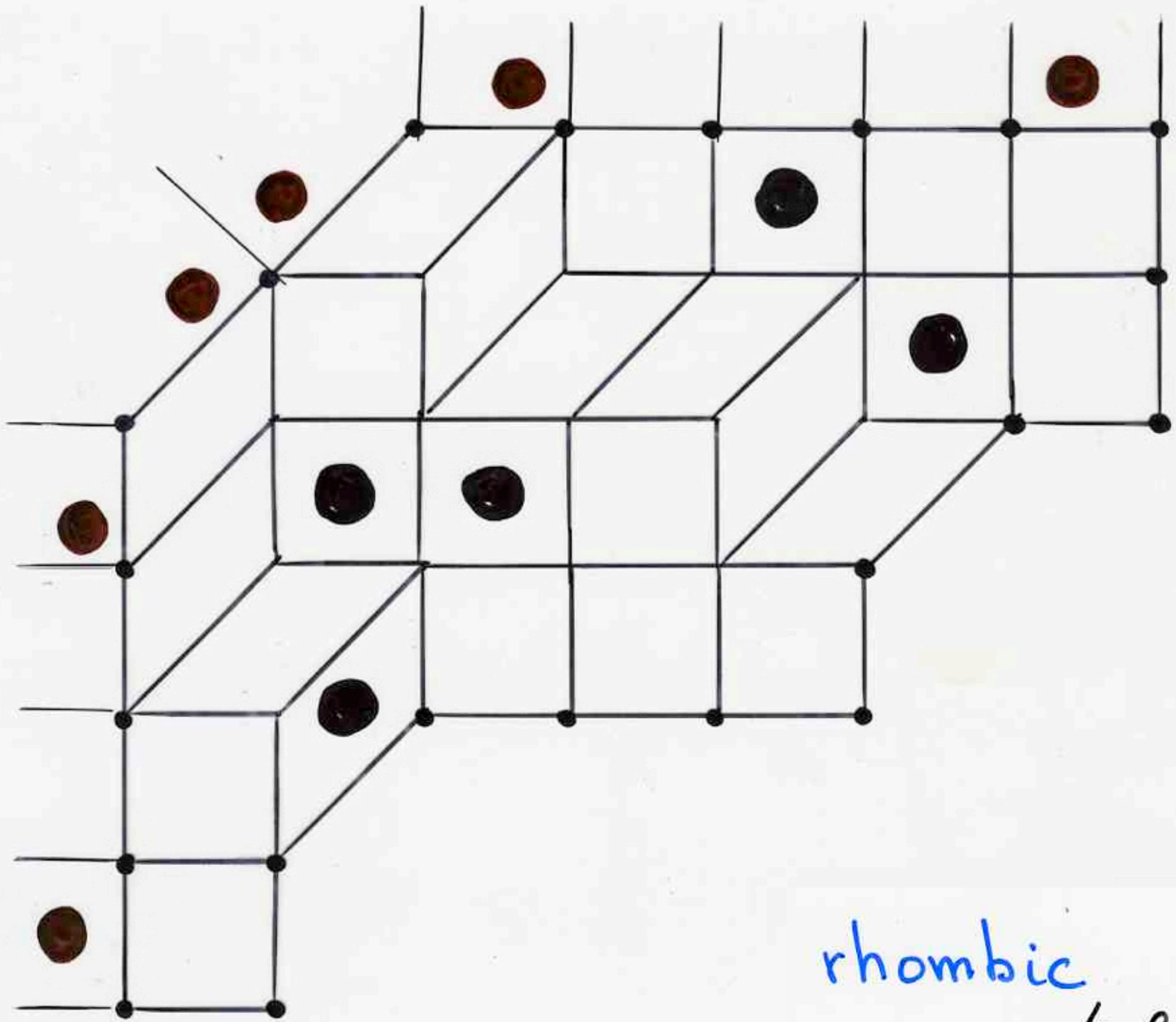
$$\frac{r+1}{n+1} \binom{n+1}{k} \binom{n+1}{\ell}$$

tree-like rhombic tableaux

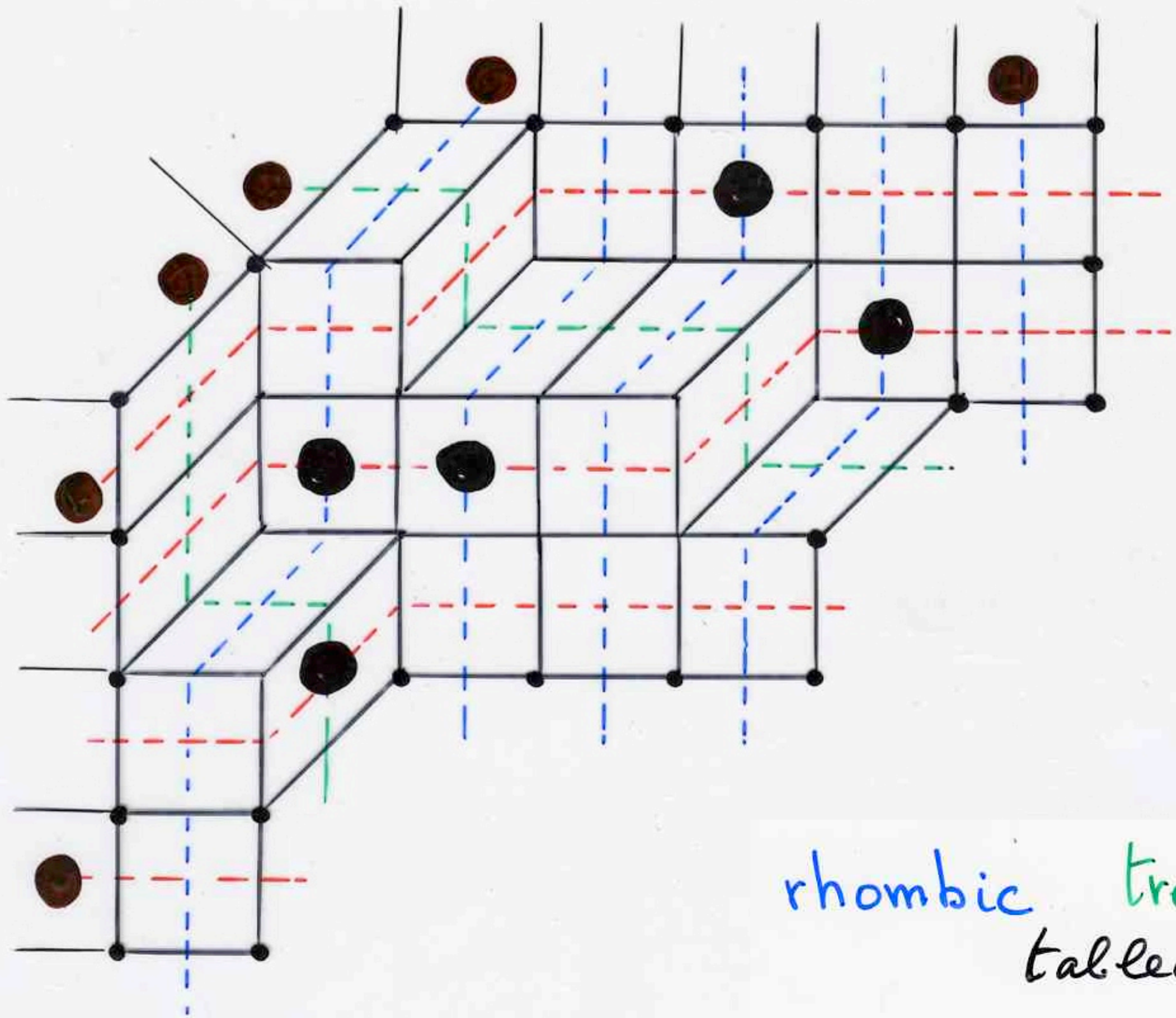

α
α
α
β
β
rhombic tree-like
tableaux



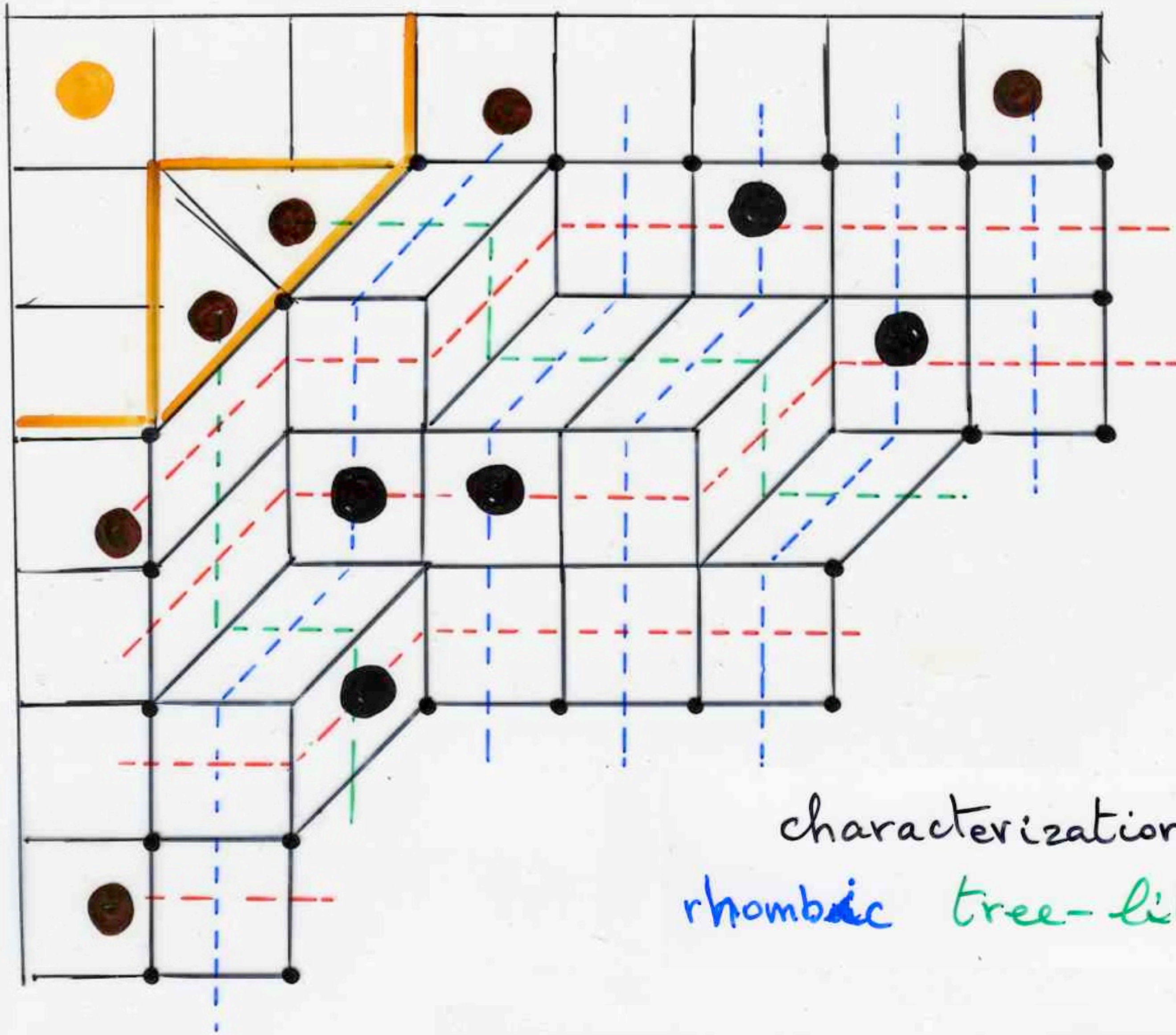
α
α
α
β
β
rhombic tree-like
tableaux



rhombic tree-like
tableaux



rhombic tree-like
tableaux



characterization of
rhombic tree-like tableaux

From the diagram $\Gamma(X)$, add a column on the left, a row above, which intersect in a cell where we add the point ● (the root)

characterization of
rhombic tree-like tableaux

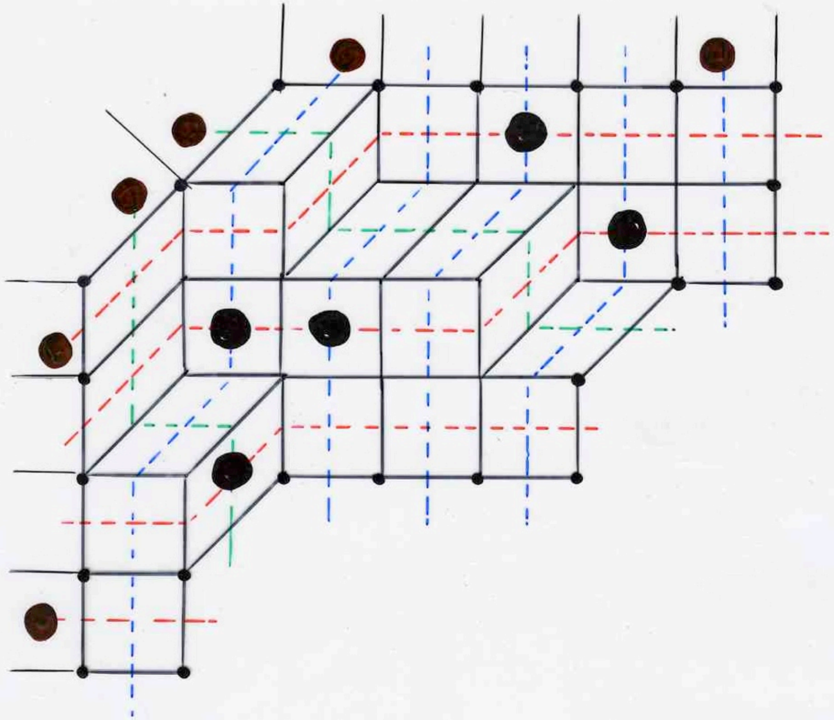
(i) for each diagonal step of the NW border of $\Gamma(X)$, there exist a point ● in the cell just above, and there are no other points in the part of the extended diagram delimited by yellow lines (except the root ●)

characterization of rhombic tree-like tableaux

(ii) for any point ●, other than the points in (i), there exist a point on the left, or above, following the lines (blue, red, or green) which cross the point ●

(iii) only one of the two possibilities of (ii) is possible

(iv) in the case the point ● of (ii) is on a green line, the only possibility is following this green line.



rhombic tree-like
tableaux

- analog of the insertion process
  for (square) tree-like tableaux
  (Aval, Boussicault, Nadeau)
- interpretation of the parameter
  q with assemblées of permutations

?

relation with
Koorwinder-Macdonald polynomials

$K_\lambda(x_1, \ldots, x_n ; q, t)$ Koornwinder-Macdonald polynomials

λ partition of n

$\lambda = [n]$

$AW(\alpha, \beta, \gamma, \delta ; q)$
Askey-Wilson

all "classical" orthogonal polynomials

(q, t) polynomials
Macdonald
roots system

Jack polynomials

Schur functions
$S_\lambda(x_1, \ldots, x_n)$

Koornwinder polynomials (1992)

$P_\lambda(\mathbf{z}; a, b, c, d \mid q, t)$

$z = (z_1, \dots, z_m)$
$\lambda = (\lambda_1, \dots, \lambda_m)$

Macdonald polynomials (1995)

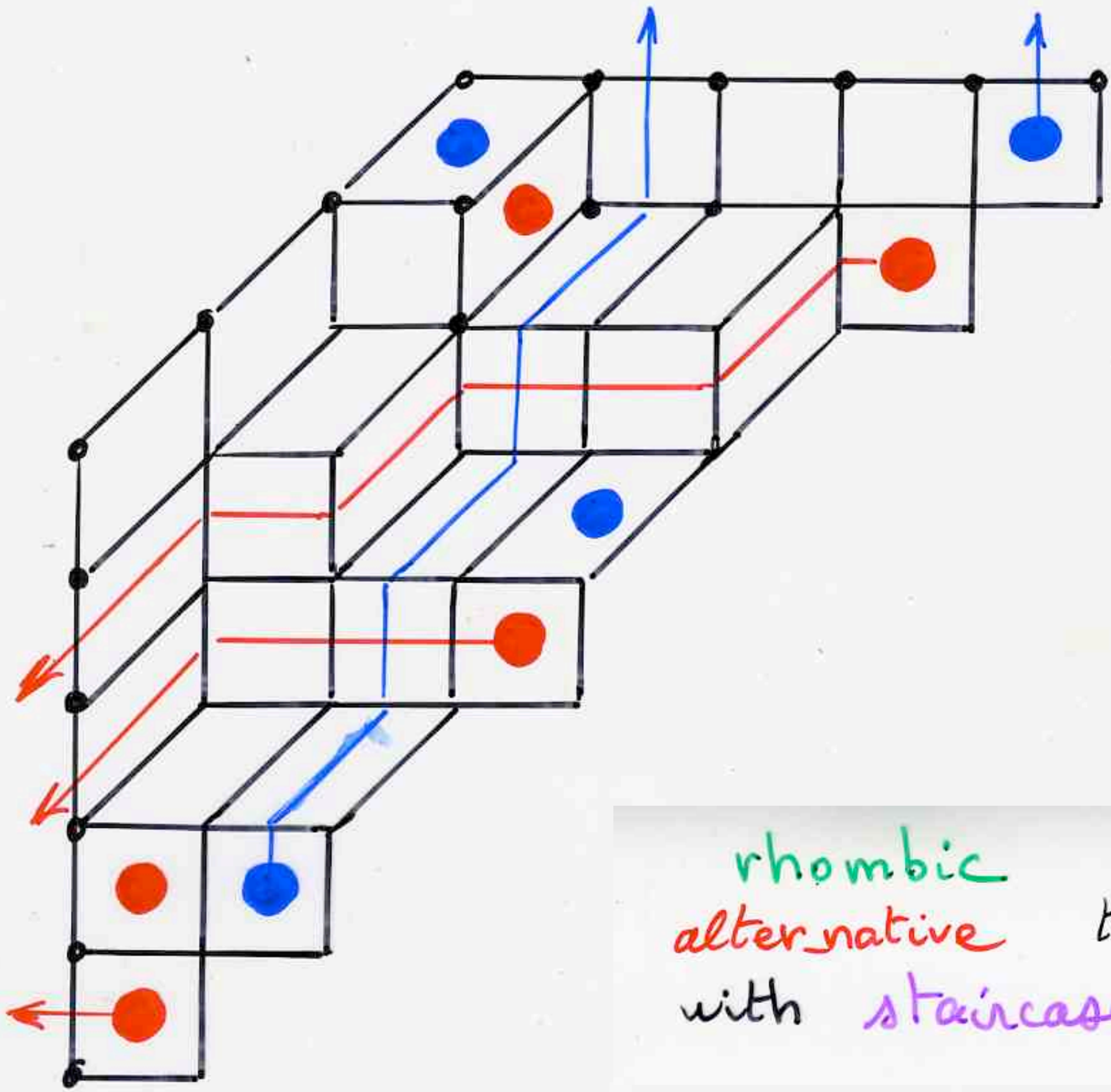
type BC root system

$\mu_n = f(x_n)$ moments

$M_\lambda = I_k(\Delta_\lambda(x_1, \dots, x_m); a, b, c, d; q, q)$

$= \int x^n d\psi$

Schur function



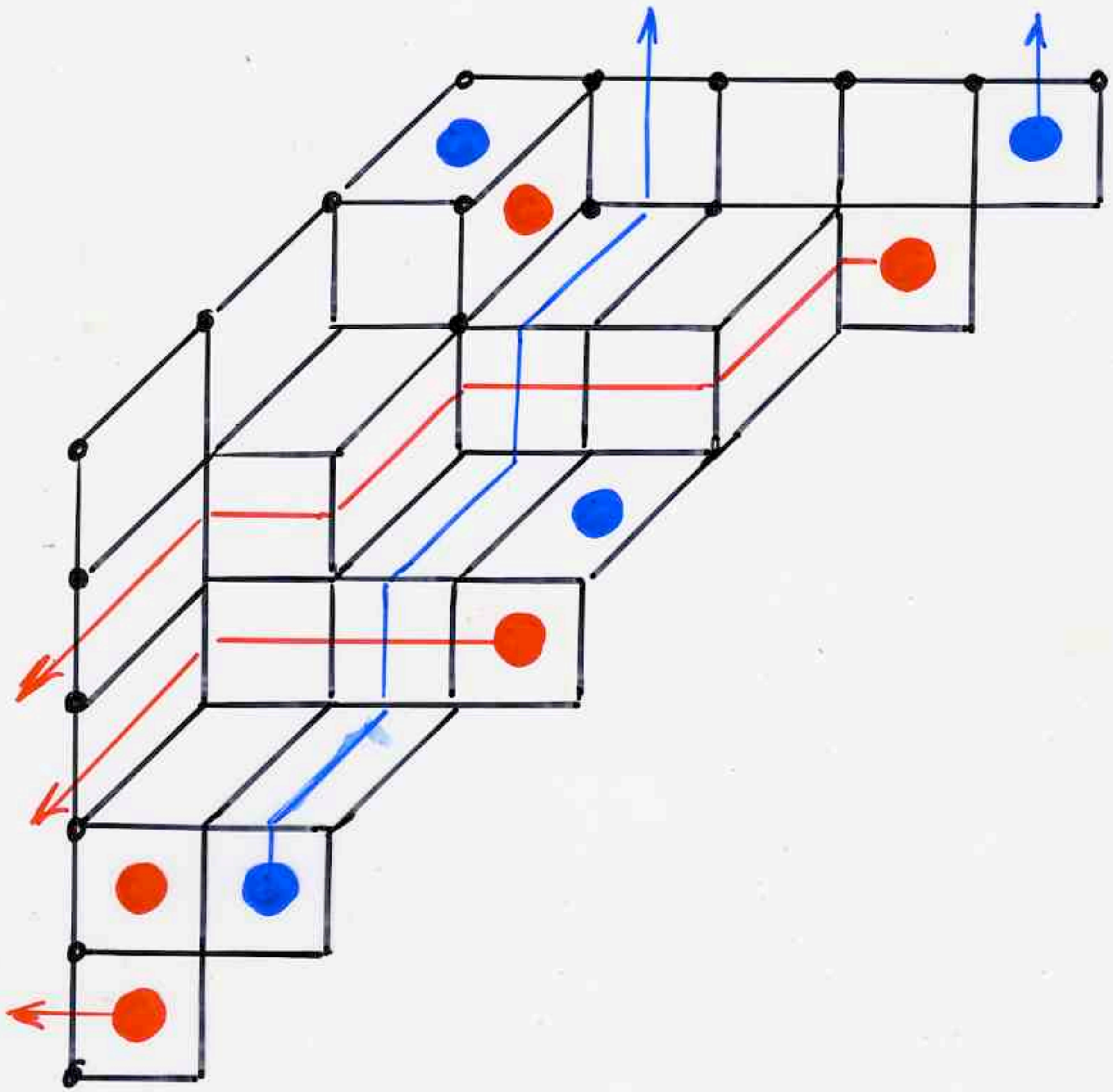
rhombic alternative tableaux with staircase shape

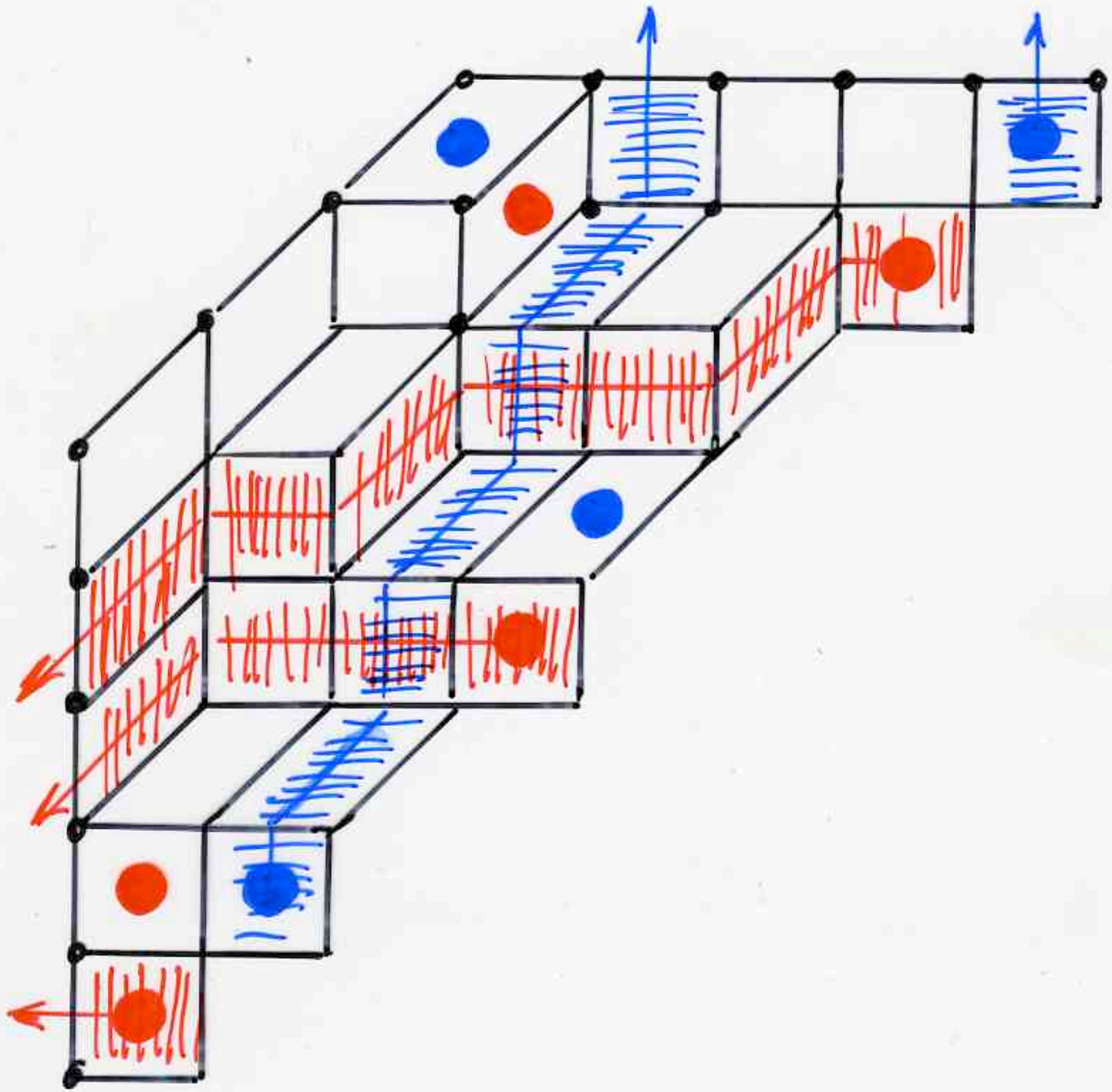
bijection

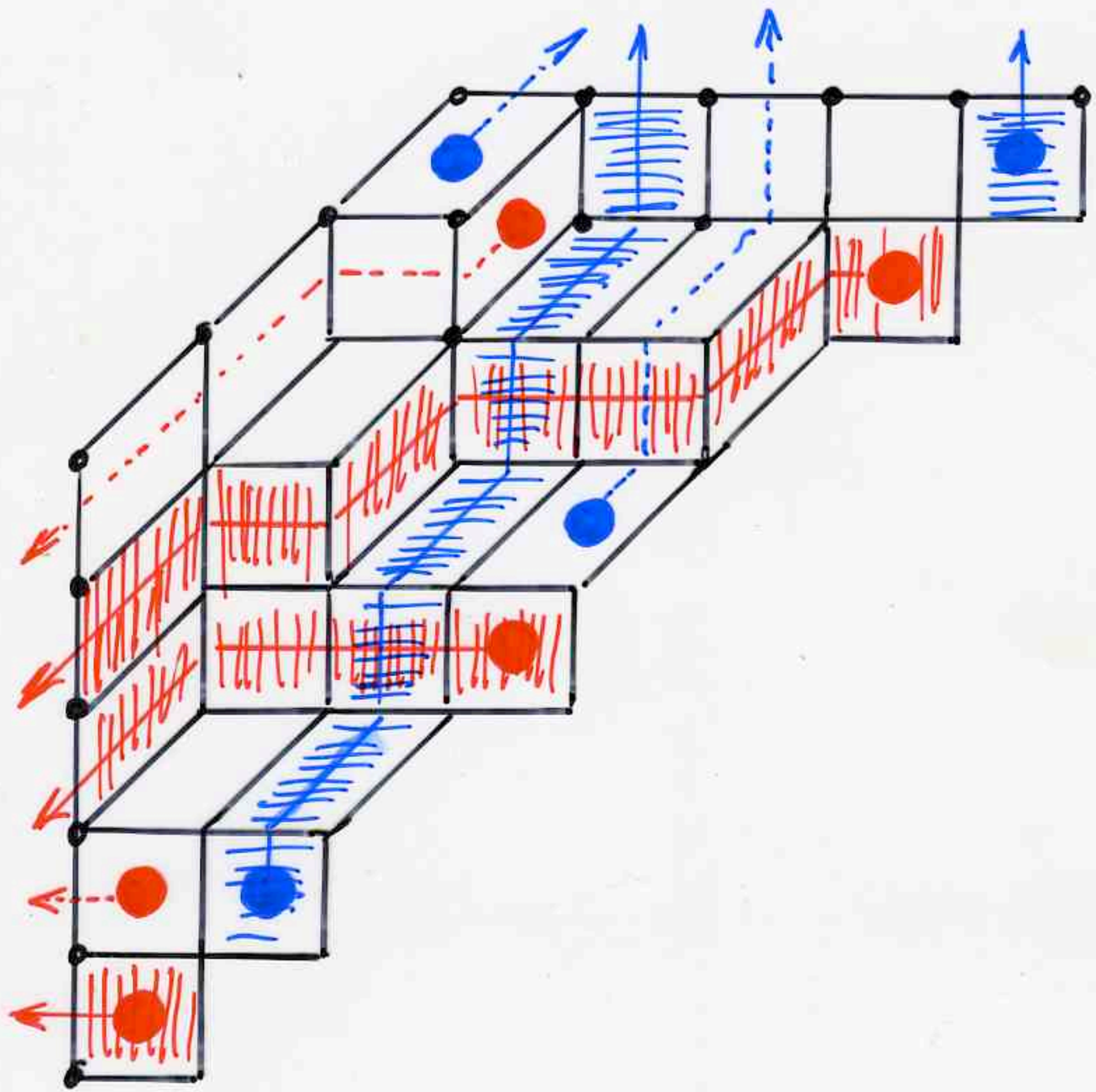
rhombic
alternative
tableaux

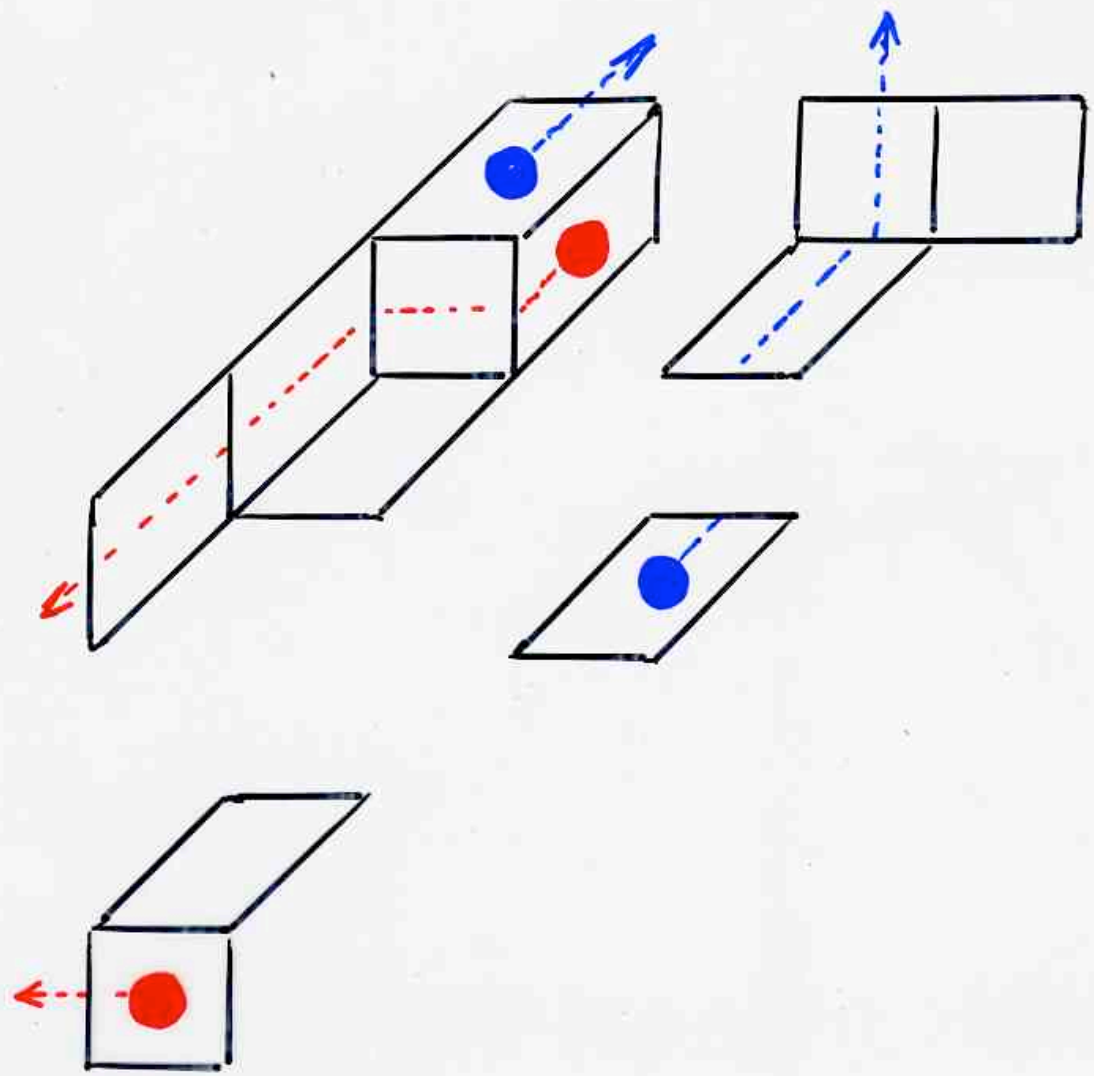
$\longleftrightarrow$

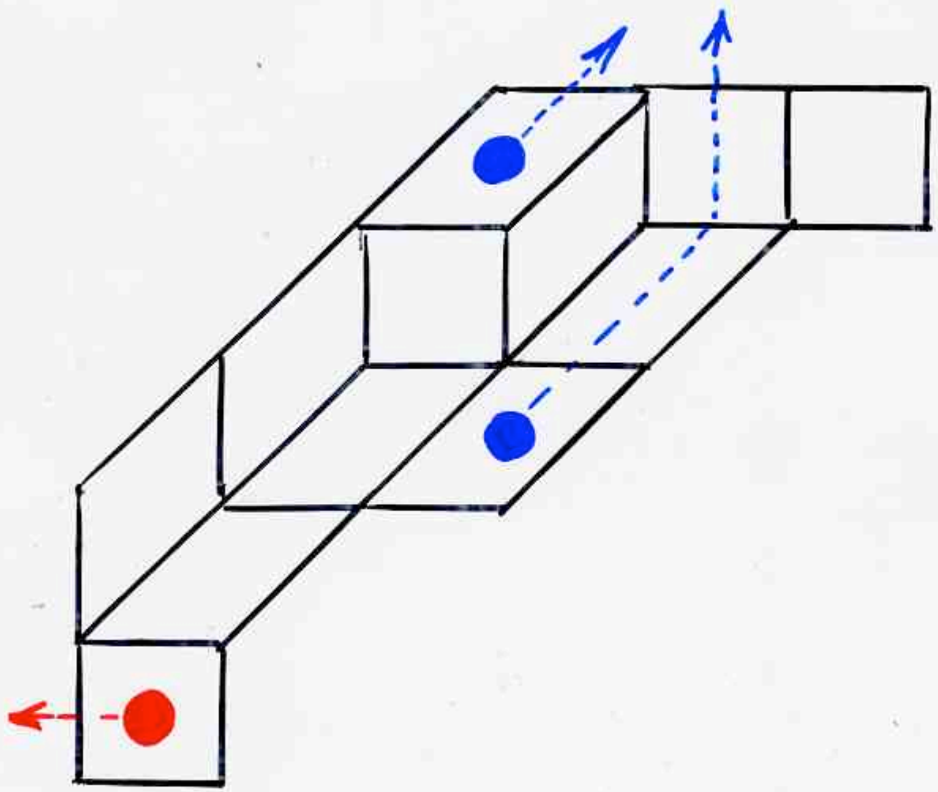
rhombic
alternative
tableaux
staircase shape
+ square cells on the SE border are colored

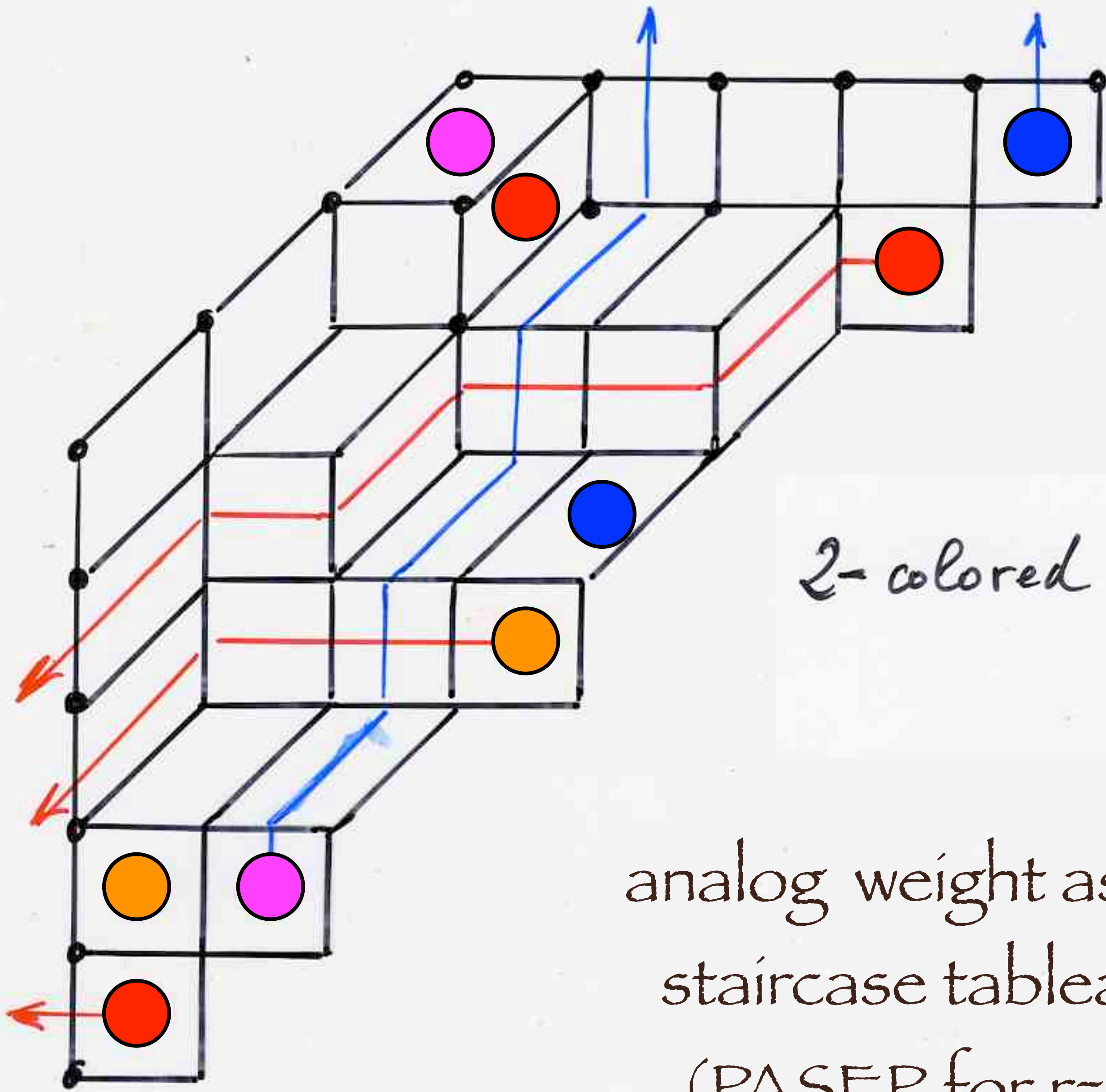


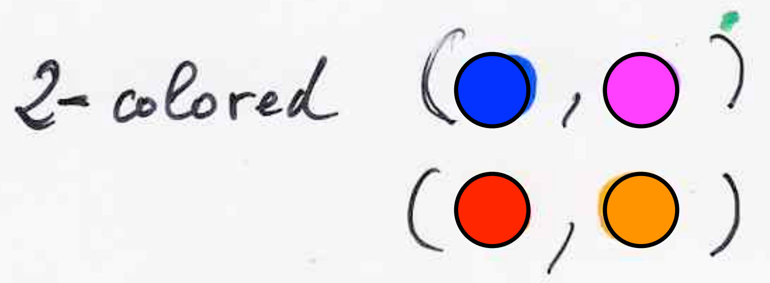












analog weight as for staircase tableaux (PASEP for r=0) nb of 2-colored rhombic alternative tableaux

$$2^{n-r}\binom{n}{r}\frac{n!}{r!}$$

nb of "staircase" rhombic tableaux

$$4^{n-r}\binom{n}{r}\frac{n!}{r!}$$

Koorwinder moments

$$M_\lambda = I_k\left(s_\lambda(x_1, \ldots, x_m)\,;\, a, b, c, d\,;\, q, q\right)$$

Schur function

$$\prod_{1\le i<j\le m} \frac{(z_iz_j\,,\, z_i/z_j\,,\, z_j/z_i\,,\, 1/z_iz_j\,;\, q)_\infty}{(t\,z_iz_j\,,\, t\,z_i/z_j\,,\, t\,z_j/z_i\,,\, t/z_iz_j\,;\, q)_\infty}$$

Koornwinder density

$$\prod_{1\le i\le m} \frac{(z_i^2\,,\, 1/z_i^2\,;\, q)_\infty}{(az_i\,,\, a/z_i\,,\, bz_i\,,\, b/z_i\,,\, cz_i\,,\, c/z_i\,,\, dz_i\,,\, d/z_i\,;\, q)_\infty}$$

$$M_\lambda(a,b,c,d\,|\,q) = \left(\frac{1-q}{2i}\right)^{|\lambda|} K_\lambda(-1;\alpha,\beta,\gamma,\delta;q)$$

$$|\lambda| = \sum_i \lambda_i$$

$$\mathbf{M}_\lambda(a,b,c,d\,|\,q) = \left(\frac{1-q}{2i}\right)^{|\lambda|} \mathbf{K}_\lambda(-1;\alpha,\beta,\gamma,\delta;q)$$

$$|\lambda| = \sum_i \lambda_i$$

$$\mathbf{K}_{(N-r,0,\dots,0)}(\xi) = \frac{Z_{N,r}(\xi)}{(1-q)^r \prod_{i=0}^{N-r-1} (\alpha\beta - q^{i+2r}\gamma\delta)}$$

Sylvie Corteel, Olya Mandelshtam,
Lauren Williams, arXiv: 1510.05023

Koorwinder moments (for $q=t$) $\lambda=[n]$
rhombic staircase tableaux

Luigi Cantini
arXiv: 1506.00284

$$K_\lambda(\xi) = \frac{\det\left(\bar{Z}_{\lambda_i+m-i+m-j}(\xi)\right)_{i,j=1}^{m}}{\det\left(\bar{Z}_{2m-i-j}(\xi)\right)_{i,j=1}^{m}}$$

$$\bar{Z}_N(\xi) = \prod_{i=0}^{N-1}\left(\alpha\beta - q^i\gamma\delta\right)^{-1} Z_N(\xi)$$

$$P_\lambda(z; a, b, c, d \,|\, q, q) = \text{const.}\ \frac{\det\left(p_{m-j+\lambda_j}(z_i; a, b, c, d \,|\, q)\right)_{i,j=1}^{m}}{\det\left(p_{m-j}(z_i; a, b, c, d \,|\, q)\right)_{i,j=1}^{m}}$$

The end of the bijective course III

two more complementary lectures: Ch4b, 4c

Monday 16 April, Thursday 19 April

Next year: the bijective course IV

Combinatorial theory of orthogonal polynomials

and continued fractions

Thank you very much !

for all of you, students, professors, friends,

For the videos:
Gayathri and Kirubananth

special thanks to Amri Prasad

ॐ सरस्वत्यै नमः।