Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 4
Trees and tableaux

Ch4a

IMSc, Chennai
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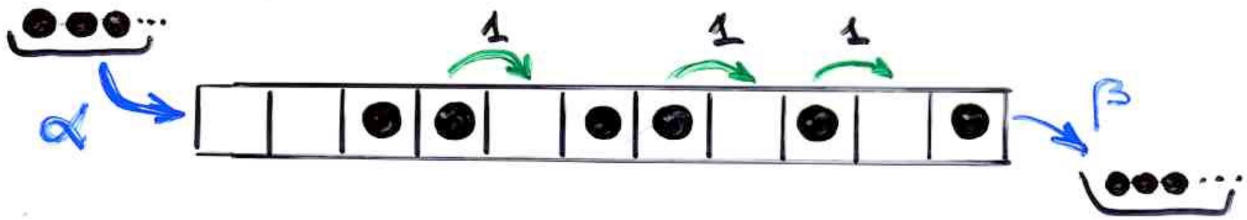
TASEP
and
Catalan alternative tableaux

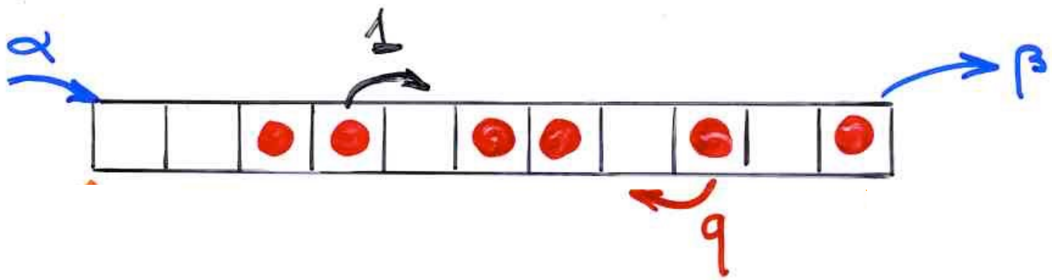
TASEP (α, β)

"totally asymmetric exclusion process"



$q = 0$

PASEP with 3 parameters q, α, β



PASEP

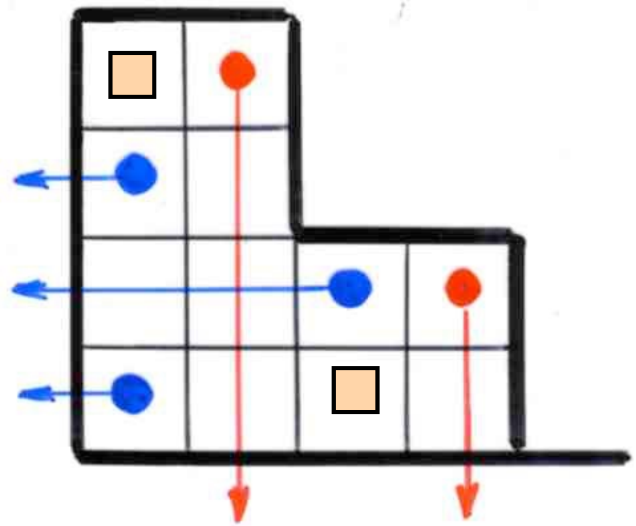
$$DE = qED + E + D$$

In the **PASEP** algebra

any word $w(E,D)$ can be uniquely written

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T: alternative tableaux, profile w

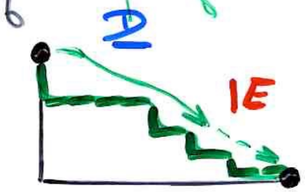


$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without

Def - profile of an alternative tableau
word $w \in \{E,D\}^*$



alternative tableau

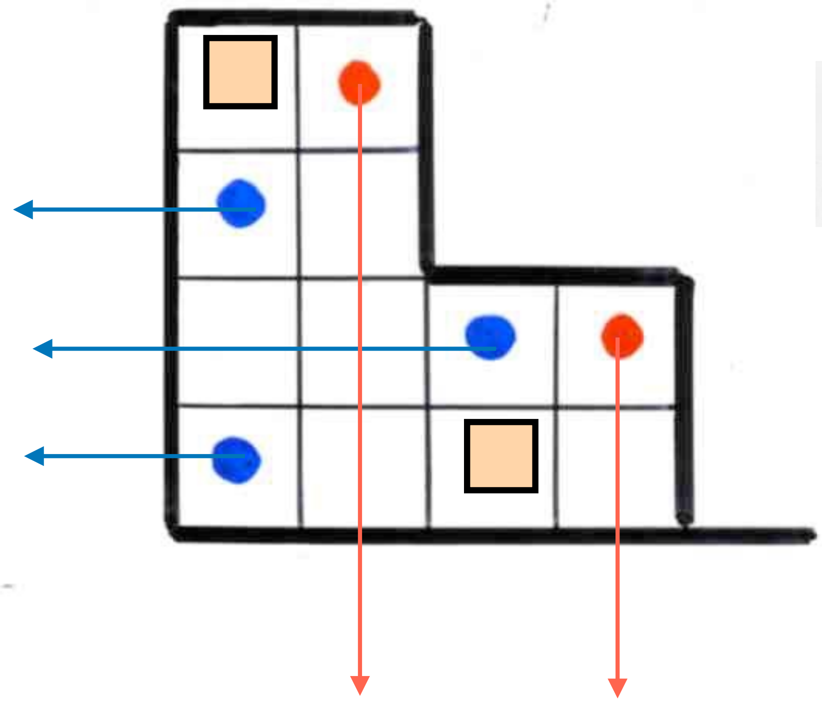
Definition

Ferrers diagram **F**

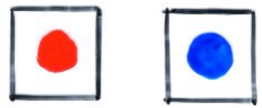
with possibly empty rows or columns

size of **F**

$\mathbf{n} = (\text{number of rows}) + (\text{number of columns})$



(i) some cells are coloured red or blue



(ii) • no coloured cell at the left of a blue cell

• no coloured cell below a red cell

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

colum vector V

row vector W

$q=0$ TASEP (α,β)

$$\begin{cases} DE = D+E \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

$k(T)$ = nb of cells

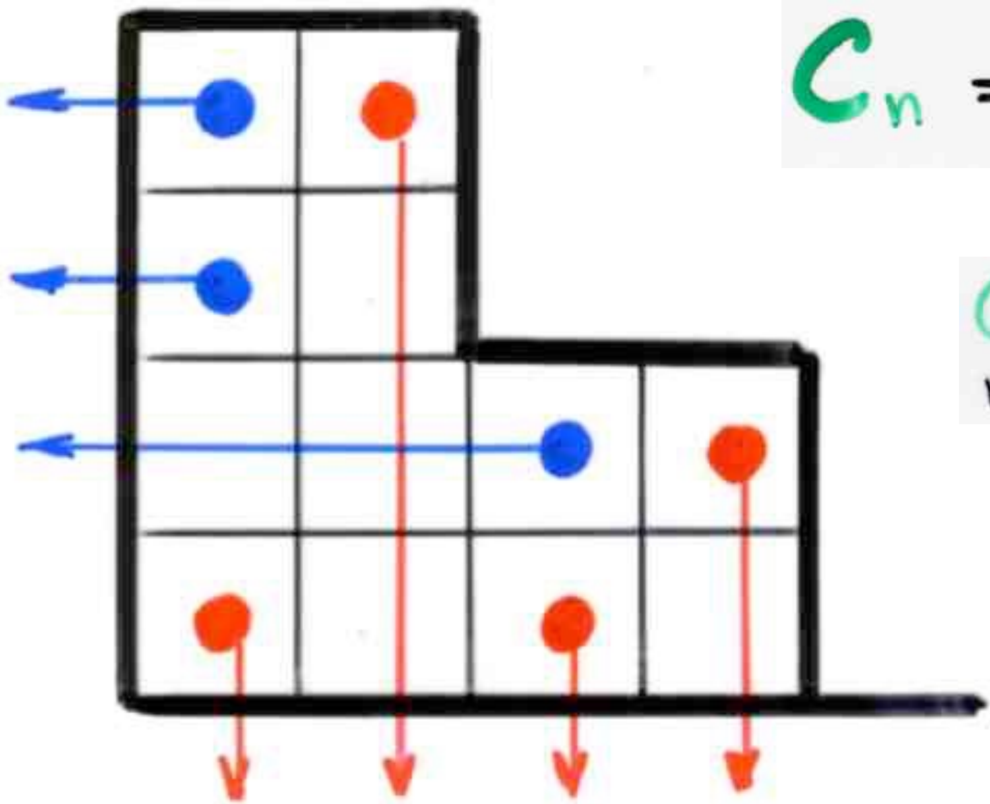
$i(T)$ = nb of rows without

$j(T)$ = nb of columns without

$q = 0$

Catalan alternative tableaux

Def Catalan alternative tableau T
alt. tab. without cells $\square$
i.e: every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

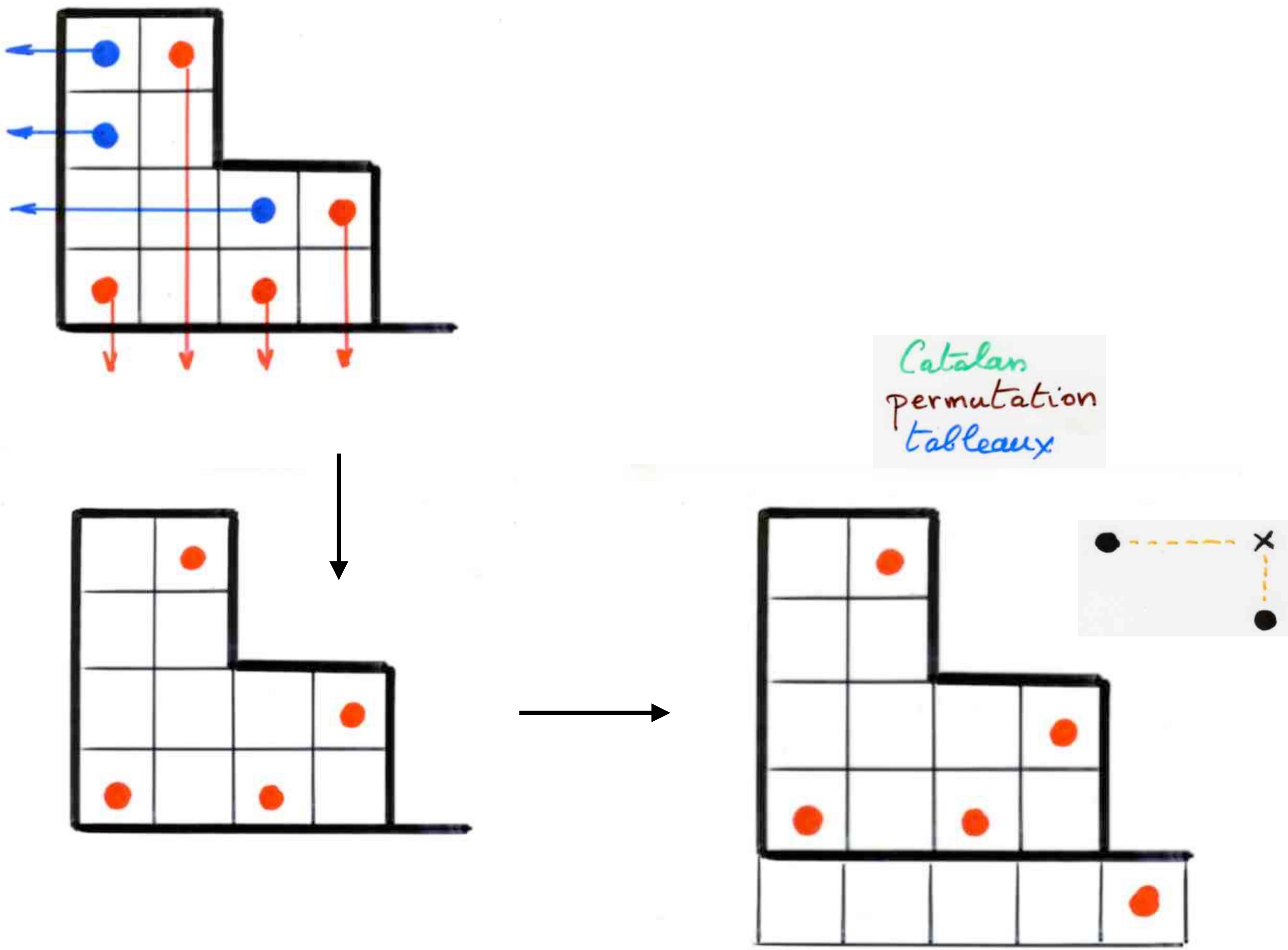
$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} \alpha^{-i(T)} \beta^{-j(T)}$$

Catalan alternative tableaux profile τ

$i(T)$ = nb of rows without ●

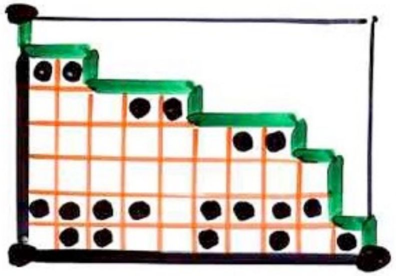
$j(T)$ = nb of columns without ●

Characterisation of alternative Catalan tableaux


Catalan
permutation
tableaux

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells with 0 and 1

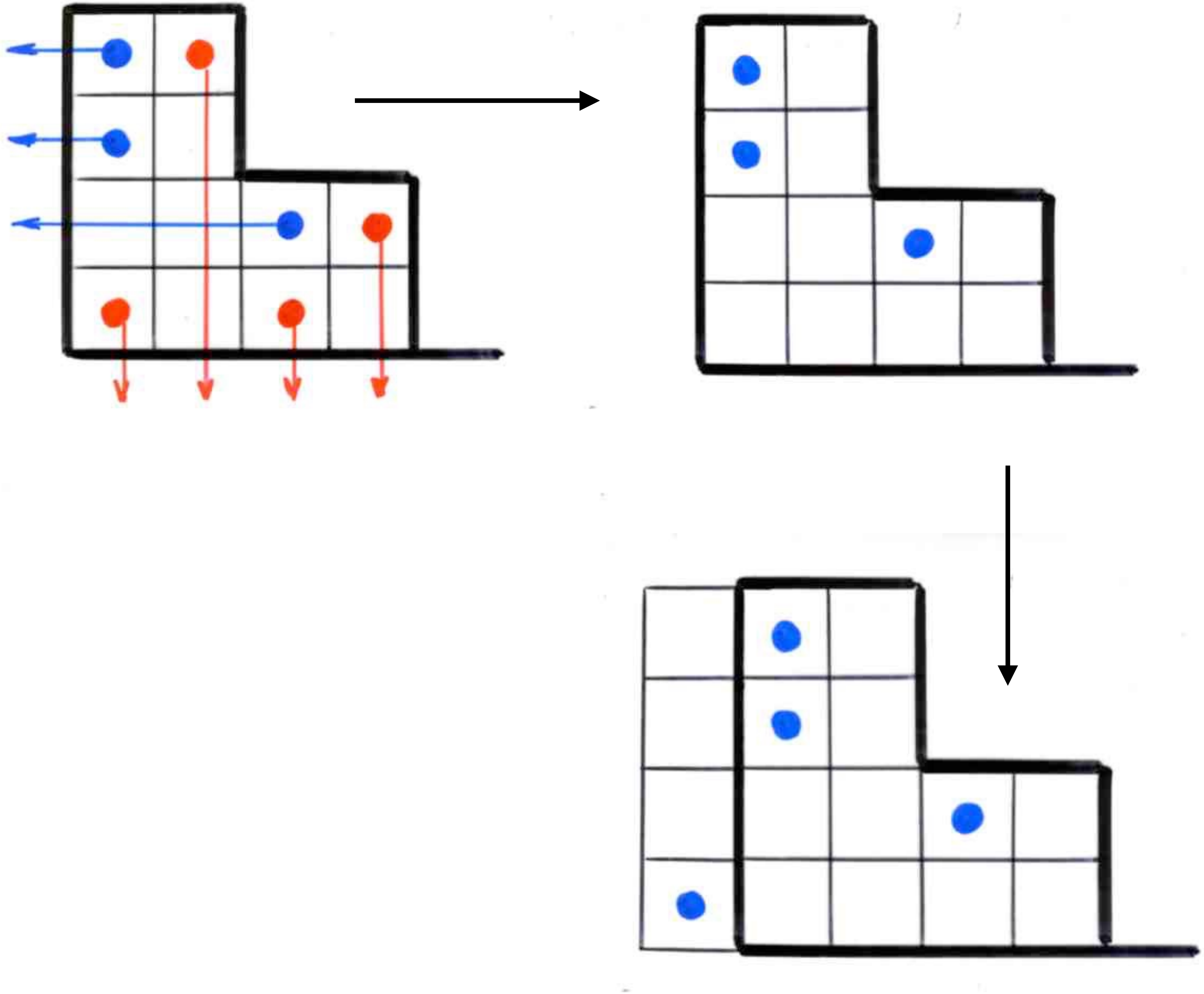
(i) in each column : at least one 1

□ = 0 ⊡ = 1

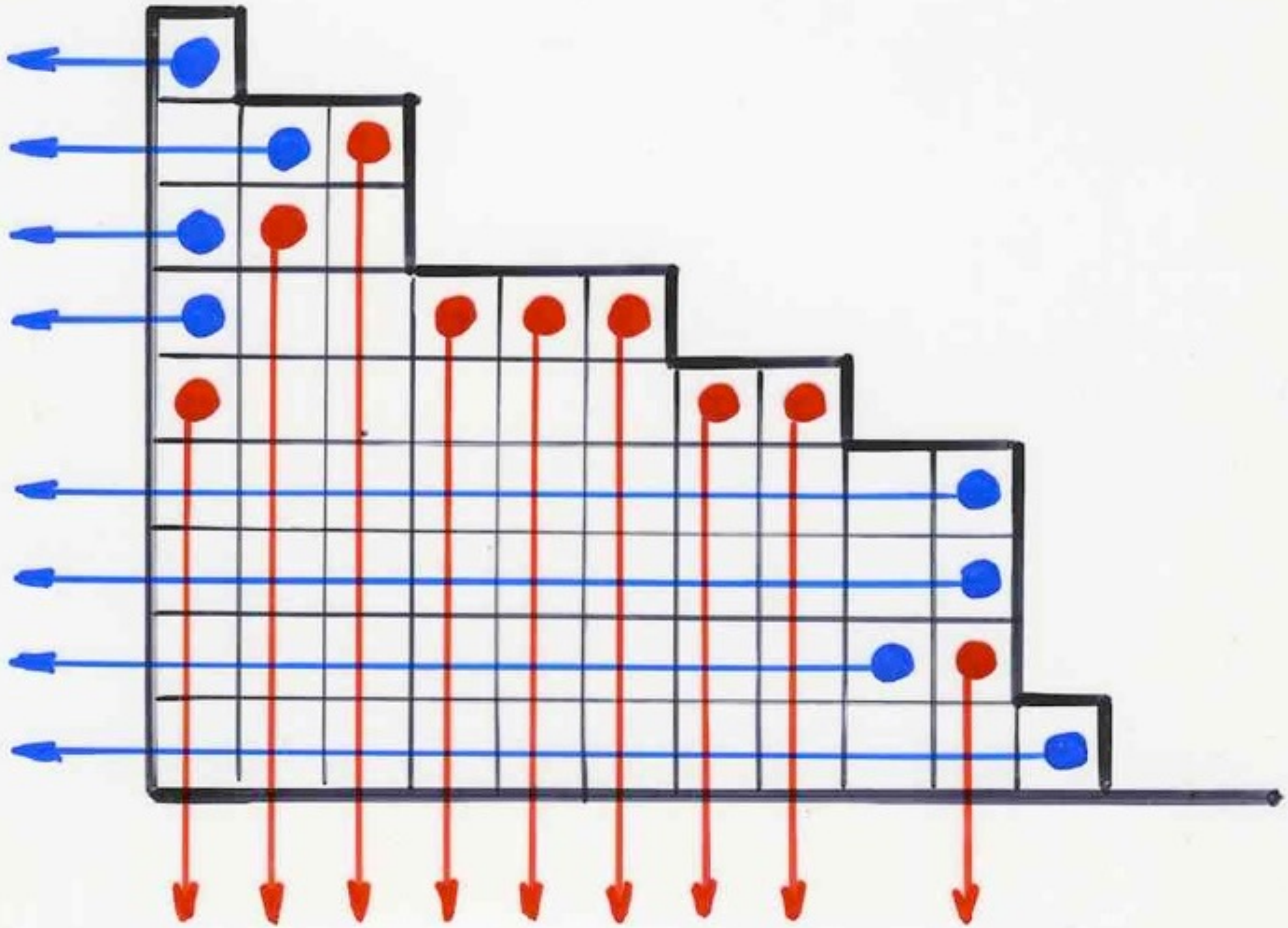
(ii) $\begin{matrix} 1 & \cdots & 0 \\ & & \vdots \\ & & 1 \end{matrix}$ forbidden

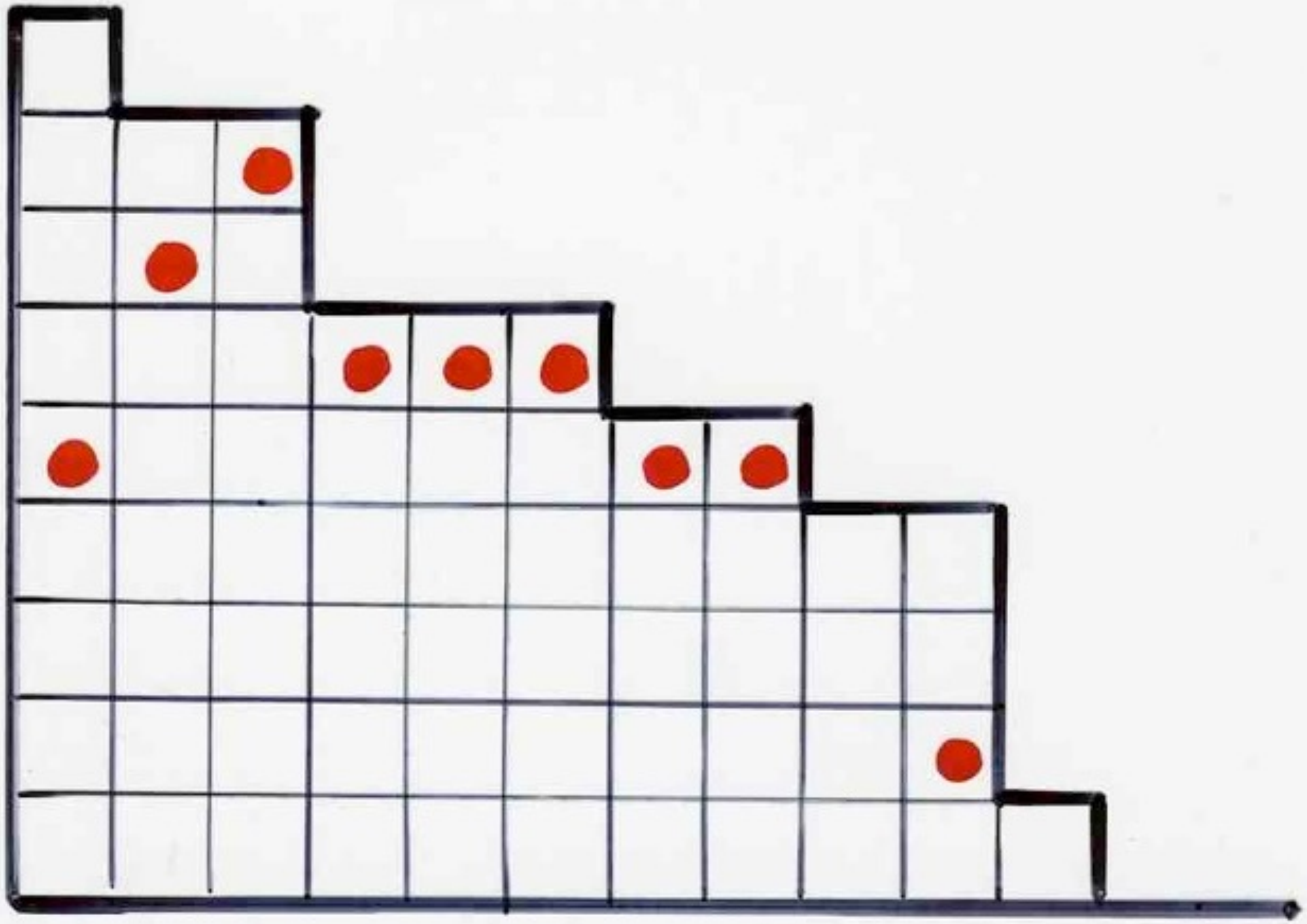
Catalan permutation tableaux

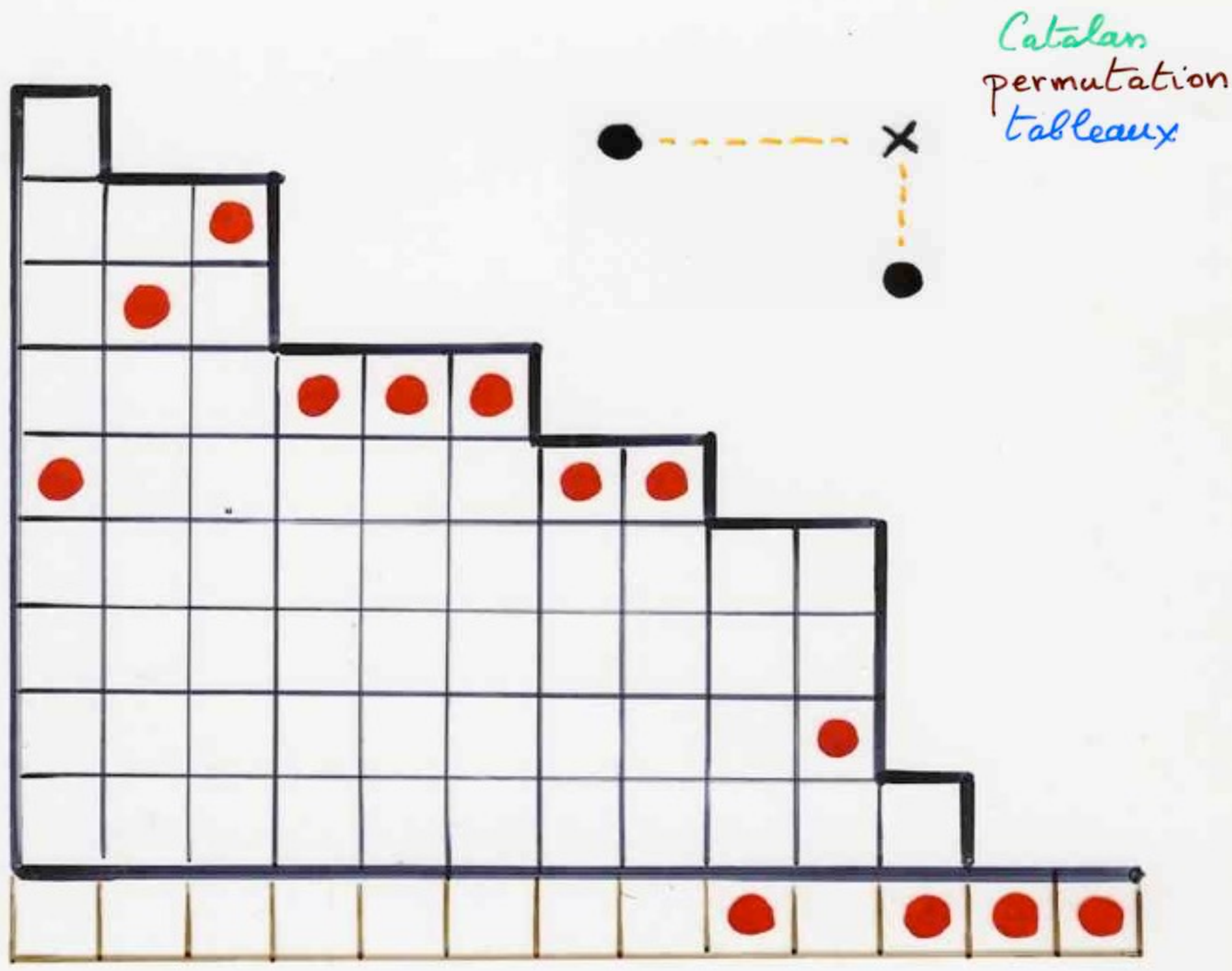
(iii) only one 1 in each column

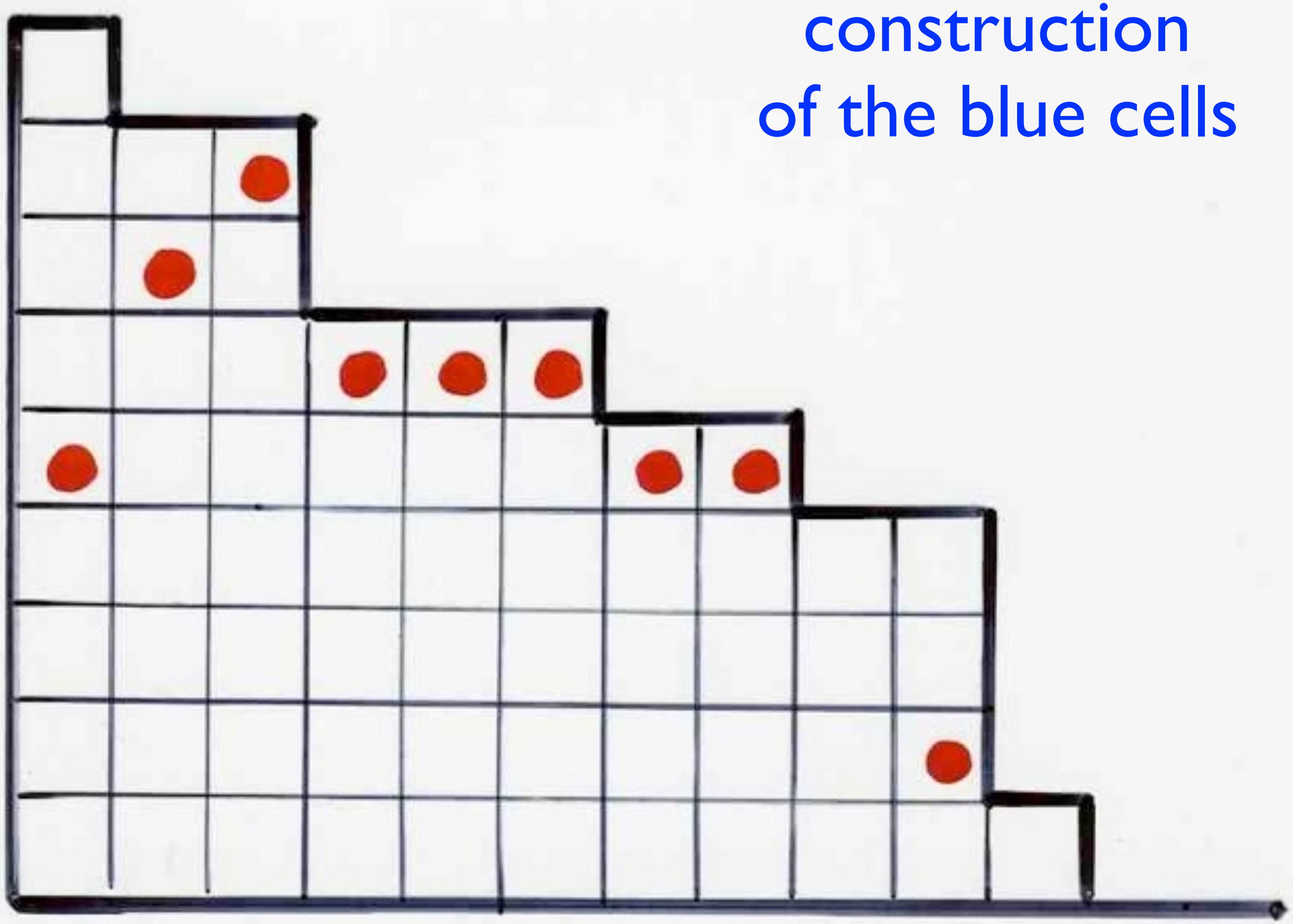


example

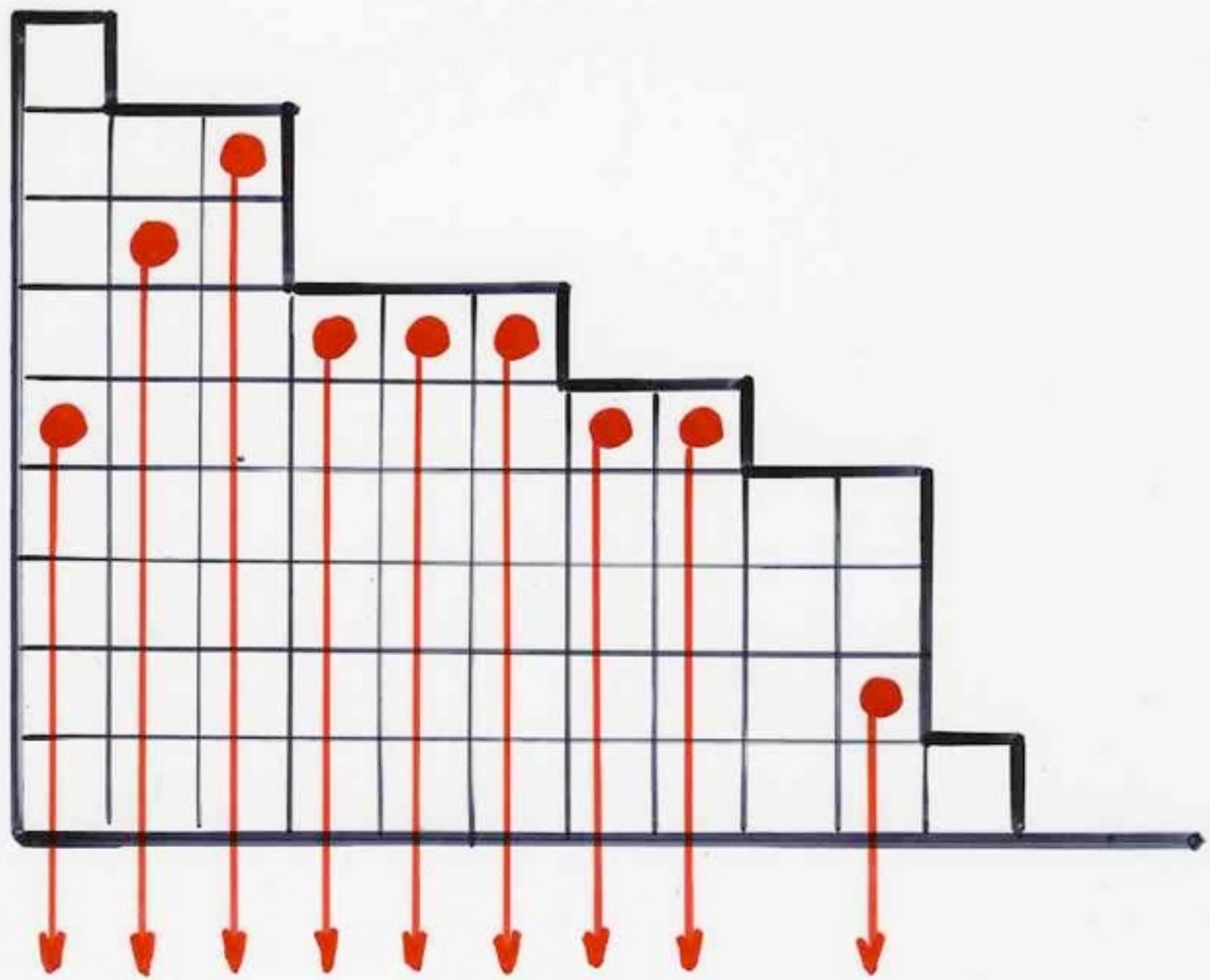


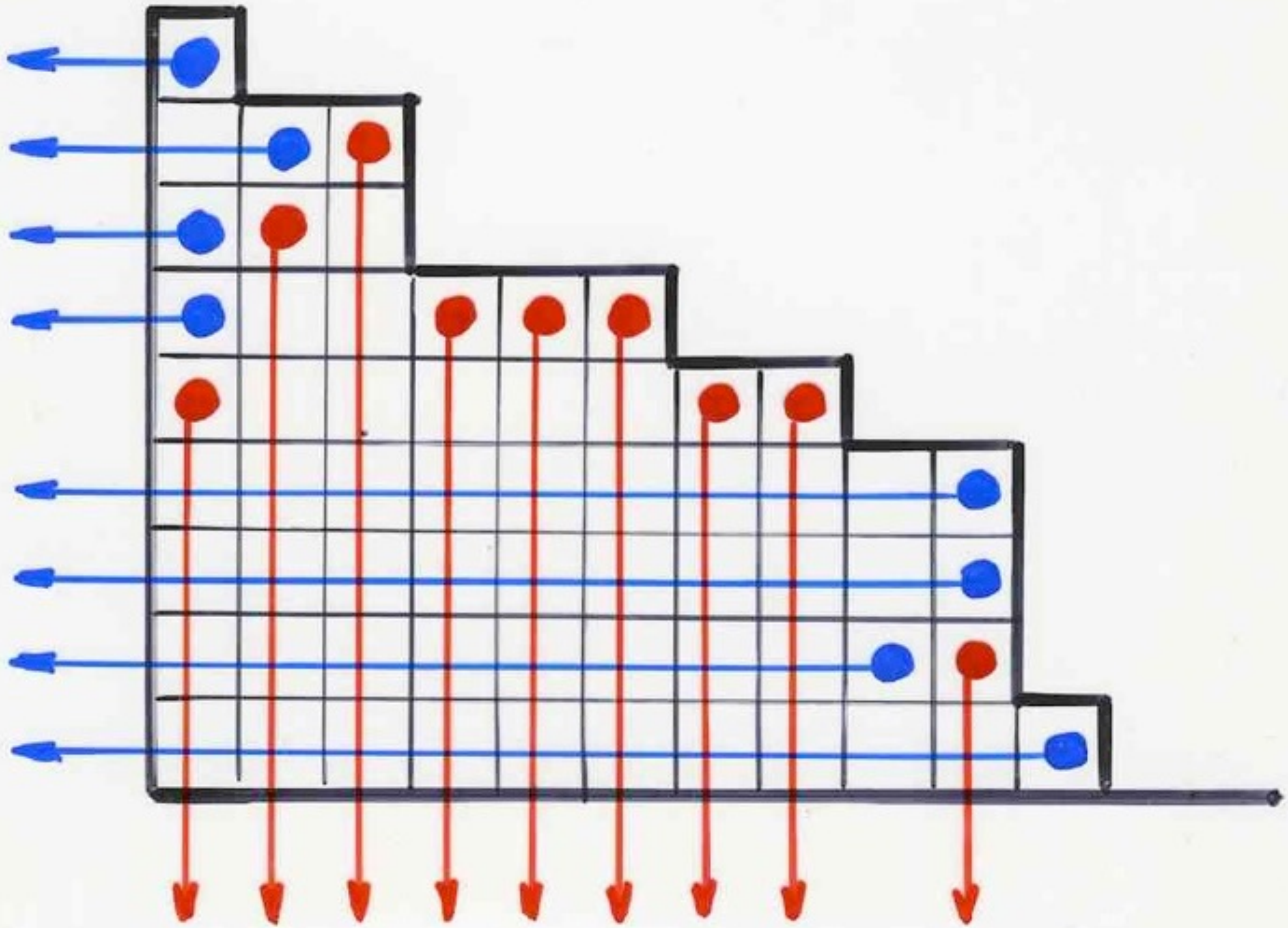


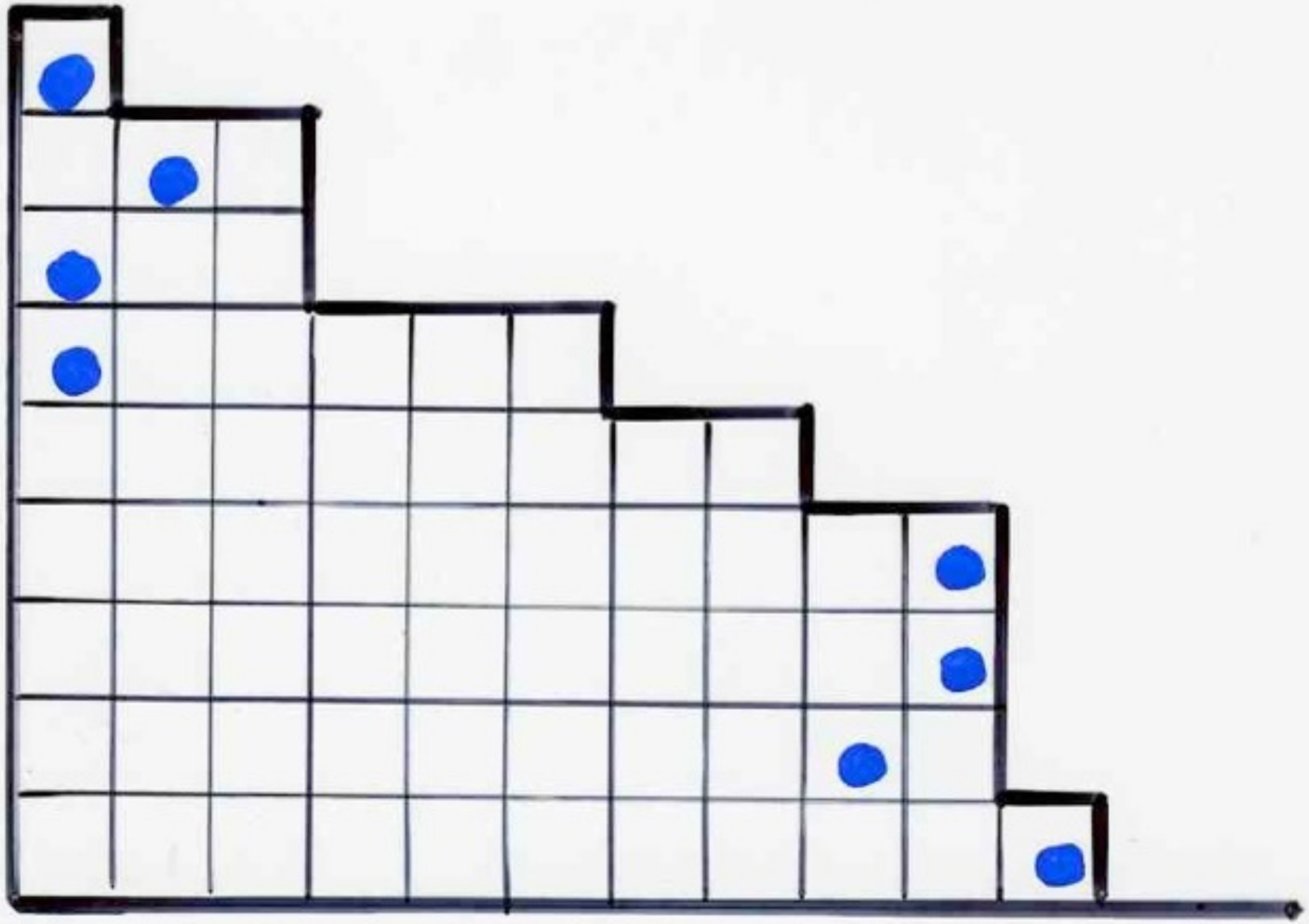

Catalan
permutation
tableaux

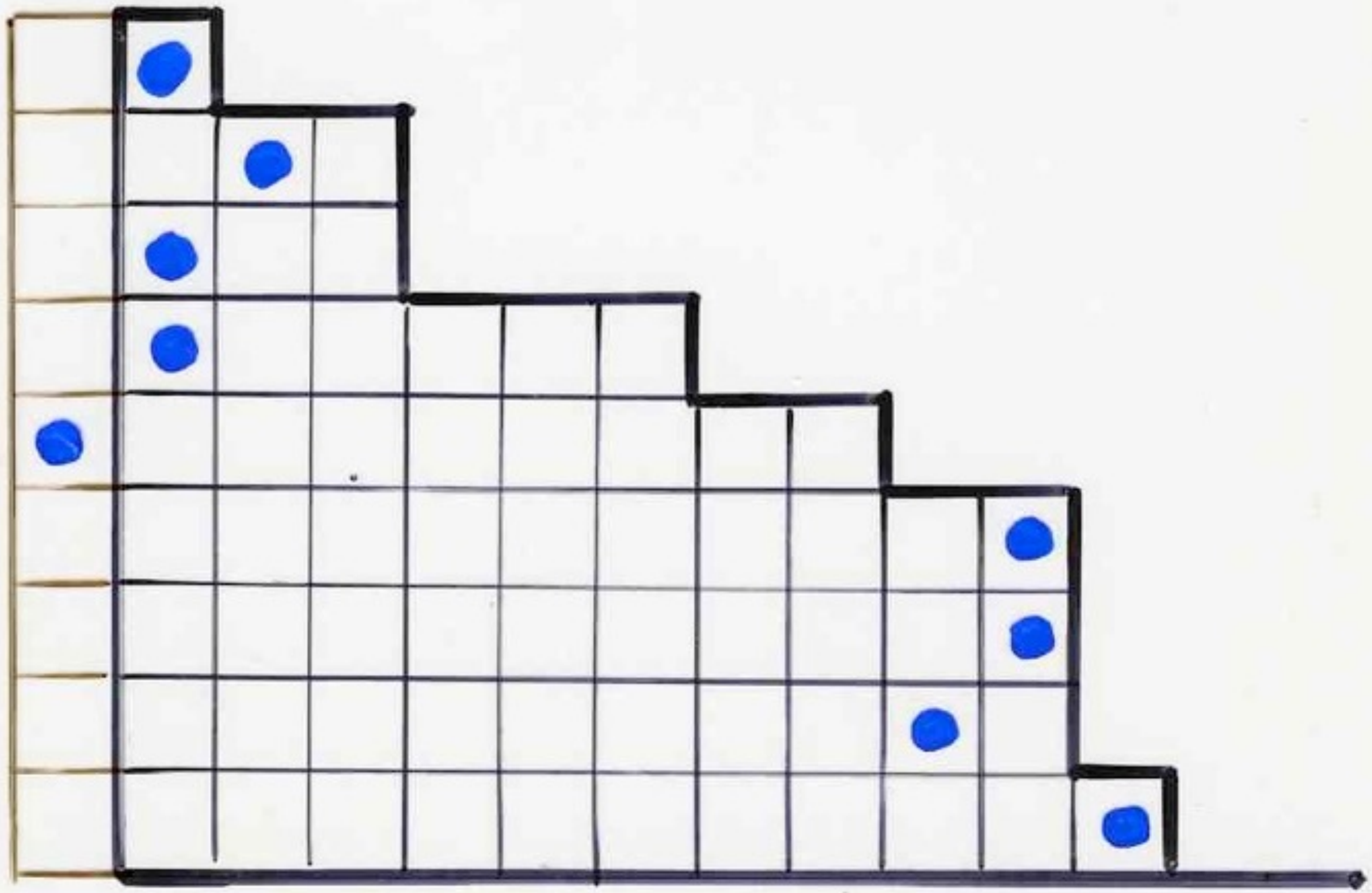


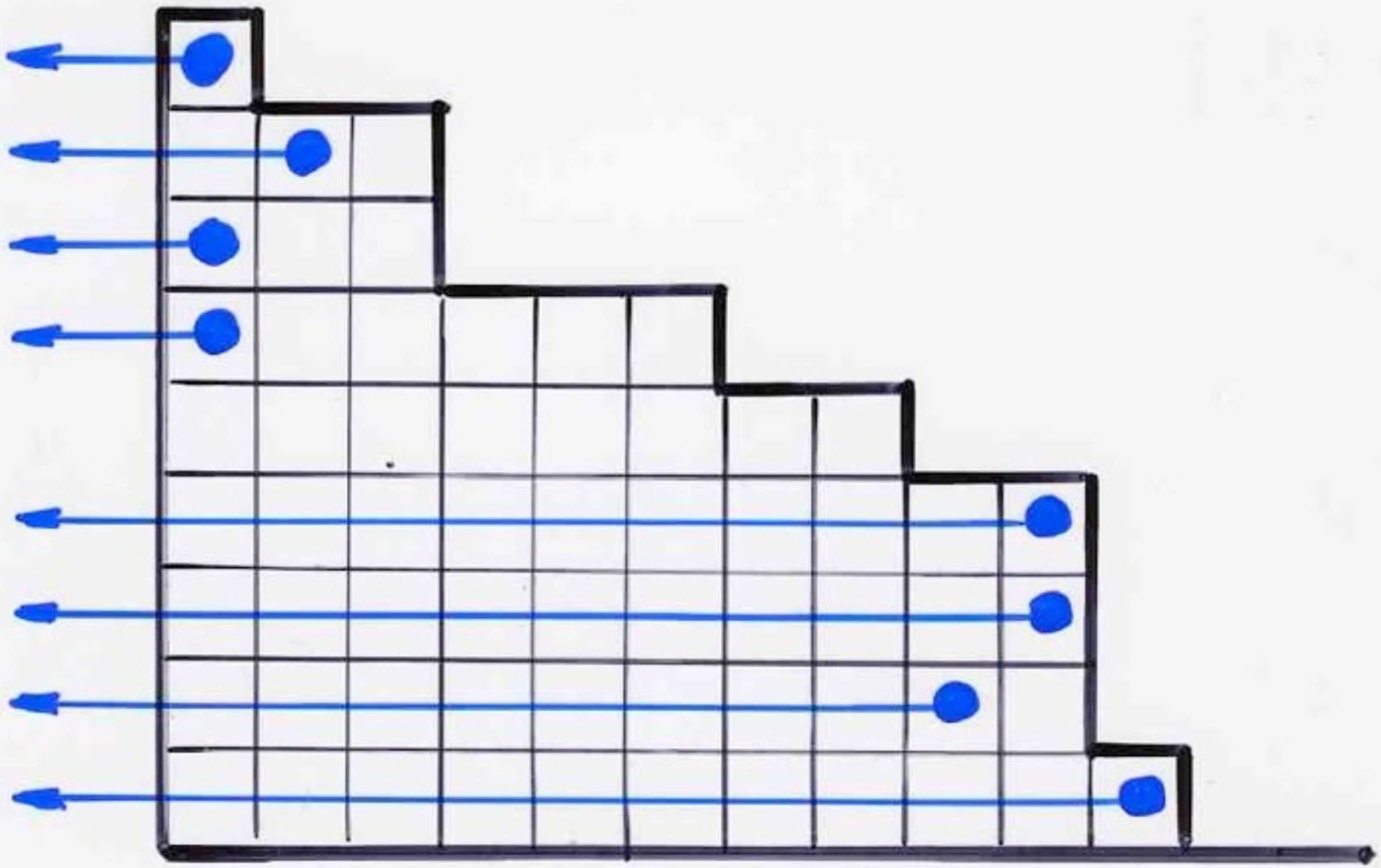
construction
of the blue cells

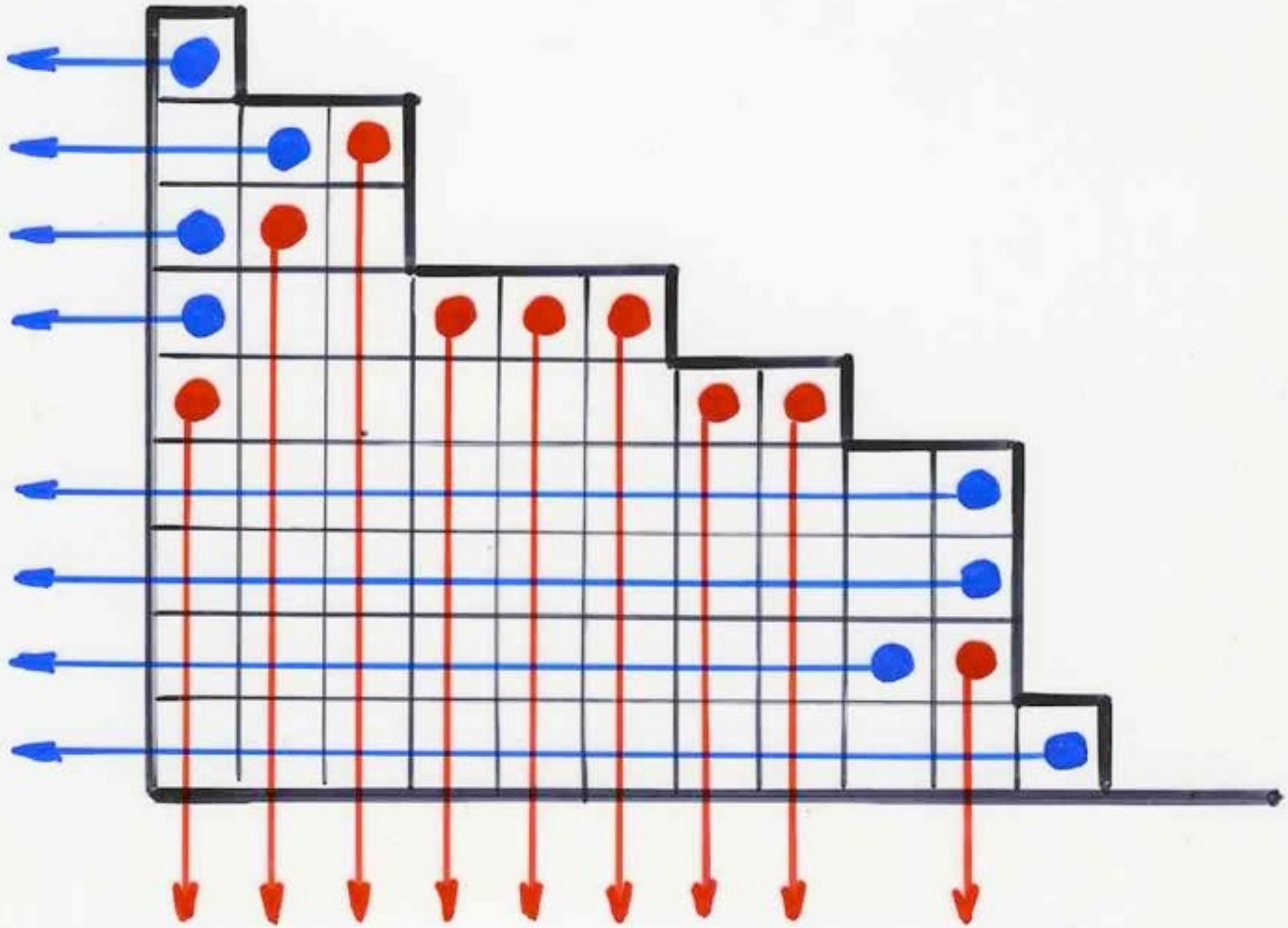












Catalan alternative tableaux

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

bijection with:

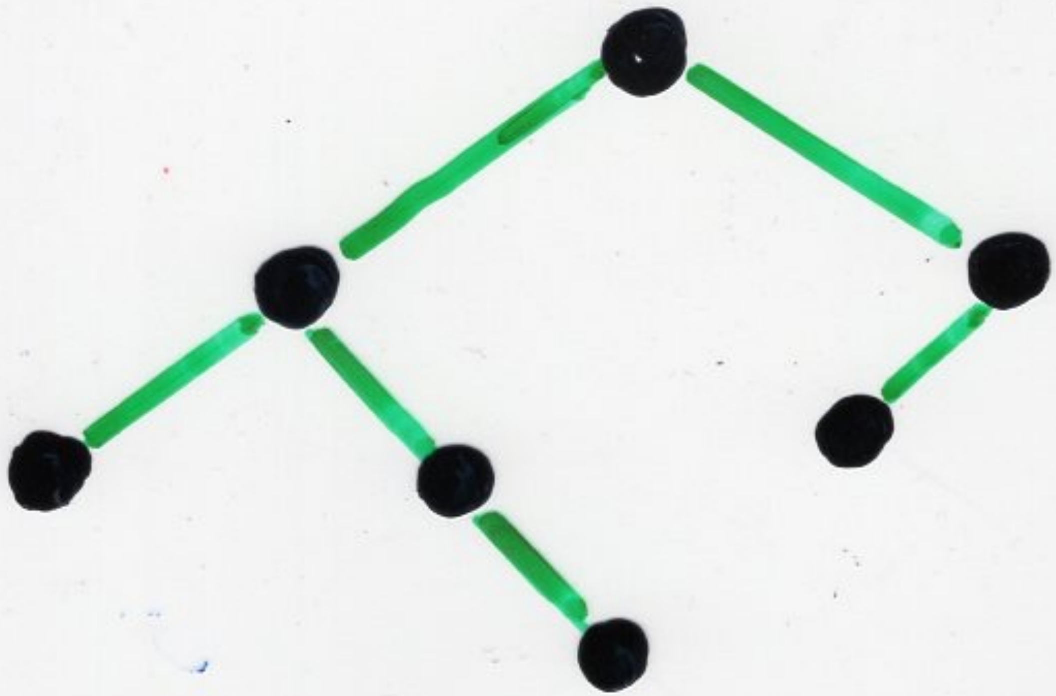
- binary trees
- pairs (u,v) of paths

Binary trees

and

complete binary trees

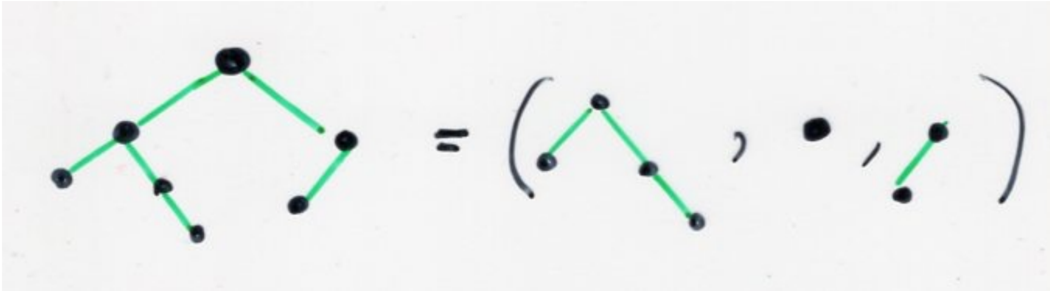
binary tree

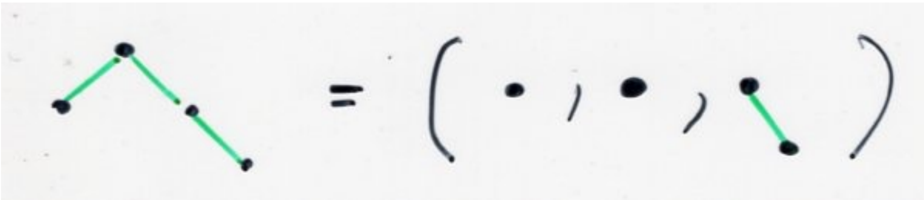


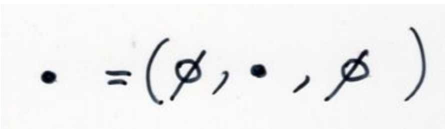
$$\begin{cases} B = (L, r, R) \\ \text{or} \\ B = \emptyset \end{cases}$$

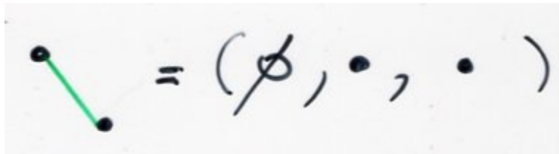
L, R binary trees

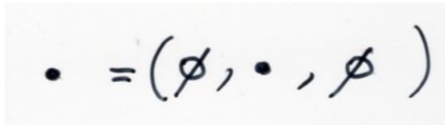
r root











$\mathbf{C}(t)$ generating function of Catalan numbers

$$\mathbf{C}(t) = \sum_{n \geq 0} C_n t^n$$

Catalan number

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

$$y = 1 + t y^2$$

→ BJC I Ch 1 , Ch 2

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$C_0 = 1$

$\Longleftrightarrow$

$y = 1 + t y^2$

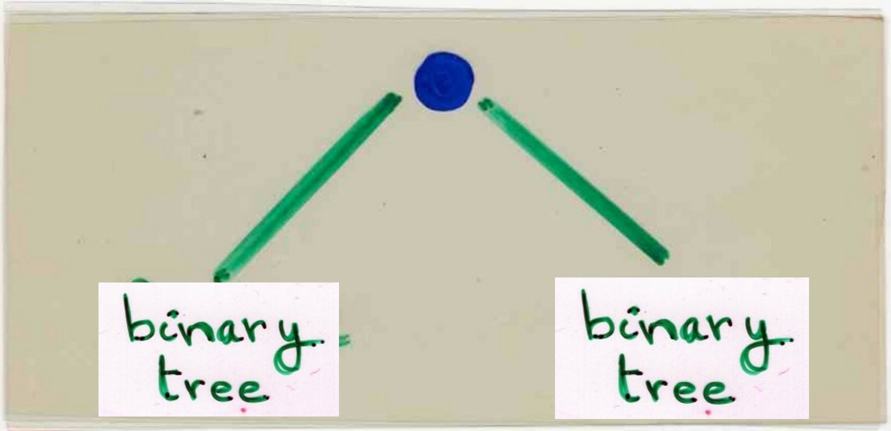
algebraic equation

classical
enumerative
combinatorics

modern enumerative combinatorics

binary tree

= • +



$$B = \{\bullet\} + (B \times \underset{\text{root}}{\bullet} \times B)$$

binary tree

$$y = 1 + t\,y^2$$

algebraic equation

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

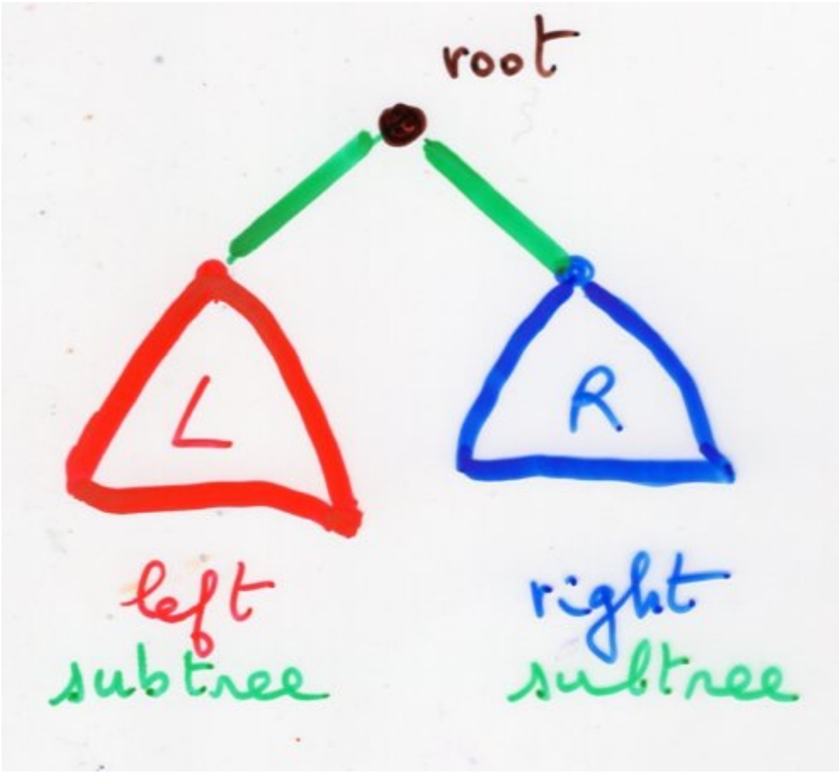
$$= \frac{(2n)!}{(n+1)!\ n!}$$

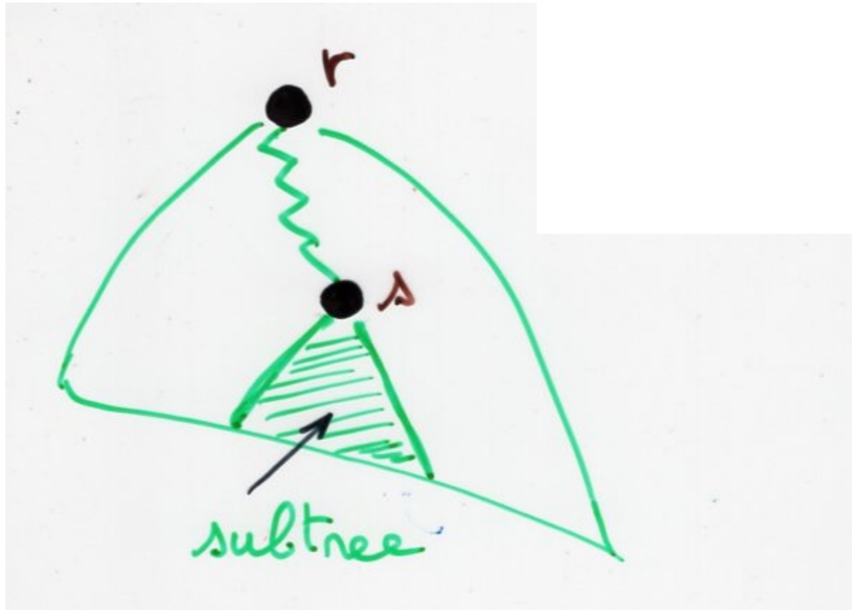
$$n! = 1 \times 2 \times \dots \times n$$

exercise

bijective proof of

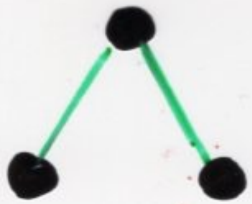
$$2(2n+1)C_n = (n+2)C_{n+1}$$


root
L
R
left
subtree
right
subtree


r
s
subtree


left
son
daughter


right
son
daughter



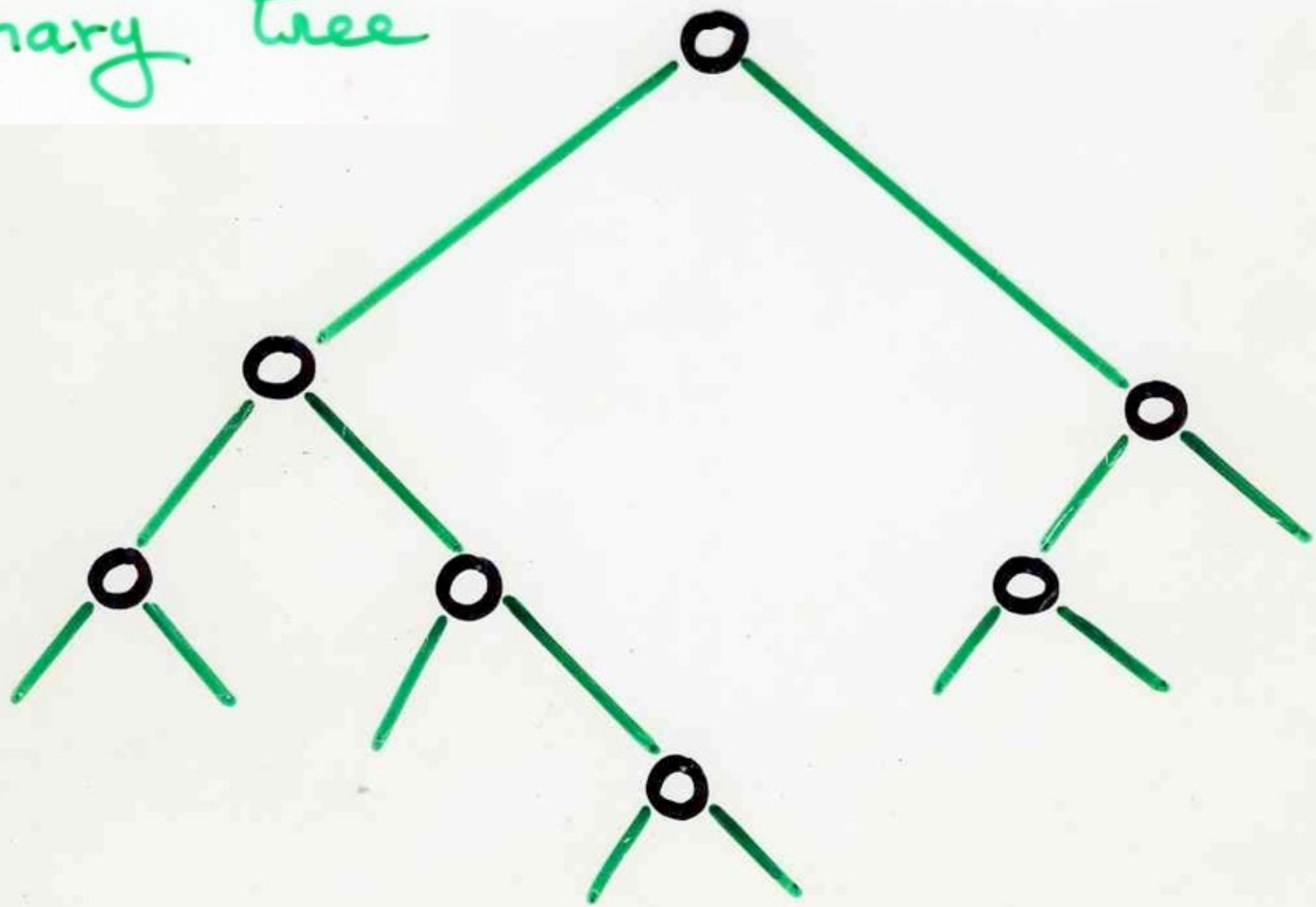
double
vertex

right
simple
vertex

left
simple
vertex

leaf

complete
binary tree



binary trees

n vertices

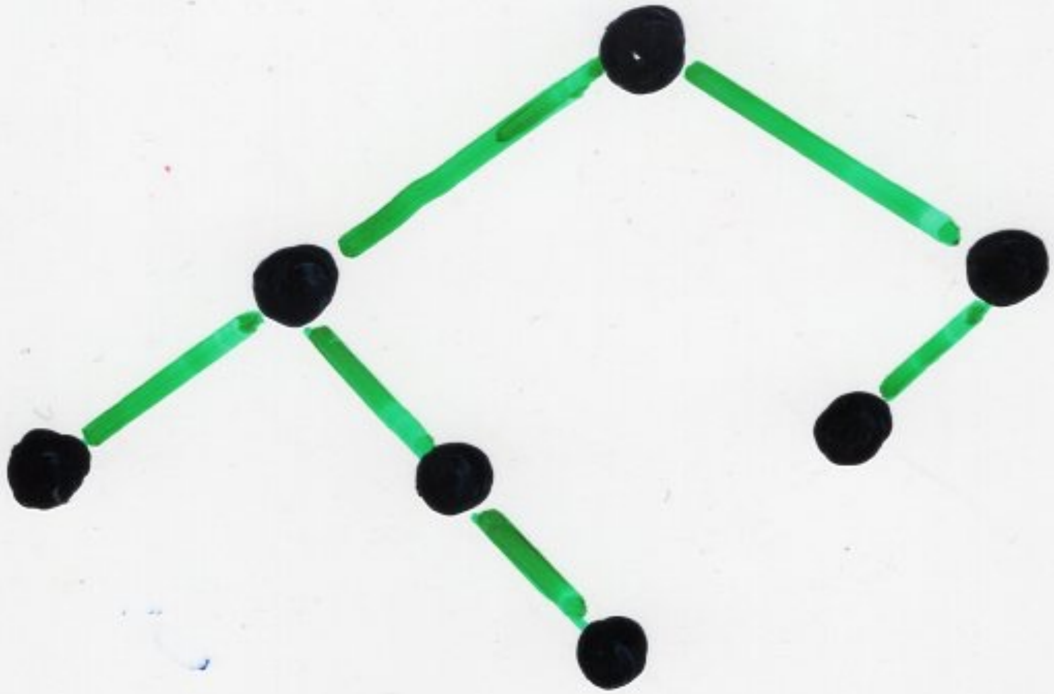
$\Updownarrow$ bijection

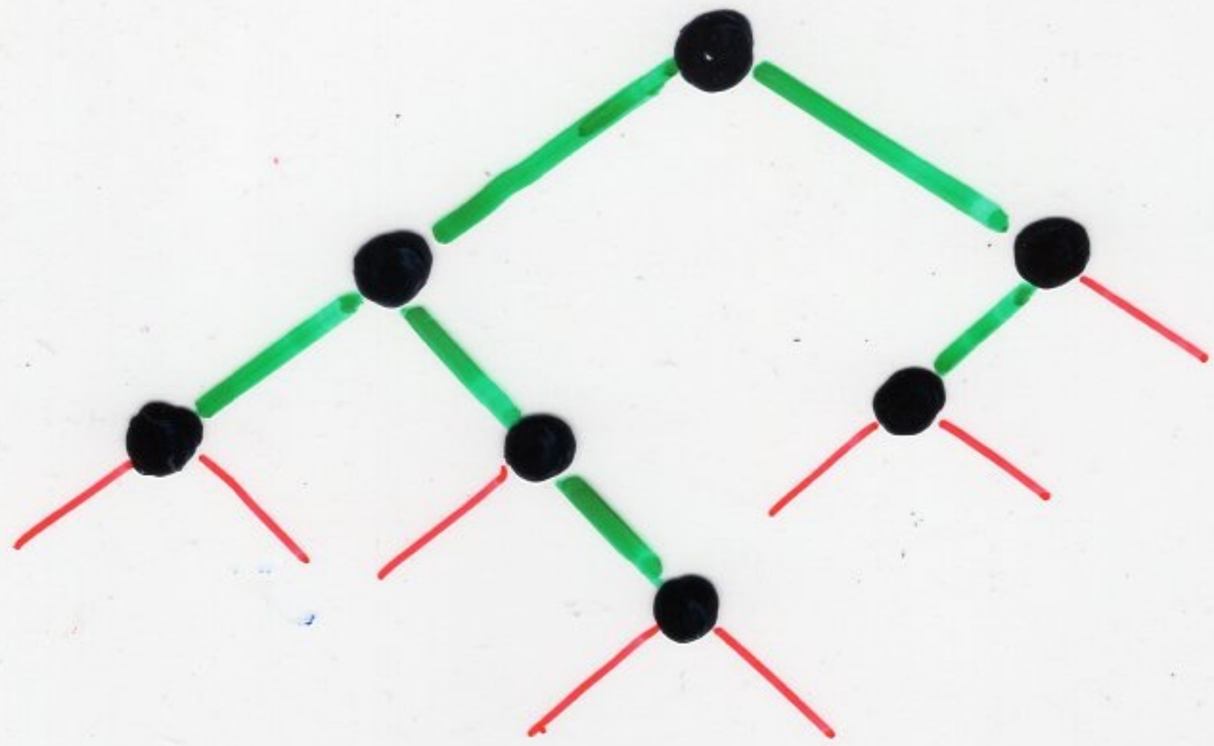
complete

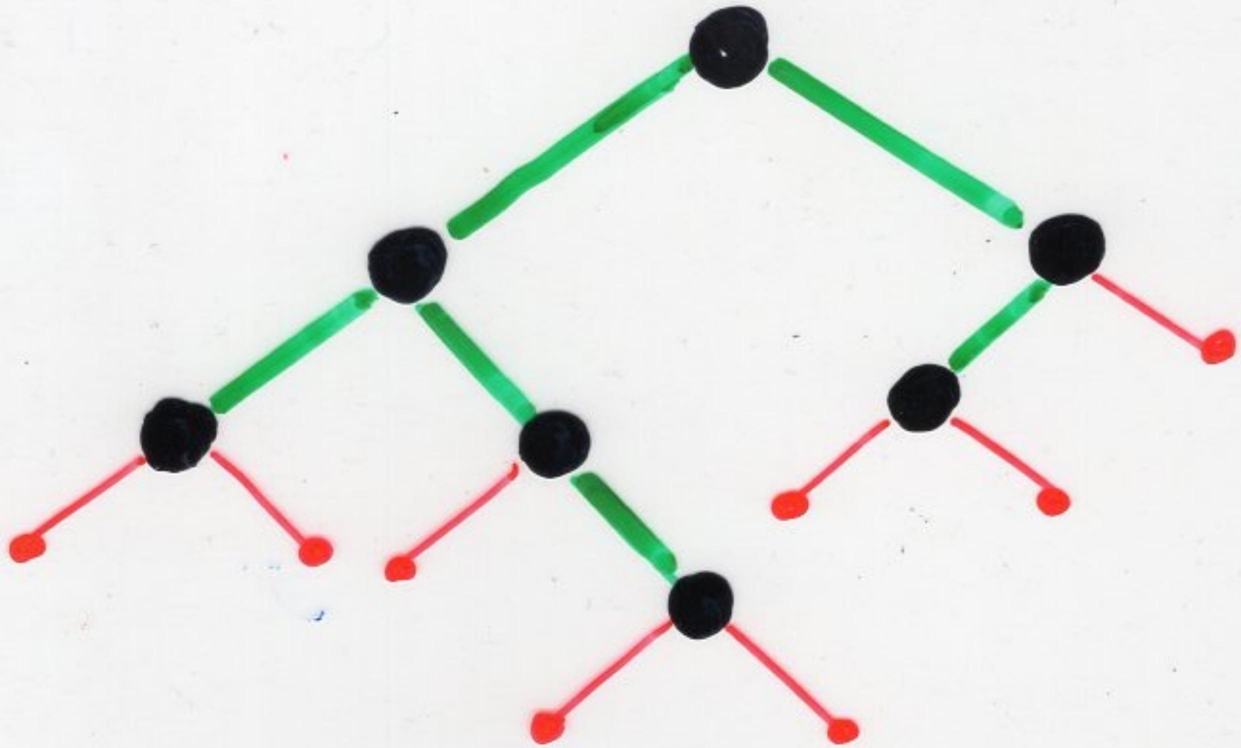
binary trees

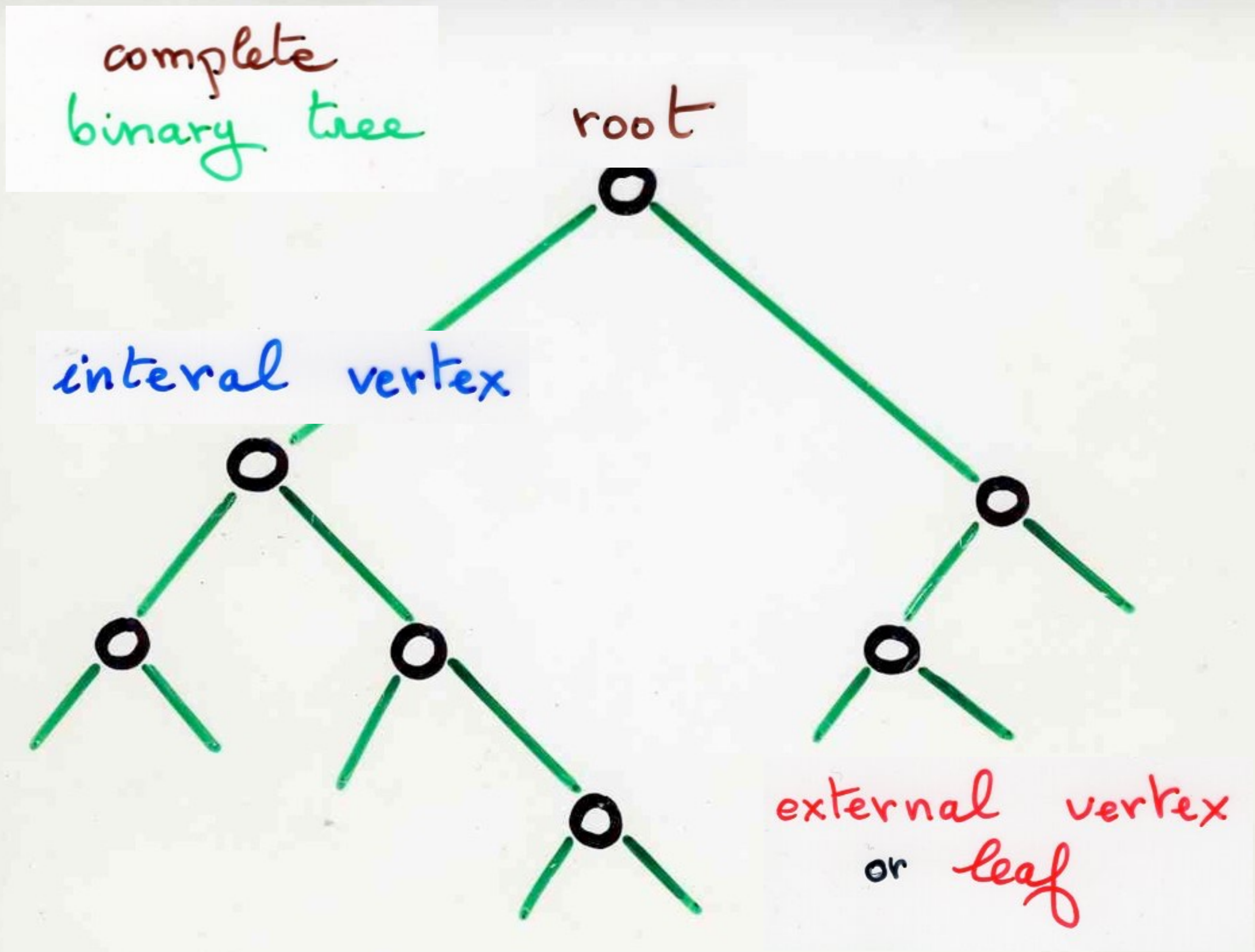
$(2n+1)$ vertices

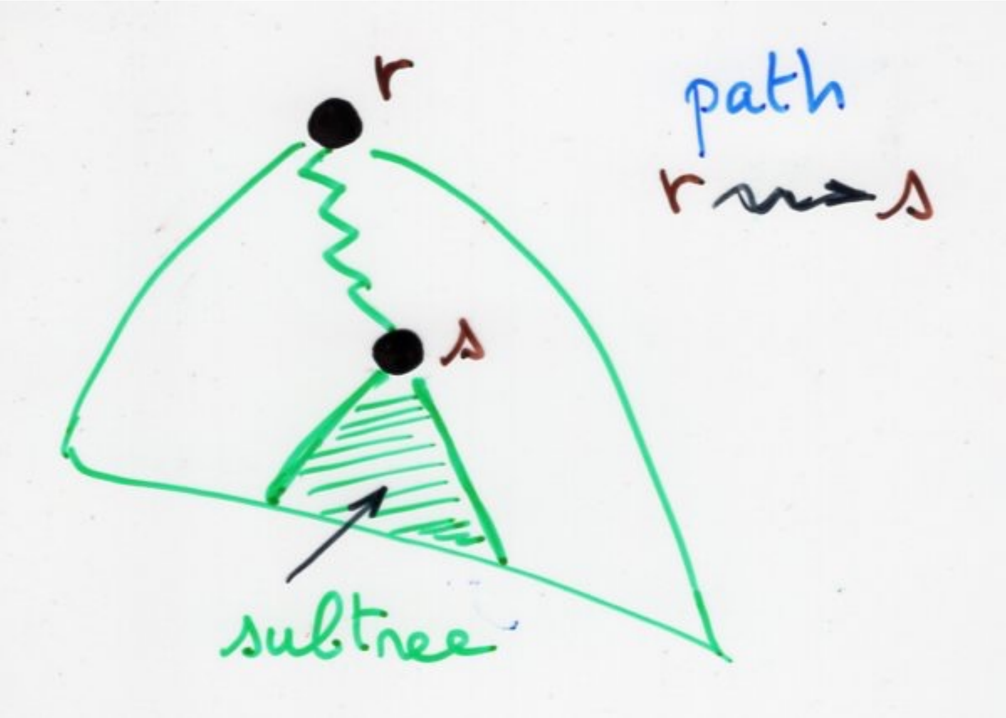
$\left\{\begin{array}{ll} n & \text{internal} \\ n+1 & \text{external} \end{array}\right.$ vertices







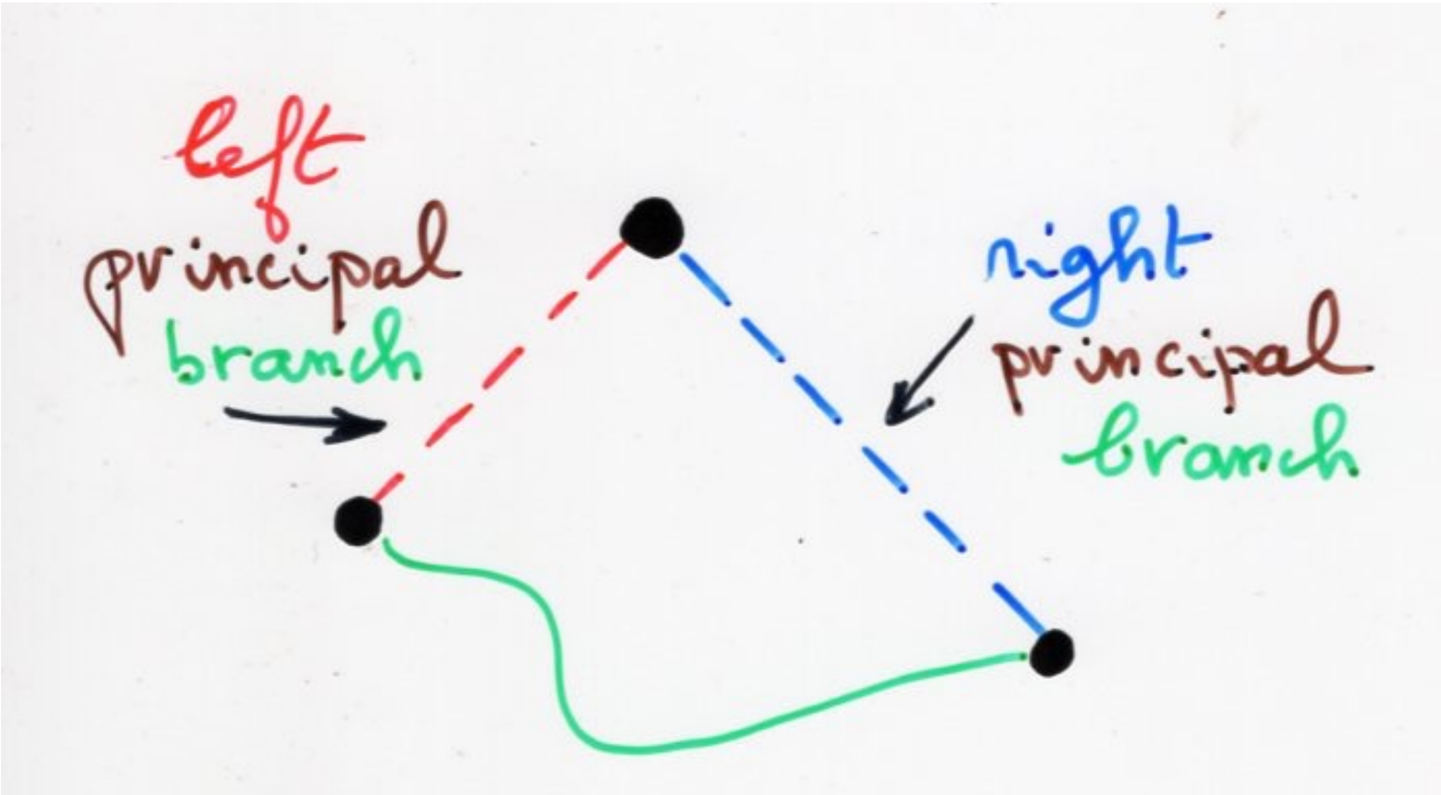

complete
binary tree
root
interal vertex
external vertex
or leaf



height $h(s)$
of the vertex s

left-height $hl(s)$
right-height $hr(s)$

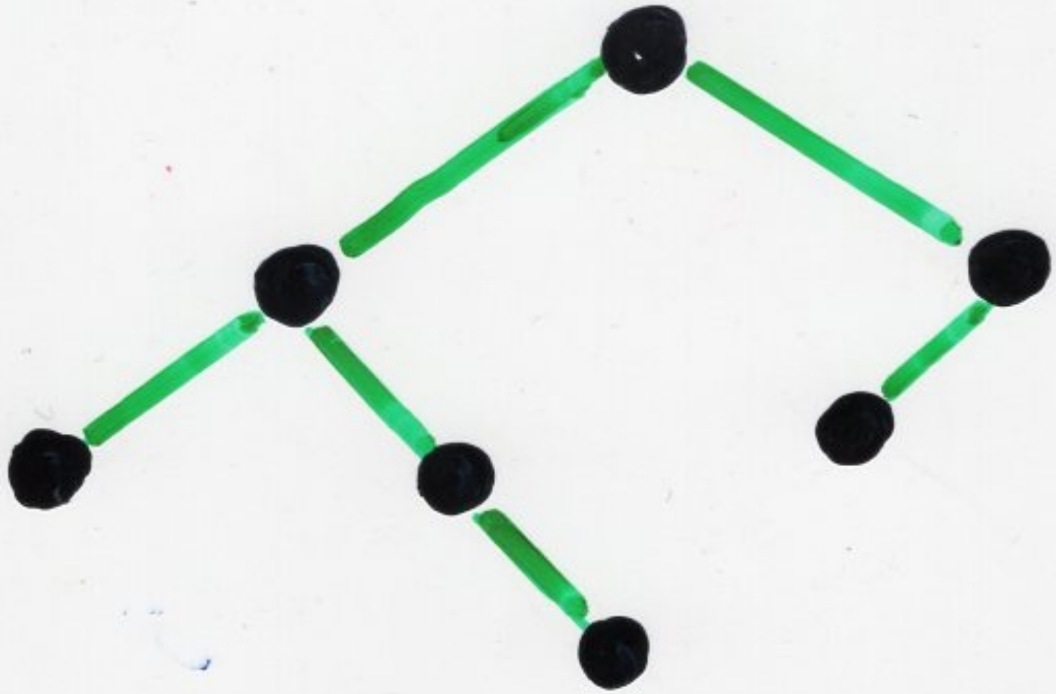
$$h(s) = hl(s) + hr(s)$$

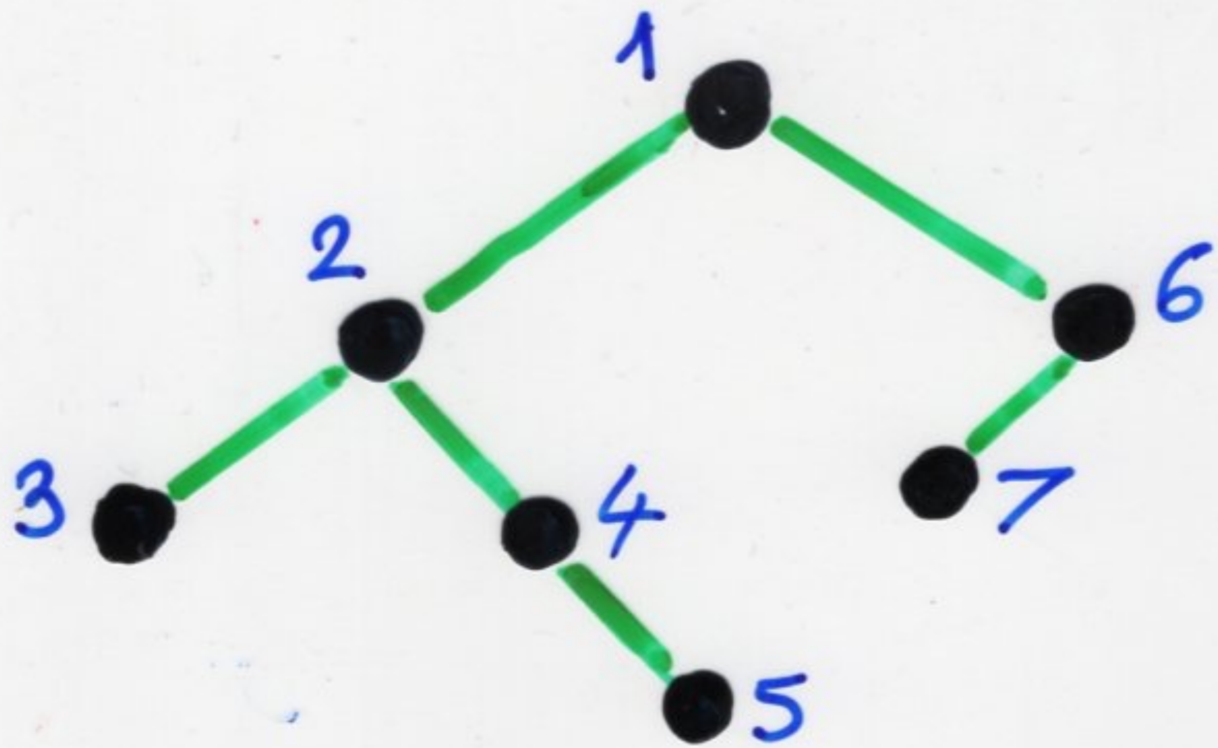

left
principal
branch
right
principal
branch

traversal of a binary tree

preorder

- visit the root (if $B \neq \emptyset$)
- then visit the left-subtree
- then visit the right subtree




1
2
6
3
4
7
5

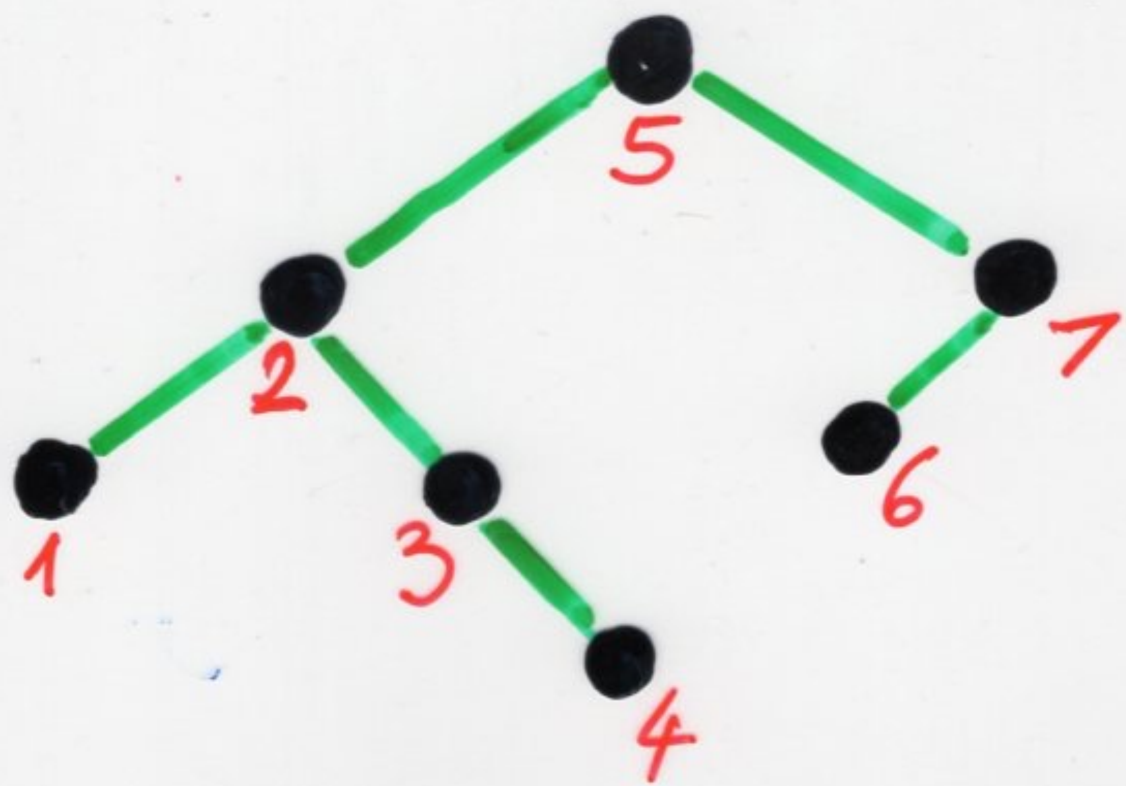
inorder

(symmetric order)

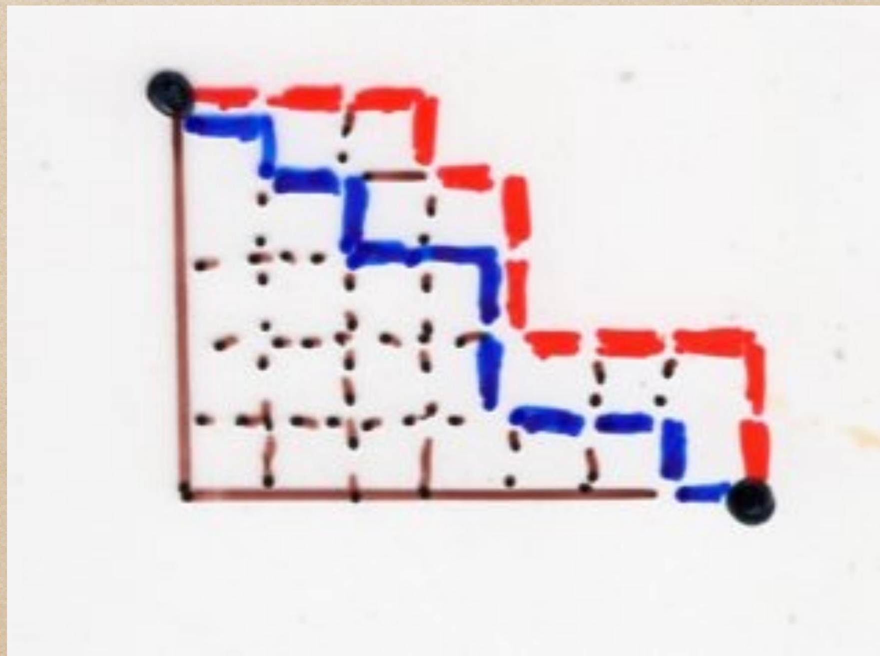
visit the left subtree

visit the root

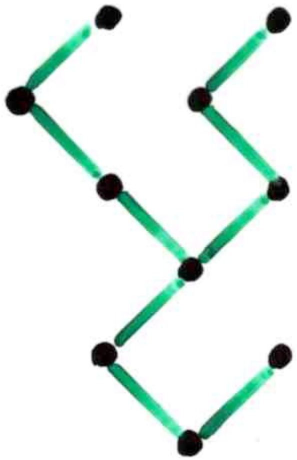
visit the right subtree

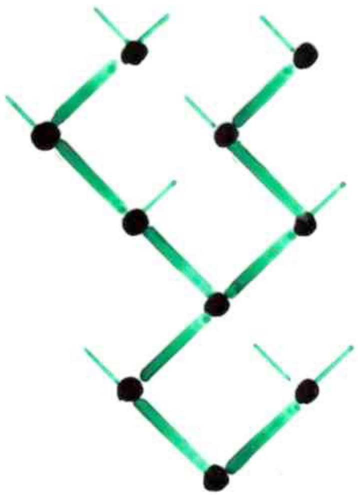

5
2
7
1
3
6
4

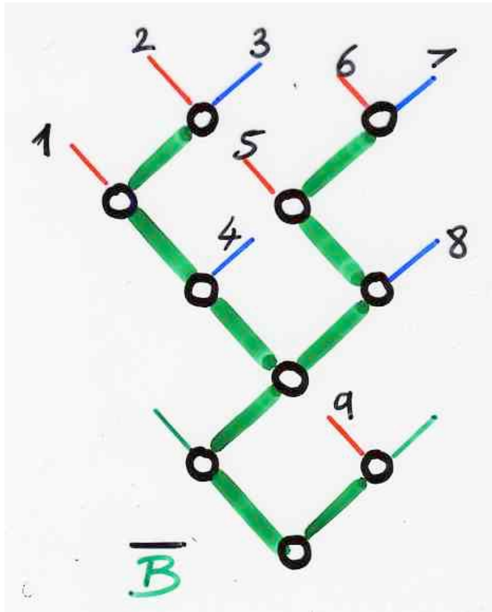
pair of paths (u,v)



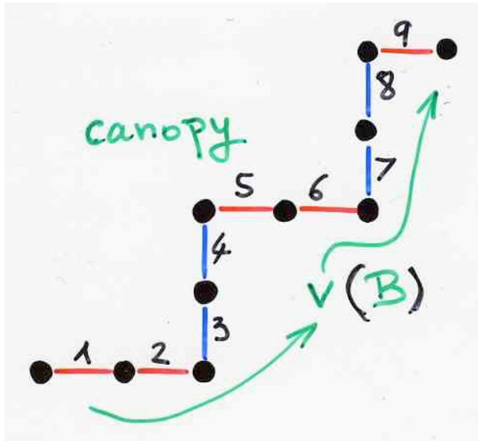
binary tree B $\longrightarrow$ pair of paths (u,v)

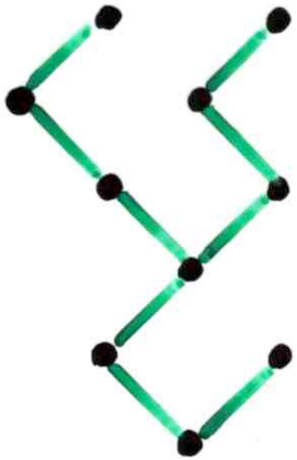


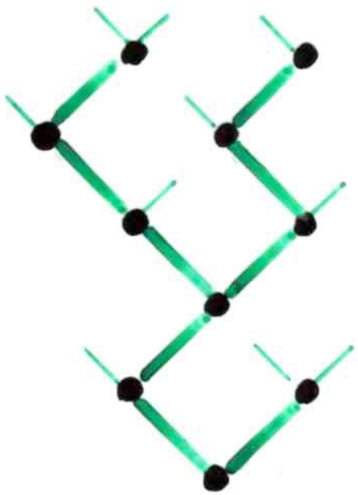


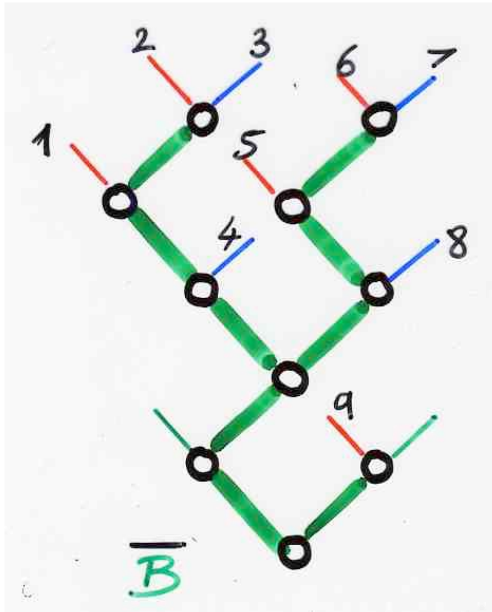


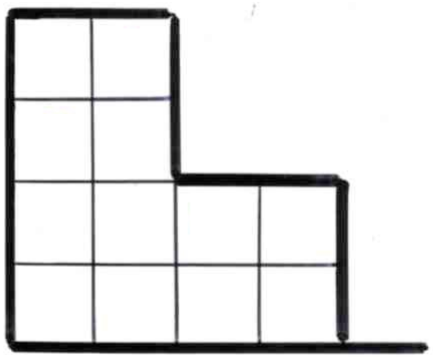
Loday, Ronco (1998)
(2012)

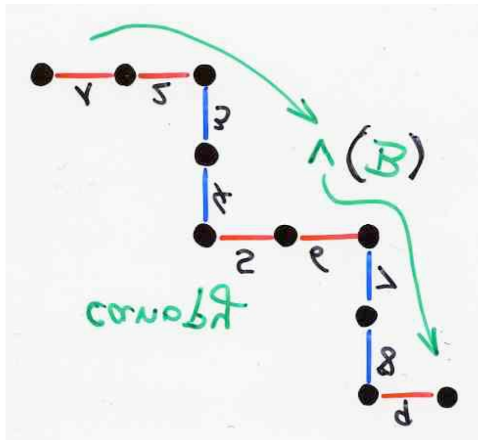


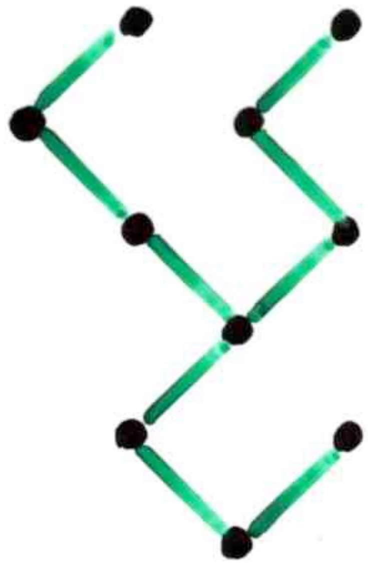


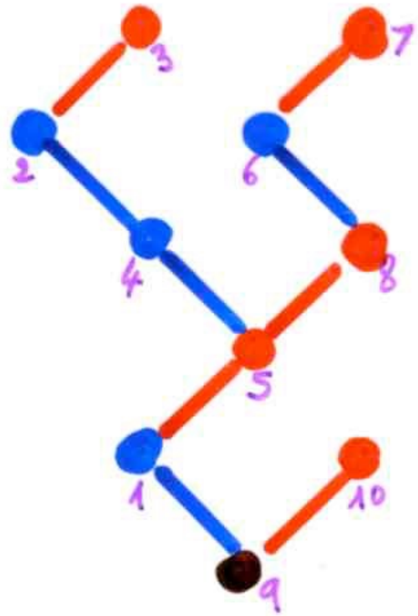



1
2
3
4
5
6
7
8
9
$\overline{B}$



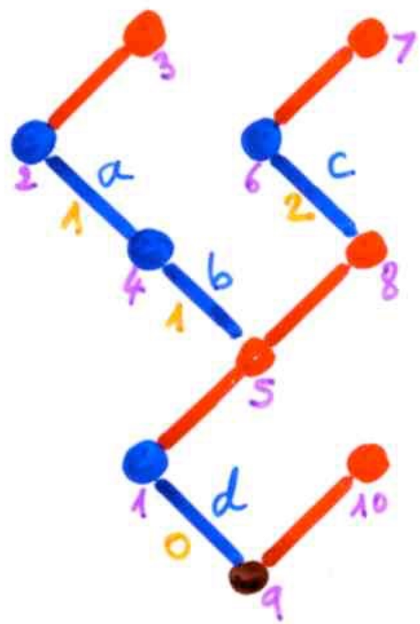

$(\overline{B})$





inorder
(= symmetric order)

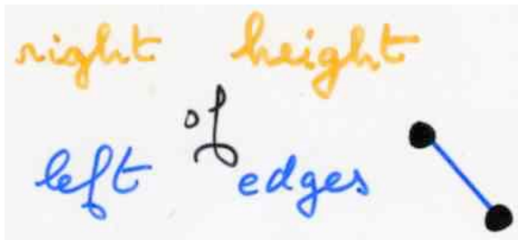
The left edges (in blue) of the binary tree are ordered according to the in-order (= symmetric order) of the first vertex of the edge.
Here the order is a, b, c, d.



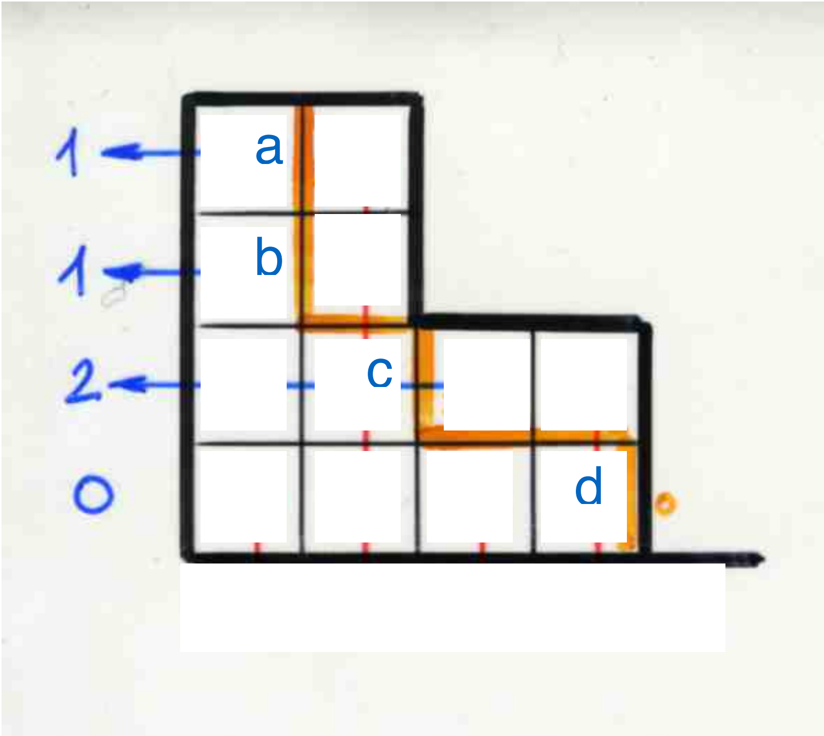
Then the right height of a left edge is the number of right edges (in red) needed to reach the vertices of that left edge.

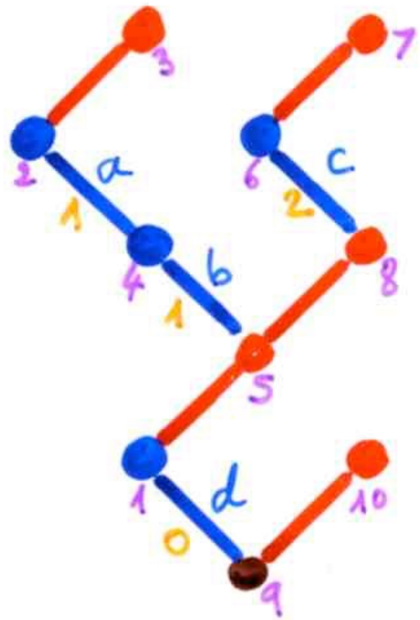
we get the vector:

a	b	c	d
1	1	2	0

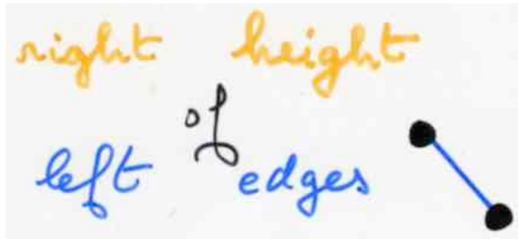


A path u (here in yellow) is uniquely defined by the following process: the South steps are ordered from top to down and associated to the order of the blue edges a,b, c, d.
The distance from each North step of u to the North-East border (the path v)
is given by the corresponding blue number (the right height of the left edge)





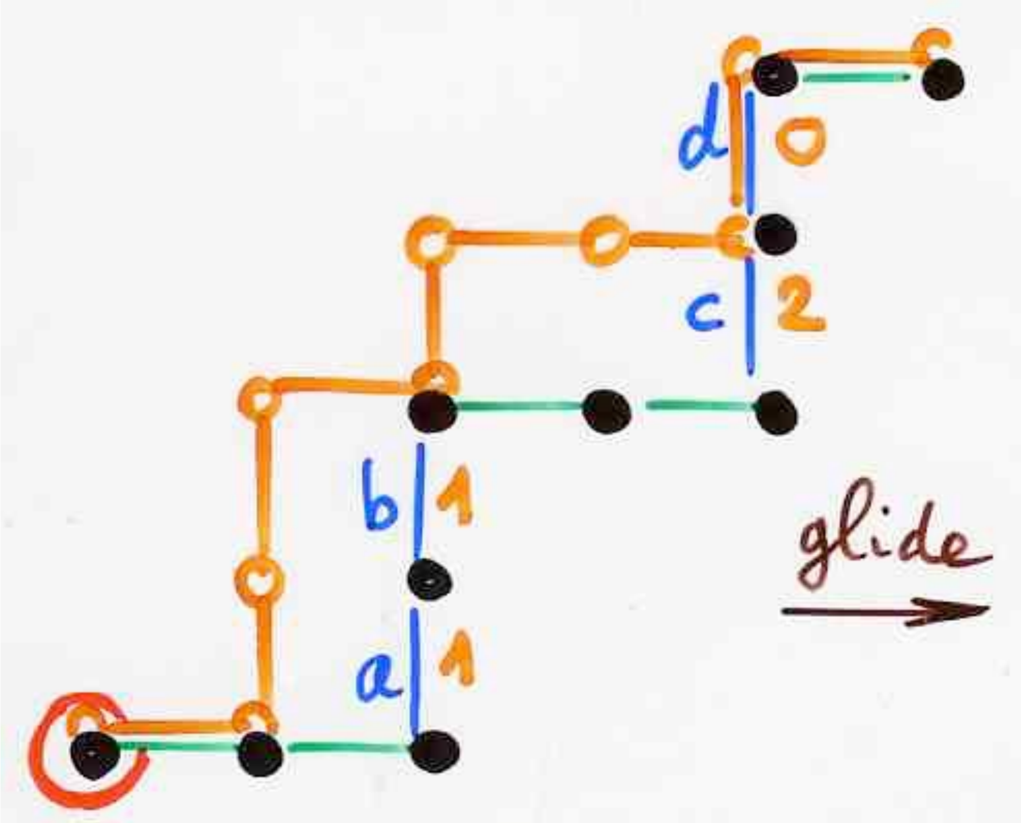
a	b	c	d
1	1	2	0



reverse bijection

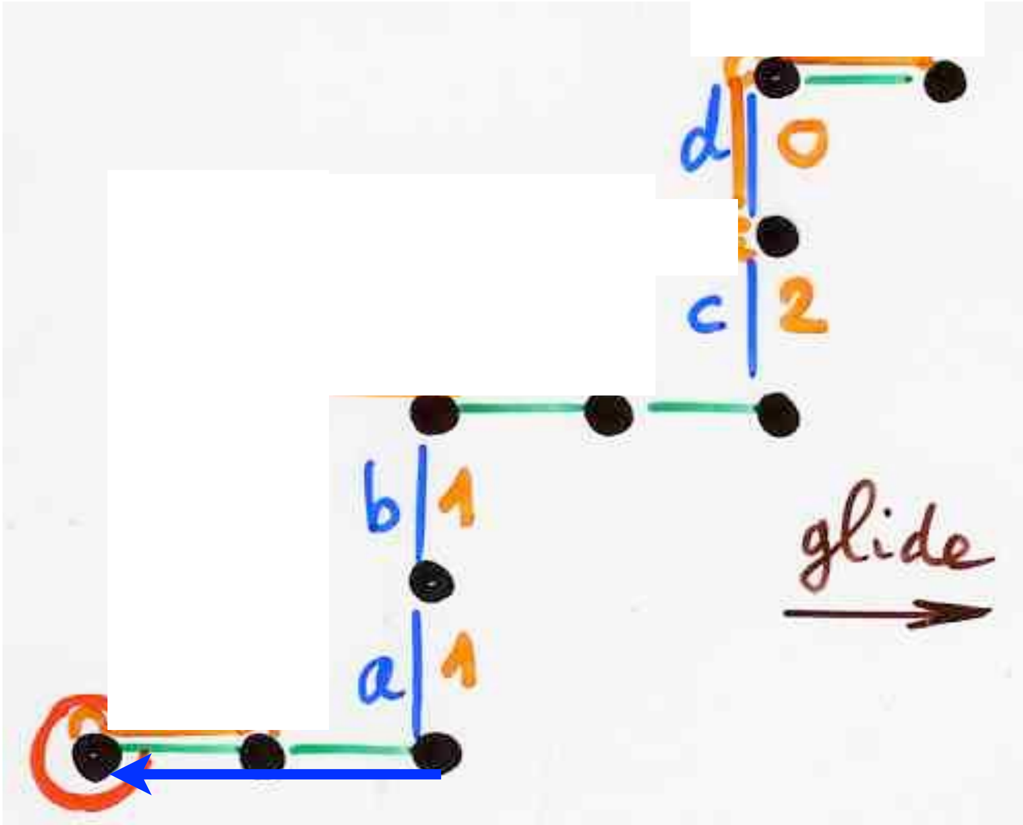
pair of paths $(u,v) \longrightarrow$ binary tree B

the «push-gliding» algorithm



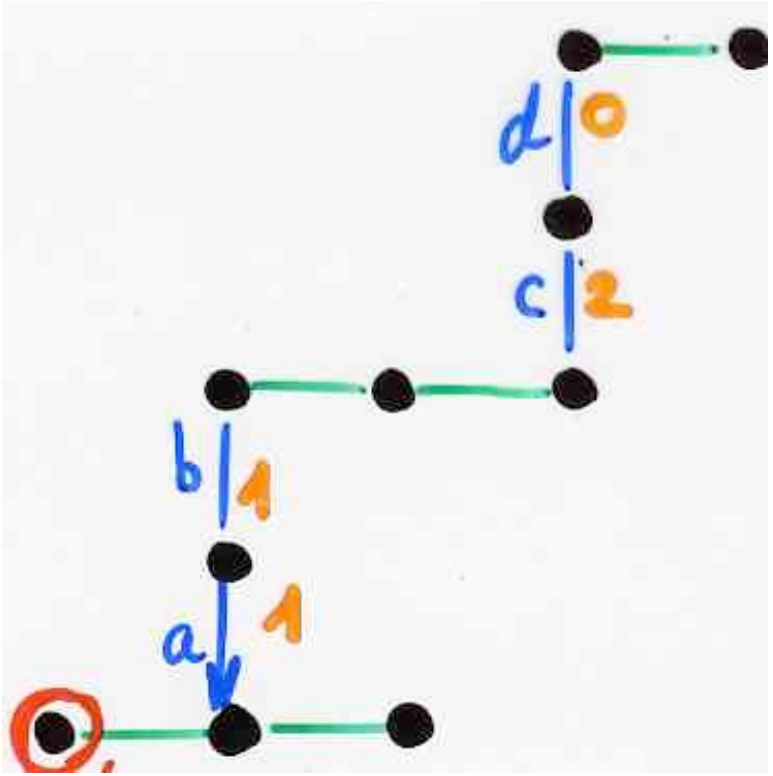
reverse bijection

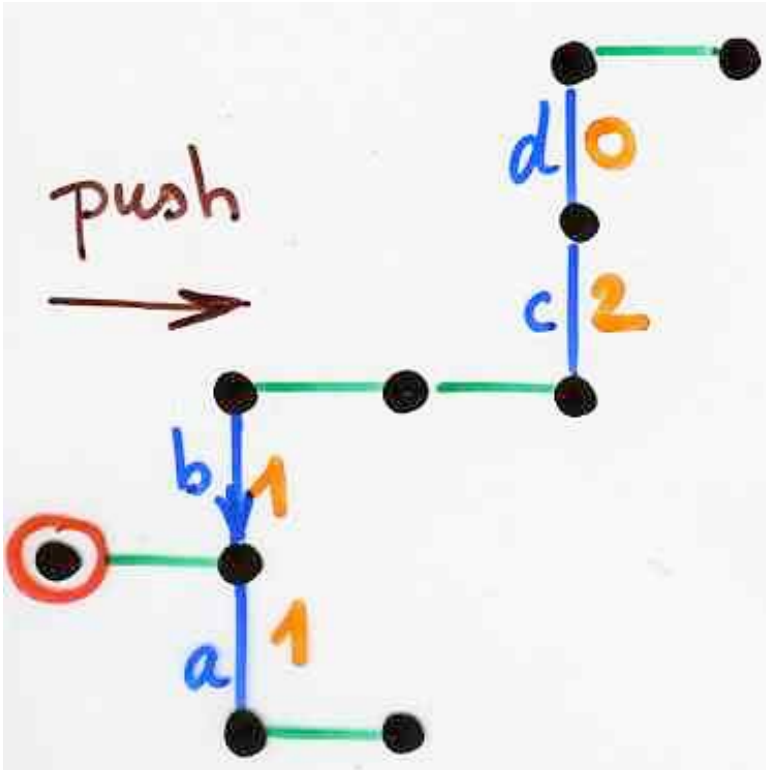
the "push-gliding" algorithm

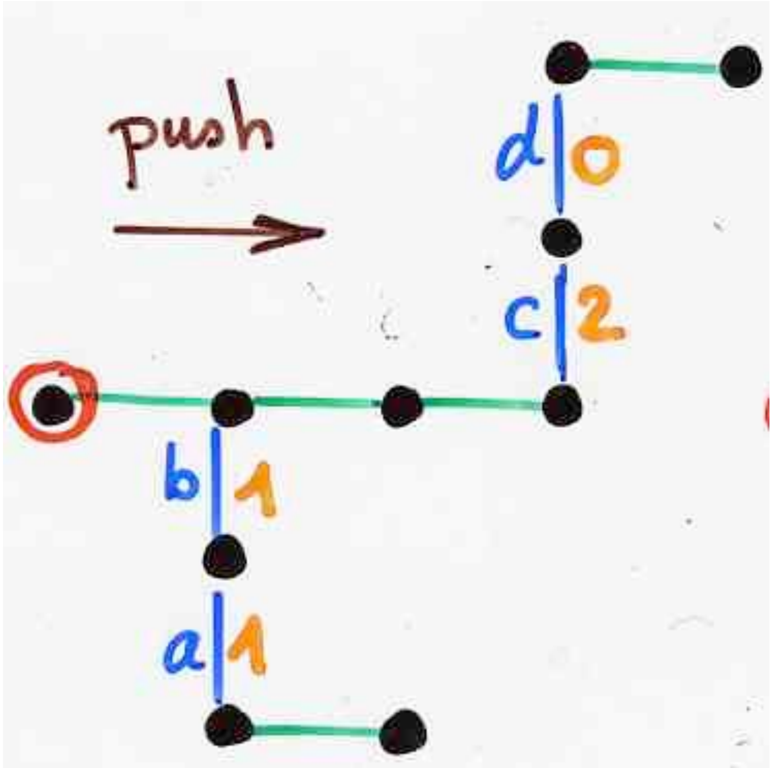


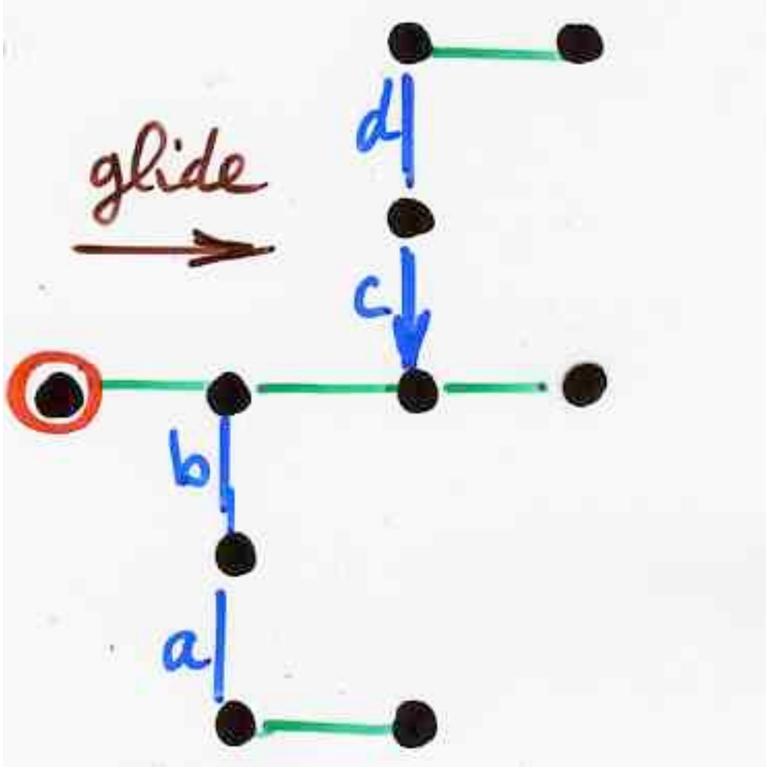
reverse bijection

the "push-gliding" algorithm

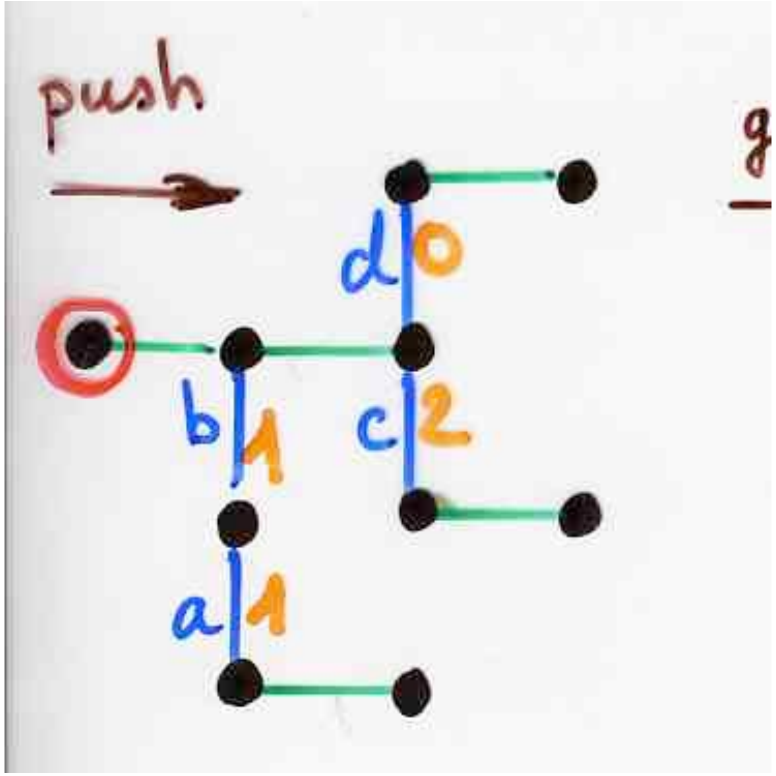

d
0
c
2
b
1
a
1

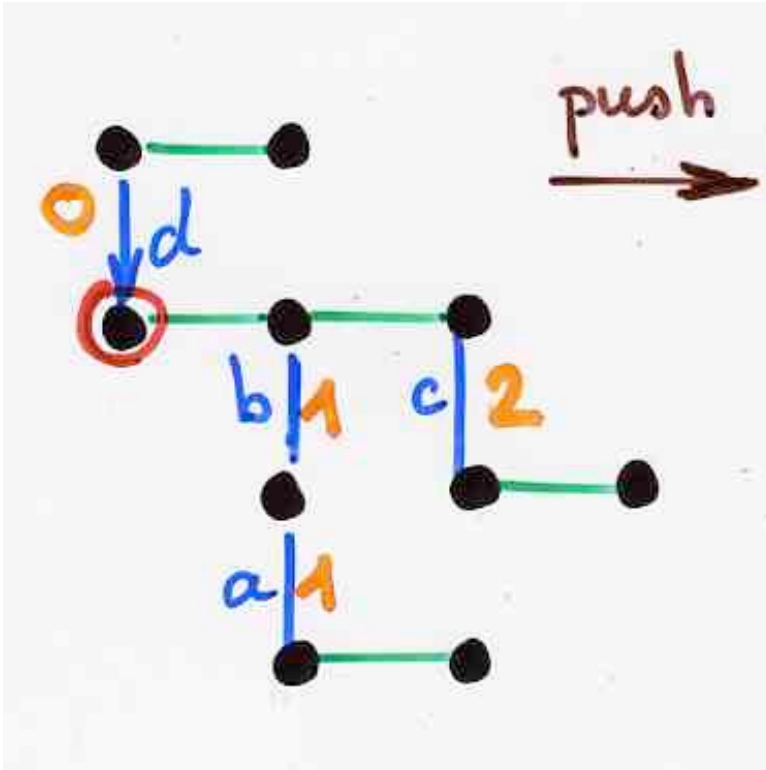

push
d
0
c
2
b
1
a
1

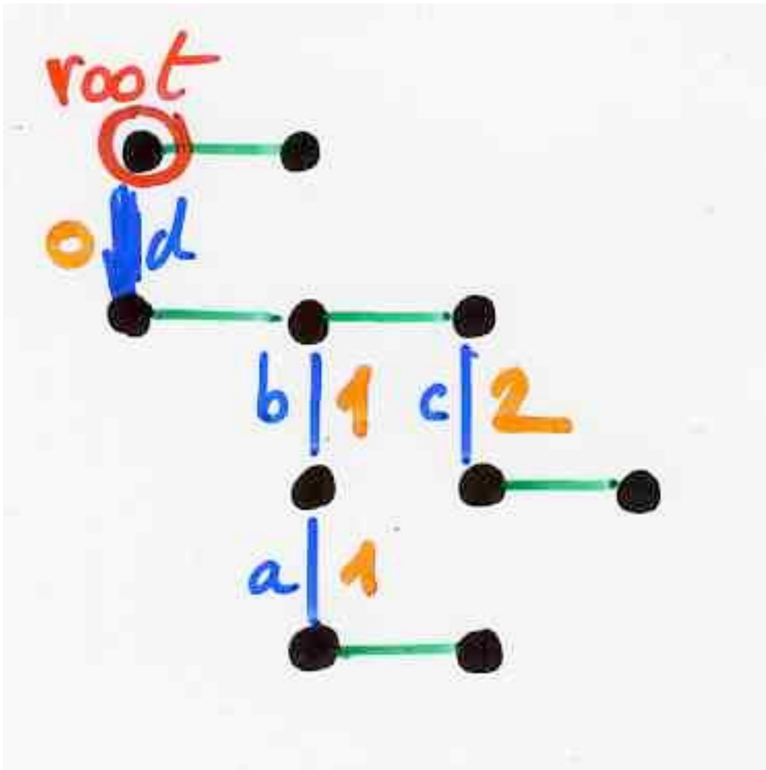

push
d
0
c
2
b
1
a
1

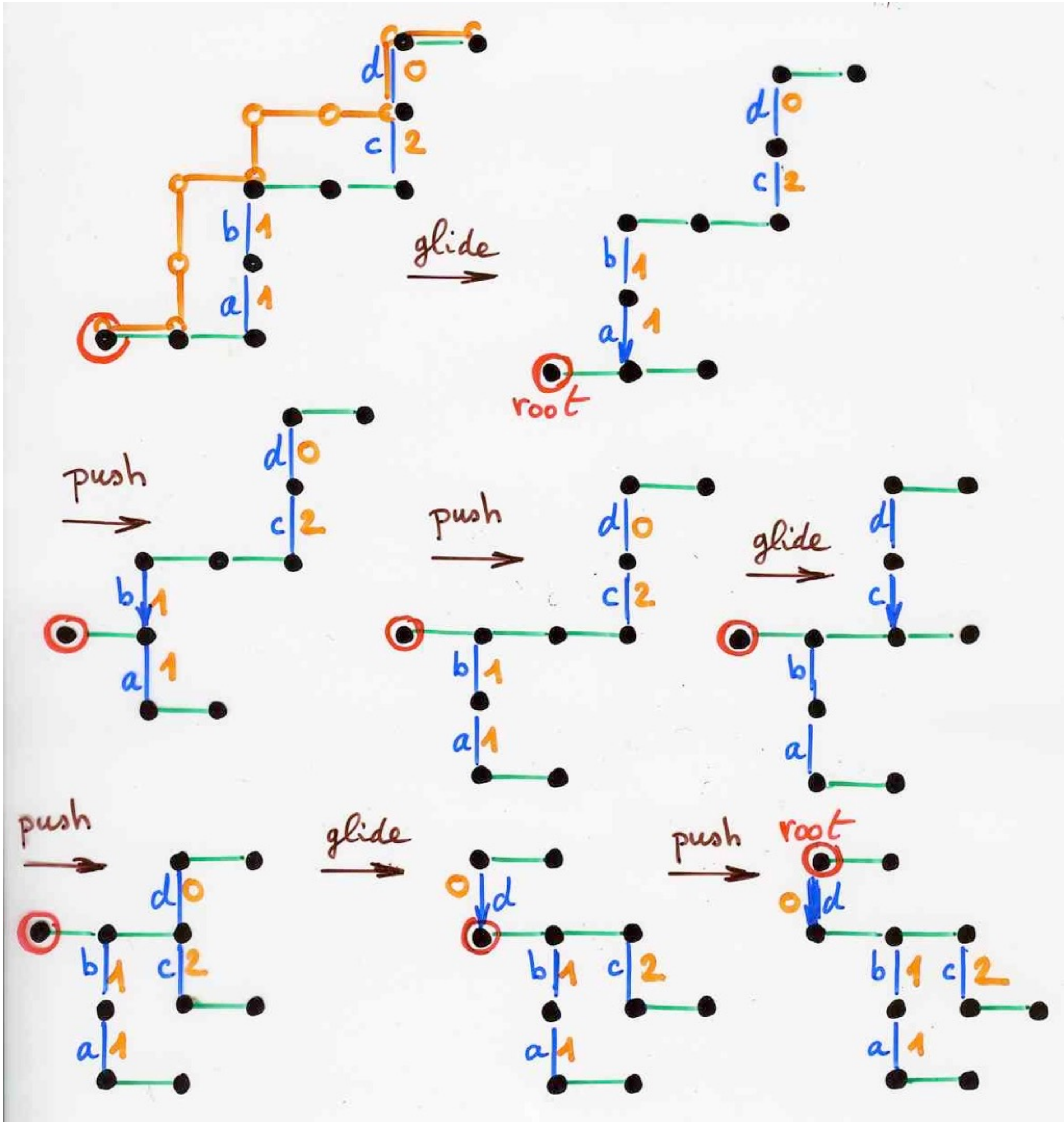


glide
d
c
b
a


push
d
0
b
1
c
2
a
1
g


push
0
d
b
1
c
2
a
1


root
0
d
b
1
c
2
a
1

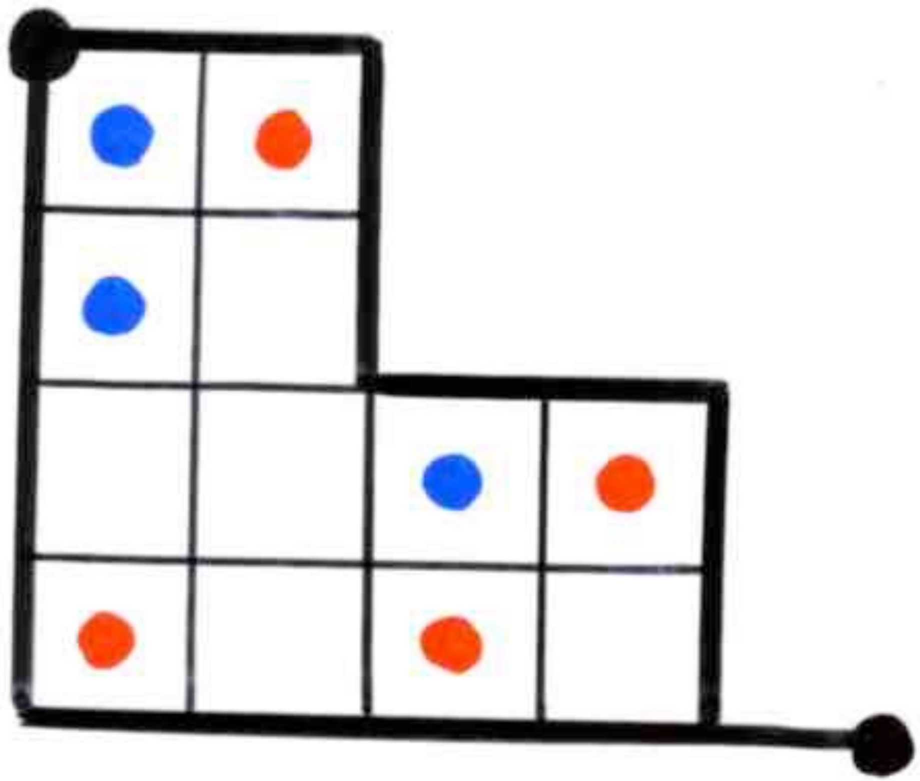


glide
root
push
push
glide
push
glide
push
root

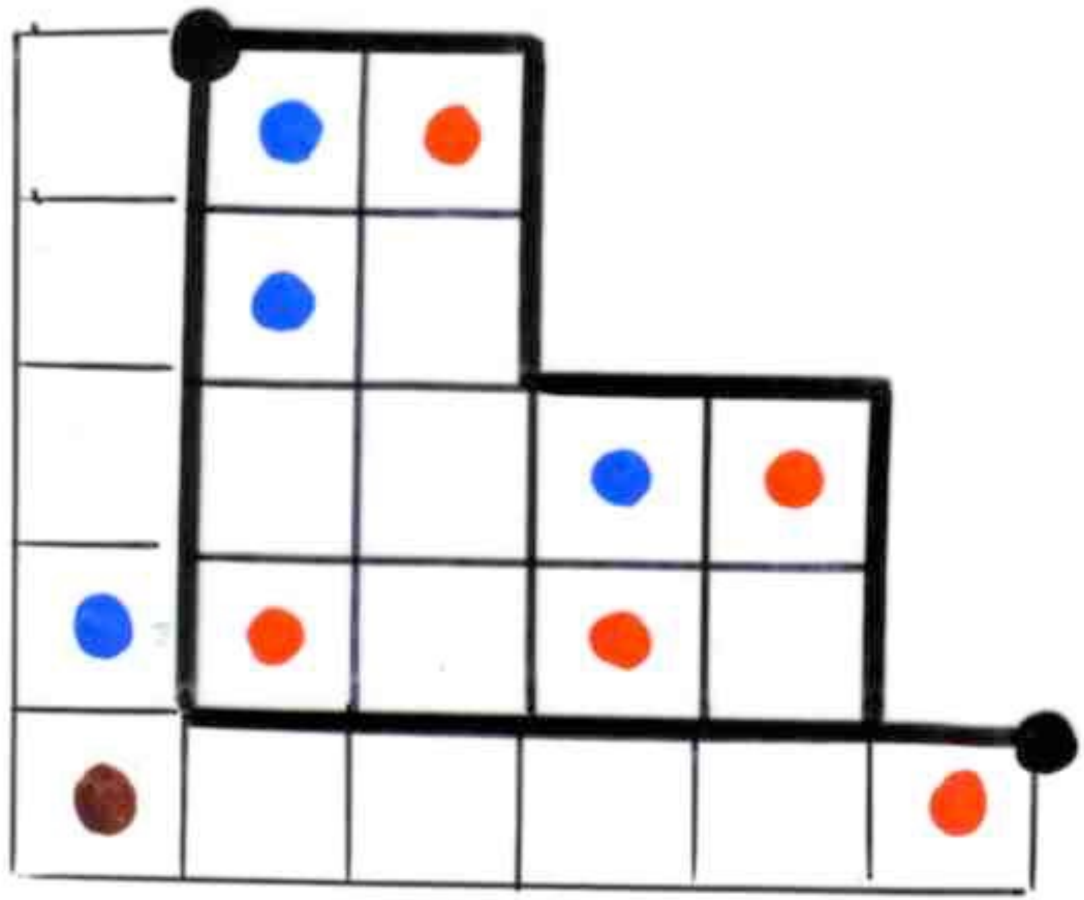
bijection

Catalan alternative tableaux

binary trees

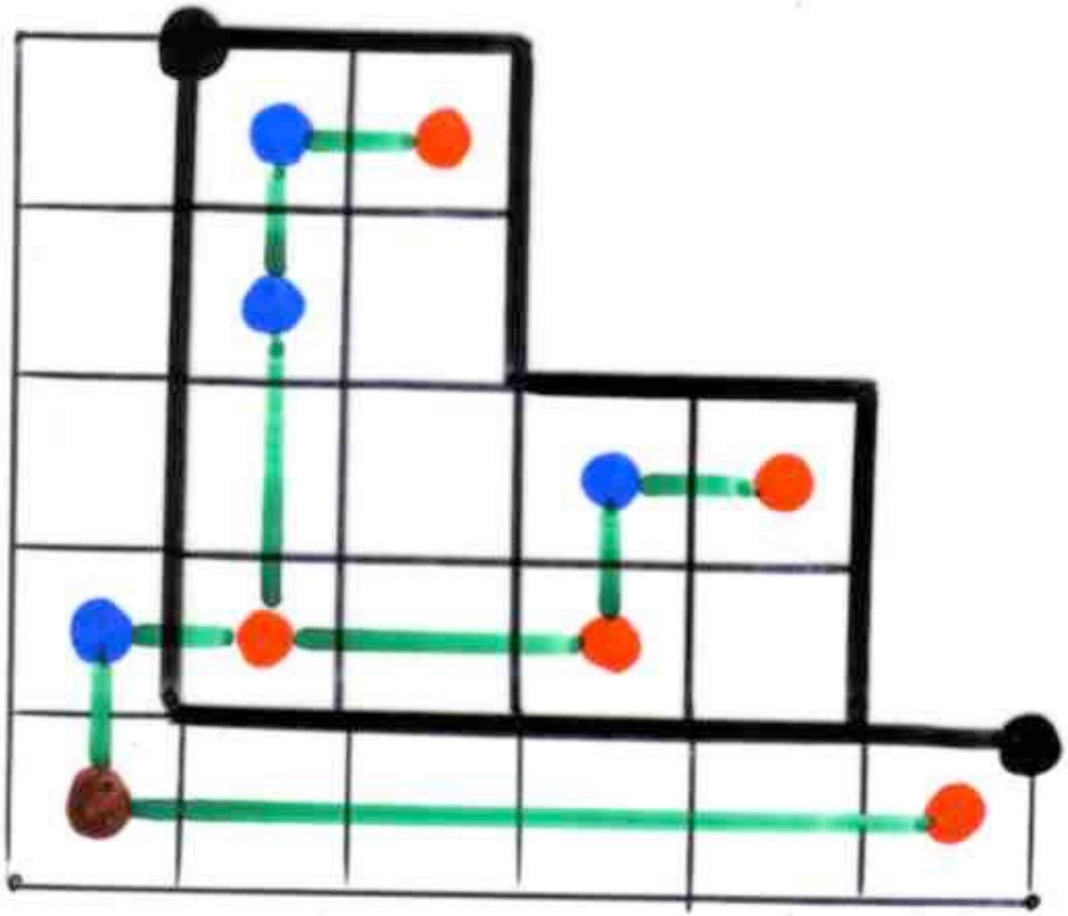


a Catalan alternative tableau

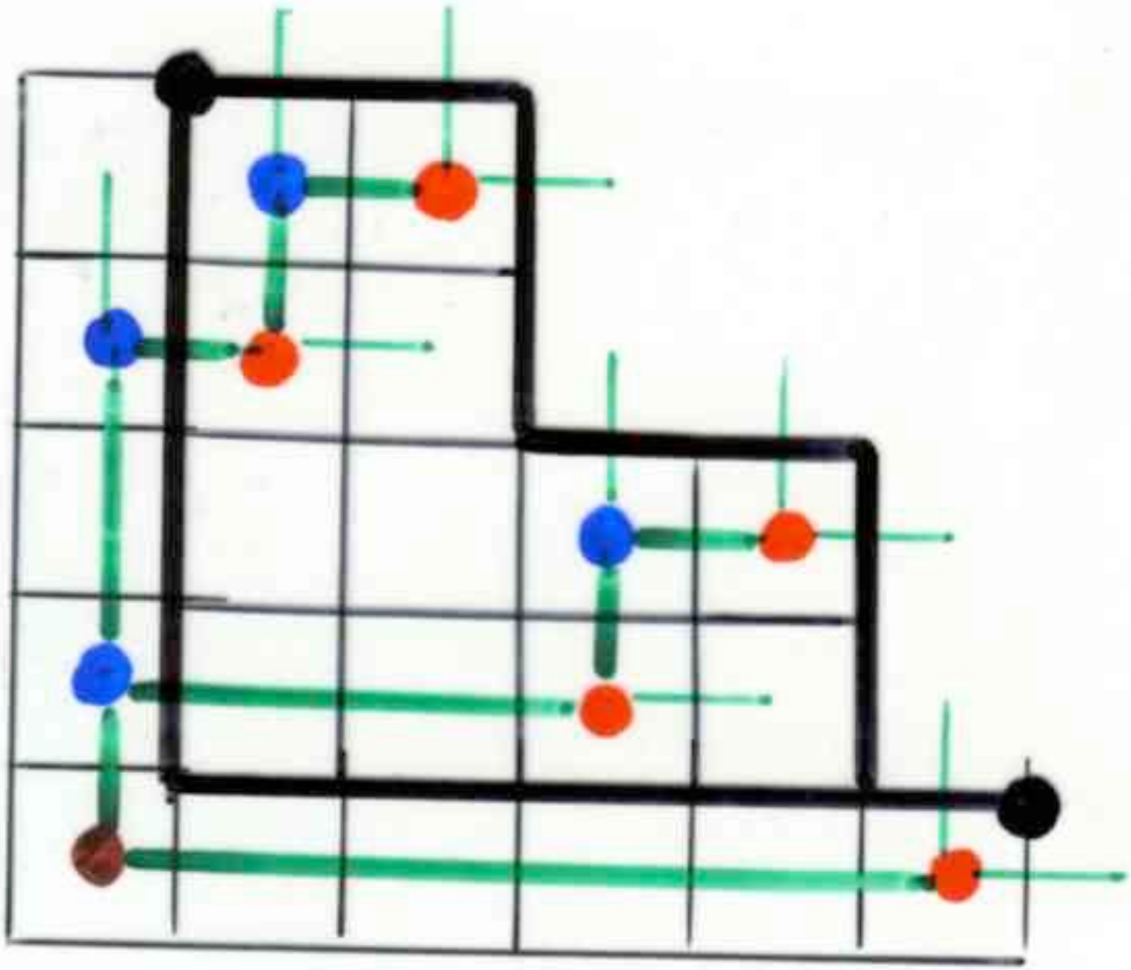


the extended Catalan alternative tableau for each blue point add a vertical (green) edge below the point

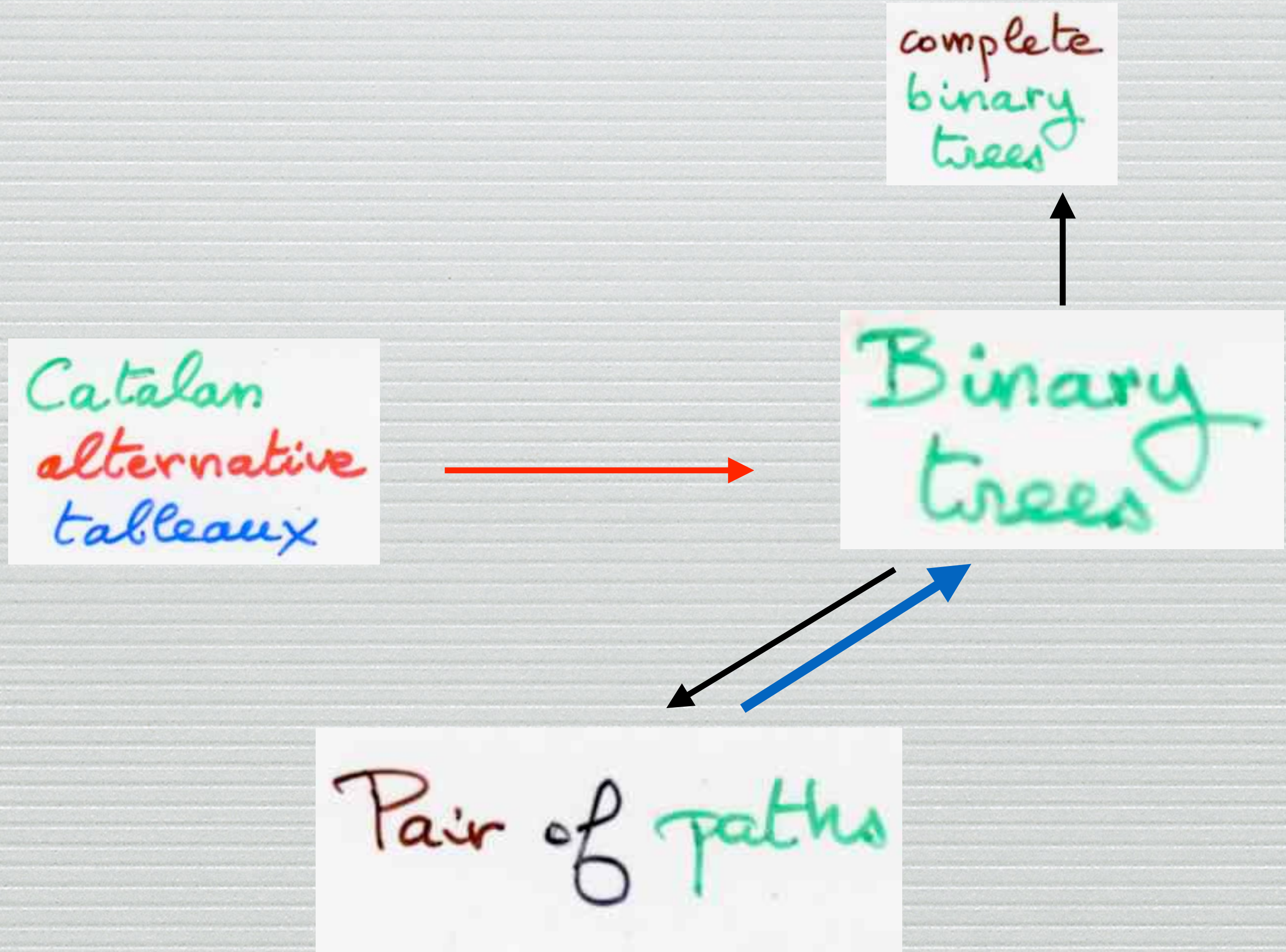
for each red point add an horizontal (green) edge at the left of he point



one get a binary tree



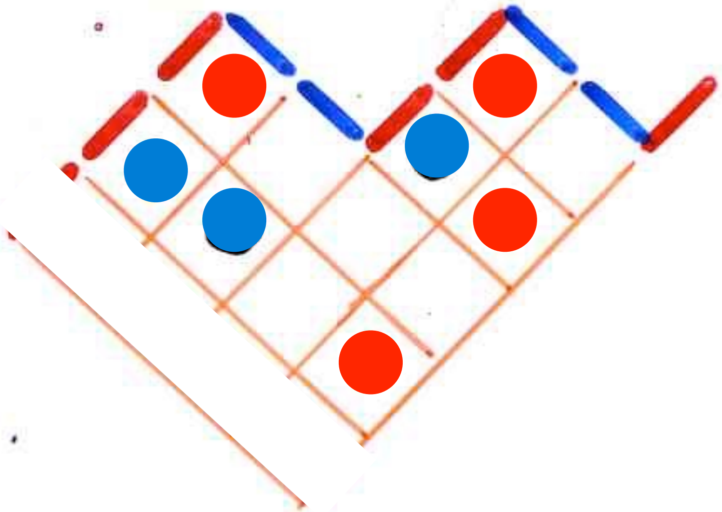
the associated extended (also called complete) binary tree

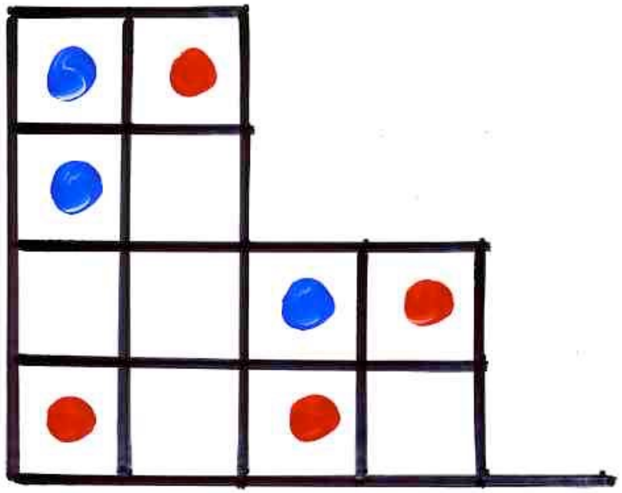

complete binary trees
Catalan alternative tableaux
Binary trees
Pair of paths

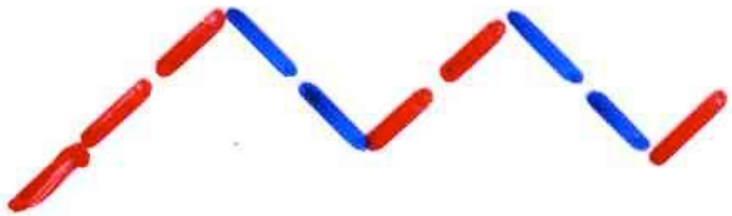
2nd bijection

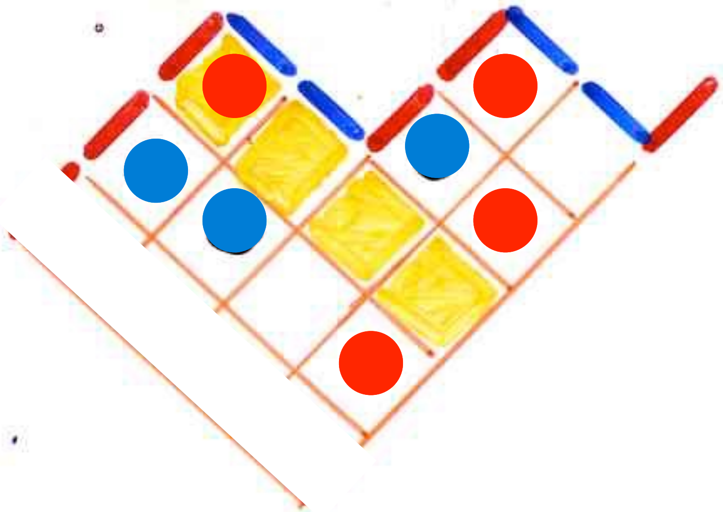
Catalan alternative tableaux

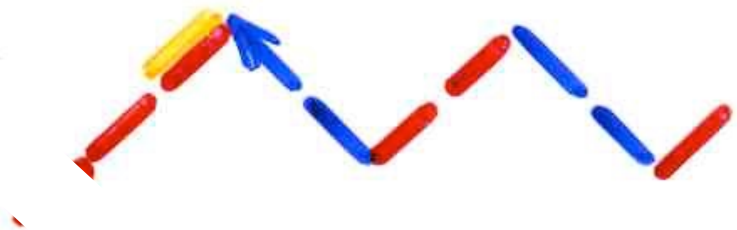
binary trees

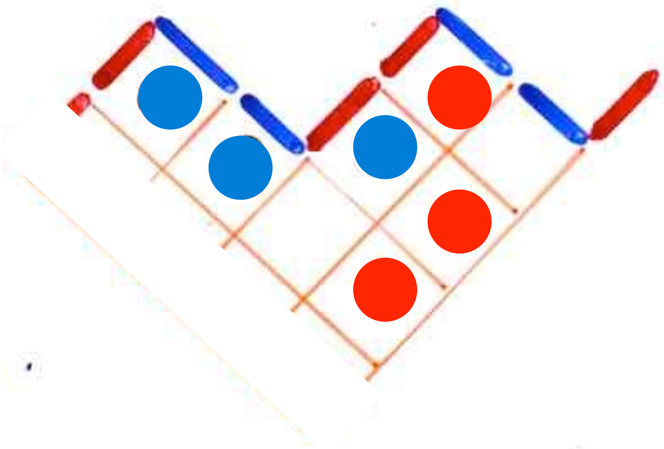


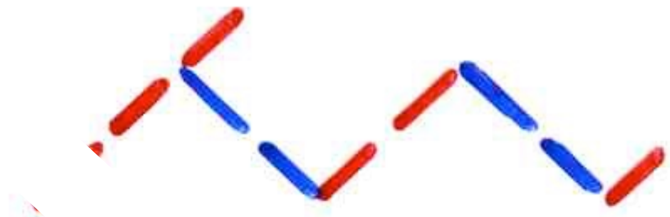


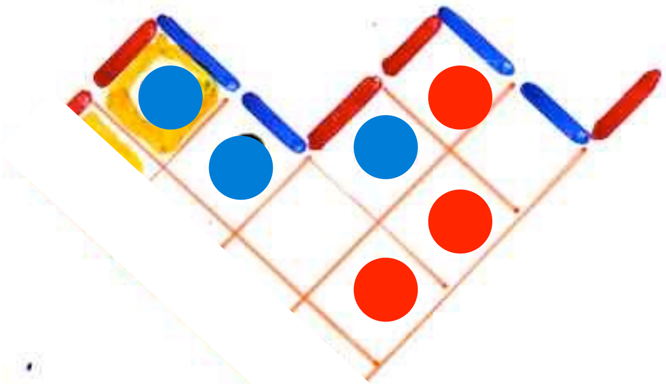


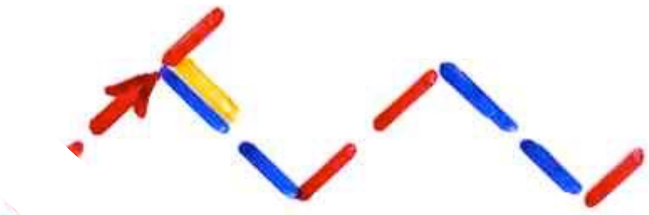


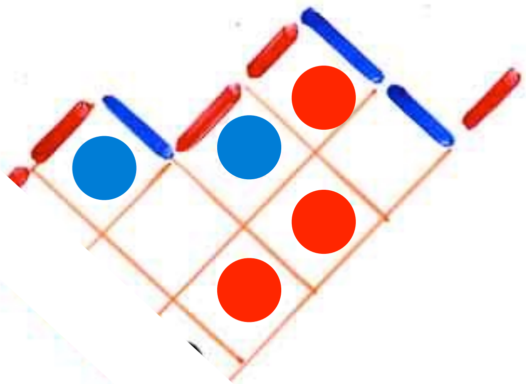


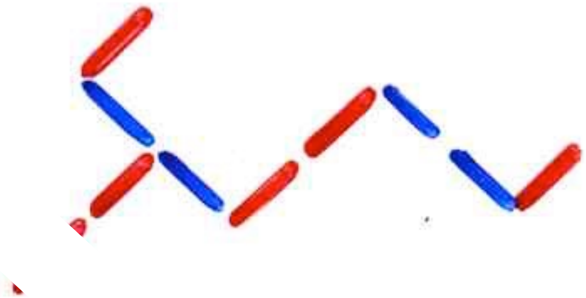


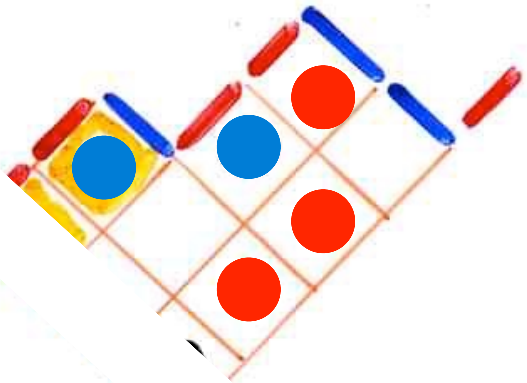
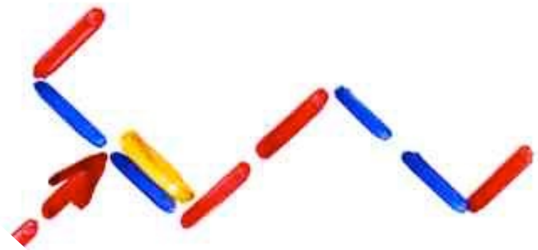


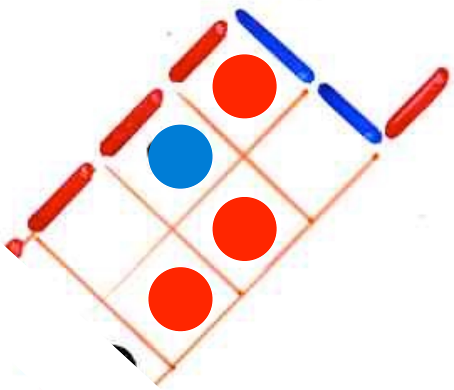


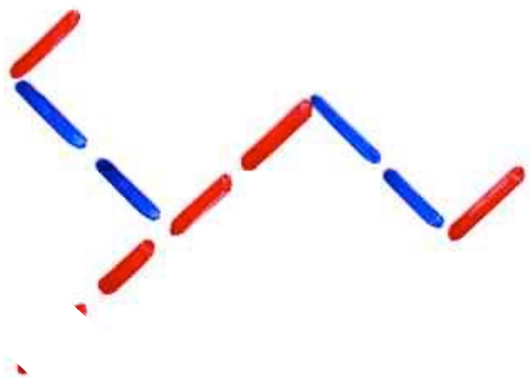


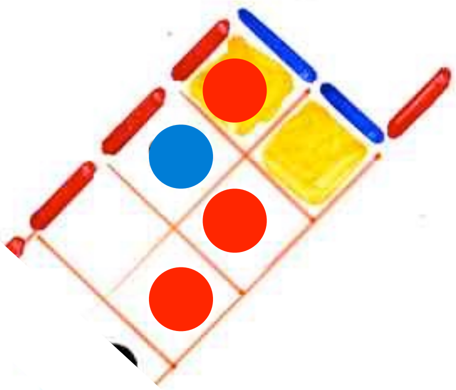


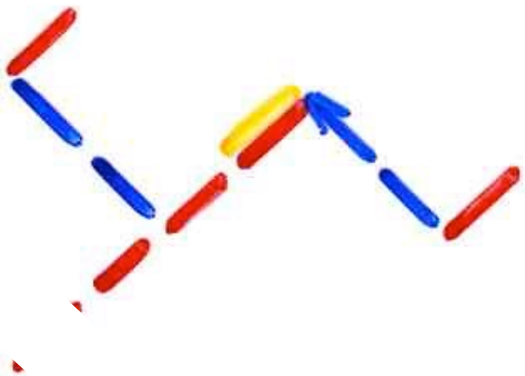


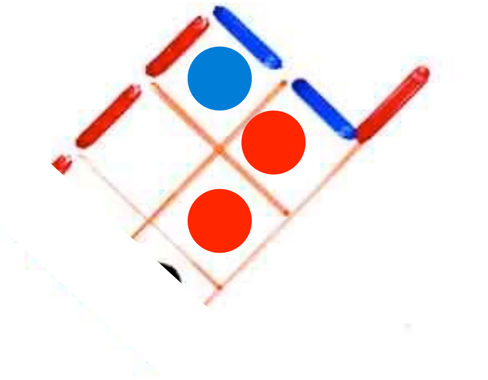
 

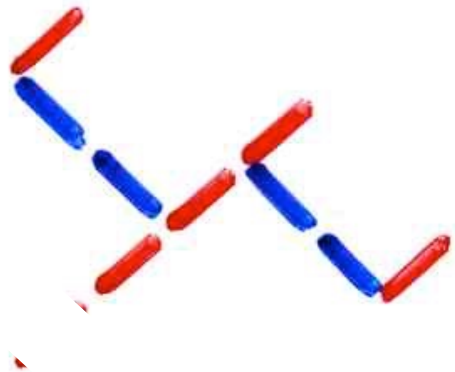


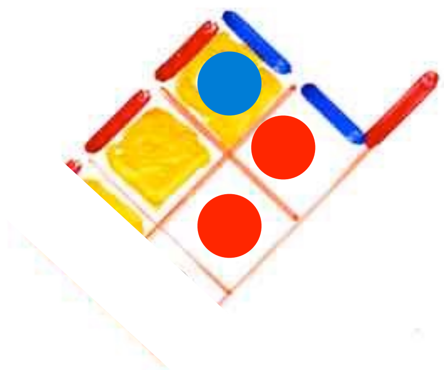


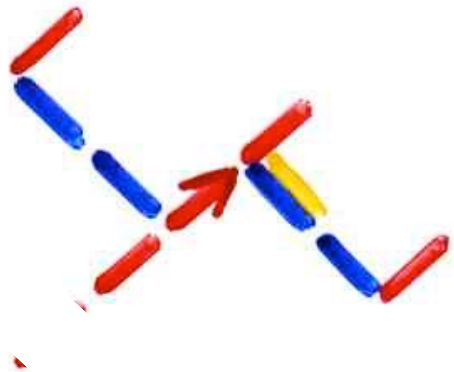


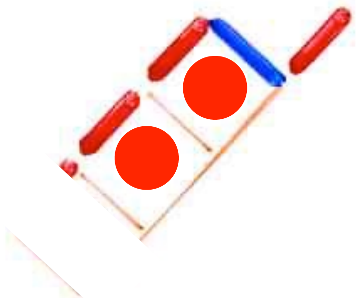


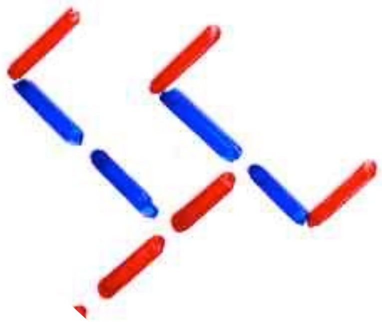


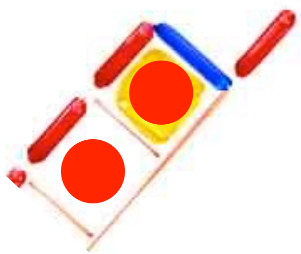


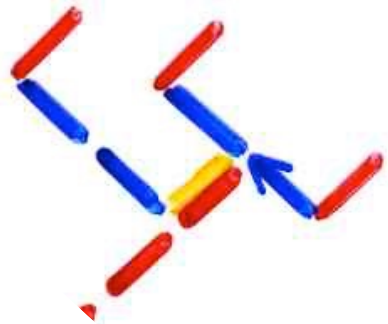


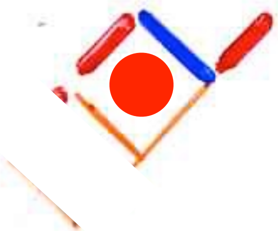


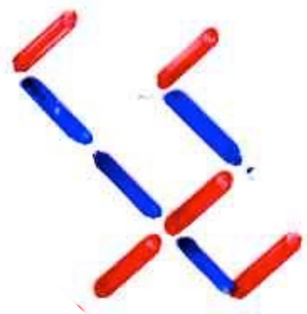


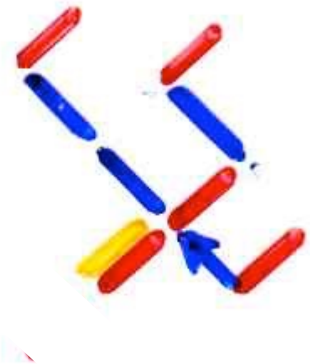


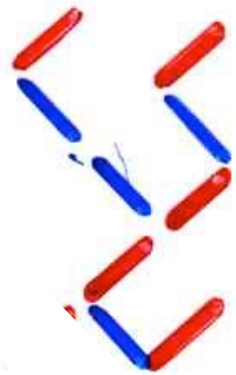






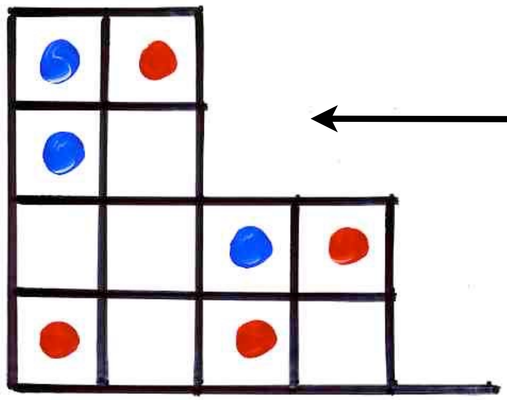


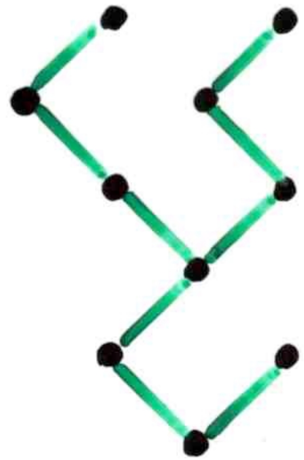


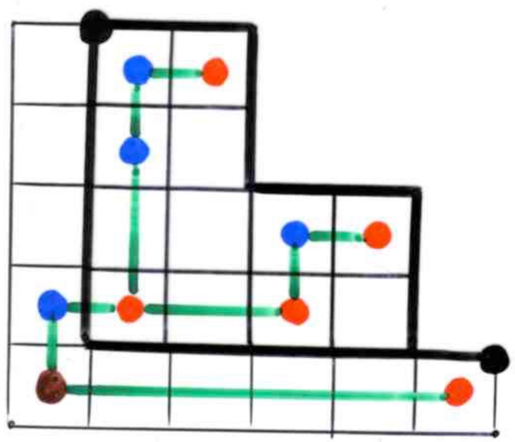


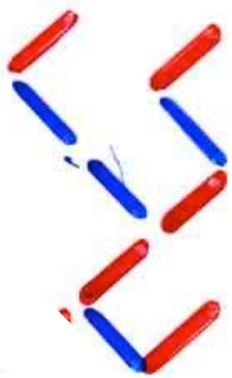
This algorithm based on a kind of « jeu de taquin » on « tableaux and trees »
is reversible. One get a bijection between Catalan alternative tableaux and binary
trees, which is the same as the one described on slide 115.

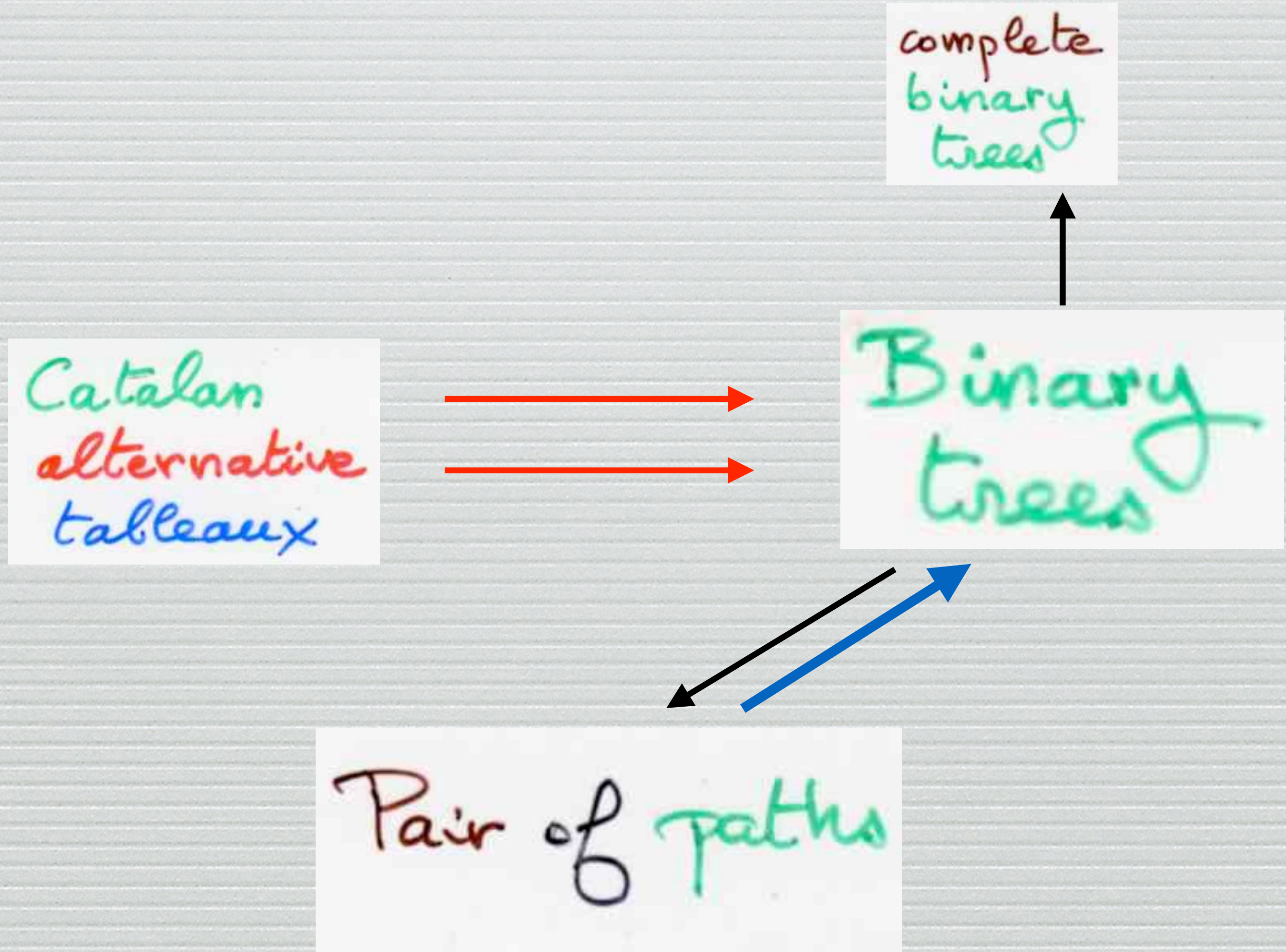
Catalan
alternative
tableaux



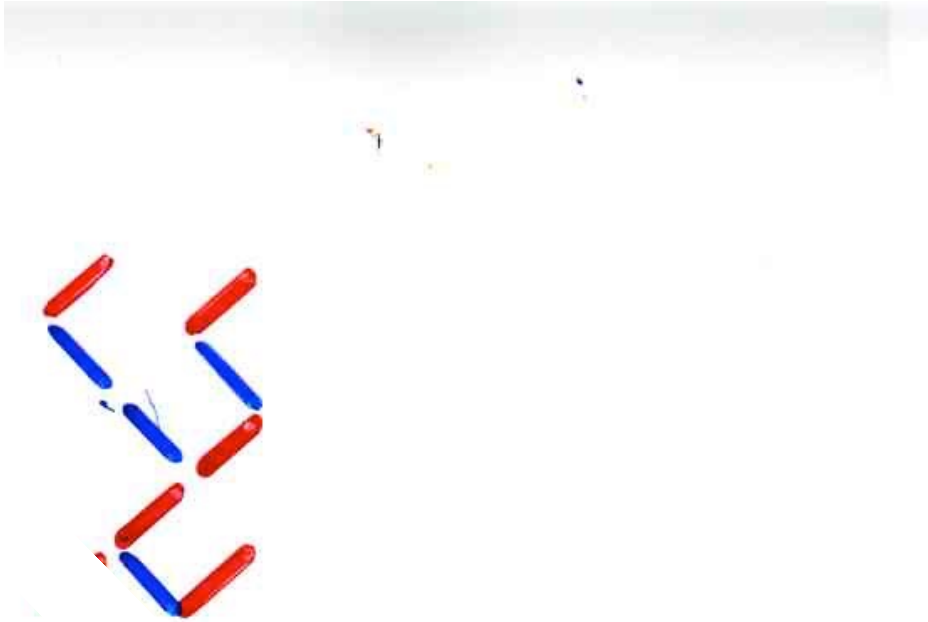


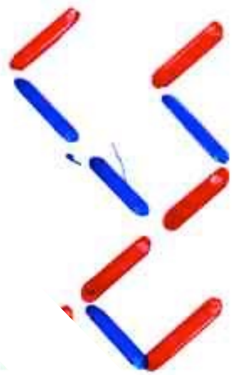


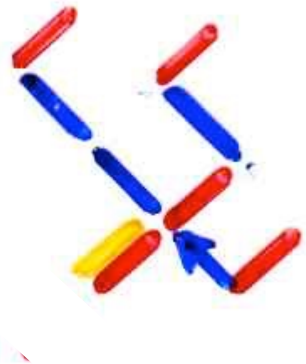


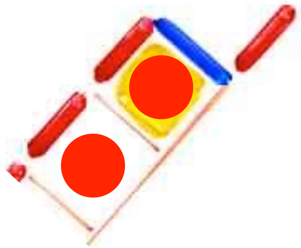

complete binary trees
Catalan alternative tableaux
Binary trees
Pair of paths

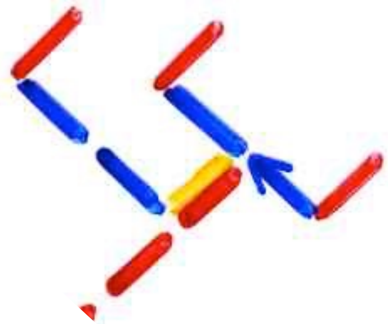
reverse bijection

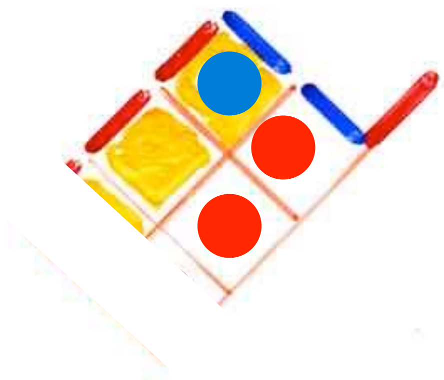


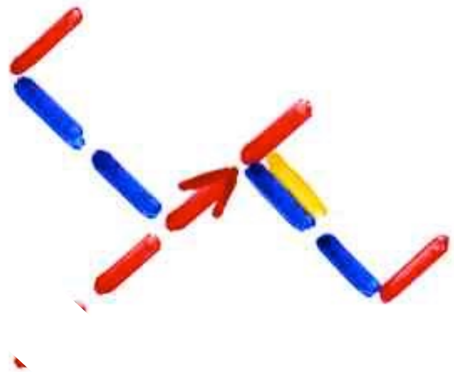


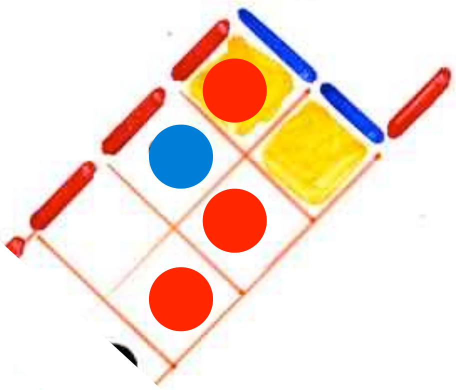


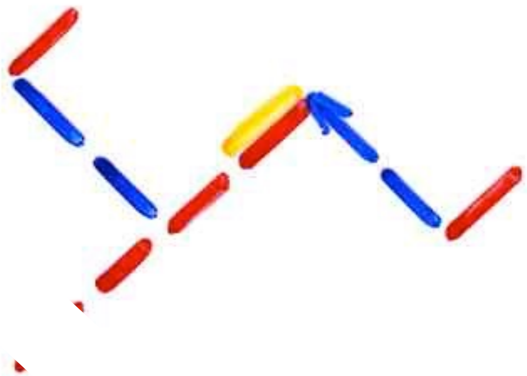


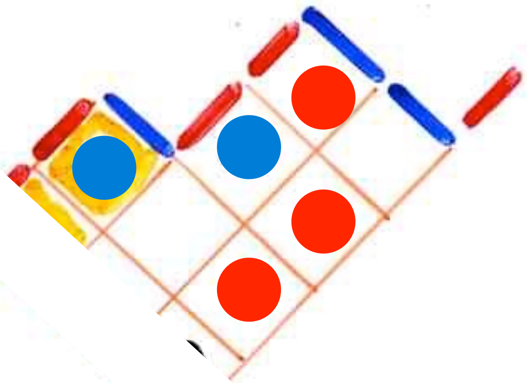


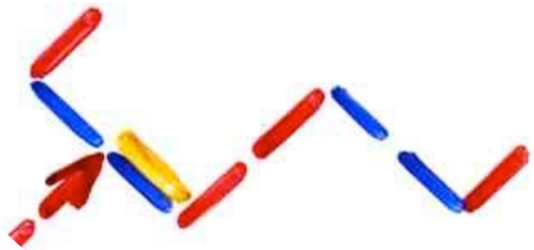


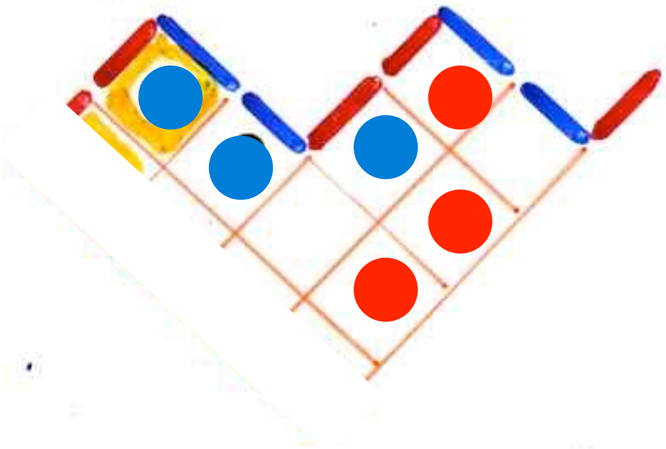


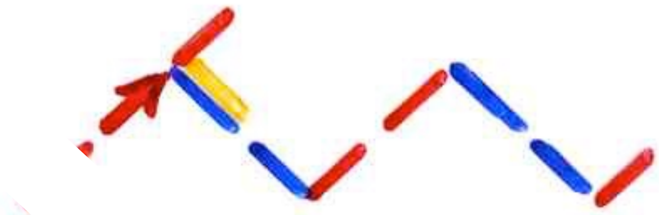


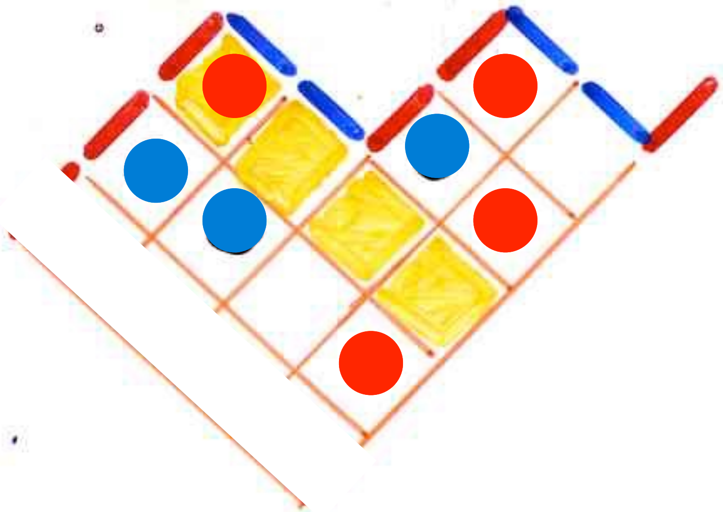


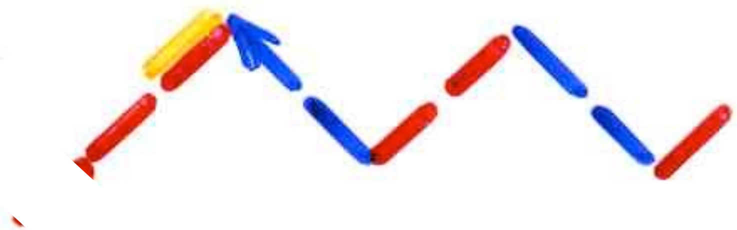




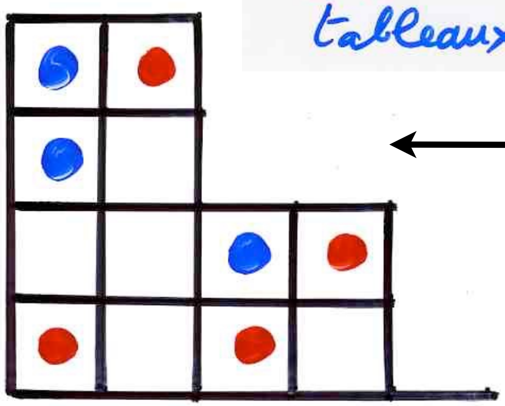


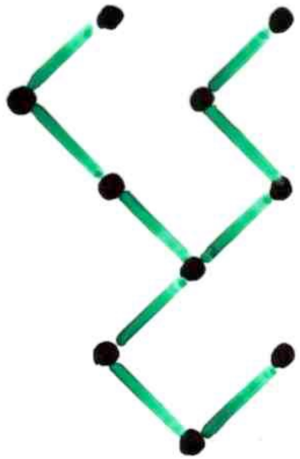


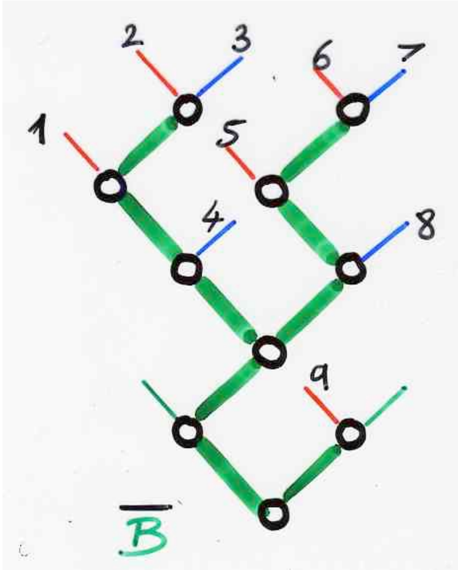




Catalan
alternative
tableaux

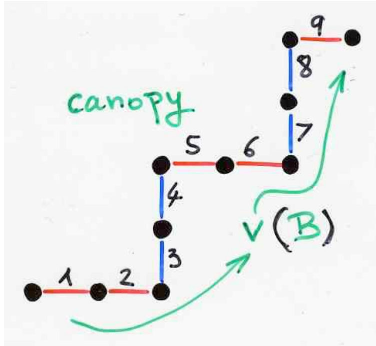


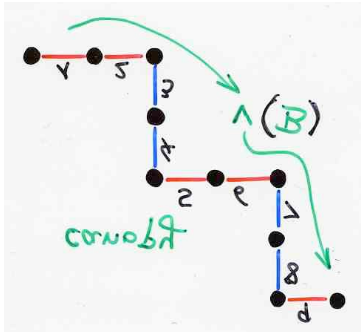


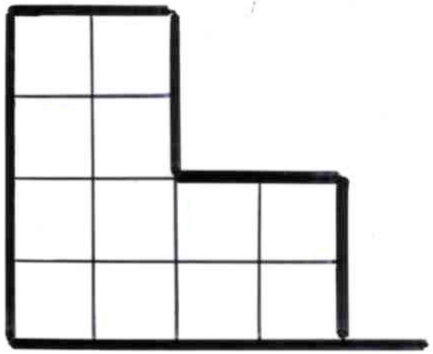


Proposition

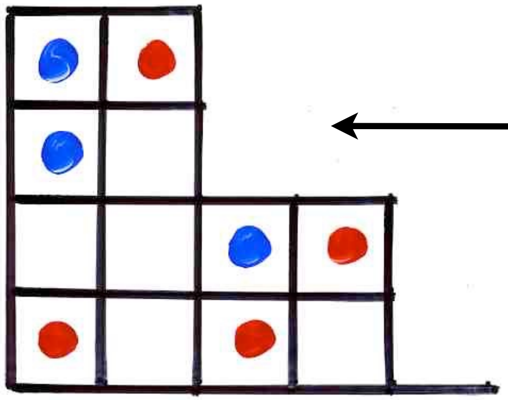
The map defined above is a bijection between alternative tableaux with profile v and binary trees with canopy v

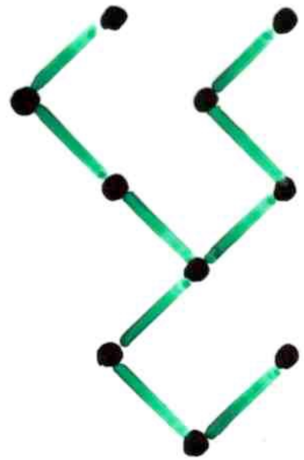


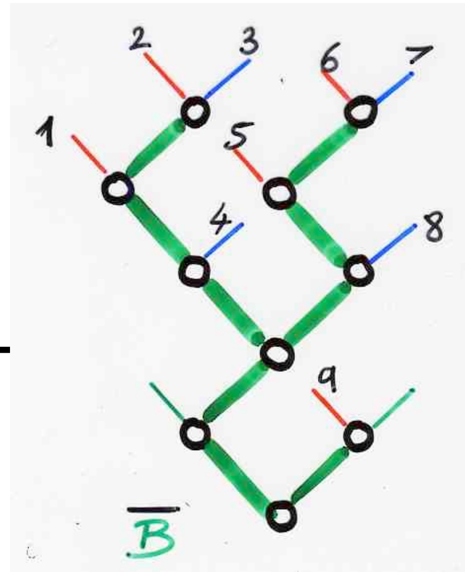


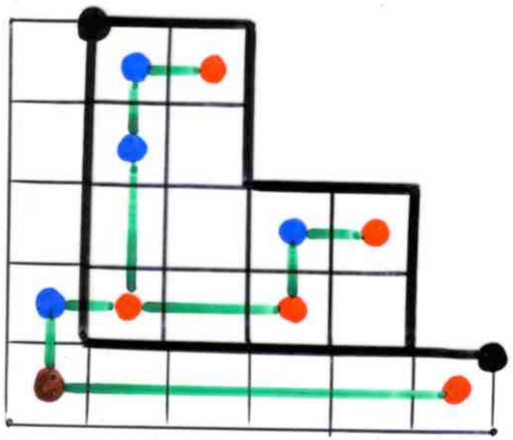


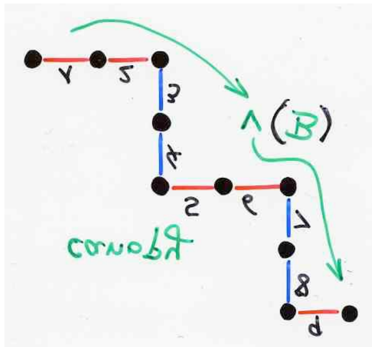
Catalan alternative tableaux

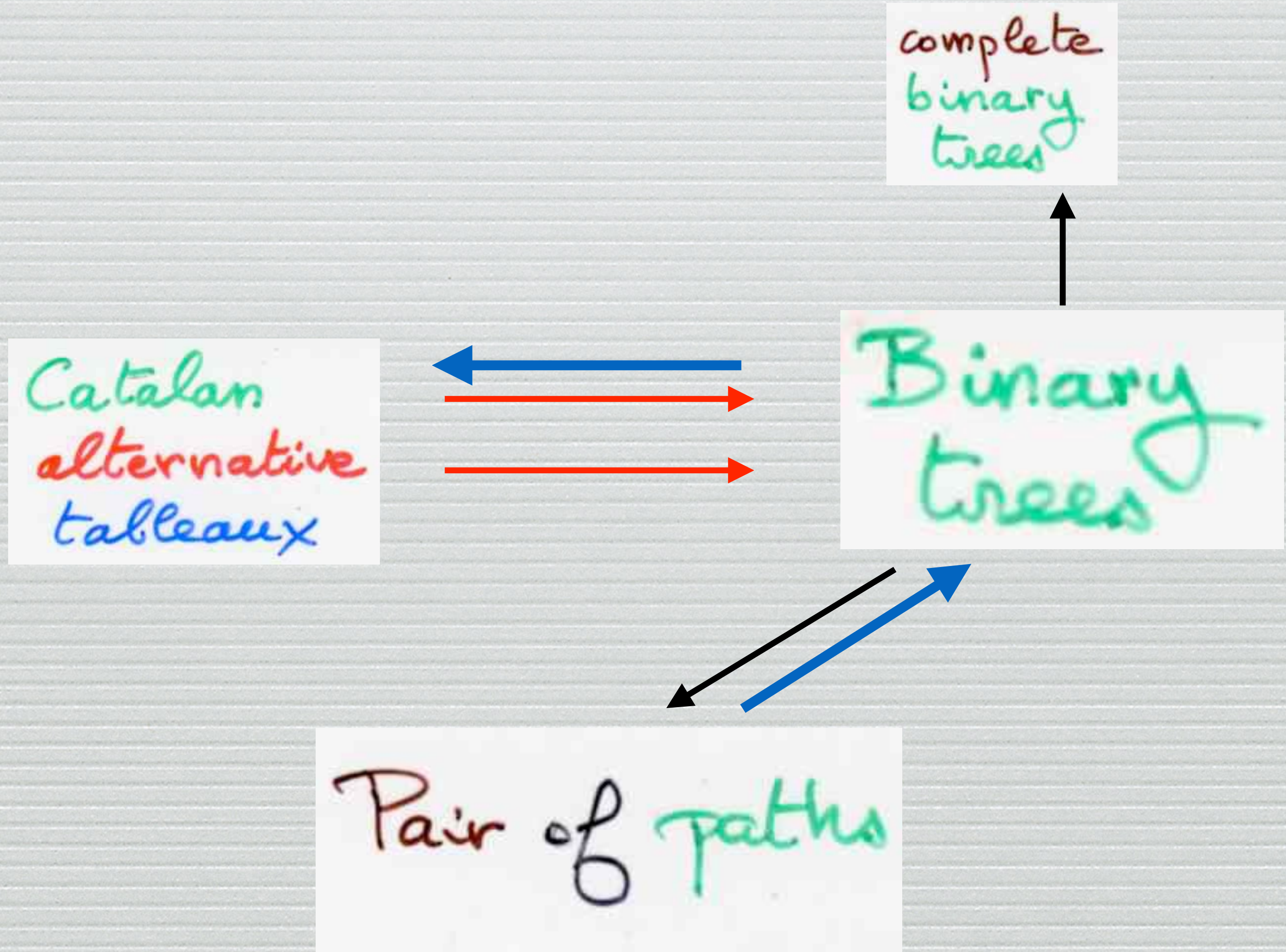











complete binary trees
Catalan alternative tableaux
Binary trees
Pair of paths

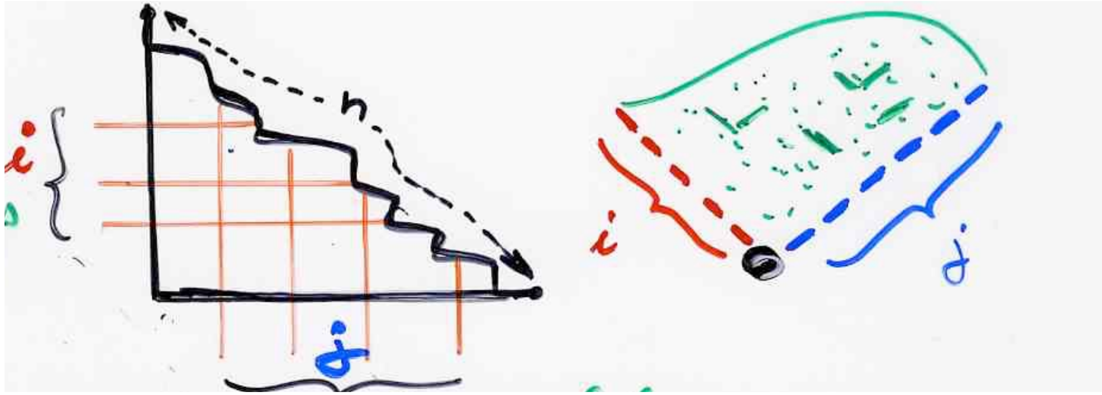
TASEP,
alternative Catalan tableaux
and
binary trees

Proposition

The map defined above is a bijection between alternative tableaux with profile v and binary trees with canopy v

$i(T)$ = nb of rows without ●

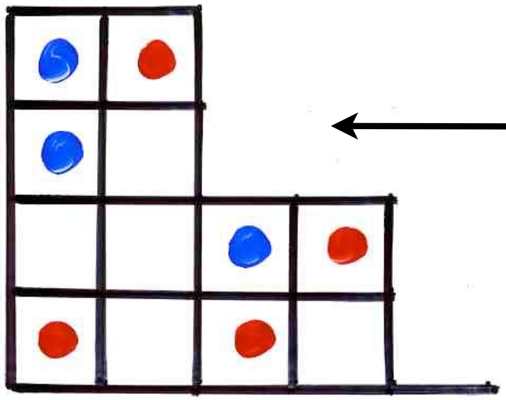
$j(T)$ = nb of columns without ●

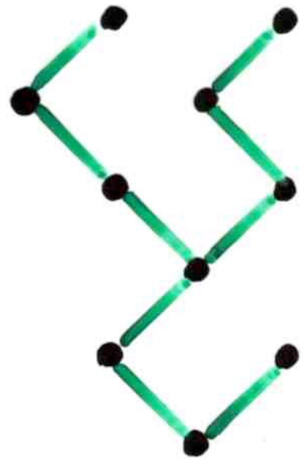


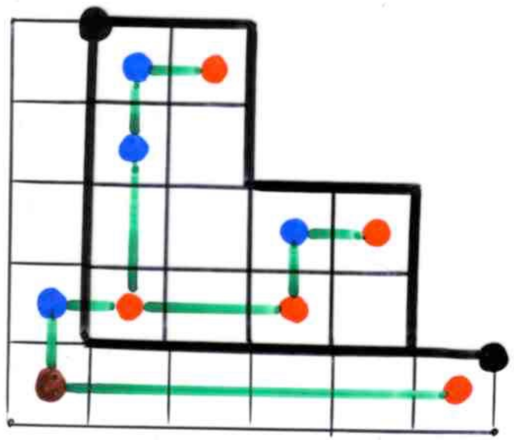
$i(T) = lpb(B)$ length of left principal branch

$j(T) = rpb(B)$ length of right principal branch

Catalan
alternative
tableaux







$T \longrightarrow B$

Catalan alternative tableau — binary tree

profile v of T = canopy v of B

$$i(T) = lpb(B)$$

$$j(T) = rpb(B)$$

$$s = (\tau_1, \dots, \tau_n)$$

$$P_n(s ; \alpha \ \beta) = \frac{1}{Z_n} \sum_{B} \alpha^{-lpb(B)} \beta^{-rpb(B)}$$

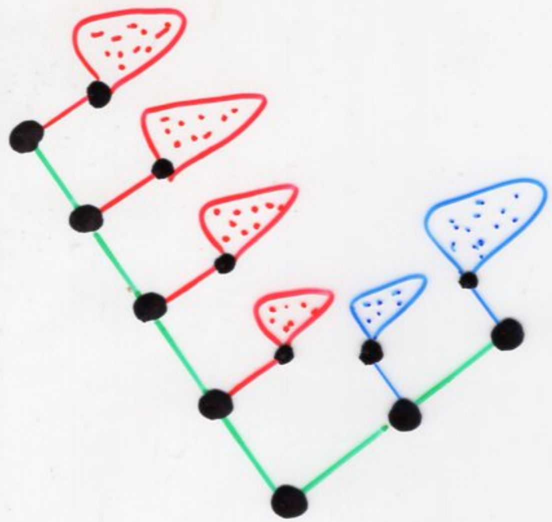
binary tree
canopy s

$$\underset{\text{partition function}}{Z_n} = \sum_{1 \leq i \leq n} \frac{i}{(2n-i)} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

$$\bar{\alpha} = \alpha^{-1}$$

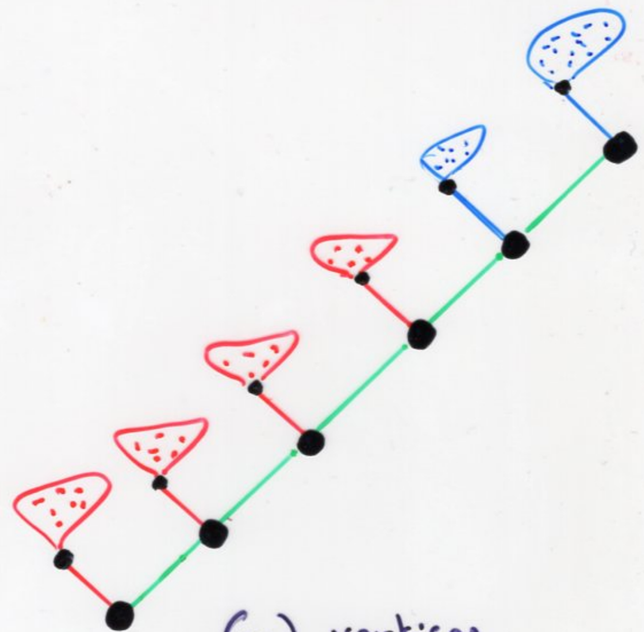
$$\bar{\beta} = \beta^{-1}$$

$$\sum_{B} \alpha^{-lpb(B)} \beta^{-rpb(B)}$$



(n+1) vertices

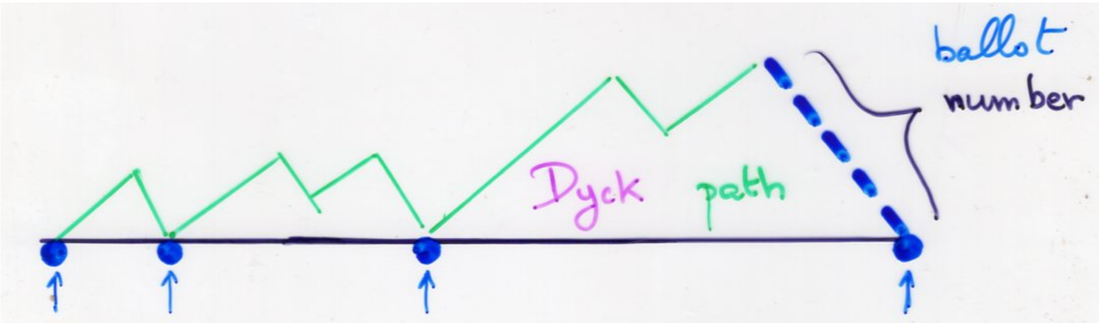
$$\longleftrightarrow$$

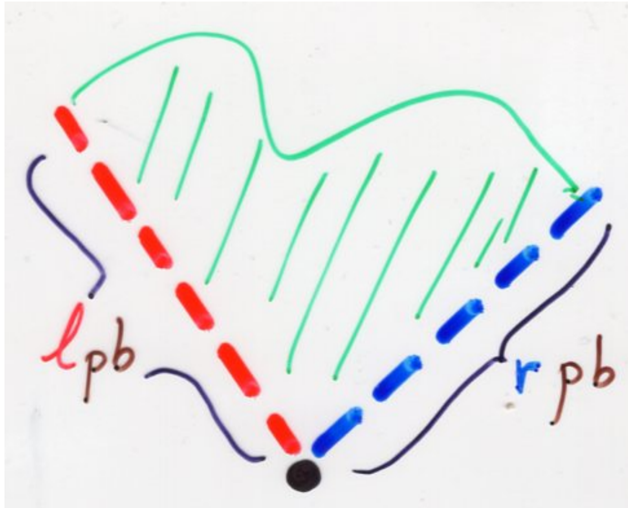


(n) vertices

$$\frac{i}{(2n-i)}\binom{2n-i}{n} \qquad \left[\bar{\alpha}^{(i)} + \bar{\alpha}^{(i-1)}\bar{\beta} + \dots + \bar{\alpha}\,\bar{\beta}^{(i-1)} + \bar{\beta}^{(i)}\right]$$

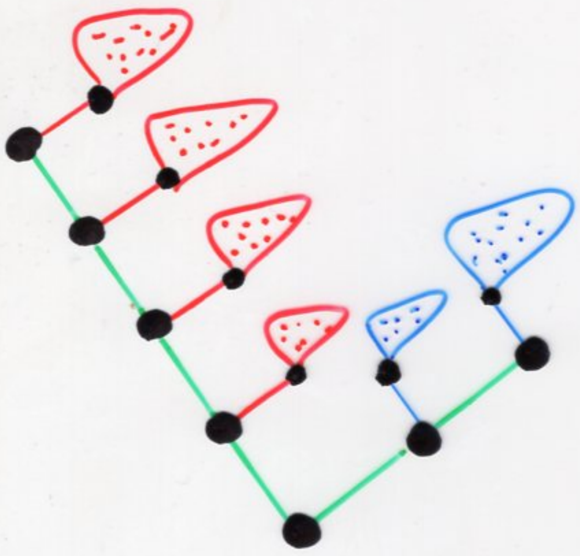
$$\frac{i}{(2n-i)}\binom{2n-i}{n}$$





$$Z_n = \sum_{1\leq i\leq n} \frac{i}{(2n-i)}\binom{2n-i}{n}\,\frac{\bar{\alpha}^{(i+1)}-\bar{\beta}^{(i+1)}}{\bar{\alpha}-\bar{\beta}}$$

partition function



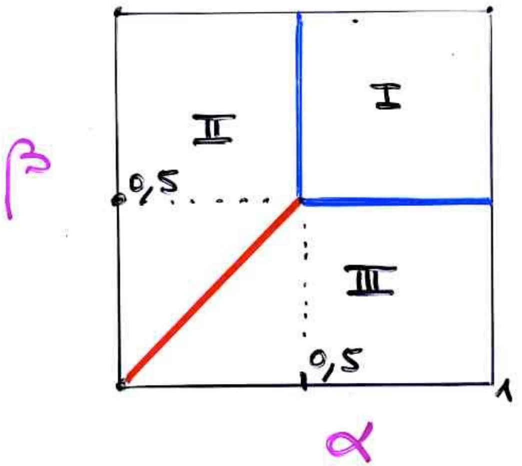
$$\sum_{n\geq 0} Z_n t^n = \frac{1}{(1-\bar{\alpha}C(t))} \times \frac{1}{(1-\bar{\beta}C(t))}$$

$C(t)$ generating function of Catalan numbers

→ BJC I Ch 1, Ch 2

$$C(t) = \sum_{n\geq 0} C_n t^n$$

$$y = 1 + t y^2$$



$n \to \infty$

$\rho = \langle \tau_i \rangle$ average occupation

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1-\beta$

$$\langle \tau_i \rangle_N = \frac{\langle W | (D+E)^{i-1} D (D+E)^{N-i} | V \rangle}{\langle W | (D+E)^N | V \rangle}$$

$$\begin{cases} DE = D + E \\ D|V\rangle = \bar{\beta}\, |V\rangle \\ \langle W| E = \bar{\alpha}\, \langle W| \end{cases}$$

seminal paper

"matrix ansatz"

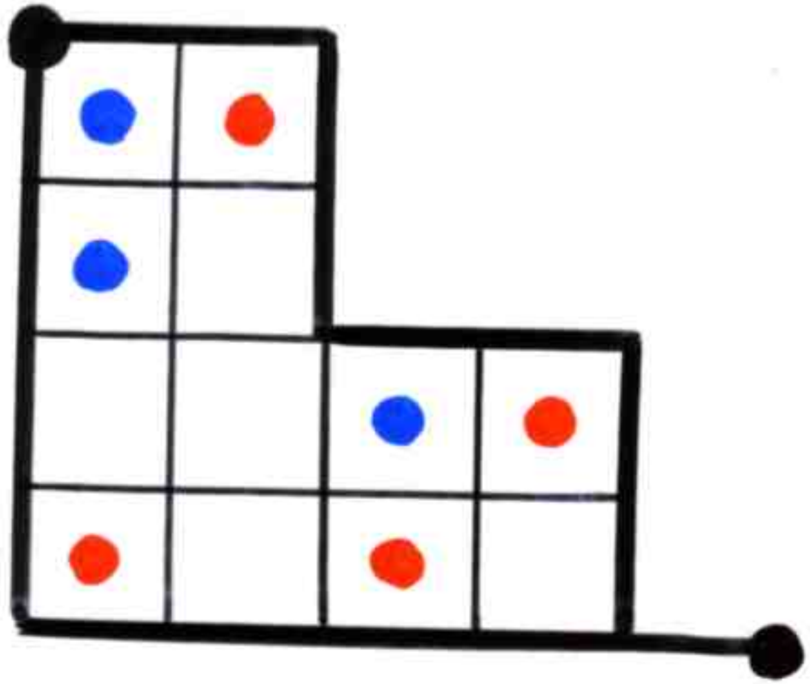
Derrida, Evans, Hakim, Pasquier (1993)

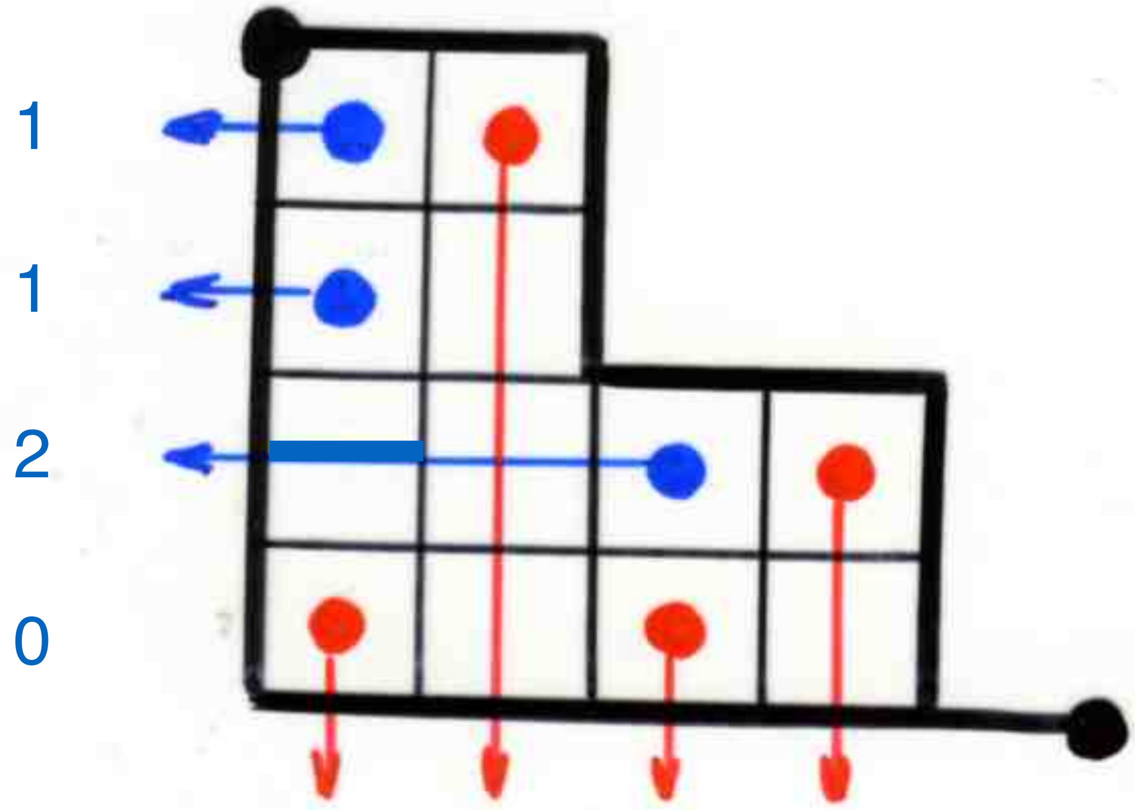
$q=0$ TASEP (α, β)

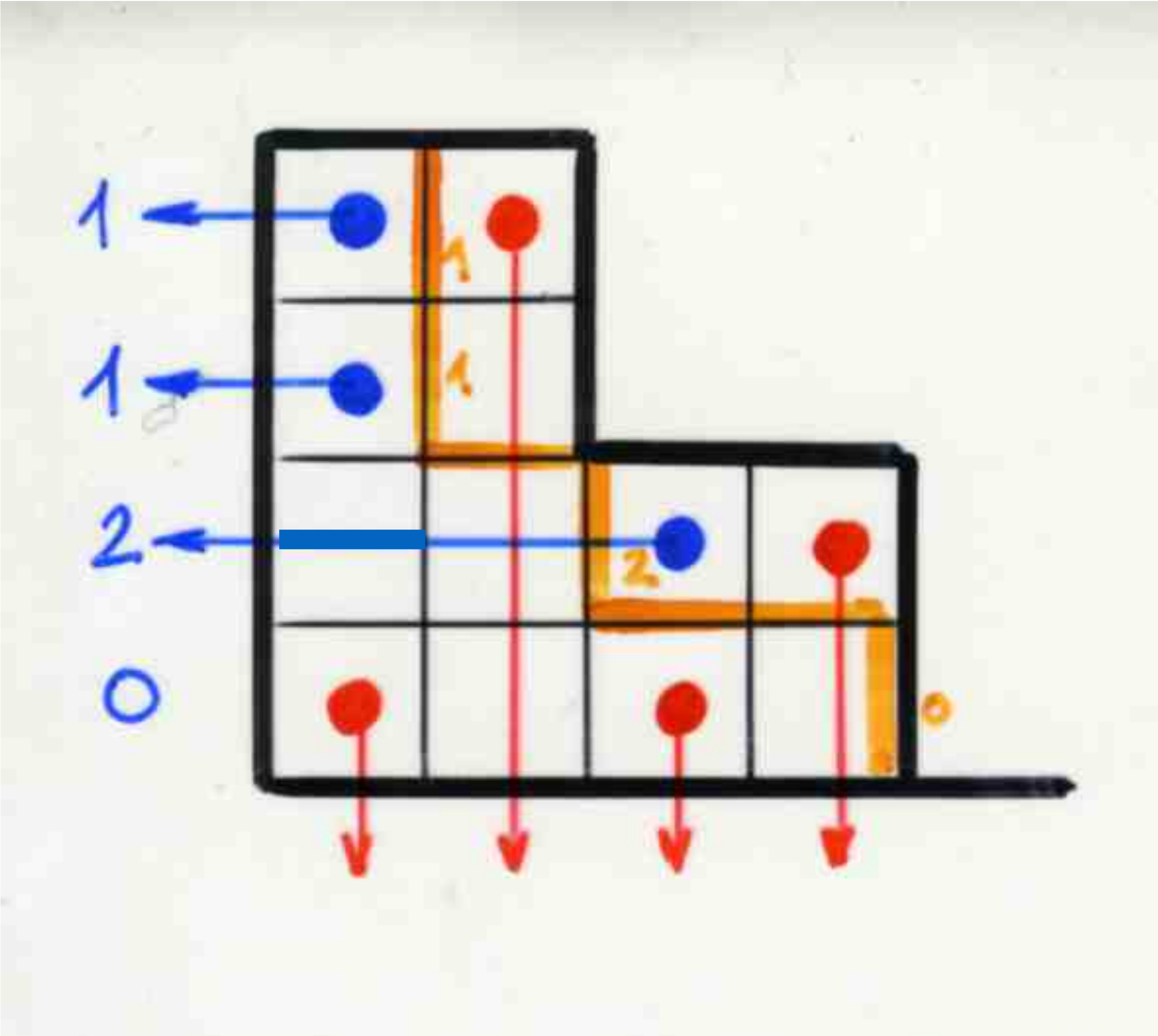
bijection

Catalan alternative tableaux

pair of paths



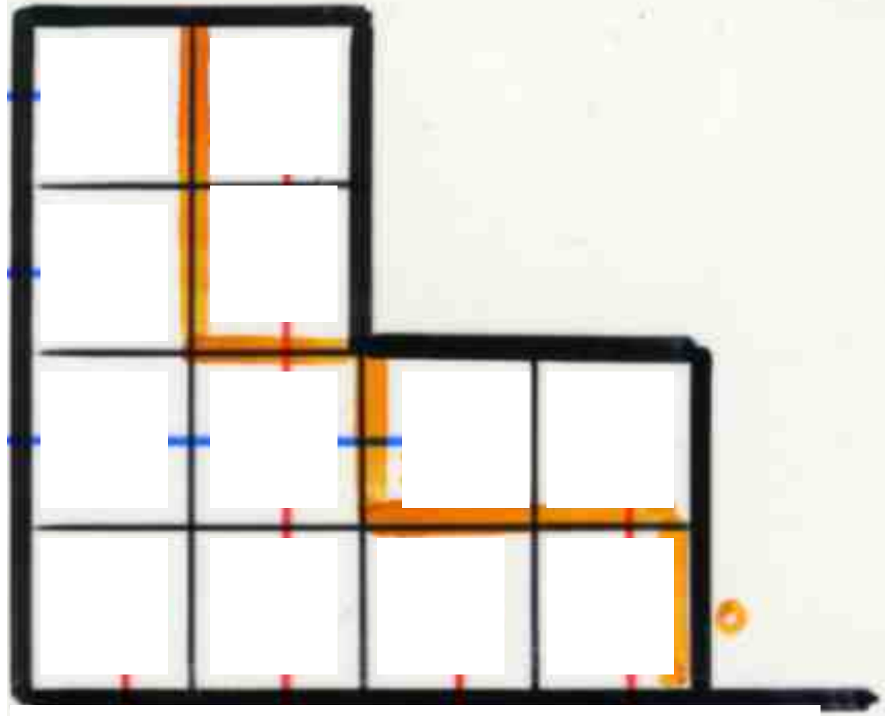

1
1
2
0

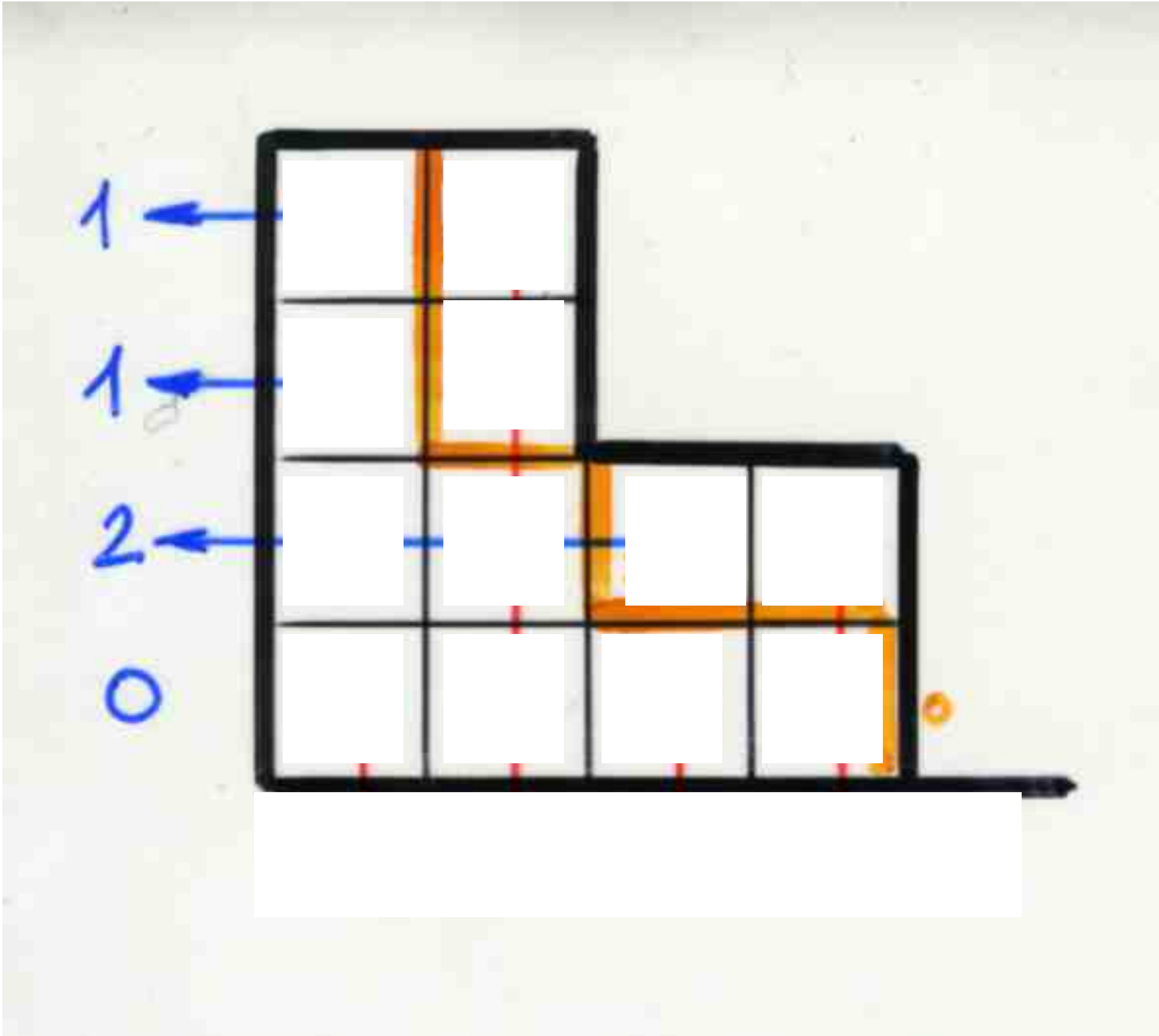

1
1
2
0
1
1
2
0

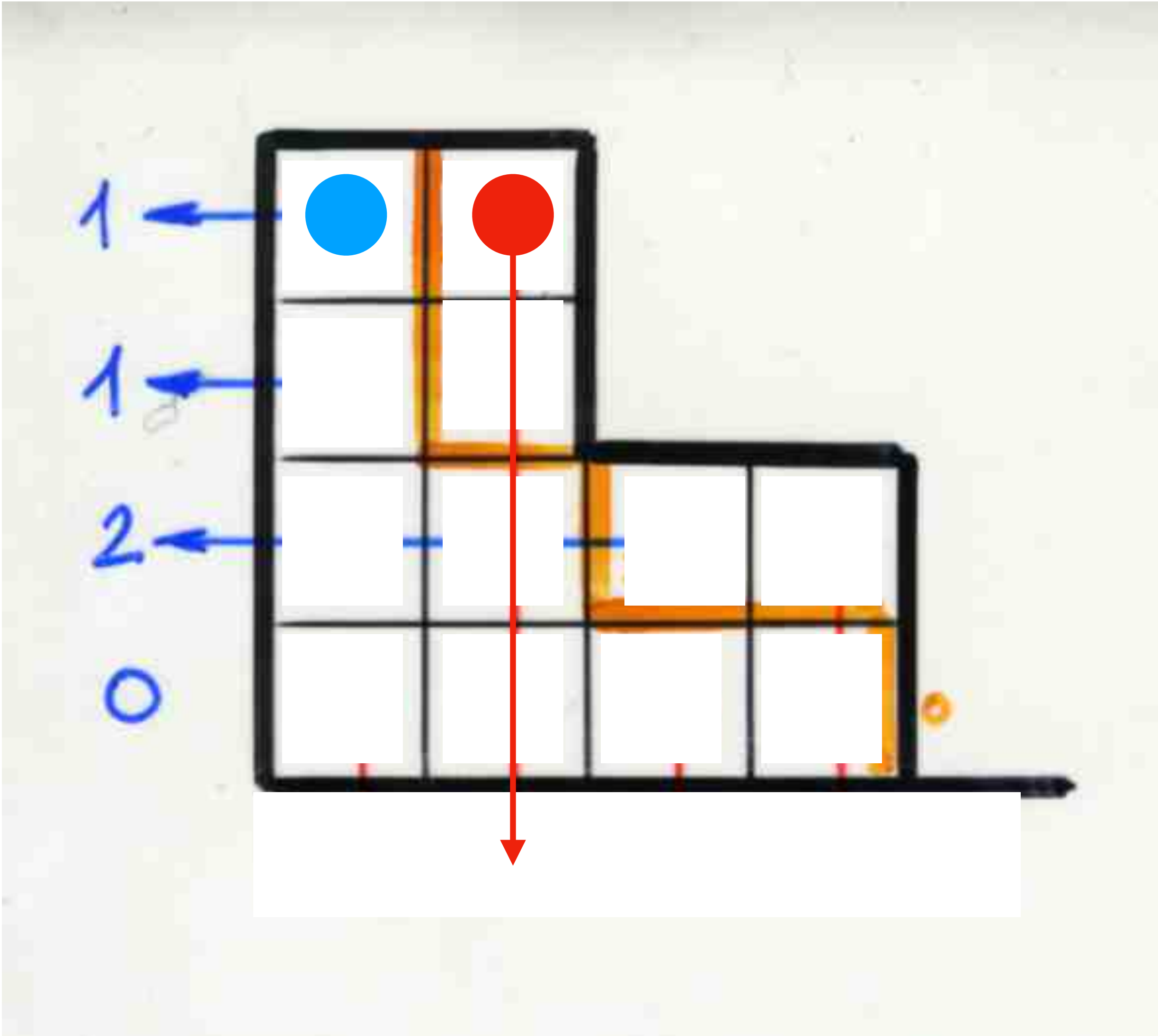
reverse bijection

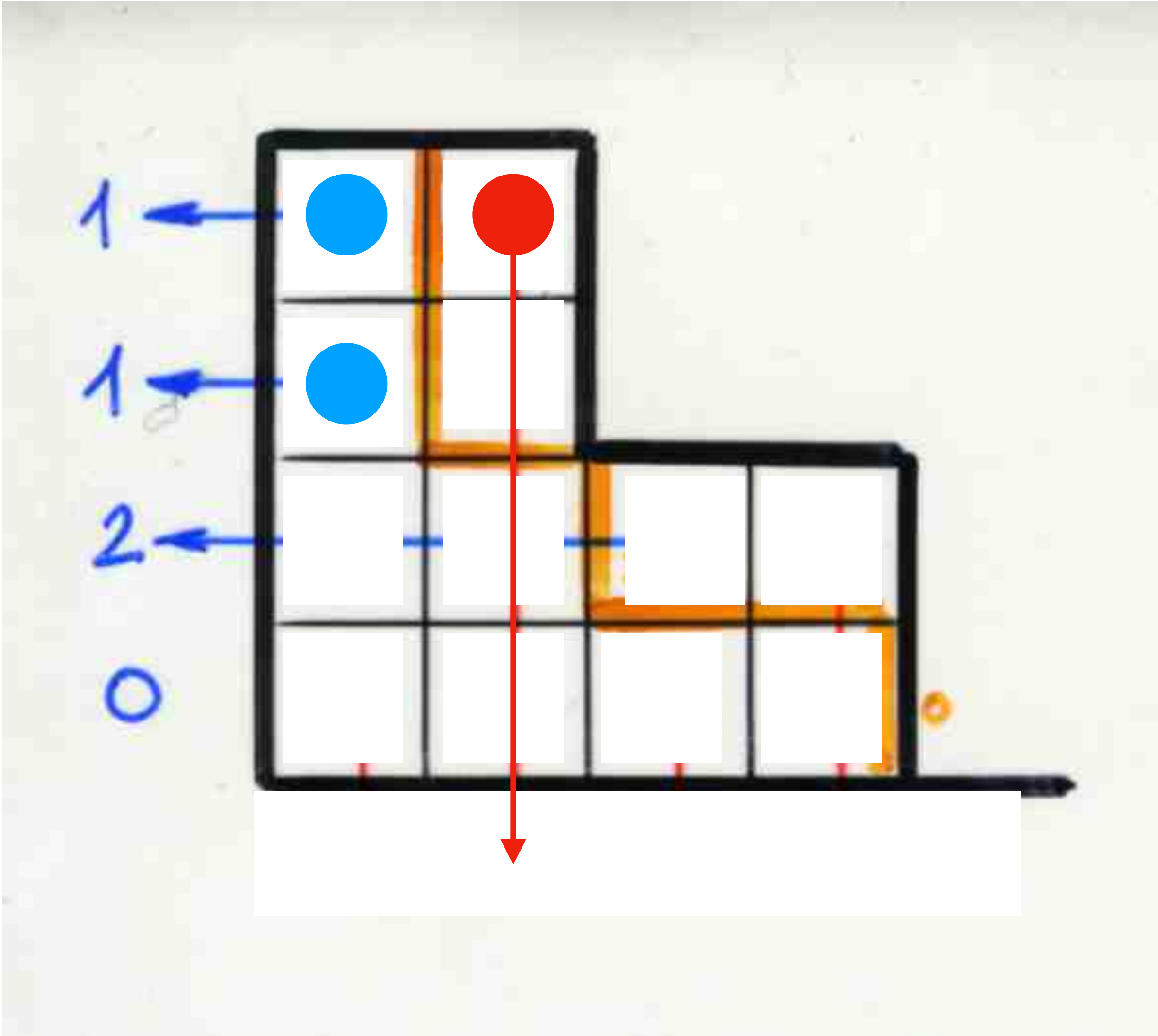
pair of paths

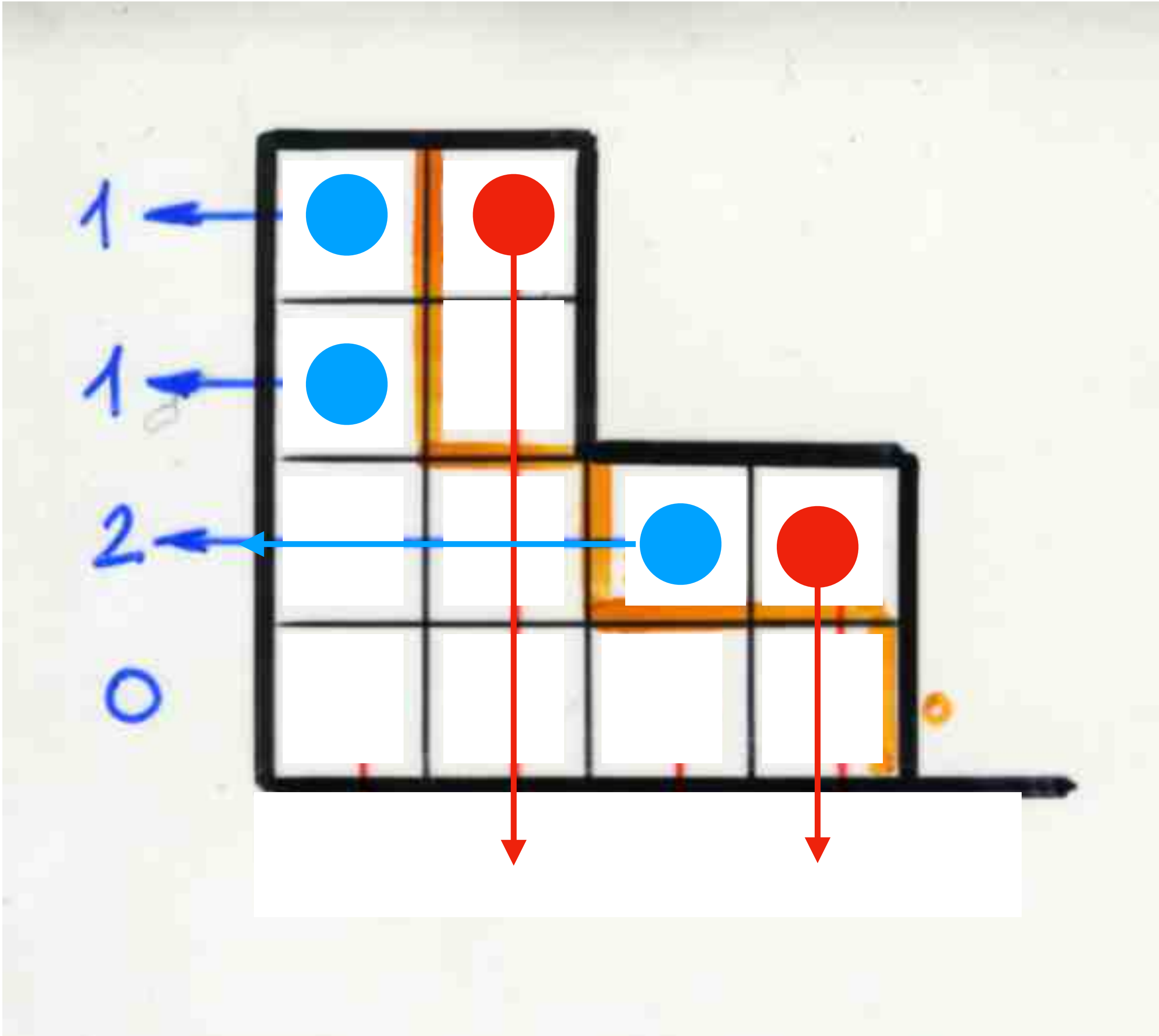
Catalan alternative tableaux

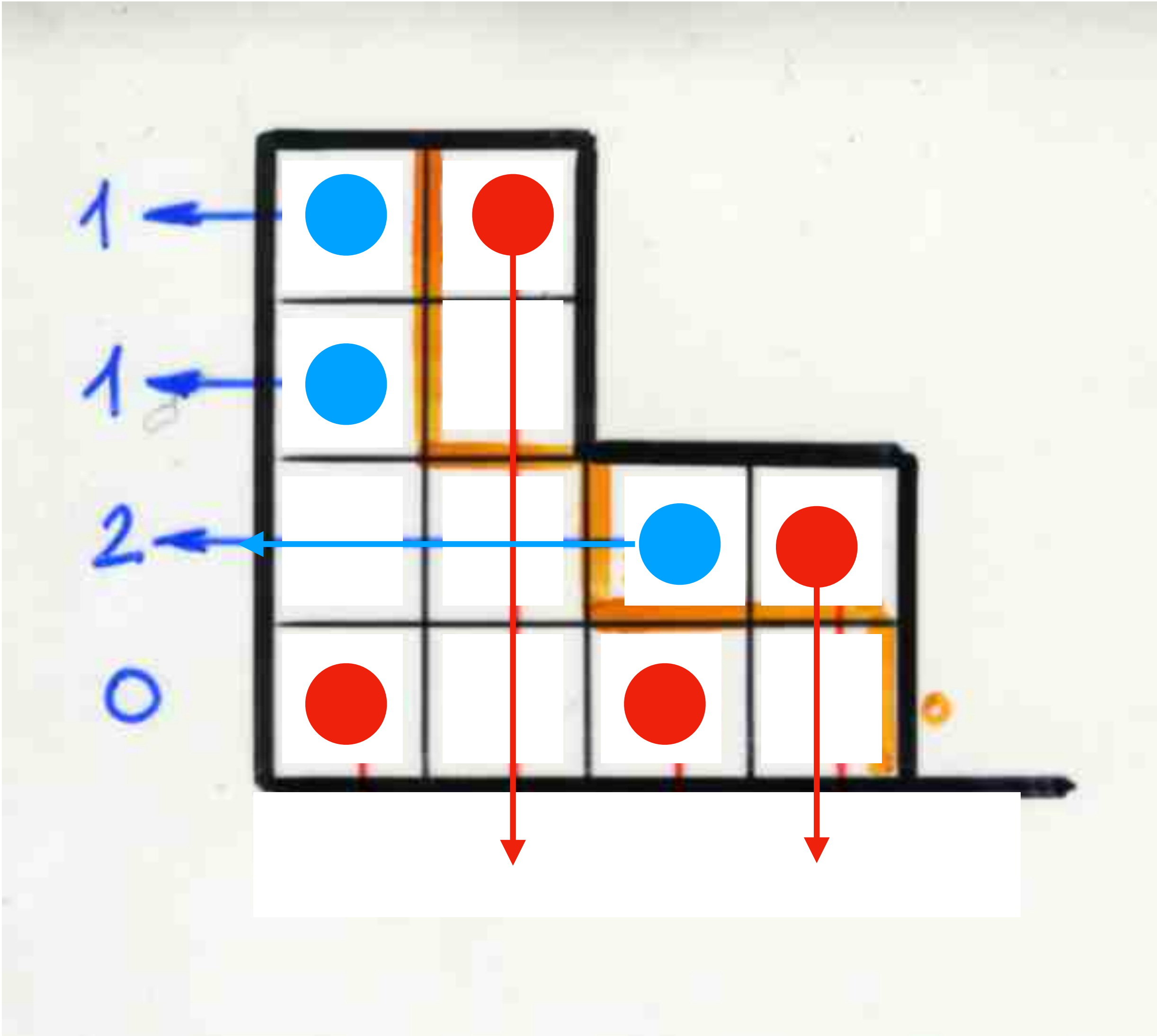


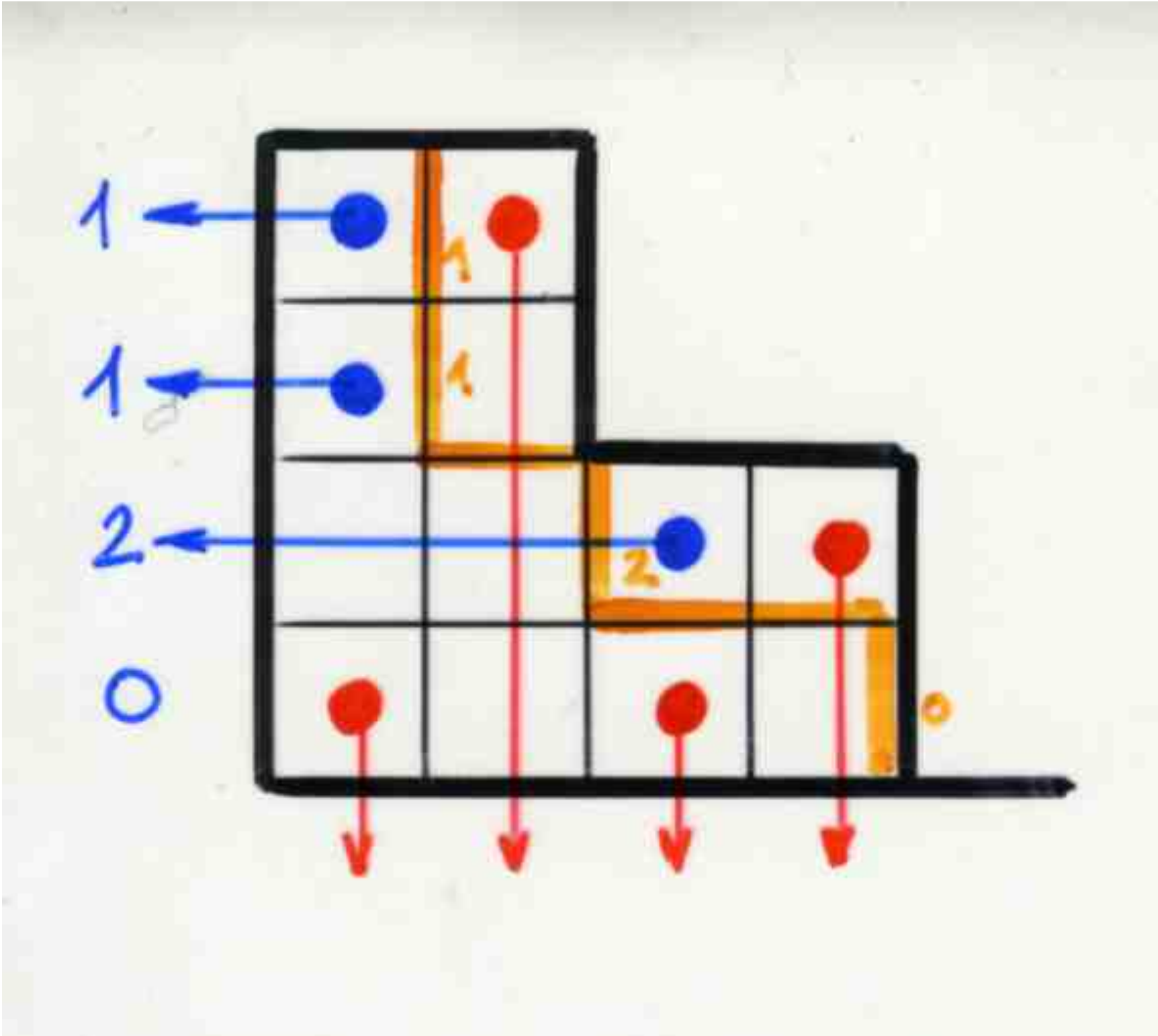

1
1
2
0

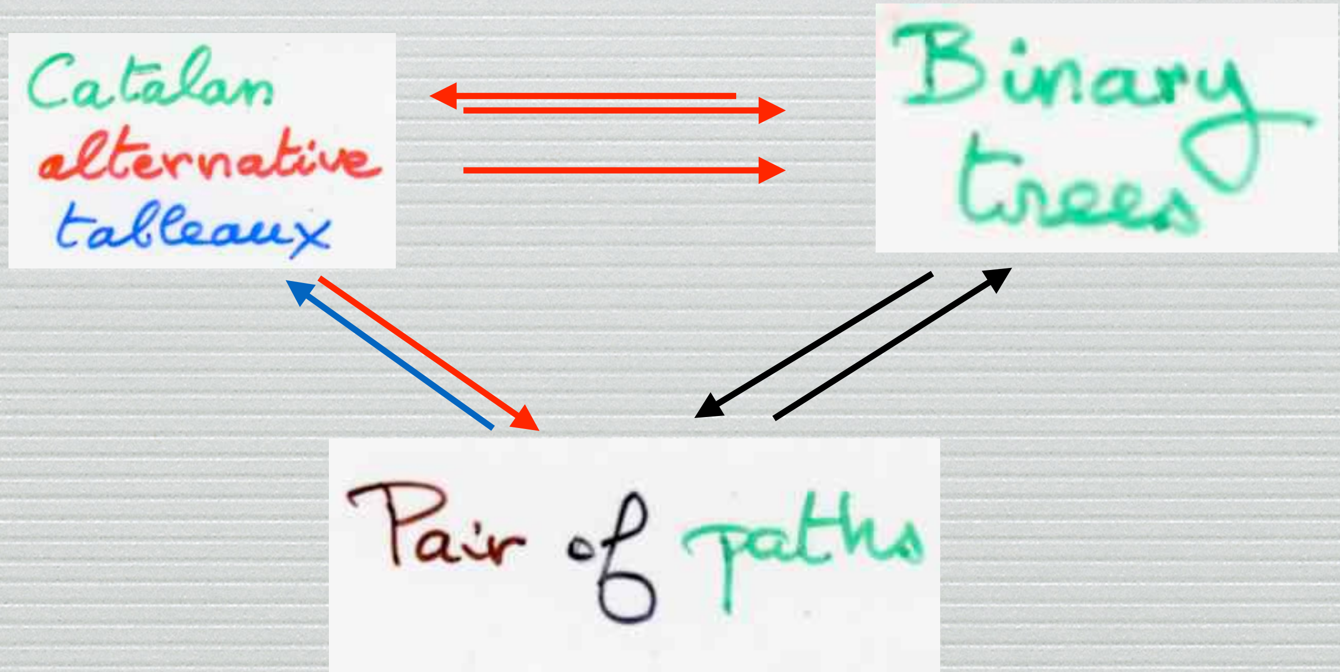

1
1
2
0


1
1
2
0

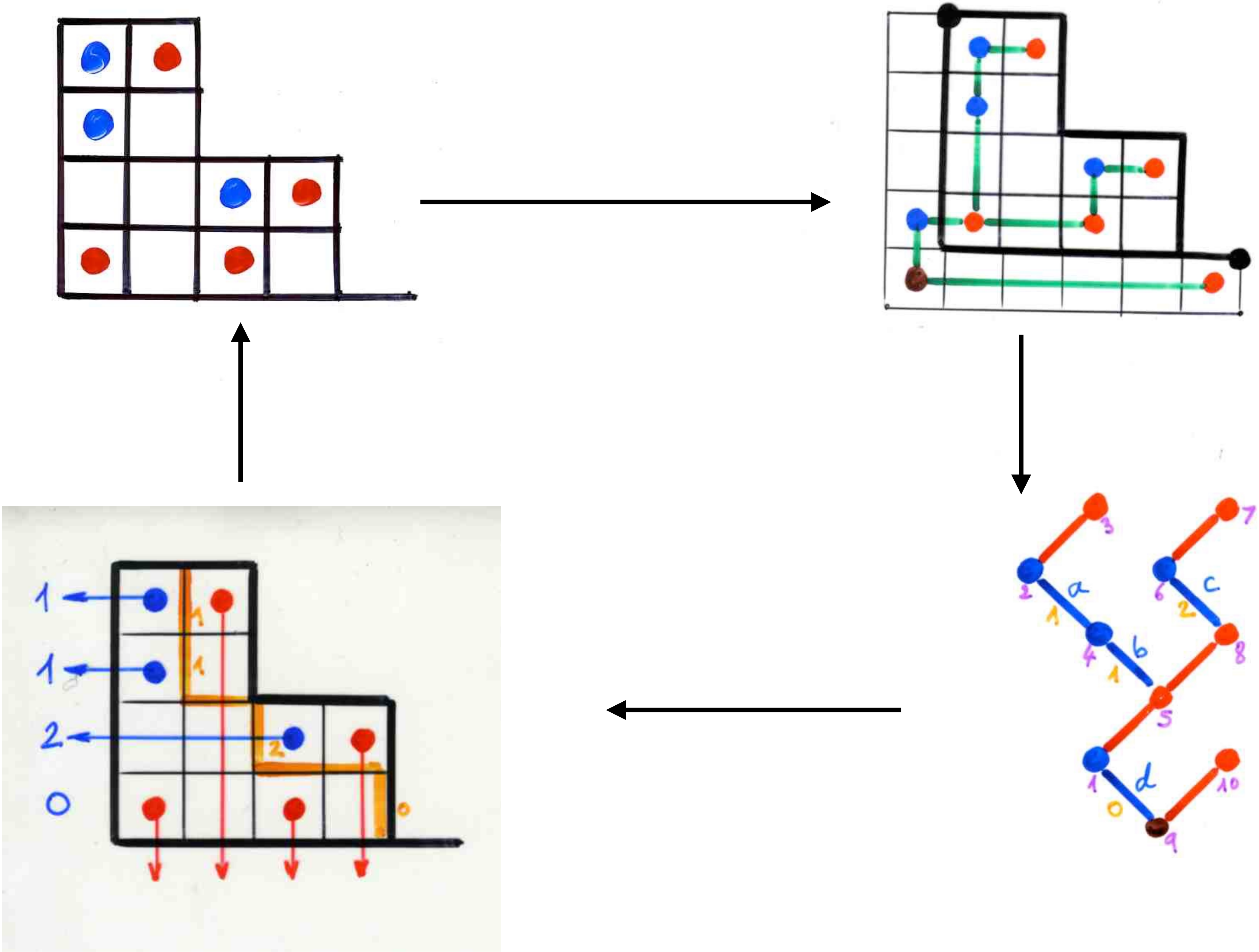

1
1
2
0


1
1
2
0


1
1
2
0
1
1
2
0



commutative diagram



1
1
2
0
a
b
c
d
1
2
3
4
5
6
7
8
9
10

The Adela bijection

demultiplication
In the PASEP algebra

PASEP algebra

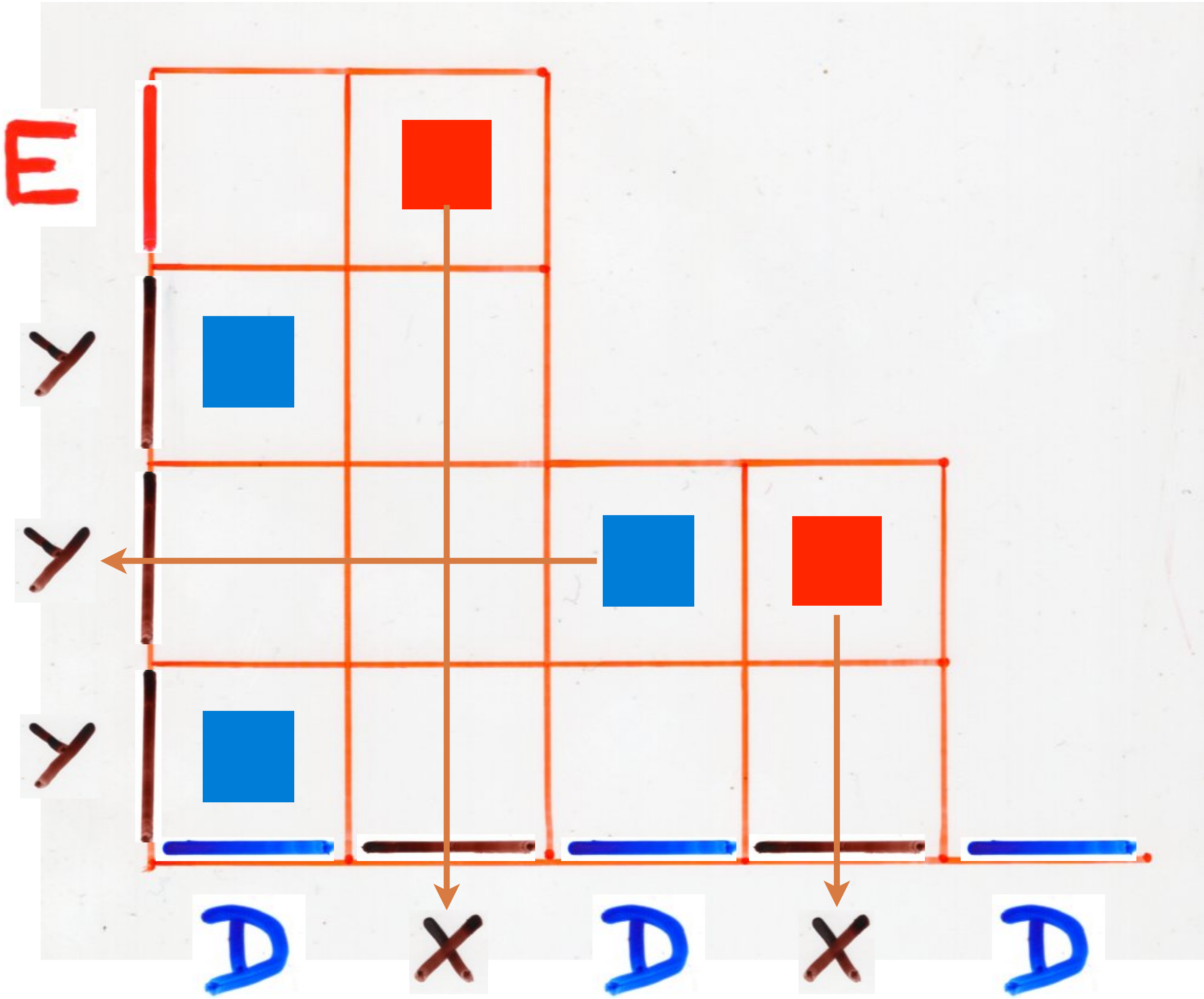
$$Q\begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

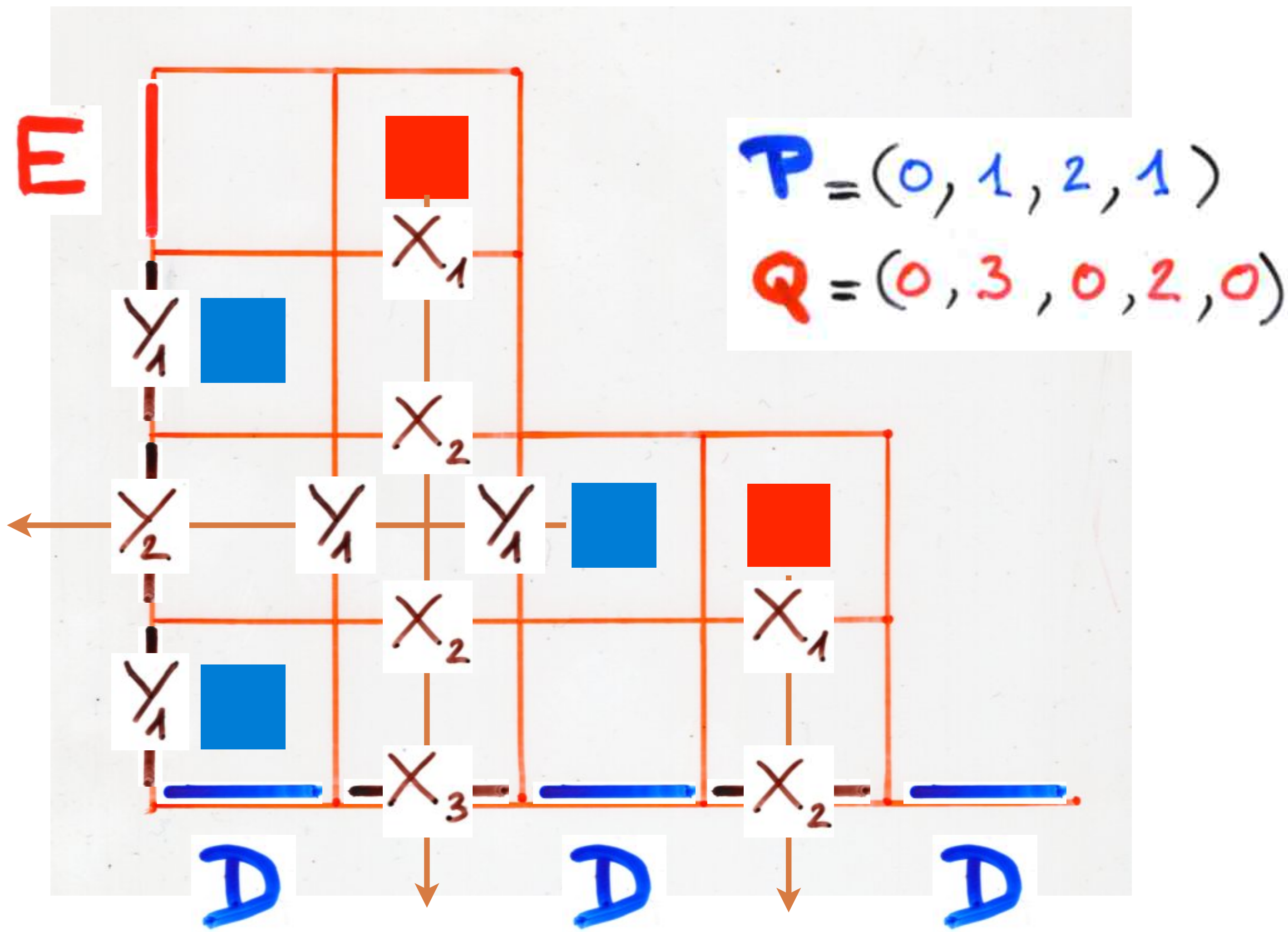
see Ch 2c, p3-8
duplication of equations in quadratic algebrea
Ch 2c, p9-15
duplication in the PASEP algebra

$$DE = ED + EX_1 + Y_1 D$$

$$\begin{cases} X_1 E = E X_2 \\ \cdots \\ X_i E = E X_{i+1} \\ \cdots \end{cases} \qquad \begin{cases} D Y_1 = Y_2 D \\ \cdots \\ D Y_i = Y_{i+1} D \\ \cdots \end{cases}$$

$$X_i Y_j = Y_j X_i$$


E
Y
Y
Y
D
X
D
X
D



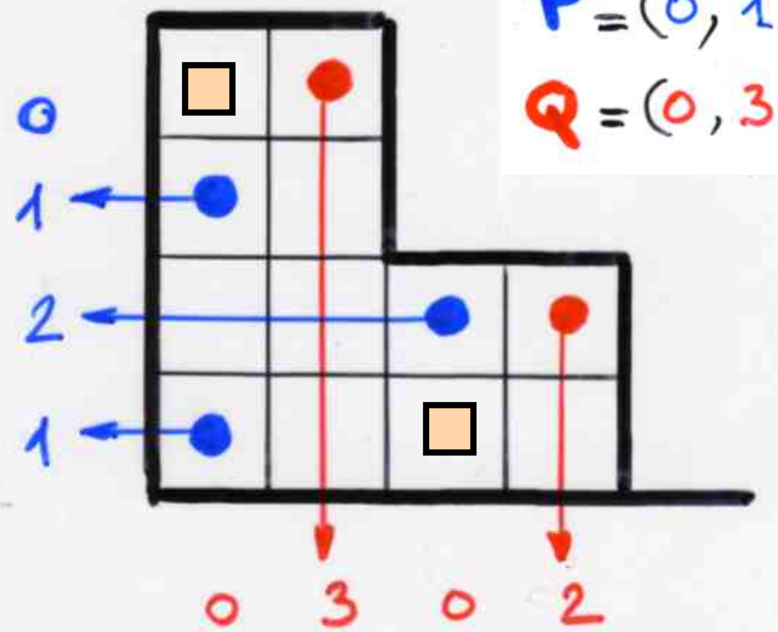
E
P = (0, 1, 2, 1)
Q = (0, 3, 0, 2, 0)
X1
X2
X2
X3
X1
X2
Y1
Y2
Y1
Y1
Y1
D
D
D

Adela bijection

$$\text{Adela}(T) = (P, Q)$$

$$P(T) = (a_1, a_2, \ldots, a_k)$$

$$Q(T) = (b_1, \ldots, b_\ell)$$



$P = (0, 1, 2, 1)$

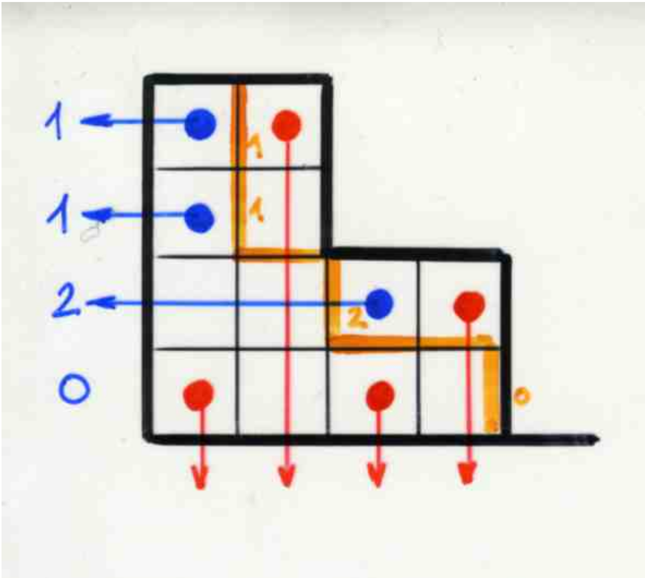
$Q = (0, 3, 0, 2, 0)$

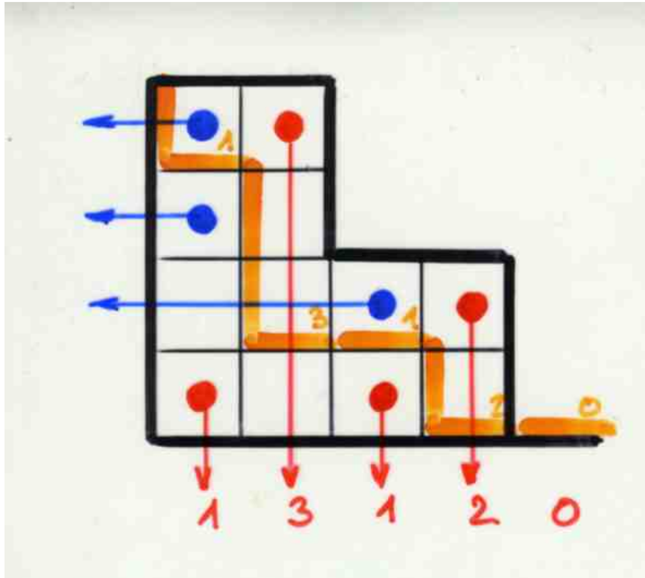
$a_i = \begin{cases} 0 & \text{if no } \bullet \text{ in row } i \\ 1 + \text{number of cells} & \text{in row } i \end{cases}$

$b_j = \begin{cases} 0 & \text{if no } \bullet \text{ in the } j^{th} \text{ column} \\ 1 + \text{number of cells} & \text{in the } j^{th} \text{ column} \end{cases}$

the Catalan case

Adela(T) = (P, Q)

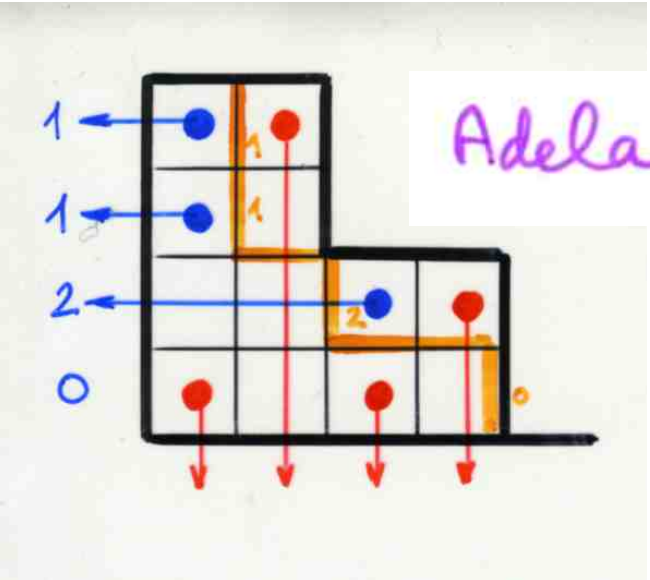




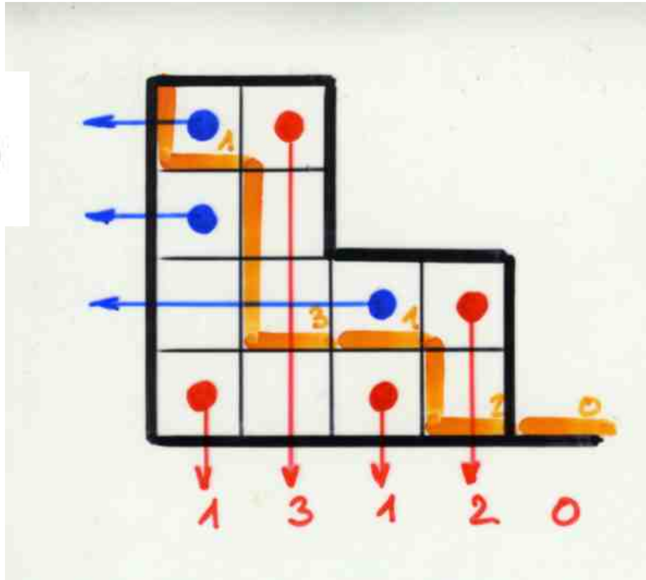
Catalan alternative tableaux ⟷ Pair of paths

The "Adela duality"

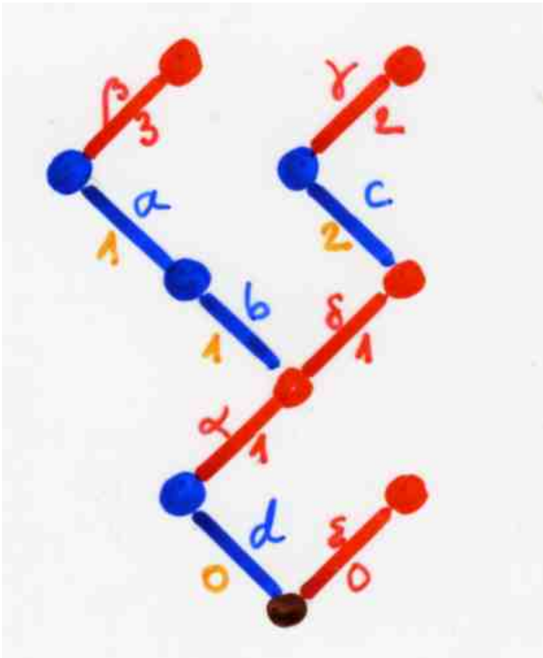
P (T) ⟷ Q (T)



$\text{Adela}(T) = (P, Q)$



the Catalan case



Adela duality

a	b	c	d
1	1	2	0

α	β	γ	δ	ε
1	3	1	2	0

right height of left edges 

inorder (= symmetric order)

left height of right edges 

TASEP with Catalan tableaux

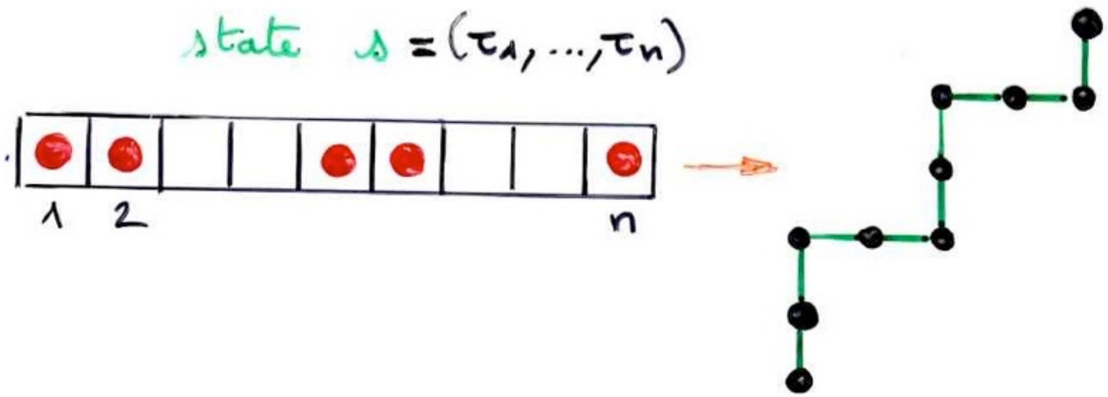
relation with Shapiro-Zeilberger interpretation

TASEP (α, β)

$\alpha = \beta = 1$

Shapiro, Zeilberger (1982)

state $s = (\tau_1, \dots, \tau_n)$



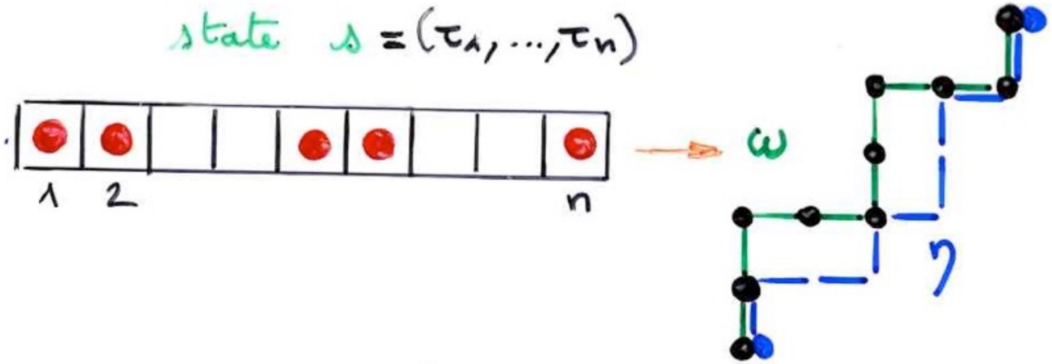
$\mathcal{P}_n(s) =$

TASEP (α, β)

$$\alpha = \beta = 1$$

Shapiro, Zeilberger (1982)

state $s = (\tau_1, \dots, \tau_n)$



$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

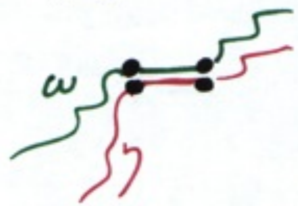
$s = (\tau_1, \dots, \tau_n) \longrightarrow \omega$ path

state

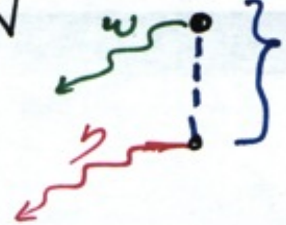
$$P_n(s; \alpha, \beta) = \frac{1}{Z_n} \sum_{\eta} \alpha^{f(\omega,\eta)} \beta^{g(\omega,\eta)}$$

path

$f(\omega,\eta)$ = nb of "contacts" horizontal



$g(\omega,\eta)$ = nb of steps N



$$\mathcal{P}_{\{\lambda_1,\ldots,\lambda_k\}}(\alpha,\beta) = \det A_\lambda^{\alpha,\beta}$$

O. Mandelshtam
(2015)

$$A_\lambda^{\alpha,\beta} = \left(A_{i,j}\right)$$

$$A_{i,j} = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \beta^{j-i}\alpha^{\lambda_i-\lambda_{j+1}}\left(\binom{\lambda_j+1}{j-i} + \binom{\lambda_j+1}{j-i+1}\right) & \\ \quad + \beta^{j-i}\alpha^{\lambda_i-\lambda_j}\displaystyle\sum_{\ell=0}^{\lambda_j-\lambda_{j+1}-1}\alpha^{\ell}\left(\binom{\lambda_j-\ell-1}{j-i-1} + \binom{\lambda_j-\ell-1}{j-i}\right) & \text{for } j \geq i \end{cases}$$

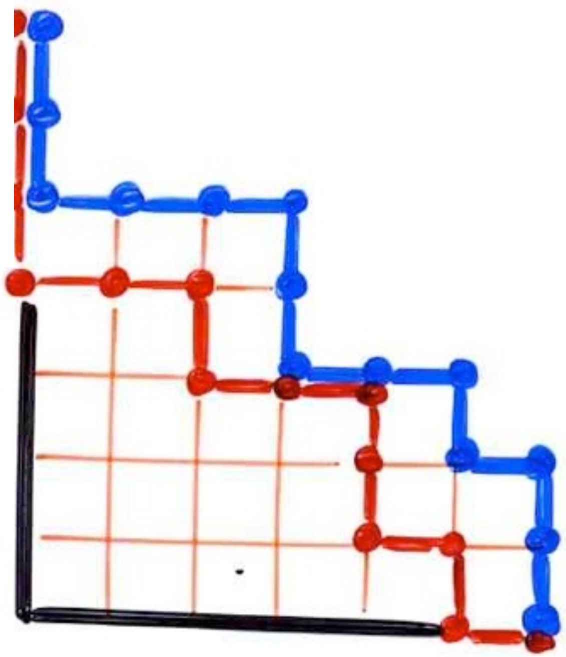
Kreweras's determinant

Narayana (1955)

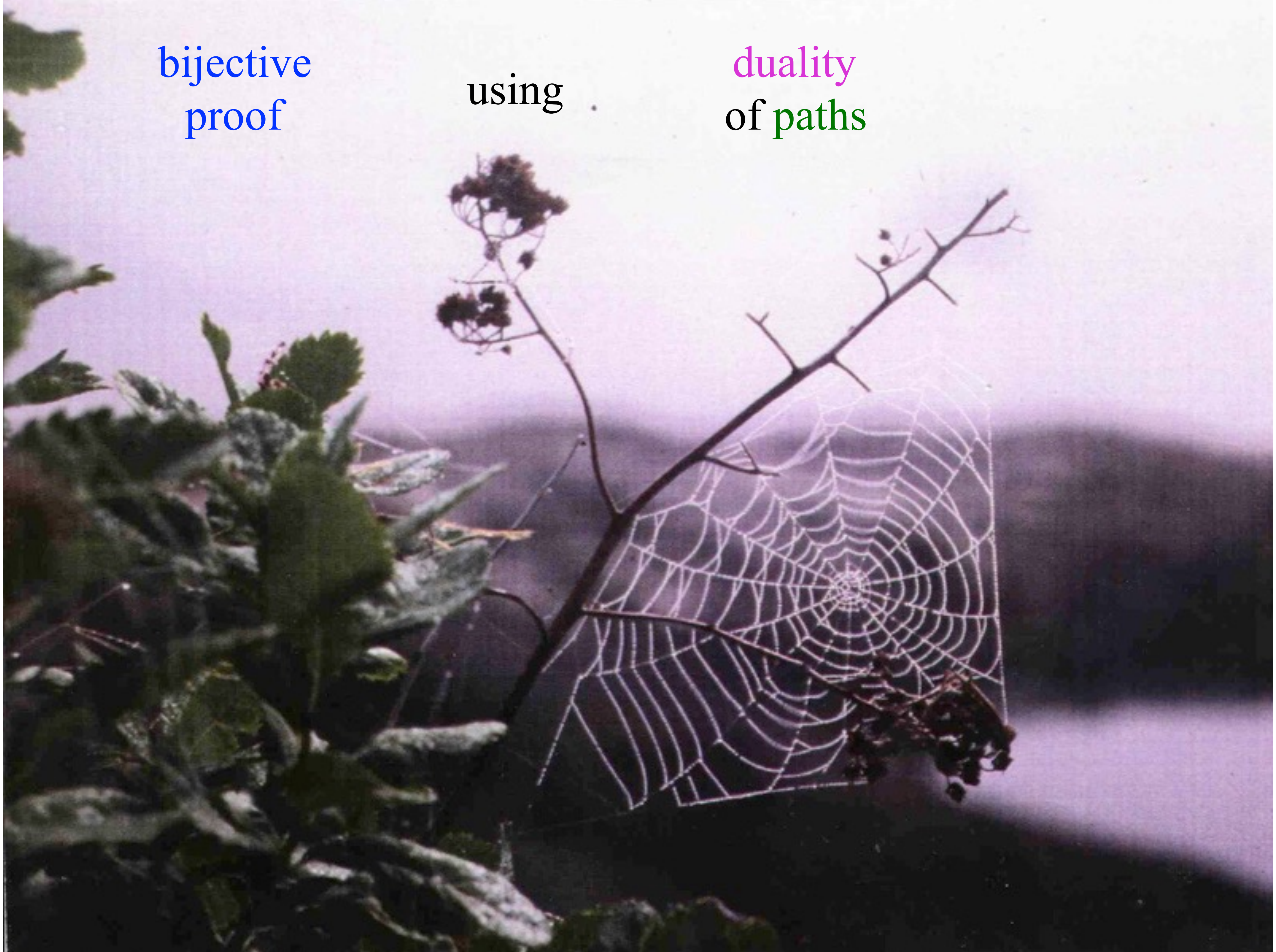
$$\det \begin{pmatrix} \lambda_i + 1 \\ j - i + 1 \end{pmatrix}_{1 \le i, j \le k}$$

k = nb of 0's in s

λ_i = nb of 1's to the left of the i^{th} zero



$\lambda = (0, 0, 3, 3, 5, 6, 6)$


bijective proof
using
duality of paths

bijection with dual configuration of paths
giving a bijective proof of Narayana determinant