An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc
January-March 2016

Chapter 4

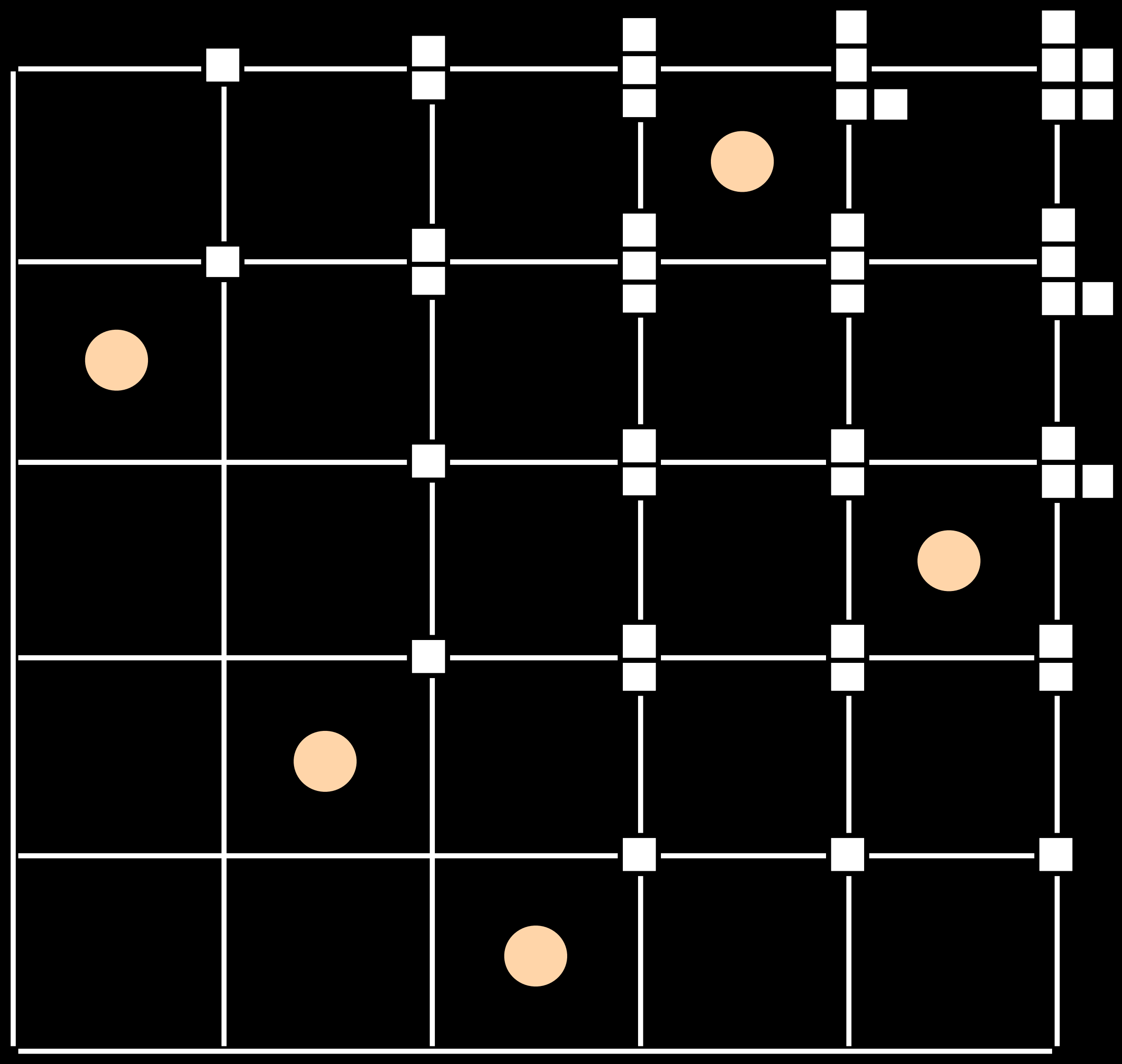
The n! garden
(4)

IMSc
25 February 2016

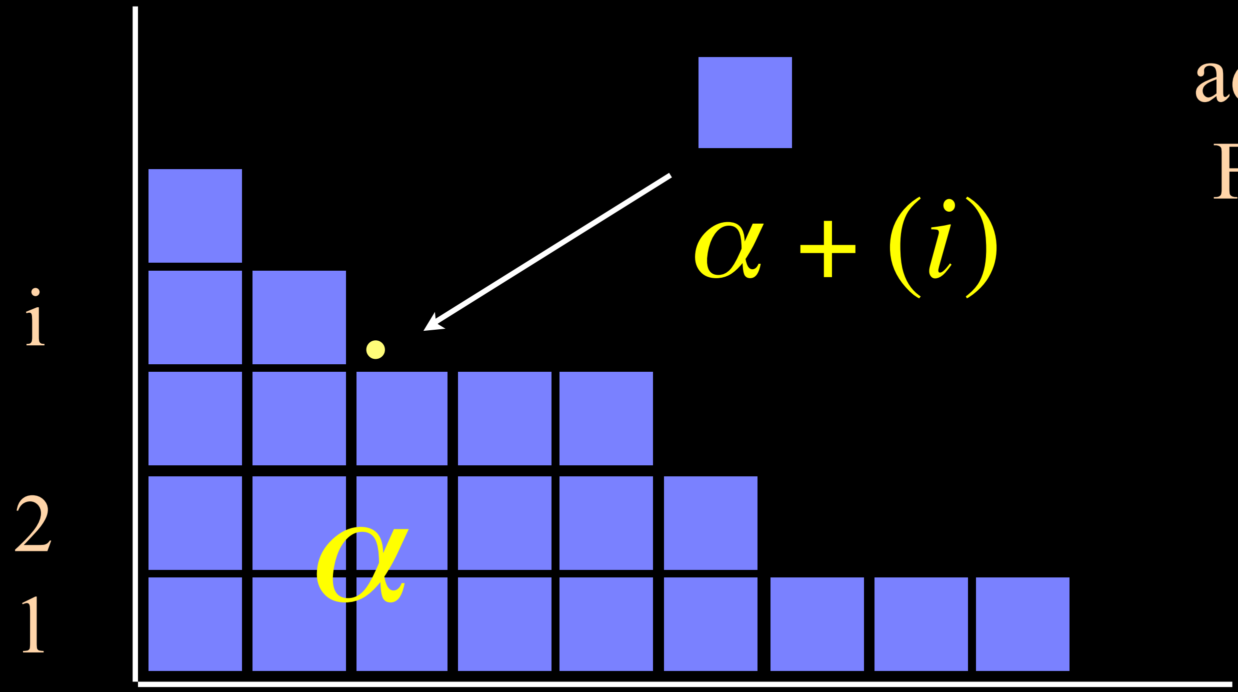
“local” algorithm on a grid or “growth diagrams”

S. Fomin, 1986, 1994

M. van Leeuwen, 1996



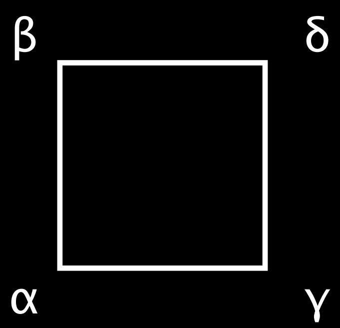
notations Operator U_i



adding a cell in a Ferrers diagram at row i

$$U_i(\alpha) = \alpha + (i)$$

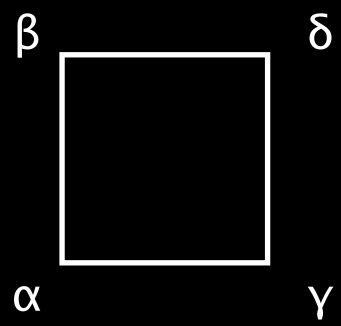
$\beta \neq \gamma$



$\delta = \beta \cup \gamma$

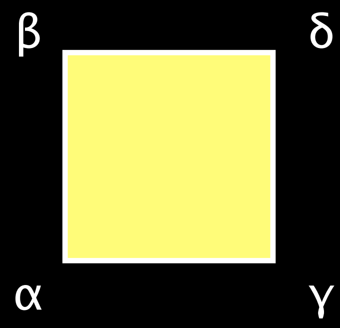
$\beta = \gamma$

$\beta = \gamma$
$\alpha \neq \beta$



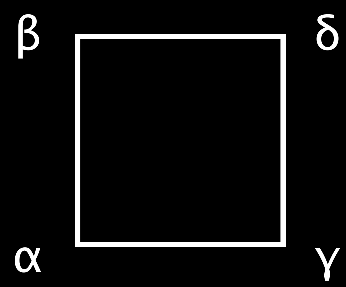
$\beta = \gamma = \alpha + (i)$
$\delta = \beta + (i+1)$

$\alpha = \beta = \gamma$



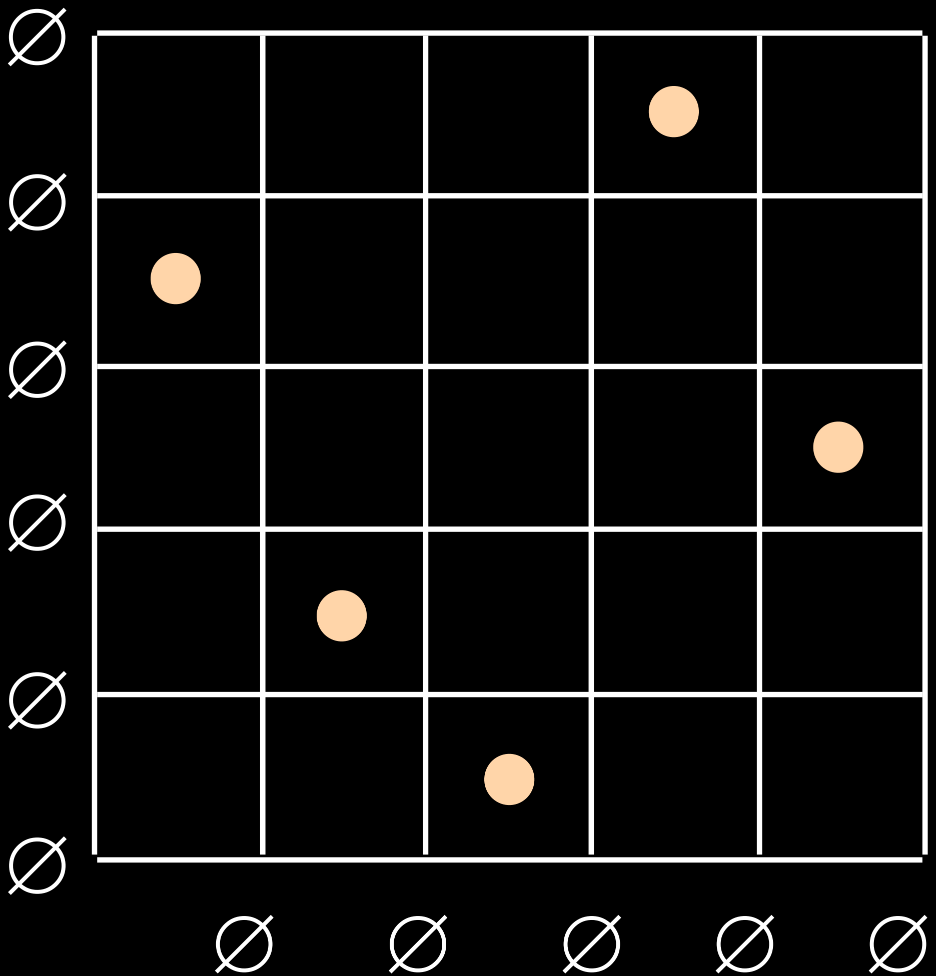
$\delta = \alpha + (1)$

$\alpha = \beta = \gamma$

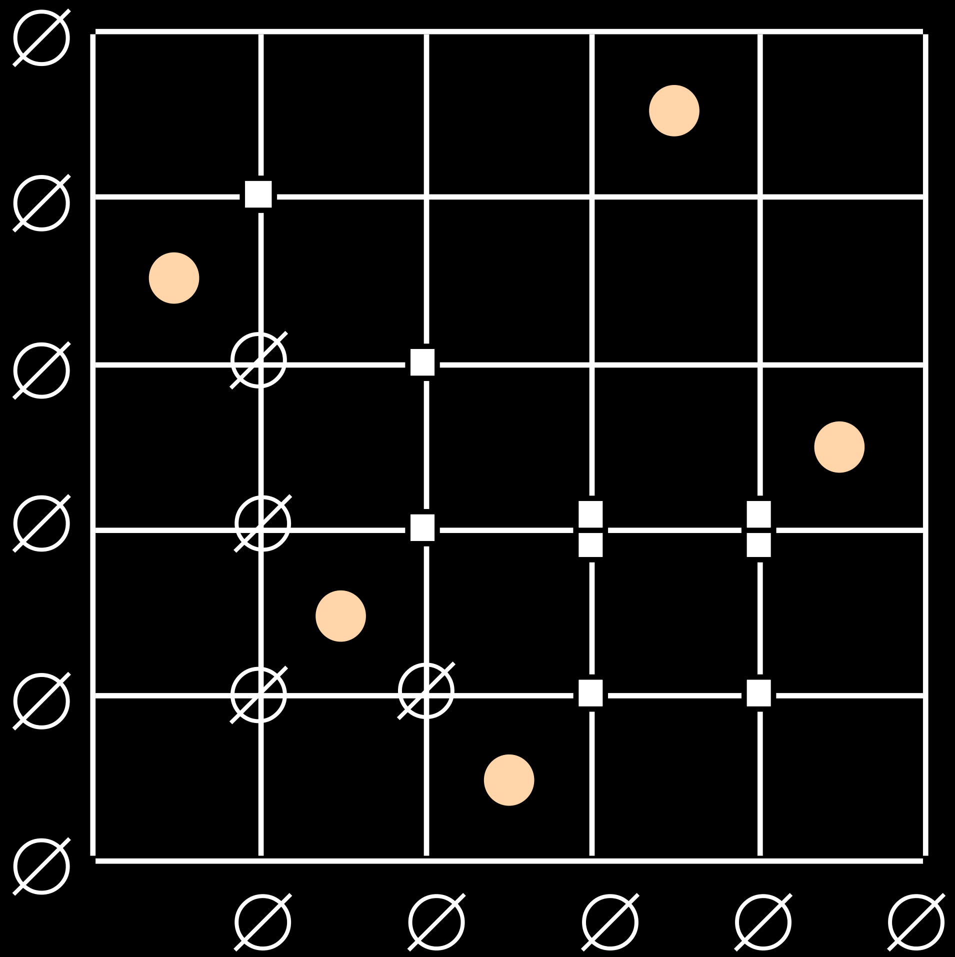


$\delta = \alpha = \beta = \gamma$

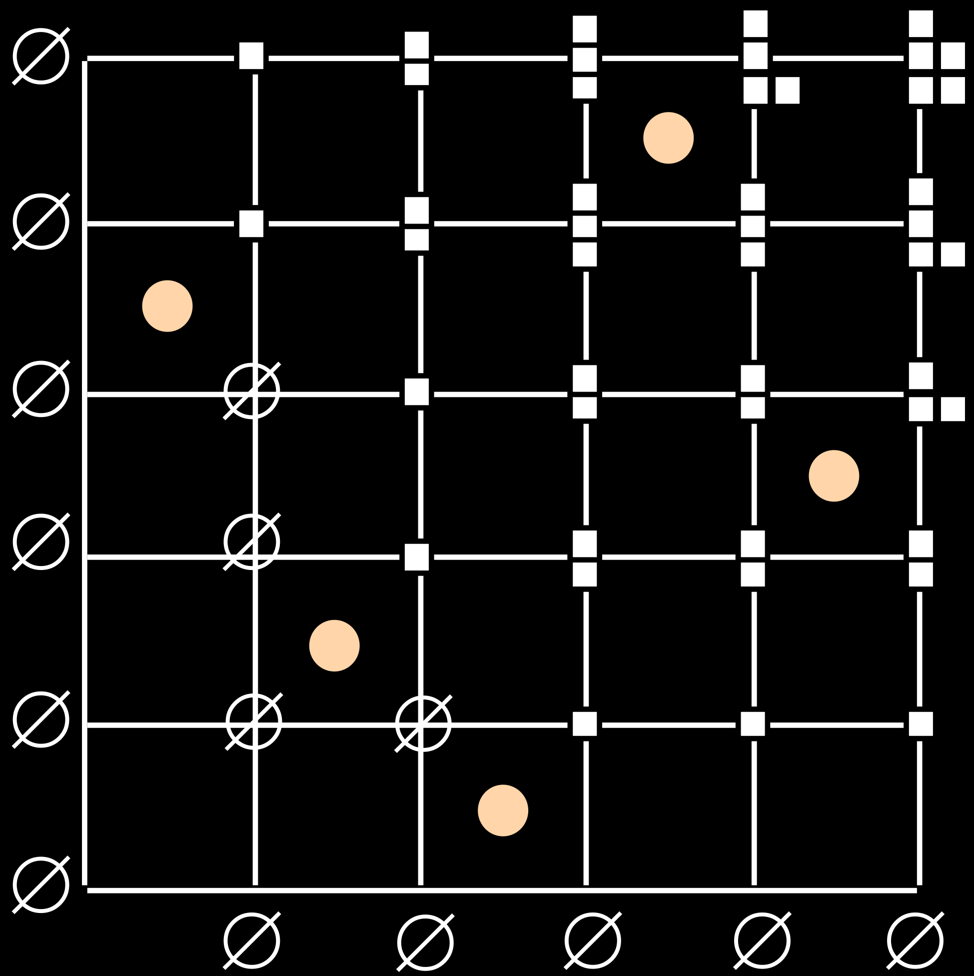
initial state



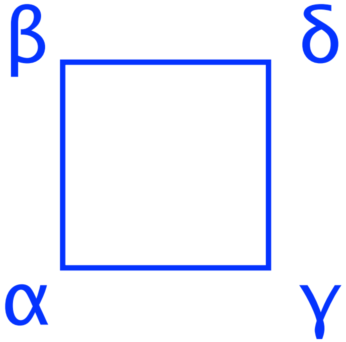
during the labeling process



final state



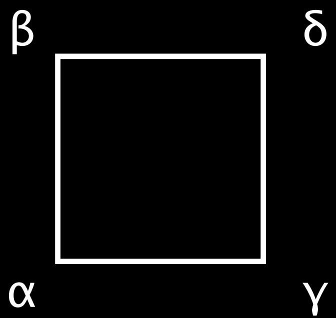
- in the labeling process of the vertices of the grid with Ferrers diagrams :
  independance of the order in which the labeling is done



- in the grid, for cell, δ is obtained from β by adding a cell, or is equal to β

- in the grid, for cell, δ is obtained from γ by adding a cell, or is equal to γ

$\beta \neq \gamma$



$\delta = \beta \cup \gamma$

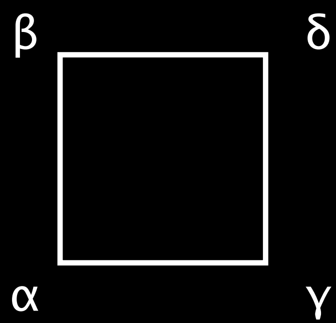
$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$

$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$

$\beta = \gamma$

$\beta = \gamma$
$\alpha \neq \beta$



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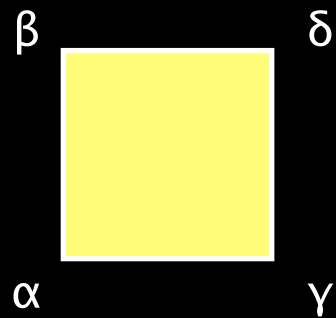

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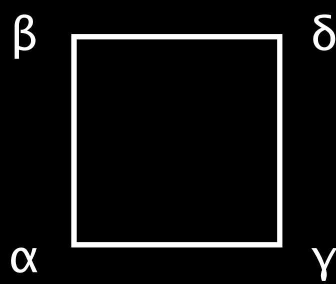

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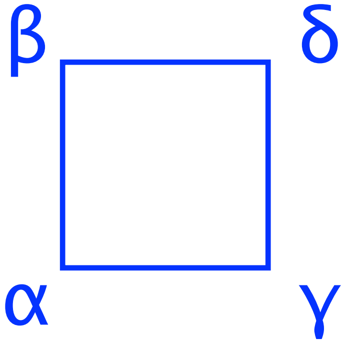

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- in the last rows and last columns:
- we get maximal chains of Ferrers diagrams

poset $\preccurlyeq$

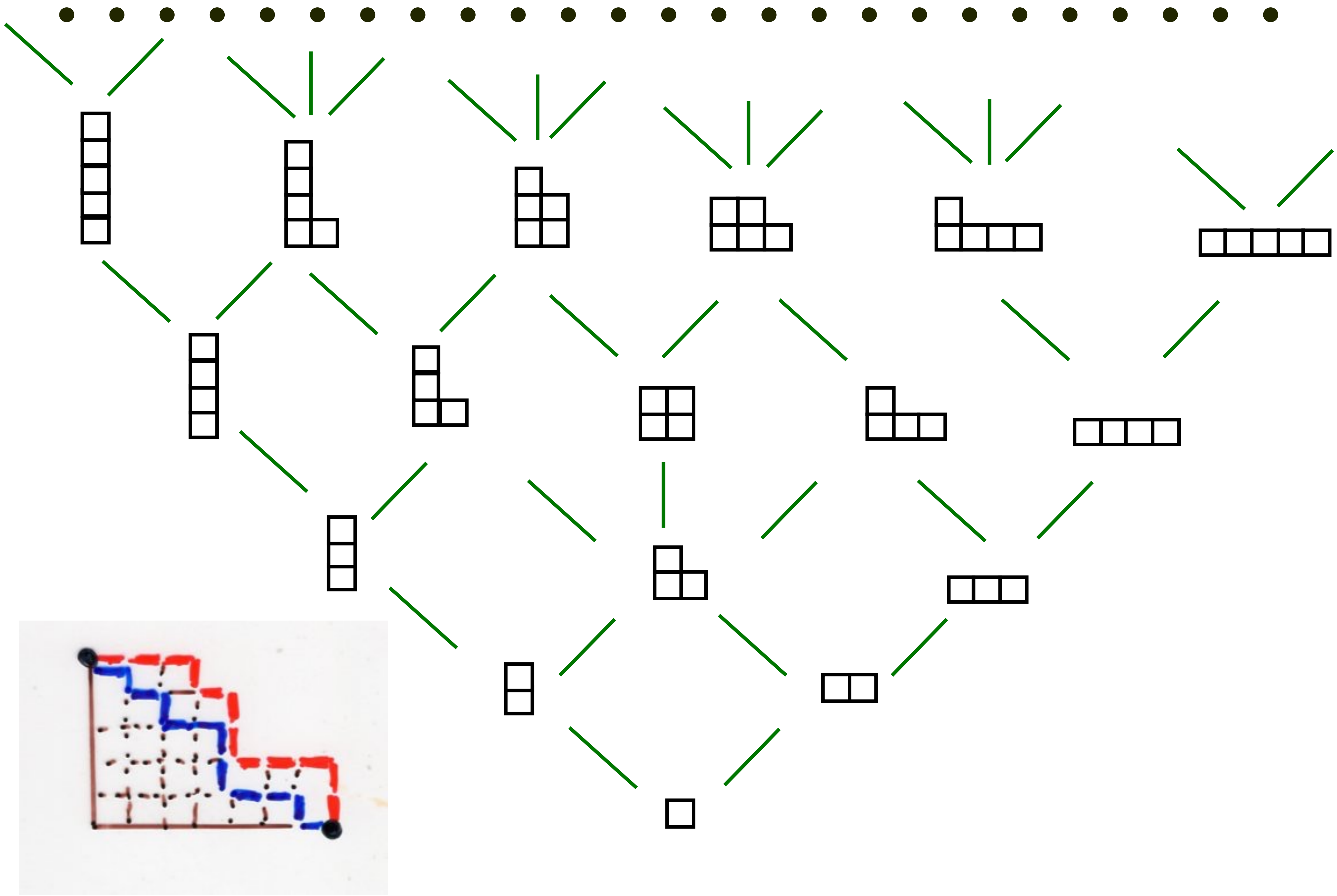
partially ordered set

covering relation

$\alpha \preccurlyeq \beta$
no γ between
α and β

Hasse diagram

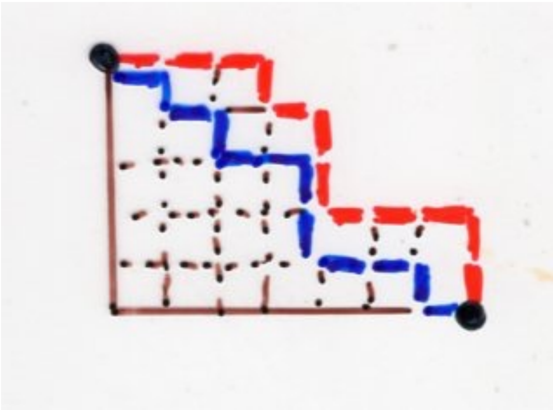
Young lattice



lattice

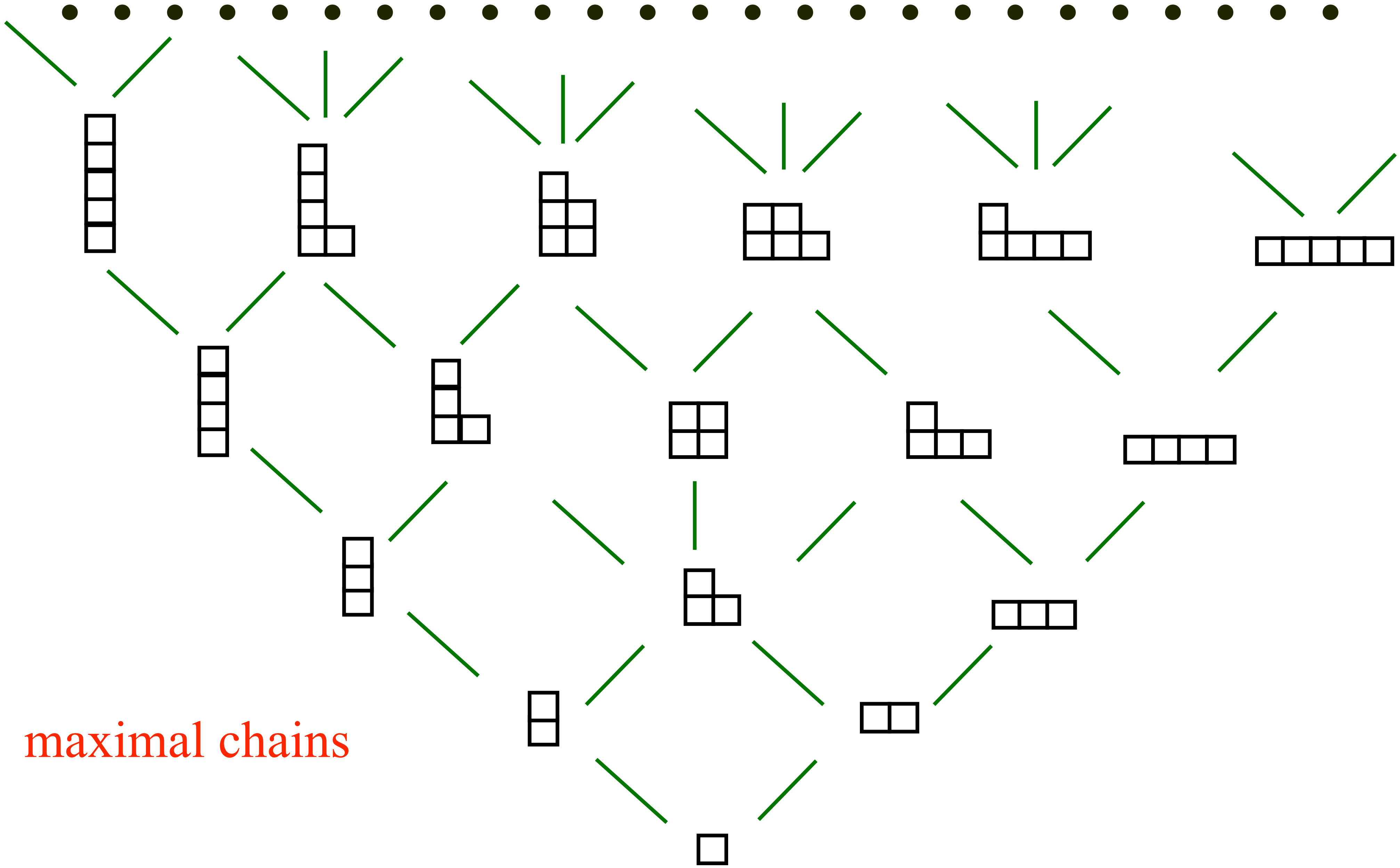
every two elements
have a unique
least upper bound (join)

and a unique
greatest lower bound
(meet)



Young lattice

Young lattice



maximal chains

- in the labeling process of the vertices of the grid with Ferrers diagrams :
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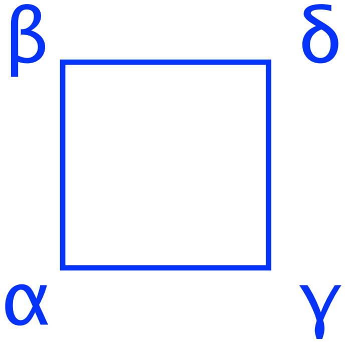

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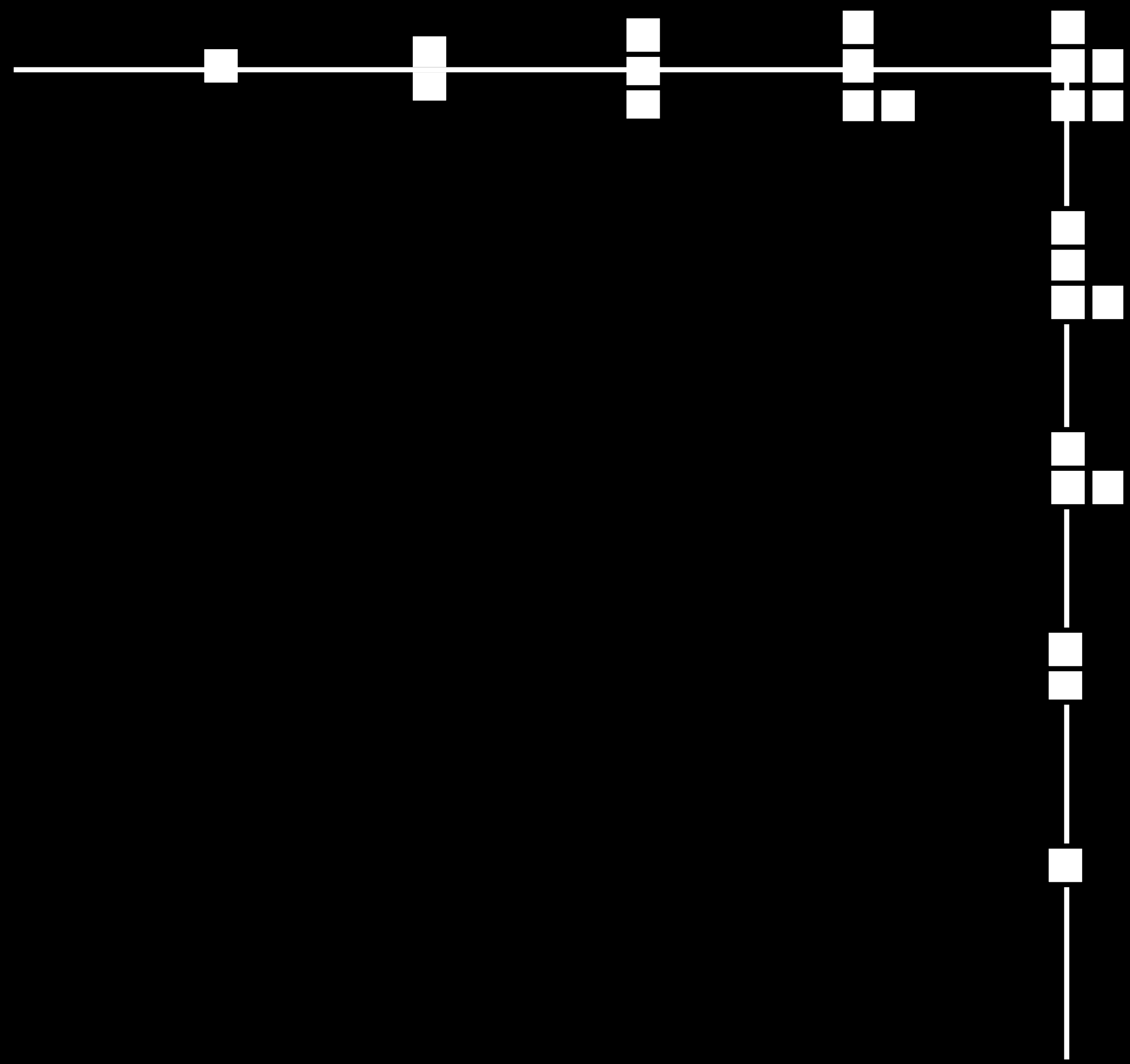

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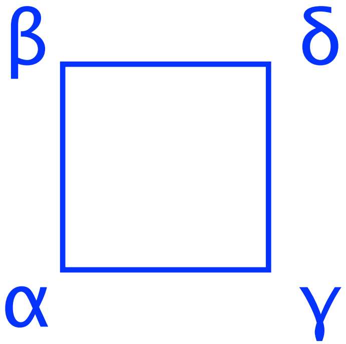

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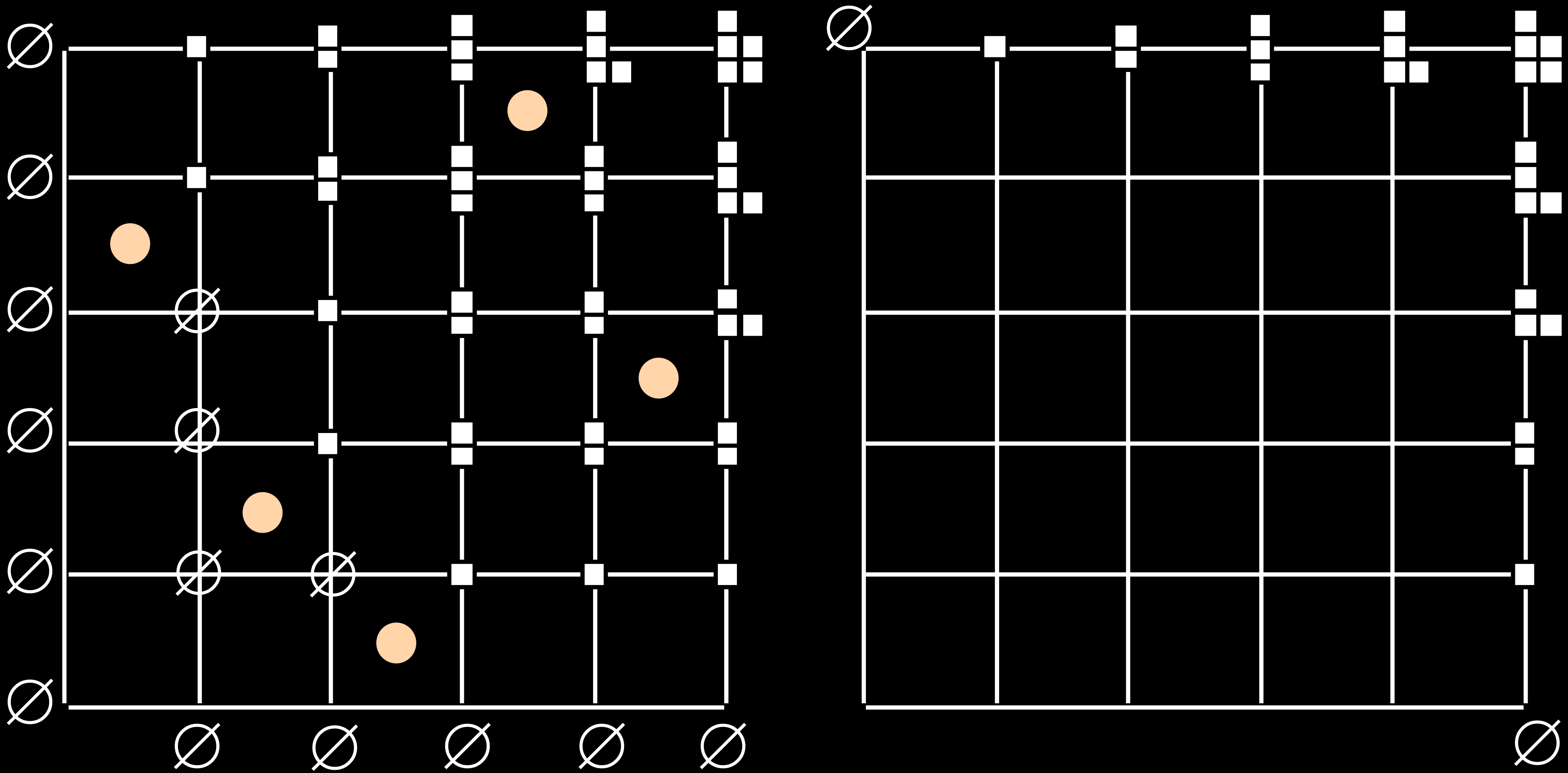
- these maximal chains encode a pair (P,Q) of Young tableaux of the same shape final state

the pair (P,Q)



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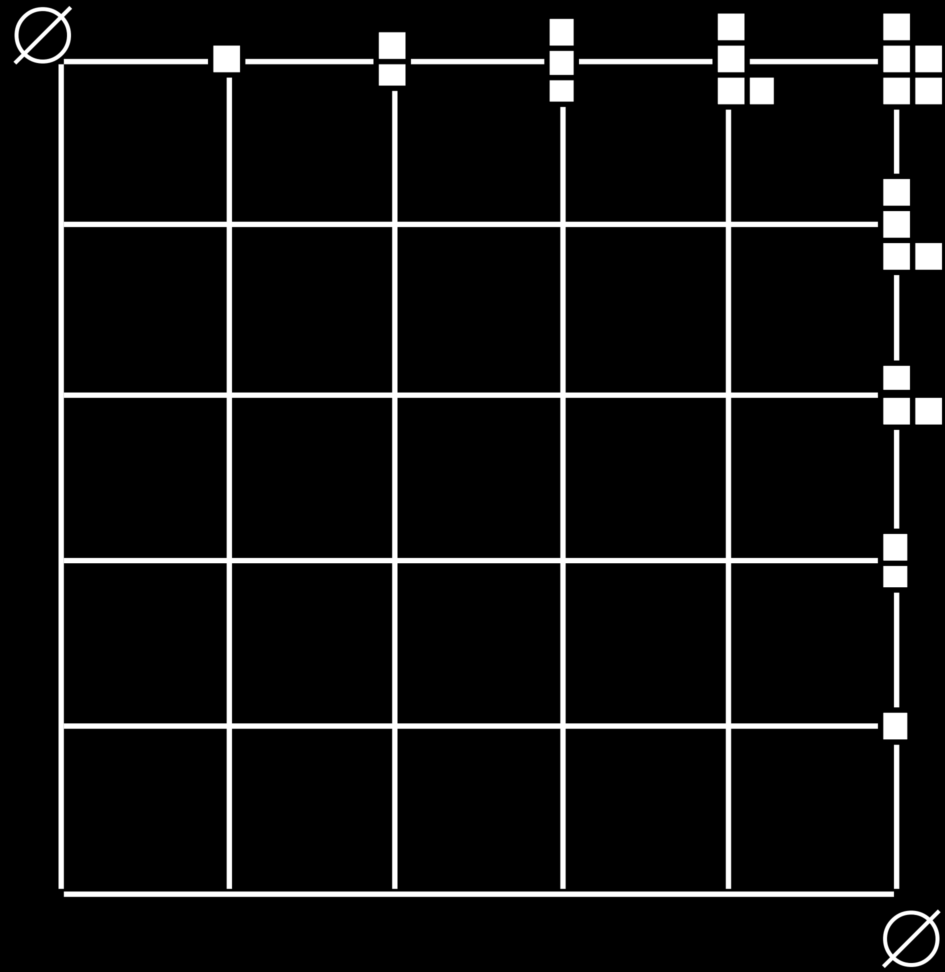
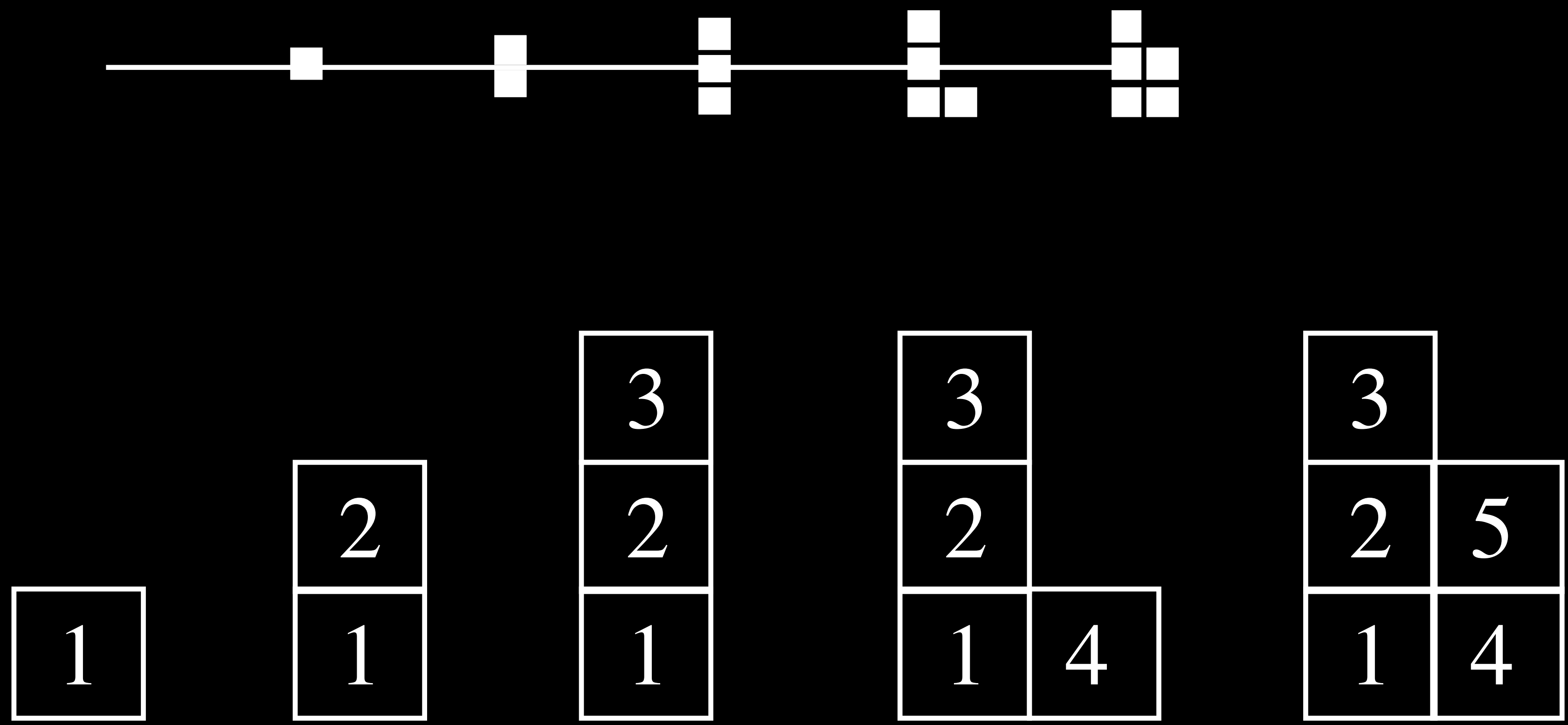
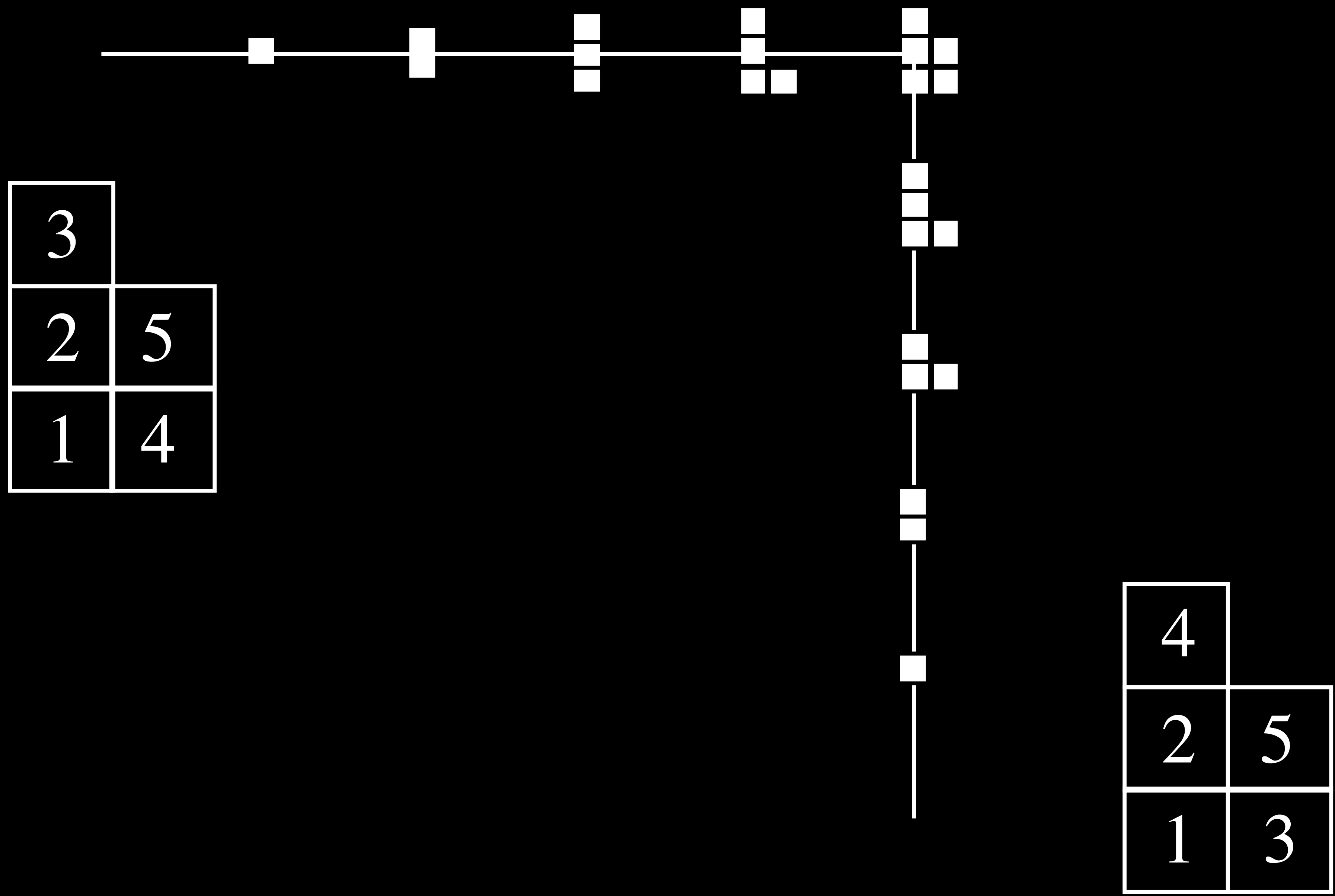



1
2 1
3 2 1
3 2 1 4
3 2 5 1 4



3
2 5
1 4
4
2 5
1 3

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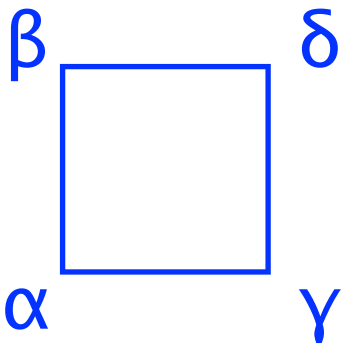

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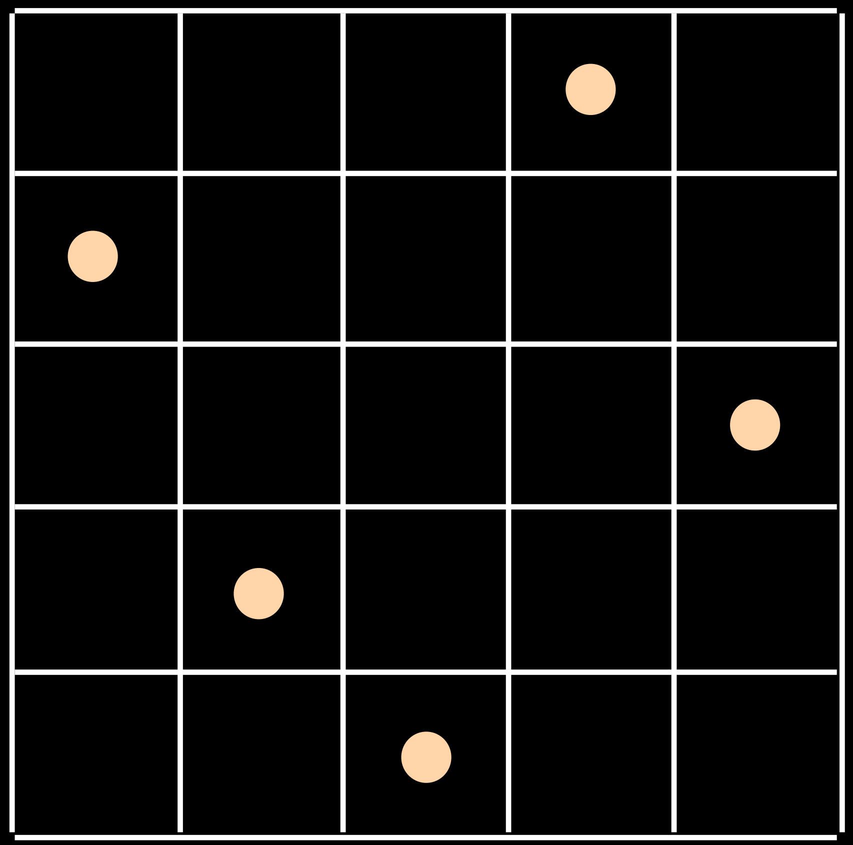
- in the last rows and last columns: we get maximal chains of Ferrers diagrams

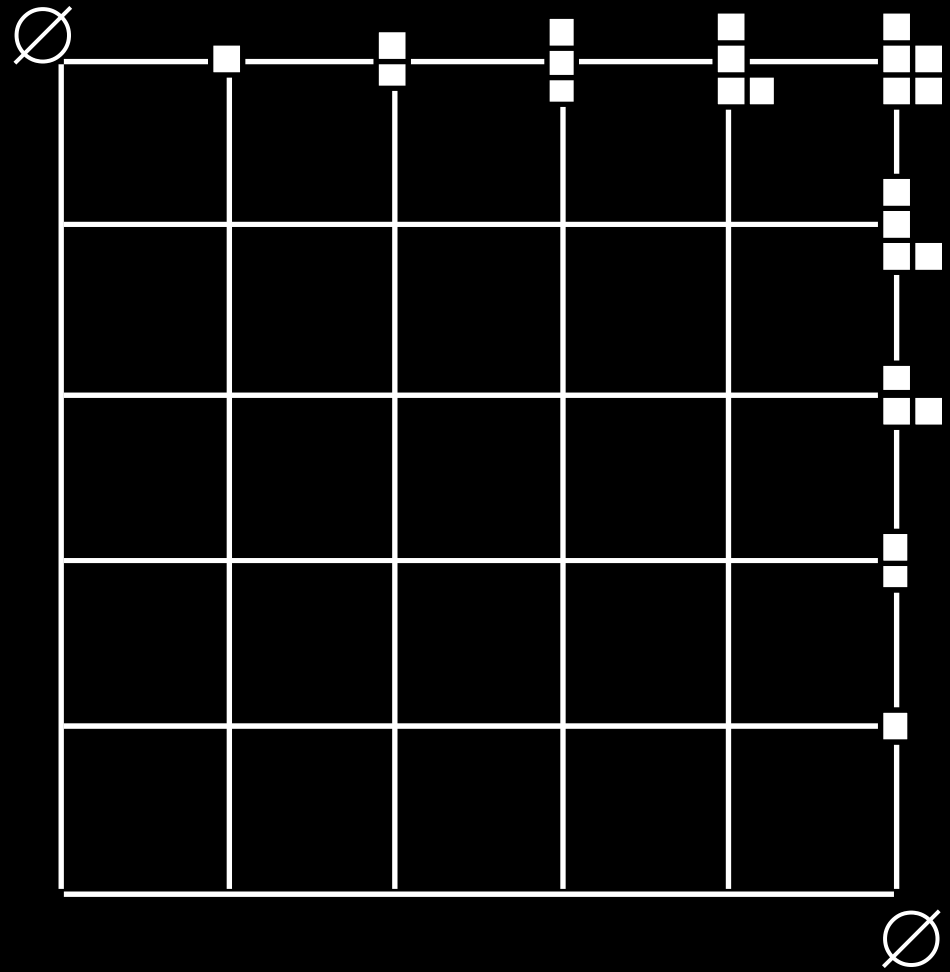
- these maximal chains encode a pair (P,Q) of Young tableaux of the same shape

- the process can be reversed, from the pair (P,Q), get back the permutation

permutation



the pair (P,Q)



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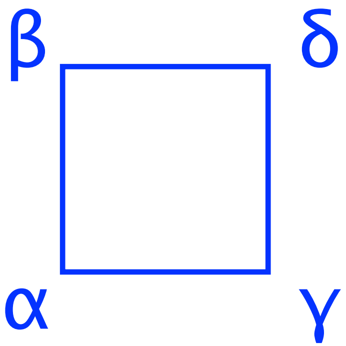

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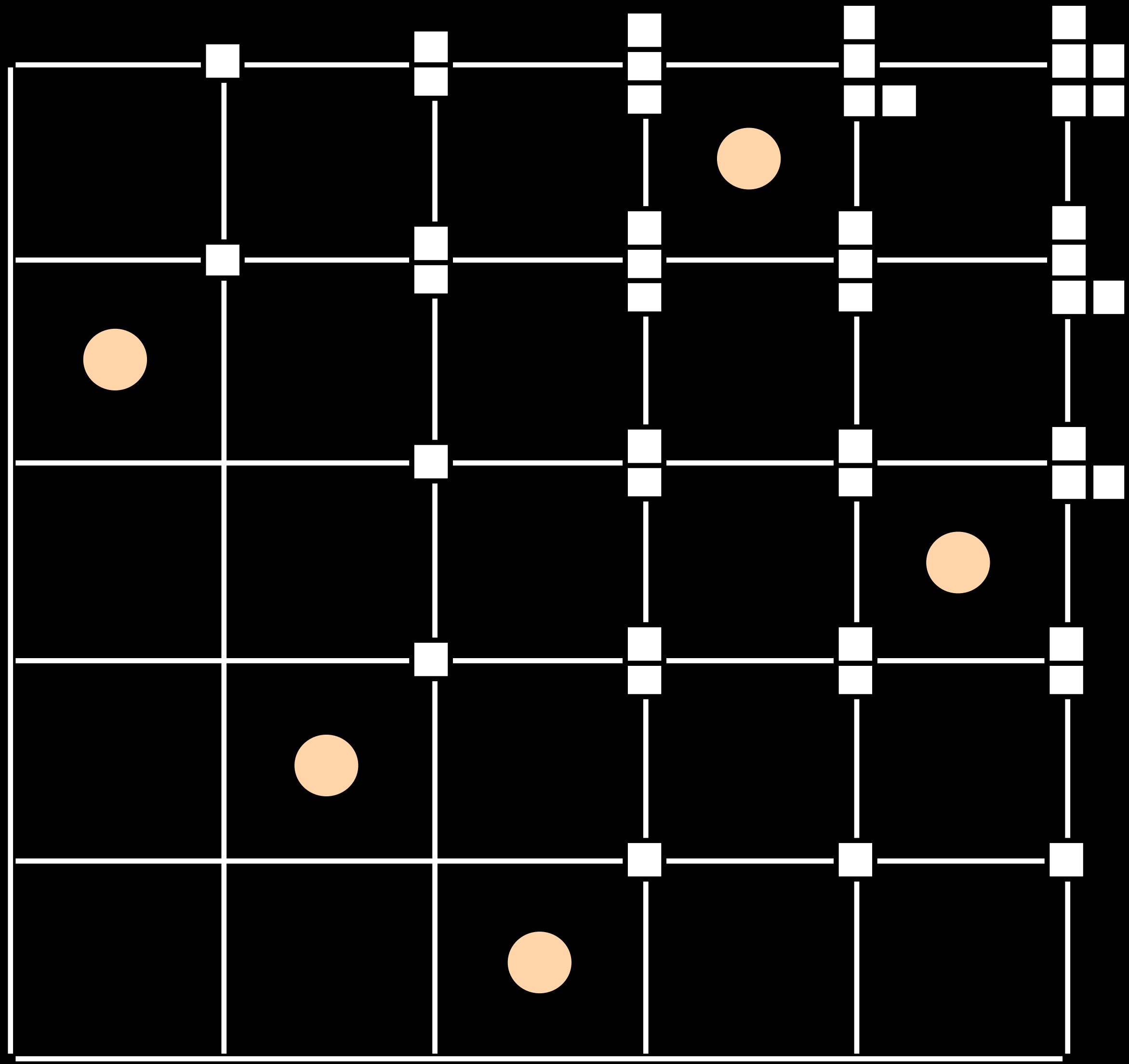
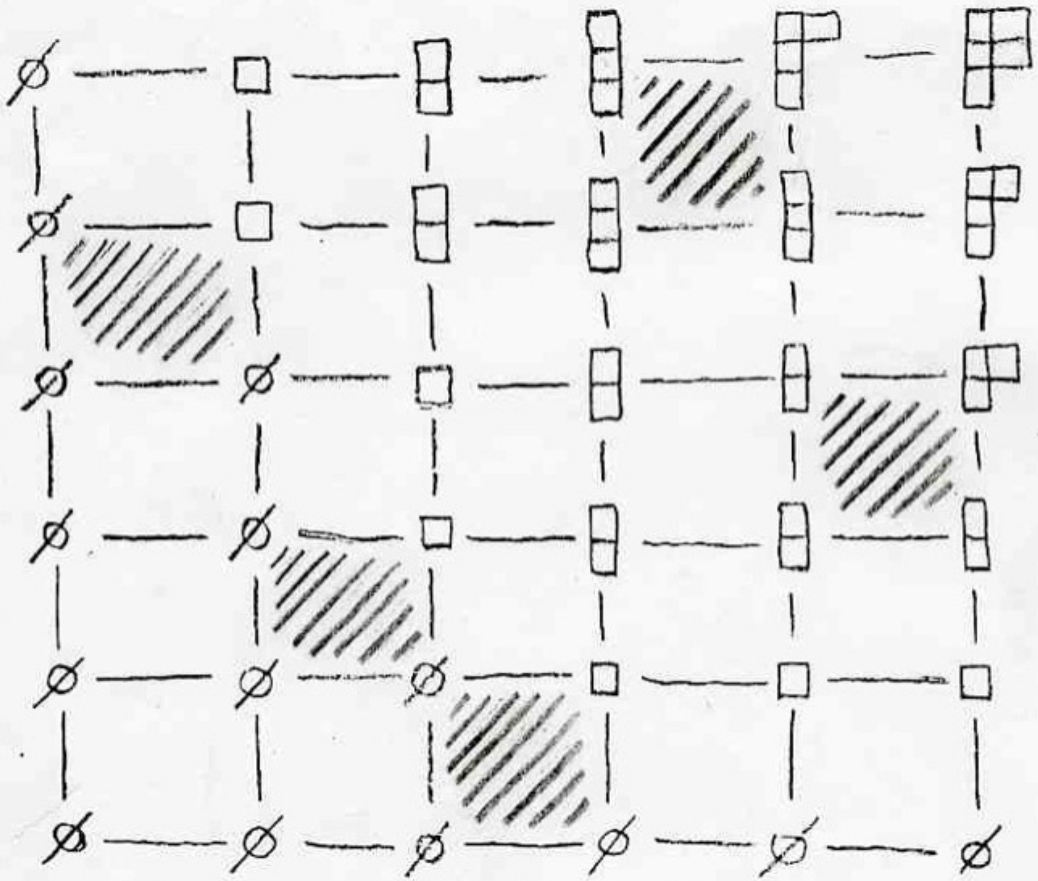
- this bijection is the same as the Robinson-Schensted correspondance



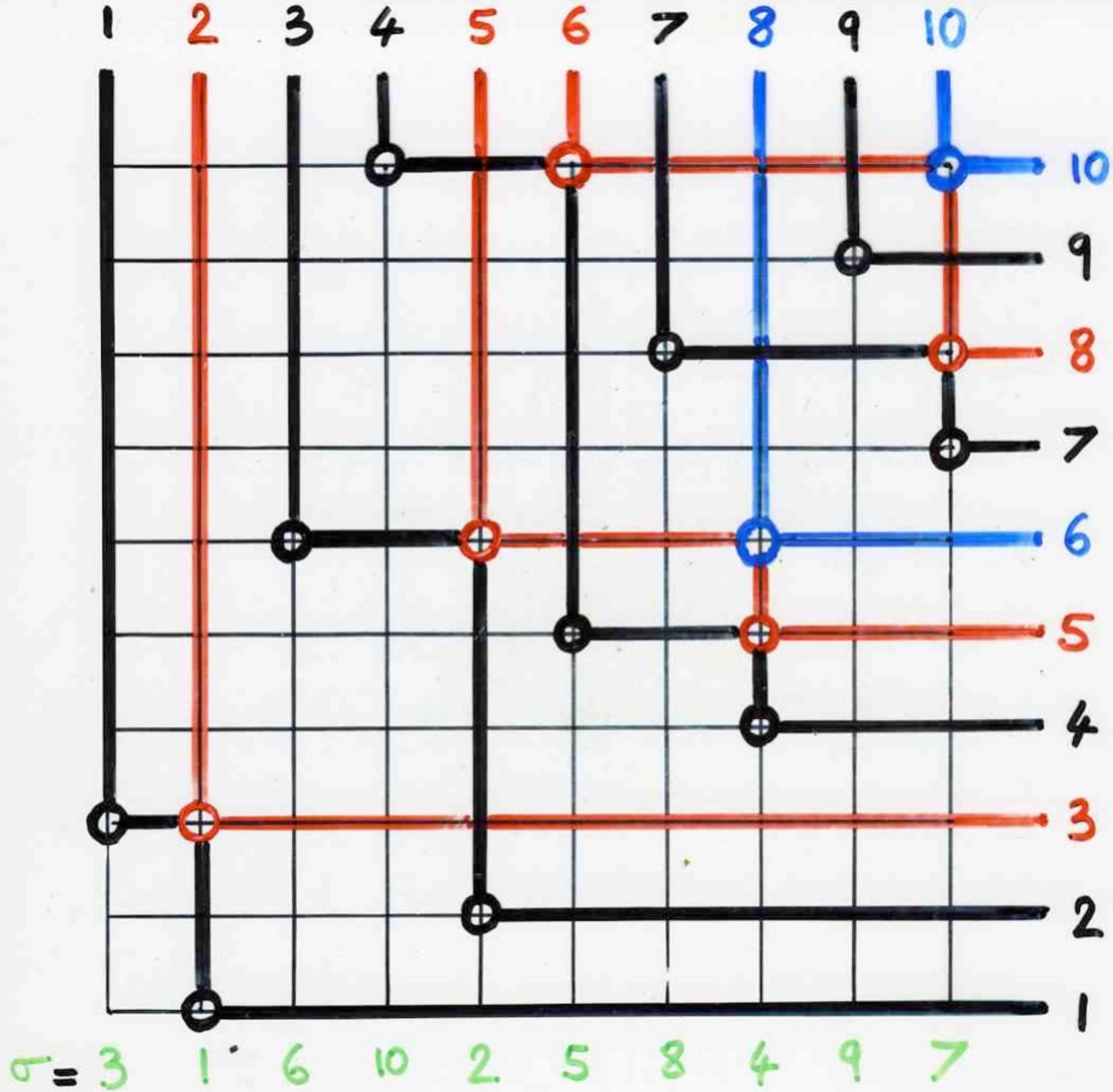
dessin fait par S. FOMIN

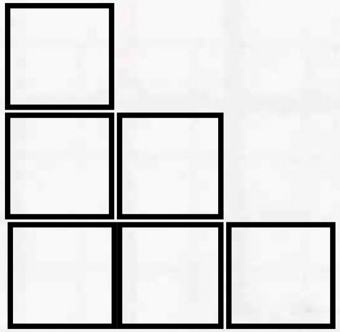


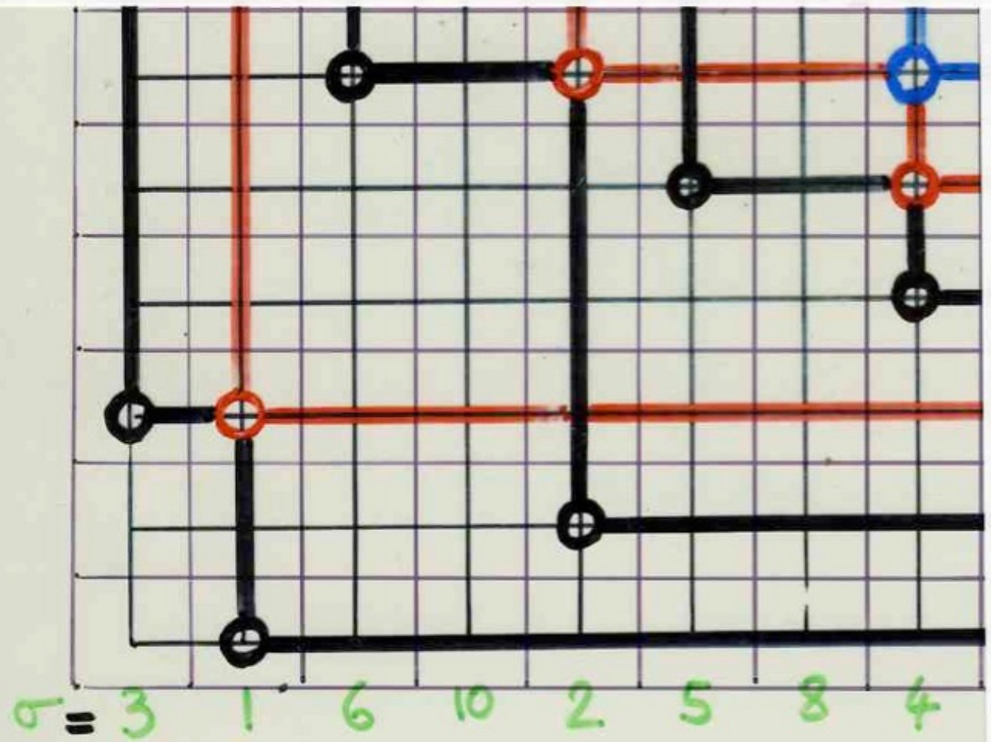
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

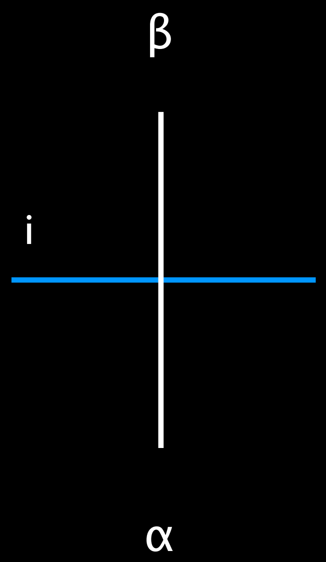
permutation associée

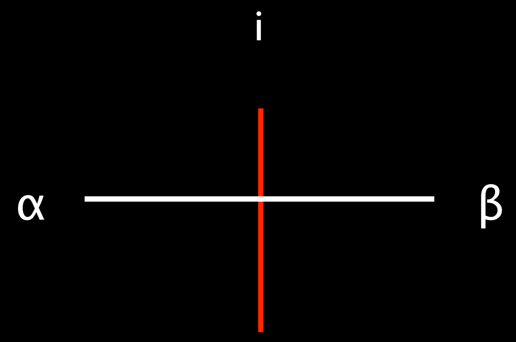
proof of the equivalence
local RSK and geometric RSK

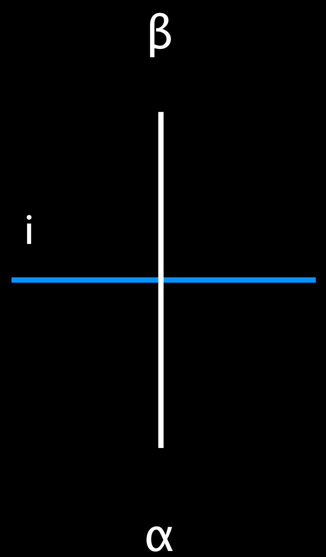

1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7



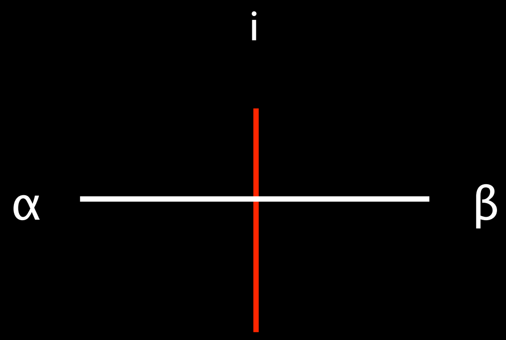

σ = 3 1 6 10 2 5 8 4

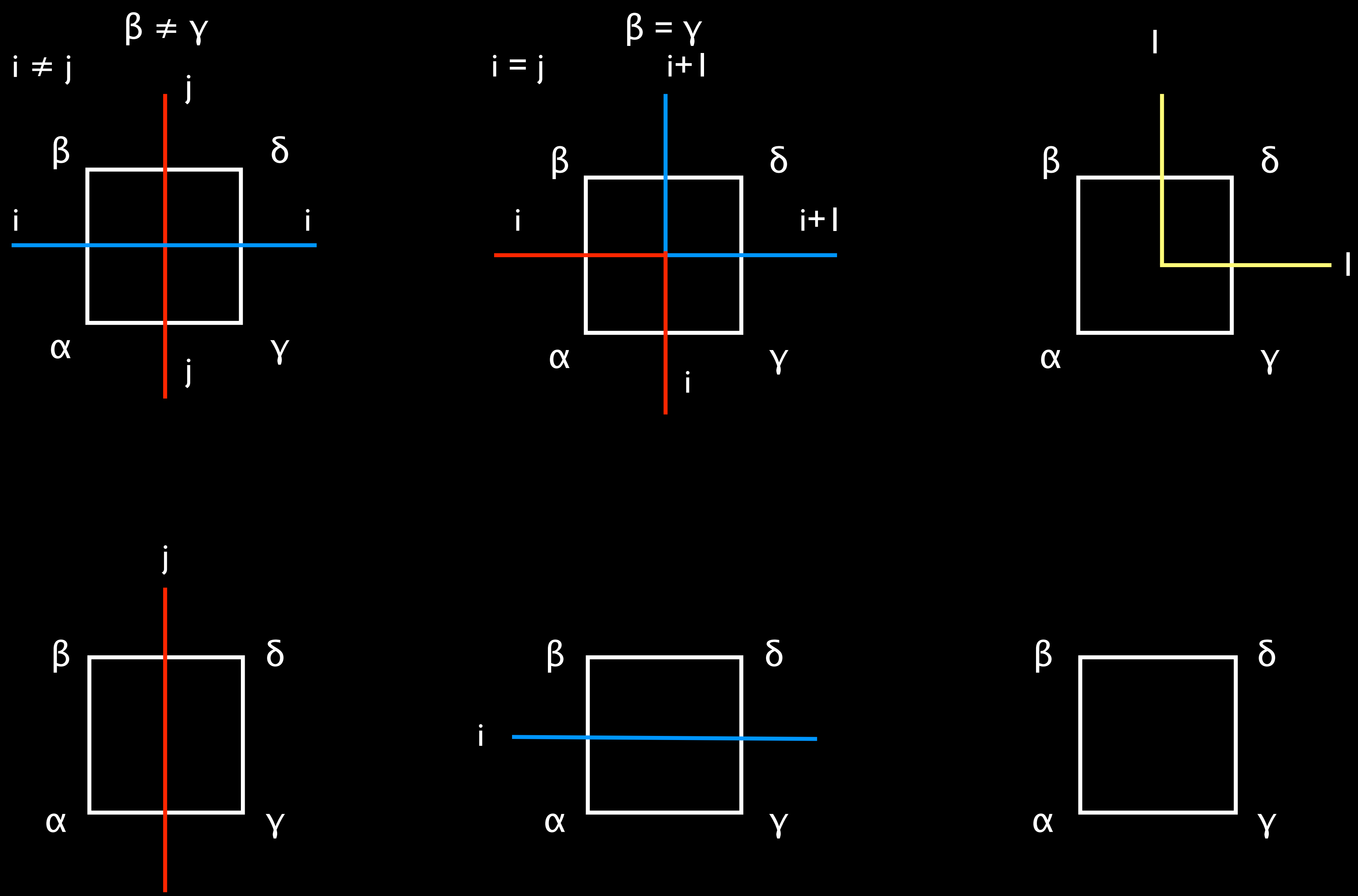

β
i
α


i
α
β

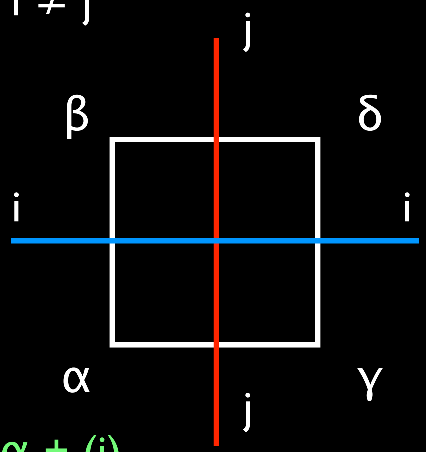


$\beta = \alpha + (i)$



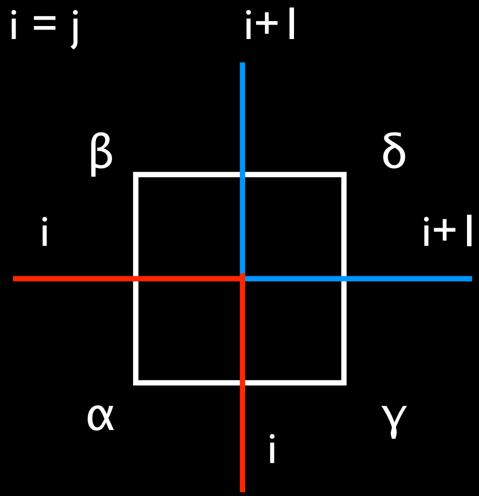

β ≠ γ
i ≠ j
j
β
δ
i
i
α
j
γ
β = γ
i = j
i+1
β
δ
i
i+1
α
i
γ
1
β
δ
1
α
γ
j
β
δ
α
γ
β
δ
i
α
γ
β
δ
α
γ

$i \neq j$

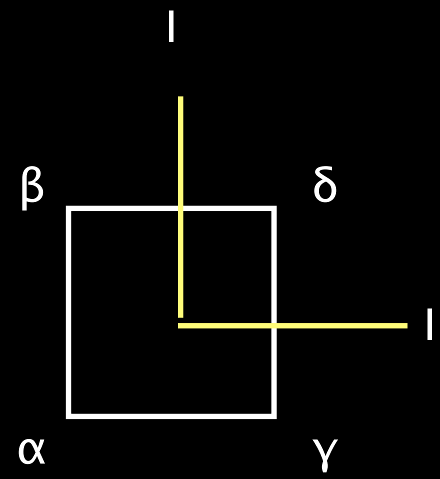


$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

$i = j$



$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i+1)$



$\beta = \gamma = \alpha$
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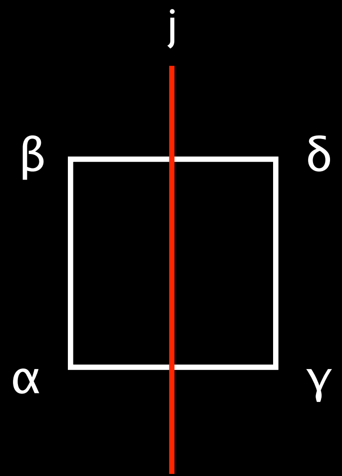
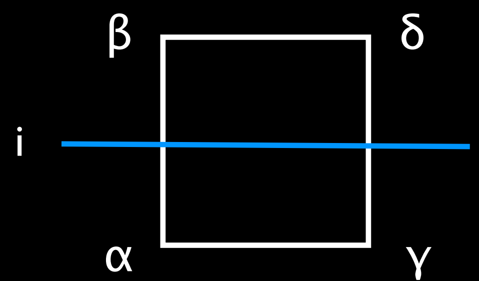


$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$



$\gamma = \alpha$
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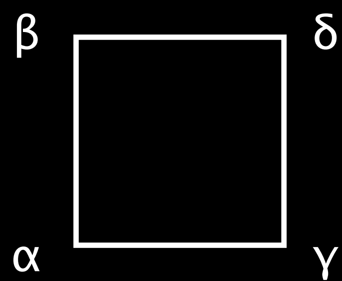
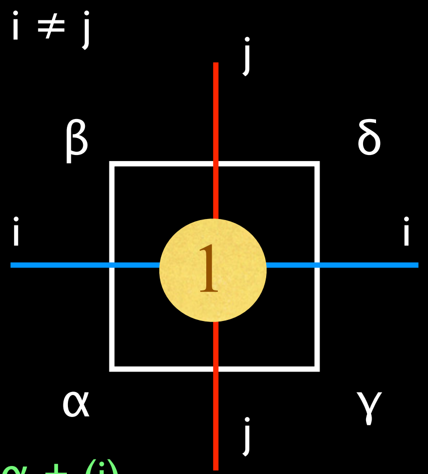


$\delta = \beta = \gamma = \alpha$

$\beta \neq \gamma$



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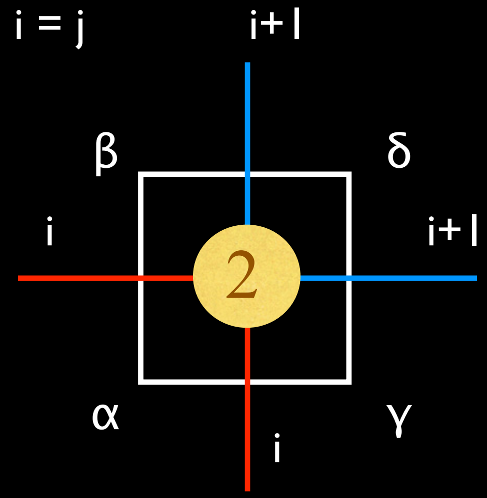
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

$\beta = \gamma$



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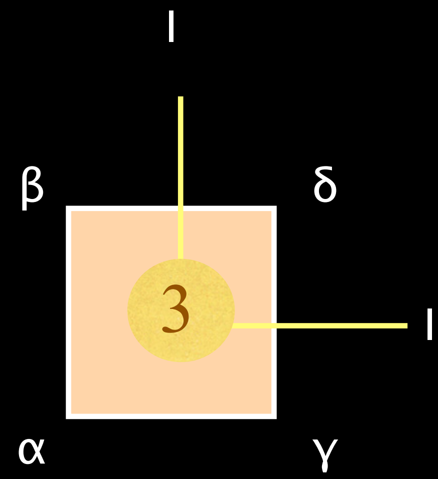

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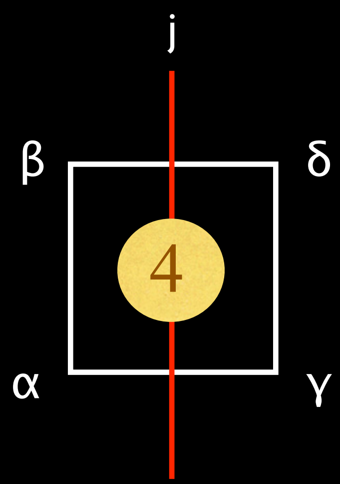

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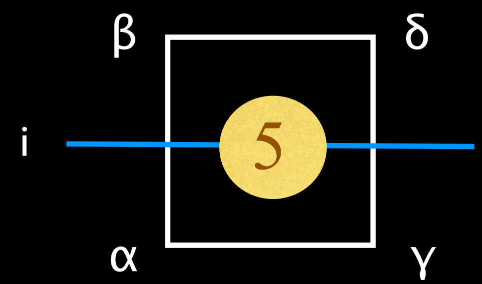

$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$

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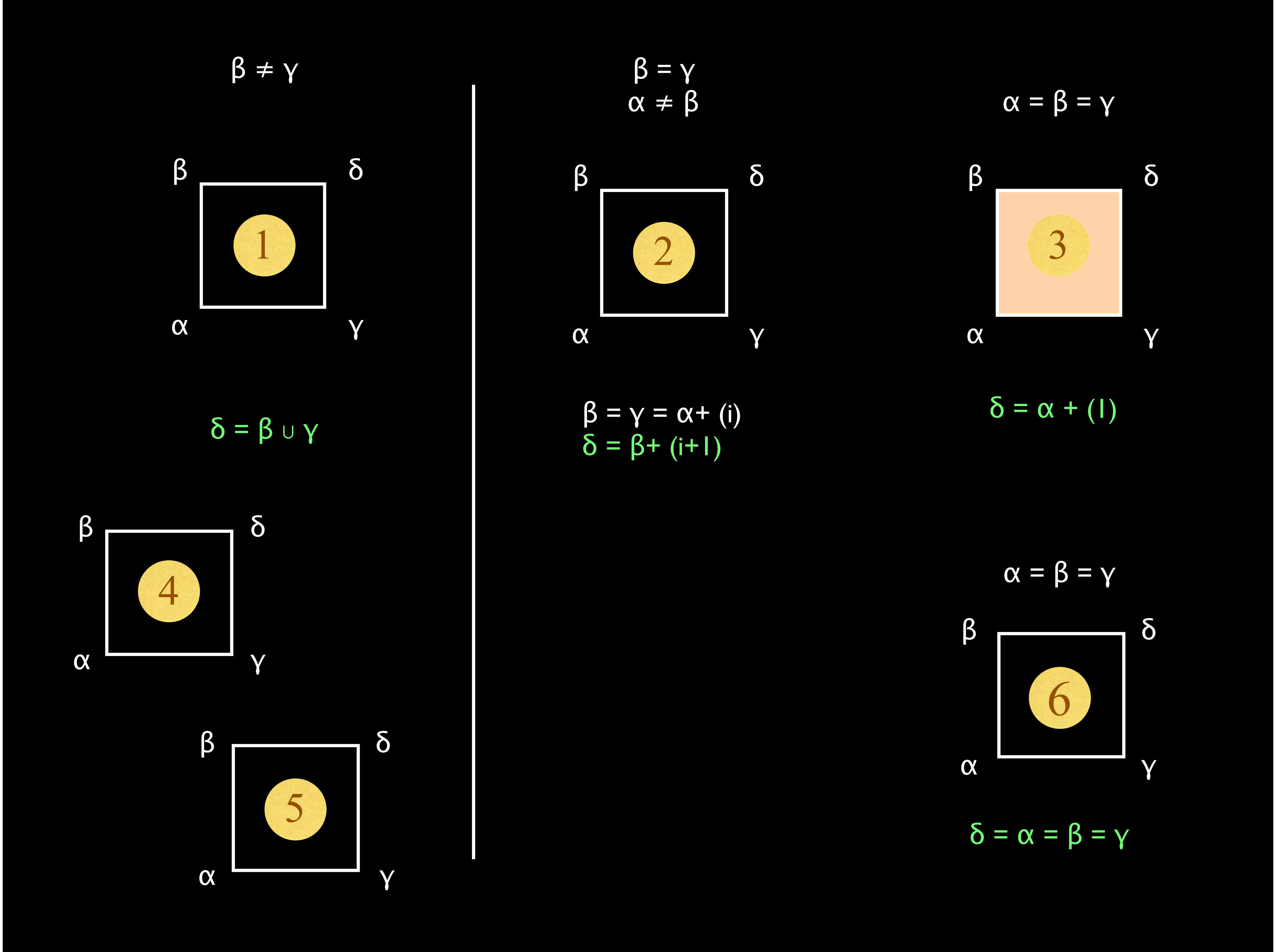
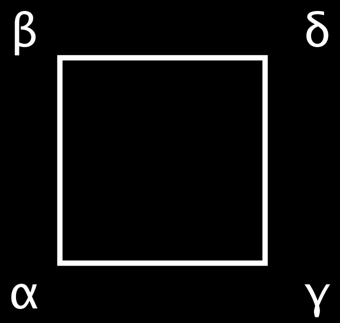

β ≠ γ
β
δ
1
α
γ
δ = β ∪ γ
β
δ
4
α
γ
β
δ
5
α
γ
β = γ
α ≠ β
β
δ
2
α
γ
β = γ = α+ (i)
δ = β+ (i+1)
α = β = γ
β
δ
3
α
γ
δ = α + (1)
α = β = γ
β
δ
6
α
γ
δ = α = β = γ

$\beta \neq \gamma$



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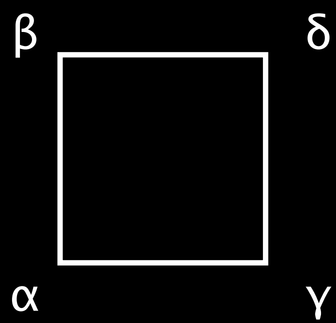

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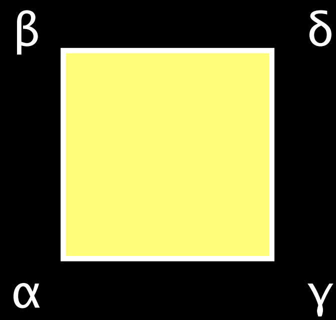

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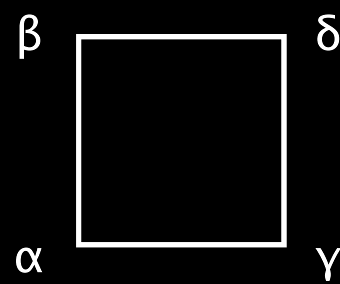

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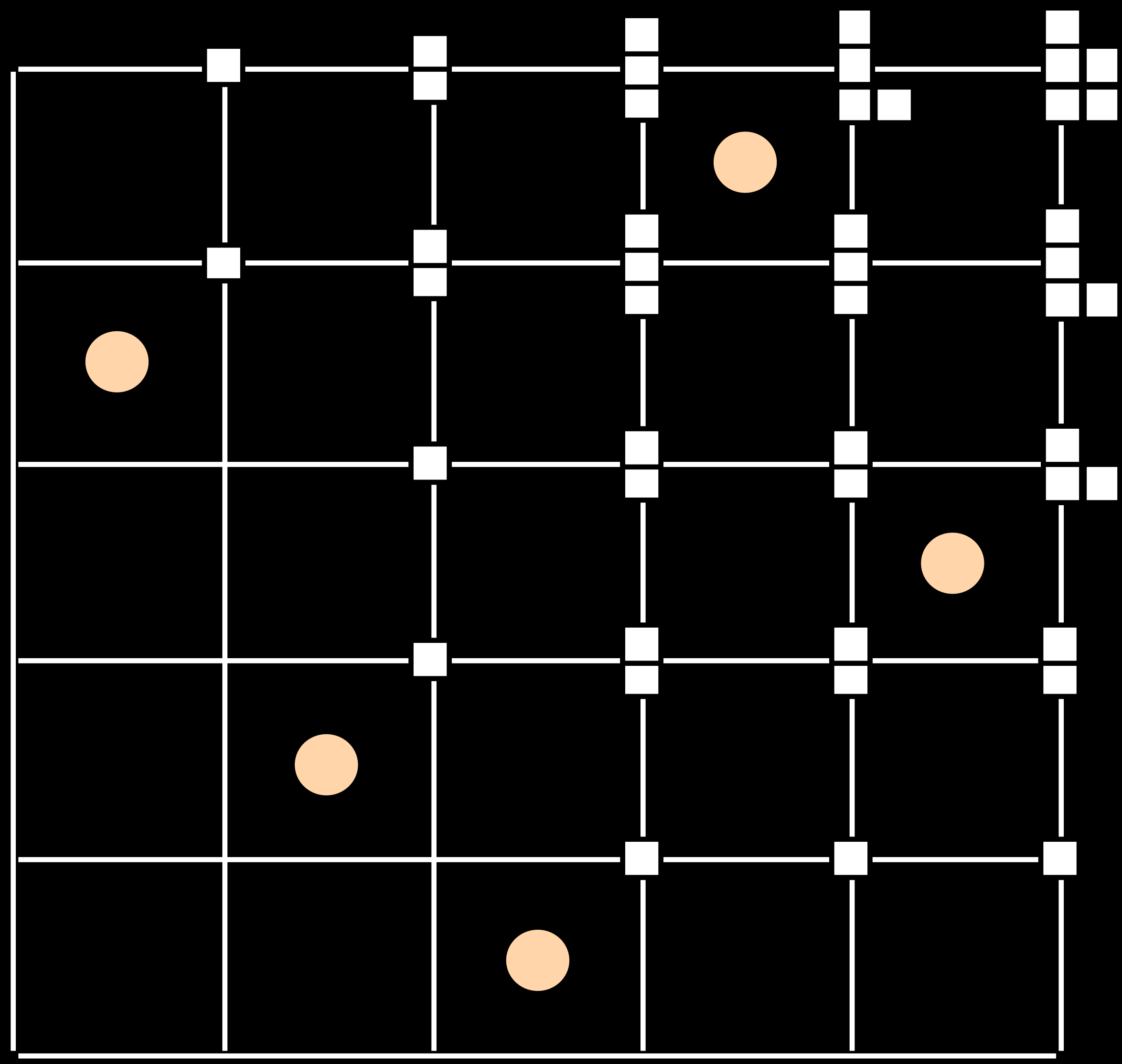
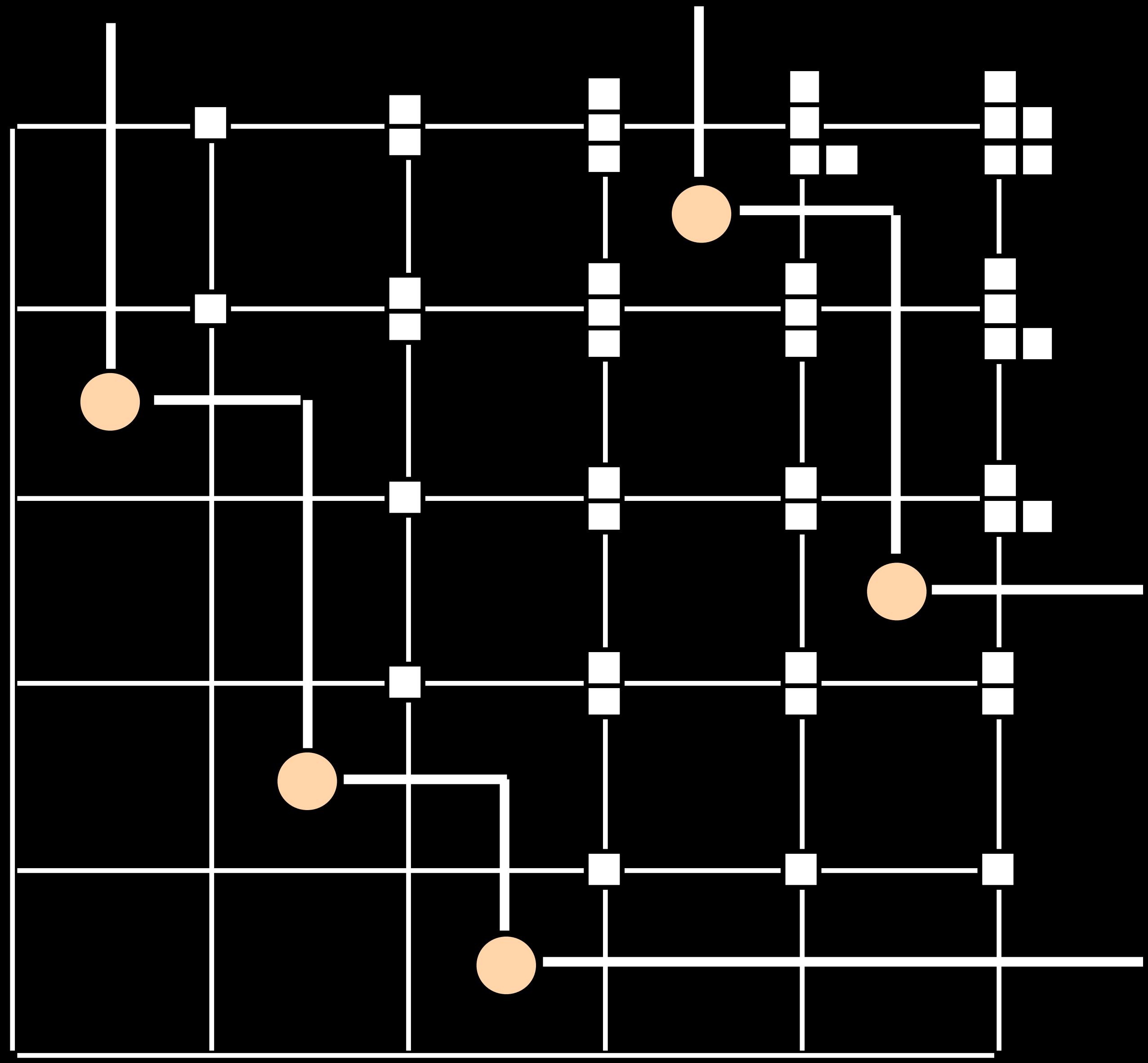
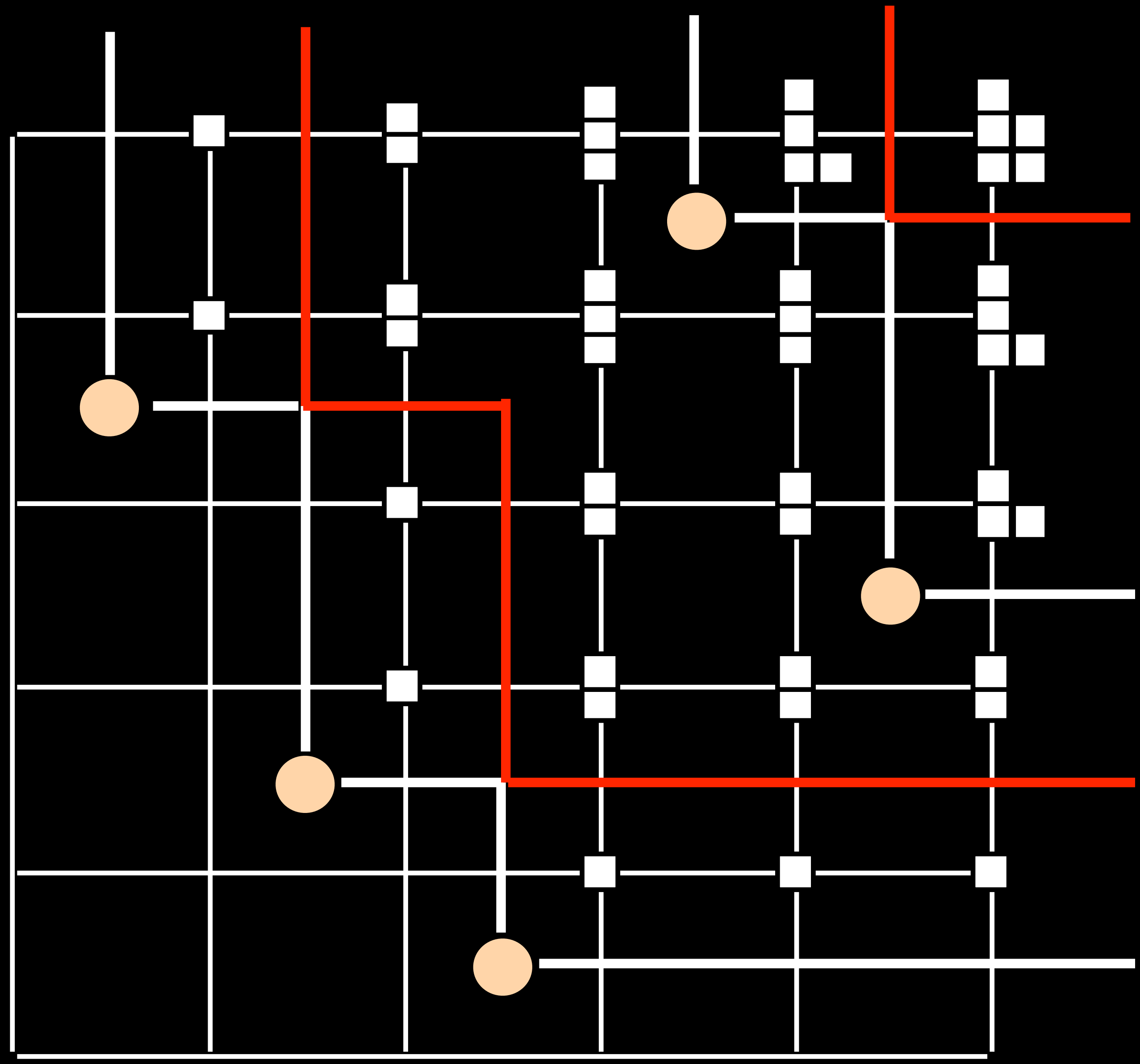
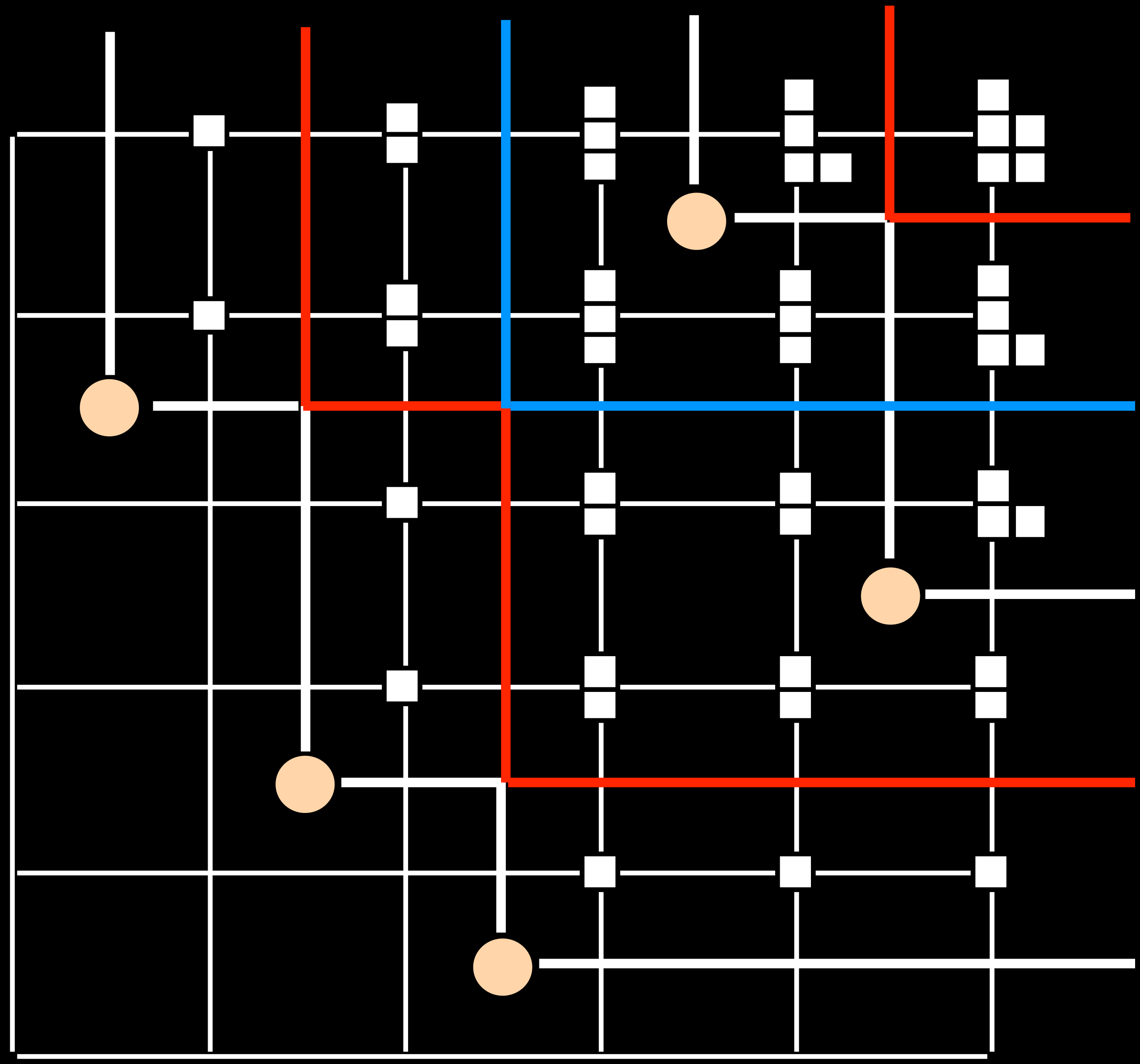
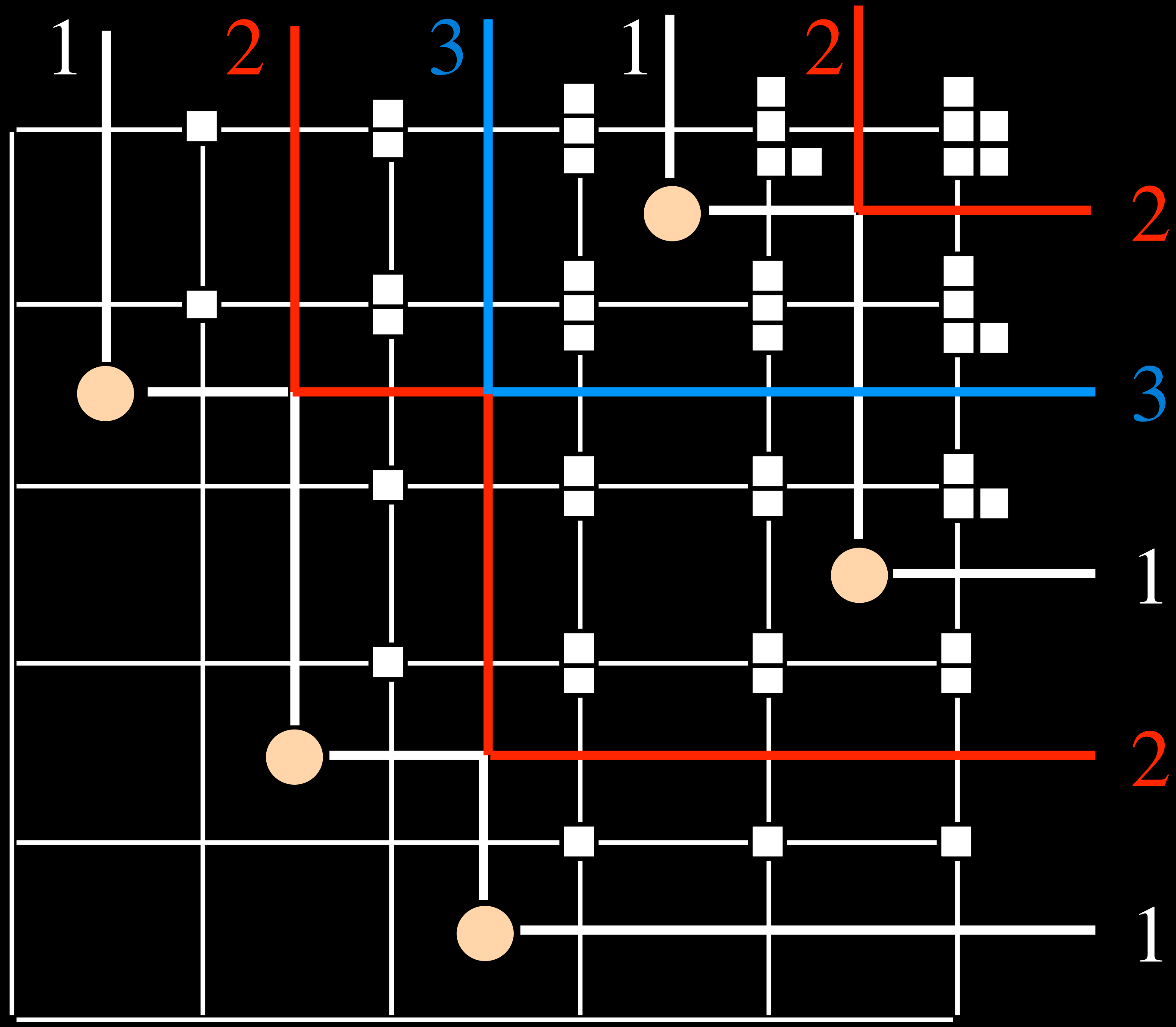


4 2 1 5 3

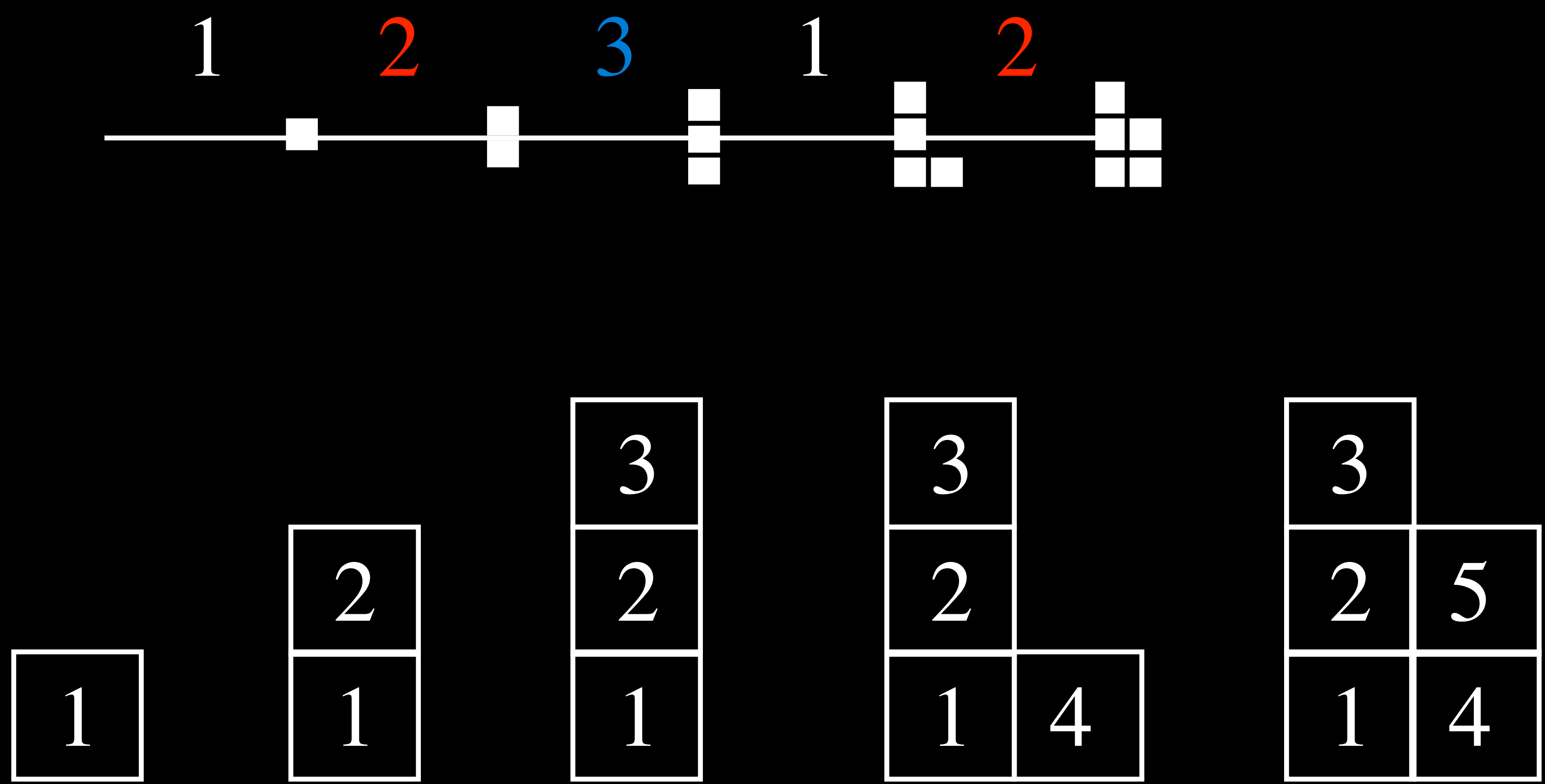






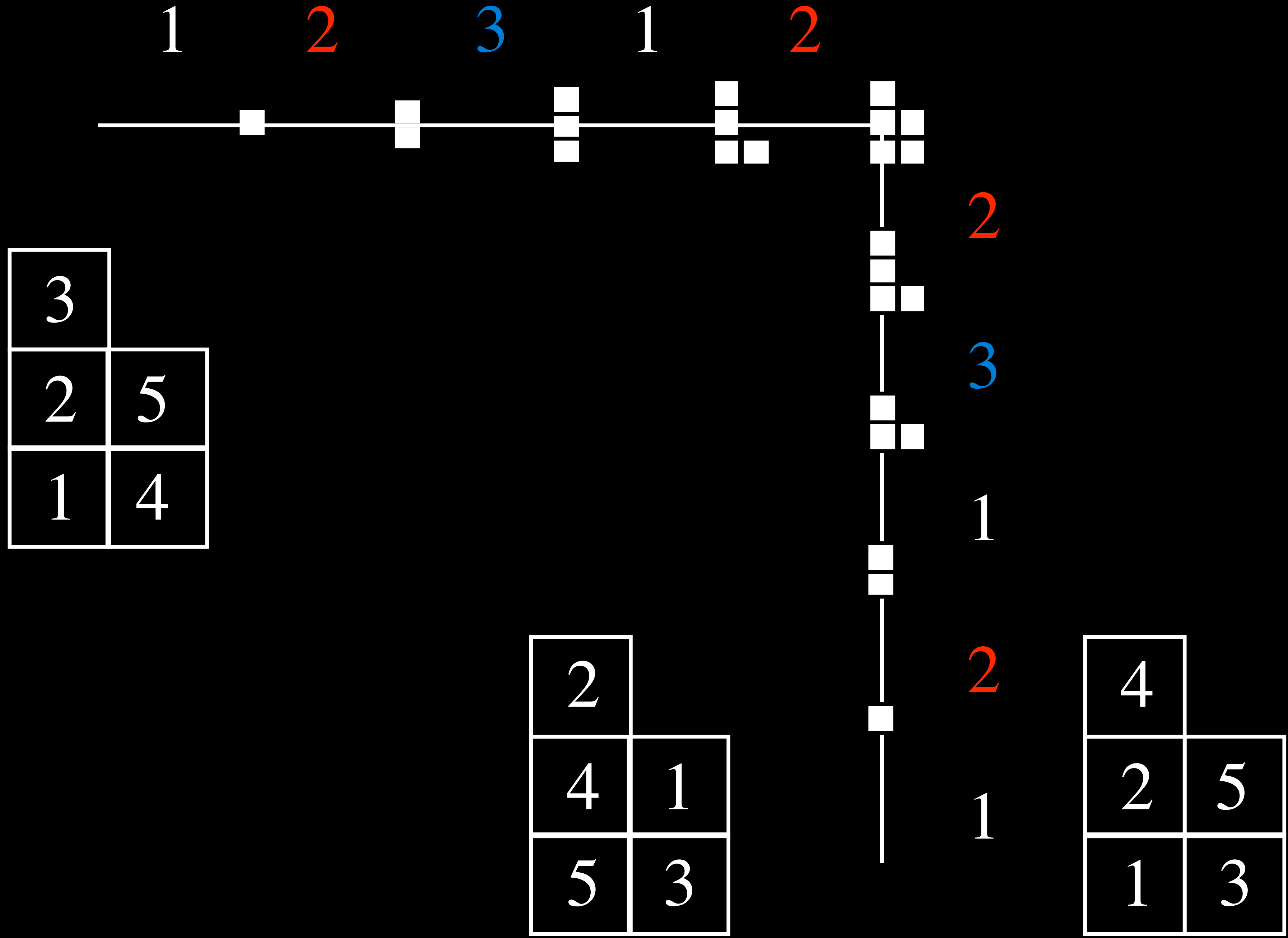


1
2
3
1
2
2
3
1
2
1



w = 1 2 3 1 2

Yamanuchi word


1 2 3 1 2
2
3
1
2
1
3
2 5
1 4
2
4 1
5 3
4
2 5
1 3

complement:

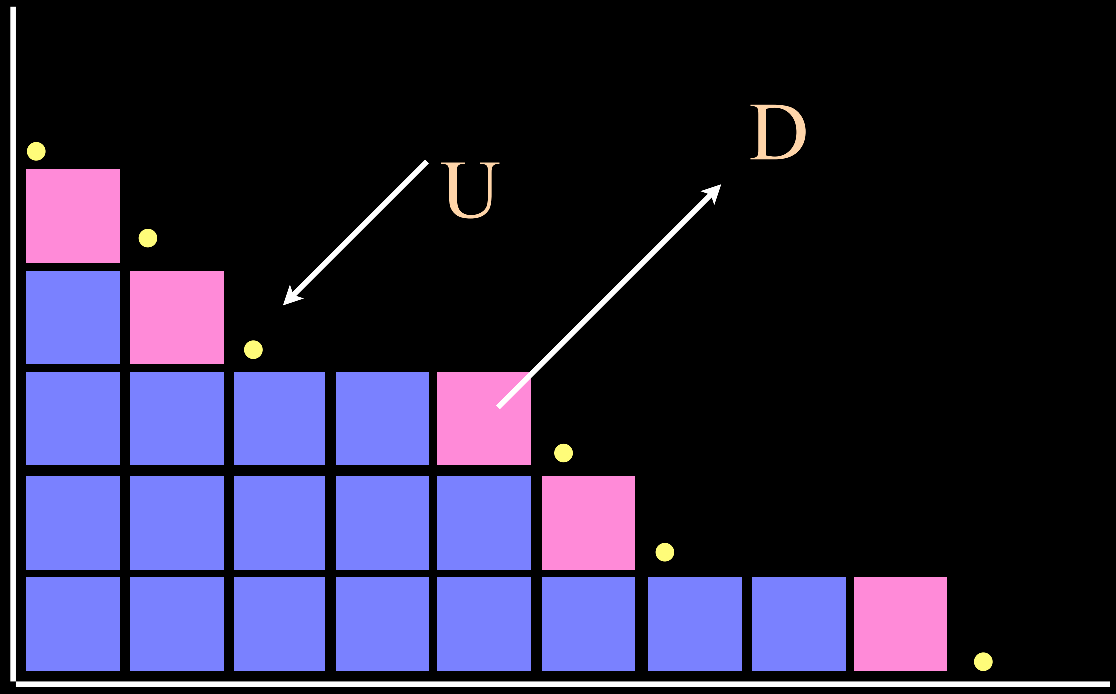
combinatorial representation of the algebra

$DU = UD + Id$

Sergey Fomin

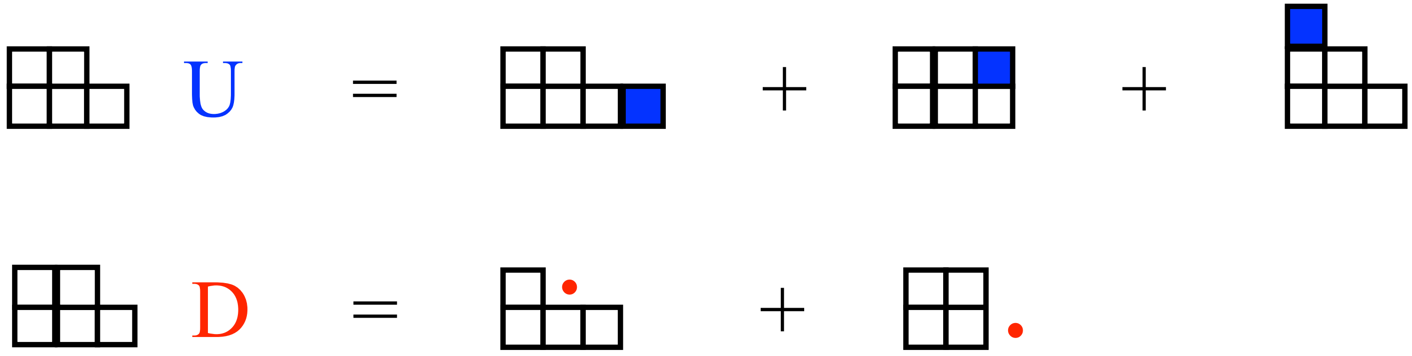
(with C. K.)

Operators U and D

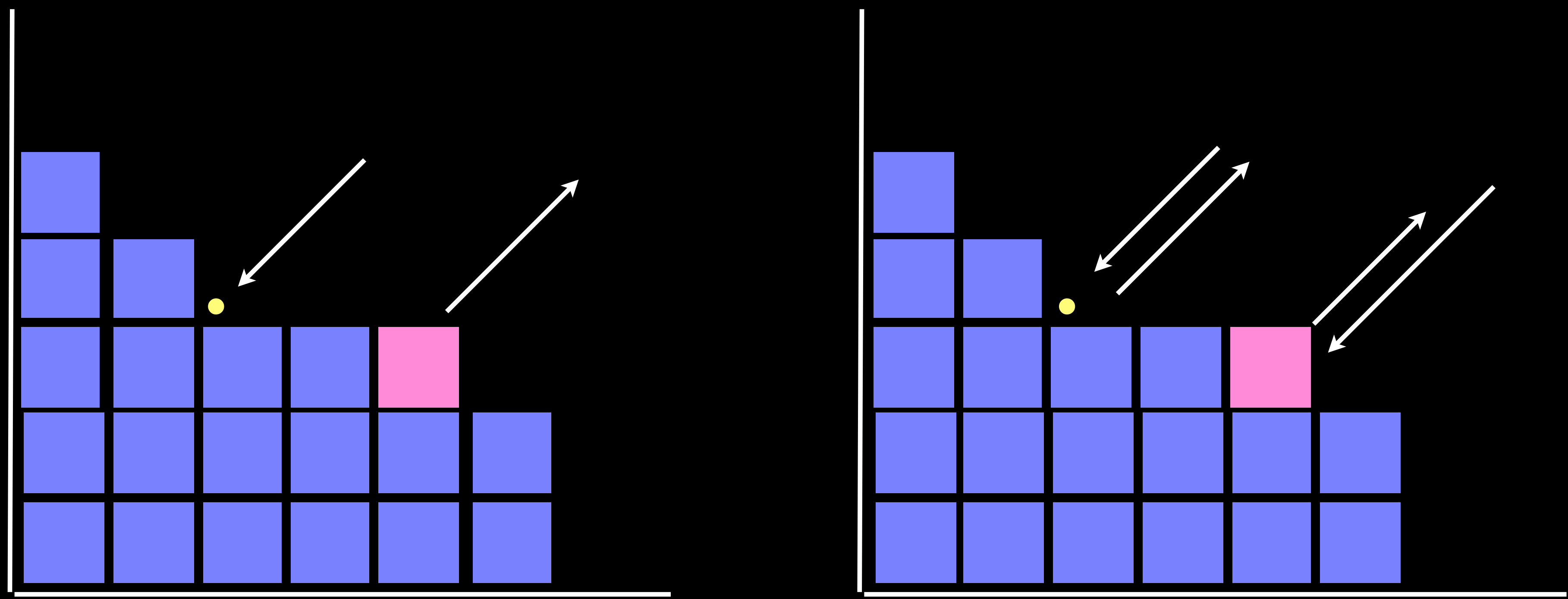


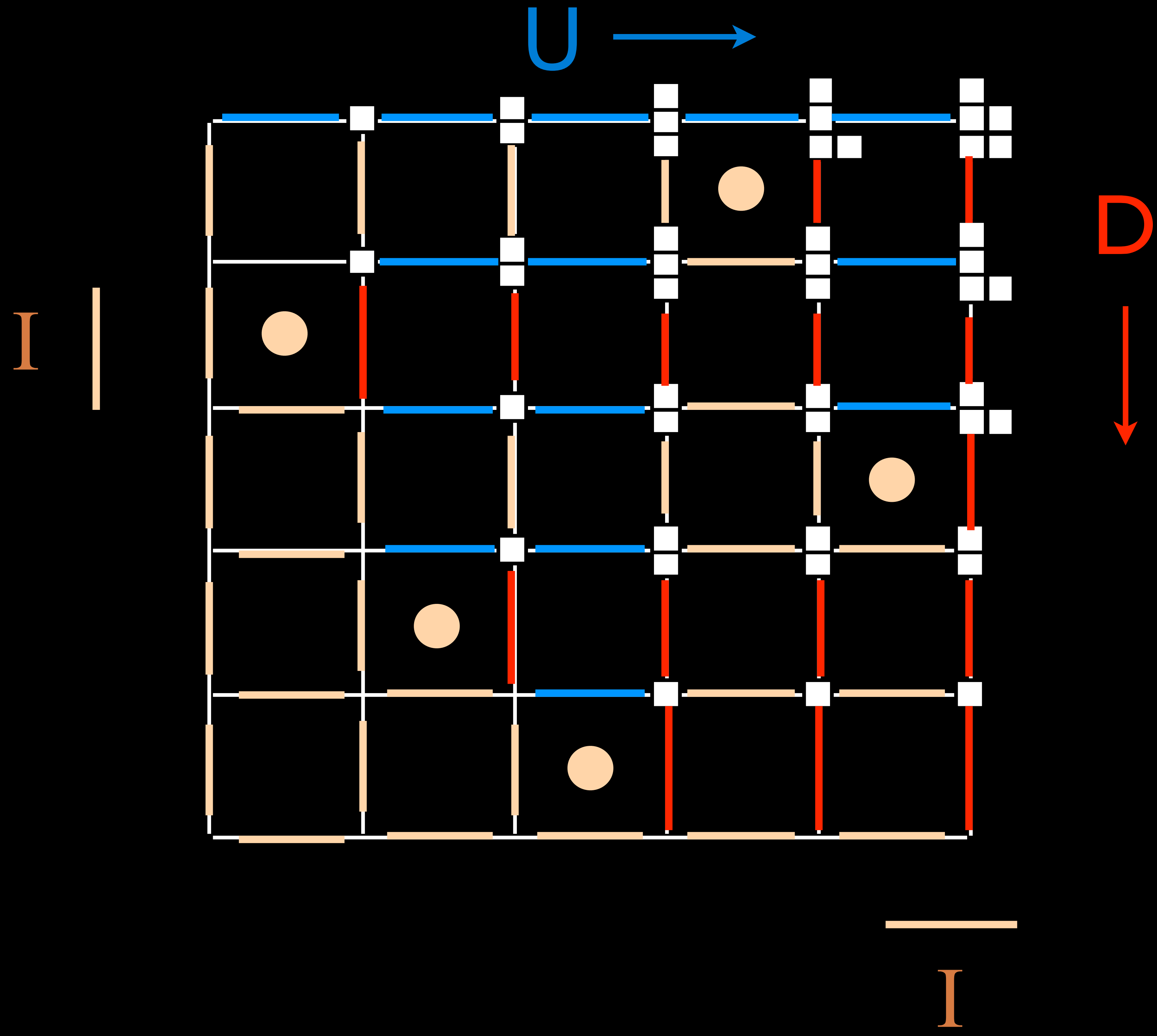
adding
or deleting
a cell in
a Ferrers
diagram

Young lattice


U
=
+
+
D
=
+

$$UD = DU + I$$




U
D
I
I

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

RSK

permutations $\longleftrightarrow$ pairs of Tableaux Young

quadratic algebra Q

Q-tableaux

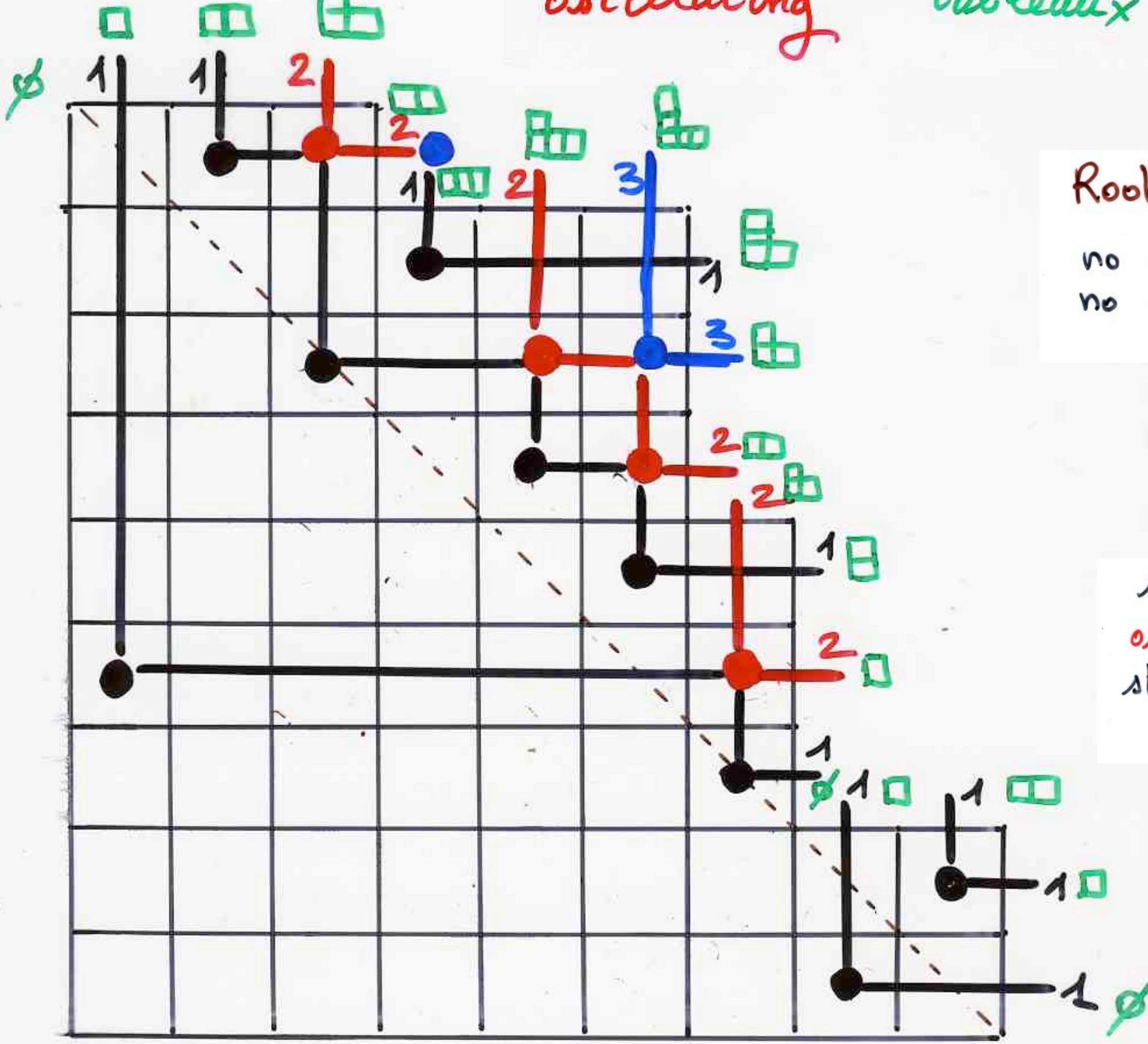
commutations
rewriting rules

planarization

$\longrightarrow$ see the course
« Quadratic algebra
and combinatorics »

oscillating tableaux

oscillating tableaux

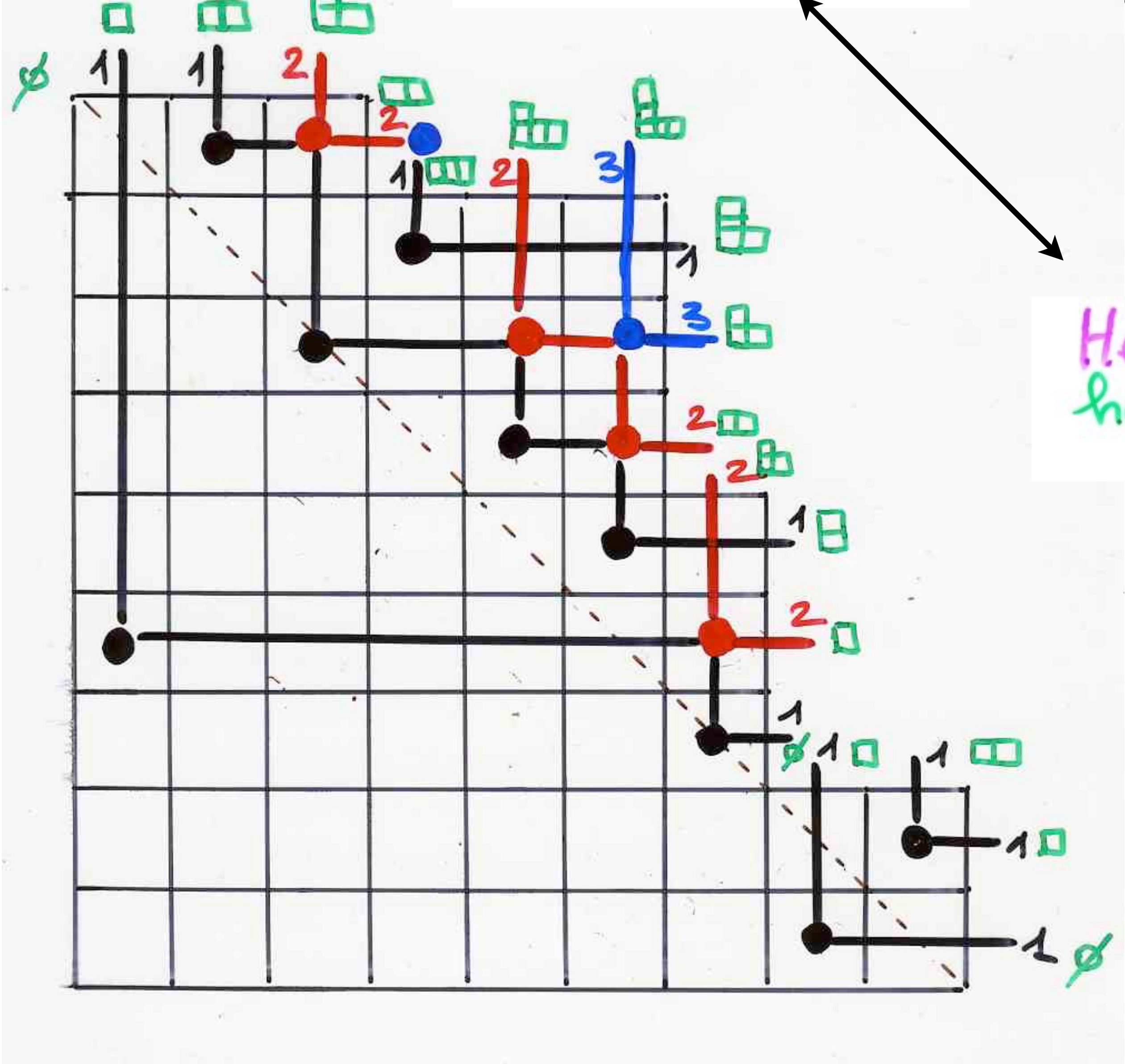


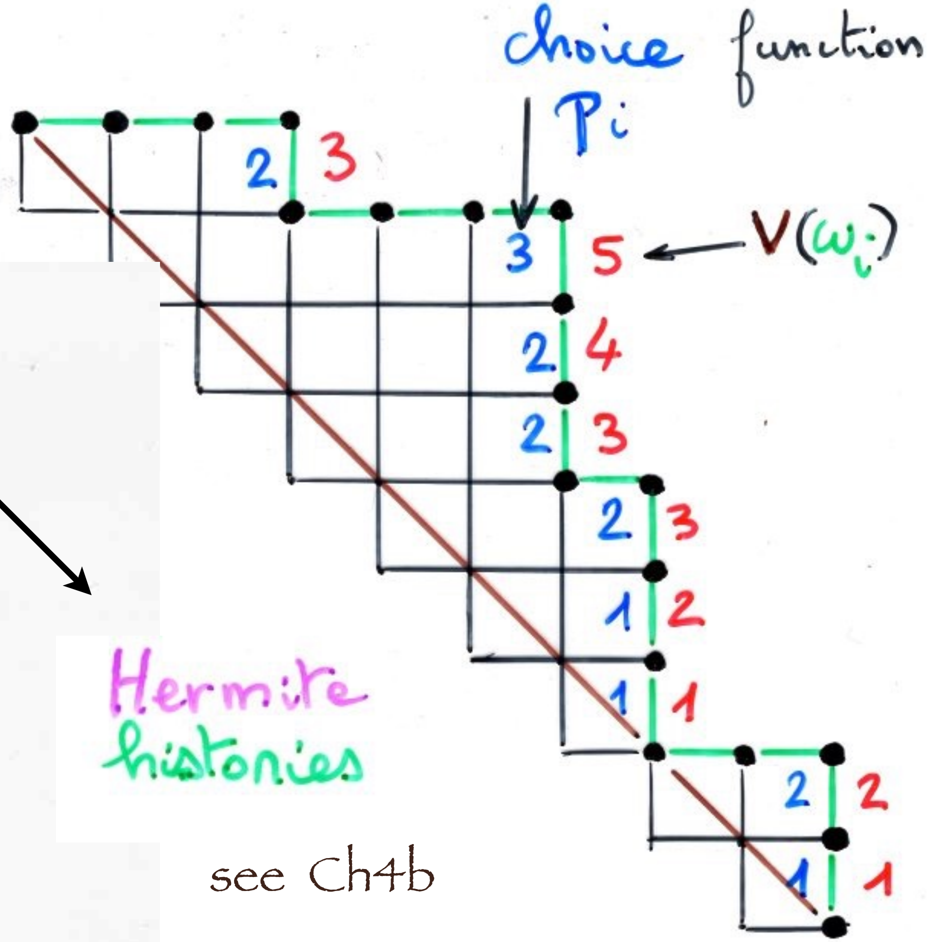
Rook placements
with
no empty row
no empty column

↕

sequences of oscillating tableaux
starting and ending
at ∅

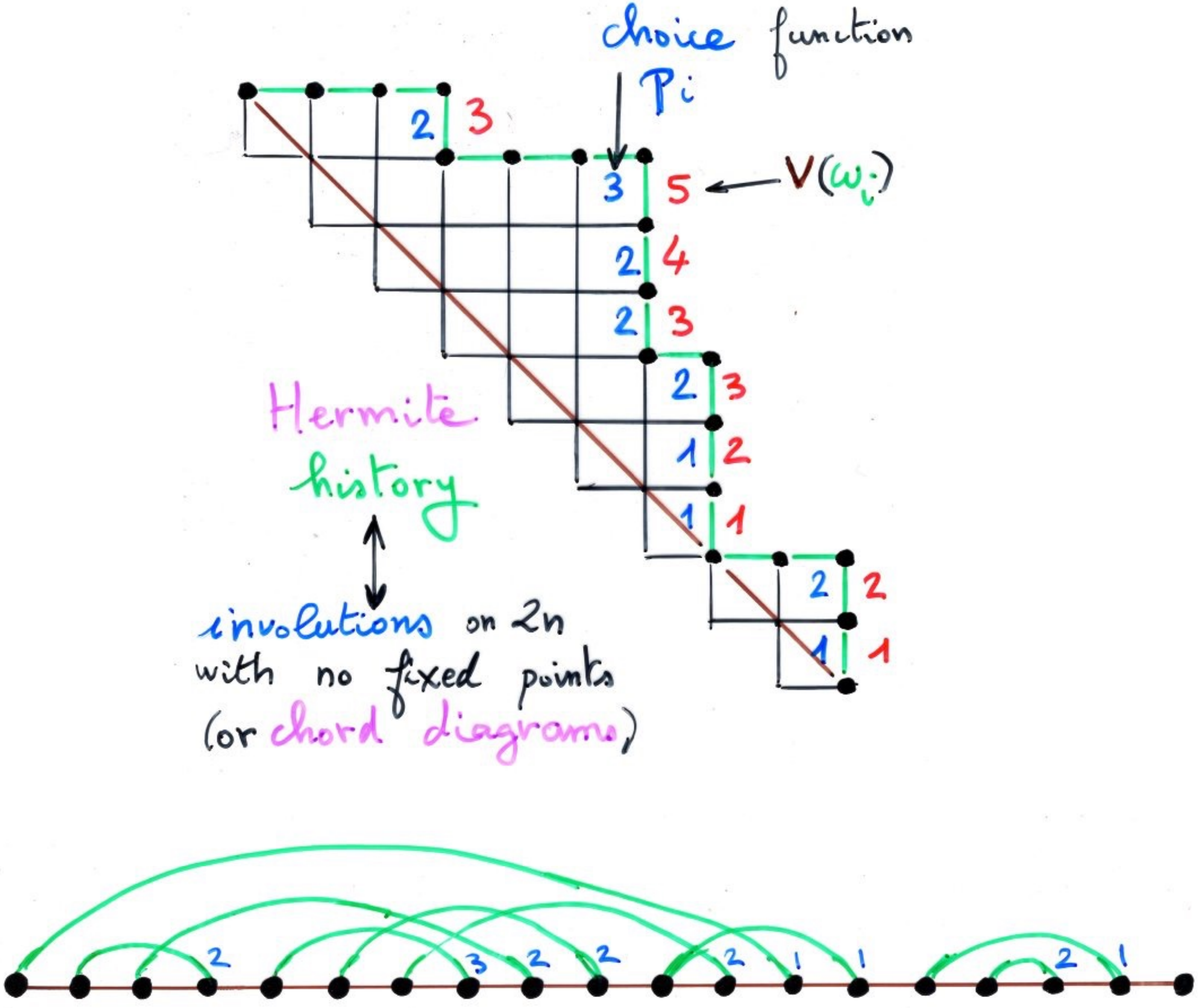
Rook placements
with
no empty row
no empty column





Hermite histories

see Ch4b


choice function
p_i
$V(\omega_i)$
Hermite history
involutions on 2n
with no fixed points
(or chord diagrams)

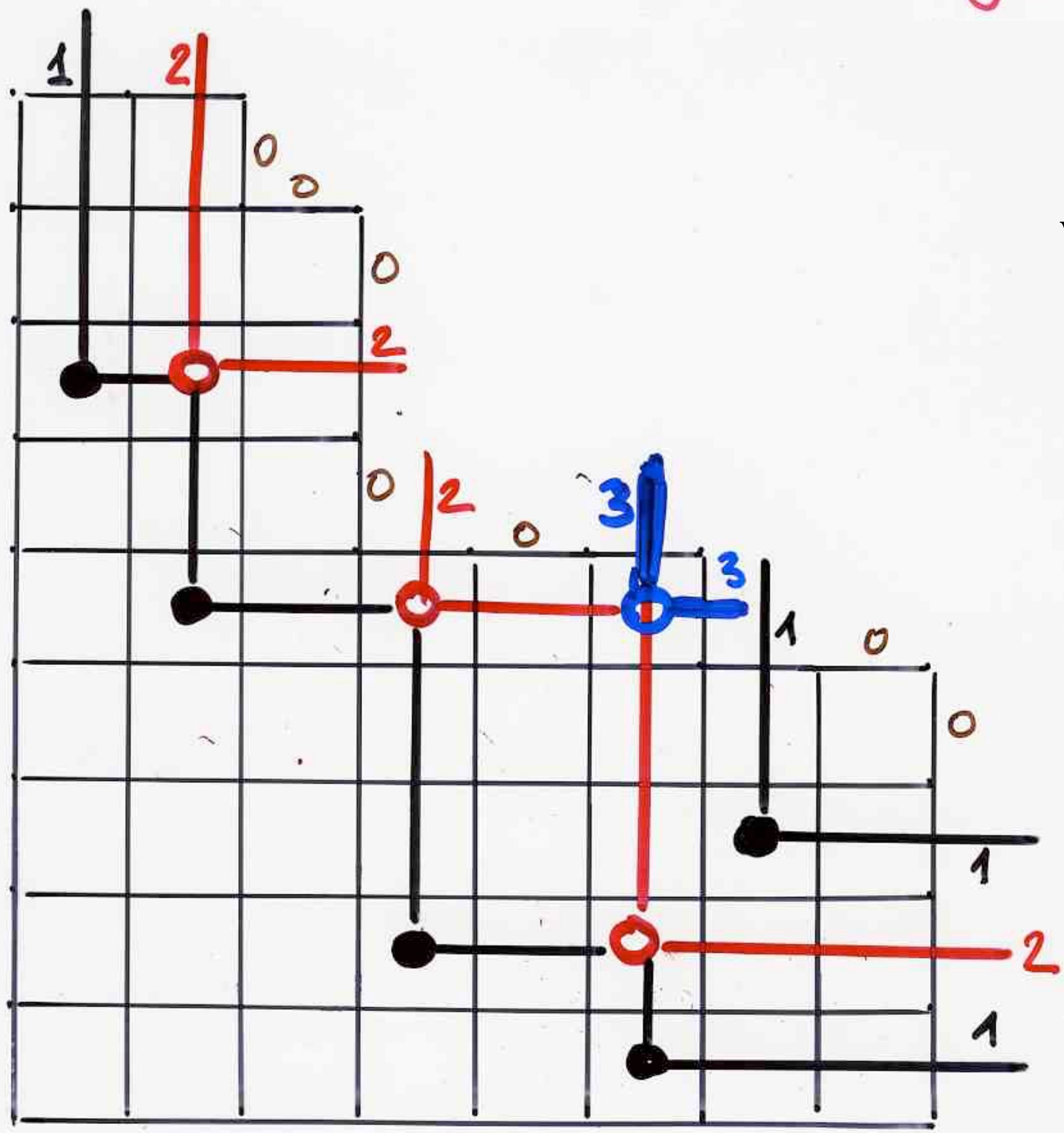
sequences of oscillating tableaux
starting and ending at $\emptyset$

Rook placements with
no empty row
no empty column

involutions on $2n$
with no fixed points
(or chord diagrams)

Hermite histories

2-colored oscillating tableaux



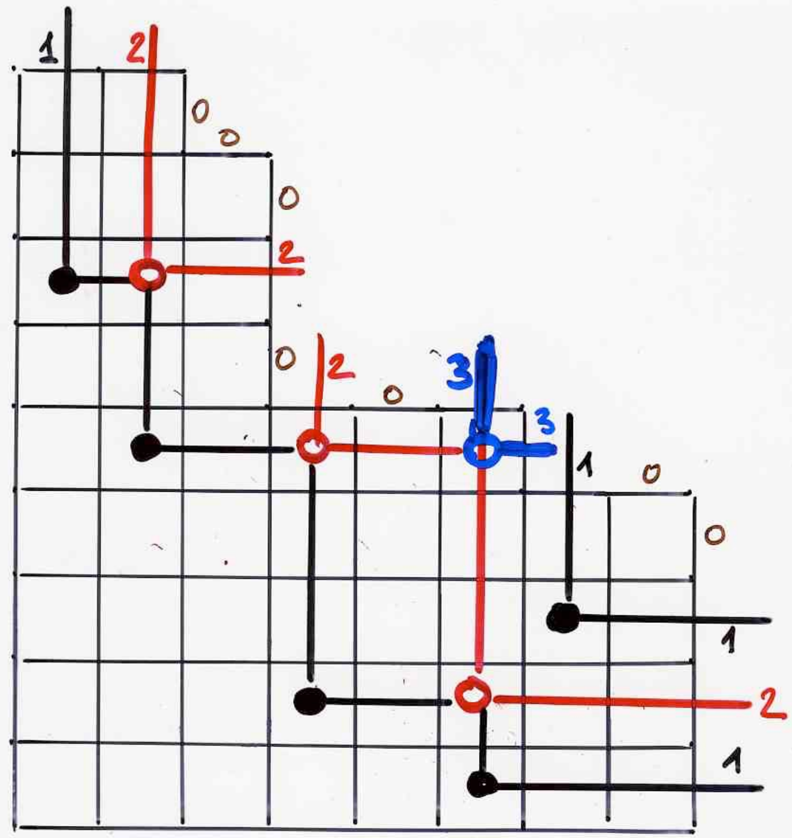
involutions on 2n
with 2-colored fixed points

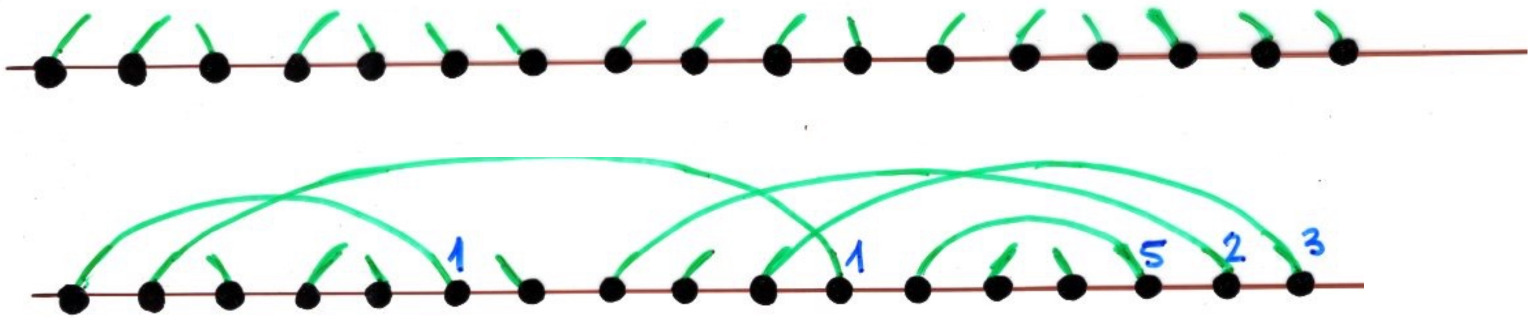
↕

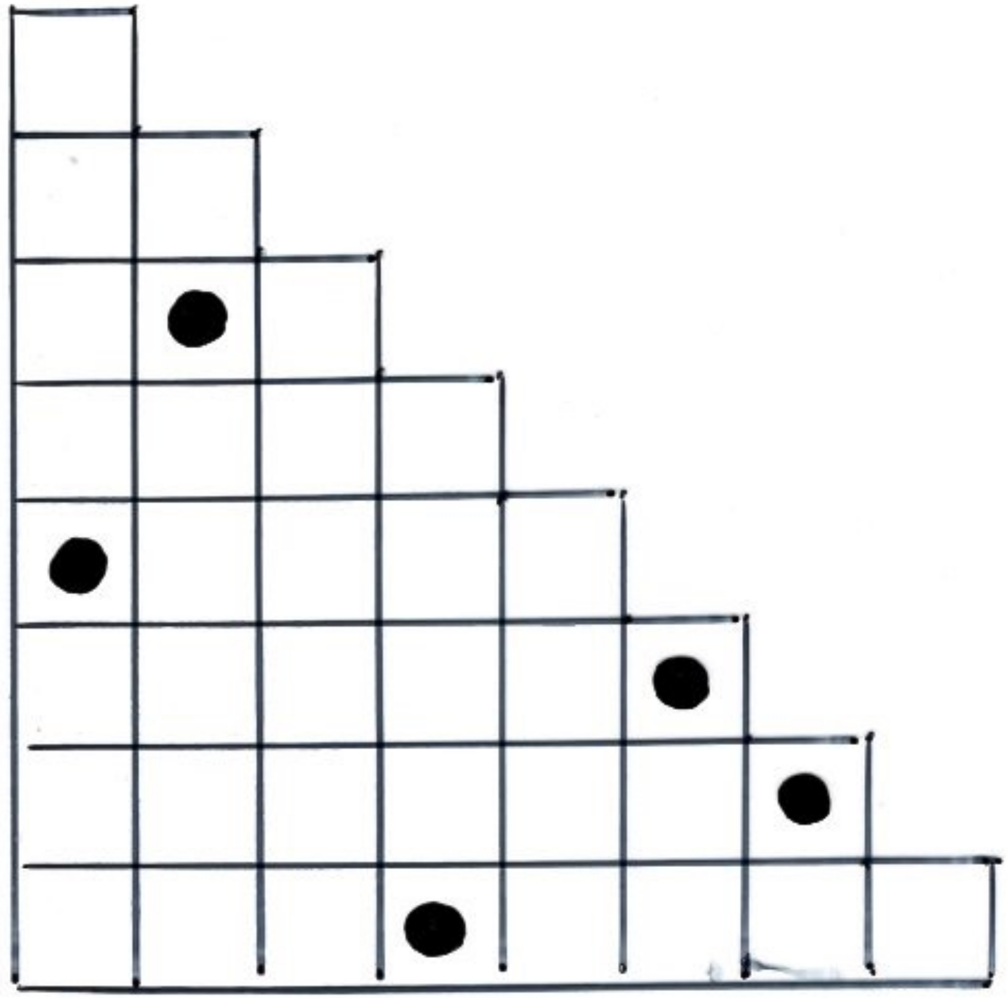
Rook placements

↕

sequences of 2n
2-colored oscillating tableaux
starting and ending at ∅





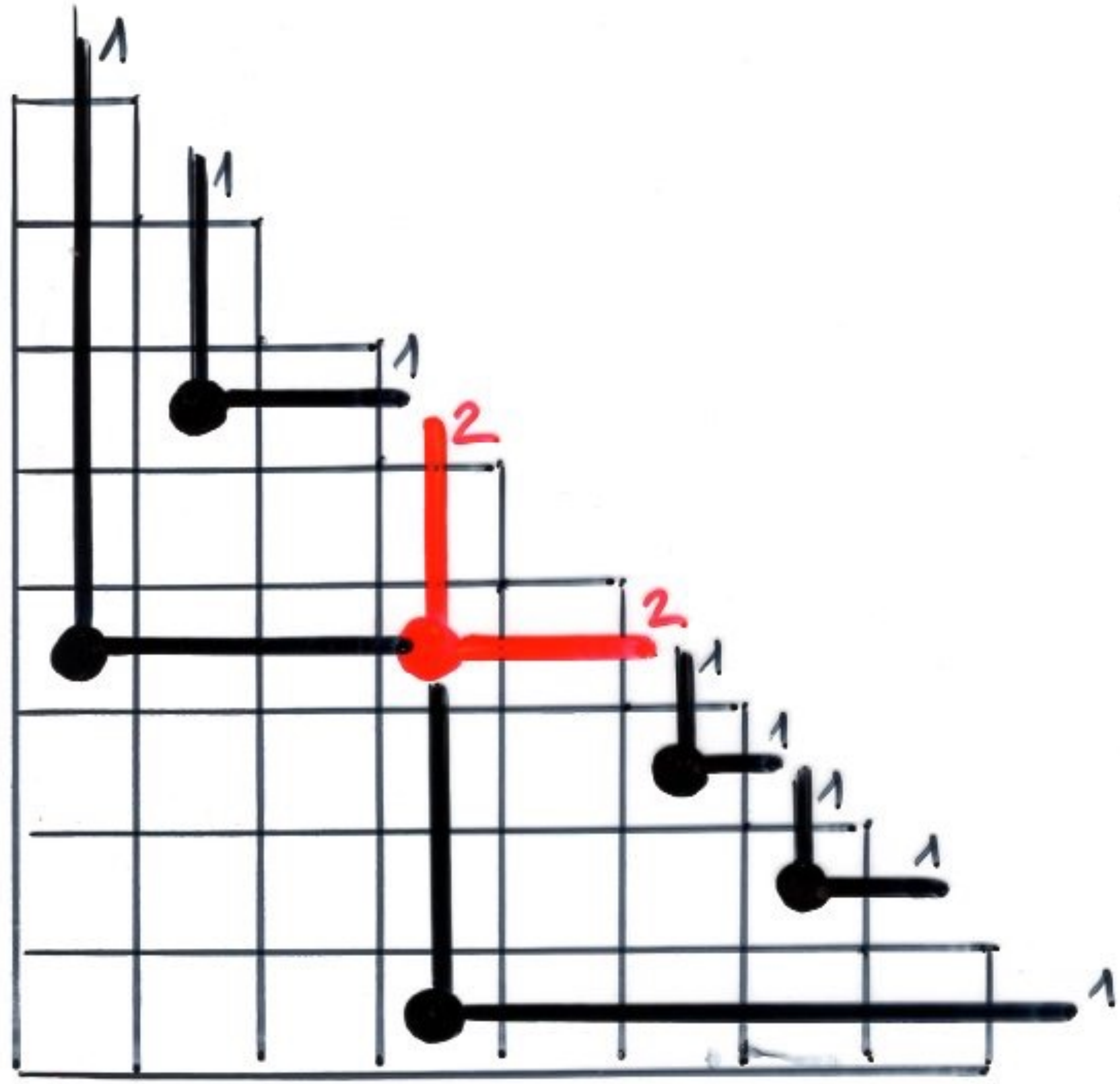


exercise Find a bijection between Rook placements in a staircase shape and partitions (of sets)

shape with n rows and columns
k rooks

$\updownarrow$

partition on $(n+1)$ elements
with $(n+1-k)$ blocks



1
1
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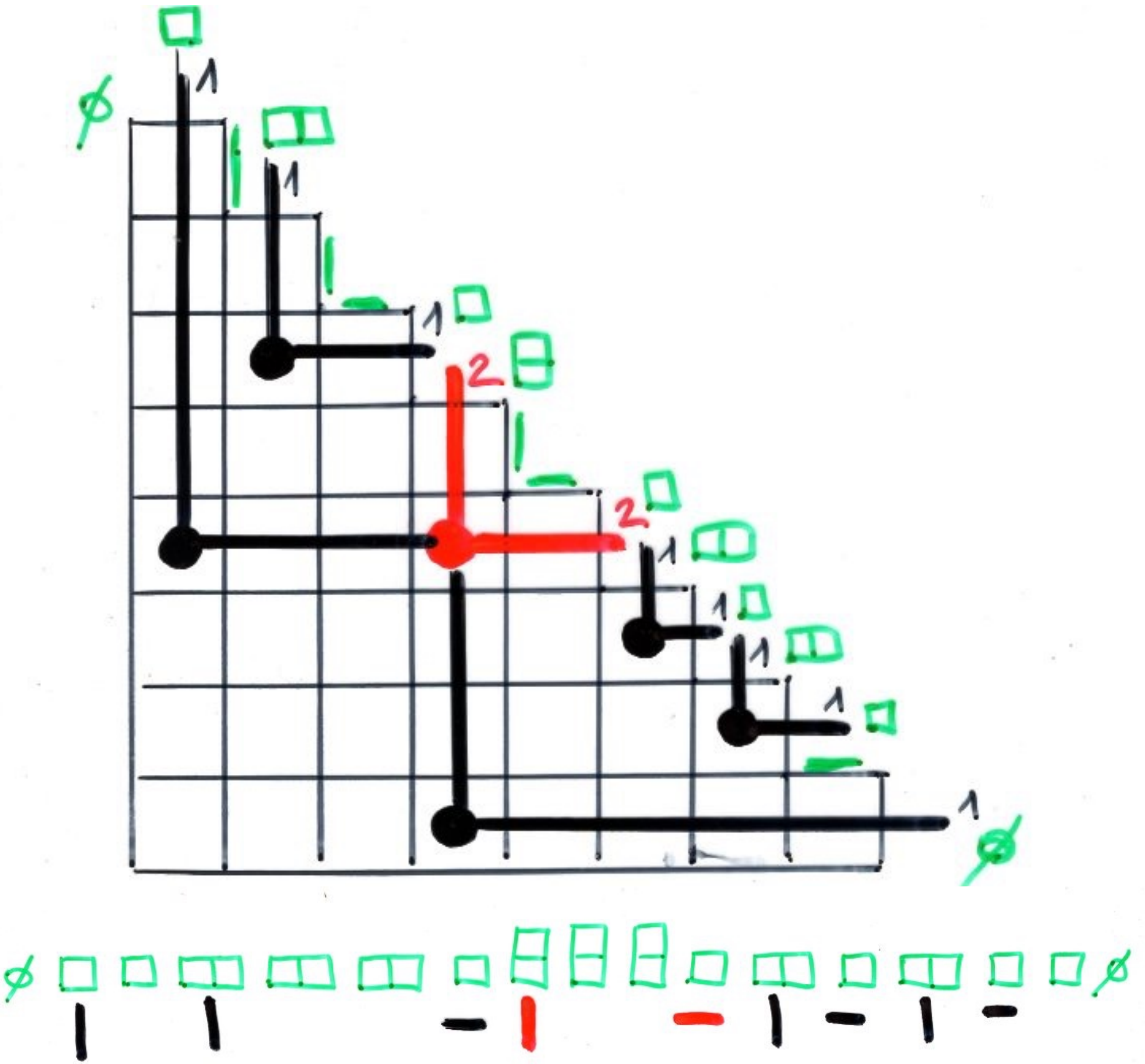
1
1
1



vacillating tableaux

hesitating tableaux

<u>exercise</u> Read (part of) the paper

W.Chen, E.Deng, R. Du, R. Stanley, C. Yan
arXiv:math.CO/0501230. Trans.A.M.S. (2005)

and reprove the fact that rook placements in a staircase shape are in bijection with sequences of vacillating (resp. hesitating) tableaux

[and thus with set partitions]

stammering tableaux

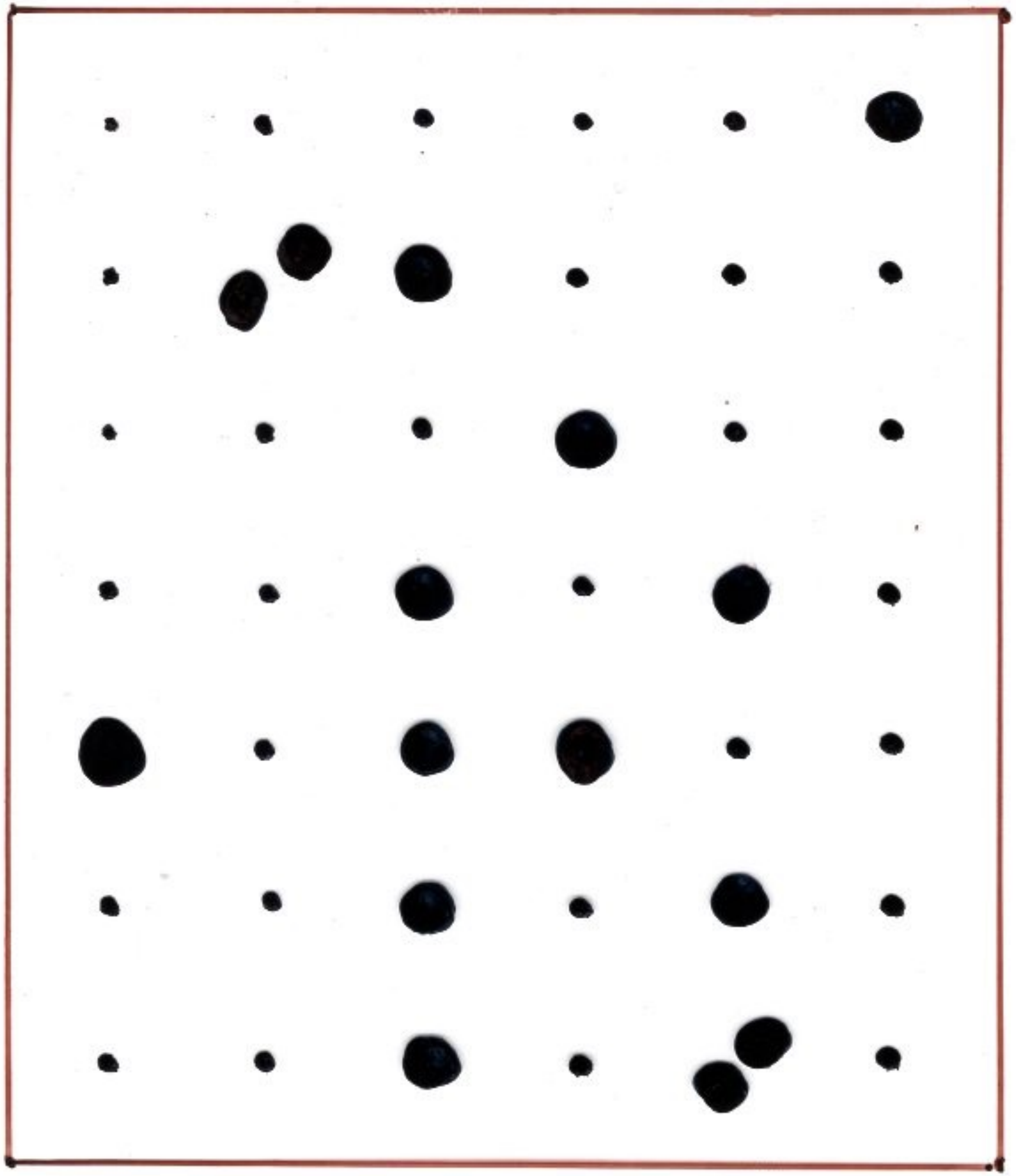
Josuat-Vergès arXiv:1601.02212 [math.CO]

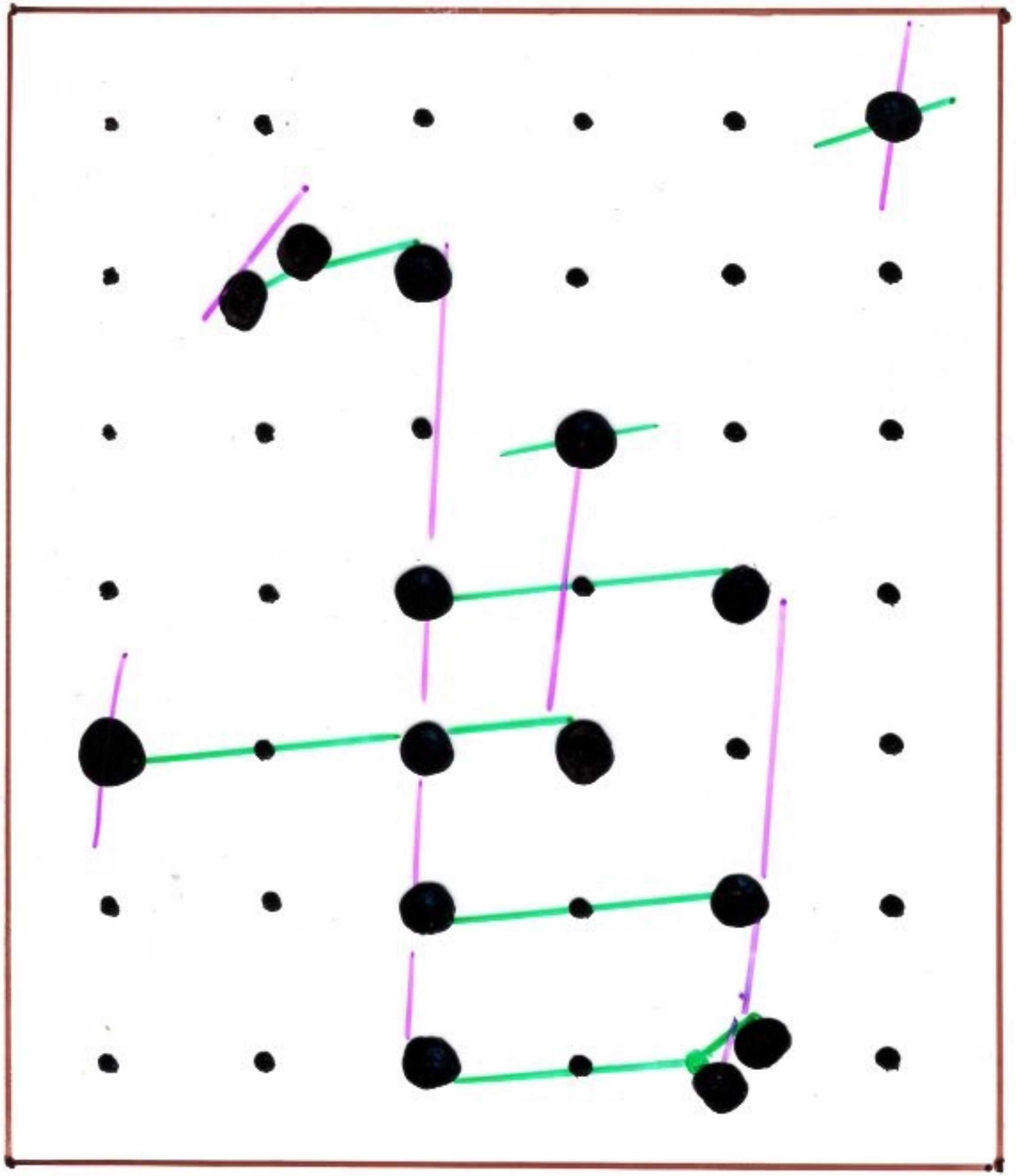
RS à RSK

extension
to matrices

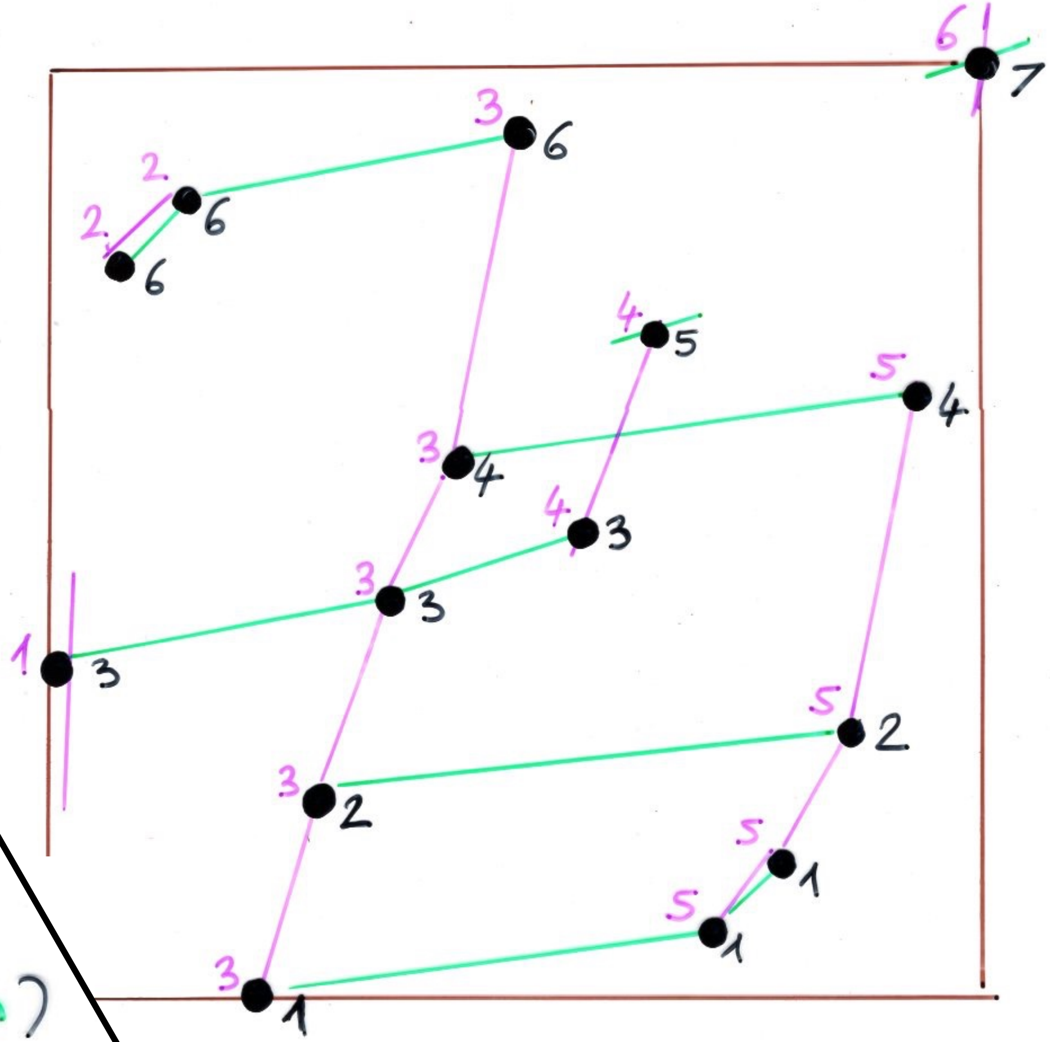
D. Knuth, 1970

$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{bmatrix}$$



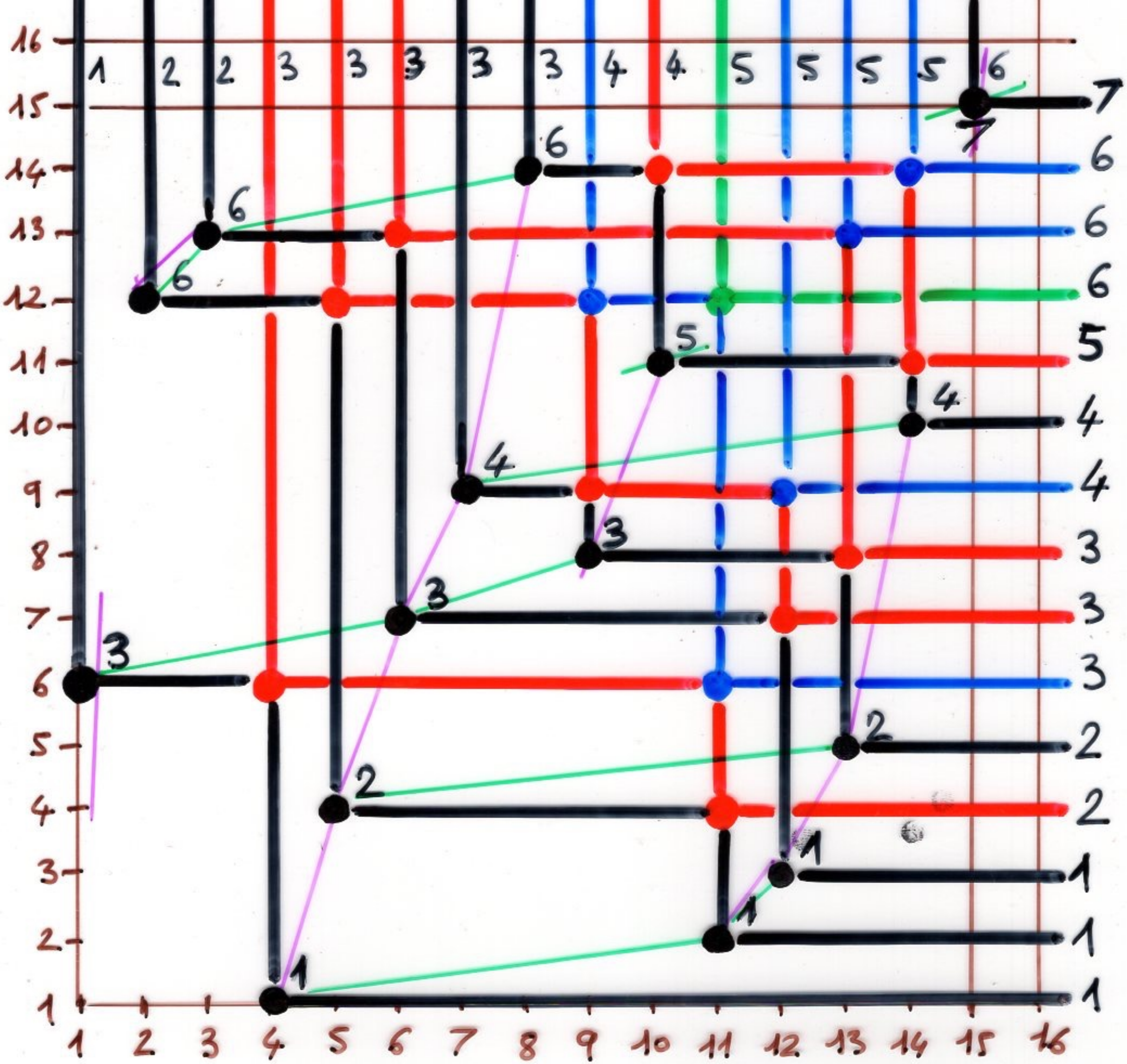


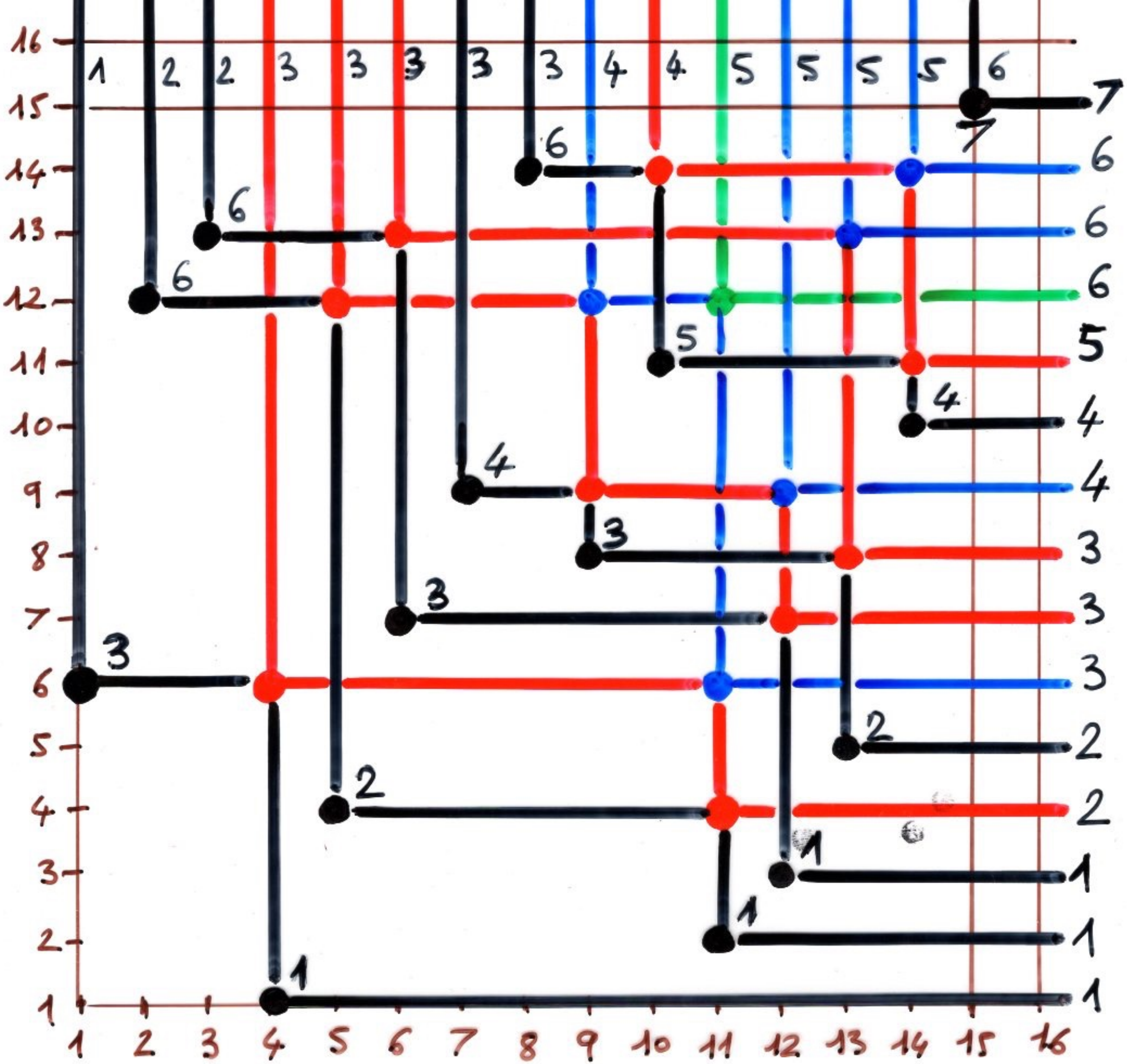
$$\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 2 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot \\
1 & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 2 & \cdot
\end{array}$$

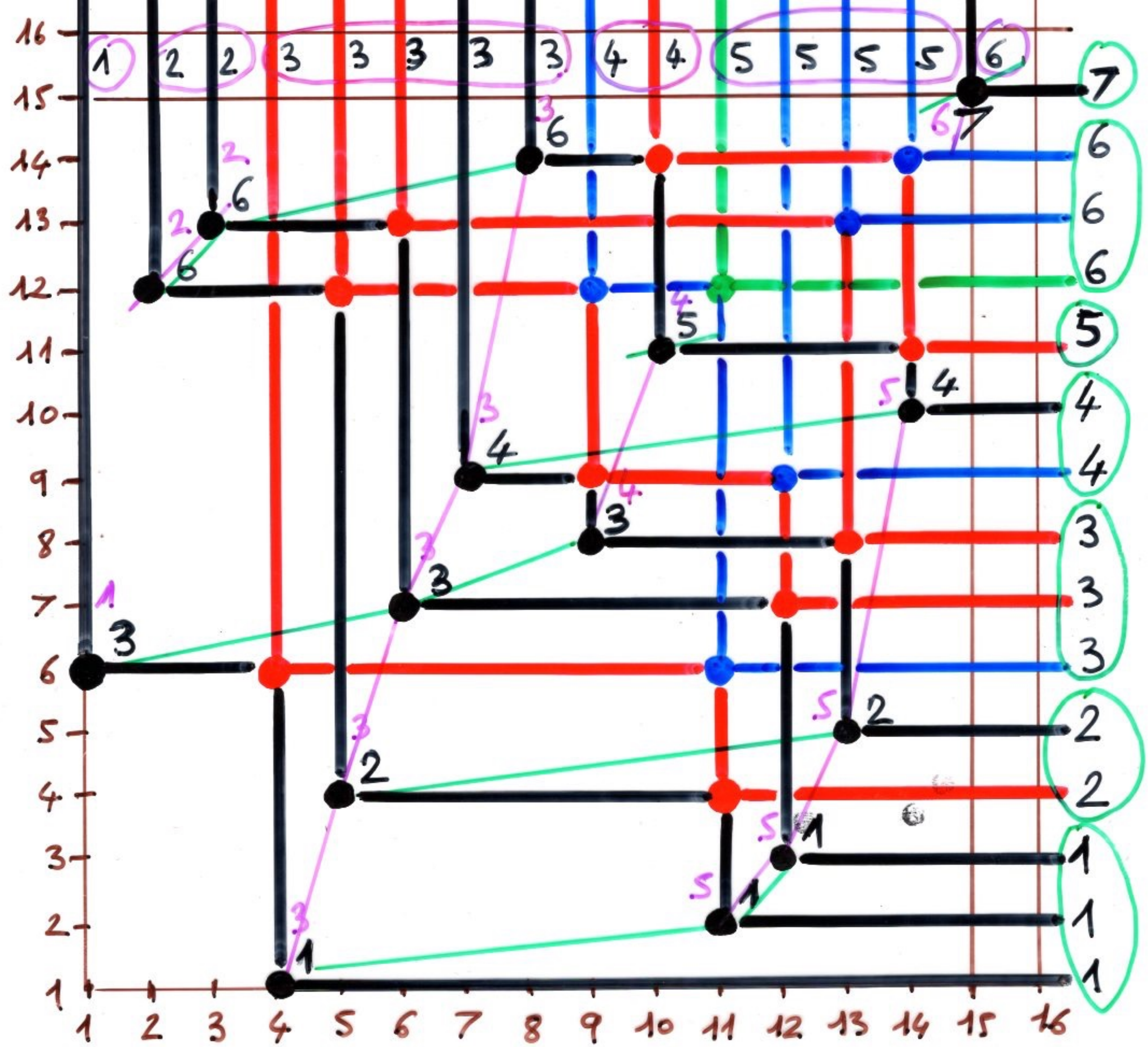


two-line array
(or generalized permutation)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{c|cc|ccccc|cc|cccc|c} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{array} \right)$$






1 2 2 3 3 3 3 3 4 4 5 5 5 5 6
7 6 6 6 5 4 4 3 3 3 2 2 1 1 1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 2 & \cdot \end{bmatrix}$$

Fulton

"matrix balls" construction

Amri Prasad

"VRSK algorithm"

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{c|cc|ccccc|cc|cccc|c} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{array}\right)$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 6 \\ \cline{1-1} 3 & 4 & 6 & 6 \\ \cline{1-4} 2 & 3 & 3 & 5 \\ \hline 1 & 1 & 1 & 2 & 4 & 7 \\ \hline \end{array}$$

$P(M)$

$$\begin{array}{|c|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 5 & 5 & 5 \\ \cline{1-4} 3 & 3 & 3 & 4 \\ \hline 1 & 2 & 2 & 3 & 3 & 6 \\ \hline \end{array}$$

$Q(M)$

Lauren Kelly Williams

App for iPad

available on the AppStore

some references about RSK

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Permutations, matrices and generalized Young tableaux, Pacific J. Math. 34 (1970) 709-727
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Jeu de taquin

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Le monoïde plaxique, in "*Non-commutative structures in algebra and geometric*", A. de Lucas ed., Quaderni de la Ricerca Scientifica n°109, 1981, p.129-156

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Sergey Fomin

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- **Schensted algorithms for dual graded graphs**, *Journal of Algebraic Combinatorics* **4** (1995), 5-45.
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- **Finite posets and Ferrers shapes** (with T.Britz, 41 pages)*Advances in Mathematics* **158** (2000), 86-127.

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- **Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.),** The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145-165.

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Sergey Fomin **Knuth equivalence, jeu de taquin and the Littlewood-Robinson rule** (30 pages)
Appendix 1 to Chapter 7 in: R.P.Stanley, *Enumerative Combinatorics, vol.2*, Cambridge University Press, 1999.

Amritanshu Prasad, Representaion Theory, A combinatorial Viewpoint, Cambridge University Press, 2015

Softwares

Guoniu Han

Autour de la correspondance de Robinson-Schensted http://math.u-strasbg.fr/~guoniu/software/rsk/index.html

Exposé au SLC 52 et LascouxFest, 29/03/2004

on the Apple store: **RSK** (by L . Williams)