Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

# commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.xavierviennot.org

# Chapter 2

## Heaps generating function

## Ordinary generating functions

(1)

more in Ch 2

course IMSc 2016

IMSc, Chennai

13 January 2017

intuitive introduction to

ordinary generating functions

formal power series

# Catalan numbers

1 2 5 14 42

Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \quad + \cdots$$

polynomial

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \cdots + C_n t^n + \cdots$$

generating function

Catalan numbers

$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

generating function

# Formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$

a little exercise

$$\frac{1}{1-(t+t^2)} = \ ?$$

$$\frac{1}{1-(t+t^2)} = ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \dots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \dots$$

$$+ (t^5 \cdots$$

$$\sum_{i \geqslant 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \dots$$
$$+ (t^5 \text{ — — — }$$

$$1 \quad 2 \quad 3 \quad 5 \quad 8$$

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$t + t + t + \ldots\ldots t + \ldots$

$1 + 1 + 1 + \ldots$

$$t + t + t + \dots t + \dots$$

$$1 + 1 + 1 + \dots$$

formal power series algebra

formalisation

# Formal power series
## algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \dots]$

$$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$\mathbb{K}[t]$ polynomials algebra

$$(a_0, a_1, a_2, \cdots, a_n, \cdots)$$

$$a_0 + a_1 t + a_2 t + \ldots + a_n t^n + \ldots$$

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable $t$ and coefficients in $\mathbb{K}$)

# algebra of formal power series

- sum $\quad f + g = h, \quad a_n + b_n = c_n$
- product $\quad fg = h, \quad c_n = \sum_{\substack{p+q=n \\ p,q \geq 0}} a_p b_q$
- product (by a scalar) $\quad \lambda f = h, \quad c_n = \lambda a_n$

$$f = \sum_{n \geq 0} a_n t^n, \quad g = \sum_{n \geq 0} b_n t^n, \quad h = \sum_{n \geq 0} c_n t^n$$

generating power series

of the coefficients (numbers $a_n$)

$$\sum_{n\geqslant 0} a_n t^n = f(t)$$

(ordinary generating function)

exponential generating function

$$\sum_{n\geqslant 0} a_n \frac{t^n}{n!}$$

summable
family

$$\sum_{i \in I} f_i(t)$$

<u>Def</u>- for every $n$, the set of $i \in I$ such that the coefficient of $t^n$ in the power series $f_i(t)$ is $\neq 0$, is a finite set.

example

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \dots$$
$$+ (t^5 - - - -$$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

example

$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

justification of the notation

$$(a_0, a_1, a_2, \dots, a_n, \dots)$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

summable family

infinite product

$$\sum_{i \in I} f_i^{(t)}$$

$$\prod_{i \in I} \left(1 + g_i^{(t)}\right)$$

example

$$\prod_{i \geq 1} \frac{1}{(1-q^{i})}$$

# other operations

- substitution

$$f(t) = \sum_{n \geq 0} a_n t^n \quad , \quad g(t) = \sum_{n \geq 0} b_n t^n$$

$b_0 = 0$

$$f \circ g(t) \ ; \ f(g(t)) = \sum_{n \geq 0} a_n (g(t))^n$$

- Inverse

$$\frac{1}{1-f} = 1 + f + f^2 + \dots + f^n + \dots$$

(si ord$(f) \geq 1$)

- derivative

$$f' \qquad \frac{df}{dt} = \sum_{n \geq 1} n\, a_n t^{n-1}$$

exponential $\qquad \exp(t) = \sum_{n\geq 0} \frac{t^n}{n!}$

logarithm $\qquad \log(1-t)^{-1} = \sum_{n\geq 1} \frac{t^n}{n}$

binomial power series

$$(1+t)^{\alpha} = \sum_{n\geq 0} \binom{\alpha}{n} t^n$$

$$= \sum_{n\geq 0} \alpha(\alpha-1)\dots(\alpha-n+1)\frac{t^n}{n!}$$

$\mathrm{ord}(f) \geq 1 \qquad \exp(f) \qquad \log(1+f) \qquad (1+f)^{\alpha}$

# formal power series in several variables

$$f(t_1, t_2, \dots, t_p) = \sum_{n_1, \dots, n_p} a_{n_1, \dots, n_p} t_1^{n_1} t_2^{n_2} \cdots t_p^{n_p}$$

$\mathbb{K}[t_1, \dots, t_p]$

$\mathbb{K}[[t_1, \dots, t_p]]$

algebra

opérations

$\partial / \partial t_i$

operations on combinatorial objects

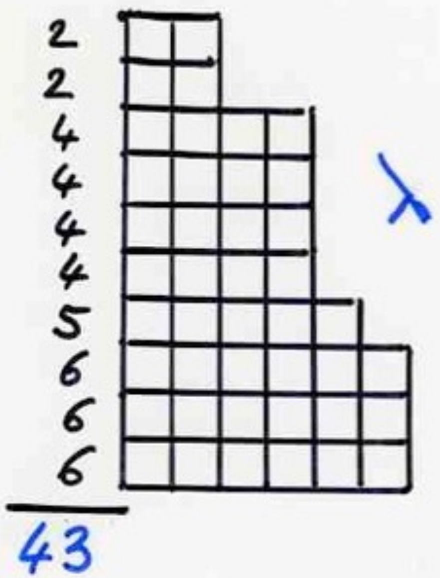
example: integers partitions

q-series

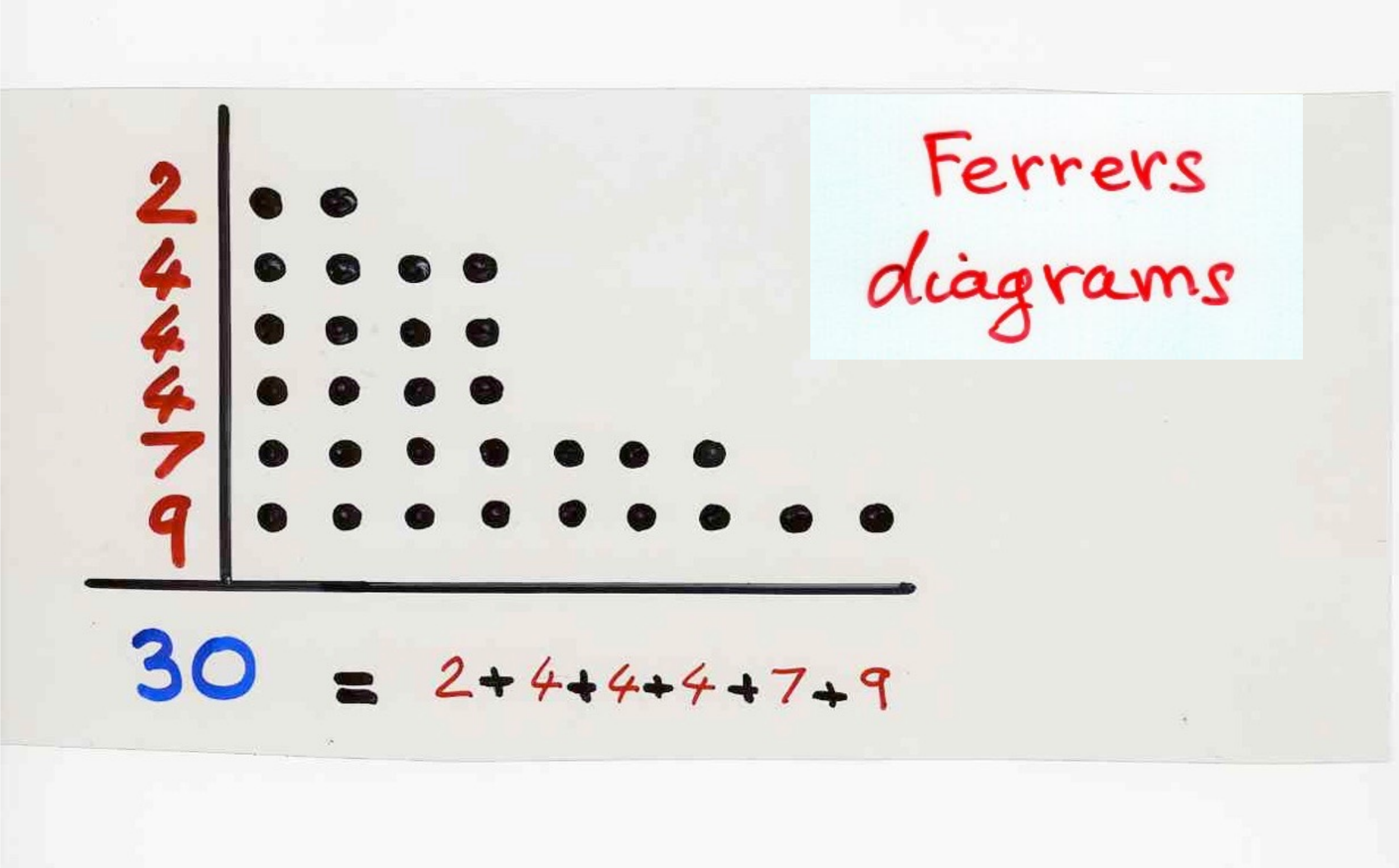
**partition** of an integer $n$

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$




Ferrers diagram


Ferrers diagrams
2
4
4
4
7
9
30 = 2+4+4+4+7+9

| | | | | |
|---|---|---|---|---|
| $1$ | $1+1$ | $1+1+1$ | $1+1+1+1$ | $1+1+1+1+1$ |
| | $2$ | $2+1$ | $2+1+1$ | $2+1+1+1$ |
| | | $3$ | $3+1$ | $2+2+1$ |
| | | | $2+2$ | $3+1+1$ |
| | | | $4$ | $3+2$ |
| | | | | $4+1$ |
| | | | | $5$ |

$$1, \quad 2, \quad 3, \quad 5, \quad 7$$

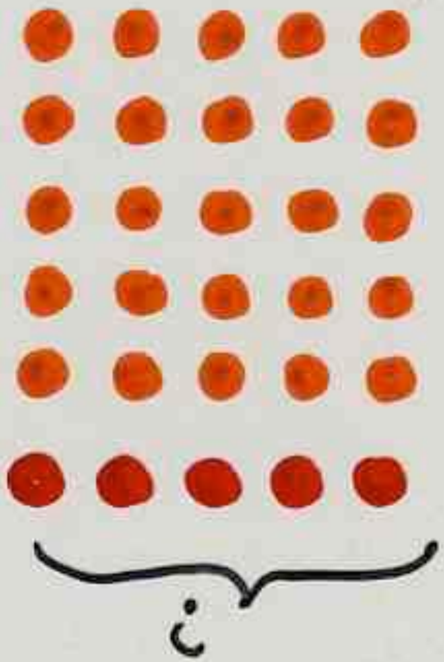
$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

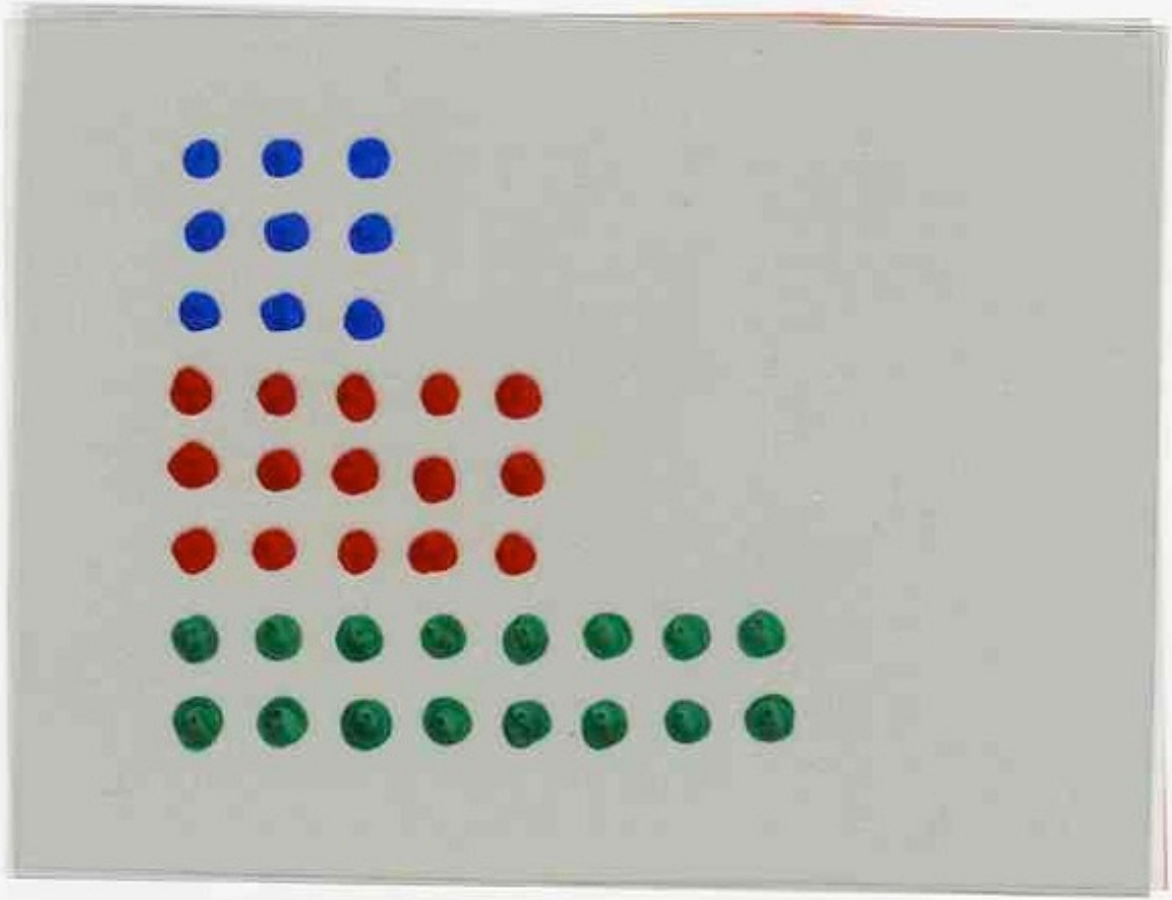
generating function for (integer) partitions

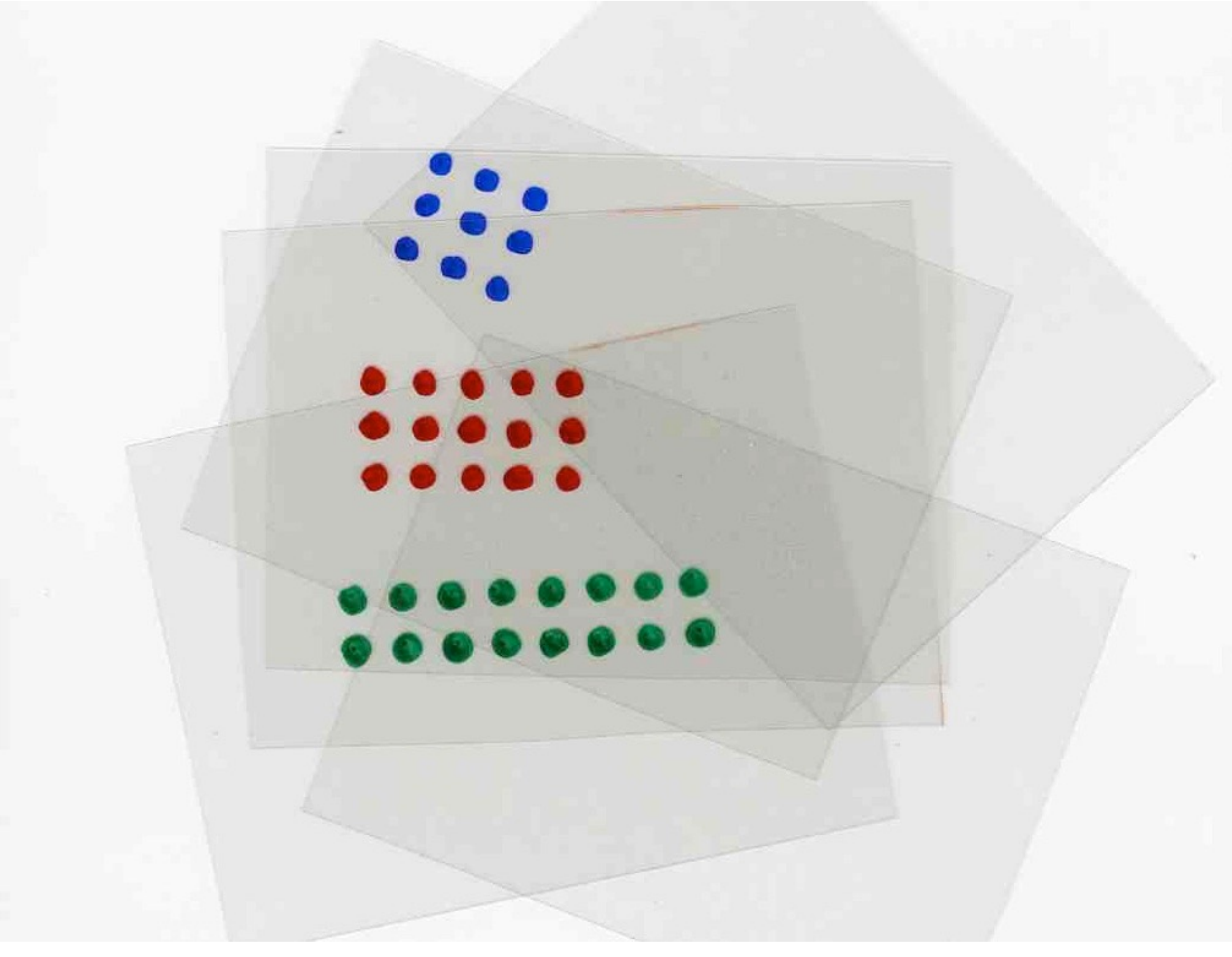
$$\sum_{n \geq 0} a_n q^n$$

$\underbrace{\bullet\ \bullet\ \bullet\ \bullet\ \bullet}_{i}$ $\qquad\qquad$ $q^{i}$




$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\ldots\cdot(1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$\prod_{i\geq 1} \frac{1}{(1-q^i)}$$

generating function
for the number of
partitions of an integer $n$

## exercise

**ex 1**

$$\sum_{n\geq 0} p(n,I)\, q^n = \prod_{i\in I} \frac{1}{1-q^i}$$

partitions
parts $\lambda_j \in I$

**ex 2**

D- partition $\lambda = (\lambda_1, \dots, \lambda_k)$

$\lambda_i - \lambda_{i+1} \geq 2$
$(1 \leq i < k)$

generating function for D- partitions

$$\sum_{m\geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

hint: find a bijection between:

partitions of $n$ with at most $m$ parts $\longleftrightarrow$ D-partitions of $n+m^2$ having exactly $m$ parts

# operations on combinatorial objects

## formalisation

- sum
- product
- seqence

# Operations on combinatorial objects

**Def.** class of valued combinatorial objects

$\mathcal{A} = (A, v)$  $A$ finite or enumerable set

$v : A \longrightarrow \mathbb{K}[X]$

valuation

(*) for $w$ monomial of $\mathbb{K}[X]$,

let $A_w = \left\{ \begin{array}{l} \alpha \in A, \text{ coeff. of } w \\ \text{in } v(\alpha) \text{ is } \neq 0 \end{array} \right\}$

then for every monomial $w$,

$A_w$ is finite

$v(\alpha)$ weight or valuation of $\alpha$

$\{v(\alpha),\ \alpha \in A\}$ is summable

Def. $f_A = \sum_{\alpha \in A} v(\alpha)$

generating power series

of objects $\alpha \in A$ weighted by $v$

ex: objects of size n

$X = \{t\}$ $v(\alpha) = t^n$

n is the size of $\alpha$, $|\alpha| = n$

$a_n = |A_{t^n}|$ (finite set)

= number of objects $\alpha \in A$ of size n

$$f_a = \sum a_n t^n$$

ex: more generally

$X = \{t\} \cup Y$ $v(\alpha) = w(\alpha)\, t^n$

in general $a_0 = 1$, only one "empty" object $\varepsilon$ with weight $v(\varepsilon) = 1$

$|\alpha| = n$, size of $\alpha$

is the number of $\alpha \in A$ such that $v(\alpha) = w(\alpha)\, t^n$

$\alpha = (A, v_A)$ $\beta = (B, v_B)$

- **sum** $\alpha + \beta = \mathcal{C}$

  $A \cap B = \emptyset$ $= (C, v_C)$

  - $C = A \cup B$ (disjoint union)
  - $v_C/_A = v_A$ $v_C/_B = v_B$

**Lemma** $f_{\mathcal{C}} = f_\alpha + f_\beta$

- **product** $\mathcal{A} \cdot \mathcal{B} = \mathcal{C}$

$$= (C, v_C)$$

- $C = A \times B$
- $(\alpha, \beta) \in C$ $\qquad v_C(\alpha, \beta) = v_A(\alpha)\, v_B(\beta)$

ex: "size" $|(\alpha, \beta)| = |\alpha| + |\beta|$

ex: binary tree

**Lemma** $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

## sequence

$\mathcal{A} = (A, v_A)$ $\qquad\qquad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \dots + \mathcal{A}^n + \dots$$

$$= \mathcal{A}^*$$

**Lemma** $f_{\mathcal{A}^*} = \dfrac{1}{1 - f_{\mathcal{A}}}$

symbolic method

Philippe Flajolet (1948–2011)

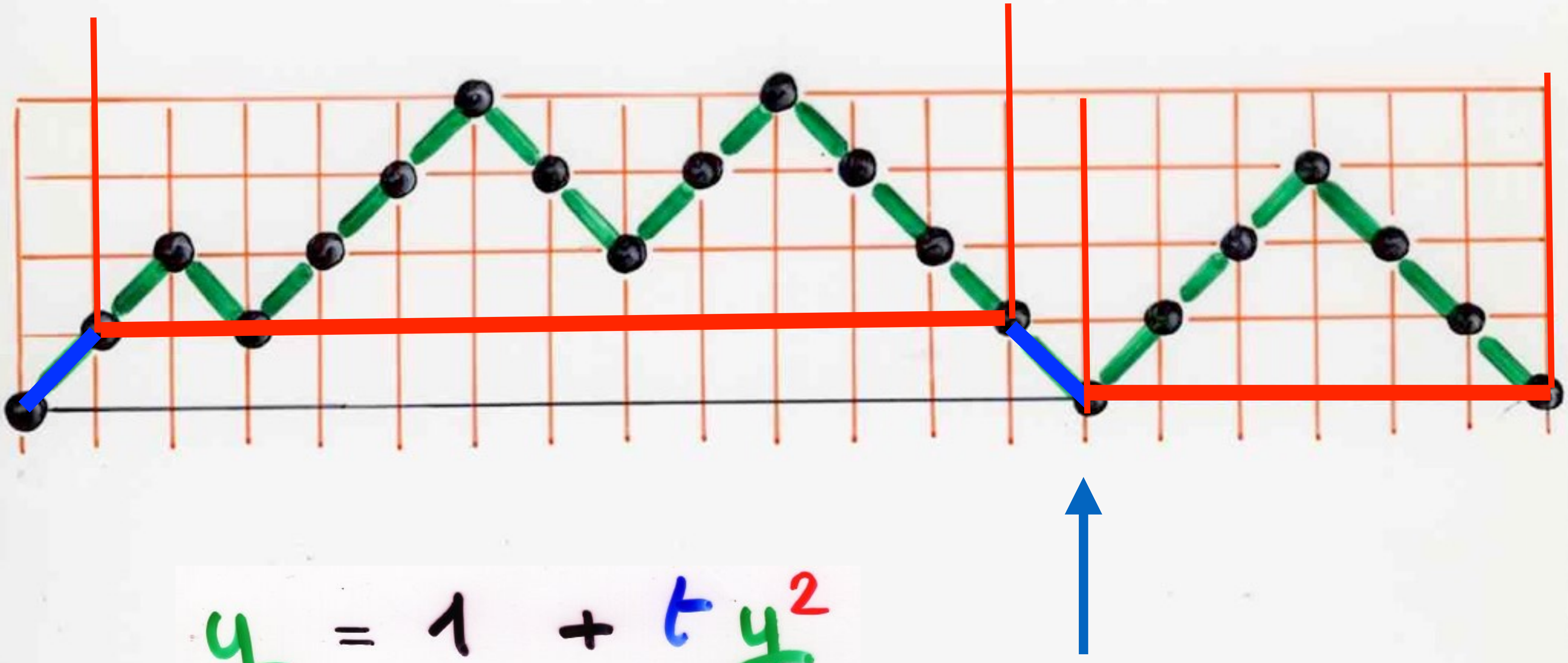
(with Robert Sedgewick)

Analytic Combinatorics

(Cambridge Univ. Press, 2008)

# Dyck paths

Dyck path



$$y = 1 + t\,y^2$$

$$y = 1 + t\,y^2$$

The number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$\Longleftrightarrow$$

classical
enumerative
combinatorics

$$y = 1 + t y^2$$

algebraic equation

$$y = 1 + t y^2$$

algebraic equation

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

(1838)

*Note sur une Équation aux différences finies ;*

**PAR E. CATALAN.**

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n}P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\dots\frac{4n-6}{n}P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\,4.\,5\dots n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3) = \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de $m$ lettres, prises $p$ à $p$; et si l'on change $n$ en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1}C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{array}{l}\frac{1}{n+1}C_{2n,n} = \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1} \\ \quad + \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \dots + \frac{1}{n}C_{2n-2,n-1}.\end{array}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$ : si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

1, 2, 6, 20, 70, 252, 924...;

qu'on les divise respectivement par

on obtiendra

(A)

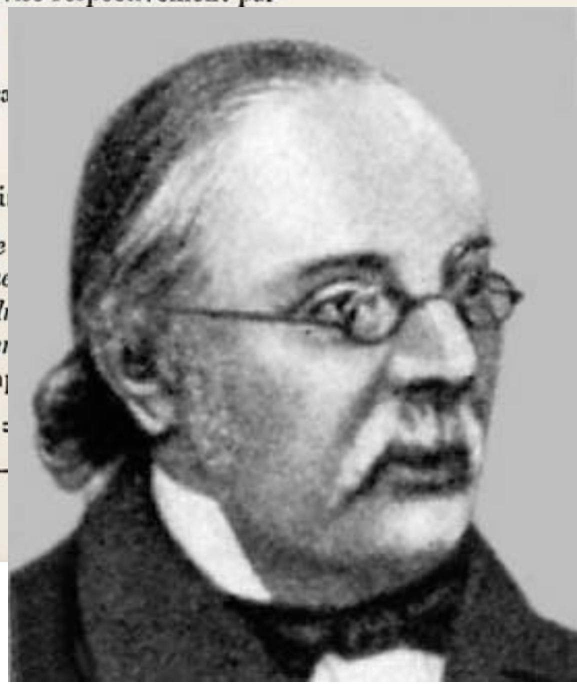
lesquels joui

*Un terme* ... *nme des*
*produits que* ... *ème, et*
*dans un ordi* ... *n multi-*
*pliant les te*

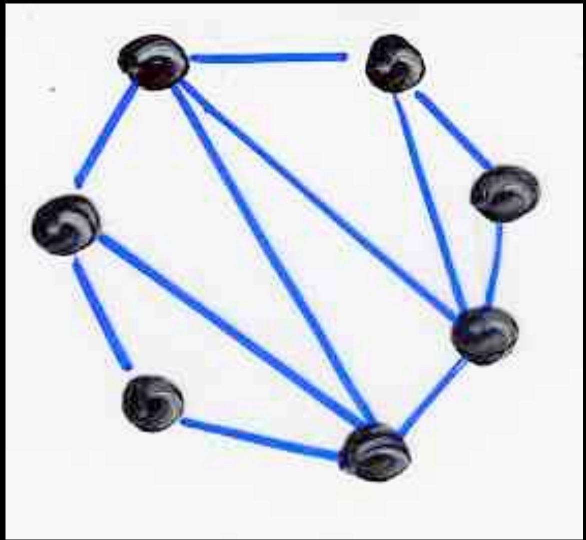
Par exem

132 =

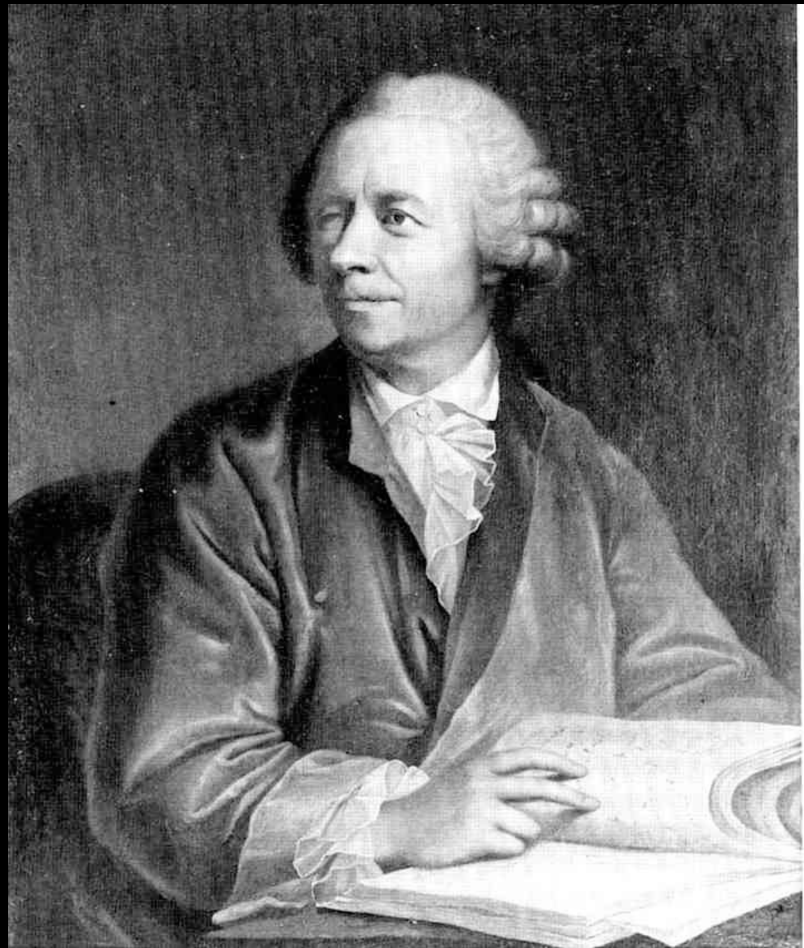




Eugène Catalan (1814-1894)

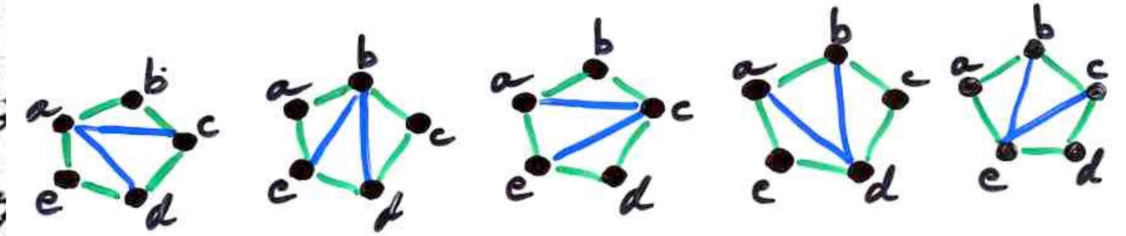


triangulation
of a convex polygon



Leonhard Euler (1707 - 1783)

theilet, und solches kan auf 5 erley verschiedene Arten geschehen nemlich
durch die Diagonalen I. $ac, ad$; II. $bd, be$; III. $ca, ce$; IV. $db, da$; V. $ec, eb$




durch $n-3$ Diagonalen in $n-2$ Triangula zerschnitten werden, auf
wie vielerley verschiedene Arten solches geschehen könne.

1, 2, 5, 14, 42, 132, 429, 1430, ...

wann $n = 3, 4, 5, 6, 7, 8, 9, 10$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter
sey

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$$14 = 5 \cdot \frac{14}{5}; \quad 42 = 14 \cdot \frac{18}{6};$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-a^2}}{2a^2}$$

$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 +$ etc

$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 +$ etc $= \frac{1-2a-\sqrt{(1-4a)}}{2aa}$

$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 +$ etc

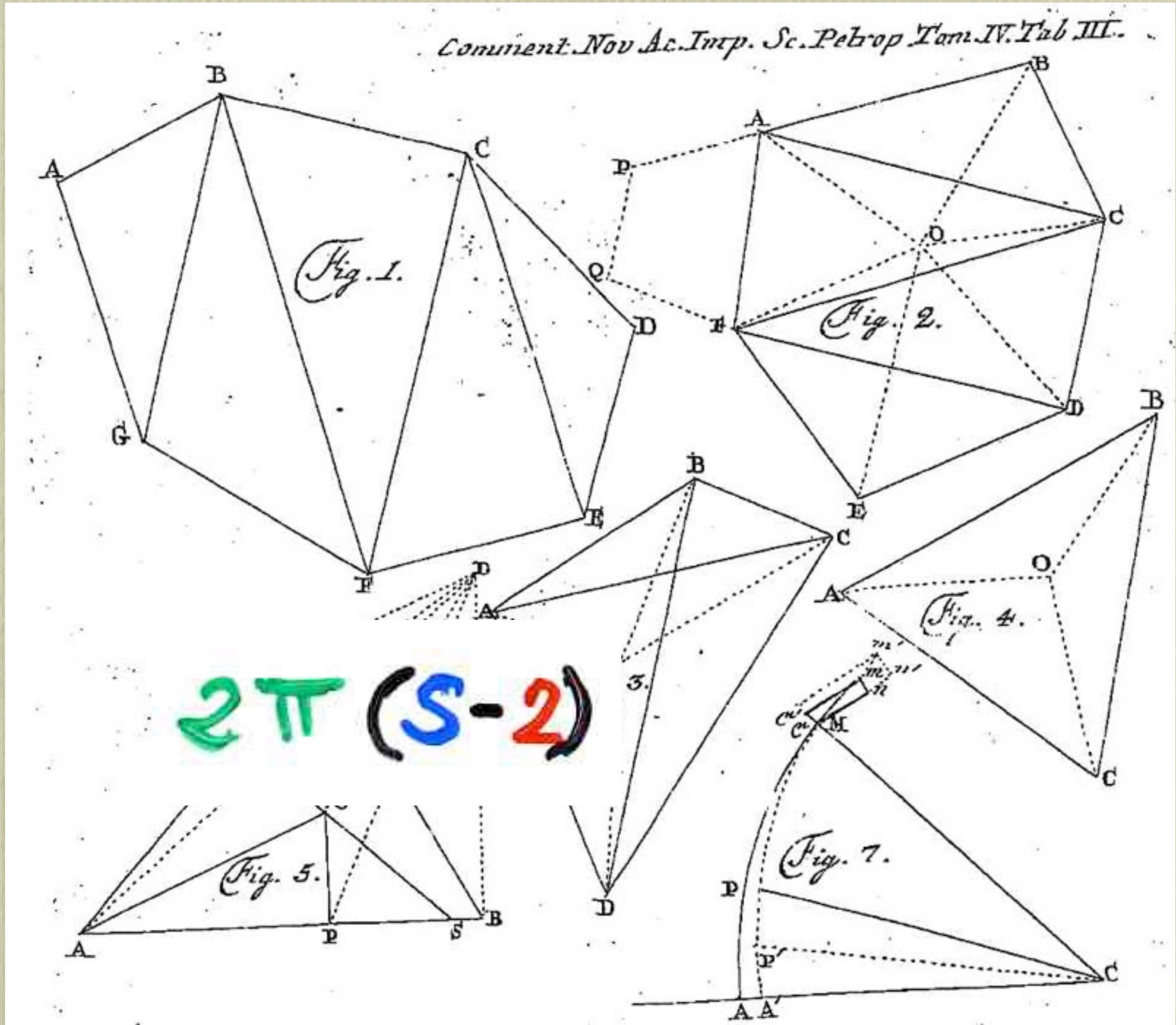
$a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} +$ etc $= 4$.

$a = \frac{1}{4}$

Berlin d. 4. Sept 1751.

4 Sept 1751
Berlin

L. Euler


Comment. Nov Ac. Imp. Sc. Petrop Tom. IV. Tab III.
Fig. 1.
Fig. 2.
3.
Fig. 4.
Fig. 5.
Fig. 7.
2π(S-2)

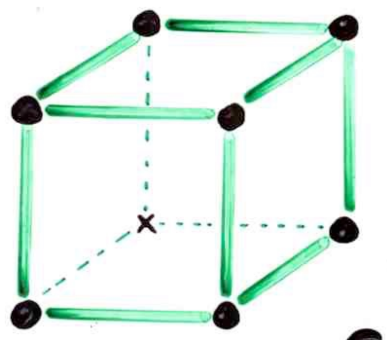
hedrarum ponatur $= H$, neque hic numerus $H$ neque numerus $S$ maior eſſe poteſt quam $\frac{2}{3}A$.

## PROPOSITIO IV.

§. 33. *In omni ſolido hedris planis incluſo aggregatum ex numero angulorum ſolidorum et ex numero hedrarum binario excedit numerum acierum.*

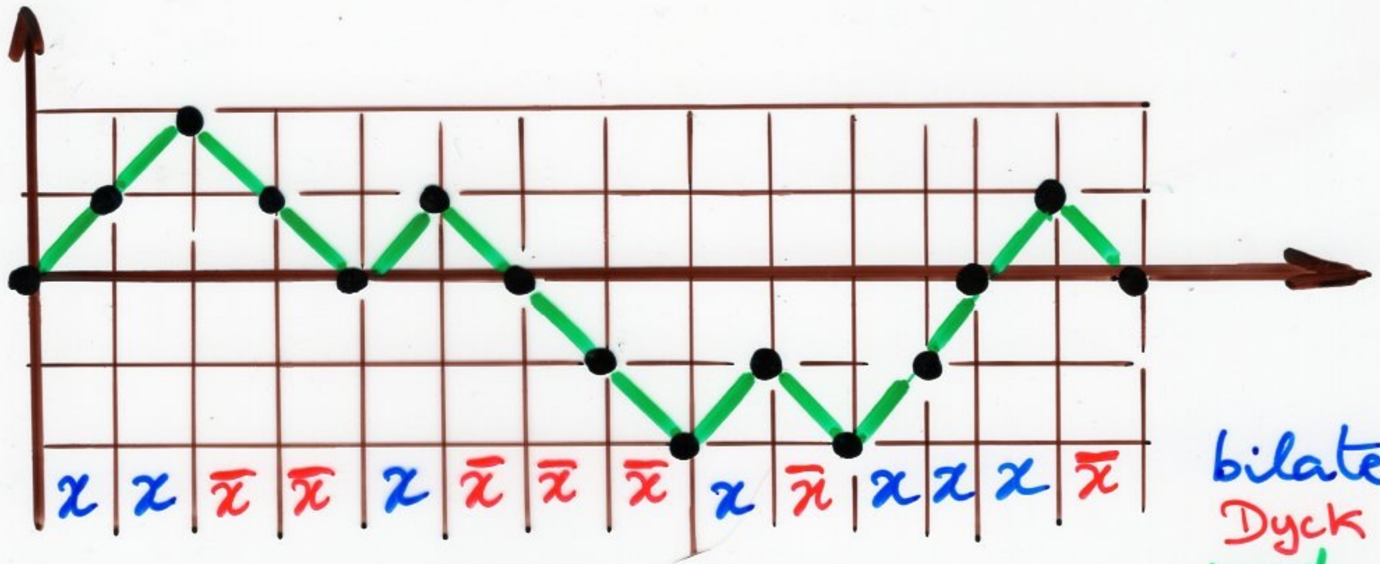
## DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum $= S$
numerus acierum - - - - $= A$
numerus hedrarum - - - $= H$
demonſtrandum eſt, eſſe $S + H = A + 2$.



$S - A + F =$
$8 - 12 + 6 = 2$

bilateral Dyck path




bilateral
Dyck
word

number of bilateral Dyck paths $\omega$ of length $2n$ $= \binom{2n}{n}$

obvious !

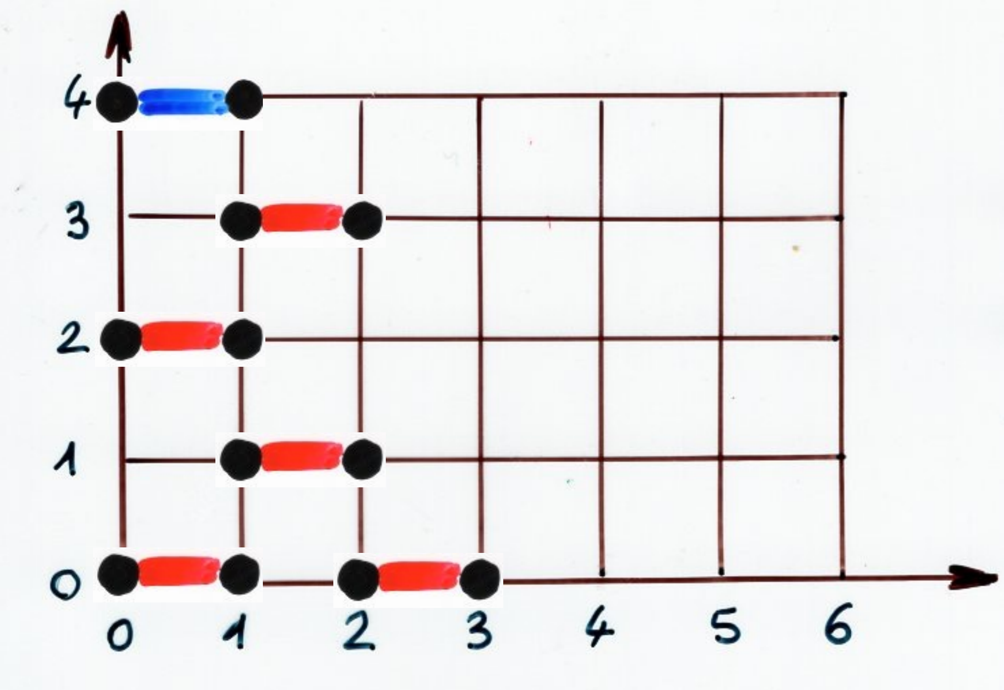
## exercise

bilateral
Dyck paths

- find an algebraic system of equations satisfied by the generating function

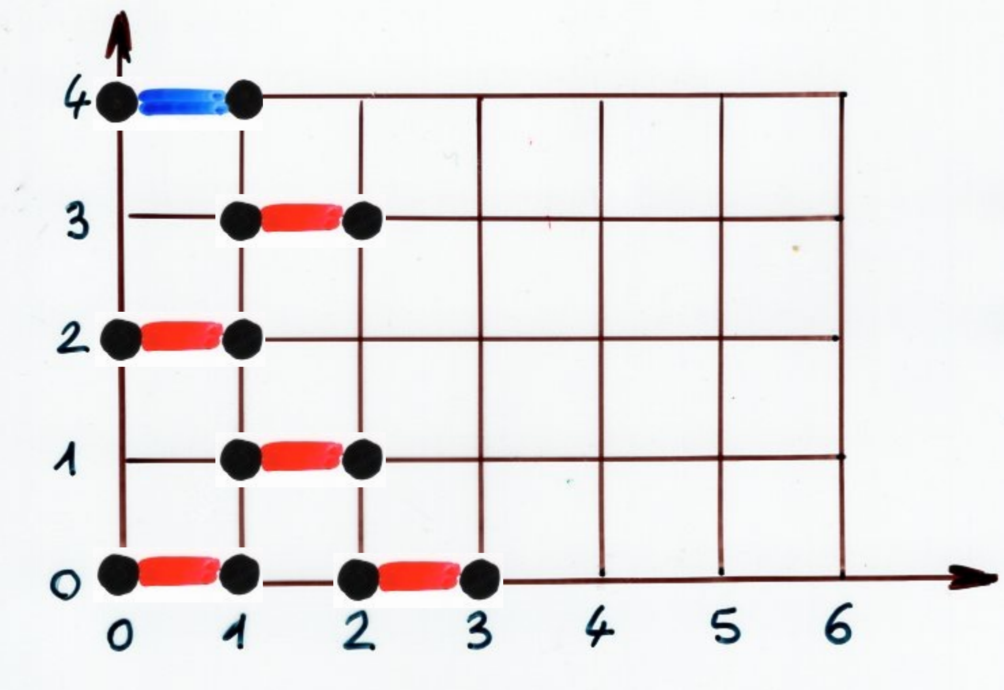
- deduce that $y = \dfrac{1}{\sqrt{1-4t}}$

# pyramids of dimers
# and
# algebraic generating functions

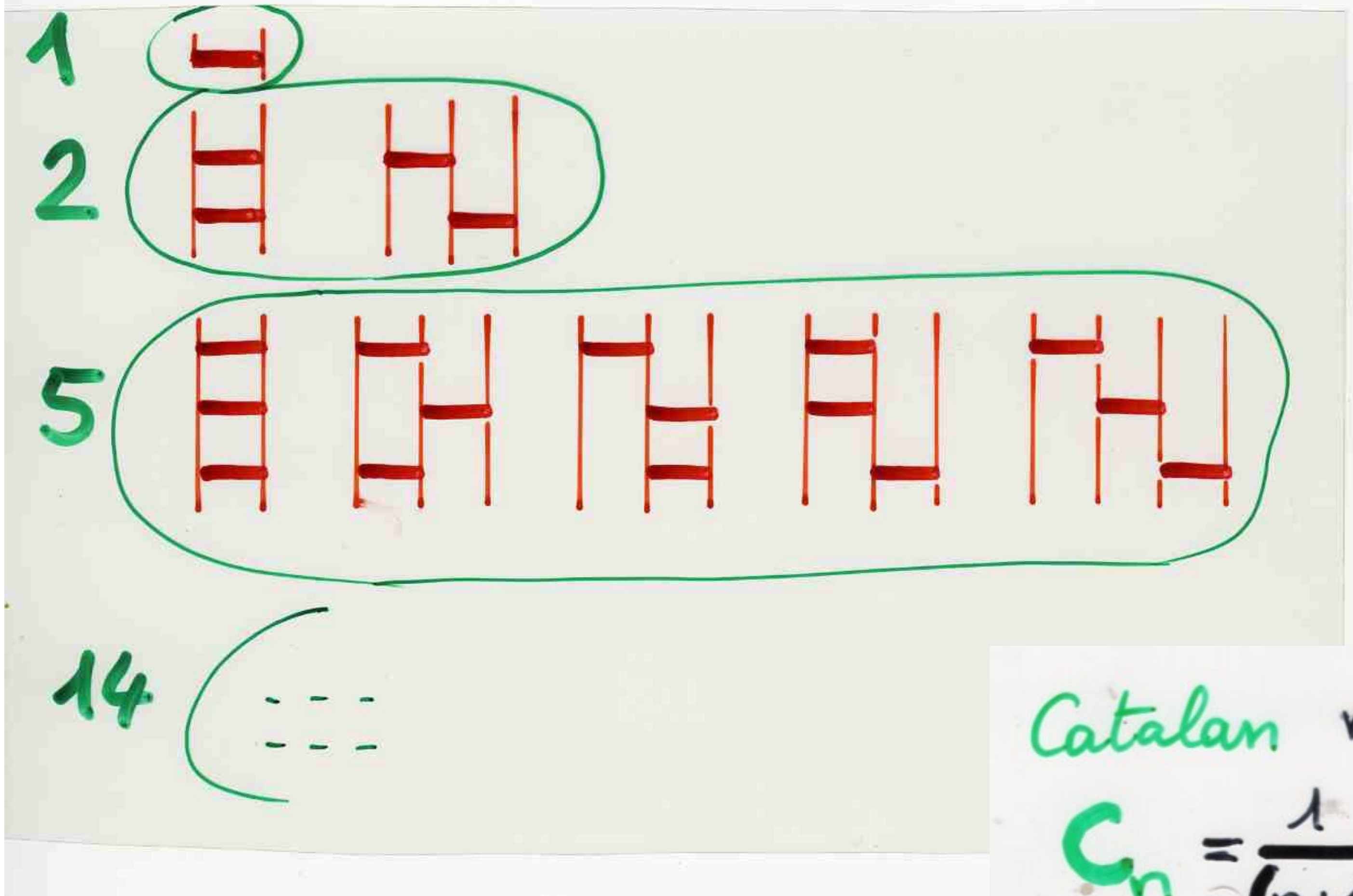



semi-pyramid of dimers on $\mathbb{N}$
the unique maximal piece
has projection [0,1]

from exercise 2, Ch 1b
bijection with Dyck paths



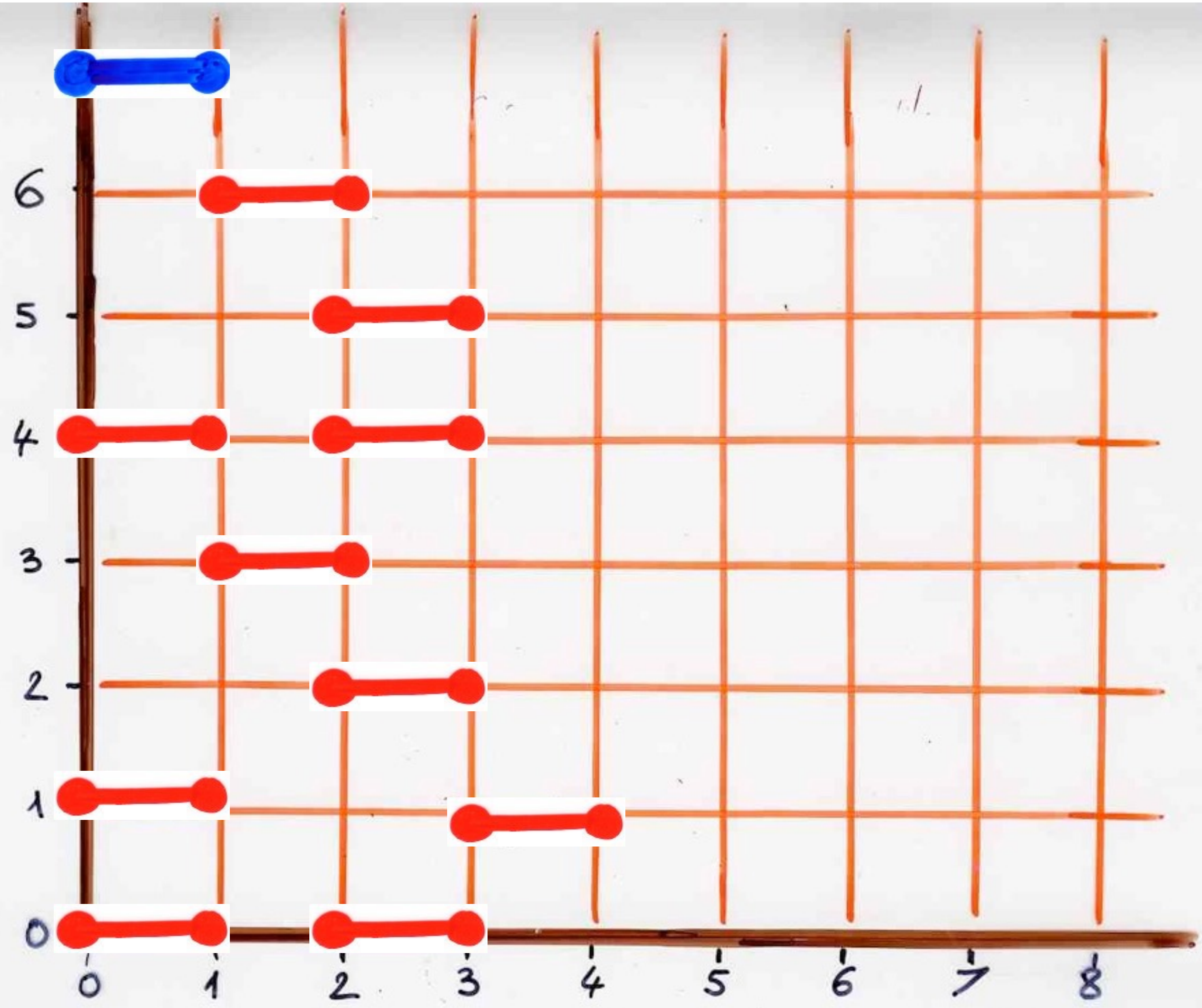

→ The number of semi-pyramids having $n$ dimers is the Catalan number $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

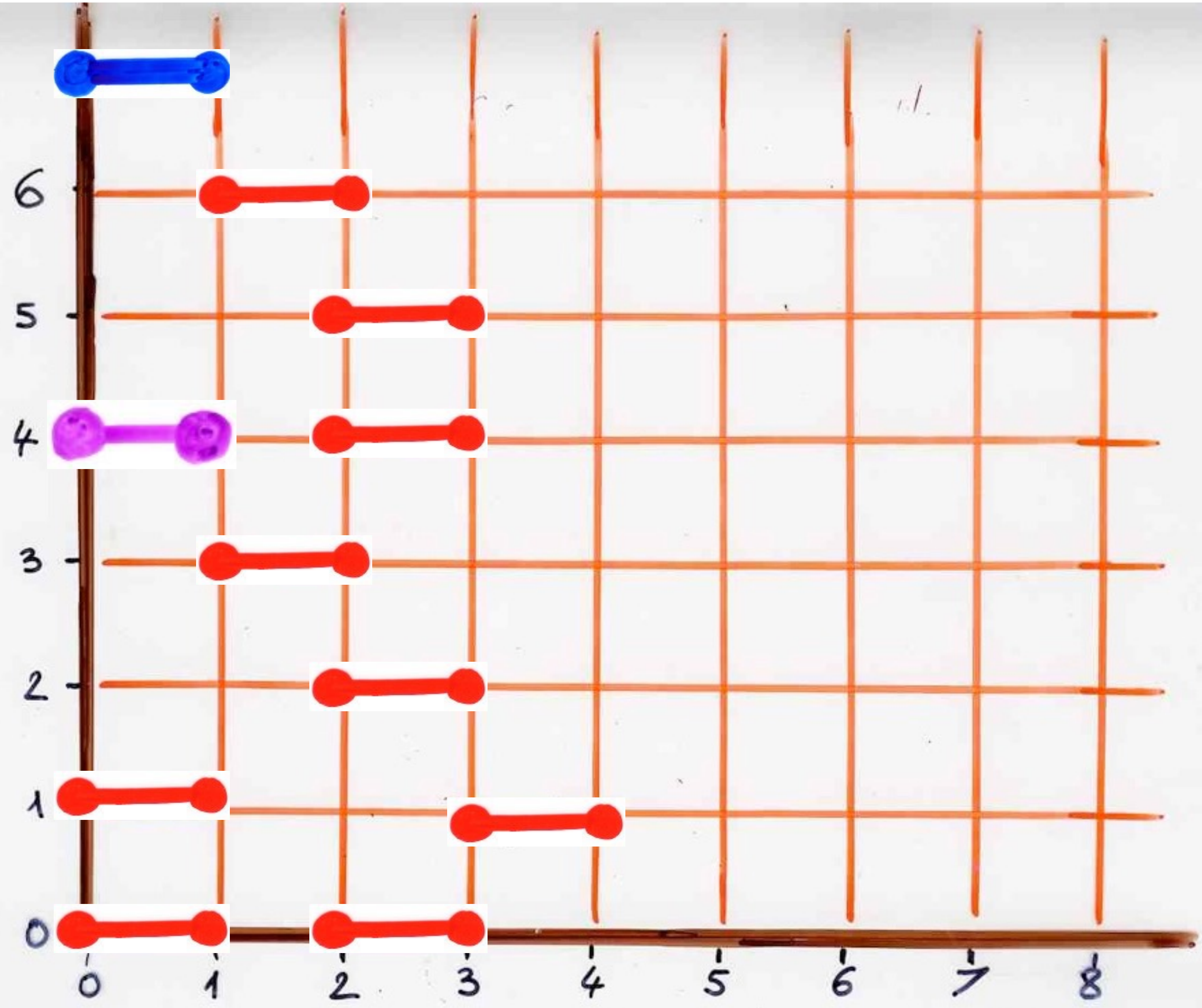



Catalan number

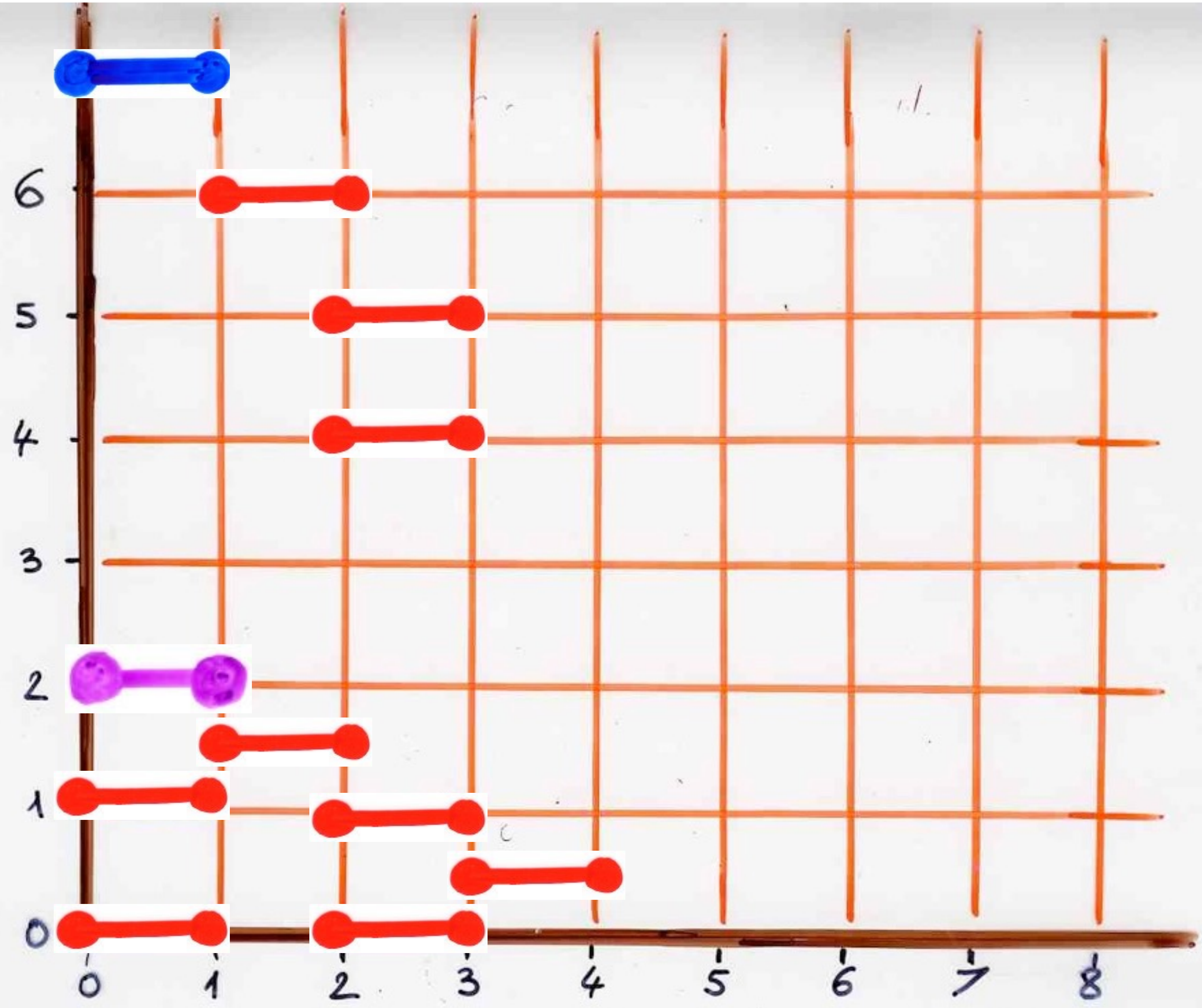
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

second proof with algebraic equation

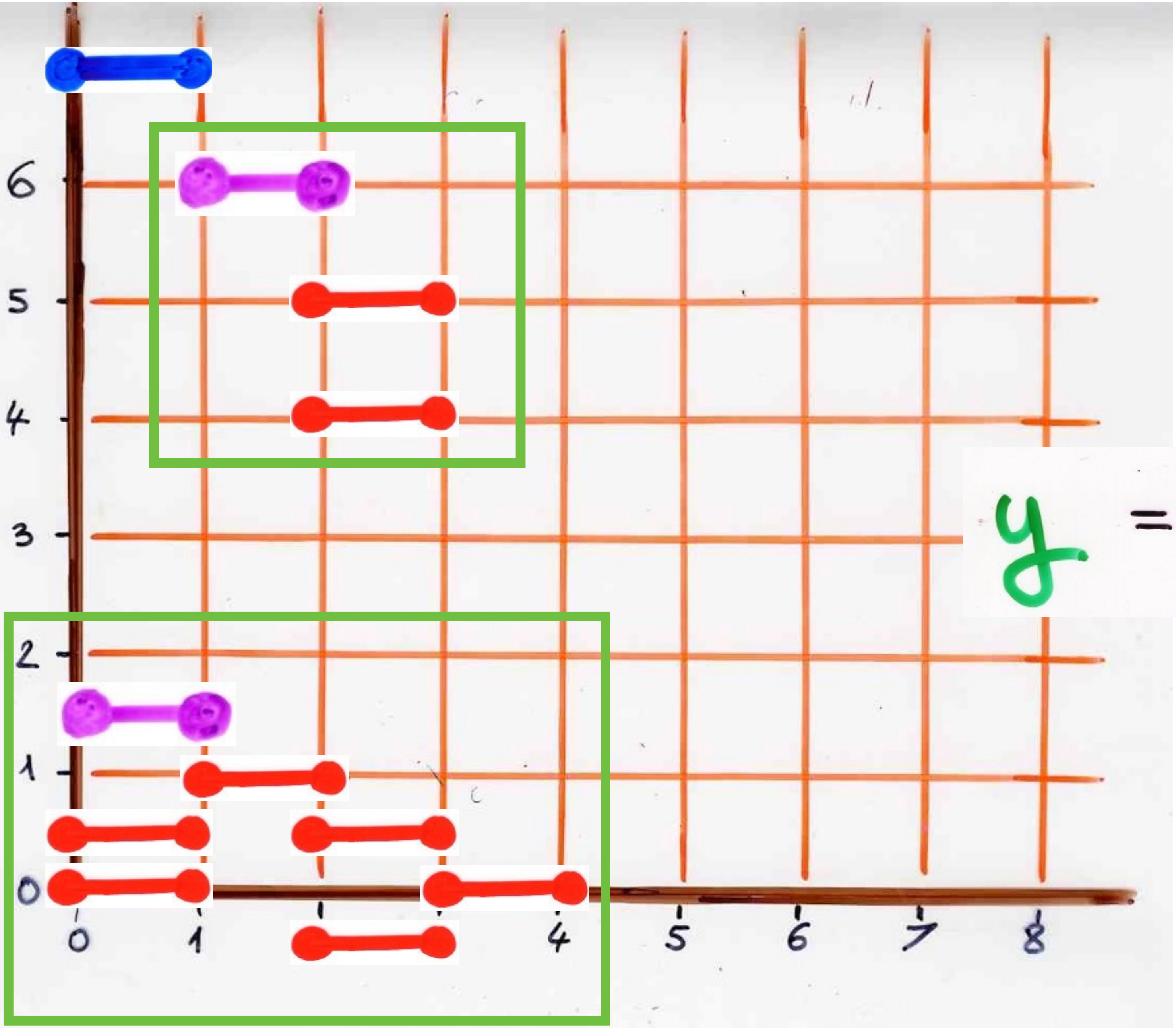

6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8



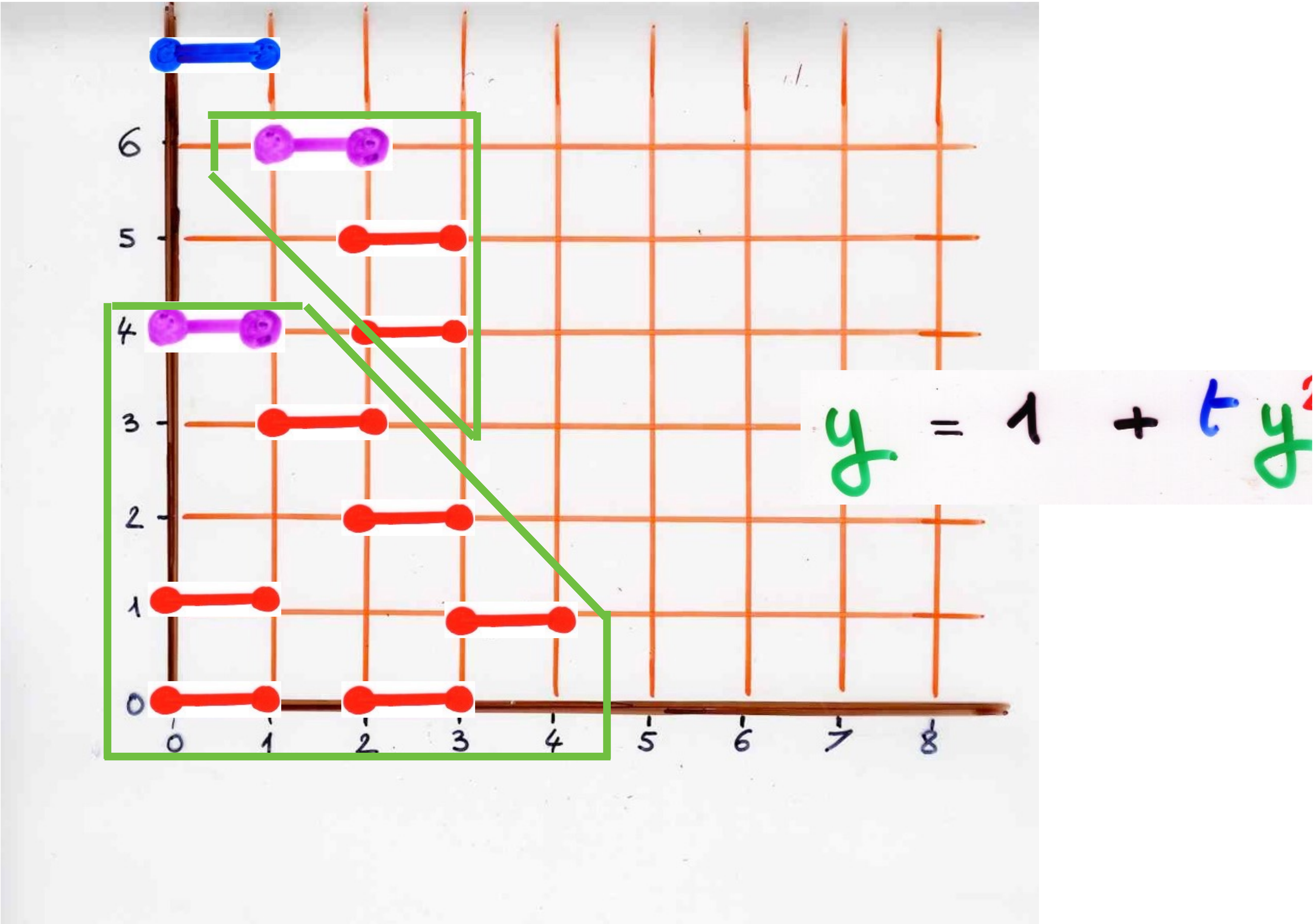
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

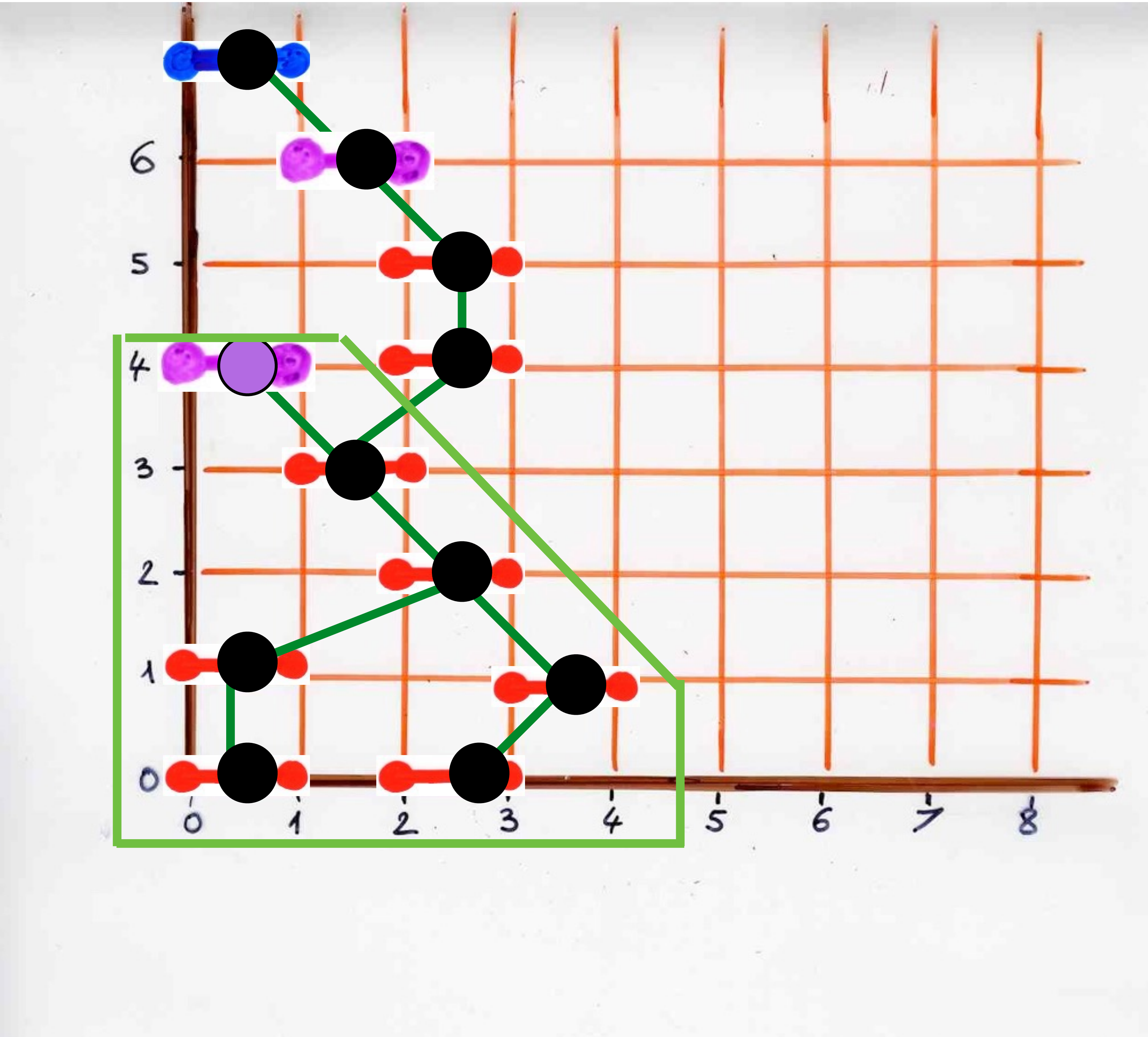


6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8

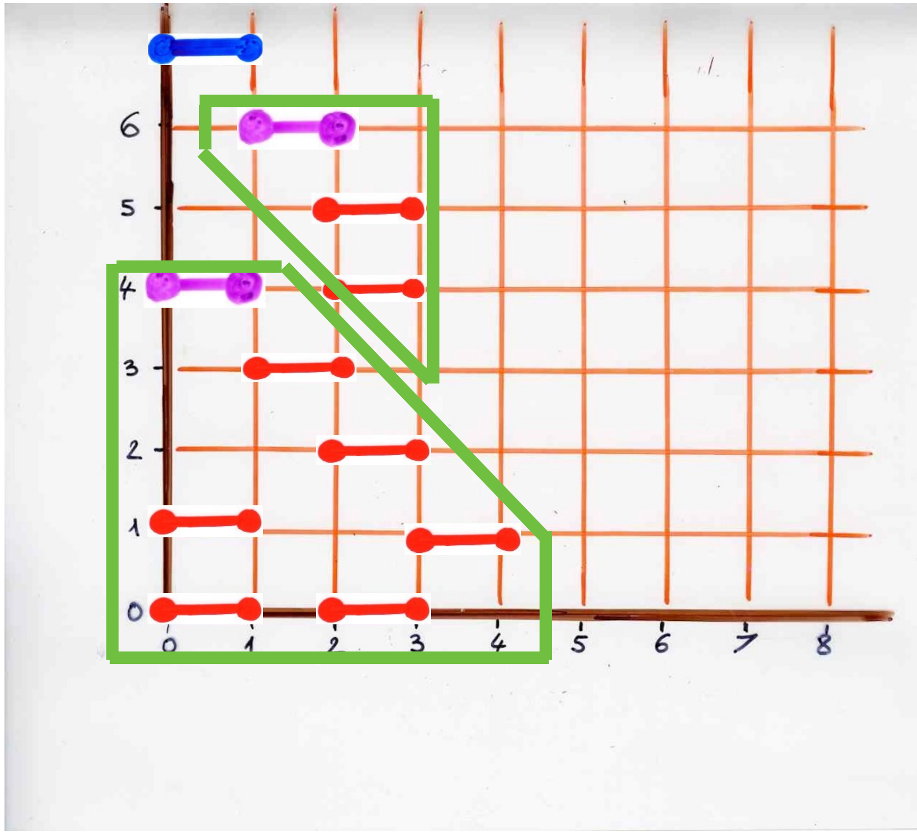



$y = 1 + t$


0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8
$y = 1 + t\,y^2$



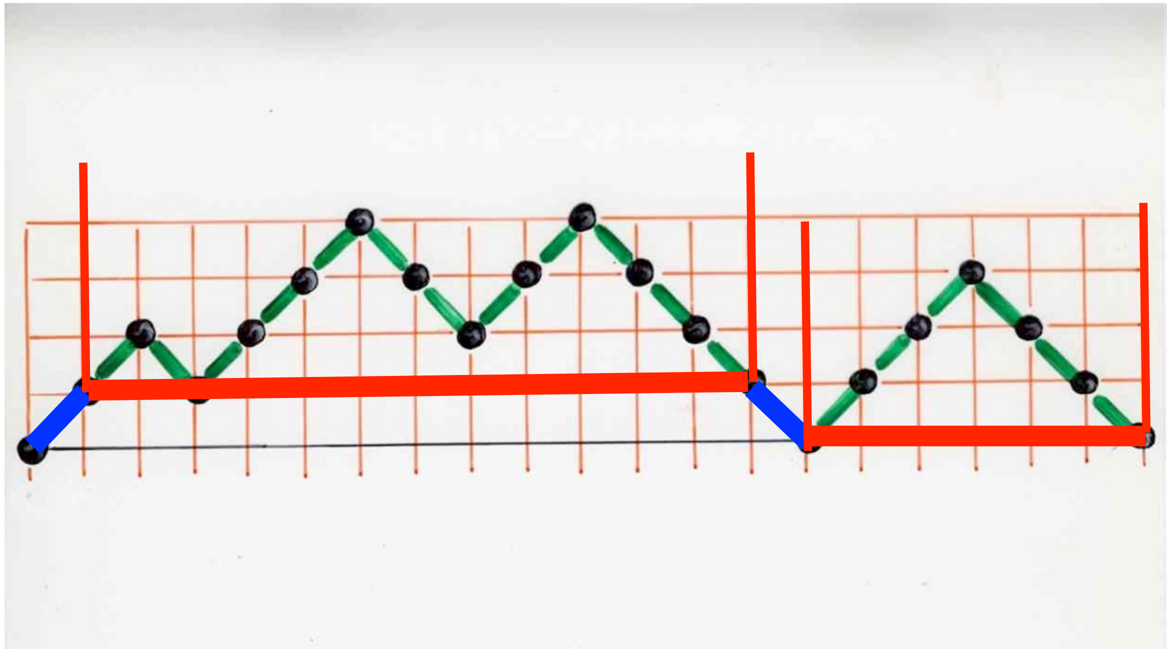
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8

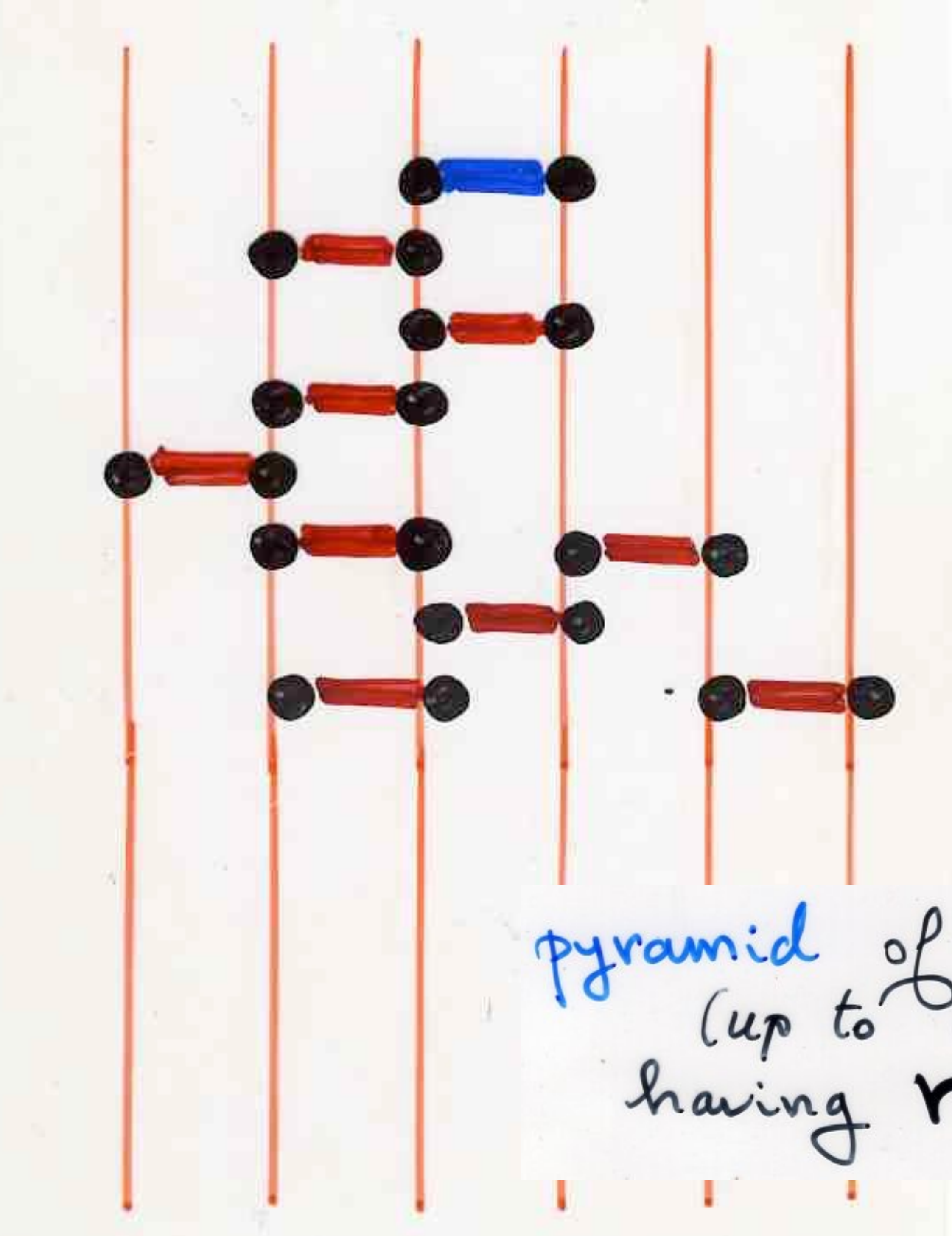



$$y = 1 + t y^2$$

## exercise

- recursive construction of a bijection
- compare with the bijection ex2, Ch1b

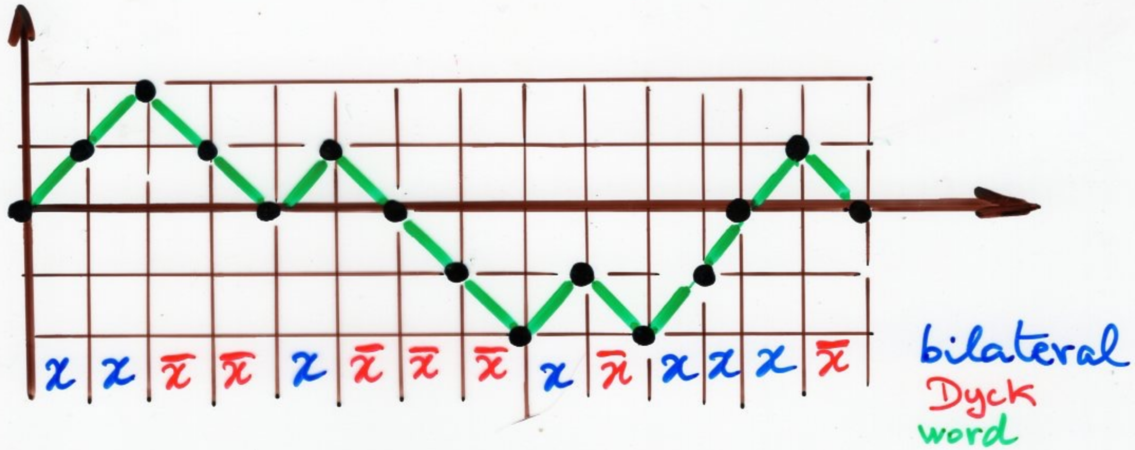




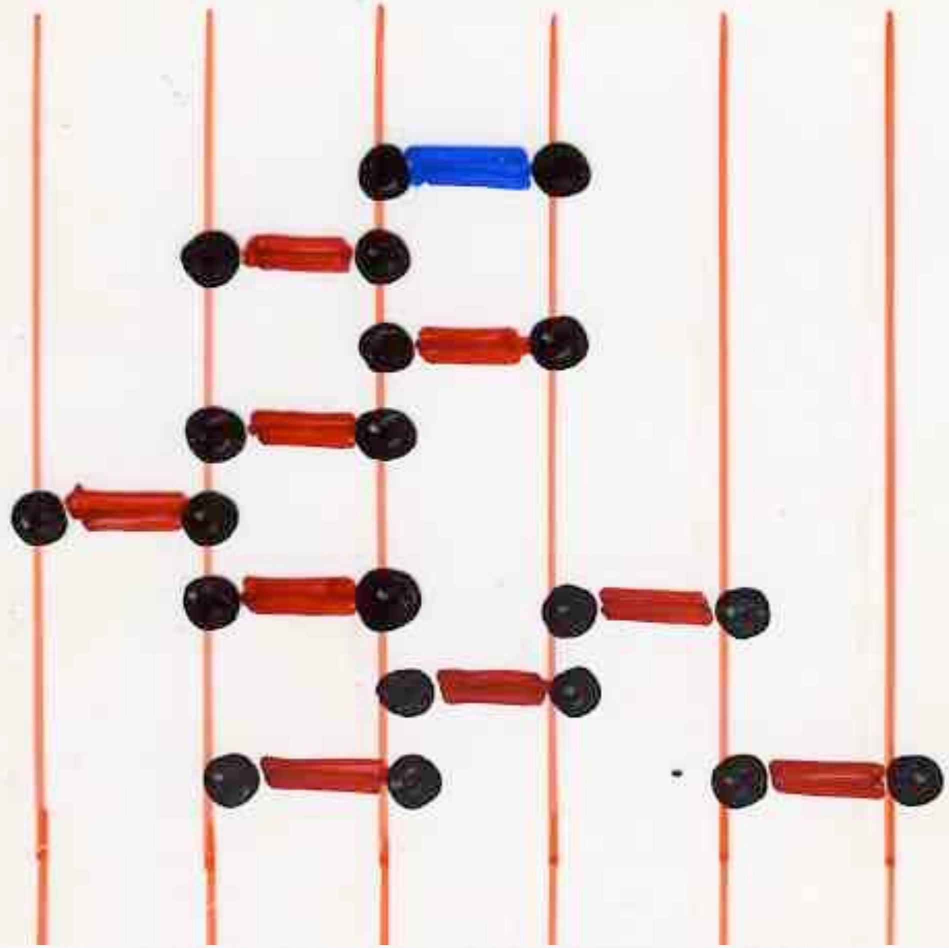
pyramid of dimers on $\mathbb{Z}$
(up to translation)
having $n$ dimers

(exercise 3, Ch 1b)
bijection with
bilateral Dyck paths

bilateral Dyck path




Thus the number of pyramids of dimers, on $\mathbb{Z}$ up to translation, having $n$ dimers is $\frac{1}{2}\binom{2n}{n}$



second proof with algebraic equation

exercise

same algebraic system of equations than bilateral Dyck paths