An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 5

Tilings, determinants and non-crossing paths

(1)

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1 March 2016

The LGV Lemma

non- intersecting
configuration
of paths

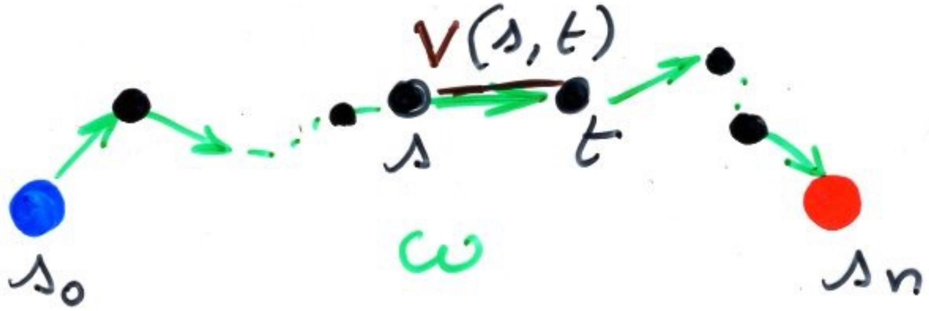
determinant

Path $\omega = (s_0, s_1, \ldots, s_n)$ $s_i \in S$

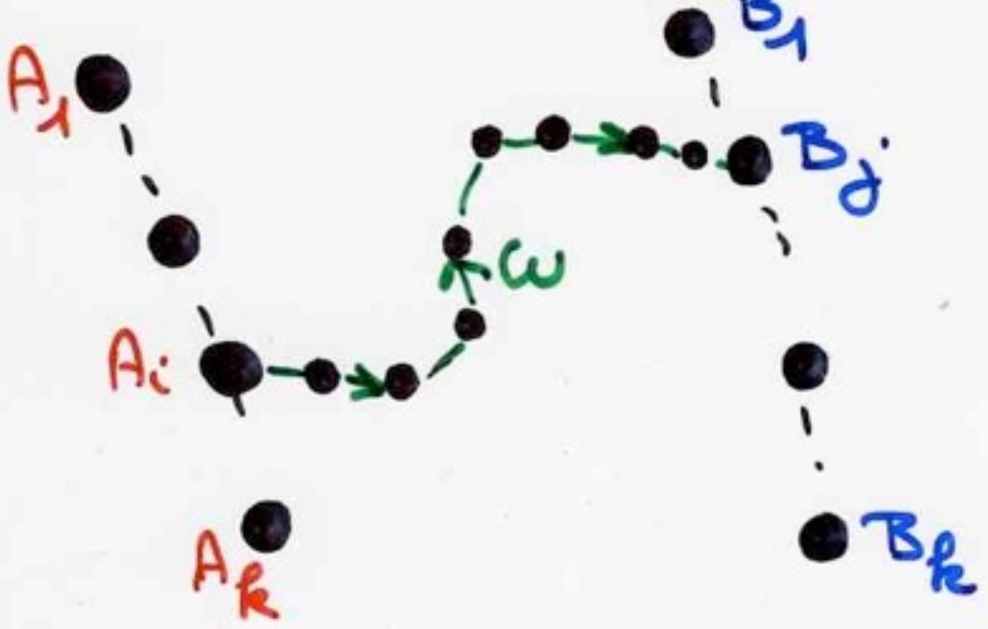
<u>notation</u> $s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation $v : S \times S \longrightarrow \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \ldots v(s_{n-1}, s_n)$$



weighted path



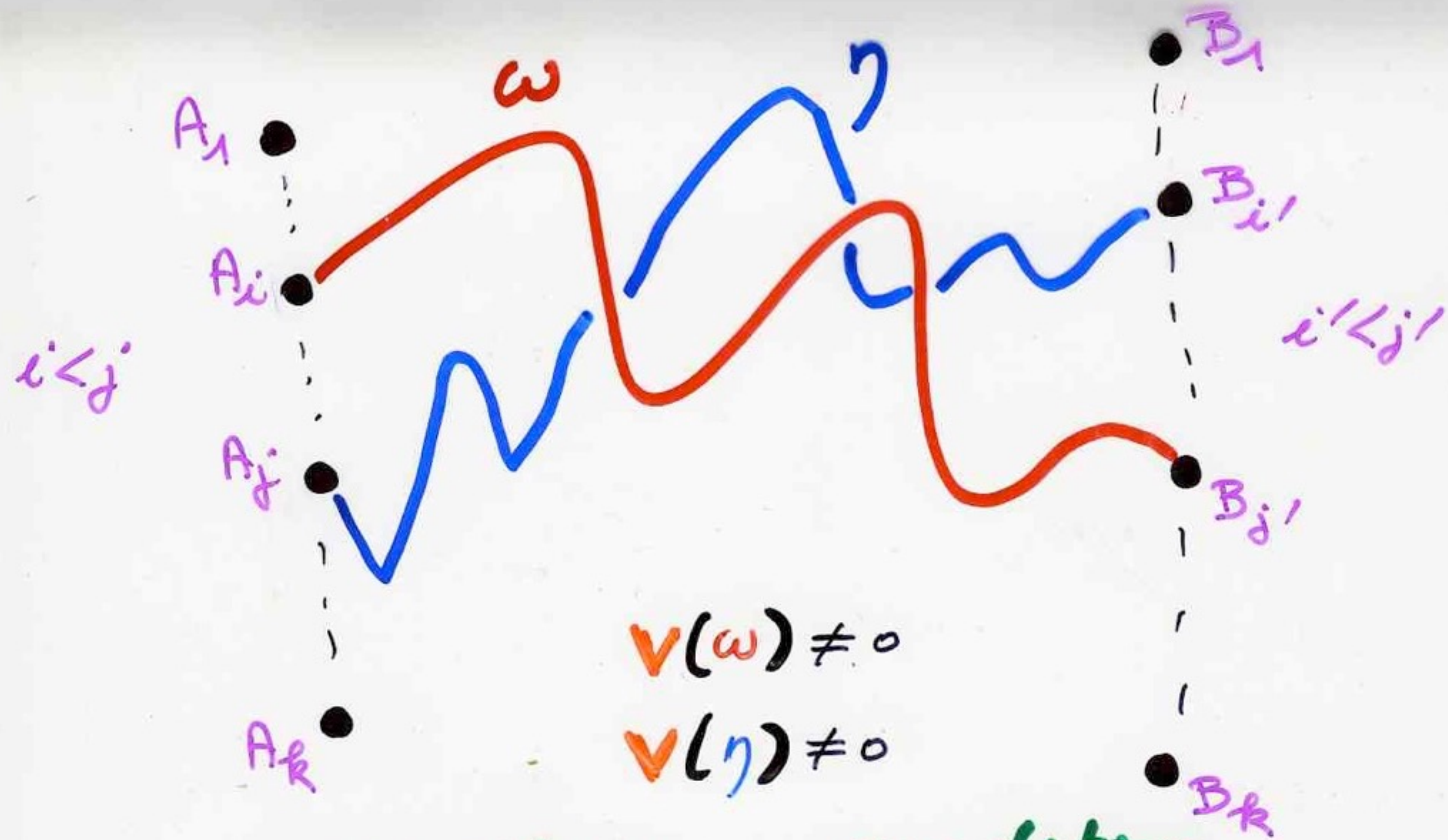
$A_1, \dots, A_k$
$B_1, \dots, B_k$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\,\omega_1,\dots,\omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1)\,\dots\,(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$



crossing condition

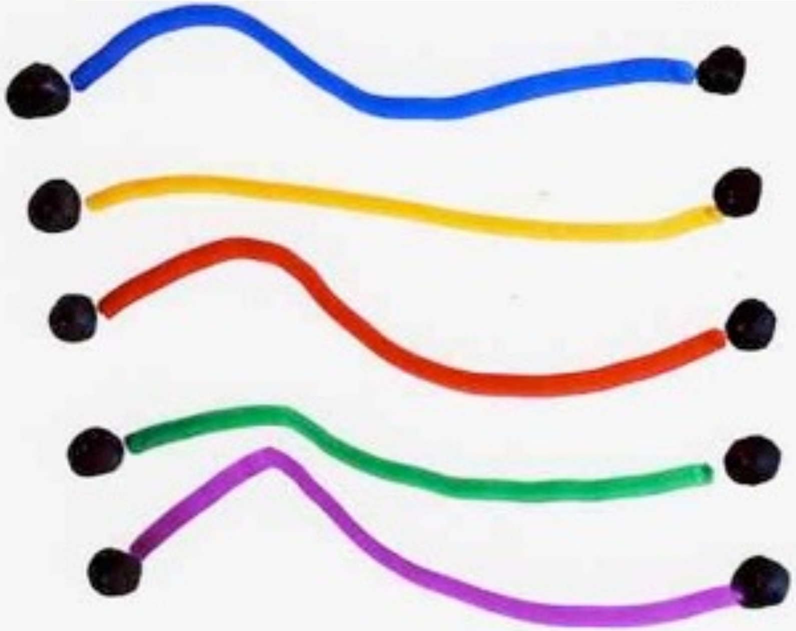
(C)

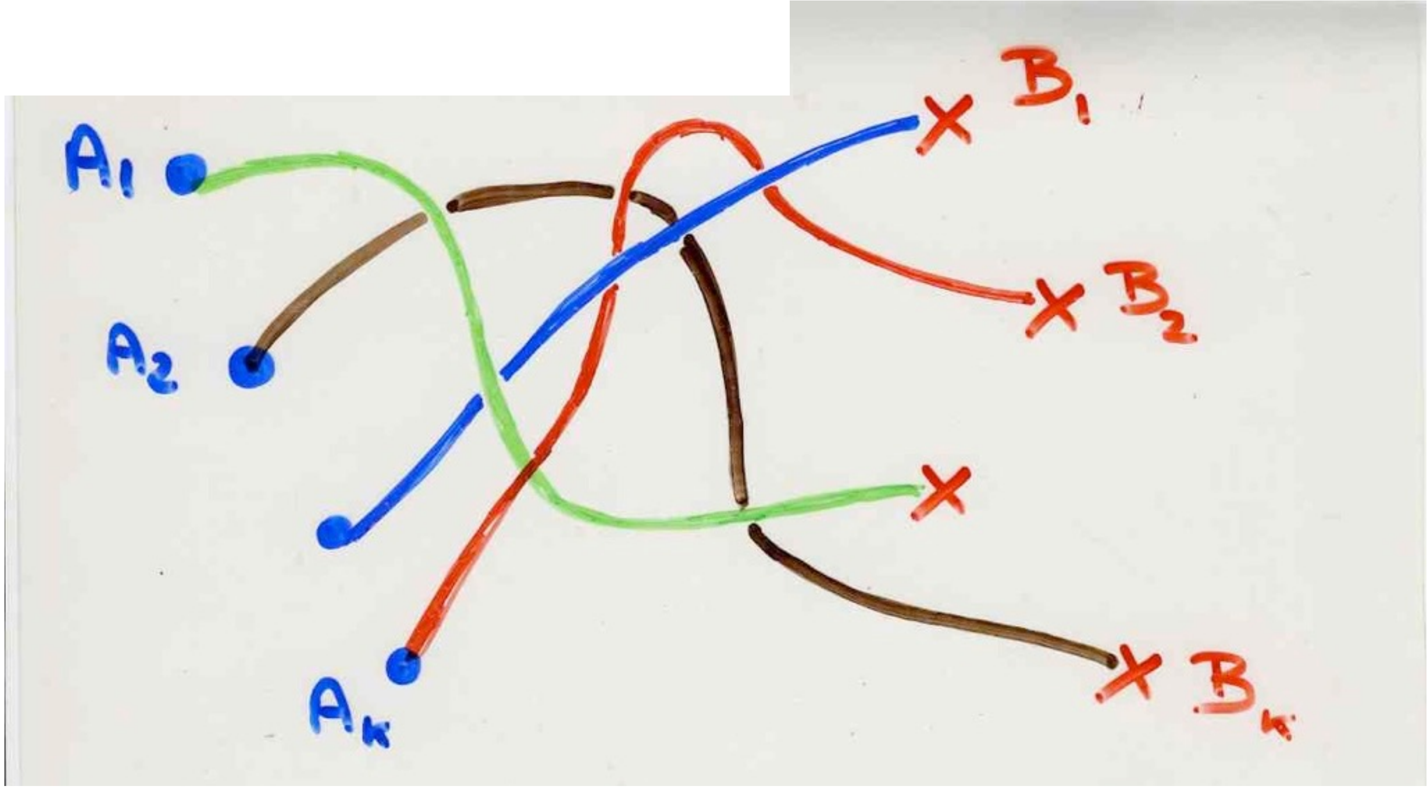
Proposition (LGV Lemma) (C) crossing condition

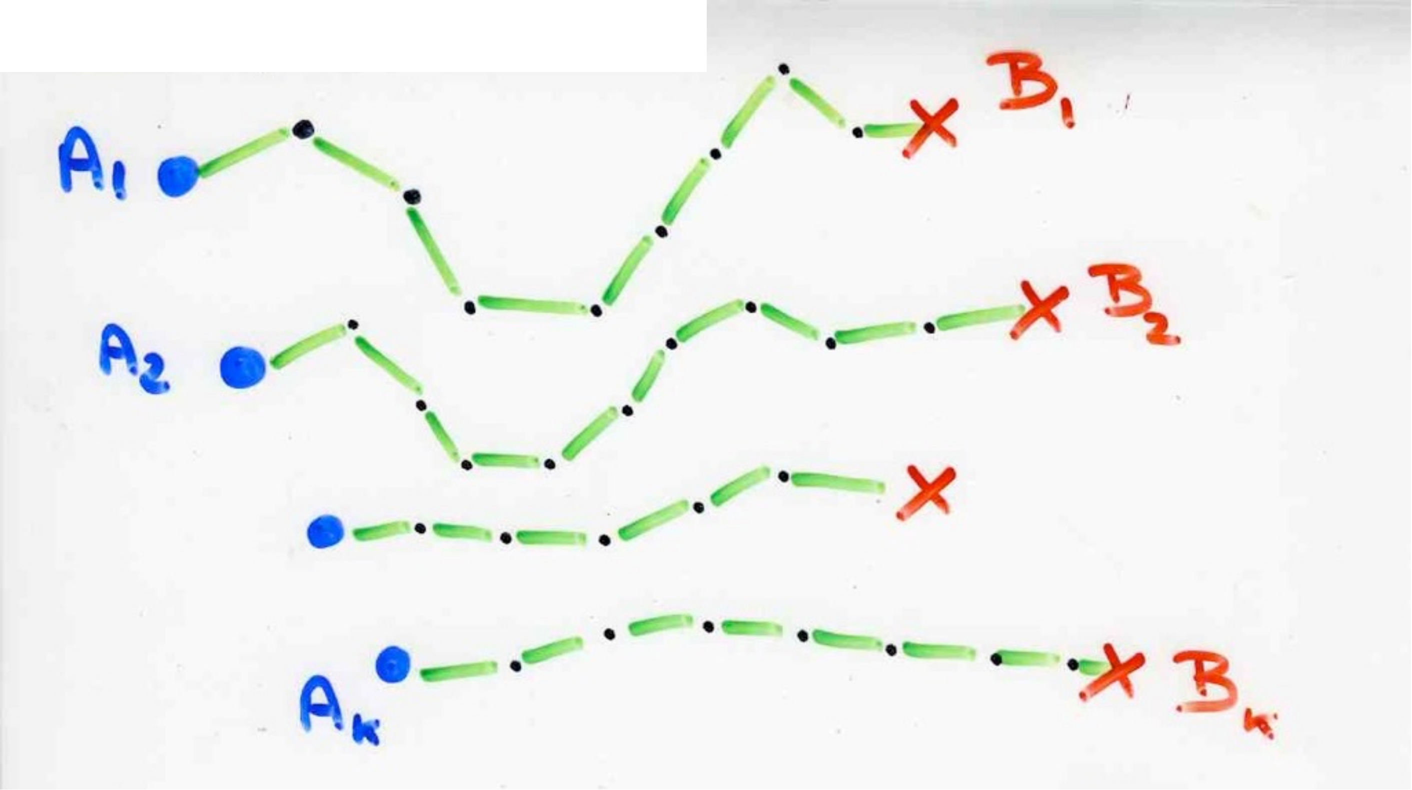
$$\det(a_{ij}) = \sum_{(\omega_1, \ldots, \omega_k)} v(\omega_1) \ldots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

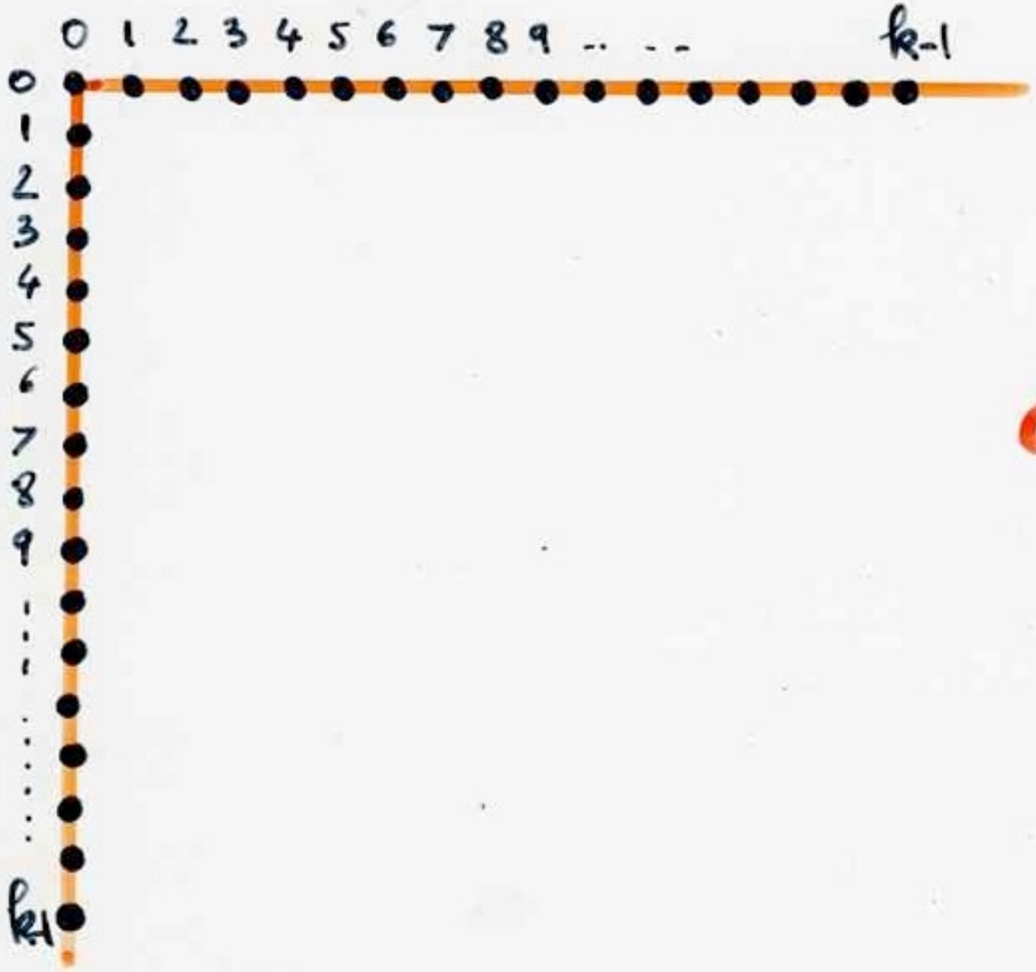
non-intersecting



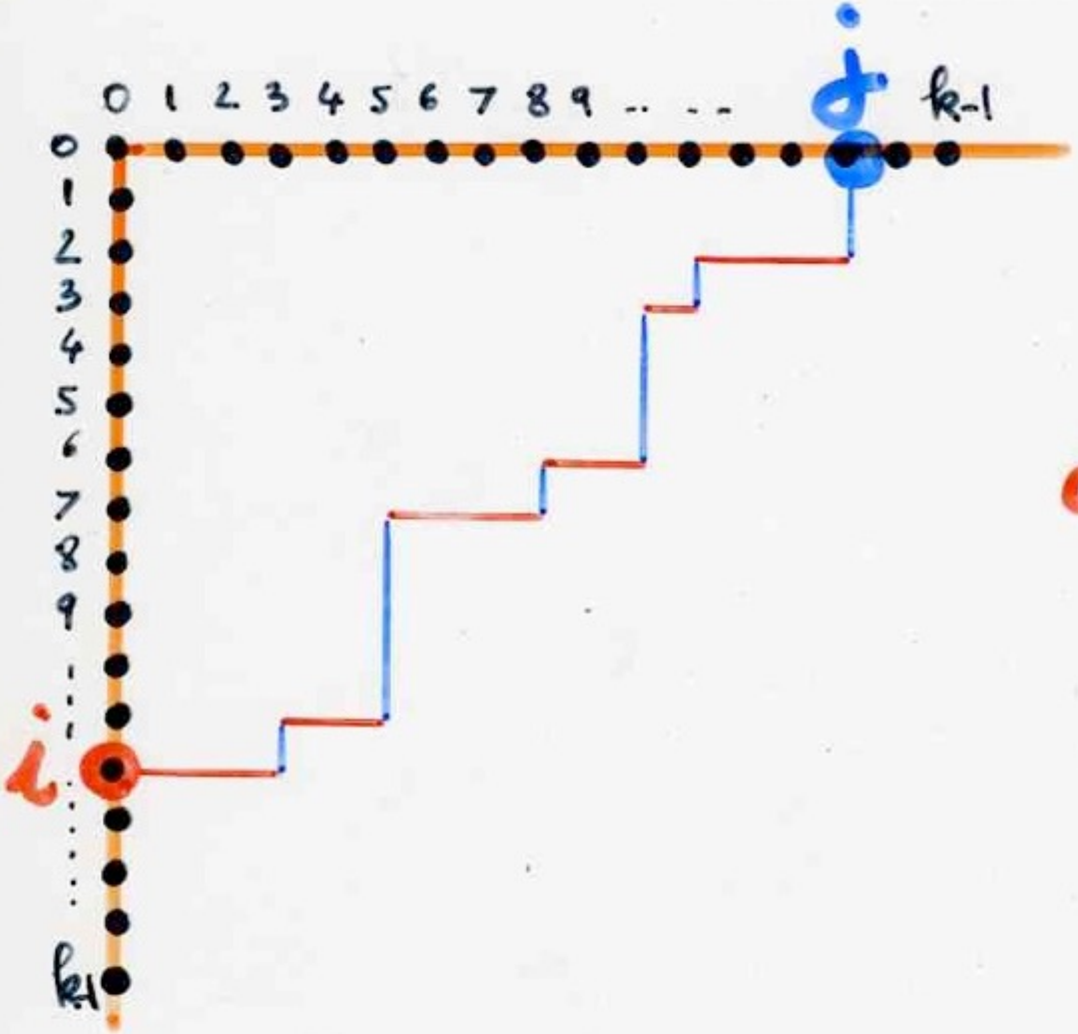

A1
A2
Ak
B1
B2
Bk


A1
B1
A2
B2
Ak
Bk

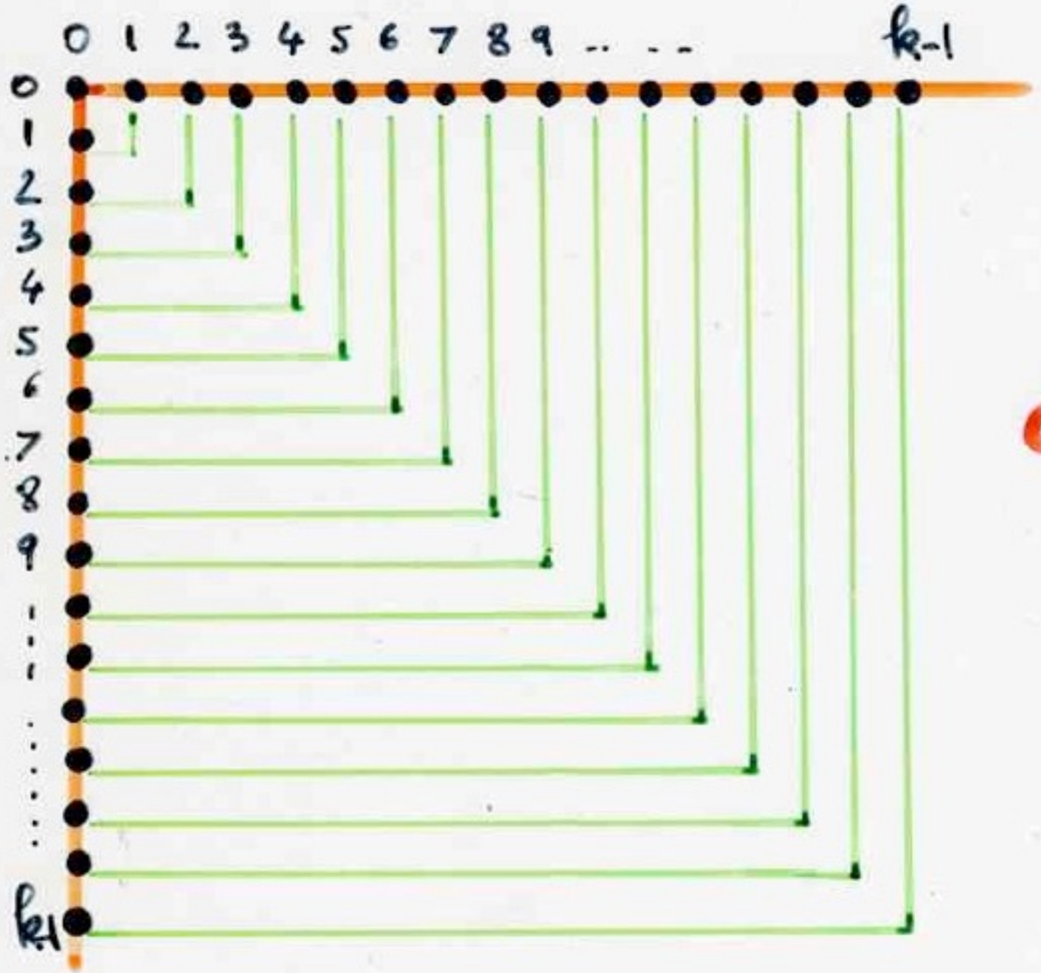
a simple example



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ 1 & 2 & 3 & 4 & 5 & \cdot & \\ 1 & 3 & 6 & 10 & \cdot & & \\ 1 & 4 & 10 & \cdot & & & \\ 1 & 5 & \cdot & & & & \\ 1 & \cdot & & & \binom{i+j}{i} & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \end{bmatrix}_{k\times k} = 1$$

proof of LGV Lemma

Proof: Involution ϕ

$$E = \left\{ (\sigma ; (\omega_1, \dots, \omega_k)) \, ; \, \begin{array}{l} \sigma \in S_n \\ \omega_i : A_i \rightsquigarrow B_{\sigma(i)} \end{array} \right\}$$

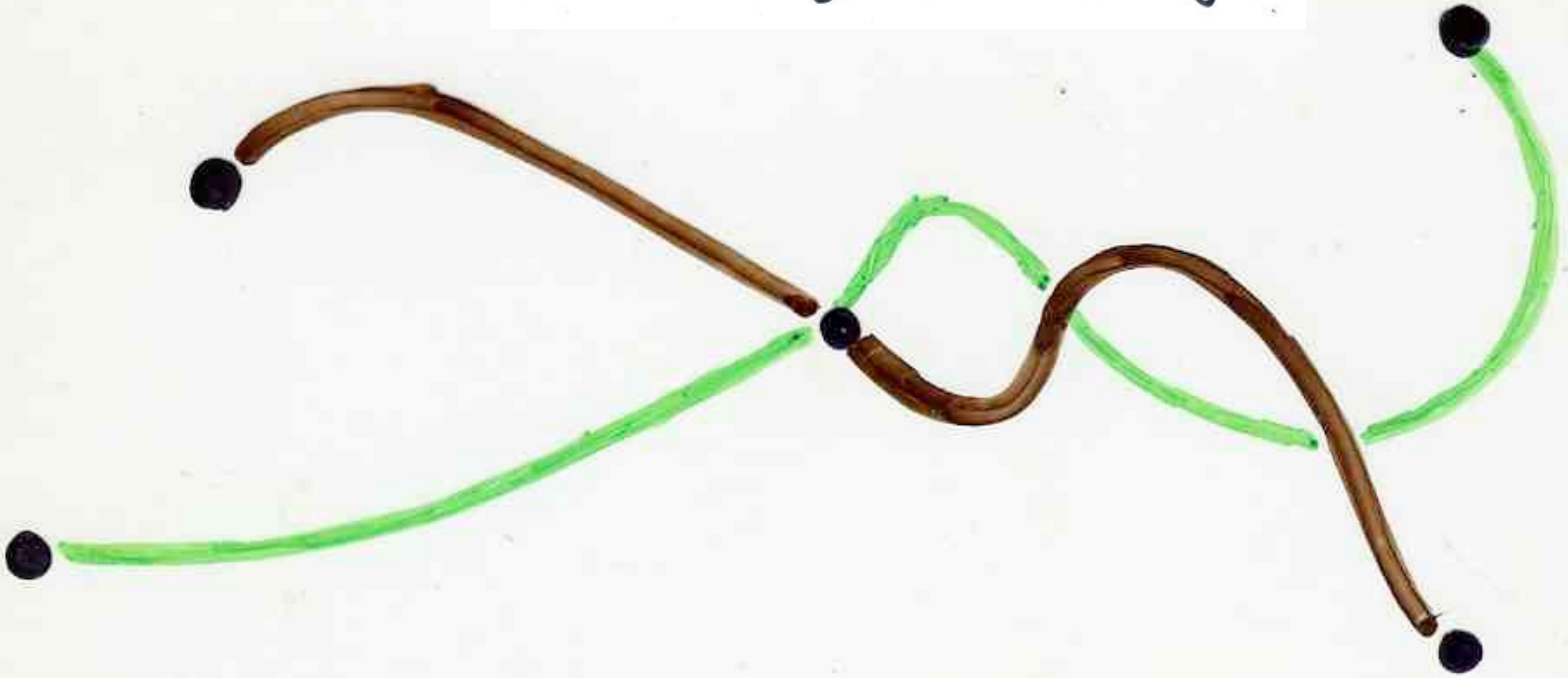
$NC \subseteq E$ non-crossing configurations

$$\phi : (E - NC) \longrightarrow (E - NC)$$

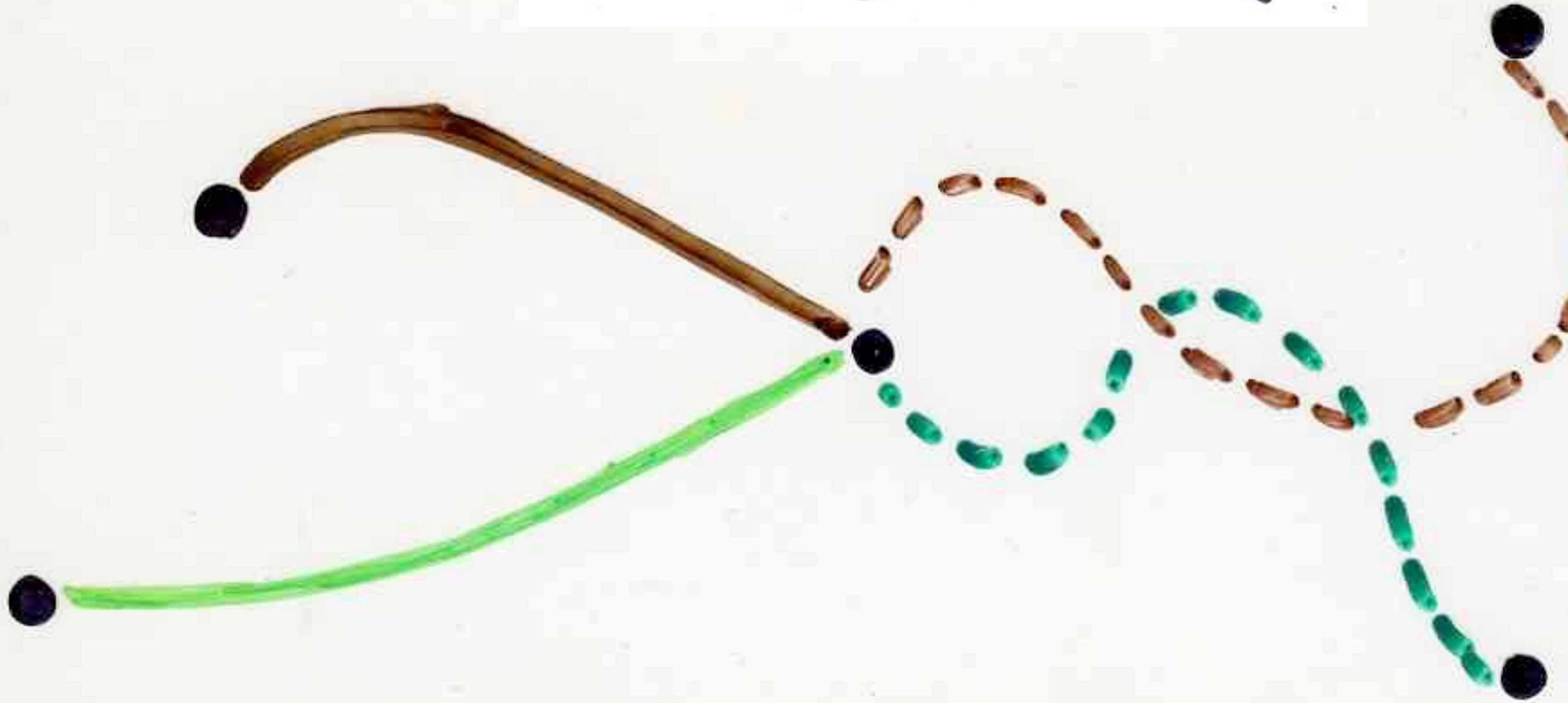
$$\phi(\sigma ; (\omega_1, \dots, \omega_k)) = (\sigma' ; (\omega'_1, \dots, \omega'_k))$$

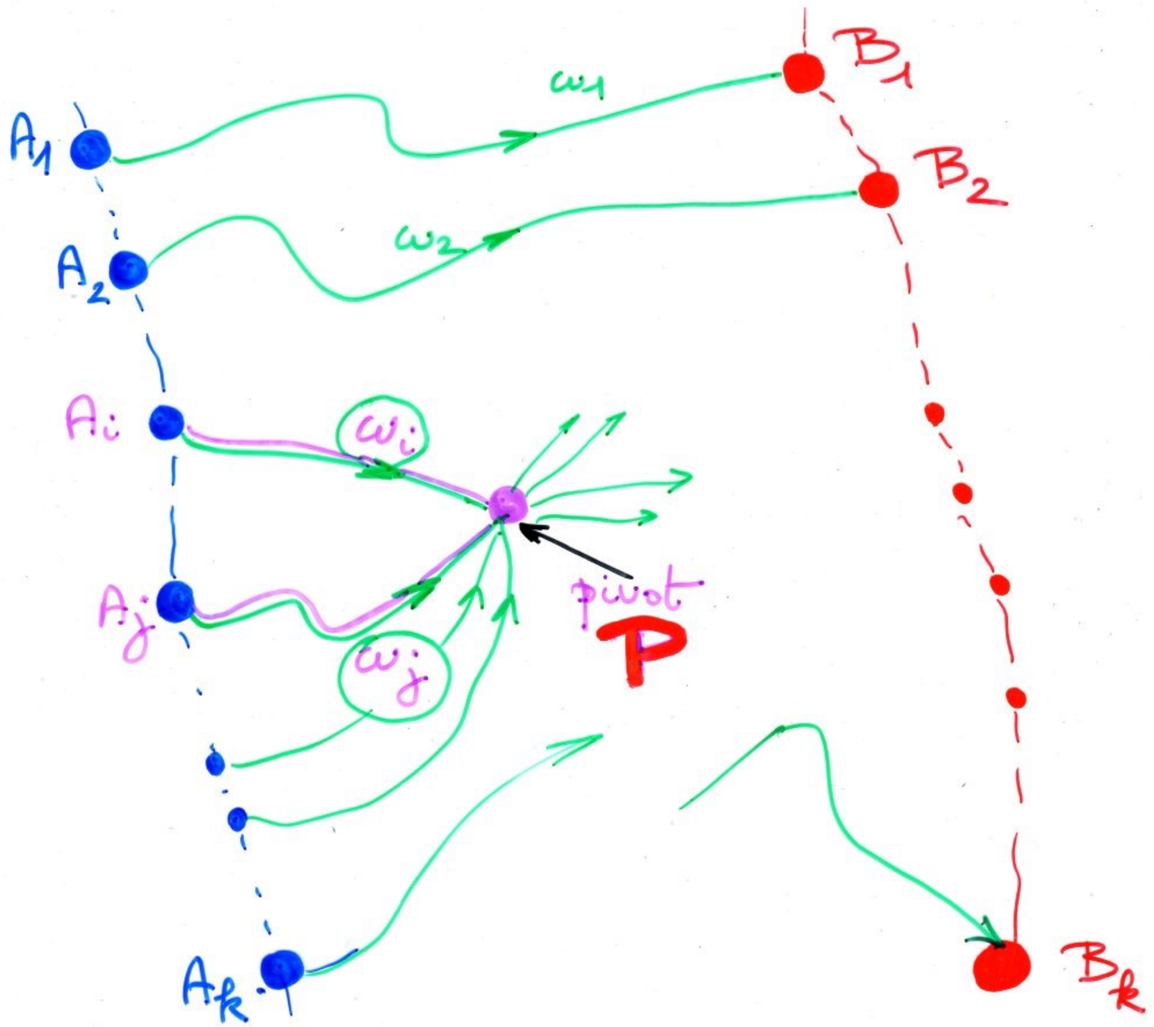
$$\left\{ \begin{array}{l} (-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')} \\ v(\omega_1) \dots v(\omega_k) = v(\omega'_1) \dots v(\omega'_k) \end{array} \right.$$

idea of the proof:



idea of the proof:

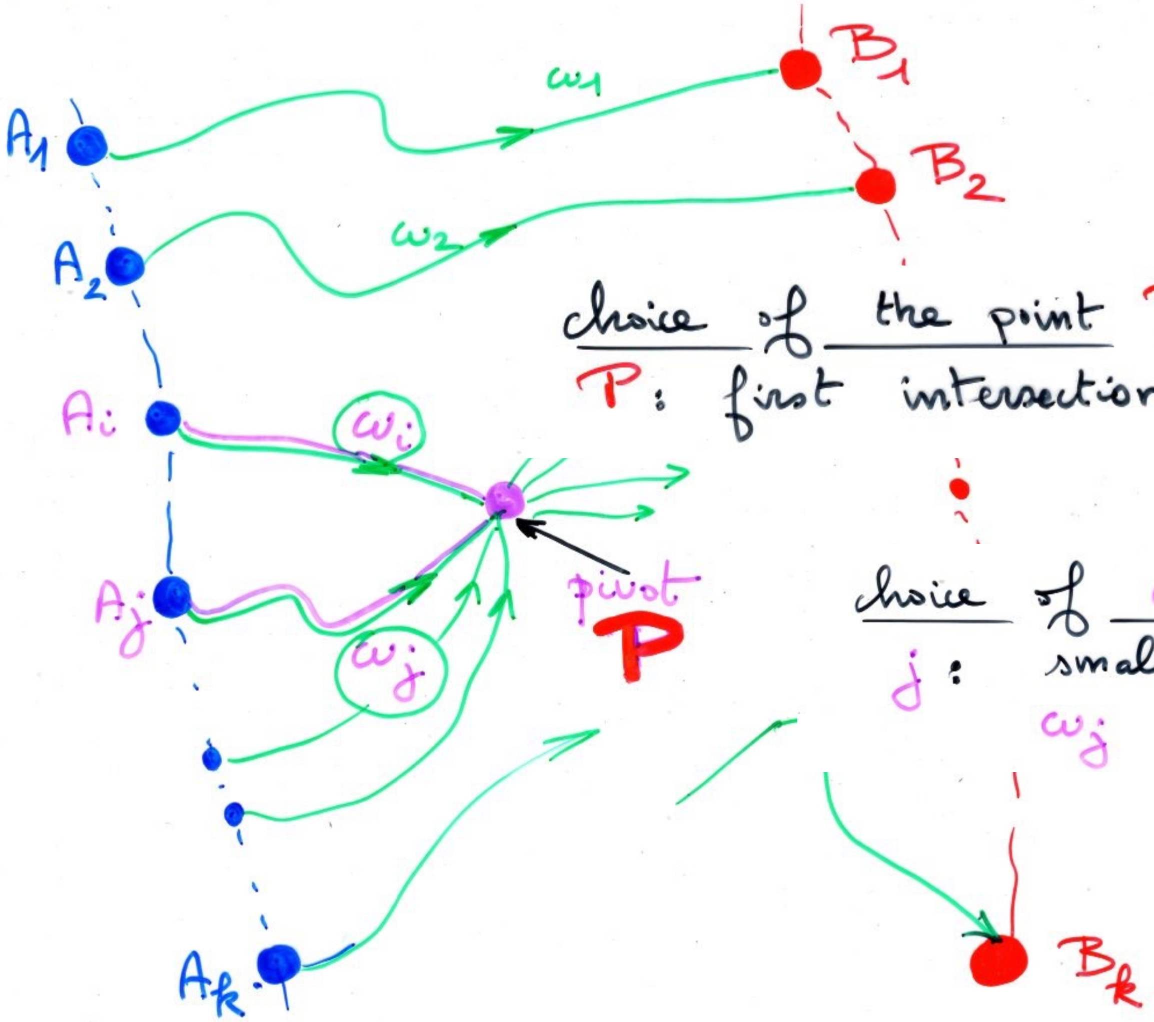




A_1
A_2
A_i
A_j
A_k
B_1
B_2
B_k
ω_1
ω_2
ω_i
ω_j
pivot
P

choice of ω_i

i : smallest i, $1 \leq i \leq k$, such that ω_i has an intersection with another path



choice of the point P

P : first intersection point on the path ω_i

choice of ω_j

j : smallest j, $i < j \leq k$ such that ω_j intersect ω_i

LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)} v(\omega_1) \dots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting

Proposition (LGV Lemma)

We consider weighted paths $\omega = (s_0, \ldots, s_n)$ in a set S with weight defined by the valuation $v : S \times S \to \mathbb{K}$ commutative ring.

$$v(\omega) = v(s_0, s_1) \ldots v(s_{n-1}, s_n)$$

Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be elements of S.

For $1 \leq i, j \leq k$ define $a_{ij} = \sum\limits_{\substack{\omega \\ A_i \rightsquigarrow B_j}} v(\omega)$

(we suppose that this sum is finite)

We assume that the crossing condition $(\mathbf{C})$ is satisfied.

Then

$$\det(a_{ij}) = \sum_{\substack{(\omega_1, \ldots, \omega_k) \\ \omega_i : A_i \rightsquigarrow B_i \\ \text{non-intersecting}}} v(\omega_1) \ldots (\omega_k)$$

Lattice paths and determinants

Chapter 29

Why « LGV **Lemma** » ?

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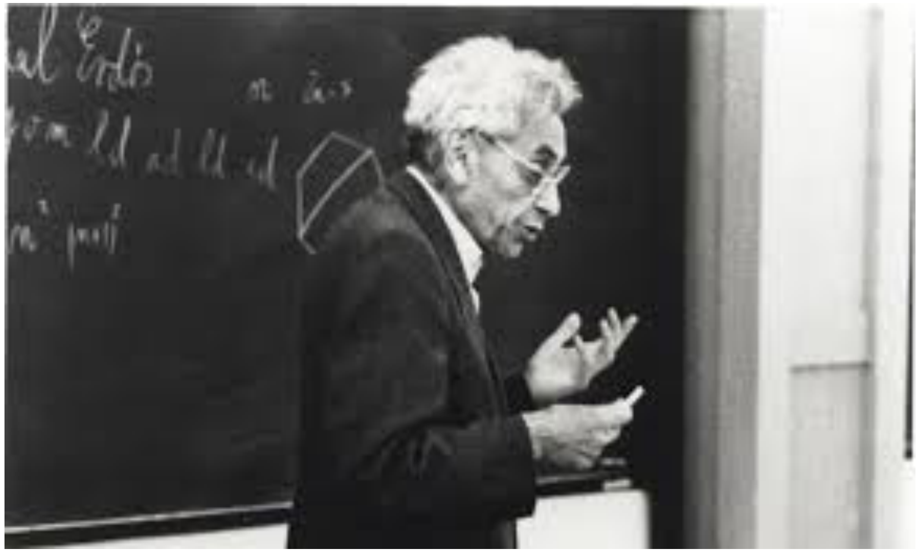
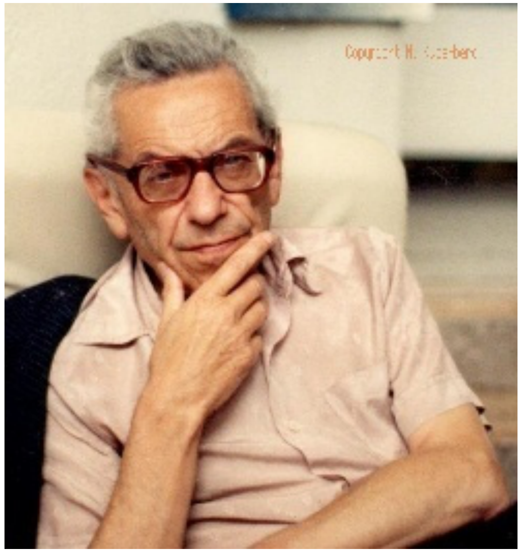
Fifth Edition
Springer



Paul Erdös liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems,

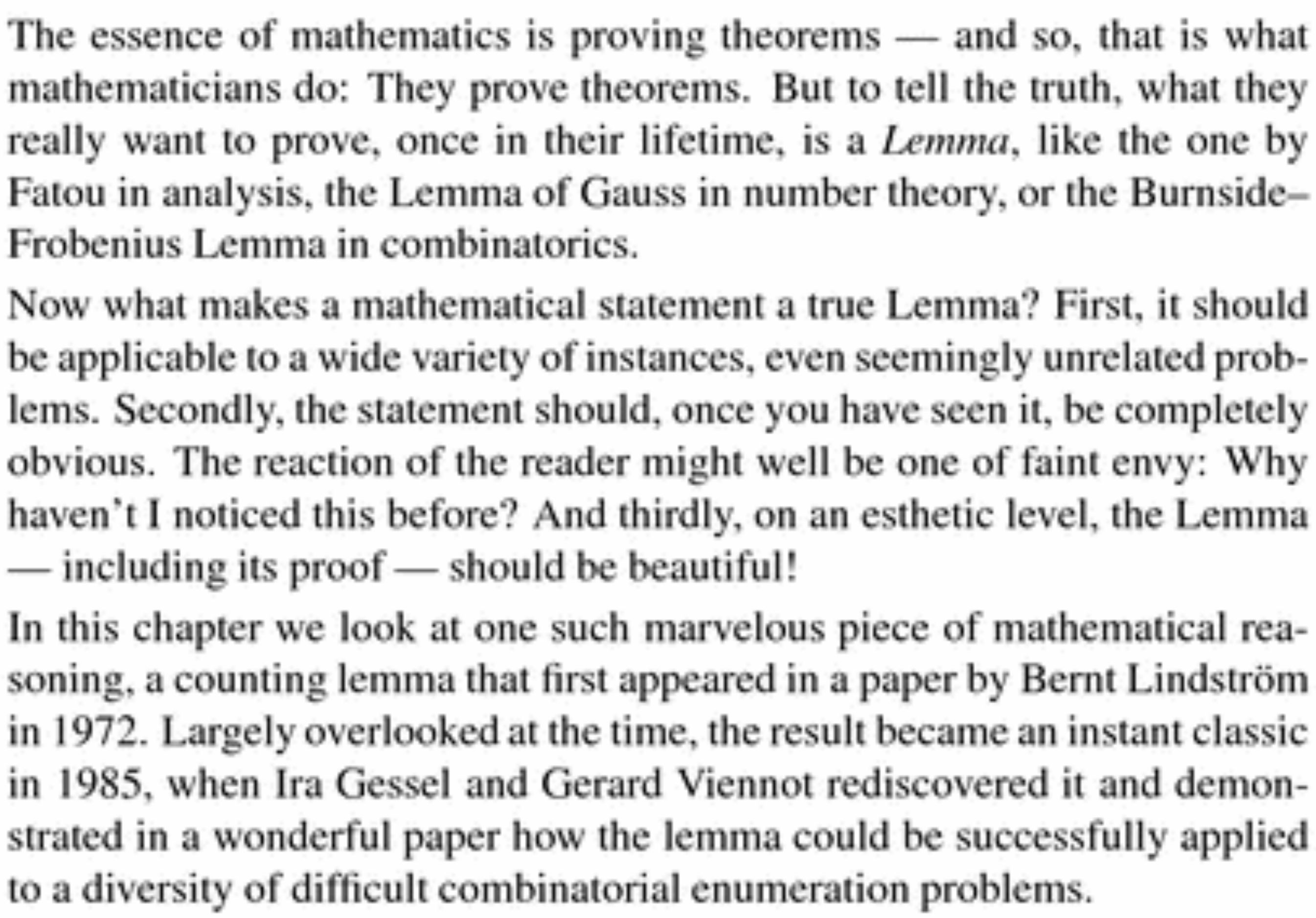


Erdös also said that you need not believe in God but, as a mathematician, you should believe in The Book.

Lattice paths and determinants

Chapter 29

Why « LGV **Lemma** » ?



The essence of mathematics is proving theorems — and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside–Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma — including its proof — should be beautiful!

In this chapter we look at one such marvelous piece of mathematical reasoning, a counting lemma that first appeared in a paper by Bernt Lindström in 1972. Largely overlooked at the time, the result became an instant classic in 1985, when Ira Gessel and Gerard Viennot rediscovered it and demonstrated in a wonderful paper how the lemma could be successfully applied to a diversity of difficult combinatorial enumeration problems.

Martin Aigner
Günter M. Ziegler
Proofs from THE BOOK
Fifth Edition
Springer

Why « **LGV** Lemma » ?

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

[4] Lindström used the term “pairwise node disjoint paths”. The term “non-intersecting,” which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

[5] By a curious coincidence, Lindström’s result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling’s bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the “Lindstrom–Gessel–Viennot theorem.” It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called “Slater determinant” in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an “ancestor” of the Lindström–Gessel–Viennot determinant.

[6] There exist however also several interesting applications of the general form of the Lindström–Gessel–Viennot theorem in the literature, see [10, 16, 51].

combinatorics

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

I. Gessel and X.G.V., *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

I. Gessel and X.G.V., *Determinants, paths and plane partitions, preprint (1989)*

statistical physics: (wetting, melting)

Fisher, Vicious walkers, Botzmann lecture (1984)

combinatorial chemistry:

John, Sachs (1985)

Gronau, Just, Schade, Scheffler, Wojciechowski (1988)

probabilities, birth and death process,

Karlin , McGregor (1959)

quantum mechanics: Slater determinant

Slater(1929) (1968), De Gennes (1968)

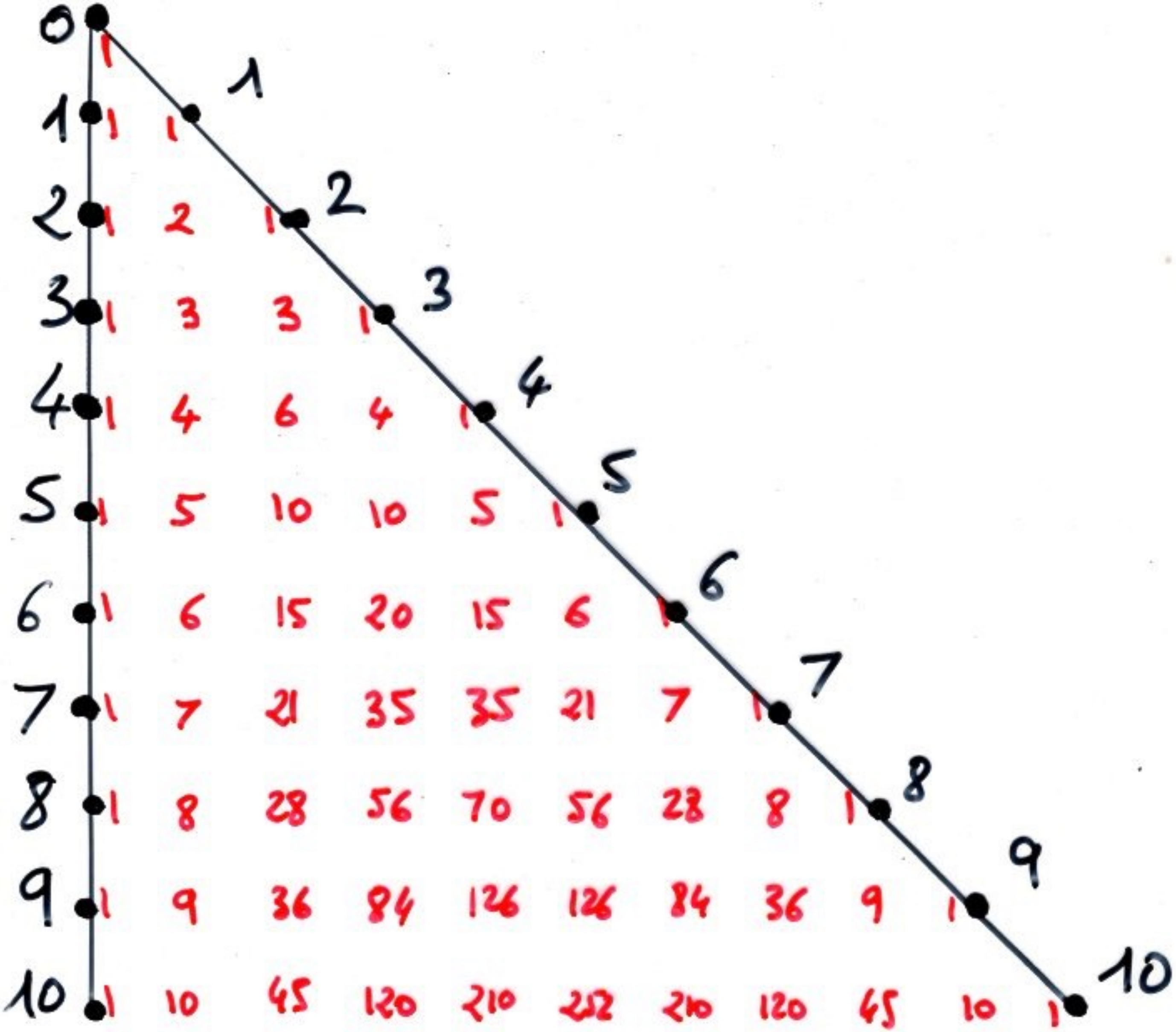
Binomial determinants

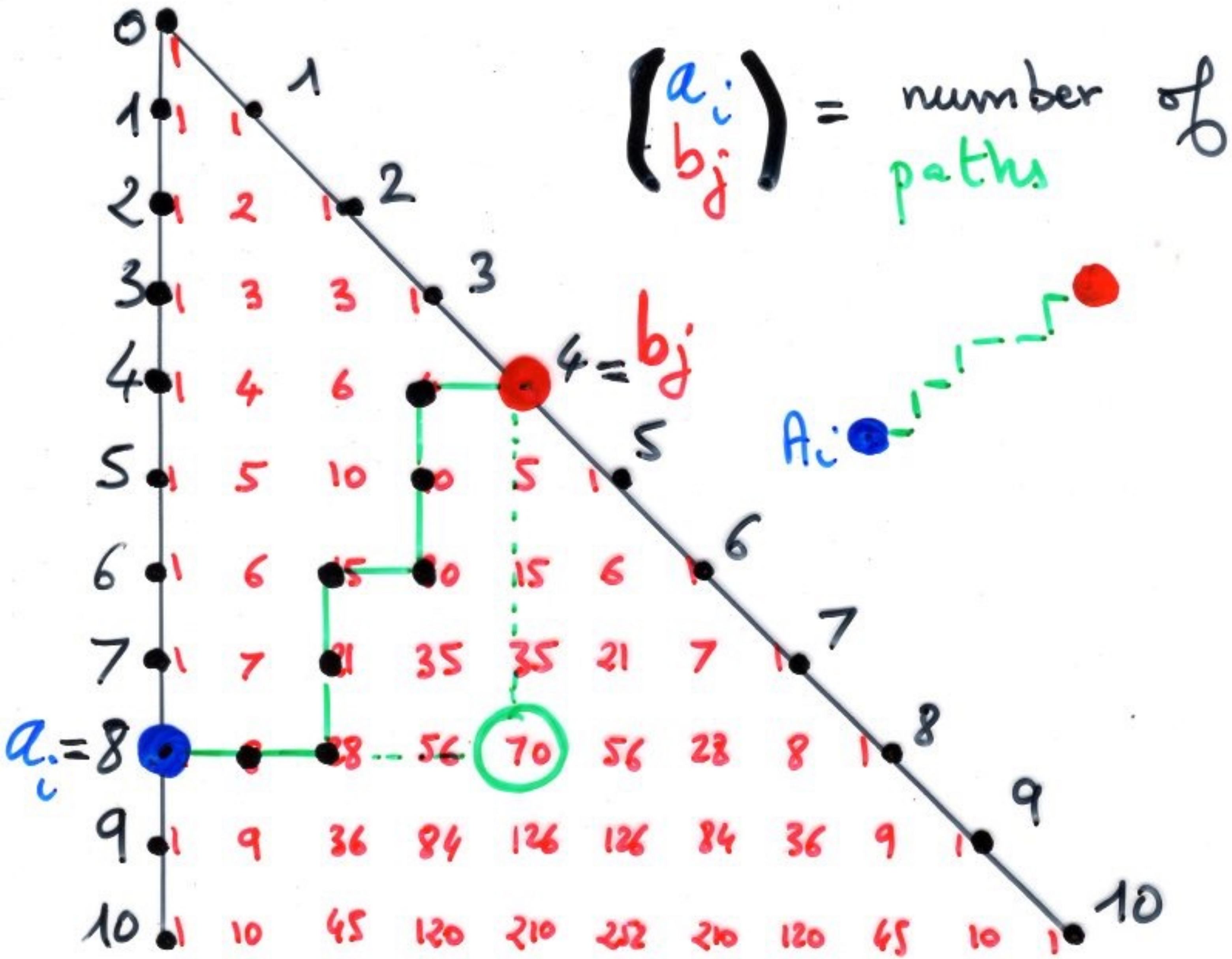
$0 \leq a_1 < \cdots < a_k$

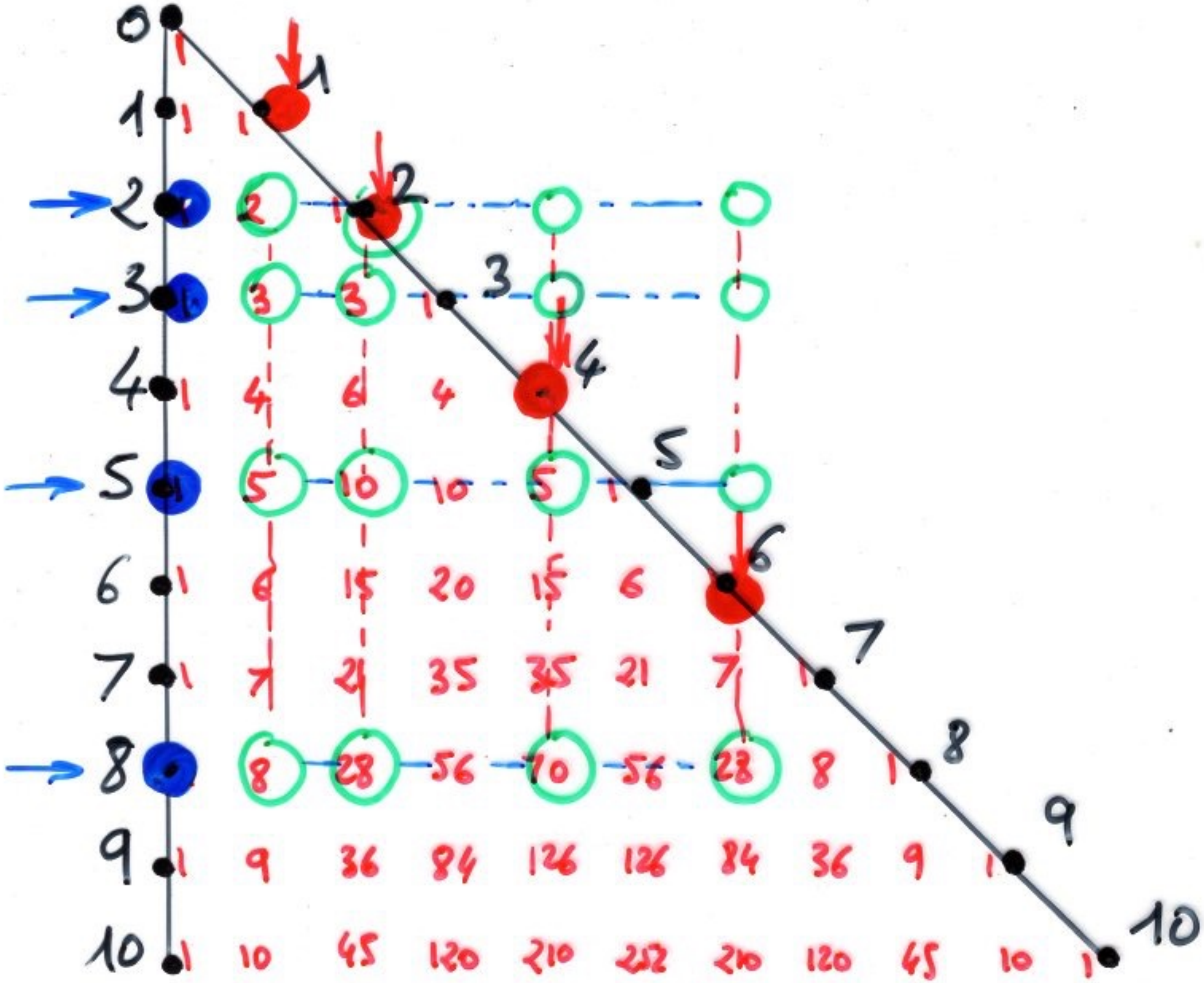
$0 \leq b_1 < \cdots < b_k$

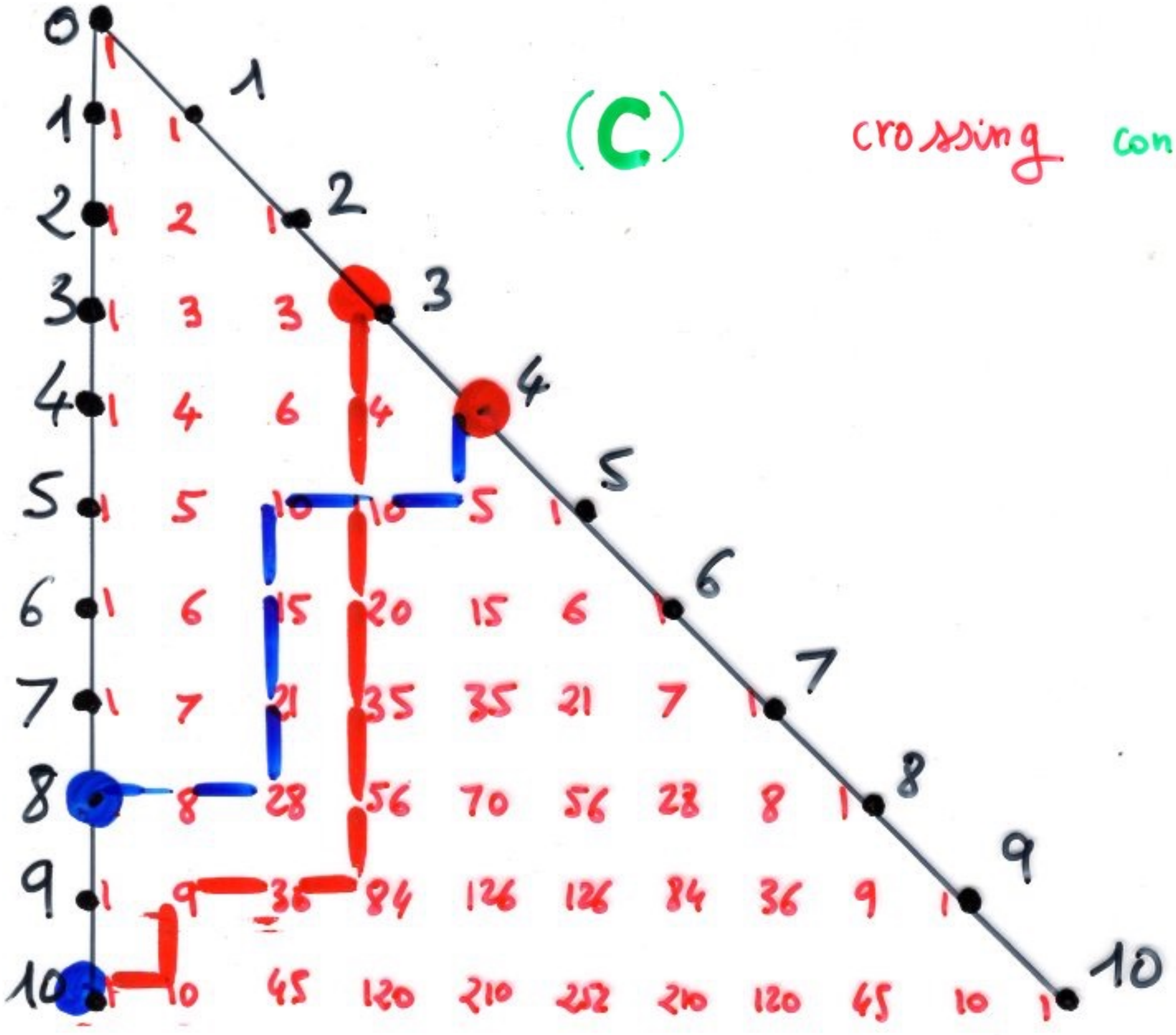
$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k}$$

$$= \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \leq \overset{i}{\underset{j}{\cdot}} \leq k}$$

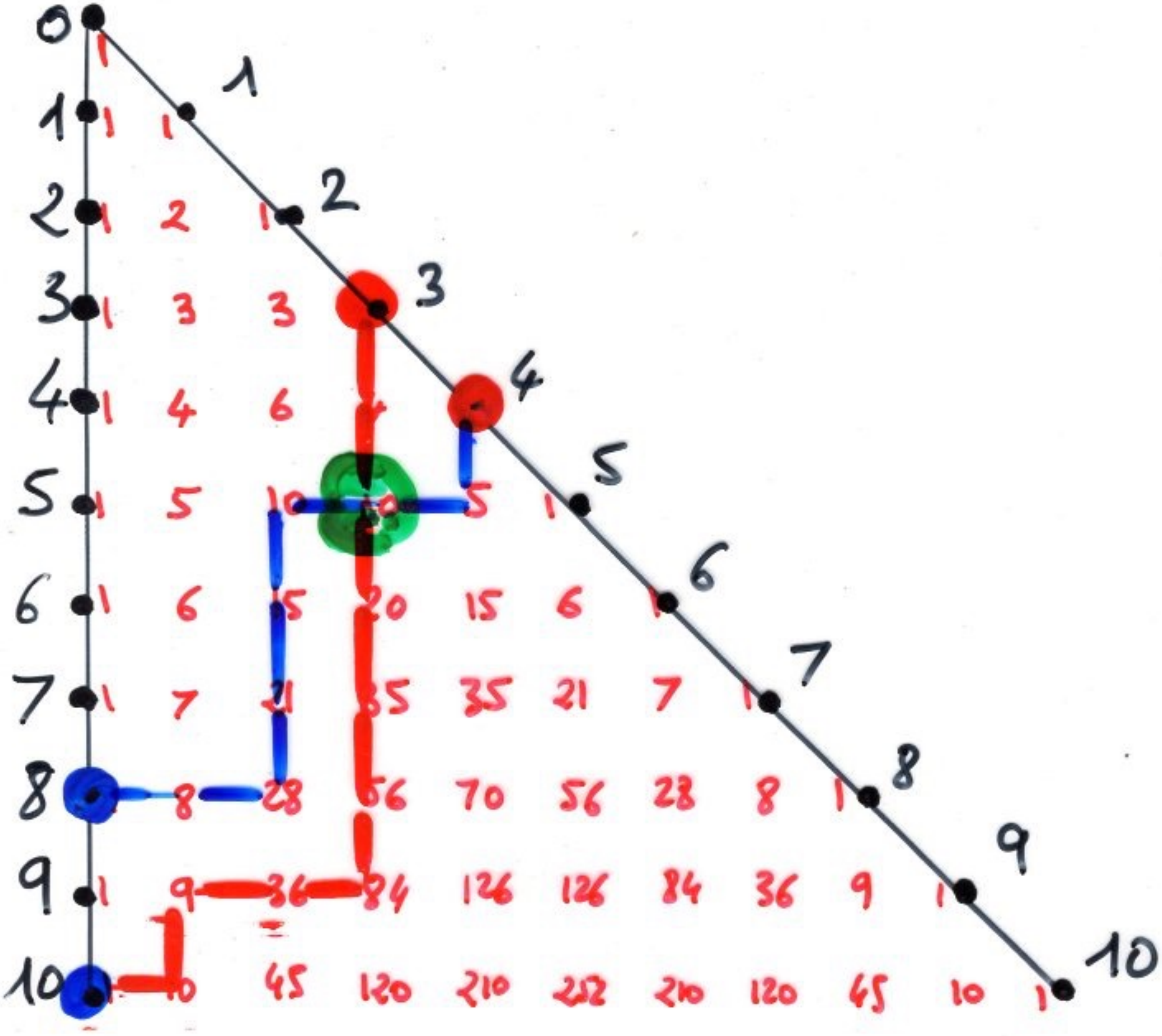

0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1
2
3
4
5
6
7
8
9
10

$\binom{a_i}{b_j}$ = number of paths
0
1
2
3
4
5
6
7
8
9
10
$a_i = 8$
$4 = b_j$
A_i
1
1 1
1 2 1
1 3 3 1
1 4 6
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1


0 1
1 1 1 1
2 2 1 2
3 3 3 1 3
4 1 4 6 4 4
5 5 10 10 5 1 5
6 1 6 15 20 15 6 6
7 1 7 21 35 35 21 7 1 7
8 8 28 56 70 56 28 8 1 8
9 1 9 36 84 126 126 84 36 9 1 9
10 1 10 45 120 210 252 210 120 45 10 1 10



(C) crossing condition


0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3
1 4 6 4
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1
2
3
4
5
6
7
8
9
10

Proposition The binomial determinant

$$\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix}$$

is the number of configurations of non-intersecting paths

$(\omega_1, \dots, \omega_k)$, $\quad \omega_i : A_i \rightsquigarrow B_j$,

$A_i = (0, a_i)$, $\quad B_j = (b_j, b_j)$

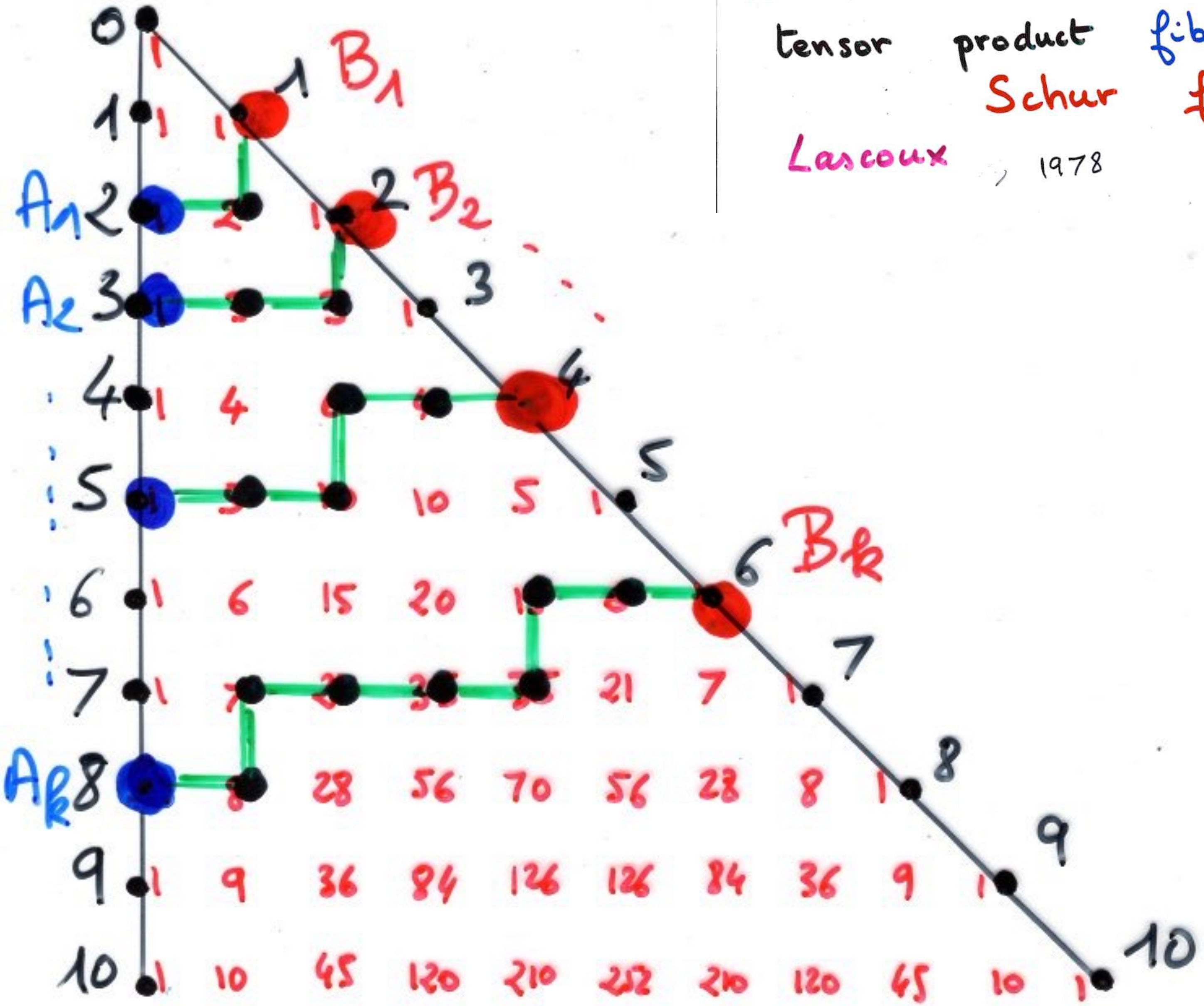
with elementary steps $\uparrow N$, $\underset{E}{\longrightarrow}$

Chern classes calculus

tensor product fiber bundles

Schur functions

Lascoux 1978

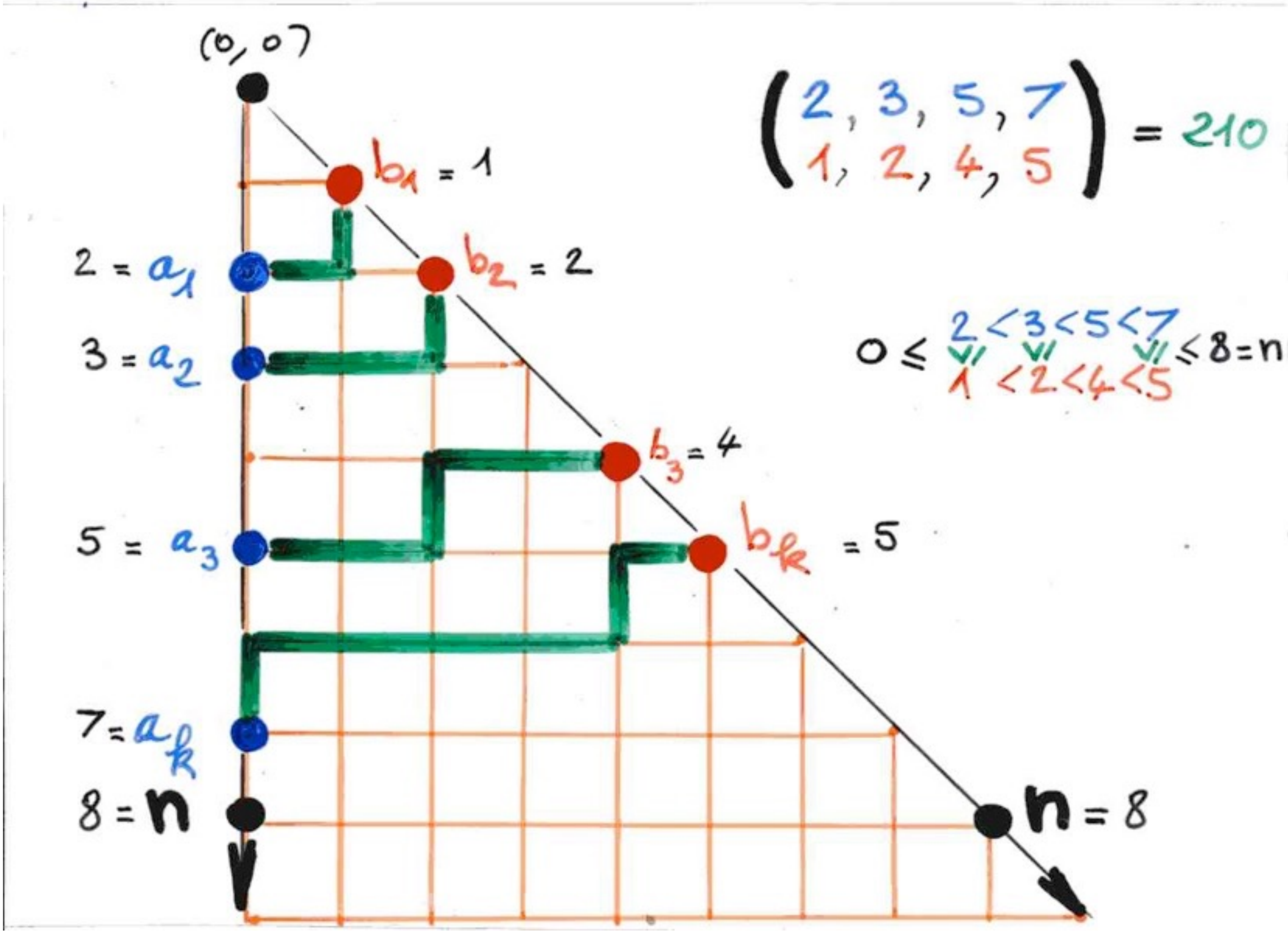


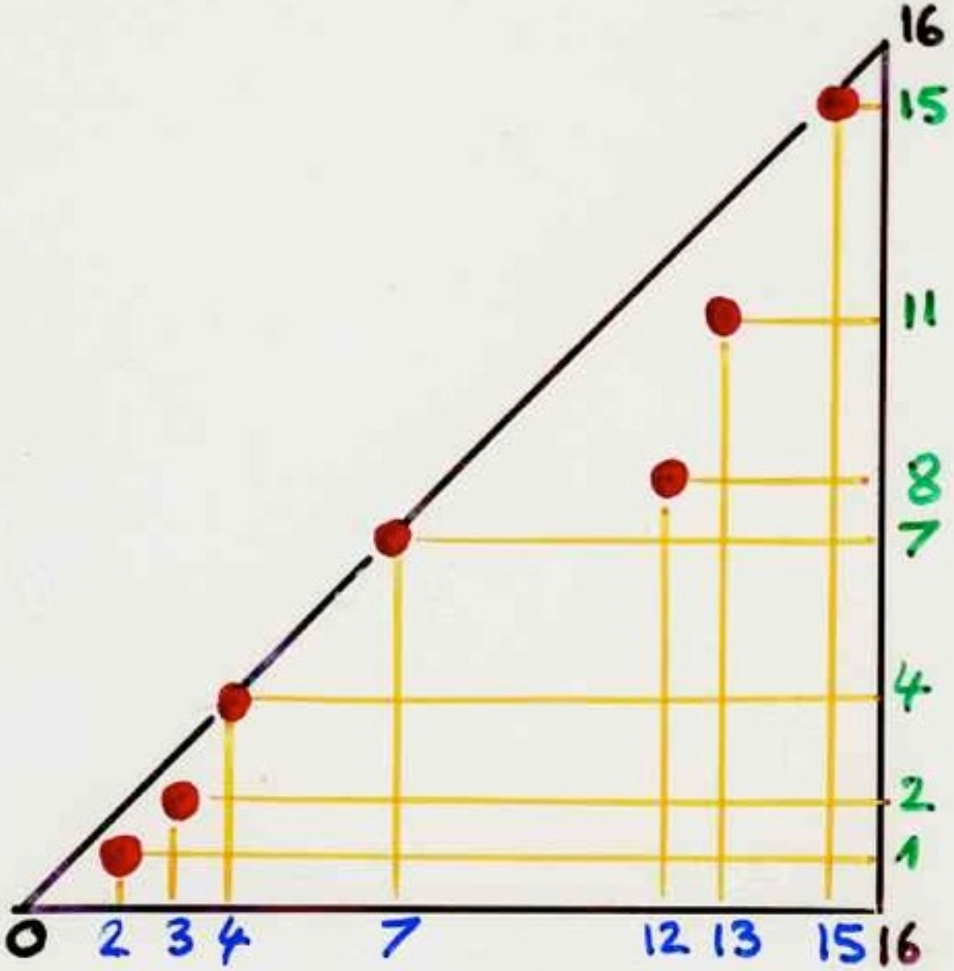
Cor 1 - $\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix} \geqslant 0$

Cor 2 - Nb of nonzero minors of $A_n = \left[\binom{i}{j}\right]_{0 \le i,j \le n}$ is C_{n+2}

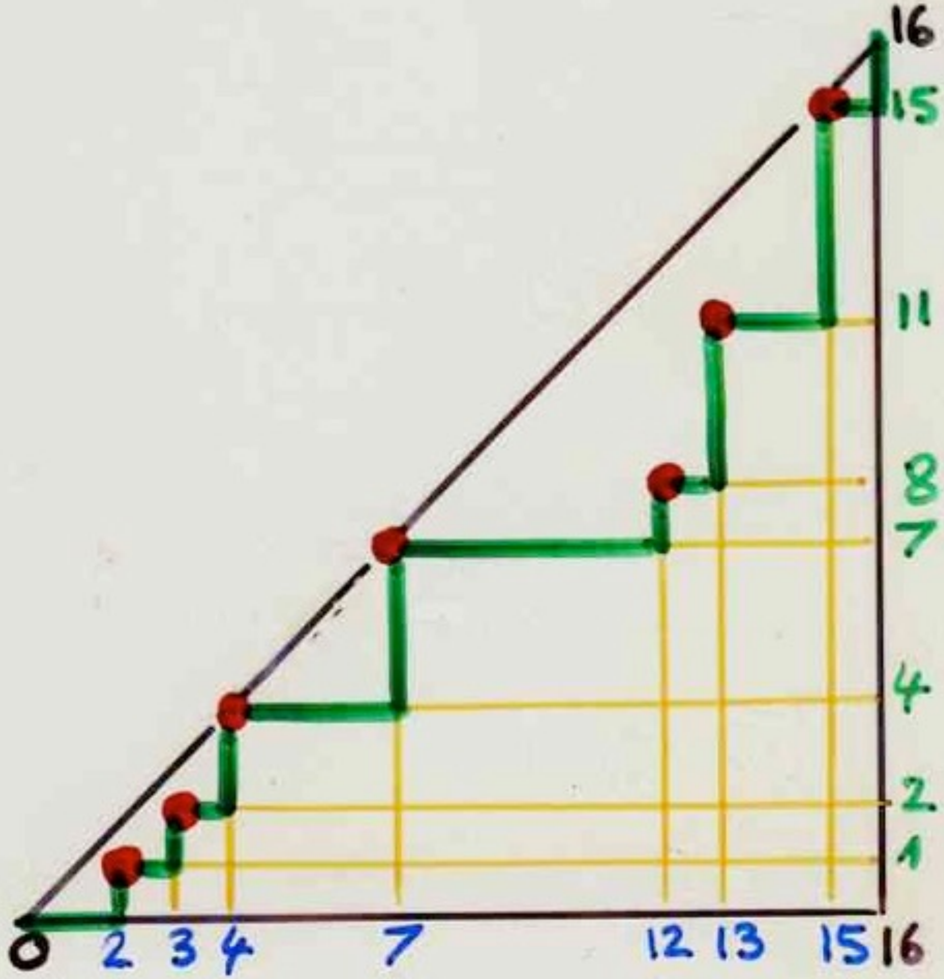
Catalan nb

$$0 \leqslant \begin{matrix} a_1 < \dots < a_k \\ b_1 < \dots < b_k \end{matrix} \leqslant n$$


(0, 0)
$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$
$b_1 = 1$
$2 = a_1$
$b_2 = 2$
$3 = a_2$
$0 \le \begin{matrix} 2 < 3 < 5 < 7 \\ \text{\rotatebox{90}{$\le$}} \quad \text{\rotatebox{90}{$\le$}} \quad \text{\rotatebox{90}{$\le$}} \\ 1 < 2 < 4 < 5 \end{matrix} \le 8 = n$
$b_3 = 4$
$5 = a_3$
$b_k = 5$
$7 = a_k$
$8 = n$
$n = 8$



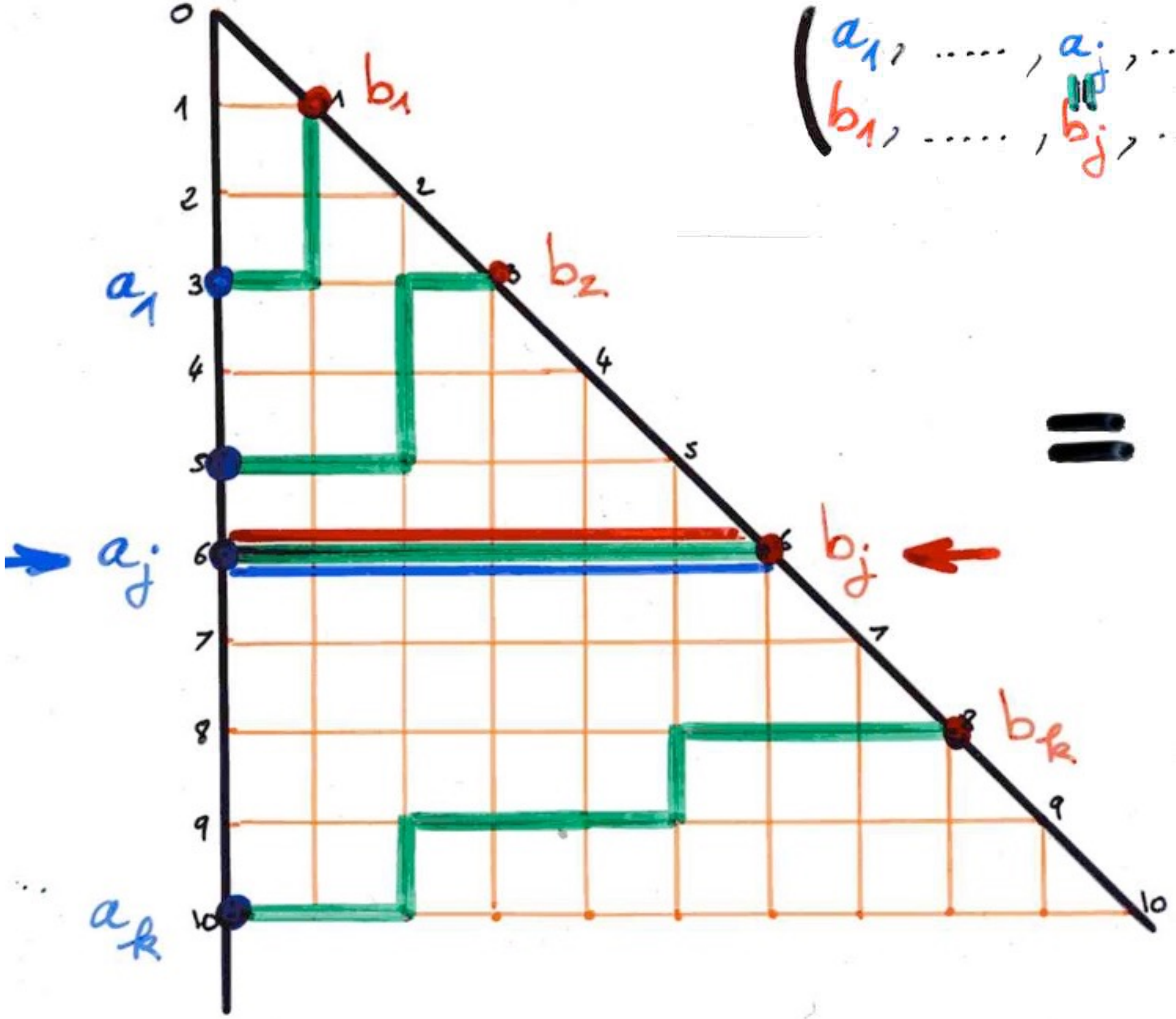
$$\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}$$



$$\begin{array}{ccccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{array}$$

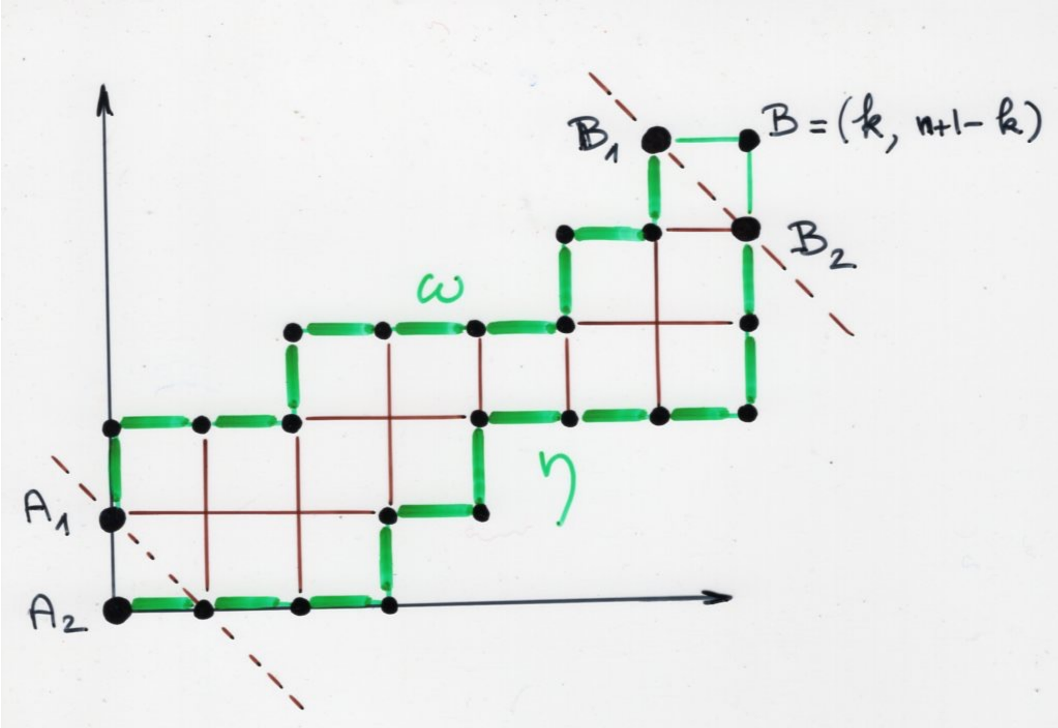
Cor 3. If $a_j = b_j$

$$\begin{pmatrix} a_1, & \dots, & a_j, & \dots, & a_k \\ & & \| & & \\ b_1, & \dots, & b_j, & \dots, & b_k \end{pmatrix}$$



$$= \begin{pmatrix} a_1, \dots, a_{j-1} \\ b_1, \dots, b_{j-1} \end{pmatrix} \begin{pmatrix} a_{j+1}, \dots, a_k \\ b_{j+1}, \dots, b_k \end{pmatrix}$$

example:
Naranaya numbers
and
Baxter permutations



$A_2 = (0,0)$ $A_1 = (0,1)$

$B_2 = (k, n-k)$ $B_1 = (k-1, n+1-k)$

$a_{ij} = \left| \mathrm{Pa}(A_i, B_j) \right|$ $1 \leq i, j \leq 2$

number of paths $A_i \rightsquigarrow B_j$ with elementary N, E steps

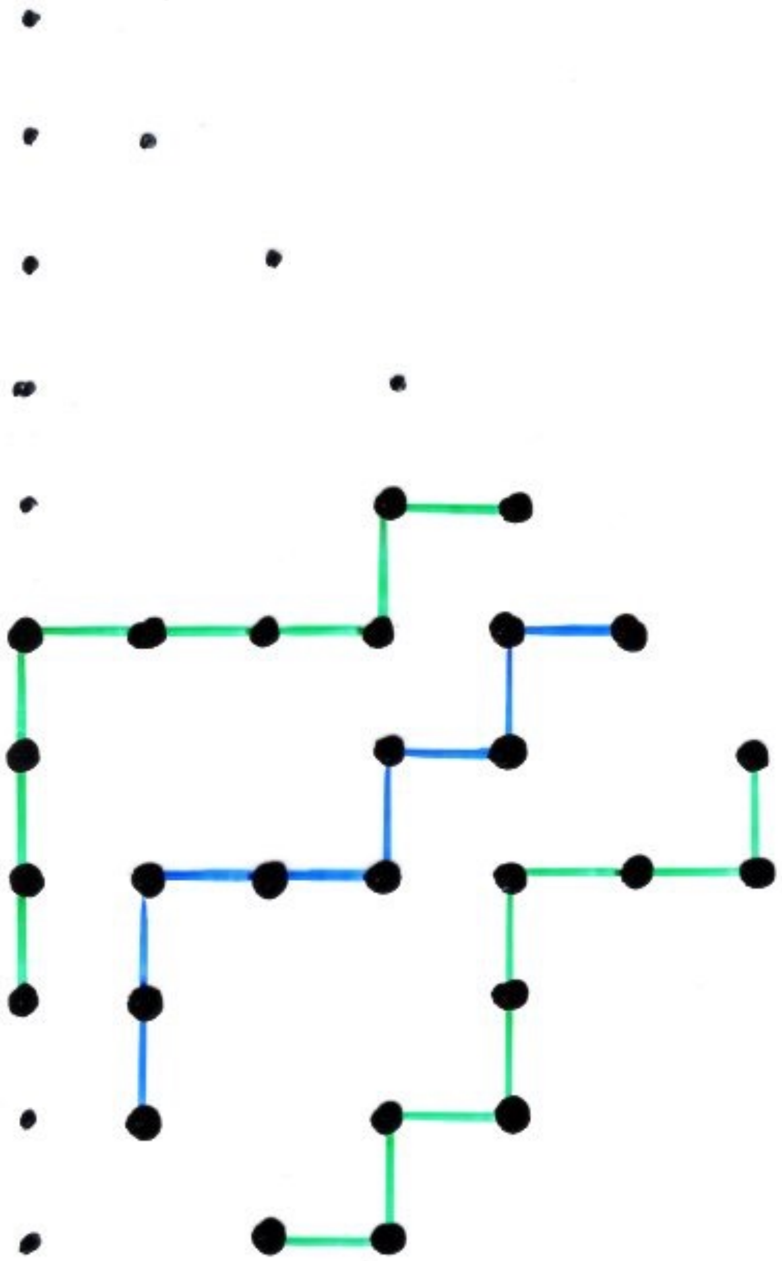
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} \\ \binom{n}{k-1} & \binom{n}{k} \end{bmatrix}$$

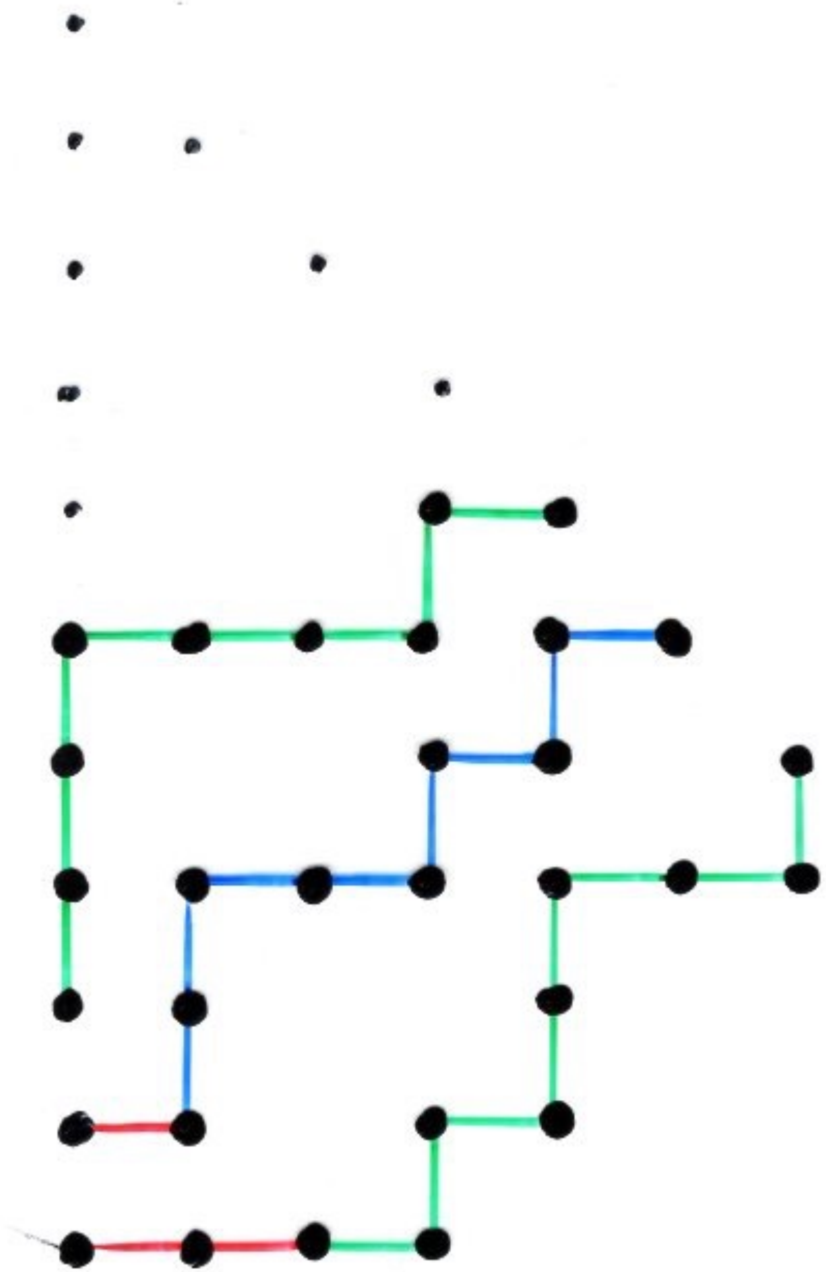
Proposition The number of pair of paths (ω, η) such that:

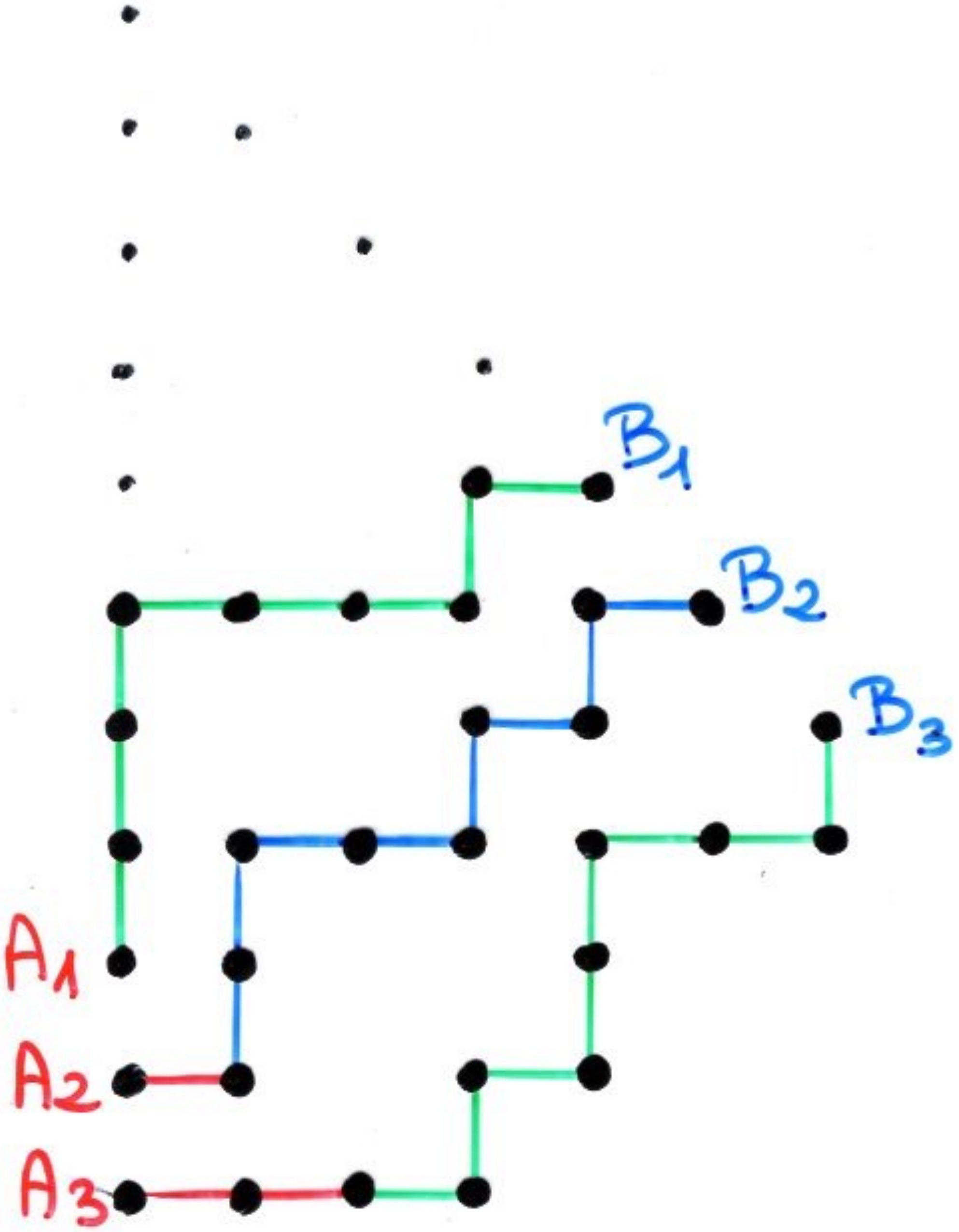
(i) • ω, η : $(0,0) \rightsquigarrow B$, elementary steps N, S

(ii) • non-intersecting (except in $(0,0)$ and B)

is the determinant : $\det(A)$

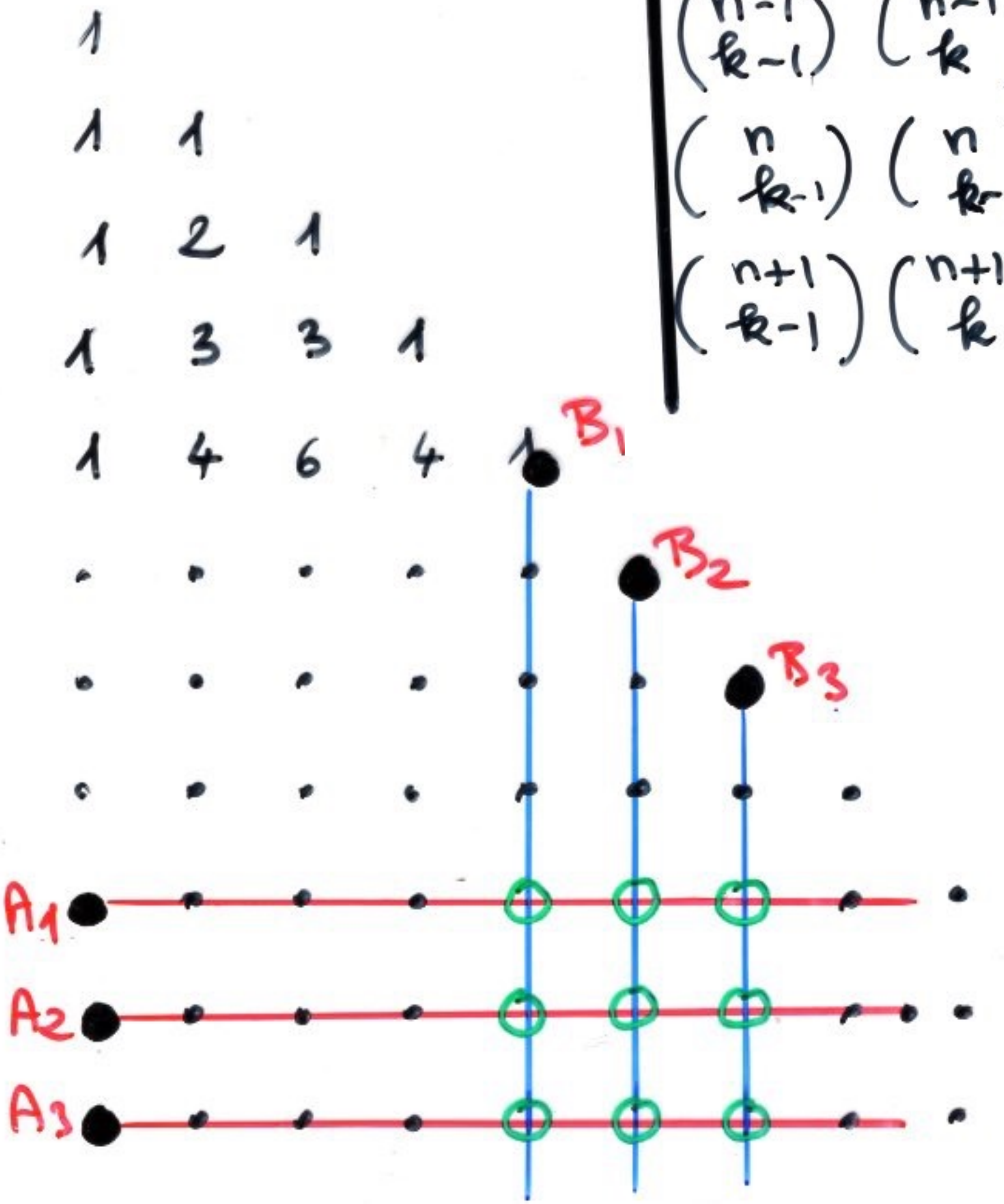






B1
B2
B3
A1
A2
A3

$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k-1} & \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$



Formulae for binomial determinant

$0 \leq a_1 < \cdots < a_k$

$0 \leq b_1 < \cdots < b_k$

$$\binom{a_1, \ldots\ldots, a_k}{b_1, \ldots, b_k}$$

$$= \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \leq i, j \leq k} = \frac{(\text{product})}{(\text{product})}$$

Lemma 1 If $b_1 \neq 0$, then

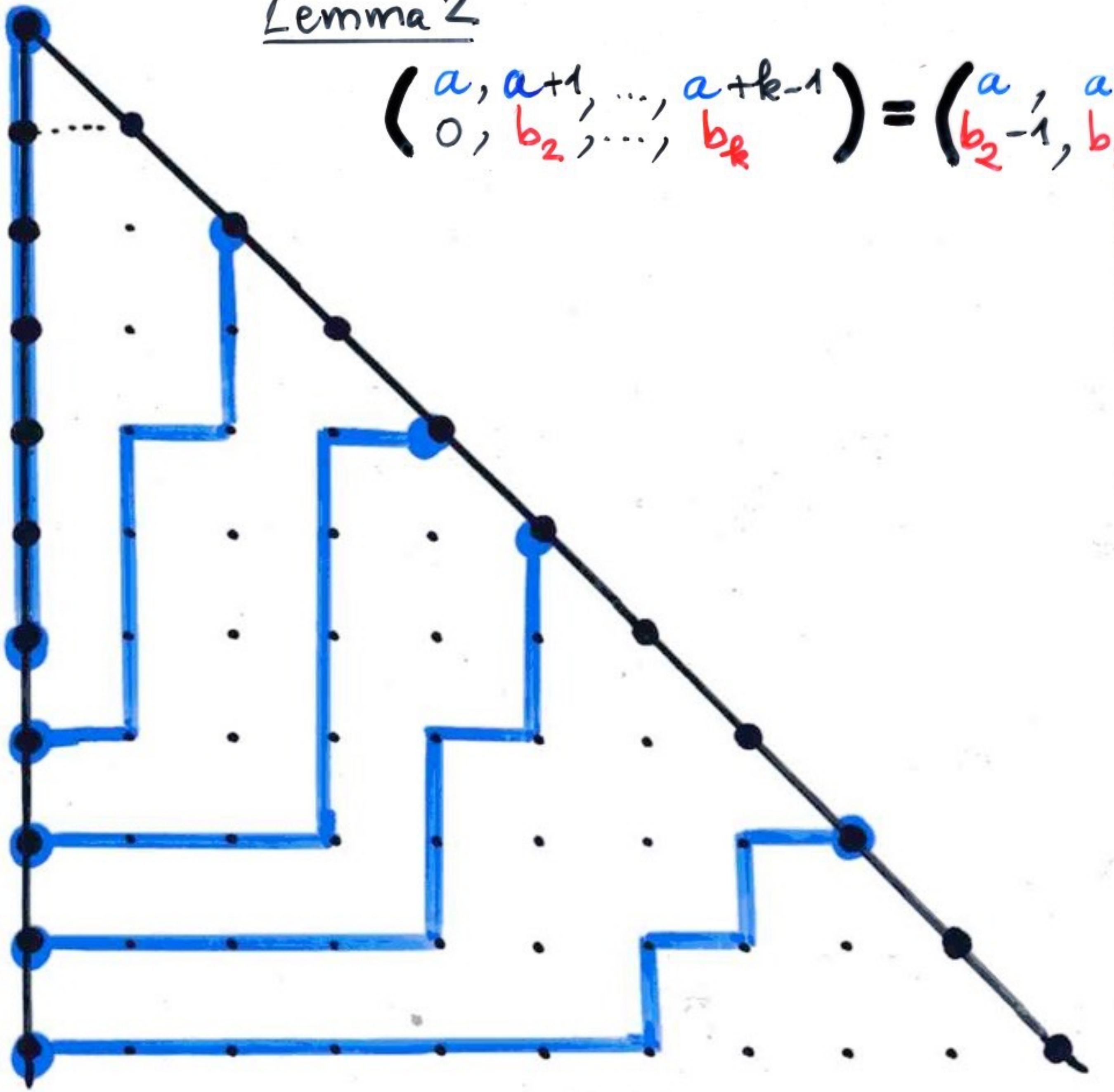
$$\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix} = \frac{a_1 \dots a_k}{b_1 \dots b_k} \begin{pmatrix} a_1-1, \dots, a_k-1 \\ b_1-1, \dots, b_k-1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a}{b} \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{matrix} \phantom{\binom{a_1}{b_1}} & & \overset{a_i}{\downarrow} & & \phantom{\binom{a_k}{b_k}} \end{matrix} \\ b_j \rightarrow & \begin{vmatrix} \binom{a_1}{b_1} & \cdots & \binom{a_i}{b_1} & \cdots & \binom{a_k}{b_1} \\ \vdots & & \vdots & & \vdots \\ \binom{a_1}{b_j} & \cdots & \binom{a_i}{b_j} & \cdots & \binom{a_k}{b_j} \\ \vdots & & \vdots & & \vdots \\ \binom{a_1}{b_k} & \cdots & \binom{a_i}{b_k} & \cdots & \binom{a_k}{b_k} \end{vmatrix} \end{array}$$

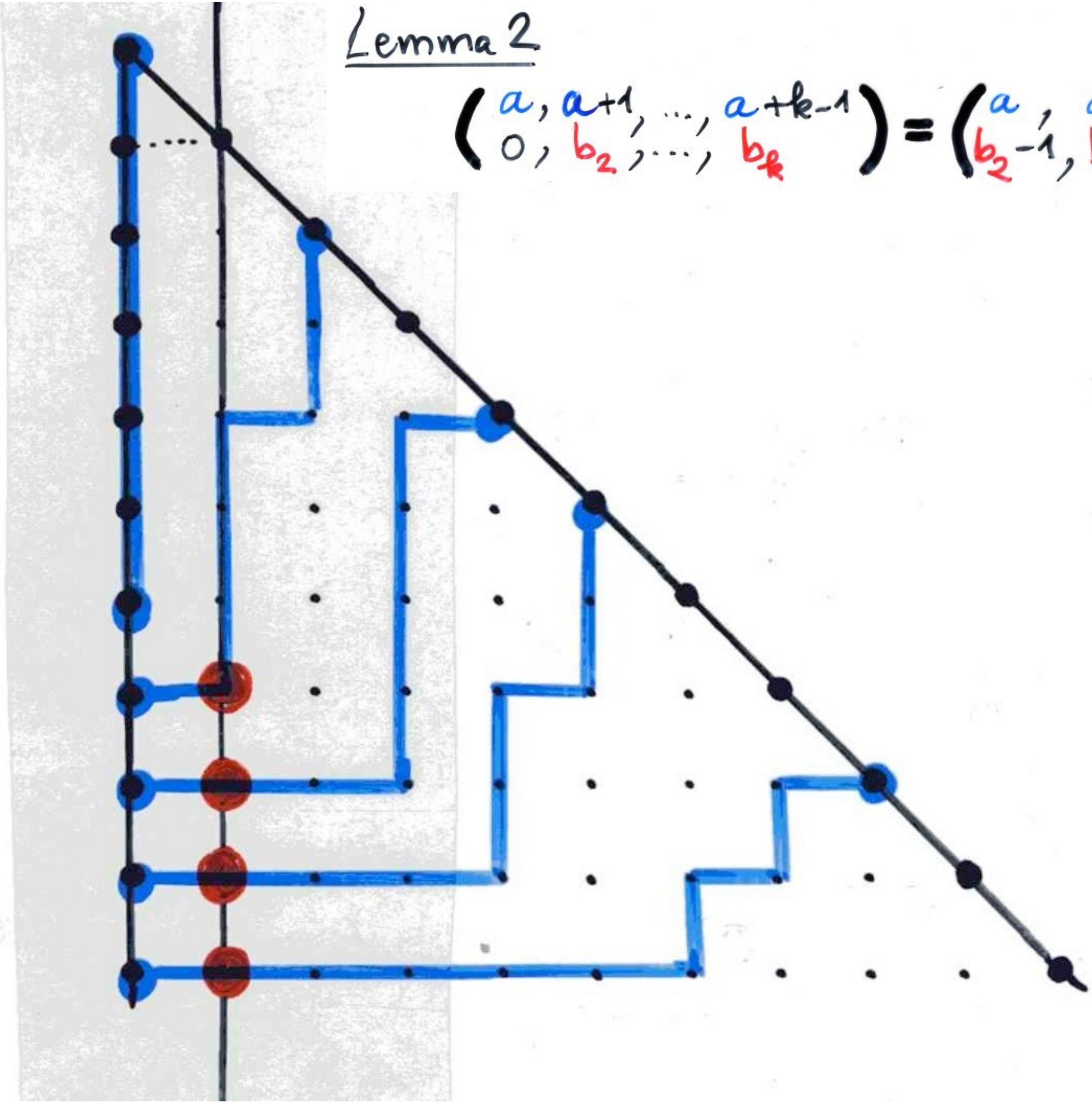
Lemma 2

$$\begin{pmatrix} a, & a+1, & \dots, & a+k-1 \\ 0, & b_2, & \dots, & b_k \end{pmatrix} = \begin{pmatrix} a, & a+1, & \dots, & a+k-2 \\ b_2-1, & b_3-1, & \dots, & b_k-1 \end{pmatrix}$$



Lemma 2

$$\begin{pmatrix} a, & a+1, & \dots, & a+k-1 \\ 0, & b_2, & \dots, & b_k \end{pmatrix} = \begin{pmatrix} a, & a+1, & \dots, & a+k-2 \\ b_2-1, & b_3-1, & \dots, & b_k-1 \end{pmatrix}$$



Proposition

$$\begin{pmatrix} a, & a+1, & \dots & , a+k-1 \\ b_1, & b_2, & \dots & , b_k \end{pmatrix} = \frac{C_a(\mu)}{H(\mu)}$$

$H(\mu)$ = product of hook-lengths of μ

$C_a(\mu)$ = product of contents of μ augmented by a

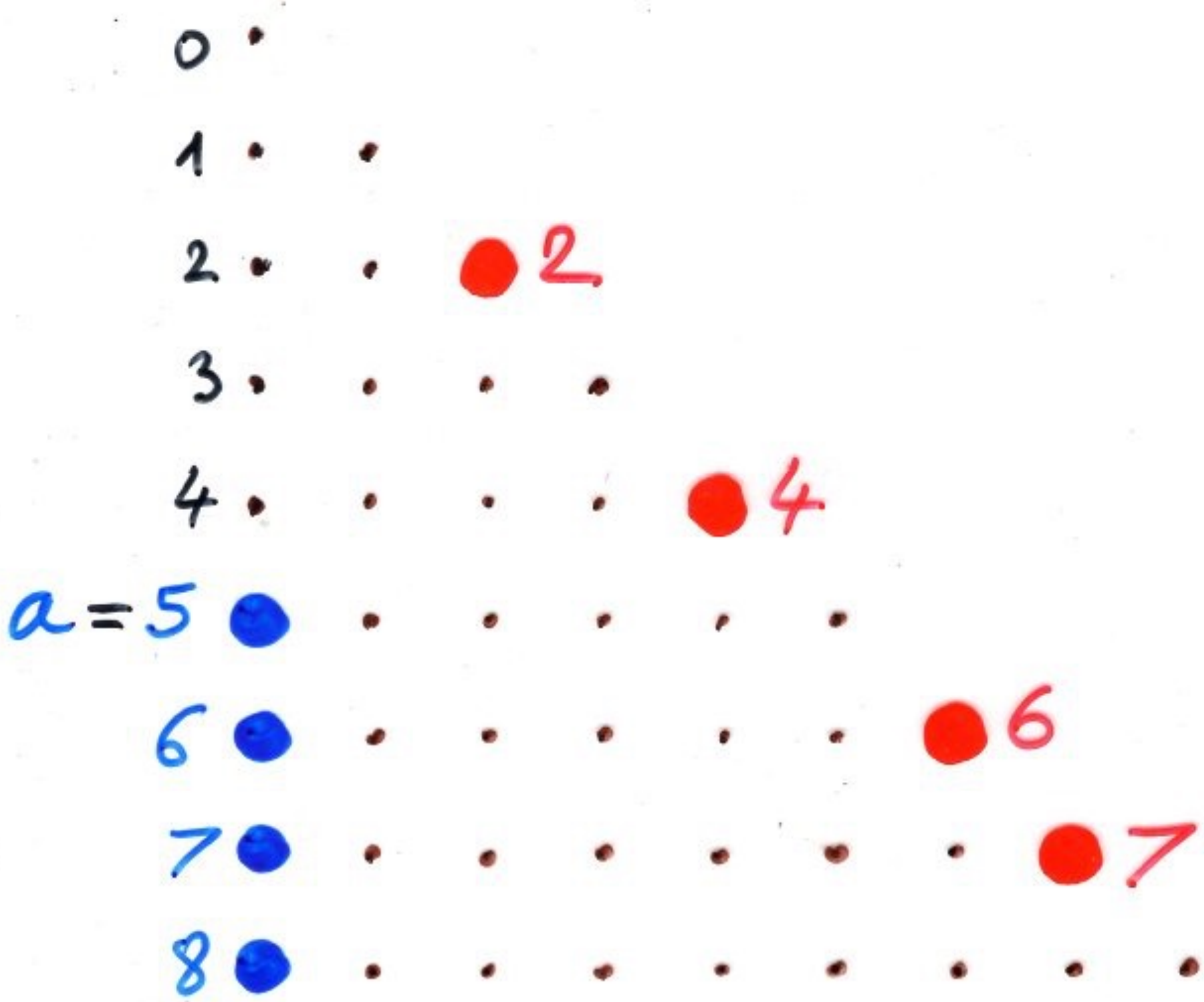
→ definitions below

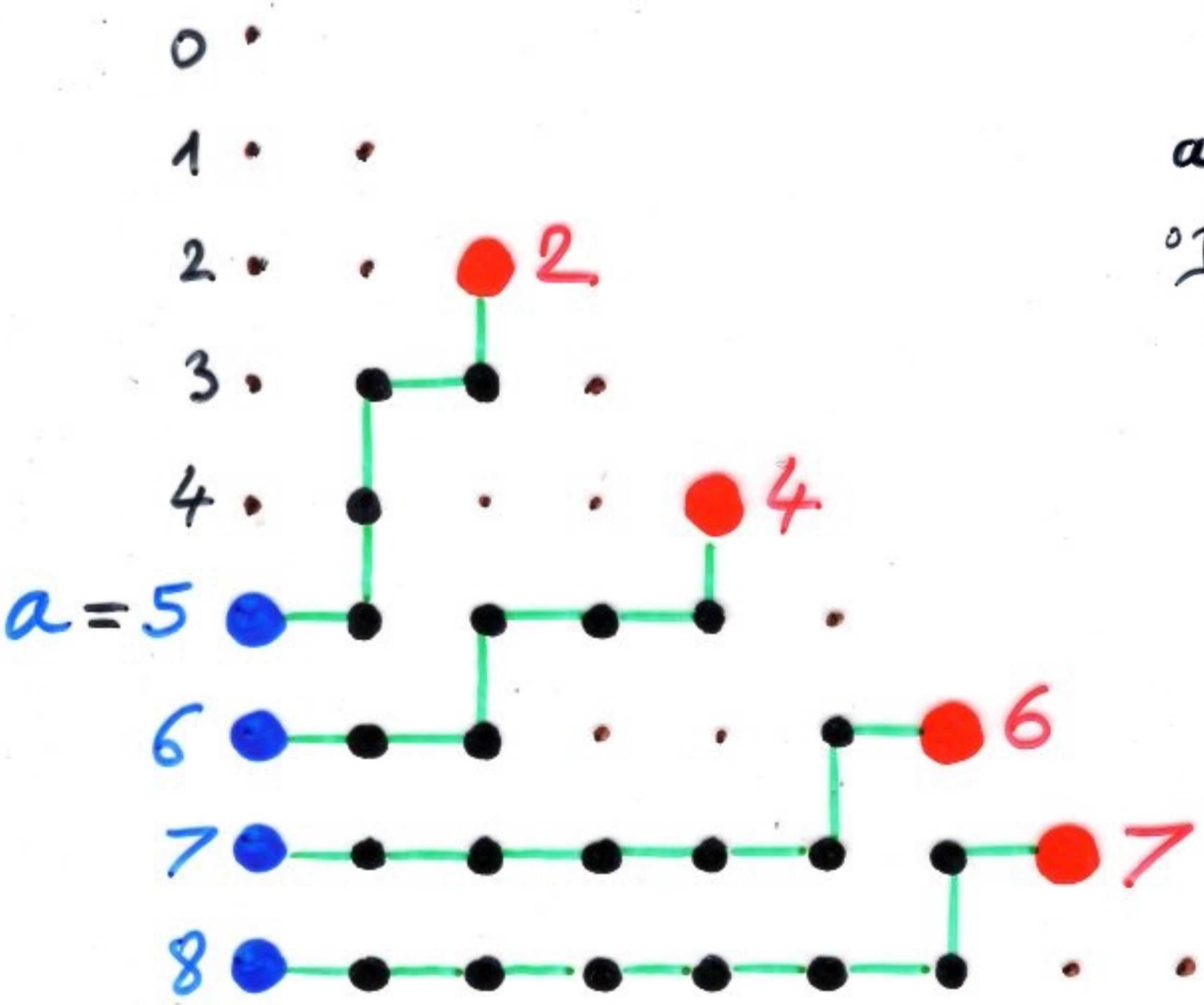
example

$$\begin{array}{r} \text{binomial} \\ \text{determinant} \end{array} \begin{pmatrix} 5,6,7,8 \\ 2,4,6,7 \end{pmatrix}$$

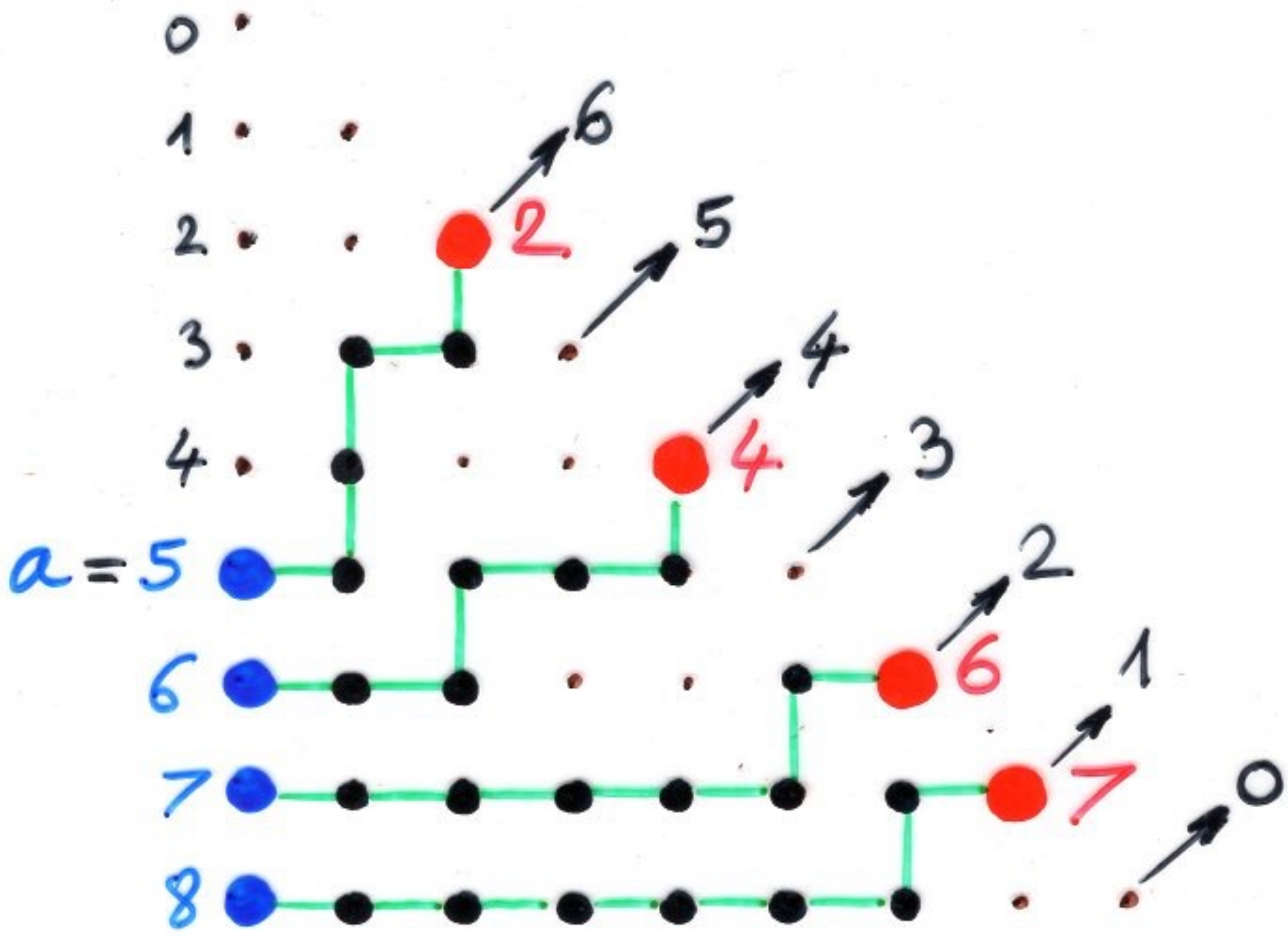
$a = 5$

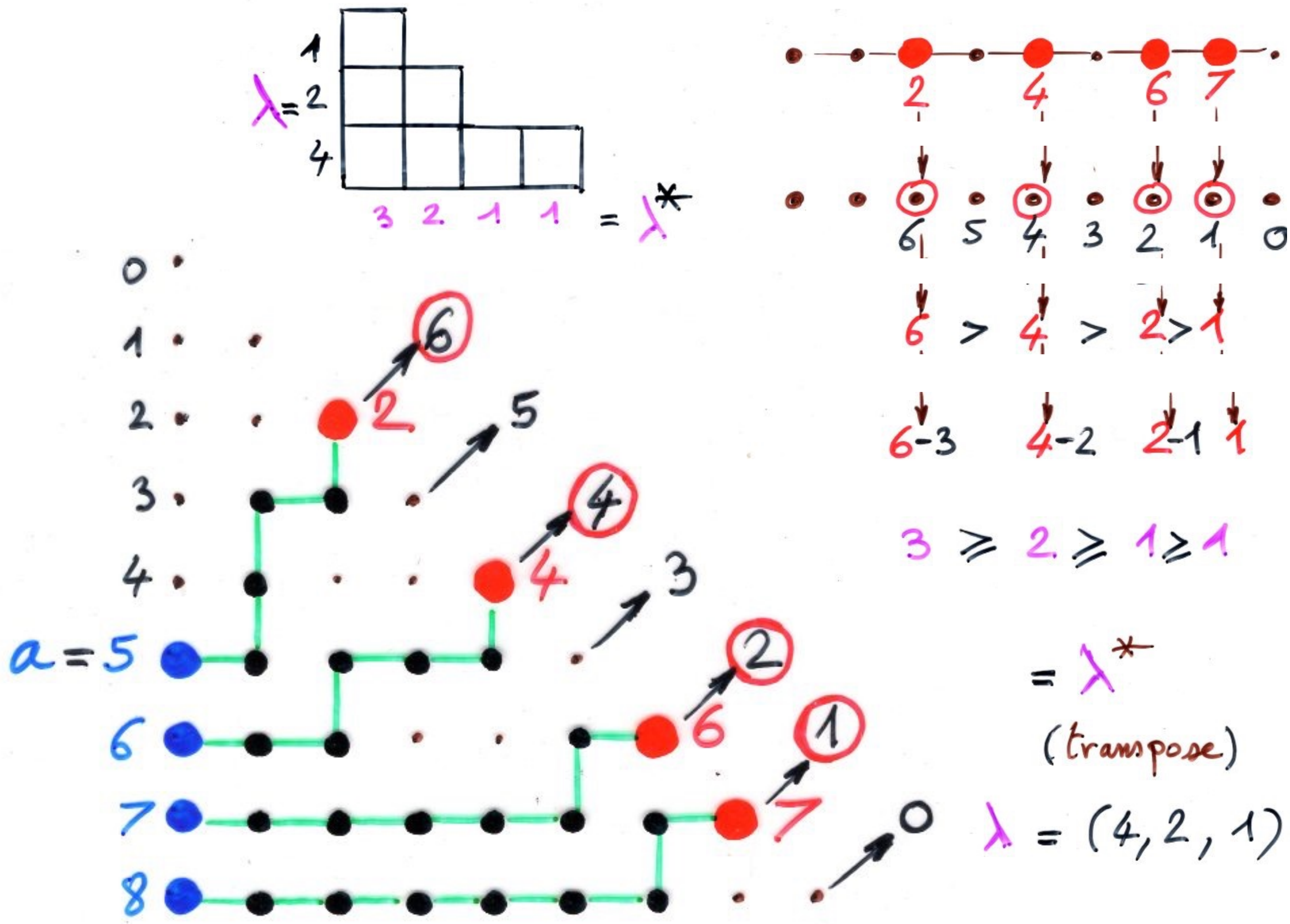
$k = 4$

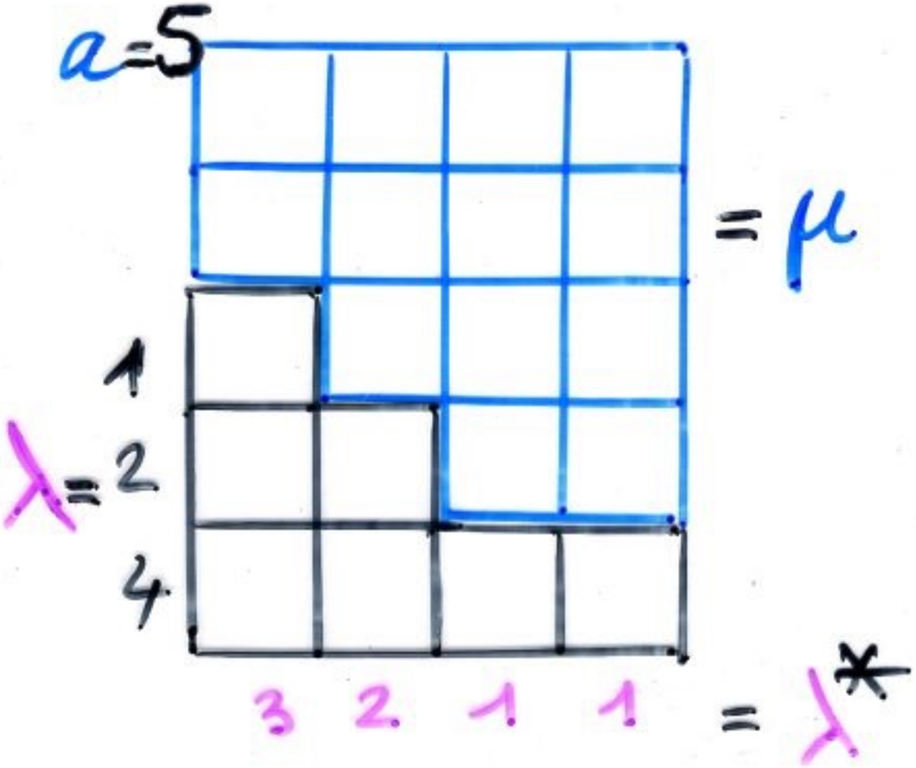


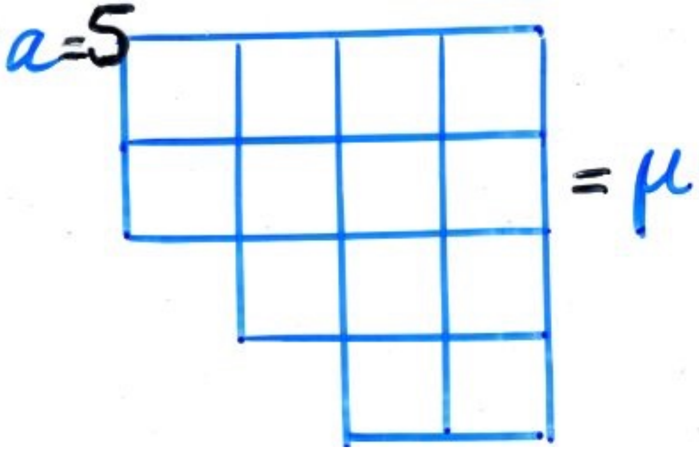


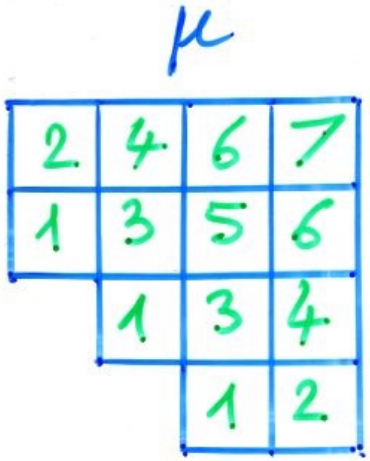
a configuration
of non-intersecting
paths related to


0
1
2
3
4
a=5
6
7
8
2
4
6
7
6
5
4
3
2
1
0

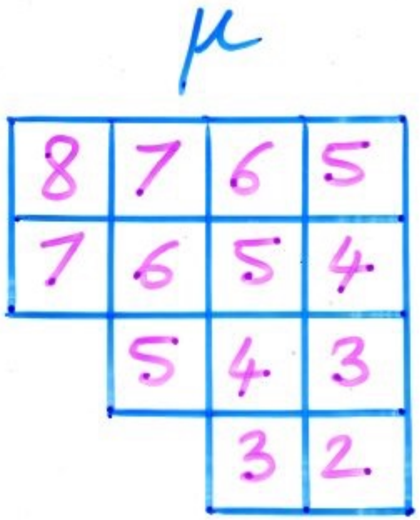

λ=
1
2
4
3 2 1 1 = λ*
2 4 6 7
6 5 4 3 2 1 0
6 > 4 > 2 > 1
6-3 4-2 2-1 1
3 ≥ 2 ≥ 1 ≥ 1
= λ*
(transpose)
λ = (4, 2, 1)
0
1
2
3
4
a = 5
6
7
8
6
2
5
4
4
3
2
6
1
7
0


a=5
= μ
1
λ= 2
4
3 2 1 1
= λ*

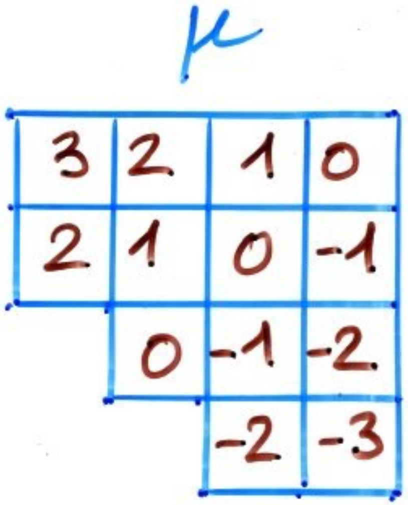

a=5
= μ



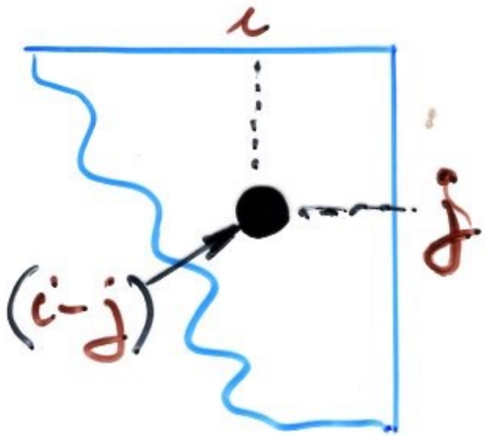
hook lengths.



contents + a



contents



Proposition

$$\begin{pmatrix} a, & a+1, & \dots & , a+k-1 \\ b_1, & b_2, & \dots & , b_k \end{pmatrix} = \frac{C_a(\mu)}{H(\mu)}$$

$H(\mu)$ = product of hook-lengths of μ

$C_a(\mu)$ = product of contents of μ augmented by a

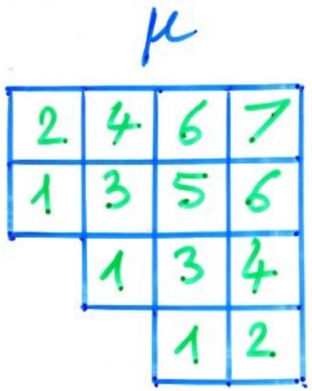
$\mathbf{C}_a(\mu)$ = product of contents of μ augmented by a

$$\left(\begin{matrix} 5, & 6, & 7, & 8 \\ 2, & 4, & 6, & 7 \end{matrix}\right) \quad \frac{\begin{array}{|c|c|c|c|} \hline 8 & 7 & 6 & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline & 5 & 4 & 3 \\ \cline{2-4} & & 3 & 2 \\ \cline{3-4} \end{array}}{\begin{array}{|c|c|c|c|} \hline 2 & 4 & 6 & 7 \\ \hline 1 & 3 & 5 & 6 \\ \hline & 1 & 3 & 4 \\ \cline{2-4} & & 1 & 2 \\ \cline{3-4} \end{array}} = 2^2 \times 5^2 \times 7 = 700$$

$\mathbf{H}(\mu)$ = product of hook-lengths of μ

$$\begin{pmatrix} 5, & 6, & 7, & 8 \\ 2, & 4, & 6, & 7 \end{pmatrix} = \frac{5\cdot6\cdot7\cdot8}{2\cdot4\cdot6\cdot7}\begin{pmatrix} 4, & 5, & 6, & 7 \\ 1, & 3, & 5, & 6 \end{pmatrix}$$

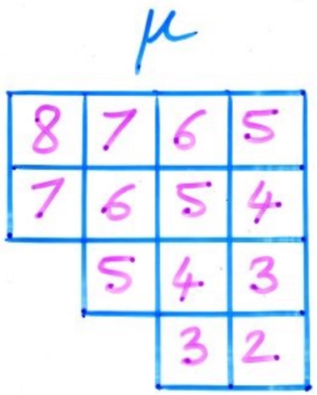
$$= \frac{4\cdot5\cdot6\cdot7}{1\cdot3\cdot5\cdot6}\begin{pmatrix} 3, & 4, & 5, & 6 \\ 0, & 2, & 4, & 5 \end{pmatrix}$$



μ

2	4	6	7
1	3	5	6
	1	3	4
		1	2

hook lengths.



μ

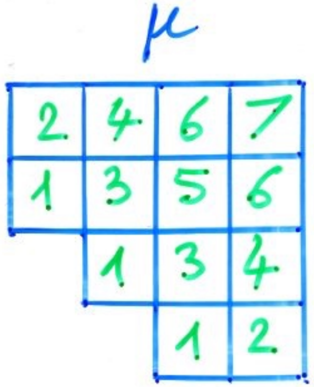
8	7	6	5
7	6	5	4
	5	4	3
		3	2

contents $+ a$

$$\binom{3,4,5,6}{0,2,4,5} = \binom{3,4,5}{1,3,4} = \frac{3\cdot4\cdot5}{1\cdot3\cdot4}\binom{2,3,4}{0,2,3}$$

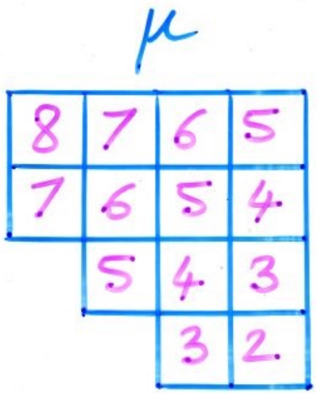
$$\binom{2,3,4}{0,2,3} = \binom{2,3}{1,2} = \frac{2\cdot3}{1\cdot2}\binom{1,2}{0,1}$$

$$= \binom{1}{0} = 1$$



$$\begin{array}{cccc} 2 & 4 & 6 & 7 \\ 1 & 3 & 5 & 6 \\ & 1 & 3 & 4 \\ & & 1 & 2 \end{array}$$

hook lengths

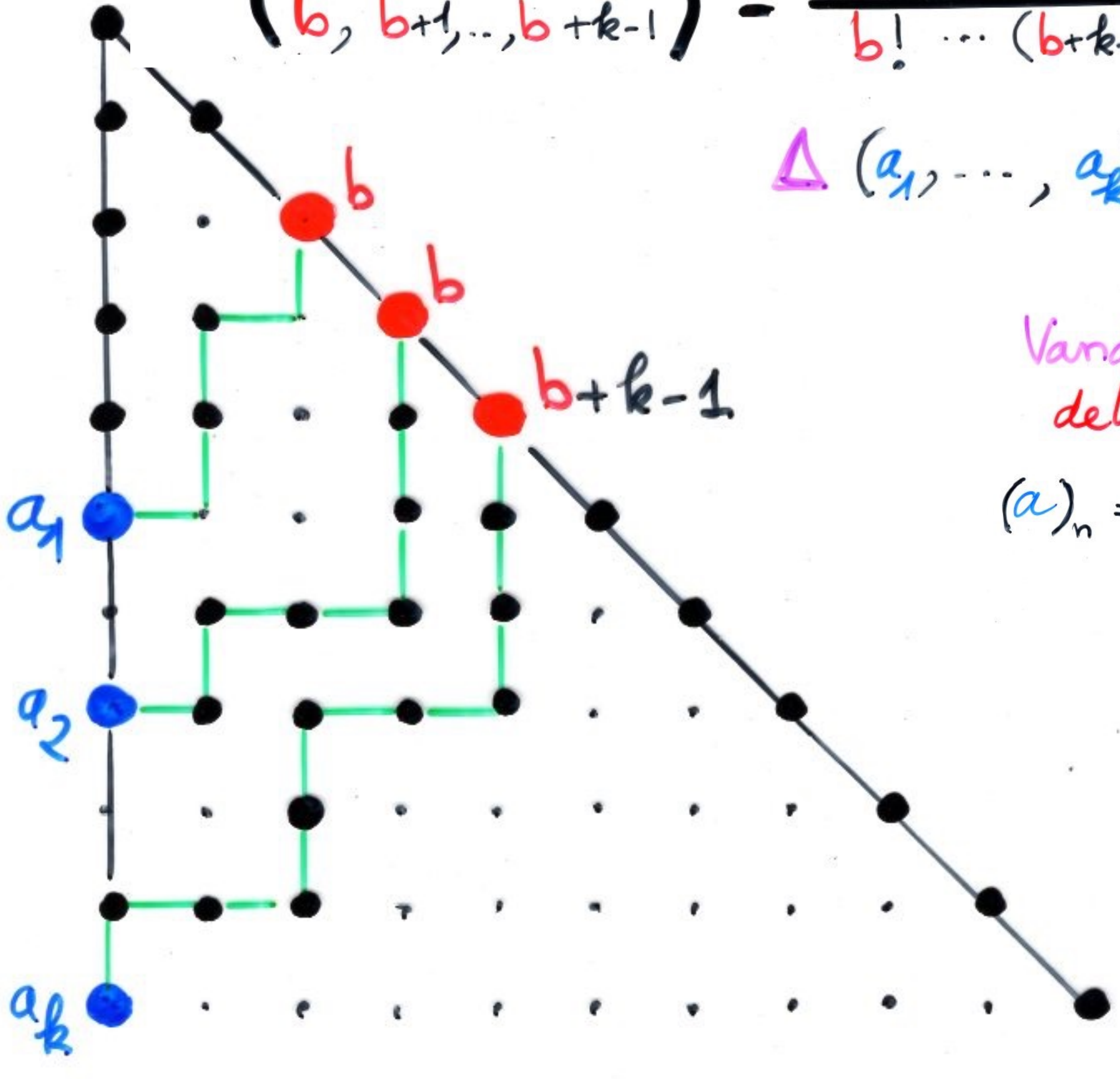


$$\begin{array}{cccc} 8 & 7 & 6 & 5 \\ 7 & 6 & 5 & 4 \\ & 5 & 4 & 3 \\ & & 3 & 2 \end{array}$$

contents + a

exercise
another formula
for binomial determinant

$$\binom{a_1, \dots, a_k}{b, b+1, \dots, b+k-1} = \frac{(a_1)_b \cdots (a_k)_b}{b! \cdots (b+k-1)!} \Delta(a_1, \dots, a_k)$$



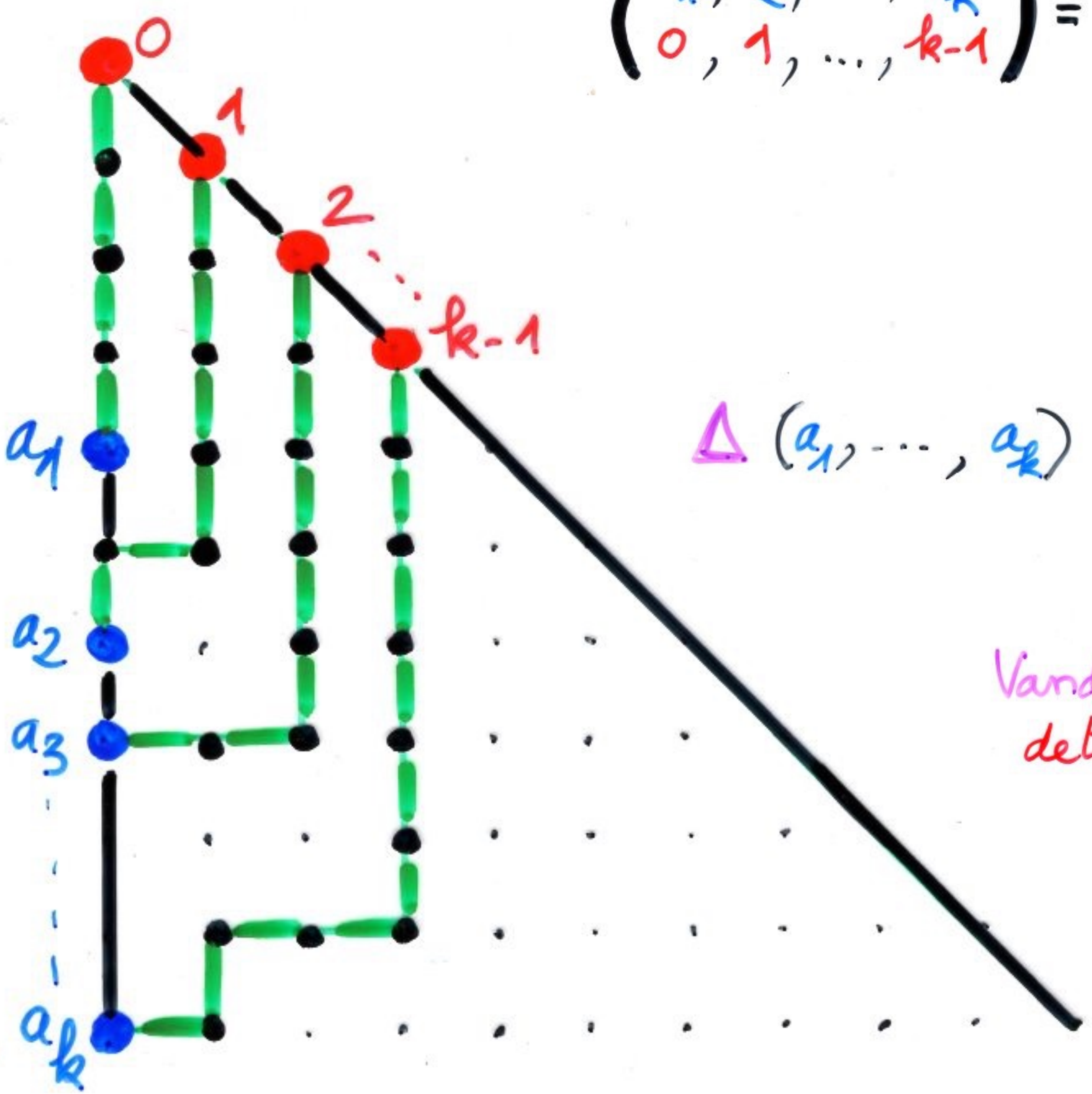
$$\Delta(a_1, \dots, a_k) = \prod_{1 \le i < j \le k} (a_i - a_j)$$

Vandermonde determinant

$$(a)_n = a(a-1) \cdots (a-n+1)$$

descending factorial

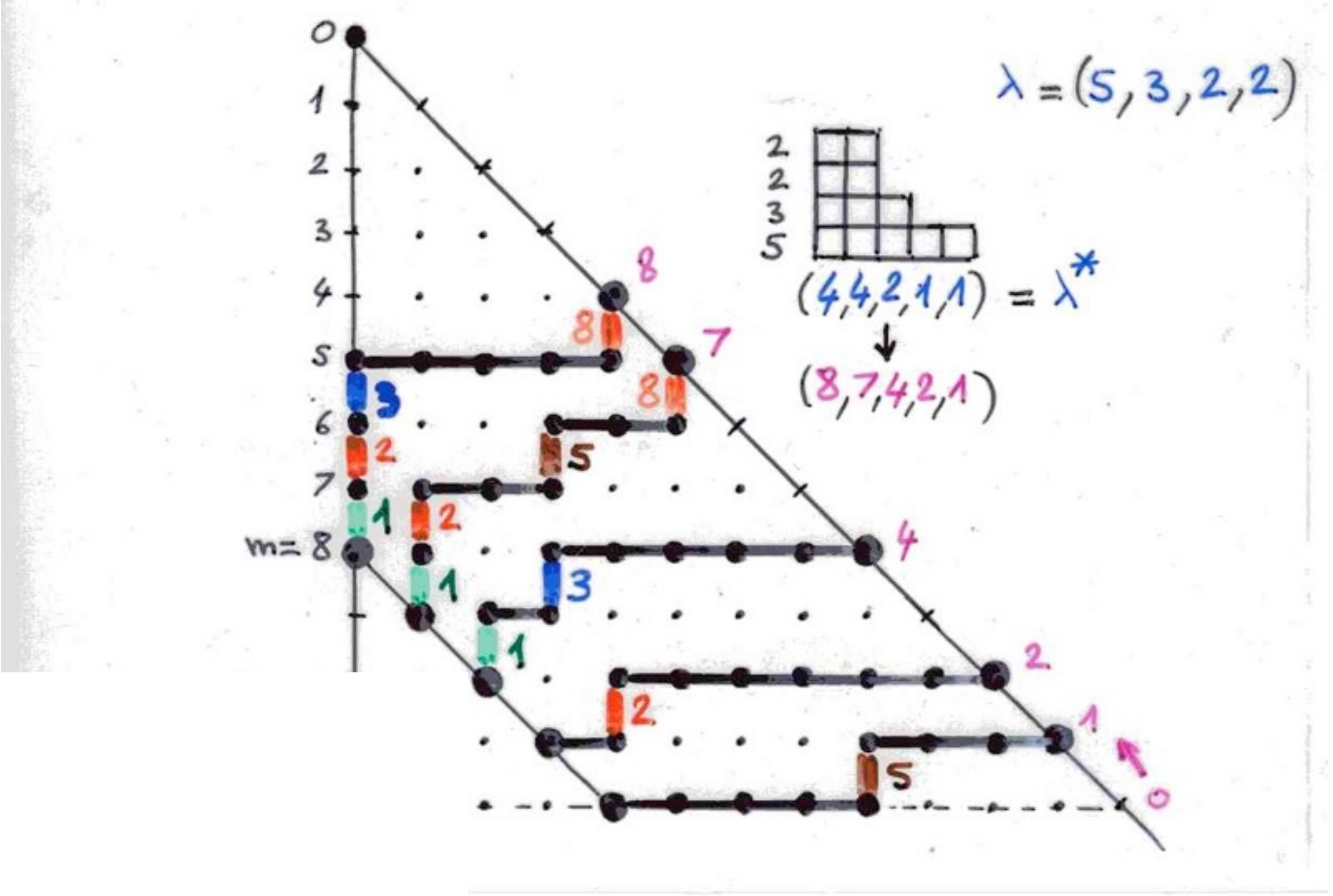
$$\begin{pmatrix} a_1, a_2, \dots, a_k \\ 0, 1, \dots, k-1 \end{pmatrix} = \frac{\Delta(a_1, a_2, \dots, a_k)}{0!\,1!\,\cdots(k-1)!}$$

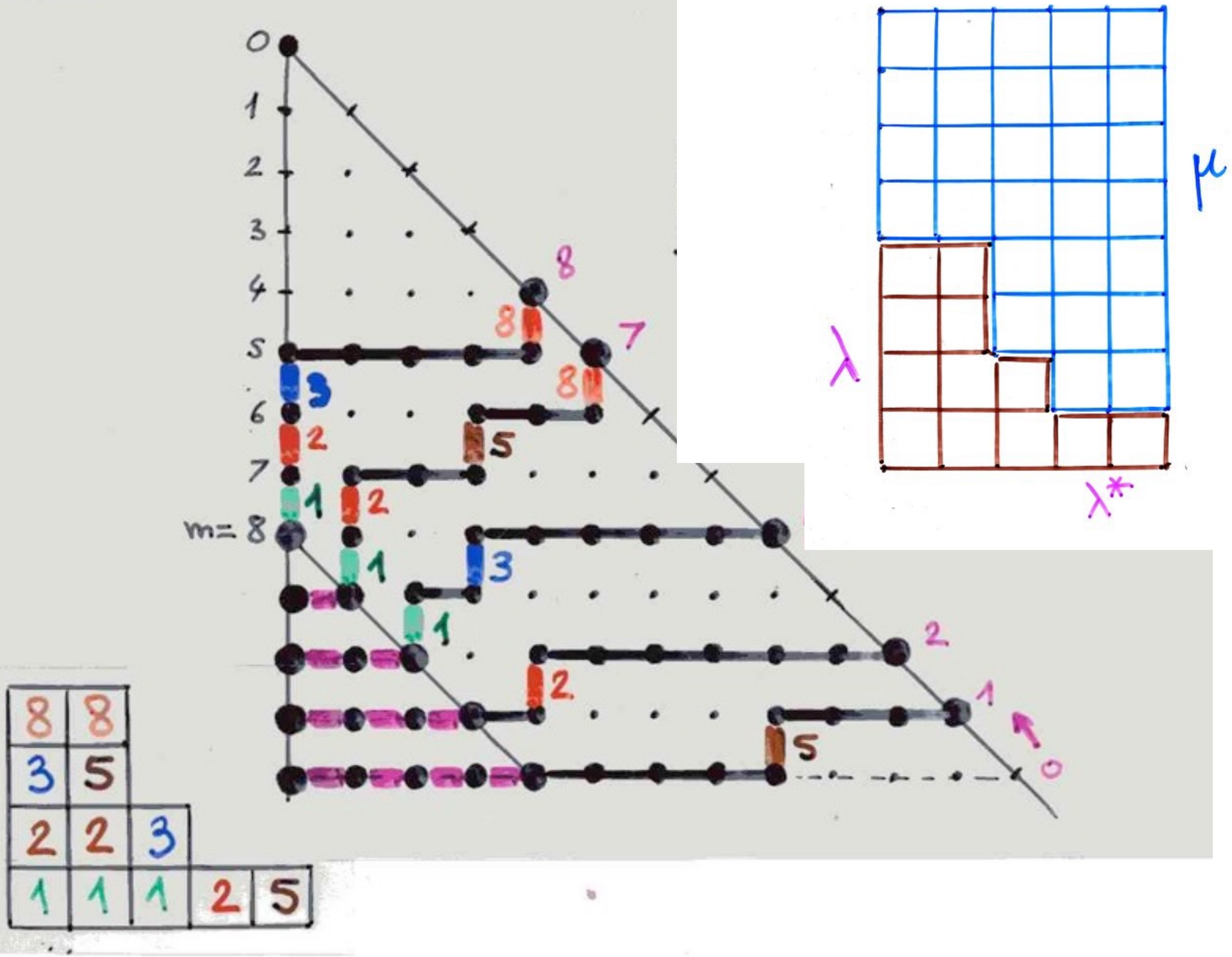


$$\Delta(a_1, \dots, a_k) = \prod_{1 \le i < j \le k} (a_i - a_j)$$

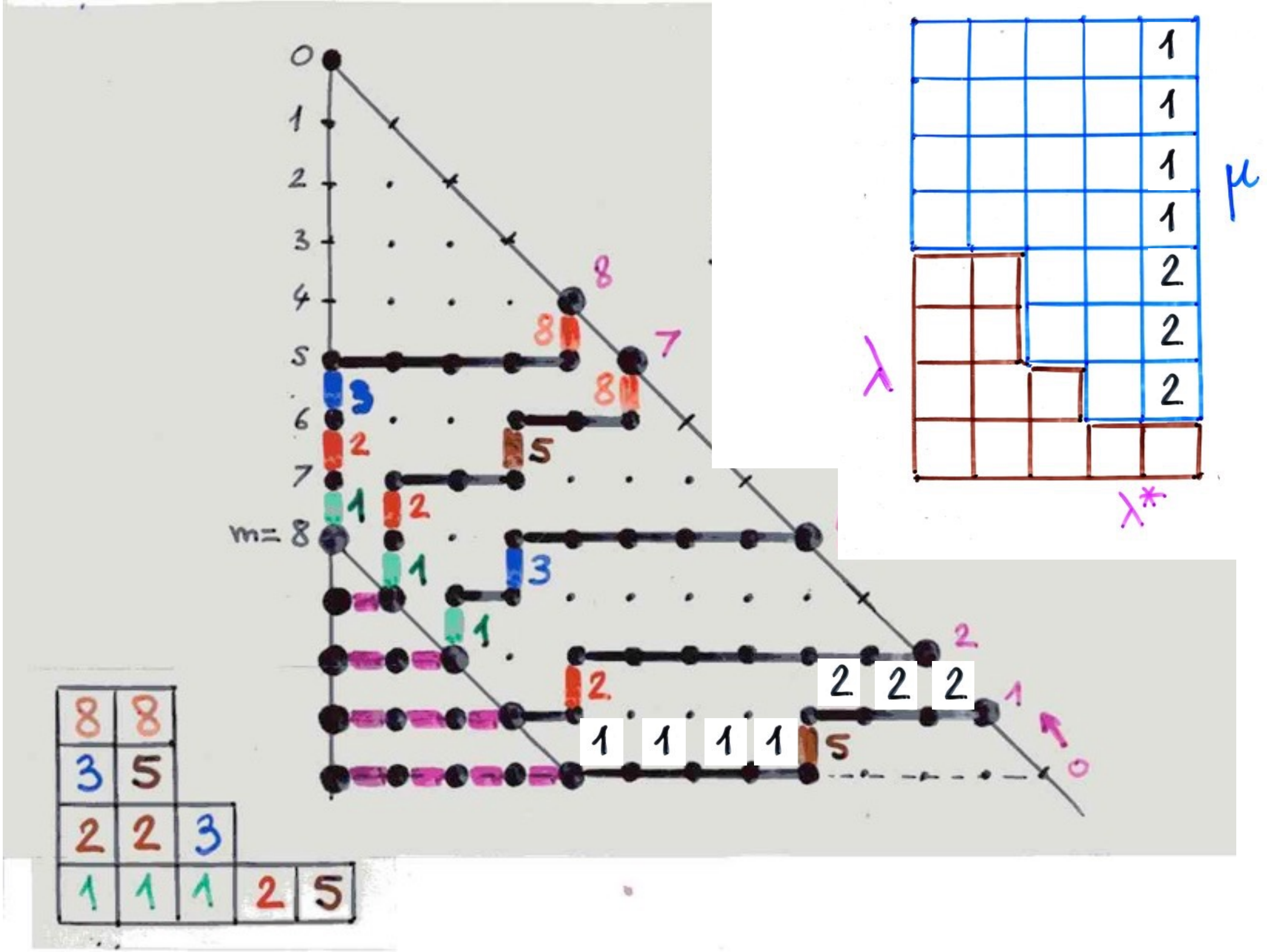
Vandermonde determinant

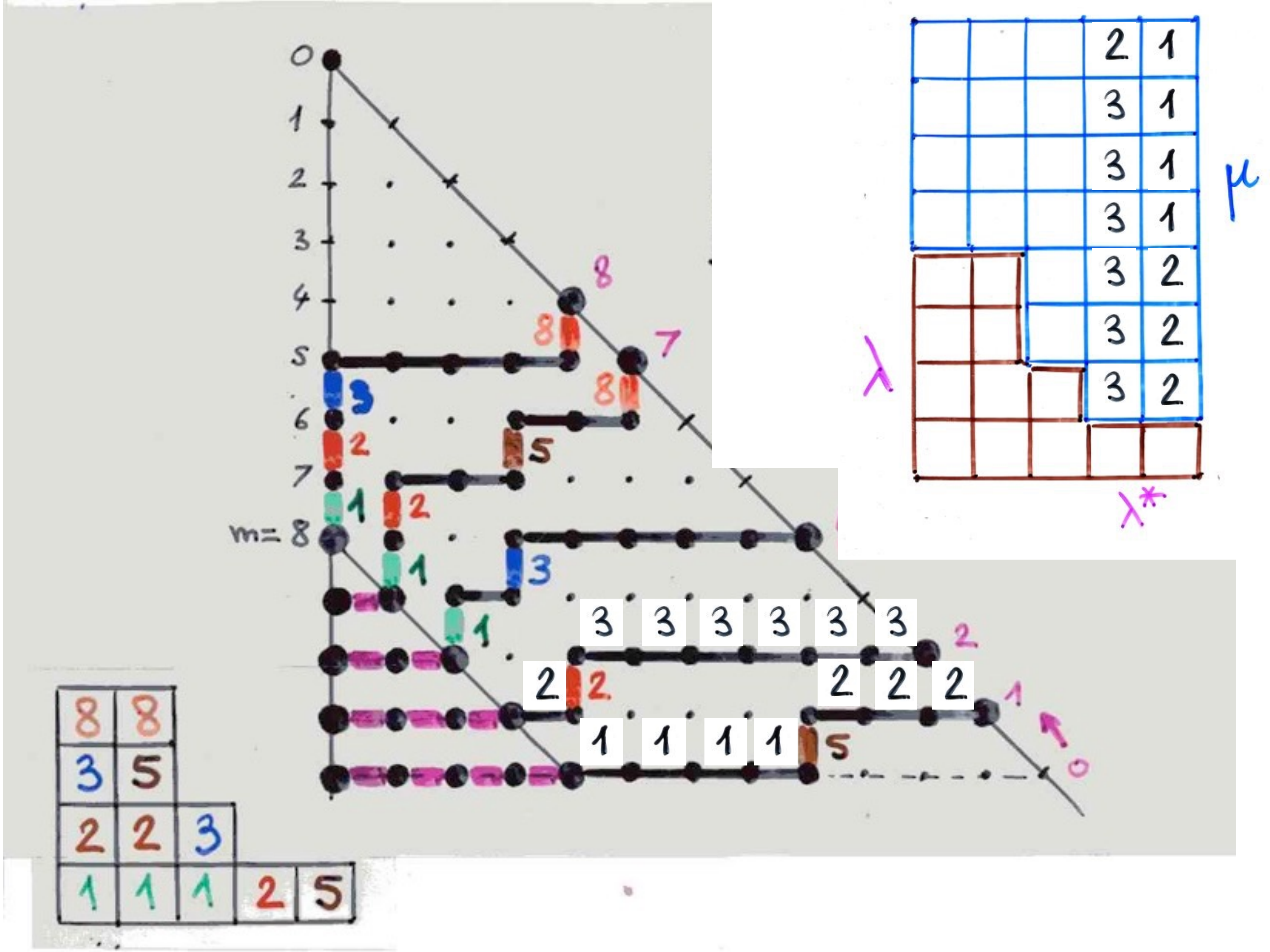
(semi-standard) Young tableaux

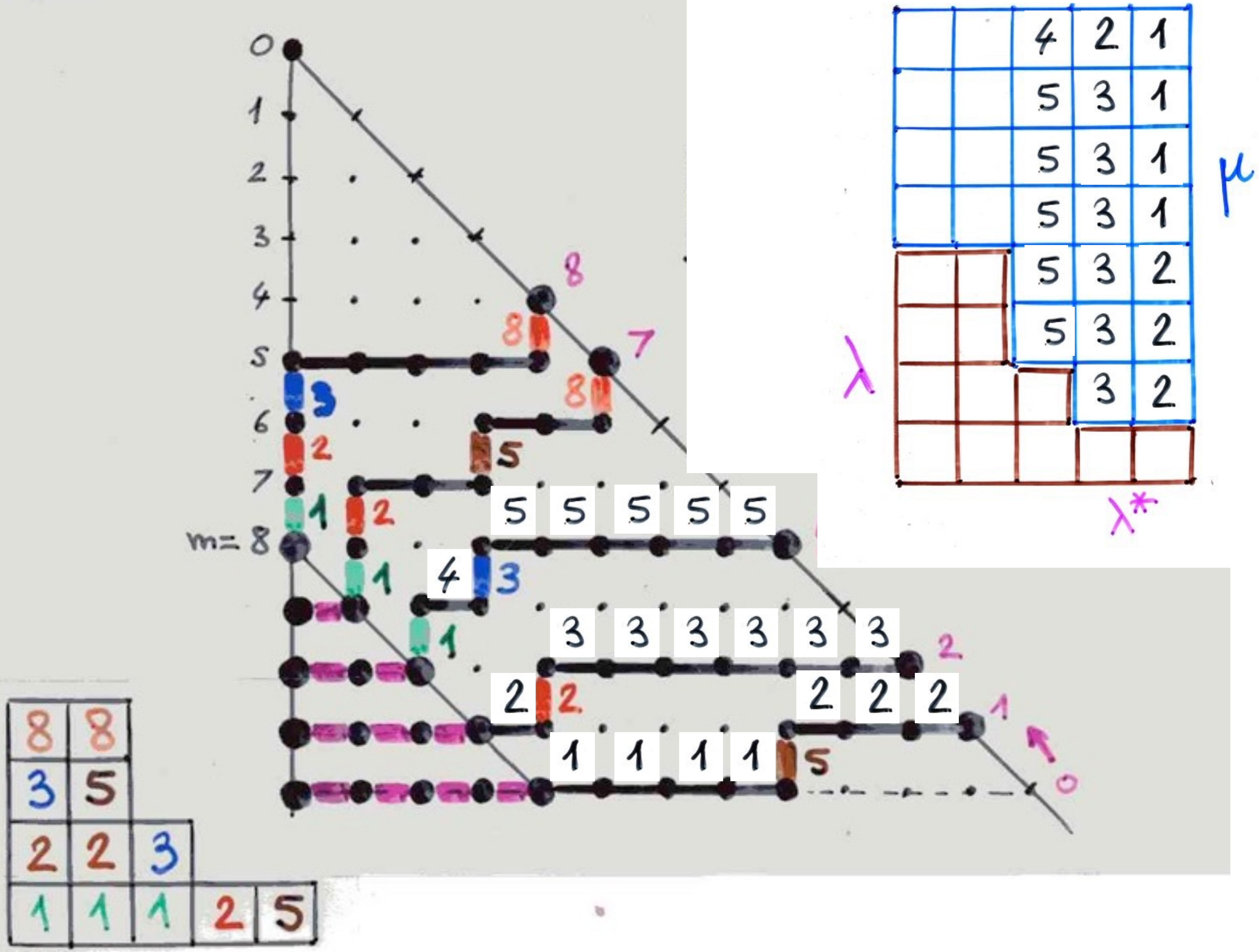

0
1
2
3
4
5
6
7
m= 8
λ = (5, 3, 2, 2)
2
2
3
5
(4,4,2,1,1) = λ*
(8,7,4,2,1)
8
7
4
2
1
0
8
8
3
2
5
1
2
2
1
3
1
2
5

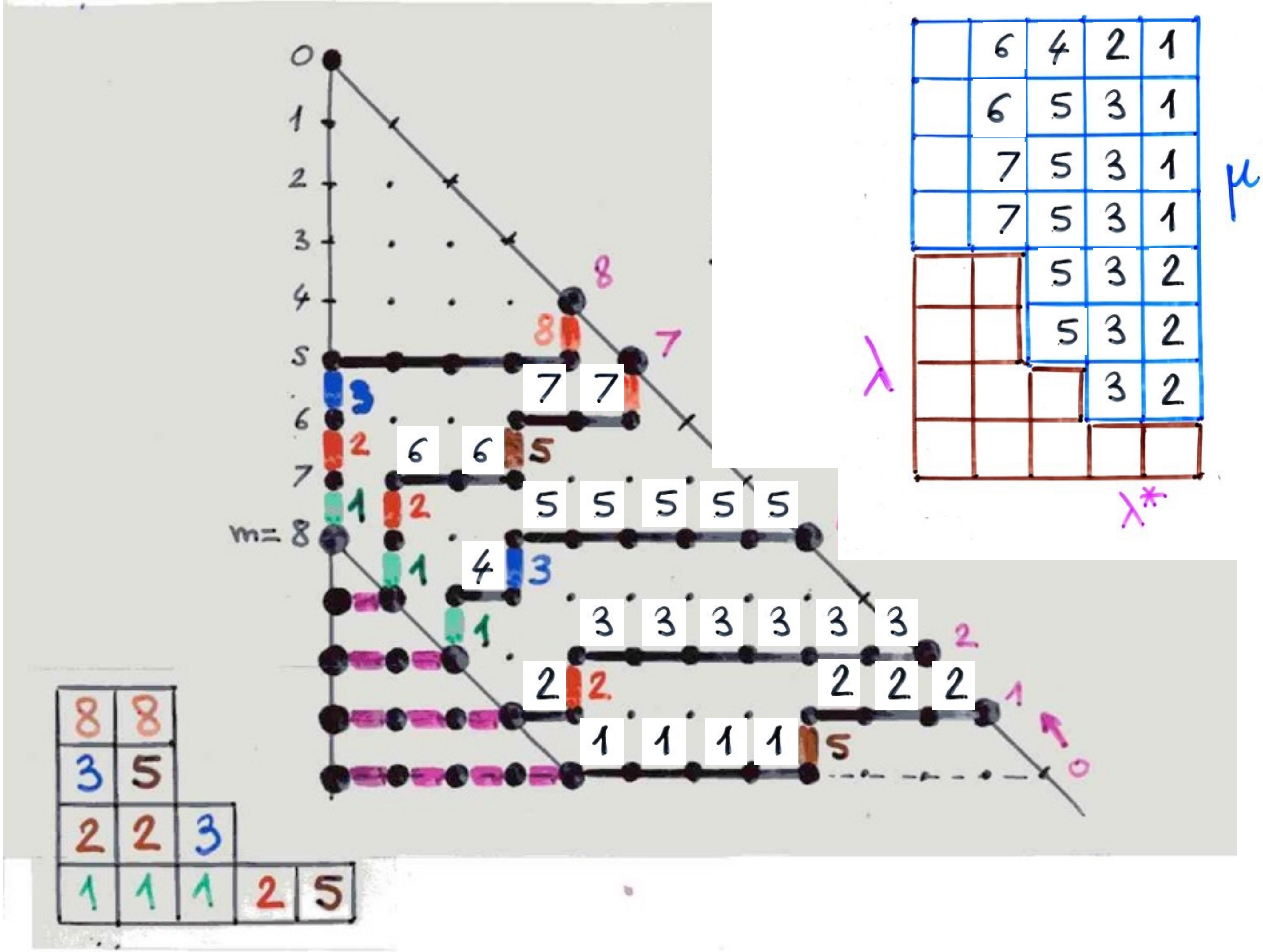


8	8			
3	5			
2	2	3		
1	1	1	2	5

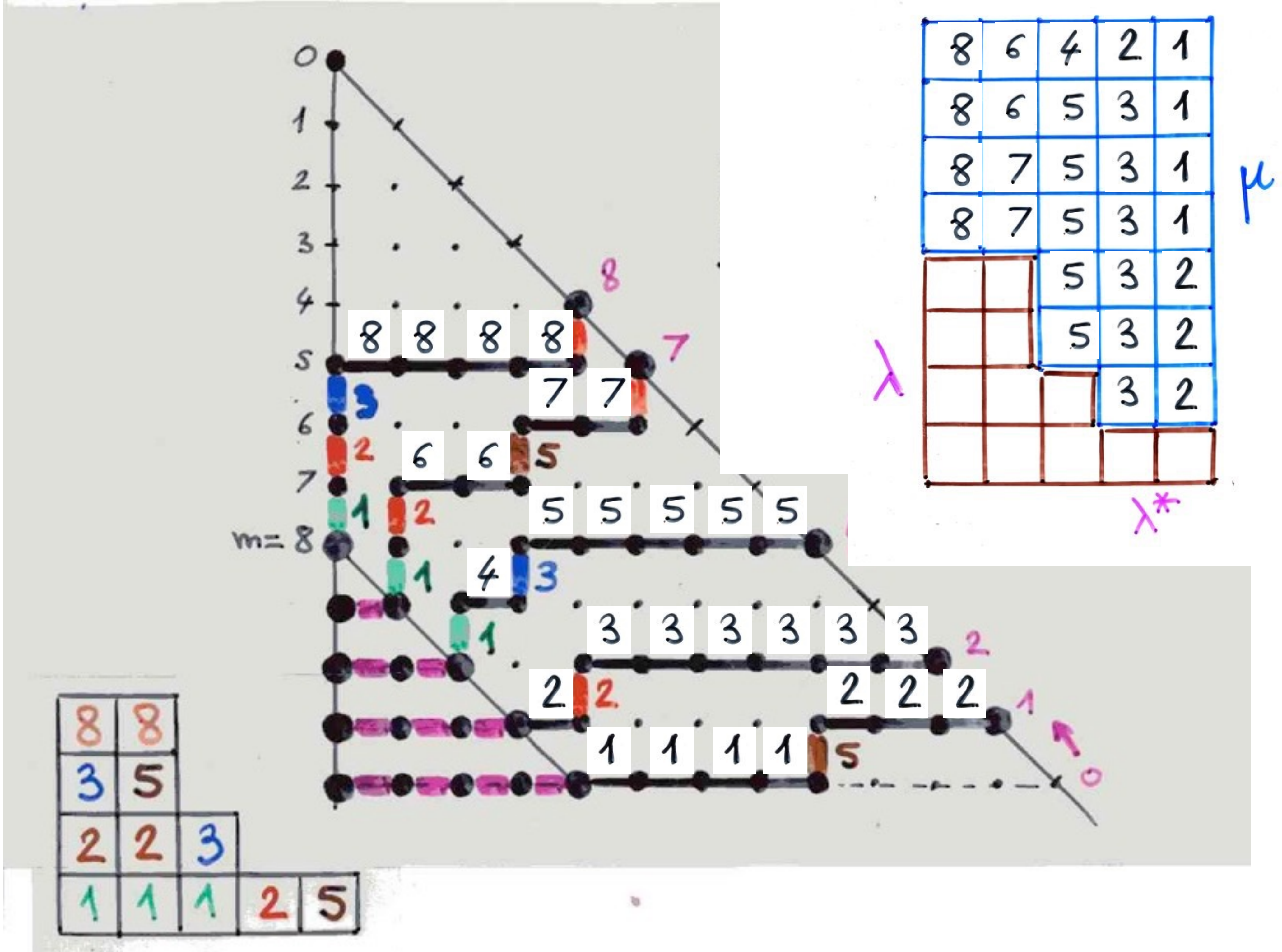

0
1
2
3
4
5
6
7
m= 8
8
7
2
1
0
8 8
3 5
2 2 3
1 1 1 2 5
1 1 1 1
2 2 2
μ
λ
λ*


0
1
2
3
4
5
6
7
m= 8
8
7
2
1
0
8
8
3
2
5
1
2
1
3
1
3 3 3 3 3 3
2
2
2 2 2
1 1 1 1
5
8 8
3 5
2 2 3
1 1 1 2 5
2 1
3 1
3 1
3 1
3 2
3 2
3 2
μ
λ
λ*


0
1
2
3
4
5
6
7
m= 8
8
7
8
8
3
2
5
1
2
1
5 5 5 5 5
4
3
1
3 3 3 3 3 3
2
2
2
2 2 2
1
1 1 1 1
5
0
4 2 1
5 3 1
5 3 1
5 3 1
5 3 2
5 3 2
3 2
μ
λ
λ*
8 8
3 5
2 2 3
1 1 1 2 5



0
1
2
3
4
5
6
7
m= 8
8
7
2
1
0
μ
λ
λ*



0
1
2
3
4
5
6
7
m= 8
8
7
2
1
0
8 8 8 8
7 7
6 6
5 5 5 5 5
4
3 3 3 3 3 3
2
2 2 2
1 1 1 1
3
2
1
2
1
5
3
1
2
5
8 6 4 2 1
8 6 5 3 1
8 7 5 3 1
8 7 5 3 1
5 3 2
5 3 2
3 2
μ
λ
λ*
8 8
3 5
2 2 3
1 1 1 2 5

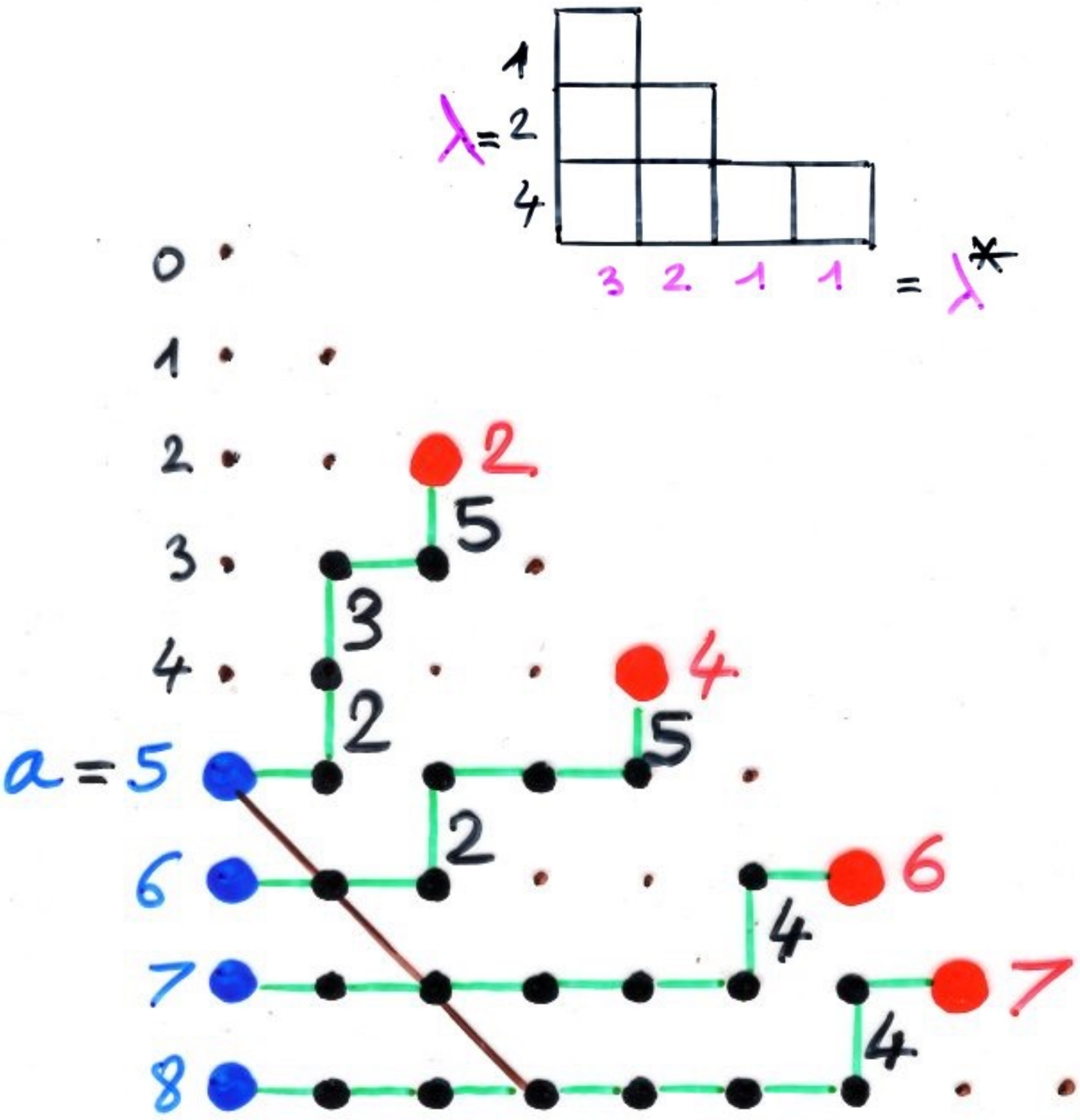
Proposition

The number of semi-standard Young tableaux with shape μ and entries in $\{1, 2, \dots, a\}$ is :

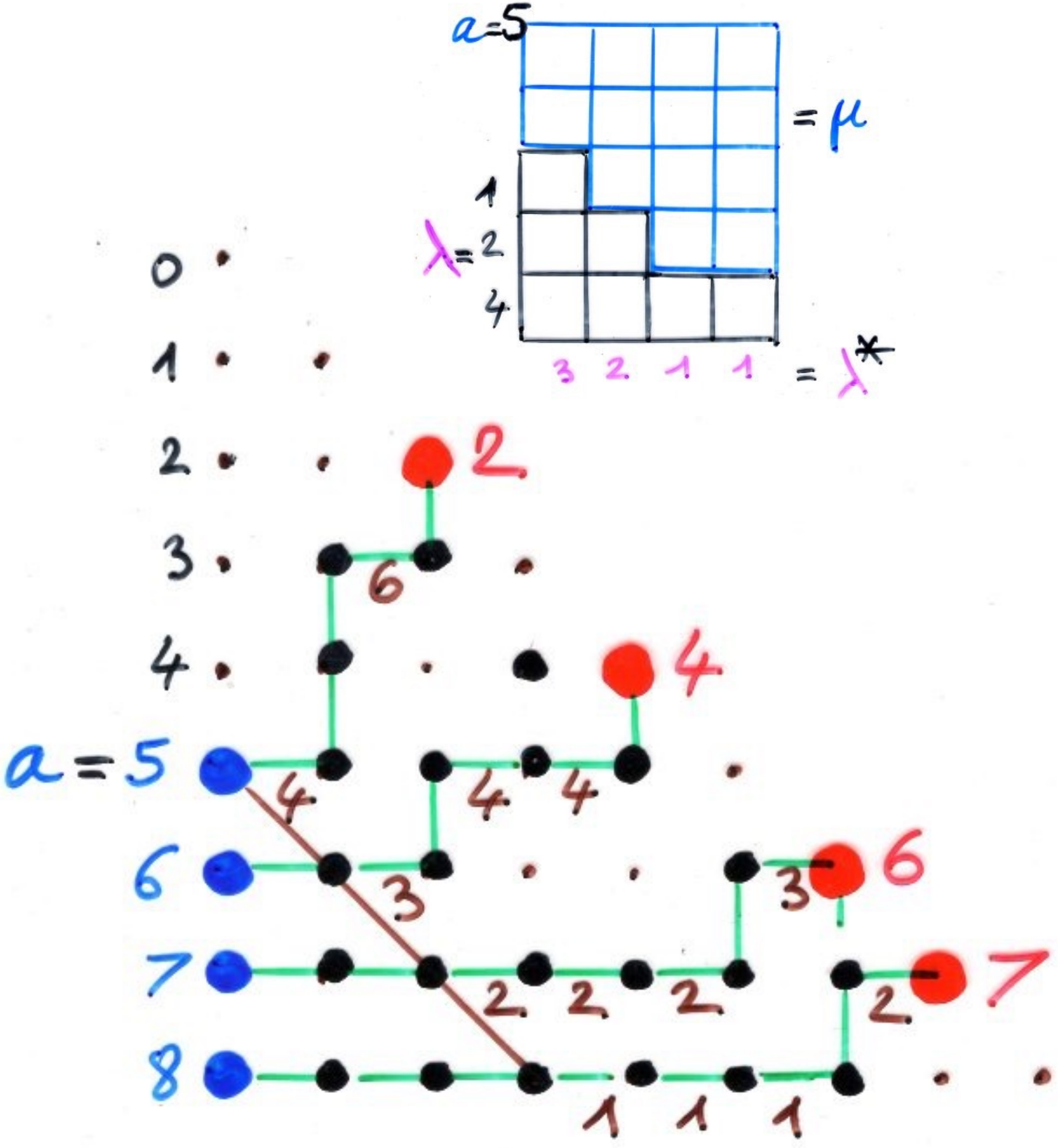
$$\frac{C_a(\mu)}{H(\mu)}$$

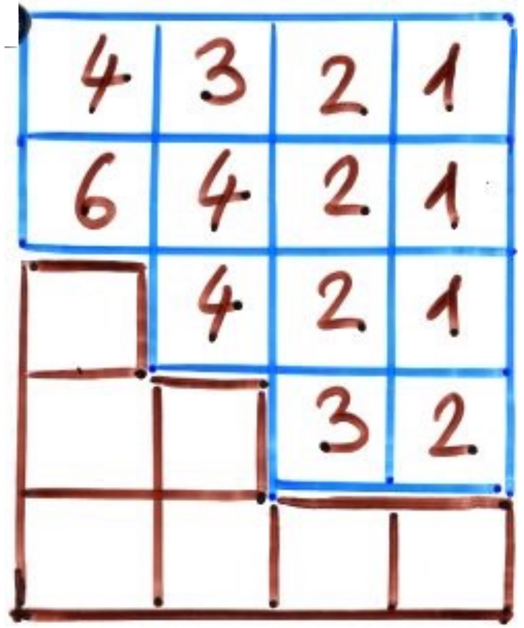
$H(\mu)$ = product of hook-lengths of μ

$C_a(\mu)$ = product of contents of μ augmented by a

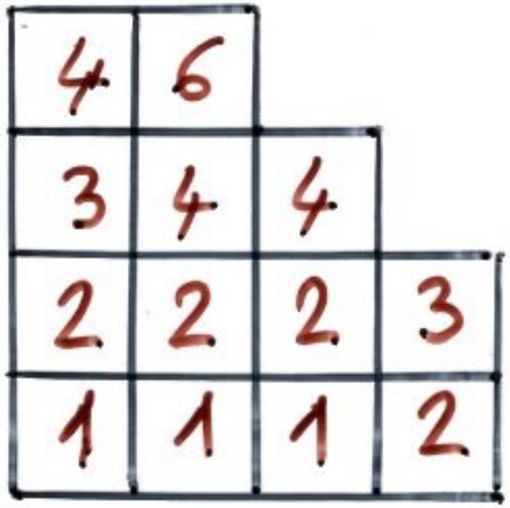


5			
3	5		
2	2	4	4

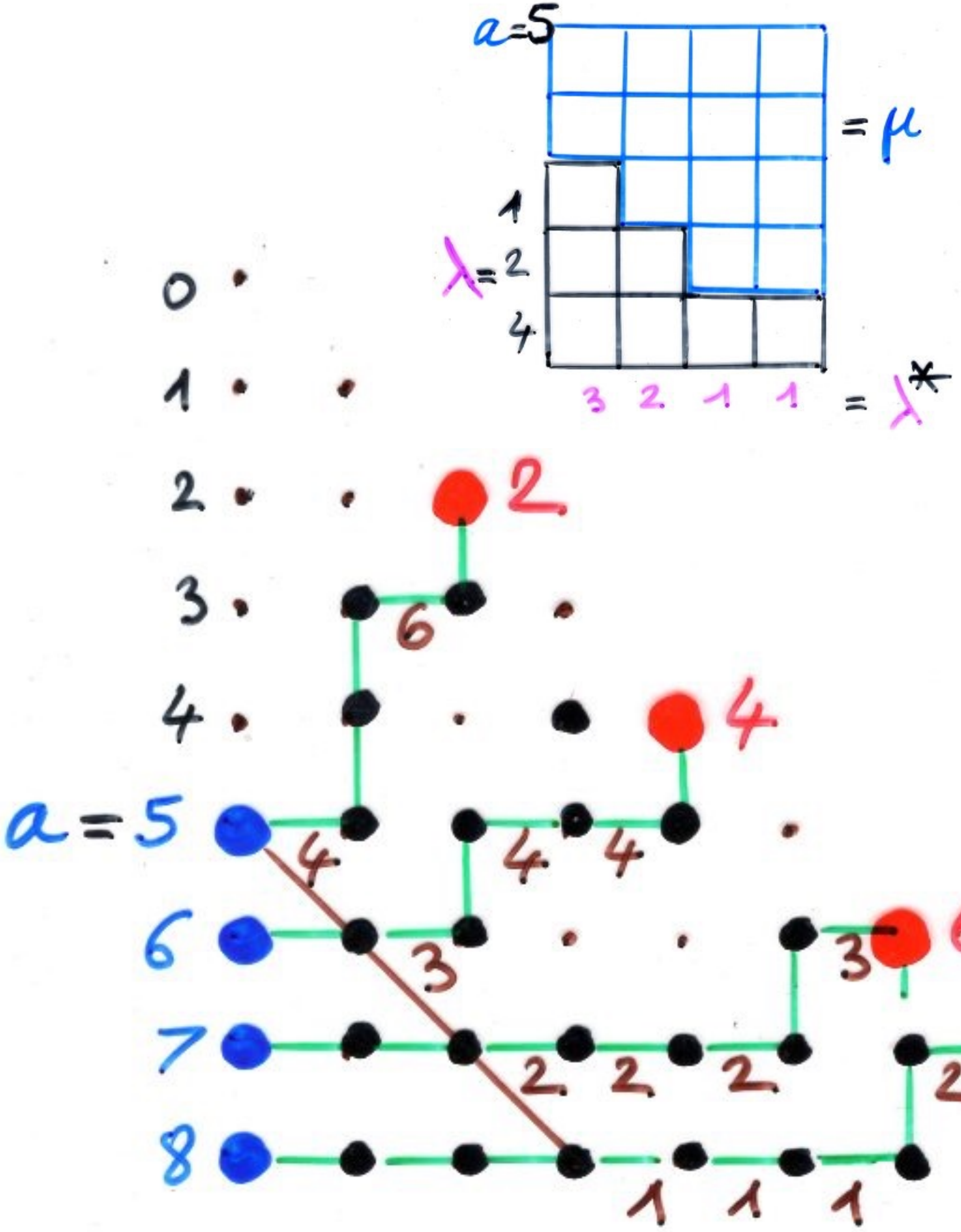


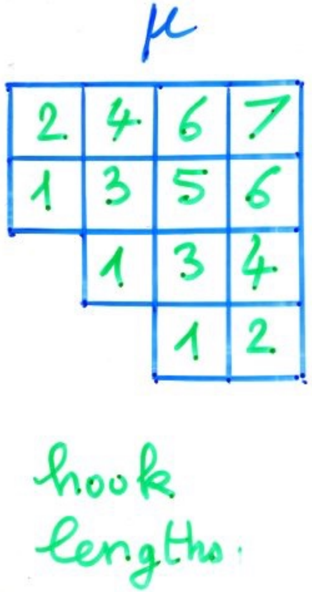


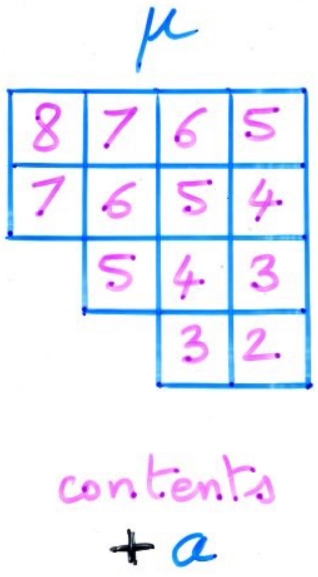
4	3	2	1
6	4	2	1
	4	2	1
		3	2

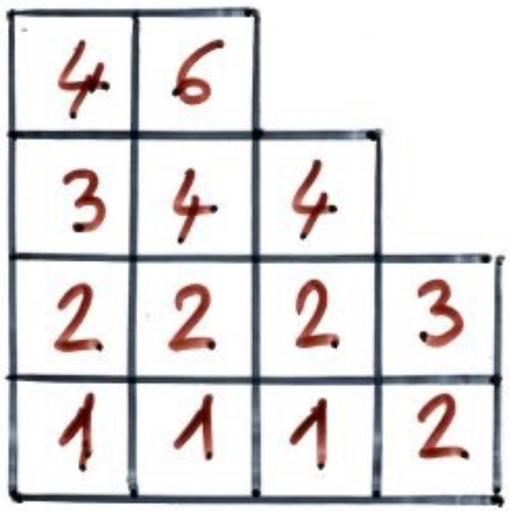


4	6		
3	4	4	
2	2	2	3
1	1	1	2

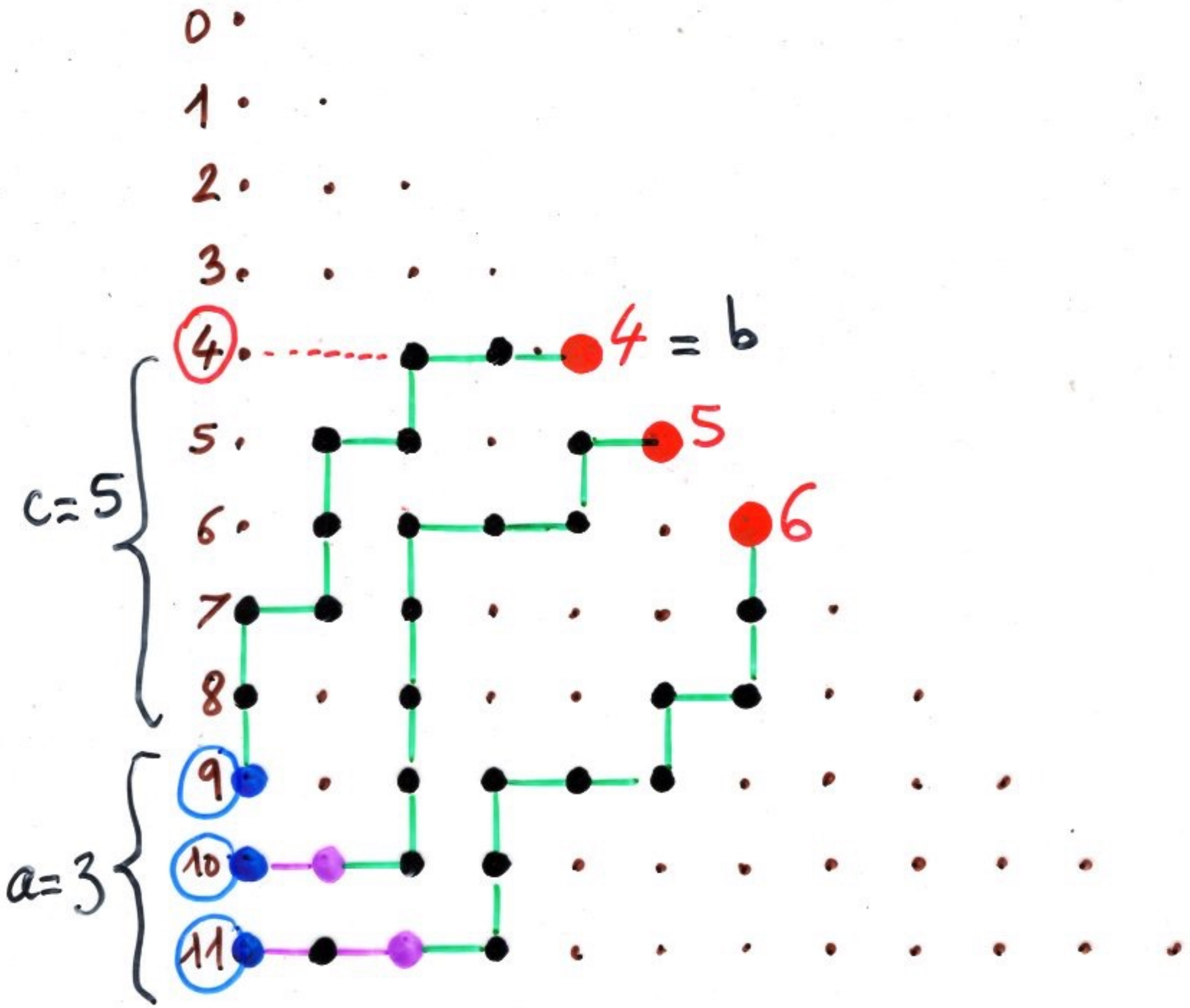

a=5
= μ
1
λ=2
4
3 2 1 1 = λ*
0
1
2
3
4
a=5
6
7
8
2
4
6
7
6
4
4 4
3
3
2 2 2
2
1 1 1

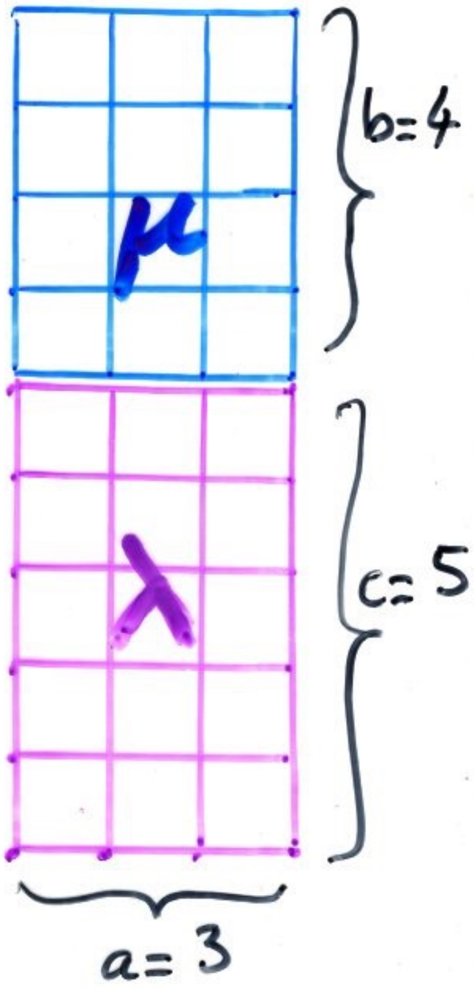

μ
2 4 6 7
1 3 5 6
1 3 4
1 2
hook lengths.

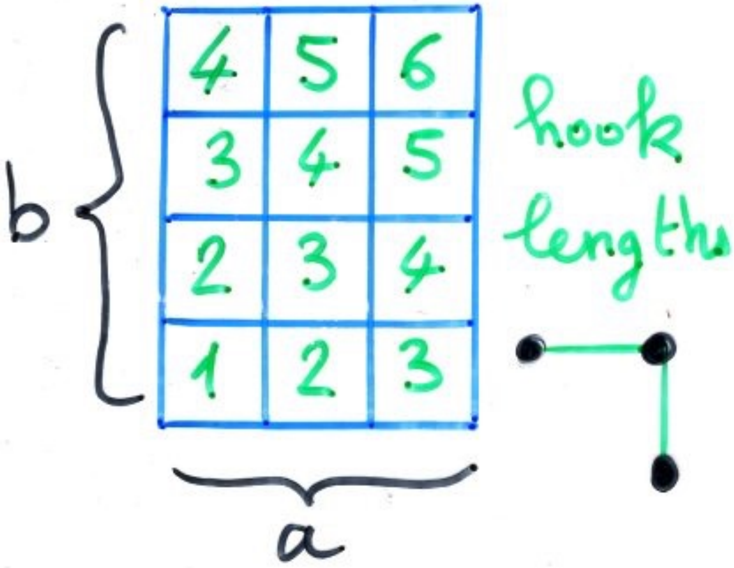

μ
8 7 6 5
7 6 5 4
5 4 3
3 2
contents + a

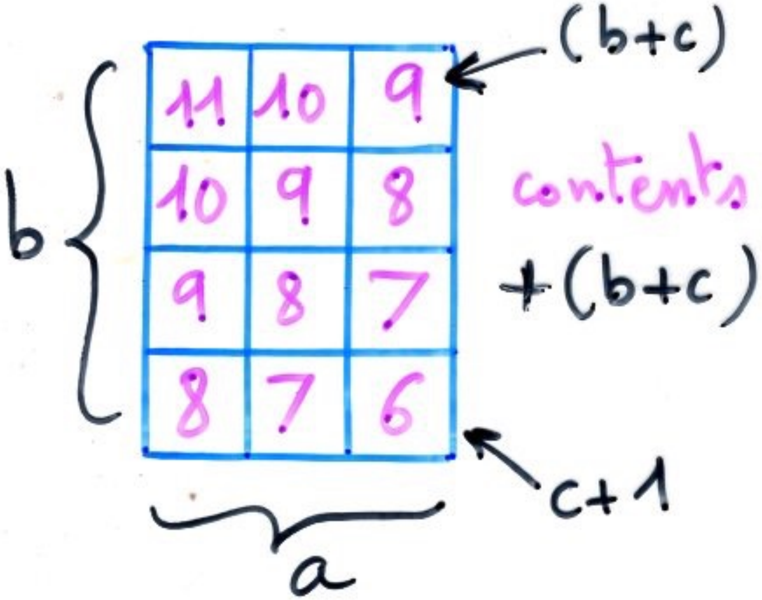

4 6
3 4 4
2 2 2 3
1 1 1 2

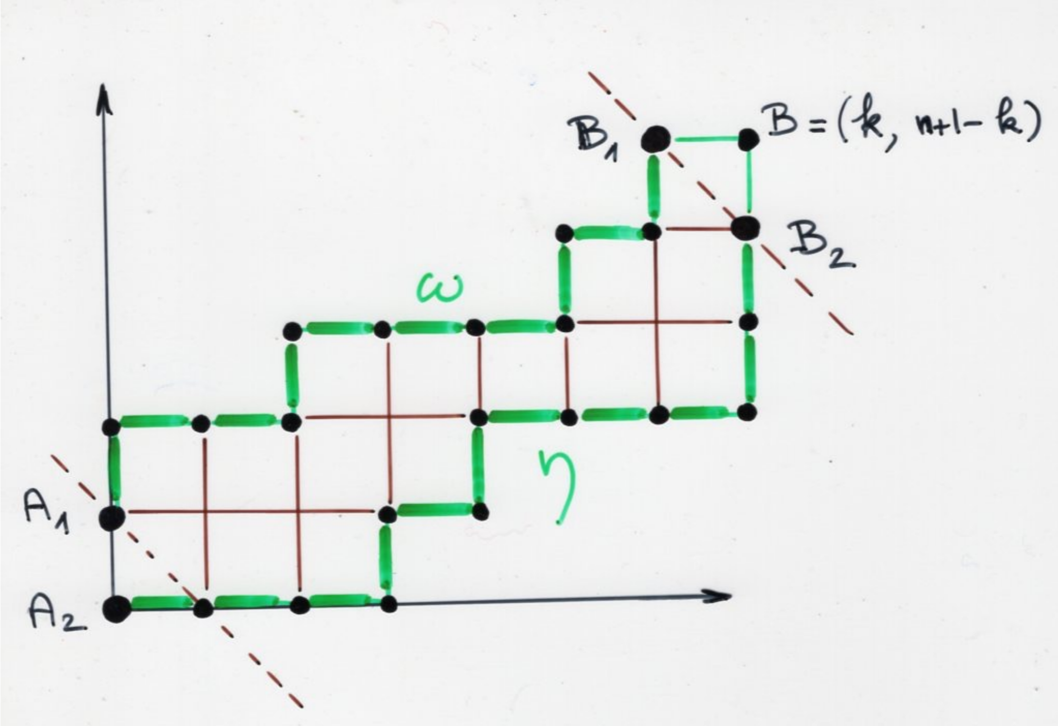
example:
Naranaya numbers
and
Baxter permutations


0
1
2
3
4
4 = b
5
5
6
6
7
8
9
10
11
c=5
a=3


b=4
μ
λ
c=5
a=3


4 5 6
3 4 5
2 3 4
1 2 3
b
a
hook lengths


(b+c)
11 10 9
10 9 8
9 8 7
8 7 6
b
a
contents
+(b+c)
c+1



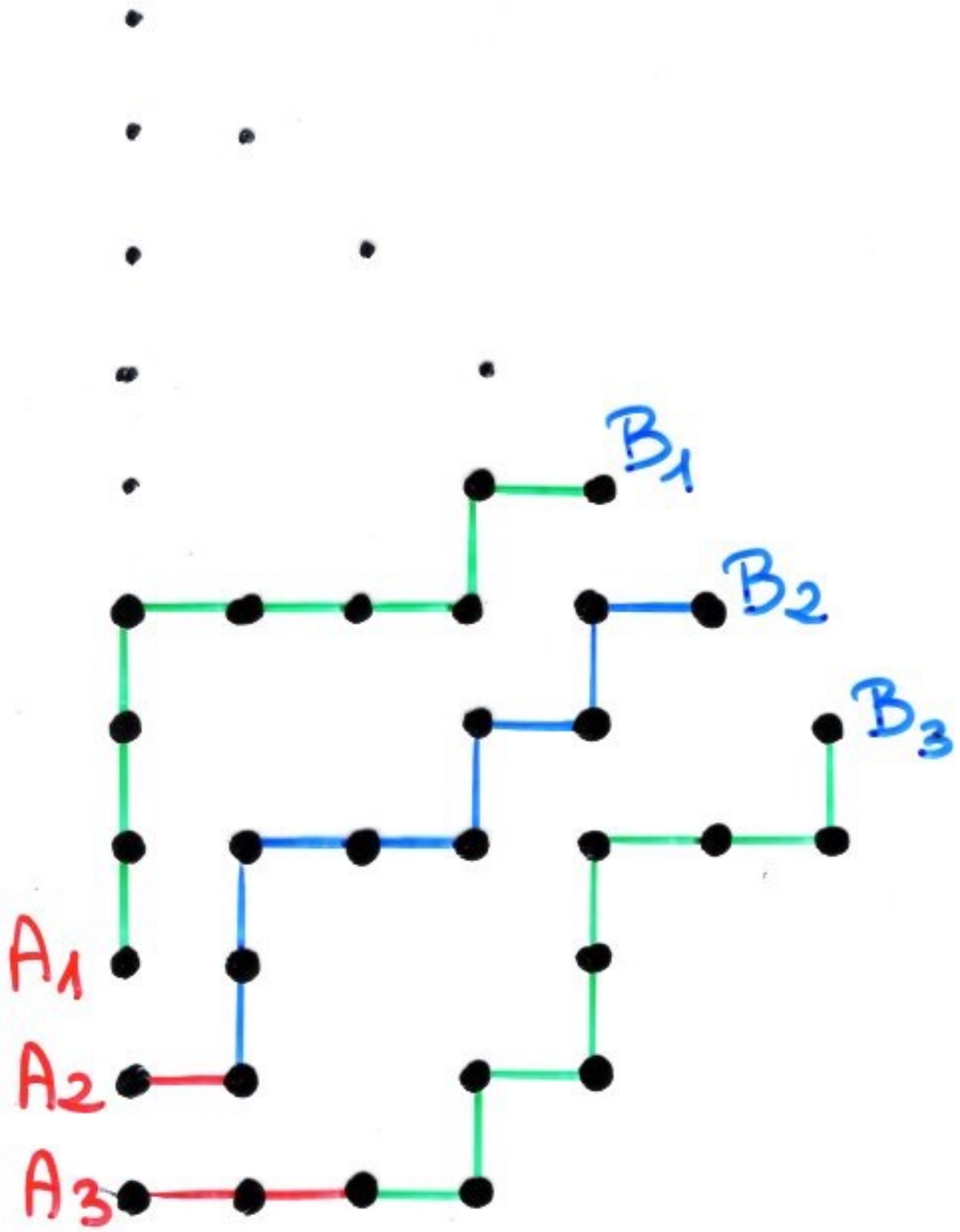
$A_2=(0,0)$ $A_1=(0,1)$

$B_2=(k, n-k)$ $B_1=(k-1, n+1-k)$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} \\ \binom{n}{k-1} & \binom{n}{k} \end{bmatrix}$$

and $$\det(A) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

Narayana numbers


B₁
B₂
B₃
A₁
A₂
A₃

1

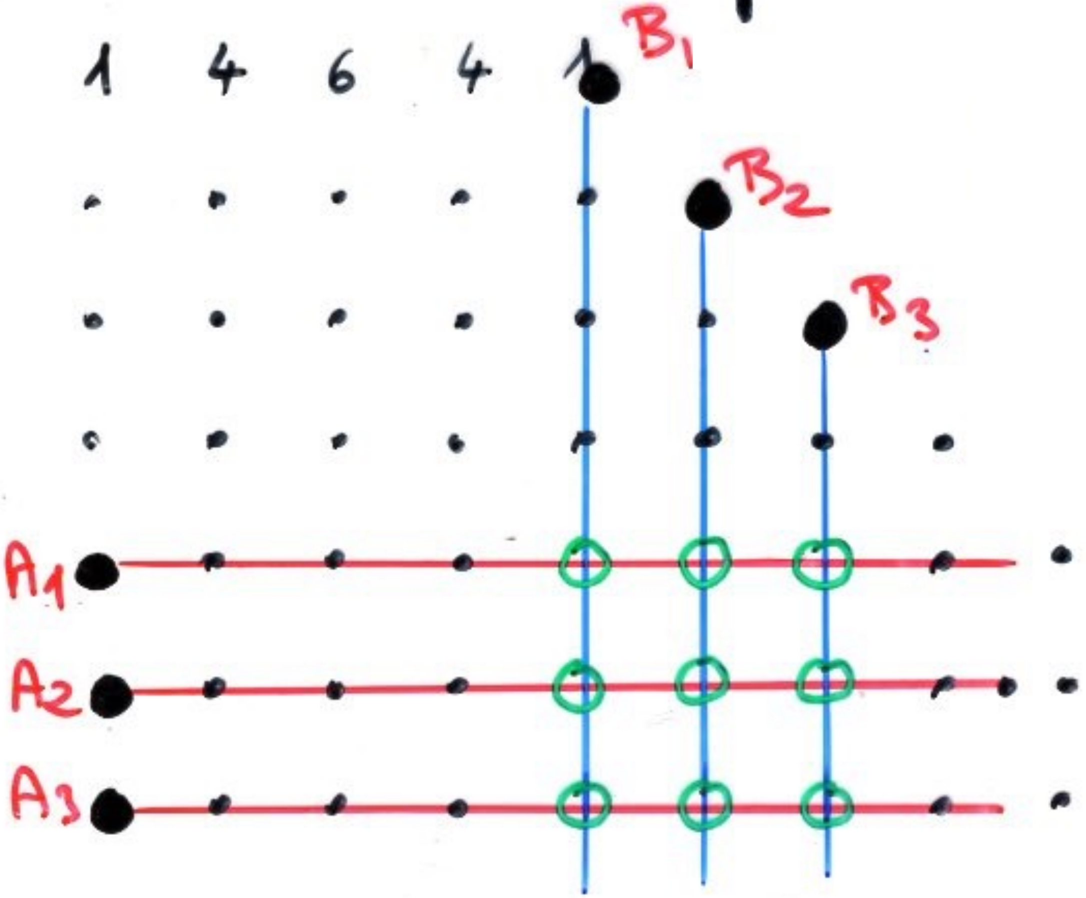
1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k-1} & \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$



Chung, Graham, Hoggatt, Kleiman (1978)

$$B(n) = \boxed{\frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}$$

Mallows (1979)

nb of Baxter permutations

having $(k-1)$ rises

$\sigma(i) < \sigma(i+1)$

$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k-1} & \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$

binomial determinants
other examples

Permutations with given up-down sequence

$$w = w_1 \dots w_{n-1}$$

$$w_i = \begin{cases} \diagup & \text{up} \\ \diagdown & \text{down} \end{cases}$$

let $a_1, .., a_k$ be the increasing sequence of indices of the letters w_i with $w_i = \diagdown$

<u>example</u> for $\sigma = 2\ 1\ 3\ 6\ 4$

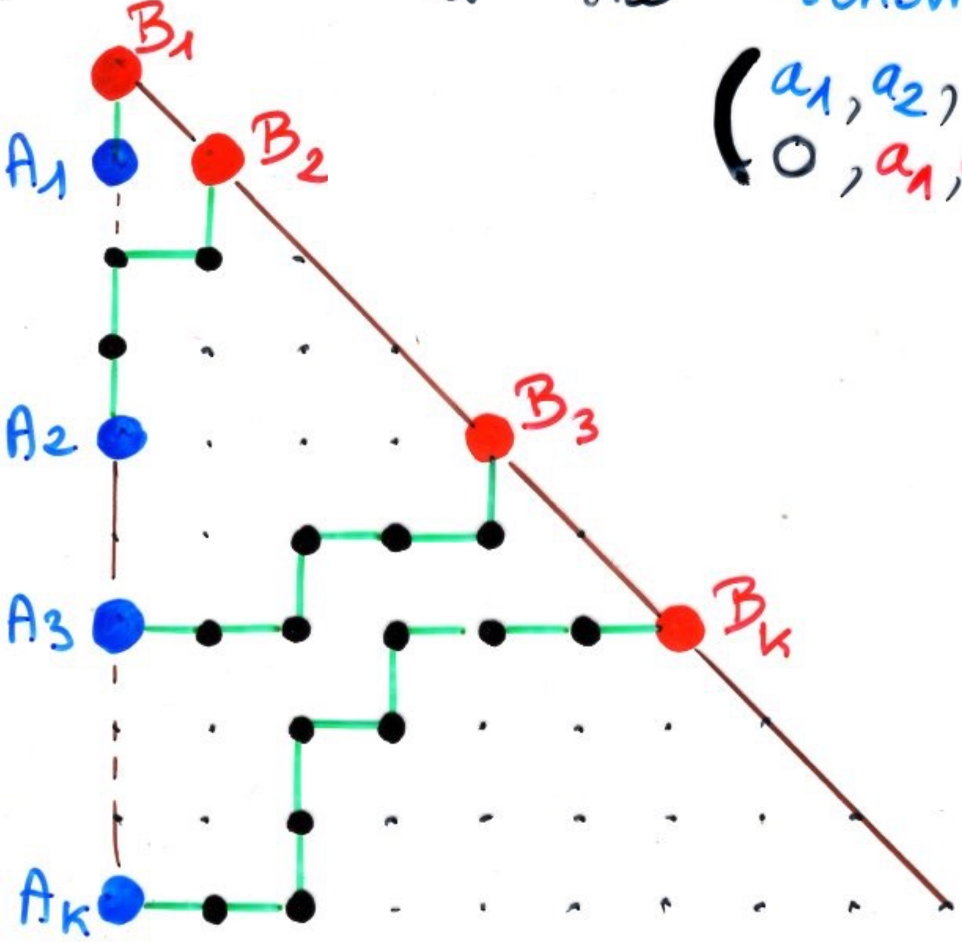
for $\sigma = 2\ 1\ 4\ 6\ 5\ 3 \in \mathfrak{S}_6$

$$w = \underset{\underbrace{w_1}}{\diagdown}\ \underset{w_2}{\diagup}\ \underset{w_3}{\diagup}\ \underset{\underbrace{w_4}}{\diagdown}\ \underset{\underbrace{w_5}}{\diagdown}$$

$$(a_1, a_2, a_3) = (1, 4, 5)$$

Proposition The number of permutations $\sigma \in \mathfrak{S}_n$ having w as up-down sequence is the binomial determinant

$$\begin{pmatrix} a_1, a_2, \dots, a_k \\ 0, a_1, a_2, \dots, a_{k-1} \end{pmatrix}$$



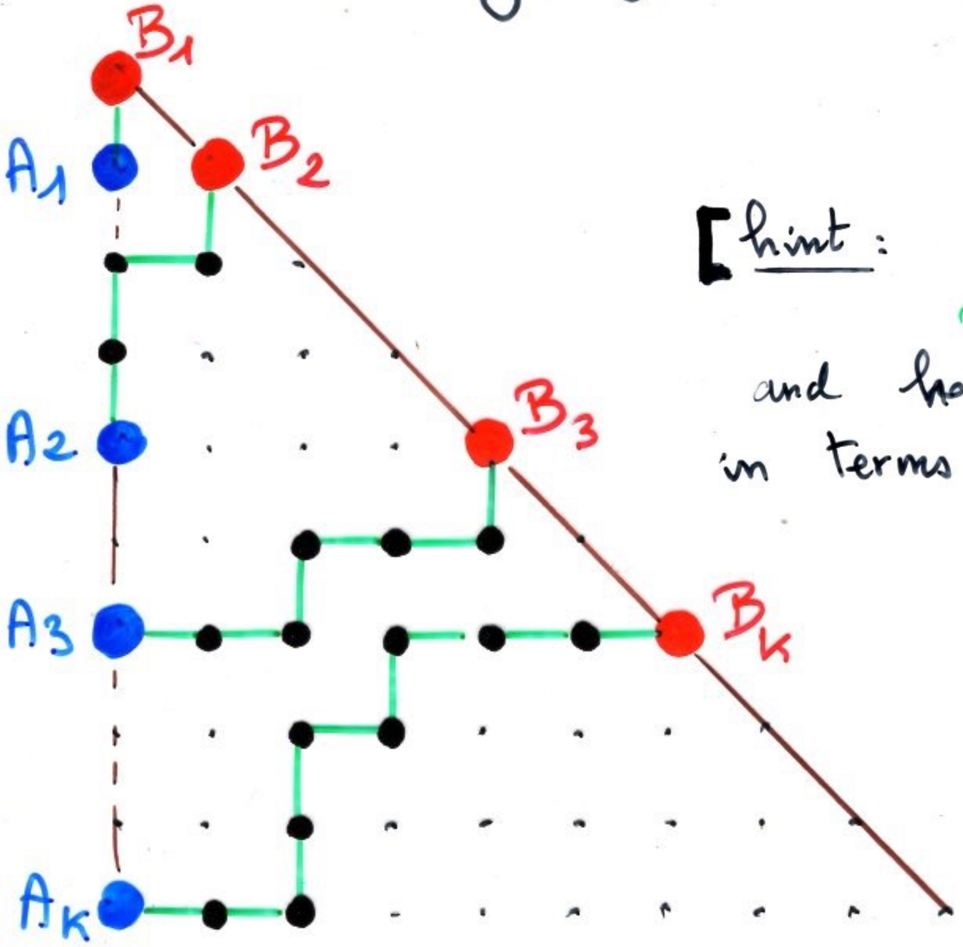
example Tangent numbers T_{2n+1}

$$T_{2n+1} = \begin{pmatrix} 1, & 3, & 5, & \dots, & 2n+1 \\ 0, & 1, & 3, & \dots, & 2n-1 \end{pmatrix}$$

alternating permutations (D. André)
(2n+1) (1880 ...)

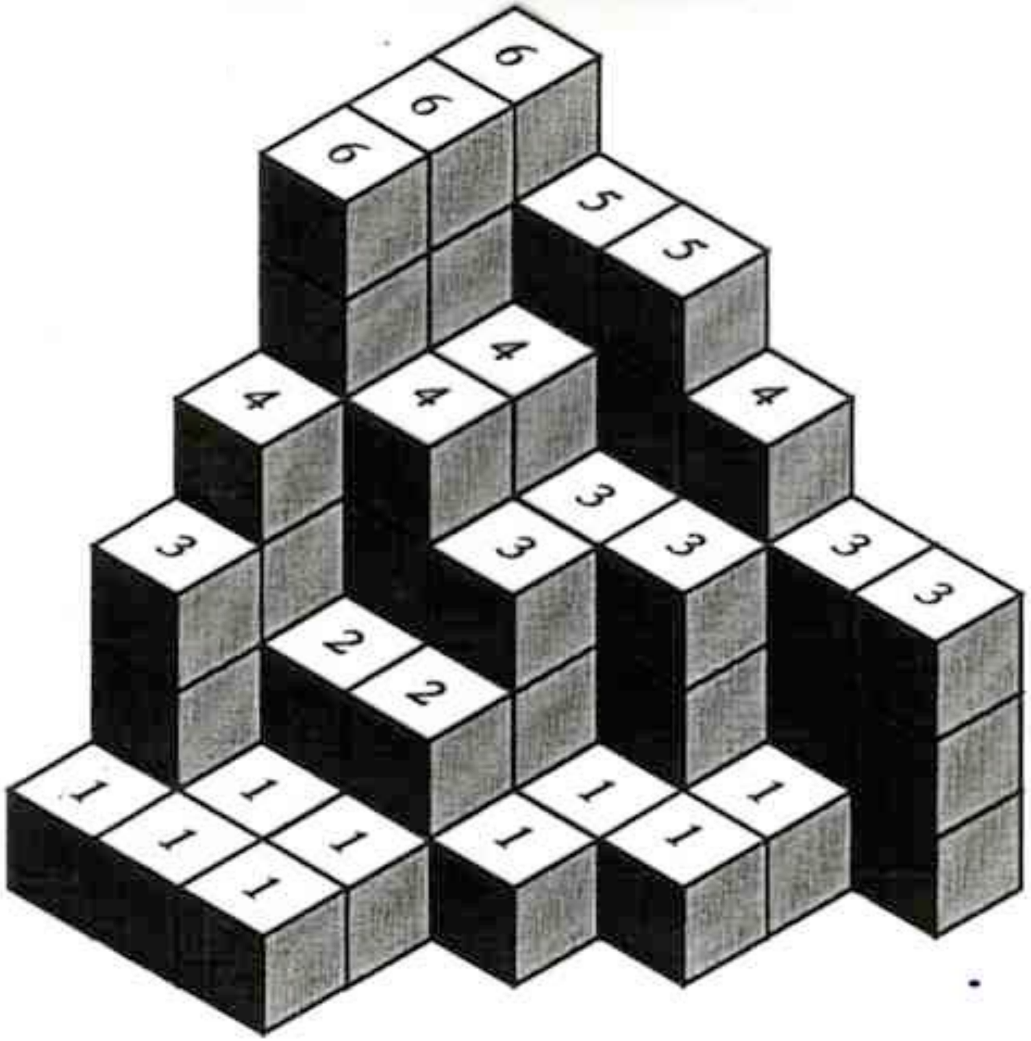
$$5 \searrow 2 \nearrow 9 \searrow 6 \nearrow 7 \searrow 1 \nearrow 4 \searrow 3 \nearrow 8$$

__exercise__ Find a bijection between permutations having $w \in \{ /\!\!/ , \backslash \}^*$ as up-down sequence and configurations of non-crossing paths $(\omega_1, \ldots, \omega_k)$ starting from $(A_1, \ldots, A_k)$ to $(B_1, \ldots, B_k)$

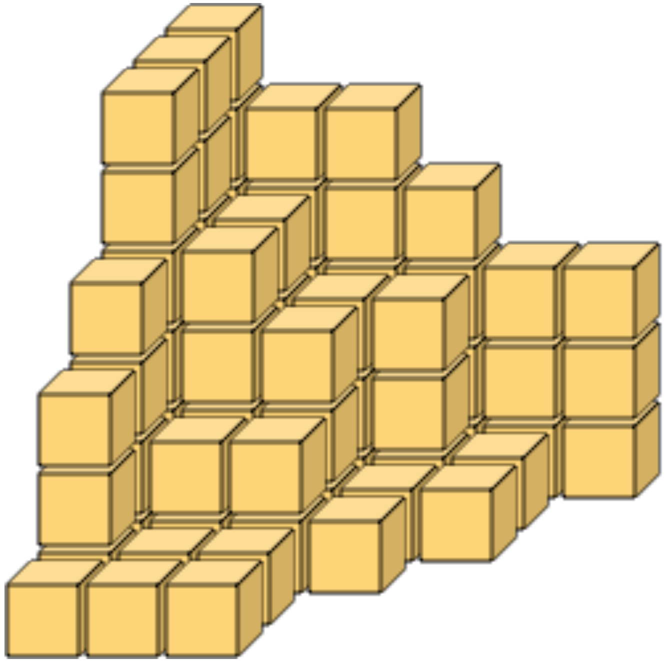


[__hint__: use an inversion table of permutations and how up-down sequences translate in terms of inversion table]

Planes partitions



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



bounded plane partitions

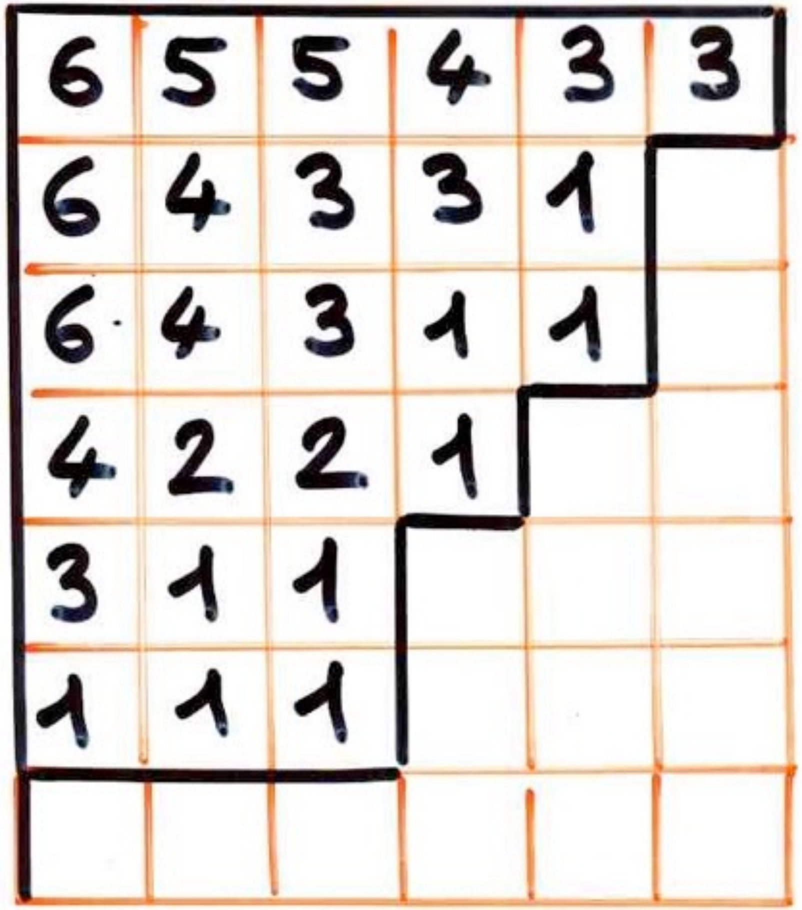
3D Ferrers diagram
in a box

$F \subseteq \mathcal{B}(a,b,c)$

$$\mathcal{B}(a,b,c) = \left\{ (i,j,k) \in \mathbb{N}^3, \begin{array}{l} 1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c \end{array} \right\}$$

$\mathcal{B}(a,b,c)$: at most a rows
at most b columns
parts $\leq c$

$\mathcal{B}(7,6,6)$

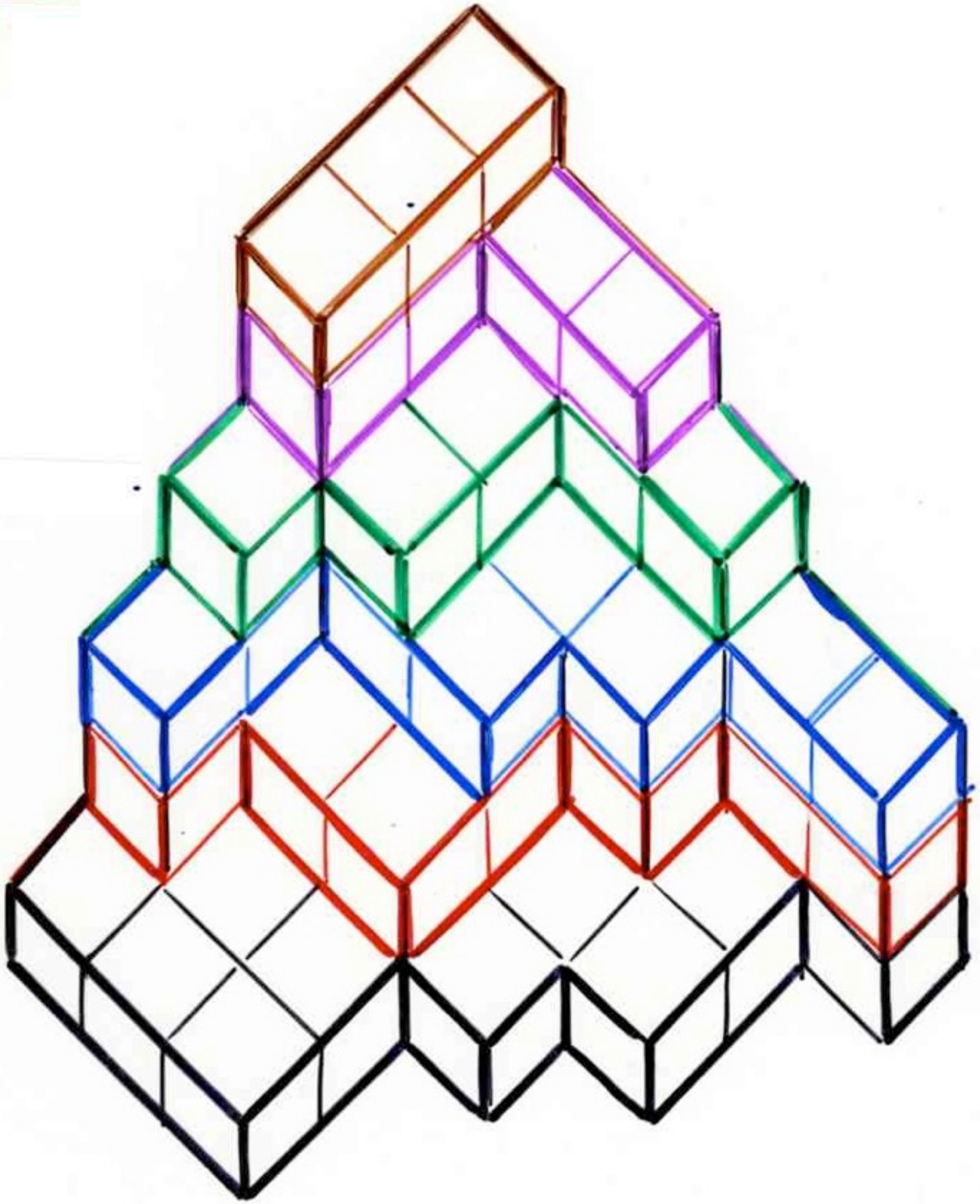


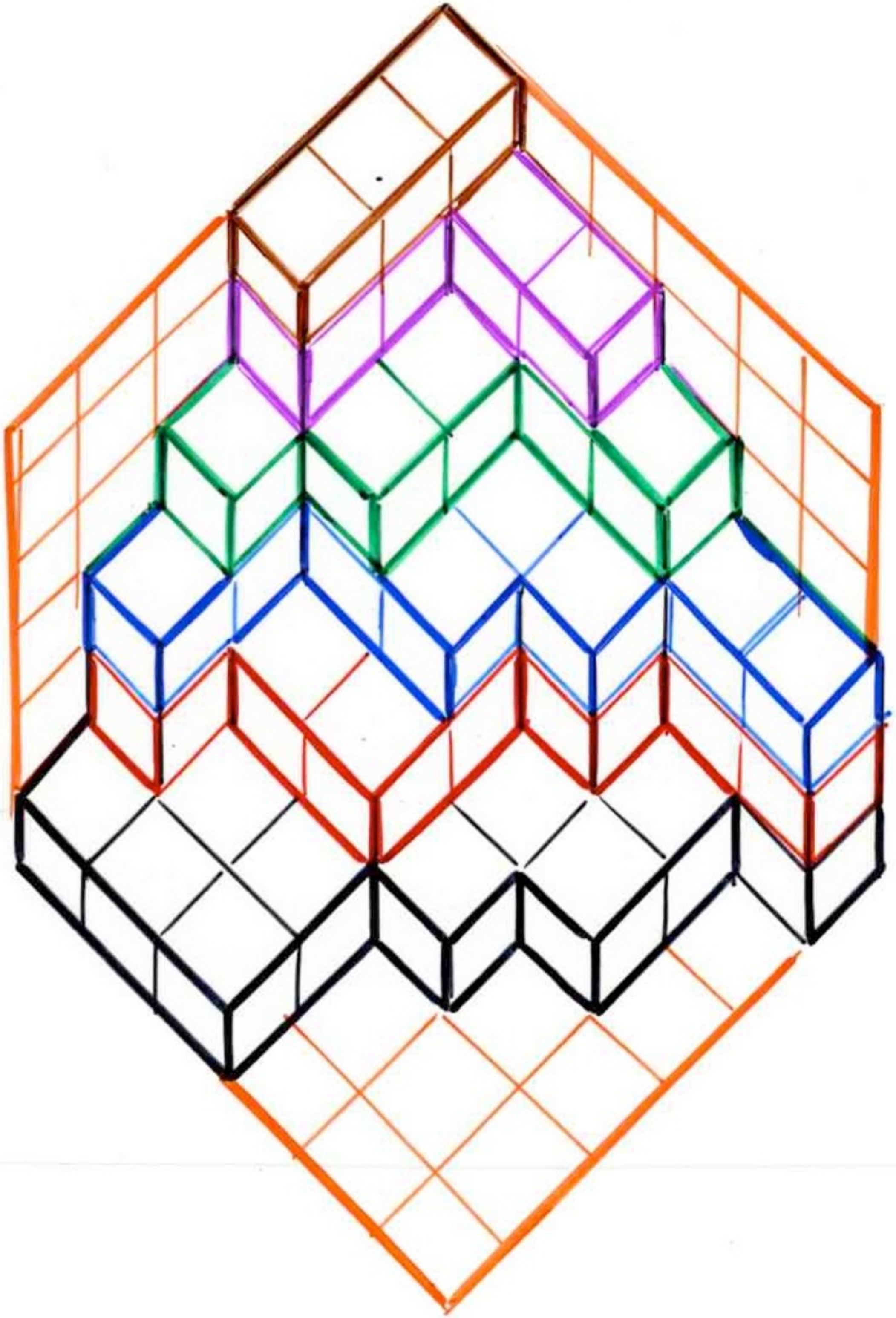
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

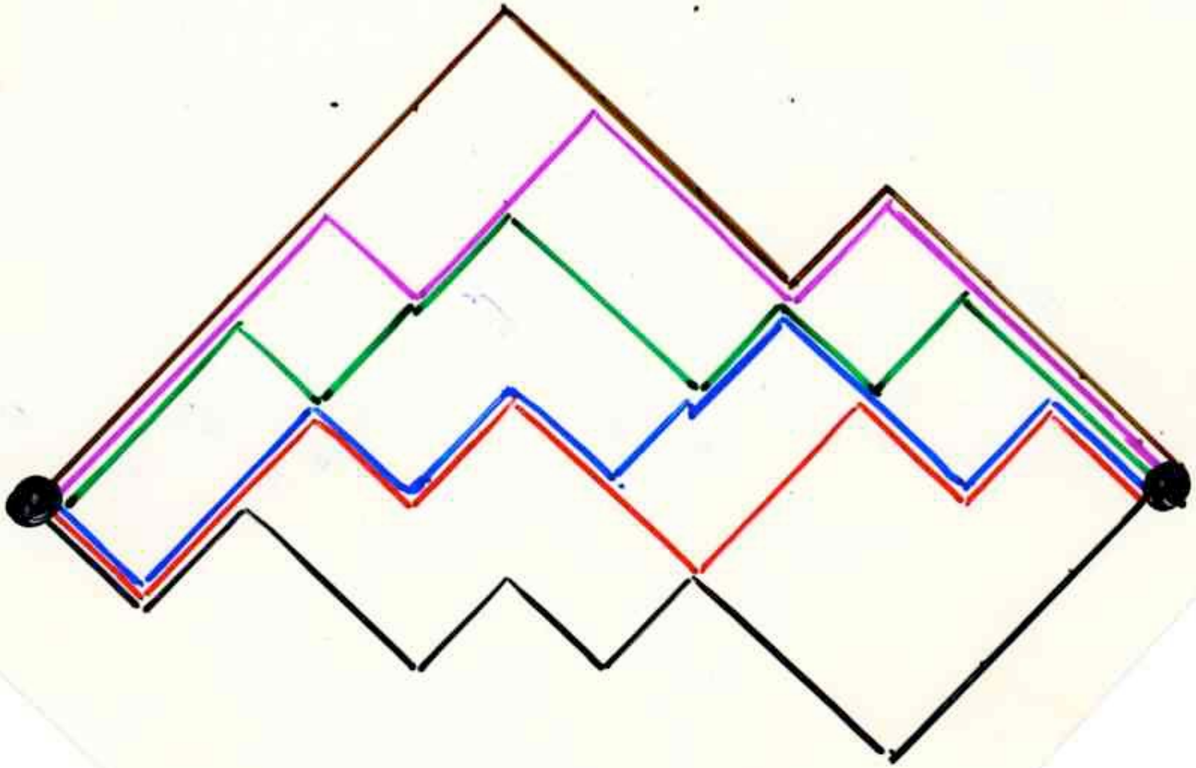
Paths for plane partitions

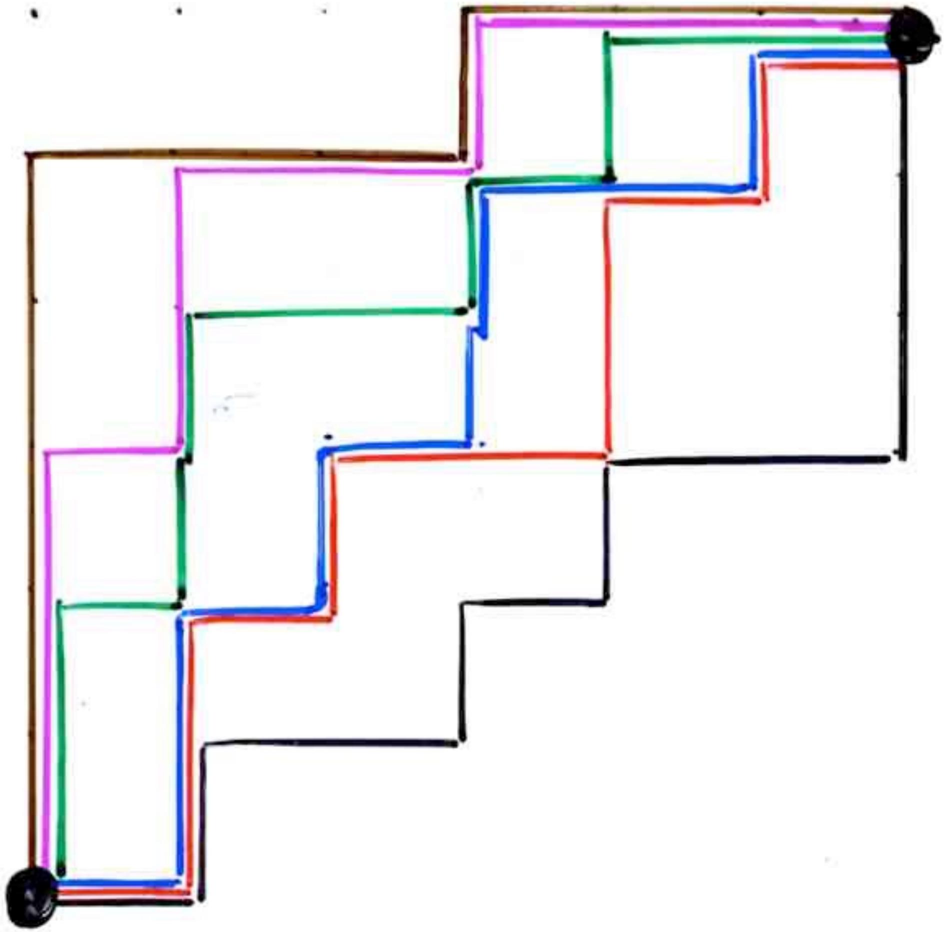
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

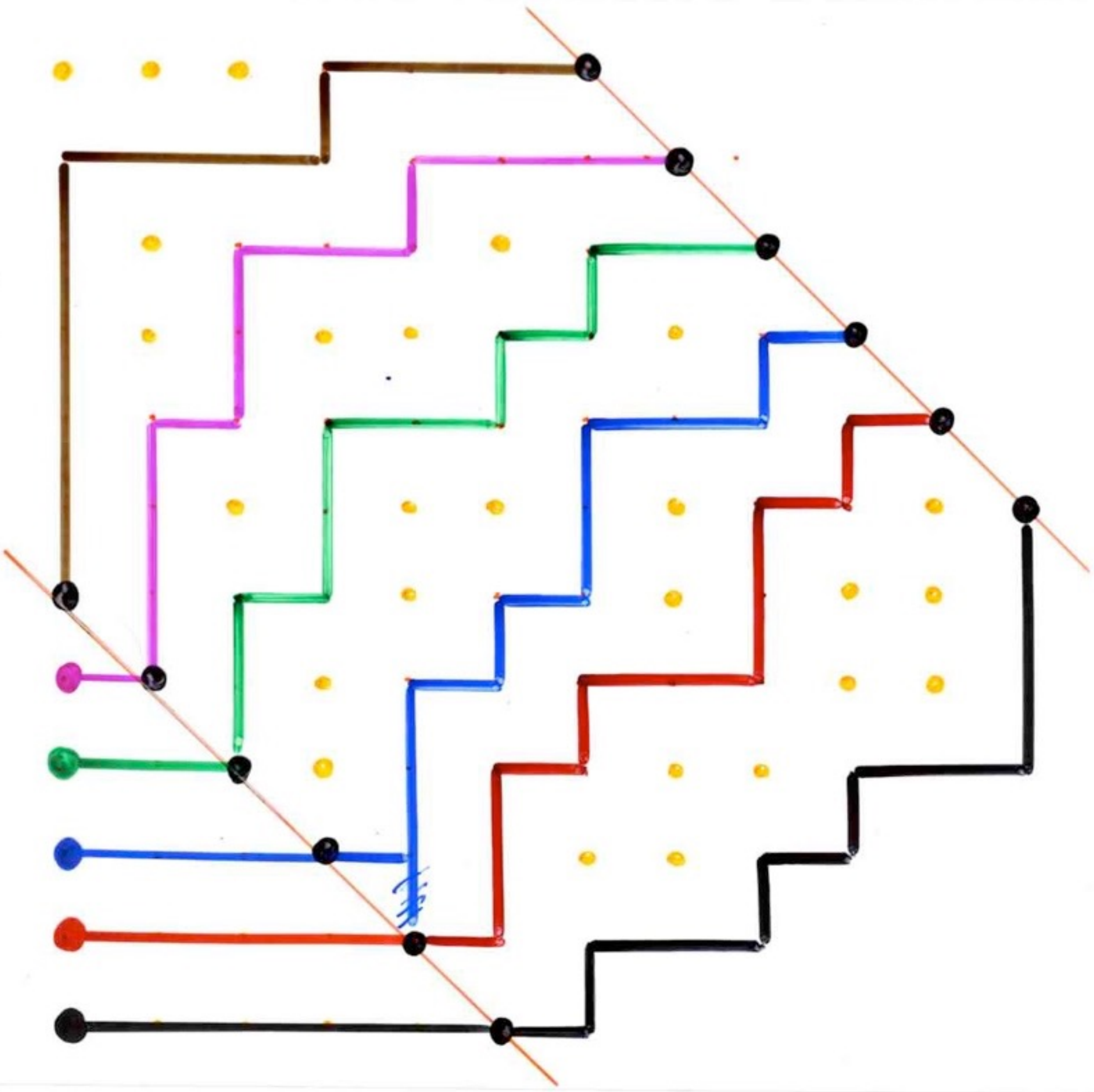


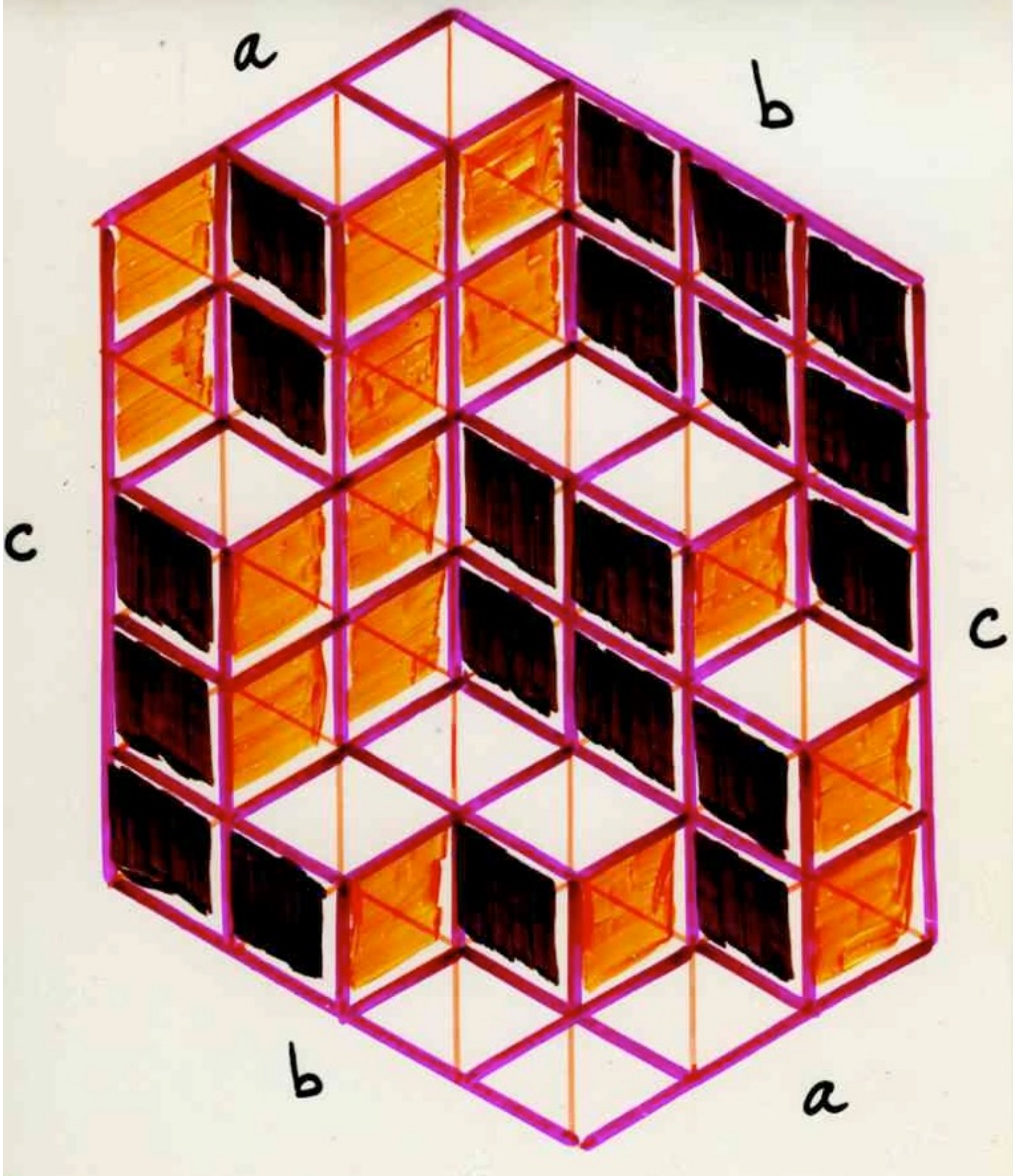


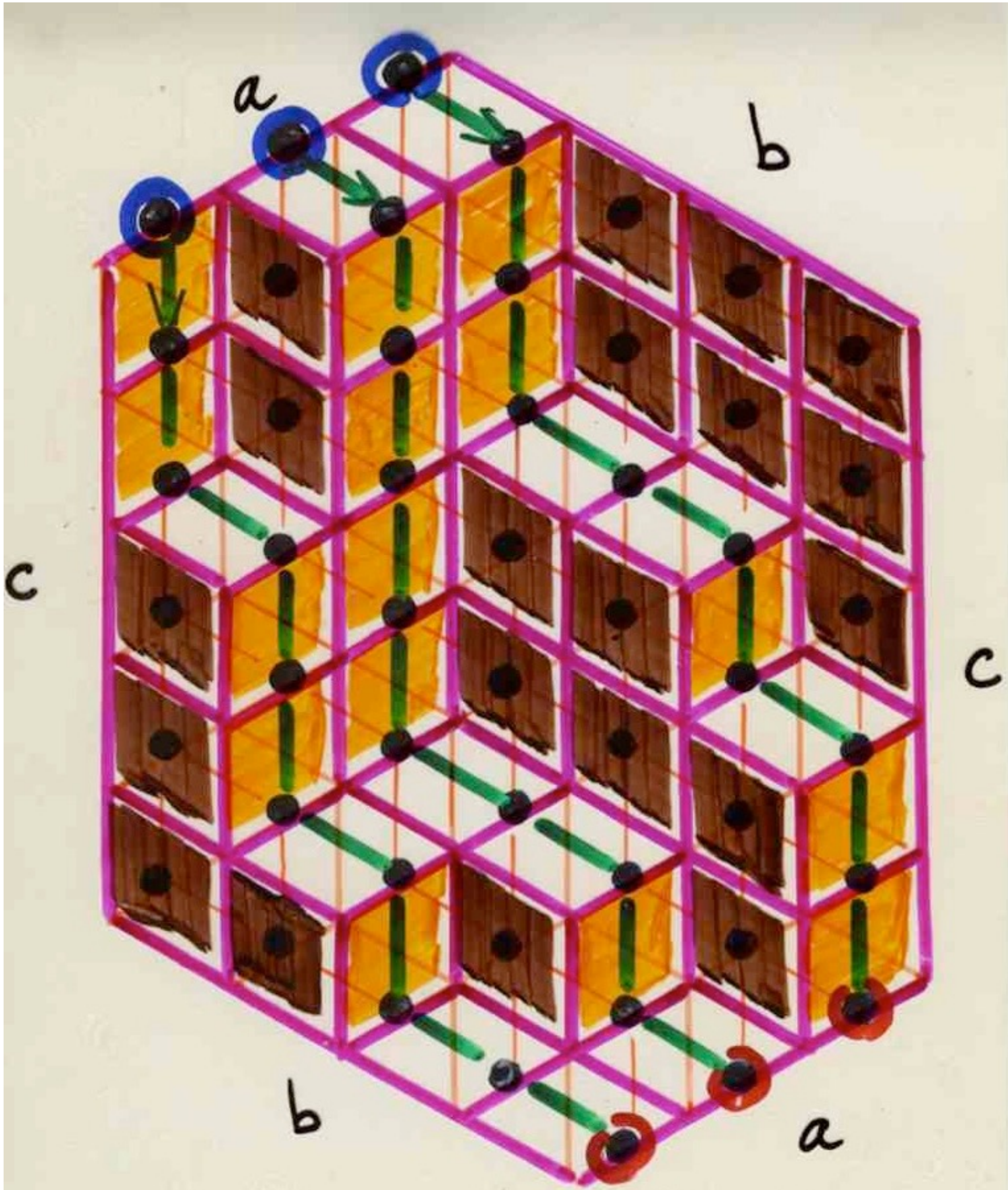
coding a plane partition
with
non-intersecting paths

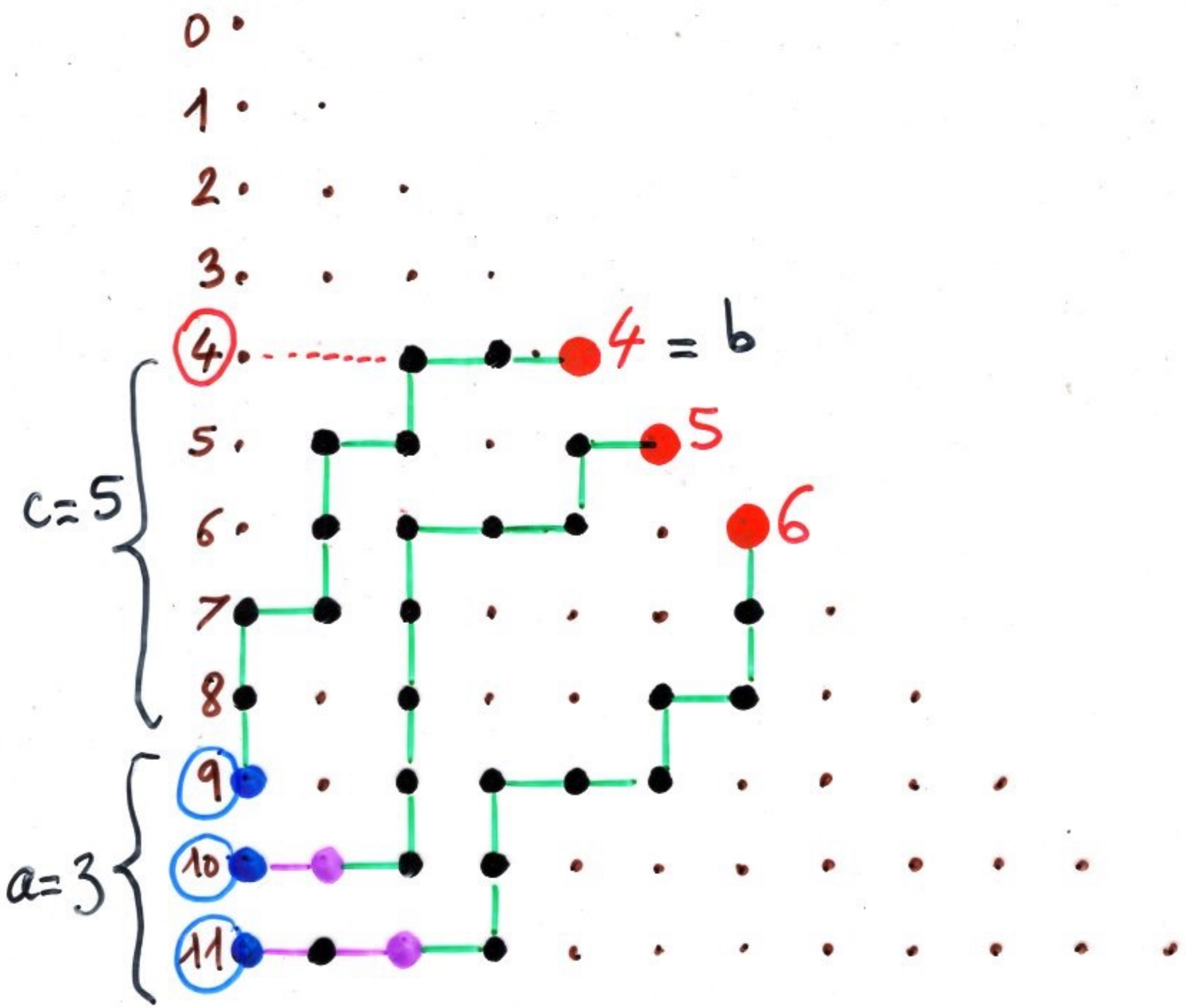


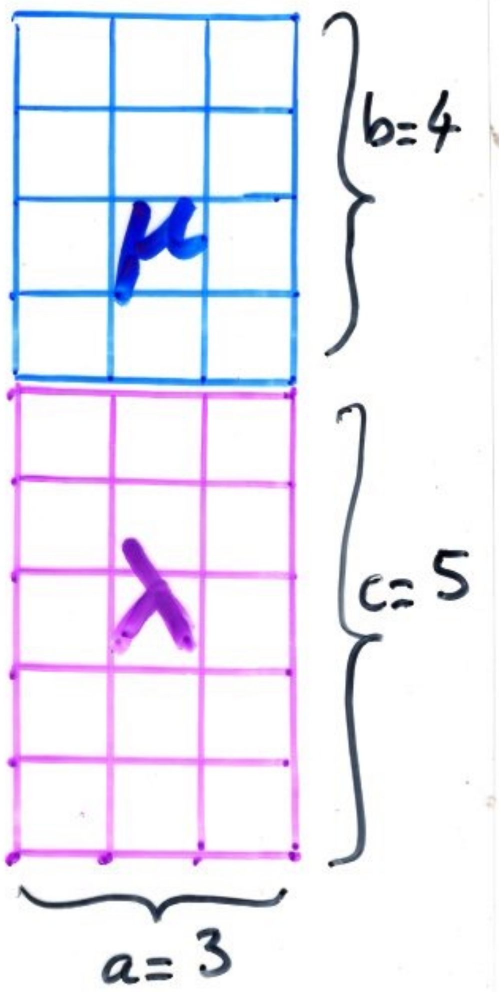


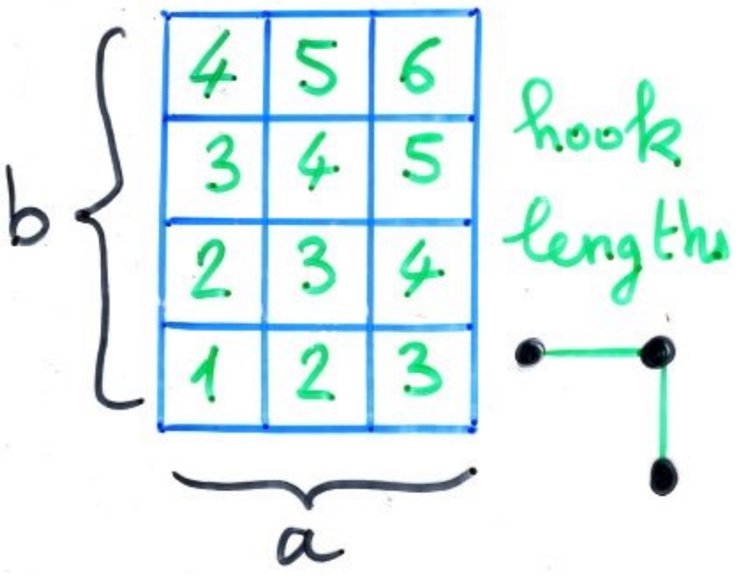


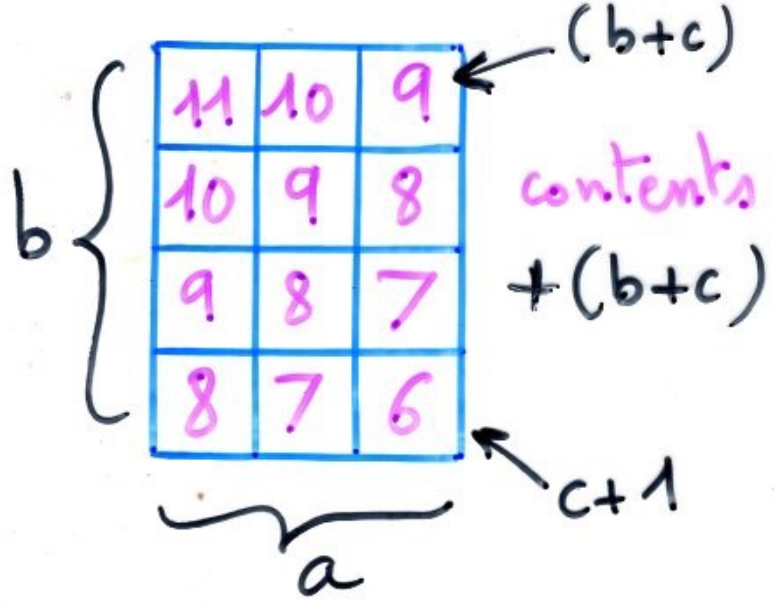

a
b
c
c
b
a


a
b
c
c
b
a

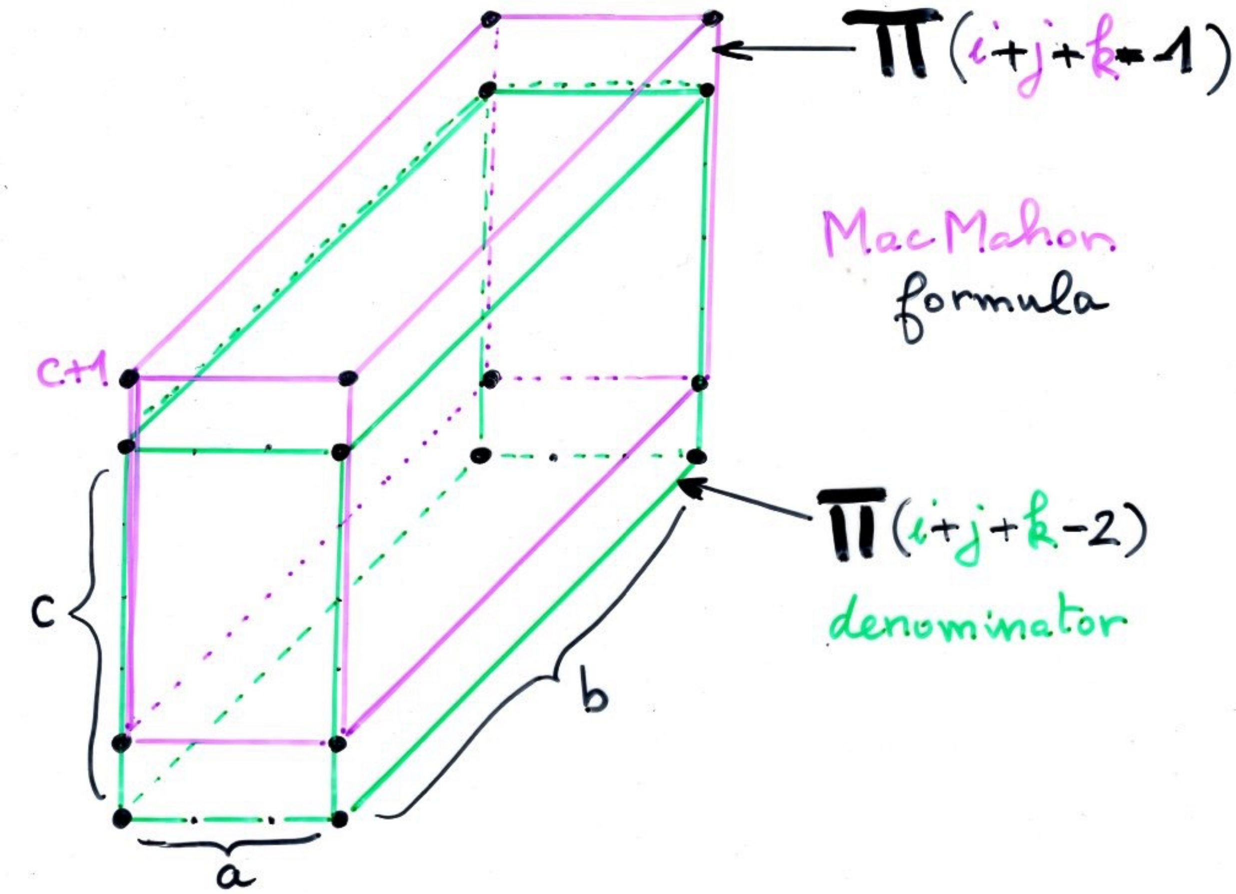

0
1
2
3
4
5
6
7
8
9
10
11
4 = b
5
6
c=5
a=3

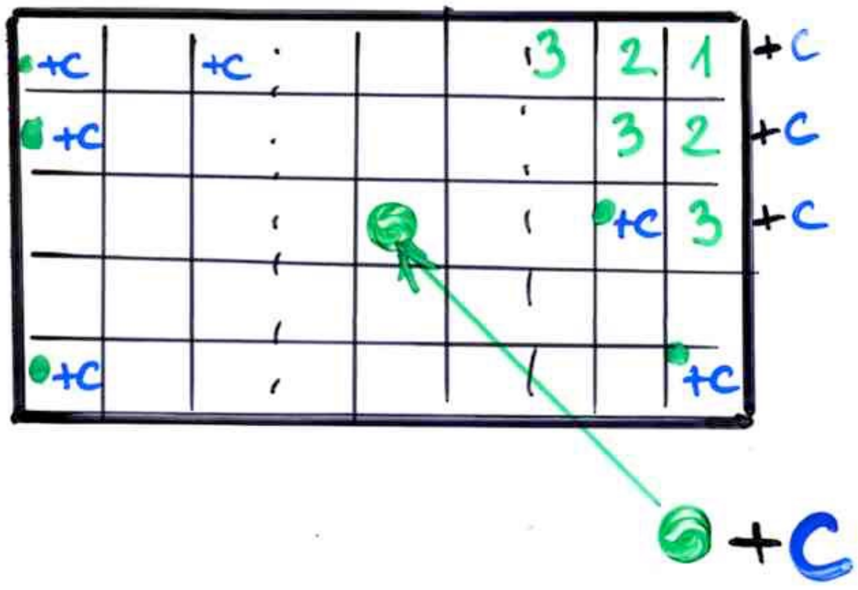

μ
λ
b=4
c=5
a=3


4 5 6
3 4 5
2 3 4
1 2 3
b
a
hook lengths

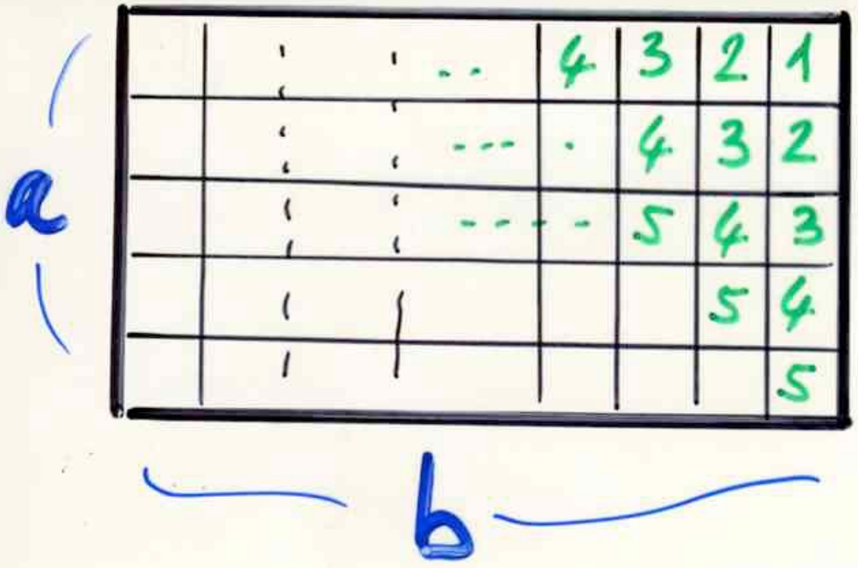

(b+c)
11 10 9
10 9 8
9 8 7
8 7 6
b
a
contents
+(b+c)
c+1

$$\prod_{\substack{1\leq i\leq a\\ 1\leq j\leq b\\ 1\leq k\leq c}} \frac{i+j+k-1}{i+j+k-2}$$


$\prod(i+j+k-1)$
MacMahon formula
$c+1$
c
$\prod(i+j+k-2)$
denominator
b
a



+c
+c
3
2
1
+c
+c
3
2
+c
+c
3
+c
+c
+c
+C



4
3
2
1
4
3
2
5
4
3
5
4
5
a
b