Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 6

Heaps and Coxeter groups
(2)

fully commutative elements
and Temperley-Lieb algebra

IMSc, Chennai
27 February 2017

from the previous lecture

Symmetric group $\mathfrak{S}_n$

S_n

$n!$ permutations

$$\sigma_i = (i, i+1) \qquad i=1,2,\dots,n-1$$

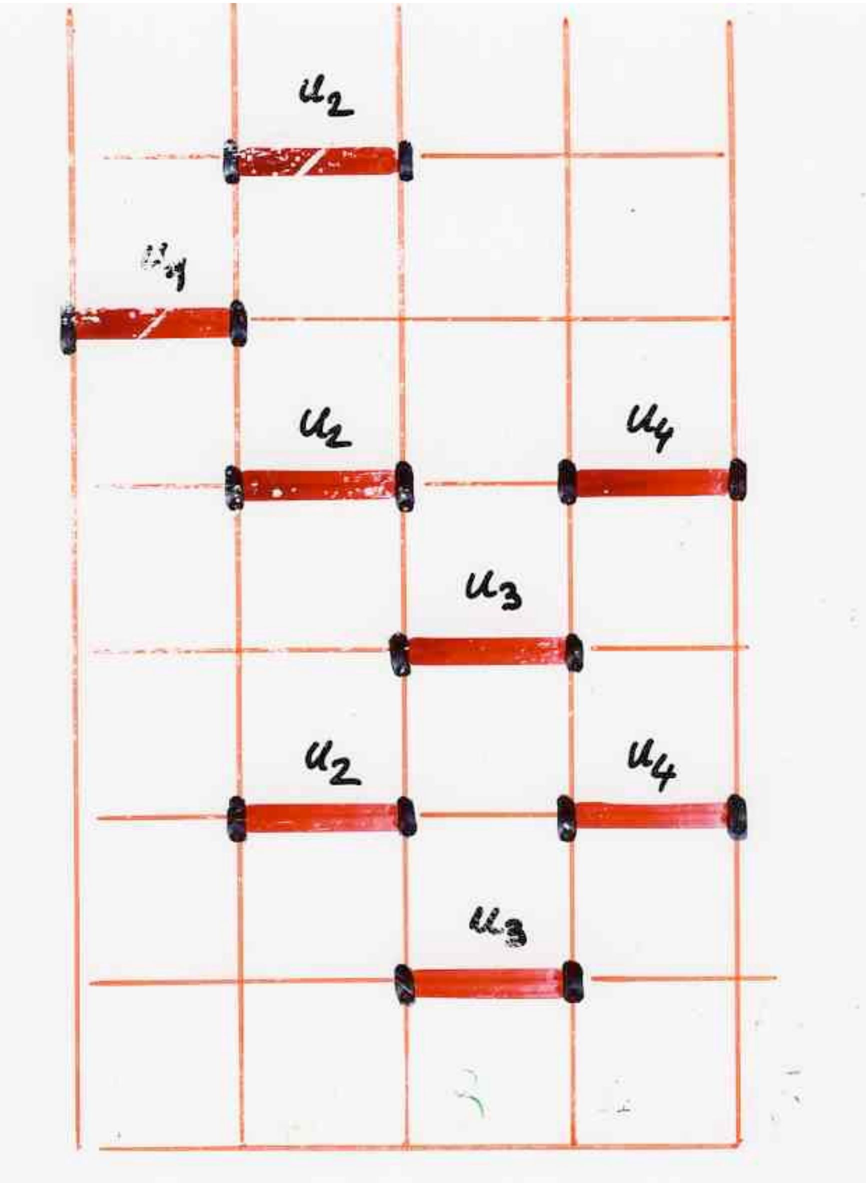
transposition of two consecutive elements

$$\begin{cases} (i) & \sigma_i \sigma_j = \sigma_j \sigma_i , \qquad |i-j| \geqslant 2 \\ (ii) & \sigma_i^2 = 1 , \\ (iii) & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} . \end{cases}$$

Moore-Coxeter

Yang-Baxter

Coxeter graph



heap of dimers $[1,n]$ $\longrightarrow$ permutation S_n

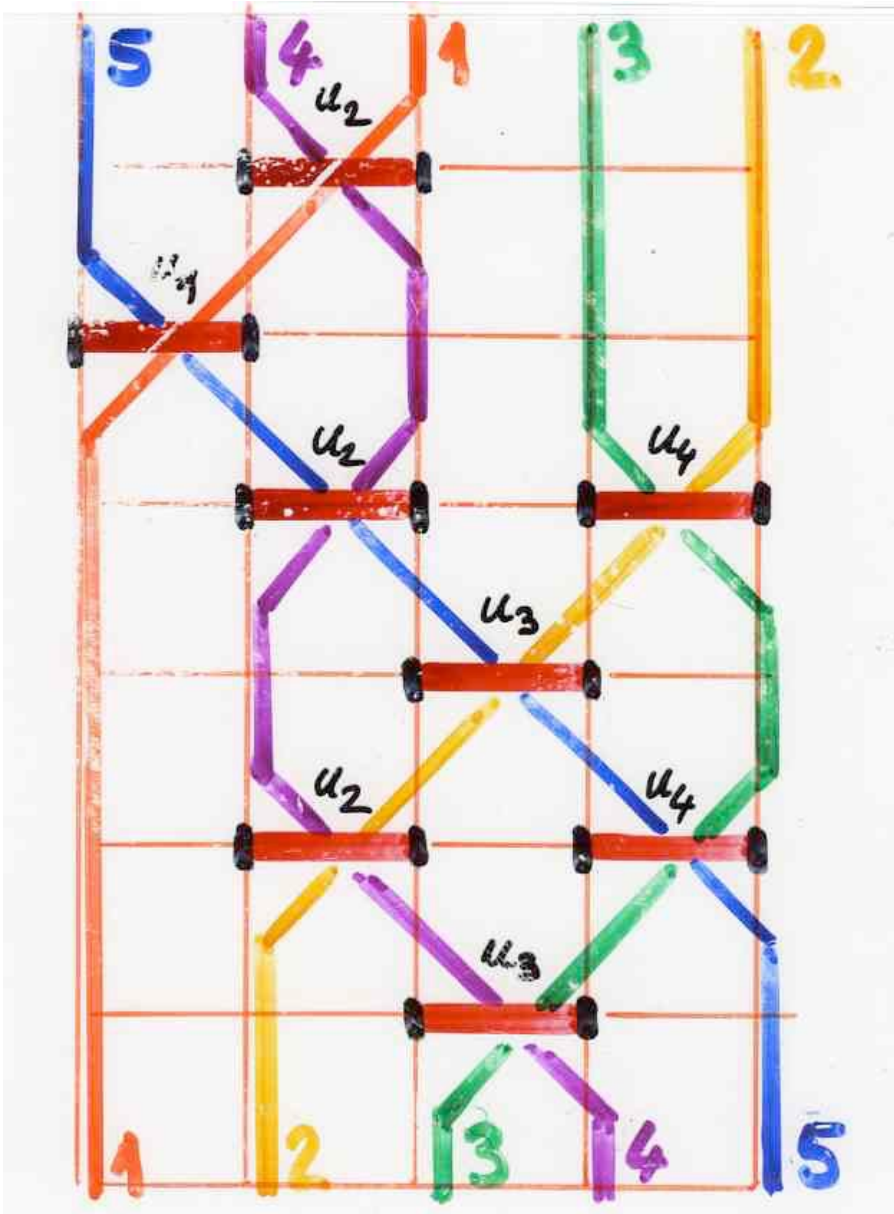
heaps of dimers $(i, i+1)$

on $\{0,1,\dots,n-1\}$

generators $\{\sigma_0, \sigma_1, \dots, \sigma_{n-1}\}$

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

iff $|i-j| \geq 2$



heap of dimers $[1,n]$ $\longrightarrow$ permutation S_n

reduced decomposition
of a permutation

$$\sigma = u_{i_1} \dots u_{i_k}$$

k minimum

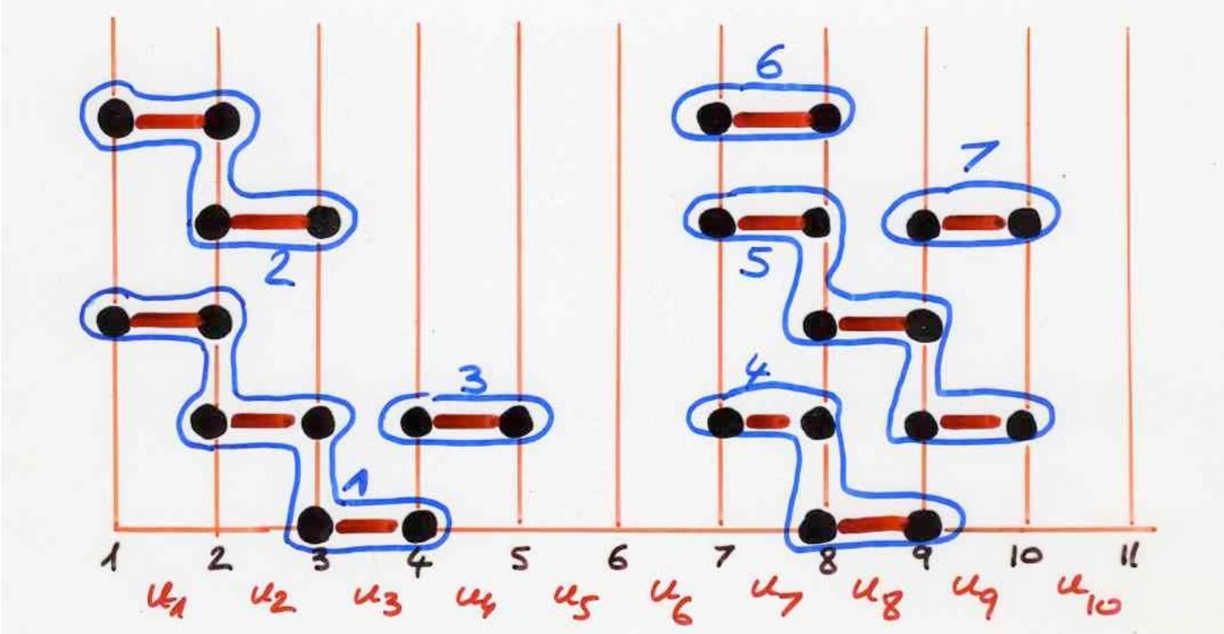
(nb of inversion)

__Lemma__. The set $R(w)$ of reduced decompositions is a disjoint union of commutation classes.

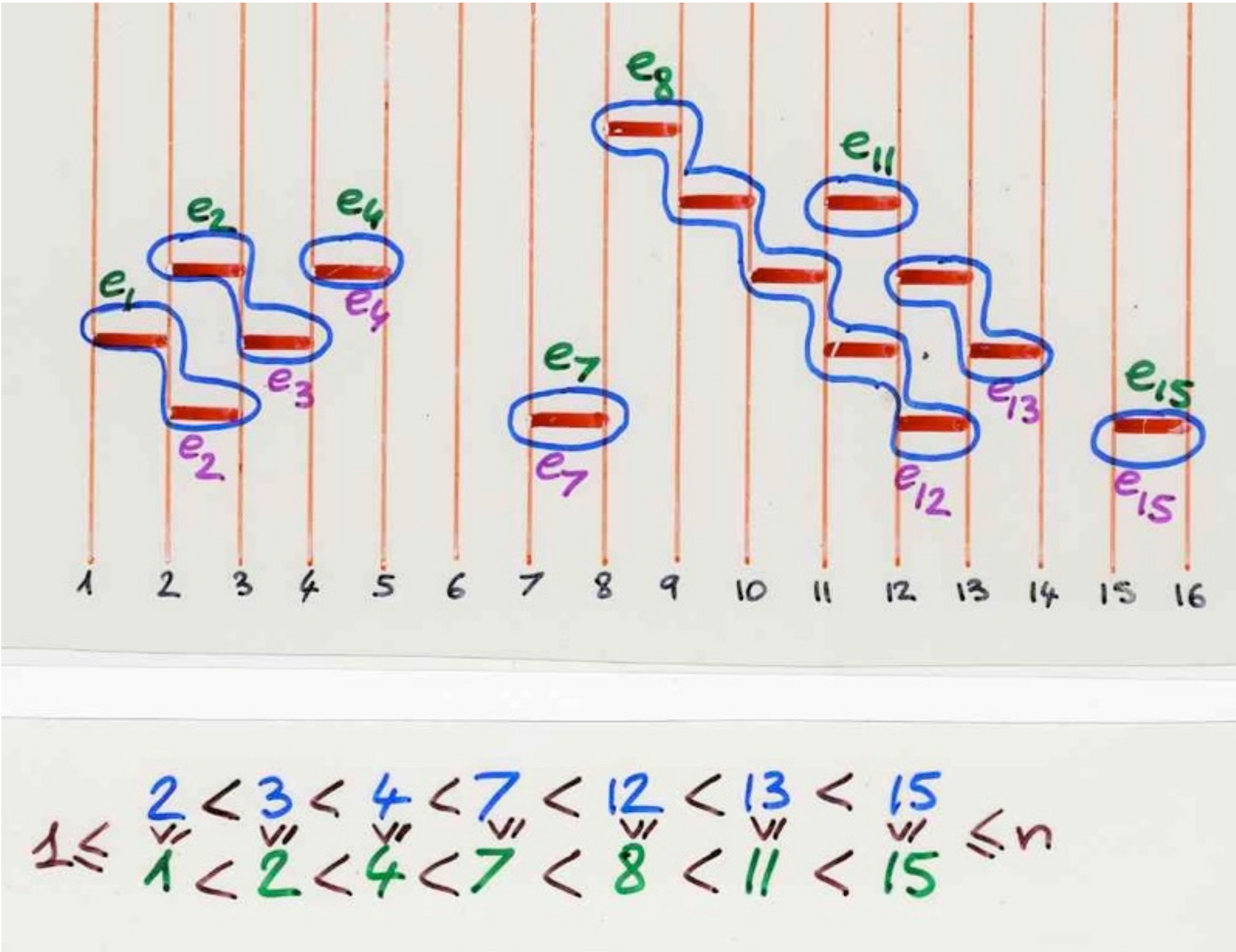
For each of them, there exist a heap $H(C)$ of $\mathbf{H}(W,S)$ such that C is exactly the set of linear extensions of the poset $H(C)$

Definition An element w of the Coxeter group W is *fully commutative* iff $R(w)$ is reduced to one commutation class.

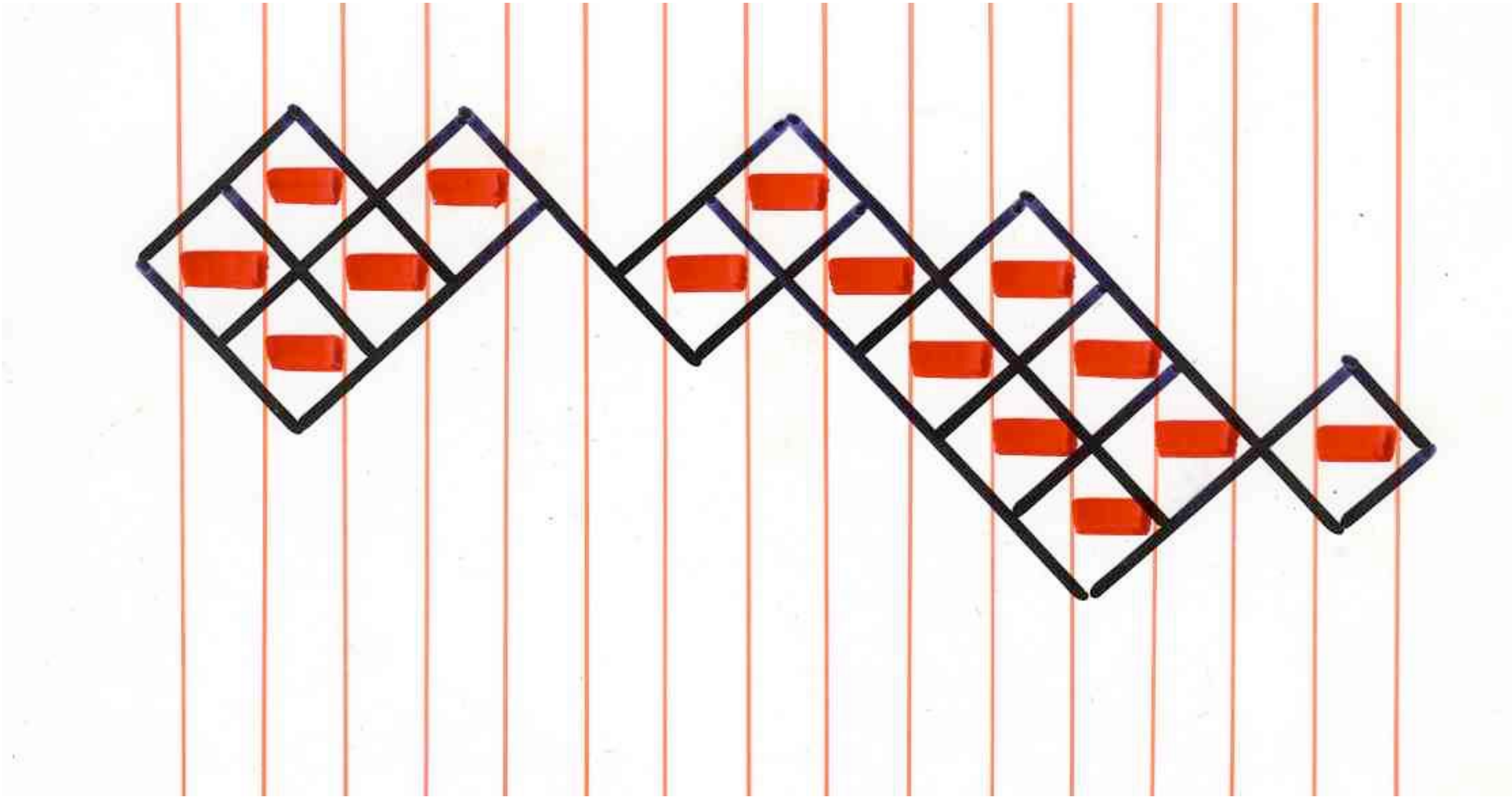
The corresponding heap $H(w)$ will also be called *fully commutative* (FC)



the stairs decomposition
of a heap of dimers



Catalan numbers



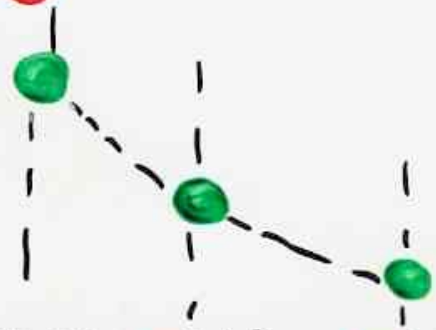
bijection

fully commutative heaps

321- avoiding permutations

(321) - avoiding permutations

no occurence of



$$\sigma(i) > \sigma(j) > \sigma(k)$$

$$i < j < k$$

Prop $\sigma \in S_n$ permutation

- (321) - avoiding

$\Updownarrow$

- only one commutation class

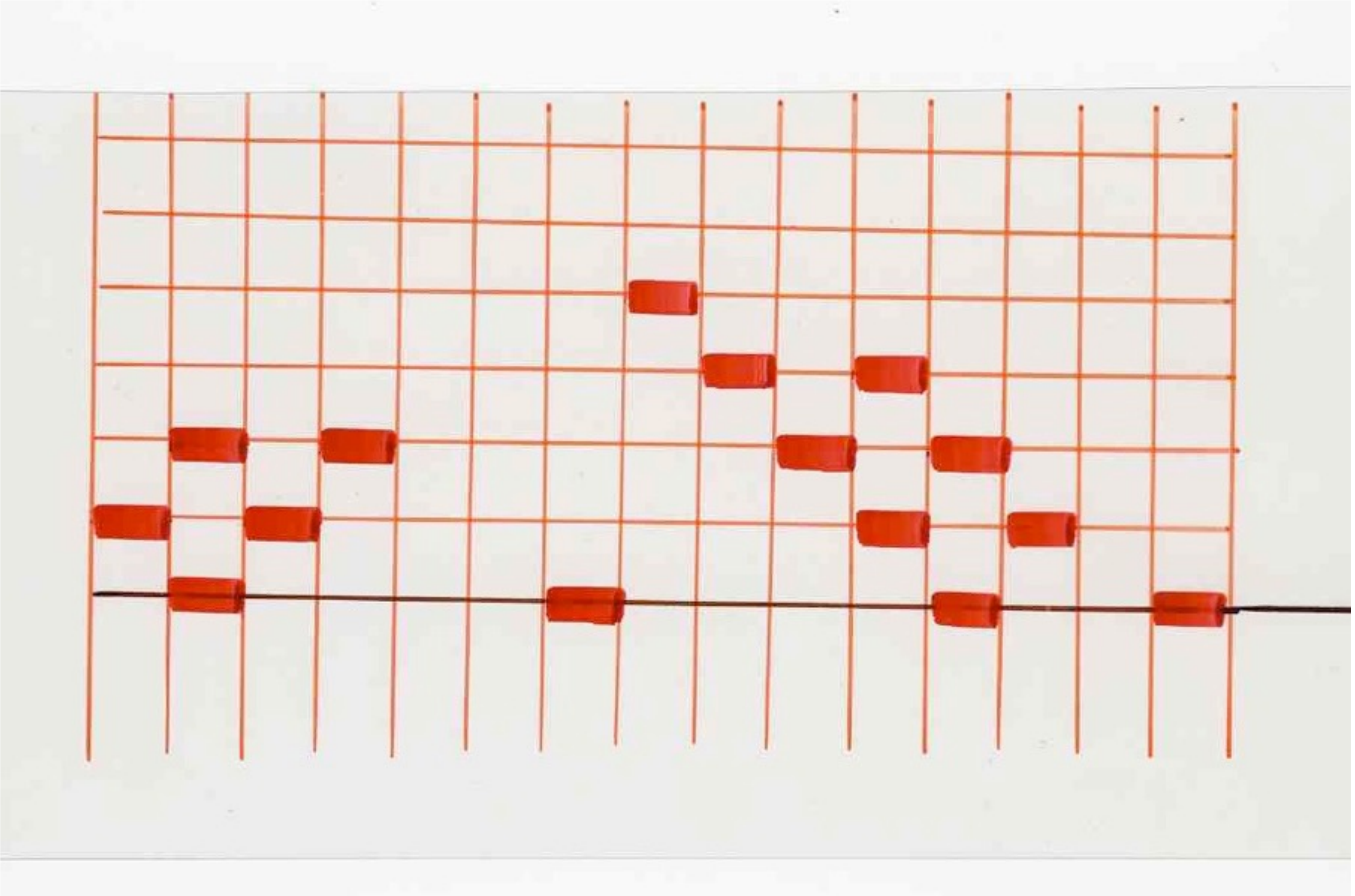
(Billey, Jockusch, Stanley) (1993)

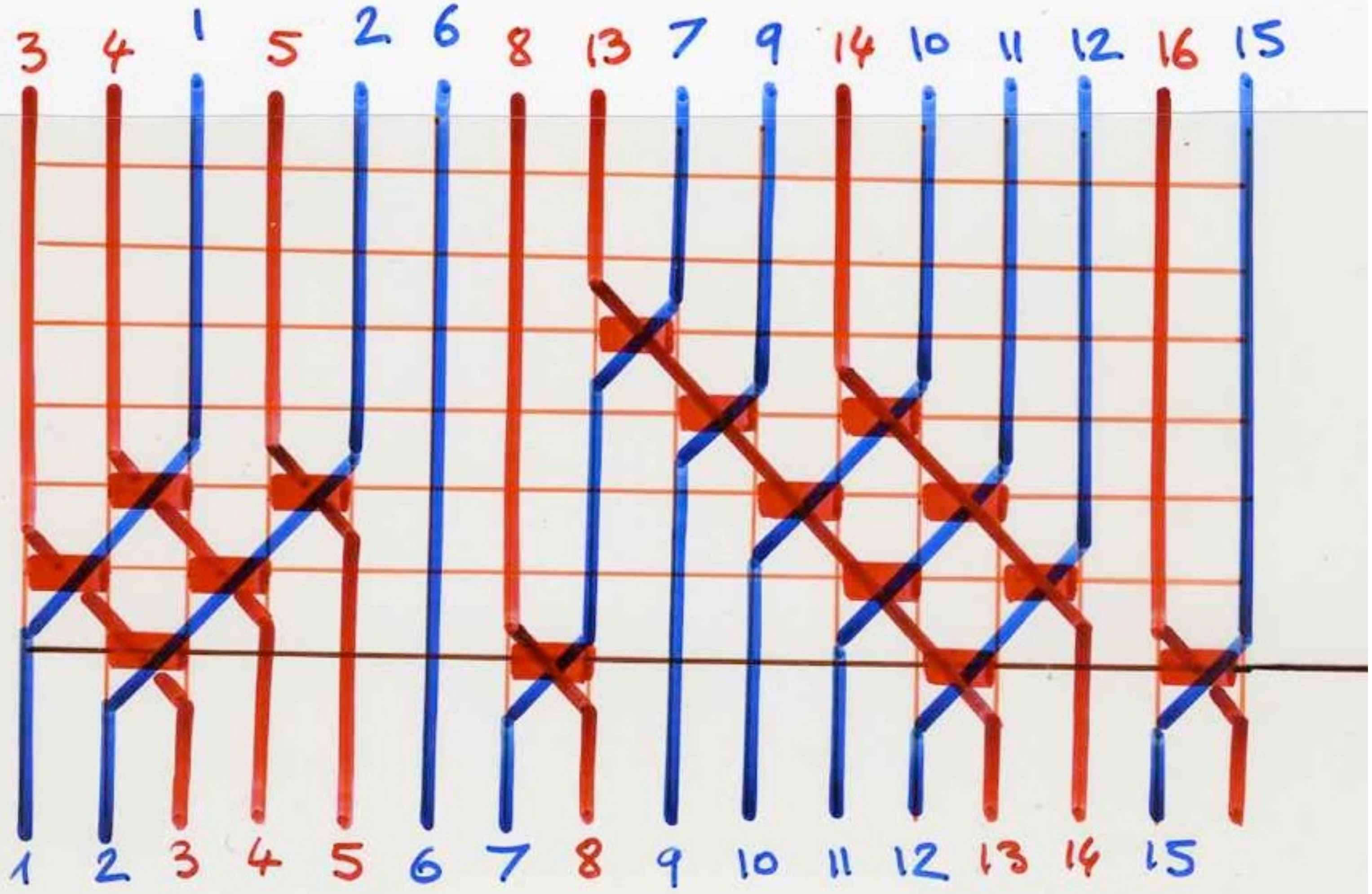
$\Updownarrow$

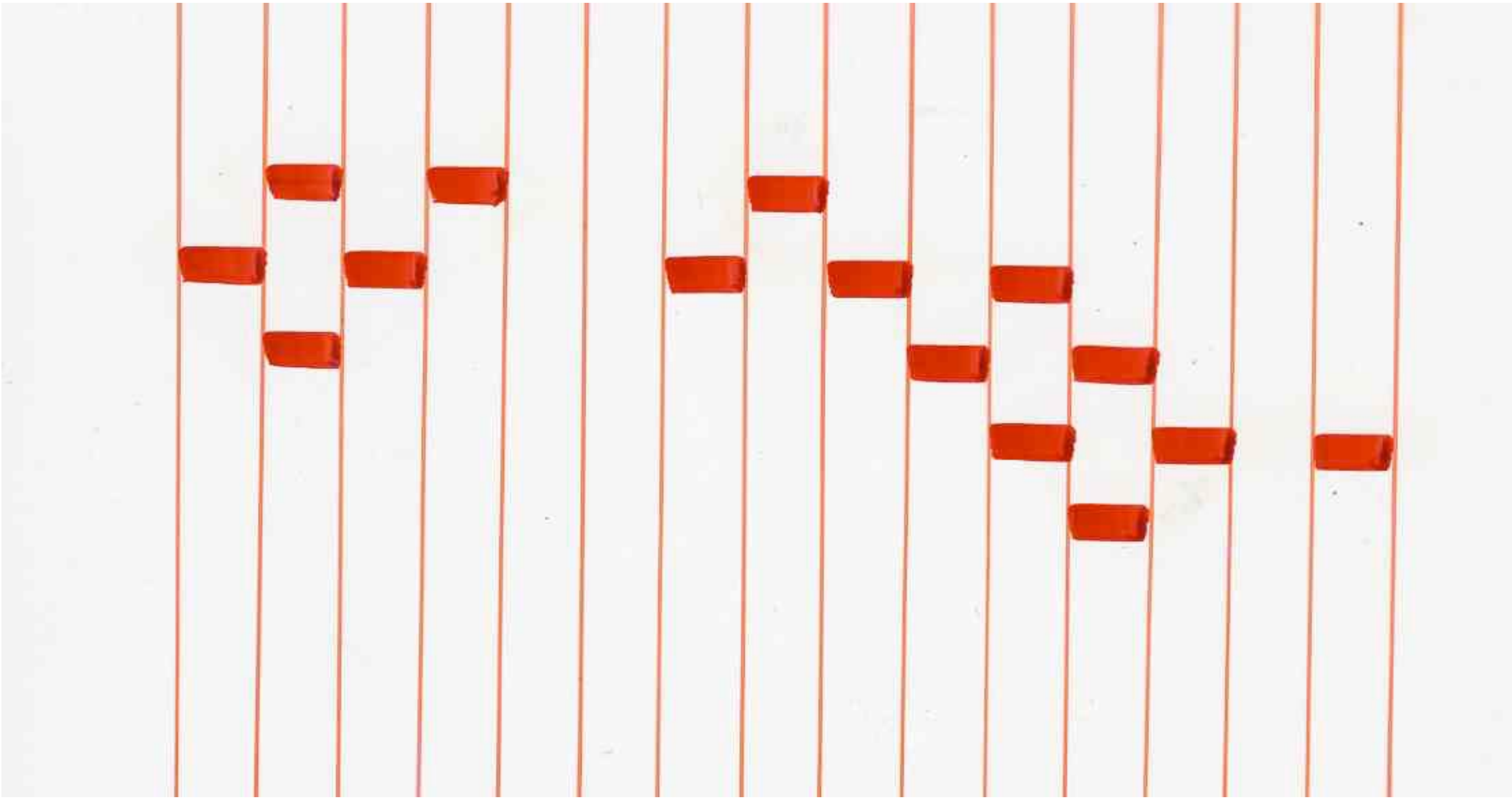
- no  or 

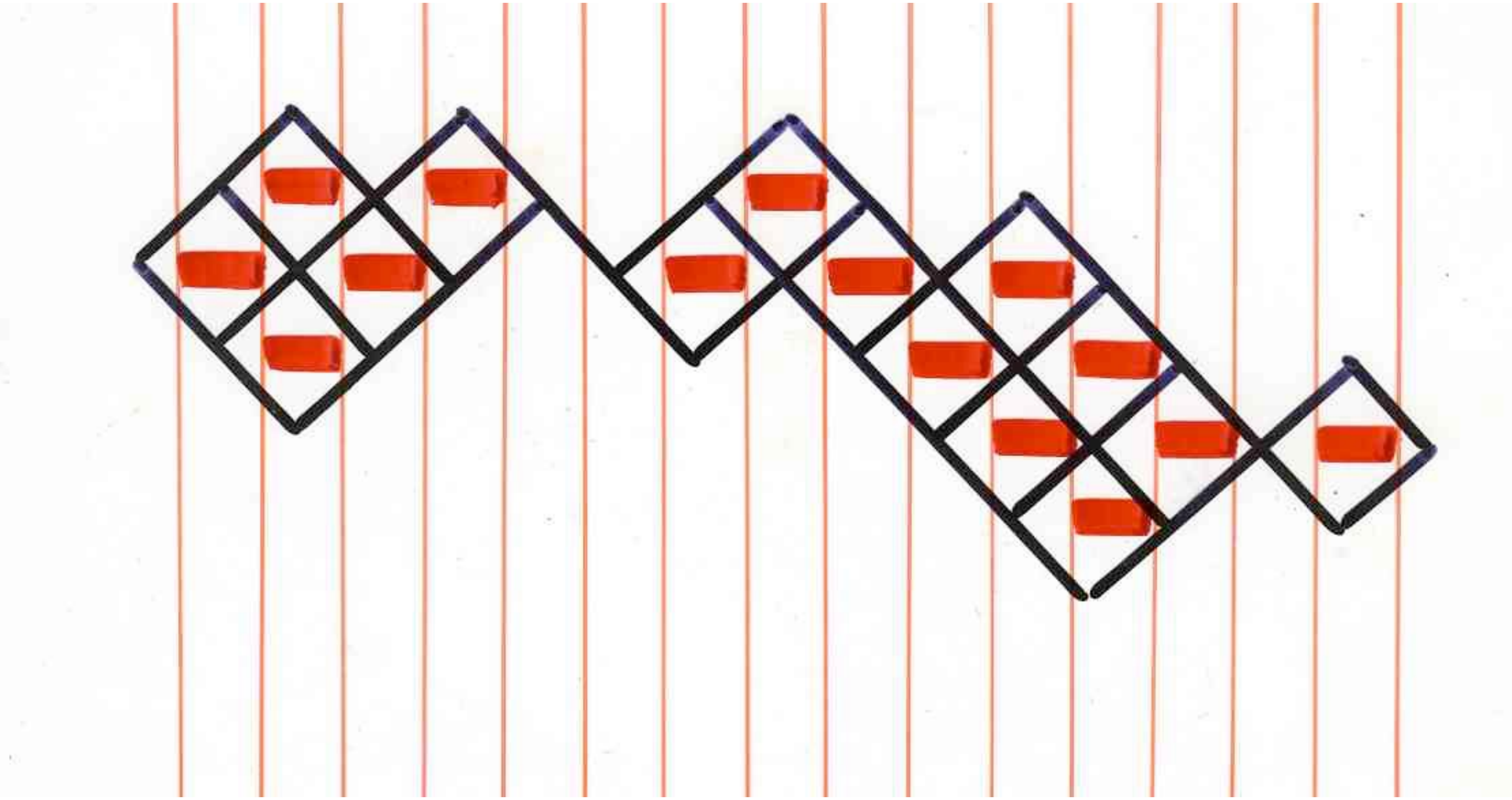
(counted by $C_n = \frac{1}{n+1}\binom{2n}{n}$

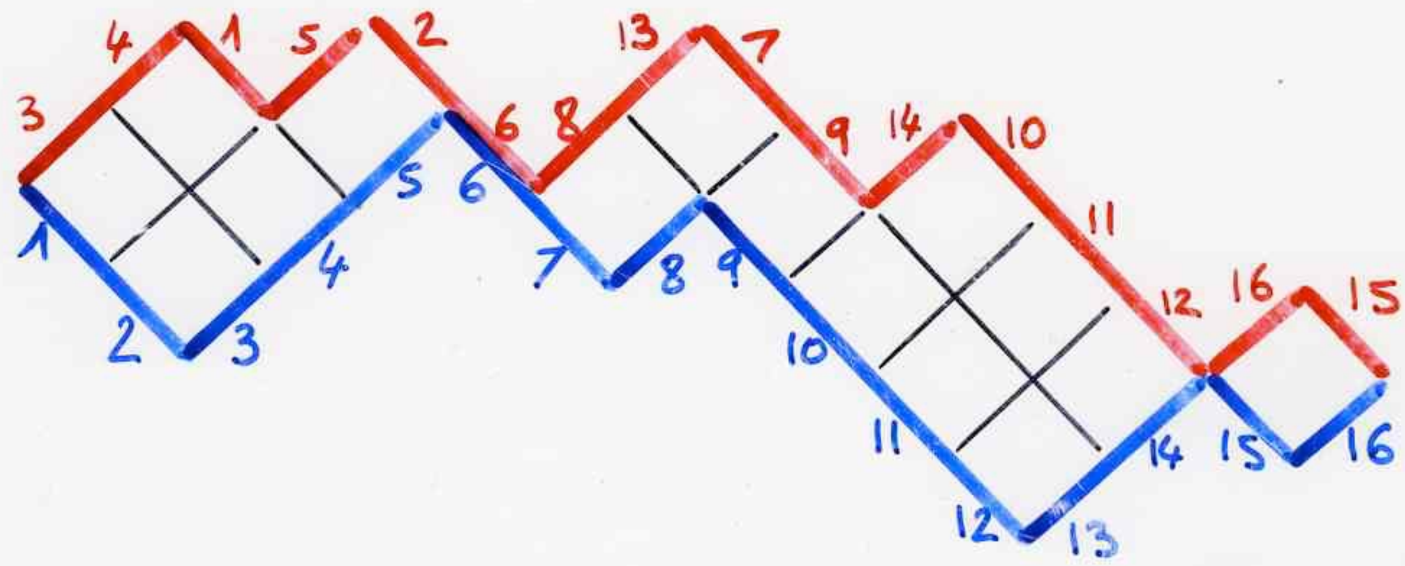
Catalan numbers




3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15



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16

Temperley-Lieb algebra

Temperley-Lieb algebra

$TL_n(\beta)$ generators $\{e_1, e_2, \dots, e_{n-1}\}$

(i) $e_i \, e_j = e_j \, e_i \qquad |i-j| \geq 2$

(ii) $e_i^2 = \beta e_i$

(iii) $\begin{cases} e_i \, e_{i+1} \, e_i = e_i \\ e_{i+1} \, e_i \, e_{i+1} = e_{i+1} \end{cases}$

β scalar

$$\left(1 + 3\, e_2 e_3 e_1 + 2\, \mathrlap{\underbrace{}_{e_2}} e_2 \overbrace{e_3\, e_2\, e_3}^{e_3}\right) \times$$

$$\left(1 - e_2 + 4\, e_3 \underbrace{e_1\, e_3}_{e_3 e_1}\right)$$

$$4\, \beta\, e_1$$

$$w \xrightarrow{*} \overline{w}$$

sequence of rewritings (reductions) and commutations

$$\begin{cases} e_i^2 \to \beta e_i \\ e_i e_{i+1} e_i \longrightarrow e_i \\ e_{i+1} e_i e_{i+1} \longrightarrow e_{i+1} \end{cases}$$

$$w = e_1\ e_2\ e_4\ e_2\ e_1\ e_3\ e_2\ e_4\ e_2$$

$$\beta \quad e_1\ e_2\ e_2\ e_4\ e_1\ e_3\ e_2\ e_4\ e_2$$

$$\beta \quad e_1\ e_2\ e_4\ e_1\ e_3\ e_2\ e_2\ e_4$$

$$e_1\ e_2\ e_4\ e_1\ e_3\ e_2\ e_4$$

$$e_1\ e_2\ e_1\ e_4\ e_3\ e_2\ e_4$$

$$e_1\ e_4\ e_3\ e_4\ e_2$$

$$e_1\ e_4\ e_2$$

$$\overline{w} = \beta^2 e_1 e_4 e_2$$

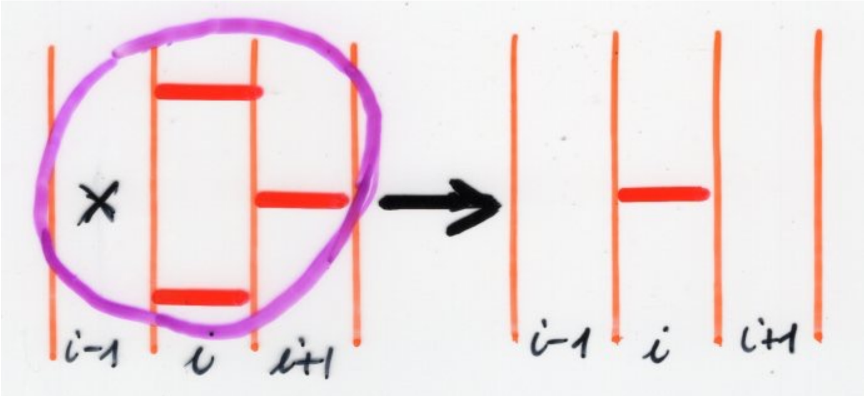
Definition A word w is reduced iff each word of the commutation class $[w]$ has no factors of the type:

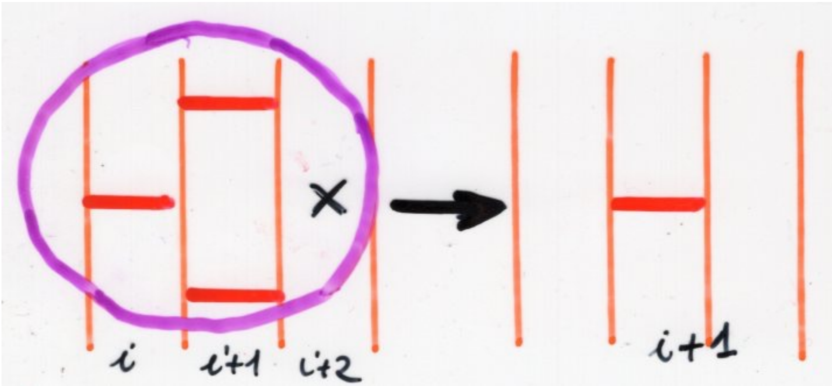
$e_i e_i$, $e_i e_{i+1} e_i$, $e_{i+1} e_i e_{i+1}$

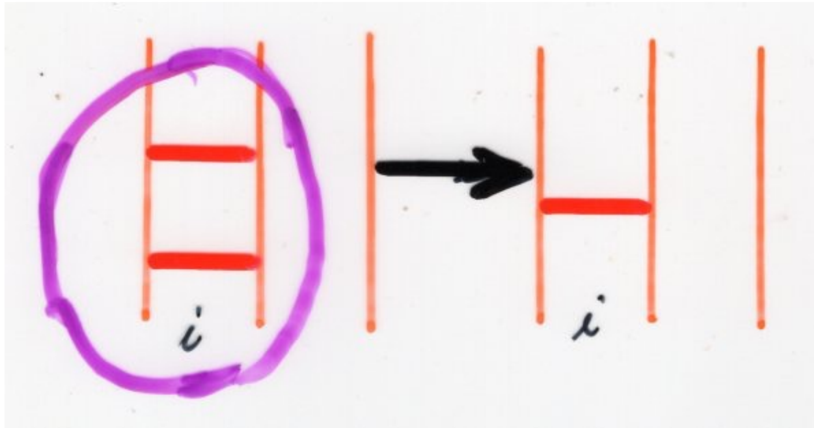
(no possible rewriting)

C commutations $e_i e_j = e_j e_i$ with $|i-j| \geq 2$

Proposition If $w \xrightarrow{*} \beta^i w_1$, $w \xrightarrow{*} \beta^j w_2$, with w_1, w_2 reduced, then $i = j$ and $w_1 \equiv_C w_2$







i+1

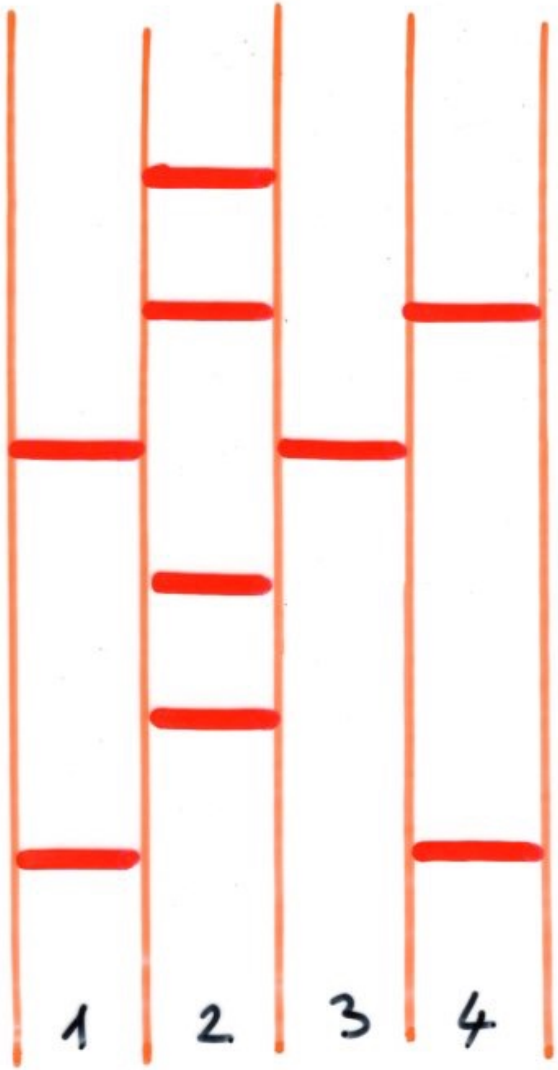
Definition H reduced heap of dimers on $\mathbb{N}$
iff no factor $H = H'FH''$ with

$F =$ 

Proposition If $\underset{\text{heap}}{H}$ $\xrightarrow{*}$ H_1 β_i, $\underset{\text{heap}}{H}$ $\xrightarrow{*}$ H_2 β_j with

H_1, H_2 reduced heaps then $H_1 = H_2$

heap H $\xrightarrow{\mathcal{D}}$ planar diagram $\mathcal{D}(H)$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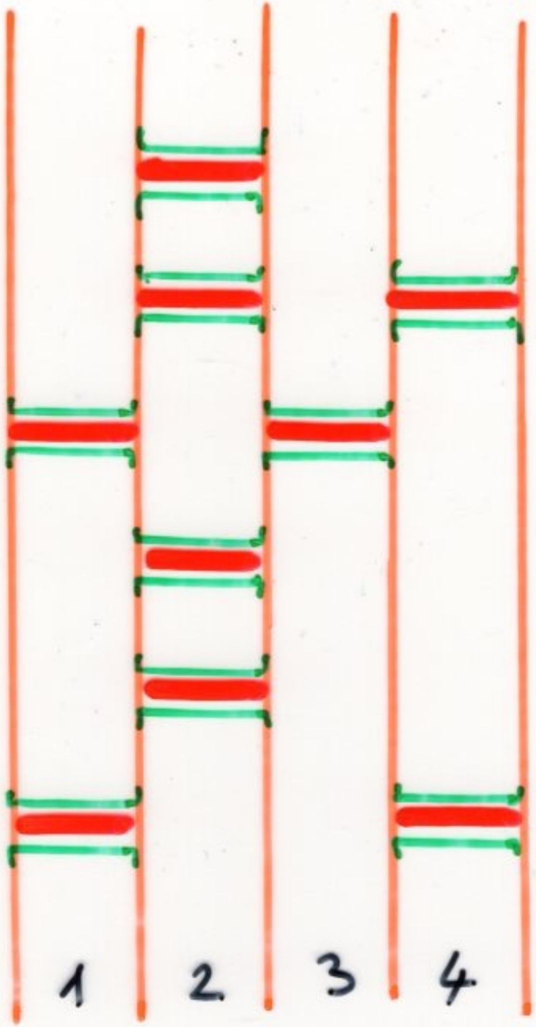
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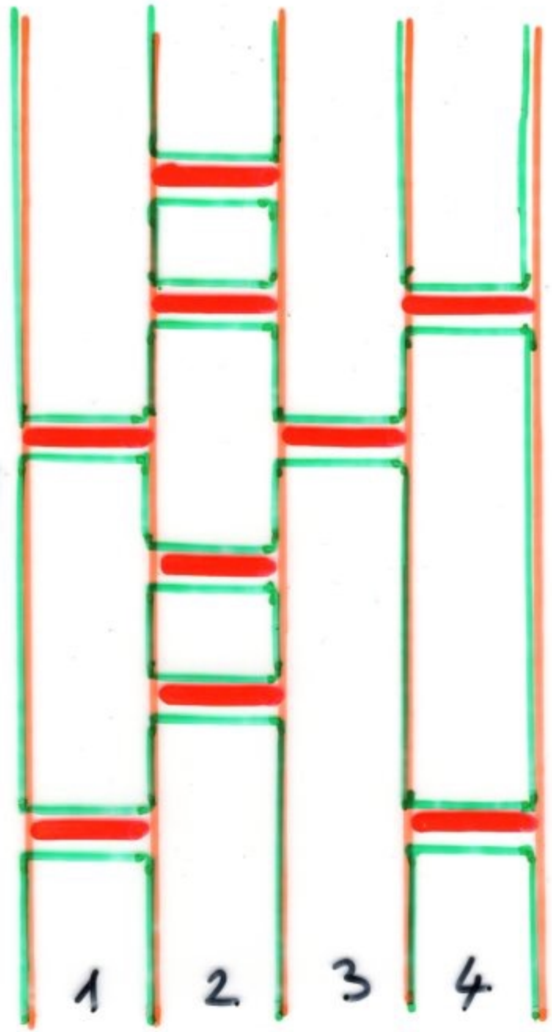
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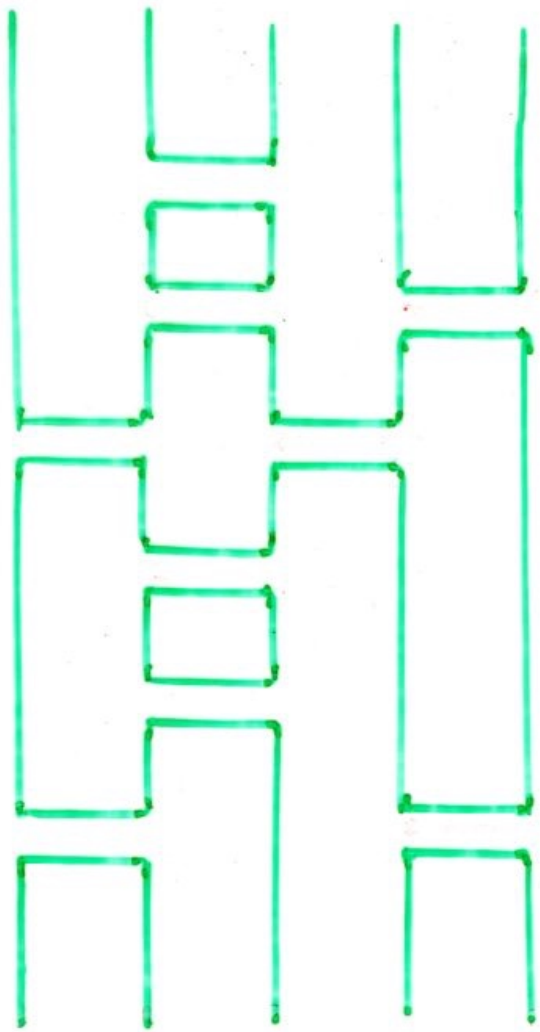
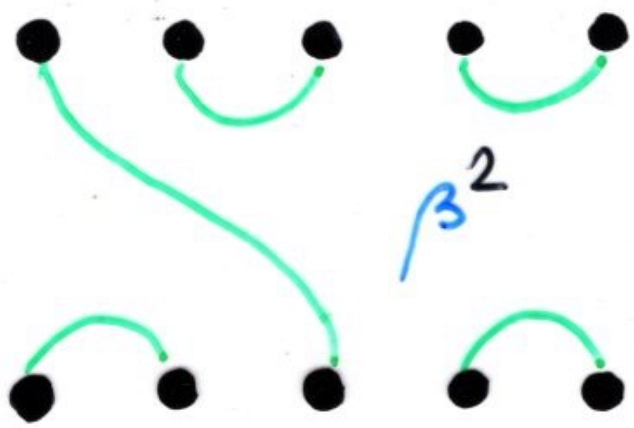
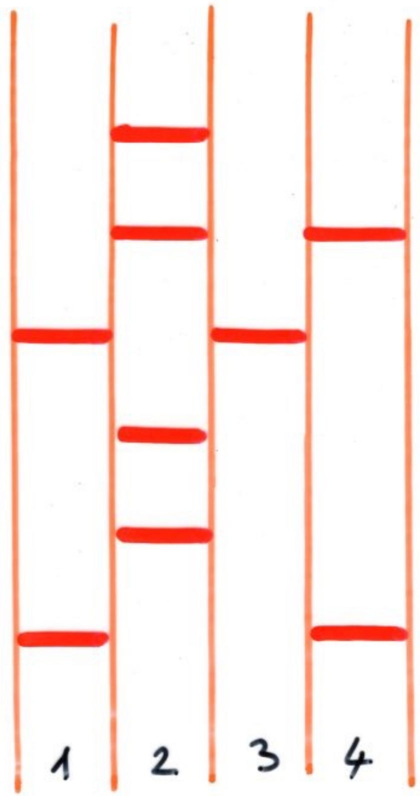
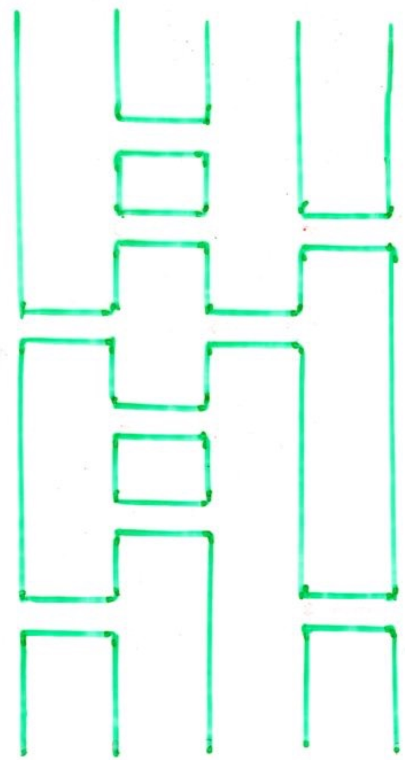
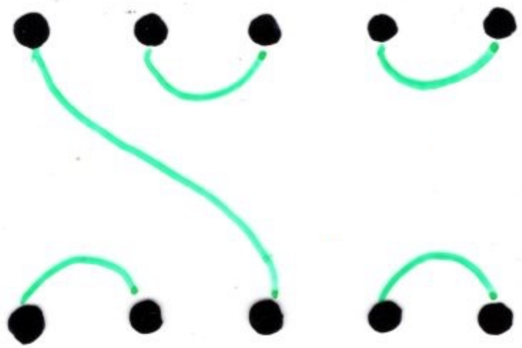


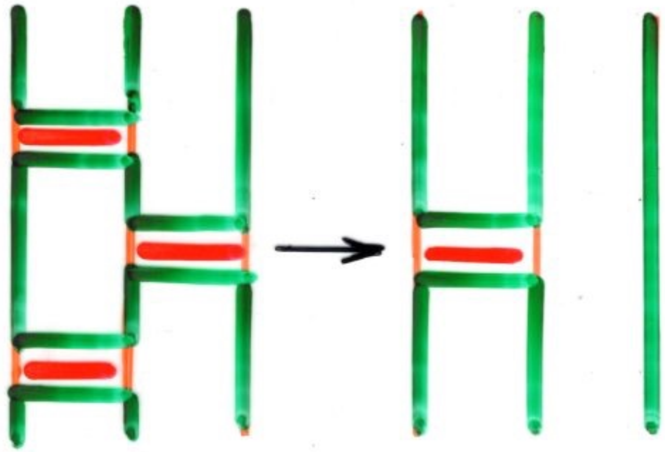

β^2

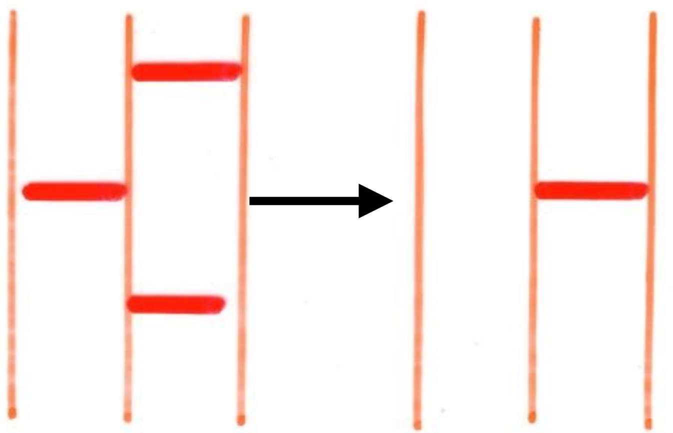
heap H $\xrightarrow{\mathcal{D}}$ planar diagram $\mathcal{D}(H)$

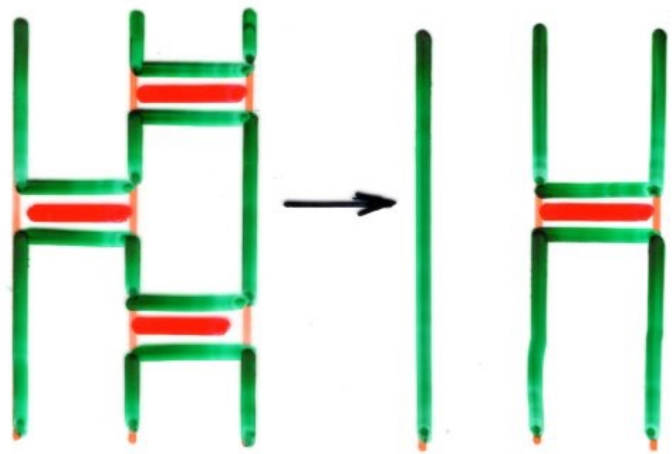


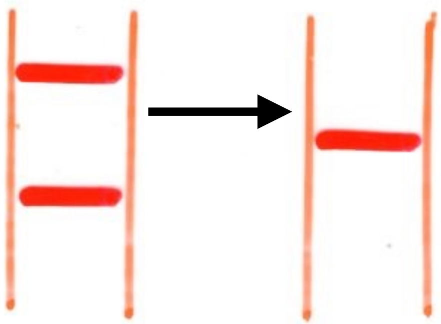


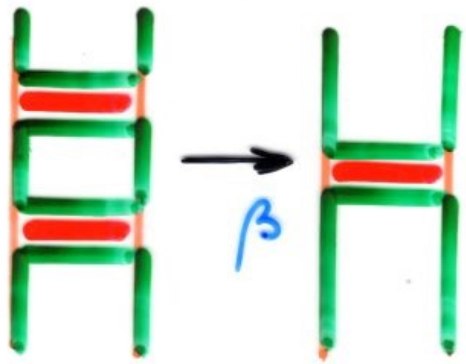











β

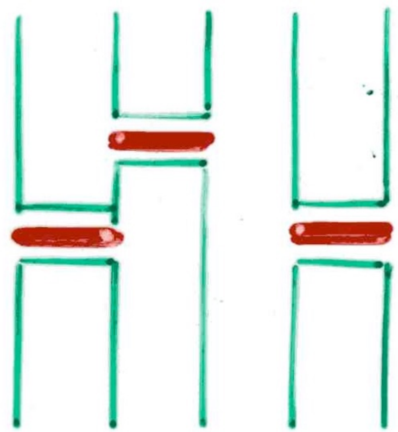
Proposition If $\underset{\text{heap}}{H}$ $\xrightarrow{*} H_1\, \beta^i$, $H \xrightarrow{*} H_2\, \beta^j$ with H_1, H_2 reduced heaps then $H_1 = H_2$

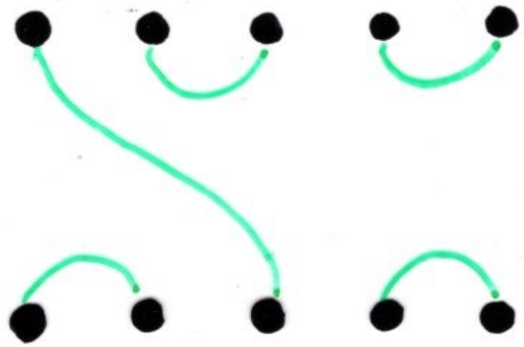
Proposition

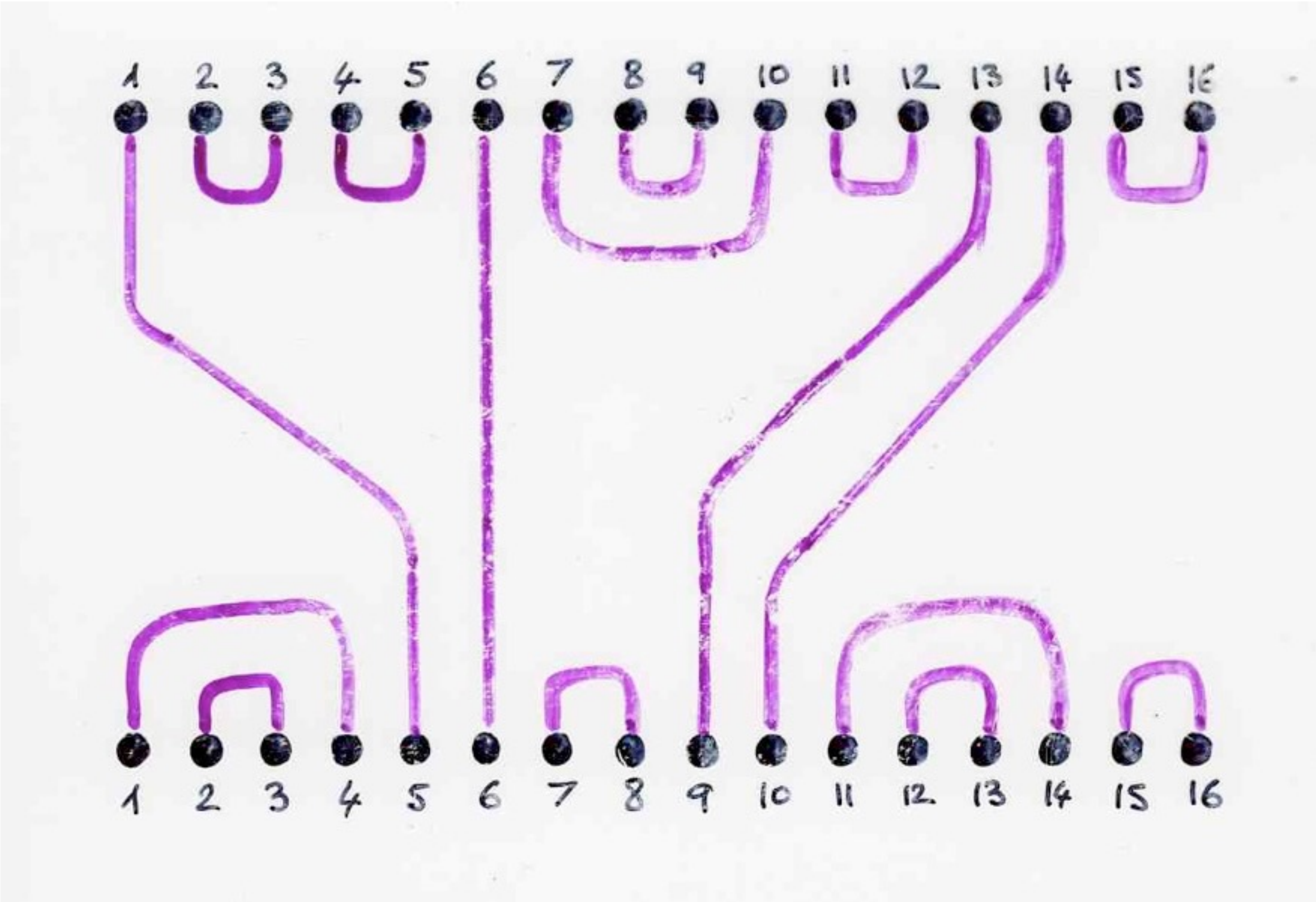
The restriction of the map D to reduced heaps is a bijection

reduced heap $\overset{D}{\longleftrightarrow}$ planar diagrams (no loops)

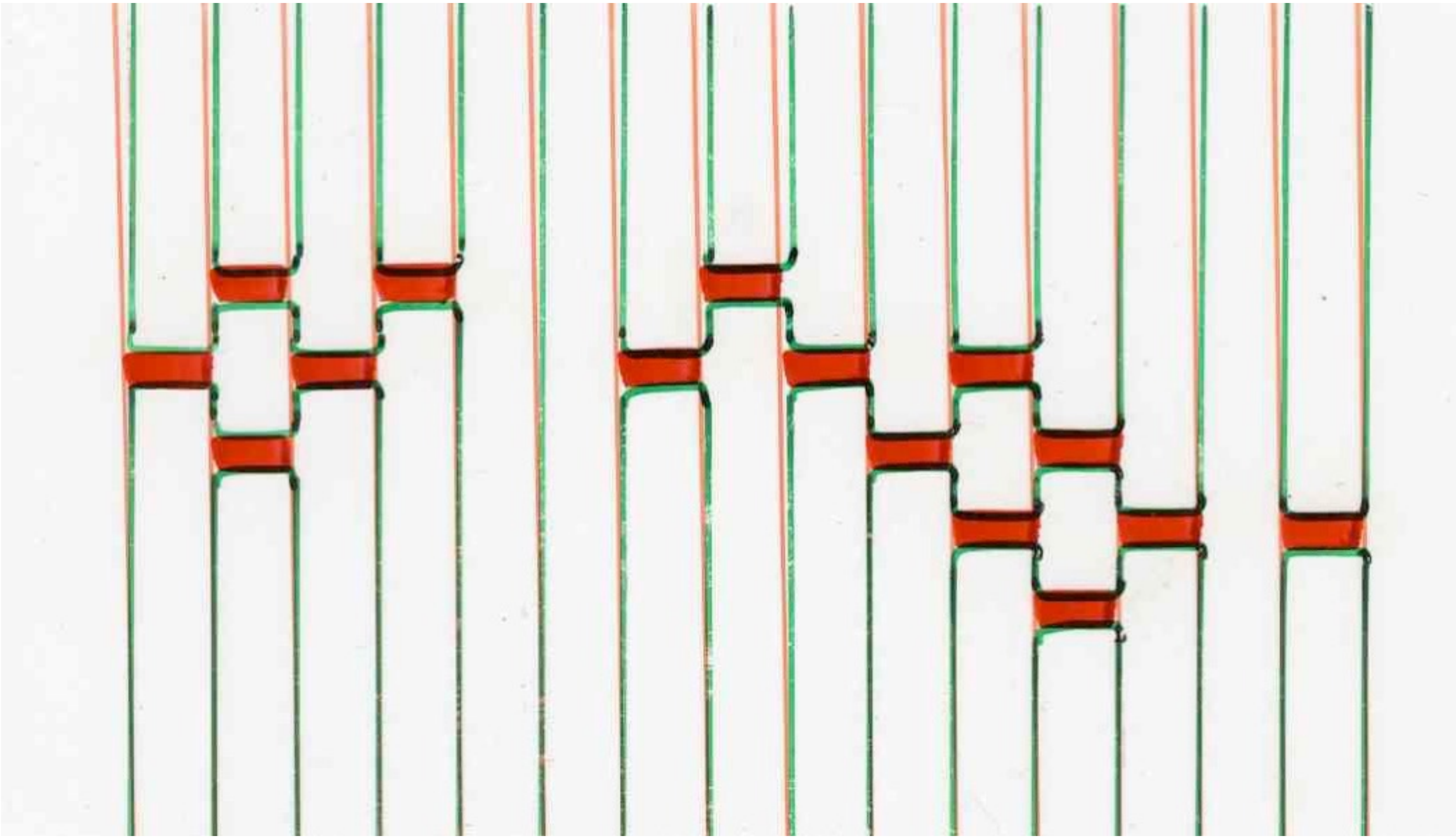
$e_1 e_4 e_2$







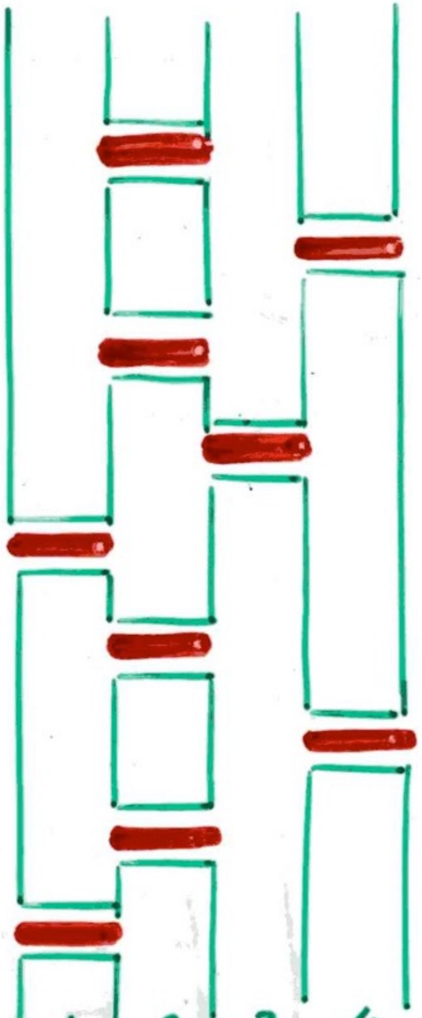
exercise: give a proof of the last proposition



Proposition If $w \overset{*}{\underset{*}{\rightrightarrows}} \begin{matrix} \beta^i w_1 \\ \beta^j w_2 \end{matrix}$, with w_1, w_2 reduced, then $i = j$ and $w_1 \equiv_C w_2$

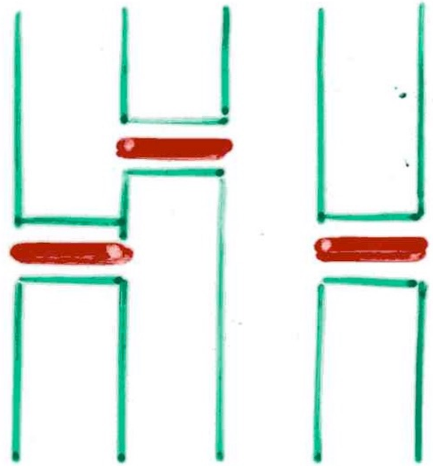
C commutations $e_i e_j = e_j e_i$ with $|i-j| \geq 2$

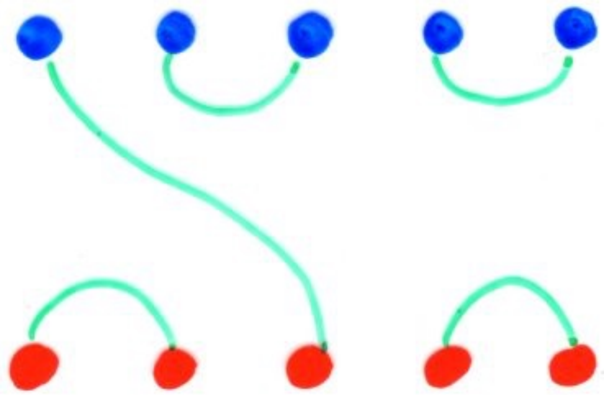
$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_4 e_2$

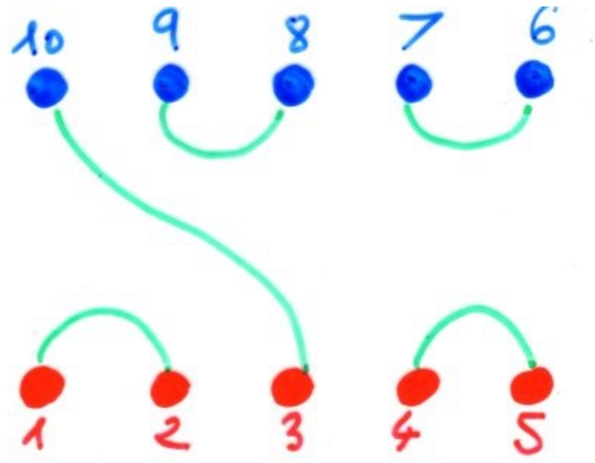


$\longrightarrow$

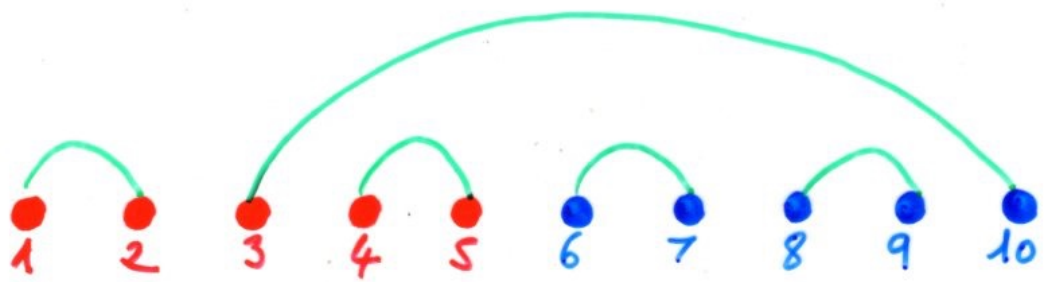
$e_1 e_4 e_2$

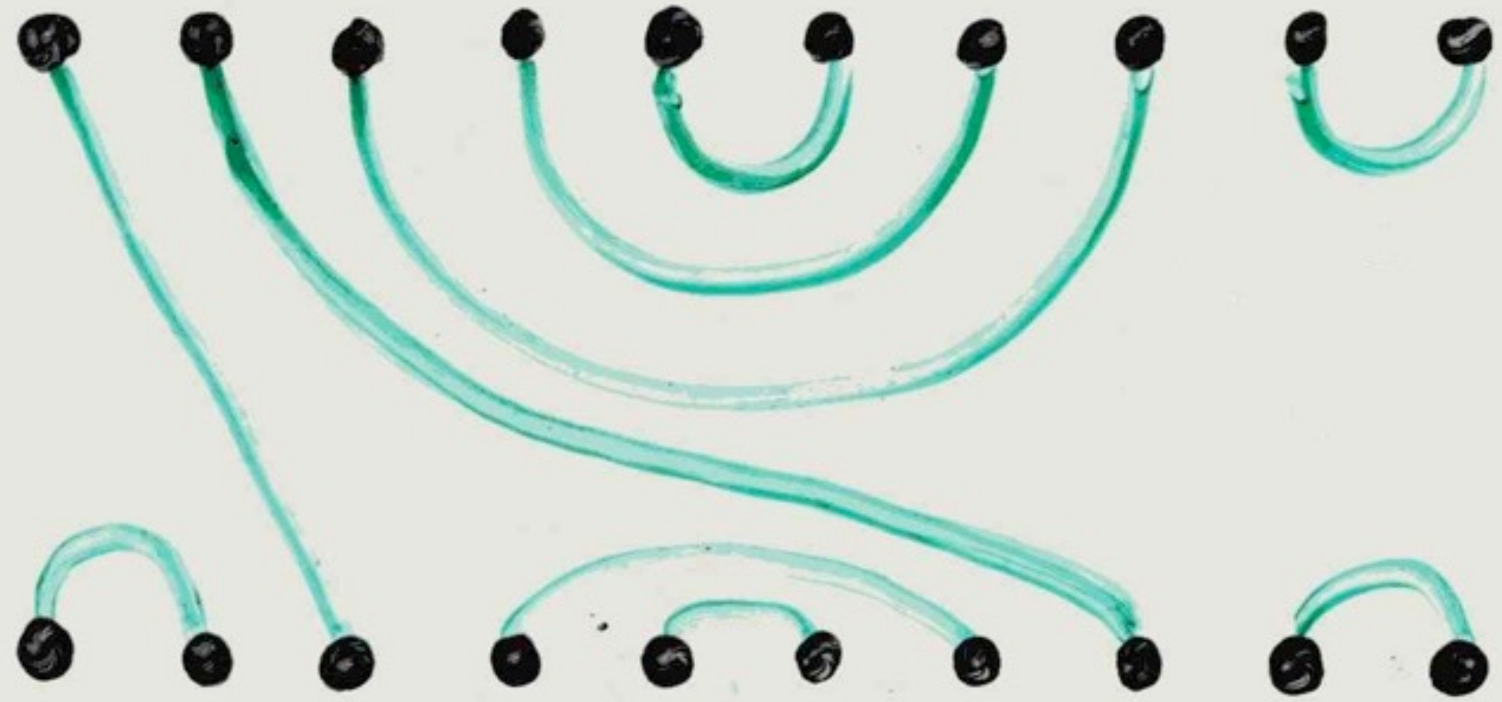


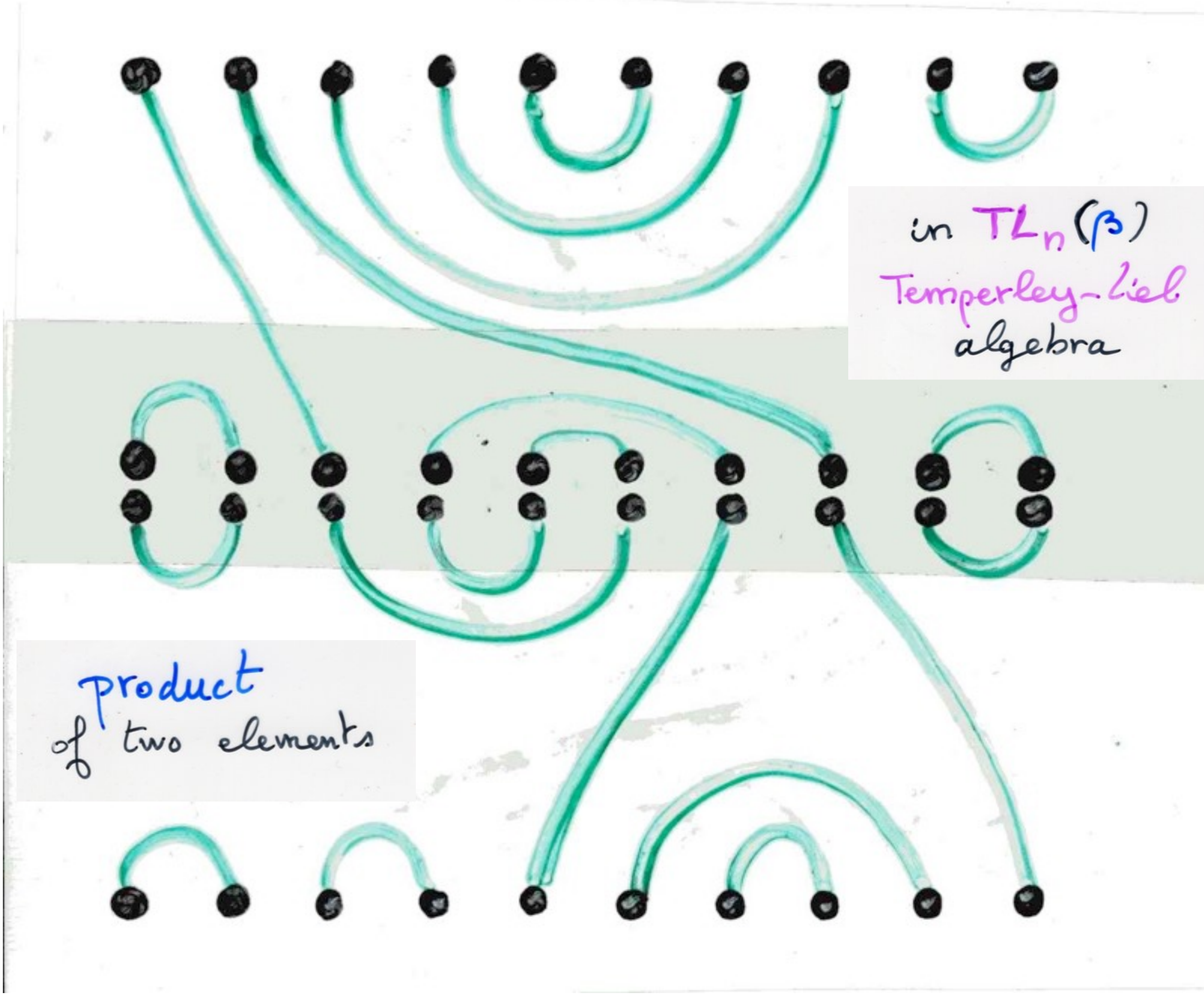




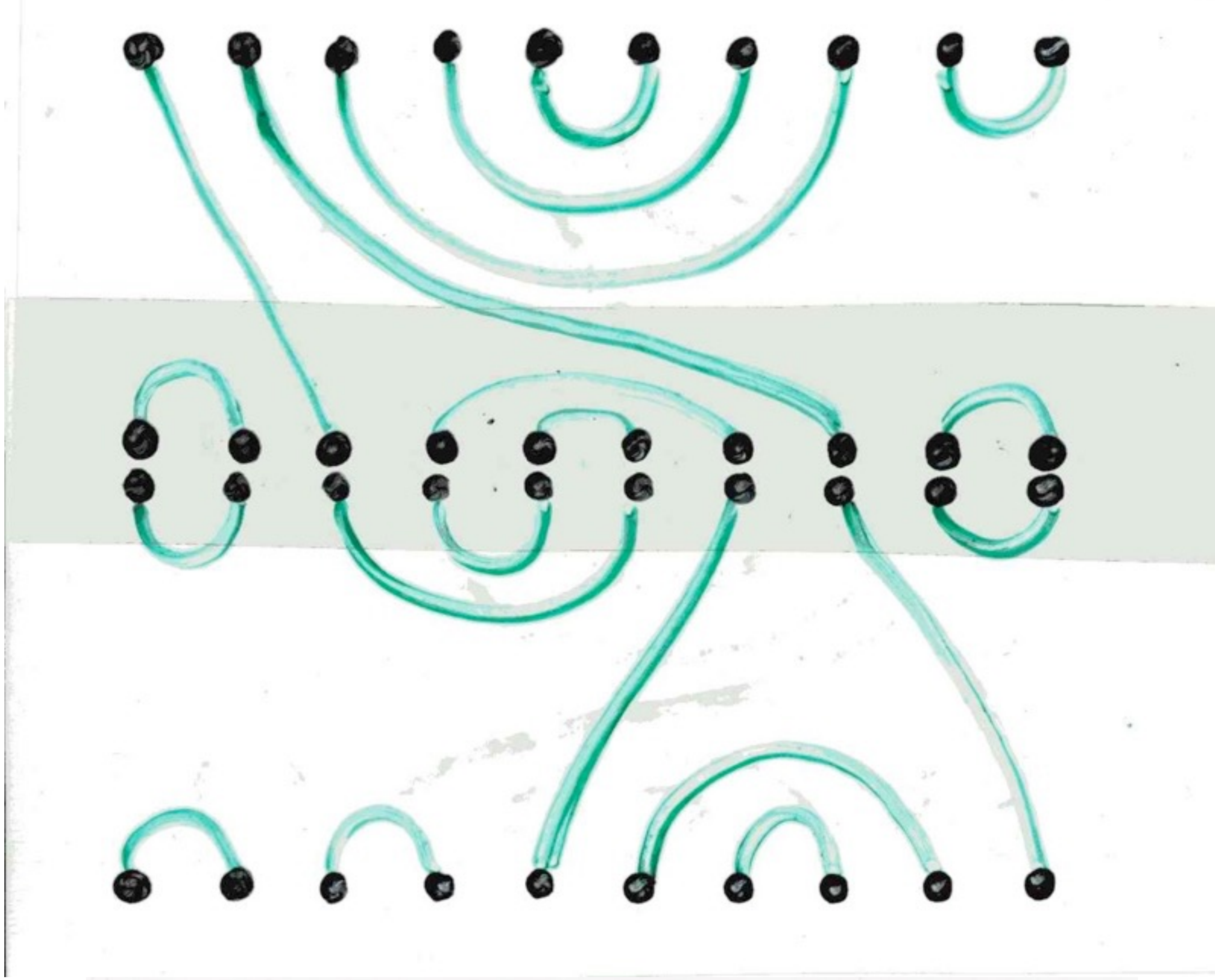
enumerated by Catalan numbers





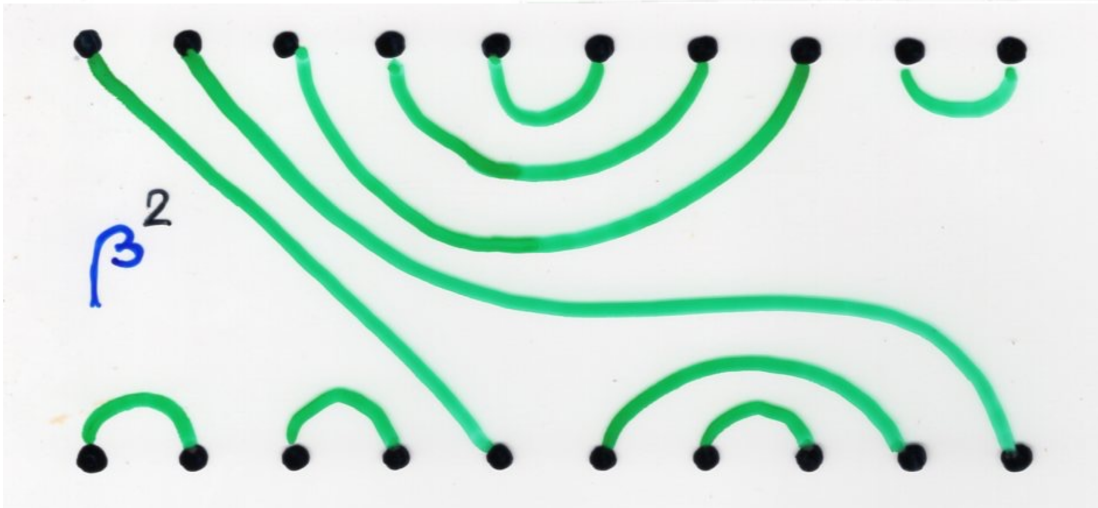

in $TL_n(\beta)$
Temperley-Lieb
algebra
product
of two elements

product of two elements



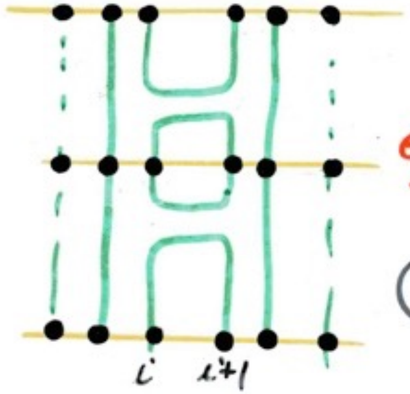
in $TL_n(\beta)$ Temperley-Lieb algebra

=



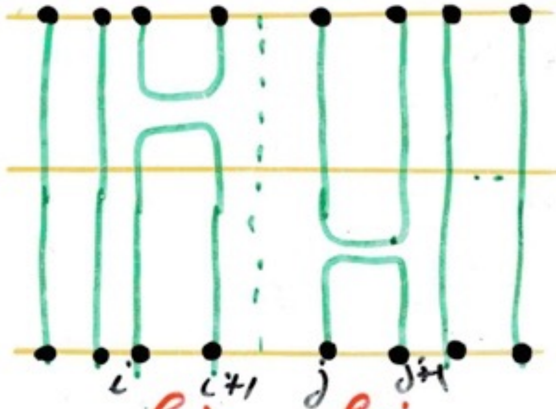
Kauffman generators

$= e_i$



$e_i^2 = \beta e_i$

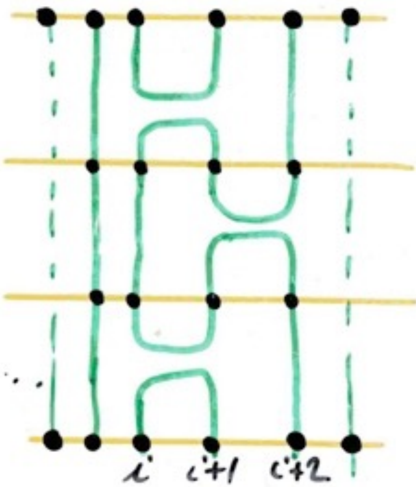
(2)



$e_i \quad e_j$

(1)

$|i-j| \geq 2$



(3)

$e_i e_{i+1} e_i = e_i$

Basis of

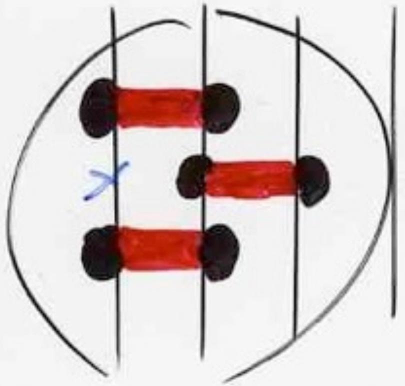
Temperley-Lieb algebra

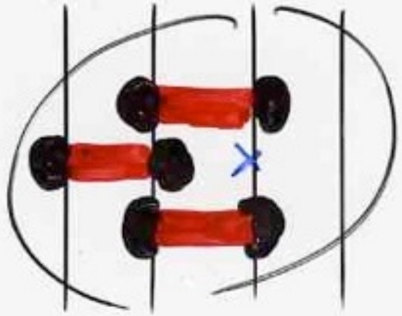
basis of (N)TL_n

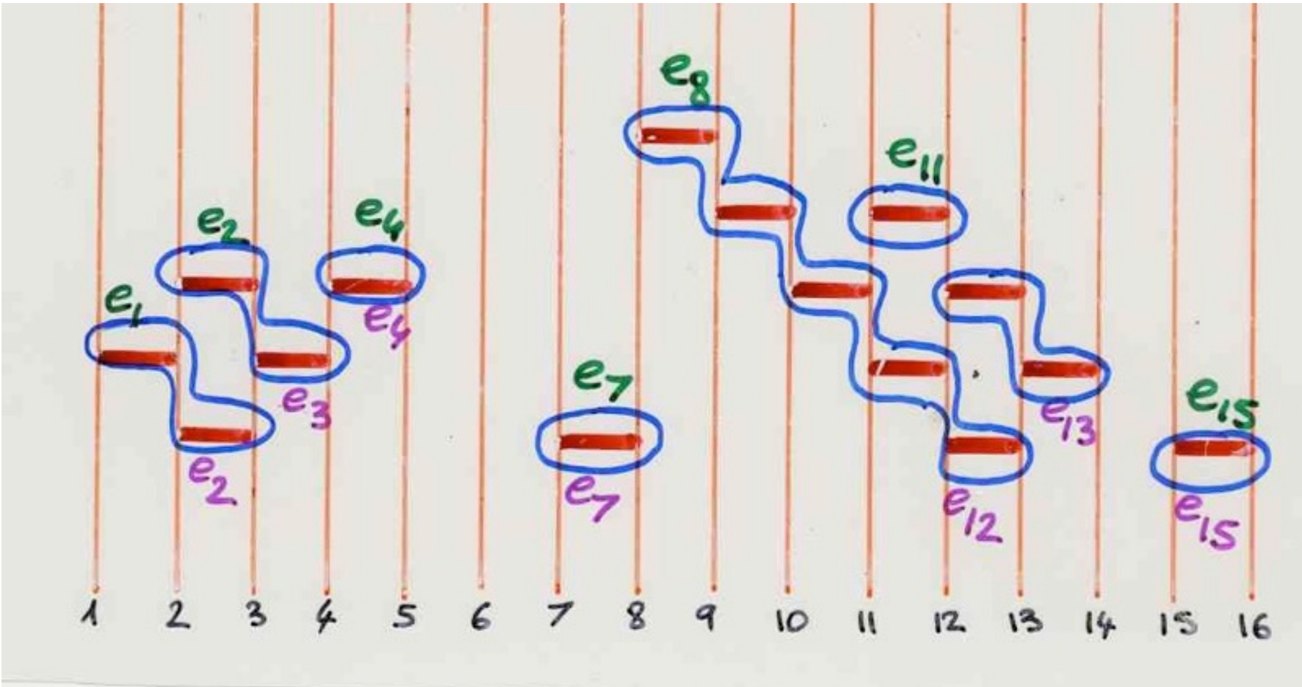
no occurrence of

u_i^2  $\longrightarrow$ strict heap

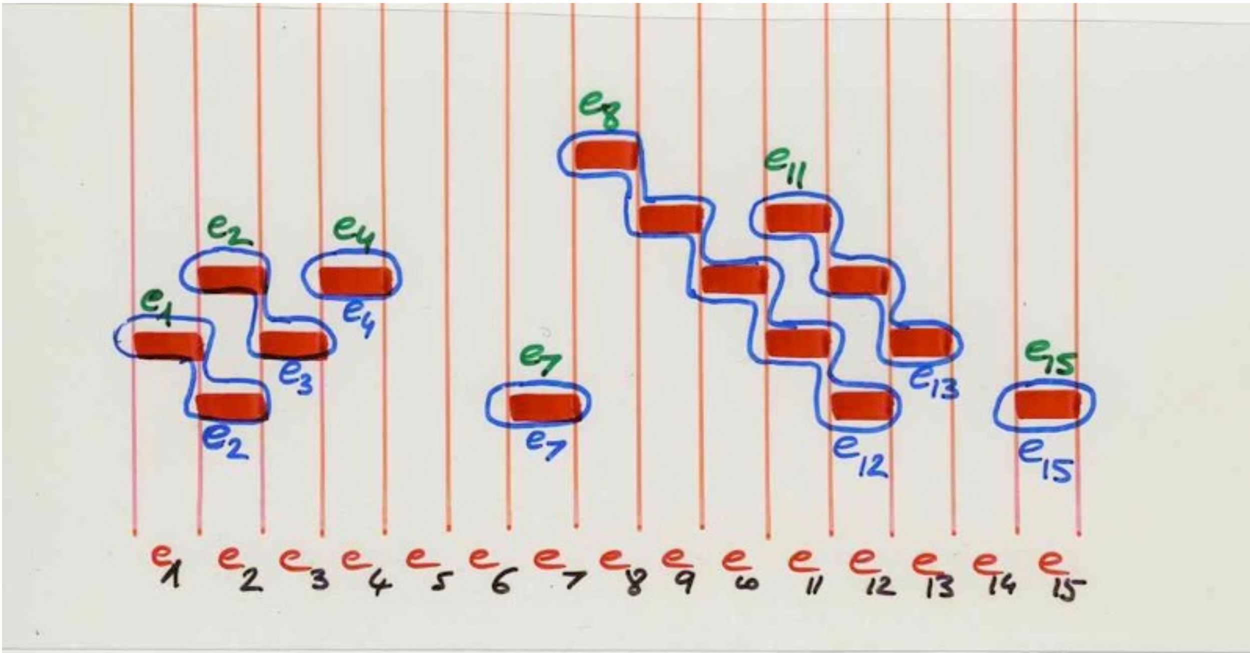
$u_i u_{i+1} u_i$ $\qquad$ $u_{i+1} u_i u_{i+1}$





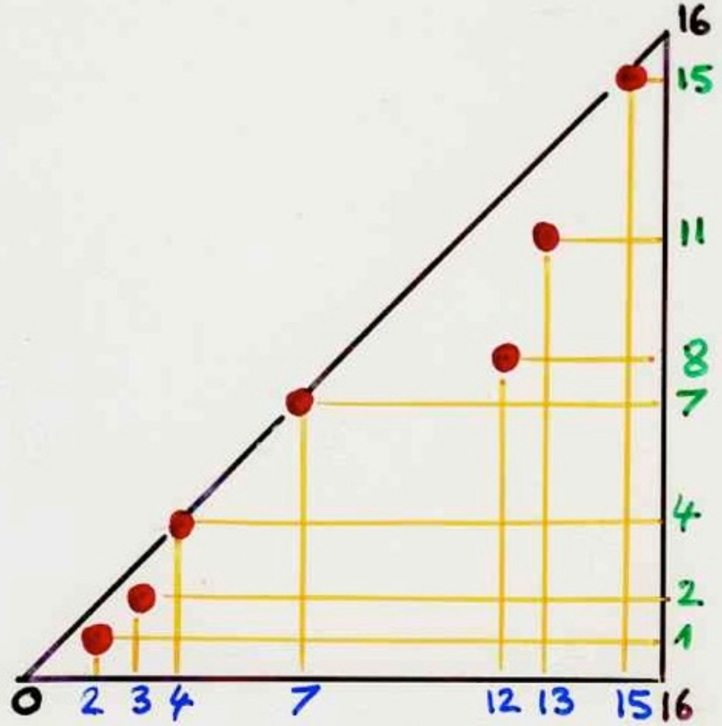


$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$



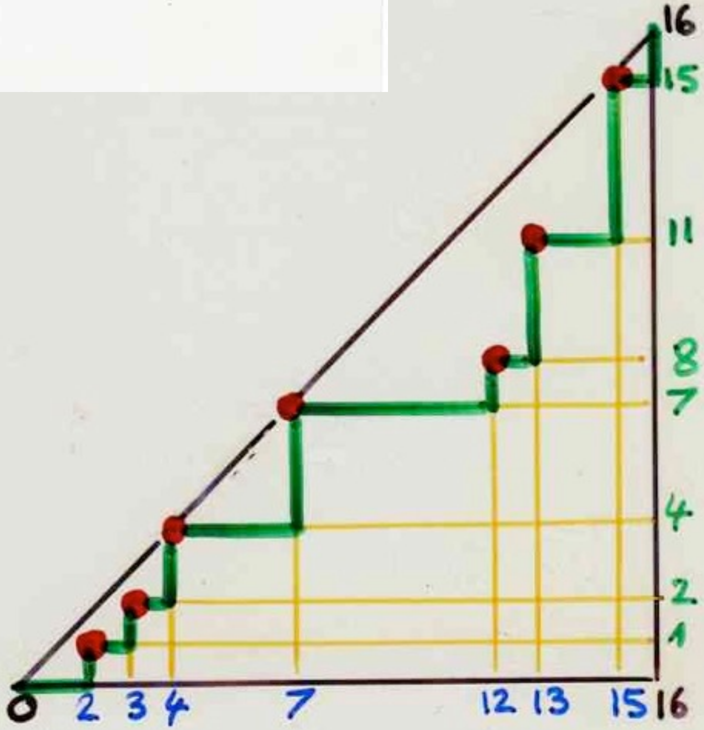
$$(e_2\, e_1)(e_3\, e_2)(e_4)(e_7)(e_{12}\, e_{11}\, e_{10}\, e_9\, e_8) \times$$

$$\times\, (e_{13}\, e_{12}\, e_{11})\,(e_{15})$$

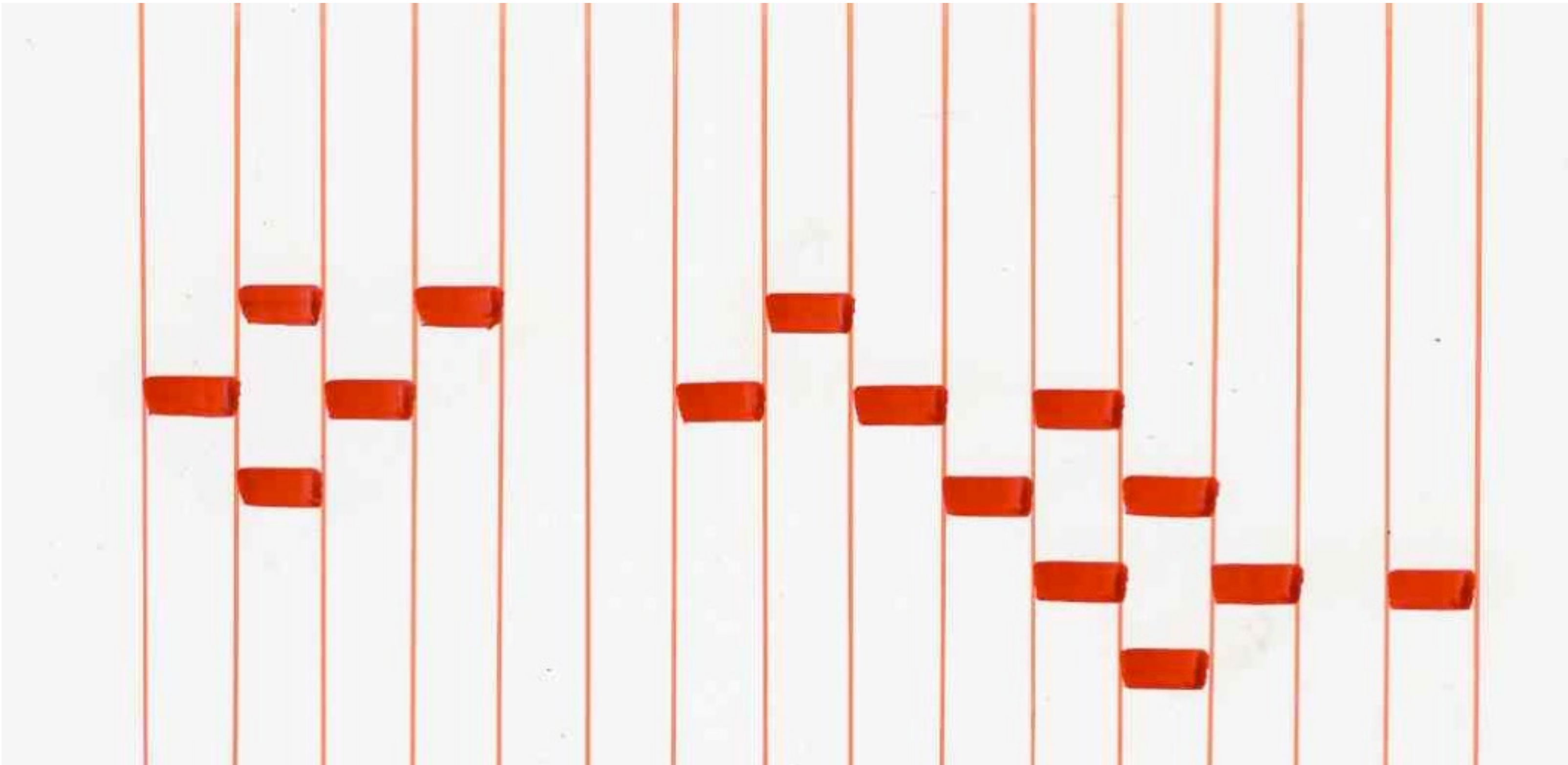


$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$

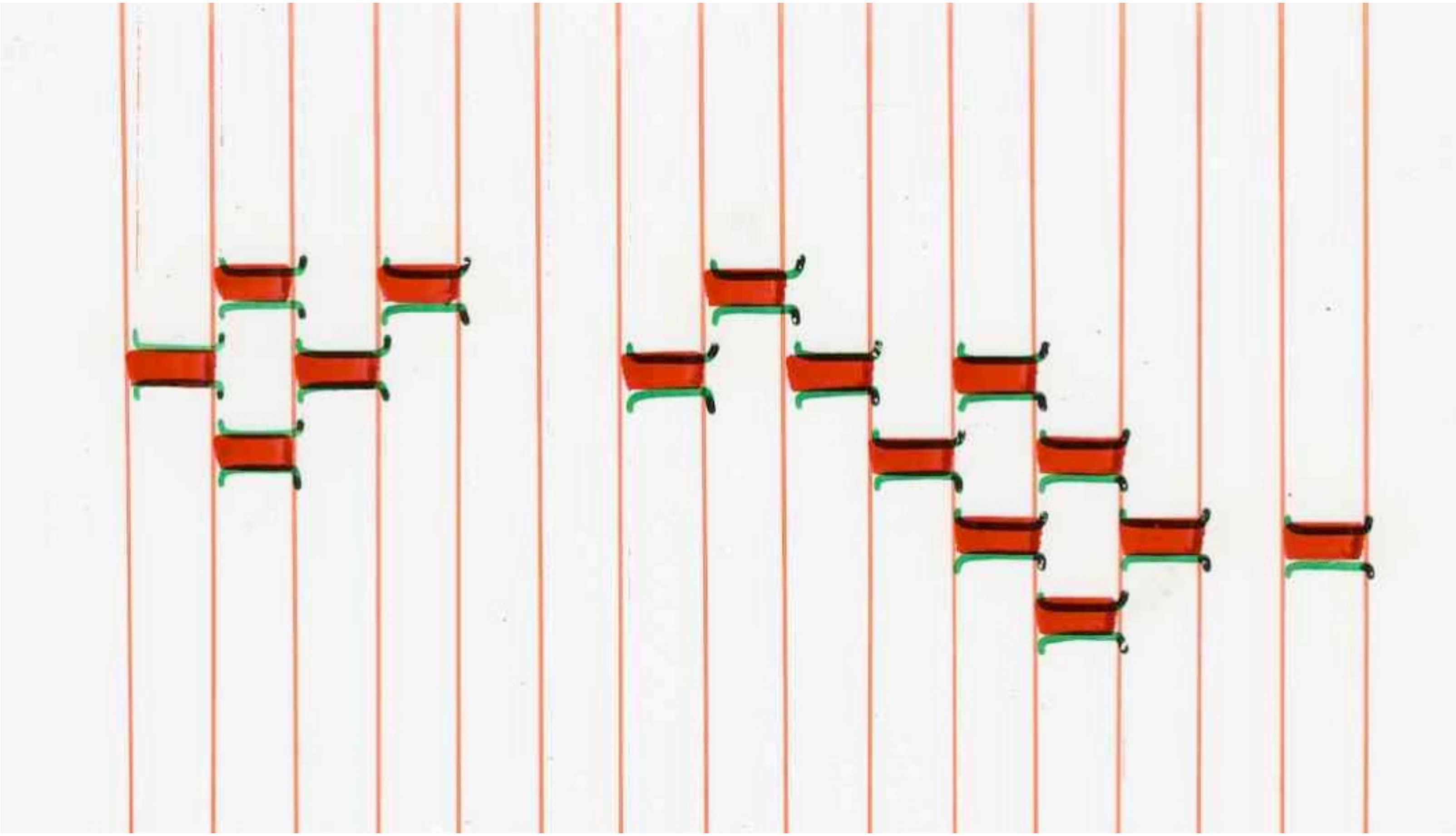
$$(e_2\, e_1)(e_3\, e_2)(e_4)(e_7)(e_{12}\, e_{11}\, e_{10}\, e_9\, e_8)\times$$
$$\times(e_{13}\, e_{12}\, e_{11})(e_{15})$$

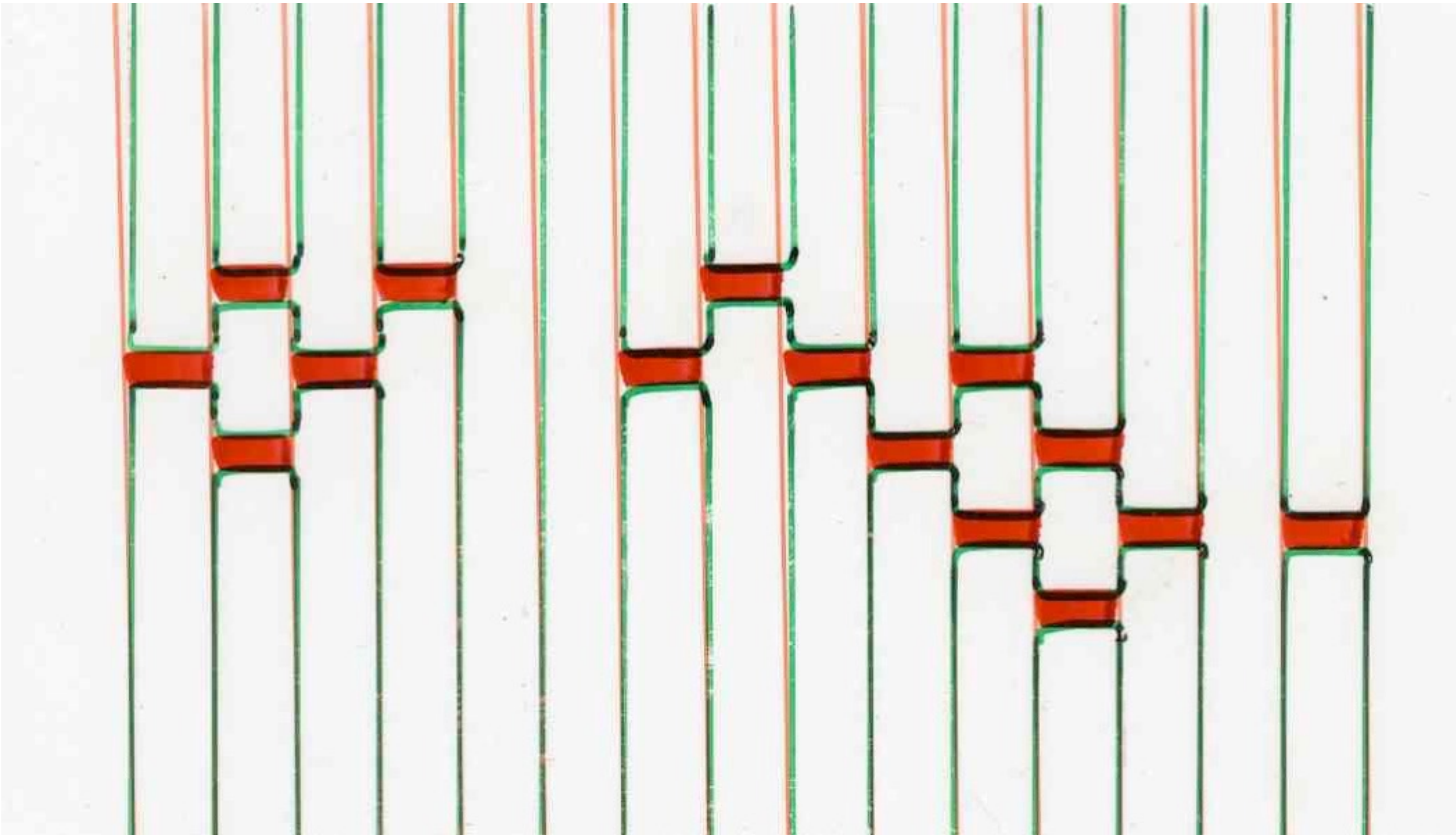


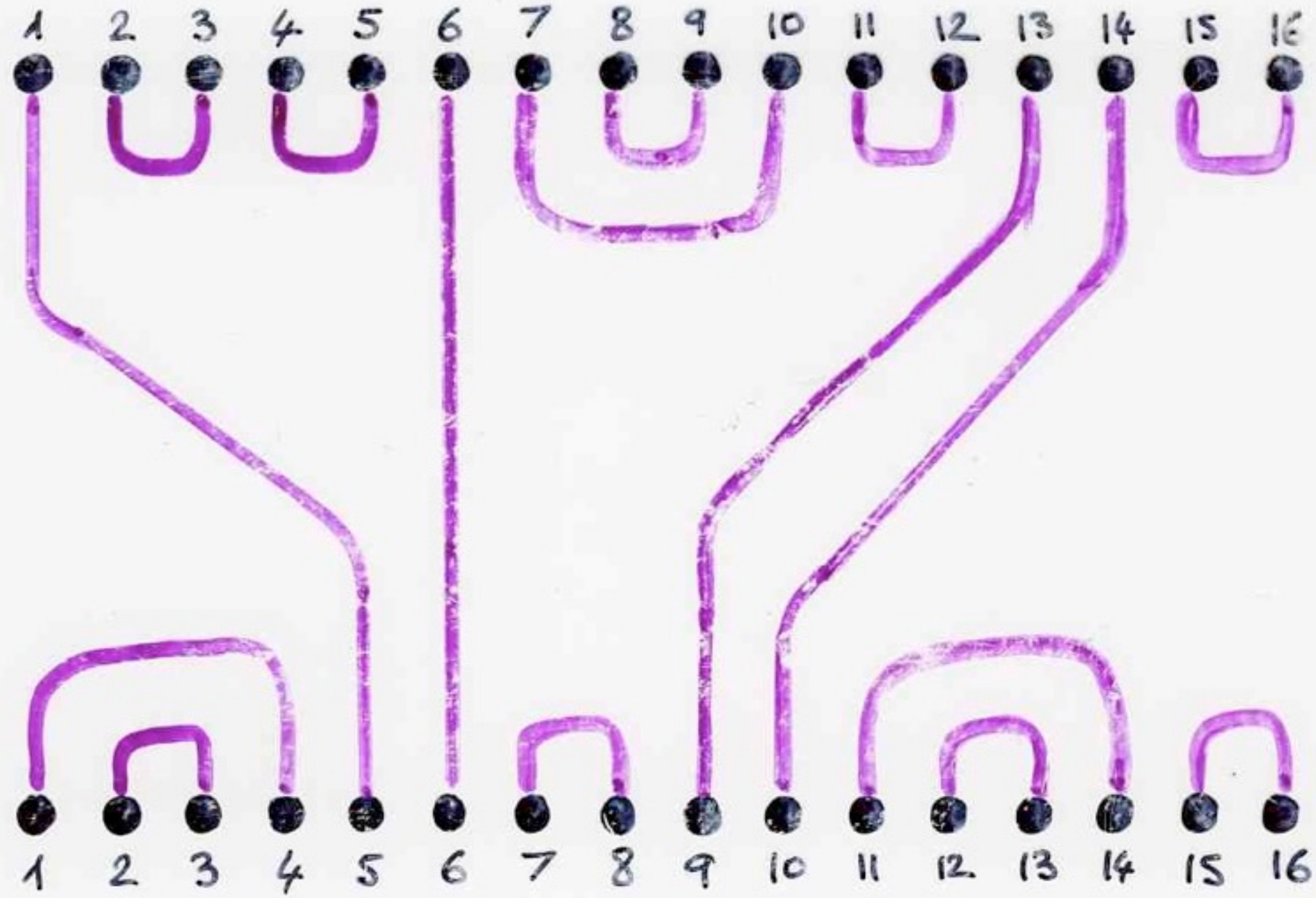
$$\begin{array}{ccccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \leq & & \leq & & \leq & & \leq & & \leq & & \leq & & \leq & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{array}$$

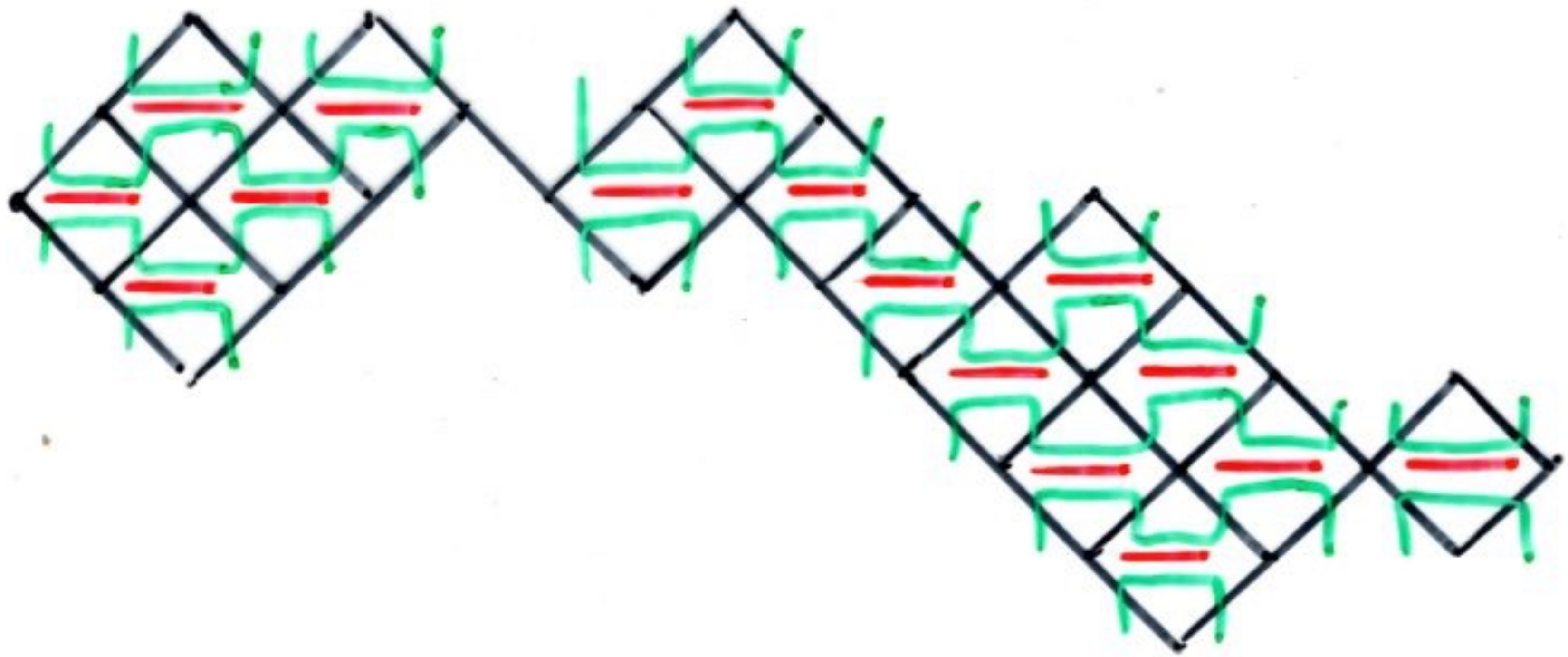


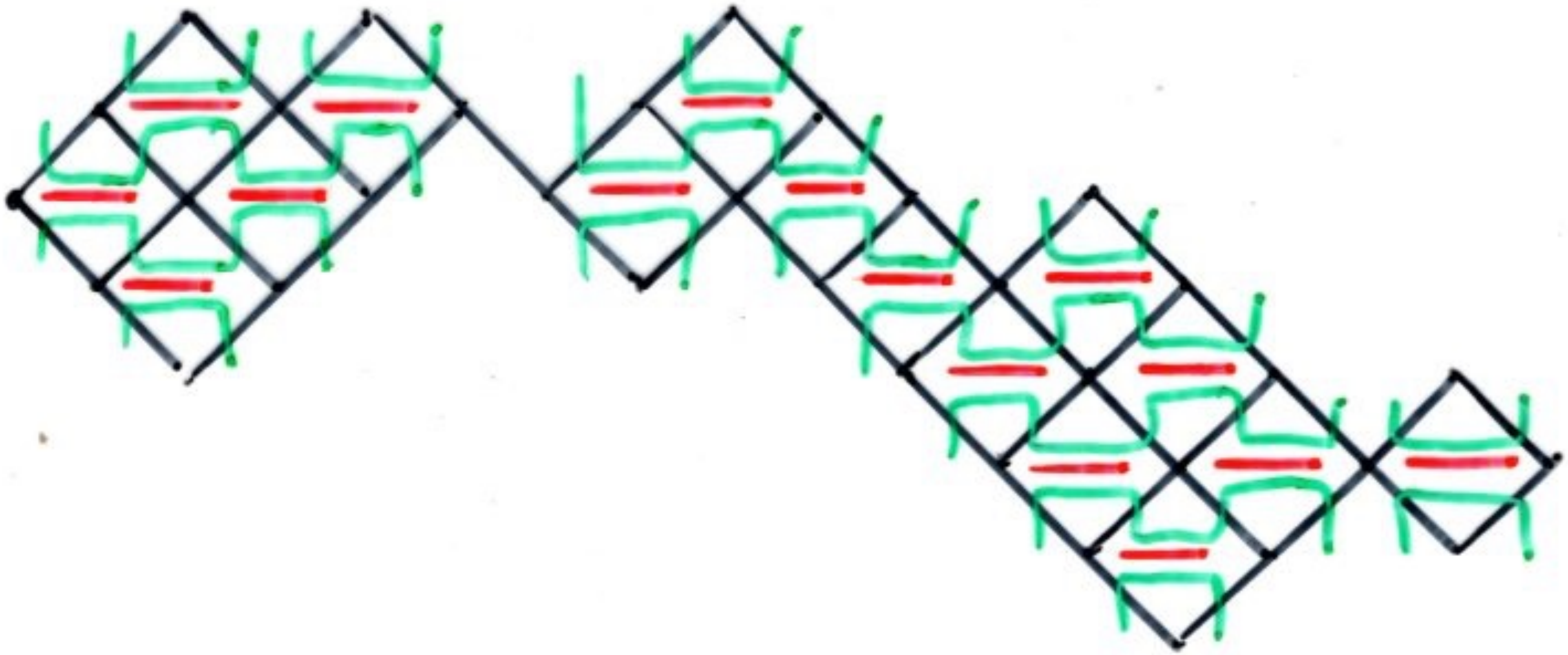
heap of dimers on $[0, n-1]$ $\longrightarrow$ element of TL_n Temperley-Lieb algebra

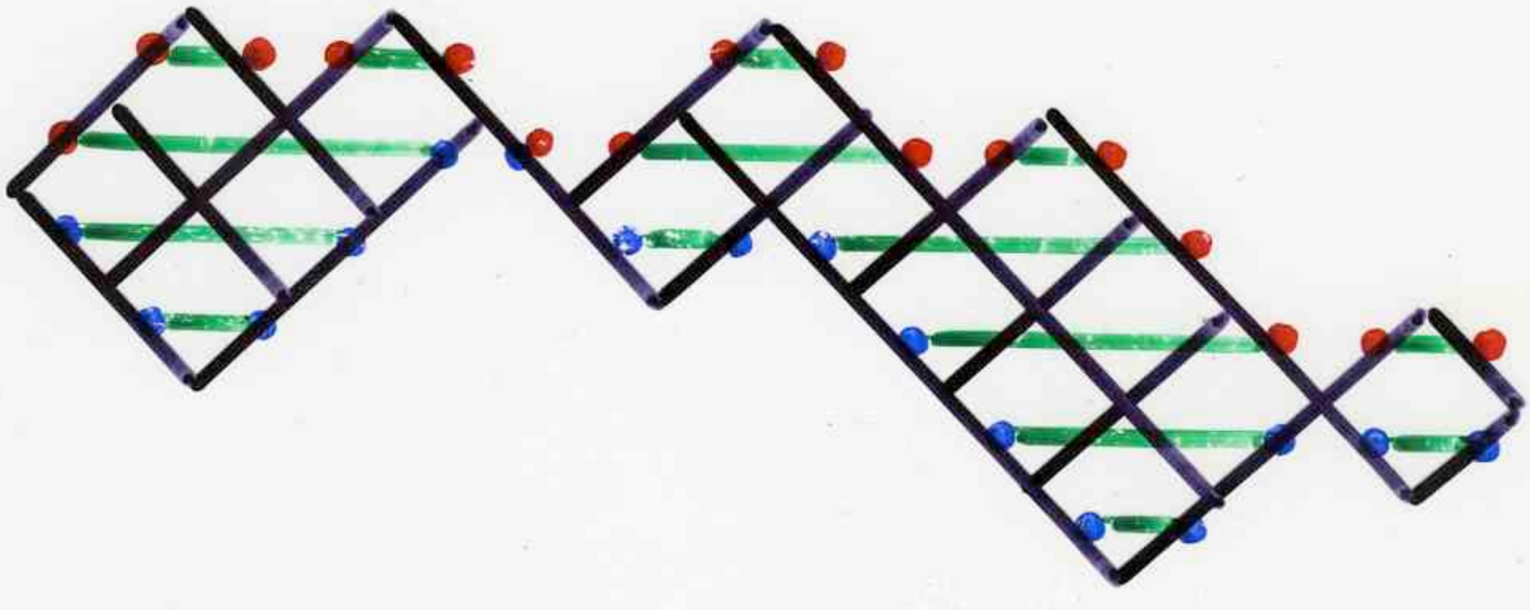





1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16



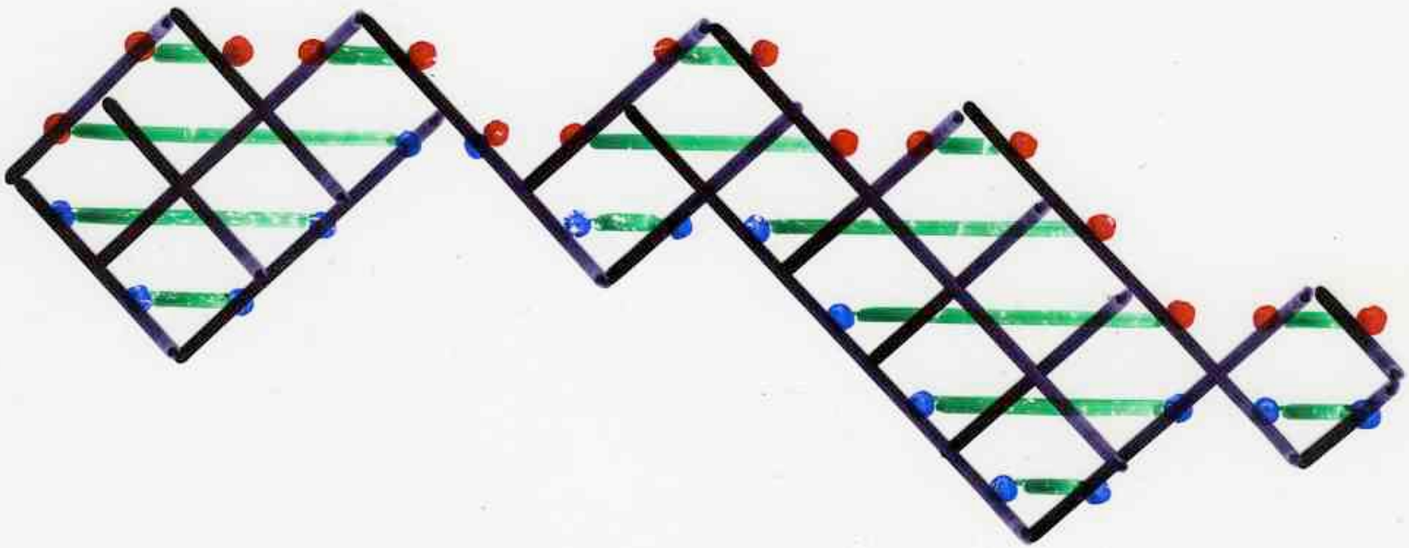


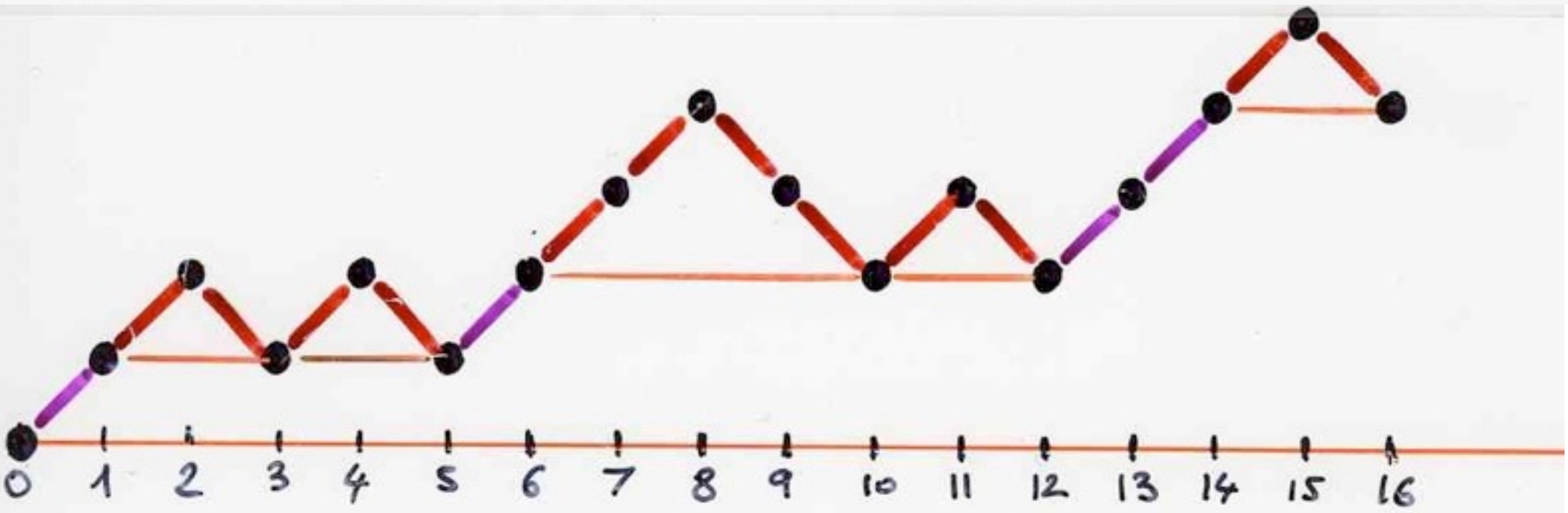


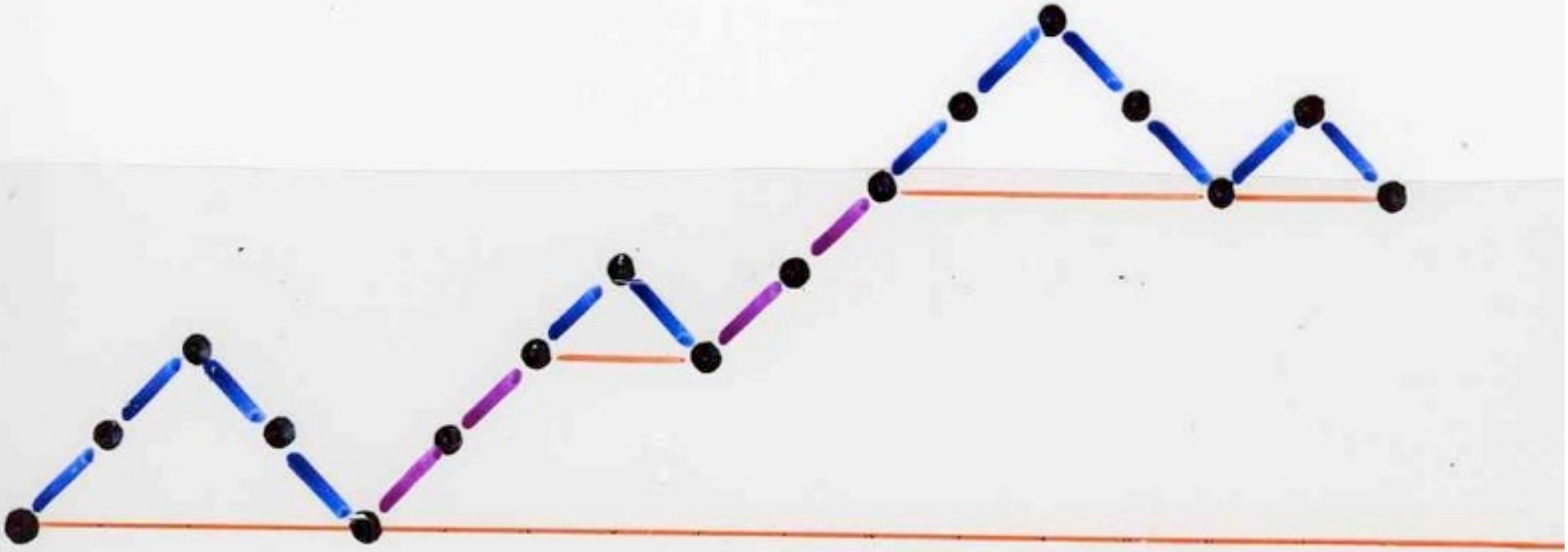
exercise:

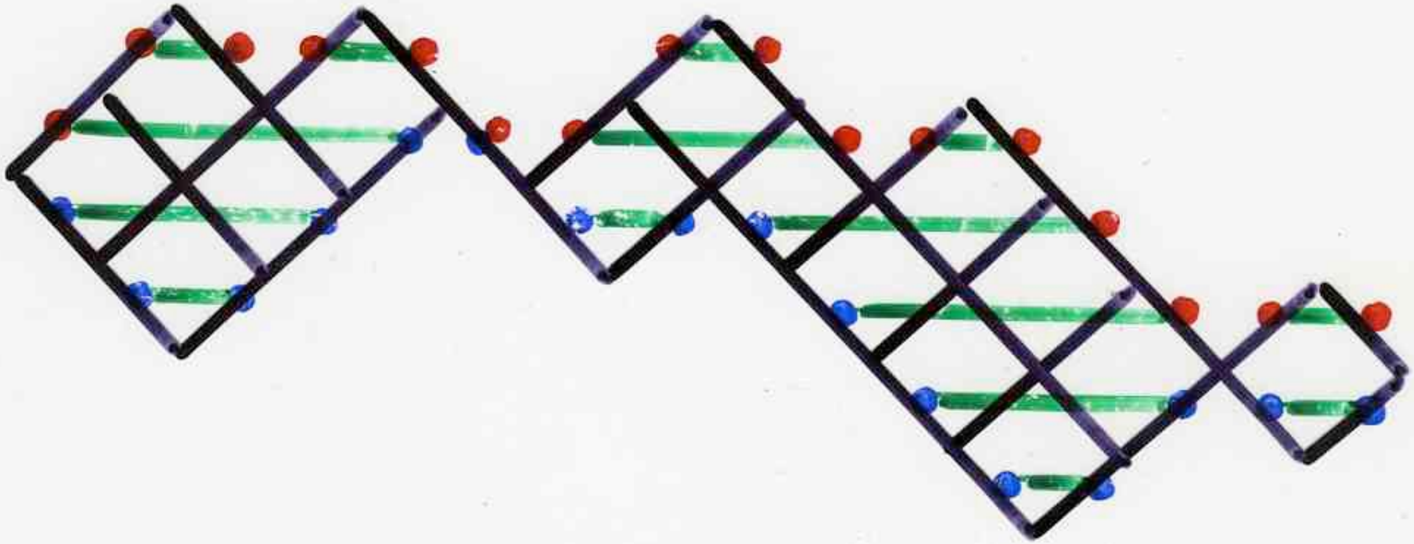
RSK

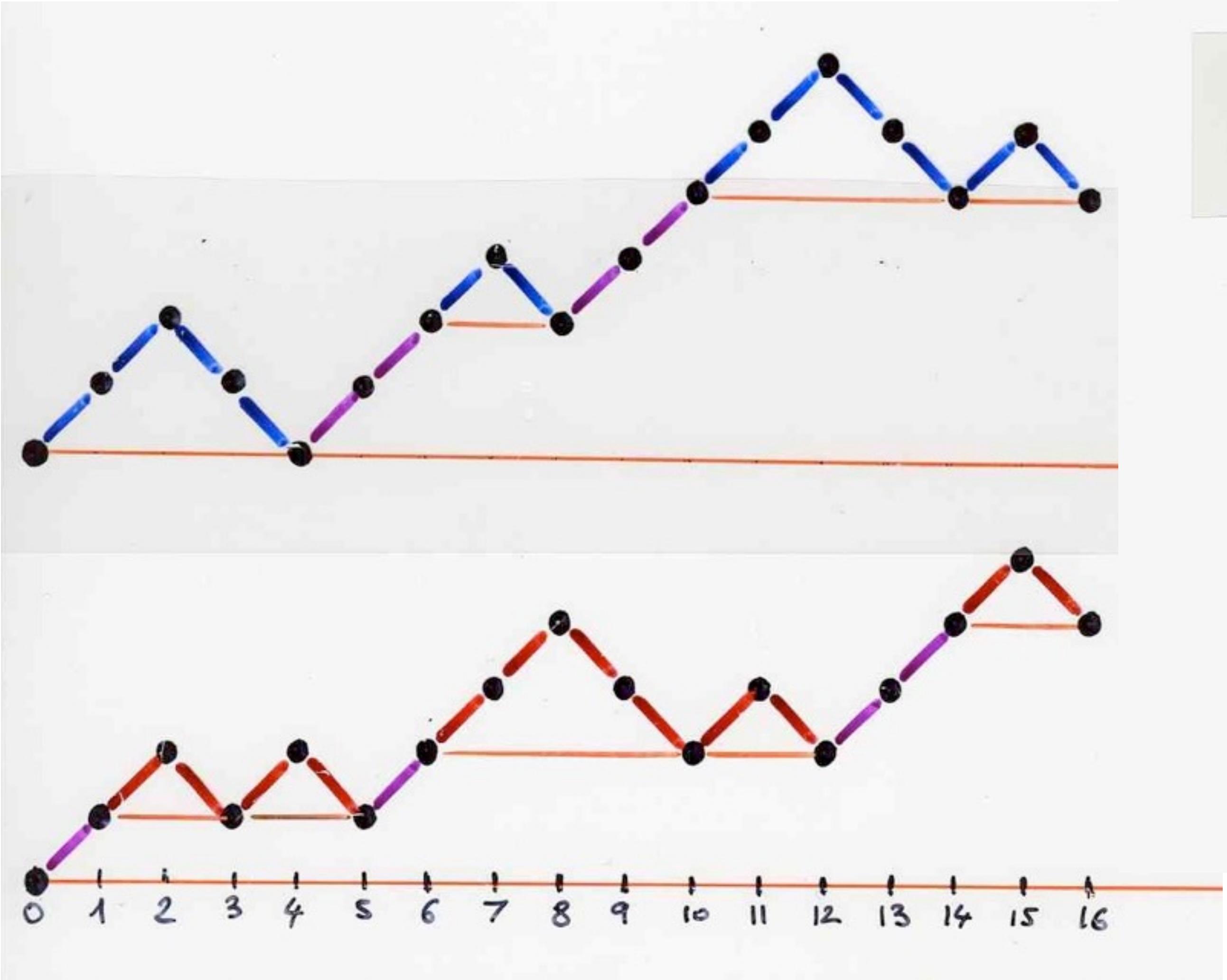
and fully commutative heaps




0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16



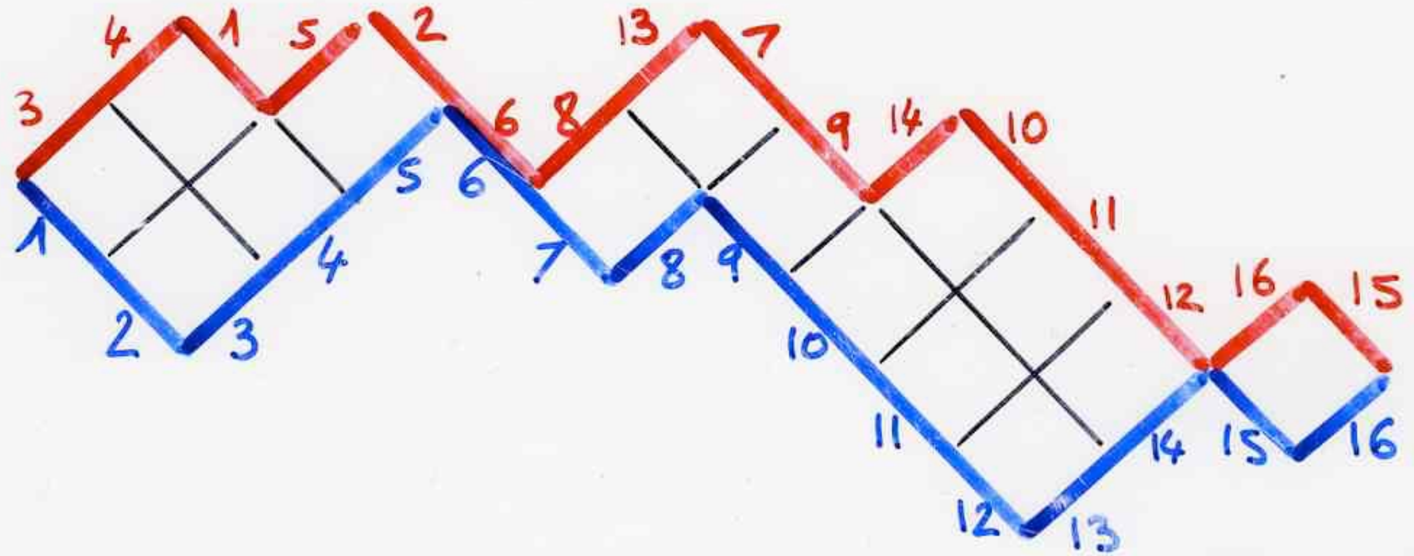




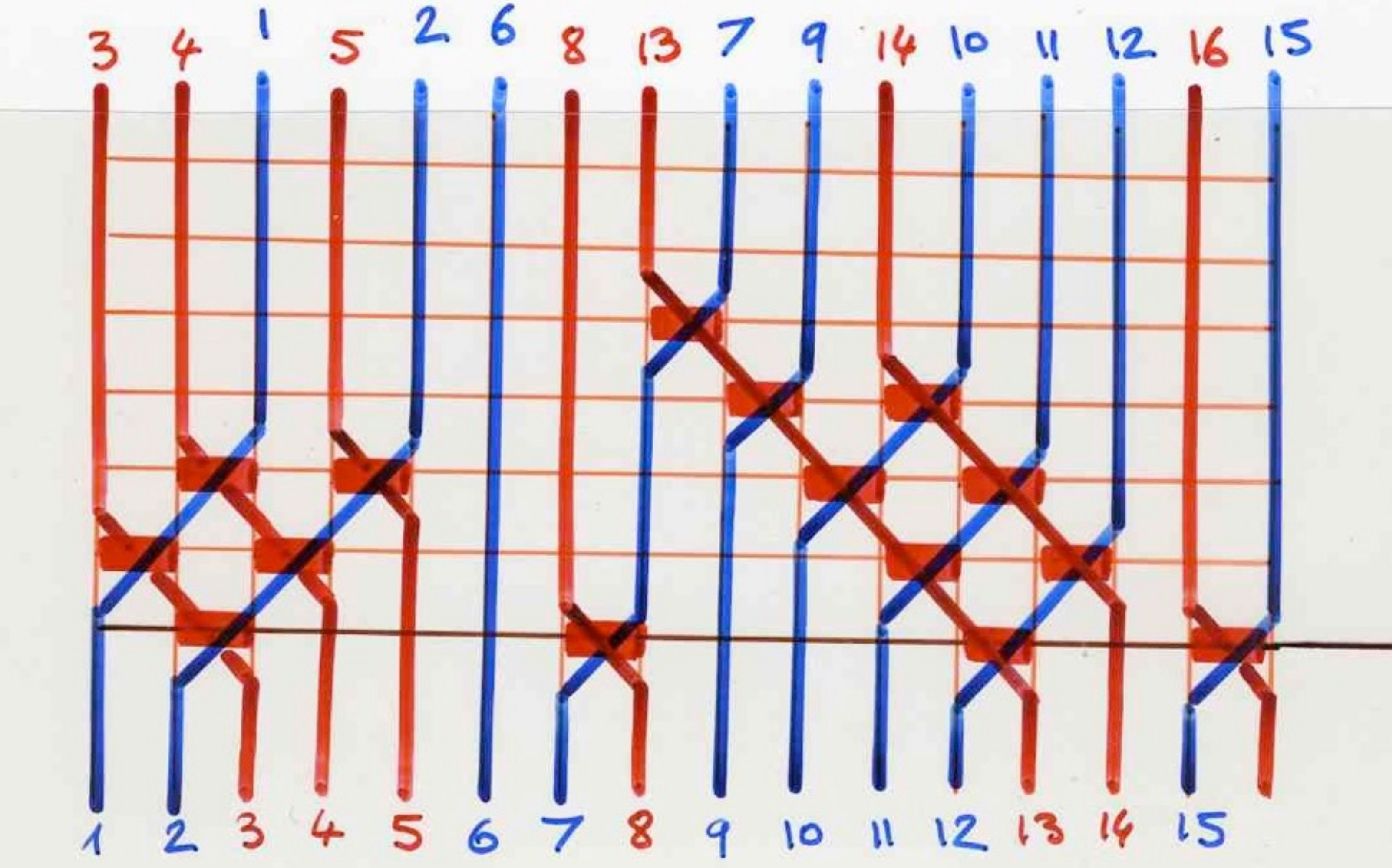
Robinson - Schensted

3	4	8	13	14	16				
1	2	5	6	7	9	10	11	12	15

3	5	9	10	12	16				
1	2	4	6	7	8	11	13	14	15



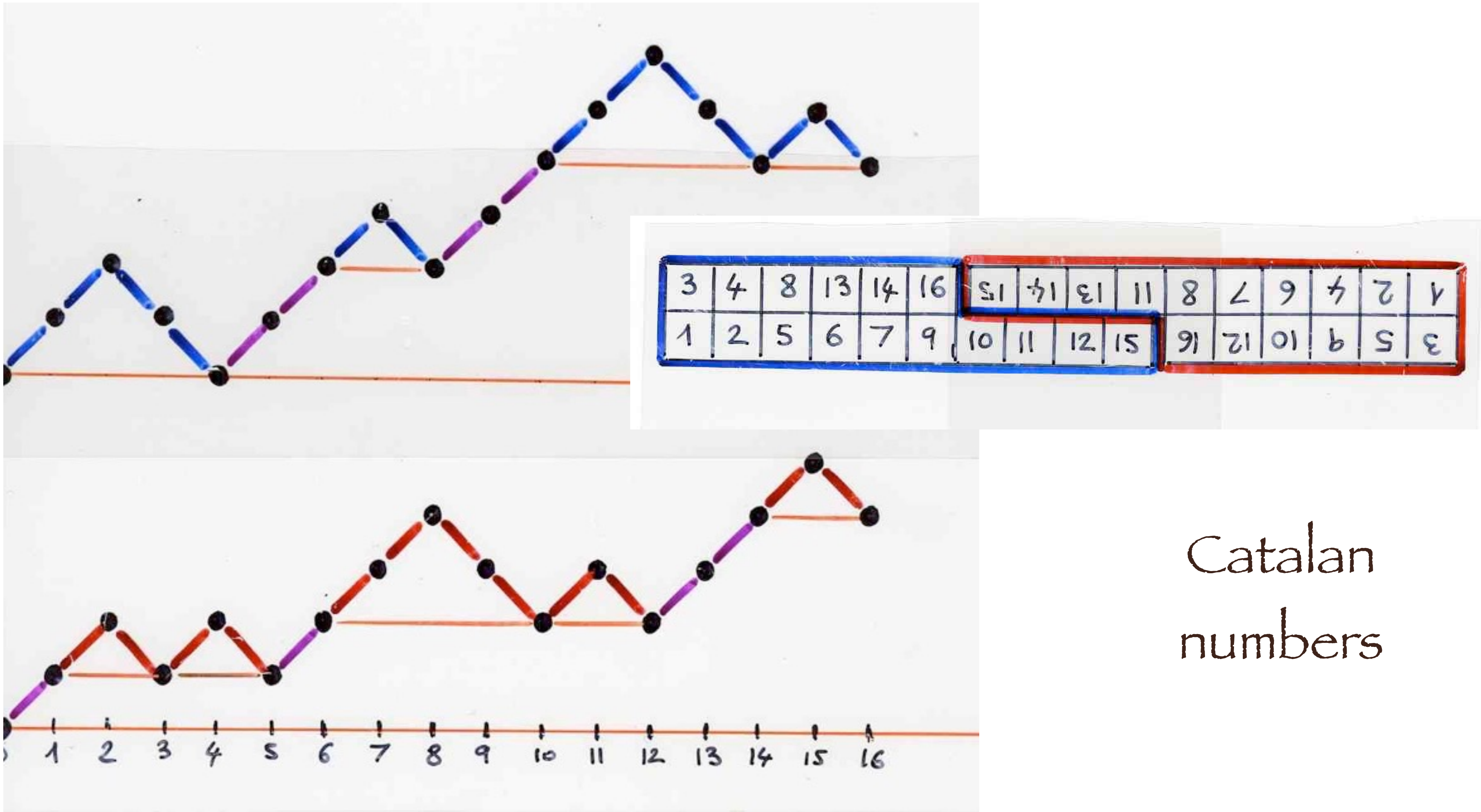
3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16


3 4 1 5 2 6 8 13 7 9 14 10 11 12 16 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Robinson - Schensted

3	4	8	13	14	16				
1	2	5	6	7	9	10	11	12	15

3	5	9	10	12	16				
1	2	4	6	7	8	11	13	14	15



Catalan numbers

nil-Temperley-Lieb algebra

nil - Temperley - Lieb algebra

NTL_n or A_n^0

(i) $e_i e_j = e_j e_i$ $|i-j| \geq 2$

(ii) $e_i^2 = 0$

(iii) $e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = 0$

nil-Temperley-Lieb algebra

NTL_n or A_n^0

(i) $e_i e_j = e_j e_i \quad |i-j| \geqslant 2$

(ii) $e_i^2 = 0$

(iii) $e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = 0$

same dimension $C_n = \frac{1}{n+1}\binom{2n}{n}$

basis

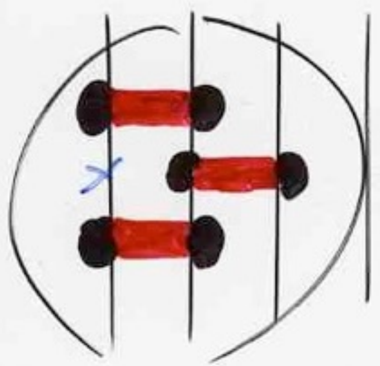
basis of (N)TL_n

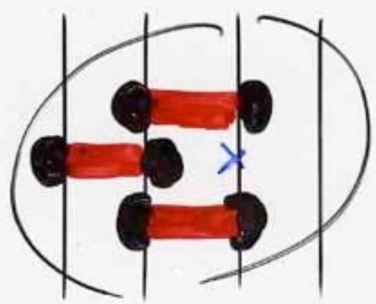
no occurrence of

u_i^2

strict heap

$u_i u_{i+1} u_i$ $\quad$ $u_{i+1} u_i u_{i+1}$



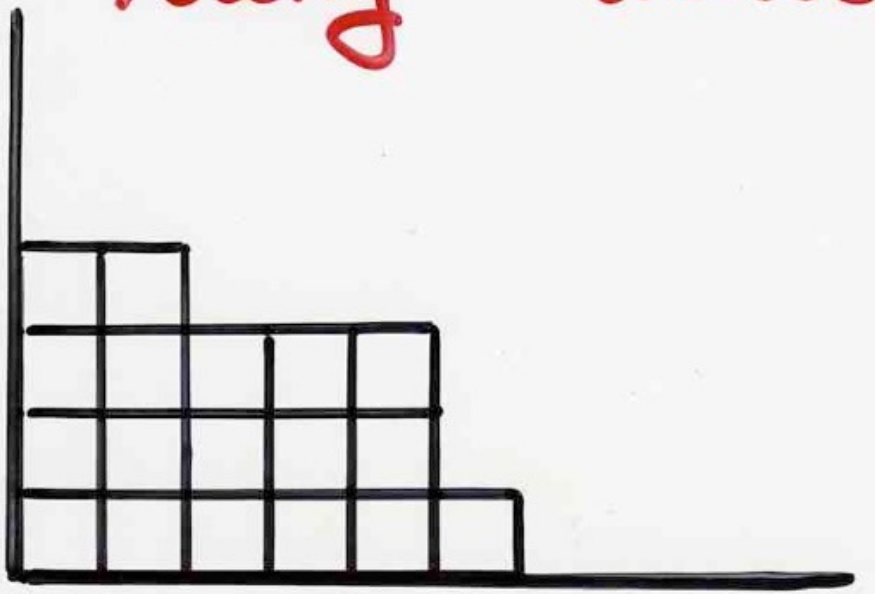


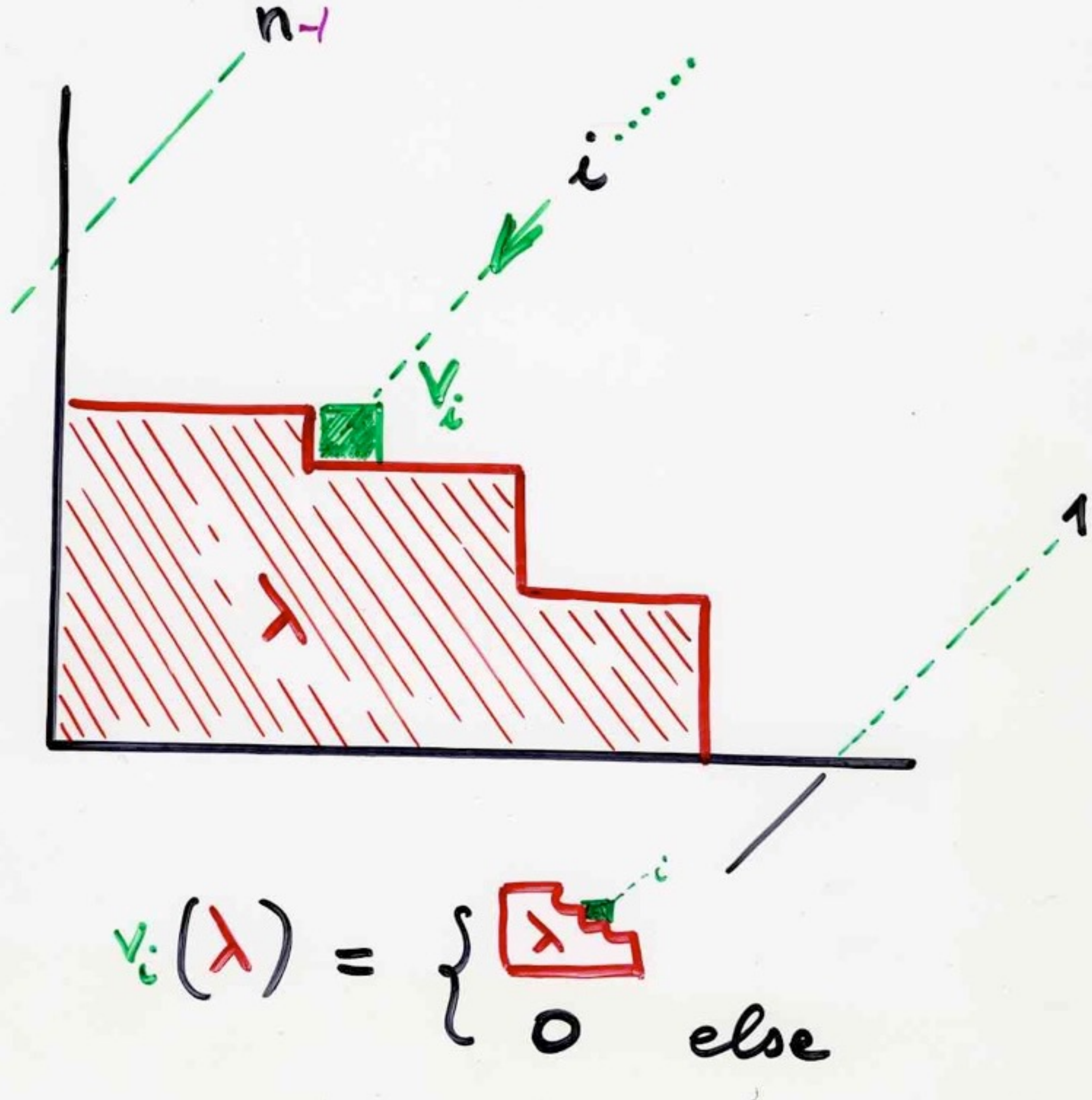
representation of NTL_n

with

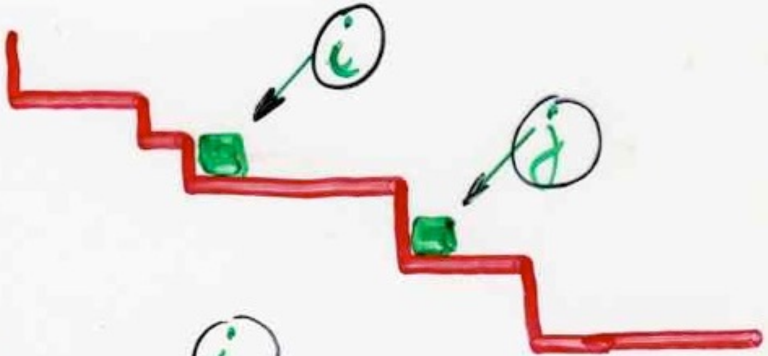
operators on the

Young lattice




n-1
i
v_i
λ
1
$v_i(\lambda) =$
0 else

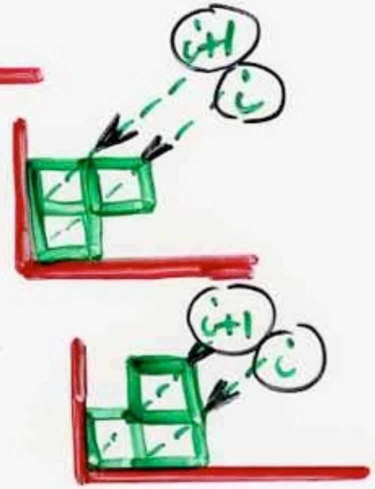
(i) $V_i \quad V_j$



(ii) V_i^2

(iii) $V_i \, V_{i+1} \, V_i$

$V_{i+1} \, V_i \, V_{i+1}$



next lecture

Chapter 7

Heaps in physics