Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 1
RSK
The Robinson-Schensted-correspondence
(Ch1c)

IMSc, Chennai

January 18, 2018

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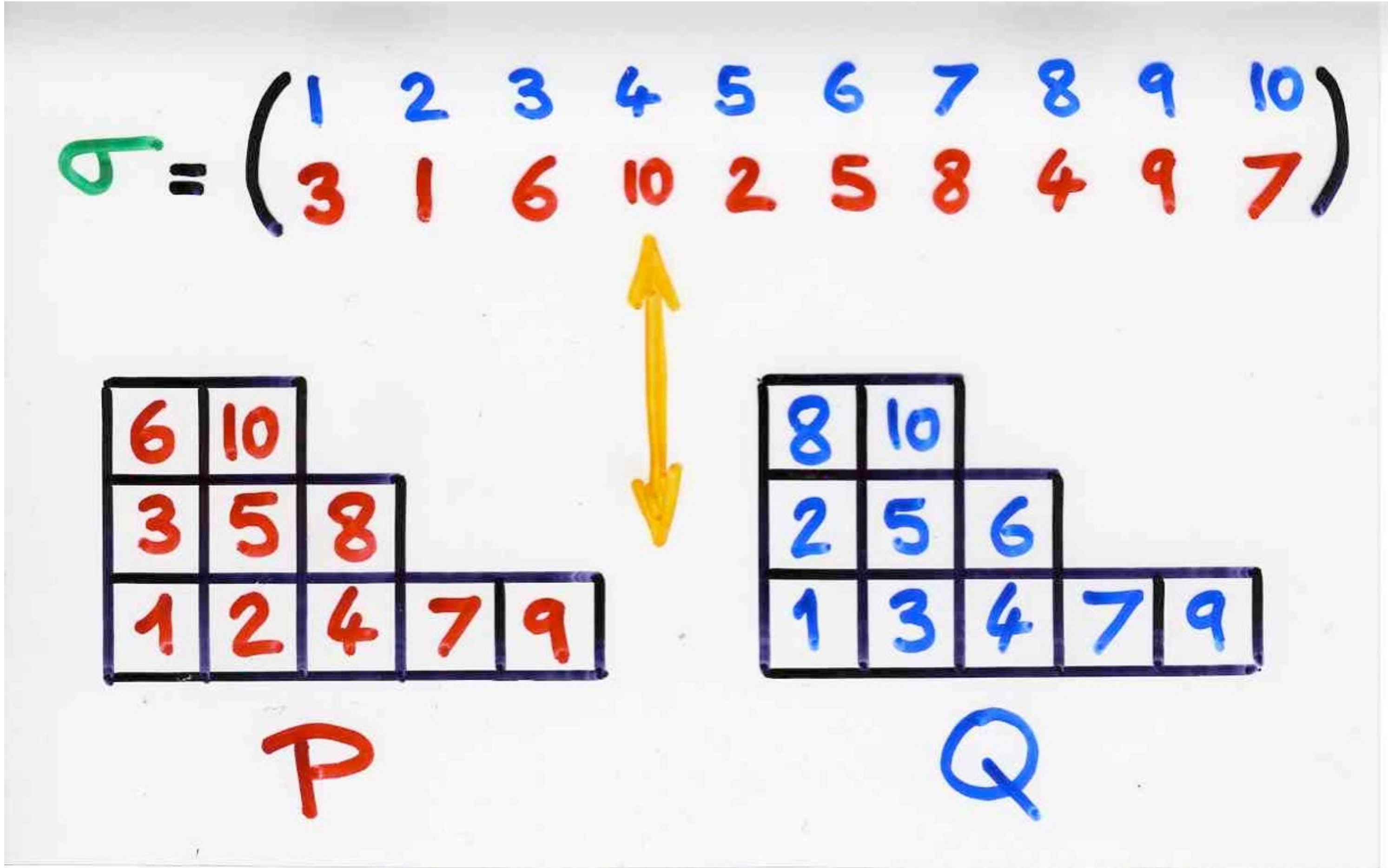
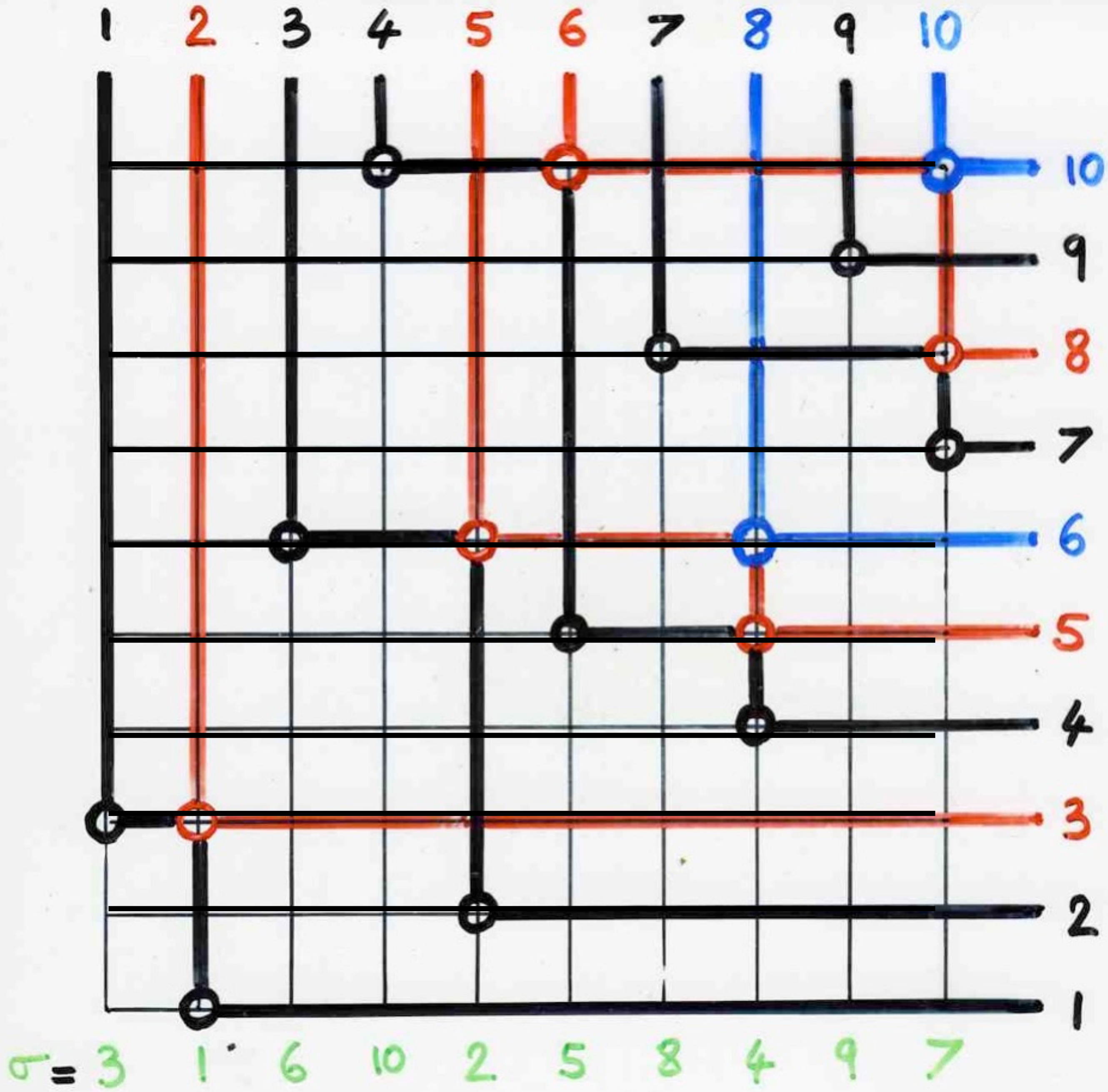
From Ch 1a, 1b:

The Robinson-Shensted correspondence

- Schensted's insertions
- geometric version with "shadow lines »
- Fomin "local rules" or "growth diagrams »

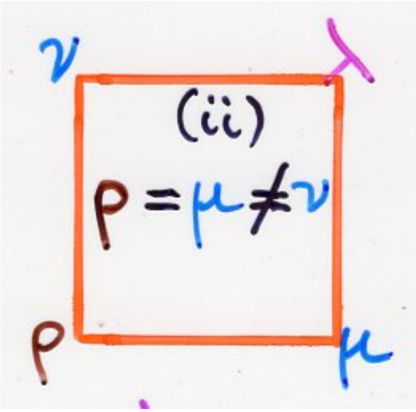


The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

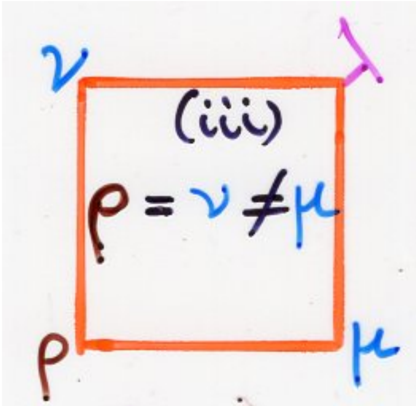


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ= 3 1 6 10 2 5 8 4 9 7

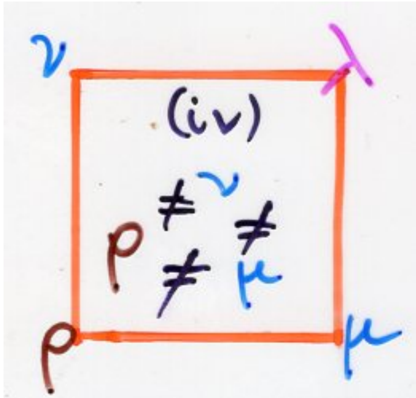
"local rules"



$\lambda = \nu$

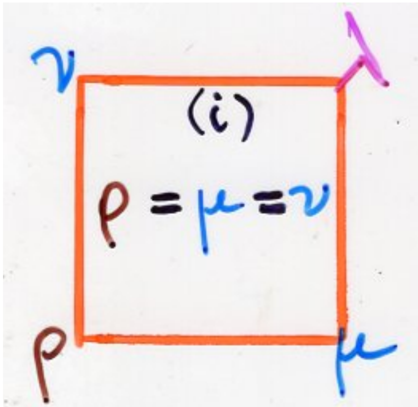


$\lambda = \mu$

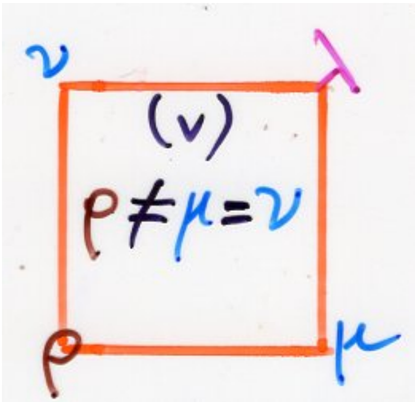


$\lambda = \mu \cup \nu$

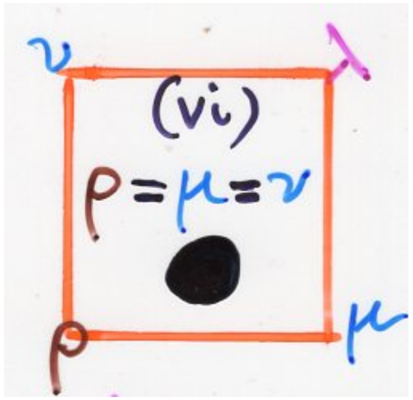
$\mu \neq \nu$



$\lambda = \rho$

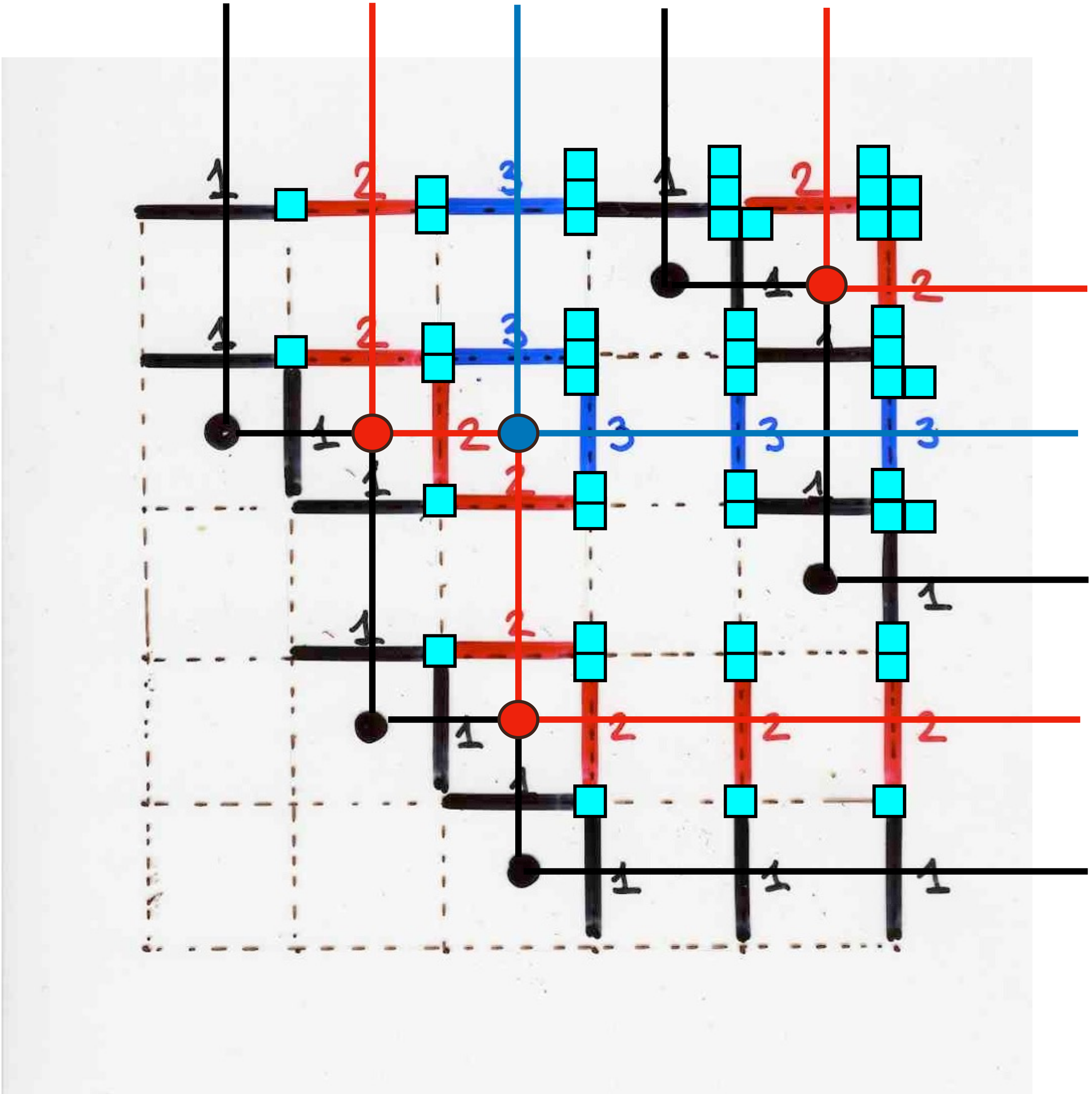


$\lambda = \left\{ \begin{matrix} \mu \\ \nu \end{matrix} \right. + (i+1)$

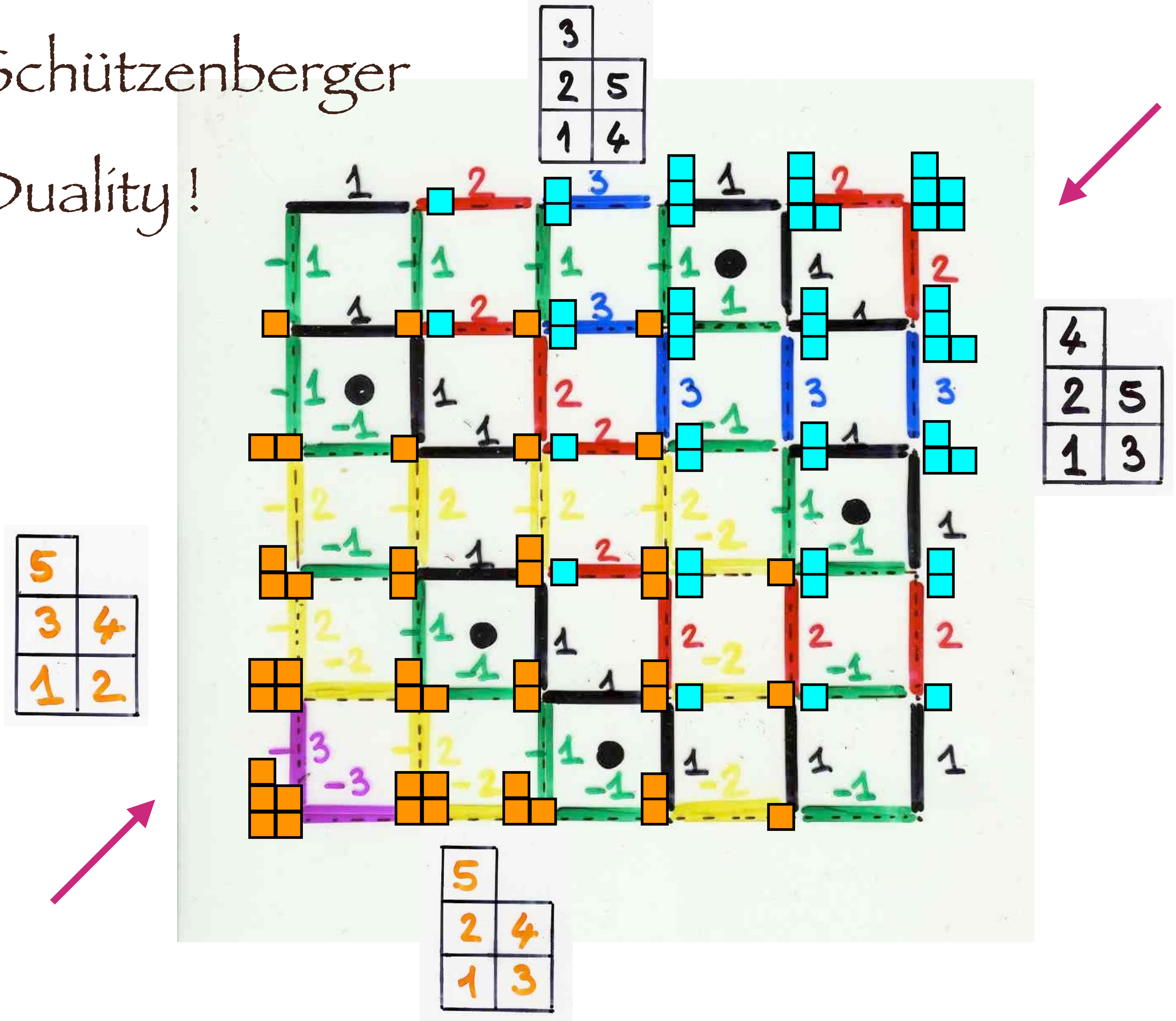


$\lambda = \left\{ \begin{matrix} \rho \\ \mu \\ \nu \end{matrix} \right. + (1)$

$\mu = \nu$



Schützenberger Duality !



"The cellular ansatz"

quadratic algebra Q

$UD = DU + Id$

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

bijections

RSK

$\longleftrightarrow$

representation of Q by combinatorial operators

pairs of Young tableaux

(i) first step

(ii) second step

commutations

rewriting rules

planarization

an example

Heisenberg operators U, D

creation and annihilation operators

quantum mechanics

$UD = DU + I_d$

commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering in physics

The monomials $\{D^i U^j\}_{i,j \geq 0}$
form a basis of the
Weyl-Heisenberg algebra

$$Q = \mathbb{C}\langle U, D \rangle \big/ J$$

non-commutative polynomials
in variable U and D
(free associative algebra)

J ideal generated by
the relation $UD = DU + I$

$UD = DU + Id$

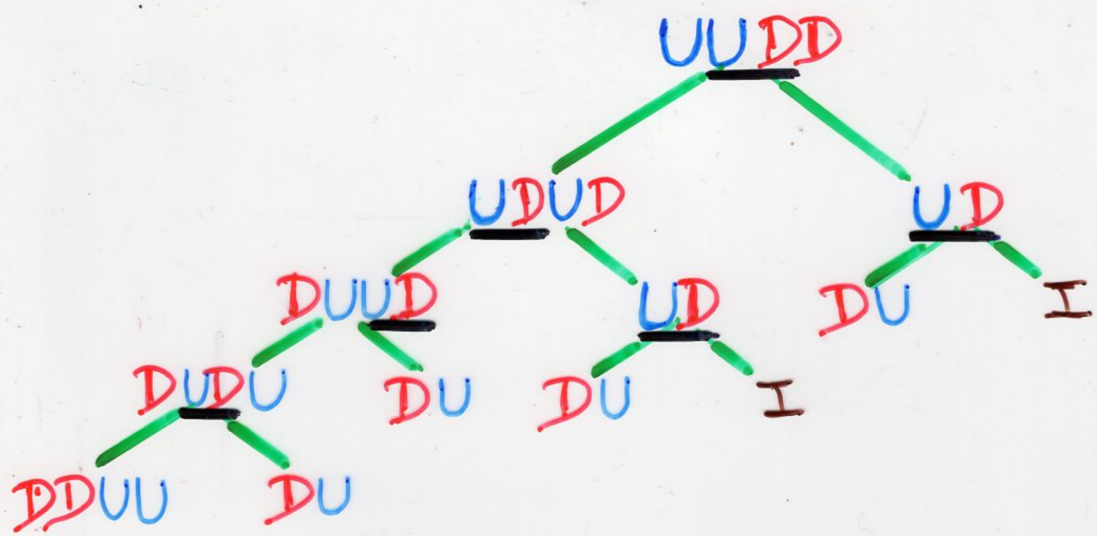
commutations

$UD \to DU$ $UD \to Id$

rewriting rules

$UUDD$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2UD \\
&= (DUDU + DU) + 2(DU + Id) \\
&= (DDUU + 2DU) + 2(DU + Id) \\
&= DDUU + 4DU + 2\,Id
\end{aligned}
$$



$$U^2D^2 = D^2U^2 + 4DU + 2I$$

this polynomial is independant
of the order of the substitutions

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

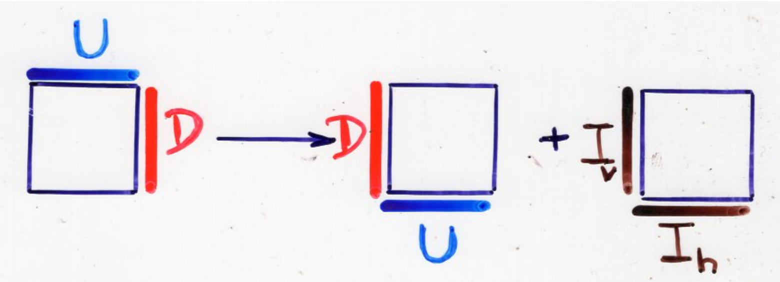
permutations

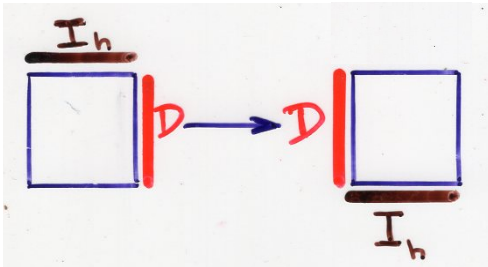
Planarization of the rewriting rules

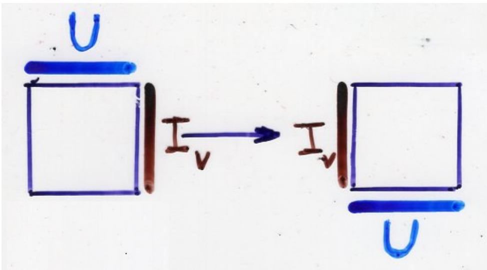
$UD = DU + Id$ commutations

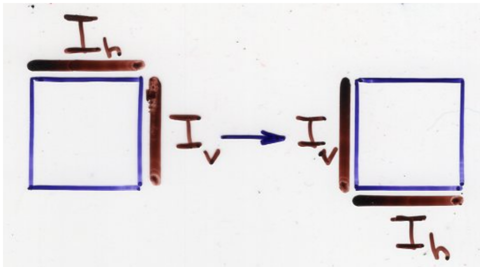
$UD \to DU$ $UD \to Id$ rewriting rules

planarization of the rewriting rules







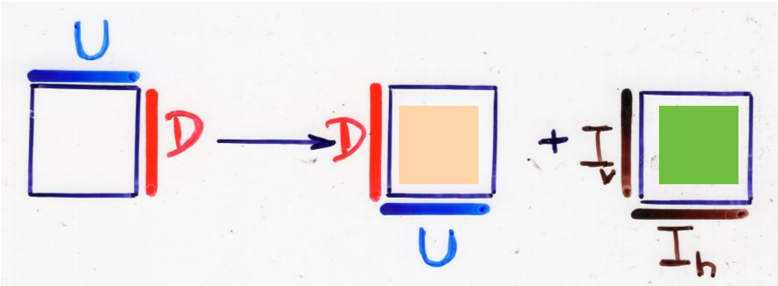


homogenization of the system of commutations relations

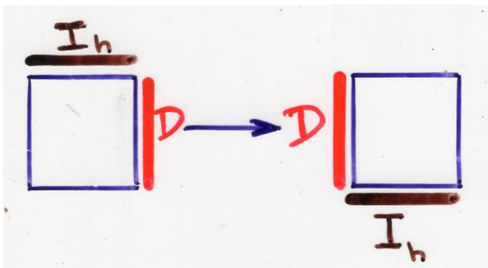
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

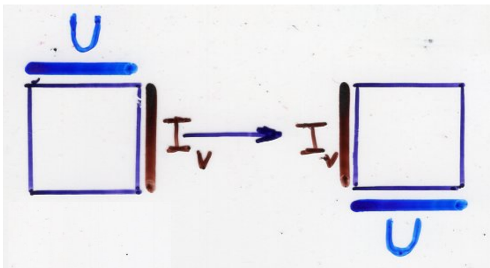
$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

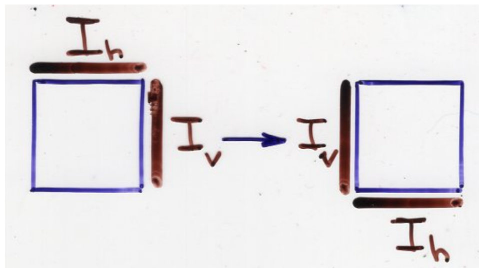
rewriting rules

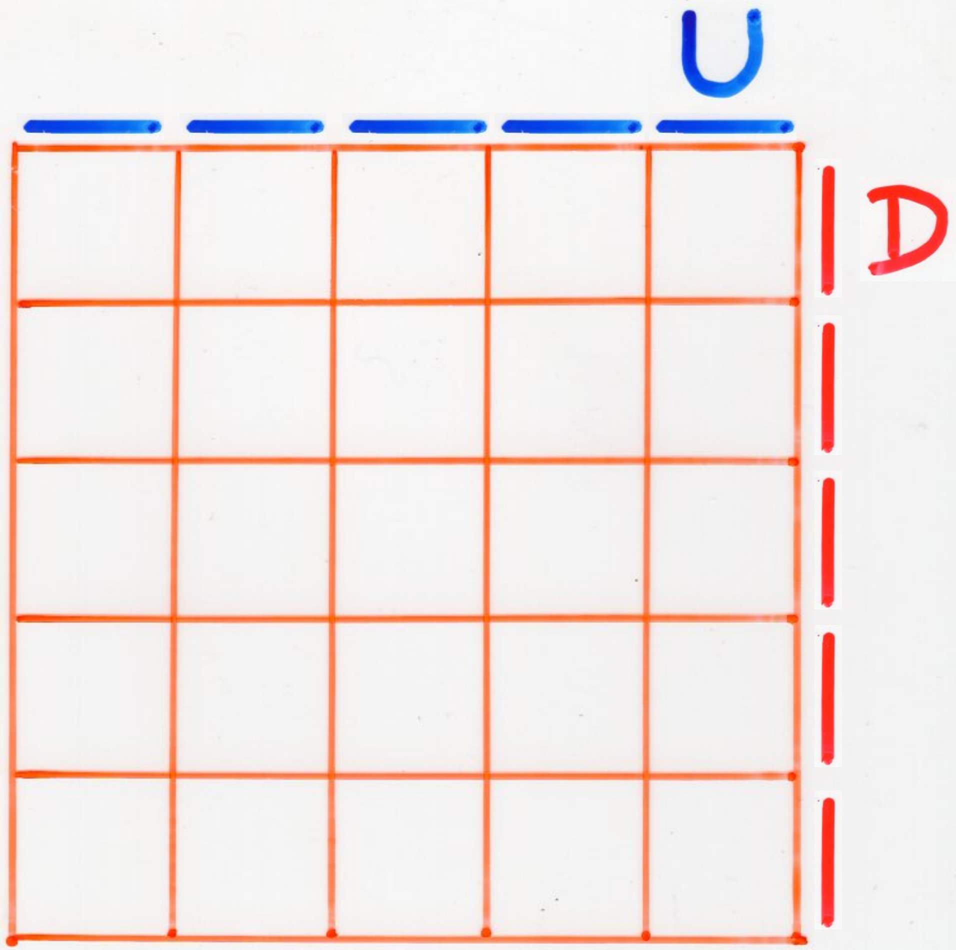


"planarization" of the "rewriting rules"





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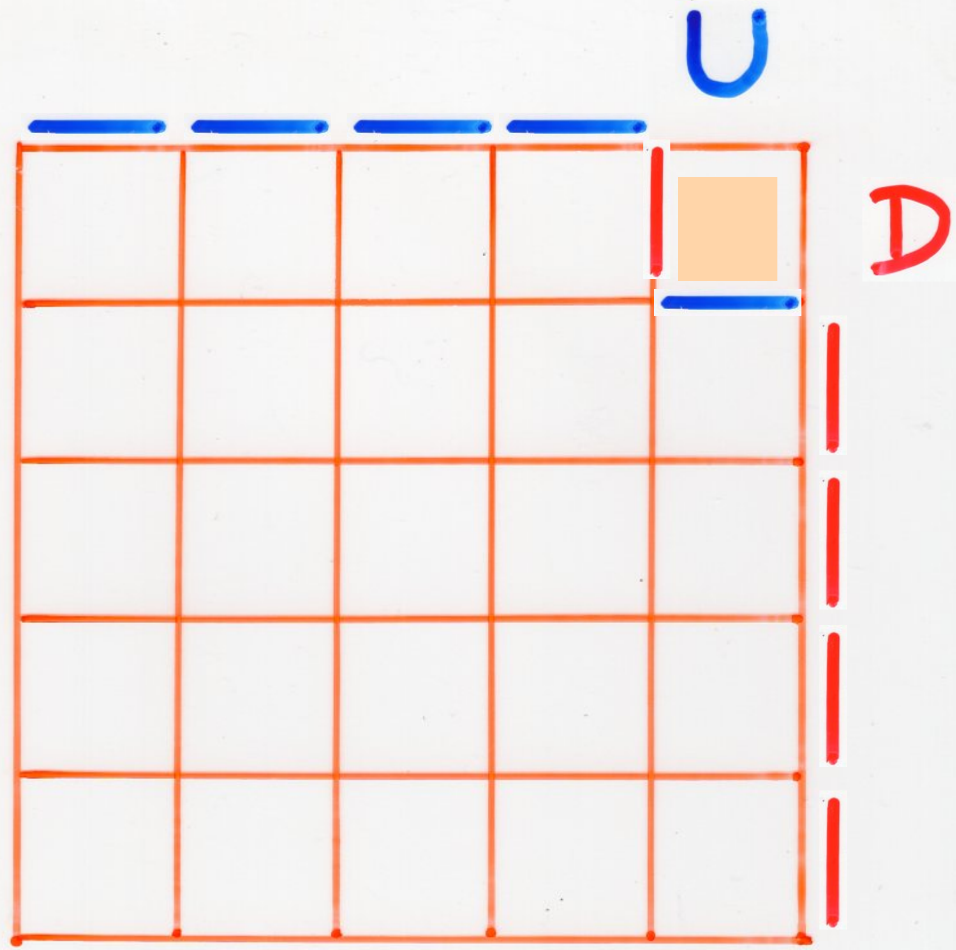
D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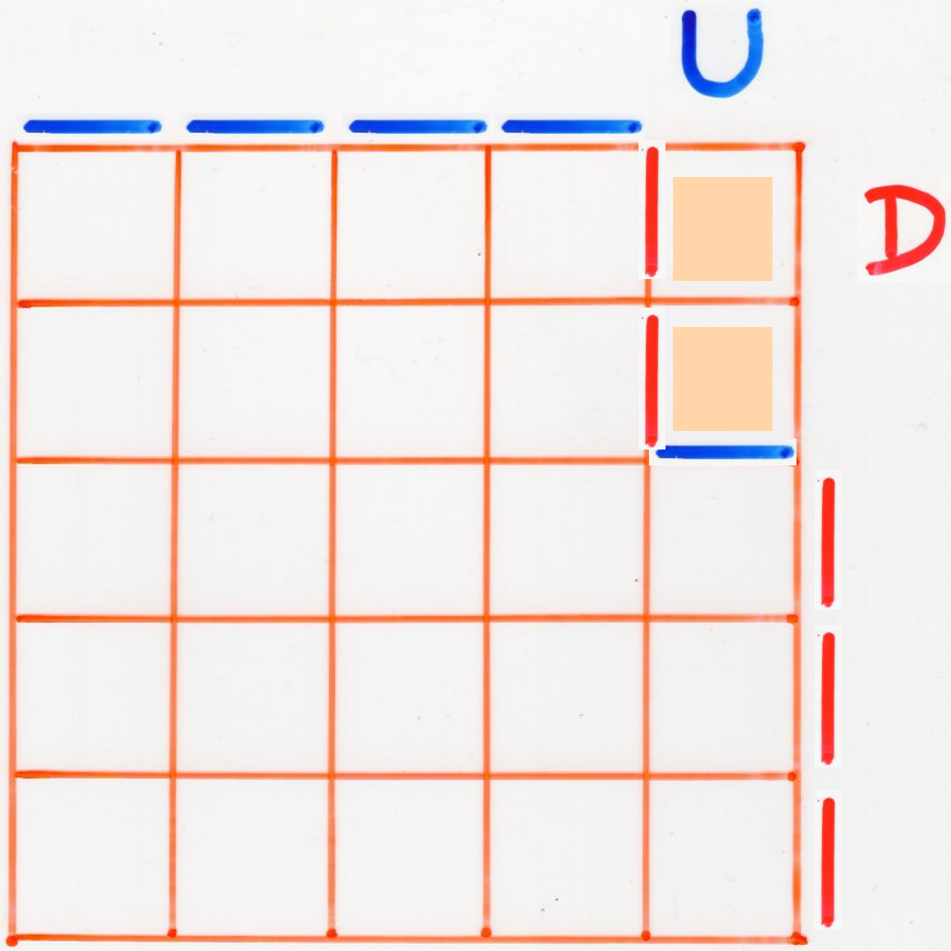
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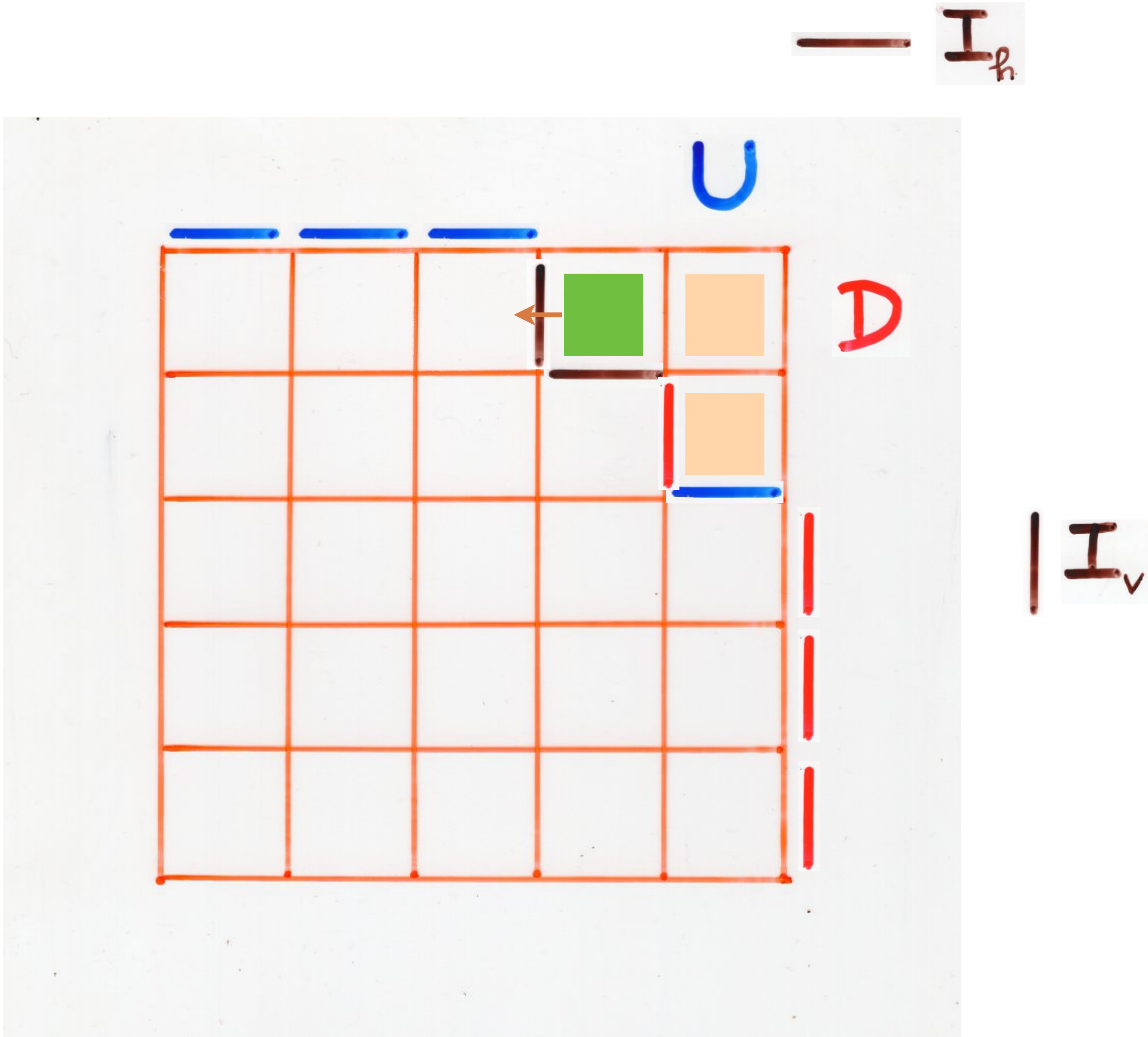
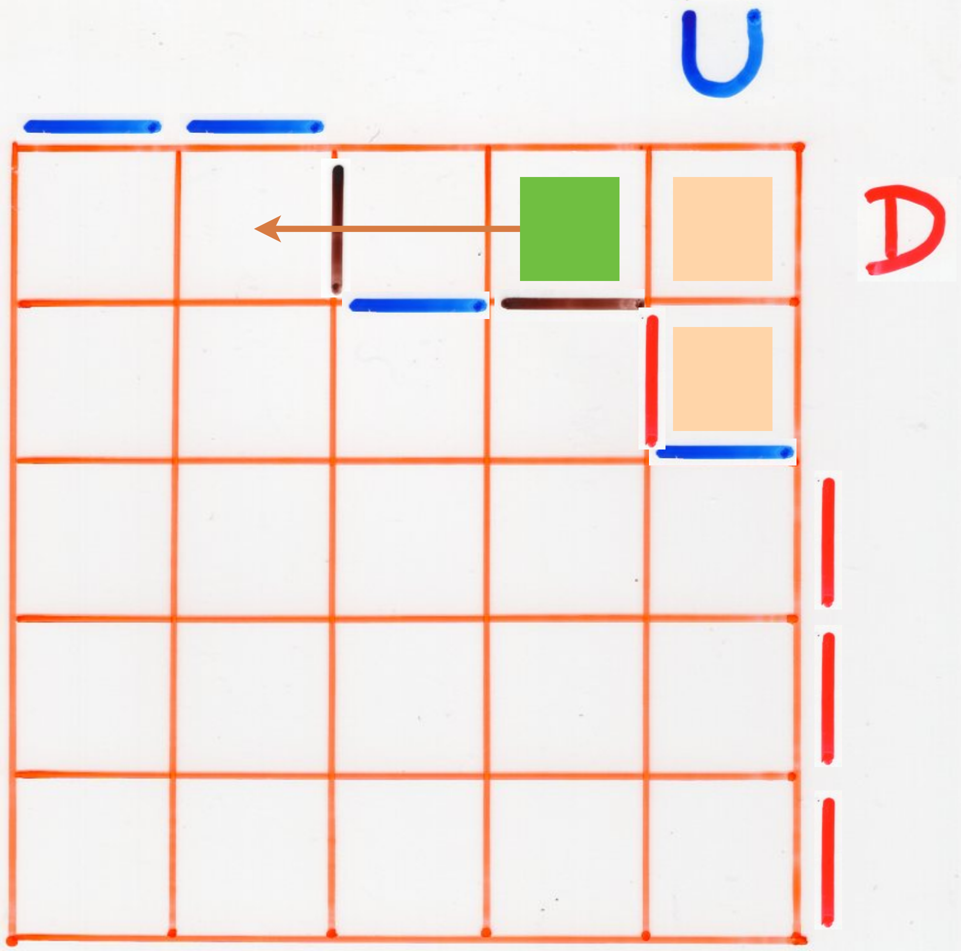

I_h
U
D
I_v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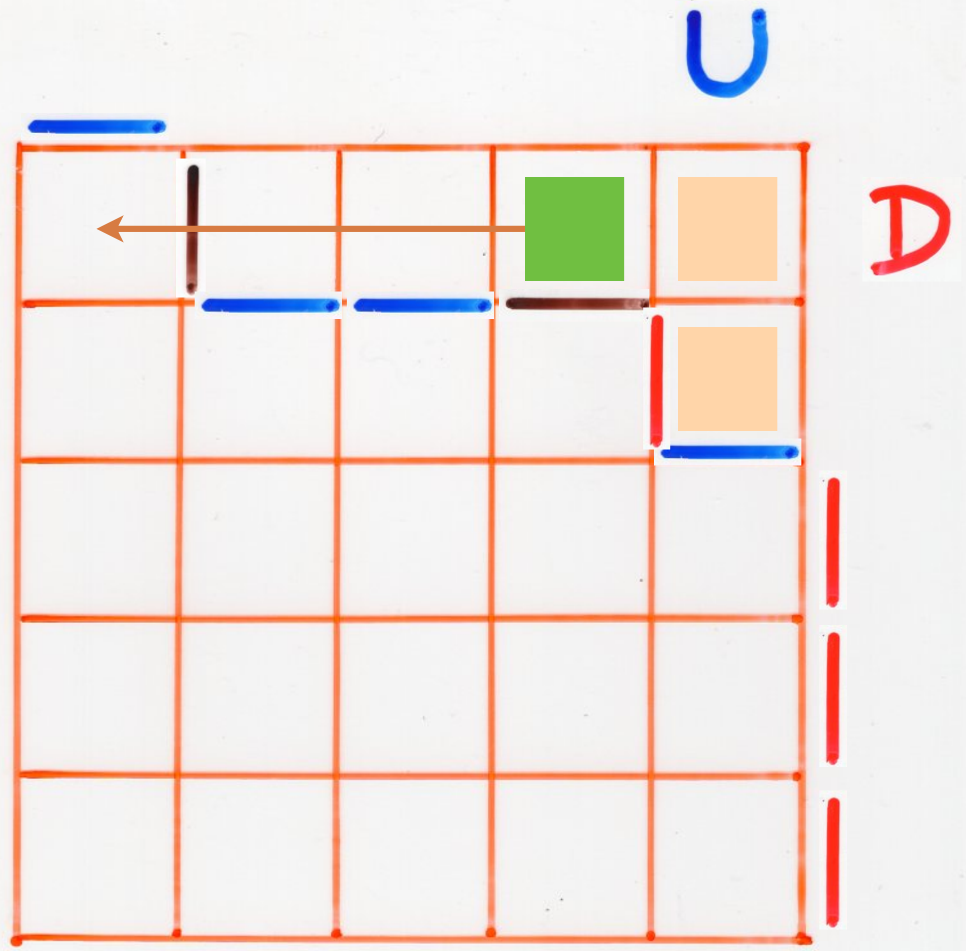
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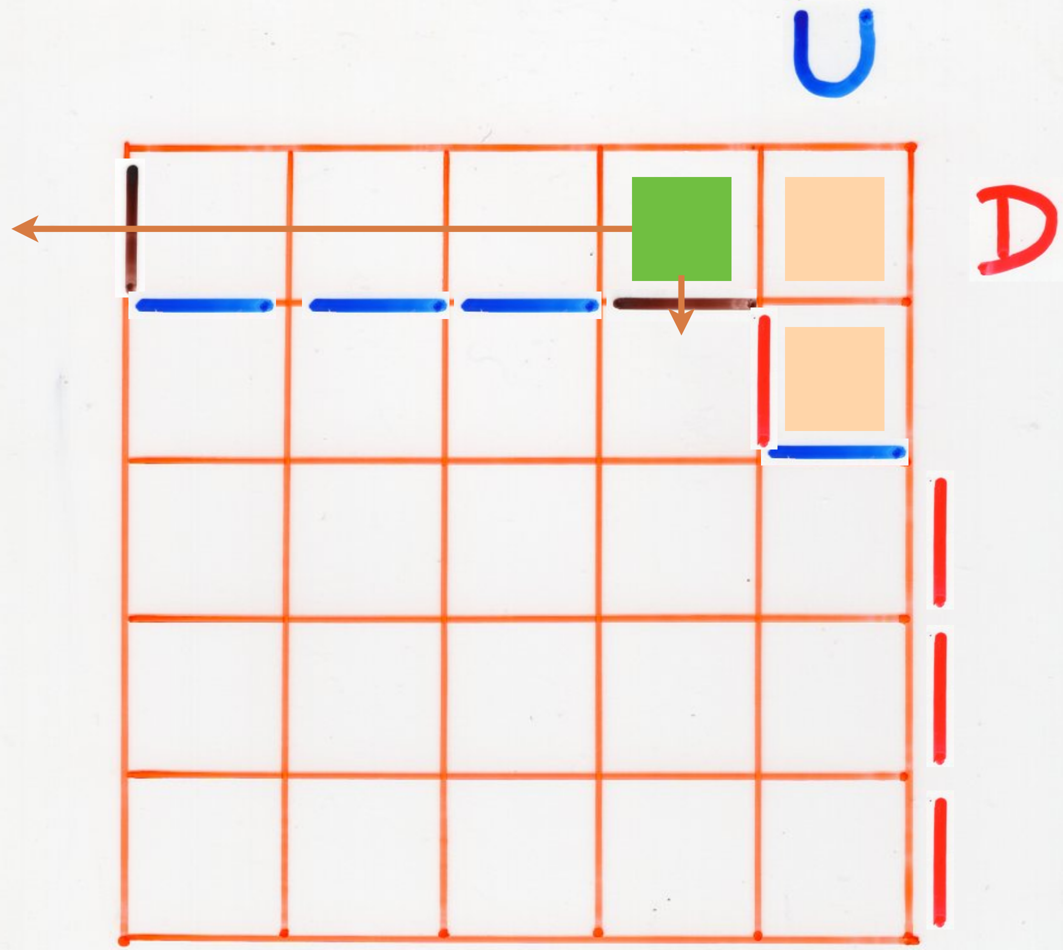
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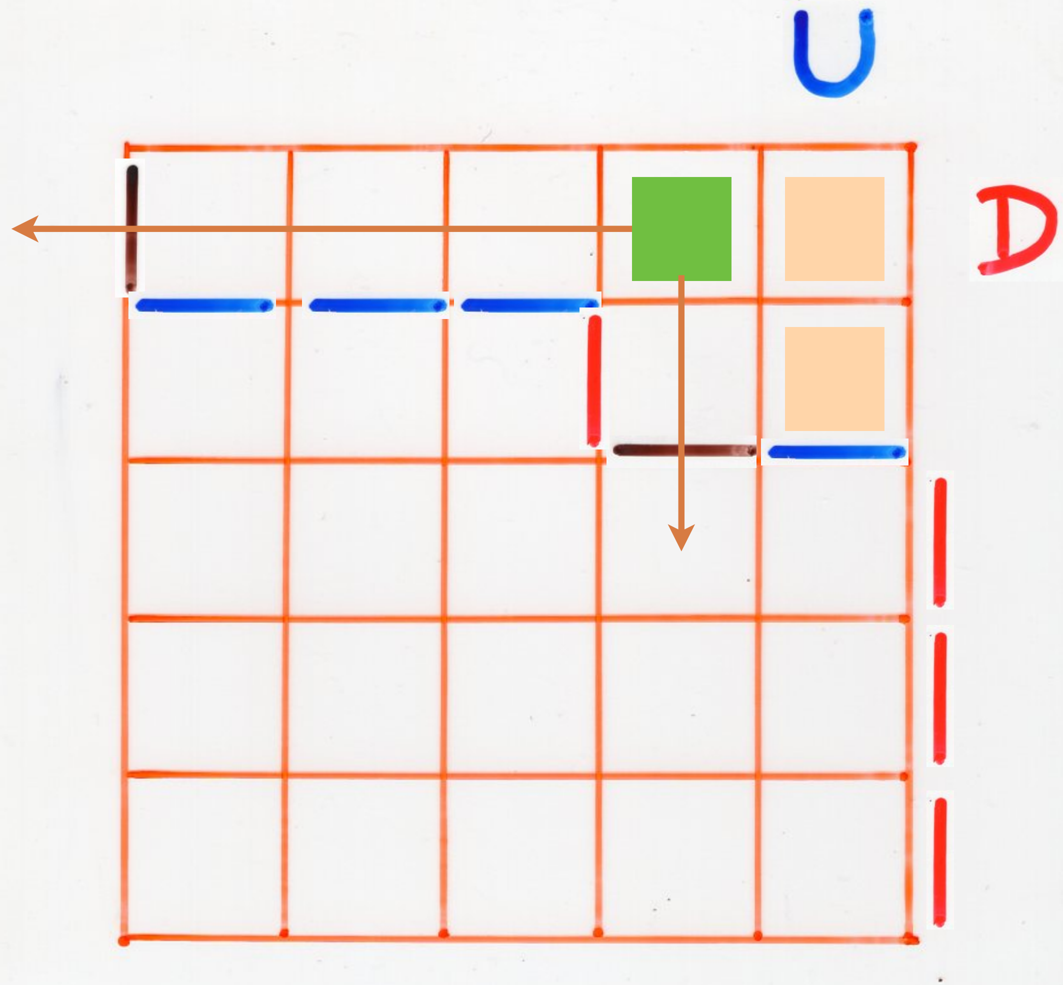
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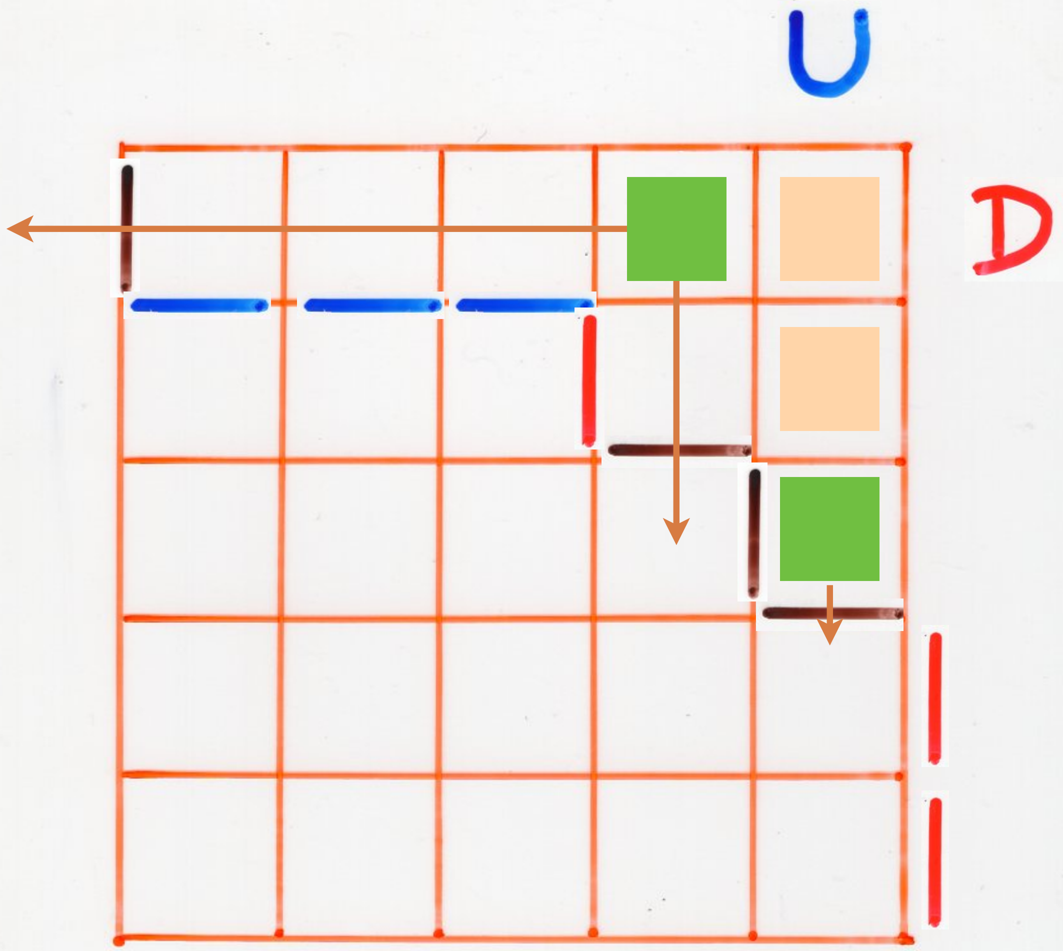
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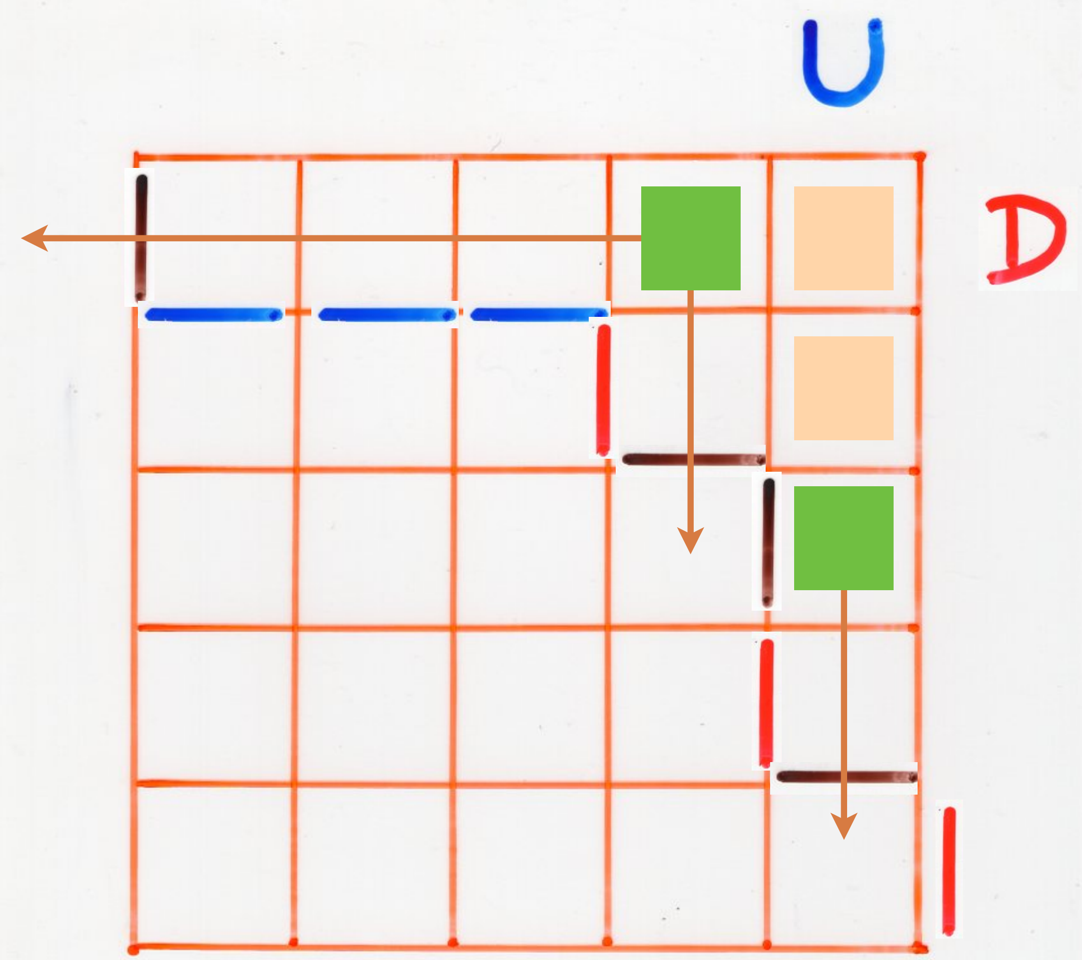
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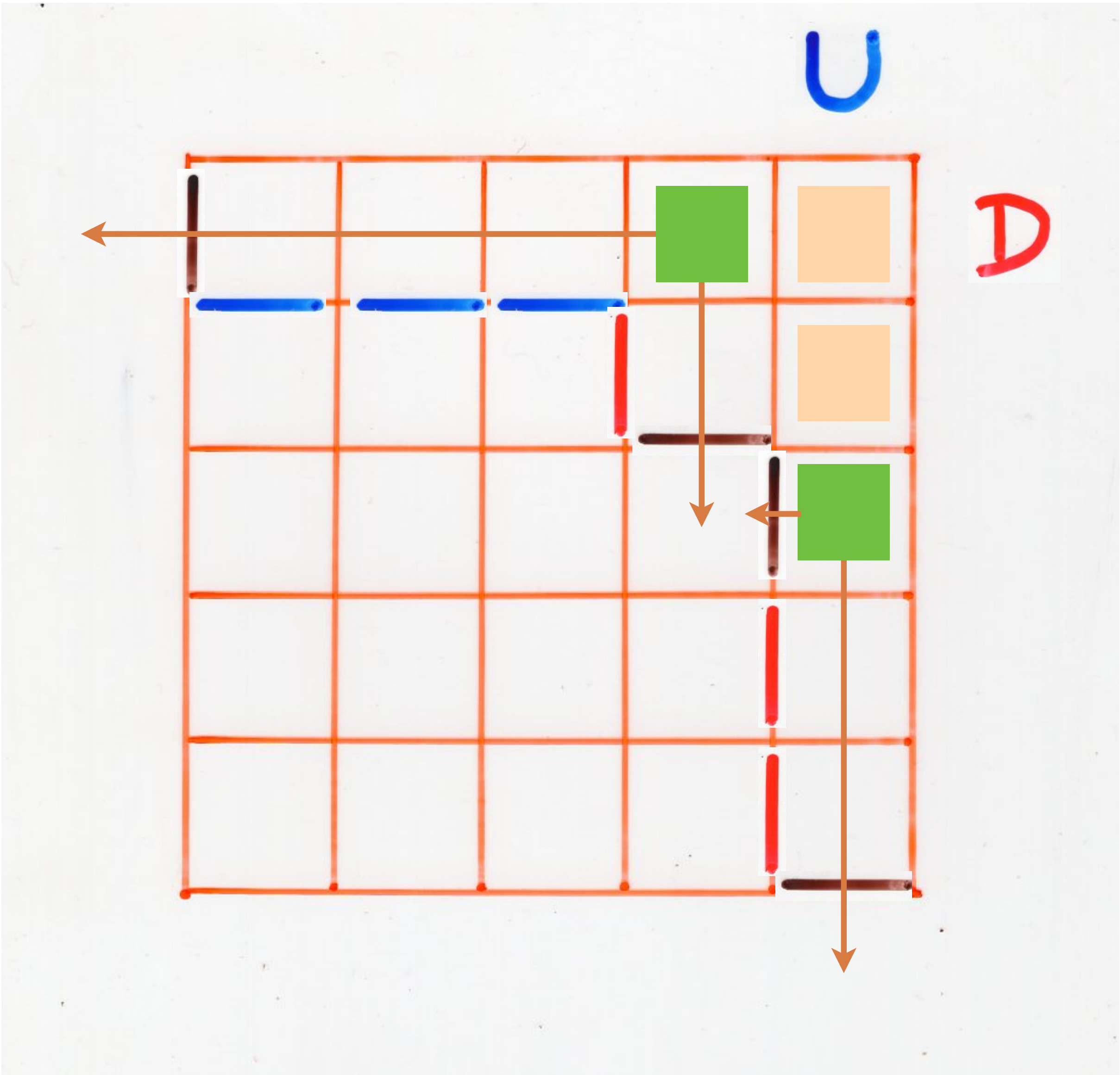
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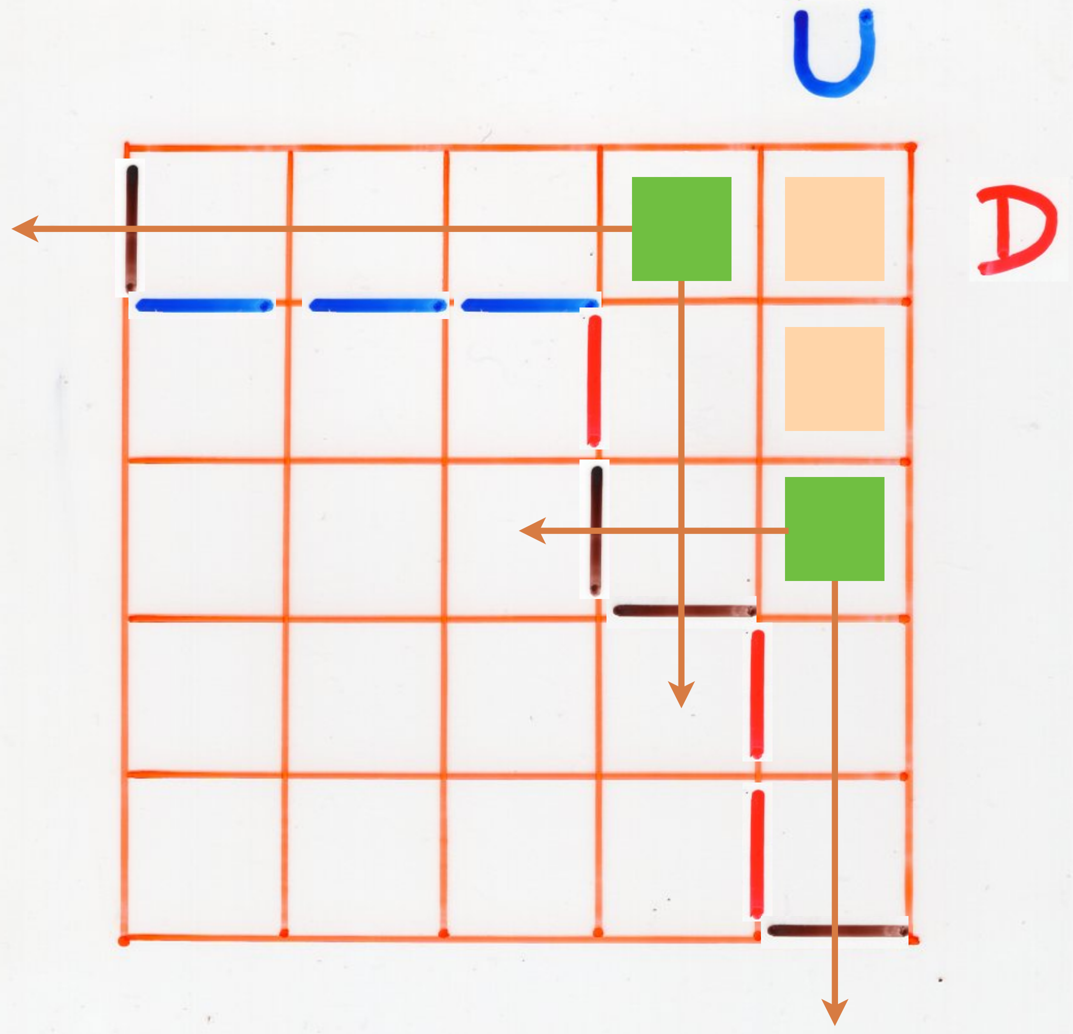
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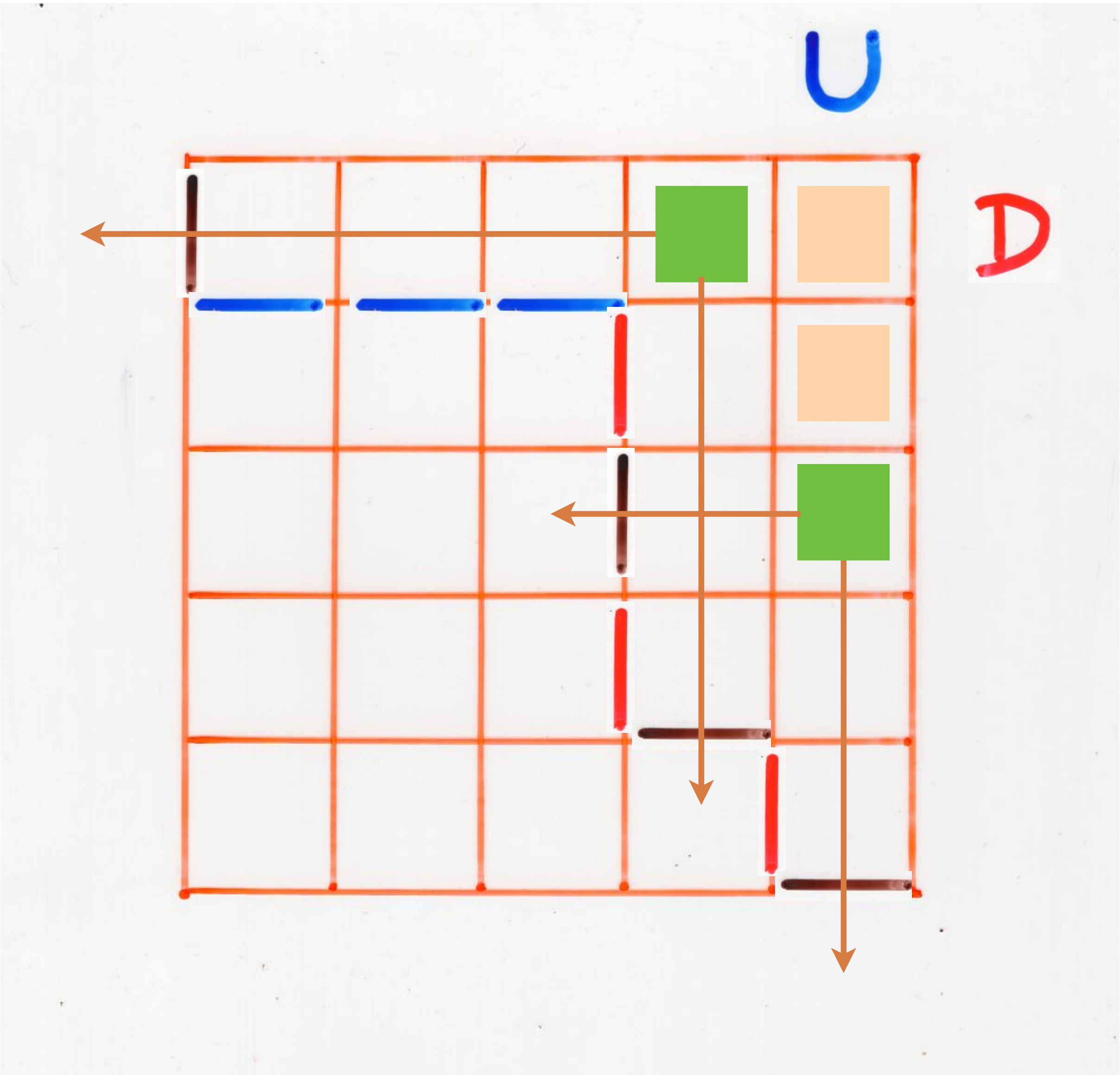
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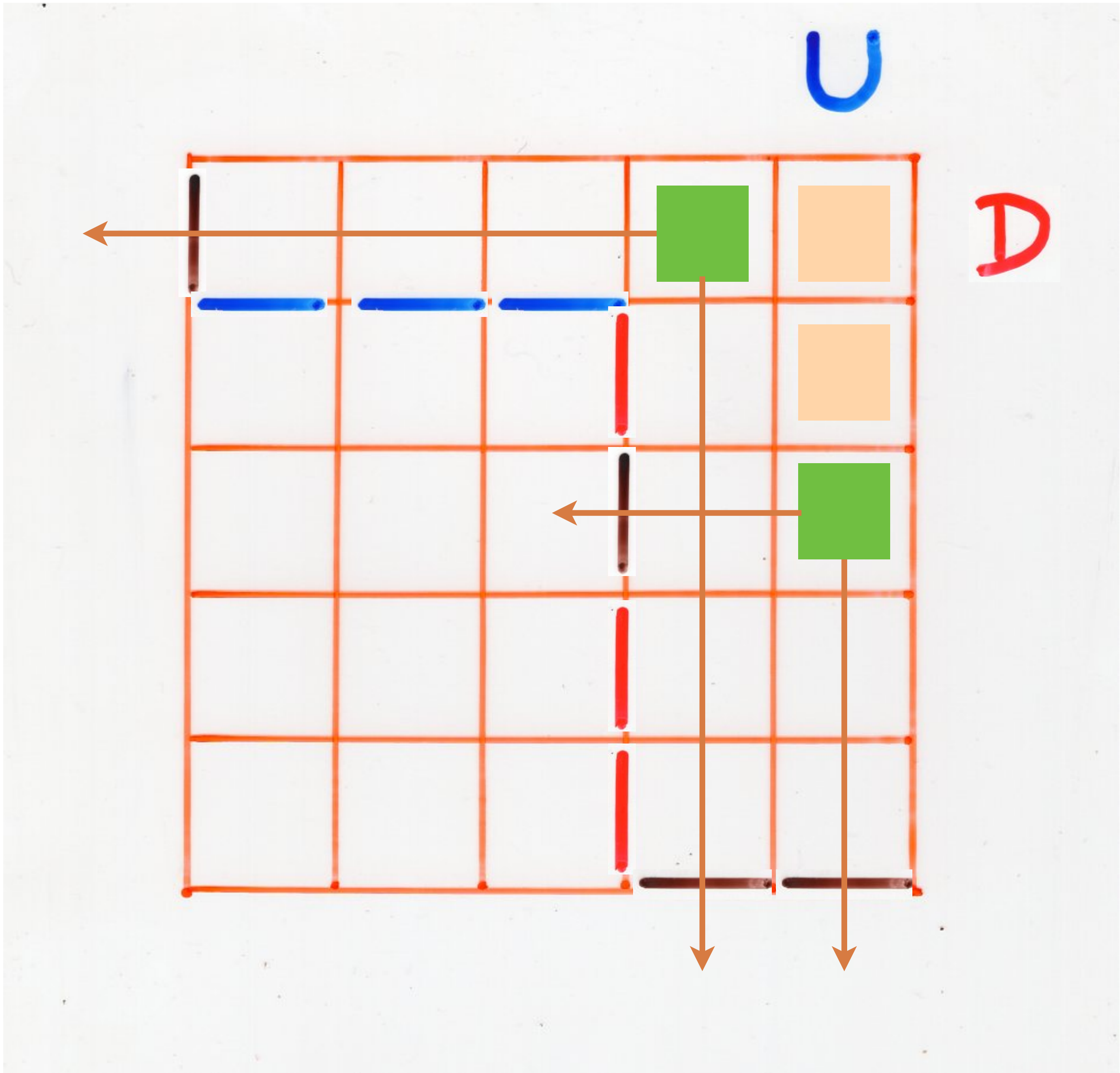
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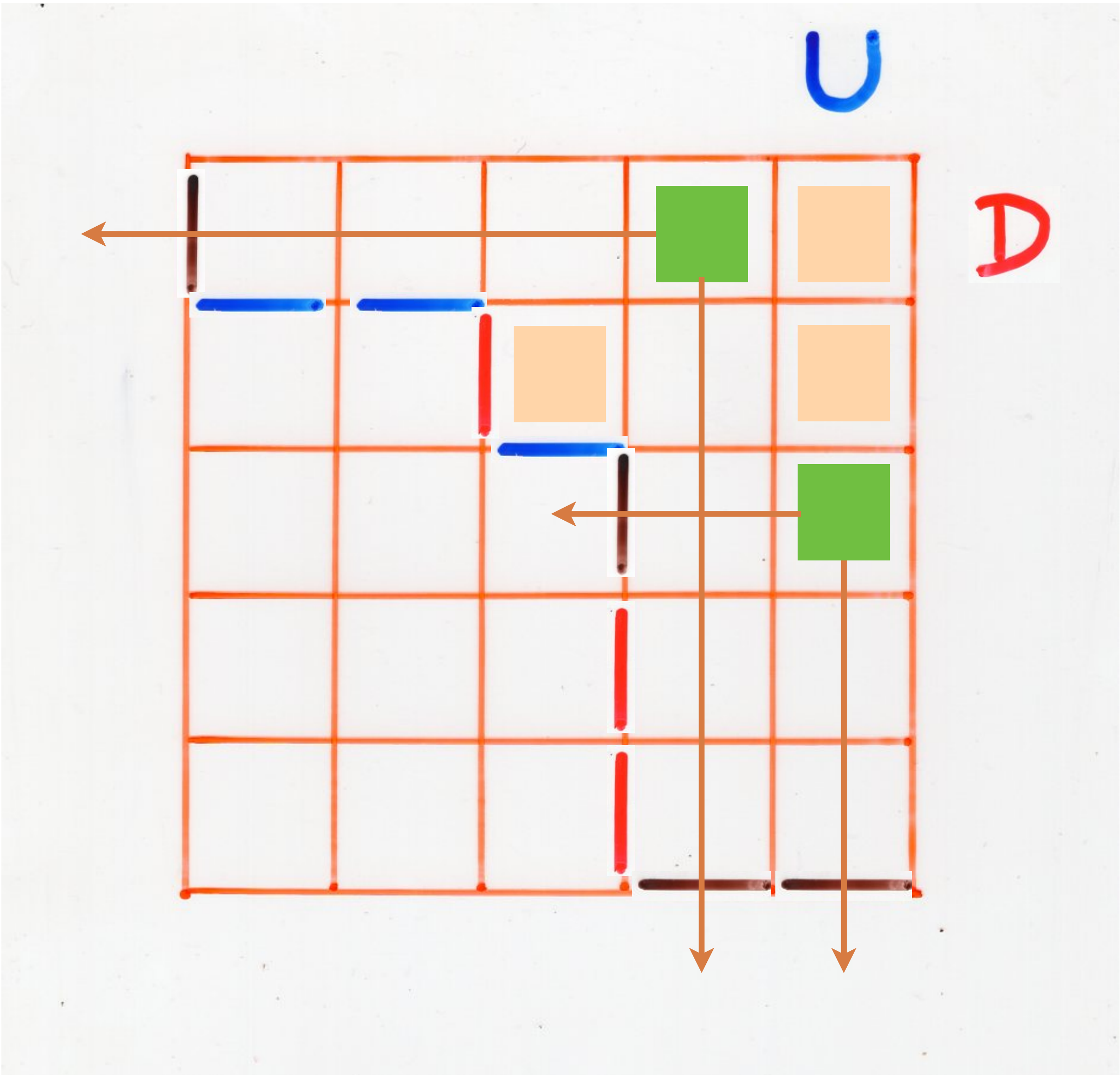
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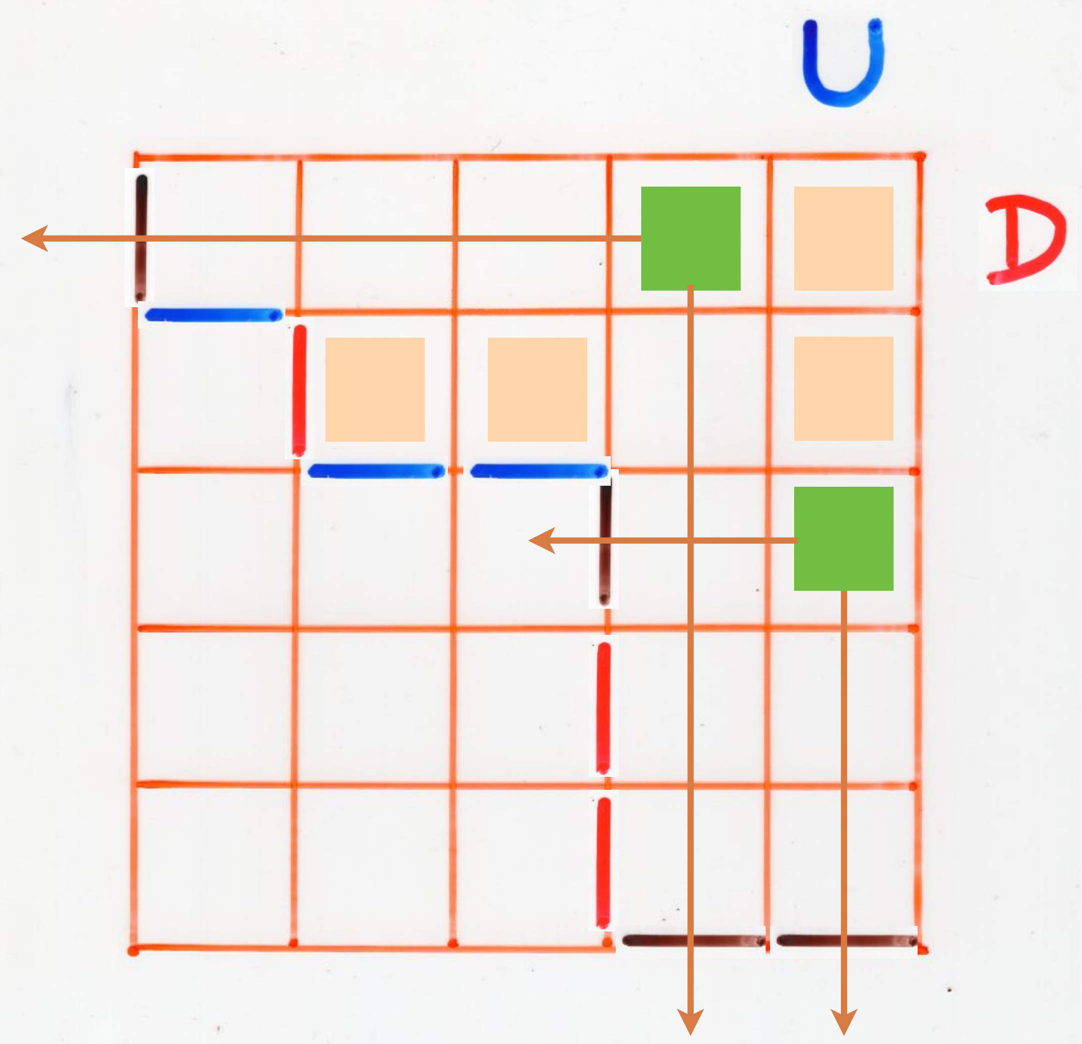
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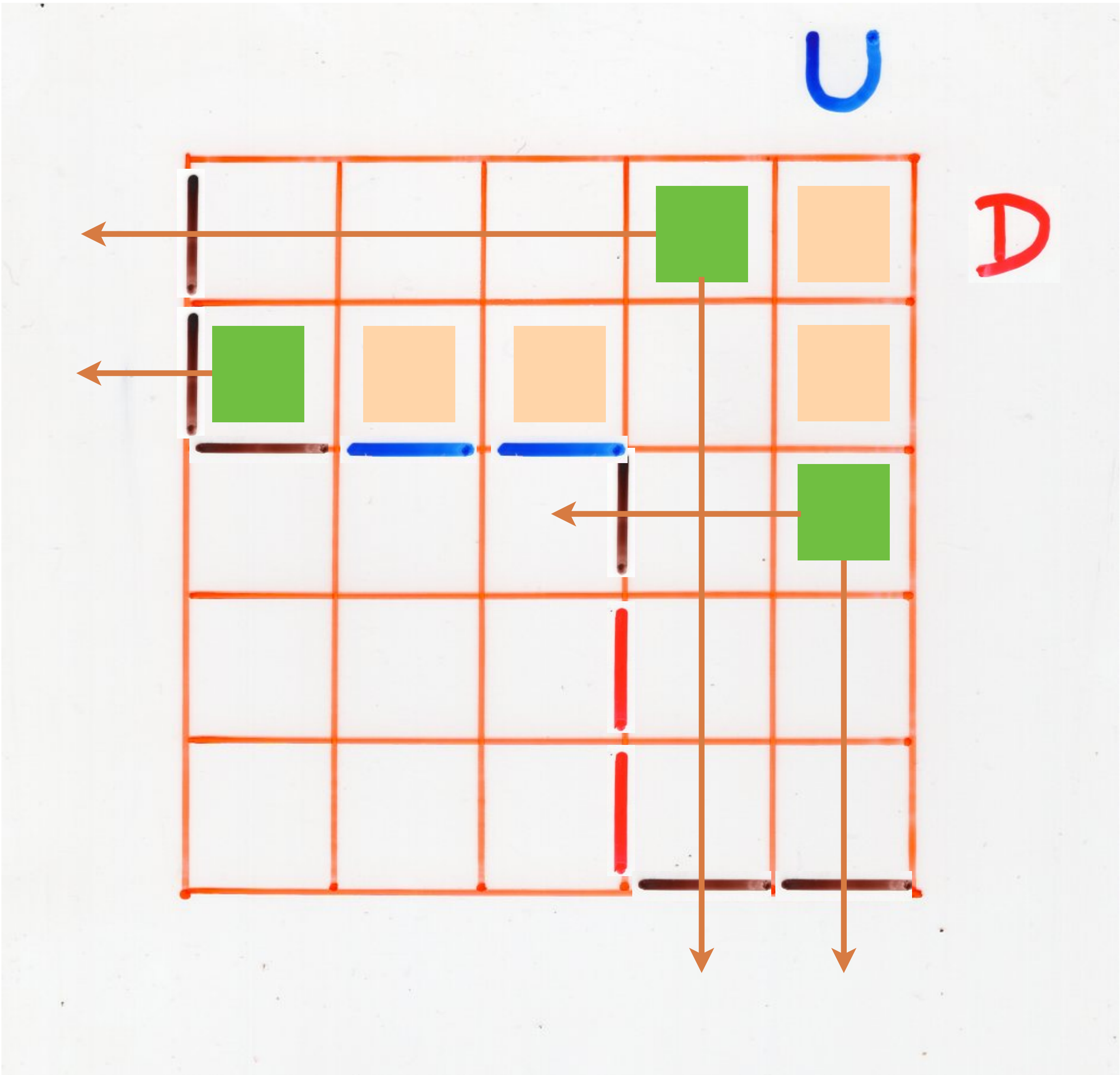
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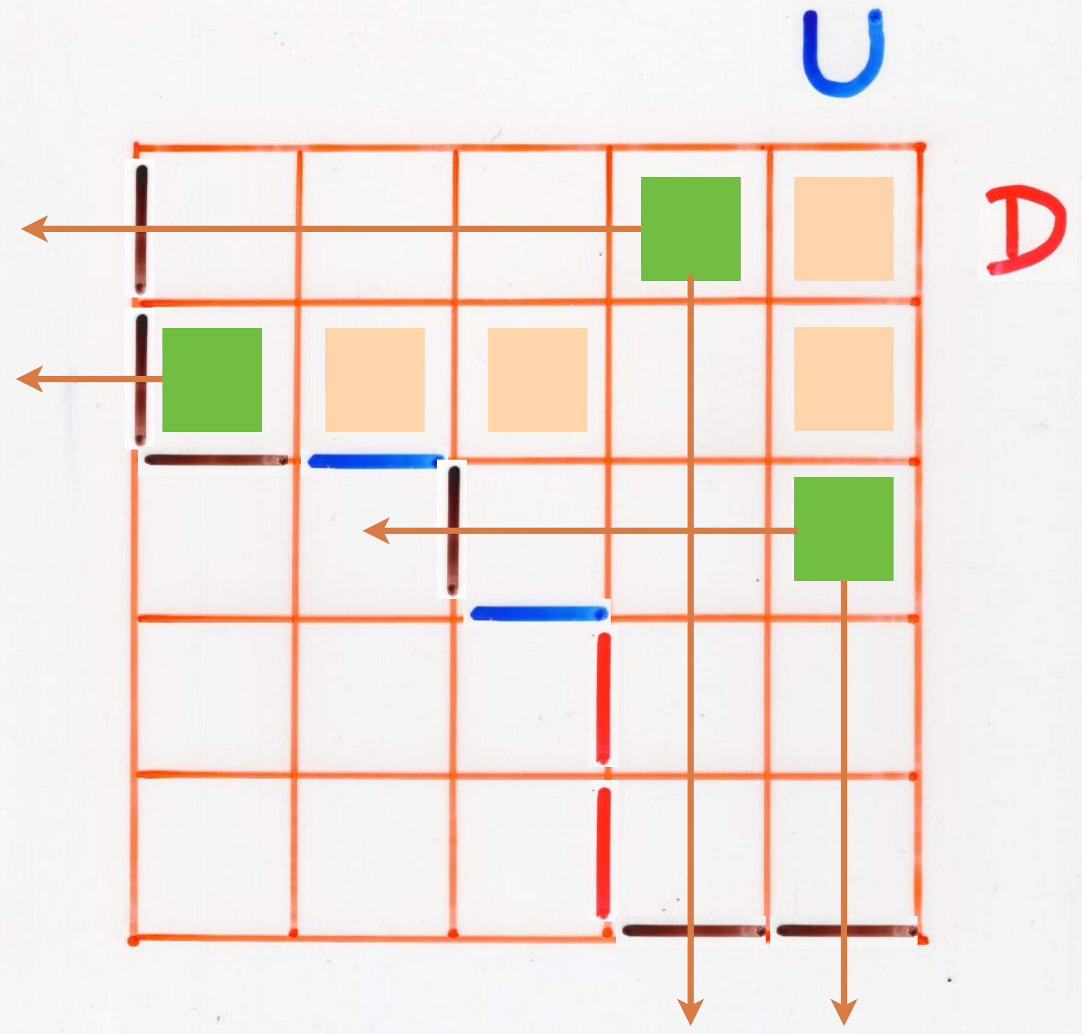
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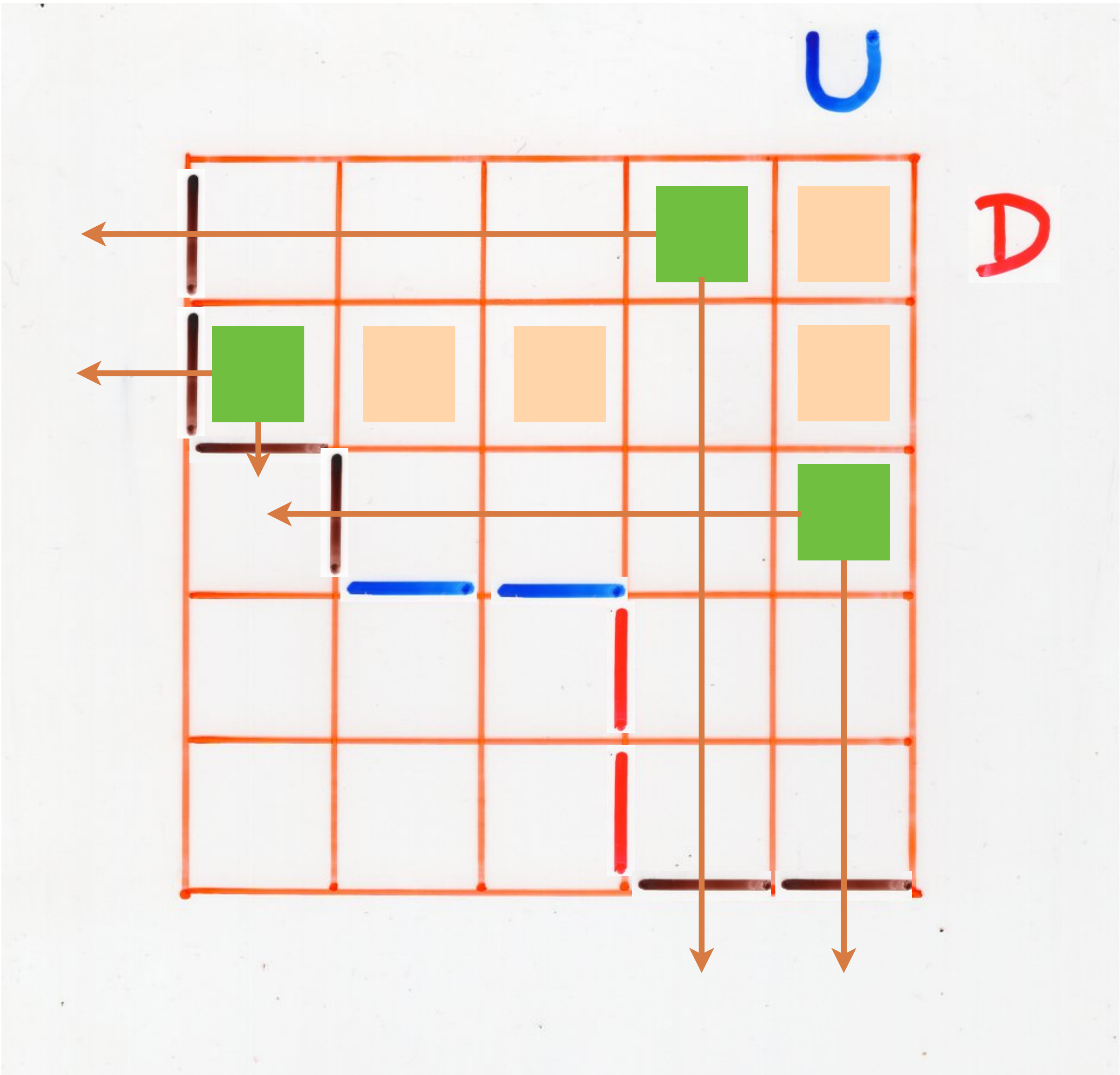
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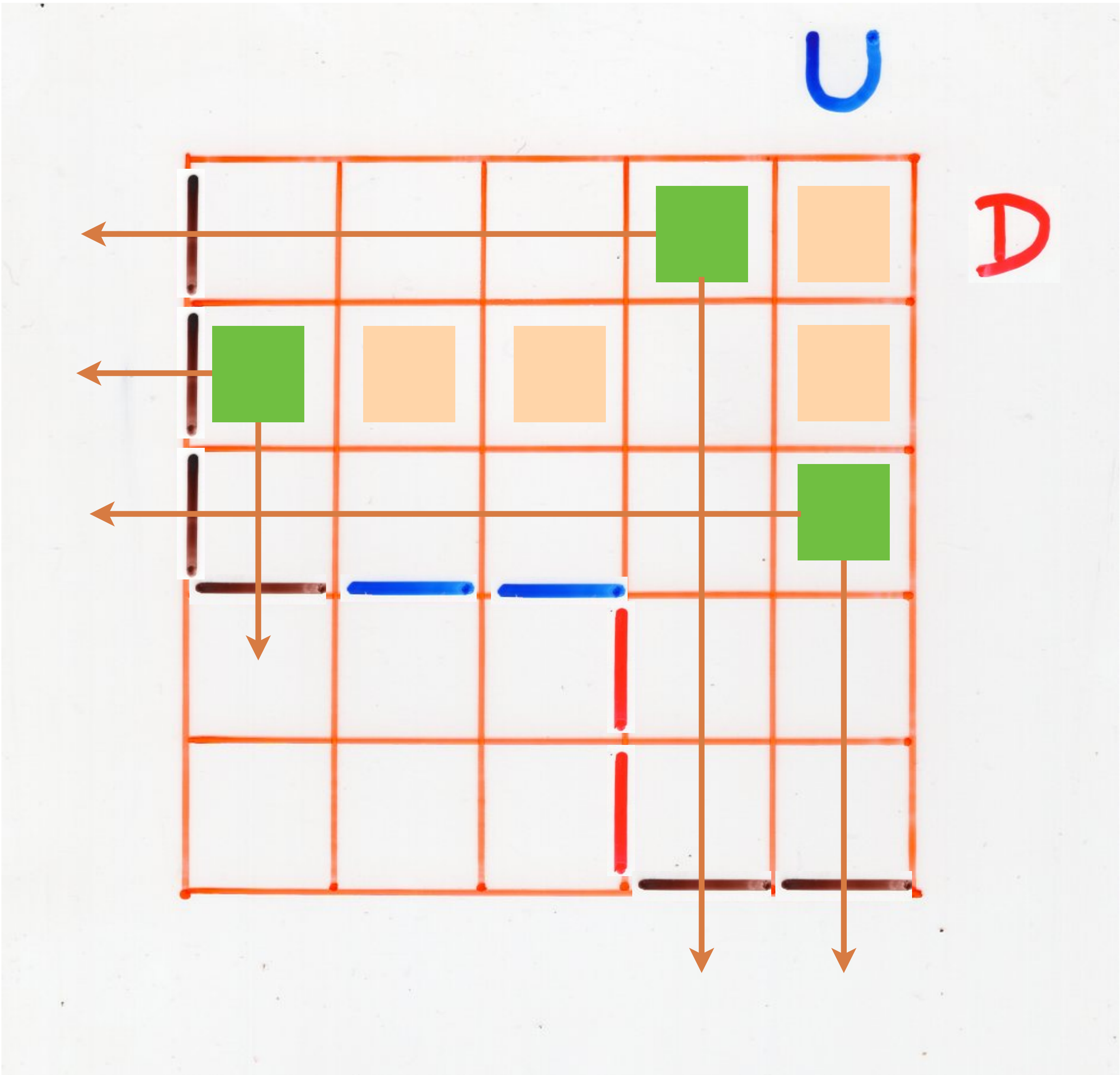
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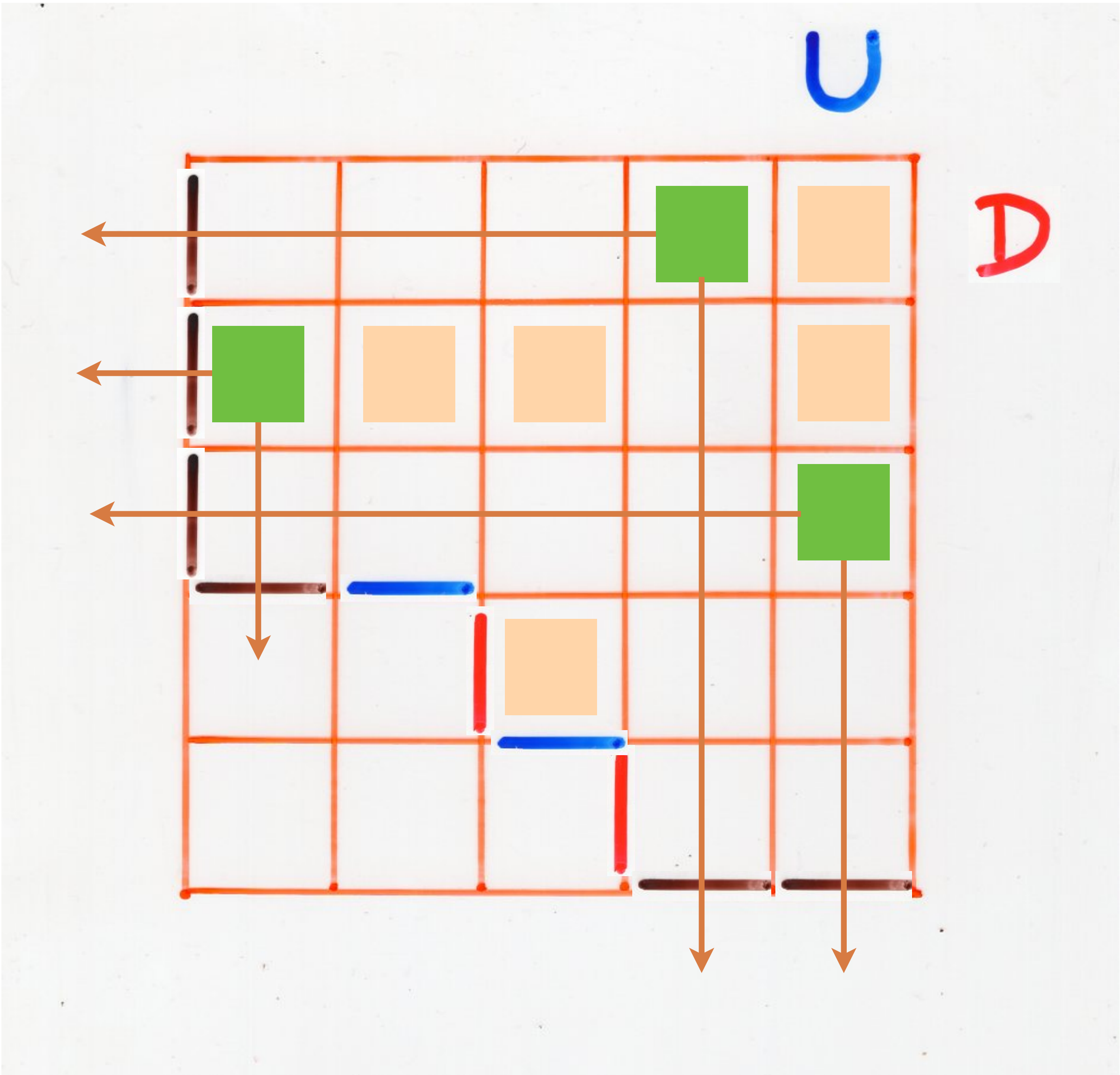
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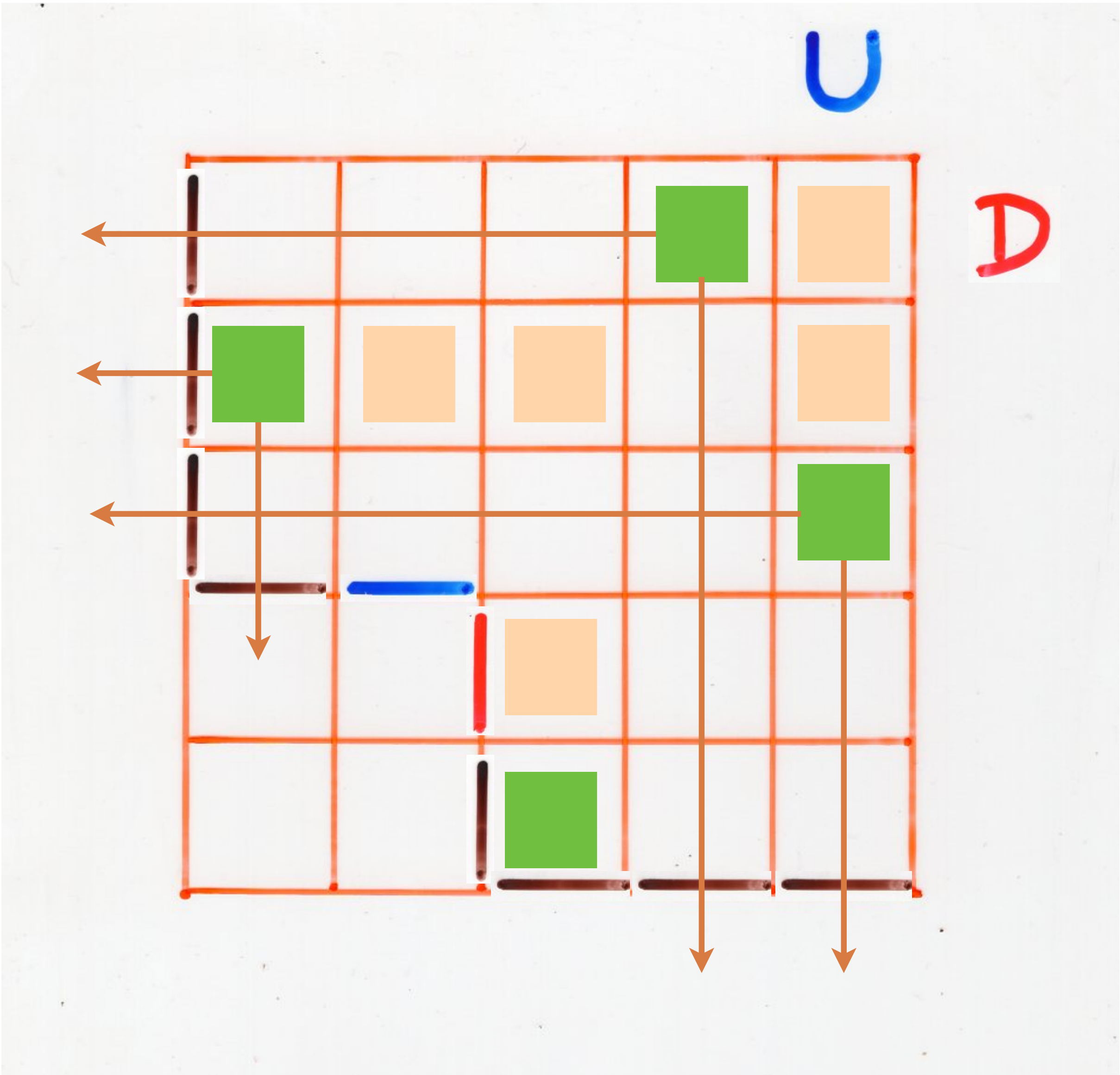
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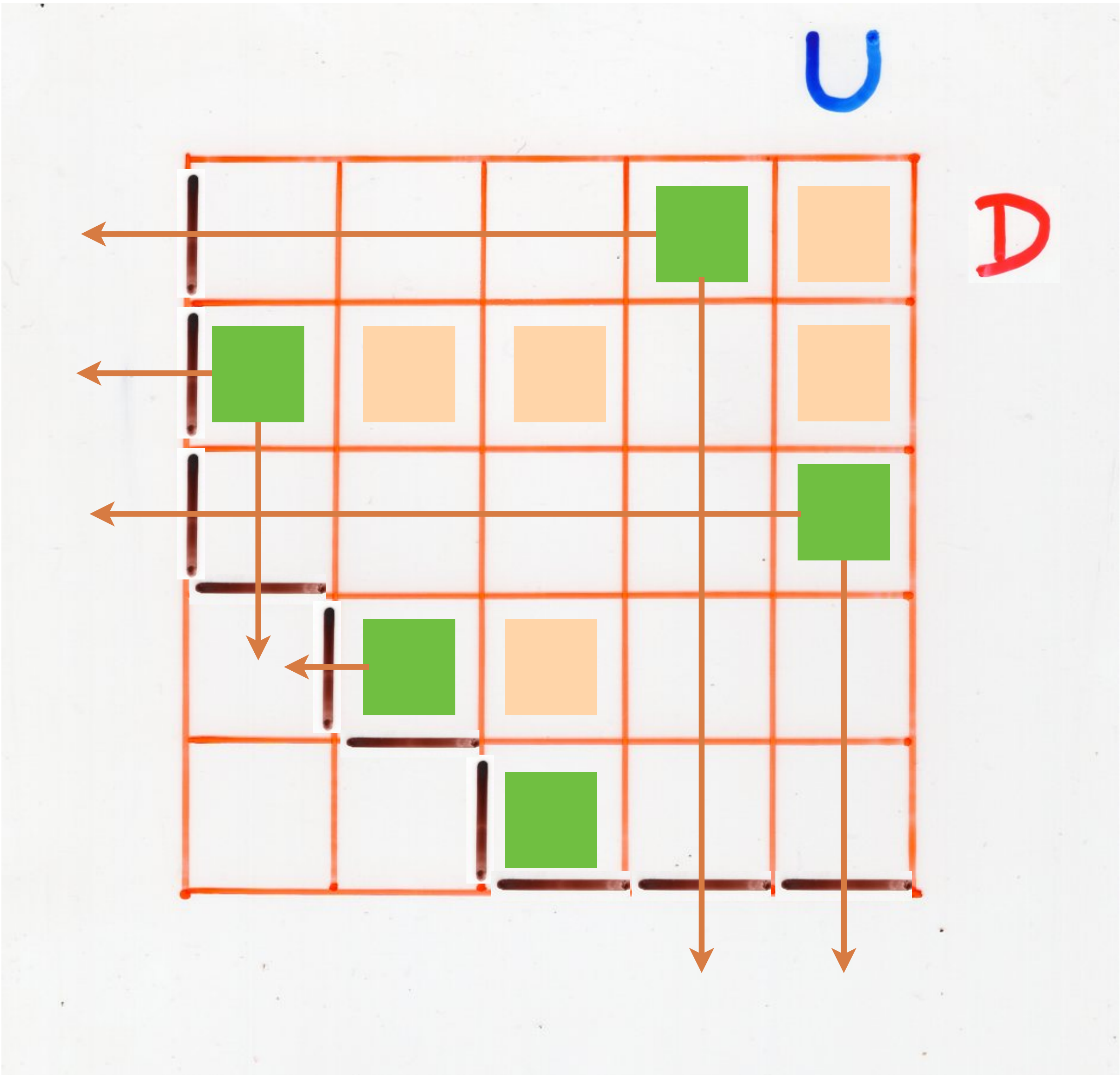
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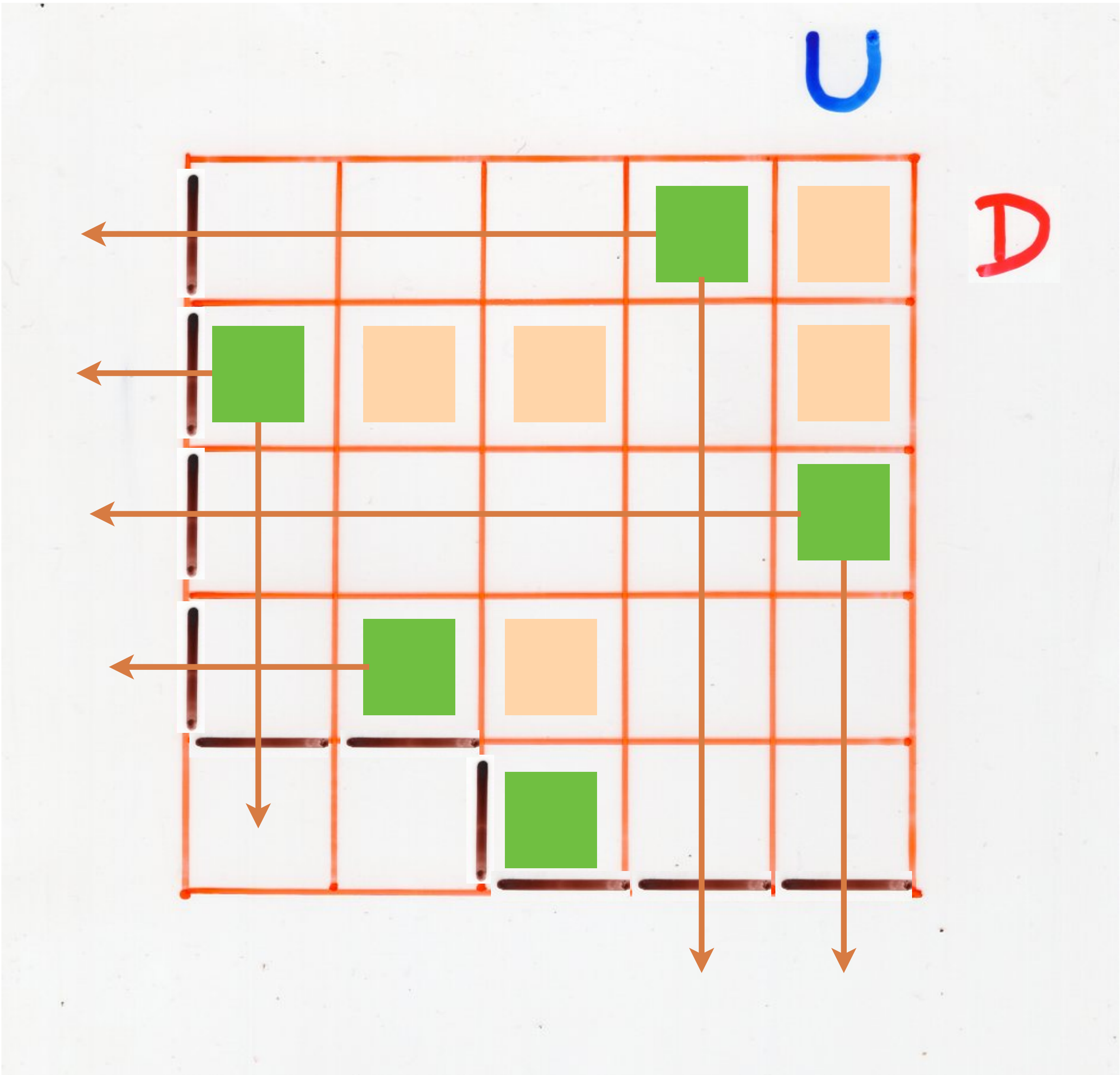
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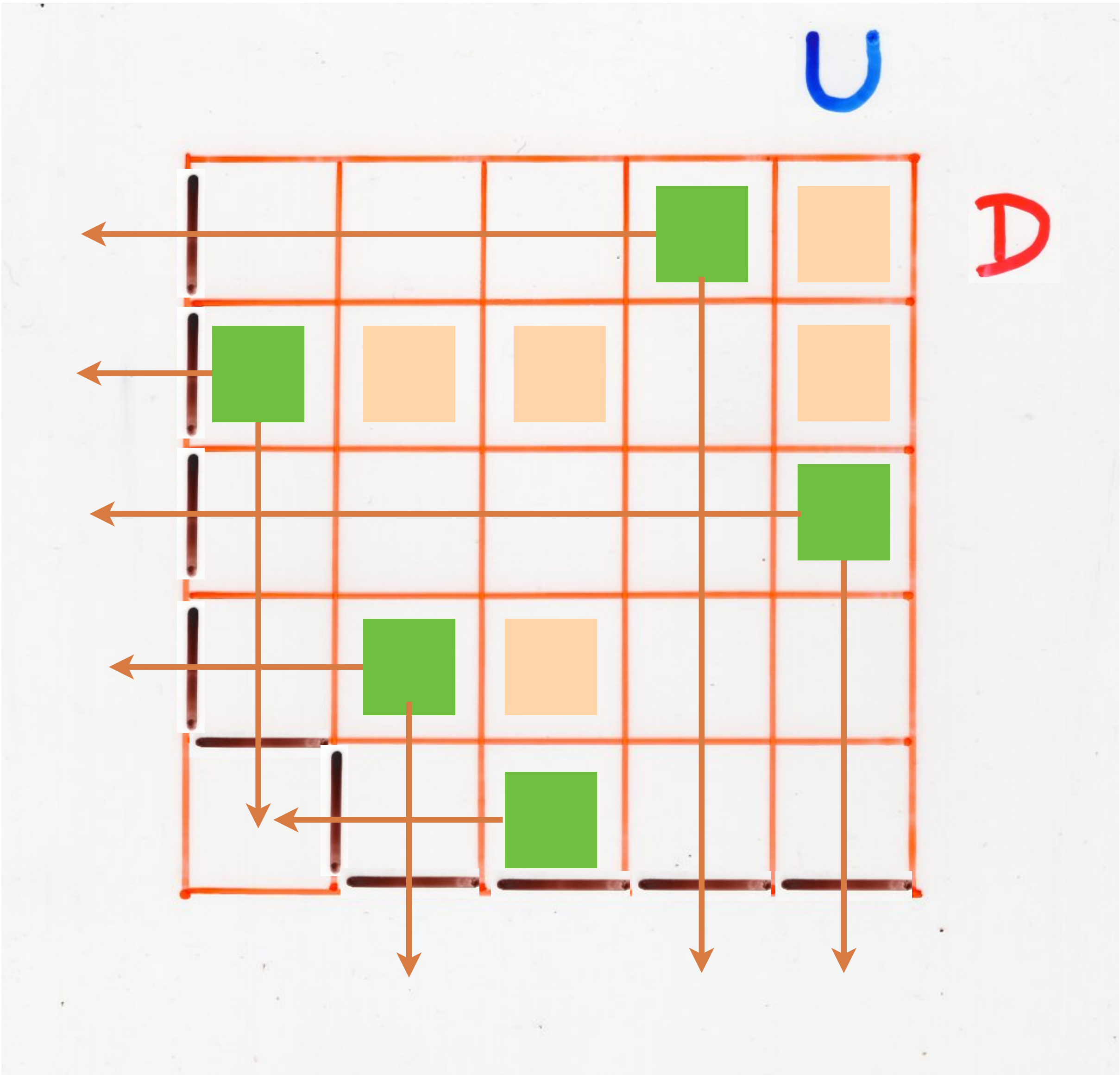

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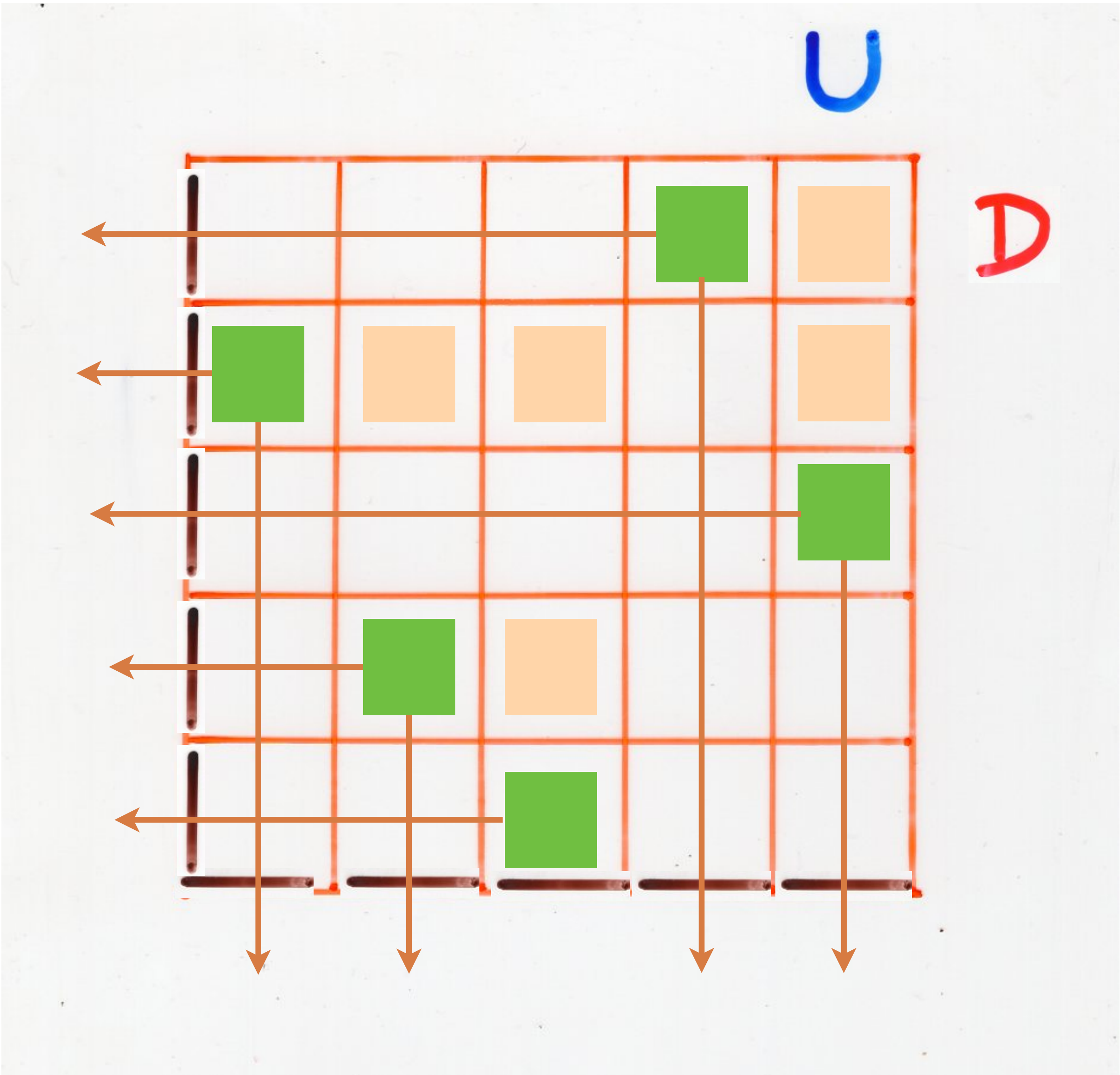
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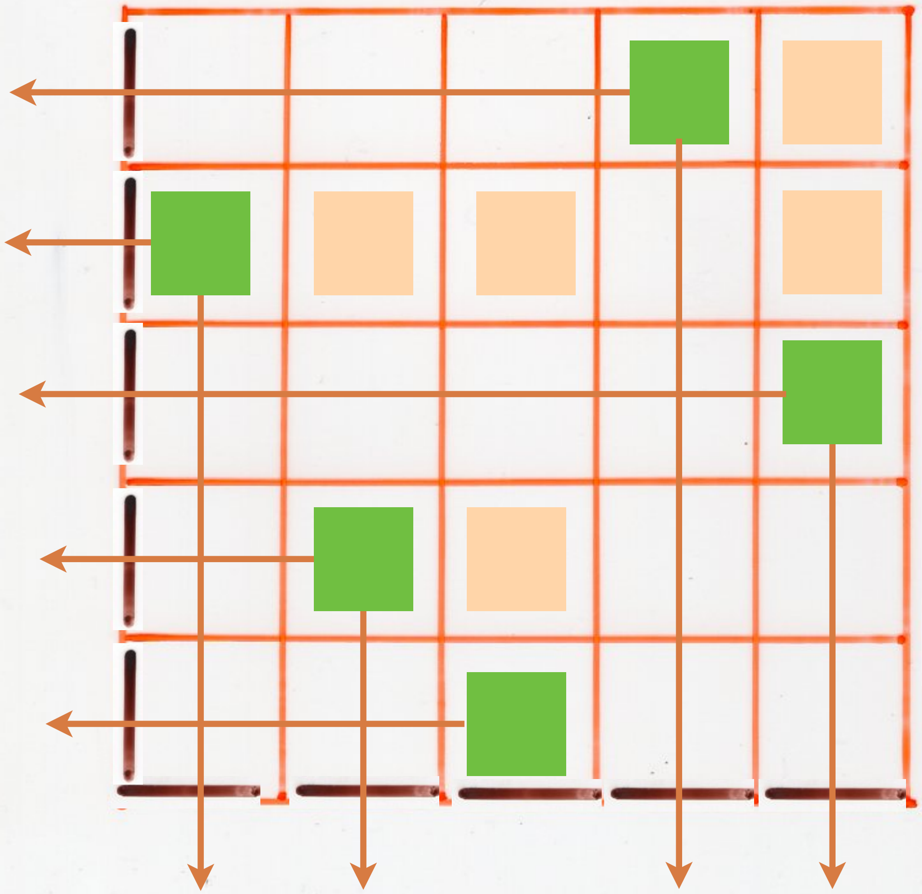
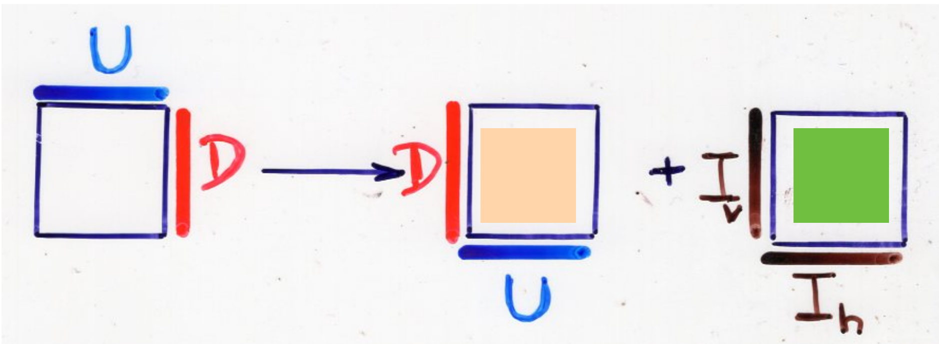


$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

"complete"
Q- tableau

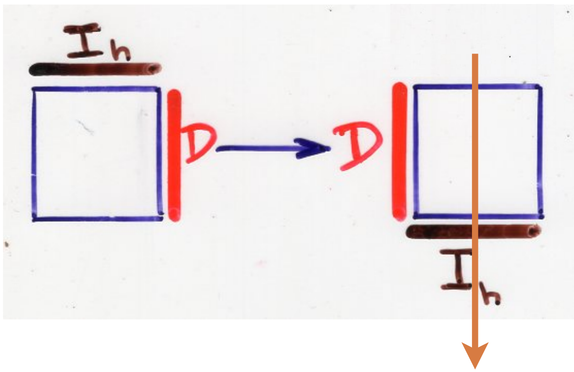
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

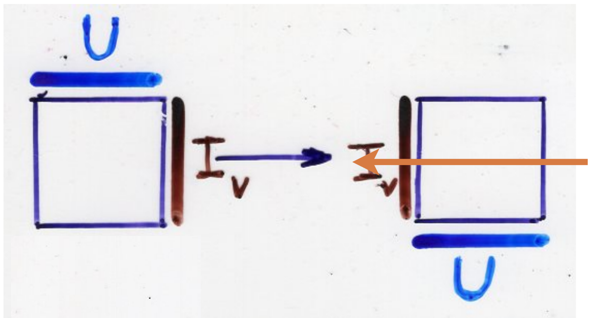
$$UD = qDU + I$$

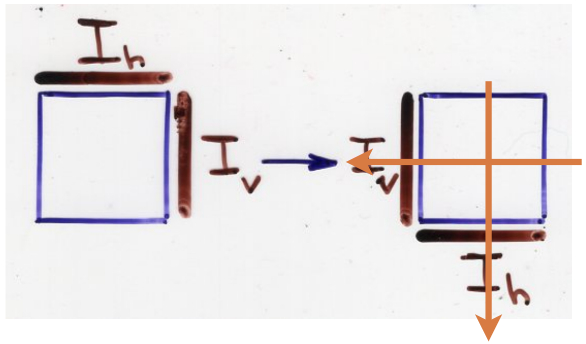


"complete"

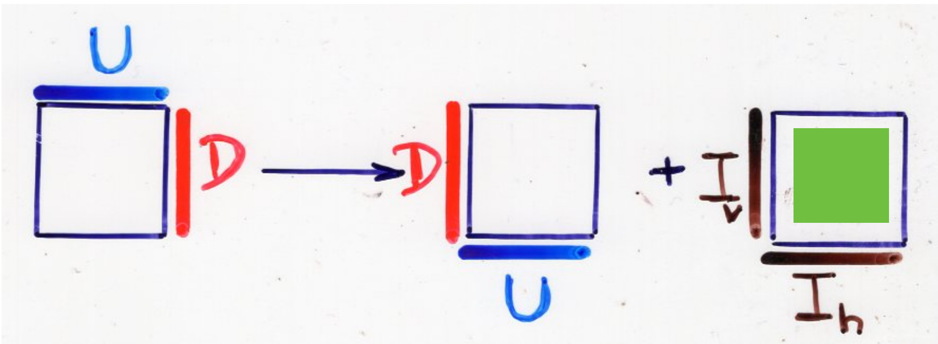
Q- tableau



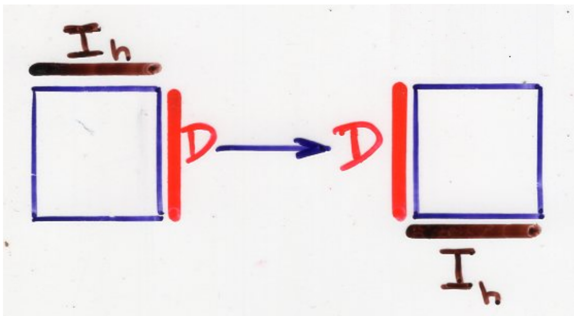


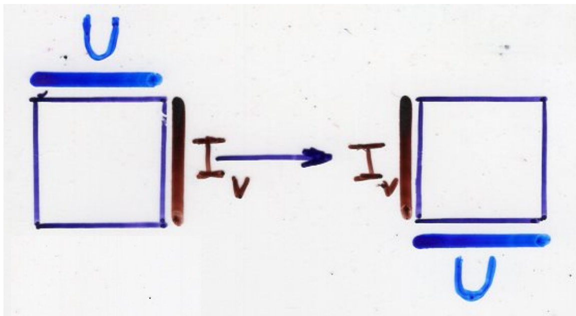


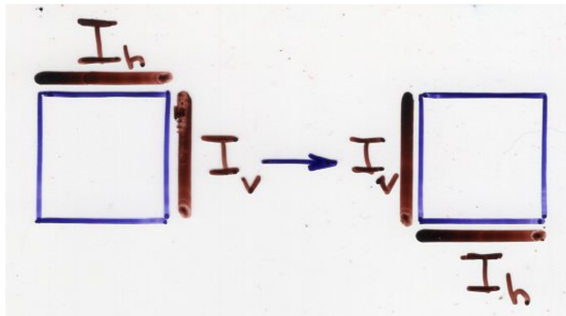
$$
\begin{cases}
UD = DU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$

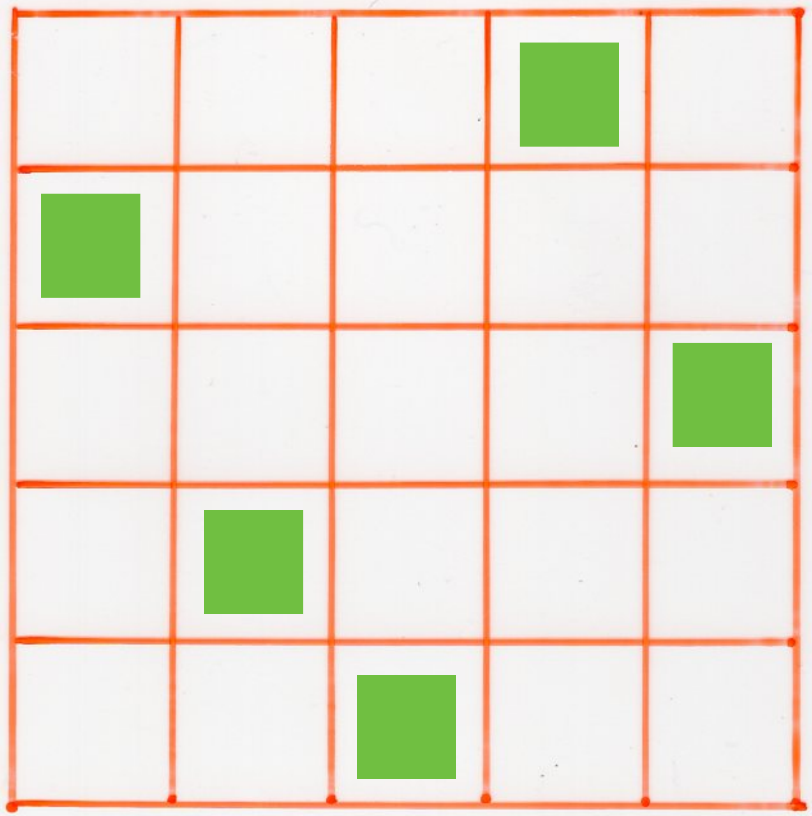


Q- tableau





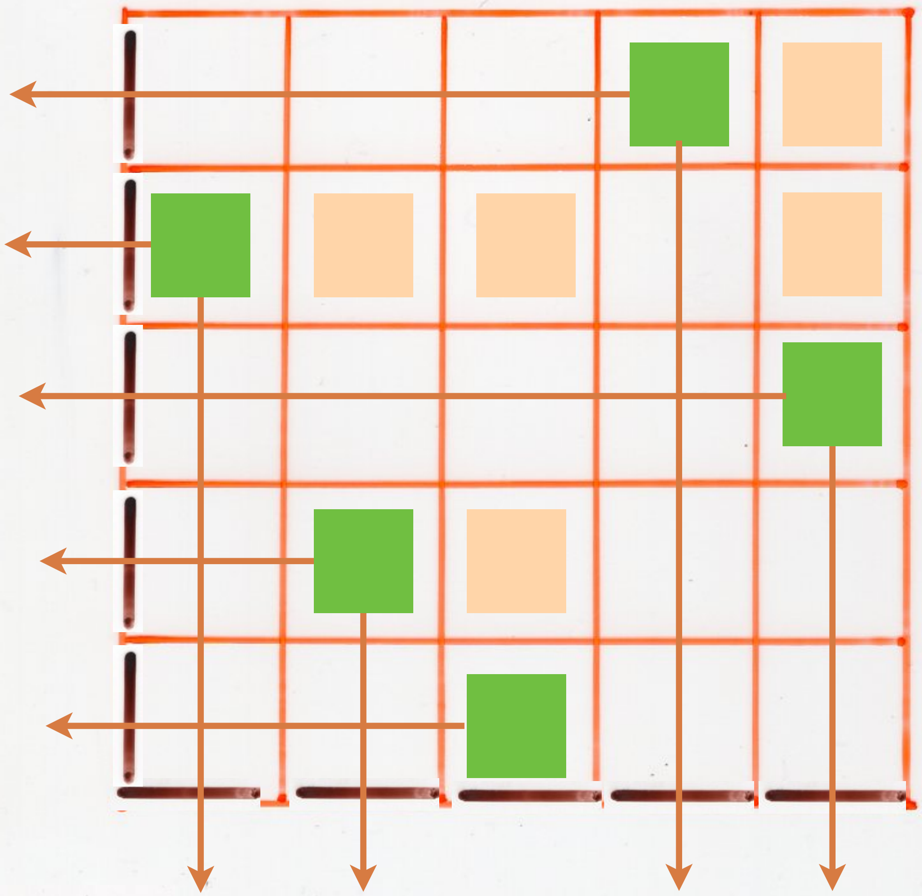




permutation
as a Q-tableau

q

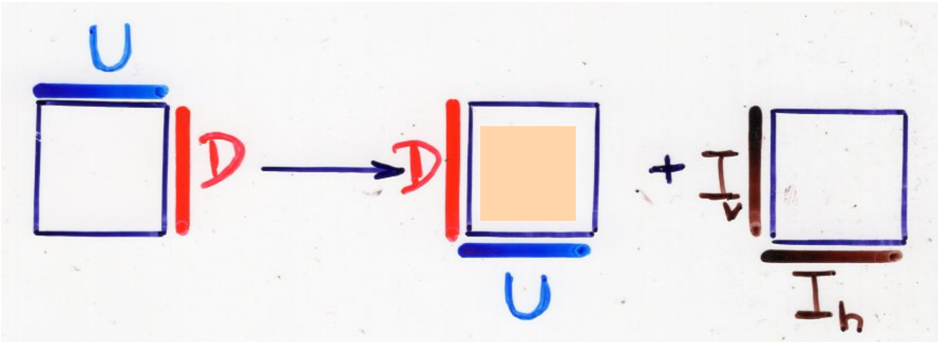
q- analog



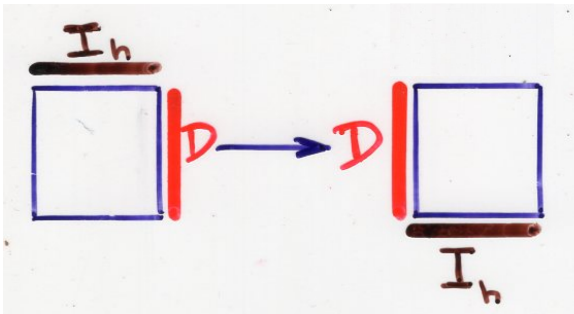
number of inversions of a permutation σ

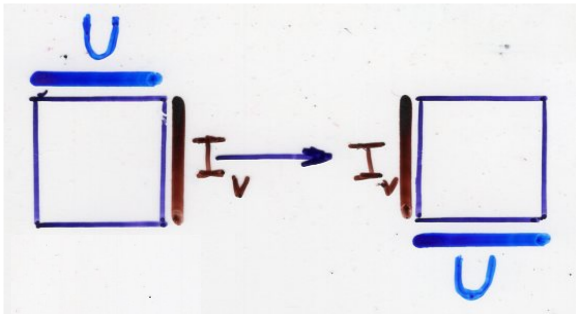
"complete" Q- tableau

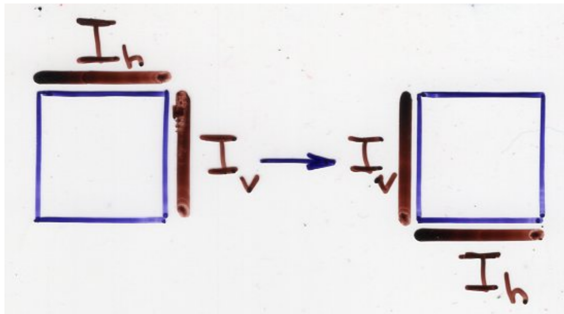
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

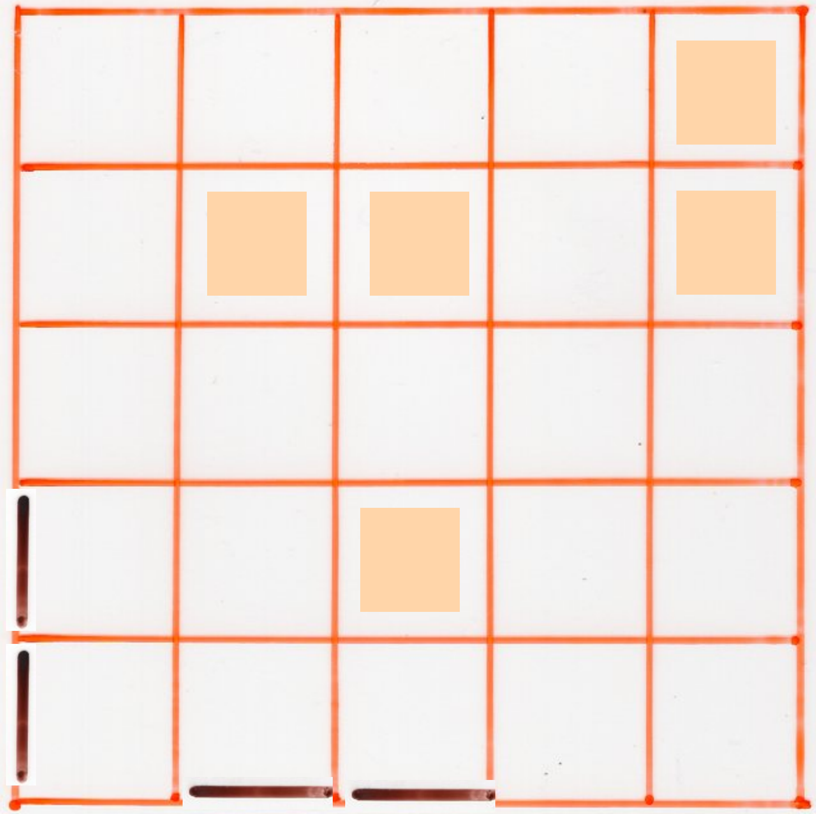


Q- tableau









another Q-tableau

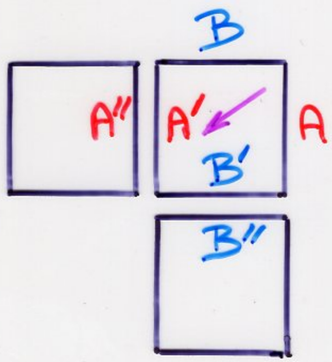
Rothe diagram
of a permutation

Definition $w(U,D)$ word of $\{U,D\}^*$

complete Q- tableau

- labeling of the cells of a Ferrers diagram $F = F(w)$ by the set R of rewriting rules
- with "compatibility" adjacent cells

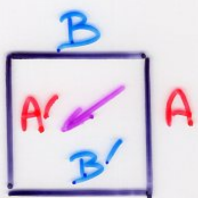
i.e.



$A, A', B, B' \in \{U, D, I_v, I_h\}$
A'', B''

$$BA \rightarrow A'B'$$

then $A' = A''$, $B' = B''$

- if the cell 

is at the NE border of F, then $B = U$, $A = D$

$\varphi : R \longrightarrow L$

map

R = set of rewriting rules of the homogenous system associated to Q

here 5 terms

L a set of "labels" (for the cell of $[n]\times[n]$)

<u>examples</u>

$L = \{\square, \square\}$

<u>examples</u>

$L = \{\square, \square\}$

φ satisfies $(*)$:

$(*)$ if $\varphi(\alpha \to \beta) = \varphi(\alpha' \to \beta')$
then $\alpha \neq \alpha'$

i.e. in a a single commutation equation

$\alpha = \beta_1 + \dots + \beta_r$

all elements $\varphi(\alpha \to \beta_i) \in L$ are $\neq$

L: set of labels

$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

if $\varphi : R \to L$
satisfies $(*)$
then

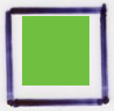
complete Q-tableaux $\xleftrightarrow{\text{bijection}}$ Q-tableaux

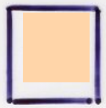
$$\varphi : \begin{array}{l} UD \to DU,\ UI_v \to I_vU, \\ I_hD \to DI_h,\ I_hI_v \to I_vI_h \end{array} \longrightarrow \square$$

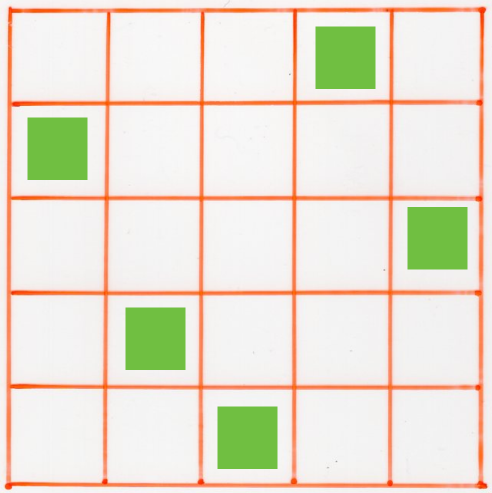
empty cell

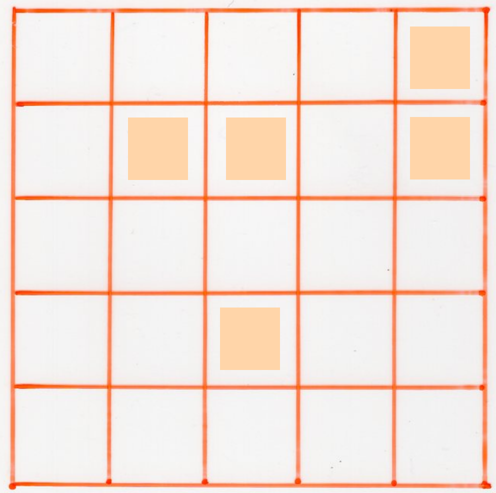
$$\varphi : \begin{array}{l} UD \to I_vI_h,\ UI_v \to I_vU, \\ I_hD \to DI_h,\ I_hI_v \to I_vI_h \end{array} \longrightarrow \square$$

empty cell

$\varphi(UD \to I_vI_h) =$ 

$\varphi(UD \to DU) =$ 





"The cellular ansatz"

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

$UD = DU + Id$

permutations

towers placements

(i) first step

commutations

rewriting rules

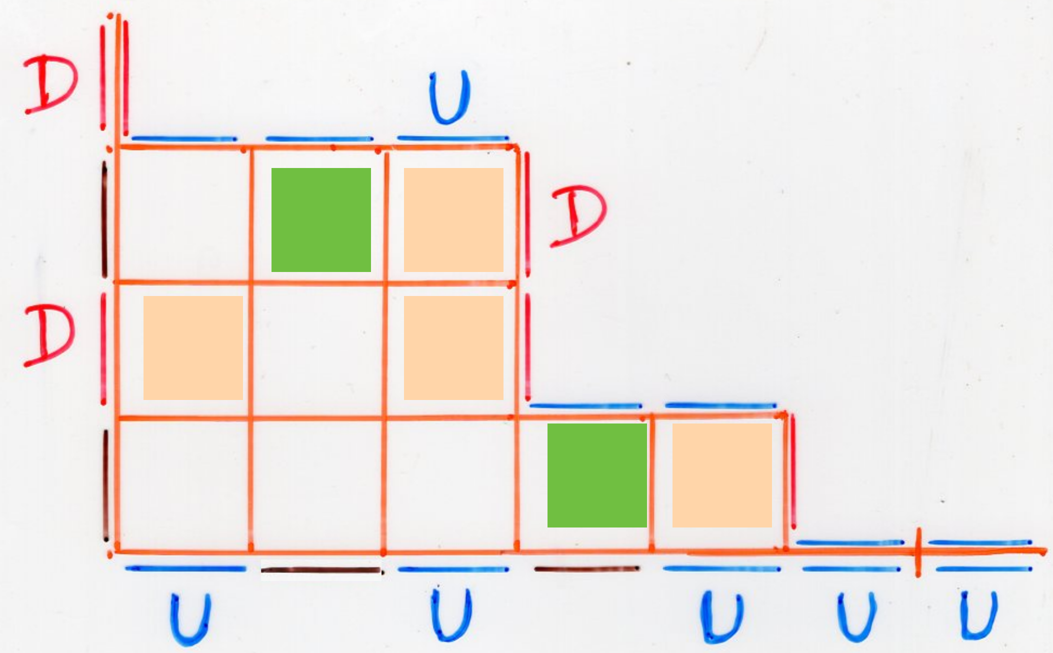
planarization

$w = D\,U^3 D^2 U^2 D\,U^2$

$w \longrightarrow F = F(w)$

F Ferrers diagram

Rooks placement



The cellular ansatz
second part:

guided construction
of a bijection
from the representation of U and D

$$UD = DU + I$$

"The cellular ansatz"

quadratic algebra Q

$UD = DU + Id$

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

representation of Q by combinatorial operators

bijections

RSK $\longleftrightarrow$

pairs of Young tableaux

(i) first step

(ii) second step

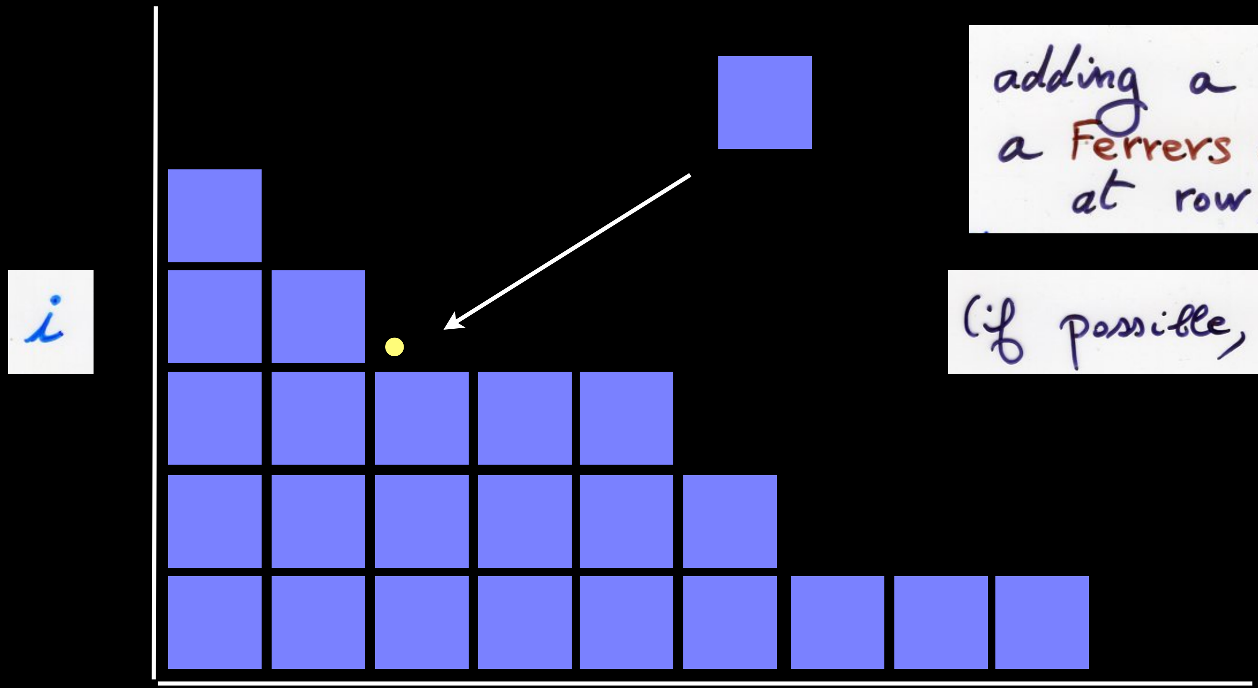
commutations

rewriting rules

planarization

notations

operator U_i



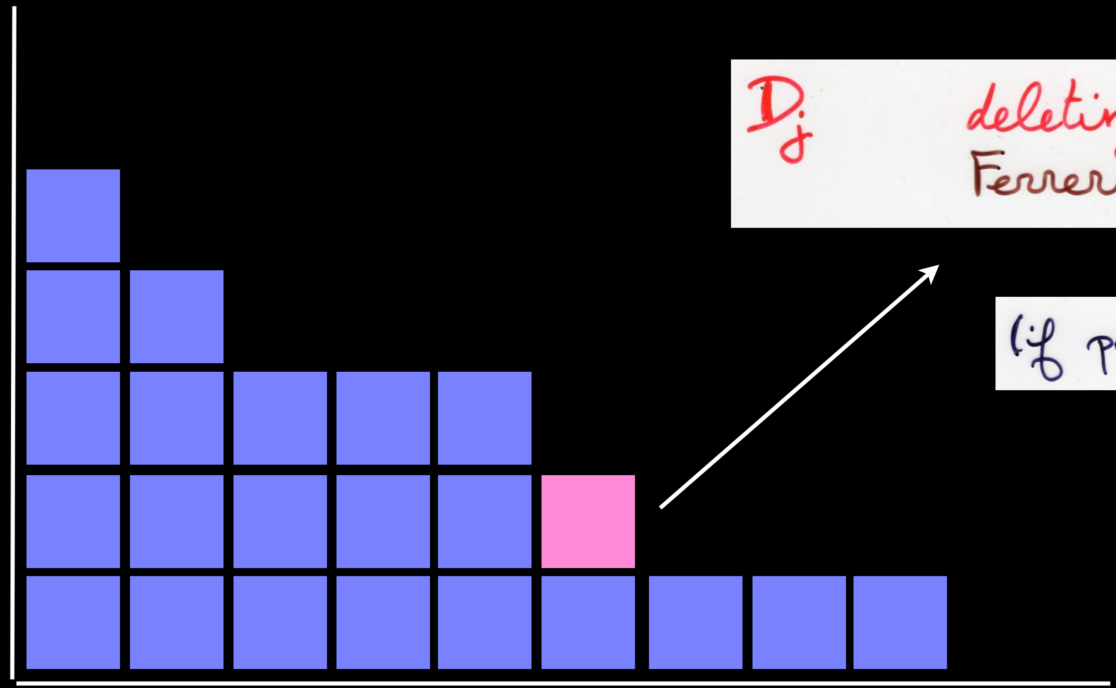
adding a cell in a Ferrers diagram ρ at row i

(if possible, else $U_i(\rho)=0$)

$$U_i(\rho) = \rho + (i)$$

$$D_j(\rho) = \rho - (j)$$

j

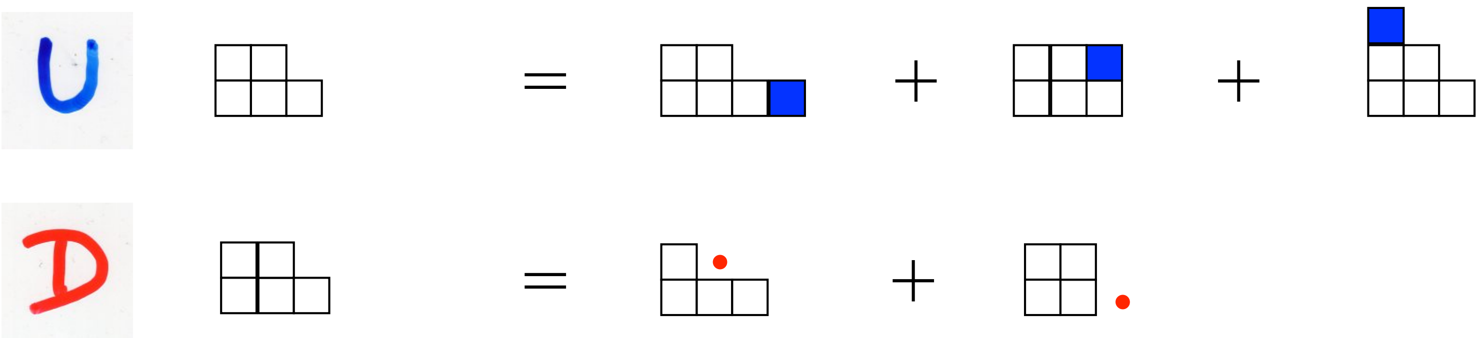


D_j deleting a cell in a Ferrers diagram ρ at row j

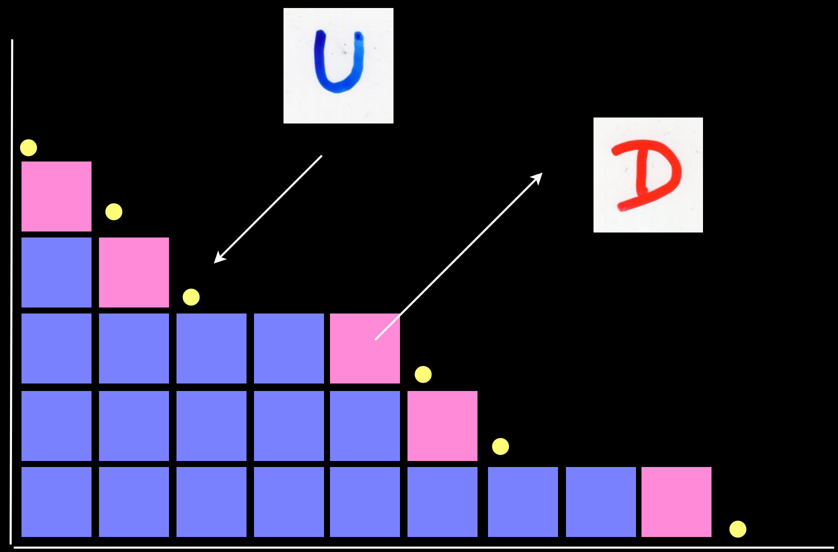
(if possible, else $D_j(\rho) = 0$)

$$U = \sum_{i \geqslant 1} U_i \qquad\qquad D = \sum_{i \geqslant 1} D_i$$

U and D are operators acting on the vector space generated by Ferrers diagrams



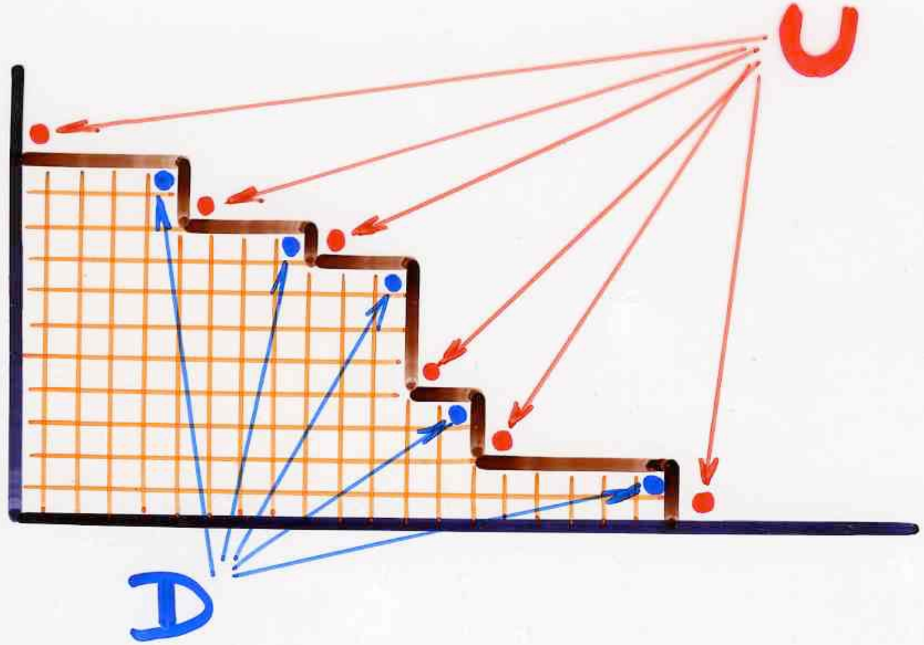
operators
U and D



Young lattice

U adding
D deleting
a cell in a Ferrers diagram

$$UD = DU + I$$



In this course, product of operators are written from left to right

$$A\,B\,(\mu) = B(A(\mu))$$

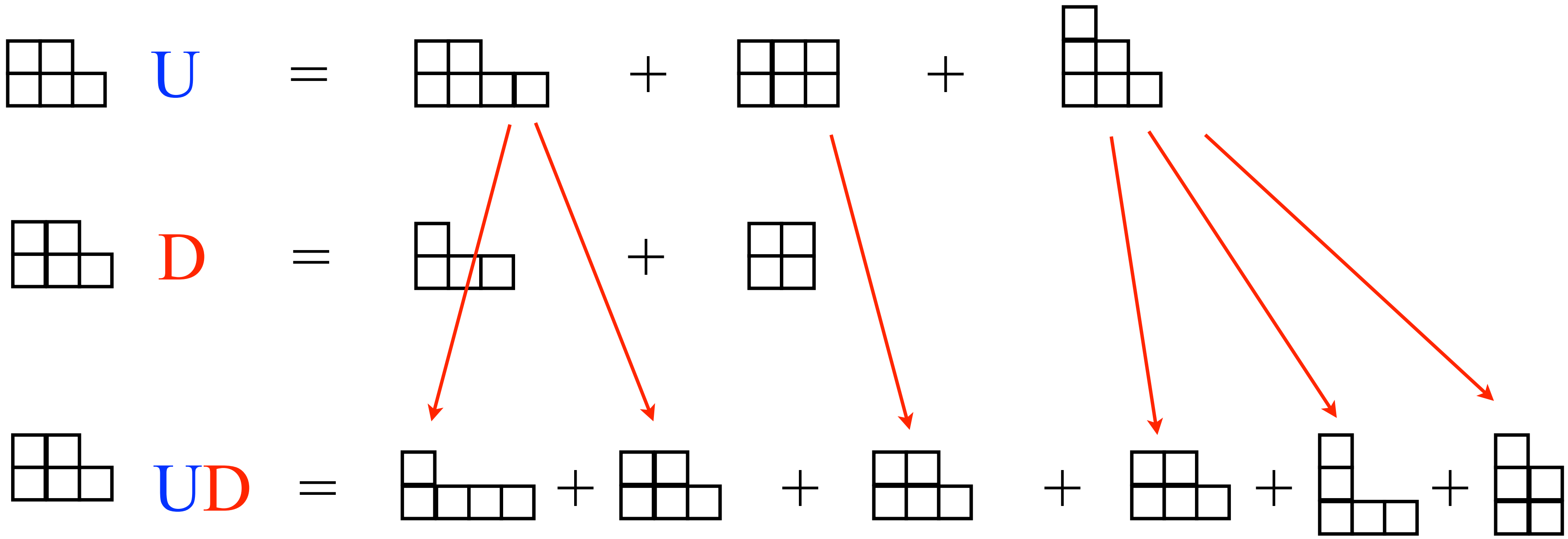
should be written $(\mu)\,A\,B$

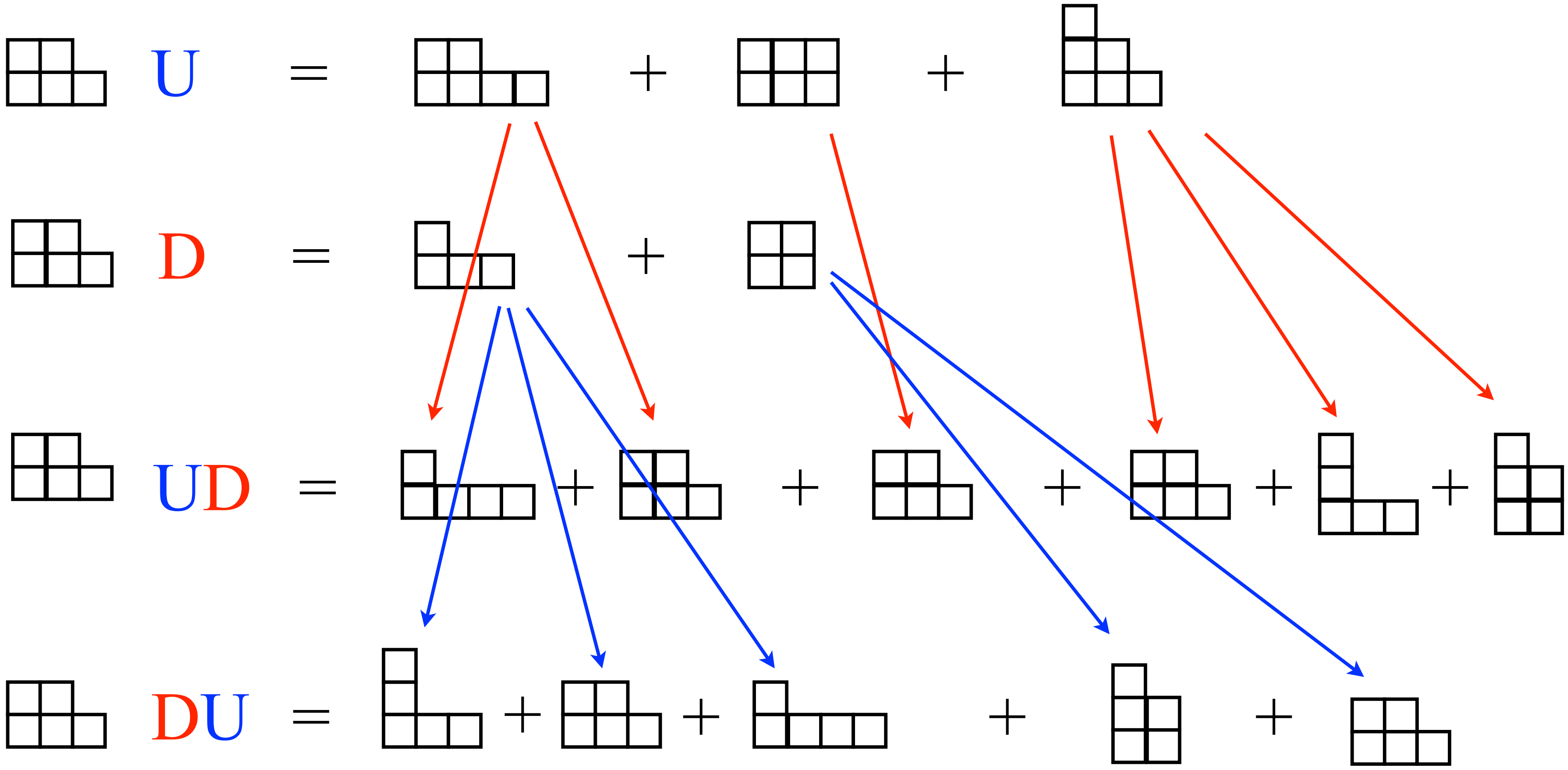
or $\langle \mu |\, A\, B$

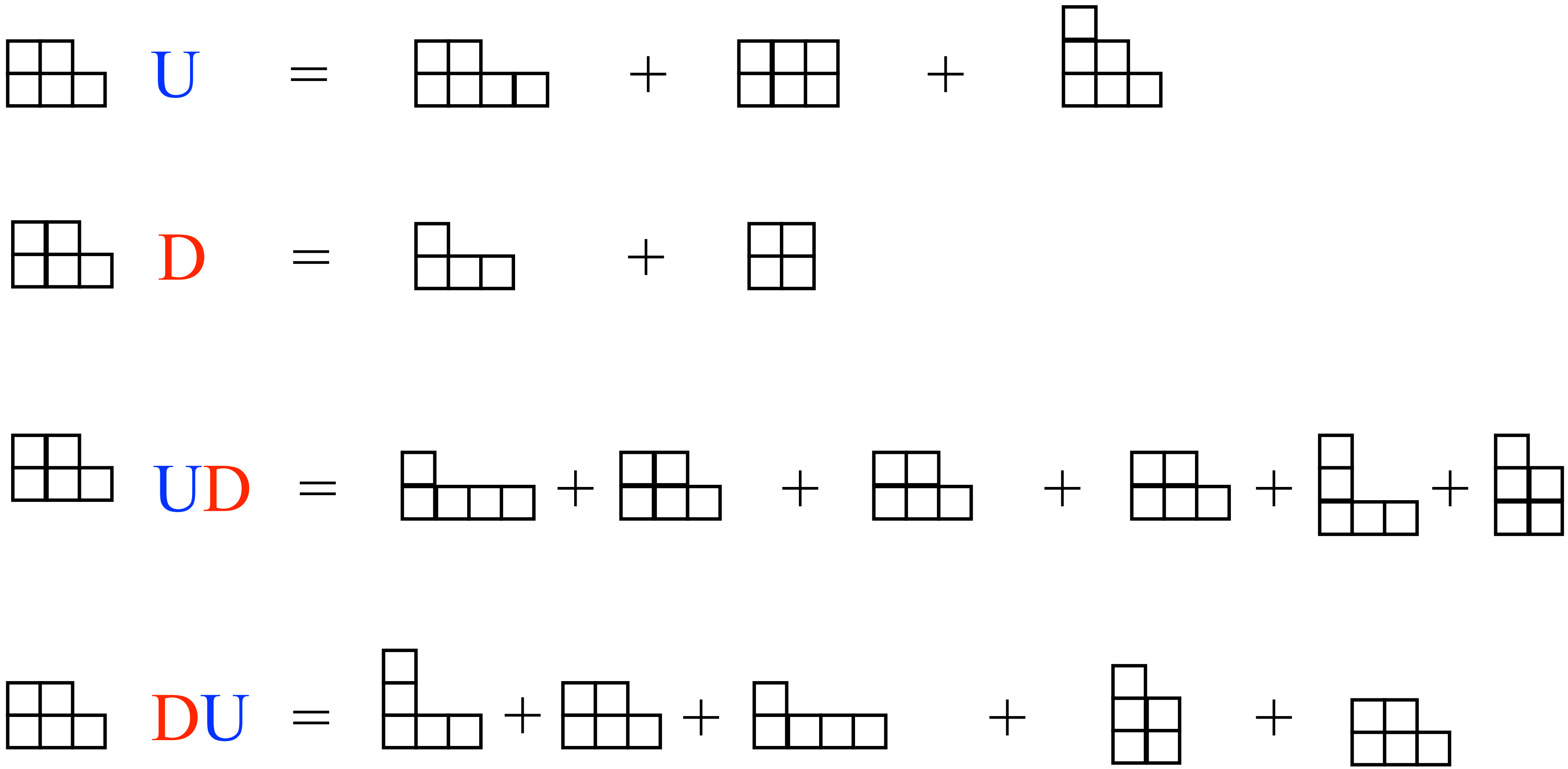
with operators written from right to left

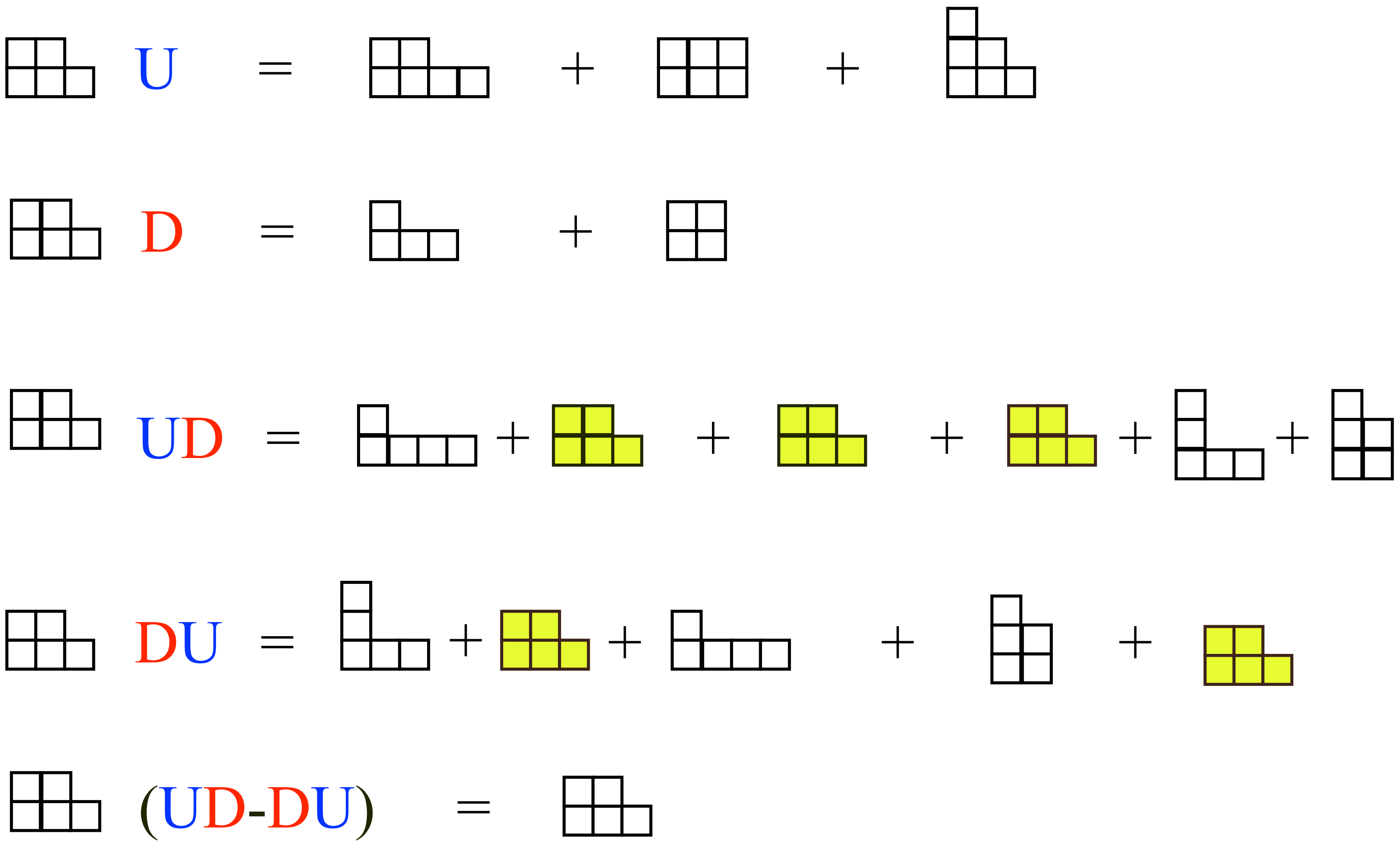
$$B\,A\,(\mu) = B(A(\mu))$$

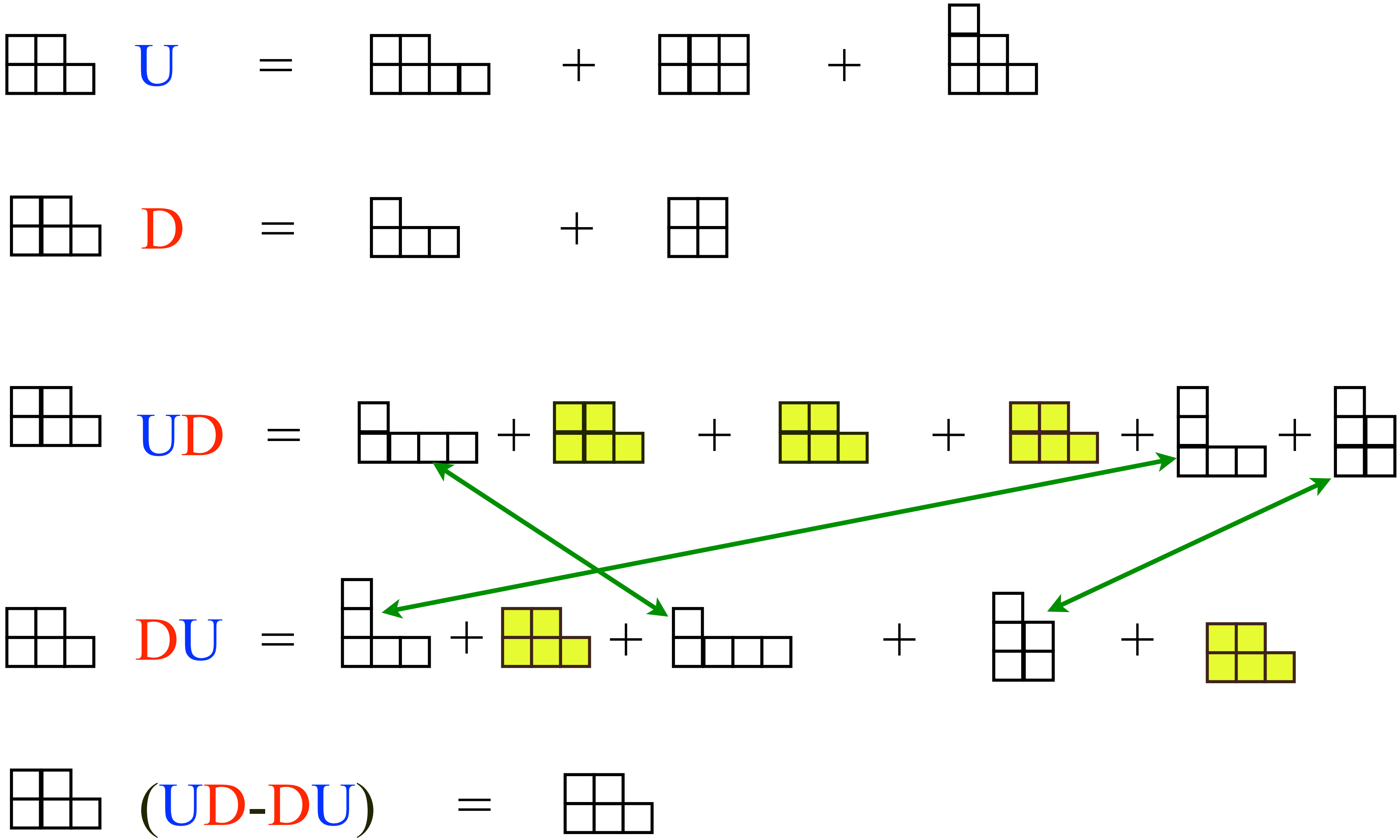
$UD - DU = I$ becomes $DU - UD = I$

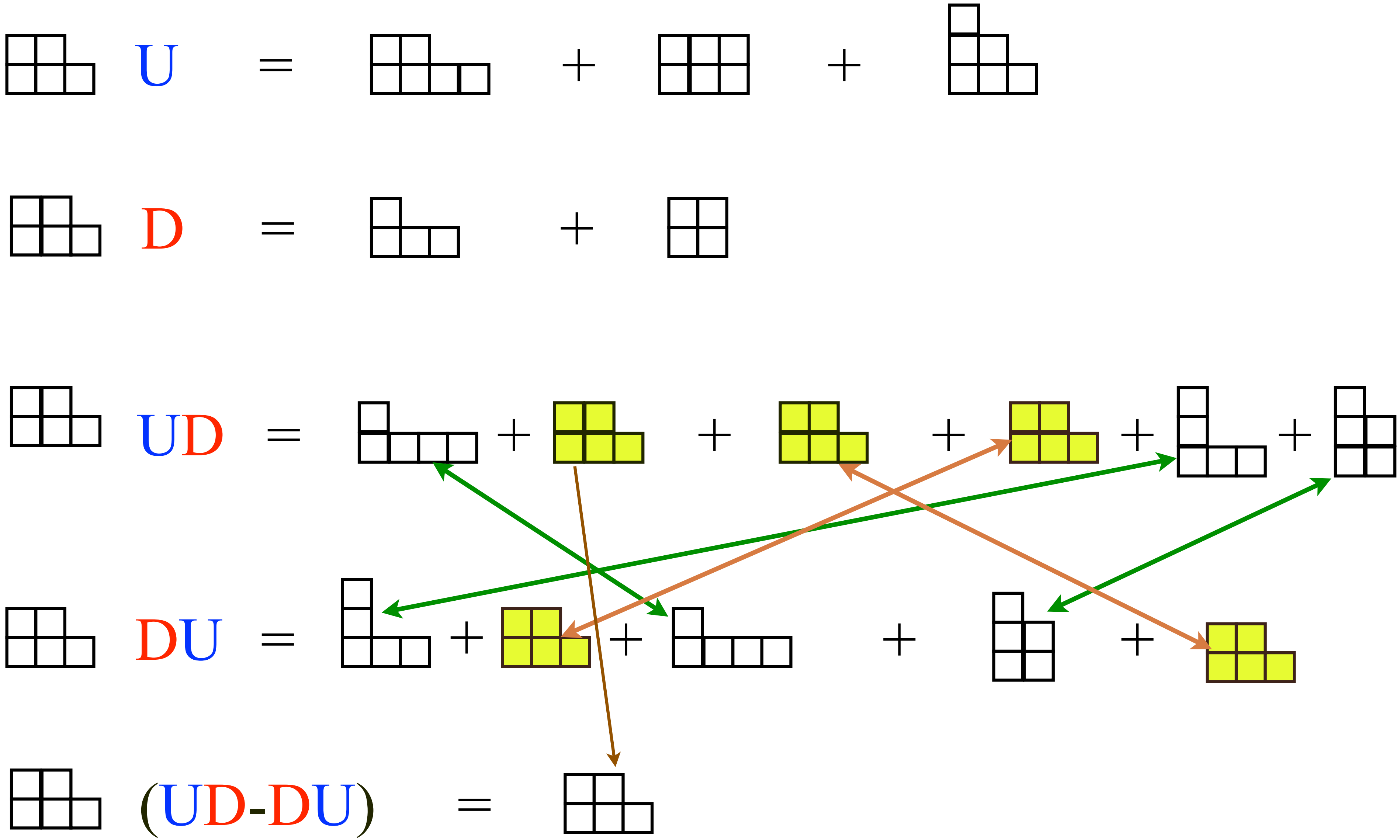

U
D
UD


U
D
UD
DU

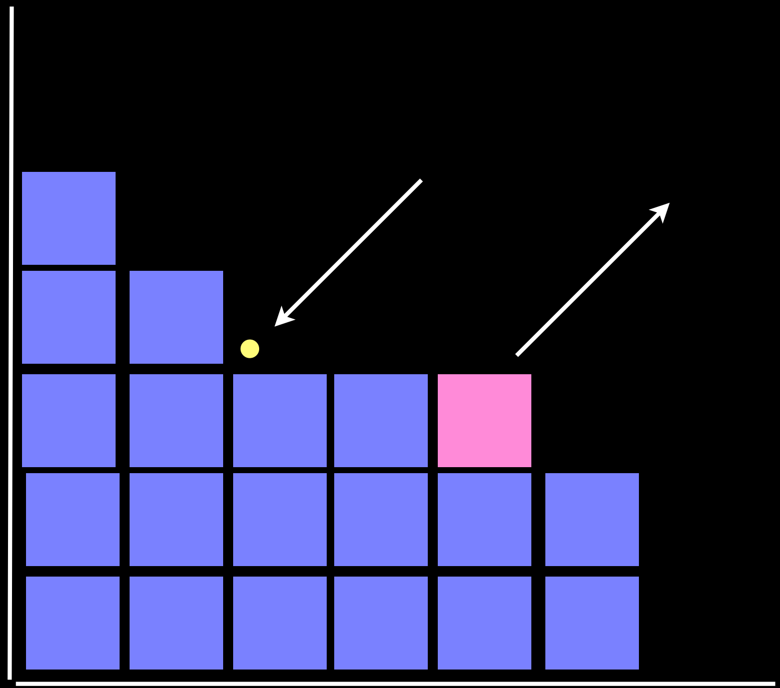

U
D
UD
DU

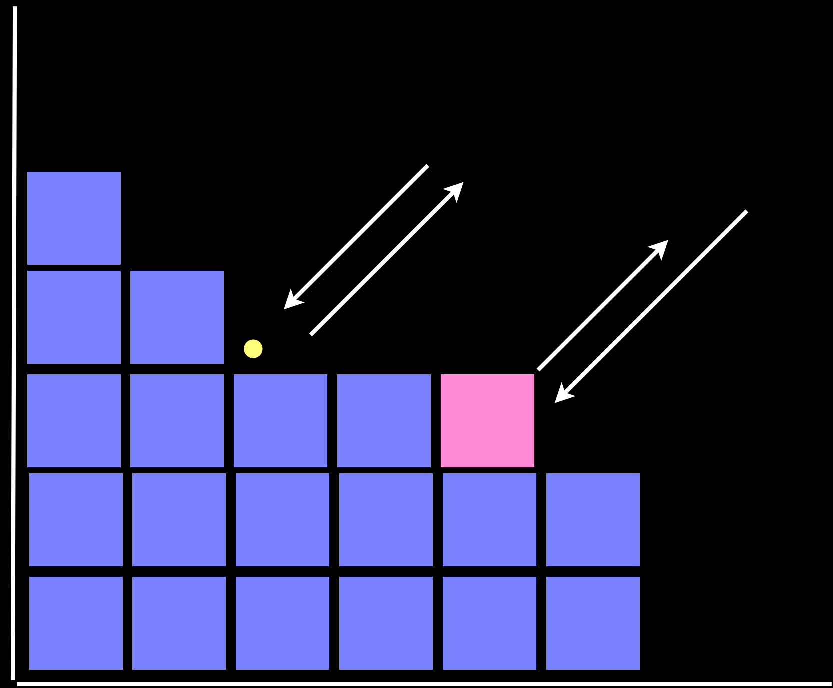

U =
D =
UD =
DU =
(UD-DU) =


U
D
UD
DU
(UD-DU)

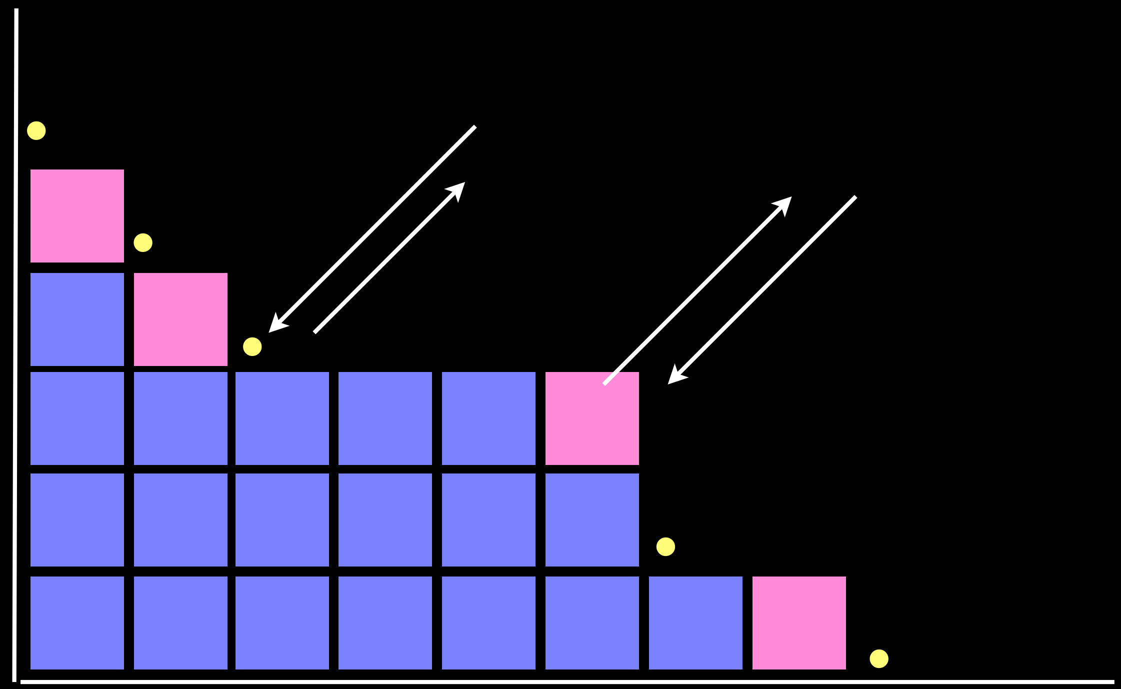

U
D
UD
DU
(UD-DU)

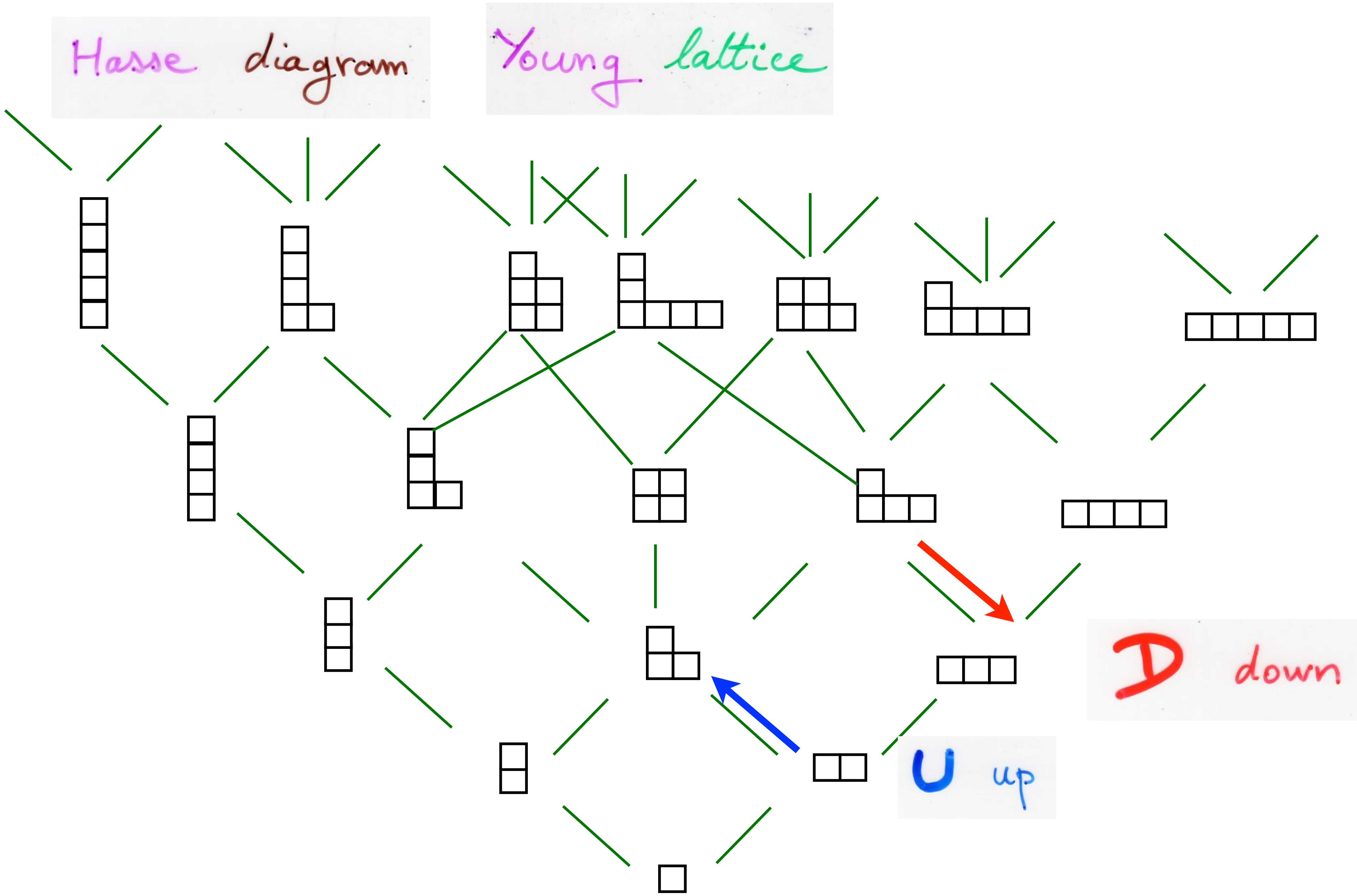
$$UD = DU + I$$





$$UD = DU + I$$




Hasse diagram
Young lattice
D down
U up

differential poset

Fomin (1992, 1995)

Stanley (1988, 1990)

Roby (1991)

$$UD = DU + I$$

U up

D down

direct proof of the identity

permutations

pairs of Young tableaux, same shape

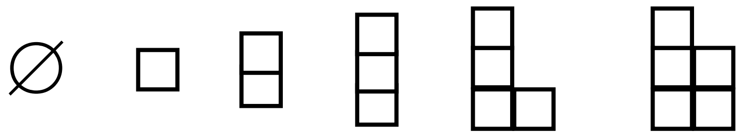
$$n! = \sum_{\lambda} (f_\lambda)^2$$

partition of n

$$UD = DU + I$$

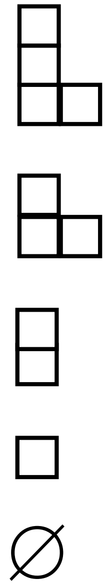
$$\langle \emptyset \,|\, U^n D^n \,|\, \emptyset \rangle$$

$\emptyset$ empty Ferrers diagram



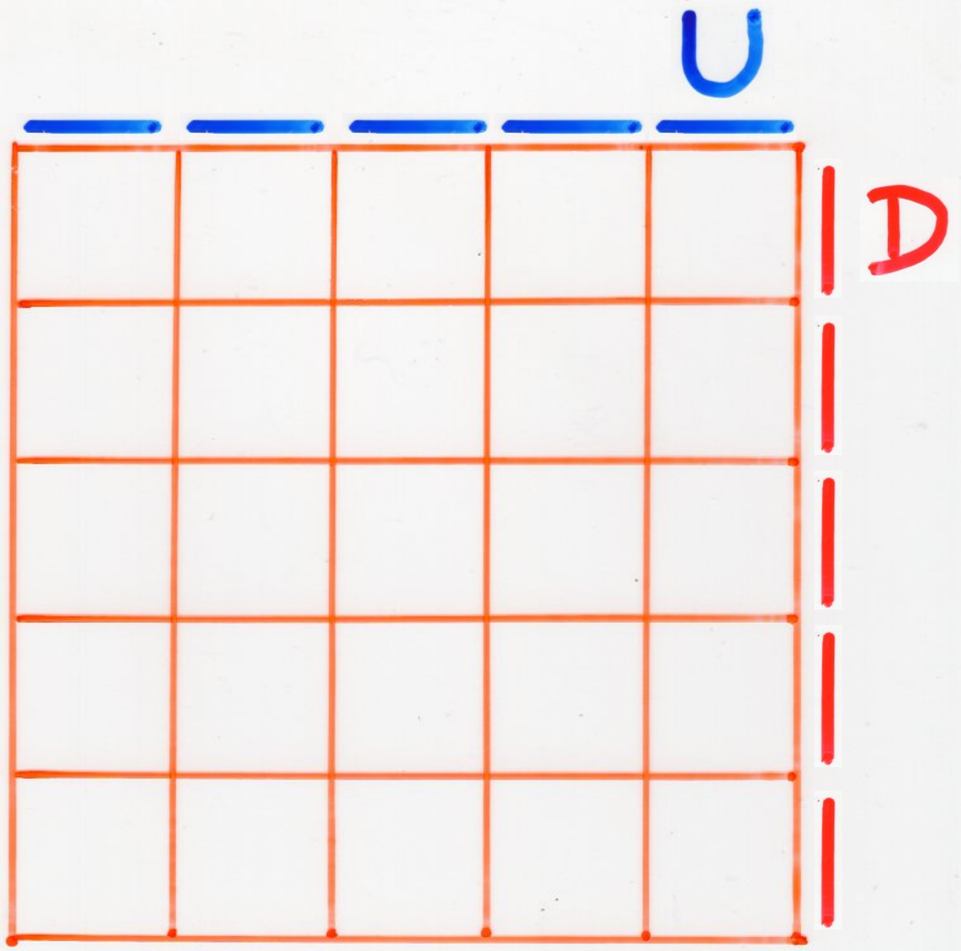
$$= \sum_{\lambda} \left(f_\lambda \right)^2$$

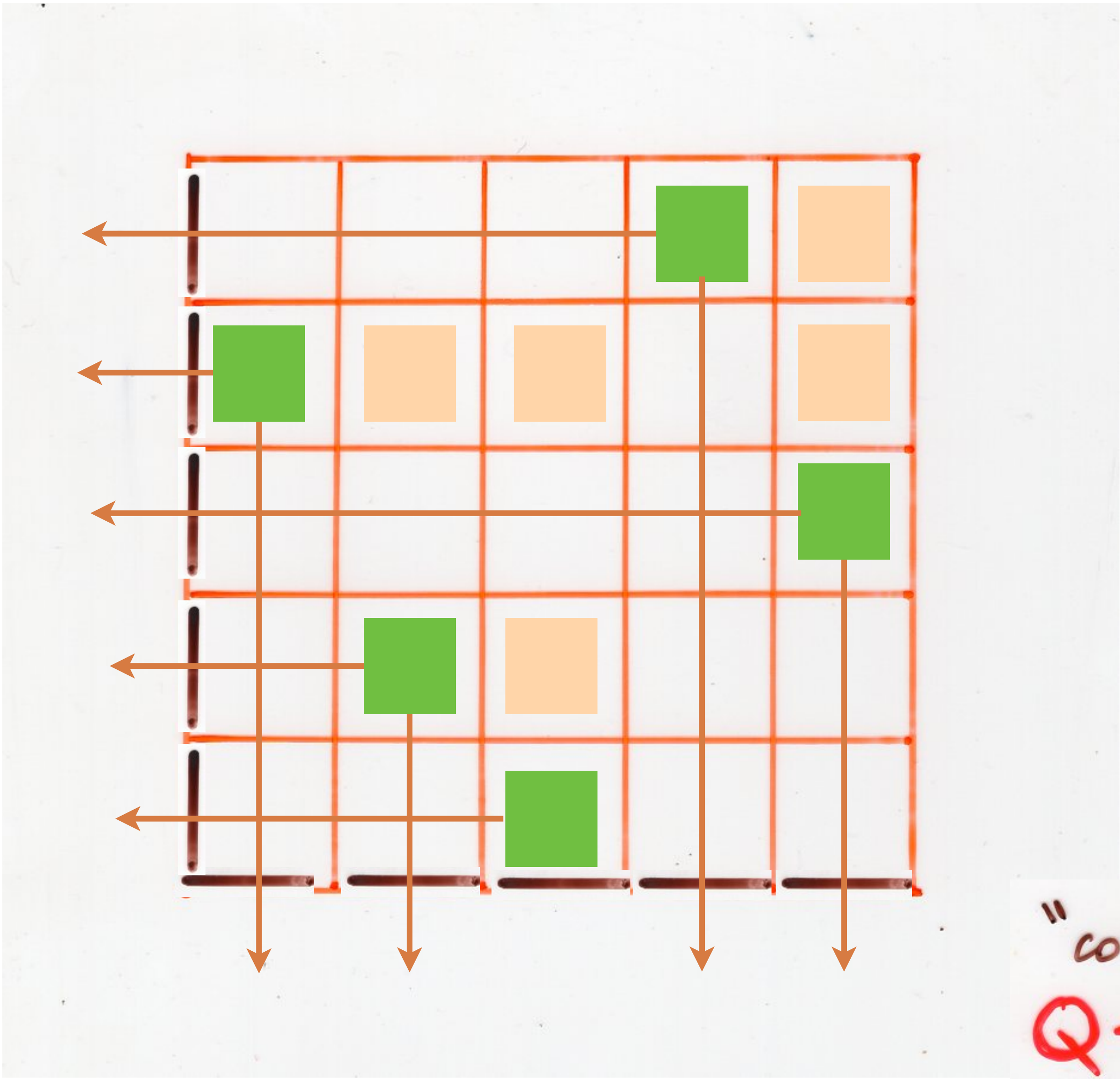
λ partition of n



$$= \sum_{i \geq 0} c_{n,i} \langle \emptyset \,|\, D^i U^i \,|\, \emptyset \rangle$$

$$= c_{n,0} \qquad = n!$$


U
D

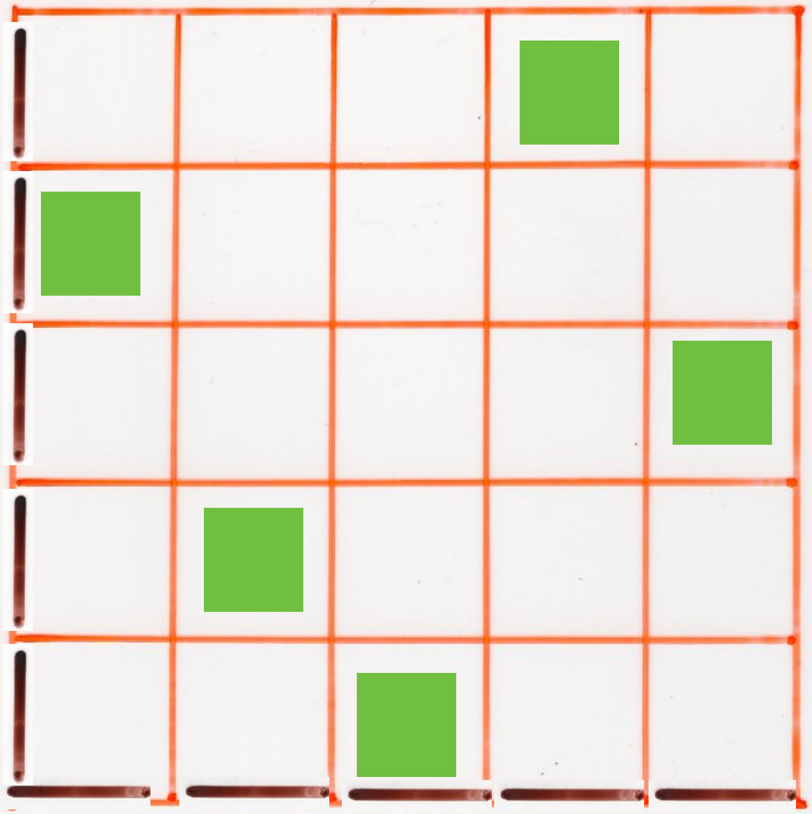


"complete"
Q - tableau

$$c_{n,0} = n!$$

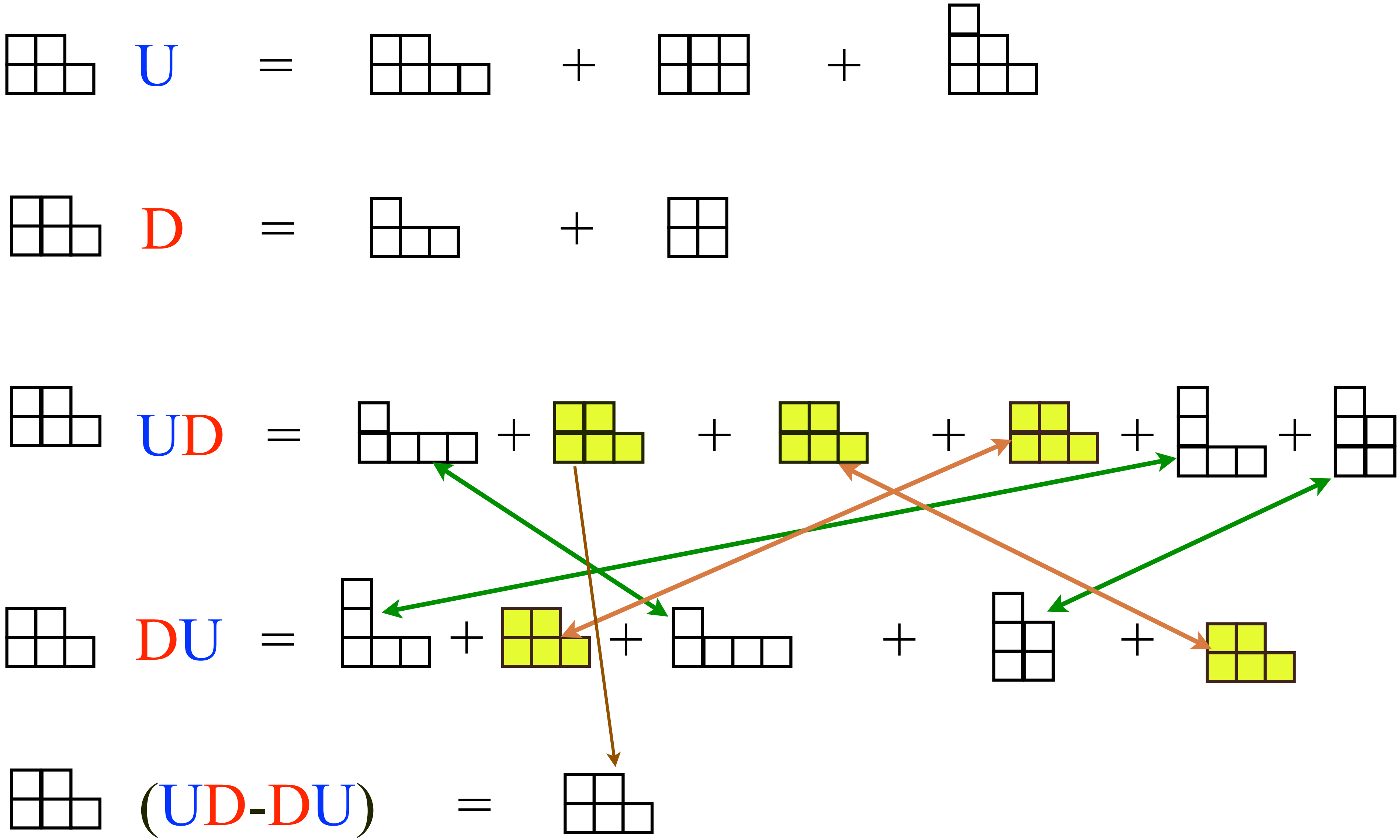
$$= \sum_{\lambda} (f_\lambda)^2$$

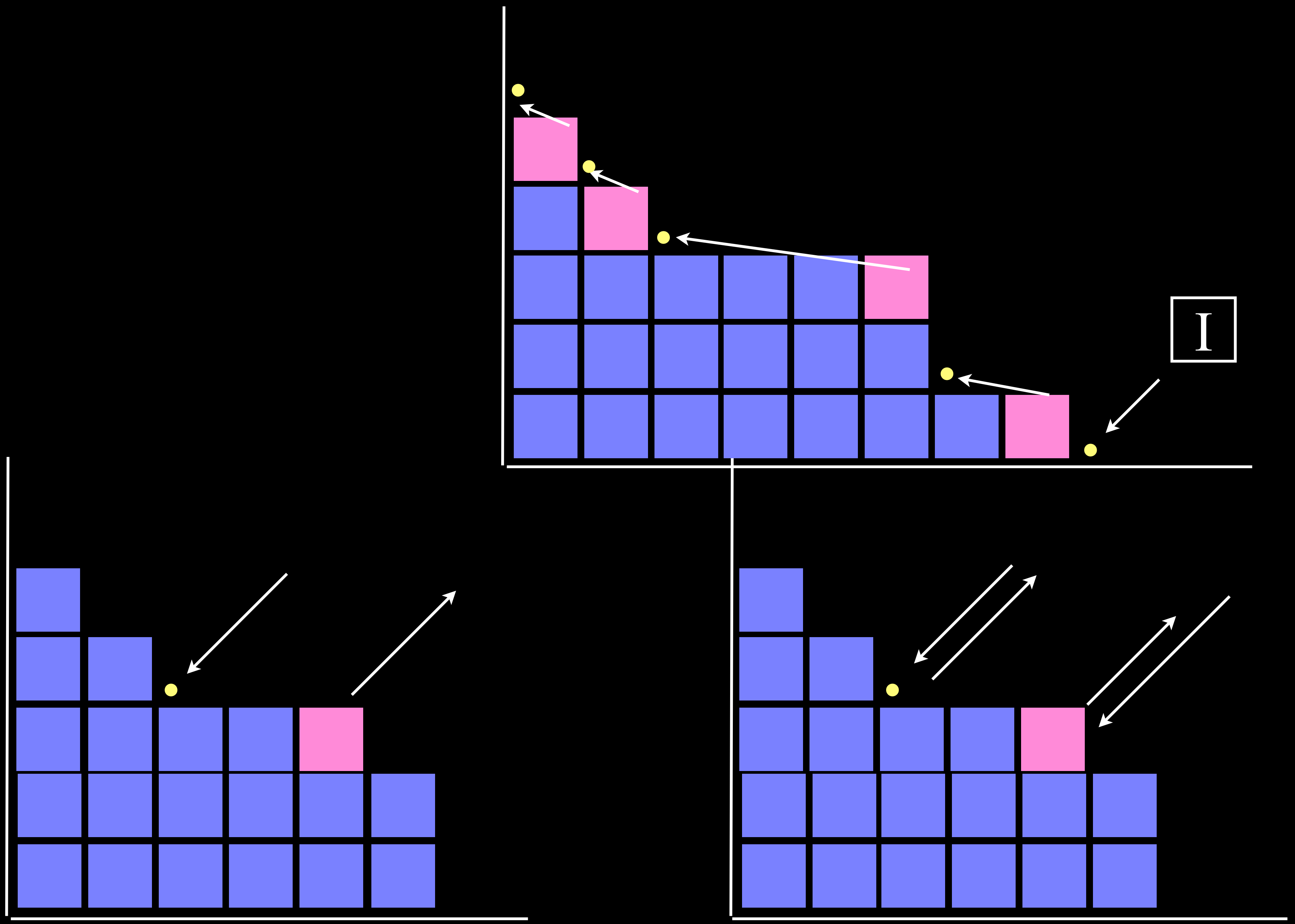
λ partition of n



permutation as a Q-tableau

construction of the RSK correspondence
by «propagation» on the grid
of an elementary «diagram bijection»
related to each cell of the grid


U =
D =
UD =
DU =
(UD-DU) =


I

$$UD = DU + I_v I_h$$

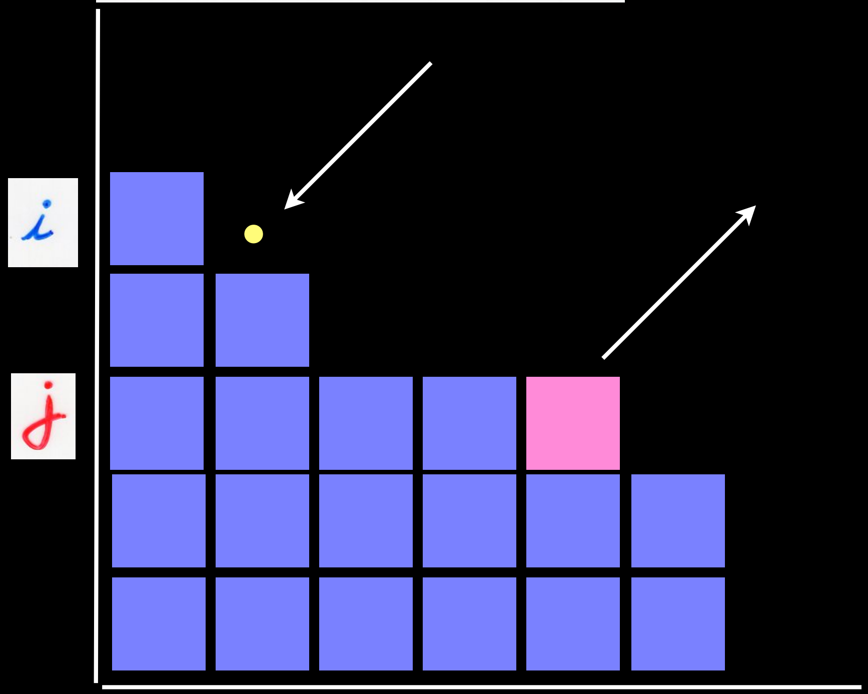
"commutation diagrams"

$$\begin{tikzcd}
\nu \arrow[d, "I_v"'] \arrow[dr, phantom, "\bullet"] & \\
\rho \arrow[r, "I_h"'] & \mu
\end{tikzcd}
\qquad
\begin{tikzcd}
(\nu, p) \arrow[d, "D"'] & \\
(\rho, m) \arrow[r, "U"'] & \mu
\end{tikzcd}
\quad \xleftrightarrow{\text{bijection}} \quad
\begin{tikzcd}
(\nu, p') \arrow[r, "U"] & (\lambda, m') \arrow[d, "D"] \\
 & \mu
\end{tikzcd}$$

p, m, p', m' are "positions"

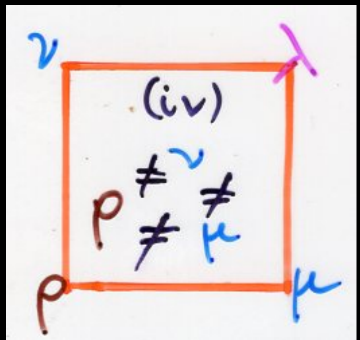
in ν, ρ, ν, λ respectively

$$\begin{array}{ccc} (\nu, p) & & (\nu, p') \xrightarrow{U} (\lambda, m') \\ \Big\downarrow D & \longleftrightarrow & \Big\downarrow D \\ (\rho, m) \xrightarrow{U} \mu & & \mu \end{array}$$



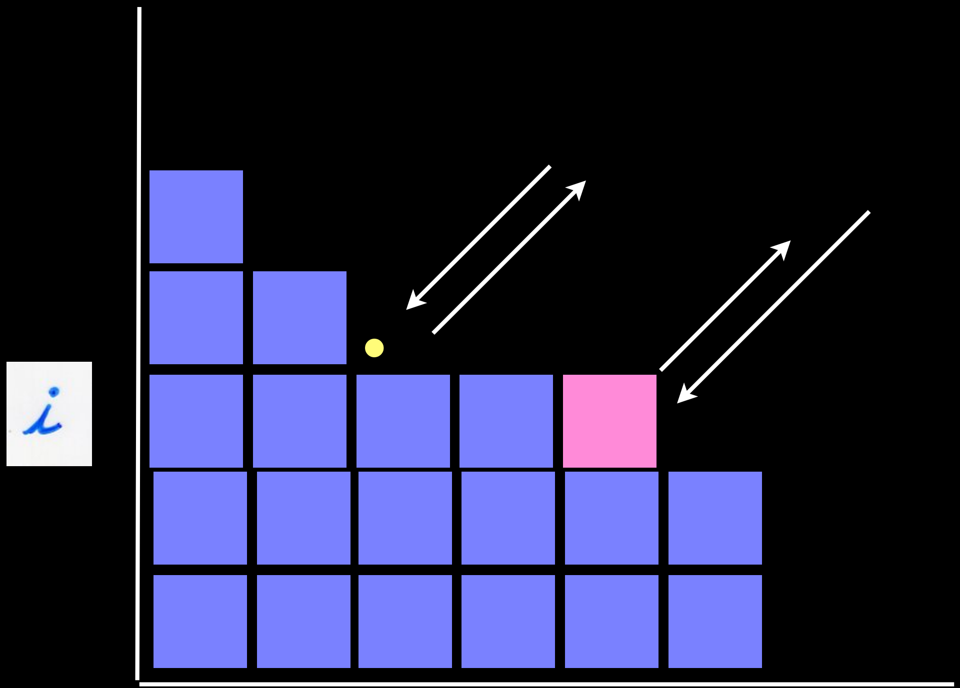
$p = j$
$m = i$

$p' = i$
$m' = j$



$\nu = \rho + (j)$
$\mu = \rho + (i)$
$\lambda = \rho + (i) + (j)$

$$\begin{tikzcd} (\nu,p) \arrow[d, "D"'] & \\ (\rho,m) \arrow[r, "U"'] & \mu \end{tikzcd} \longleftrightarrow \begin{tikzcd} (\nu,p') \arrow[r, "U"] & (\lambda,m') \arrow[d, "D"] \\ & \mu \end{tikzcd}$$

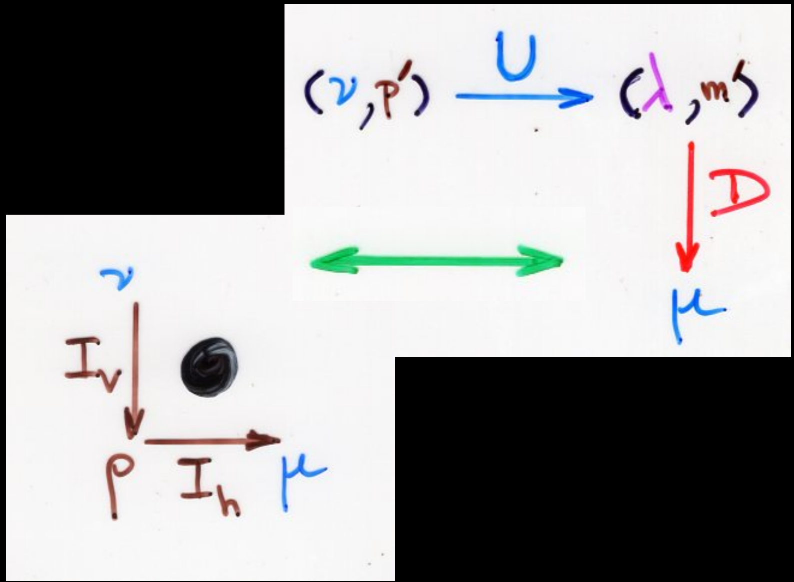


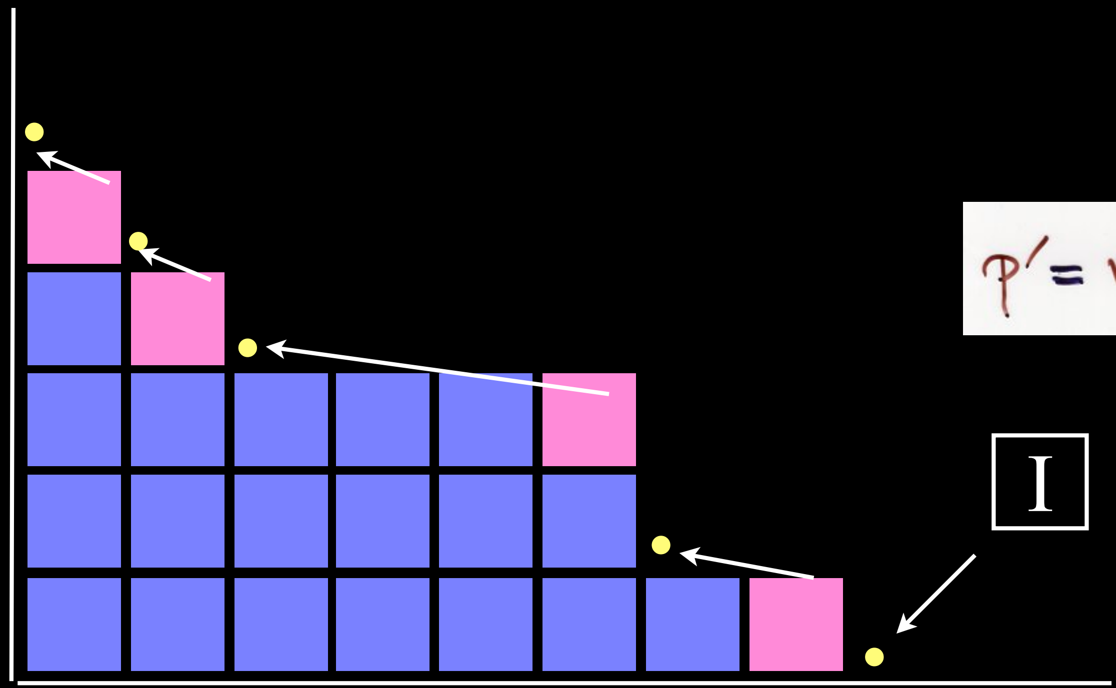
$p = i$
$m = i$

$p' = i+1$
$m' = i+1$

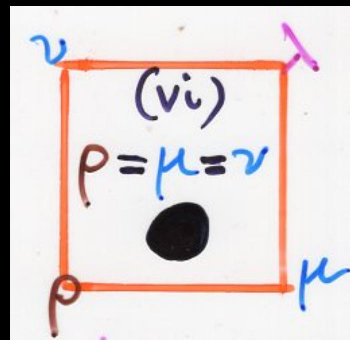
$$\begin{tikzcd} \nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\begin{matrix}(v)\\ \rho \neq \mu = \nu\end{matrix}"] & \lambda \arrow[d, dash] \\ \rho \arrow[r, dash] & \mu \end{tikzcd}$$

$\mu = \nu = \rho + (i)$
$\lambda = \mu + (i+1)$

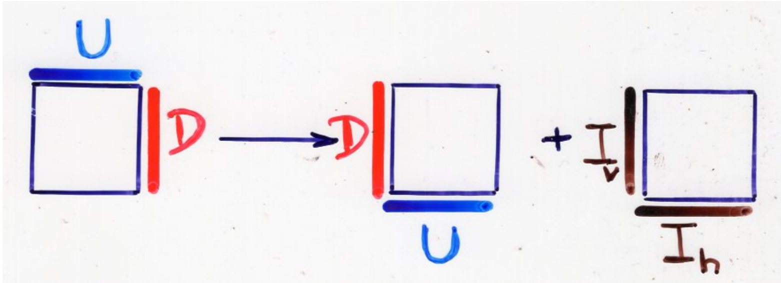


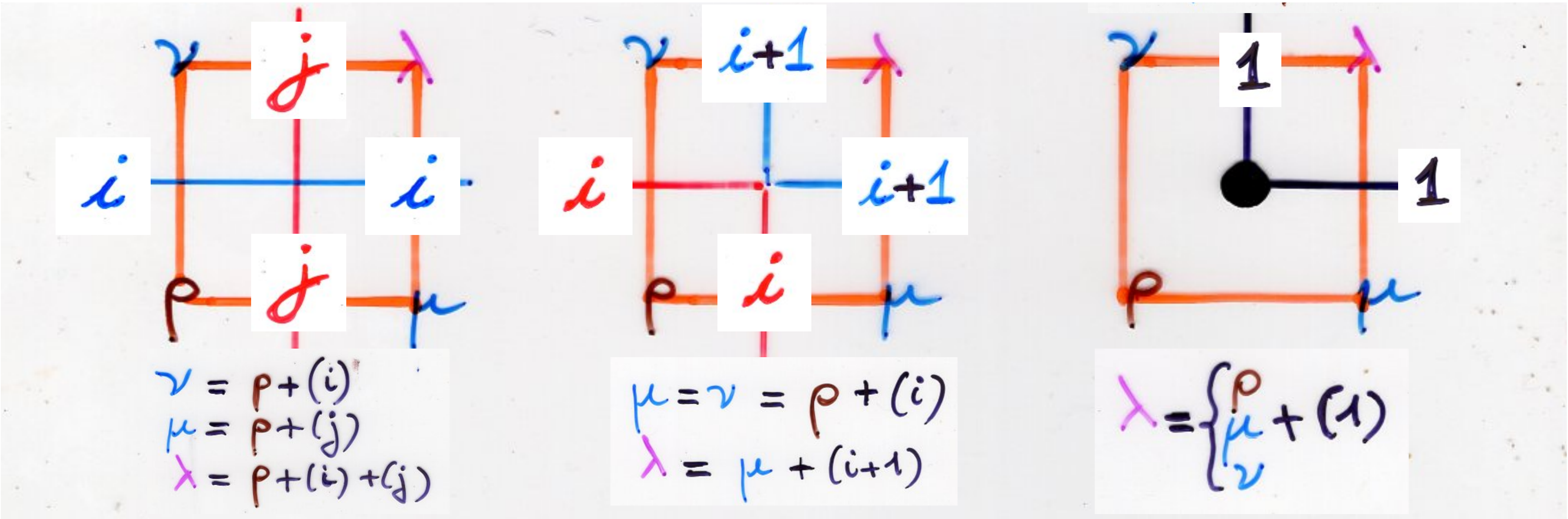


$p' = m' = 1$



$\lambda = \left\{ \begin{matrix} \rho \\ \mu \\ \nu \end{matrix} \right. + (1)$

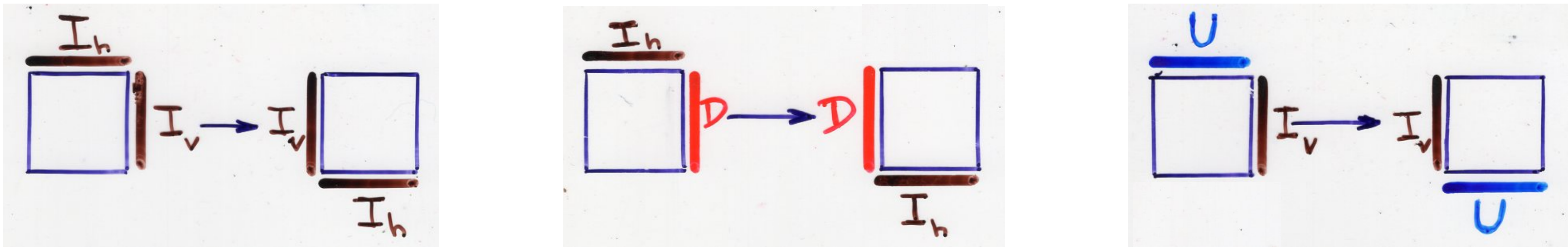

U
D
D
U
+ I_v
I_h

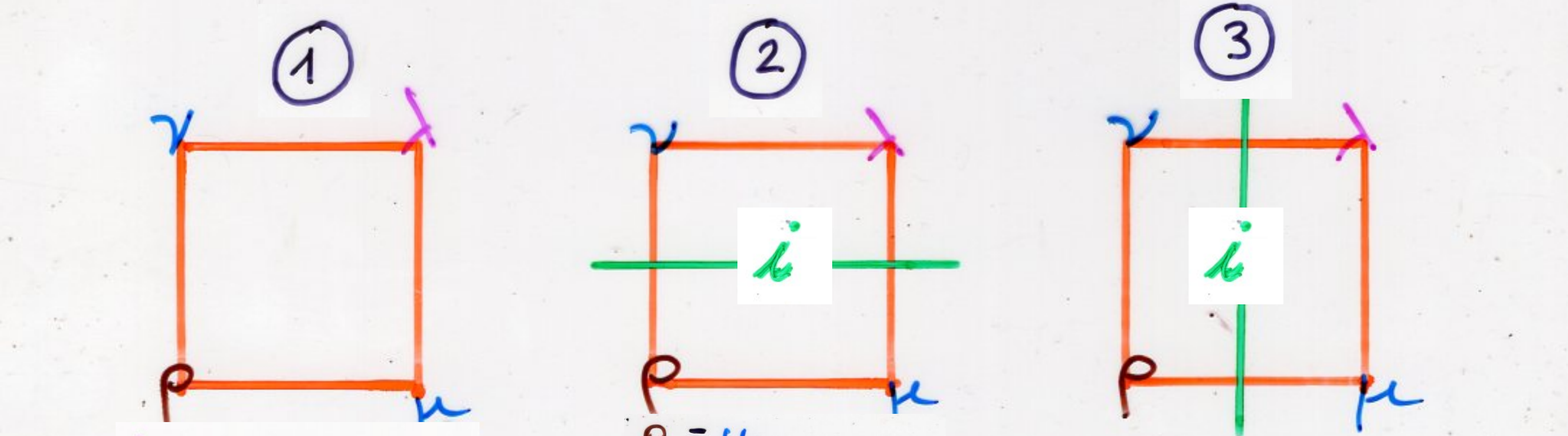

ν
j
λ
i
i
j
ρ
μ
ν = ρ + (i)
μ = ρ + (j)
λ = ρ + (i) + (j)
ν
i+1
λ
i
i+1
i
ρ
μ
μ = ν = ρ + (i)
λ = μ + (i+1)
ν
1
λ
1
ρ
μ
λ = {ρ, μ, ν} + (1)

4

5

6





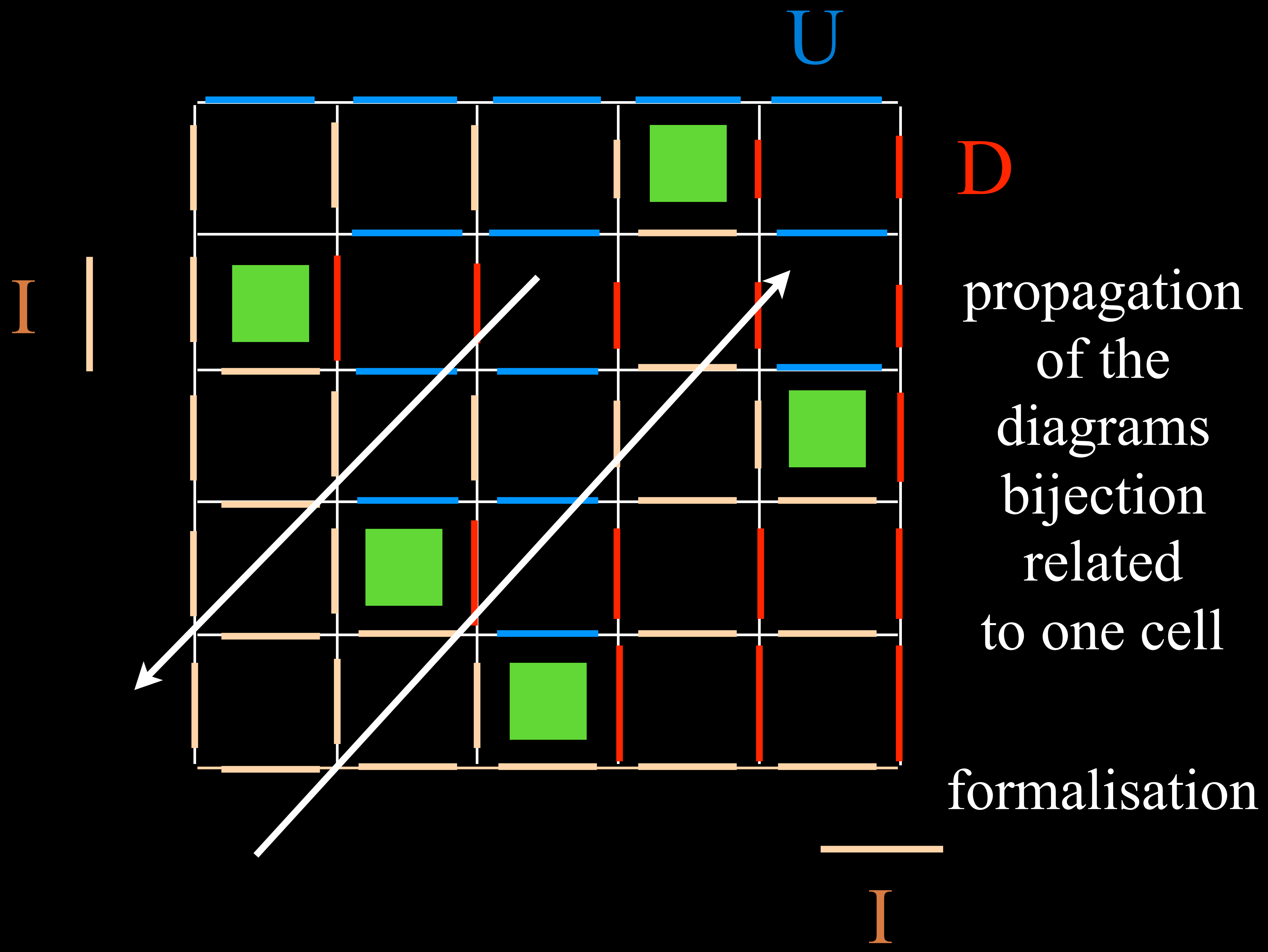
$\lambda = \rho = \mu = \nu$

$\rho = \mu$

$\lambda = \nu = \rho + (i)$

$\rho = \nu$

$\lambda = \mu = \rho + (j)$


U
D
I
propagation
of the
diagrams
bijection
related
to one cell
formalisation
I

For $w = w_1 \cdots w_n$ word of $\{U, D, I_h, I_v\}^*$

we consider sequences h

$$h = \left((\mu_1, p_1), \ldots, (\mu_n, p_n), \mu_{n+1} \right)$$

where μ_i, $i=1,..,n+1$ are partitions (Ferrers diagrams)

and for $i=1,..,n$ μ_{i+1} is obtained from μ_i by applying the operator w_i at position p_i

$$w = w(h)$$

if $w_i = I_h$ or I_v, then $(\mu_{i+1}, p_{i+1}) = (\mu_i, p_i)$

h "histories" admissible sequence

2-colored vacillating tableaux

$(P, Q) \rightarrow (\alpha, \beta) \rightarrow$ sequence h

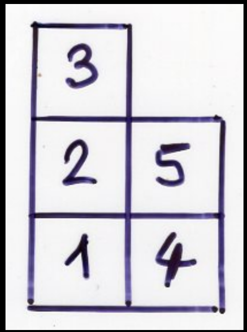
Young tableaux same shape λ

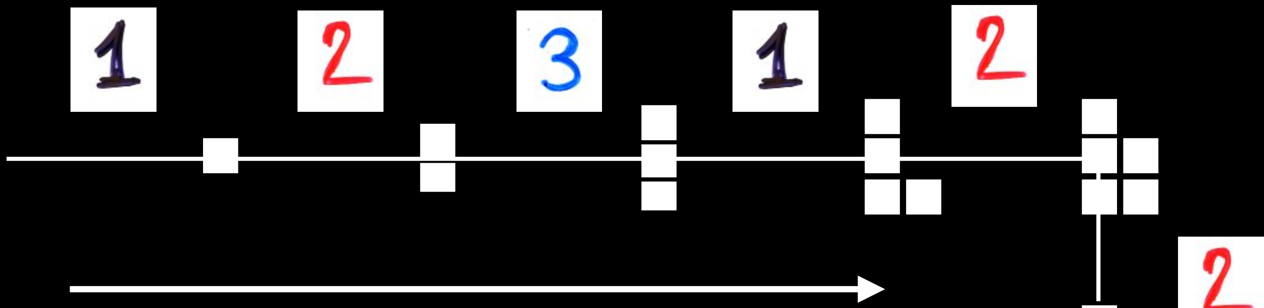
pair of maximal chains $\emptyset \rightarrow \lambda$

$$h = ((\mu_1, \rho_1), \ldots, (\mu_{2n}, \rho_{2n}), \mu_{2n+1})$$

with $\mu_1 = \mu_{2n+1} = \emptyset$ and

$$w(h) = U^n D^n$$

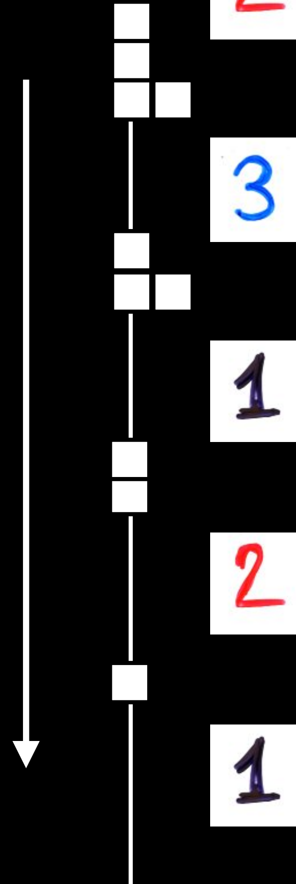


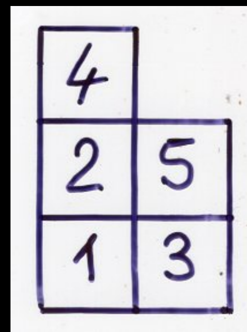


$$h = \left((\mu_1, p_1), \ldots, (\mu_{2n}, p_{2n}), \mu_{2n+1} \right)$$

with $\mu_1 = \mu_{2n+1} = \emptyset$ and

$$w(h) = U^n D^n$$

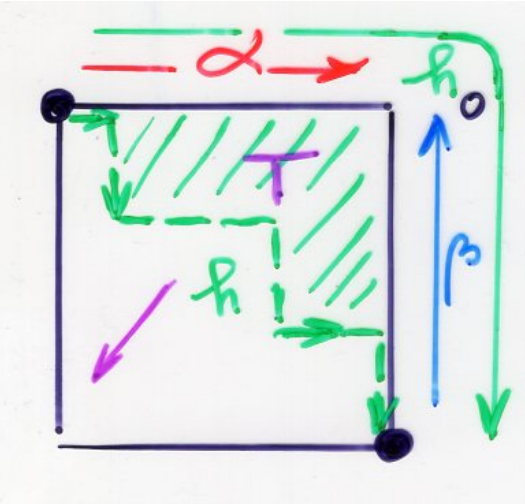




Starting from $h_0(\alpha, \beta) = h_0(P, Q)$ $T = \emptyset$

we "propagate" the "commutation diagrams" through the lattice $[n] \times [n]$.

At any step, we have a pair



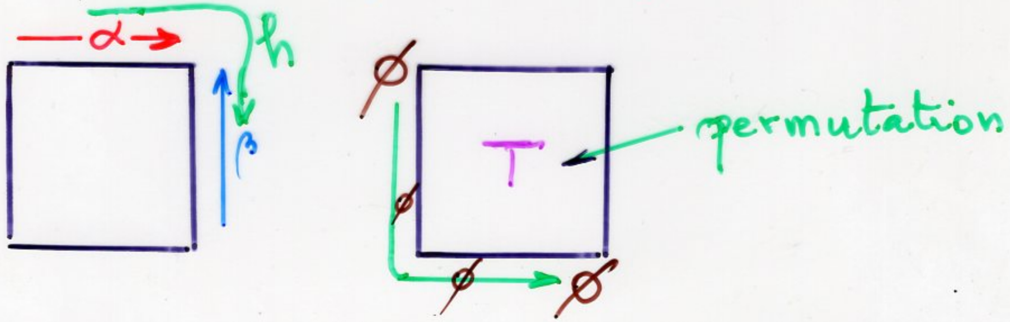
(h, T)

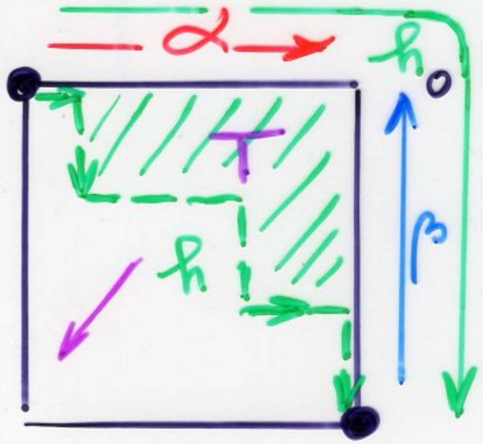
T tableau above the path associated to $w(h)$ with cells labeled by $\square$ $\boxed{\bullet}$

$(h, T) \longleftrightarrow h_0 = h(\alpha, \beta)$ are in bijection

By recurrence

Thus $h(\alpha, \beta)$ in bijection with





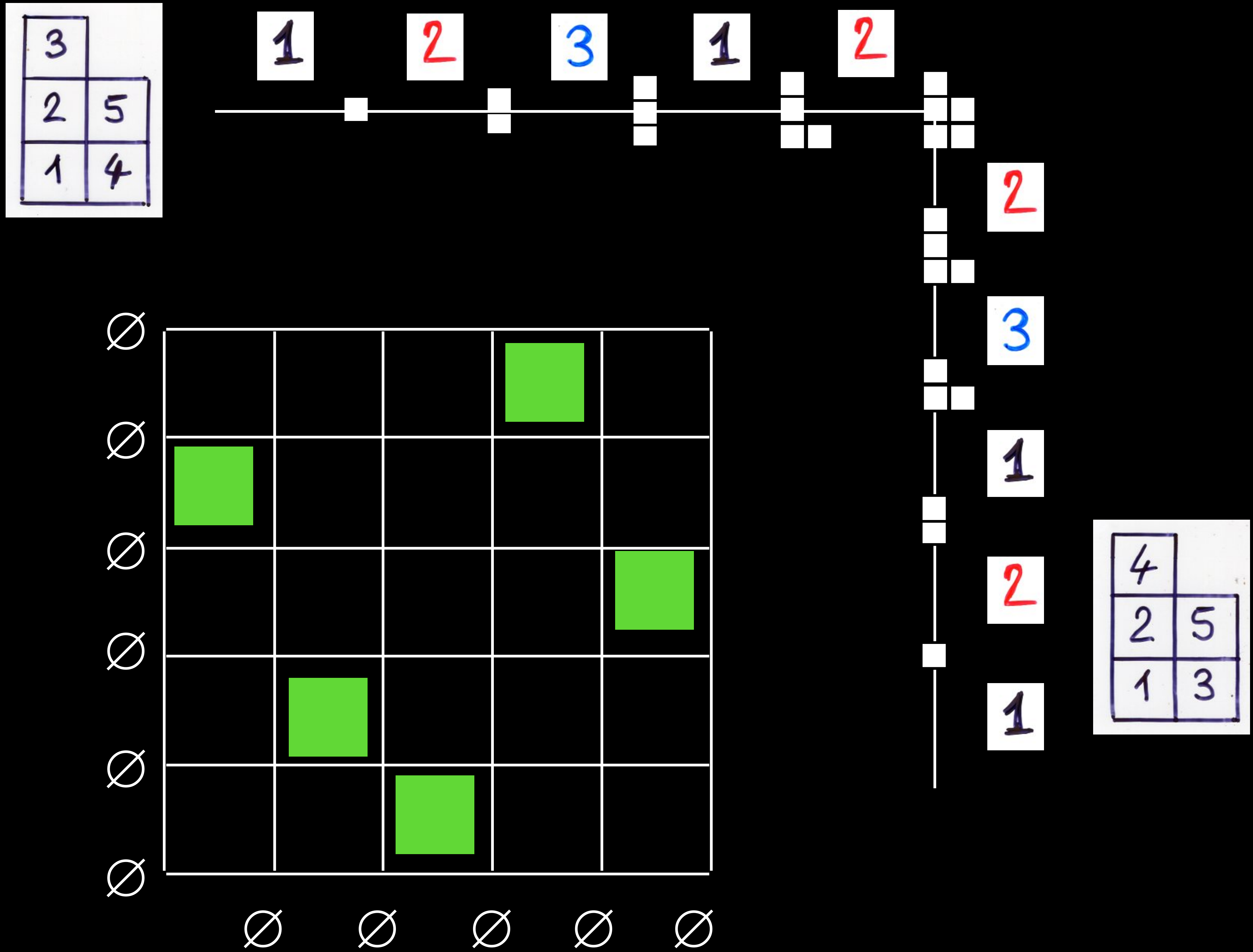
$$(h, T)$$

T tableau above the path associated to $w(h)$ with cells labeled by

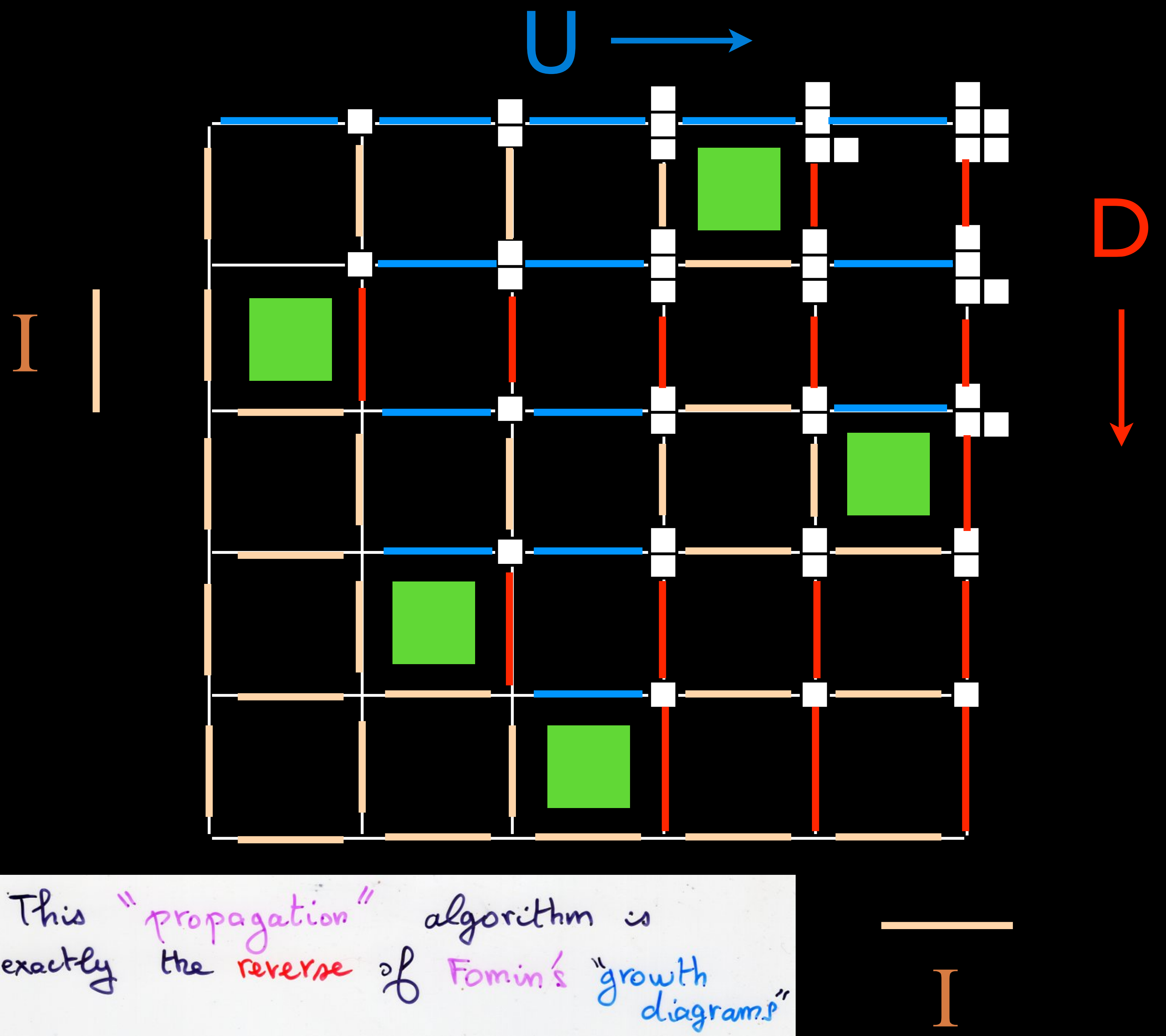
$$(h, T) \longleftrightarrow h_0 = h(\alpha, \beta)$$

are in bijection

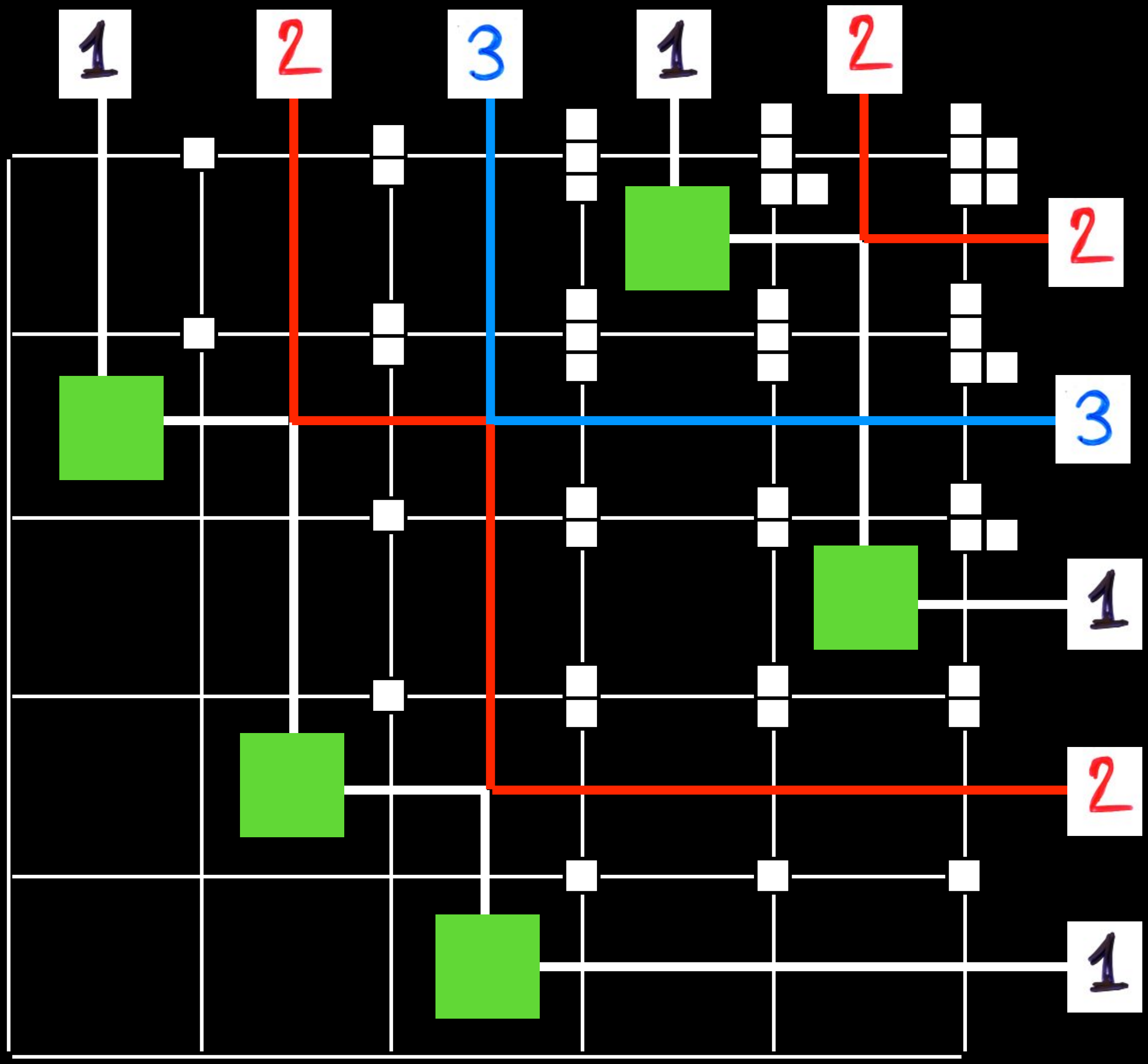
By recurrence



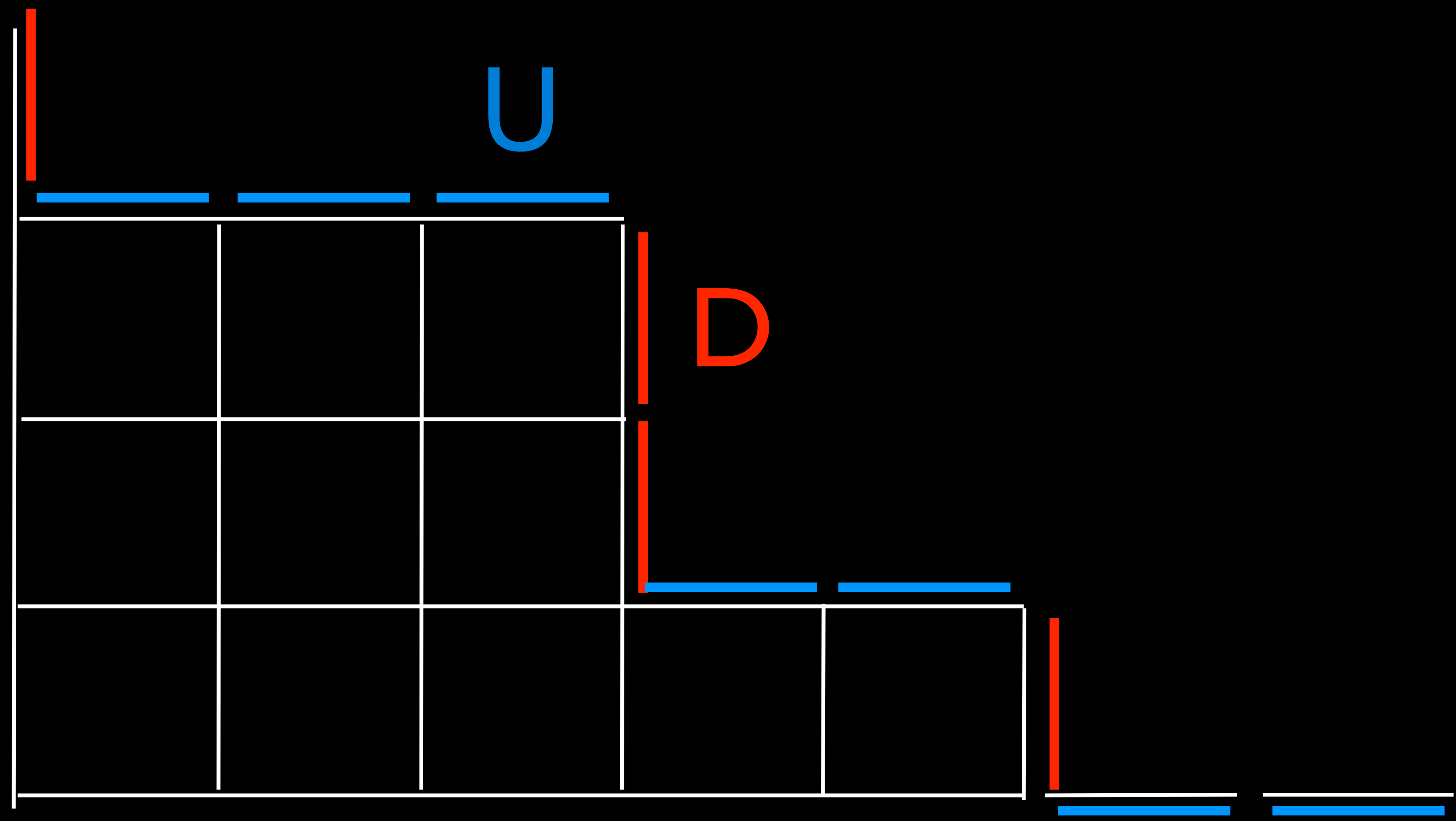
3
2 5
1 4
1
2
3
1
2
2
3
1
2
1
4
2 5
1 3
∅
∅
∅
∅
∅
∅
∅
∅
∅
∅
∅

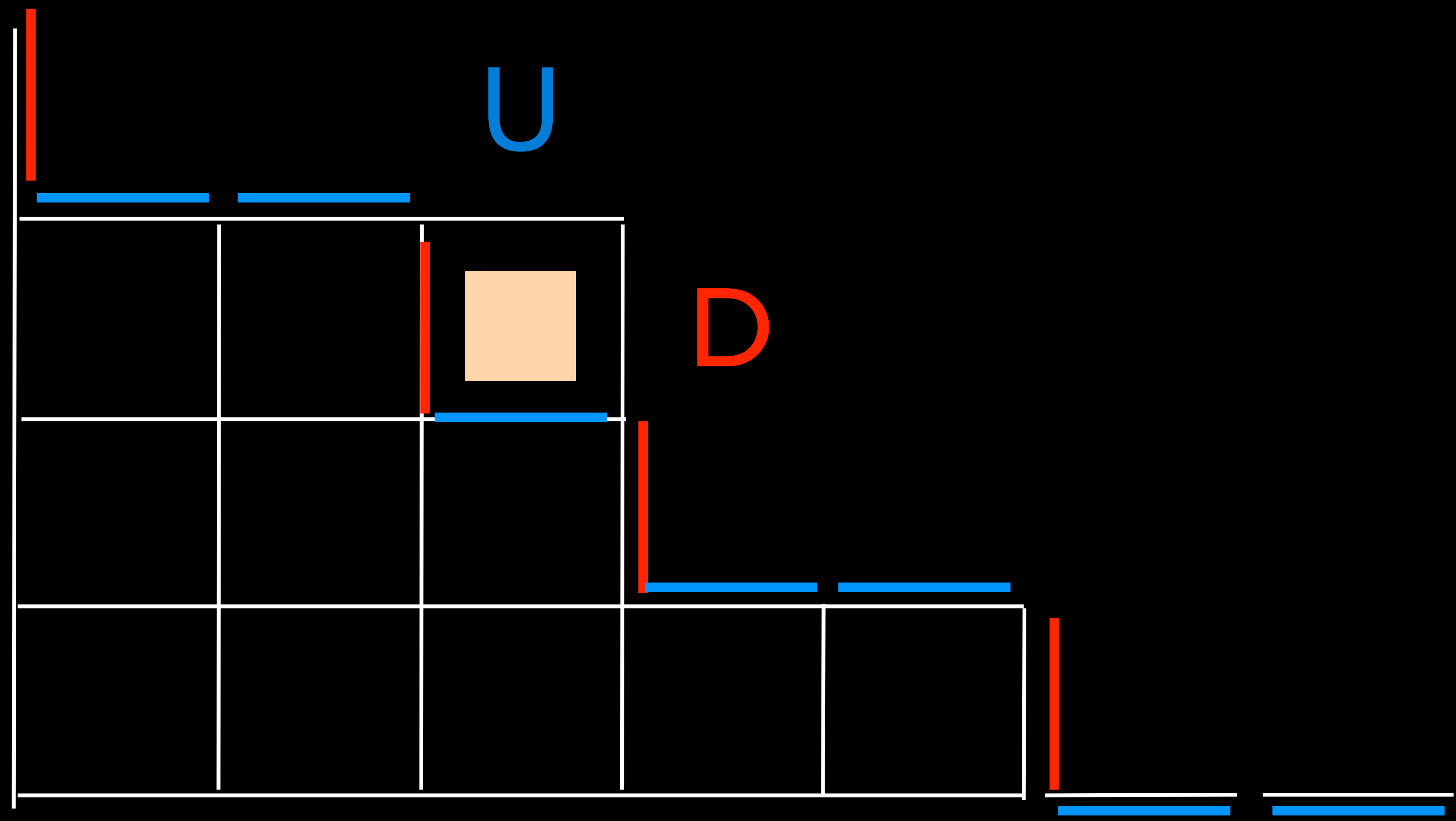


This "propagation" algorithm is exactly the reverse of Fomin's "growth diagrams"


1
2
3
1
2
2
3
1
2
1

rook placements


U
D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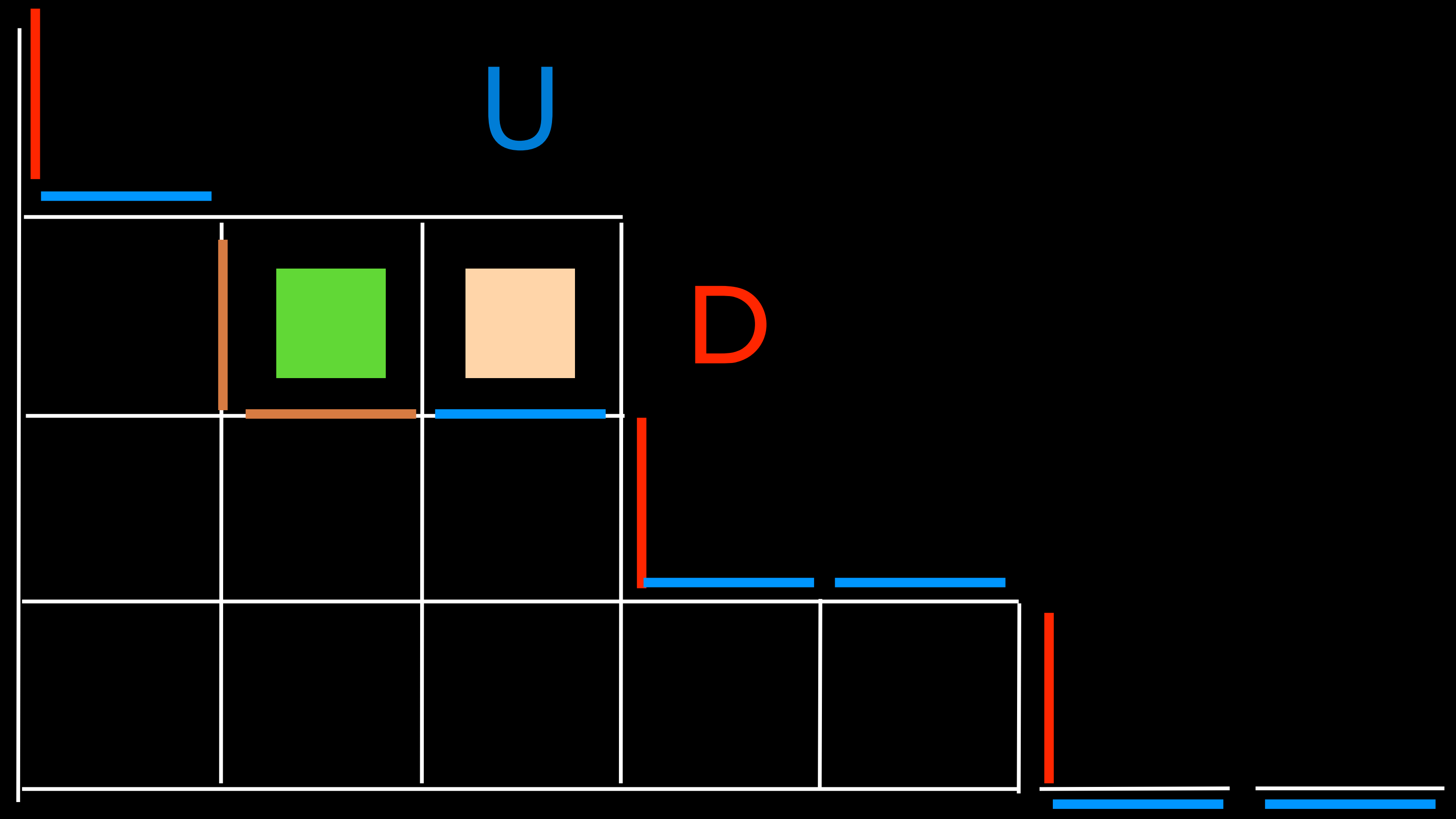
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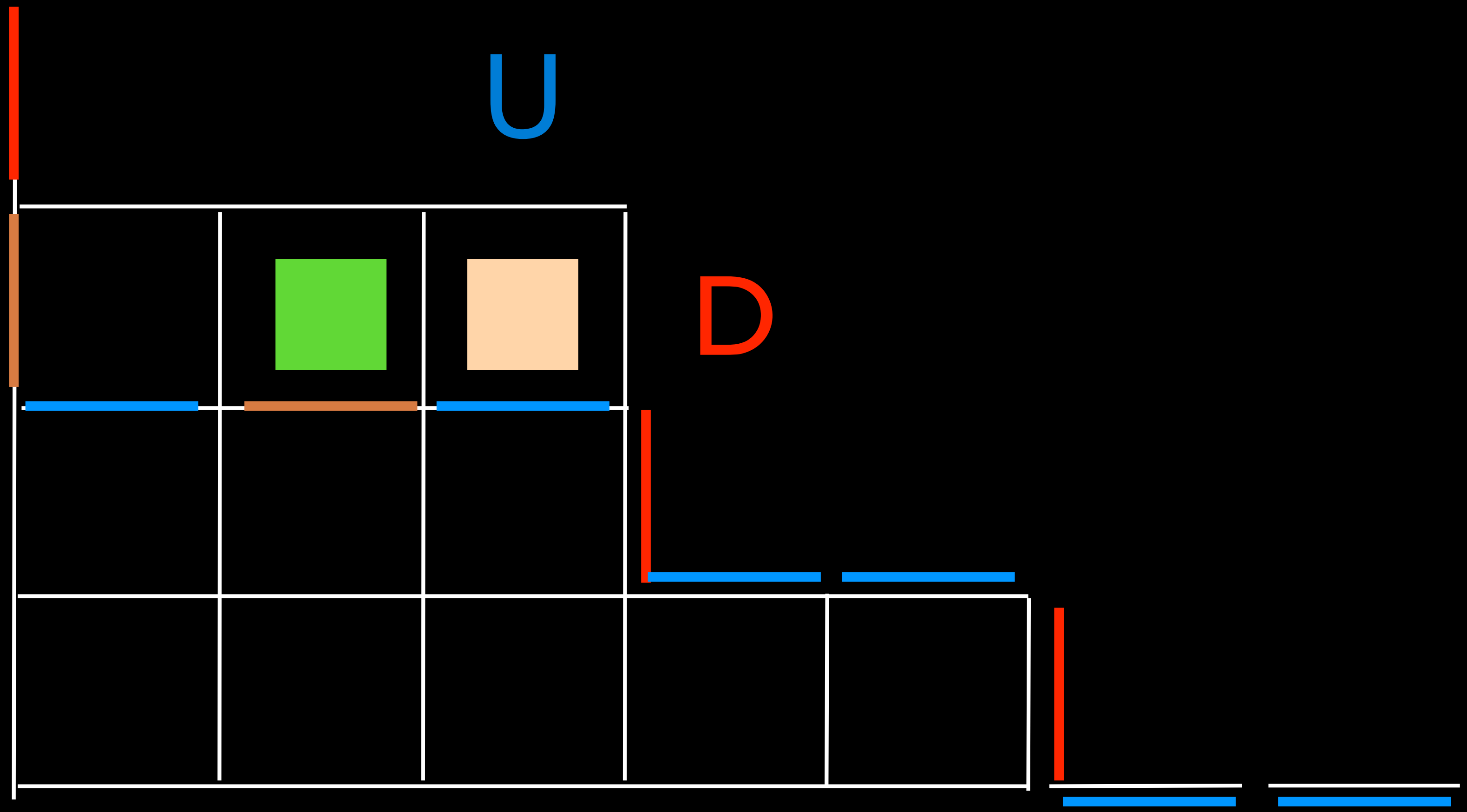
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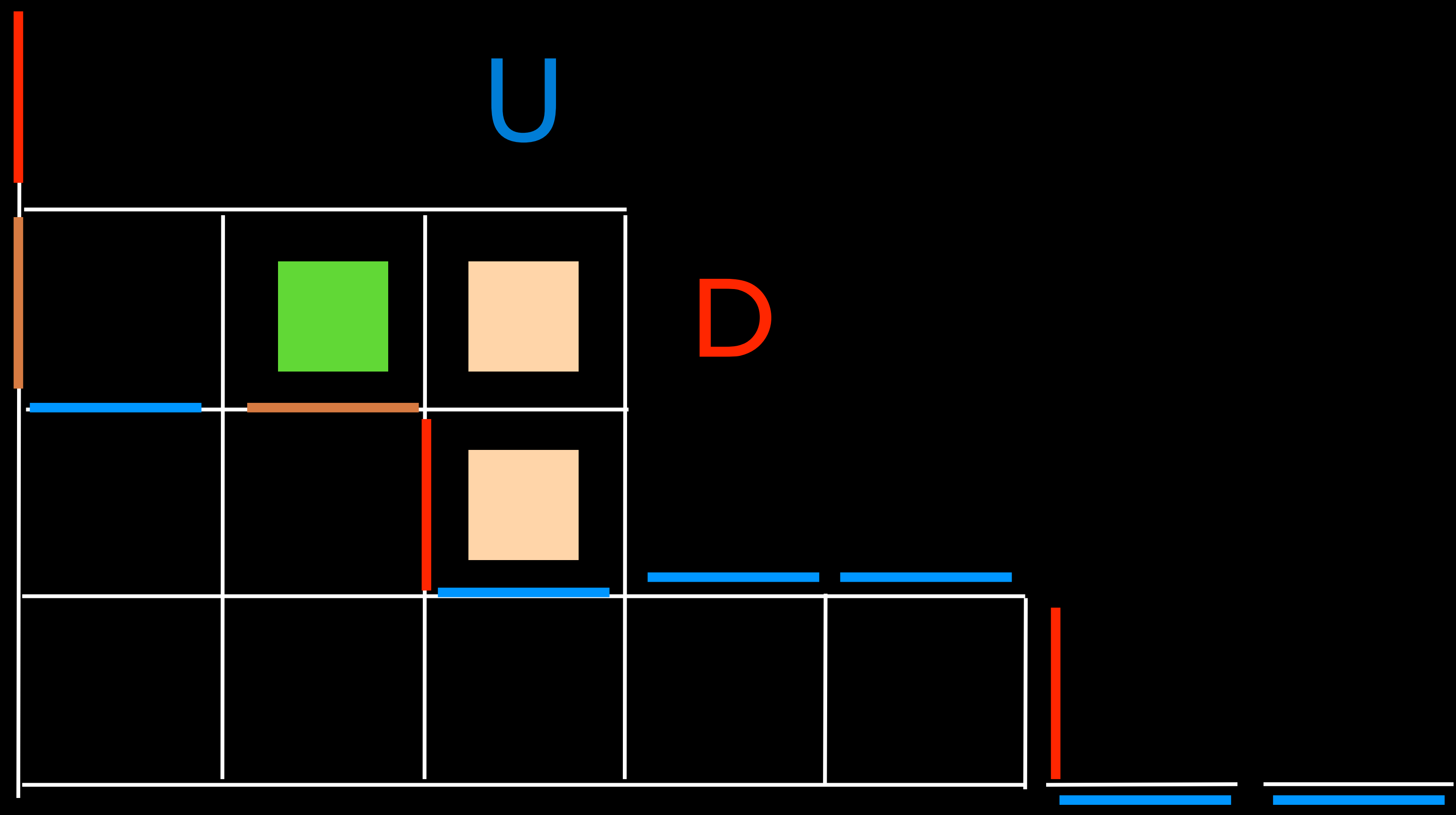
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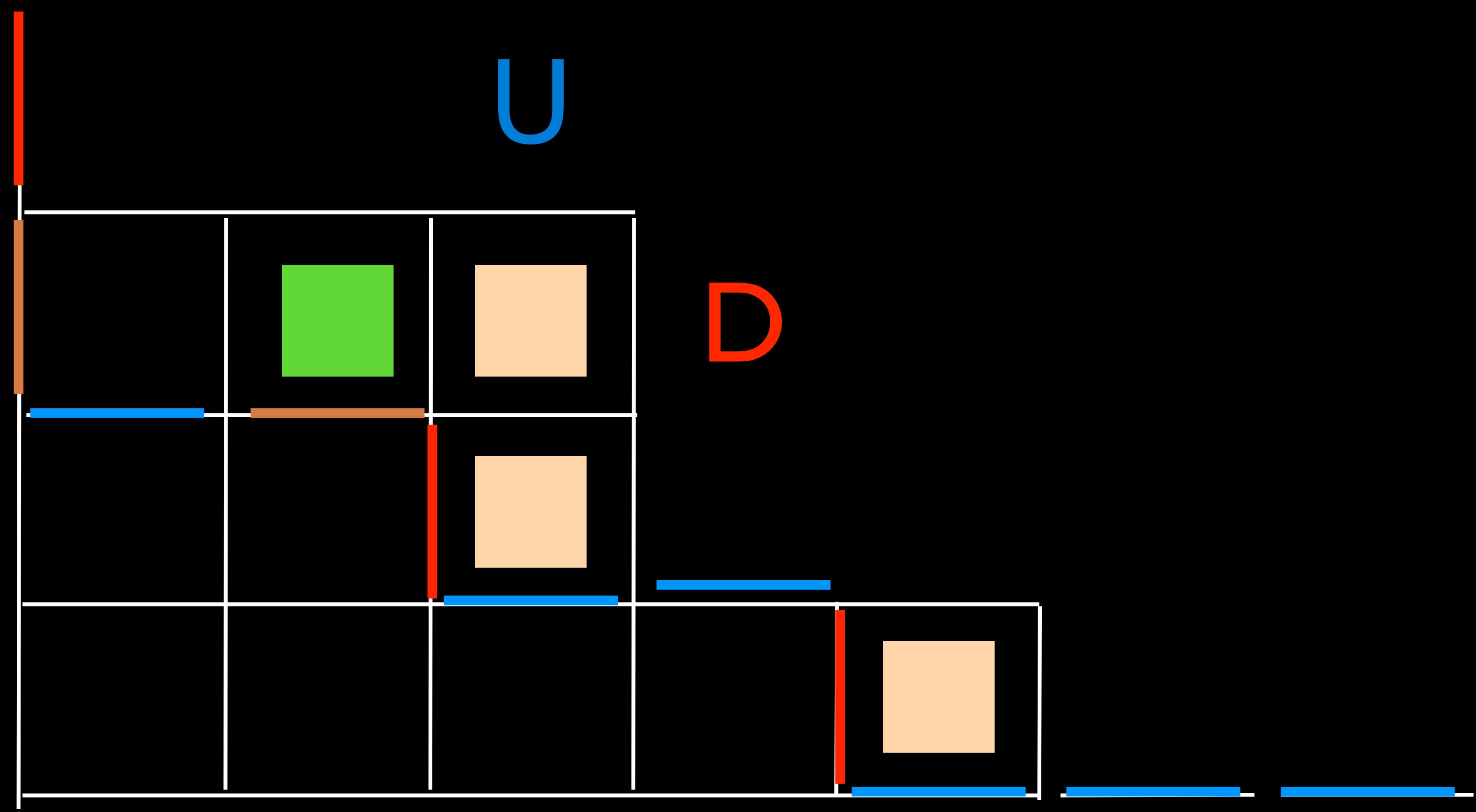
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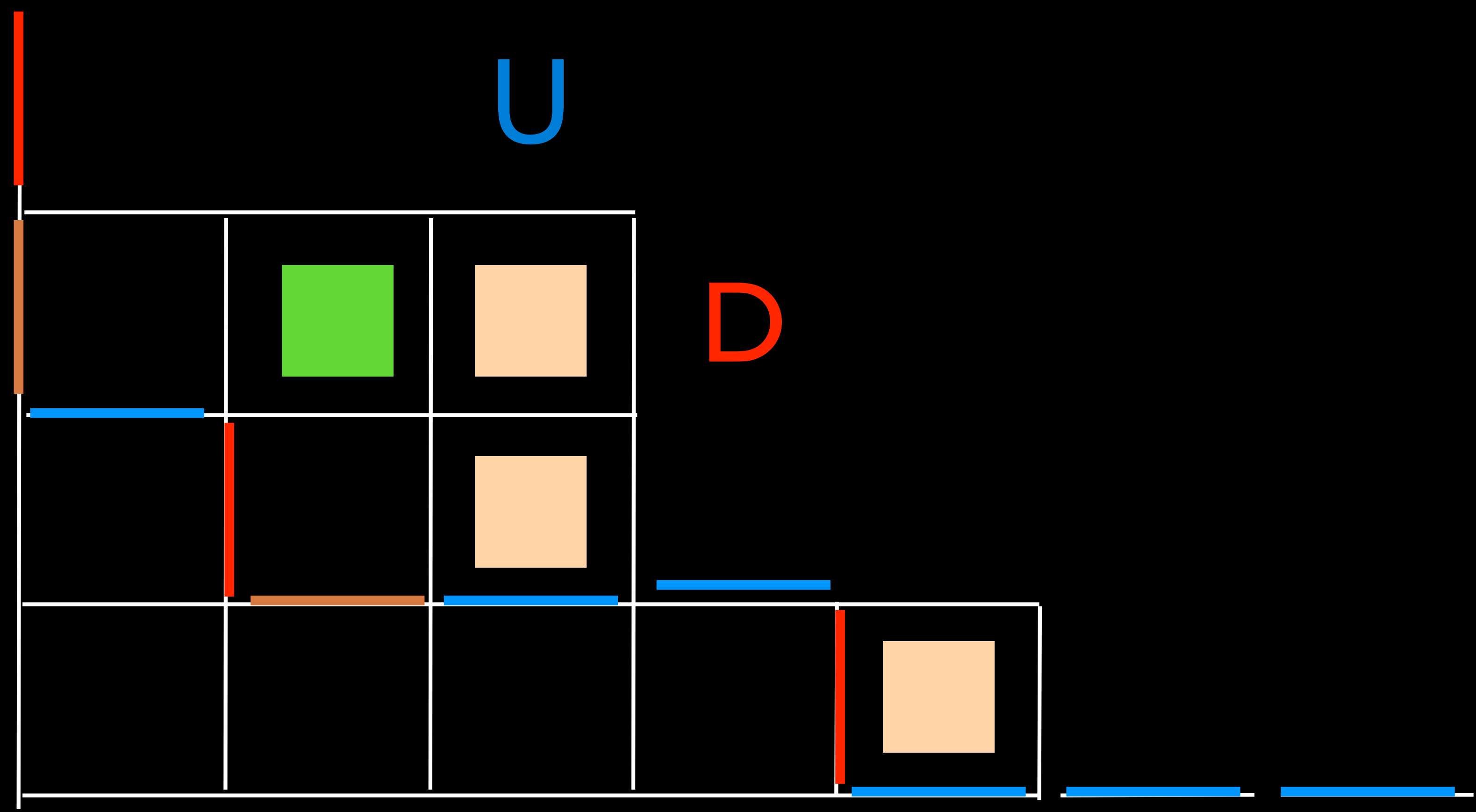
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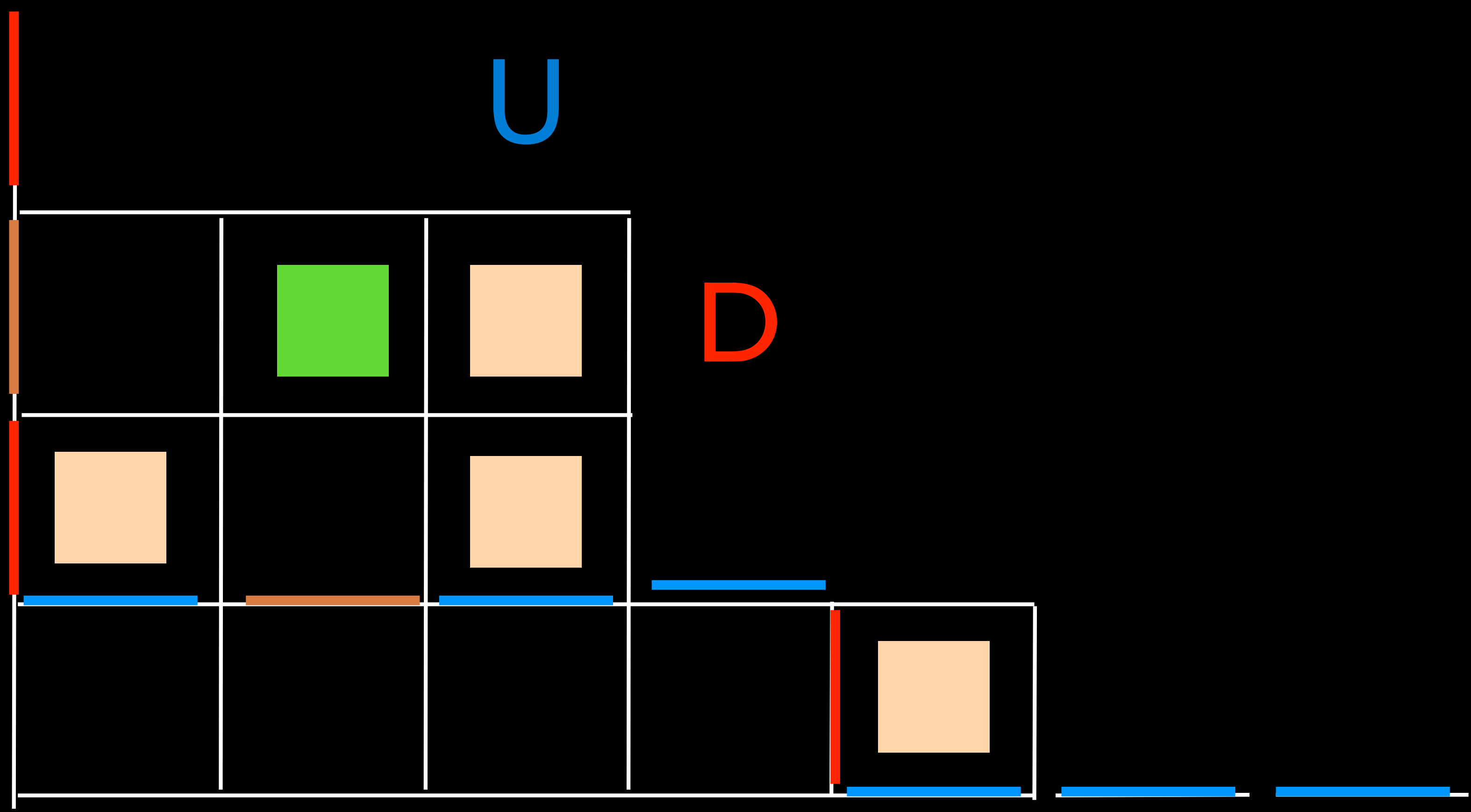
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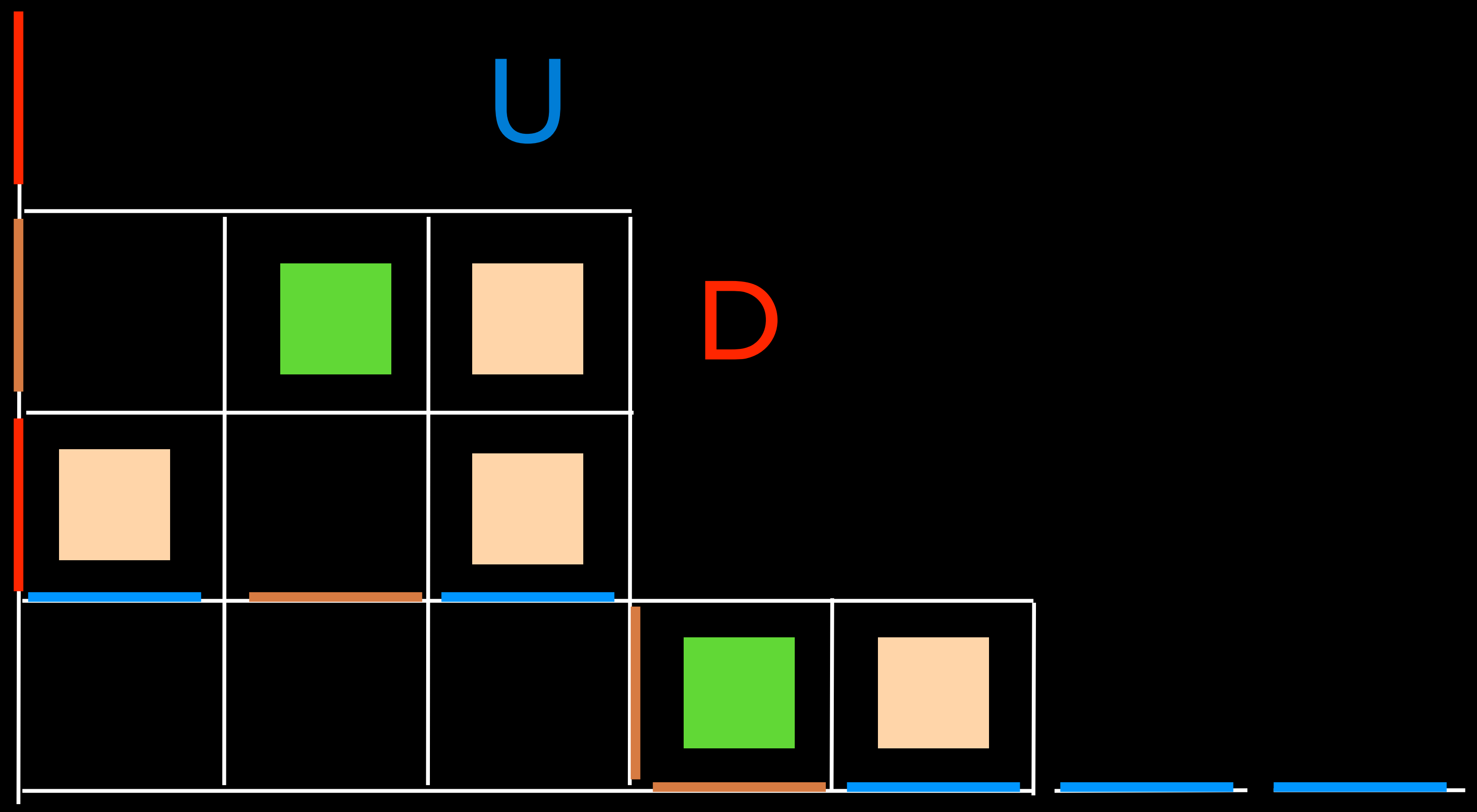
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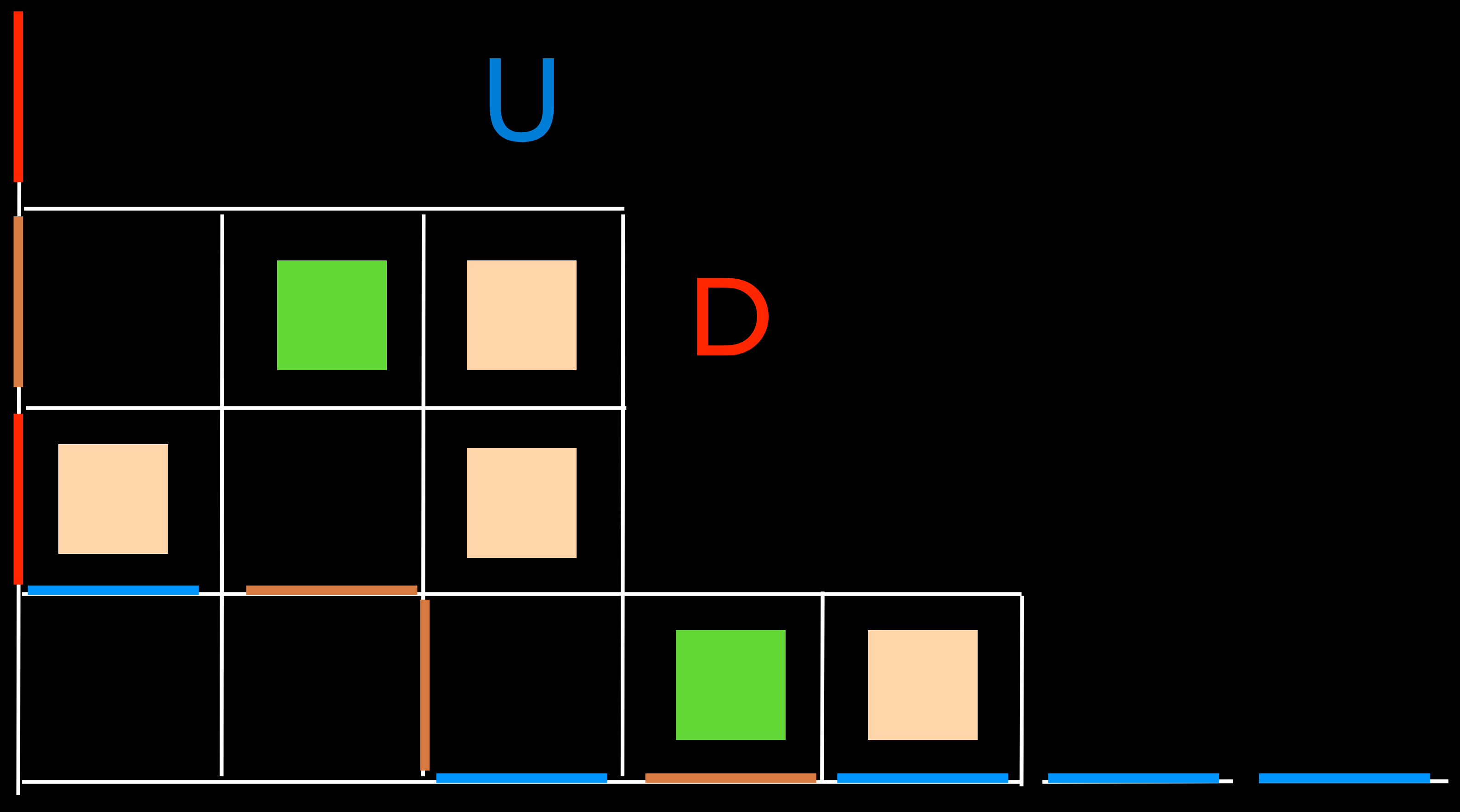

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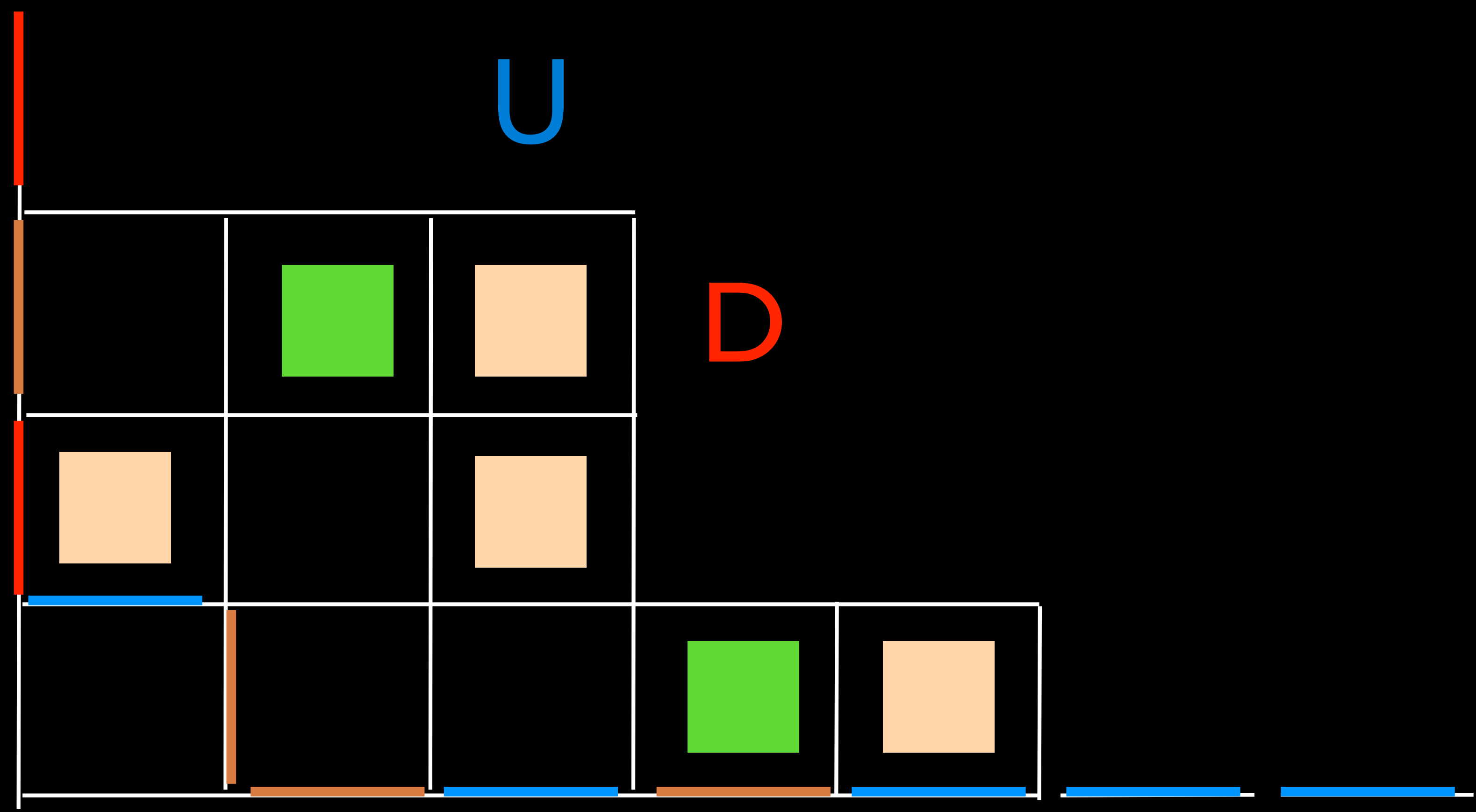
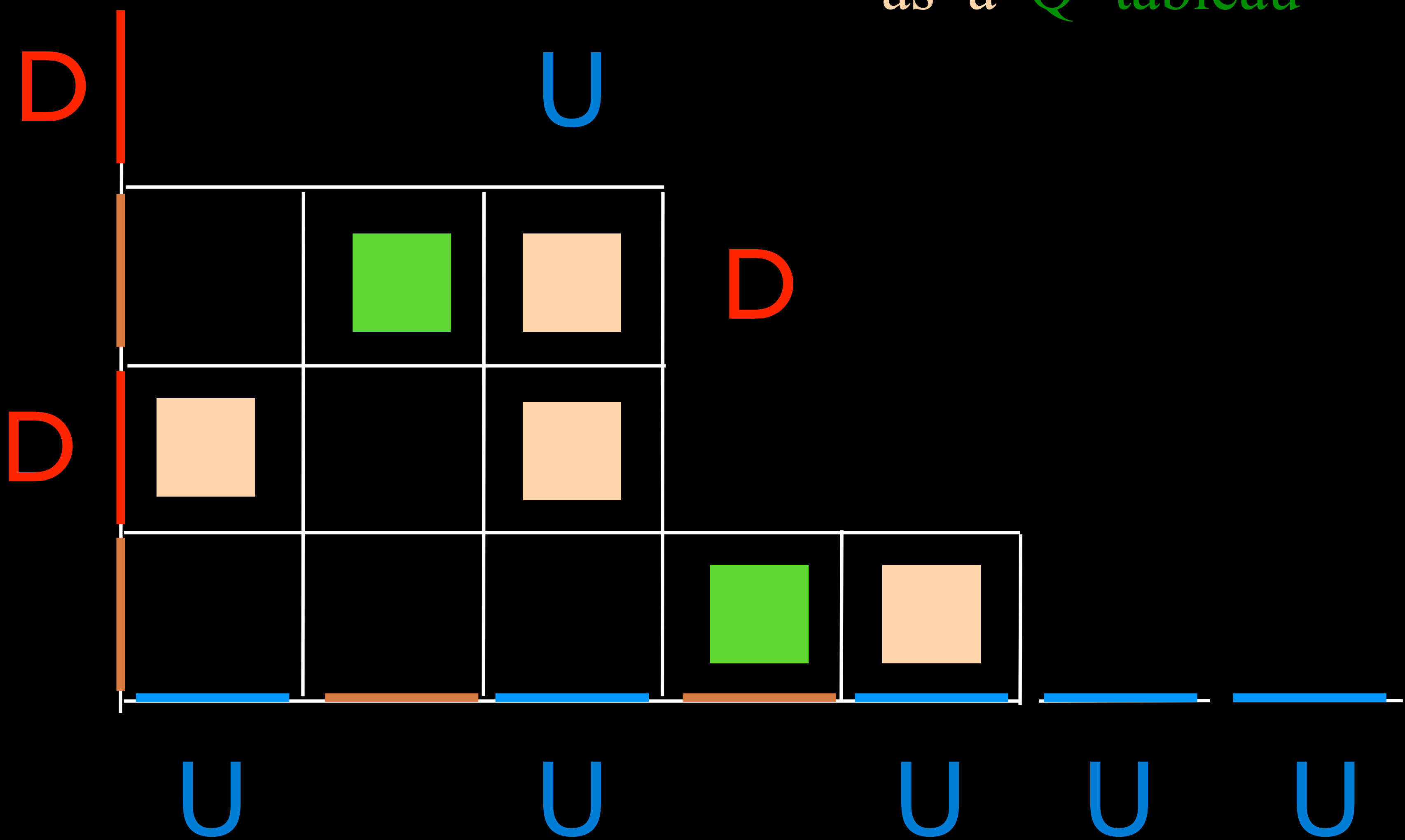
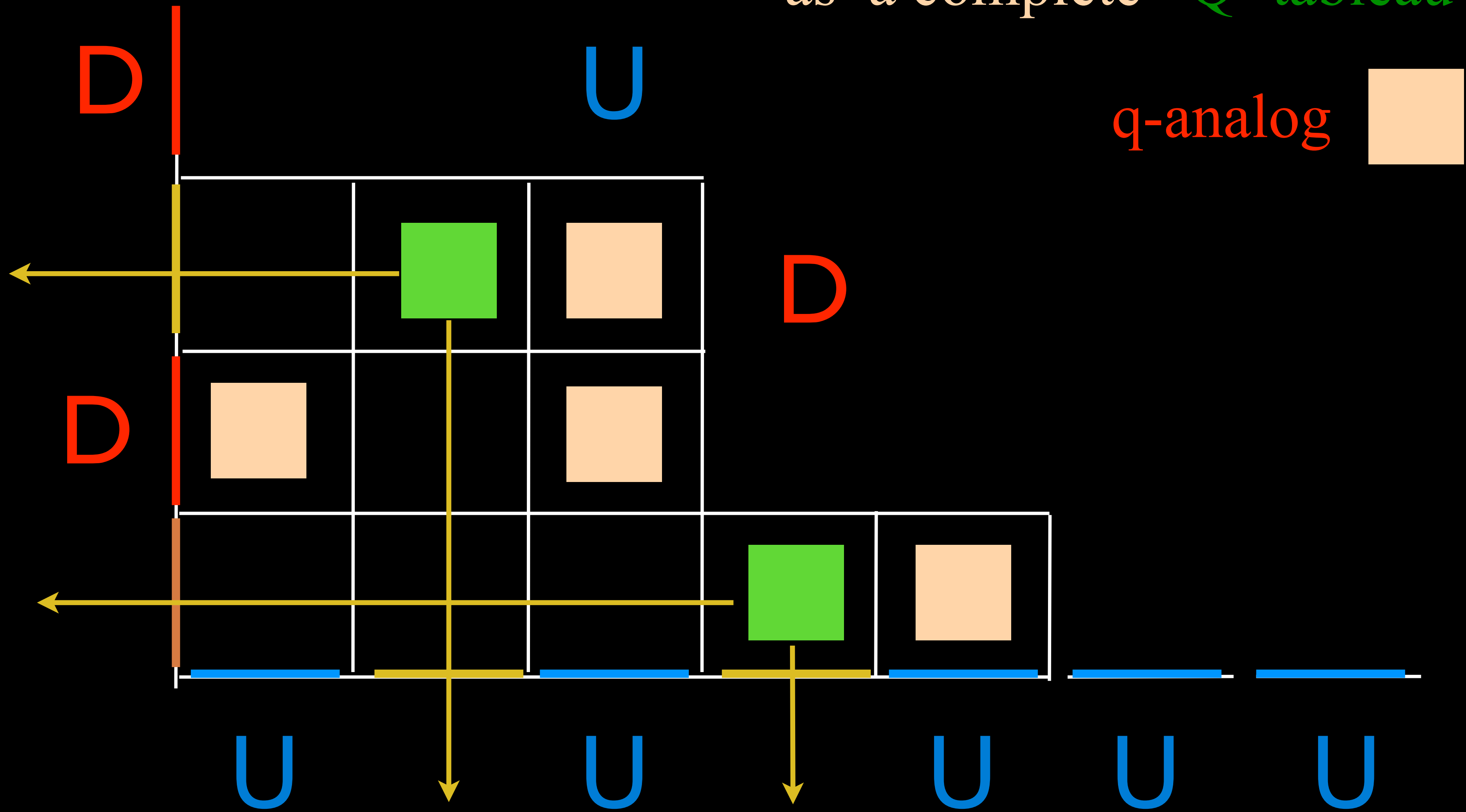

U
D

rook placement
as a Q- tableau



rook placement

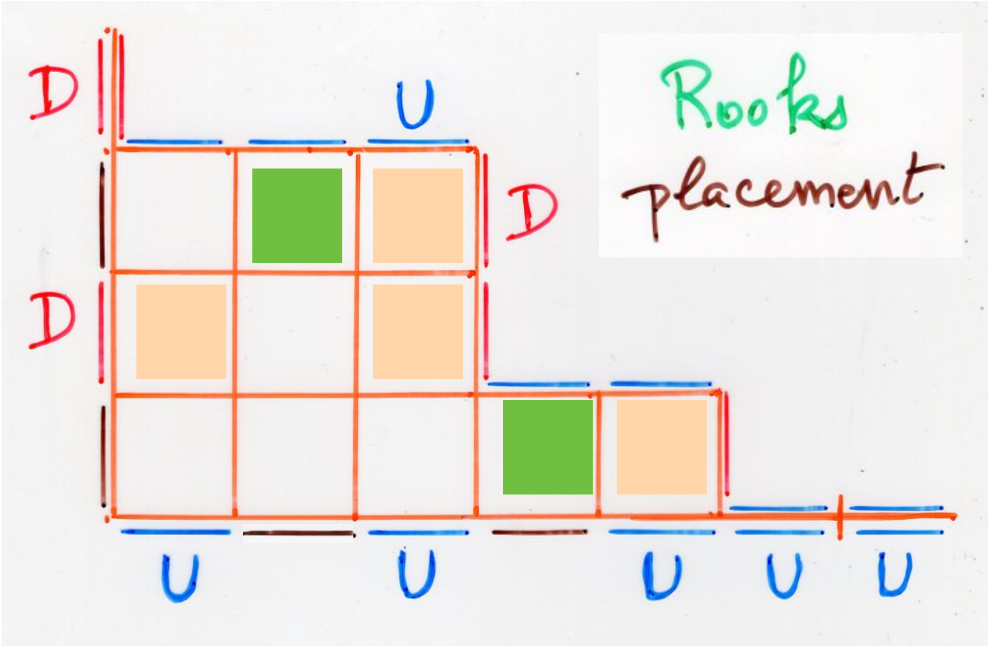
as a complete Q- tableau



$$w = D\,U^3 D^2 U^2 D\,U^2$$

$$w \longrightarrow F = F(w)$$

F Ferrers diagram



<u>Proposition</u>

$$w(U,D) = \sum_{T} D^{i(T)} U^{j(T)}$$

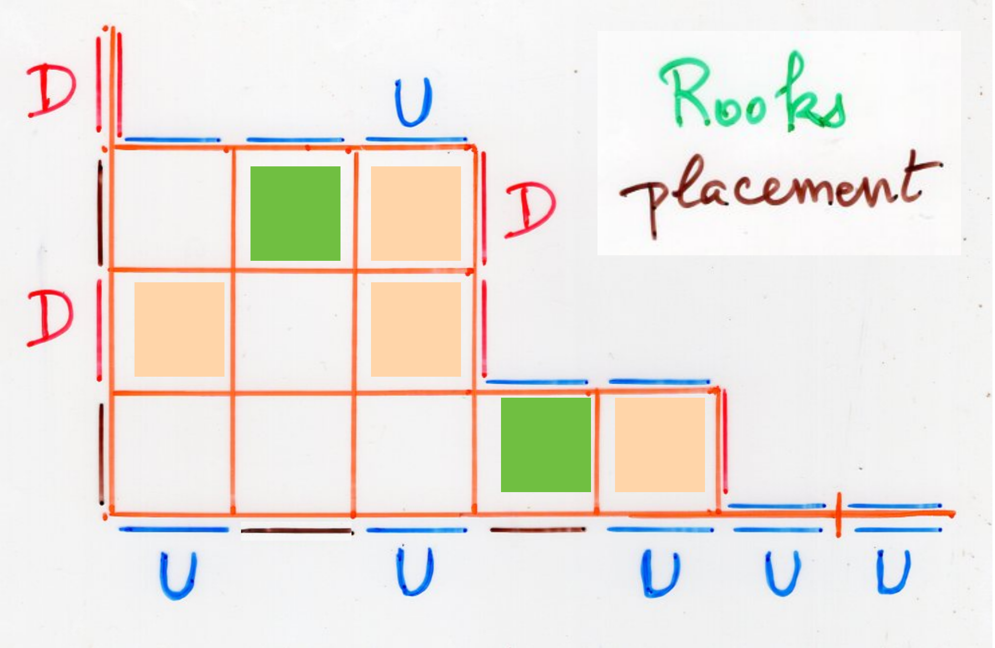
$$\begin{cases} i(T) = \text{number of rows} \\ j(T) = \text{number of columns} \end{cases} \text{with no cell labeled}$$

$$UD \longrightarrow I_v I_h$$

$$q^{k(T)}$$

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w)\, D^i U^j$$



Proposition

$c_{ij}(w)$ = number of placements of k rooks on the Ferrers "board" F

with $i = |w|_D - k$

$j = |w|_U - k$

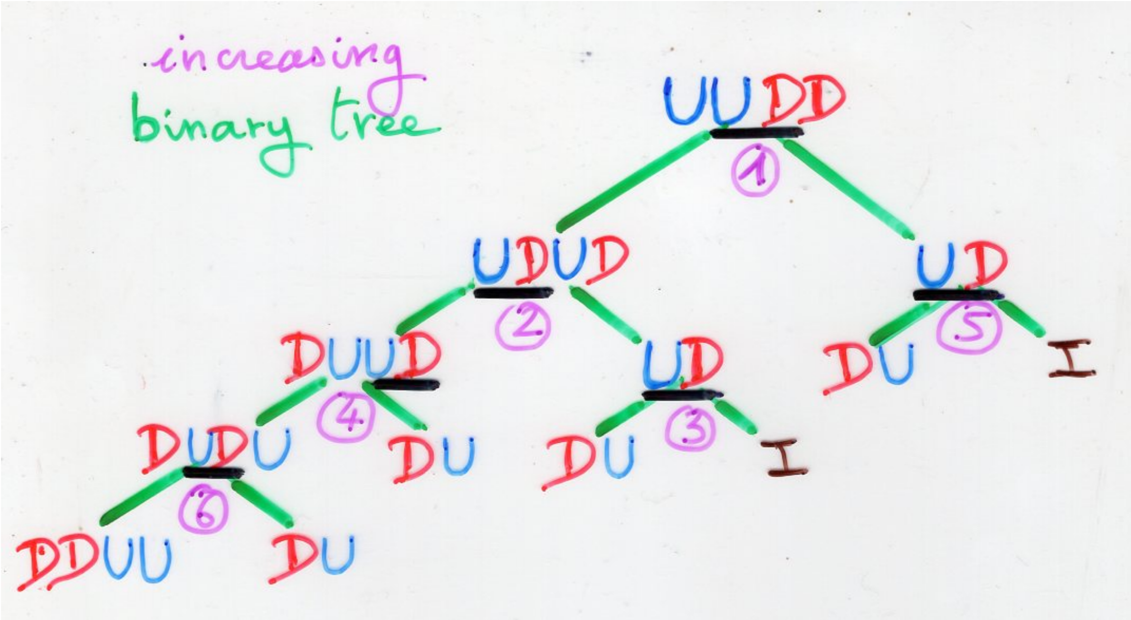
$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} \, D^i U^i$$

$$c_{n,0} = n!$$

permutations

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

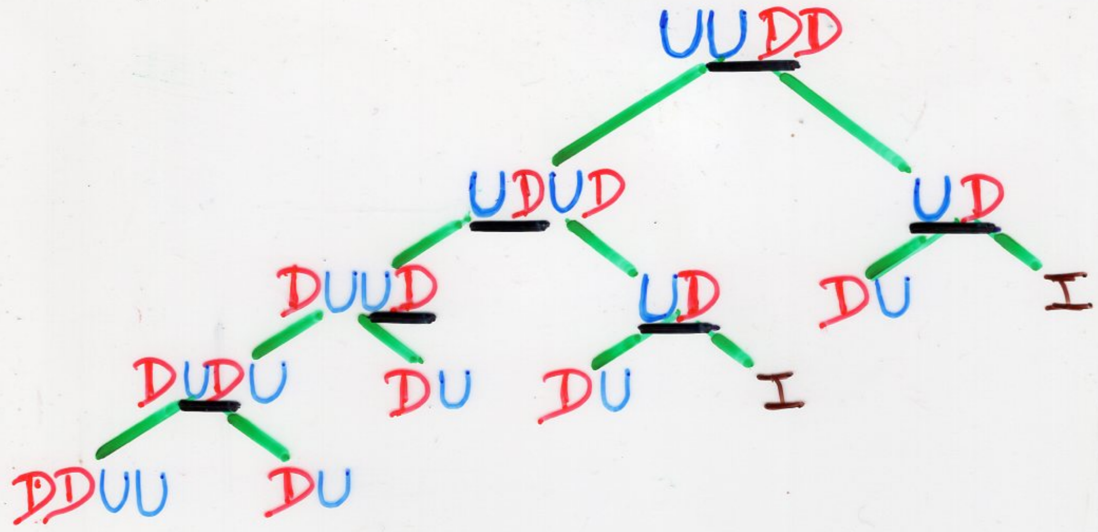
choice of the underlying grid $\binom{n}{i}^2$



binary tree T associated to a possible rewriting process

$$U^2D^2 = D^2U^2 + 4DU + 2I$$

this polynomial is independant of the order of the substitutions



binary tree T
associated
to a possible
rewriting process

$$U^2D^2 = D^2U^2 + 4DU + 2I$$

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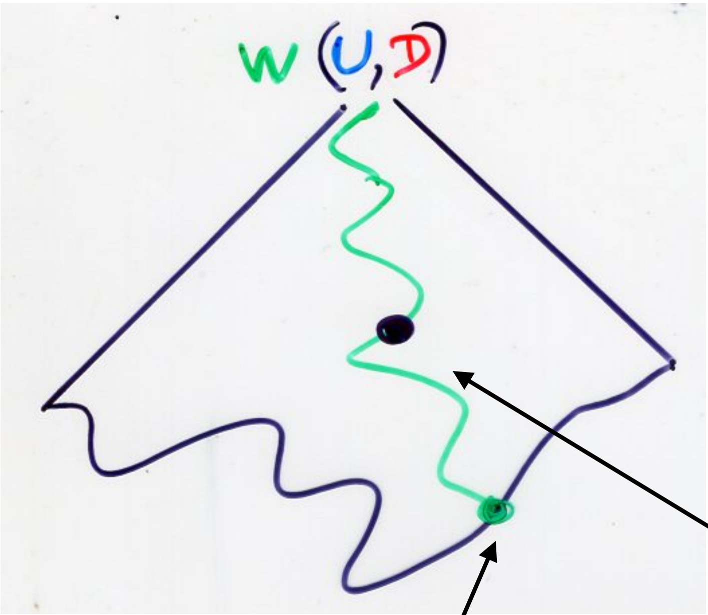


binary tree T
associated
to a possible
rewriting process

internal vertex

$\updownarrow$

skew-Ferrers

complete skew-Q-tableau

leafs of T

bijection

$\longleftrightarrow$

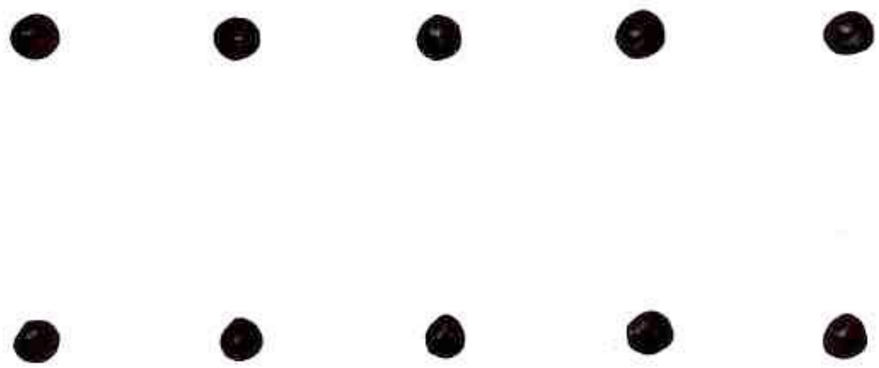
complete
Q-tableaux
shape λ

$\lambda = F(w)$

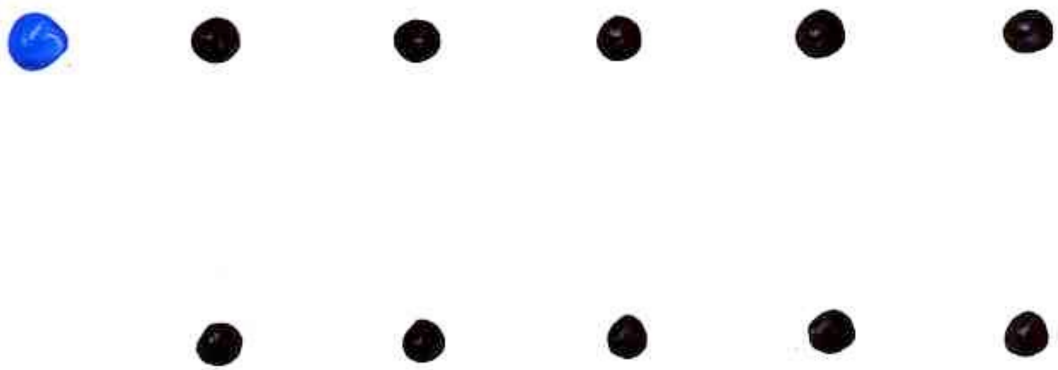
Another representation of the algebra

$$UD = DU + 1$$

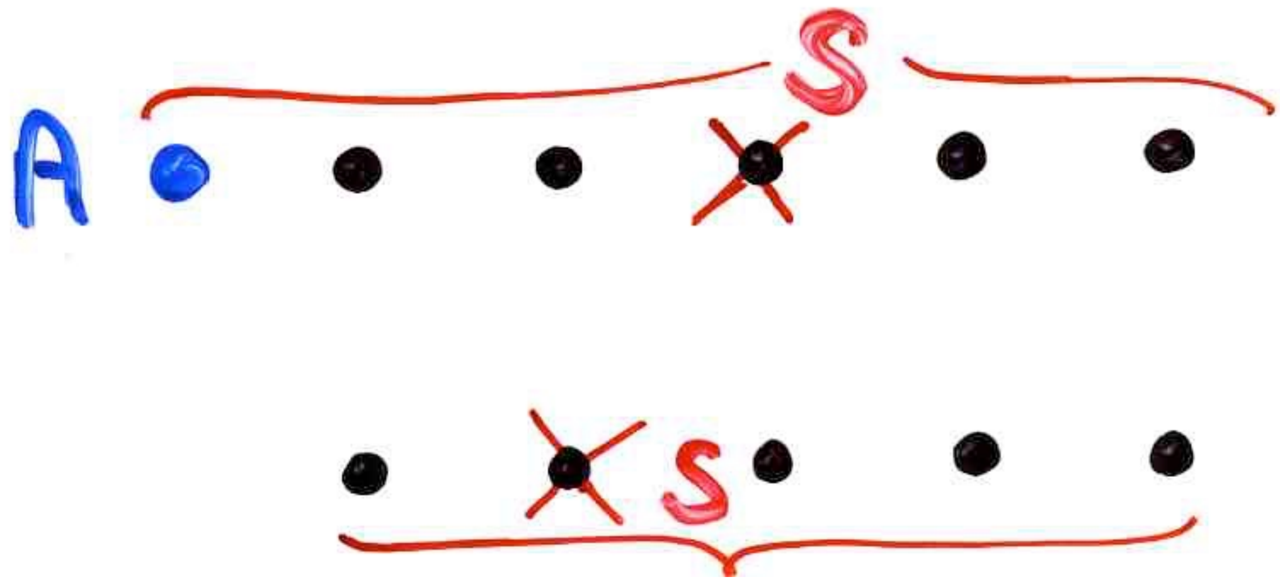
Polya urn



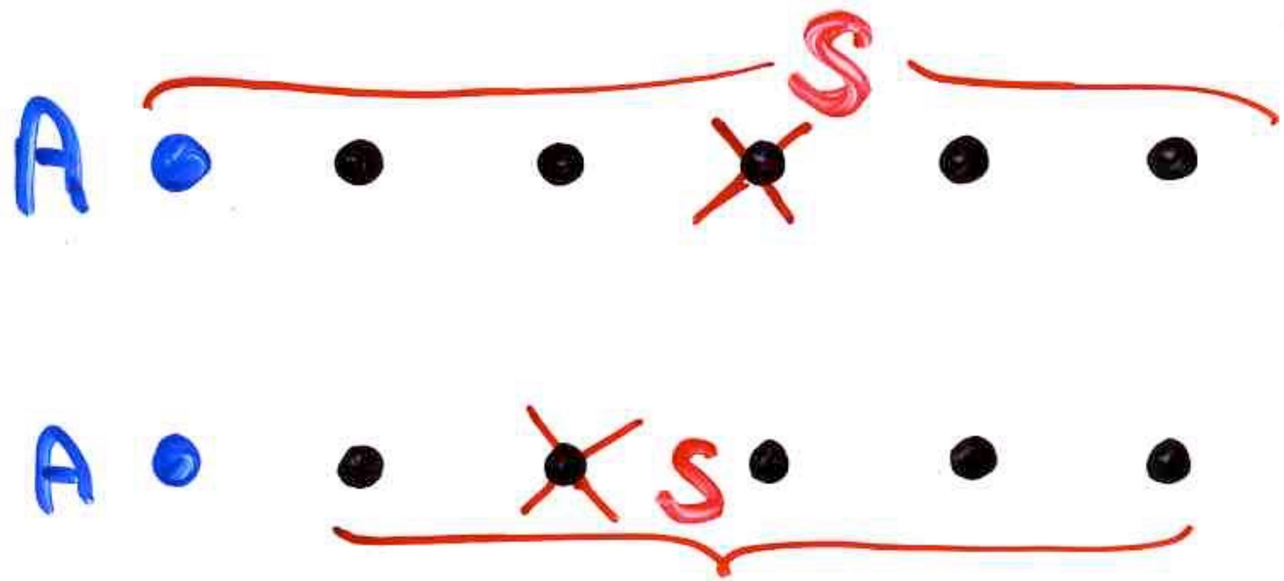
A

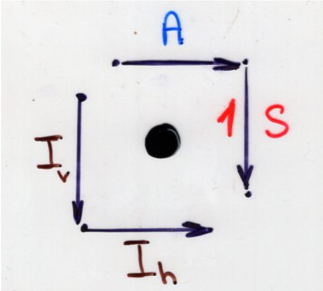


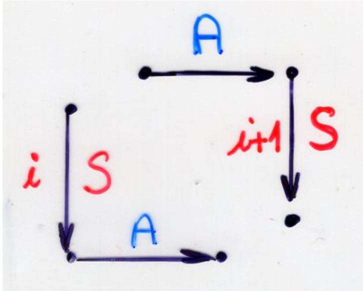

A
S

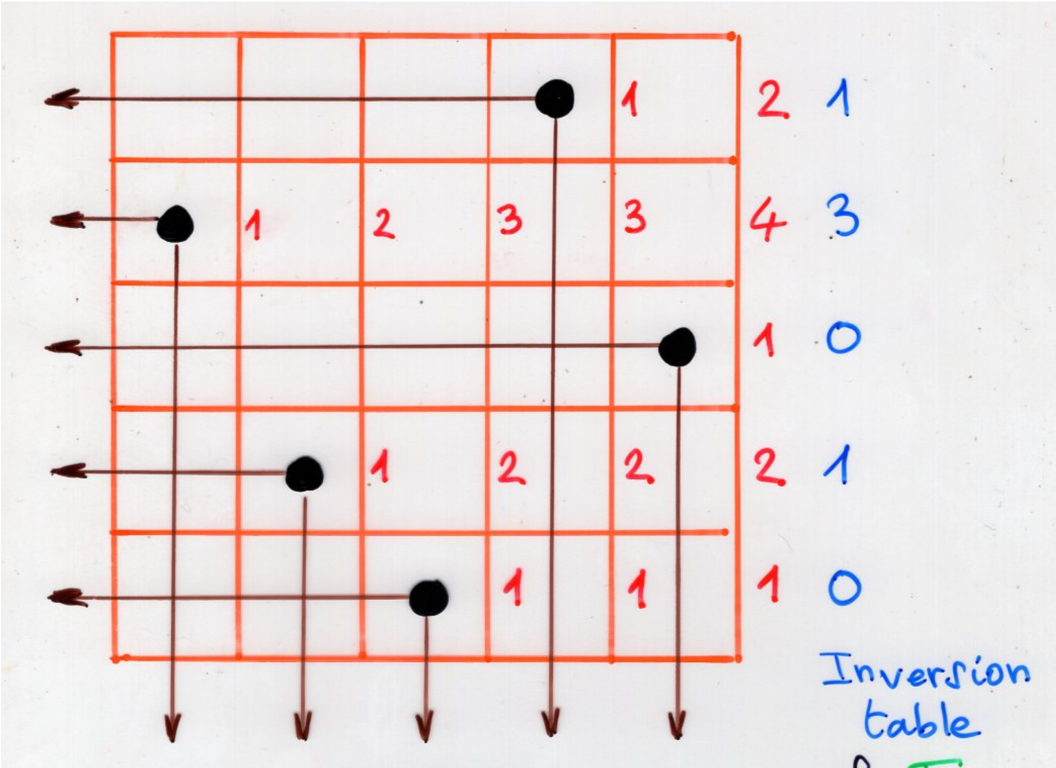

S
A
S

$$AS - SA = I$$







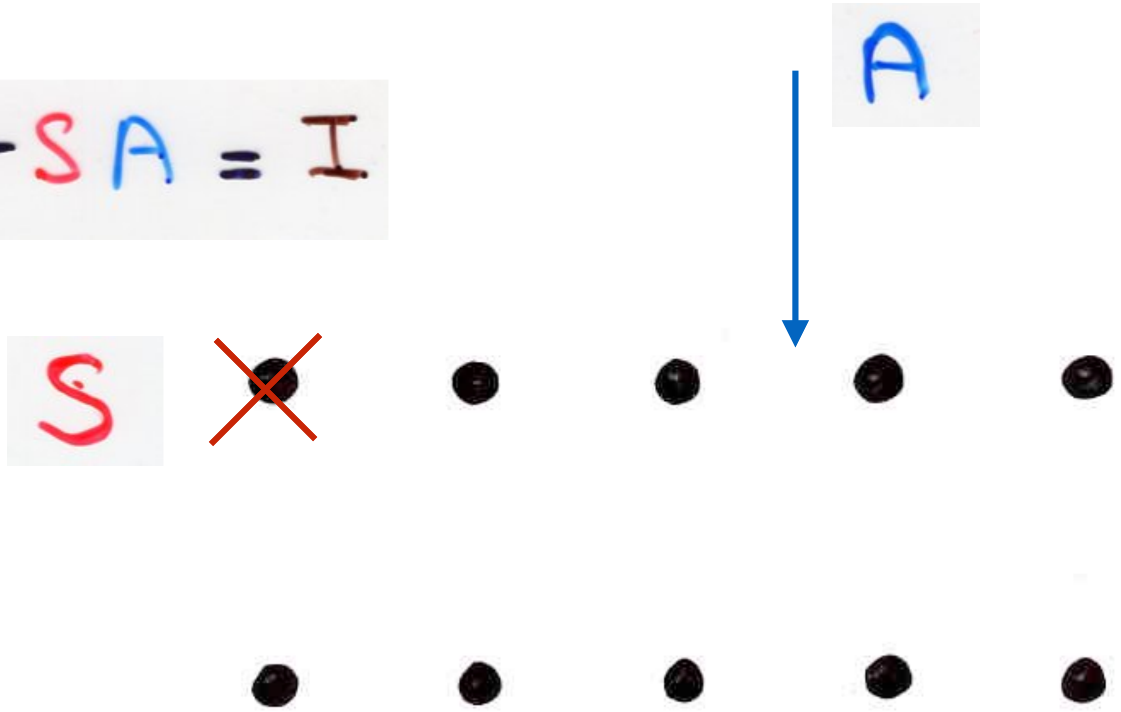


$\sigma = 4\ 2\ 1\ 5\ 3$

$\mathrm{Inv}(\sigma) = 3\ 1\ 0\ 1\ 0$

Priority queue

AS - SA = I



data structures

Computer Science