An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 4

The n! garden
(1)

IMSc
16 February 2016

permutations

classic

permutations very classic

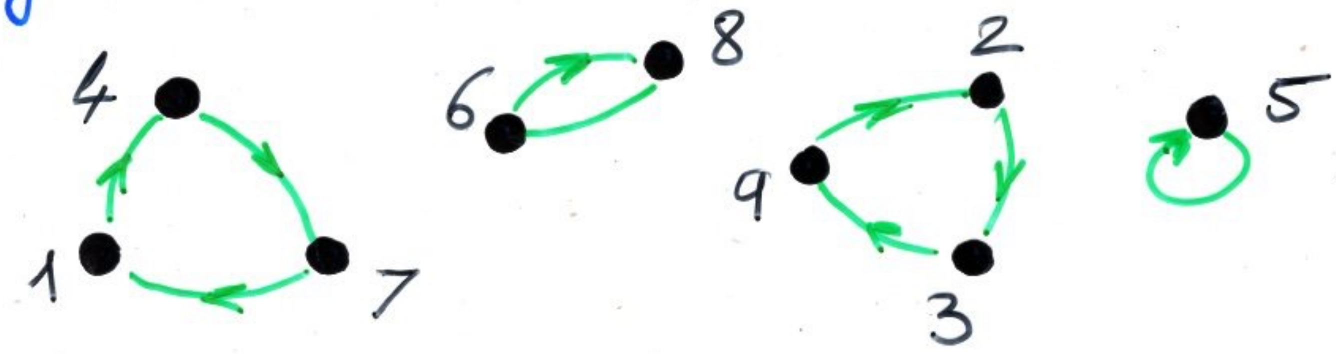
different representations

as a bijection $\{1, 2, \dots, n\} \xrightarrow{\sigma} \{1, 2, \dots, n\}$

($\mathbb{B}$-species)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

cycles notation



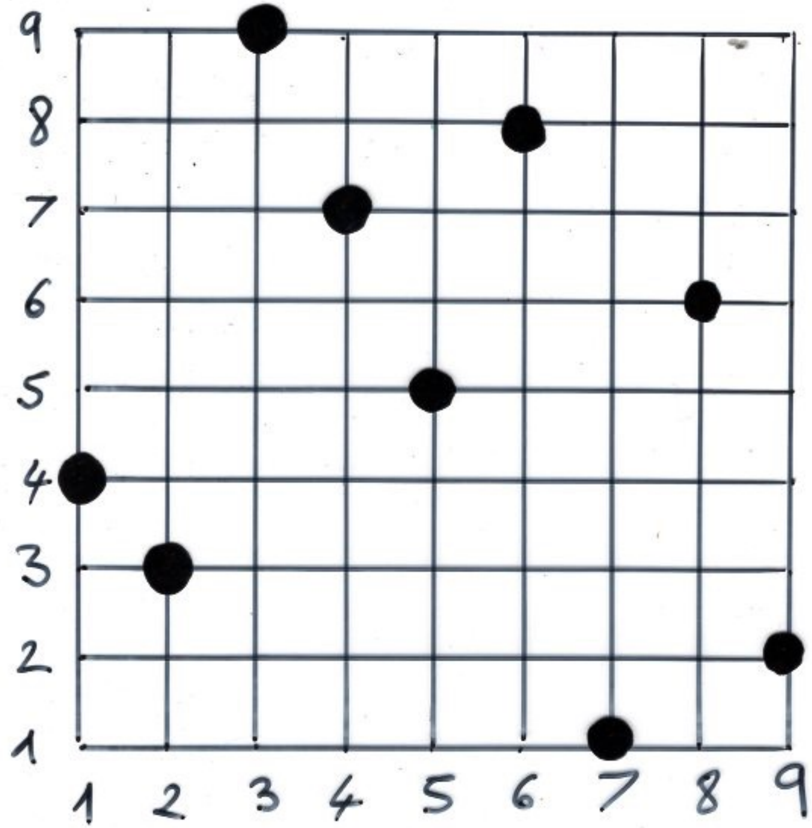
($\mathbb{L}$- species)

total order as a word

$w = 4\ 3\ 9\ 7\ 5\ 8\ 1\ 6\ 2$

graphical representation

$$\{(i, \sigma(i))\}_{1 \le i \le n}$$



$\sigma \in \mathfrak{S}_n$

symmetric group

inverse $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}$

$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$

$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$

composition (product) of two permutations

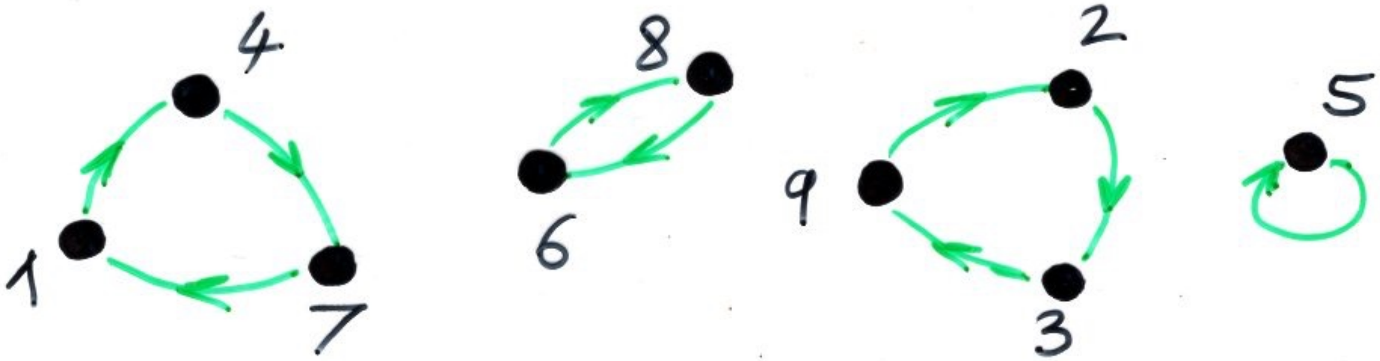
$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

neutral element

identy permutation

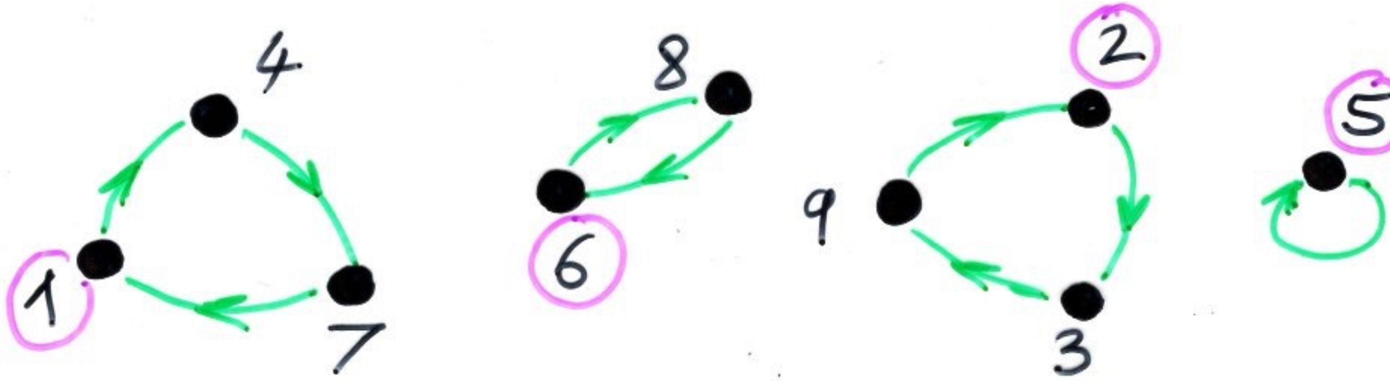
a classical bijection very classic !

σ cycles notation $\xrightarrow{f}$ word $\tau = f(\sigma)$



a classical bijection very classic !

σ cycles notation $\xrightarrow{f}$ word $\tau = f(\sigma)$



$\tau = /\, 6\ 8 \,/\, 5 \,/\, 2\ 3\ 9 \,/\, 1\ 4\ 7$

$$\tau = 6\ 8\ 5\ 2\ 3\ 9\ 1\ 4\ 7$$

$w = x_1 x_2 \dots x_n$

lr-min

word with distinct letters

left to right minimum element

$$x_i = \min(x_1, x_2, \dots, x_n)$$

$\tau = 6\ 8\ 5\ 2\ 3\ 9\ 1\ 4\ 7$

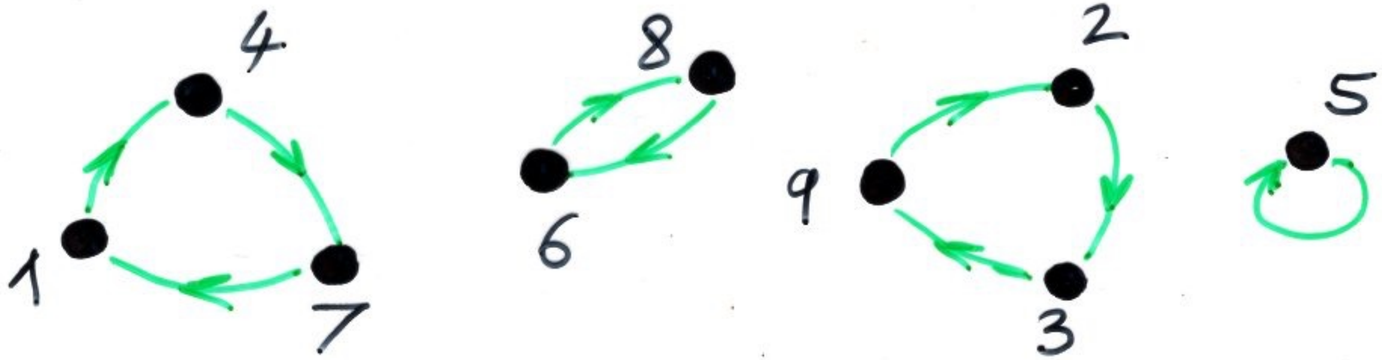
$w = x_1 x_2 \dots x_n$

word with distinct letters

lr-min

left to right minimum element

$x_i = \min(x_1, x_2, \dots, x_n)$



Foata (1965)

"transformation fondamentale"

cycles $\longrightarrow$ lr-min elements of w

distribution of lr-min elements

Stirling numbers $s_{n,k}$

$$\sum_{1\leqslant k\leqslant n} s_{n,k}\, x^k = x(x+1)\cdots(x+n-1)$$

$$\frac{1}{(1-t)^x} = \sum_{n\geqslant 0}\Big(\sum_k s_{n,k}\, x^k\Big)\frac{t^n}{n!}$$

(from ch 3)

$$(1-t)^{-x} = 1 - \frac{t}{1!}(-x) + t^2\frac{(-x)(-x-1)}{2!} - t^3\frac{(-x)(-x-1)(-x-2)}{3!} + \cdots$$

<u>exercise</u> algorithm finding the minimum of a sequence

algorithm $\min(w)$

  $\{w = a_1 .. a_n$ sequence of numbers$\}$

begin

  $m := a_1$

  for $i=2$ to n **do**

    begin **if** $a_i < m$ **then** $m := a_i$

    end

  $\min(w) := m$

end

$$\text{cost} = an + b\,s(n)$$

$s(n) = \{$number of times the instruction $m := a_i$ is done$\}$

exercise algorithm finding the minimum of a sequence

average cost $\dfrac{\sum k\, s_{n,k}}{n!} = \dfrac{s'_n(1)}{s_n(1)}$

Stirling numbers $s_{n,k}$

$$\sum_{1 \leq k \leq n} s_{n,k}\, x^k = x(x+1)\cdots(x+n-1)$$

$= H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

harmonic number

data structures

computer science

average cost $an + b\, H_n$

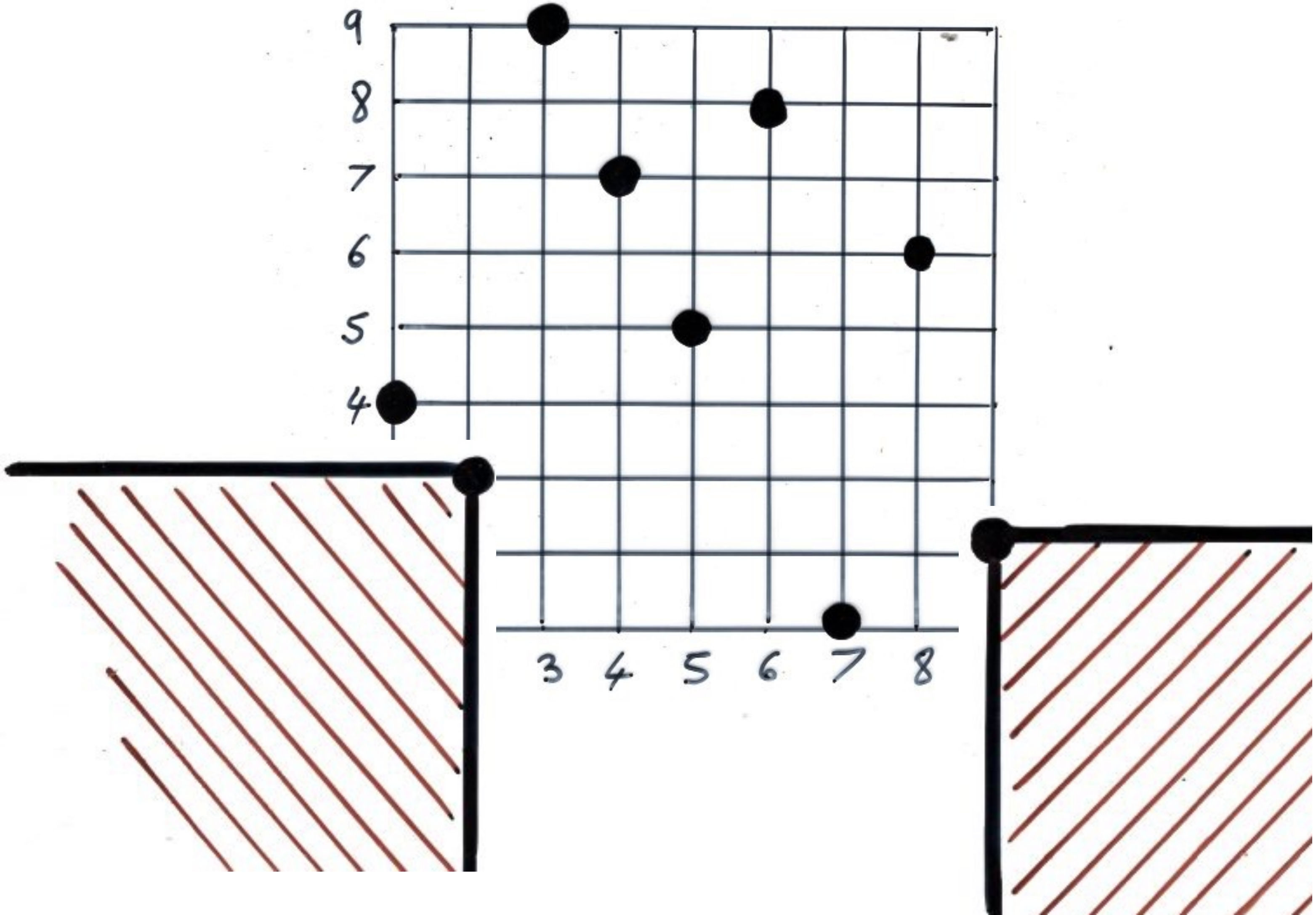
minimum $an + b\, 1$

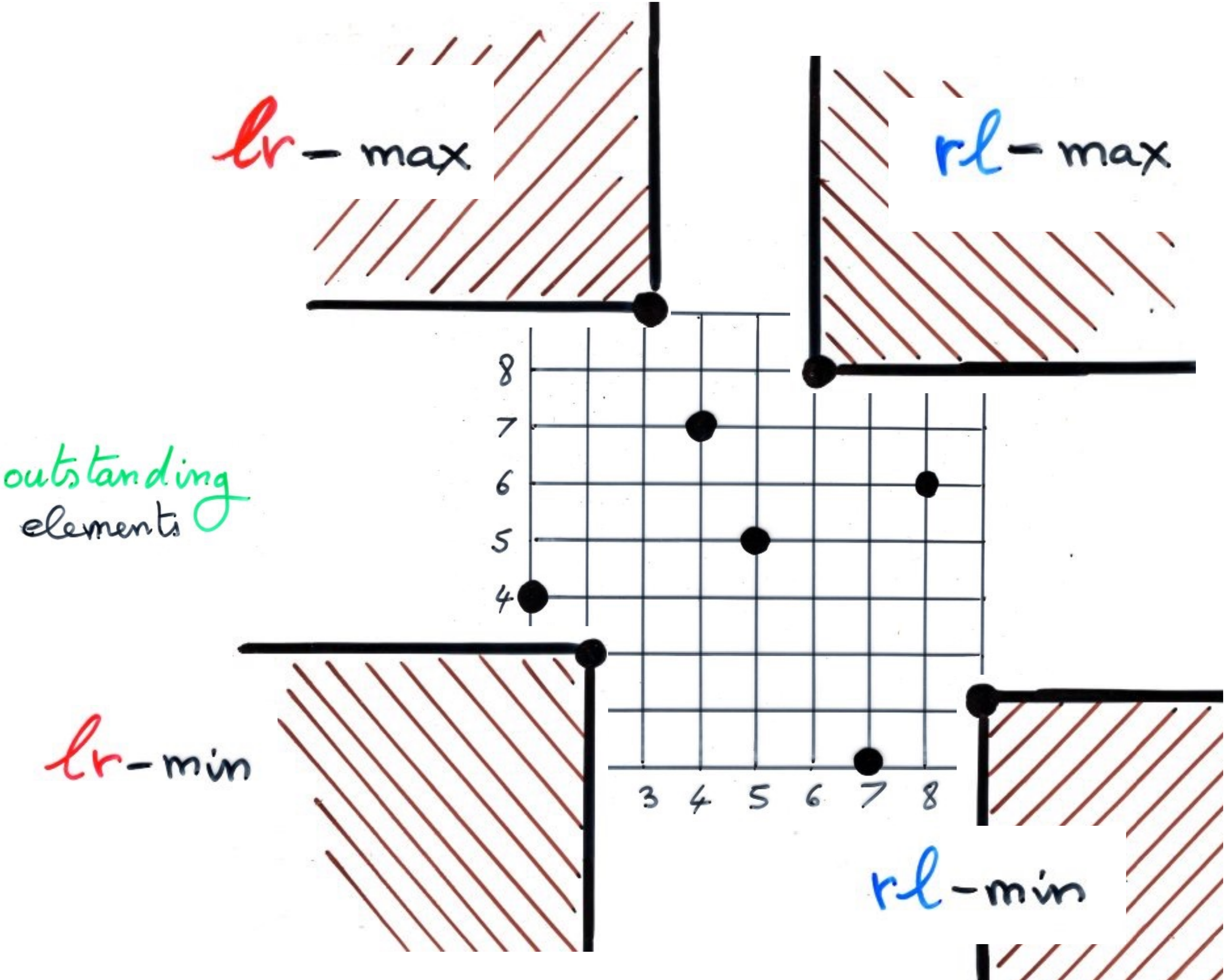
maximum $an + b\, n$

rl-min

$$x_i = \min(x_i, x_{i+1}, \ldots, x_n)$$

lr-min




lr-max
rl-max
outstanding elements
8
7
6
5
4
lr-min
3 4 5 6 7 8
rl-min

rise $\sigma(i) < \sigma(i+1)$

descent $\sigma(i) > \sigma(i+1)$

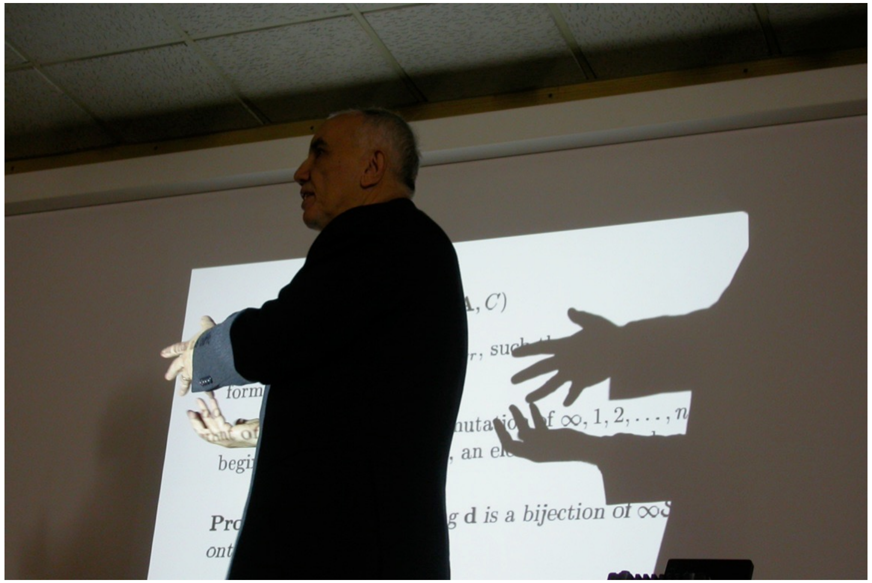
$$4\ 7\ 1\ 9\ 2\ 3\ 5\ 8\ 6$$

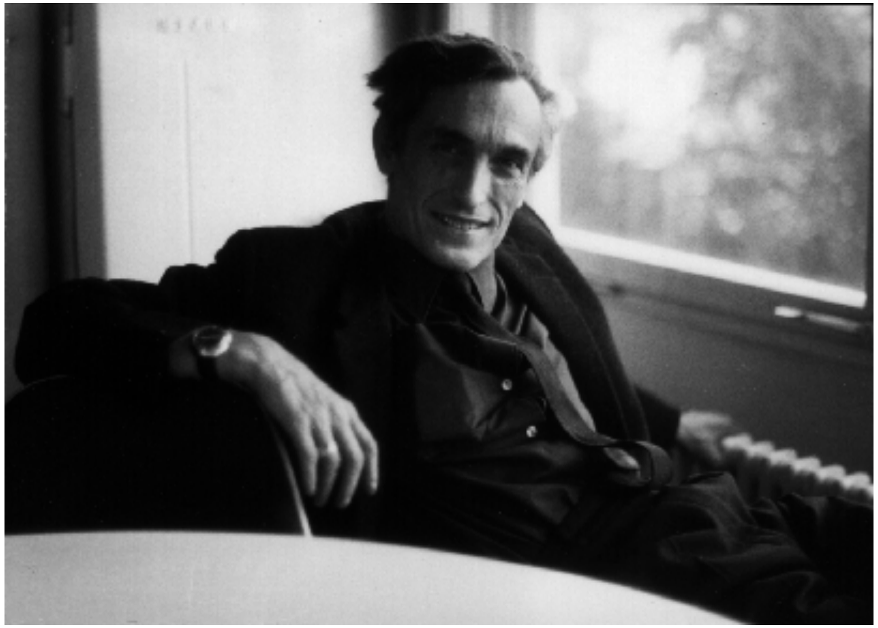
$a_{n,k}$ = number of $\sigma \in \mathfrak{S}_n$ having k rises

$$A_n(x) = \sum_k a_{n,k}\, x^k$$

Euler (1755)

eulerian polynomials





D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens"
(1970)

<u>exercise</u> bijective proof

$$A_n(x) = \sum_{0 \le k \le n-1} (n-k)!\,(x-1)^k\, S(n, n-k)$$

Frobenius

excedance

$i < \sigma(i)$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

excedance

$i < \sigma(i)$

$\mathrm{exc}(\sigma)$ = number of excedances

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{rise}(\sigma)}$$

$\mathrm{rise}(\sigma)$ = number of rises of σ

eulerian distribution

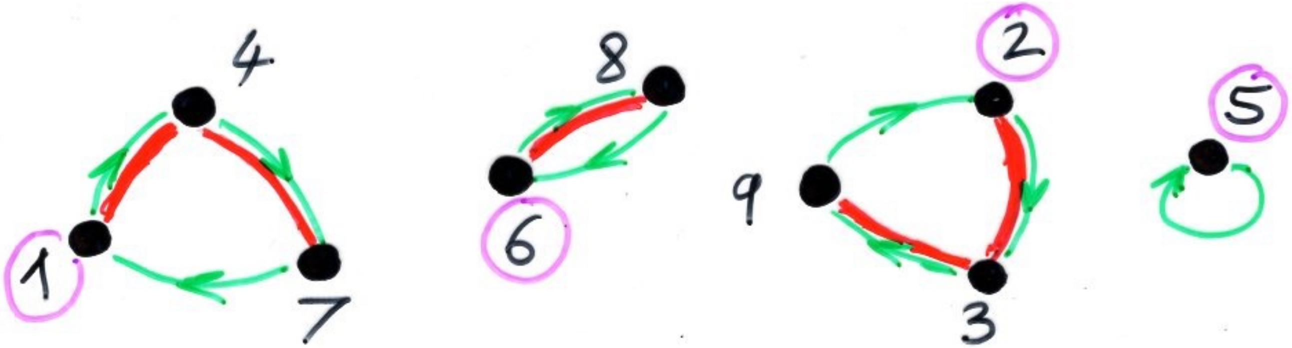
$A_n(x)$ eulerian polynomial

excedance

$i < \sigma(i)$

$\mathrm{exc}(\sigma)$ = number of excedances

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$



$\tau = / 6 - 8 / 5 / 2 - 3 - 9 / 1 - 4 - 7$

up-down sequence of a permutation

4 7 1 9 2 3 5 8 6

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$

<u>exercise</u> number of permutations with a given up-down sequence

example: / \ \ / /

1
/ 0 1
1 1 0 \
2 1 0 0 \
/ 0 2 3 3 3
/ 0 0 2 5 8 11 → 26

1
1 0 ←
→ 0 1 1
2 2 1 0 ←
→ 0 2 4 5 5
16 16 14 10 5 0 ←
→ 0 16 32 46 56 61 61
272 0

tangent and secant numbers

Inversion table
q-analogue

<u>Definition</u>

sub-excedante functions

$$f : [1, n] \longrightarrow [0, n-1]$$

pour tout $1 \leqslant i \leqslant n$, $0 \leqslant f(i) < i$

$\mathcal{F}_n$ set of sub-excedante functions

$$|\mathcal{F}_n| = n!$$

$$\sum_{f \in \mathcal{F}} q^{\mathrm{sum}(f)} = 1(1+q)\cdots(1+q+q^2+\cdots+q^{n-1})$$

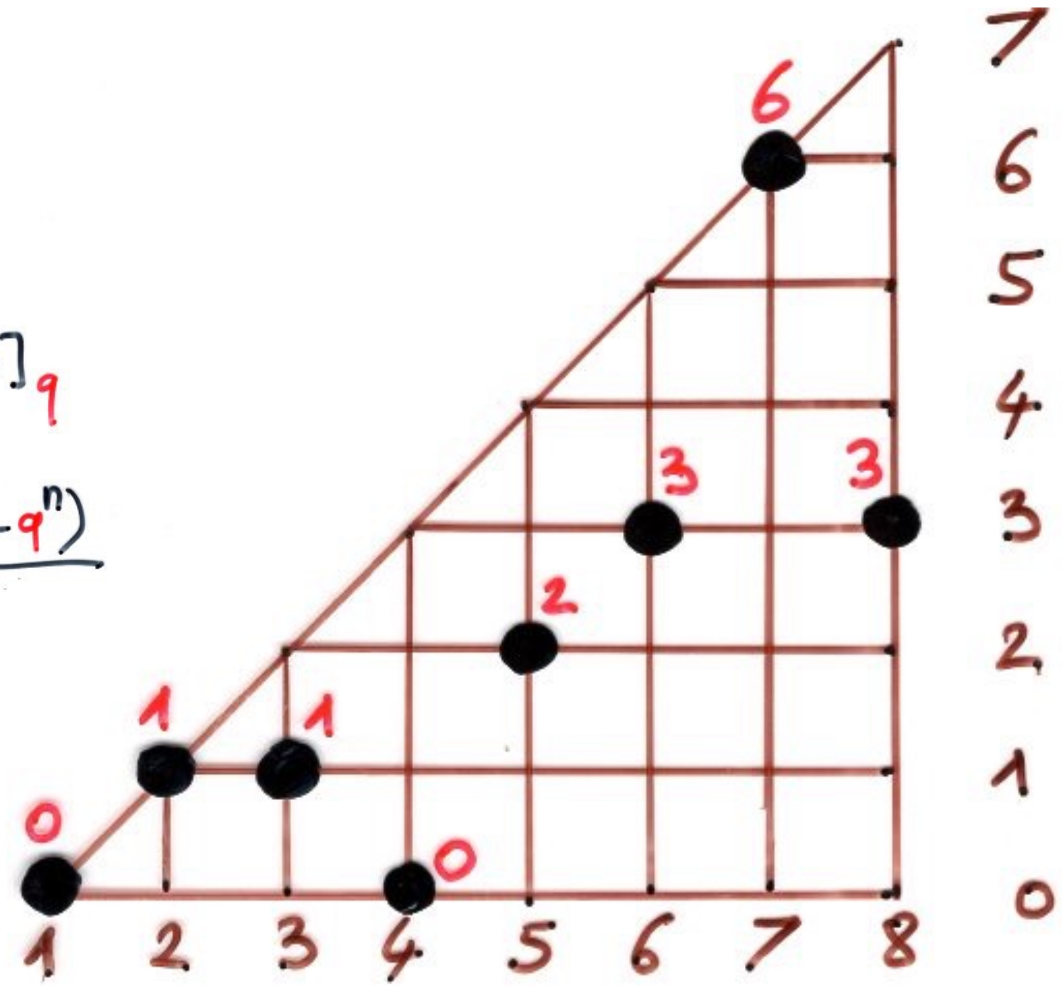
$$= [n!] \text{ or } [n]! \text{ or } [n!]_q$$

$$[i]_q = 1+q+\cdots+q^{i-1}$$

$$= \frac{1-q^i}{1-q}$$

$$[n!]_q = [1]_q \times [2]_q \times \cdots \times [n]_q$$

$$= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$



2 bijections $\mathfrak{S}_n \longleftrightarrow \mathcal{F}_n$

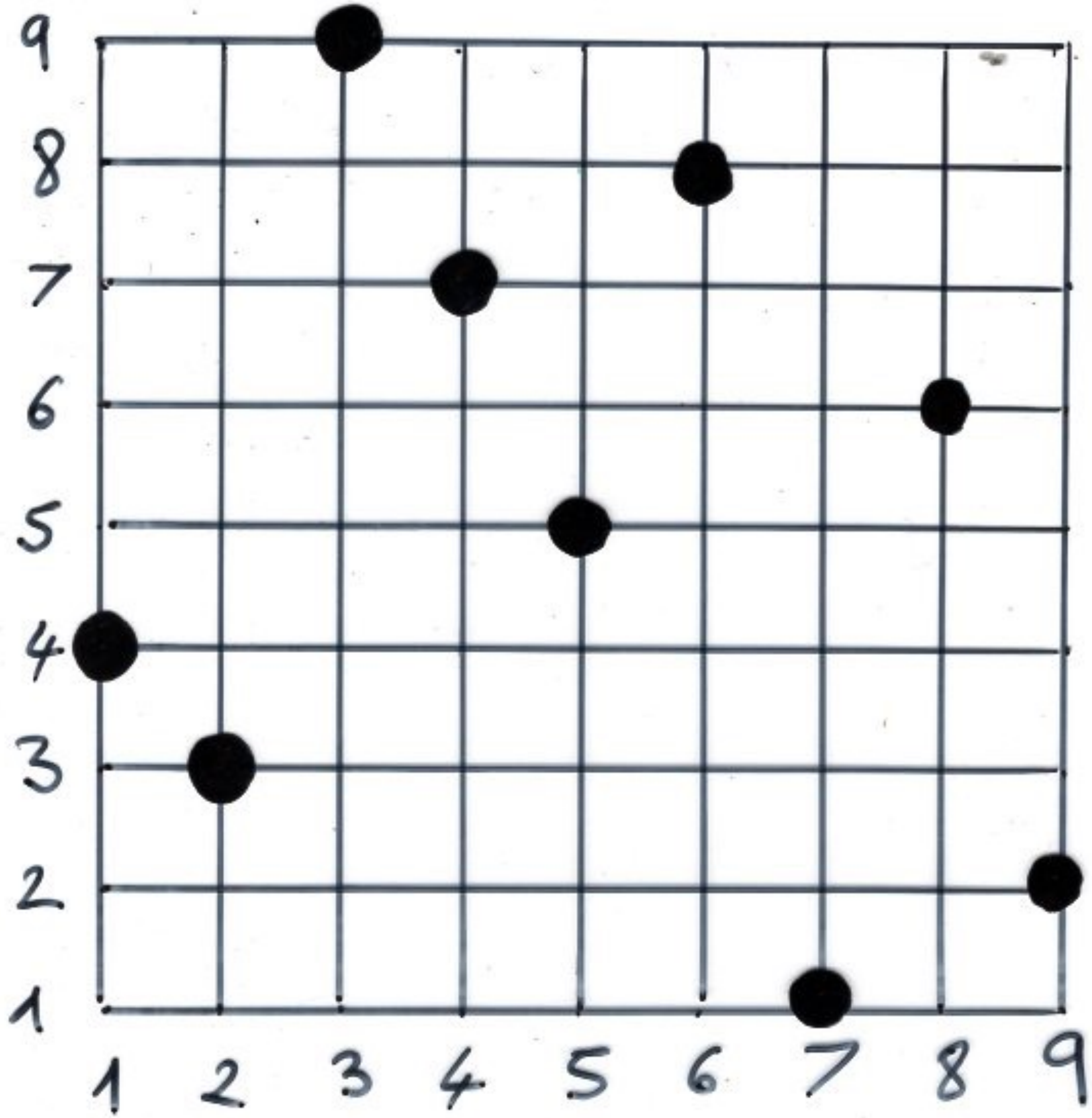
- inversion table
- "assemblée" of increasing arborescences

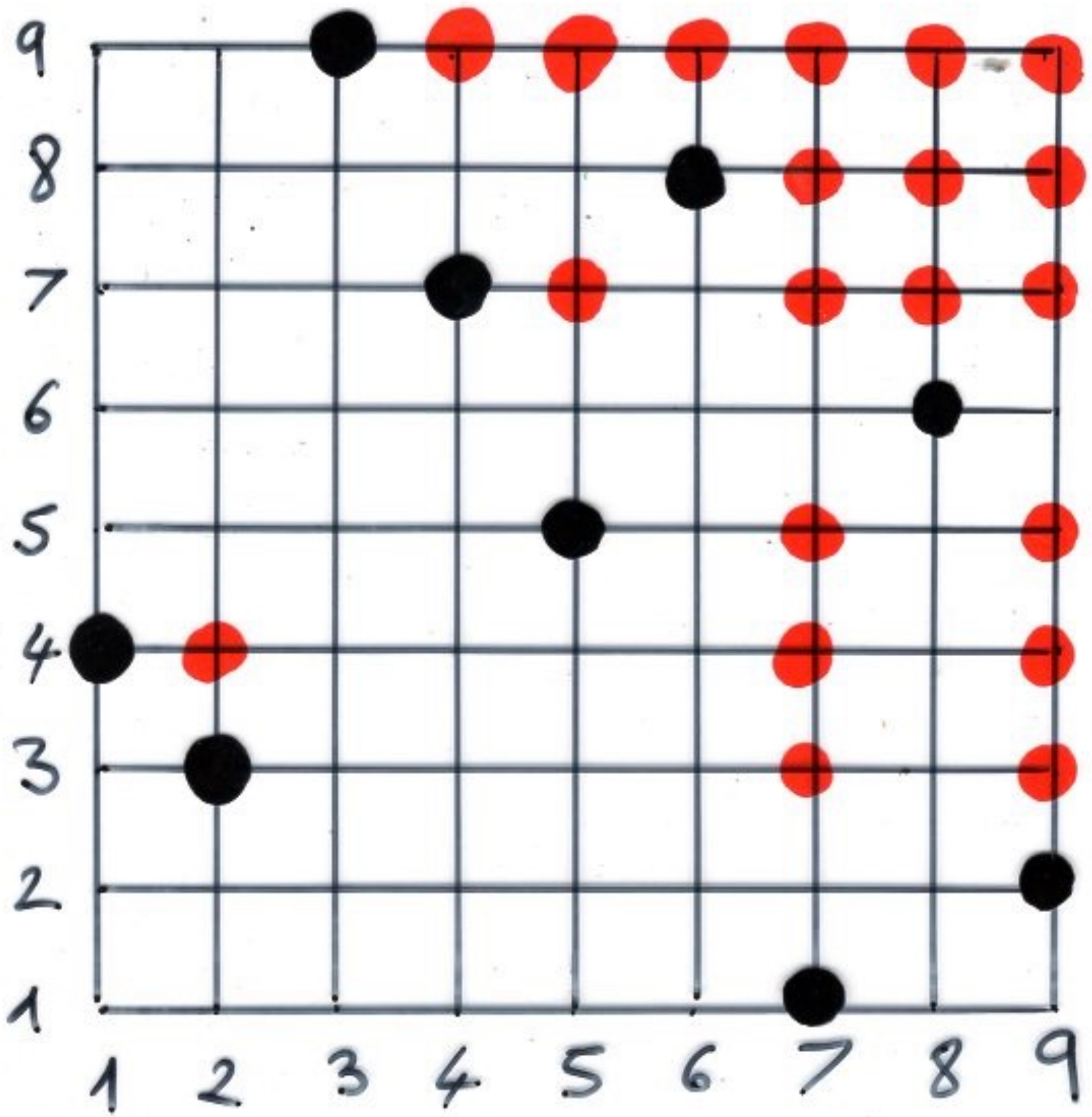
inversion of σ

(i, j) $\quad 1 \leq i < j \leq n$

$\sigma(i) > \sigma(j)$

$inv(\sigma)$ = number of inversions


9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9



Rothe diagram
(1800)

$$\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$$

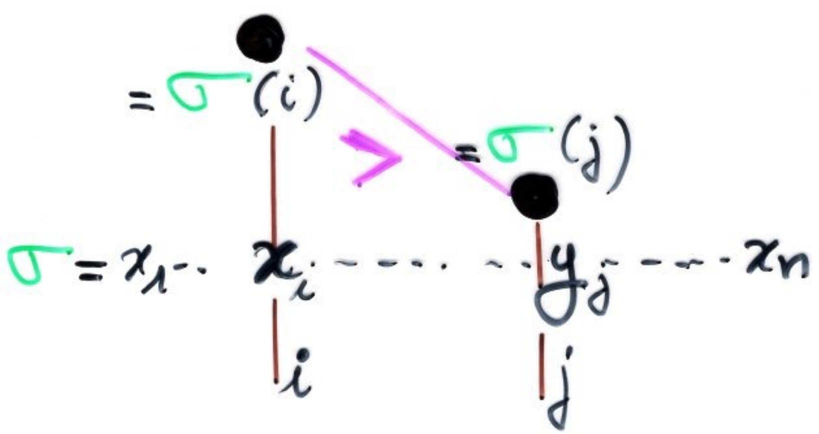
$\sigma \in \mathfrak{S}_n \longrightarrow f \in \mathcal{F}_n$ **Inversion table**

$$\sigma = \frac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

$1 \leq x \leq n$ $\quad x = \sigma(i)$

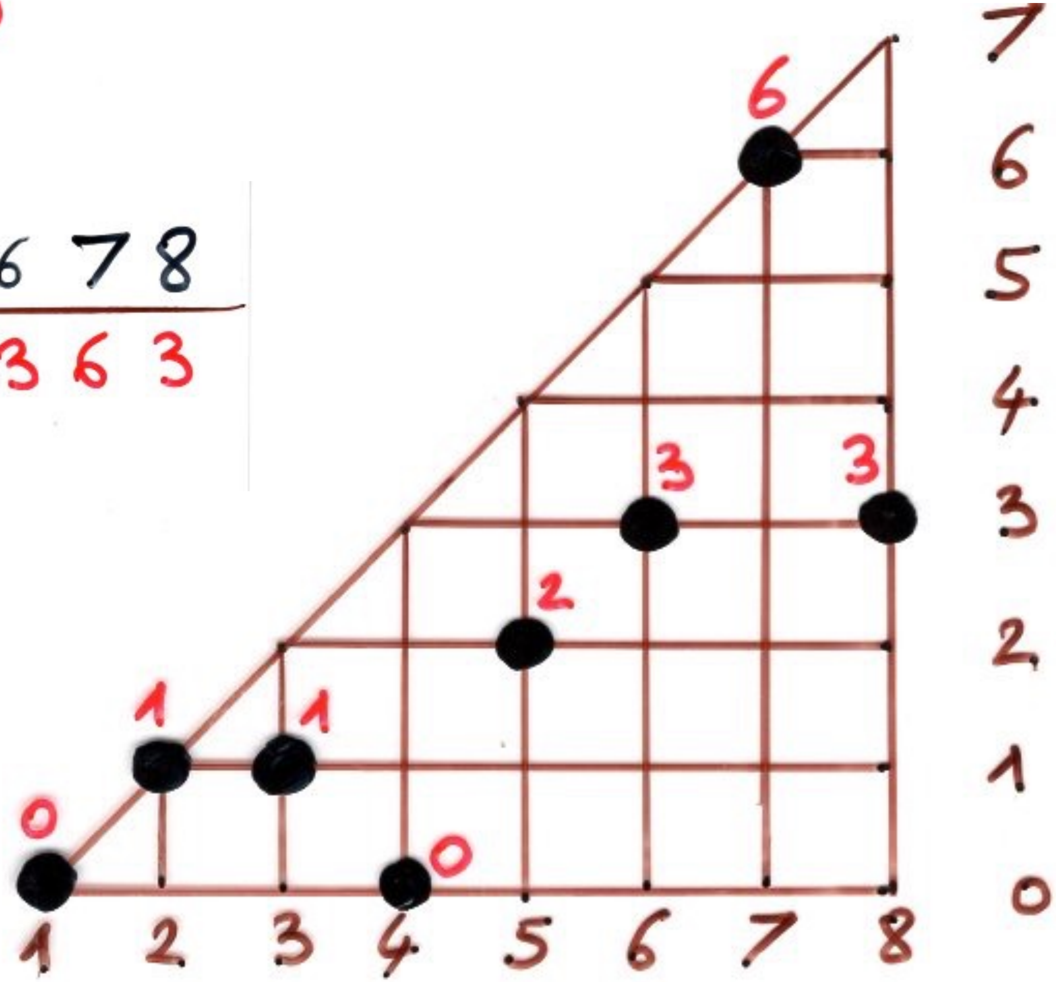
$f(x) =$ number of j, $i < j \leq n$
with $\sigma(j) < \sigma(i)$

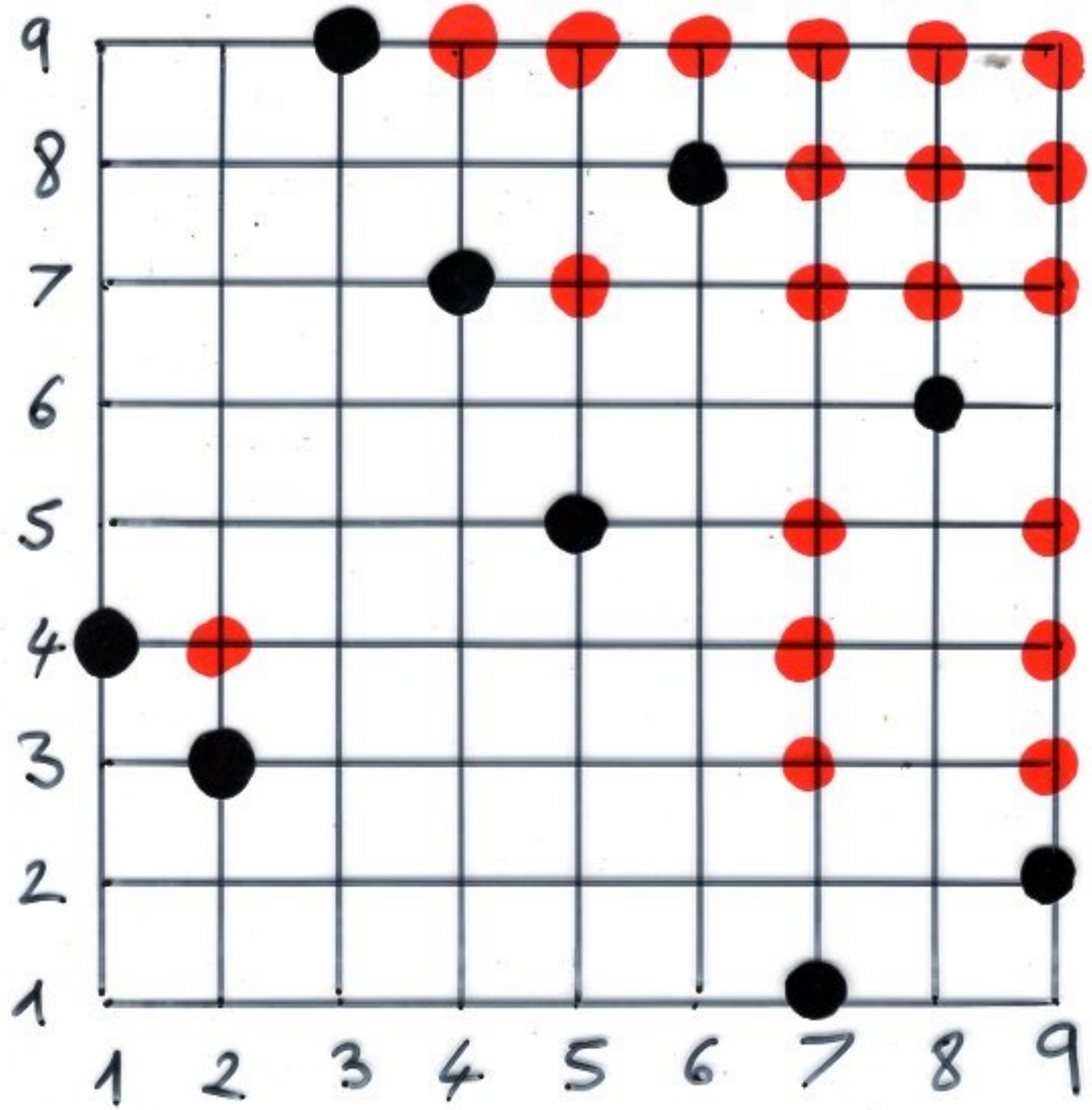


Inversion table

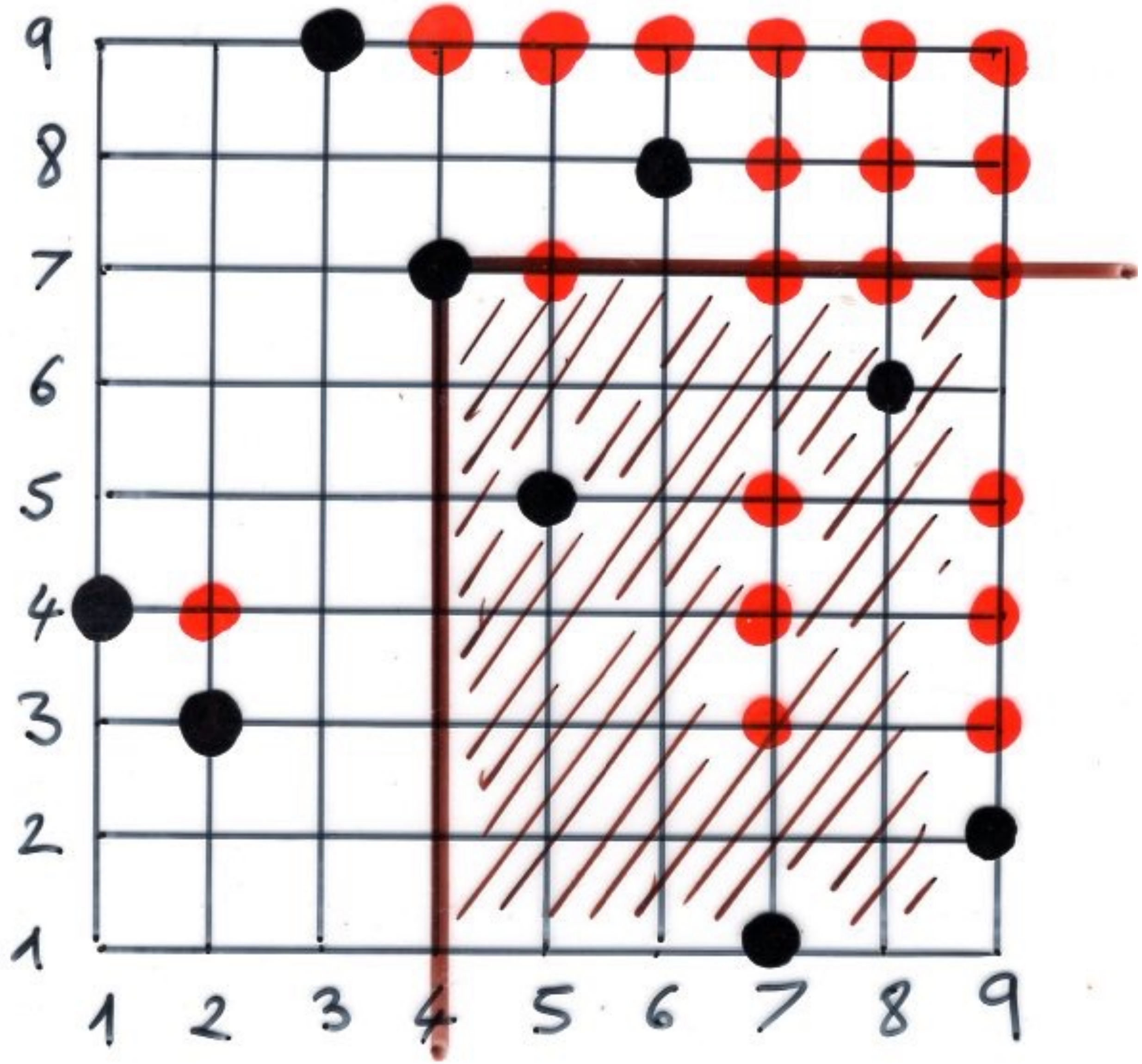
$$\sigma = \begin{array}{cccccccc} 7 & 2 & 3 & 6 & 8 & 5 & 1 & 4 \\ \hline 6 & 1 & 1 & 3 & 3 & 2 & 0 & 0 \end{array}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3





9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9


9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9

reverse bijection

and

q-analogs of histories

1
0

9°

9°

1

0

1

1 0

$\underbrace{1}_{0}$ 9^0

$\underbrace{2}_{1}$ 1 $\underbrace{\quad}_{0}$ 9^1

$$\underbrace{1}_{0} \qquad 9^0$$

$$\underbrace{2}_{1} \quad 1 \quad \underbrace{\qquad}_{0} \qquad 9^1$$

$$\underbrace{\qquad}_{2} \quad 2 \quad \underbrace{\qquad}_{1} \quad 1 \quad \underbrace{\qquad}_{0}$$

$$\underbrace{1}_{0}$$

$$\underbrace{2}_{1}\ 1\ \underbrace{\quad}_{0}$$

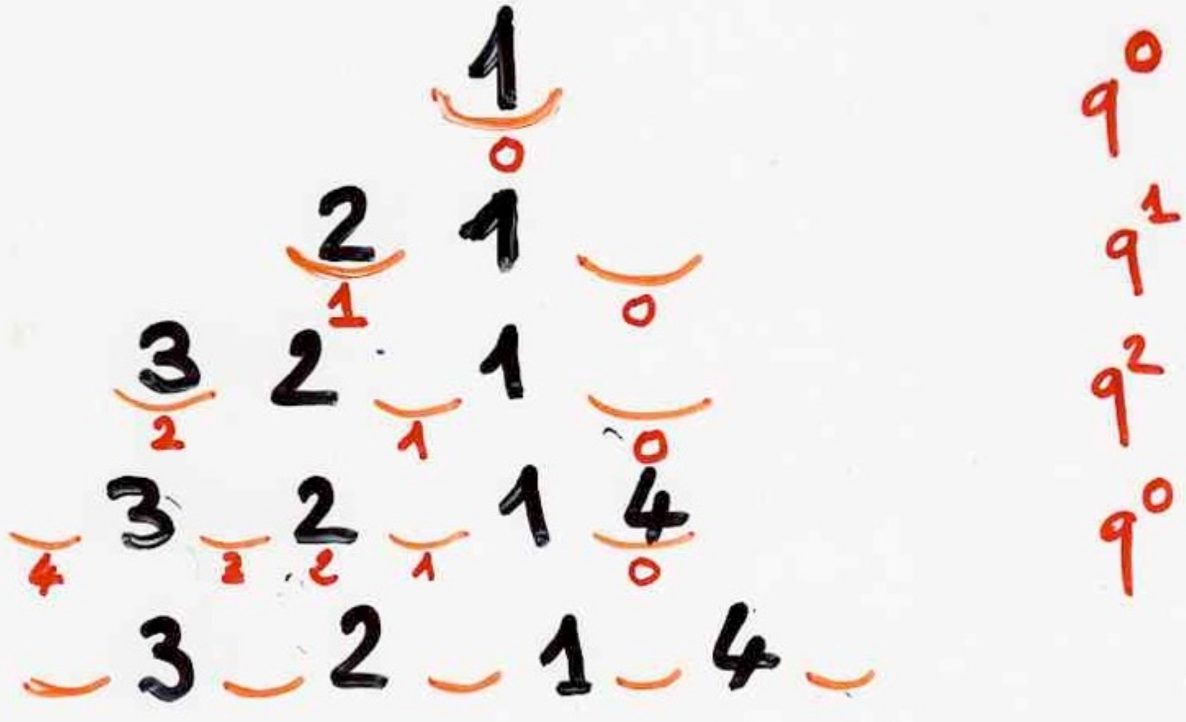
$$\underbrace{3}_{2}\ 2\ \underbrace{\quad}_{1}\ 1\ \underbrace{\quad}_{0}$$

$$\underbrace{\quad}\ 3\ \underbrace{\quad}\ 2\ \underbrace{\quad}\ 1\ \underbrace{\quad}$$

9^0

9^1

9^2


1
0
2 1
1 0
3 2 1
2 1 0
3 2 1 4
4 3 2 1 0
3 2 1 4
q^0
q^1
q^2
q^0

1 $\qquad q^0$

0

2 1 $\qquad q^1$

1 0

3 2 1 $\qquad q^2$

2 1 0

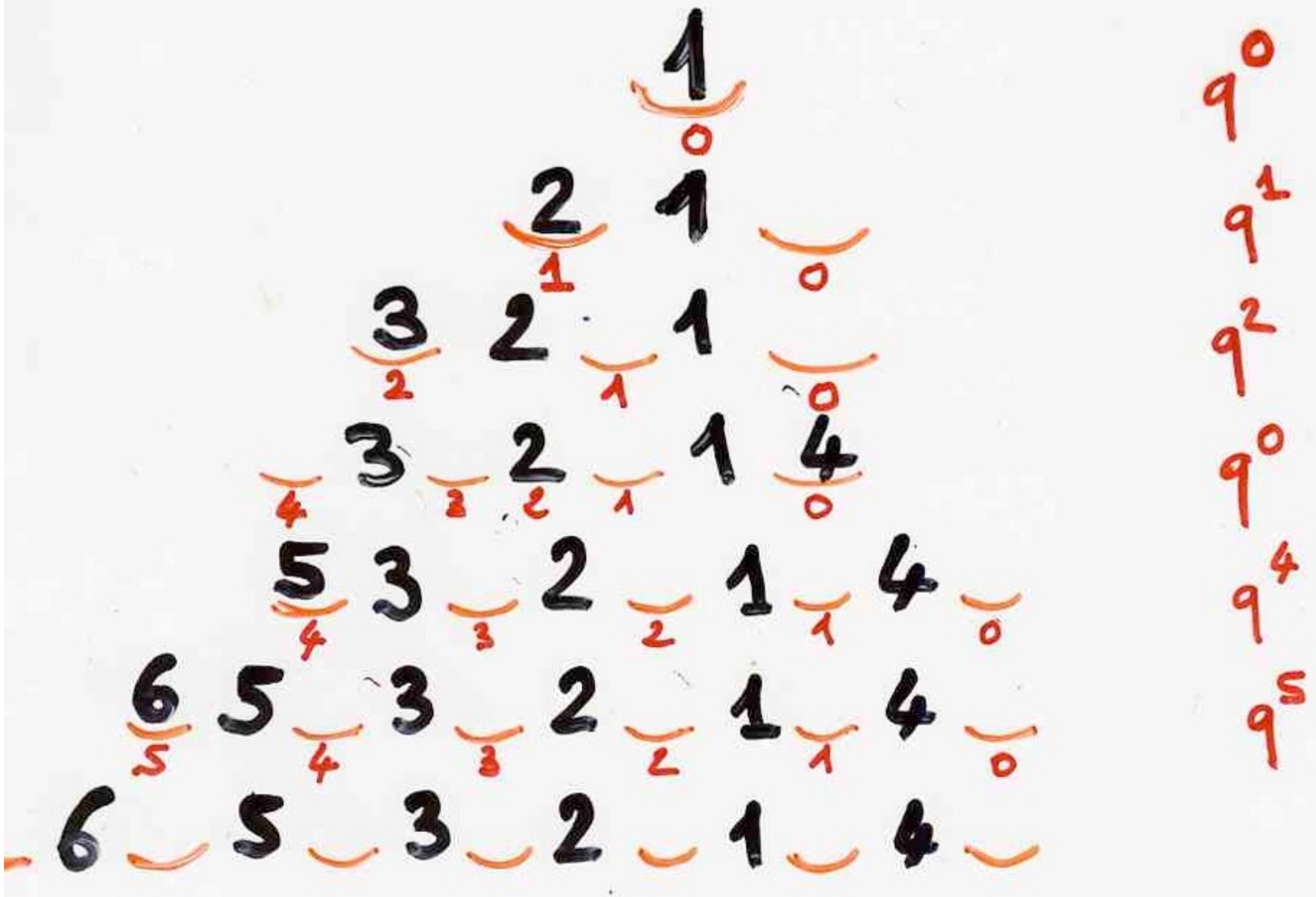
3 2 1 4 $\qquad q^0$

4 3 2 1 0

5 3 2 1 4 $\qquad q^4$

4 3 2 1 0

5 3 2 1 4


1
0
2 1
1 0
3 2 1
2 1 0
3 2 1 4
4 3 2 1 0
5 3 2 1 4
4 3 2 1 0
6 5 3 2 1 4
5 4 3 2 1 0
6 5 3 2 1 4
q^0
q^1
q^2
q^0
q^4
q^5

1 $\quad q^0$

0

2 1 $\quad q^1$

1 0

3 2 1 $\quad q^2$

2 1 0

3 2 1 4 $\quad q^0$

4 3 2 1 0

5 3 2 1 4 $\quad q^4$

4 3 2 1 0

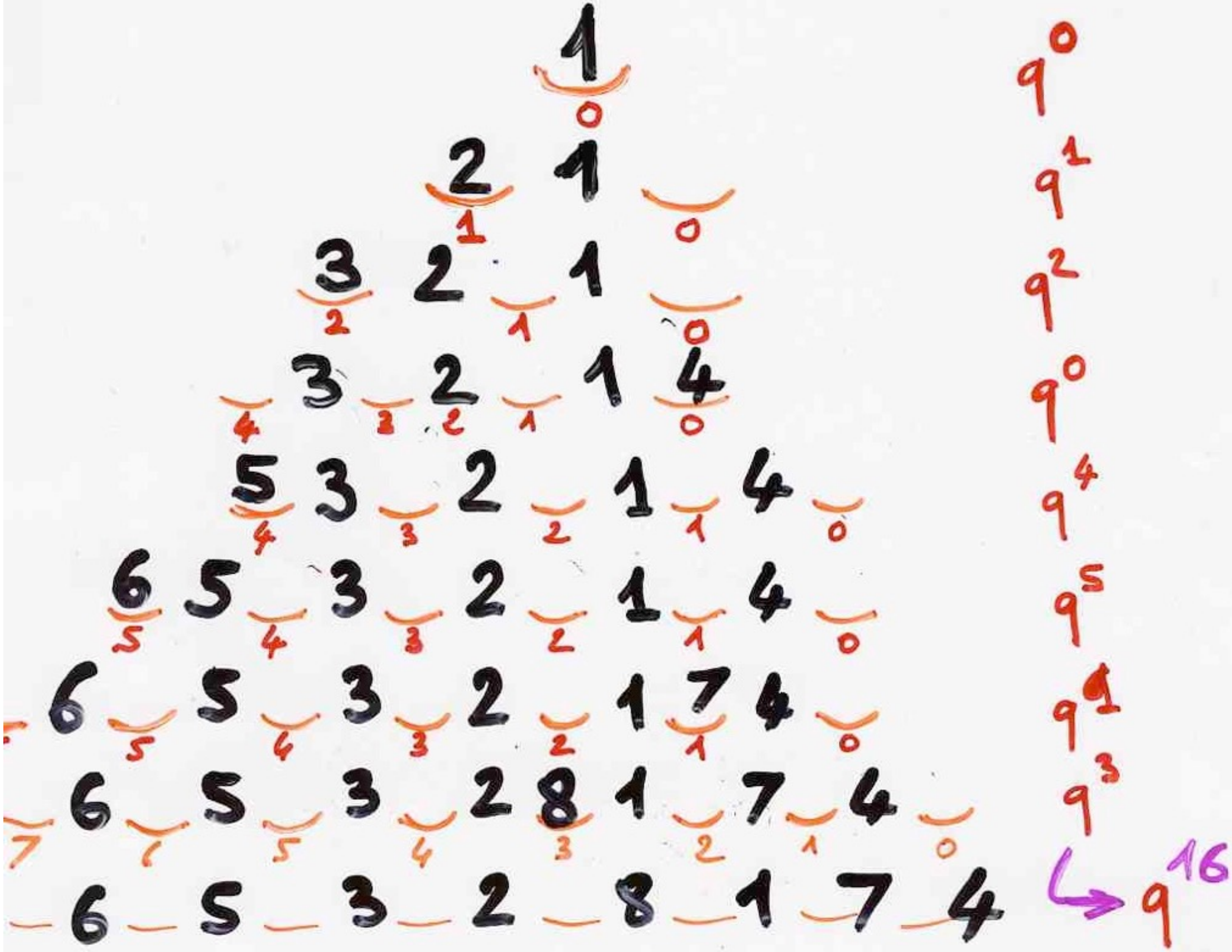
6 5 3 2 1 4 $\quad q^5$

5 4 3 2 1 0

6 5 3 2 1 7 4 $\quad q^6$

5 4 3 2 1 0

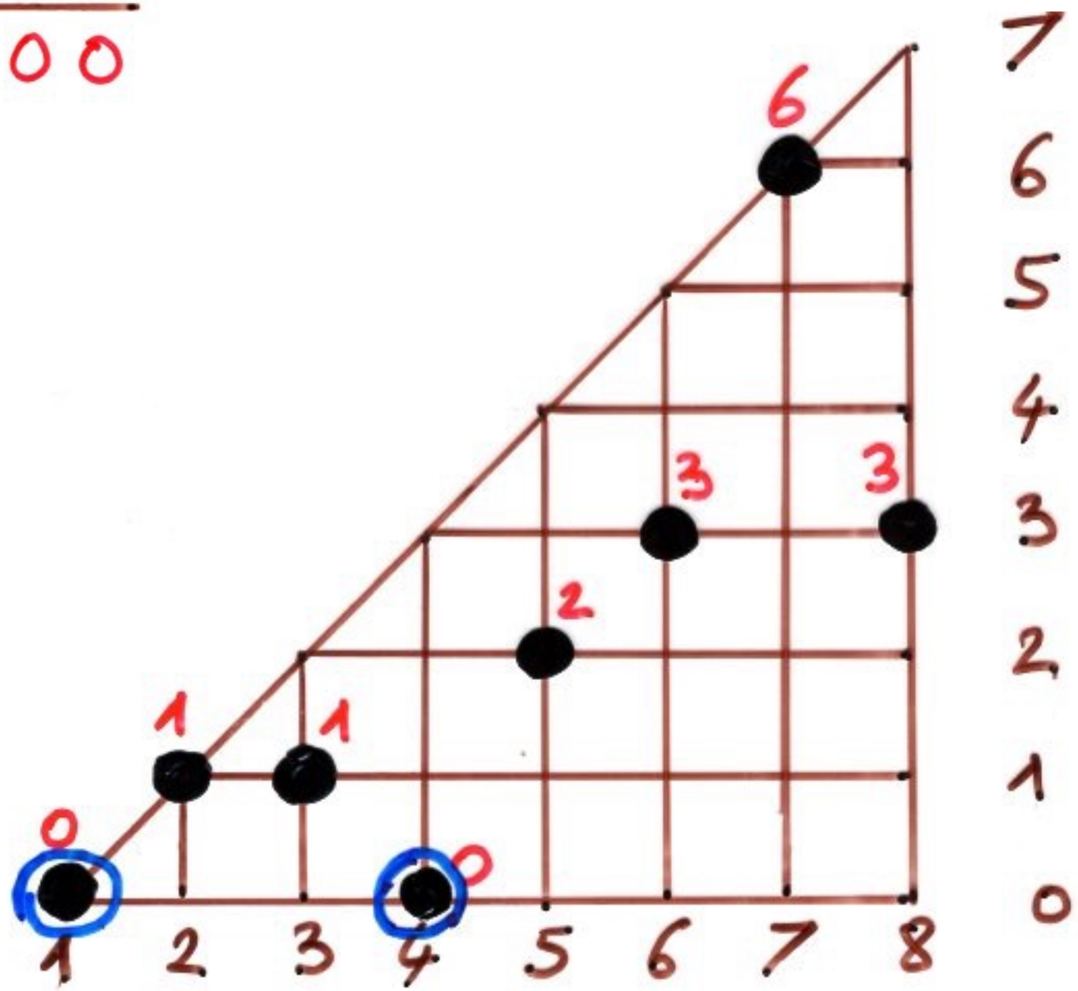
6 5 3 2 1 7 4


1 q0
2 1 q1
3 2 1 q2
3 2 1 4 q0
5 3 2 1 4 q4
6 5 3 2 1 4 q5
6 5 3 2 1 7 4 q1
6 5 3 2 8 1 7 4 q3
6 5 3 2 8 1 7 4 q16

Inversion table

$$\sigma = \frac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$$

rl- min

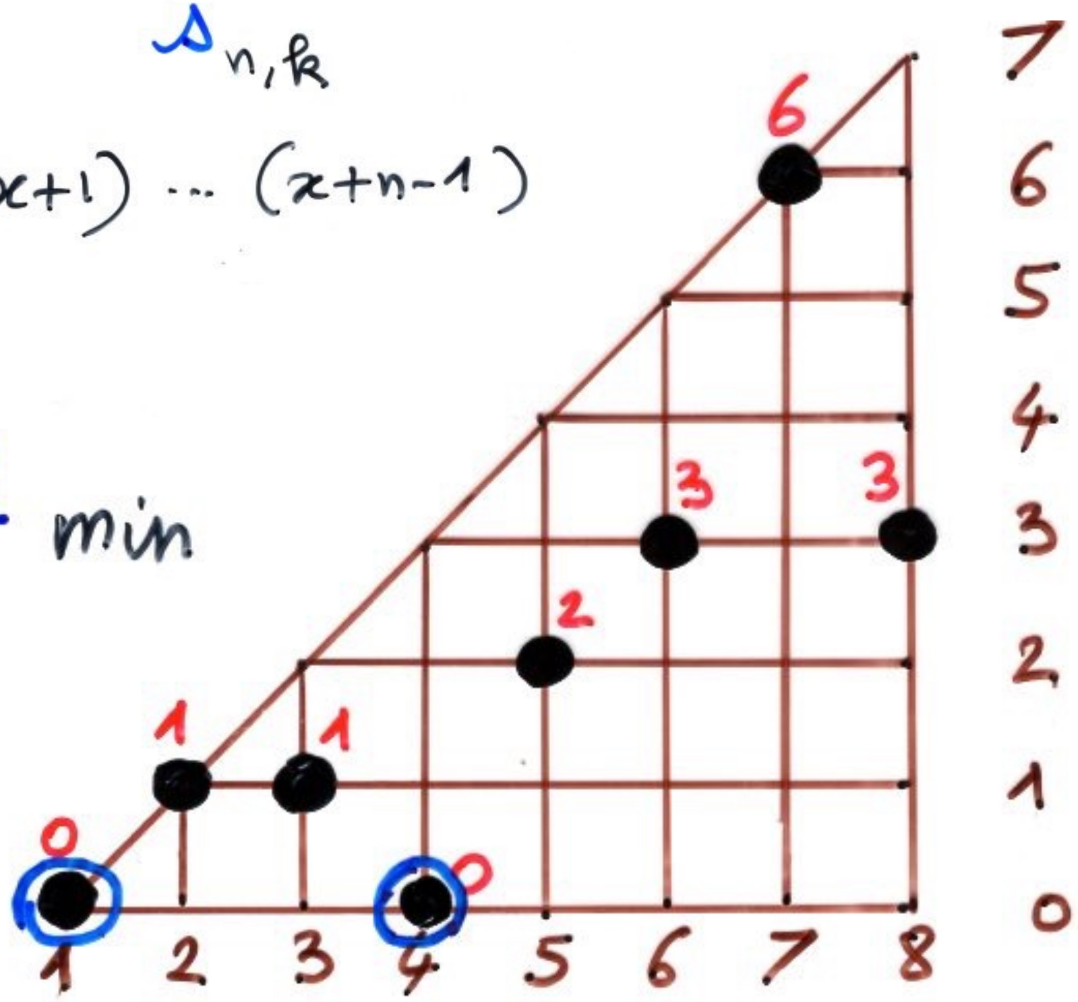


Inversion table

Stirling numbers $s_{n,k}$

$$\sum_{1 \leq k \leq n} s_{n,k}\, x^k = x(x+1) \cdots (x+n-1)$$

rl- min

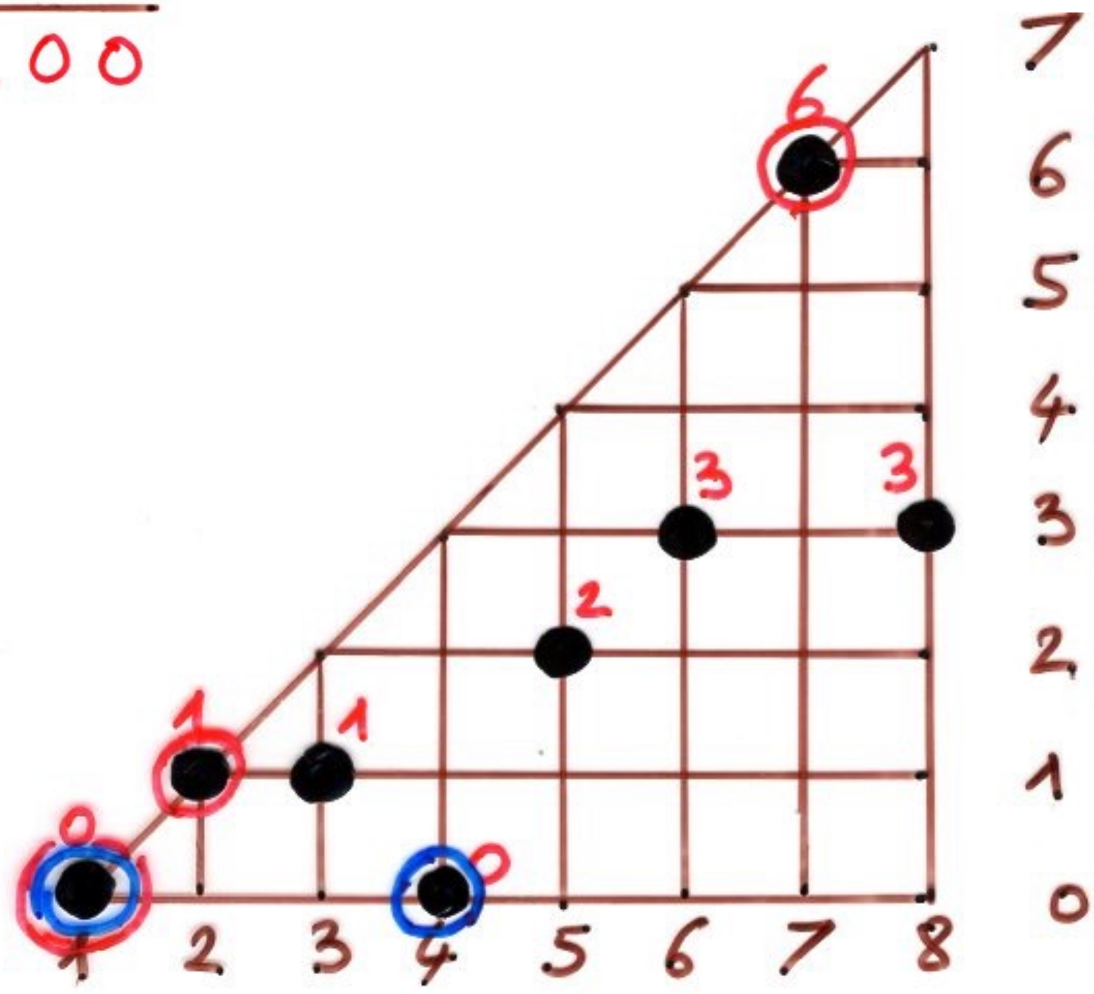


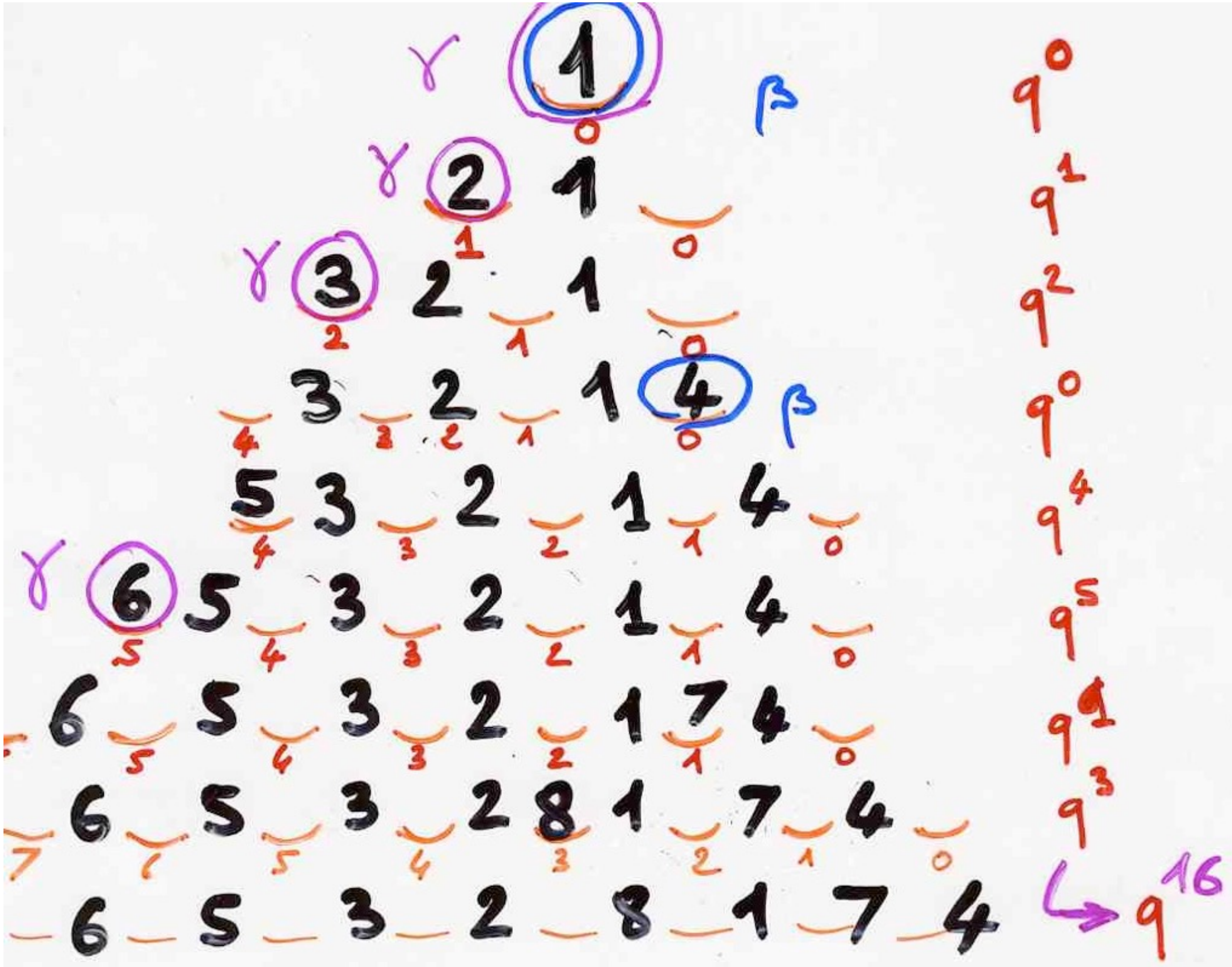
Inversion table

$$\sigma = \frac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$$

rl- min

lr- min





$$\beta + q + q^2 + \dots + q^{n-2} + \gamma q^{n-1}$$

$$[n;\beta,\gamma]_q$$

distribution of
permutations

3 parameters : $\begin{cases} \text{number of inversions} \\ \text{number of rl-min elements} \\ \text{number of lr-min elements} \end{cases}$

$$[i\,;\alpha,\beta]_q = (\alpha + q + q^2 + \dots + q^{i-2} + \beta q^{i-1})$$

$$[1\,;\alpha,\beta]_q = \alpha\beta$$

$$[n\,;\alpha,\beta]!_q = \prod_{i=1}^{n-1} [i\,;\alpha,\beta]_q$$

the maj index

MacMahon

master theorem

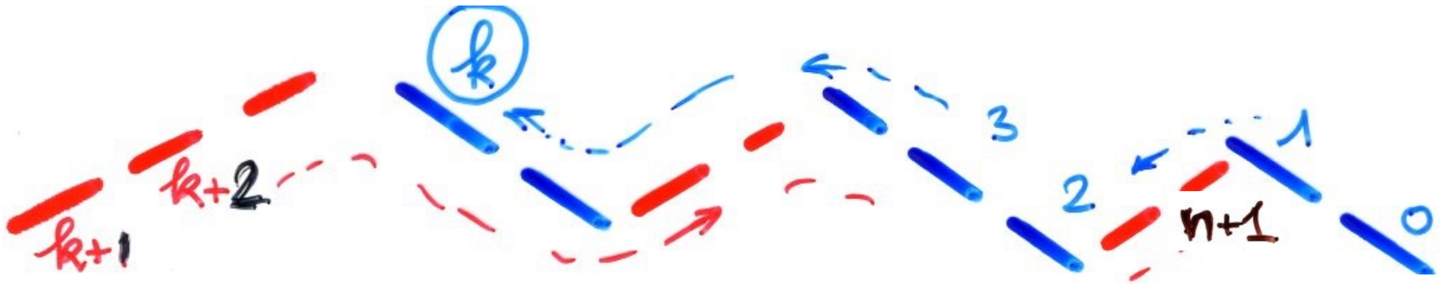
$$\frac{1}{\det(I-A)}$$

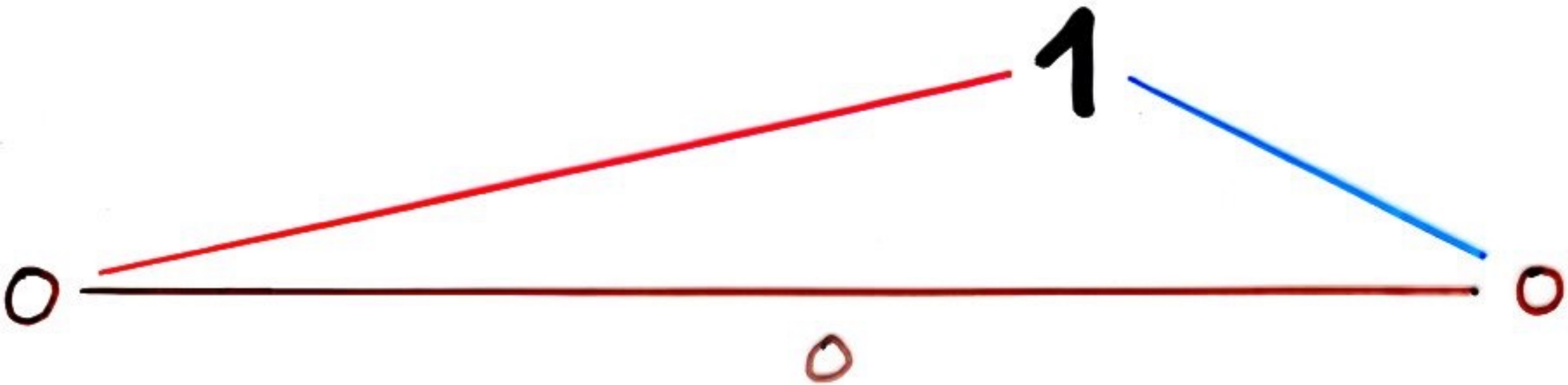
$$\mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma(i) > \sigma(i+1)}} i$$

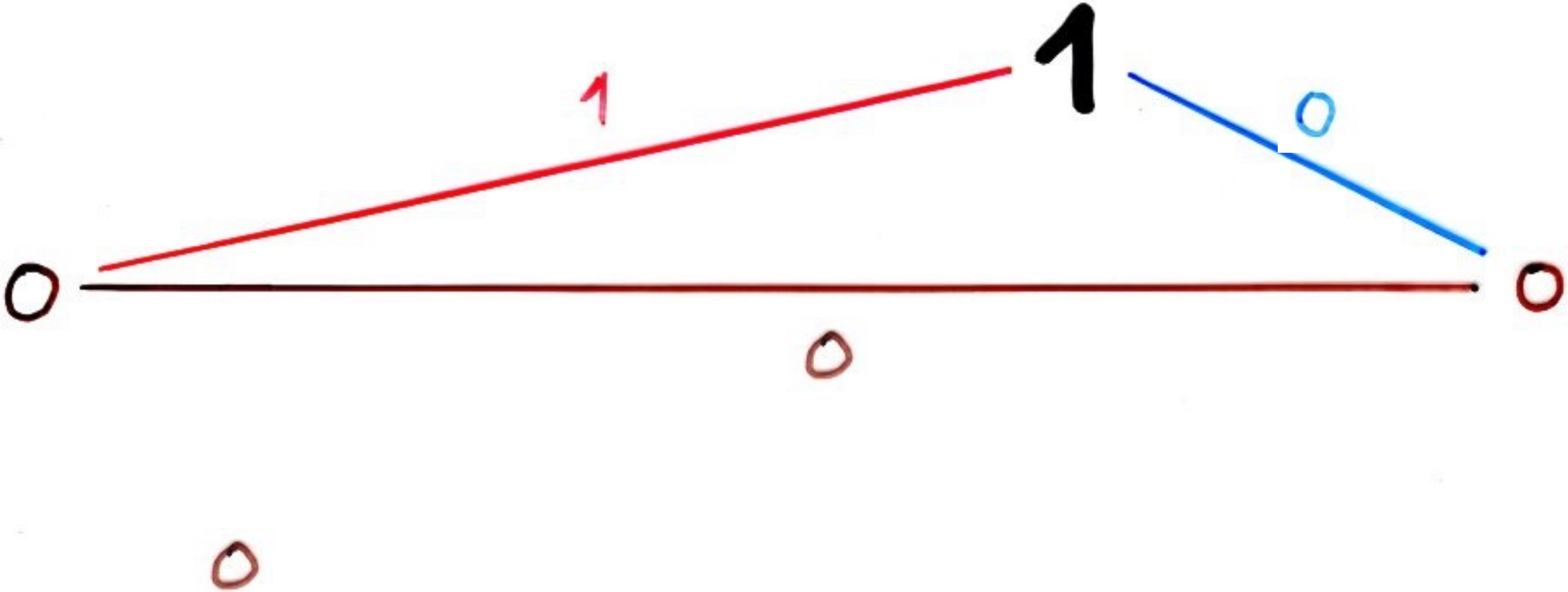
$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}$$

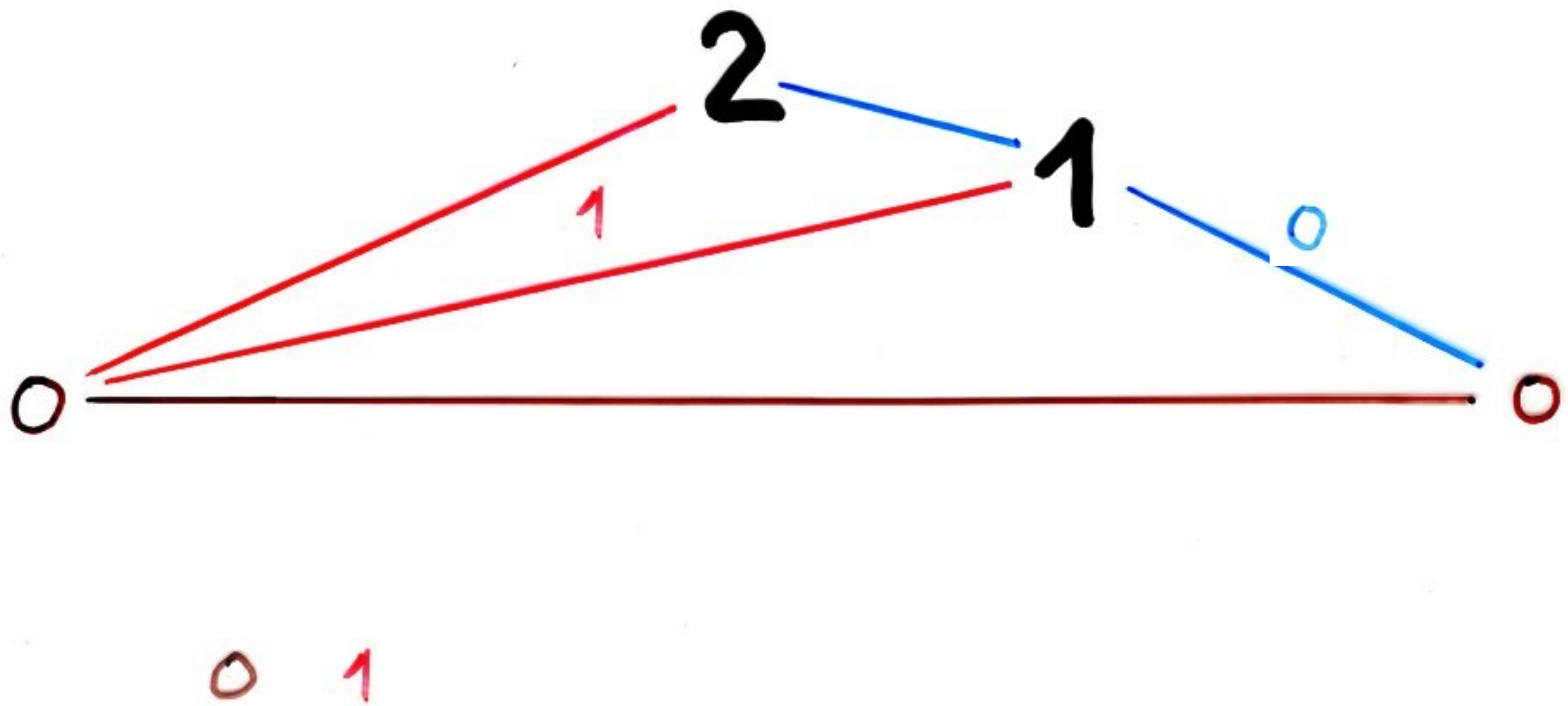
Mahonian
distribution

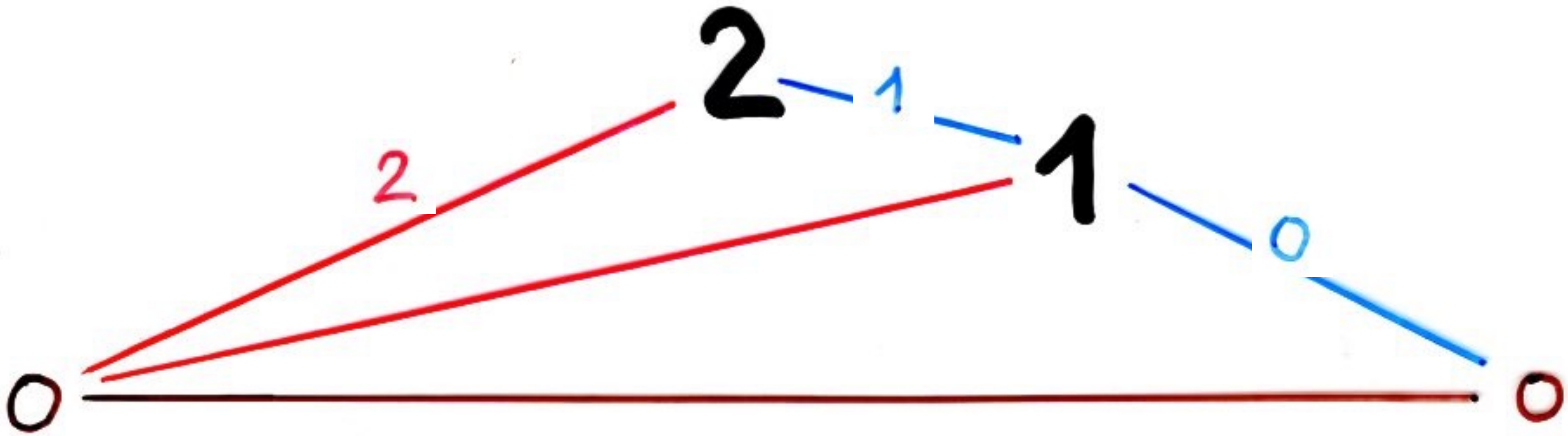

n+1
3
2
1
0


k
3
1
k+2
2
0
k+1
n+1

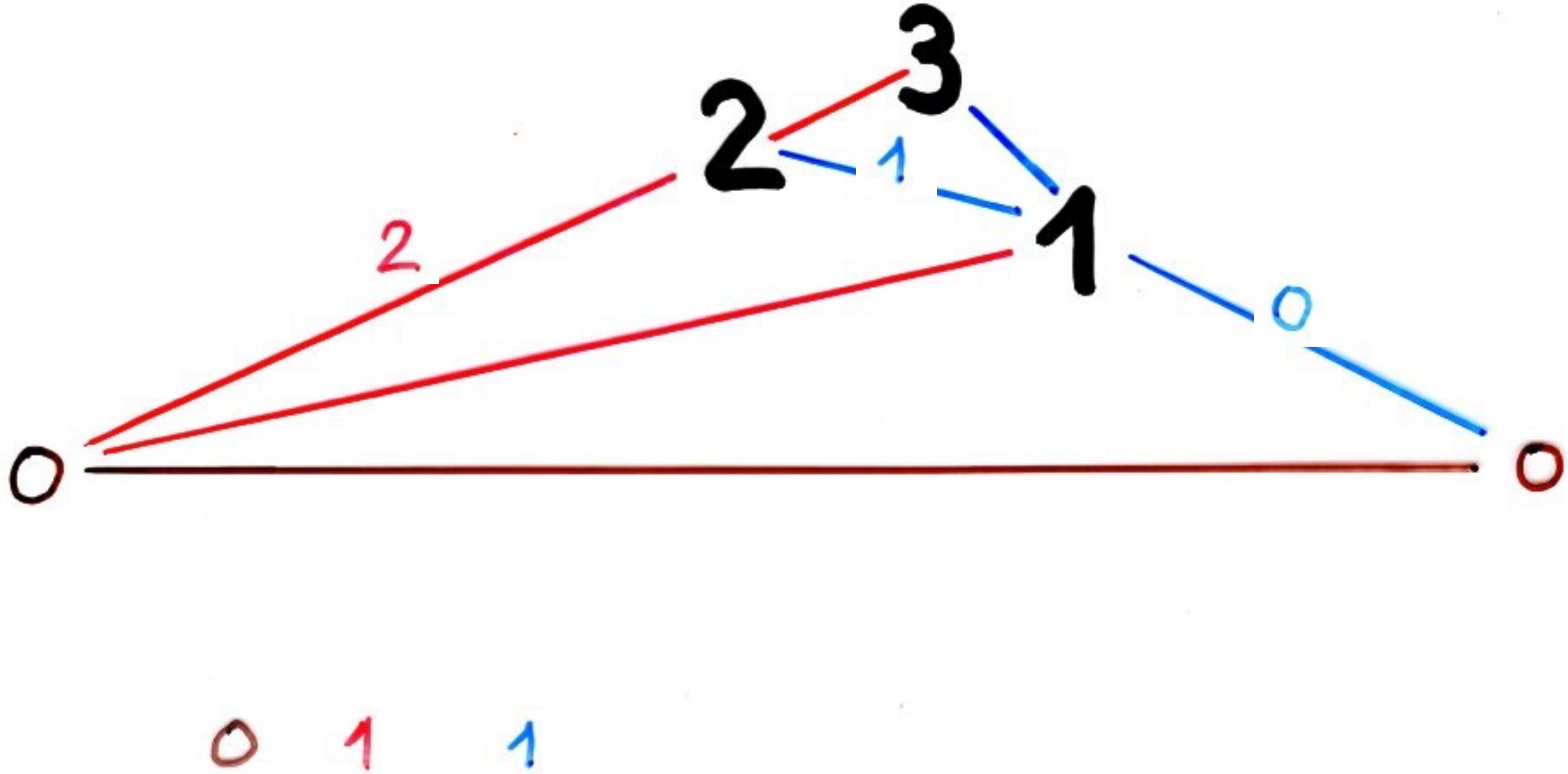

1
0
0
0


1
1
0
0
0
0
0

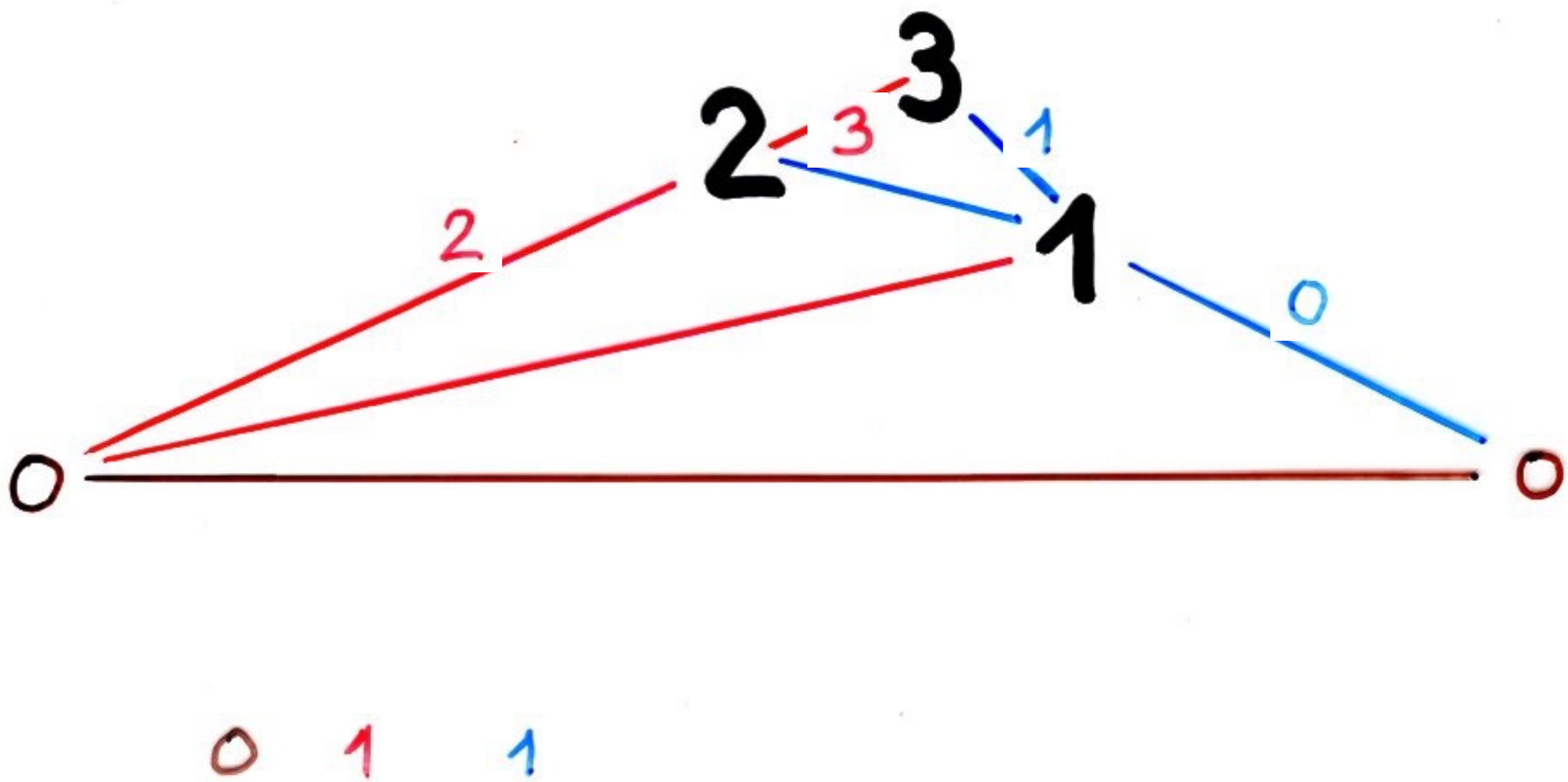

2
1
1
0
0
0
0 1



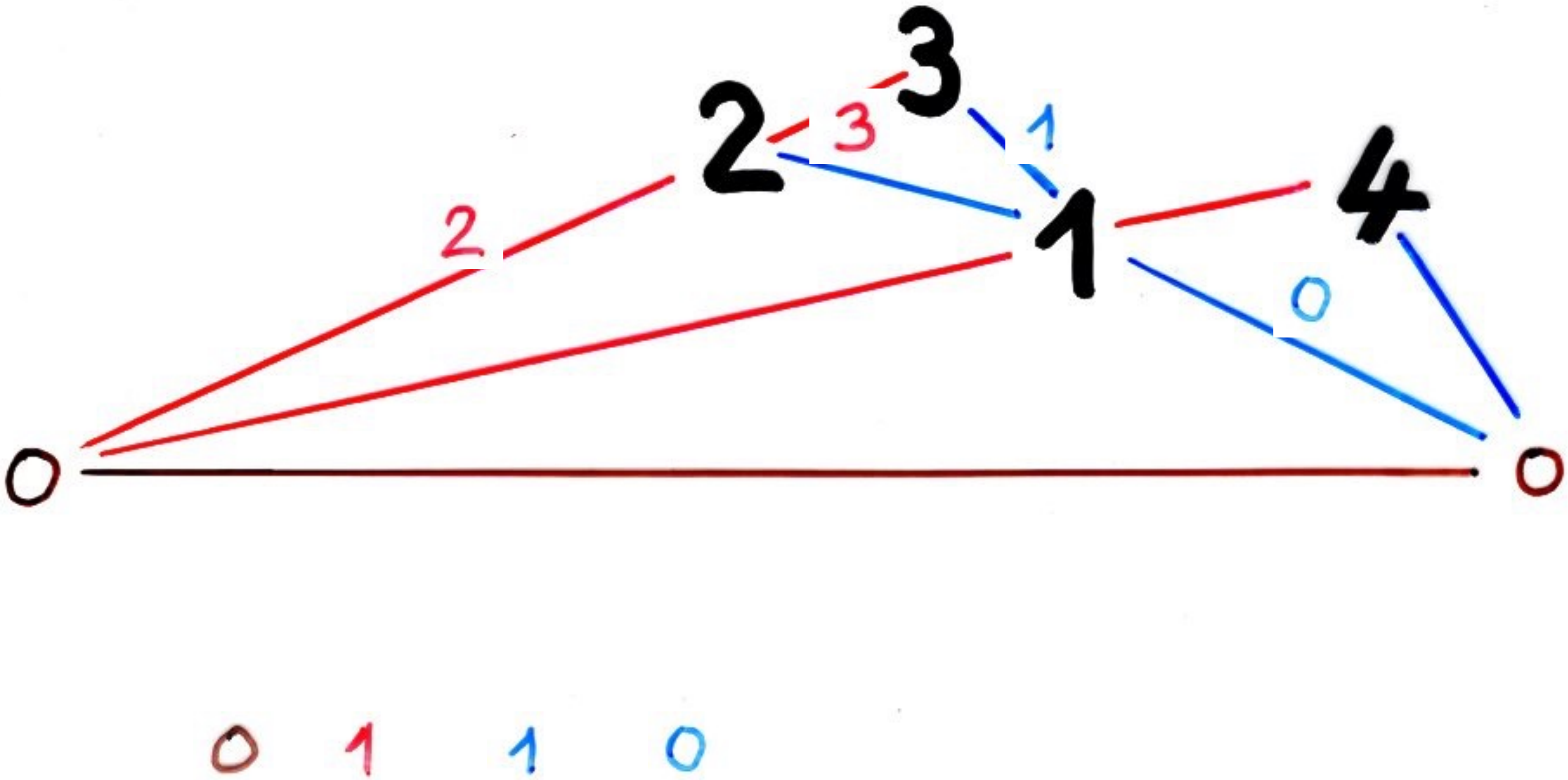
0 1

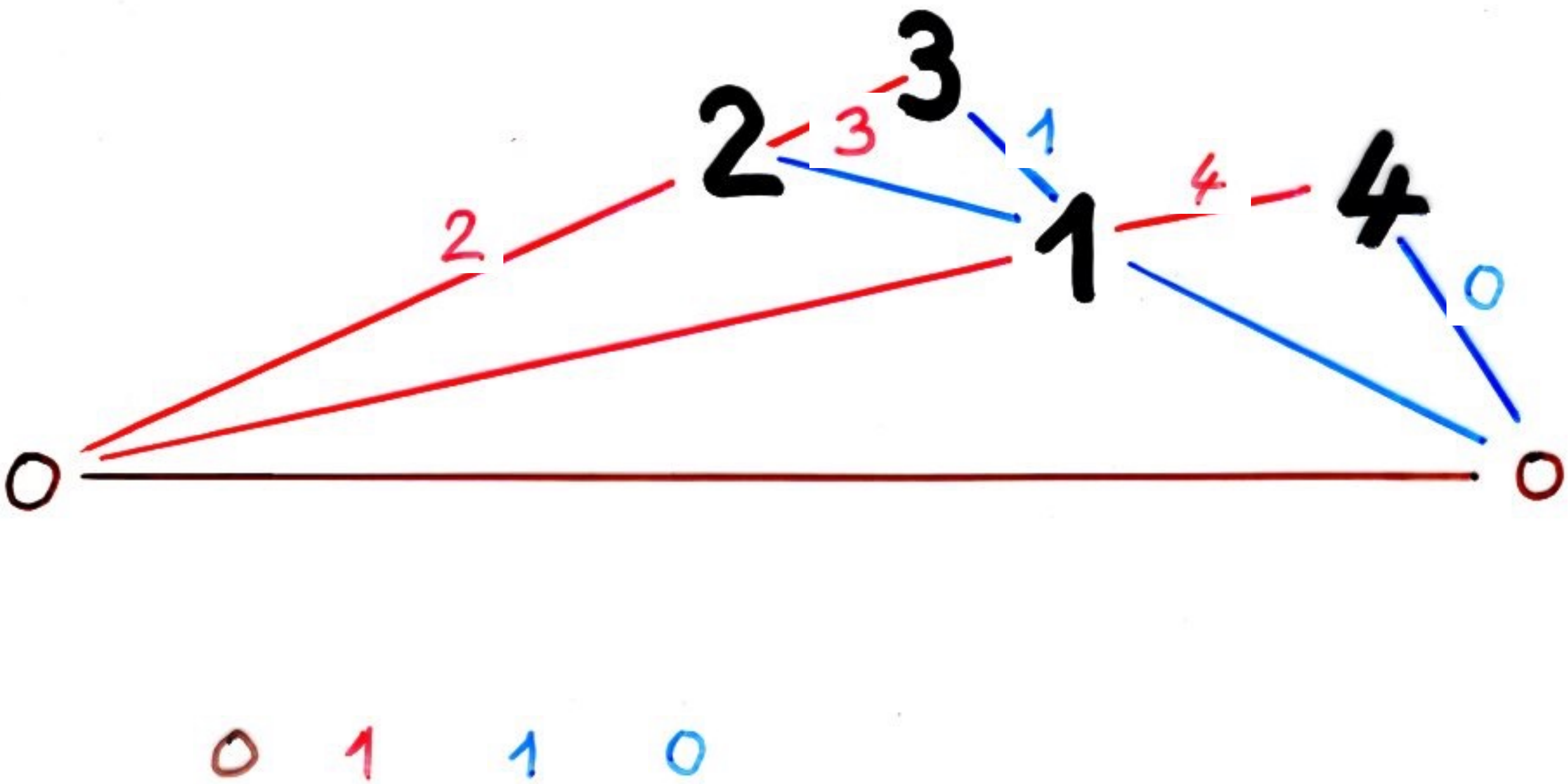


3
2
1
2
1
0
0
0
0 1 1

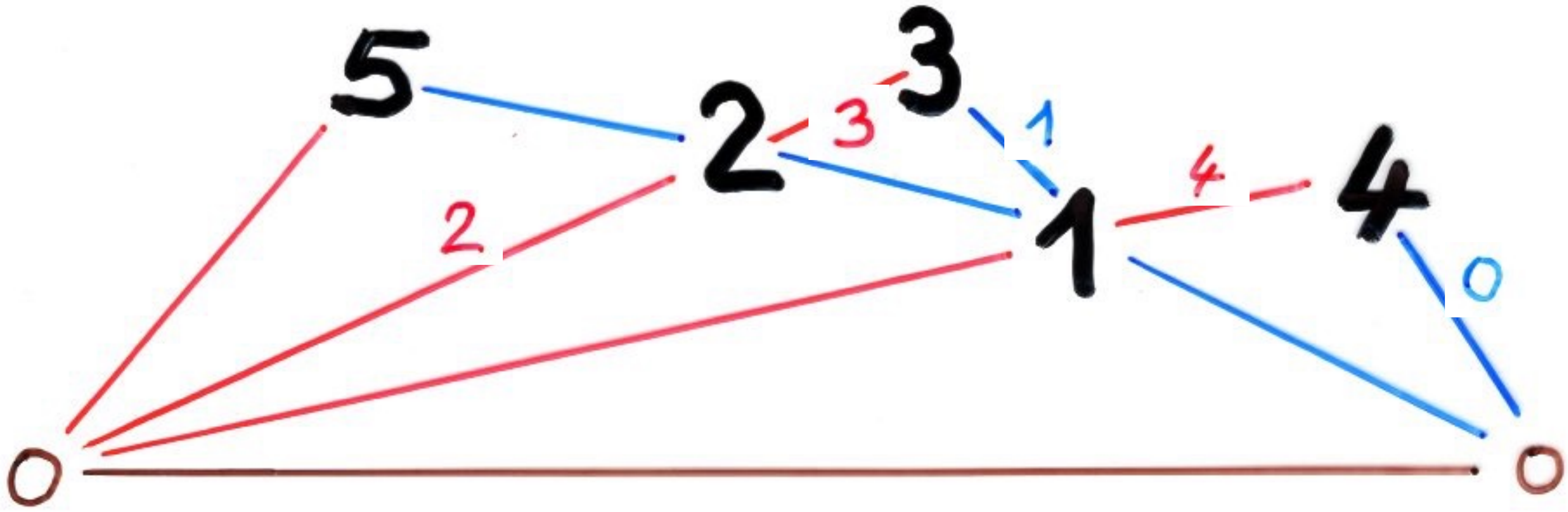


3
2
3
1
1
2
0
0
0
0
1
1

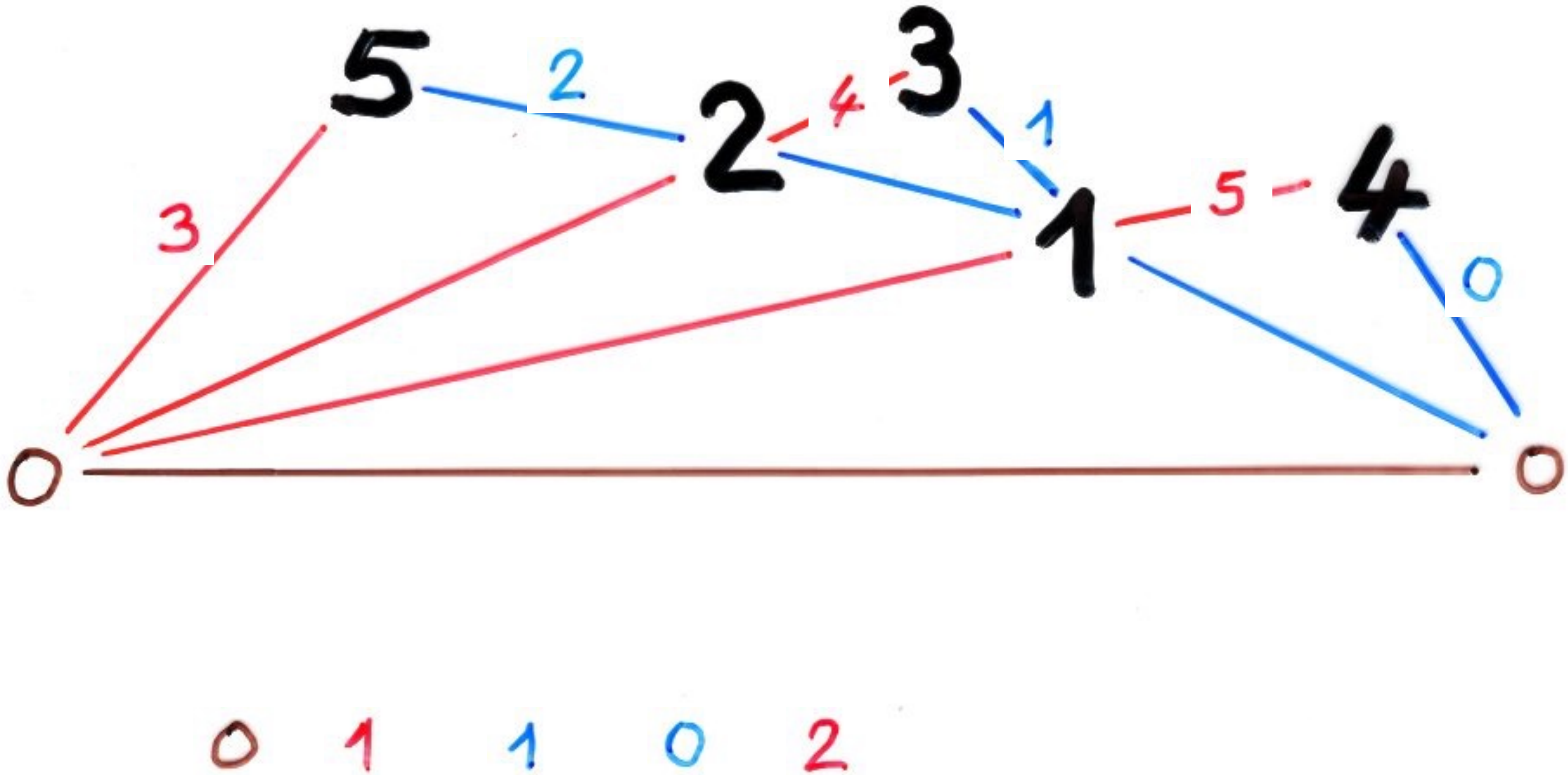

0
2
3
1
4
0
2
3
1
0
0 1 1 0

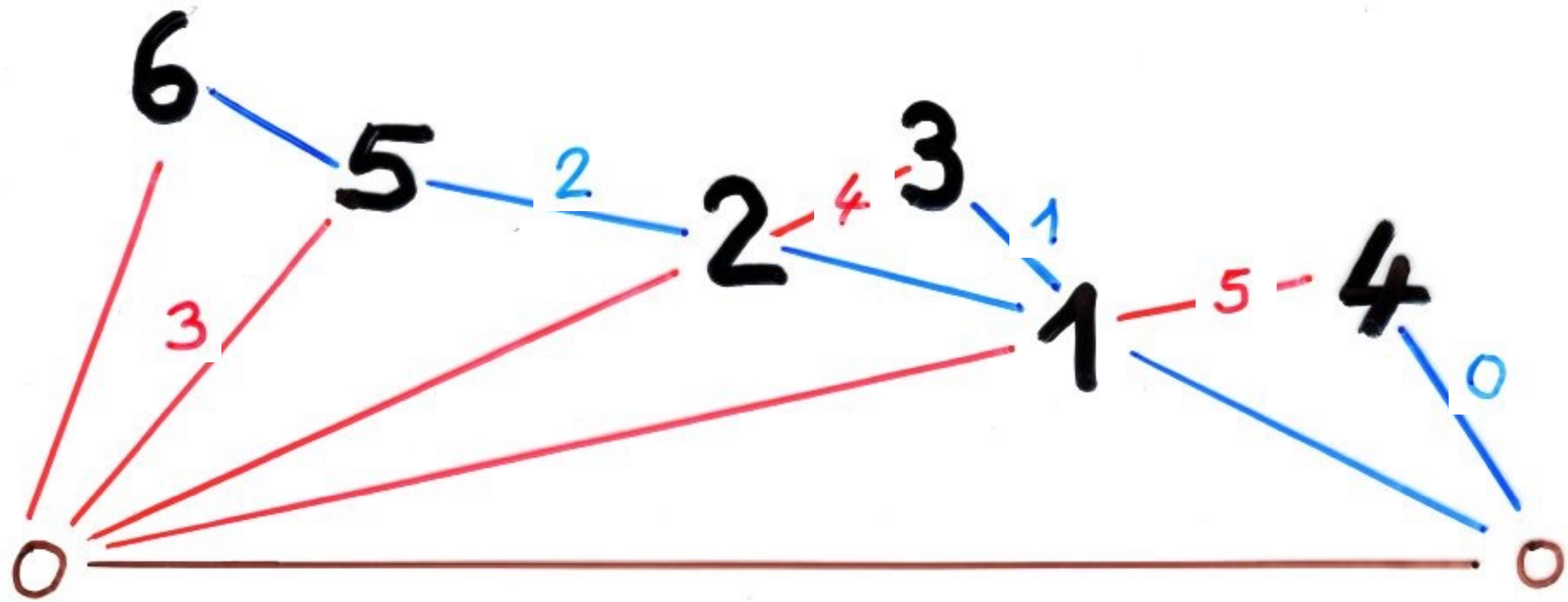


3
2
1
4
0
0
2
3
1
4
0
0 1 1 0

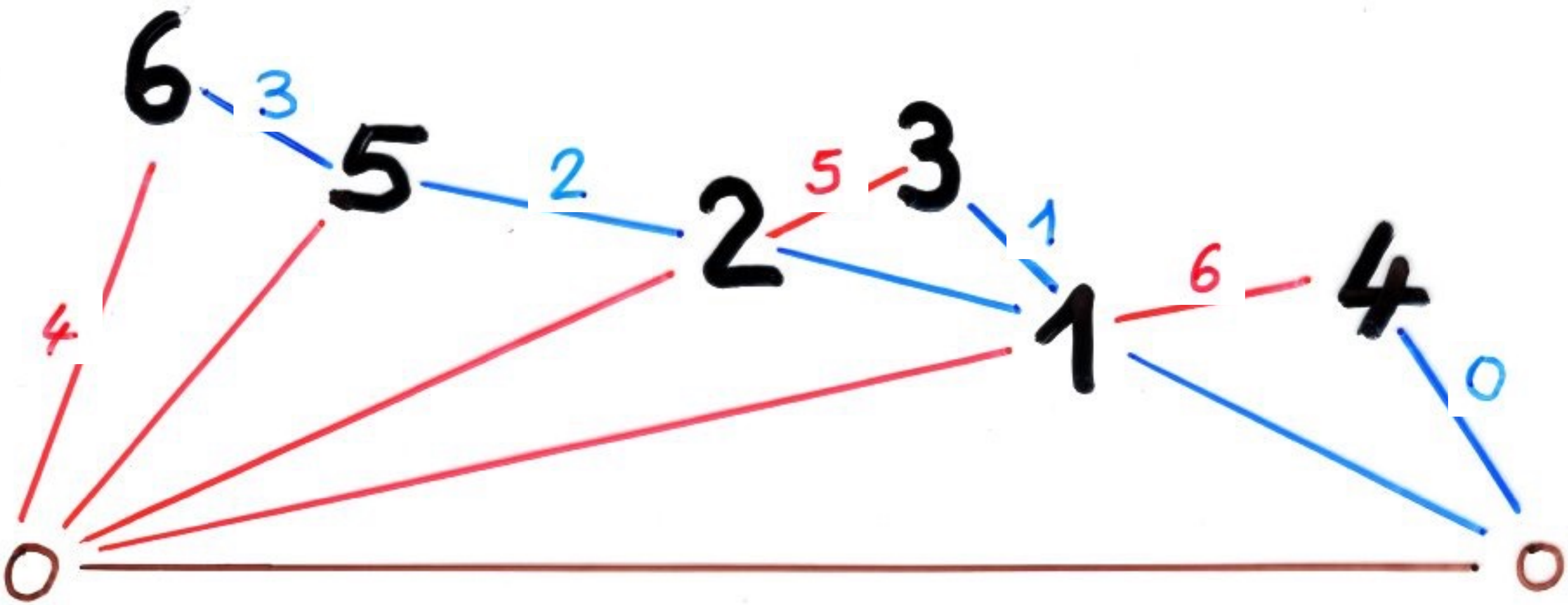


0 1 1 0 2


5
2
3
1
4
0
0
3
2
4
1
5
0
0 1 1 0 2



0 1 1 0 2 3



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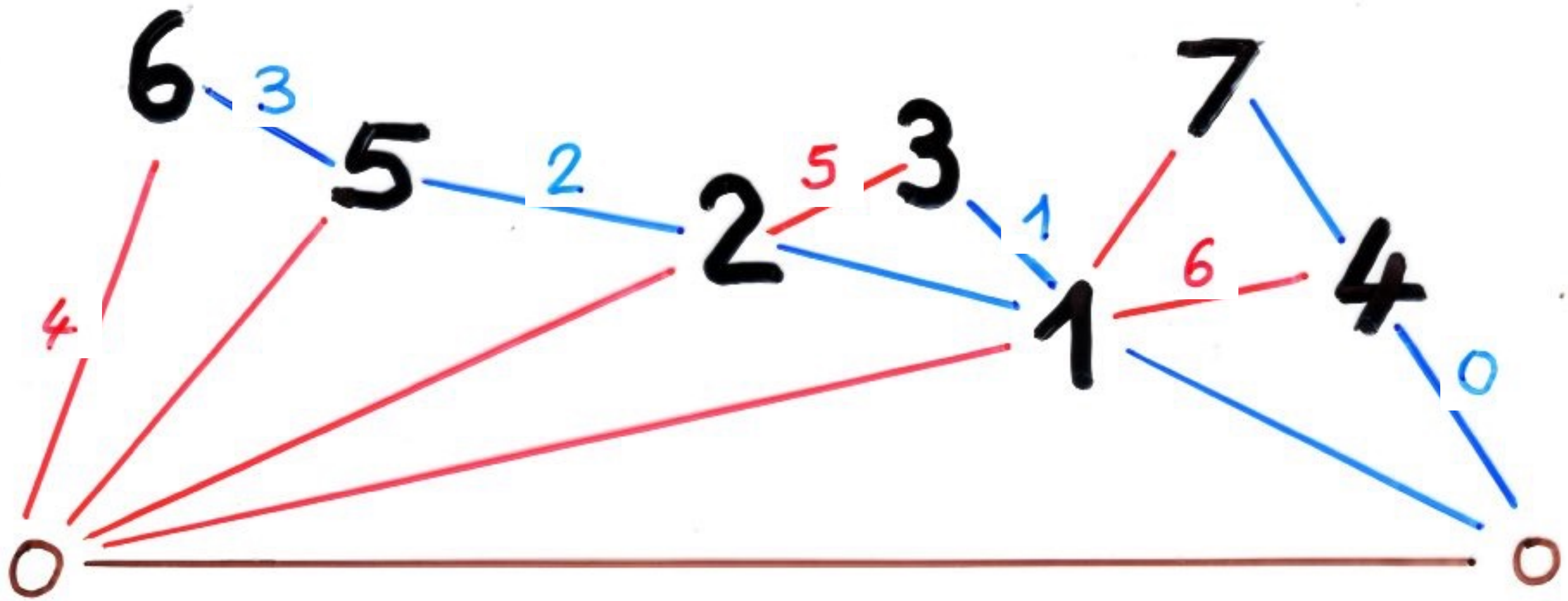

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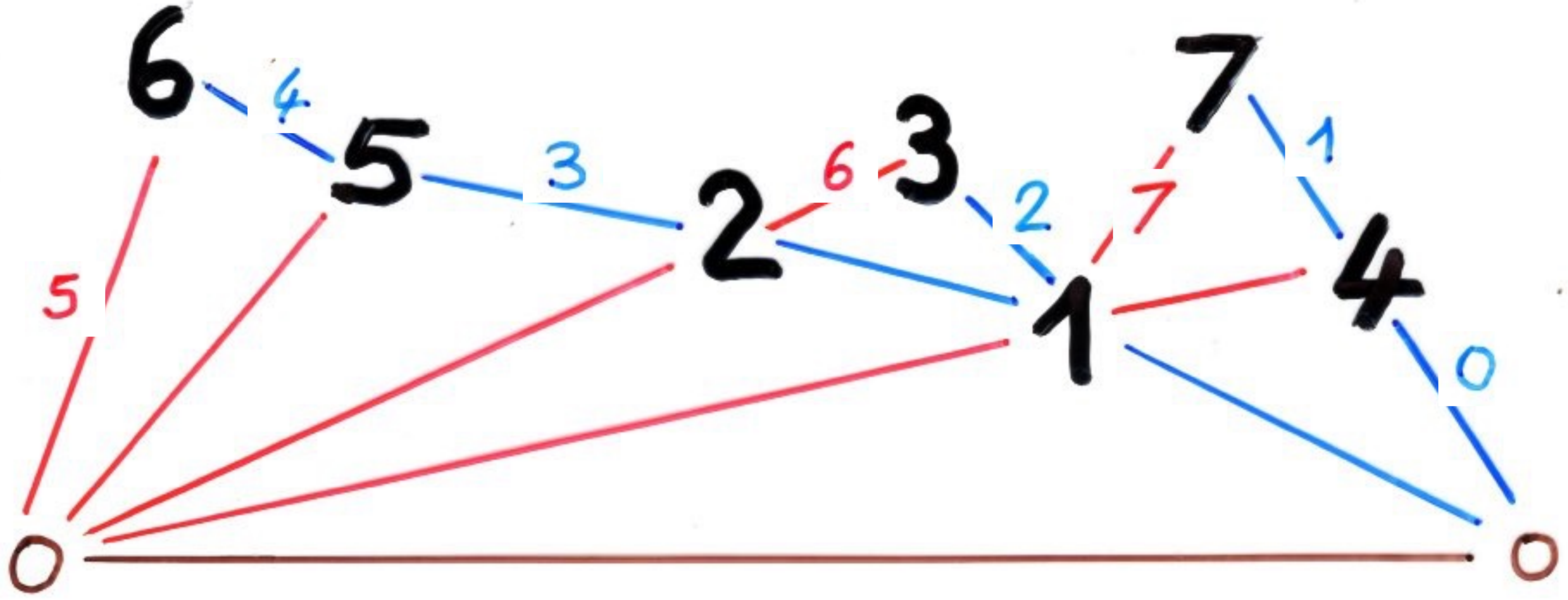

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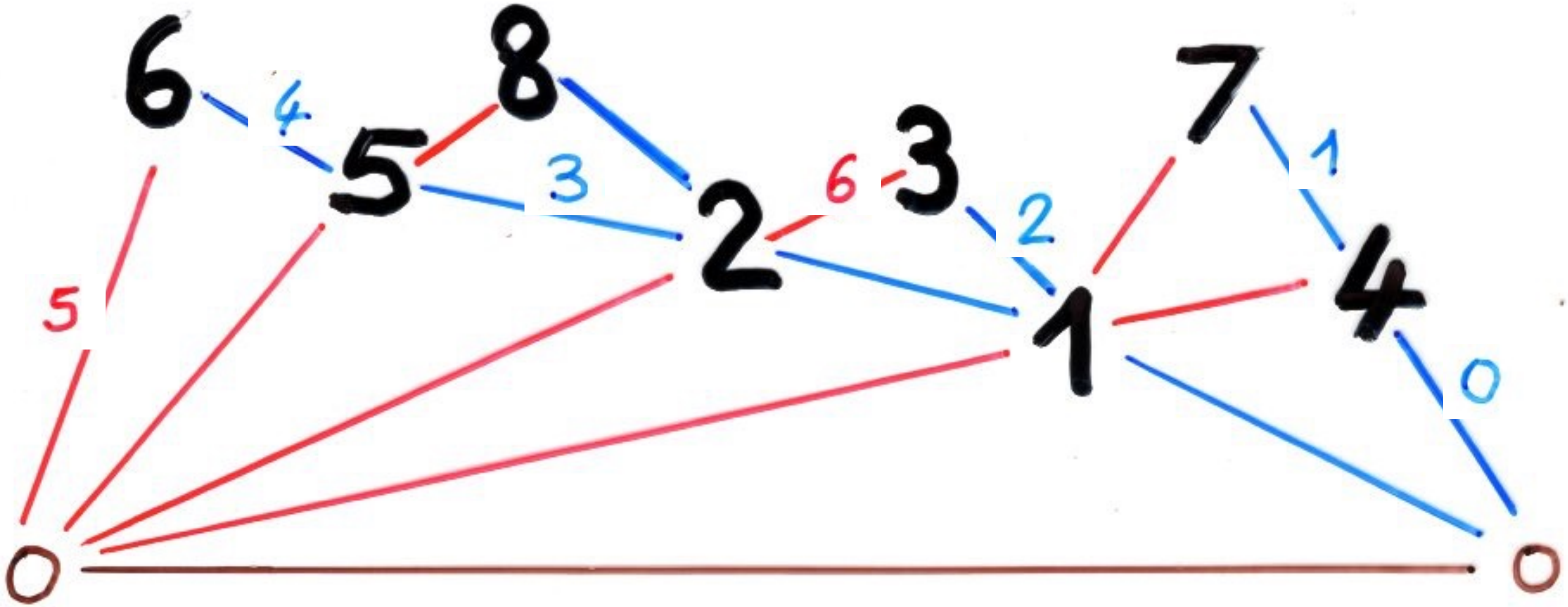

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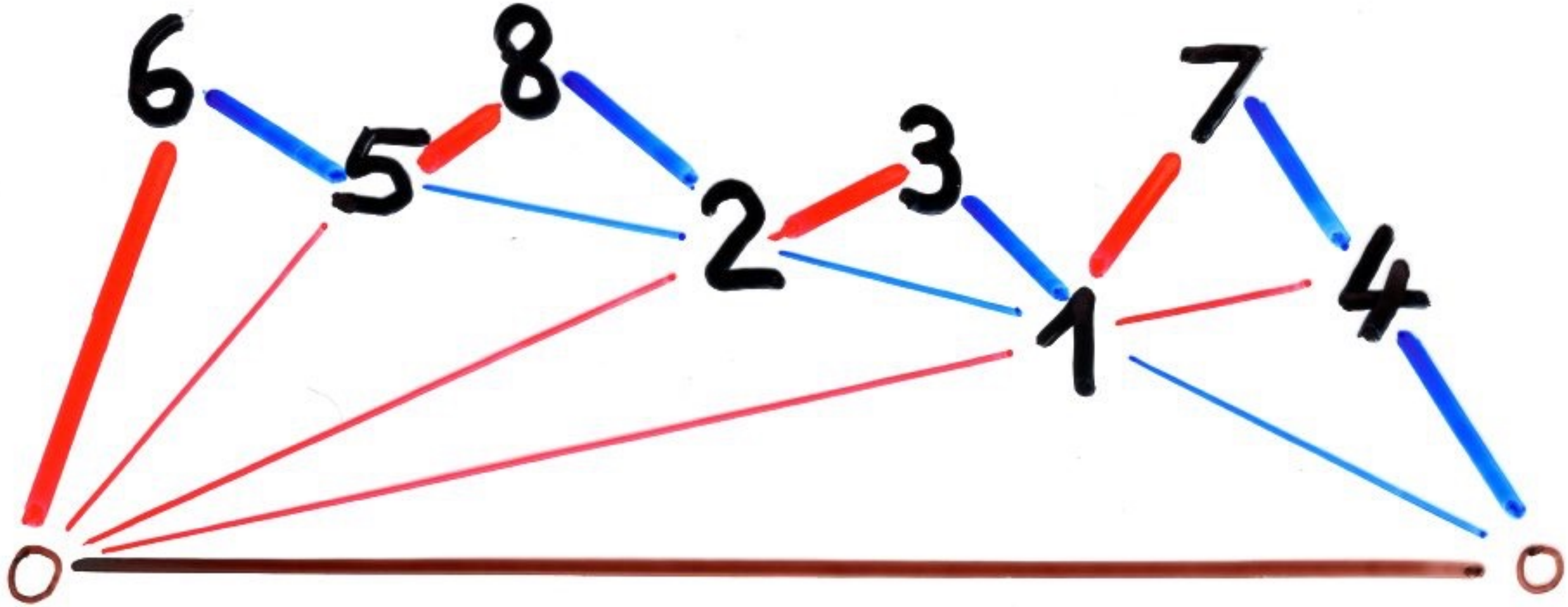

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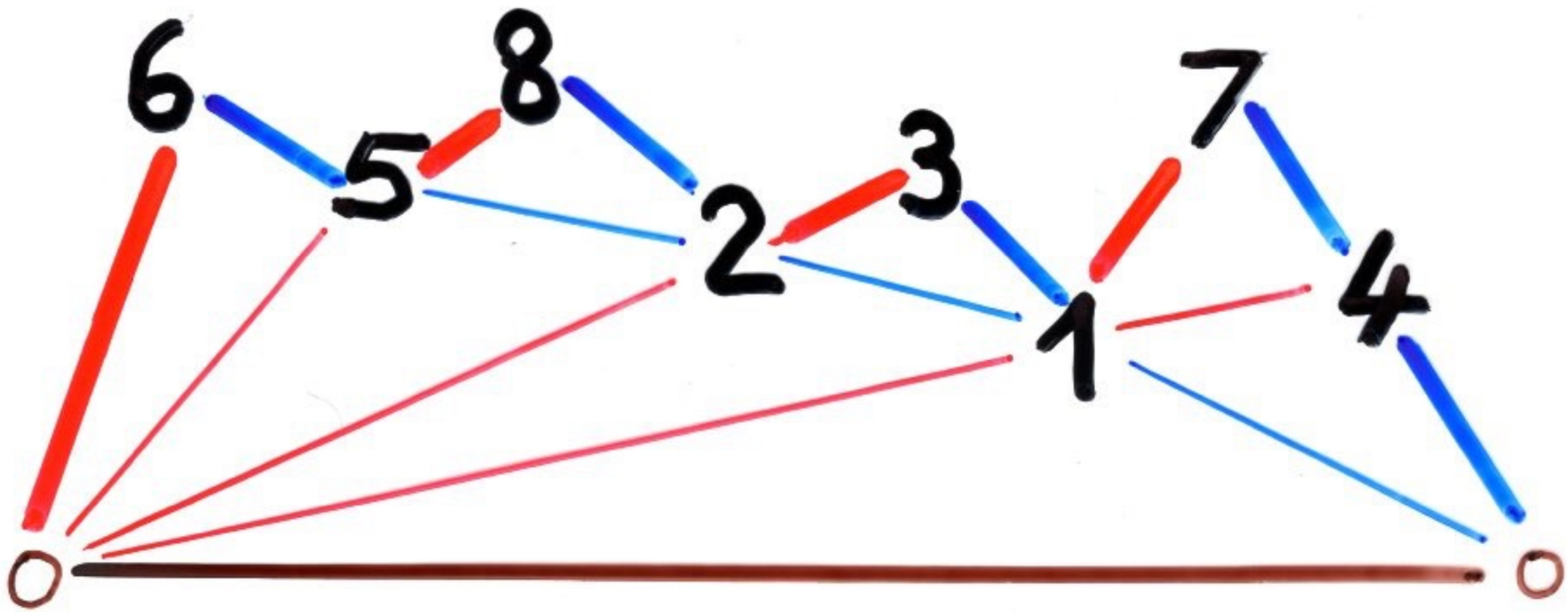
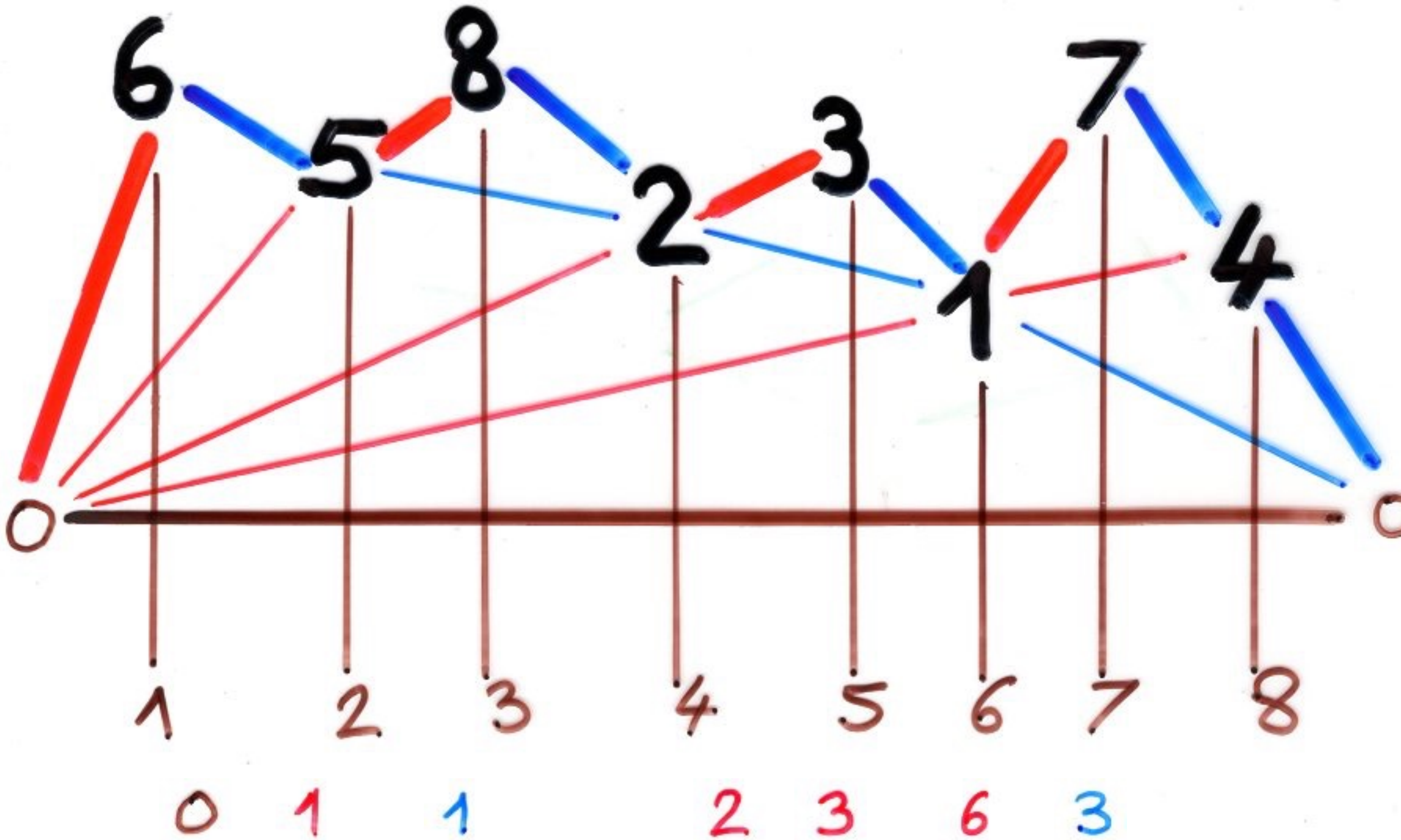


0 1 1 0 2 3 6 3



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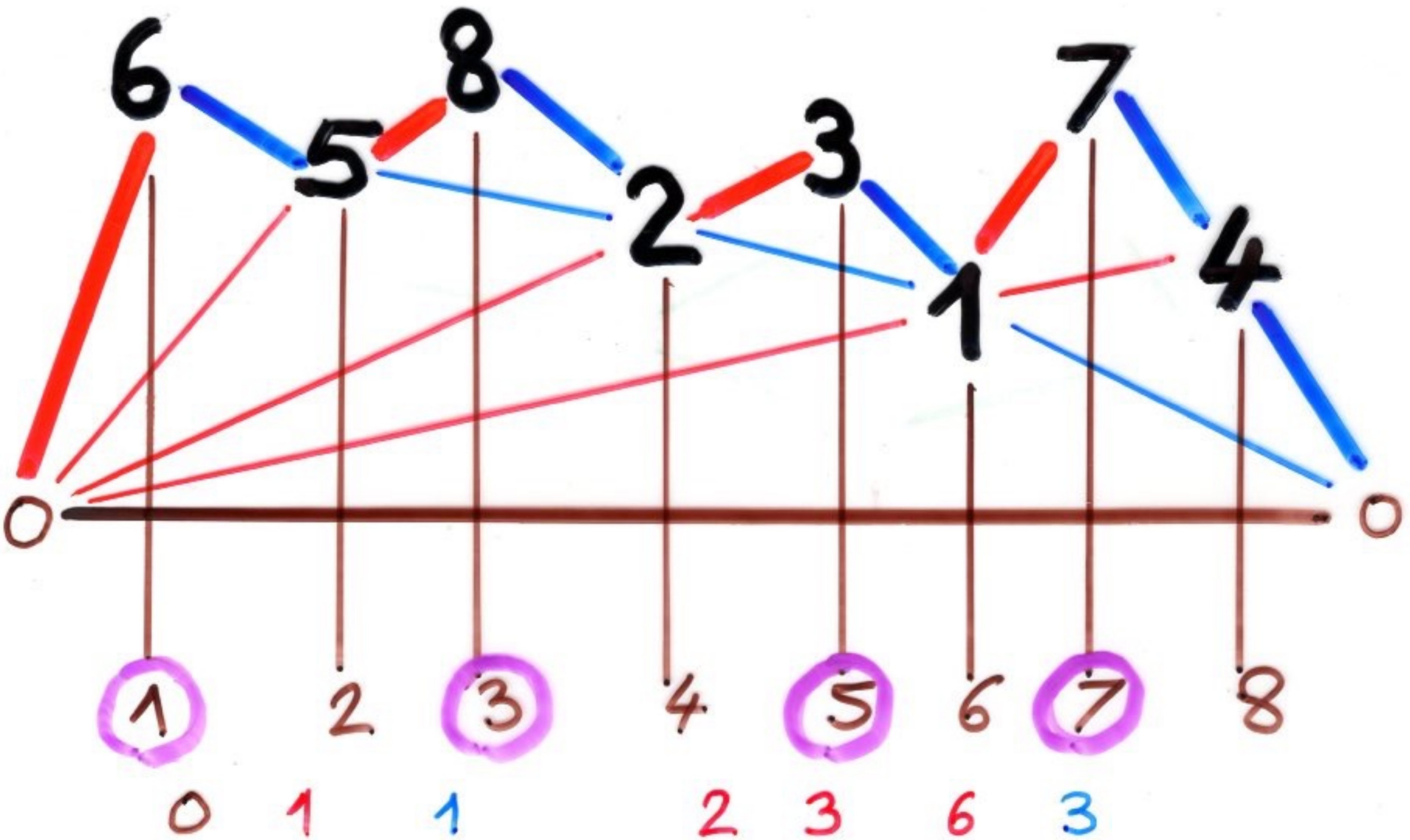
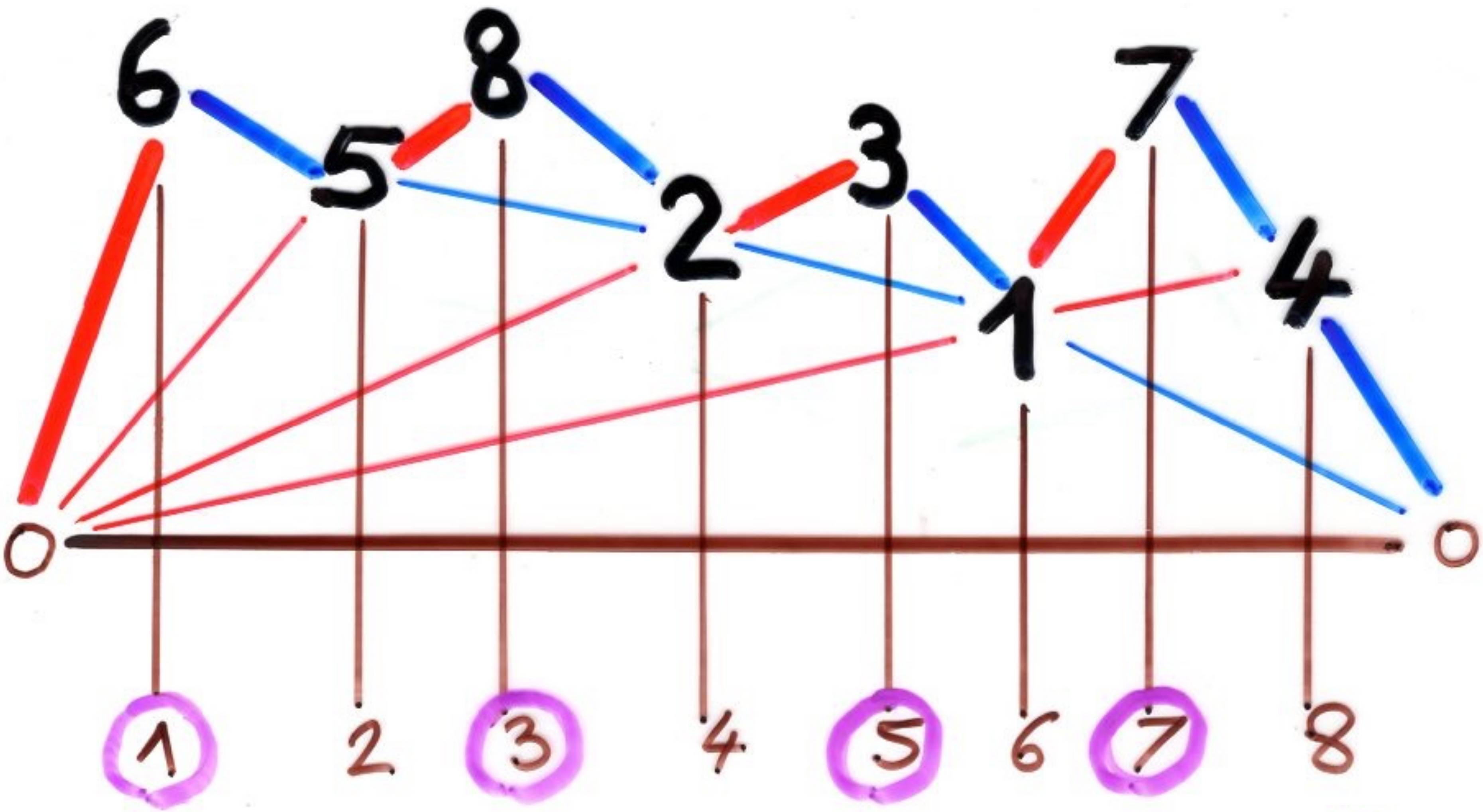
$$\mathrm{maj}(\sigma) = 1+3+5+7$$
$$= 16$$



$$\mathrm{maj}(\sigma) = 1+3+5+7 = 16$$

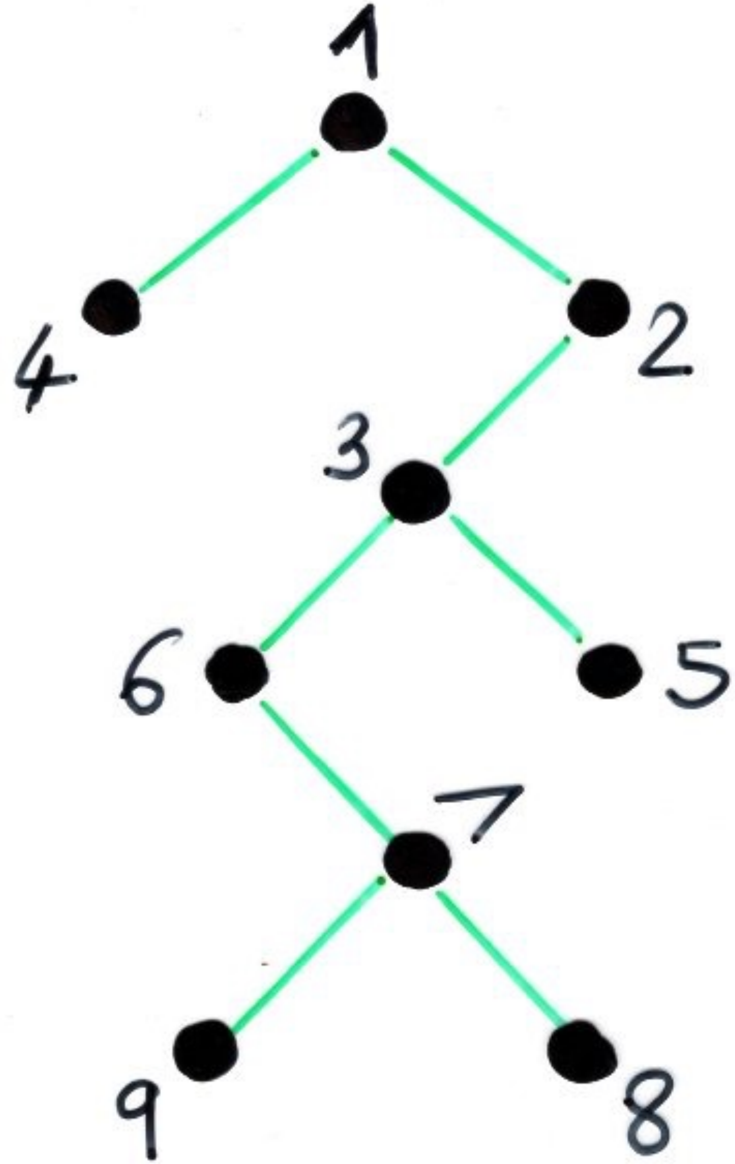


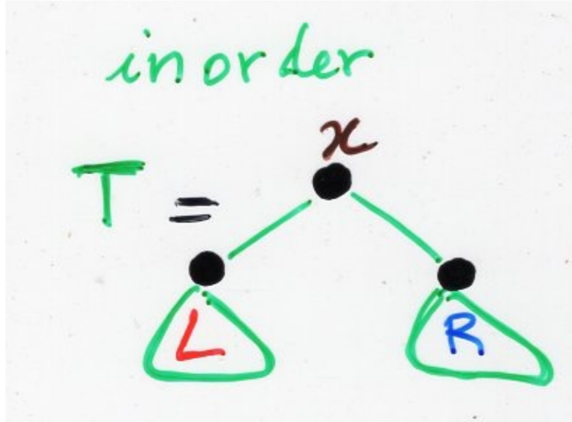
$$0 \quad 1 + 1 + 2 + 3 + 6 + 3 = 16$$

the philosophy of « histories »

and its q-analogues

Increasing binary trees





$$\pi(T) = \underbrace{\pi(L)}\, x \,\underbrace{\pi(R)}$$

projection of $T \in \mathcal{B}_n$

$$\pi(T) = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

w word of $\{1, 2, \dots, n, \dots\}^*$
all letters distinct

<u>Definition</u> $\delta(w)$ "déployé" of w

$$\begin{cases} \delta(e) = \emptyset \quad \text{(empty word)} \\ \delta(w) = (\delta(u), m, \delta(v)) \end{cases}$$

$w = umv$ where m is the minimum letter of w

<u>Proposition</u> π and δ are bijections
and $\delta = \pi^{-1}$

$$\mathfrak{S}_n \underset{\delta}{\overset{\pi}{\rightleftarrows}} \mathcal{T}_n$$

Definition x-factorization

$\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

$$\sigma = u \, \lambda(x) \, x \, \rho(x) \, v$$

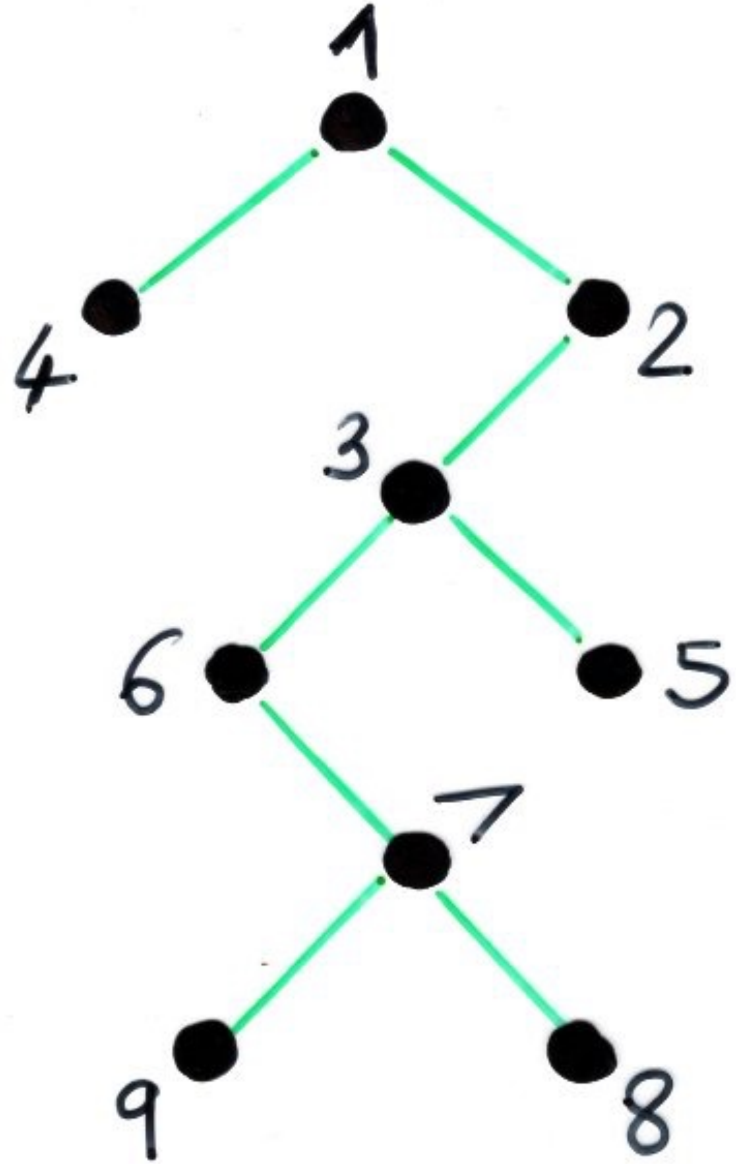
$$\begin{cases} \bullet \text{ the letters of } \lambda(x) \text{ and } \rho(x) \text{ are } > x \\ \bullet \ |\lambda(x)| \text{ and } |\rho(x)| \text{ maximum} \end{cases}$$

Lemma $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{T}_n$, $x \in [1, n]$

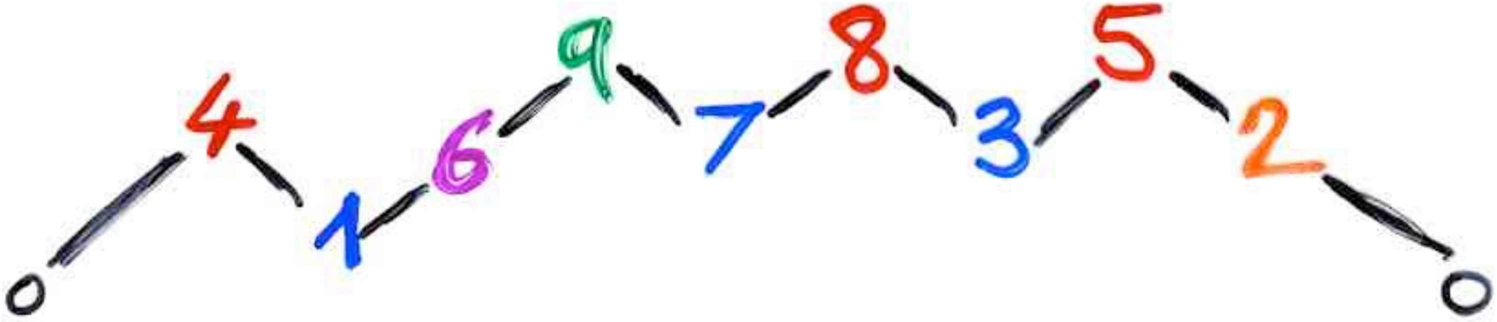
$u \, \lambda(x) \, x \, \rho(x) \, v$ x-factorization

then the left (resp. right) subtree of the vertex x in the tree $\delta(\sigma)$ is:

$\delta(\lambda(x))$ resp. $\delta(\rho(x))$



$$\pi(T) = 4\,1\,6\,9\,7\,8\,3\,5\,2$$



$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$

A through (valley)

S peak

J double rise

K double descent

in σ

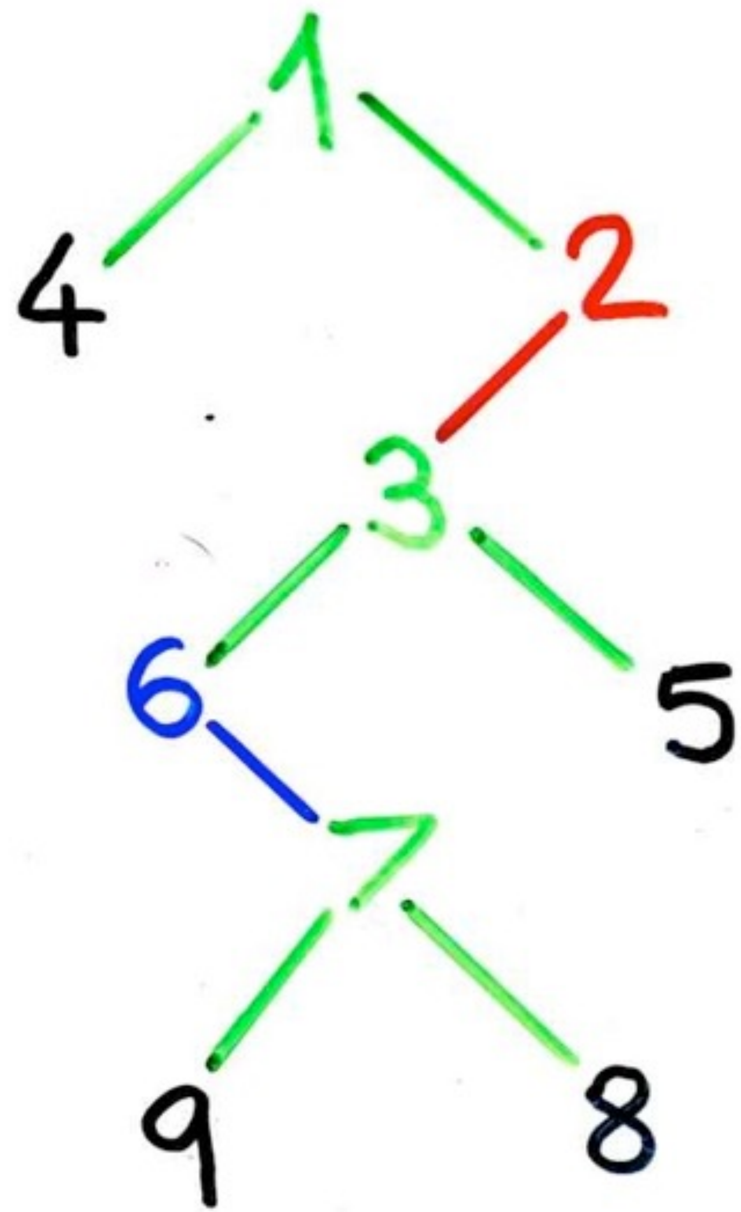
<u>Corollary</u> $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{T}_n$, $x \in [1, n]$

in σ, x is : peak, valley, double rise, double descent

iff in $\delta(\sigma)$

x is : leaf, double vertex, right simple vertex, left simple vertex

in $\delta(\sigma)$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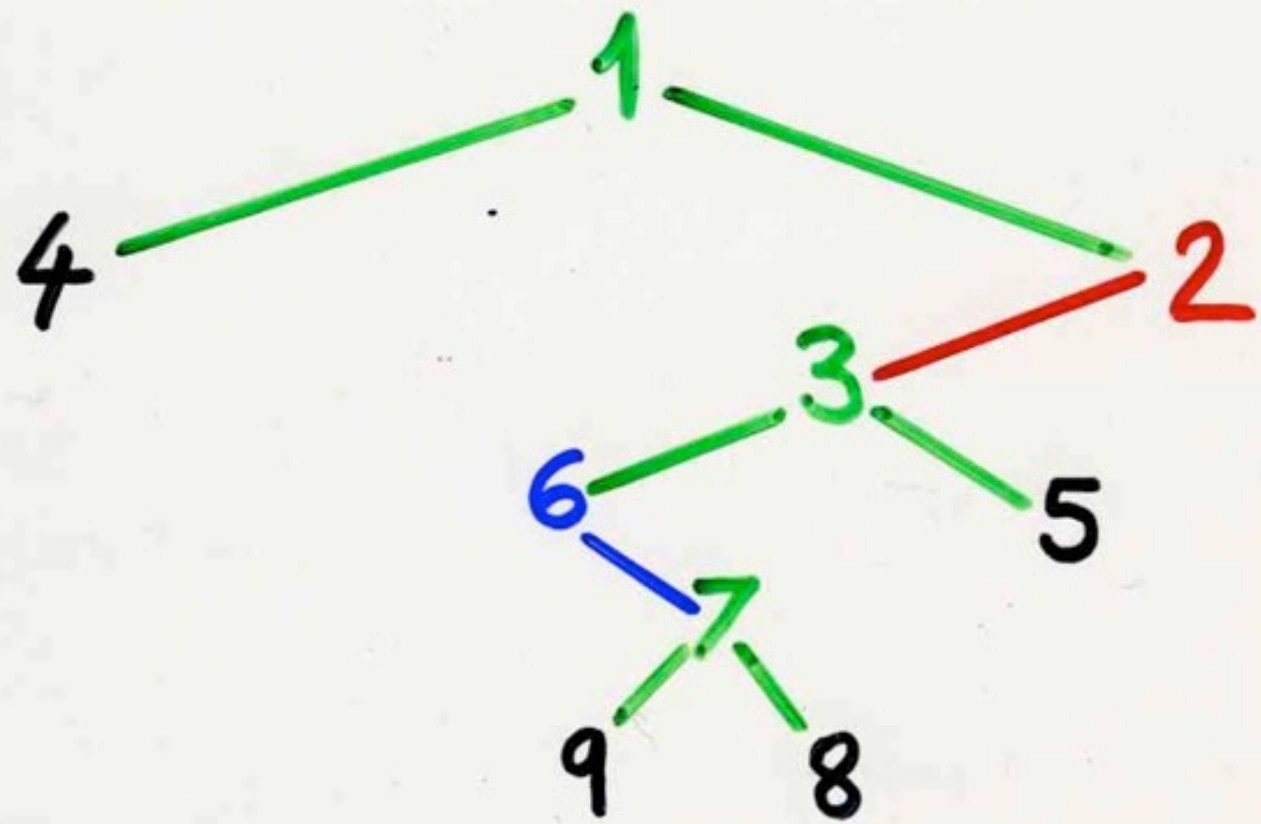
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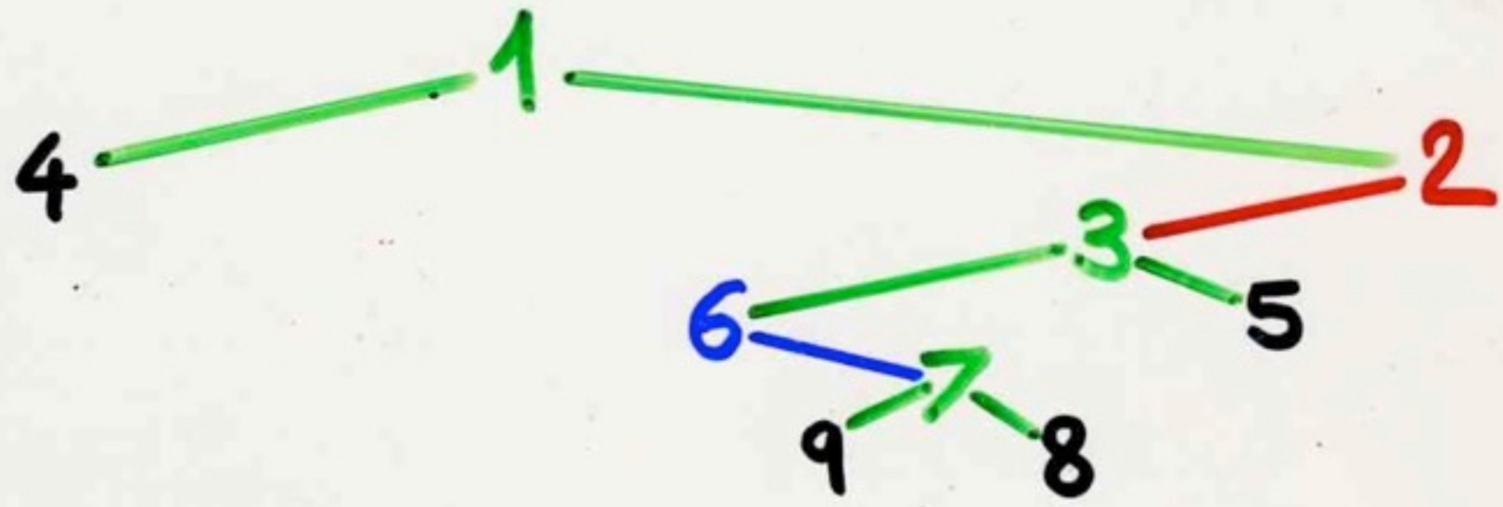
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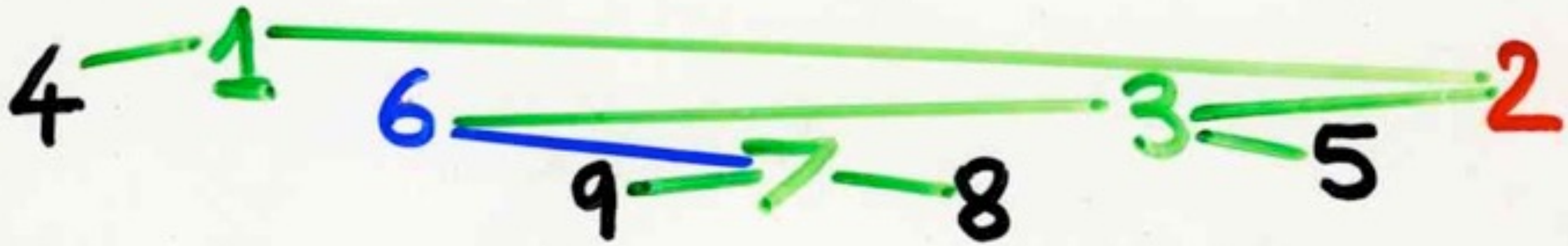
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4 1 6 9 7 8 3 5 2

$LR\text{-}\min(\sigma)$ = set of lr-min elements of σ

$RL\text{-}\min(\sigma)$ = set of rl-min elements of σ

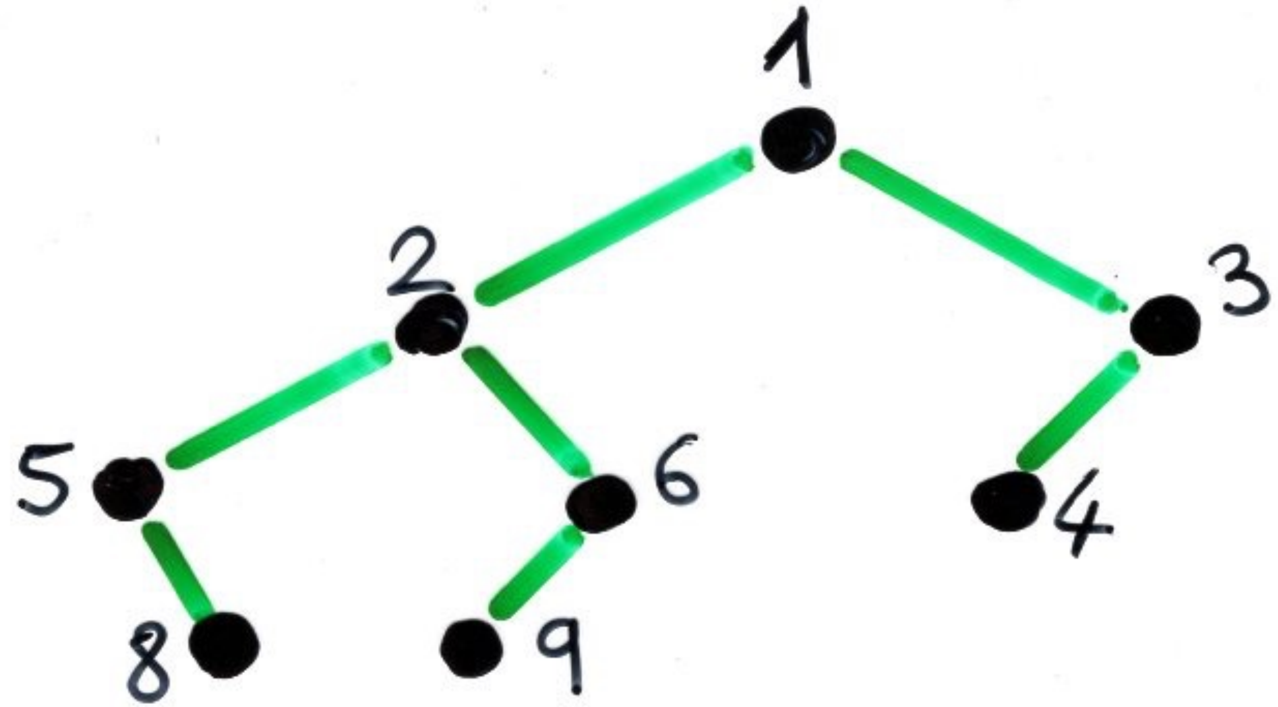
$LB(T)$ = left branch of $T \in \mathcal{B}_n$

$RB(T)$ = right branch of $T \in \mathcal{B}_n$

<u>Proposition</u> $\sigma \in \mathfrak{S}_n$, $T = \delta(\sigma) \in \mathcal{B}_n$

$$LR\text{-}\min(\sigma) = LB(T)$$

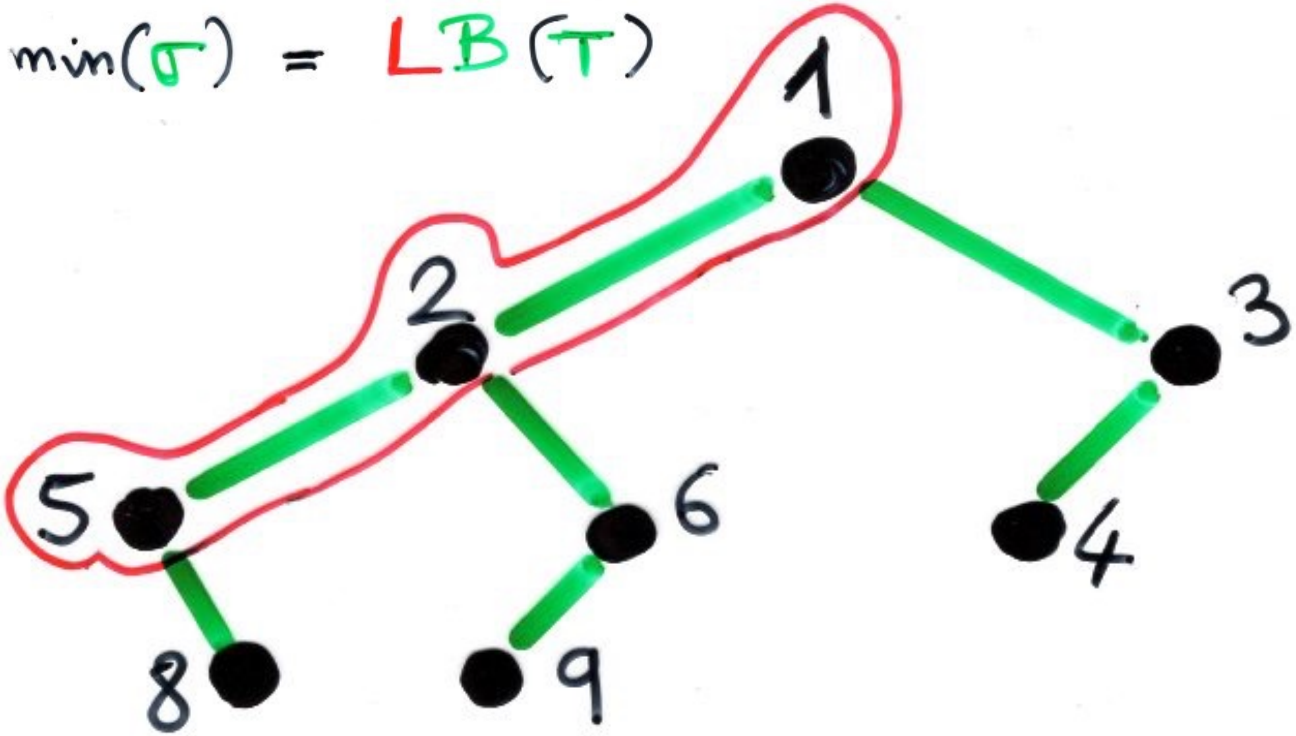
$$RL\text{-}\min(\sigma) = RB(T)$$



$\pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$

$T = \delta(\sigma)$

$LR\text{-}\min(\sigma) = LB(T)$

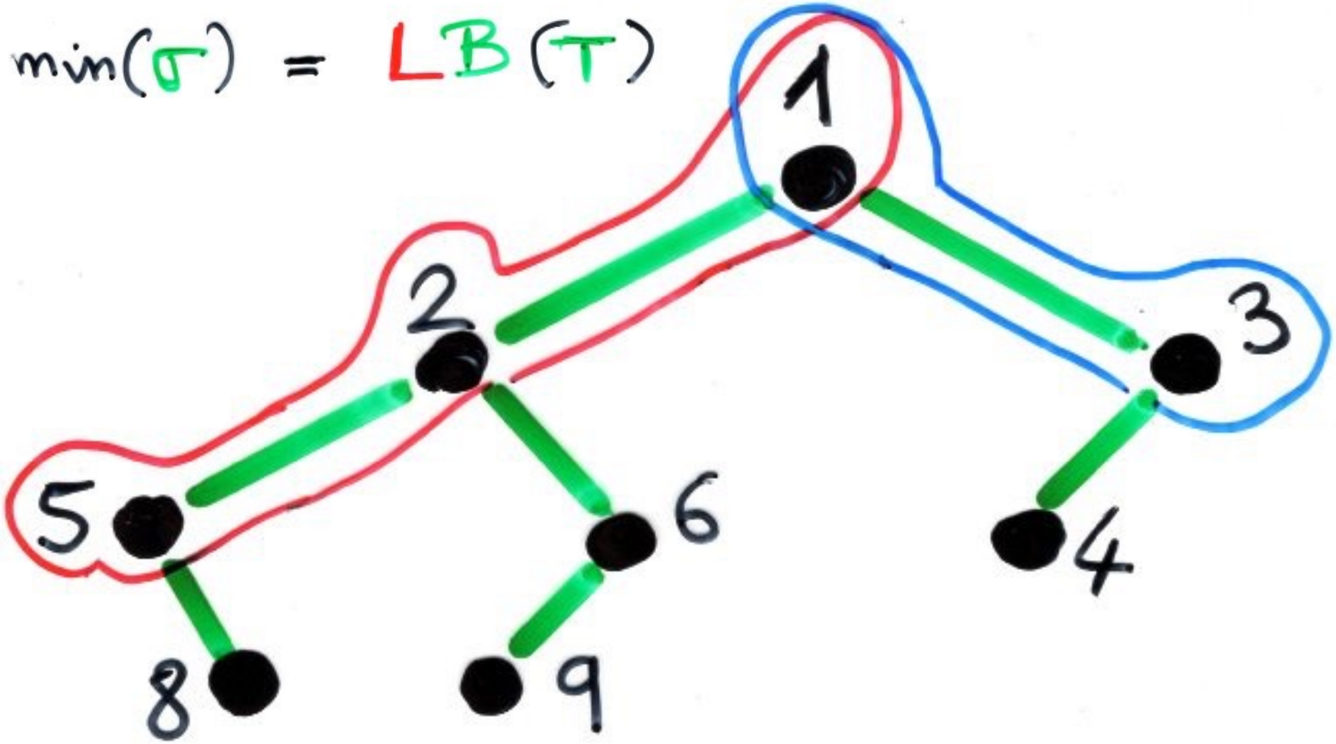


$\pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$

$T = \delta(\sigma)$

$RL\text{-}\min(\sigma) = RB(T)$

$LR\text{-}\min(\sigma) = LB(T)$

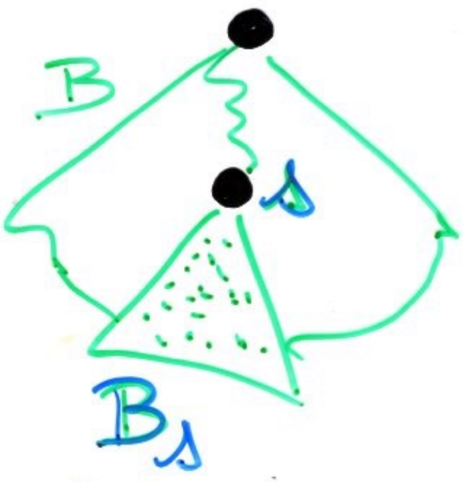


$$\Pi(T) = 5\ 8\ 2\ 9\ 6\ 1\ 4\ 3$$

$$T = \delta(\sigma)$$

exercise "hook length" formula

B binary tree n vertices



The number of increasing labelling of B (number of $T \in \mathcal{T}_n$ with B underlying binary tree) is:

$$\frac{n!}{\prod_{s \text{ vertex of } B} |B_s|}$$

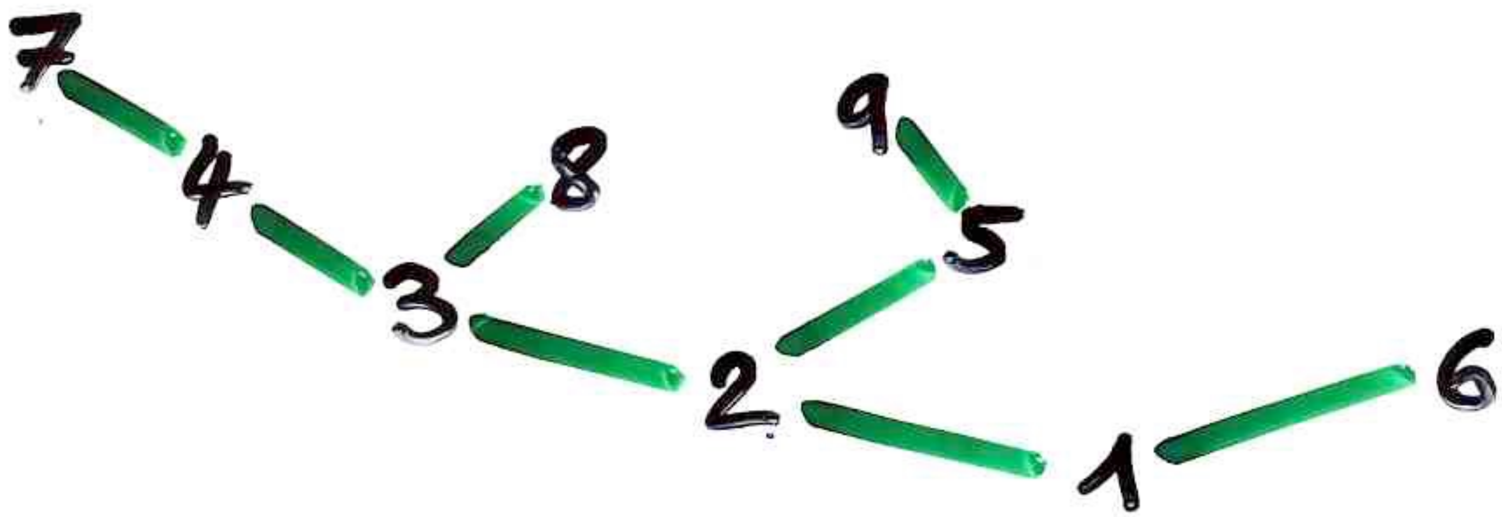
<u>exercise</u> let $\sigma = uv \in \mathfrak{S}_n$

knowing $\delta(u)$ and $\delta(v)$ describe a procedure to deduce $\delta(uv)$

[hint: get inspiration from Proposition p. 117 related to exercises on binary search trees]

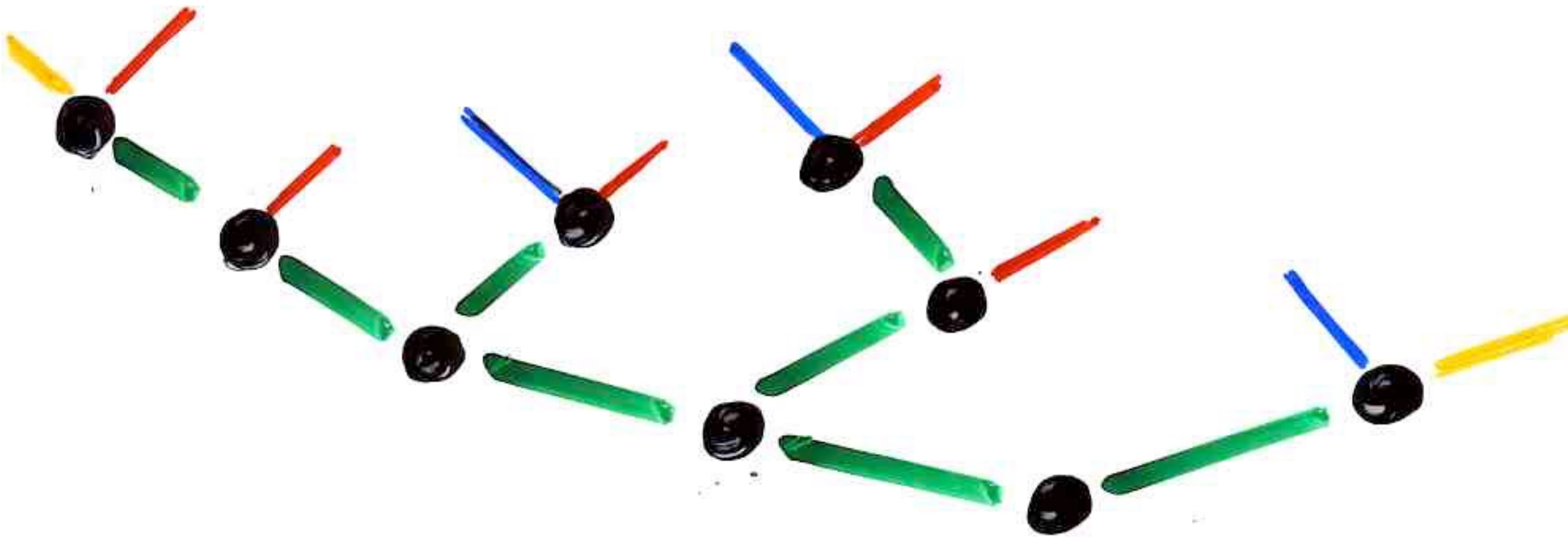
exercise

$$\begin{tikzcd}
\underset{\text{permutation}}{\sigma} \arrow[r] \arrow[dd] & T(\sigma) \ \text{increasing binary tree} \arrow[d] \\
 & B \ \text{binary tree} \arrow[d] \\
\underset{\text{up-down sequence}}{UD(\sigma)} \arrow[r, phantom, "="] & \underset{\text{canopy}}{c(B)}
\end{tikzcd}$$



$$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6\ \ldots$$

up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$



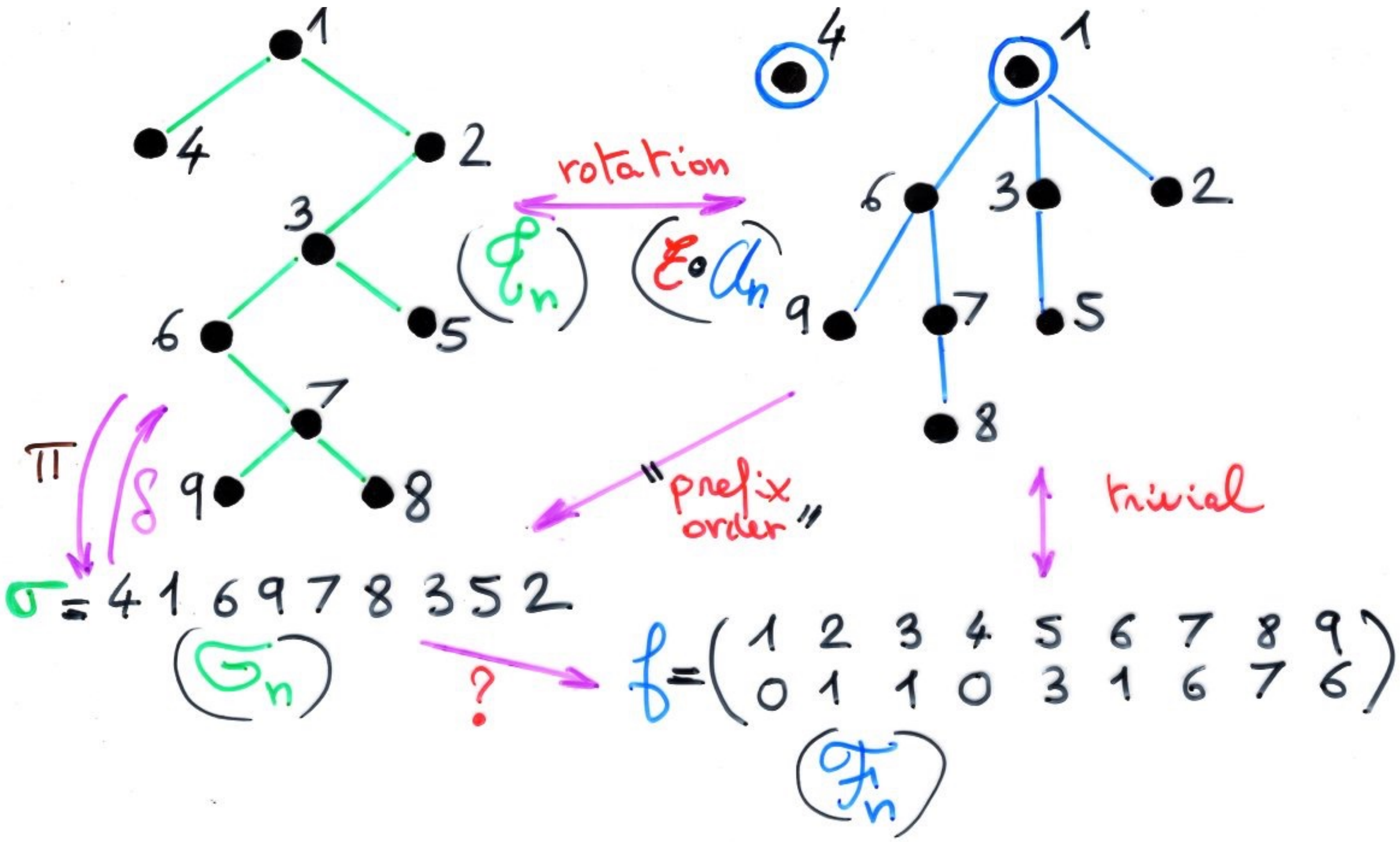
$$\sigma = 7\,4\,3\,8\,2\,9\,5\,1\,6\,\ldots$$

up-down sequence $\quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$

assemblée of

increasing arborescences

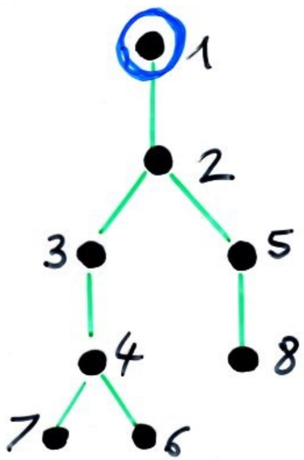
bijection: permutations $\longleftrightarrow$ (assemblée of increasing arborescences)



<u>exercise</u> (?) give a direct definition of the map $\mathfrak{S}_n \rightarrow \mathcal{F}_n$

__exercise__ A **1-2** arborescence is such that every vertex has only 1 or 2 childs.
Give a differential equation satisfied by 1-2 increasing arborescences and deduce that the number a_n of such arborescence is :

$$\begin{cases} a_{2n} = E_{2n} & \text{secant} \\ & \quad\text{numbers} \\ a_{2n+2} = T_{2n+1} & \text{tangent} \end{cases}$$



<u>exercise</u> Jacobi permutations

$\underline{\text{ex}}$: $\sigma = 7\,2\,8\,5$

A permutation $\sigma \in \mathfrak{S}_n$ is called Jacobi iff for every $x \in [1, n]$, the word $\rho(x)$ of the x-factorization of σ has even length

Let Y (resp. Z) the $\mathbb{L}$-species of Jacobi permutations on an odd (resp. even) number of elements.

(i) Show that Y and Z satisfies

$$\begin{cases} Y' = Z^2, & Y[\emptyset] = \emptyset \\ Z' = Y Z, & Z[\emptyset] = \{\emptyset\} \end{cases}$$

Thus the generating functions

$y = Y(t)$ and $z = Z(t)$ satisfies :

$$\begin{cases} y' = z^2, & y(0) = 0 \\ z' = yz, & z(0) = 1 \end{cases}$$

having unique solution $y = \tan t$, $z = \dfrac{1}{\cos t}$

(ii) Deduce that the number of assembleé of increasing arborescences on $[1, m]$ such that every vertex has an even number of childs is:

E_{2n} (secant numbers) for $m=2n$

T_{2n+1} (tangent numbers) for $m=2n+1$

[give a bijection with Jacobi permutations]

[hint: use the right rotation in the bijections

assembleé increasing arborescences $\longrightarrow$ increasing binary trees $\longrightarrow$ permutations]

[or exchange $\rho(x)$ and $\lambda(x)$ in the x-factorisation]

(iii) Deduce a combinatorial proof of the identity

$$\exp\left(\int_0^t \frac{du}{\cos u}\right) = \tan t + \frac{1}{\cos t}$$

Conclusion

Thus we get 3 different combinatorial interpretations of tangent and secant numbers: alternating permutations, Jacobi permutations, and 1-2 increasing arborescences, related to 3 different systems of differential equations

→ André permutations (Foata - Schützenberger) are easily put in bijection with 1-2 increasing arborescences computer science

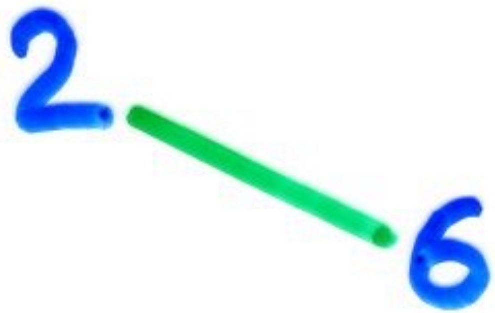
binary search trees

data structures

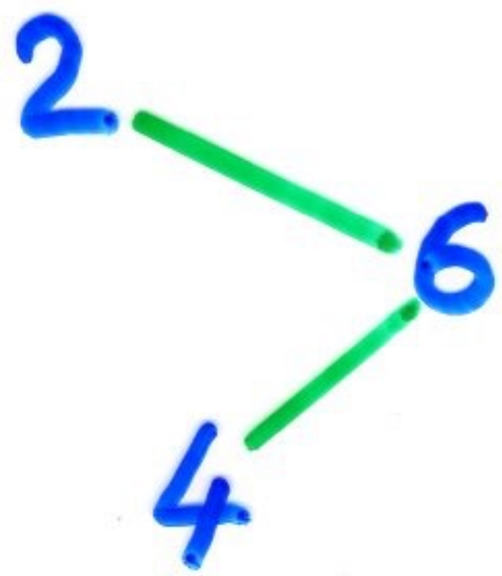
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree

2

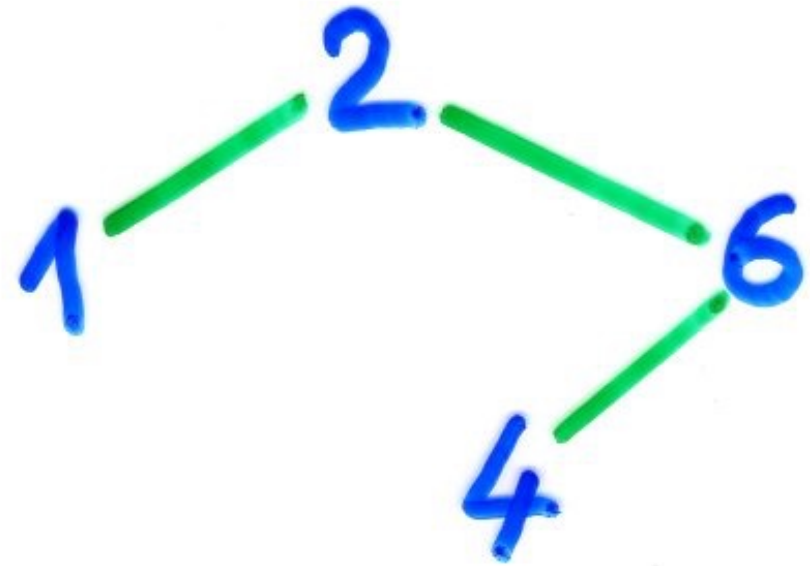
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



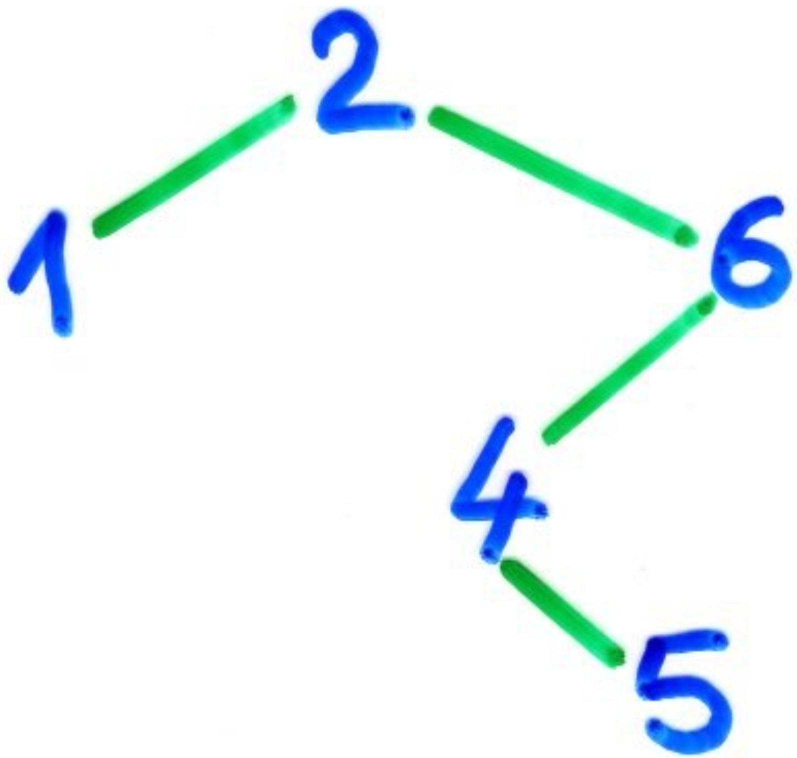
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



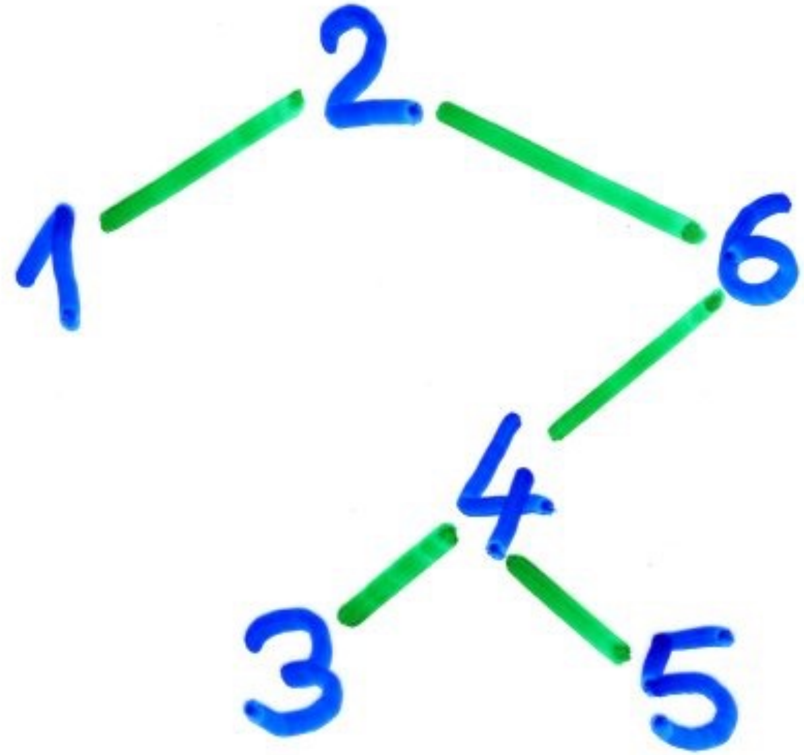
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



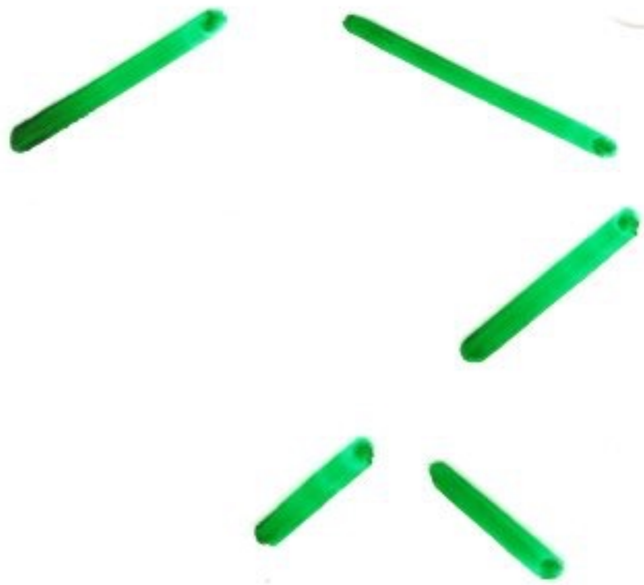
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



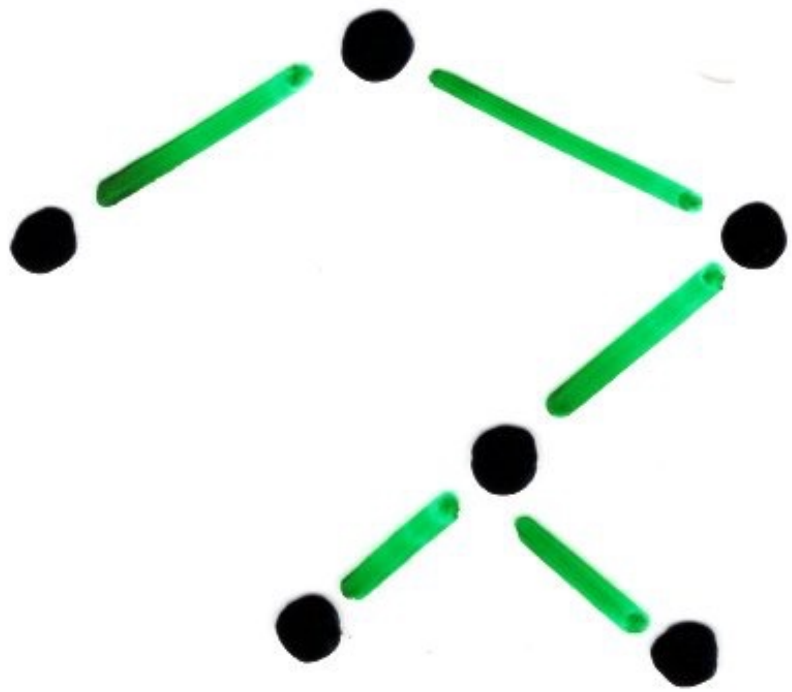
$\Pi(B) = 1\ 2\ 3\ 4\ 5\ 6$

$=$ identity permutation

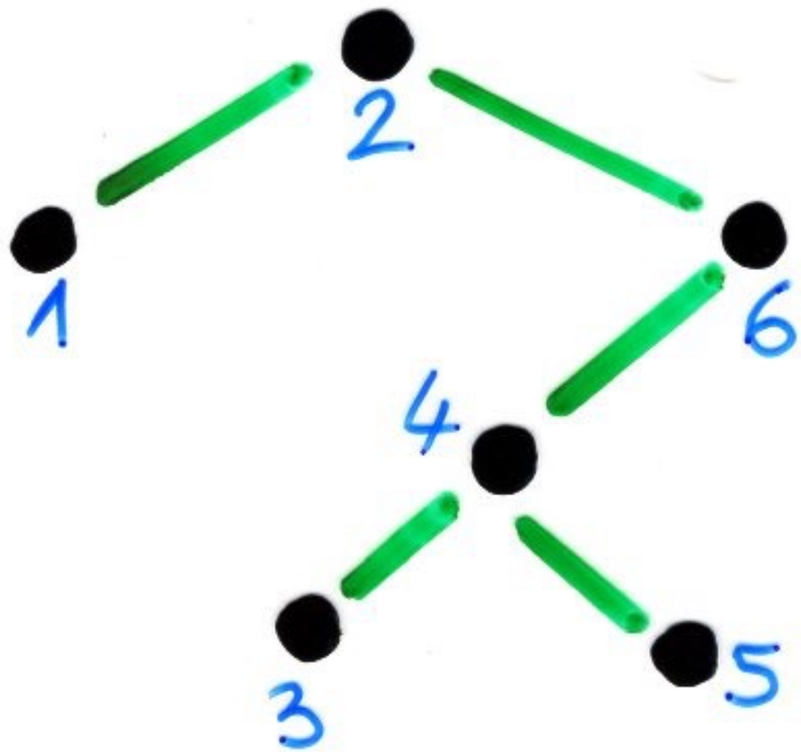
binary search tree



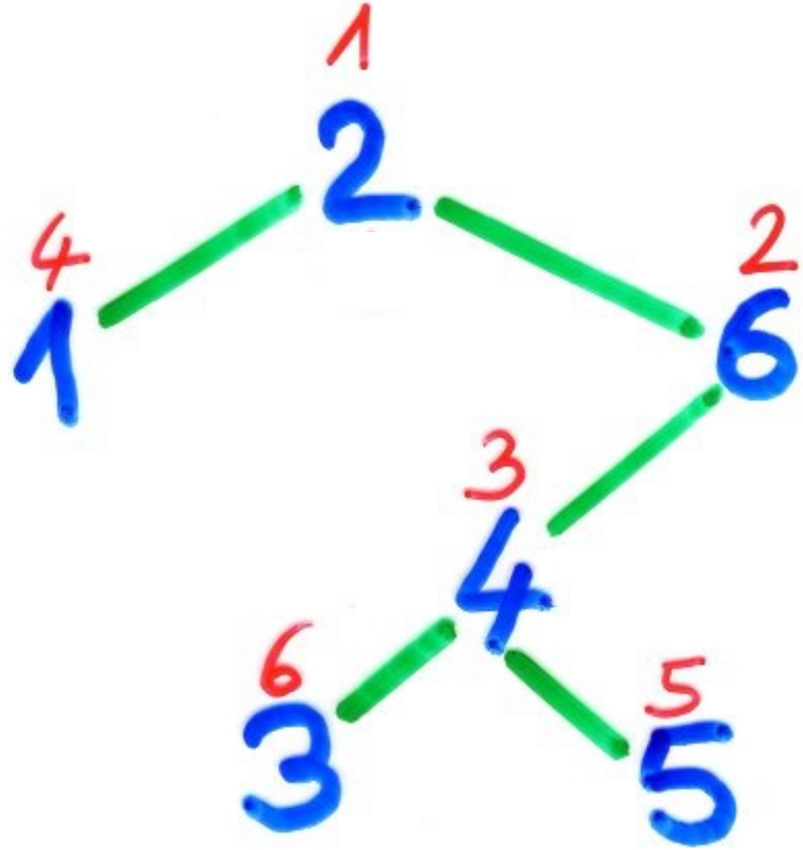
binary search tree



binary search tree

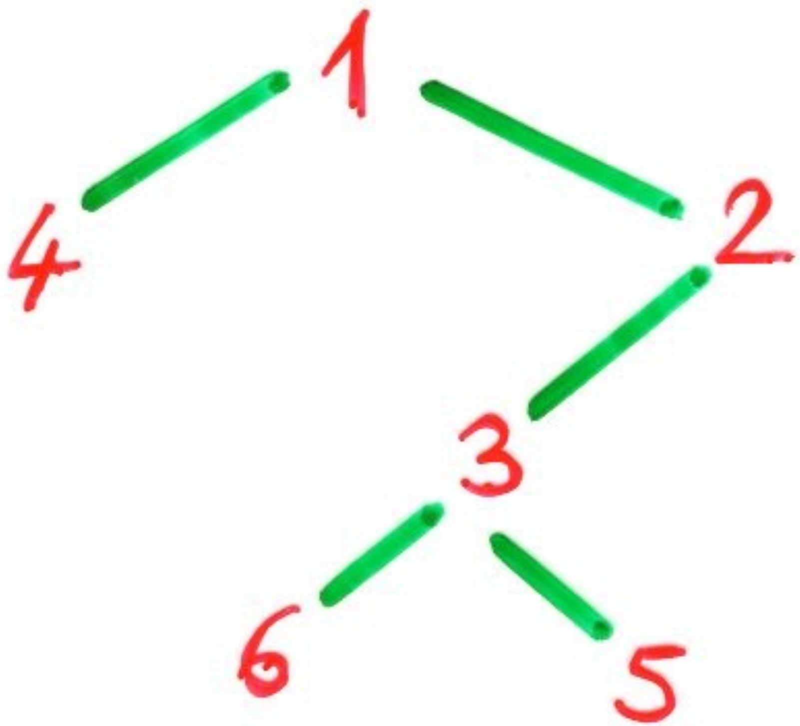


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$



$$\tau = 4\ 1\ 6\ 3\ 5\ 2$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$



$$\tau = \sigma^{-1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 3 & 5 & 2 \end{pmatrix}$$

analysis of the insertion in

binary search trees

Problem

average cost of an insertion in a random binary search tree

parameter : number of comparisons for the insertion of the last elements

= height of $\mathbf{n}$ in a random increasing binary tree on $[1, n]$

We want to prove that the average of this height is :

$$2\,(H_n - 1)$$

Proposition $\sigma \in \mathfrak{S}_n$, $T = \delta(\sigma) \in \mathcal{T}_n$, $x \in [1, n]$

(i) $\mathrm{Path}(1, x) = RL\text{-}\min(x_1 x_2 .. x_i) \cup LR\text{-}\min(x_i .. x_n)$

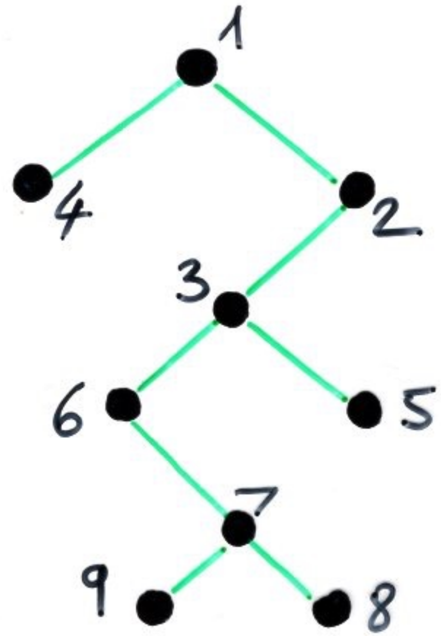
with $\sigma = x_1 x_2 .. x_i .. x_n$, $x_i = x$

(ii) the vertex y has a right child iff

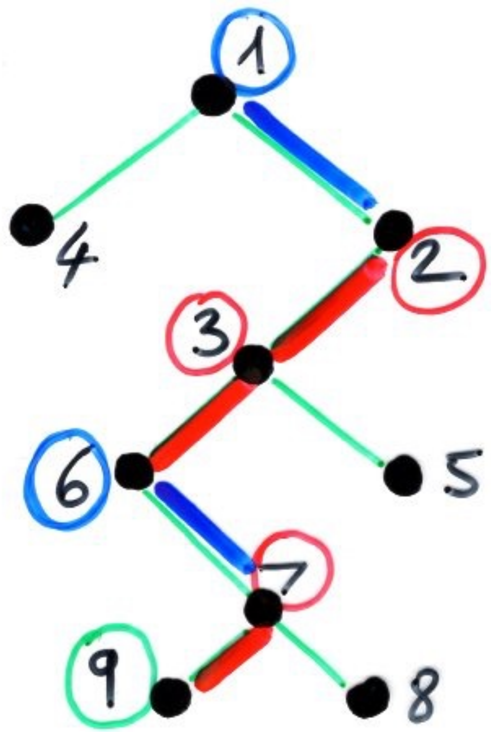
$y \in RL\text{-}\min(x_1 .. x_i)$

(ii)' ------ y -- a left child iff

$y \in LR\text{-}\min(x_1 .. x_i)$



$$\sigma = \underbrace{4\ 1\ 6}_{u}\ \underbrace{9}_{x}\ \underbrace{7\ 8\ 3\ 5\ 2}_{v}$$



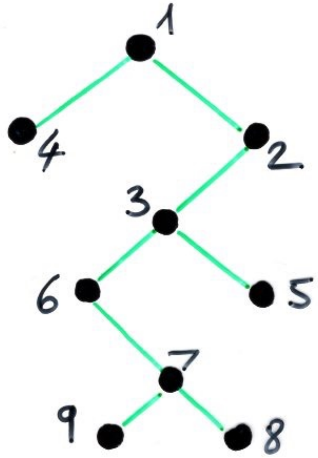
exercise define the map θ by:

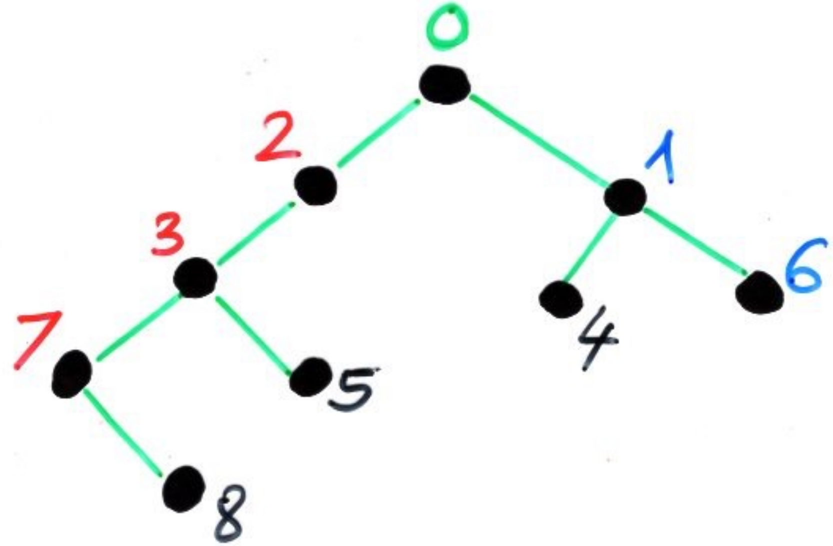
for $\sigma = u\, n\, v$, $\quad \theta(\sigma) = v\, 0\, u$

$$\sigma \in \mathfrak{S}_n \longrightarrow \theta(\sigma) \in \mathfrak{S}_{[0,n-1]}$$

The vertices of the path Path$(1,n)$ in $\delta(\sigma)$ are in correspondence with

$$LR\text{-min}(\theta(\sigma)) \cup RL\text{-min}(\theta(\sigma))$$





$$\sigma = \underbrace{4\ 1\ 6}_{u}\ 9\ \underbrace{7\ 8\ 3\ 5\ 2}_{v}$$

$$\theta(\sigma) = 7\ 8\ 3\ 5\ 2\ 0\ 4\ 1\ 6$$

$$\theta(\sigma) = \underbrace{7\ 8\ 3\ 5\ 2}_{LR\text{-min}}\ 0\ \underbrace{4\ 1\ 6}_{RL\text{-min}}$$

<u>Prove:</u> the number of increasing binary trees T on [1,n] such that

$$|LB(T)| + |RB(T)| - 2 = k$$

is :

$$2^k s_{n-1,k}$$

[proof directly on the tree, or using sub-excedante functions (inversion tables)] deduce that the number of increasing binary trees on $[1,n]$ such that the height of n is $k \geqslant 0$, is

$$2^k s_{n-1,k}$$

<u>Conclusion</u> The average height of n in a random increasing binary tree on $[1,n]$ is

$$2(H_n - 1)$$