Chapter 6

Heaps and Coxeter groups (2)

complements

IMSc, Chennai
27 February 2017

complements:

relation with symmetric functions

S_n symmetric group

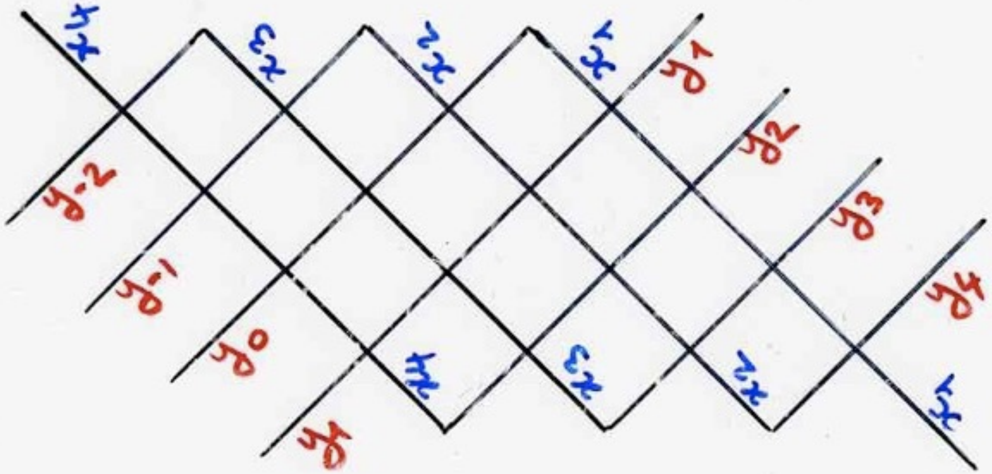
$\sigma \in S_n$ permutation

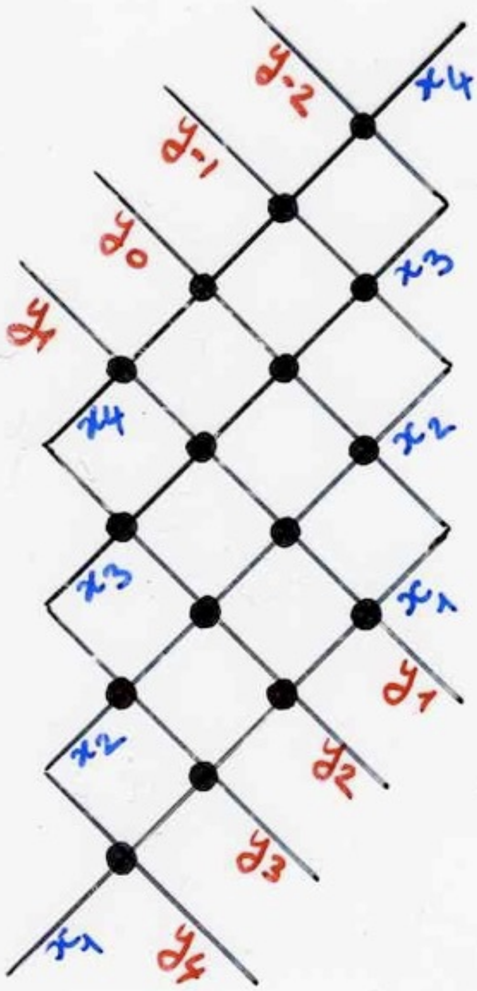
$F_\sigma(x_1, \ldots, x_m)$

polynomial

F_σ

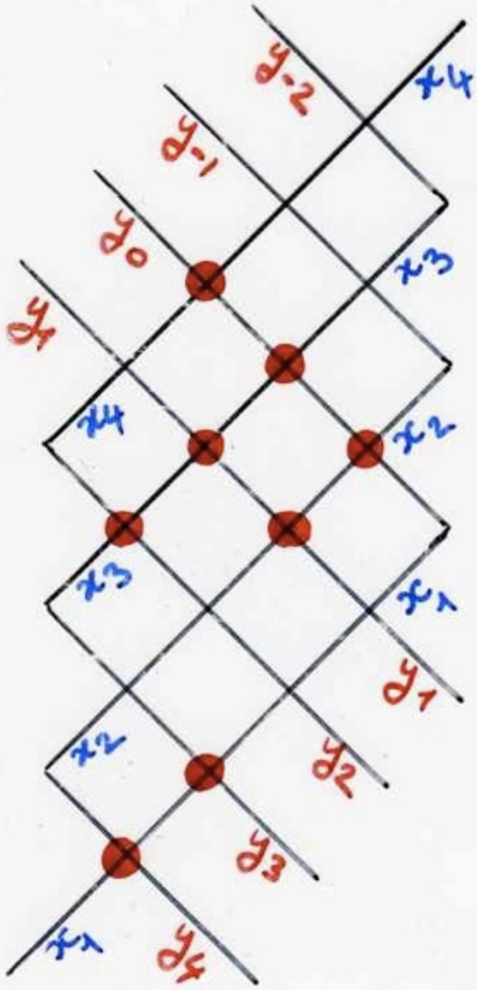
stable Schubert


x4
x3
x2
x1
y1
y2
y3
y4
y-2
y-1
y0
y5
x4
x3
x2
x1



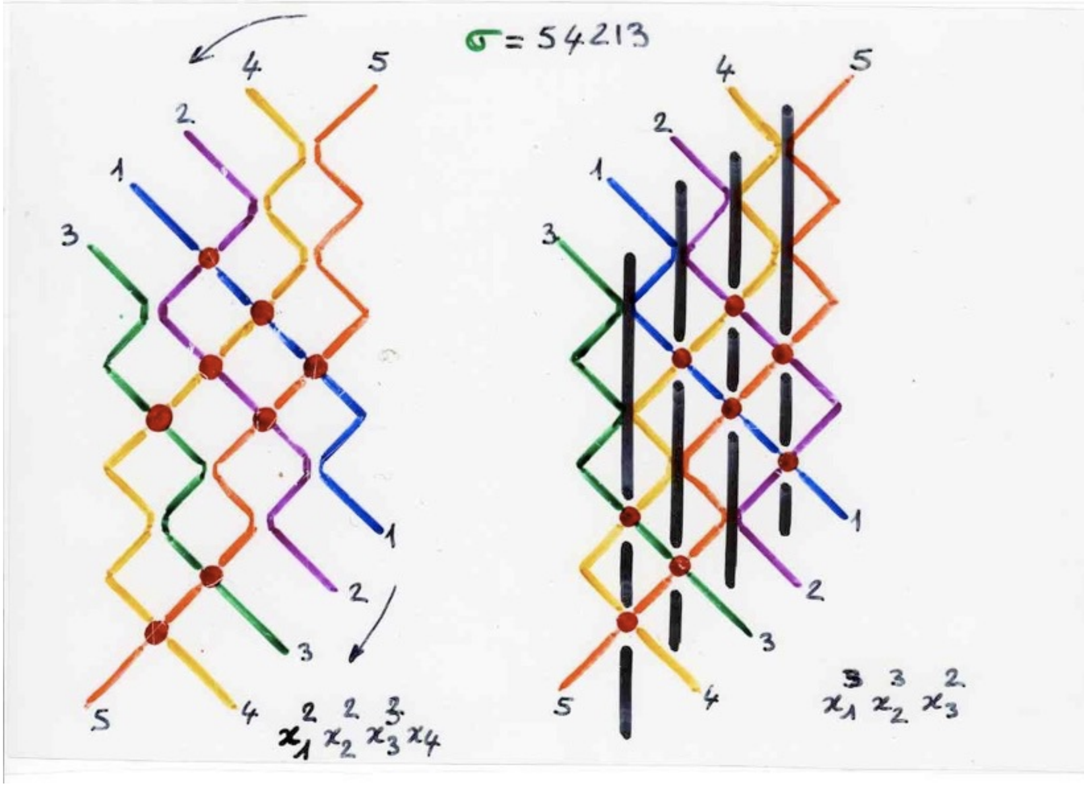
intersection points

I

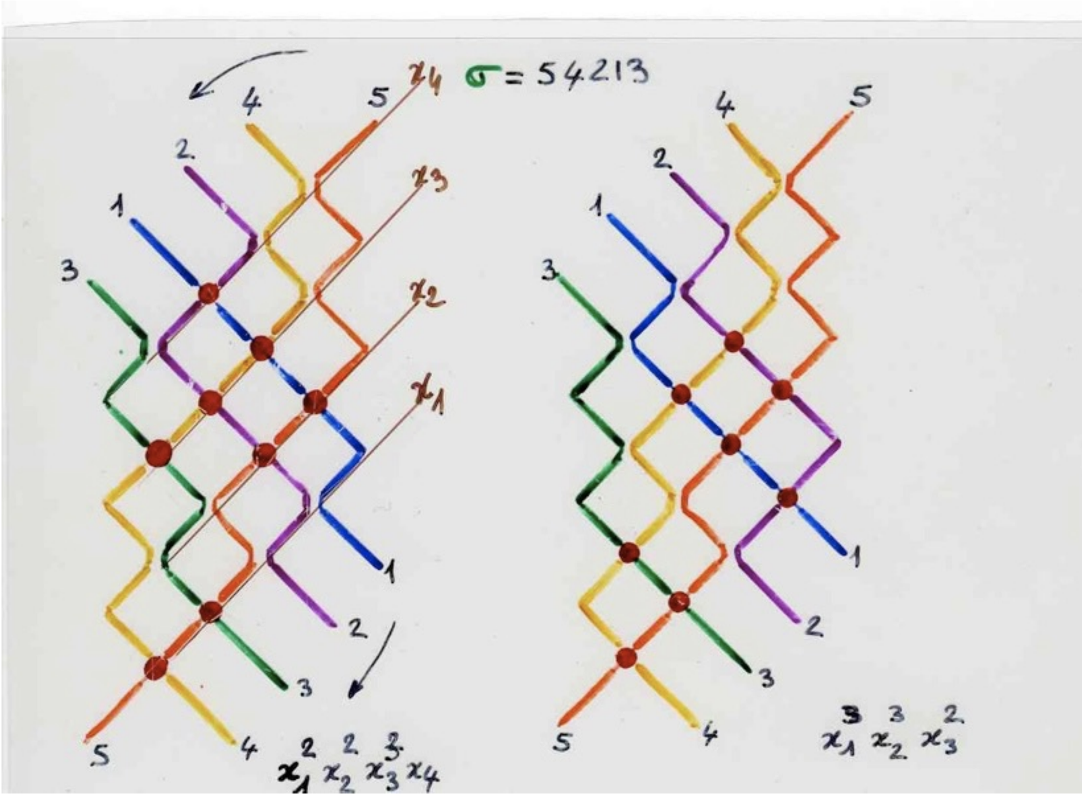


choice of
intersection
points

$U \subseteq I$



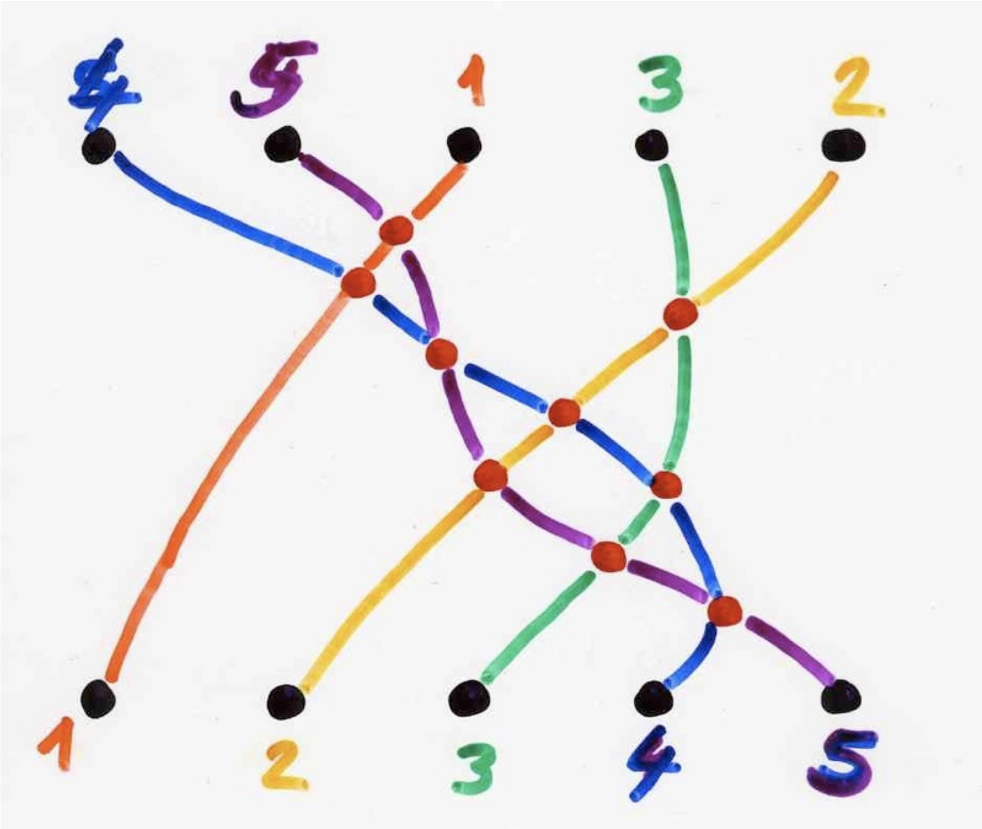
σ = 54213
$x_1^2 x_2^2 x_3^3 x_4$
$x_1^3 x_2^3 x_3^2$

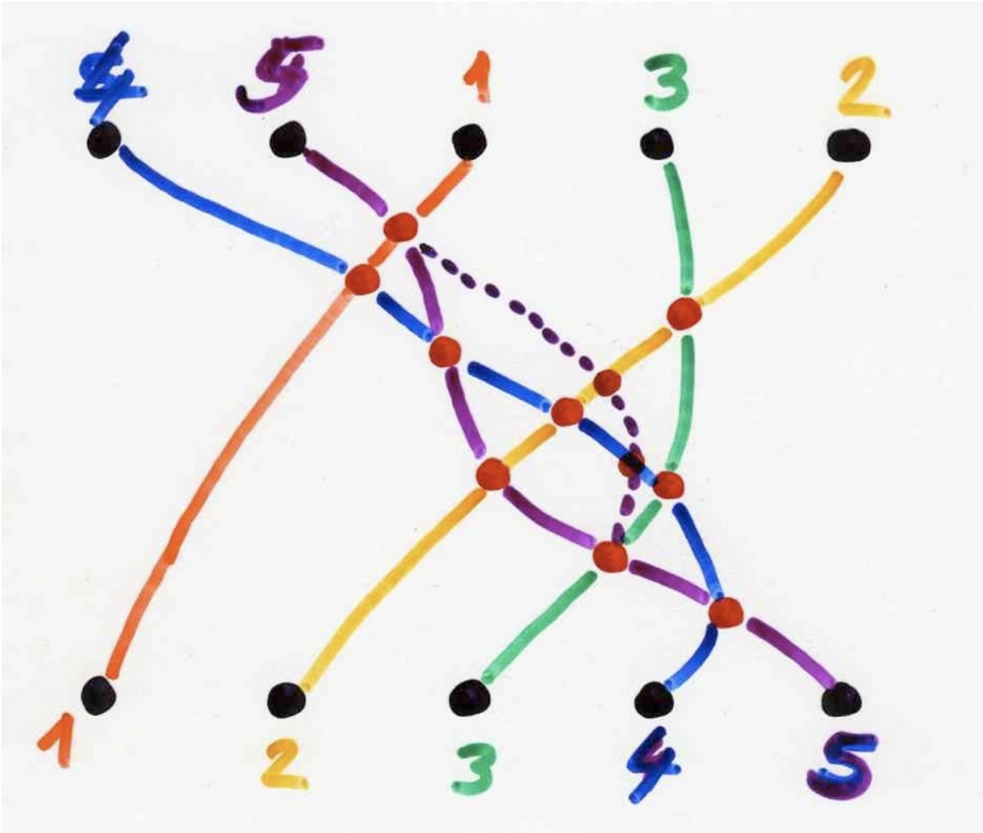

σ = 54213
x4
x3
x2
x1
x1² x2² x3³ x4
x1³ x2³ x3²

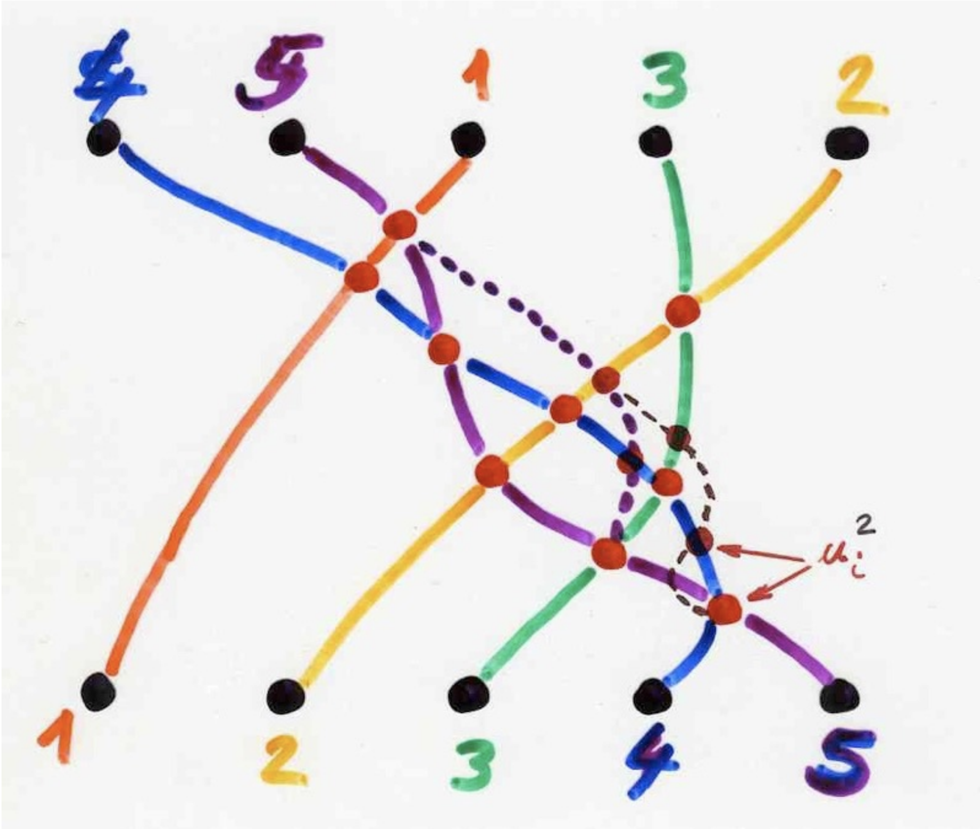
$\sigma \in S_n$ permutation

Def. $$F_\sigma(x_1, \dots, x_m) = \sum_{\substack{U \subseteq I \\ \sigma(U) = \sigma \\ (*)}} v(U)$$

(*) two threads intersect at most once


4
5
1
3
2
1
2
3
4
5


4
5
1
3
2
1
2
3
4
5



4
5
1
3
2
u_i^2
1
2
3
4
5

complements (continued)

symmetric functions and
321- avoiding permutations

Proof of $F_\sigma = S_{\lambda/\mu}$ for σ (321)-avoiding

Bijections

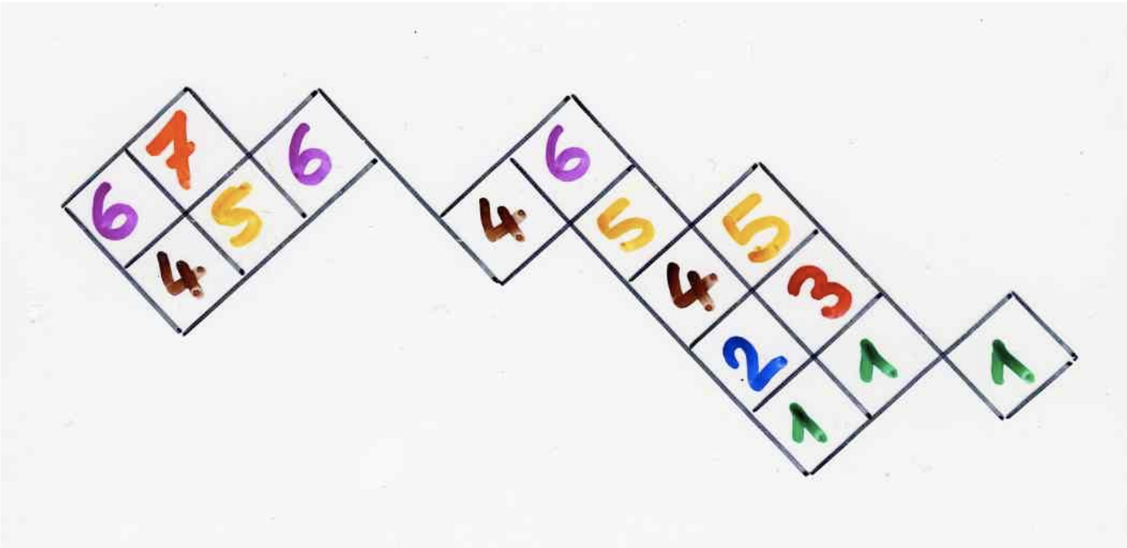
$\sigma \in S_n$ (321)-avoiding $\longleftrightarrow$ λ/μ "length" n no empty column

$U \subseteq I$ $\longleftrightarrow$ tableau shape λ/μ

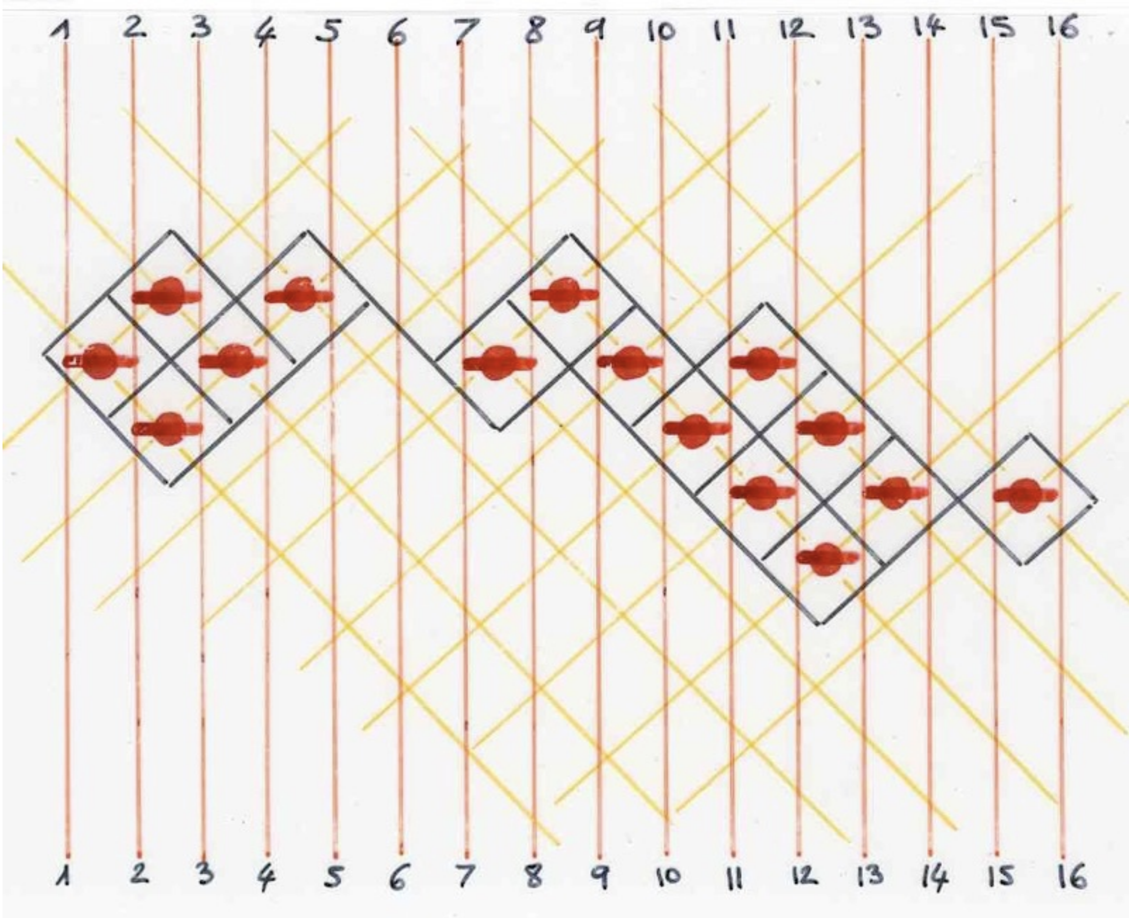
(or resolved configuration of threads)

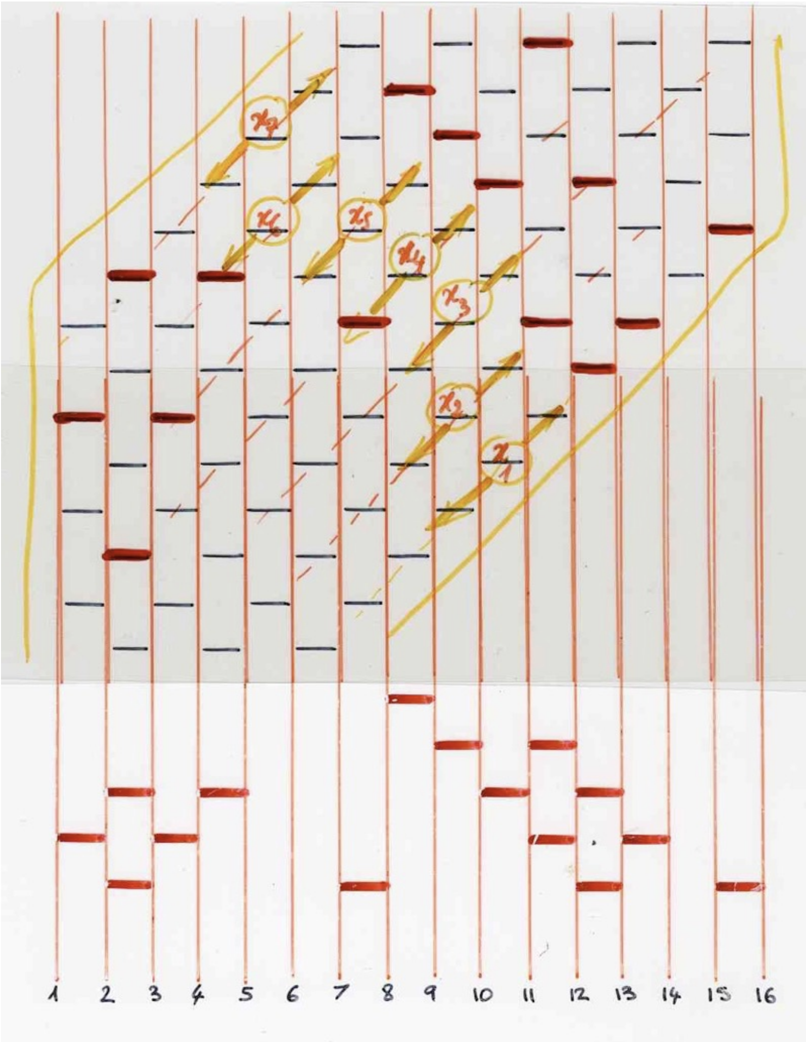
(or pre-heap)

giving σ

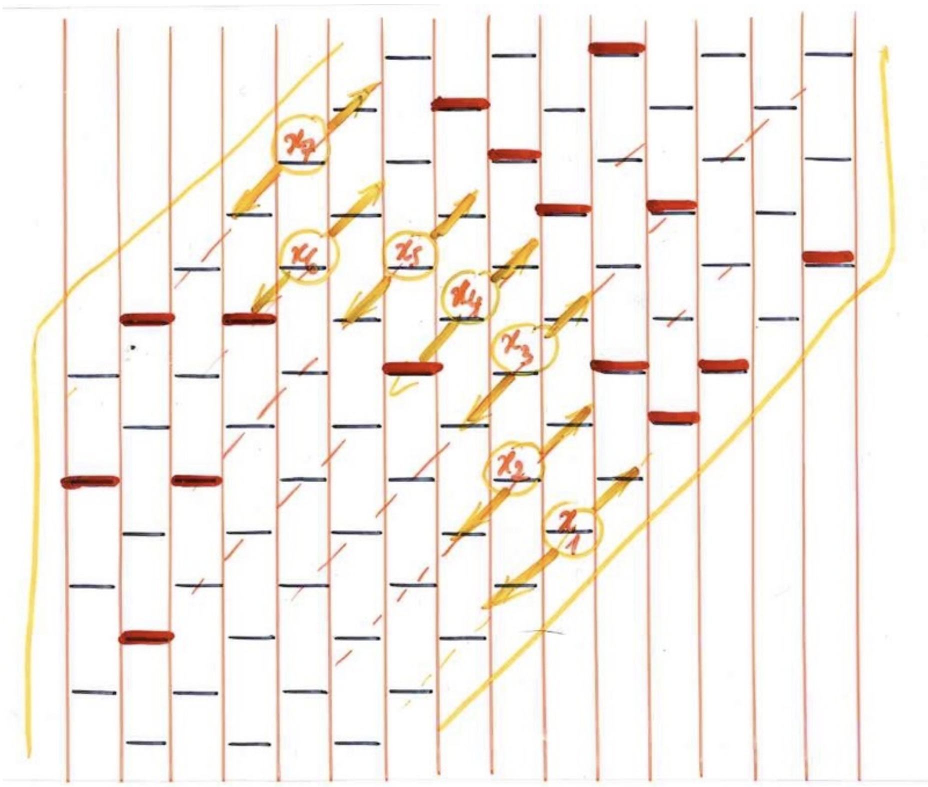


6
7
4
5
6
4
6
5
4
5
3
2
1
1
1

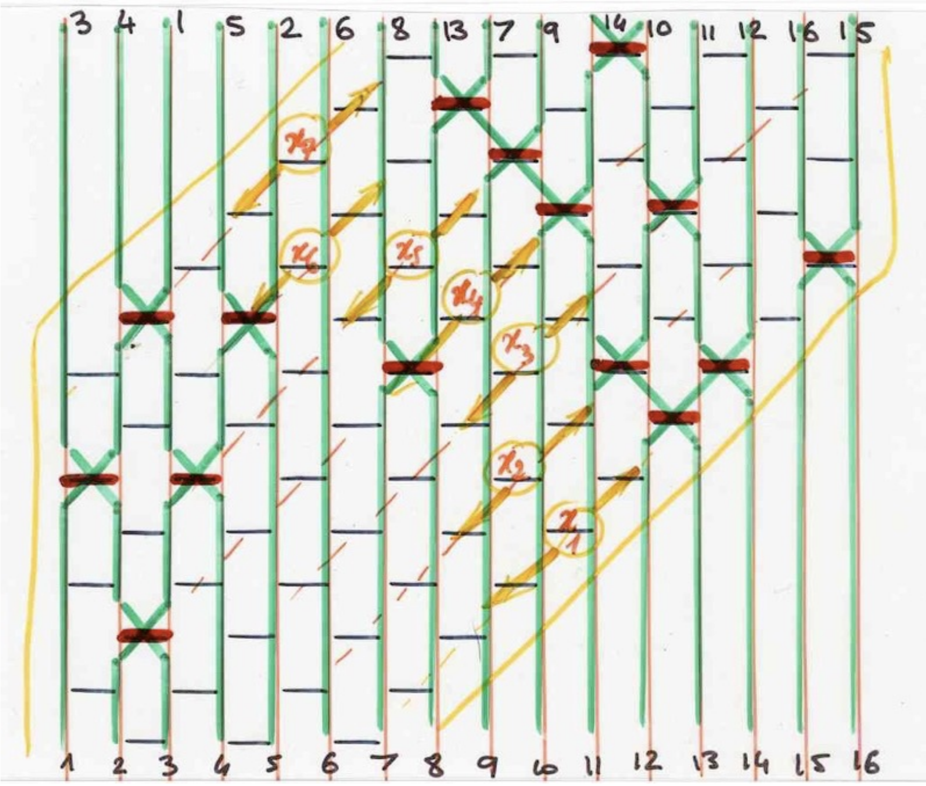

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

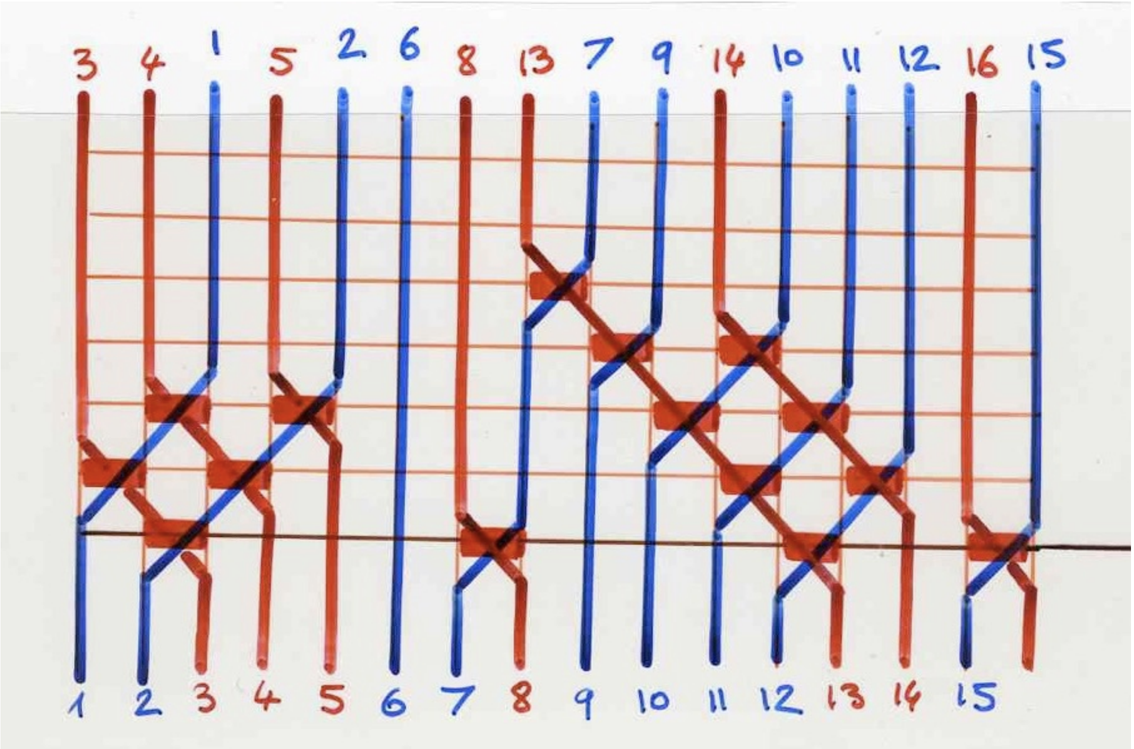


x_7
x_6
x_5
x_4
x_3
x_2
x_1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16



x_7
x_6
x_5
x_4
x_3
x_2
x_1


3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
x_7
x_6
x_5
x_4
x_3
x_2
x_1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

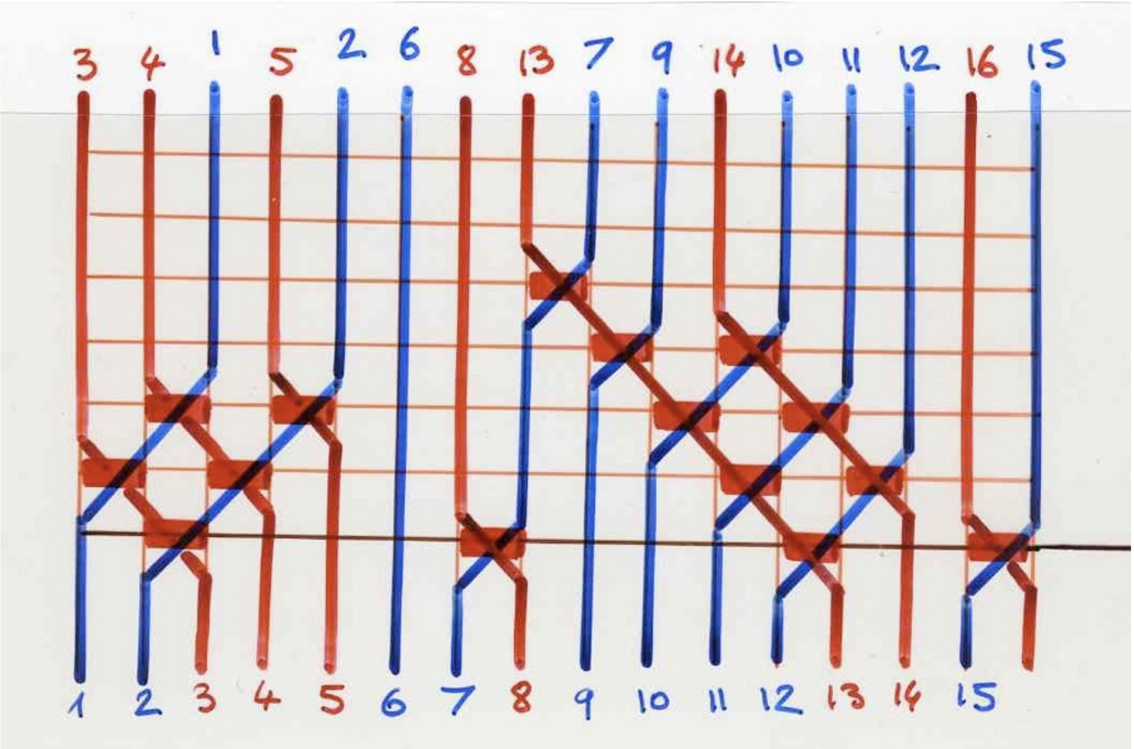

3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

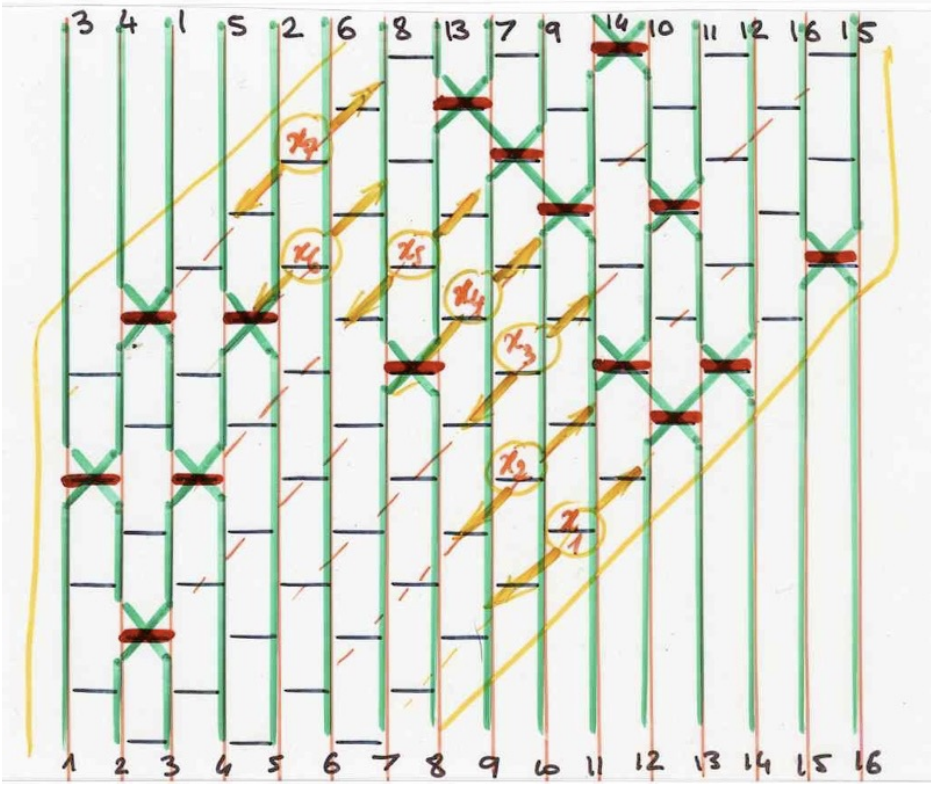
Proof of $F_\sigma = S_{\lambda/\mu}$ for σ (321)-avoiding

Bijections

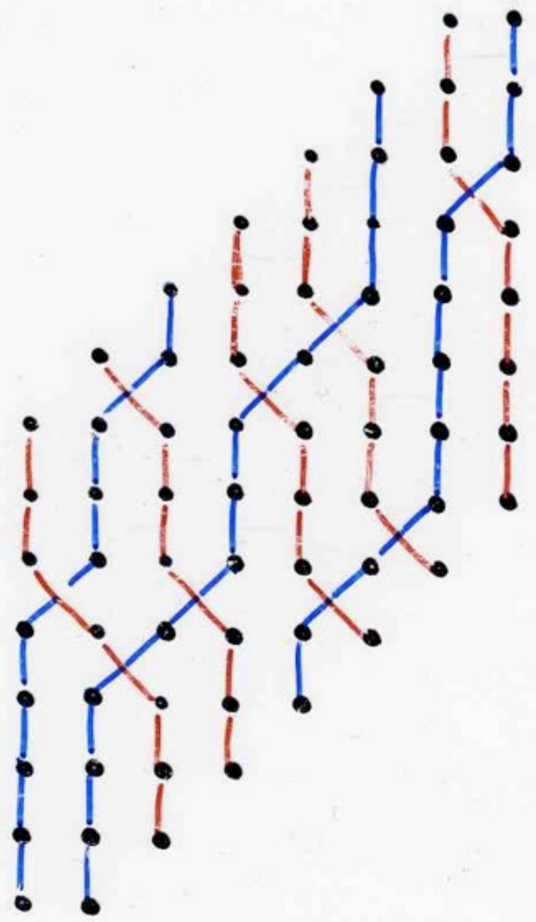
$\sigma \in S_n$ (321)-avoiding $\longleftrightarrow$ λ/μ "length" n no empty column

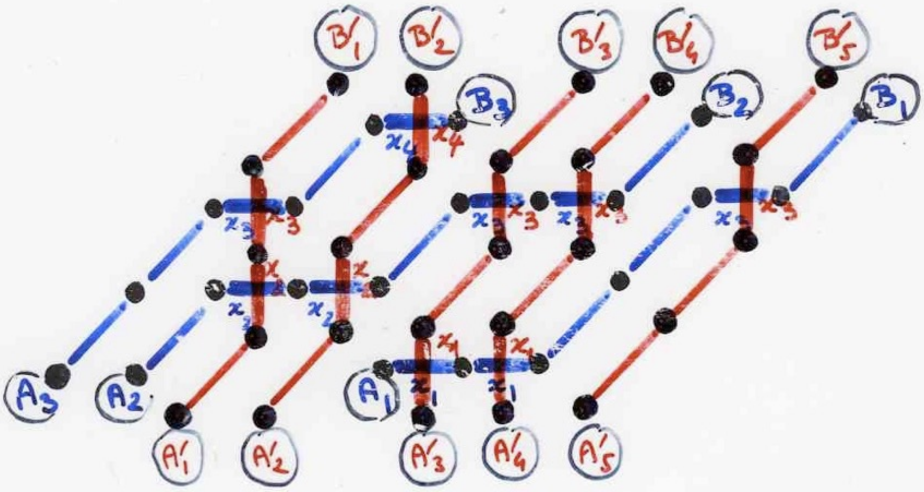
$U \subseteq I$ (or resolved configuration of threads) (or pre-heap) giving σ $\longleftrightarrow$ tableau shape λ/μ


3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15



3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
x_7
x_6
x_5
x_4
x_3
x_2
x_1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16



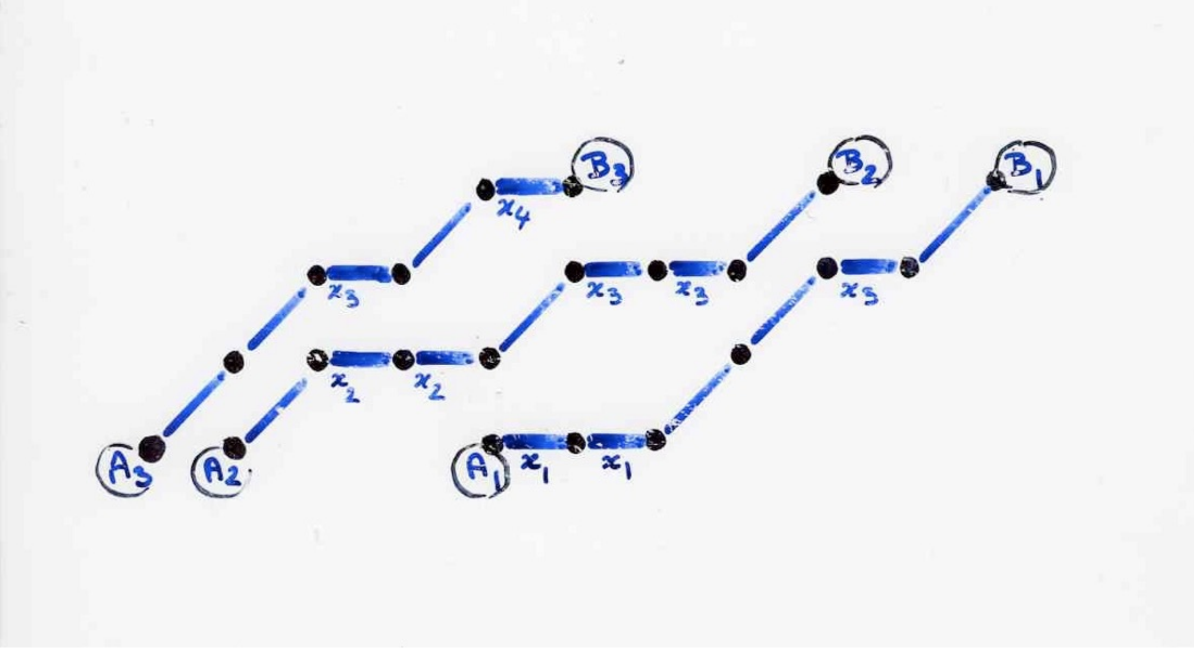


B'1
B'2
B3
B'3
B'4
B2
B'5
B1
x4
x4
x3
x3
x3
x3
x3
x3
x3
x3
x2
x2
x2
x2
x1
x1
x1
x1
A3
A2
A1
A'1
A'2
A'3
A'4
A'5

complements (continued)

Jacobi-Trudi identities

course IMSc 2016, Ch 5b, p 12-41



B3
B2
B1
x4
x3
x3
x3
x3
x2
x2
A3
A2
A1
x1
x1

$\lambda = (5, 4, 2)$

$\mu = (2, 0, 0)$

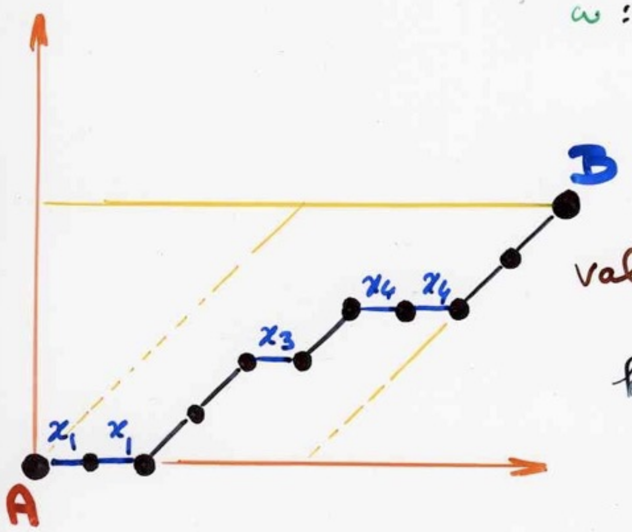
$$\det\left(h_{\lambda_i - \mu_j - i + j}\right)_{1 \le i,j \le r} = \begin{vmatrix} h_3 & h_6 & h_7 \\ h_1 & h_4 & h_5 \\ h_{-2} & h_1 & h_2 \end{vmatrix}$$

$\tilde{H}$ transpose

<u>Lemme</u> $$h_p(x_1, .., x_m) = \sum_{\omega} v(\omega)$$

Motzkin path

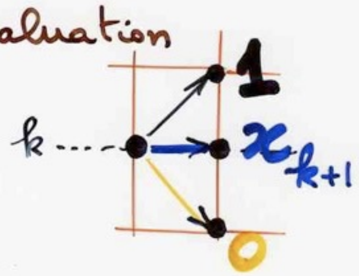
$\omega : A \rightsquigarrow B$

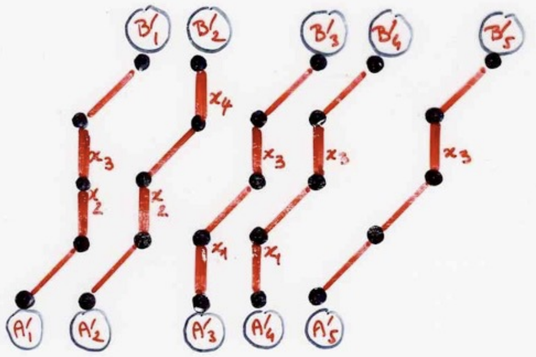


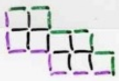
$A = (0,0)$

$B = (p+m-1, m-1)$

valuation





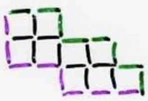


$\lambda' = (3, 3, 2, 2, 1)$

$\mu' = (1, 1, 0, 0, 0)$

$$\det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1 \le i,j \le 5} = \begin{vmatrix} e_2 & e_3 & e_5 & e_6 & e_7 \\ e_1 & e_2 & e_4 & e_5 & e_6 \\ e_{-1} & e_0 & e_2 & e_3 & e_4 \\ e_{-2} & e_{-1} & e_1 & e_2 & e_3 \\ e_{-4} & e_{-3} & e_{-1} & e_0 & e_1 \end{vmatrix}$$

$\tilde{E}$ transpose

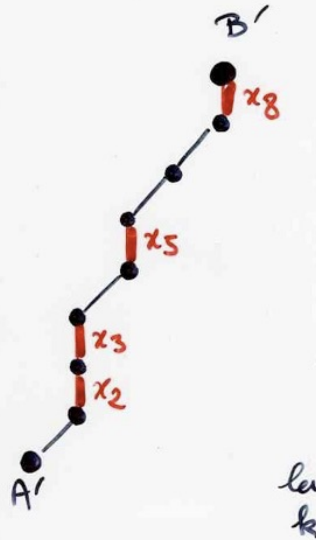


$\lambda' = (3, 3, 2, 2, 1)$

$\mu' = (1, 1, 0, 0, 0)$

$$\det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1 \leq i,j \leq 5} = \begin{vmatrix} e_2 & e_3 & e_5 & e_6 & e_7 \\ e_1 & e_2 & e_4 & e_5 & e_6 \\ e_{-1} & e_0 & e_2 & e_3 & e_4 \\ e_{-2} & e_{-1} & e_1 & e_2 & e_3 \\ e_{-4} & e_{-3} & e_{-1} & e_0 & e_1 \end{vmatrix}$$

$\widetilde{E}$ transpose



$$e_p = \sum_{\omega} v(\omega)$$

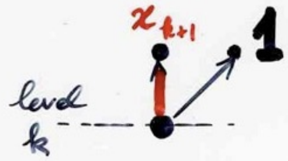
"Favard" path

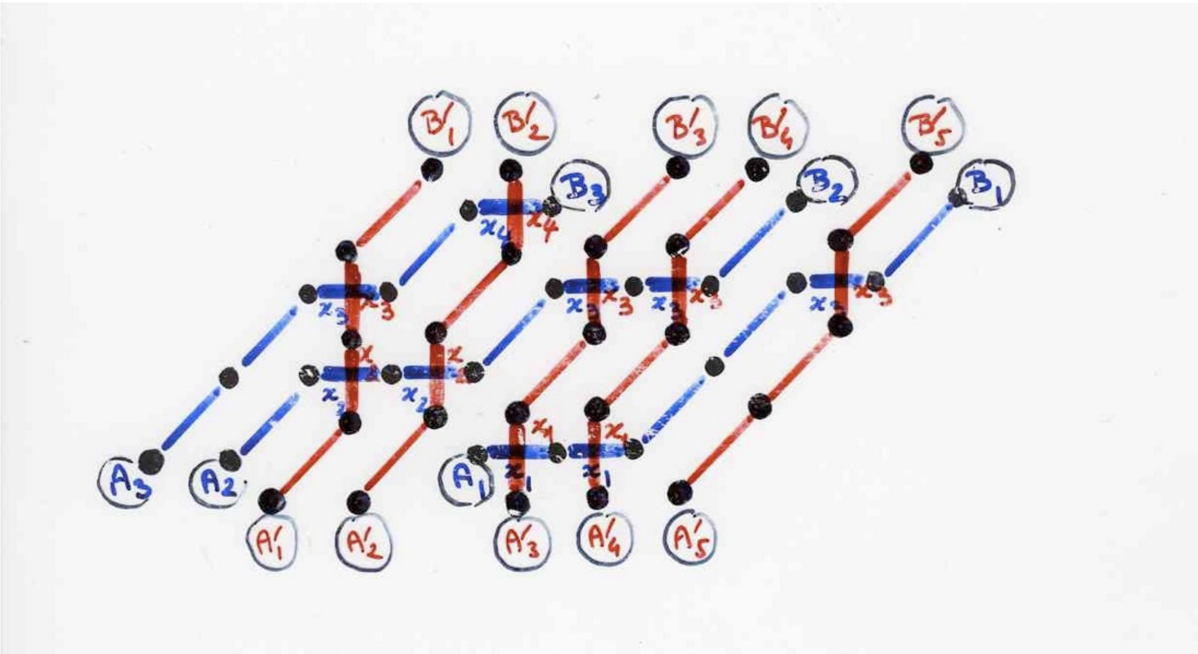
$\omega : A' \rightsquigarrow B'$

$A' = (0, 0)$

$B' = (m, m-p)$

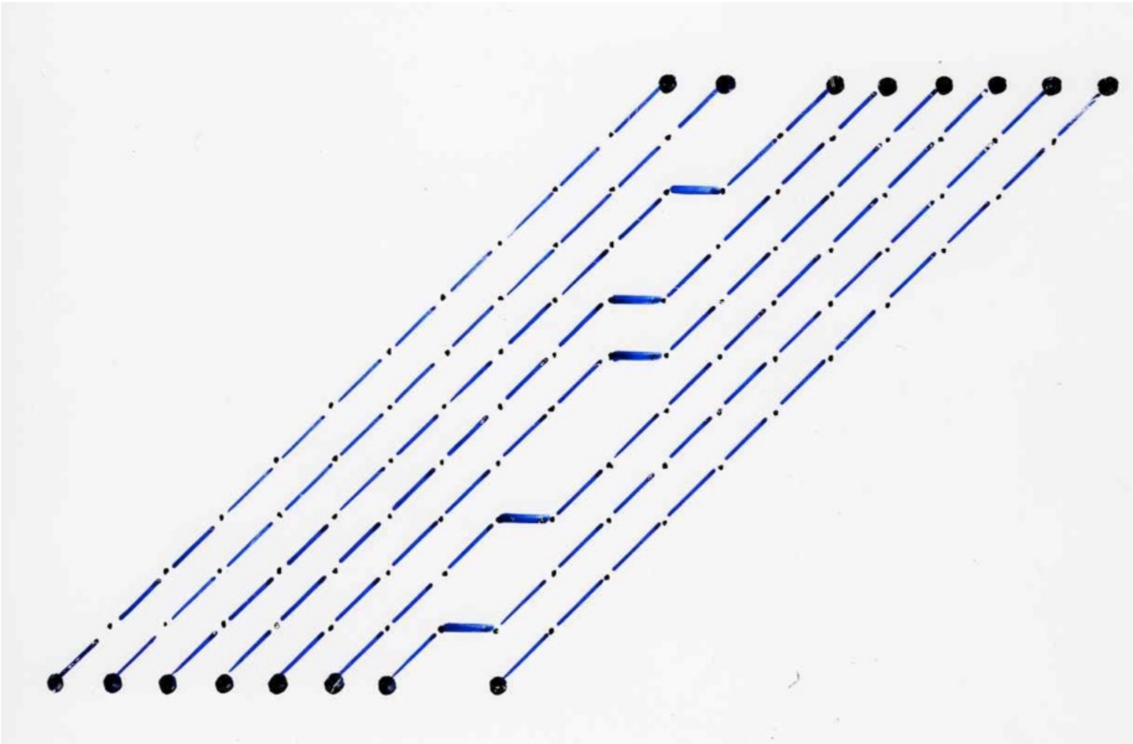
valuation :

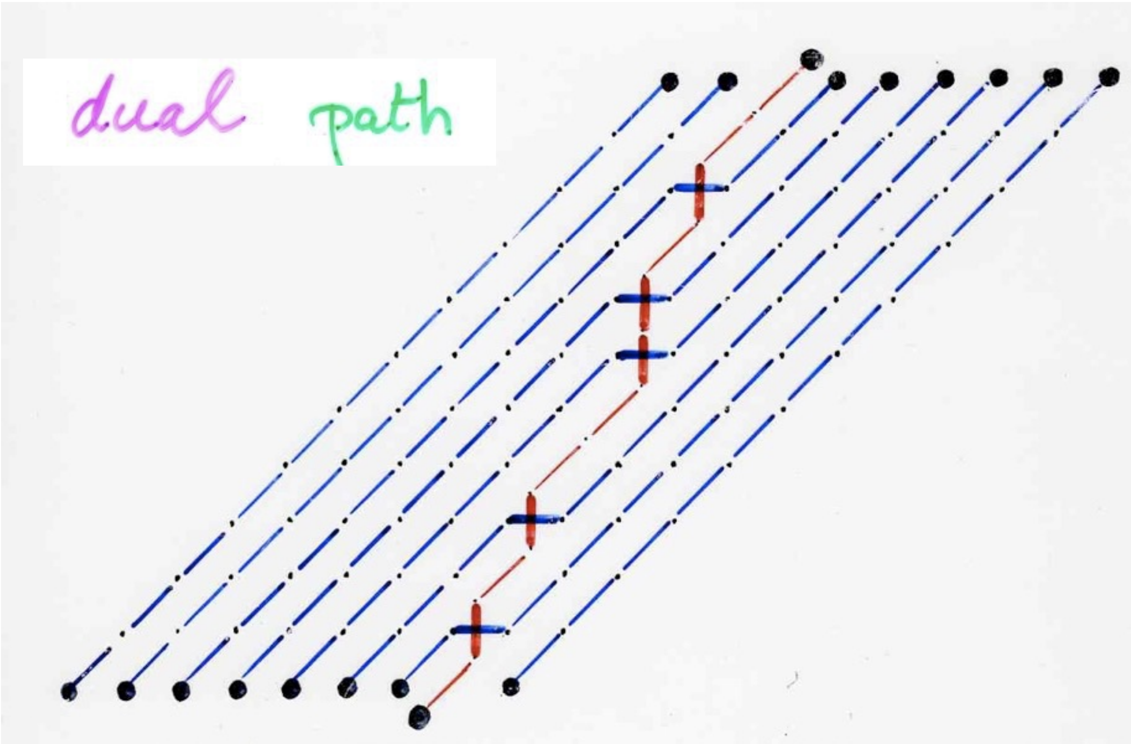


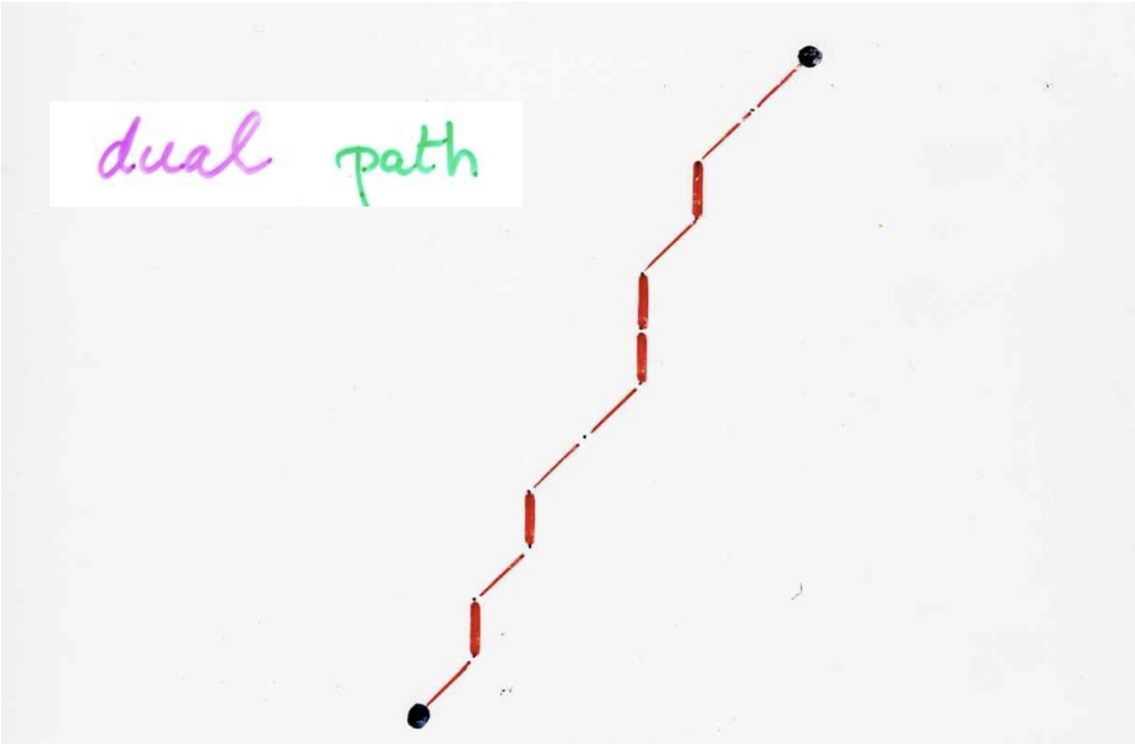


B'1
B'2
B'3
B'4
B'5
B3
B2
B1
A3
A2
A1
A'1
A'2
A'3
A'4
A'5

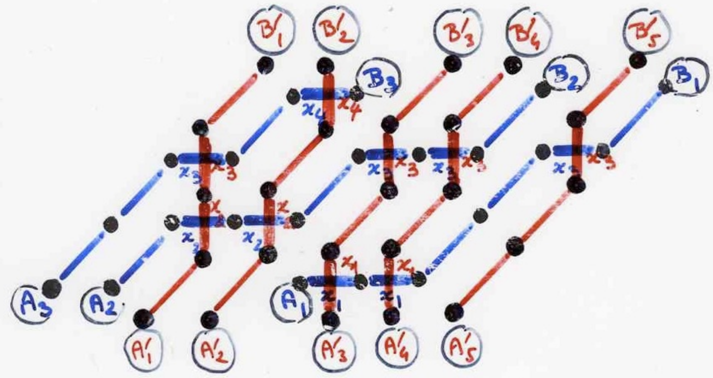

Paths duality
P. Lalonde, X.V. (1985, 1999)

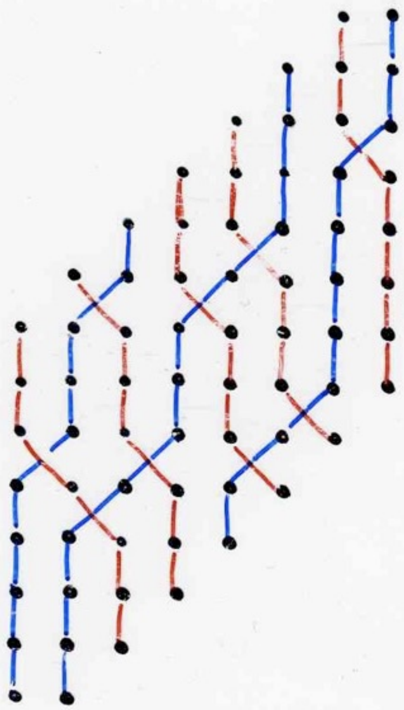


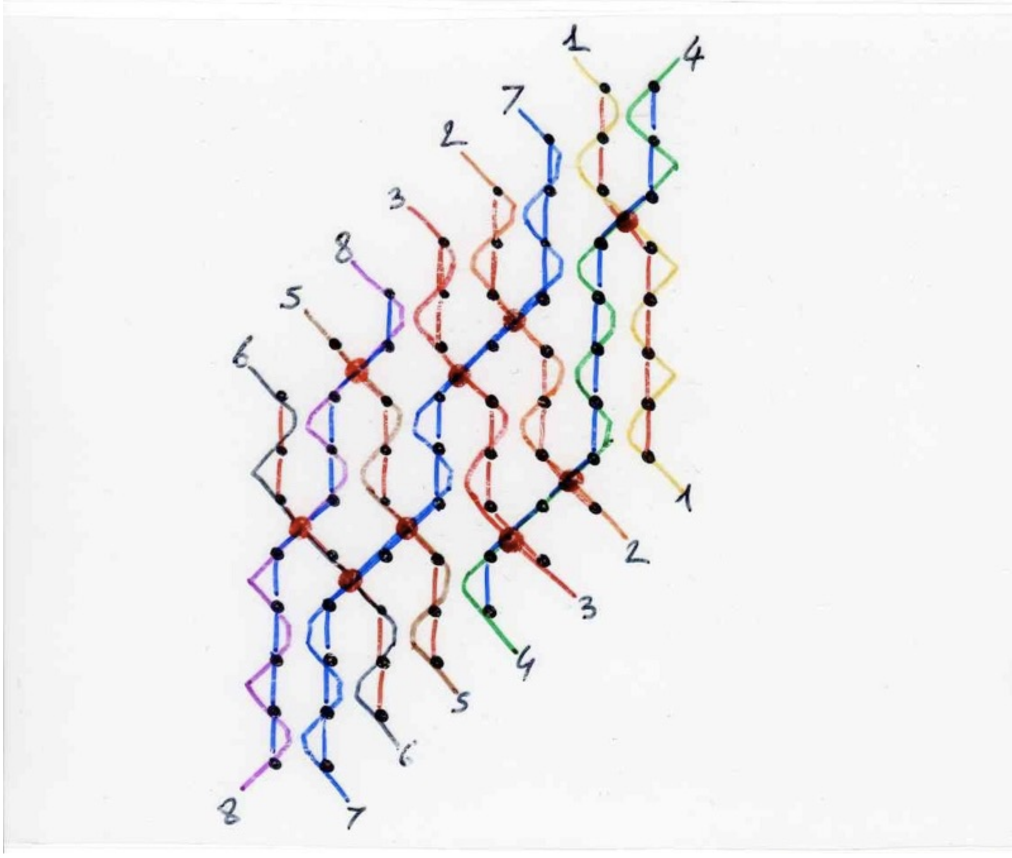

dual path

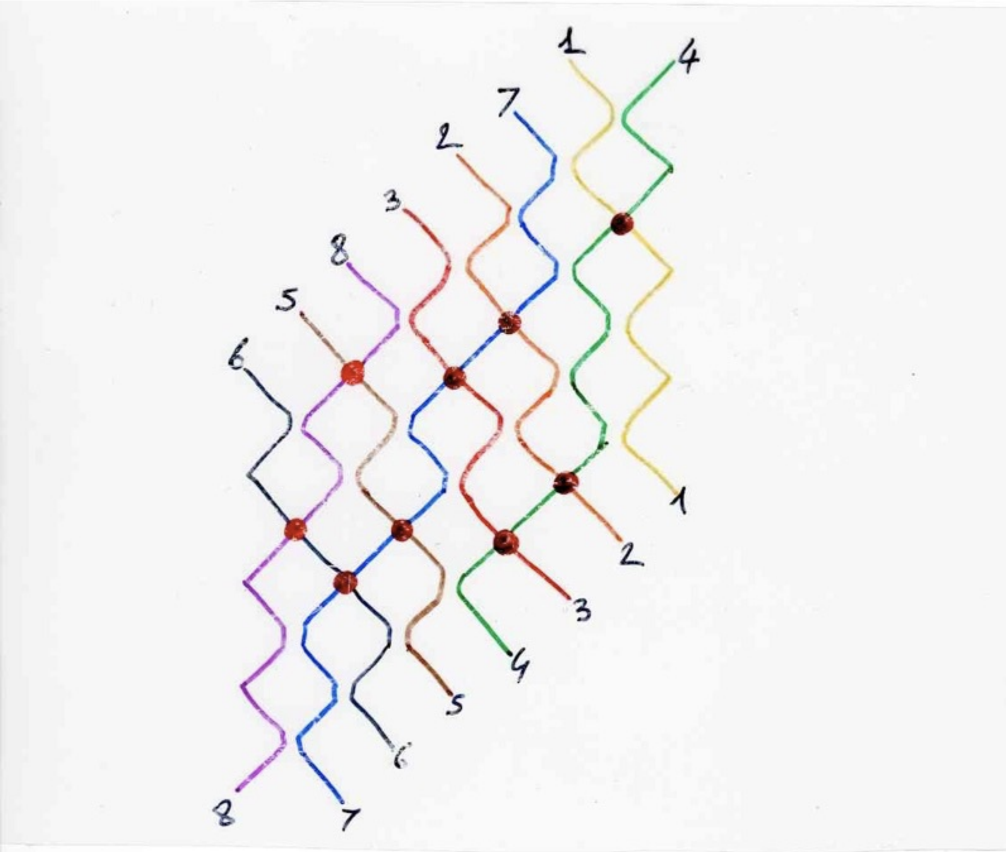

dual path

dual configurations of non-intersecting paths






1
4
7
2
3
8
5
6
1
2
3
4
5
6
8
7


1
4
7
2
3
8
5
6
1
2
3
4
5
6
8
7

$$(\text{Fomin-Kirillov}) = (\text{Jacobi-Trudi}) + (\text{Trudi-Jacobi})$$

for (321)

$S_{\lambda\backslash\mu}$

(with LGV Lemma)