An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 2

The Catalan garden

(4)

IMSc

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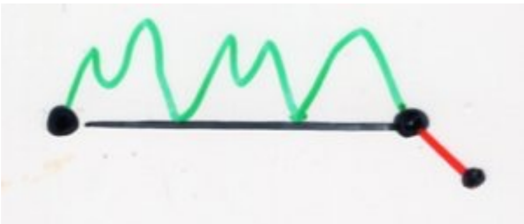
from previous lecture:

the cyclic lemma

- Definition $w, w' \in X^*$ are conjugate iff
  $w = uv \quad w' = vu$ equivalence relation

- labelled conjugate $w = x_1 \cdots x_n$, $x_i \in X$
  $(i, w_i) \quad w_i = x_i \cdots x_n \, x_1 \cdots x_{i-1}$

$A = D\bar{x} \subseteq \{x, \bar{x}\}^*$ D Dyck words



Proposition (cyclic lemma)

Let $w \in \{x, \bar{x}\}^*$, $p > 0$, with $\delta(w) = -p$

$\delta : X^* \to \mathbb{Z}_+$ $\quad \delta(x) = 1$, $\delta(\bar{x}) = -1$

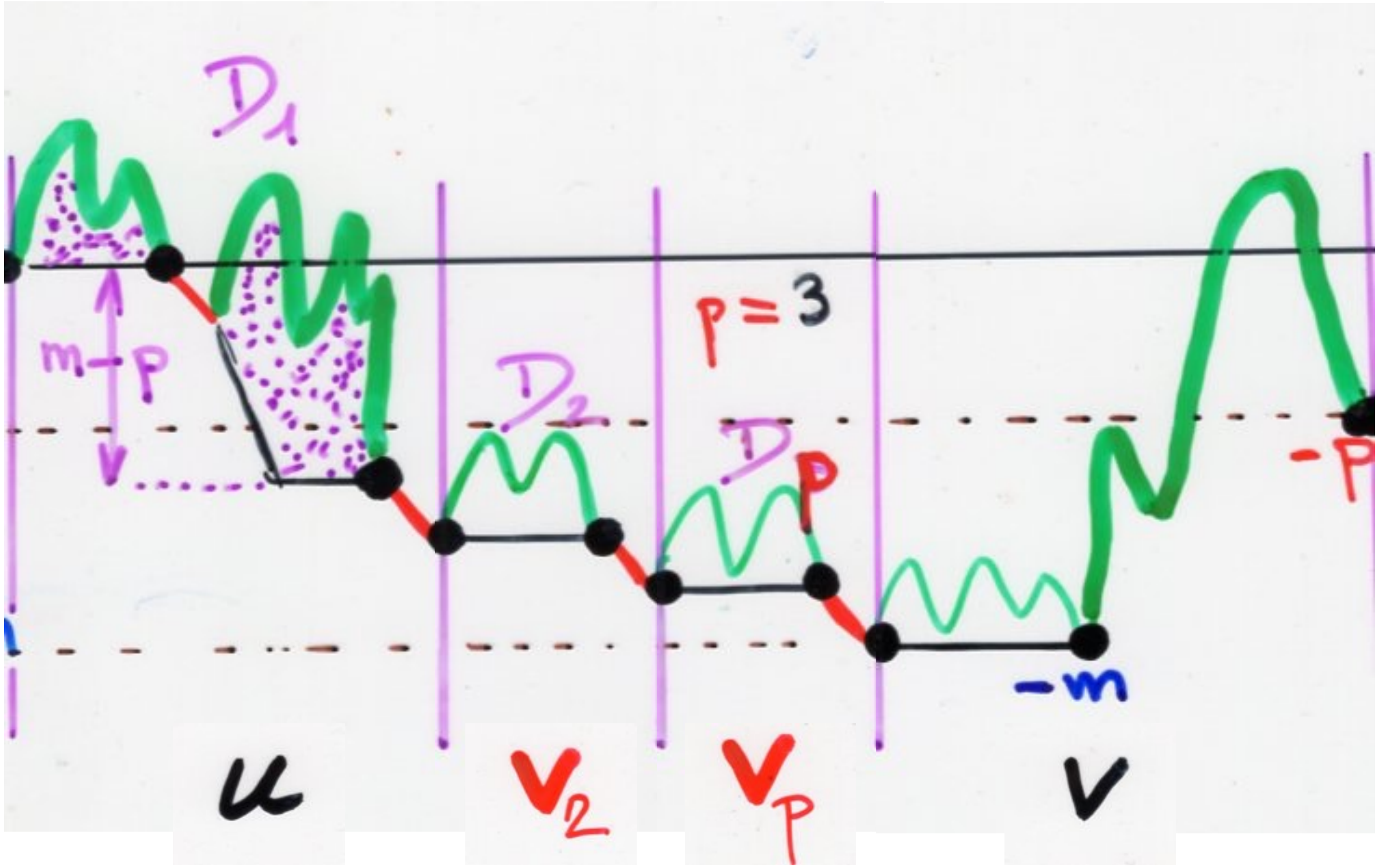
$$\delta(w) = |w|_x - |w|_{\bar{x}}$$

There are exactly p labelled conjugates (i, w_i) such that $w_i \in A^*$

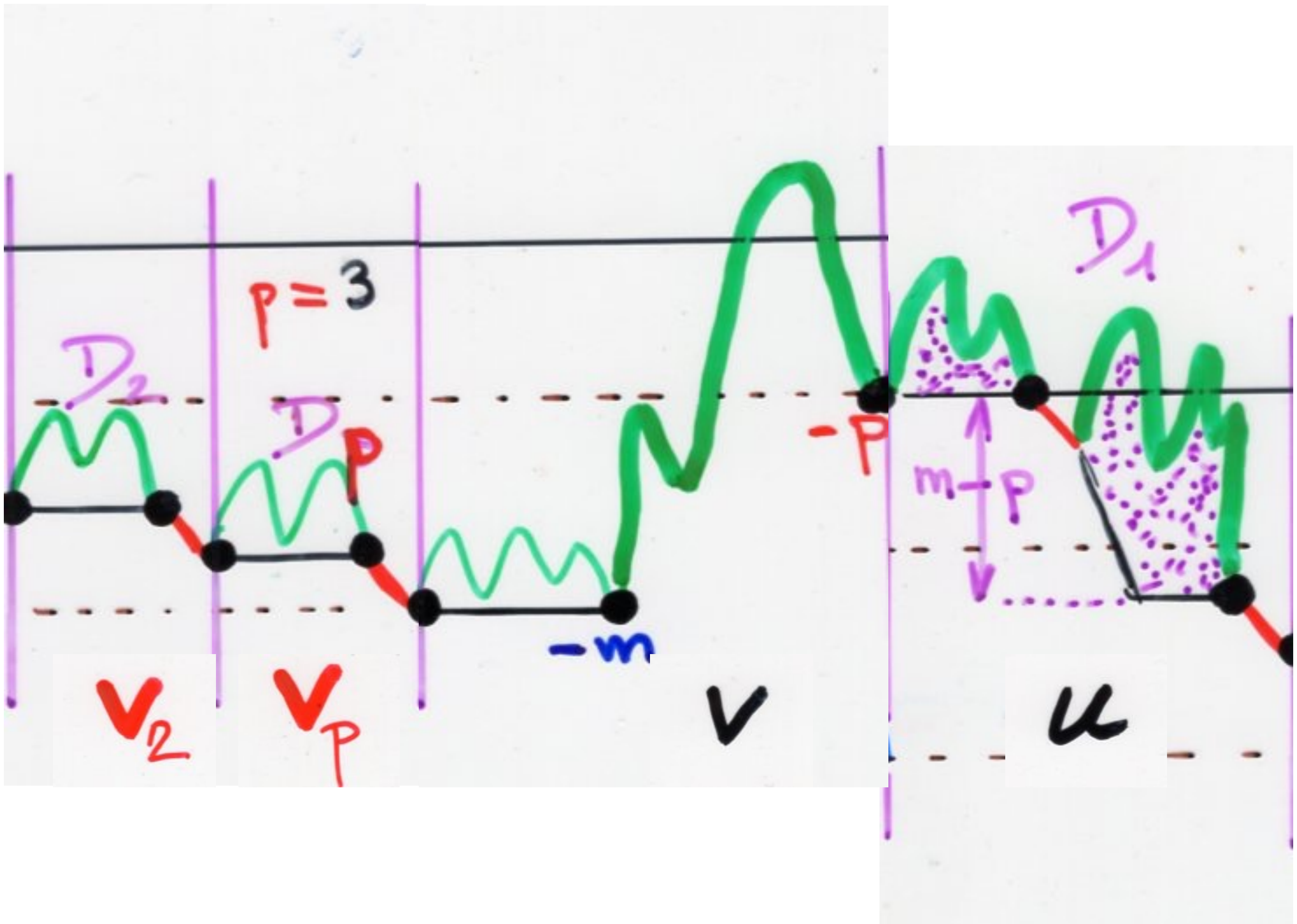
in fact $w_i \in A^p$

$w = u\, v_2 \ldots v_p\, v$ $\quad$ with $vu, v_2, \ldots, v_p \in A$

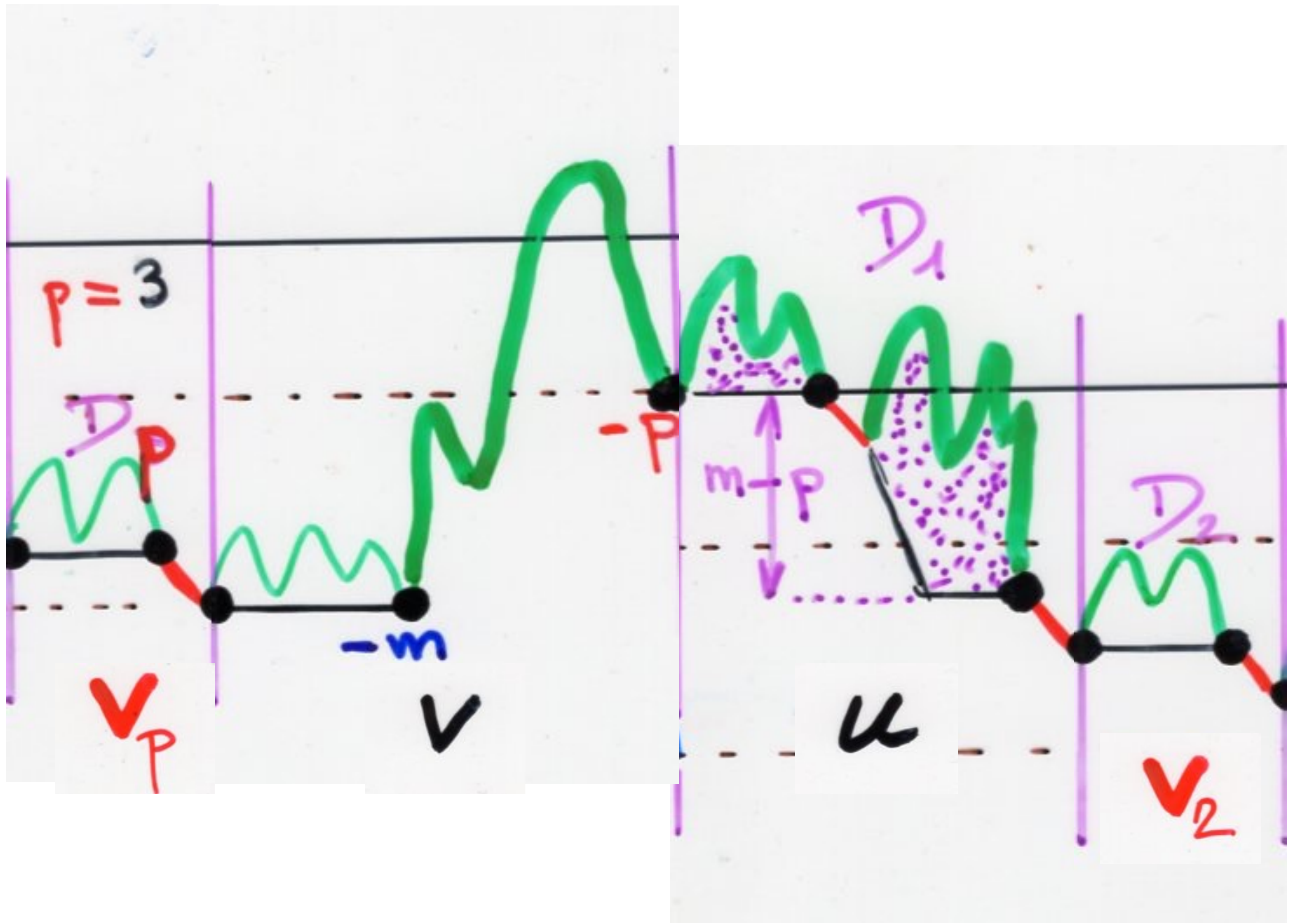
the p conjugates are
$$\begin{cases} v_2 \ldots v_p\, vu \\ v_3 \ldots v_p\, vu\, v_2 \\ \text{------} \\ vu\, v_2 \ldots v_p \end{cases}$$



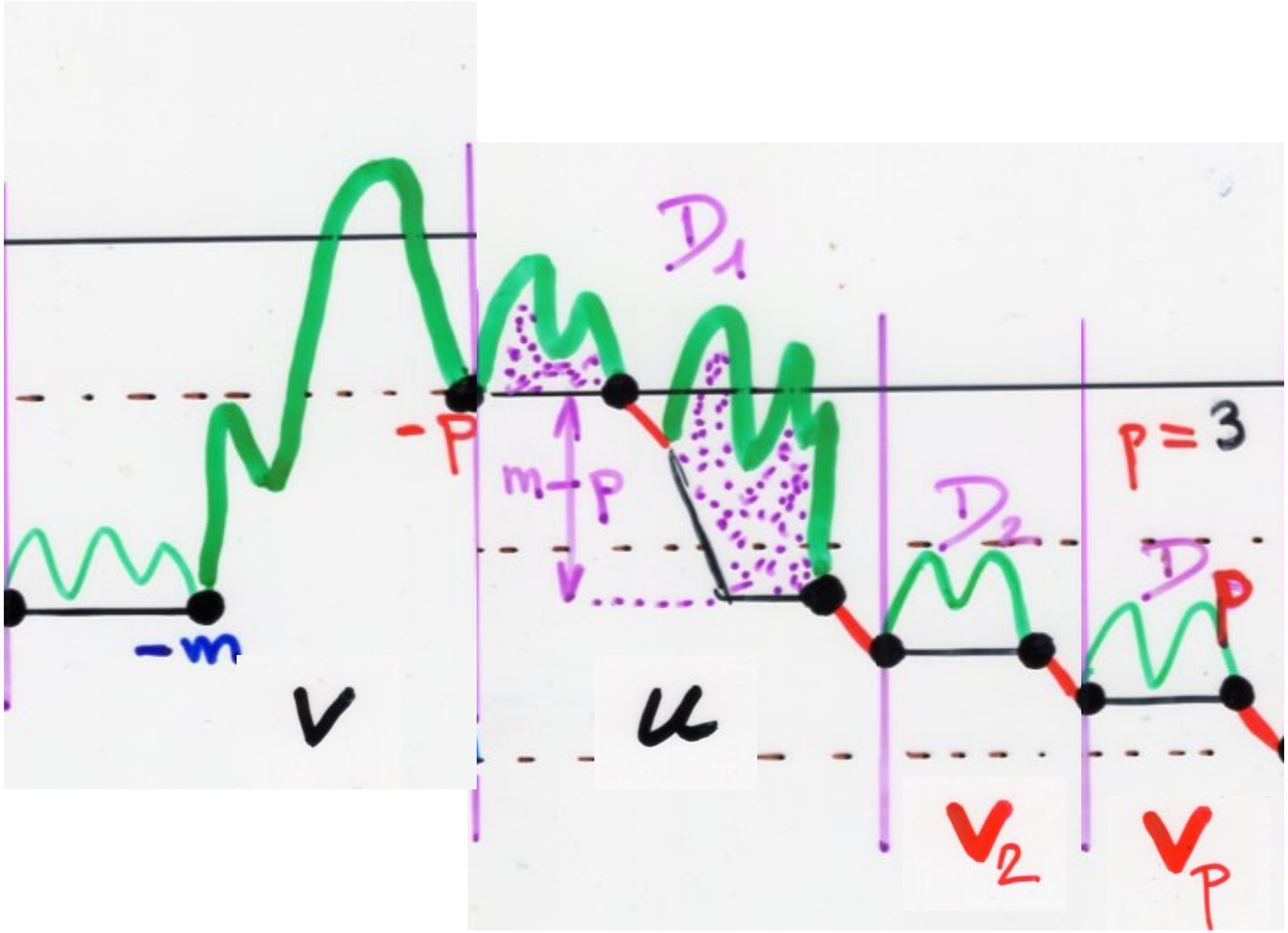
D1
m − p
D2
p = 3
Dp
−p
−m
u
v2
vp
v



p = 3
D2
D_p
p
-m
-p
D1
m-p
V2
V_p
V
u



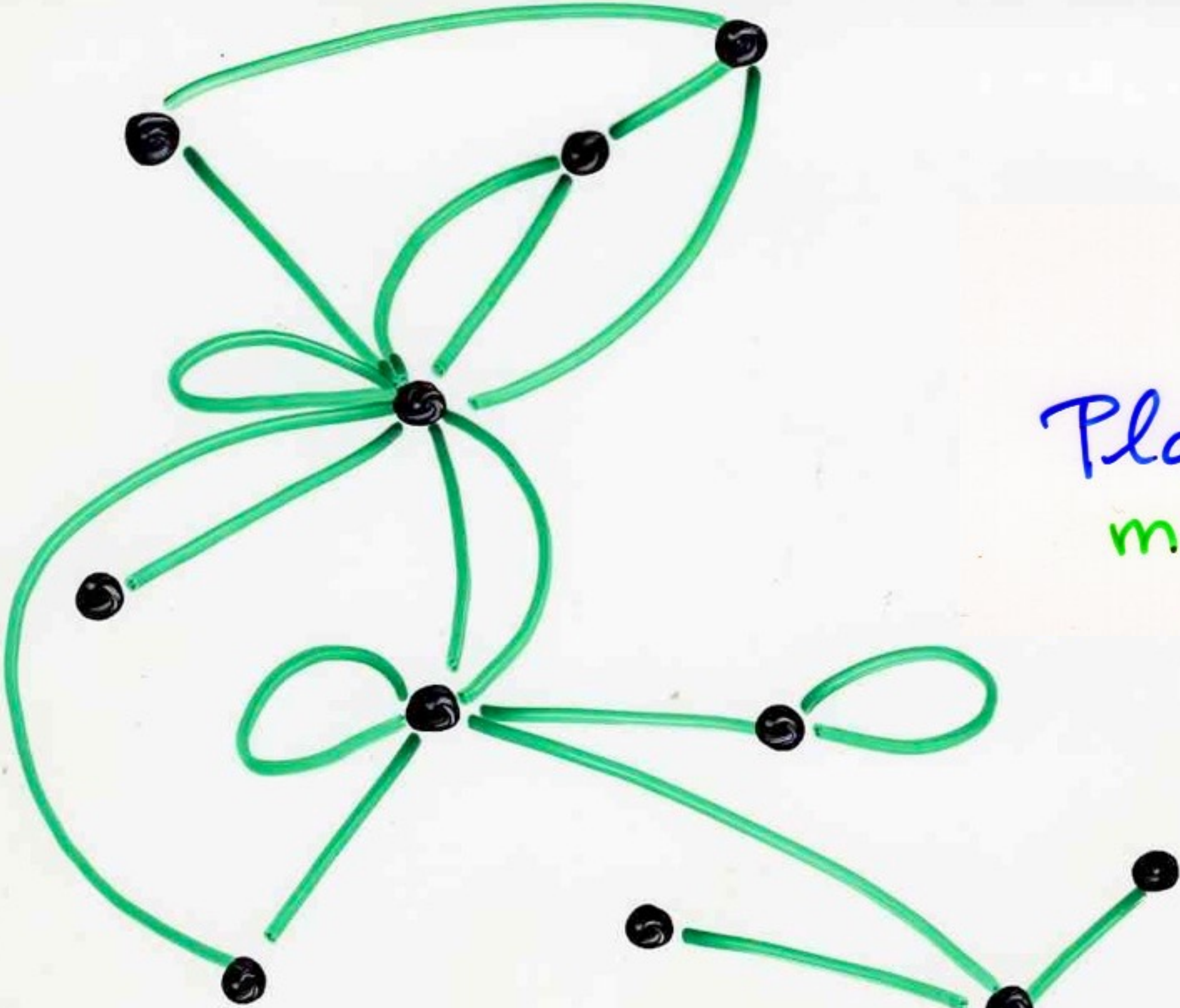
p = 3
D
p
-m
-p
D1
m - p
D2
V_p
V
U
V_2



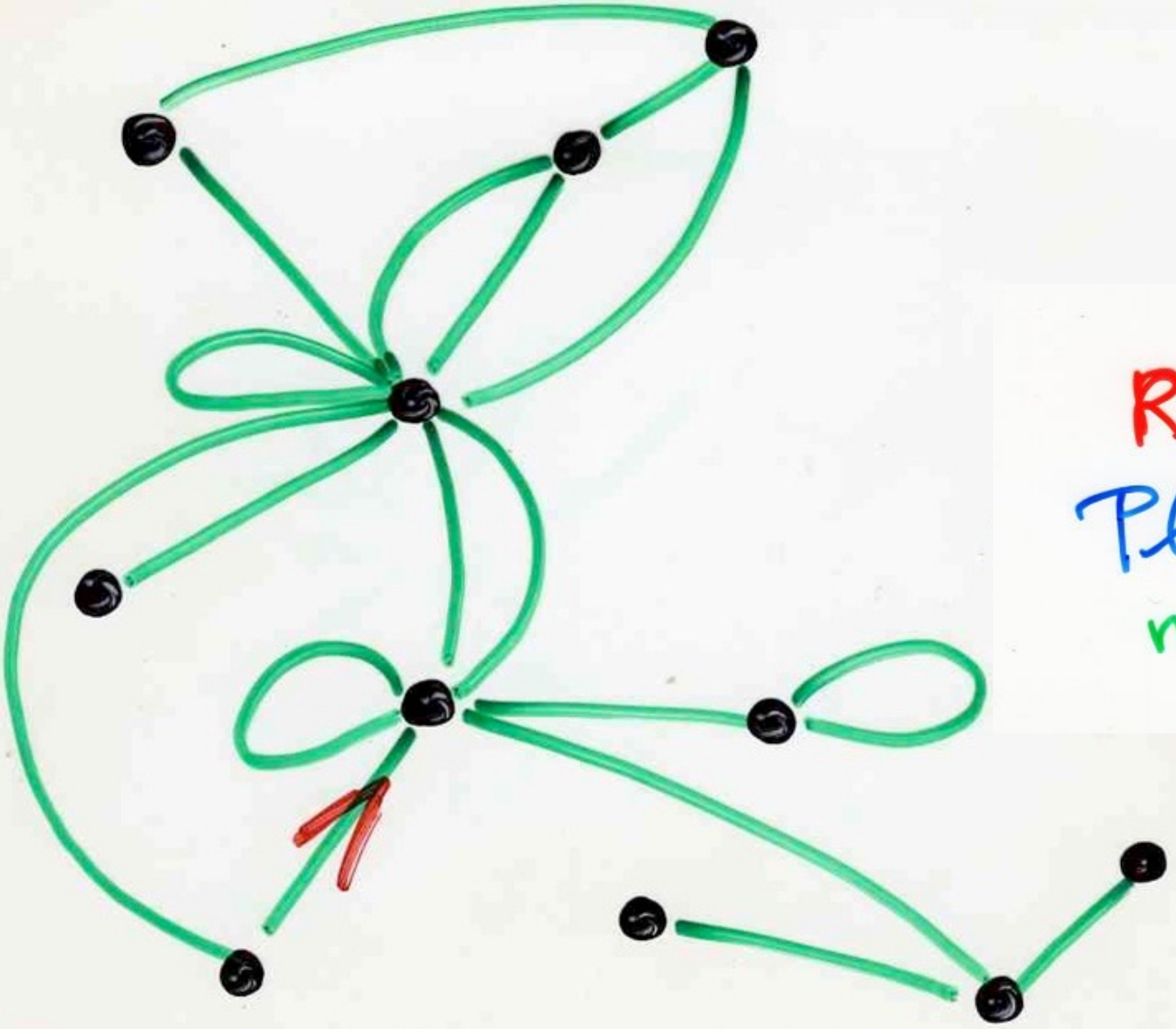
D_1
$-p$
$m-p$
$p = 3$
D_2
D_p
$-m$
v
u
V_2
V_p

planar maps

and the cyclic lemma



Planar
map



Rooted
Planar
map

Tutte (1960)

a_n number of rooted planar maps with n edges

$$y = \sum_{n \geq 0} a_n t^n$$

$$\begin{cases} h = 1 + 3t h^2 \\ y = h - t h^3 \end{cases}$$

Cori, Vauquelin (1970)

$$a_n = \frac{2 \times 3^n}{(n+2)} C_n \qquad C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Schaeffer (1997)

Schaeffer (1997)

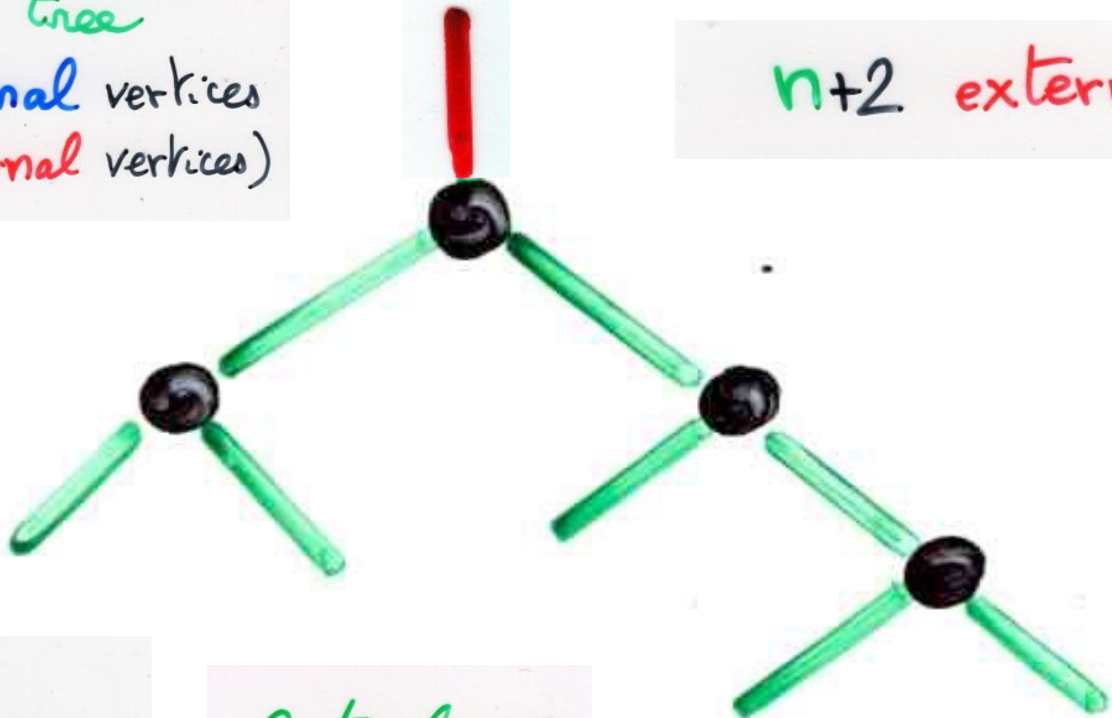
binary tree
with n internal vertices
(or $n+1$ external vertices)

$$\frac{2 \times 3^n}{(n+2)} C_n$$

Catalan number

Schaeffer (1997)

binary tree
with n internal vertices
(or $n+1$ external vertices)



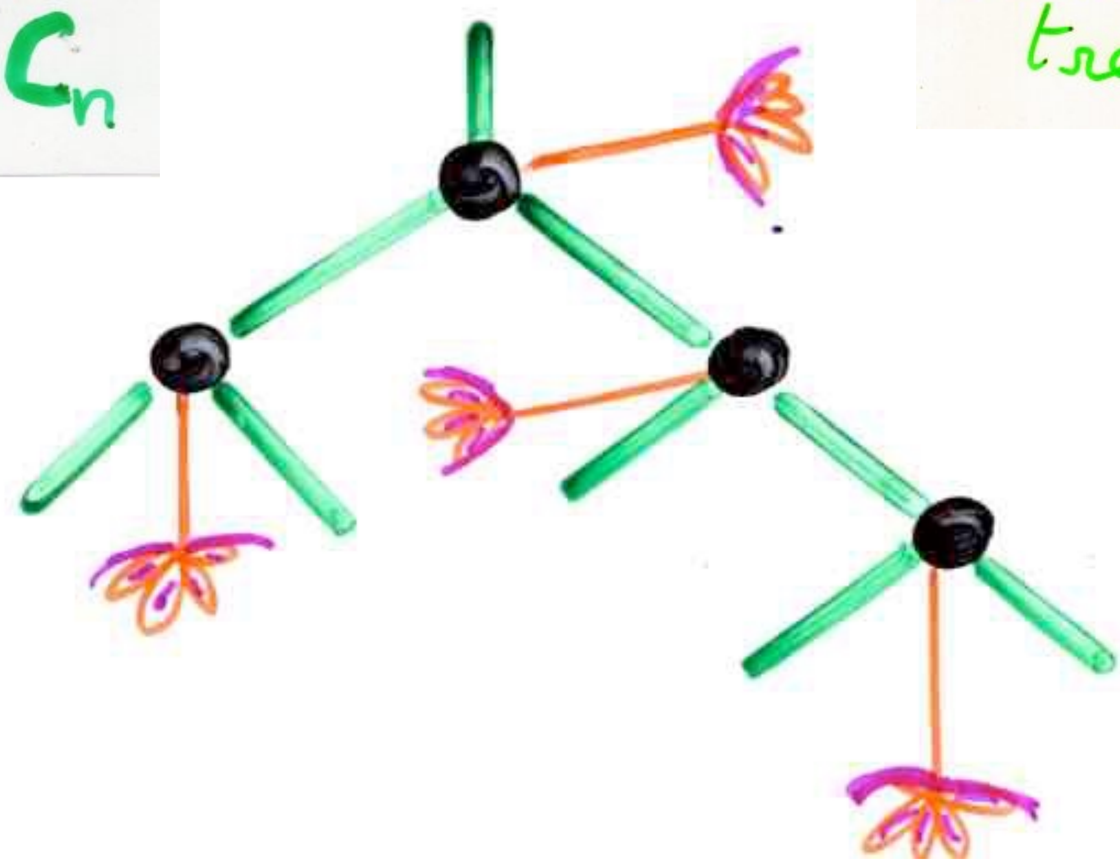
$n+2$ external edges

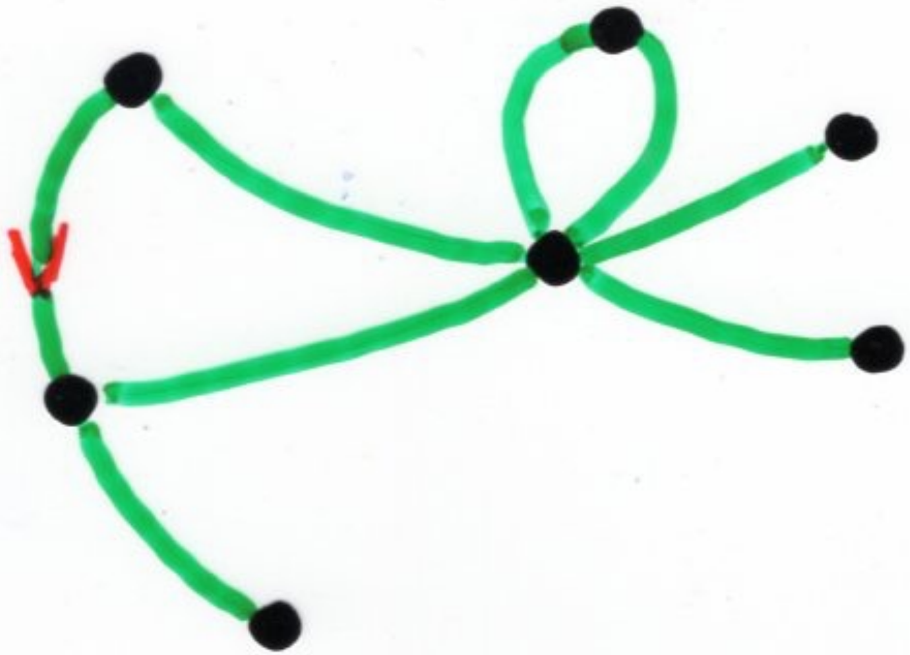
$$\frac{2 \times 3^n}{(n+2)} C_n$$

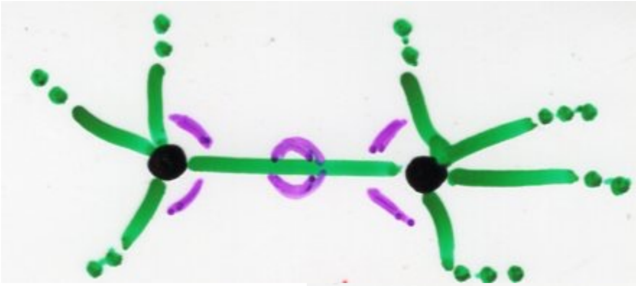
Catalan number

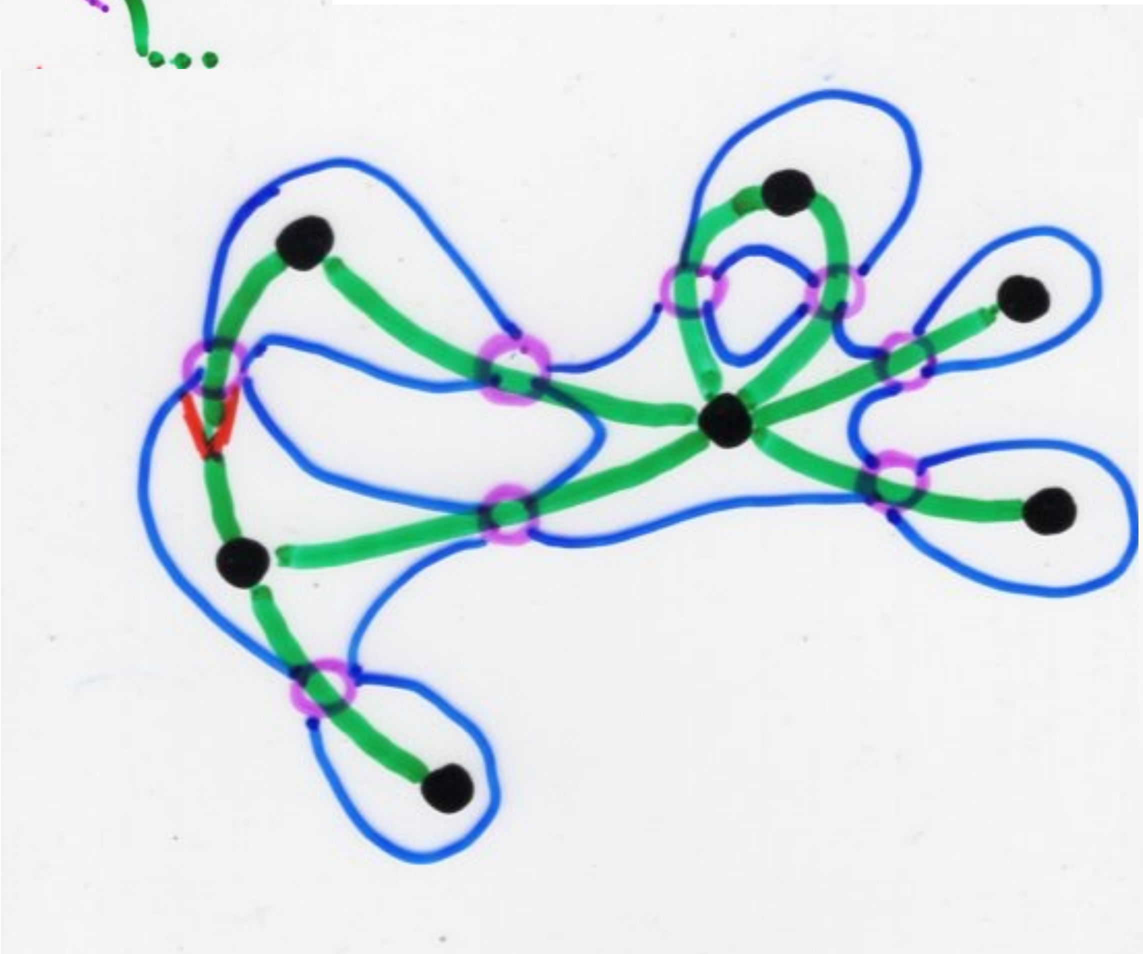
$3^n C_n$

blossoming trees





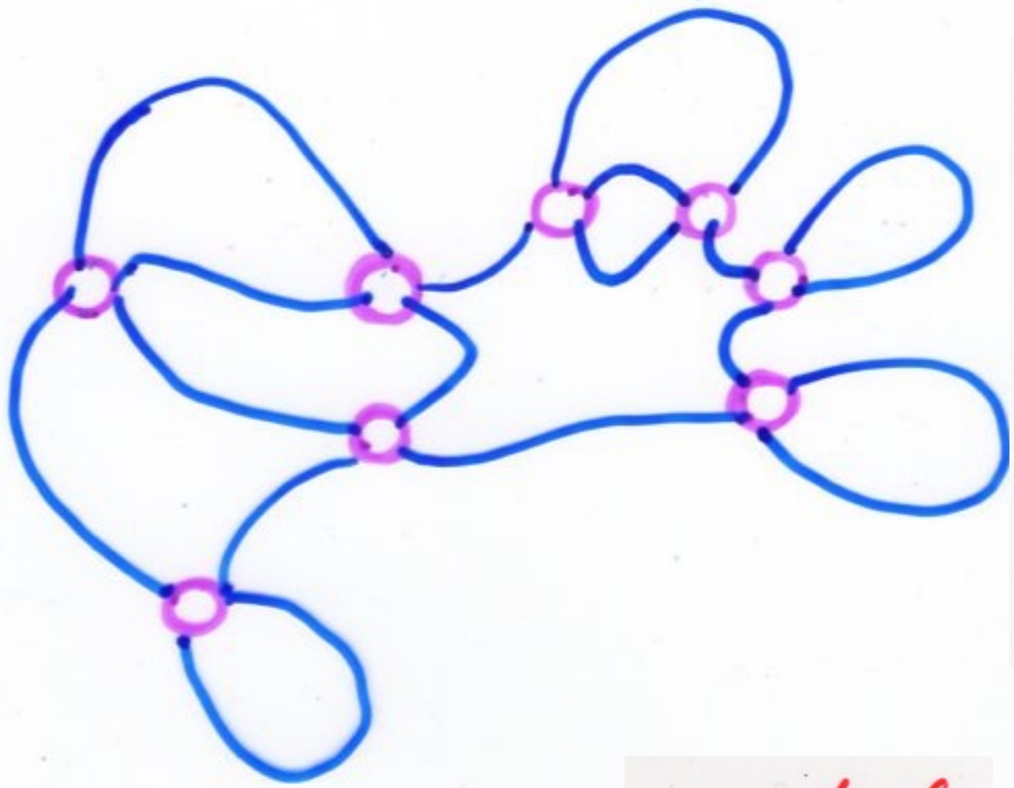




Planar map
n edges

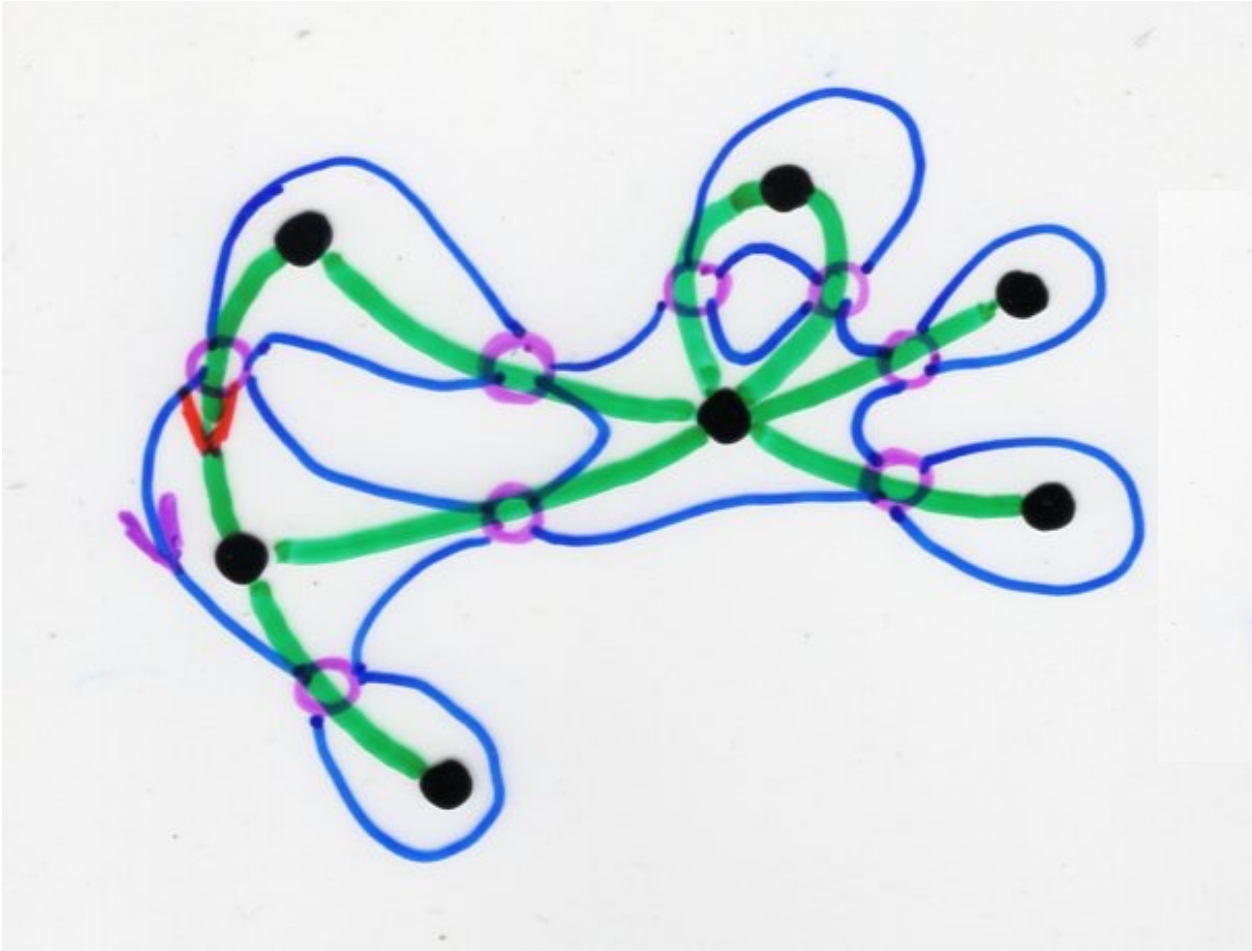
$\updownarrow$

quartic
planar map
n vertices



Planar map
n edges
$\updownarrow$
quartic
planar map
n vertices

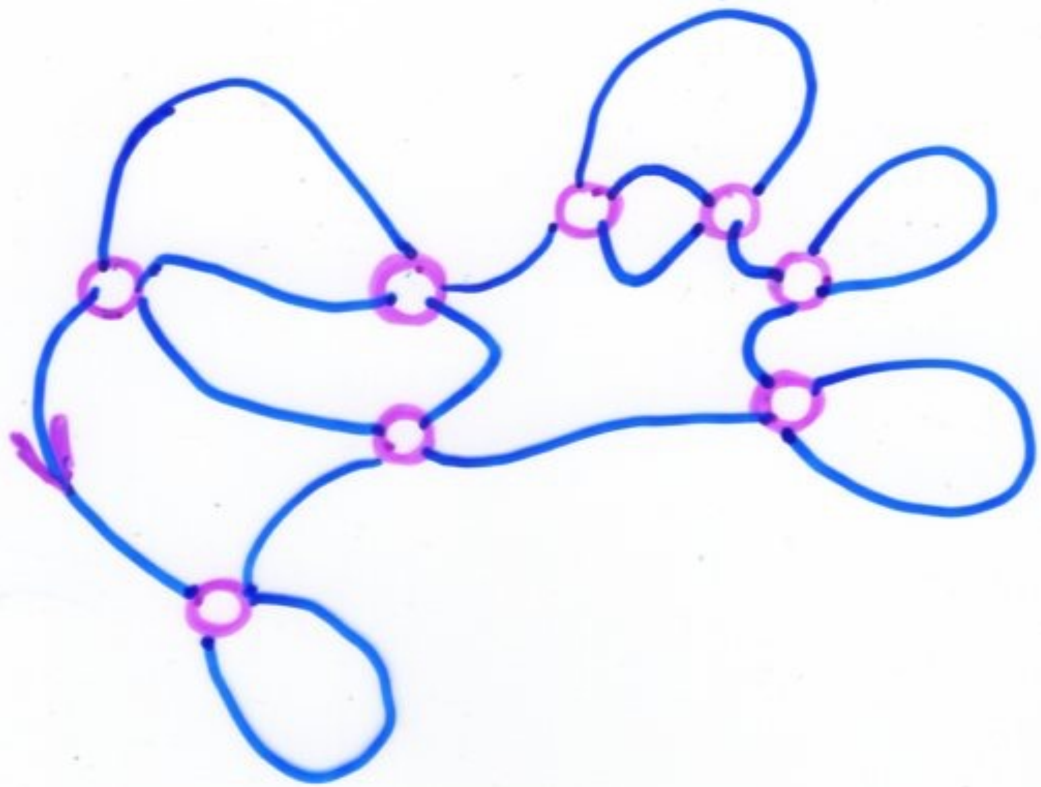
radial
map



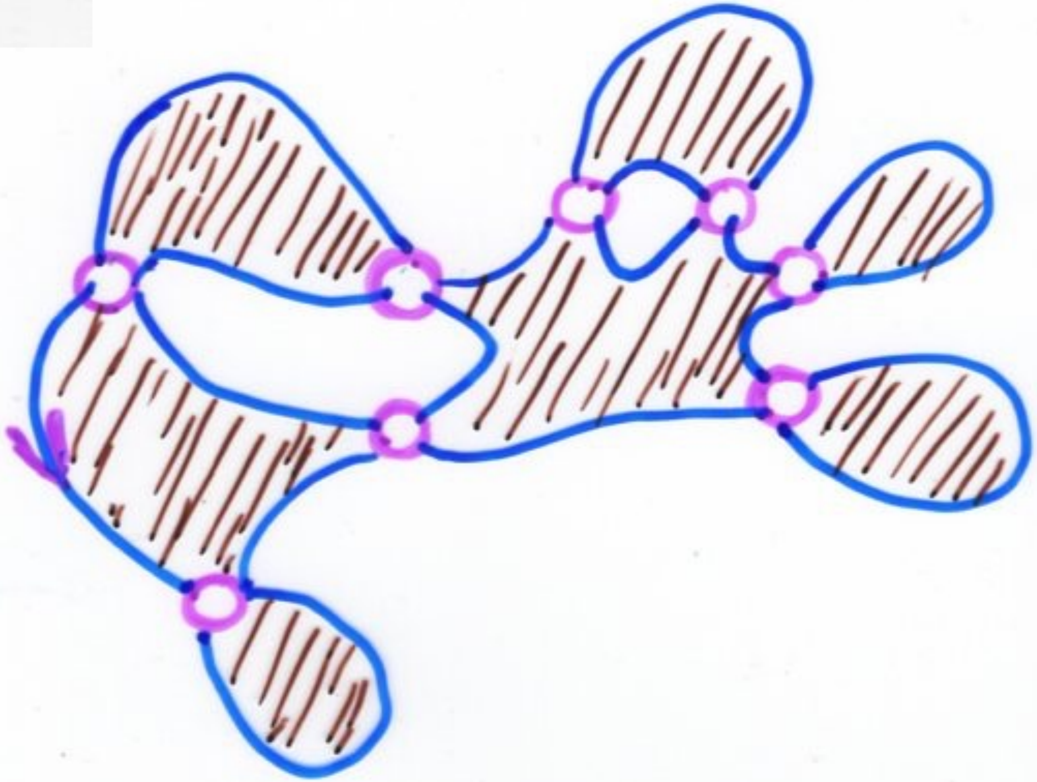
rooted
Planar map
n edges

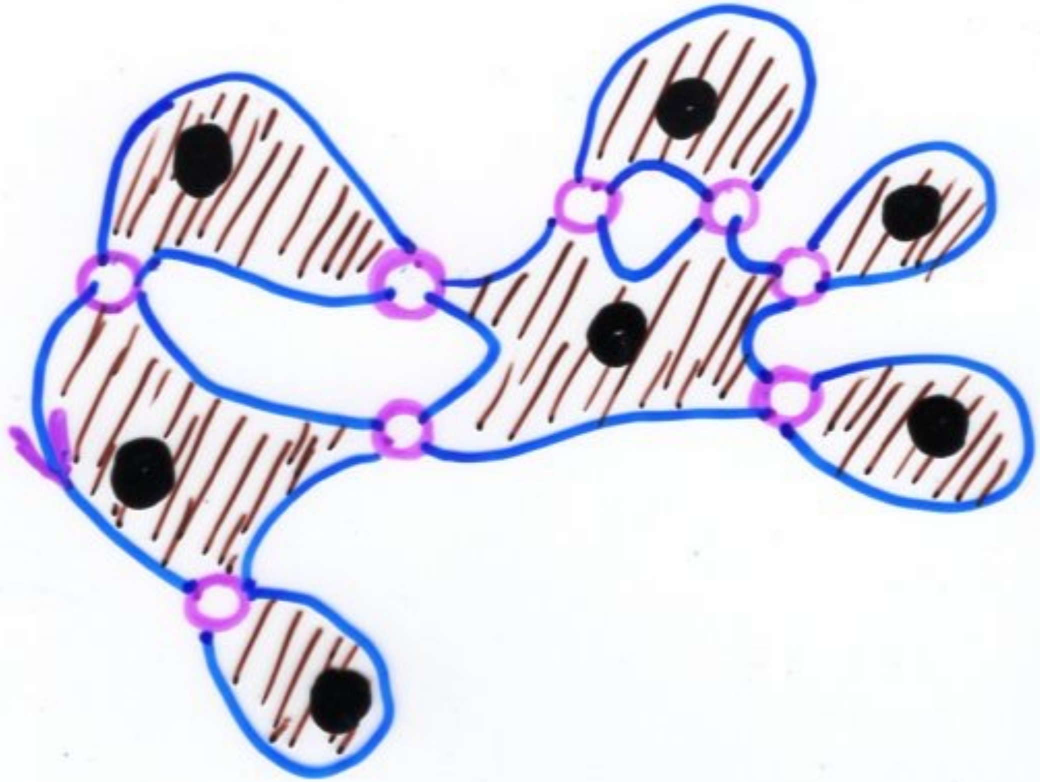
$\updownarrow$

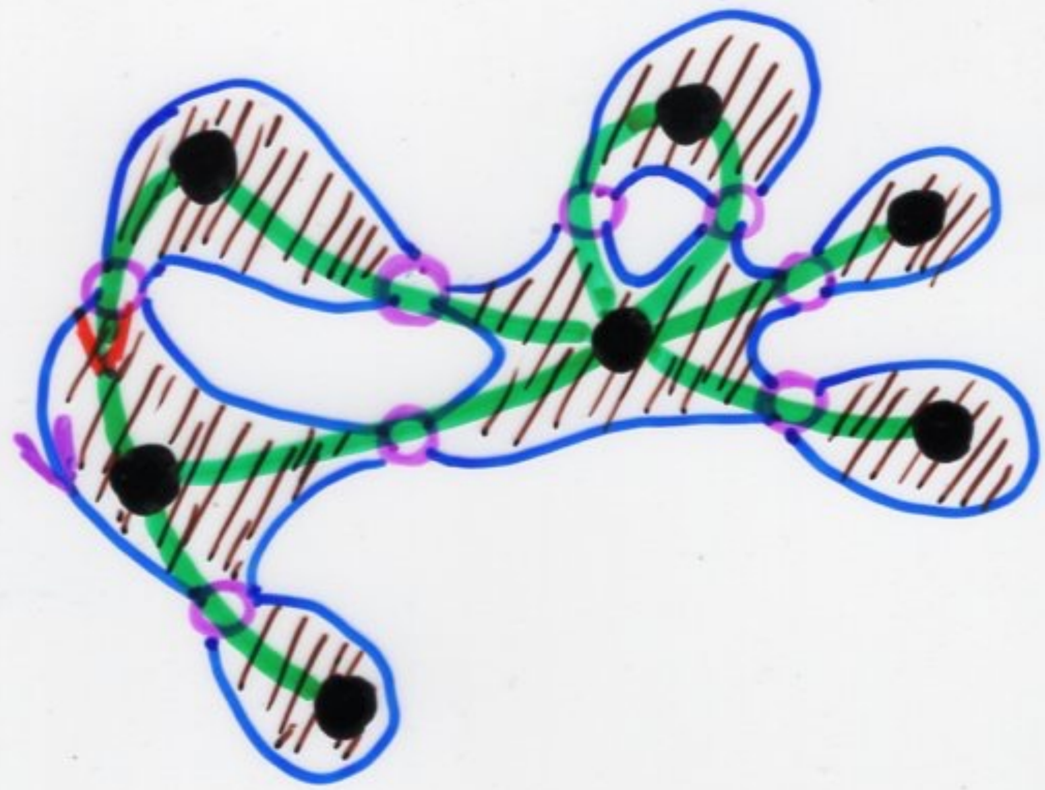
rooted
quartic
planar map
n vertices



reverse
bijection





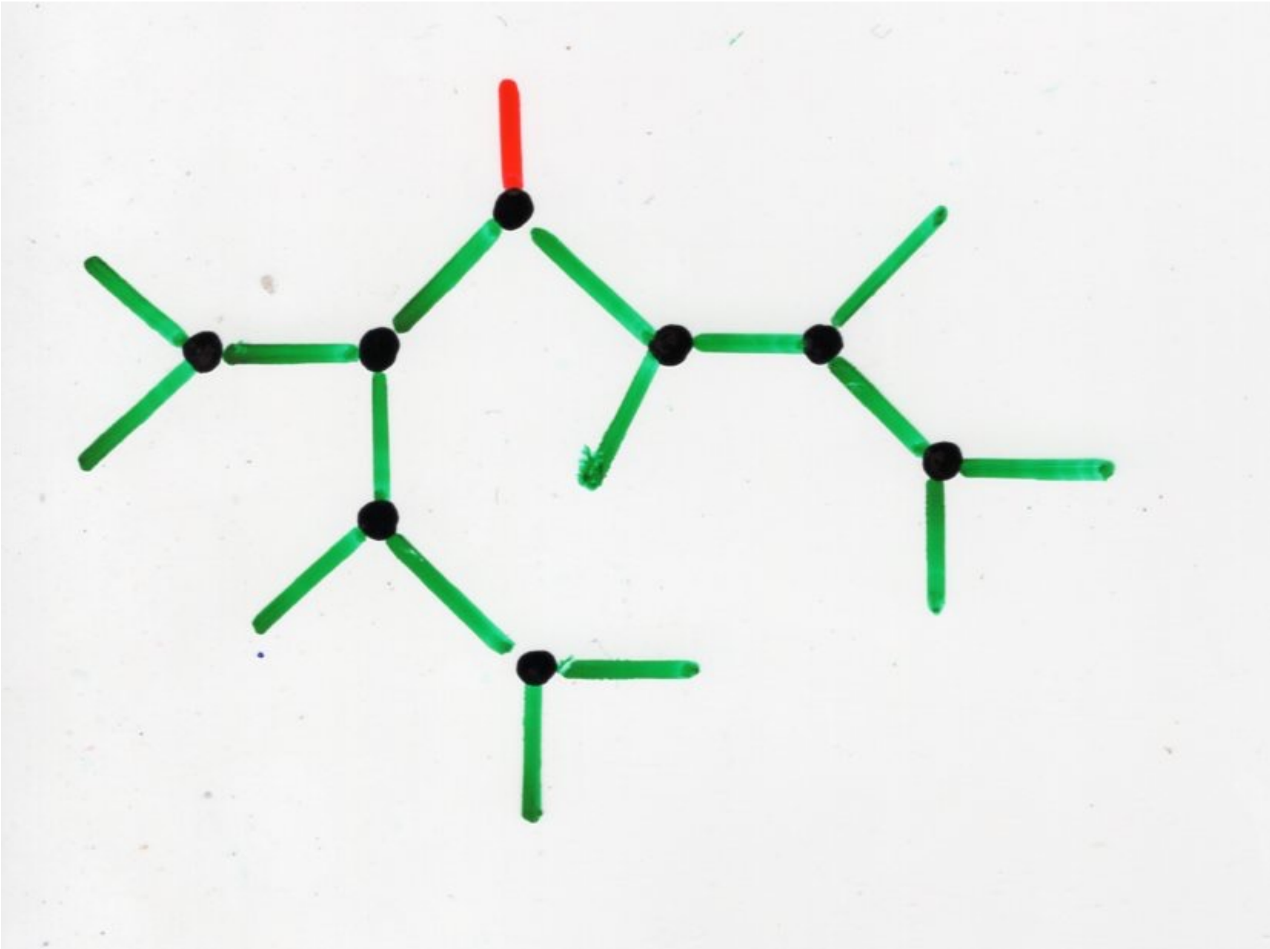


Schaeffer's bijection

well balanced blossoming trees

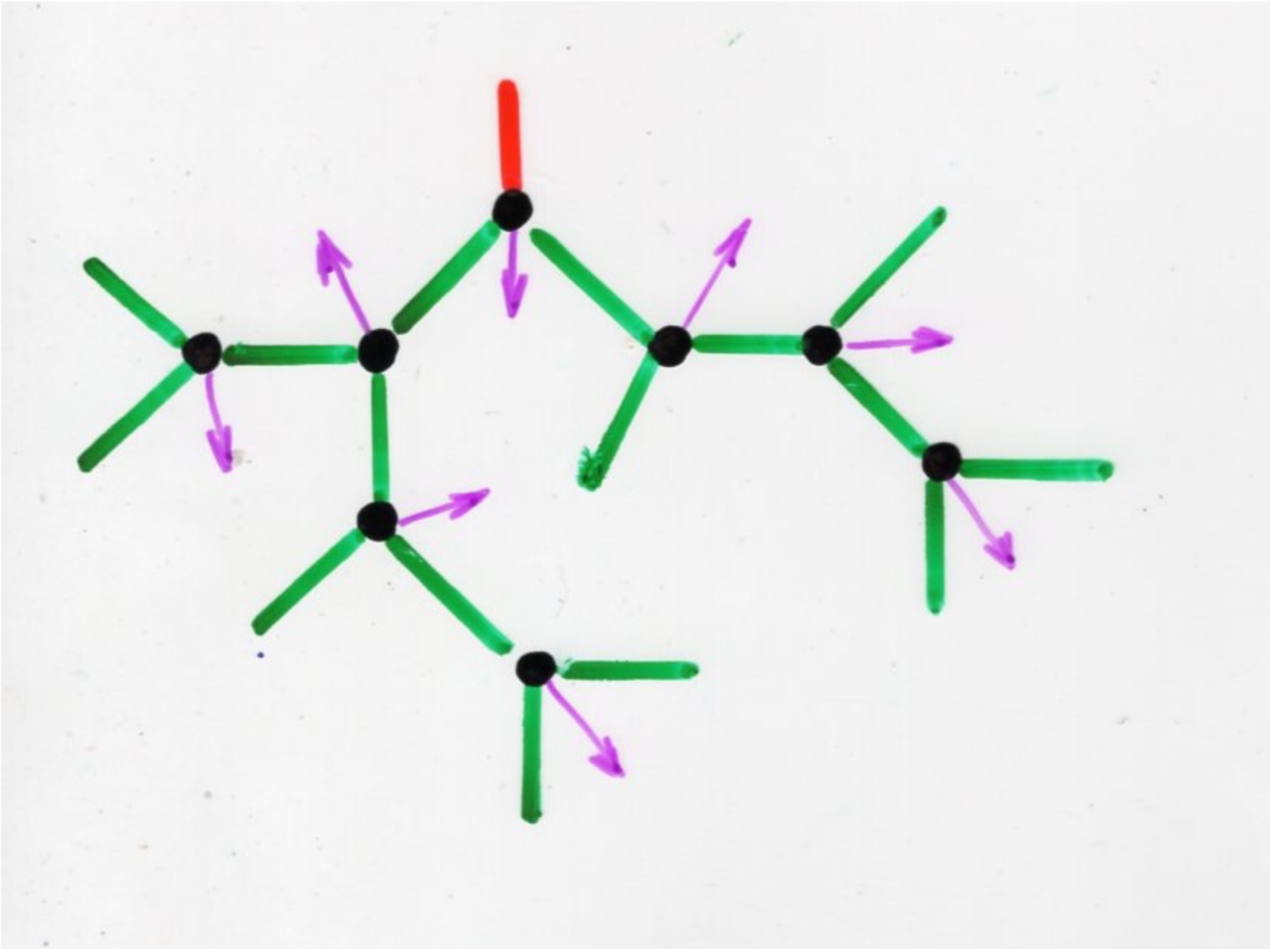
$$\downarrow$$

quartic rooted planar maps

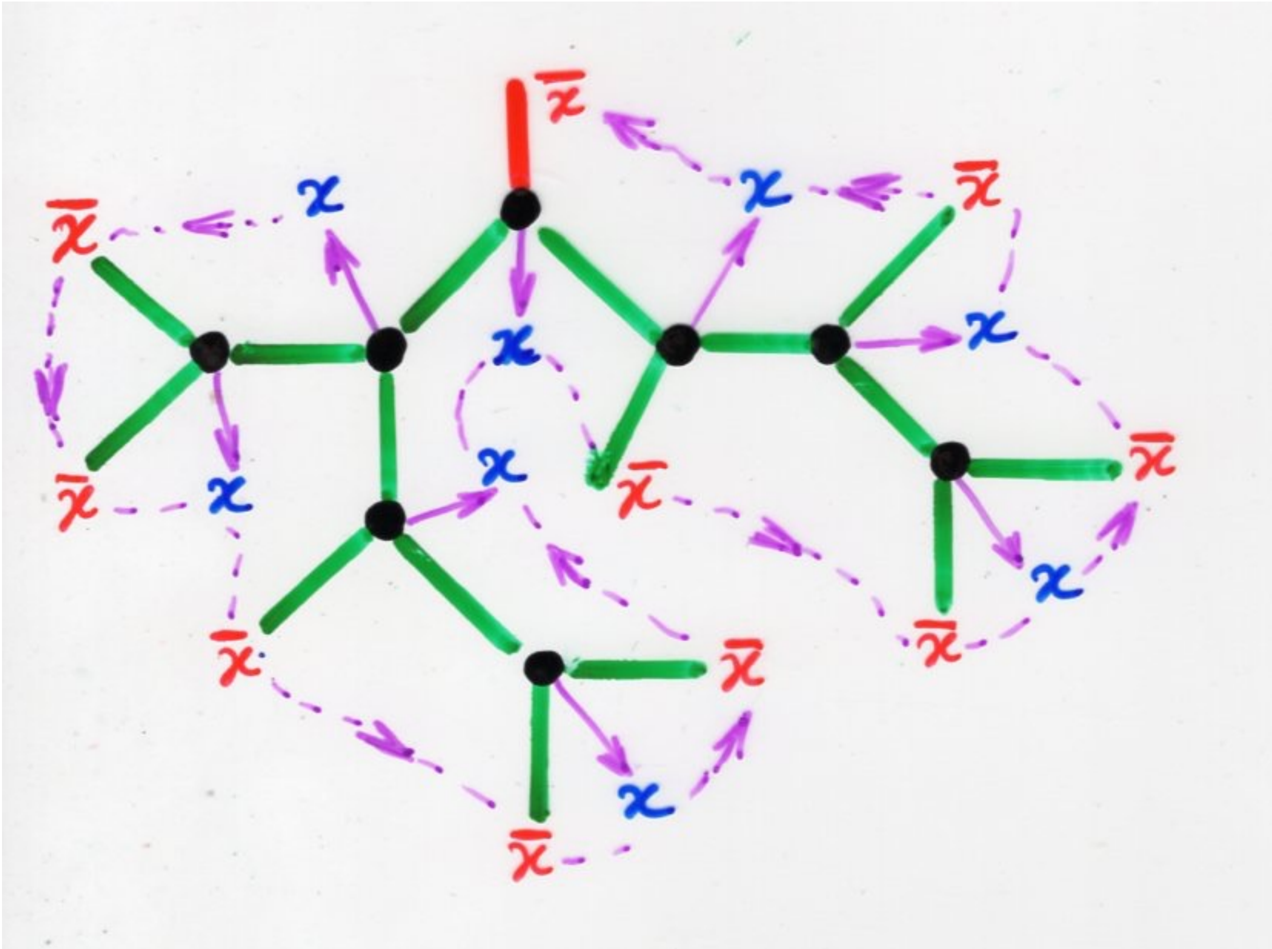


$3^n C_n$

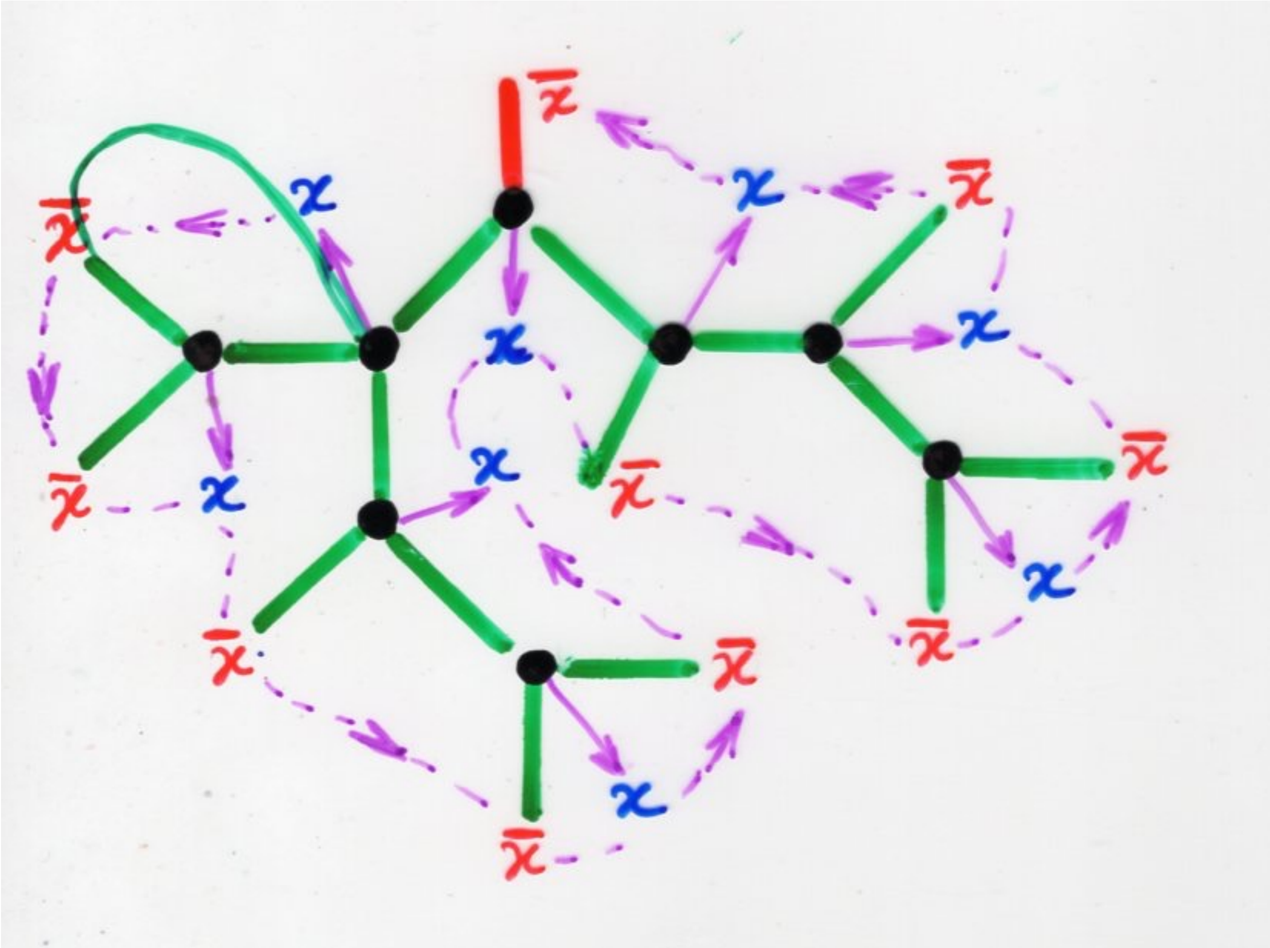
blossoming trees



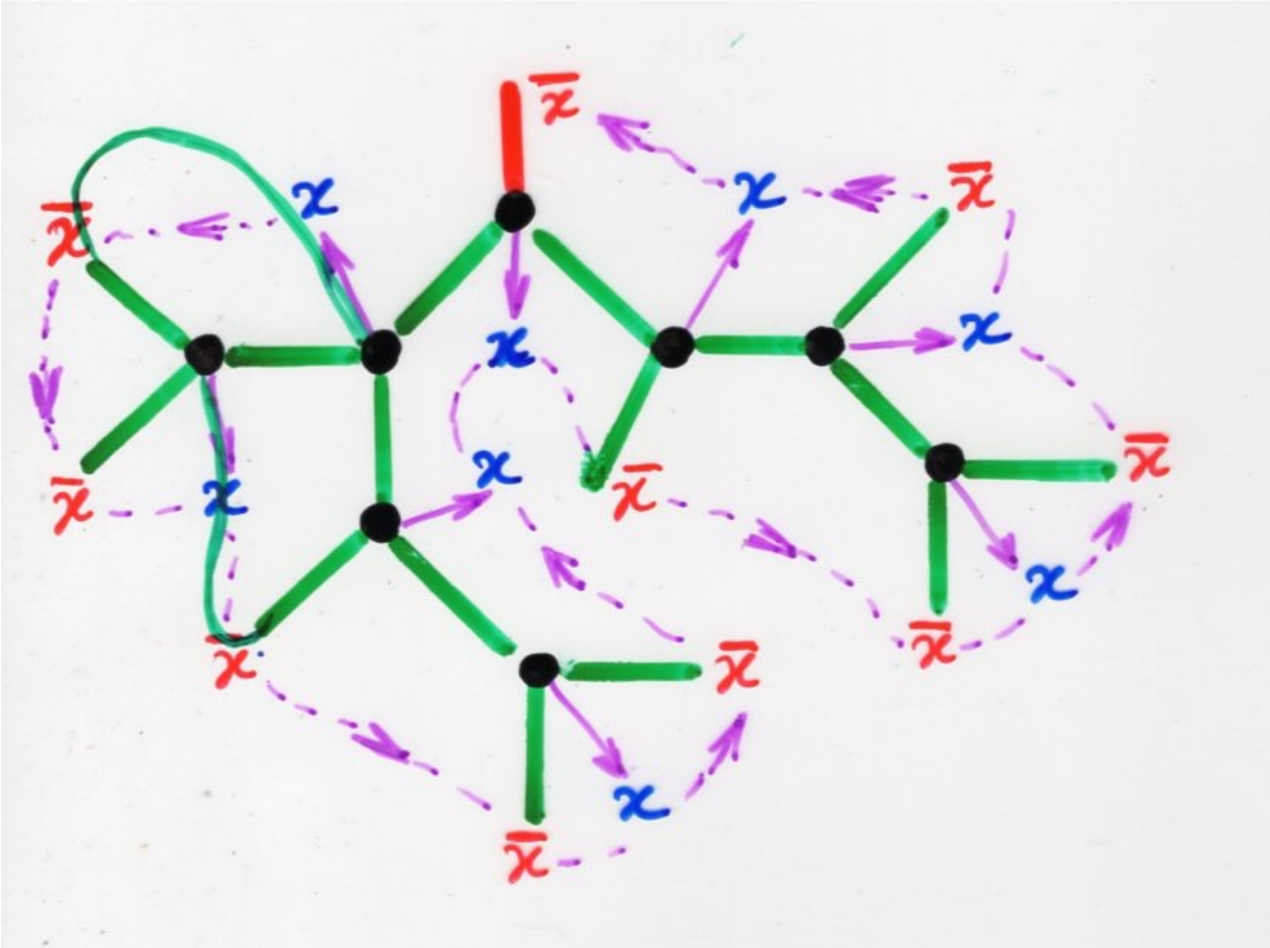
border word



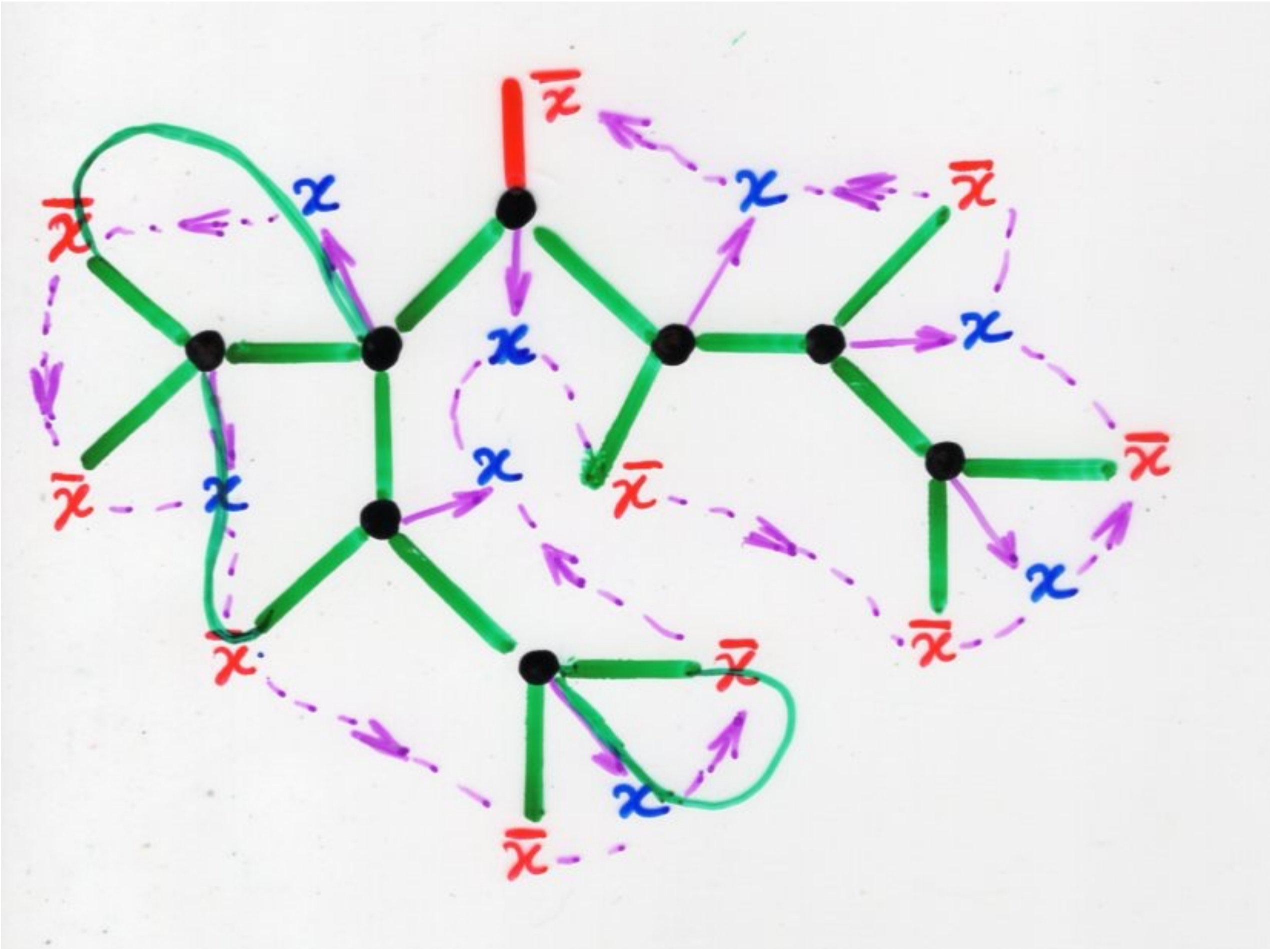
$$w = x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}$$



$\bar{x}$
x



$\bar{x}$
x


$\bar{x}$
x
$\bar{x}$
x
$\bar{x}$
x
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x
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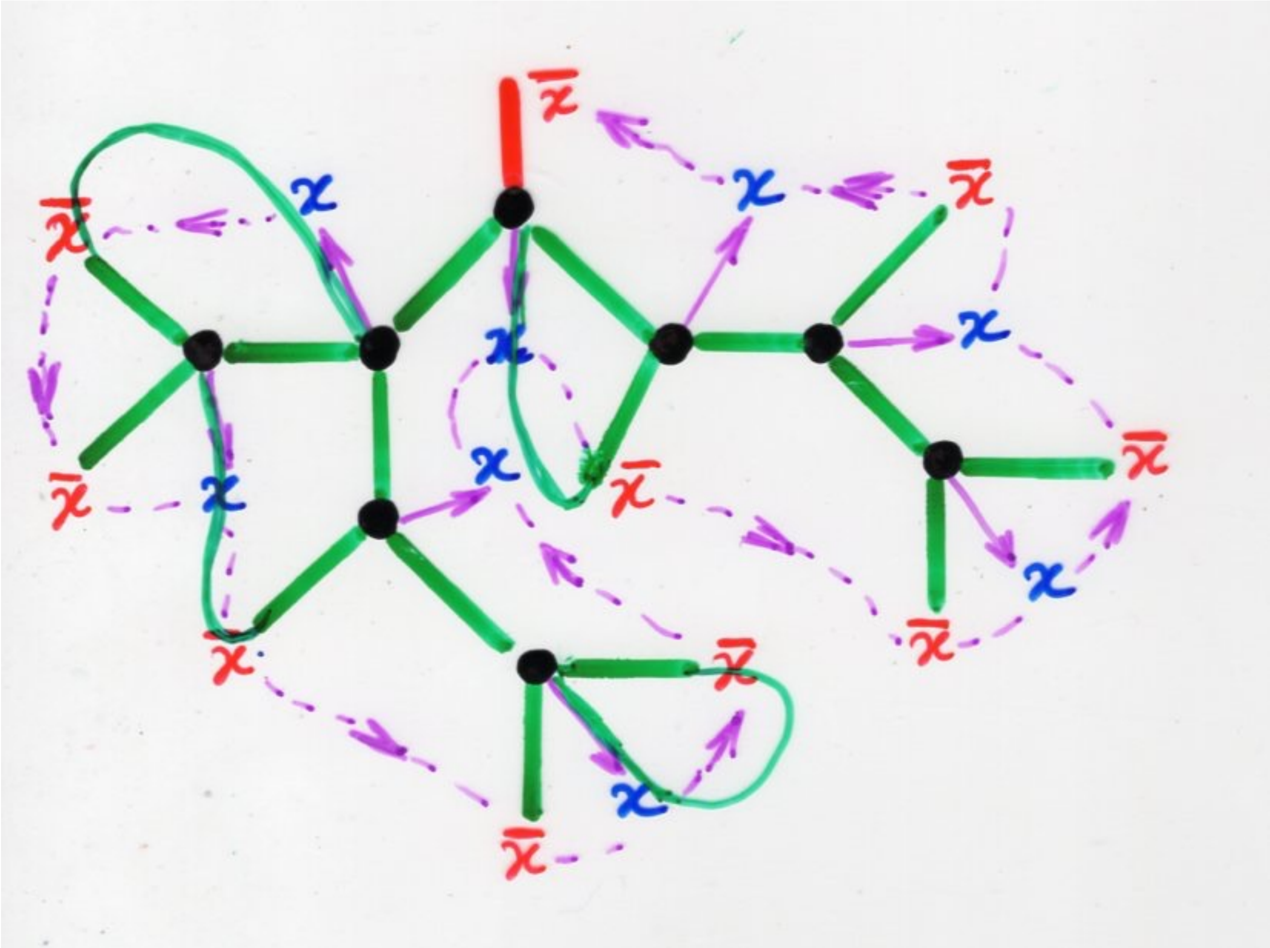
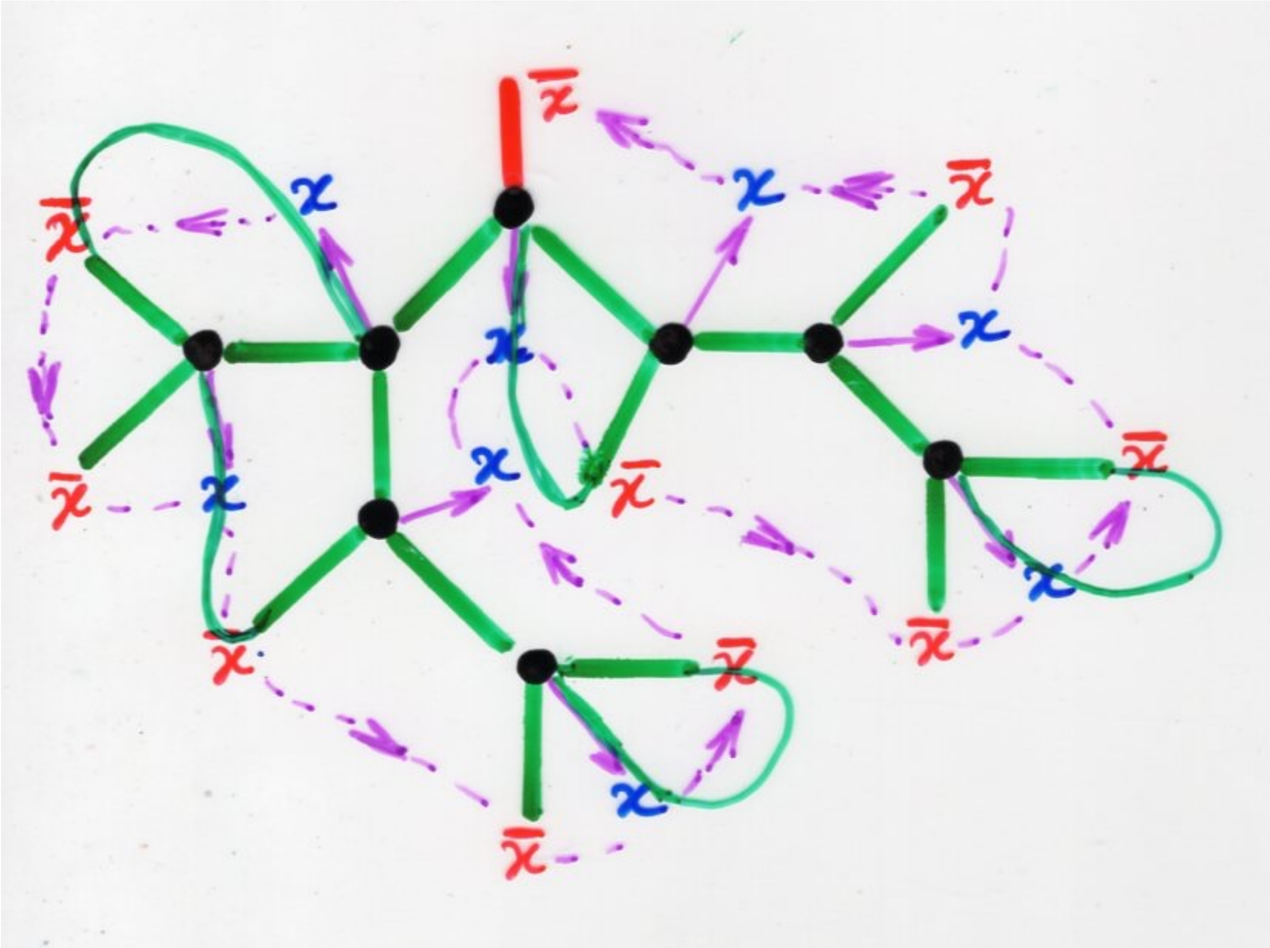
$\bar{x}$
$\bar{x}$
x
$\bar{x}$





$\bar{x}$
x
$\bar{x}$
x
$\bar{x}$
x
x
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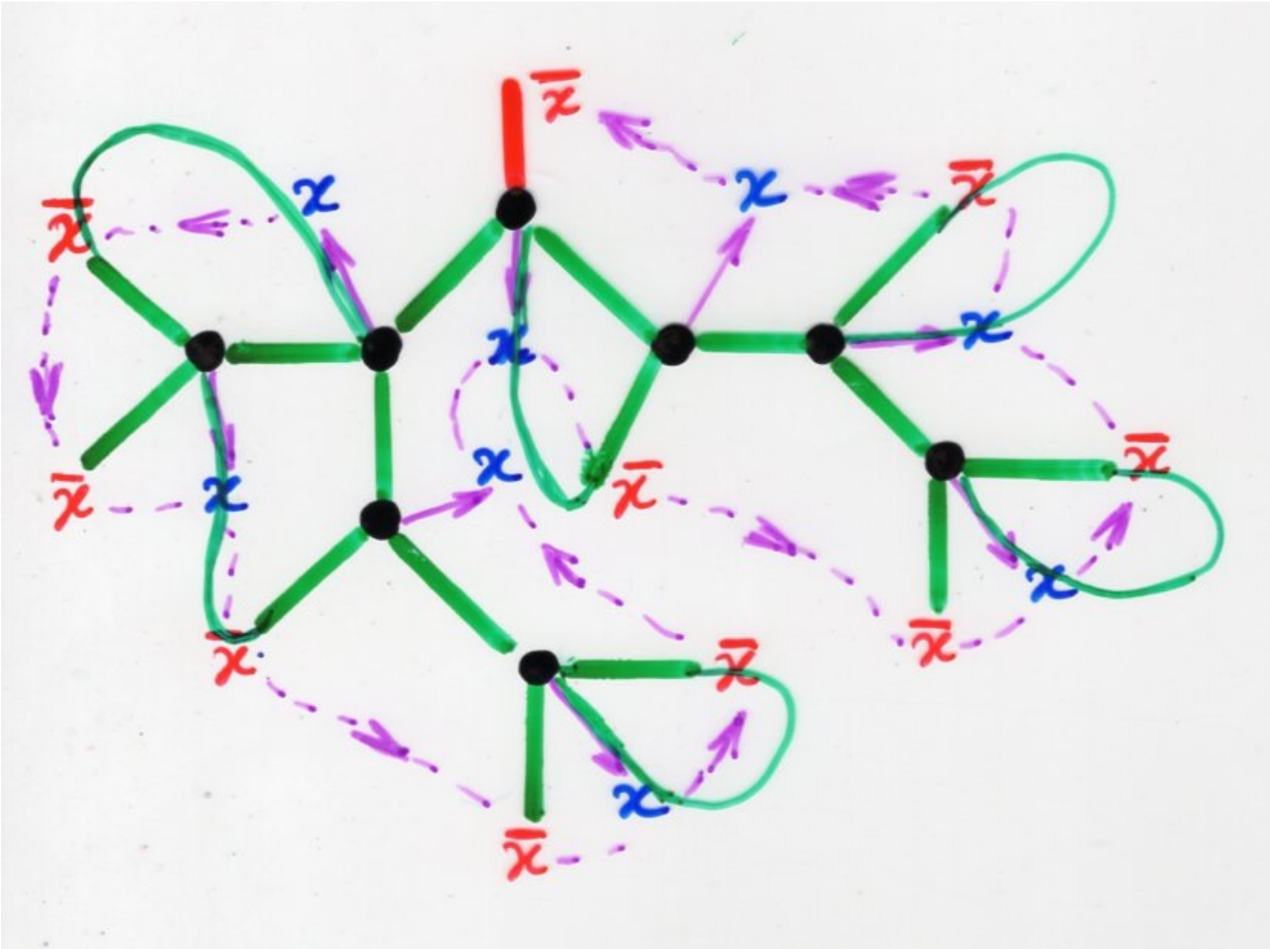
x
$\bar{x}$



$\bar{x}$
$\bar{x}$
x
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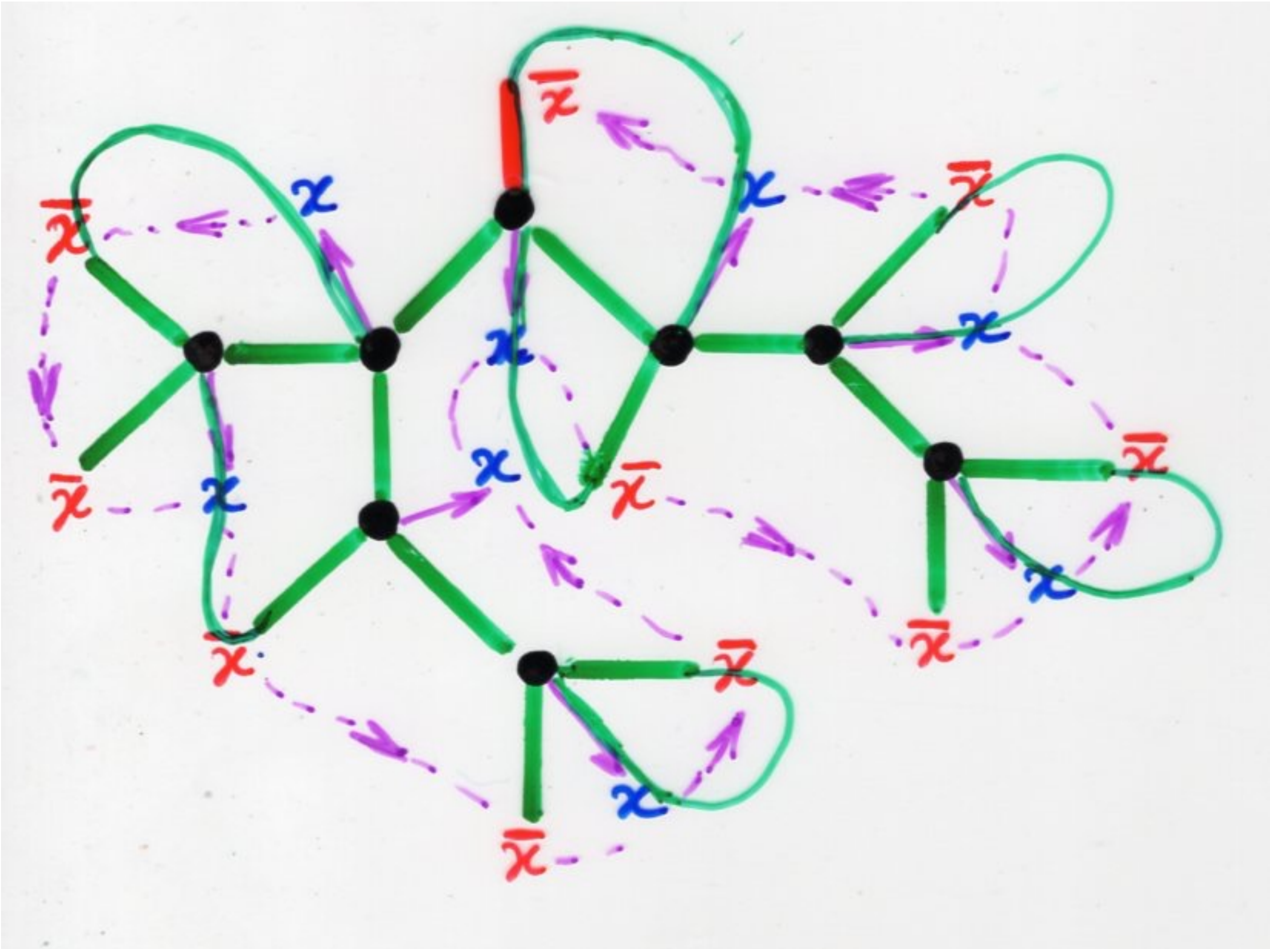
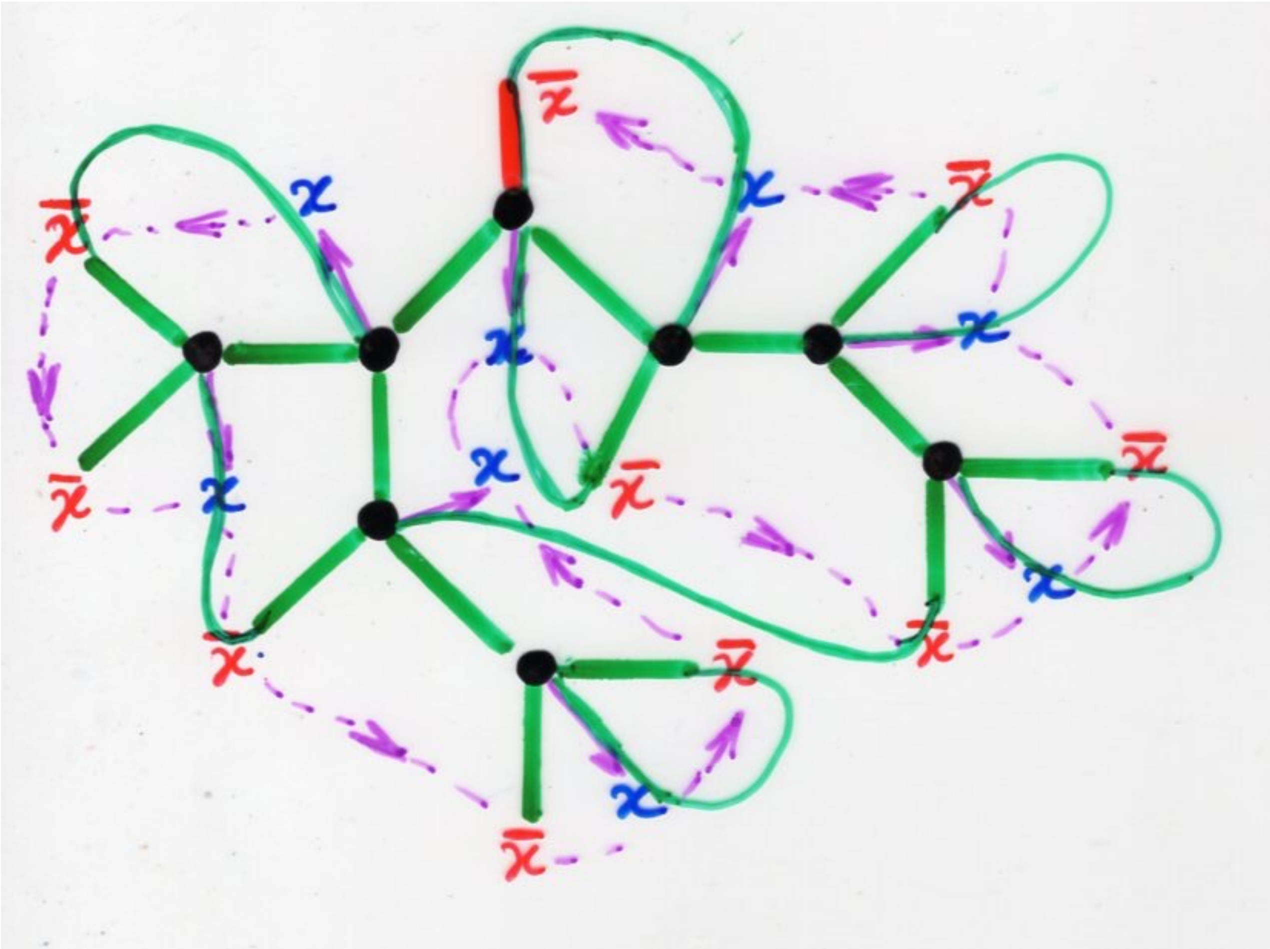
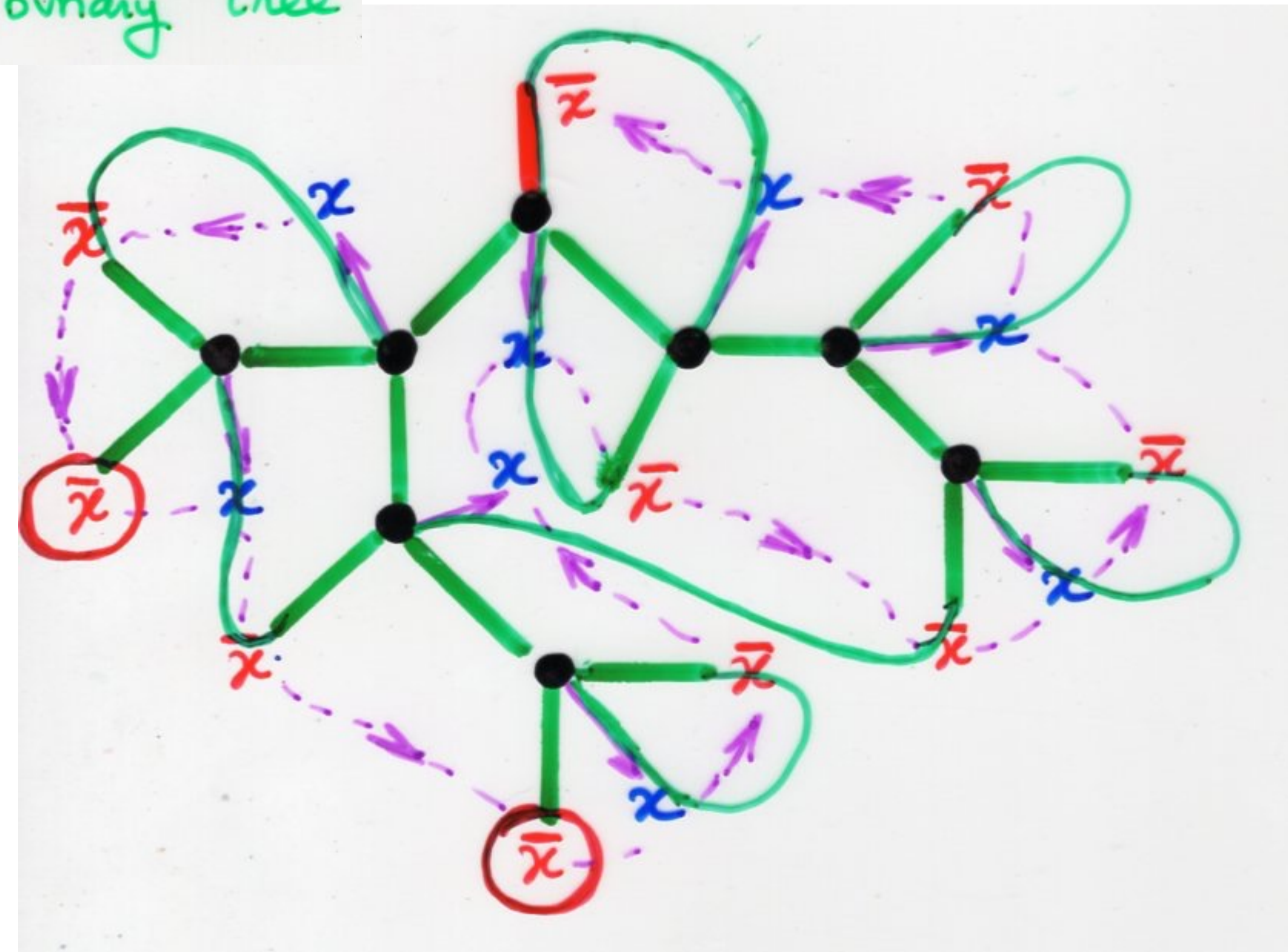
x
$\bar{x}$


x
x̄



x
$\bar{x}$

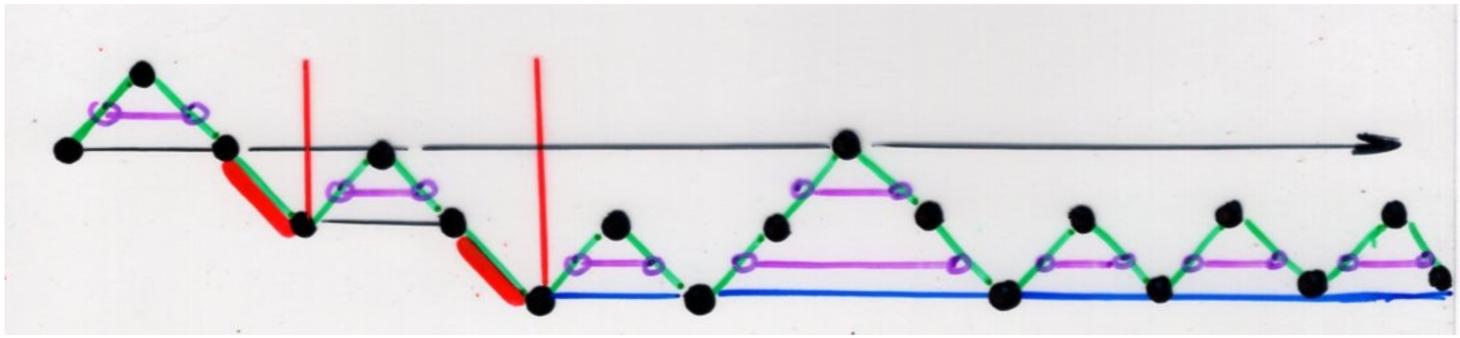
partial closure of the blossoming (complete) binary tree



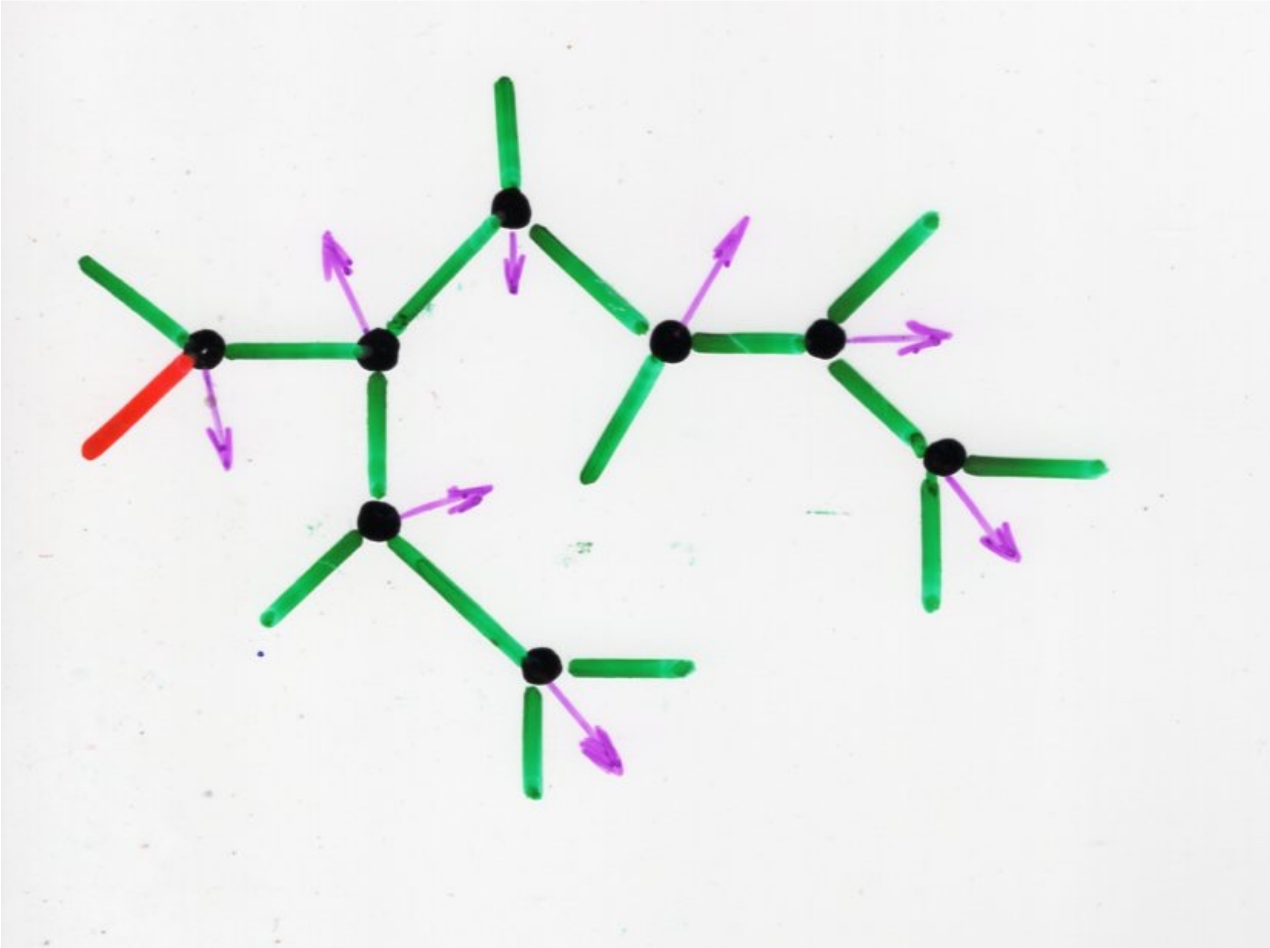
$$w = x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}$$

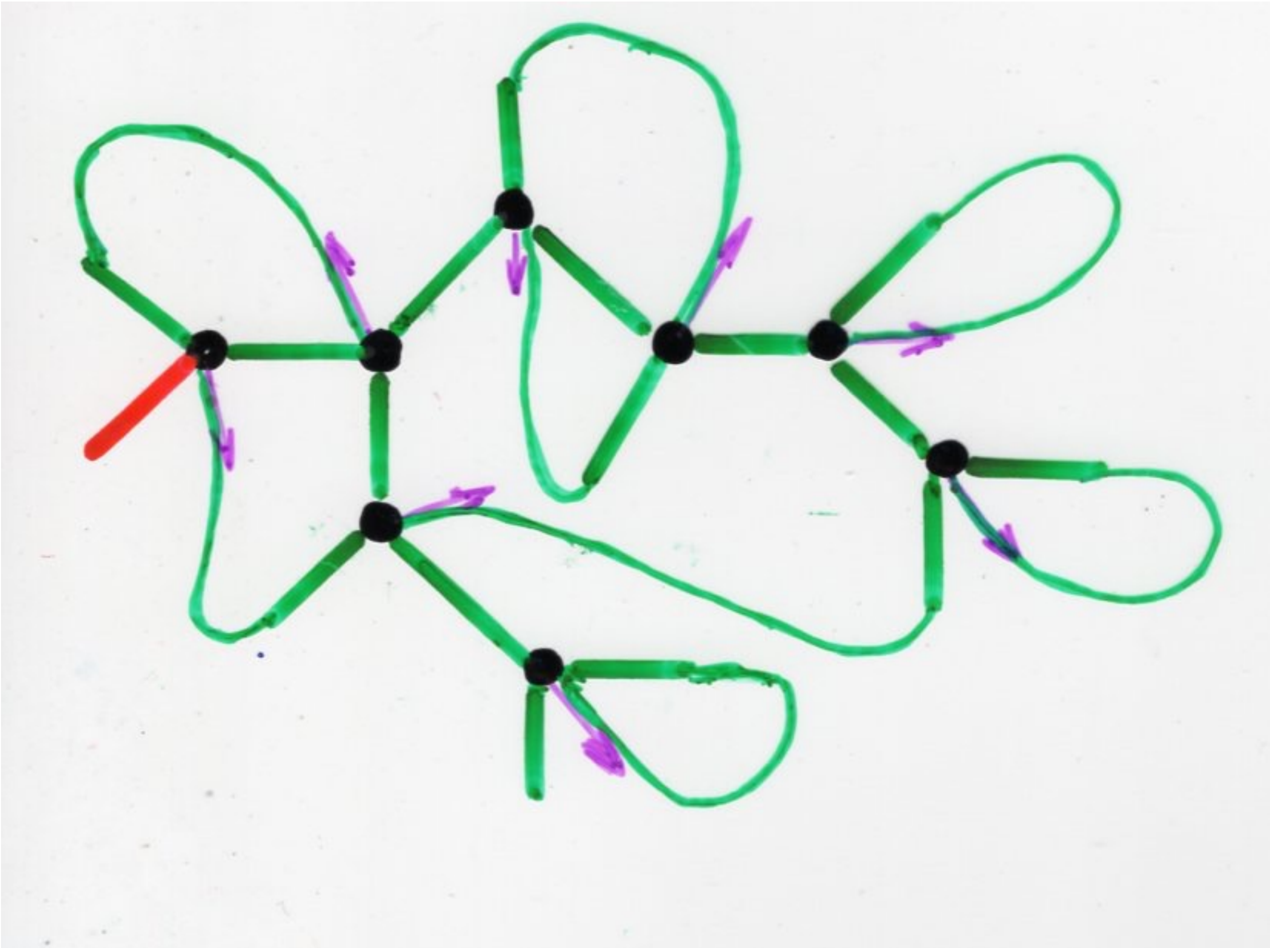
border word

$$w = x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}$$

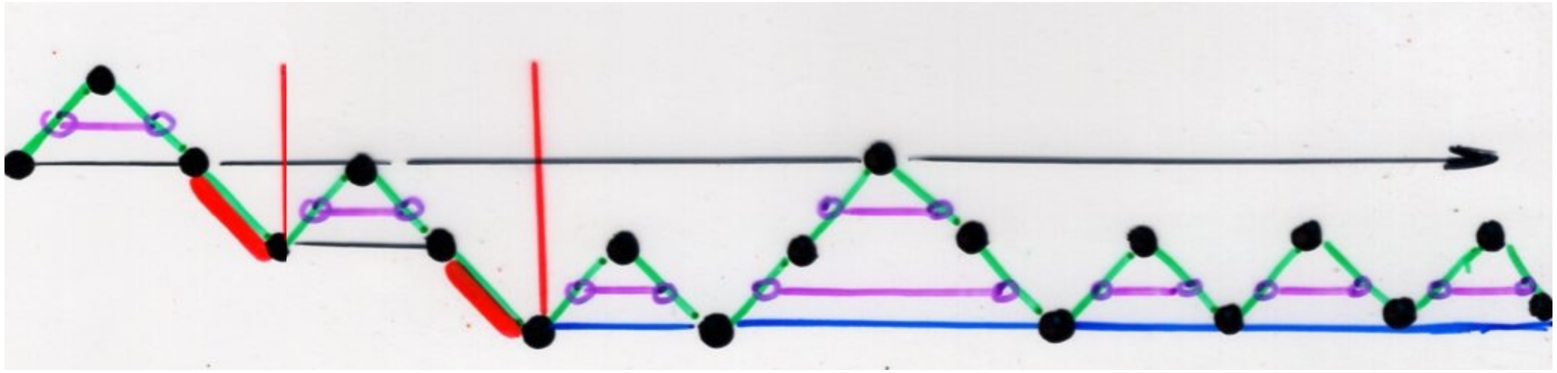


conjugate trees : changing the external root edge

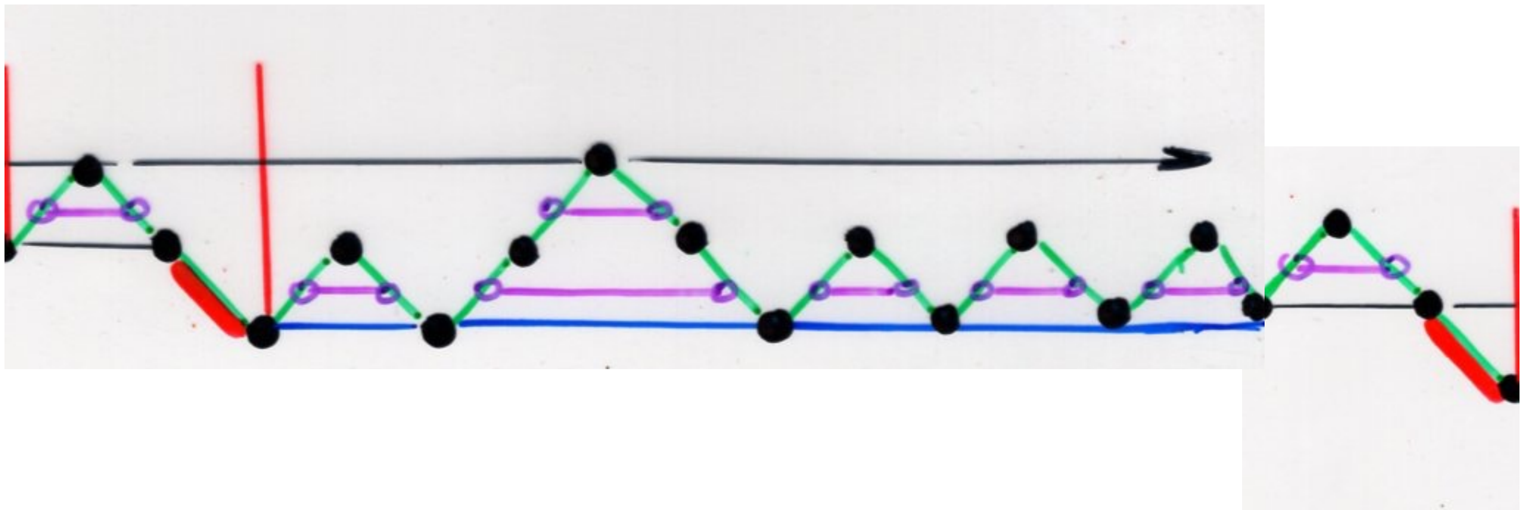




The border words of two conjugate blossoming trees are conjugate.



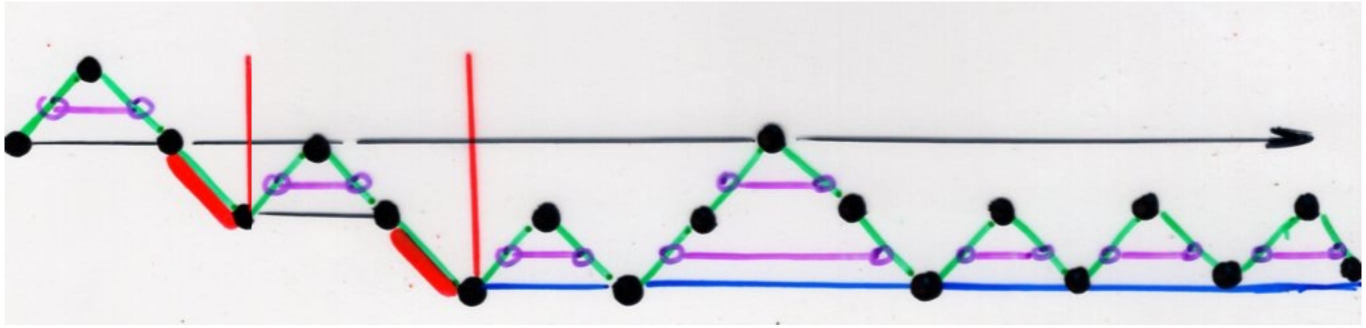
$$w = x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}$$



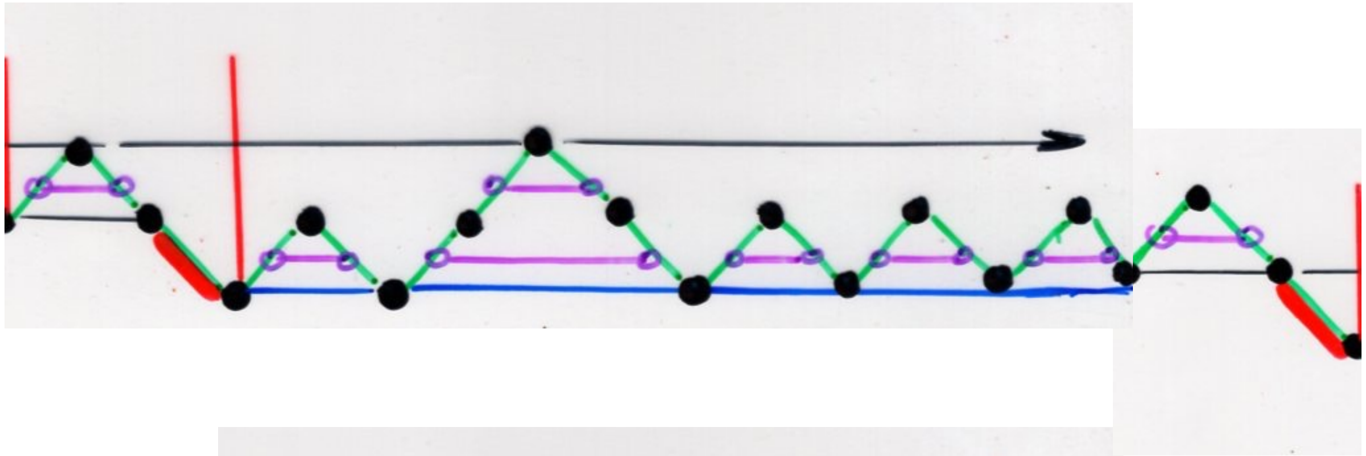
Definition A blossoming tree is well balanced iff its border word is a product of Lukacievicz words (i.e. iff its root edge is free after the partial closure)

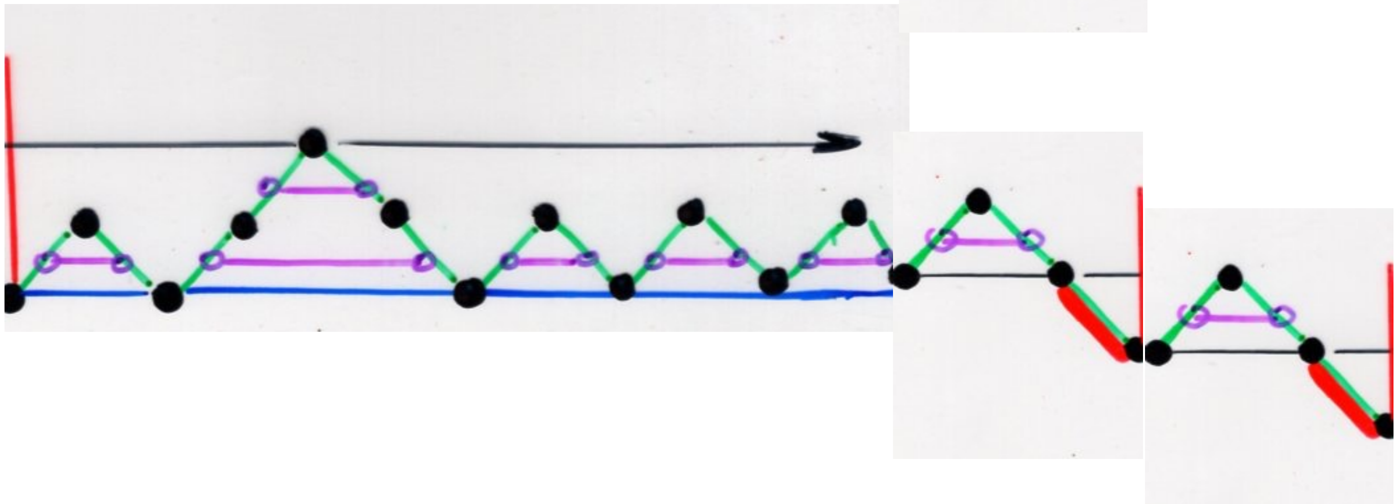
Proposition The number of well balanced blossoming trees with **n** vertices is :

$$\frac{2}{(n+2)} \cdot 3^n \cdot \frac{1}{(n+1)} \binom{2n}{n}$$

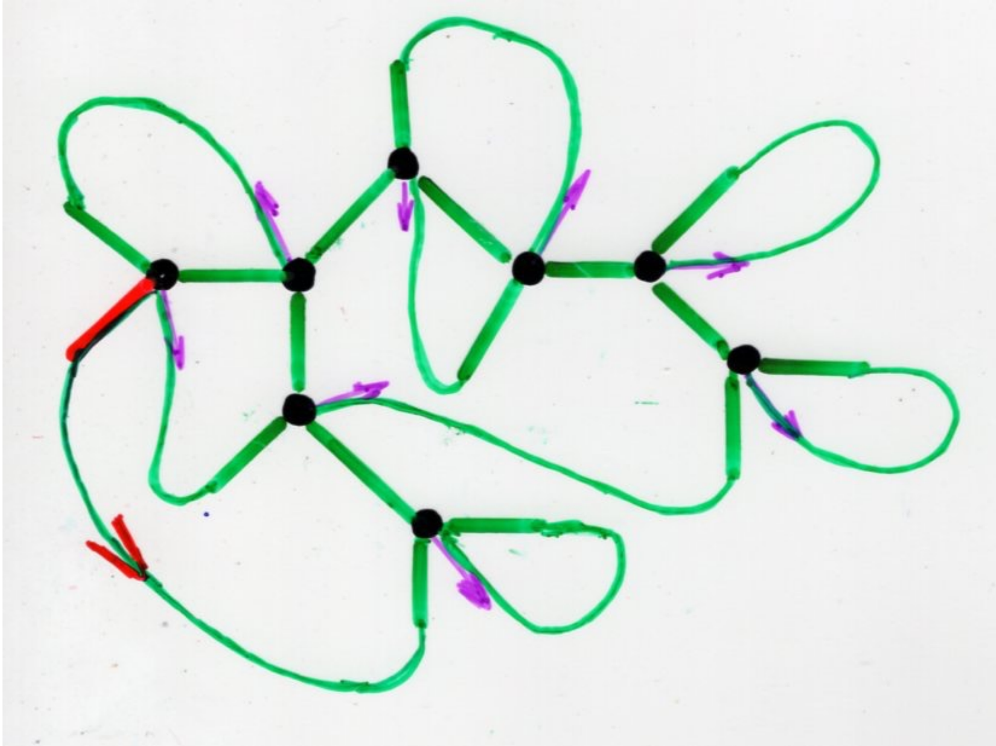


$$w = x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}$$

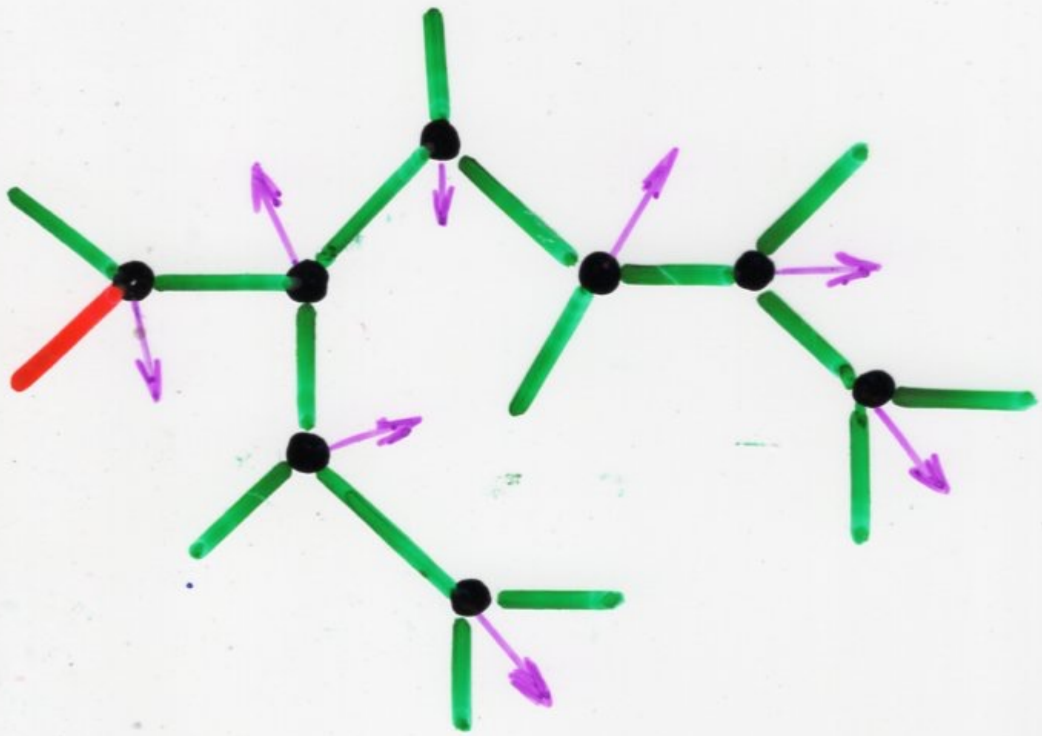


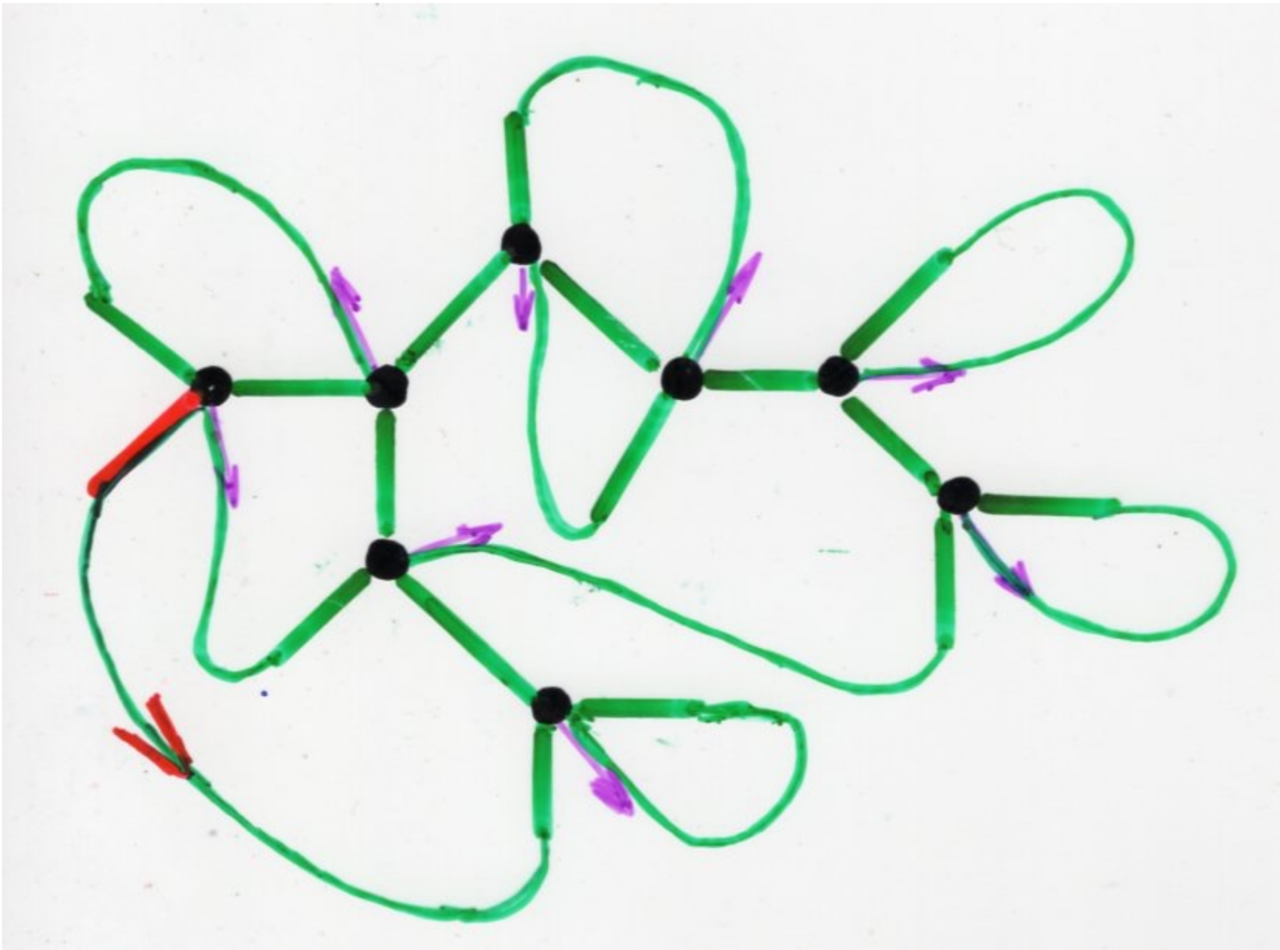


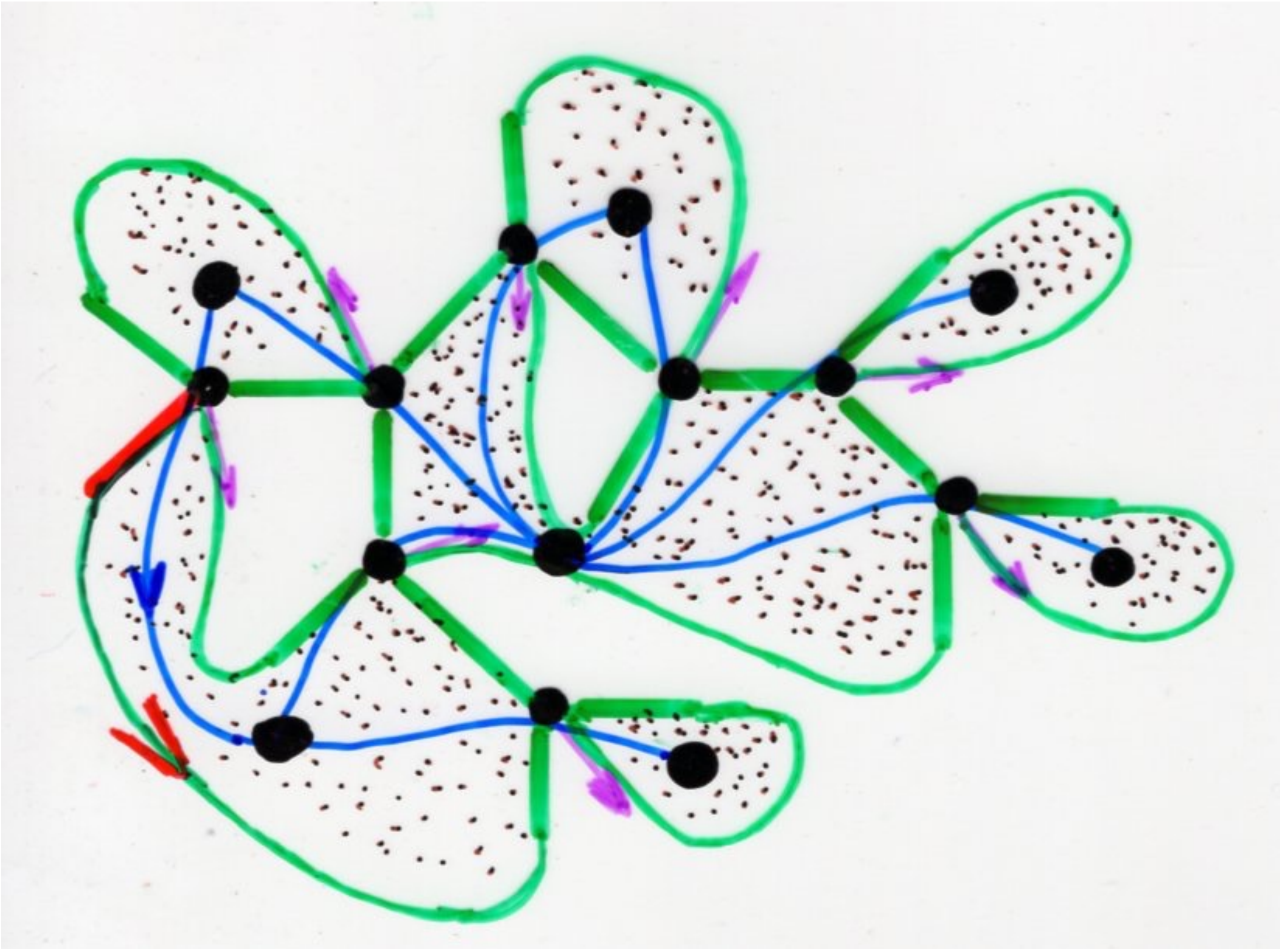
complete closure
of a well-balanced
blossoming tree

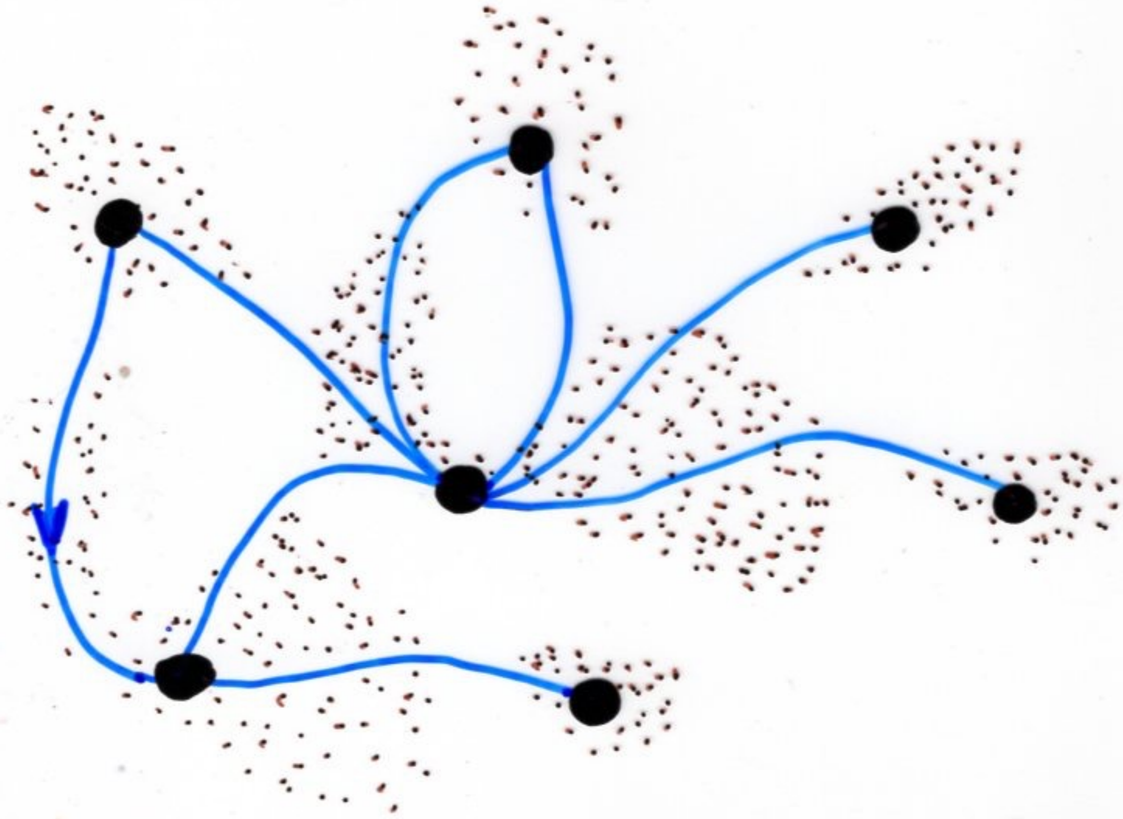


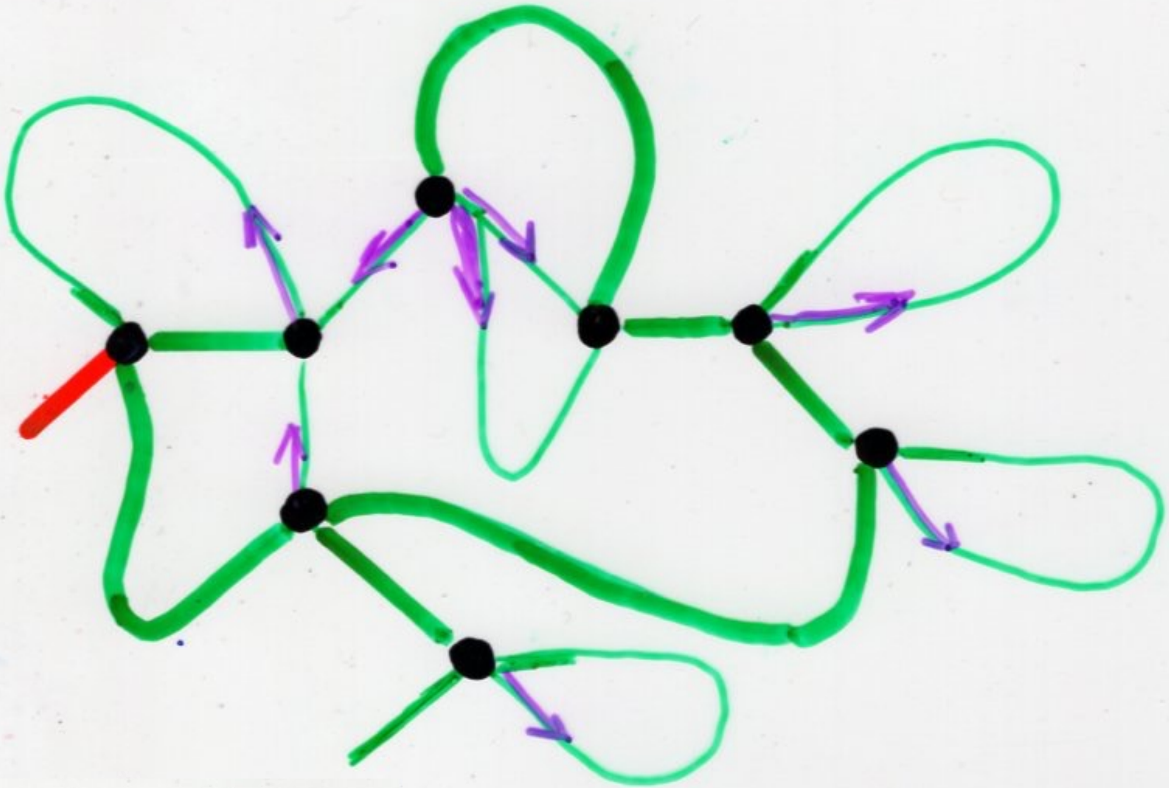
Proposition This operation is a bijection from well balanced blossoming trees (n vertices) to rooted quartic planar maps with n vertices, and thus to rooted planar maps with n edges.



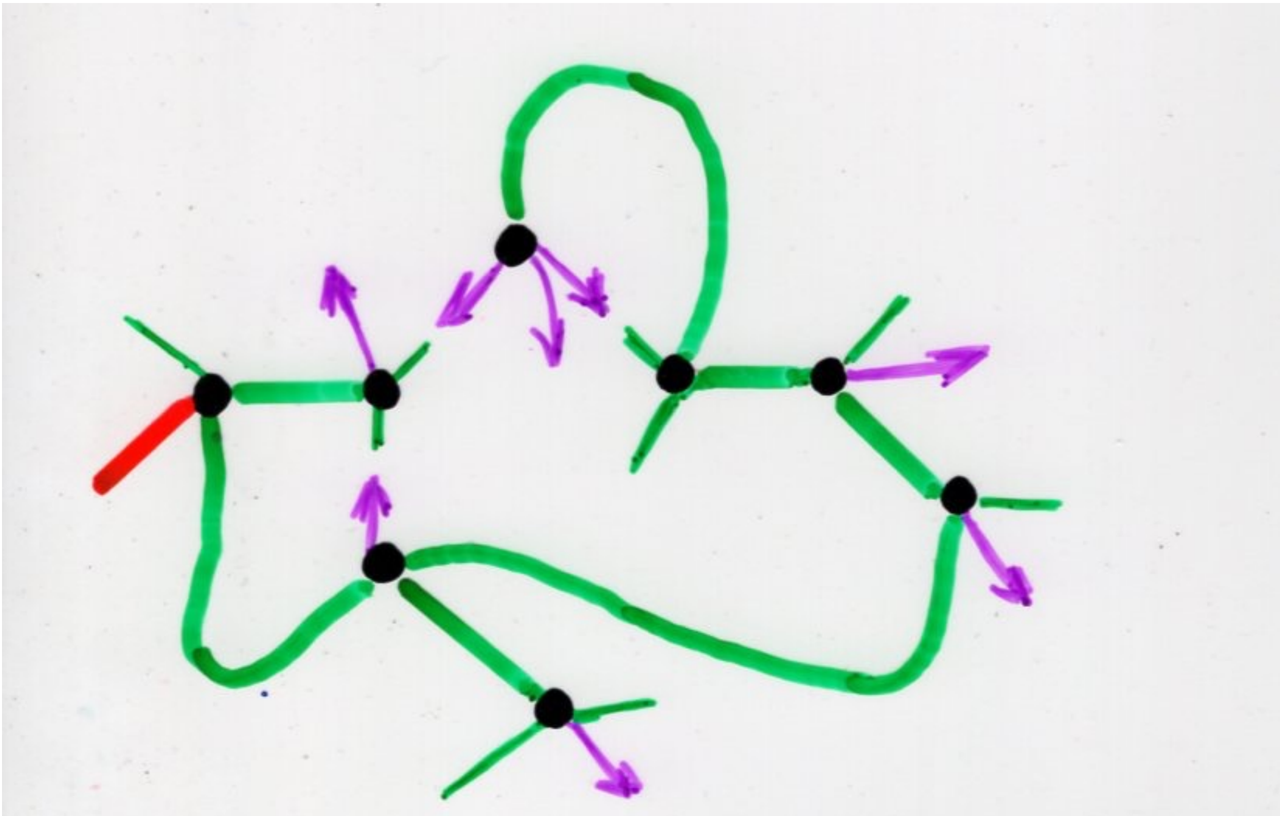








another spanning tree
which closure gives back
the same quartic map
but not blossoming !



another spanning tree
which closure gives back
the same quartic map
but not blossoming !

reverse
bijection

only one
blossoming
spanning tree
will give back
the quartic map

Cori, Vauquelin (1970)

Schaeffer (1997)

Bouttier, Di Francesco, Guitter (2002)

. . . many others

quantum
gravity

introduction to q-analogues

with q-Catalan

$$[i]_q = 1+q+\cdots+q^{i-1} = \frac{1-q^i}{1-q}$$

$$[n!]_q = [1]_q \times [2]_q \times \cdots \times [n]_q$$

$$= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

$[n!]$ or $[n]!$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n!]}{[k!]\,[(n-k)!]}$$

→ see Ch 4
The $n!$ garden

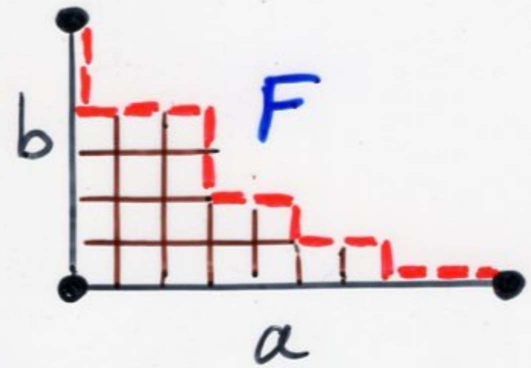
the garden of q-analogues

q-analogue of Catalan numbers

$$\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}$$

Combinatorial interpretation ?

$$\sum_{F \subseteq} q^{\mathrm{area}(F)} = \begin{bmatrix} a+b \\ a \end{bmatrix}$$

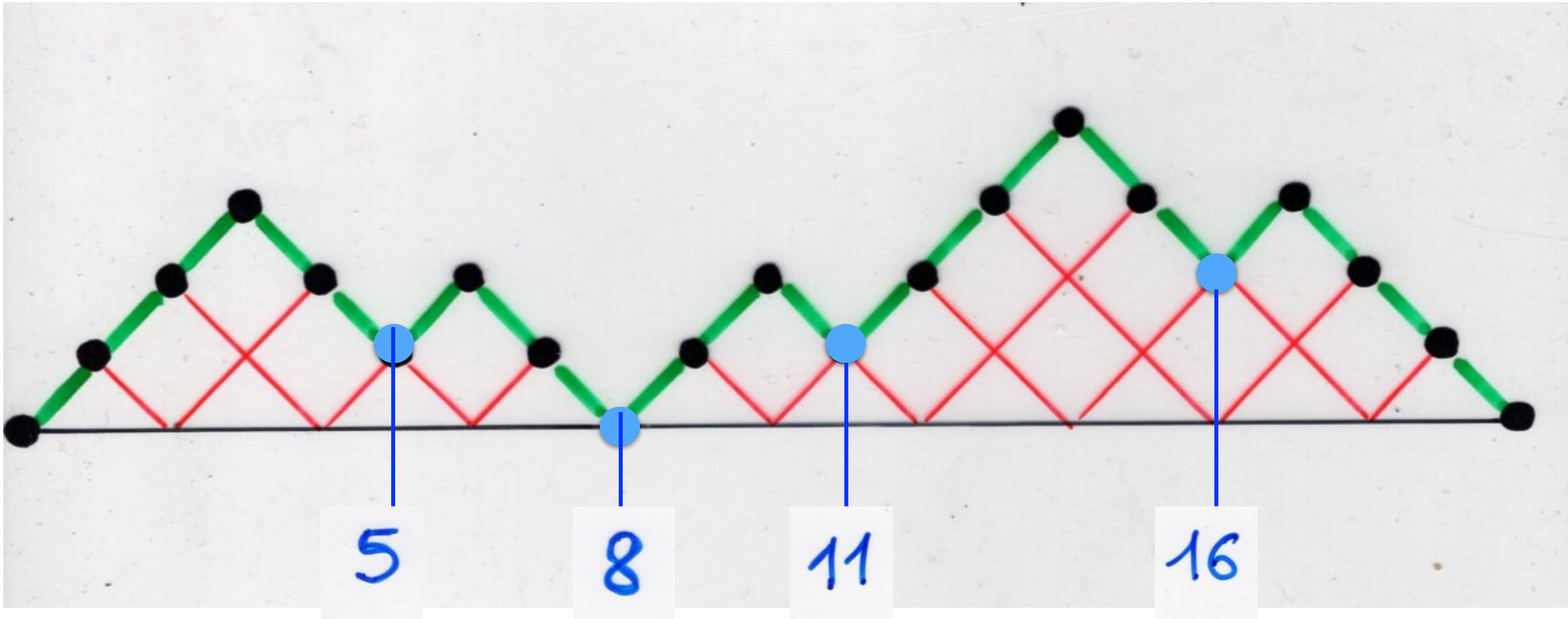


Ferrers diagram
(or Young)

→ see Ch 4
The n! garden

maj q-Catalan

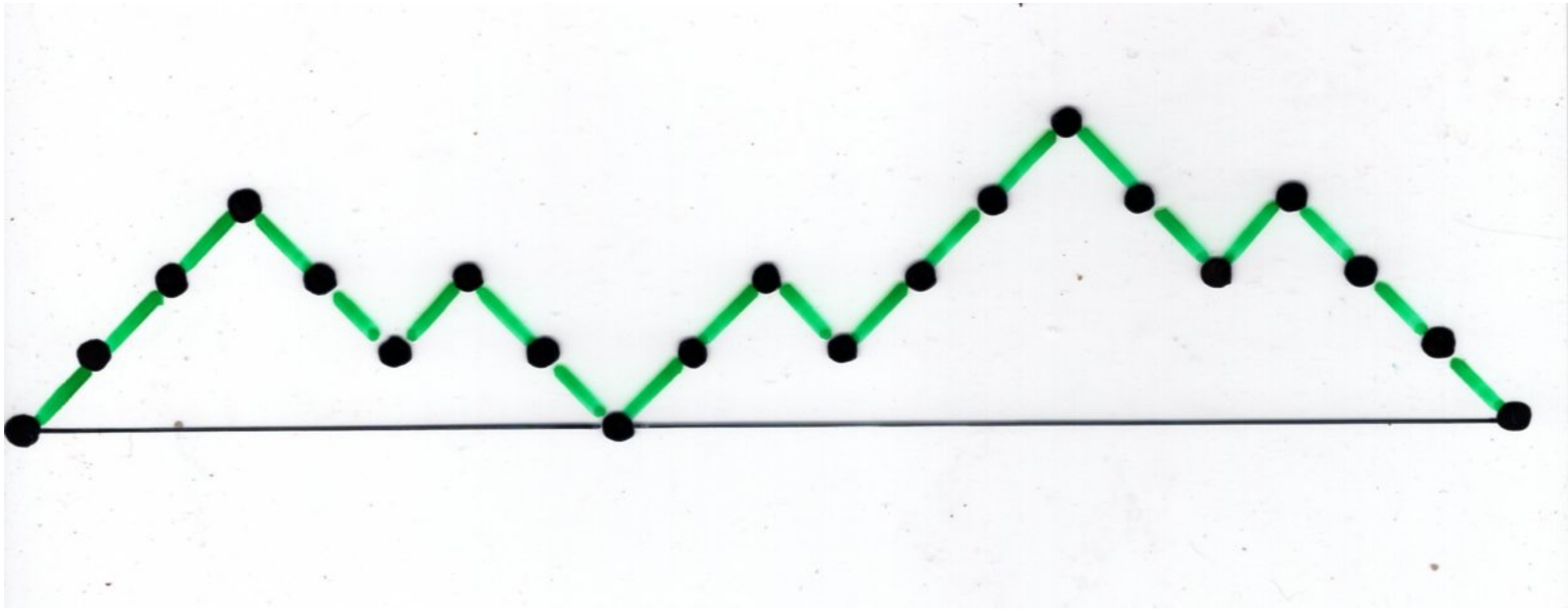
maj 5+8+11+16

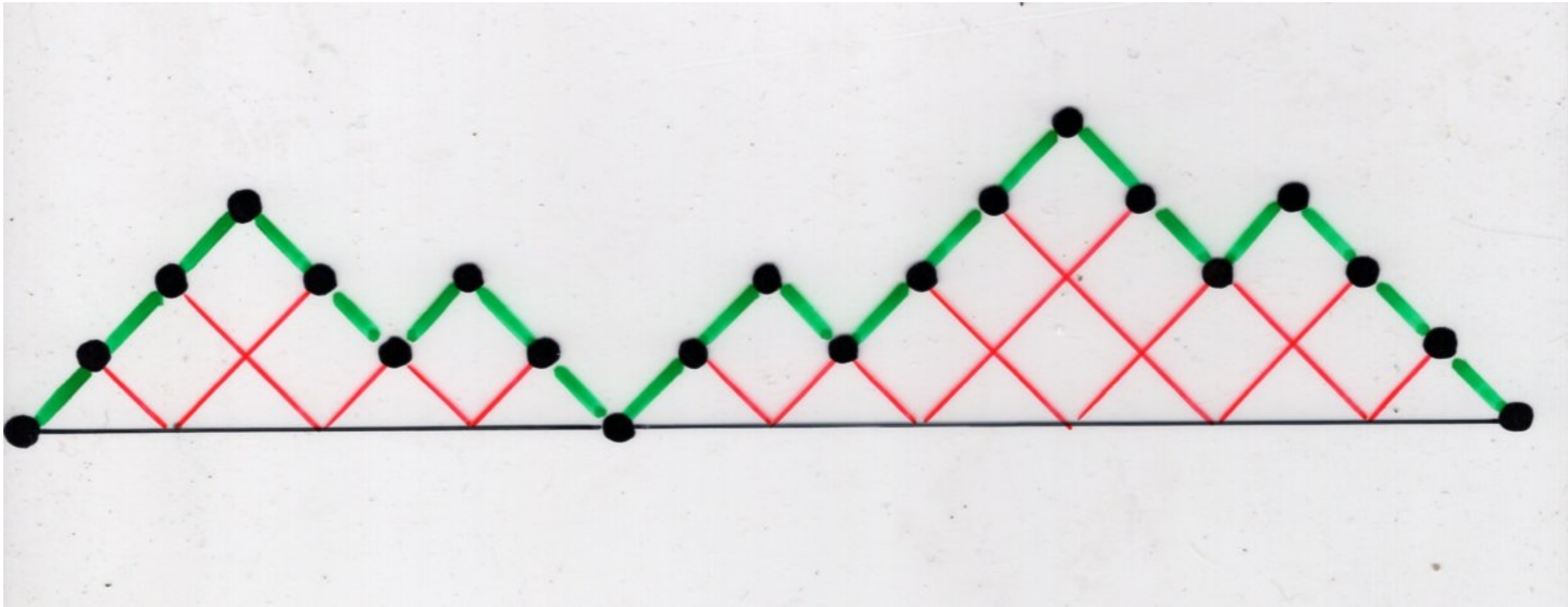


$$\sum_{\substack{\omega \\ \text{Dyck paths} \\ |\omega|=2n}} q^{\mathrm{maj}(\omega)} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}$$

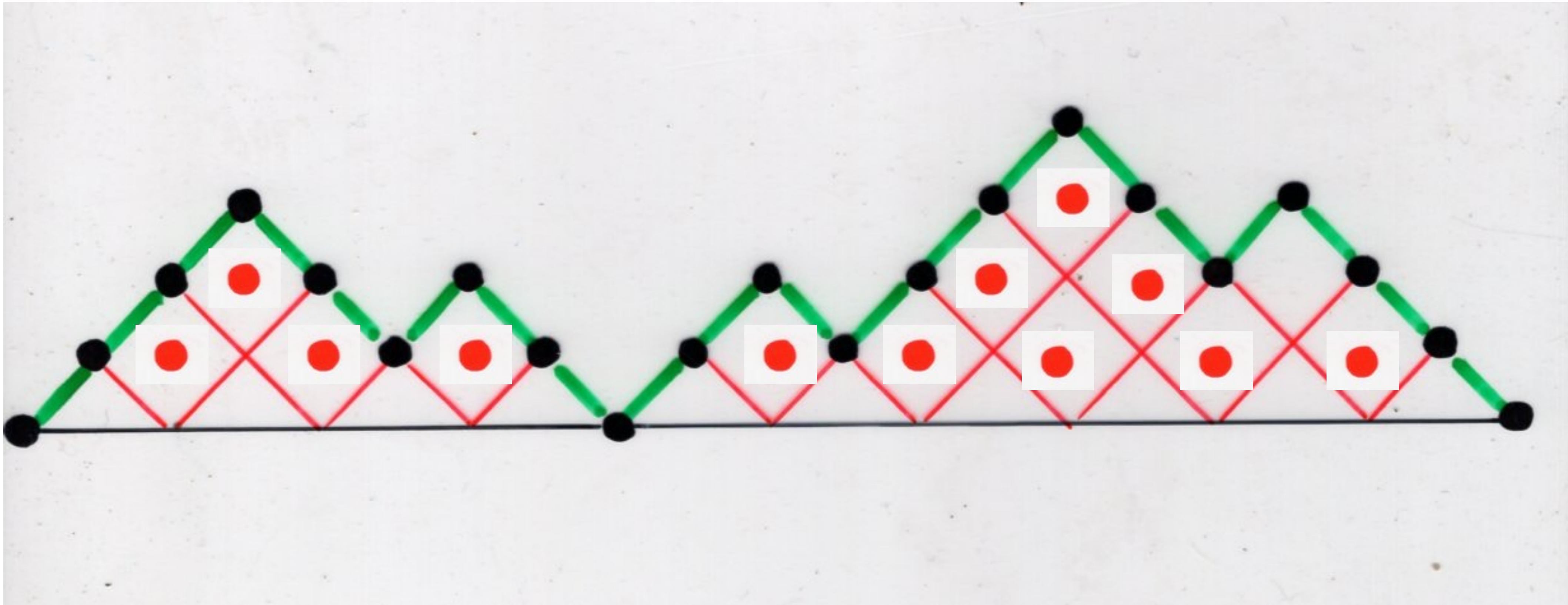
→ see Ch 4 The n! garden

another q-analogue of Catalan numbers





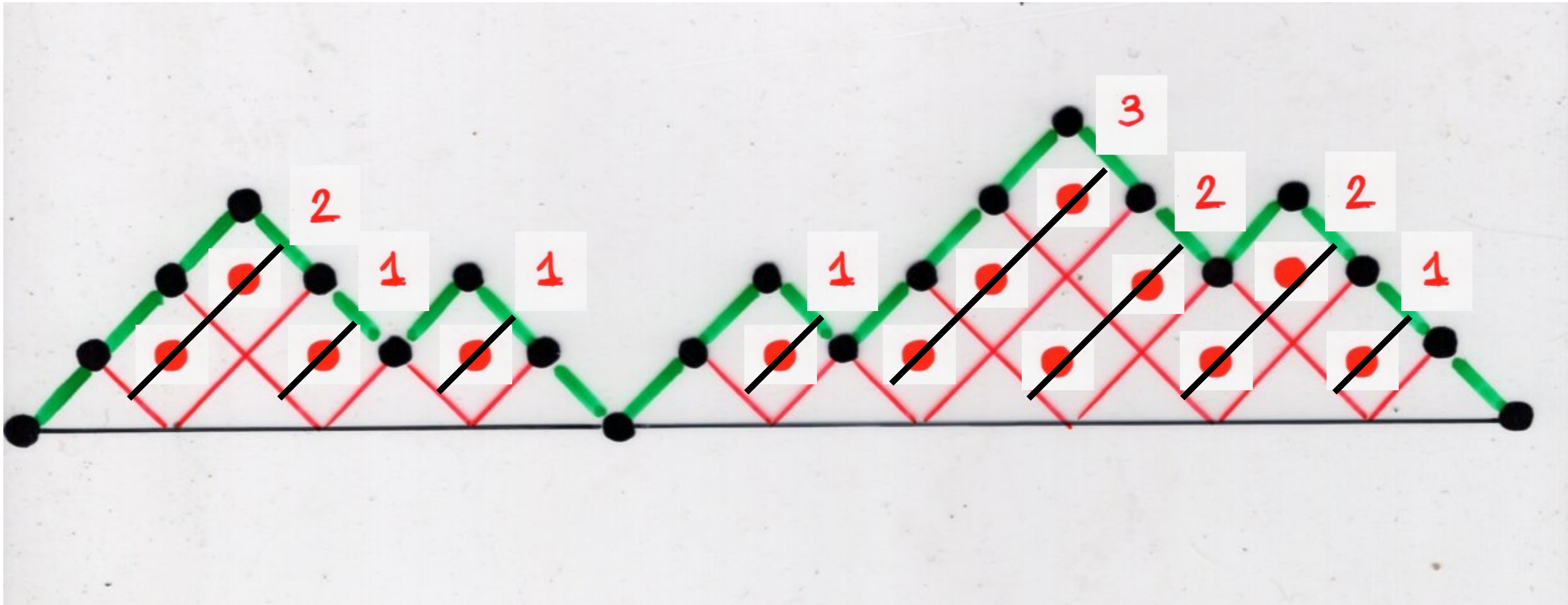
area = 13



exercise

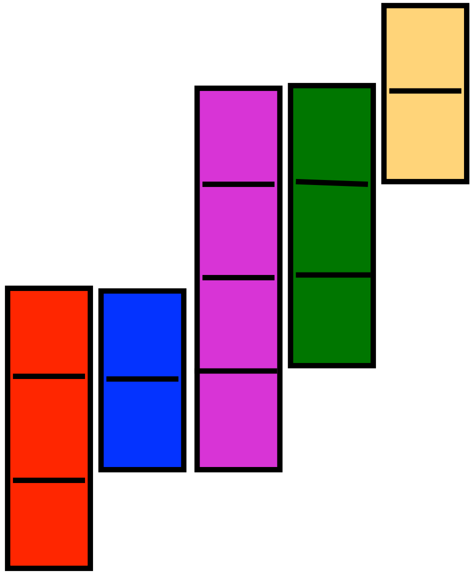
$$\sum_{\substack{\omega \\ \text{Dyck paths}}} q^{\mathrm{area}(\omega)} t^{|\omega|/2} = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{tq}{1 - \cfrac{tq^2}{\cdots \cfrac{}{1 - tq^k \cdots}}}}}$$

area = 13



a third q-analogue of Catalan numbers

Polya
q-Catalan



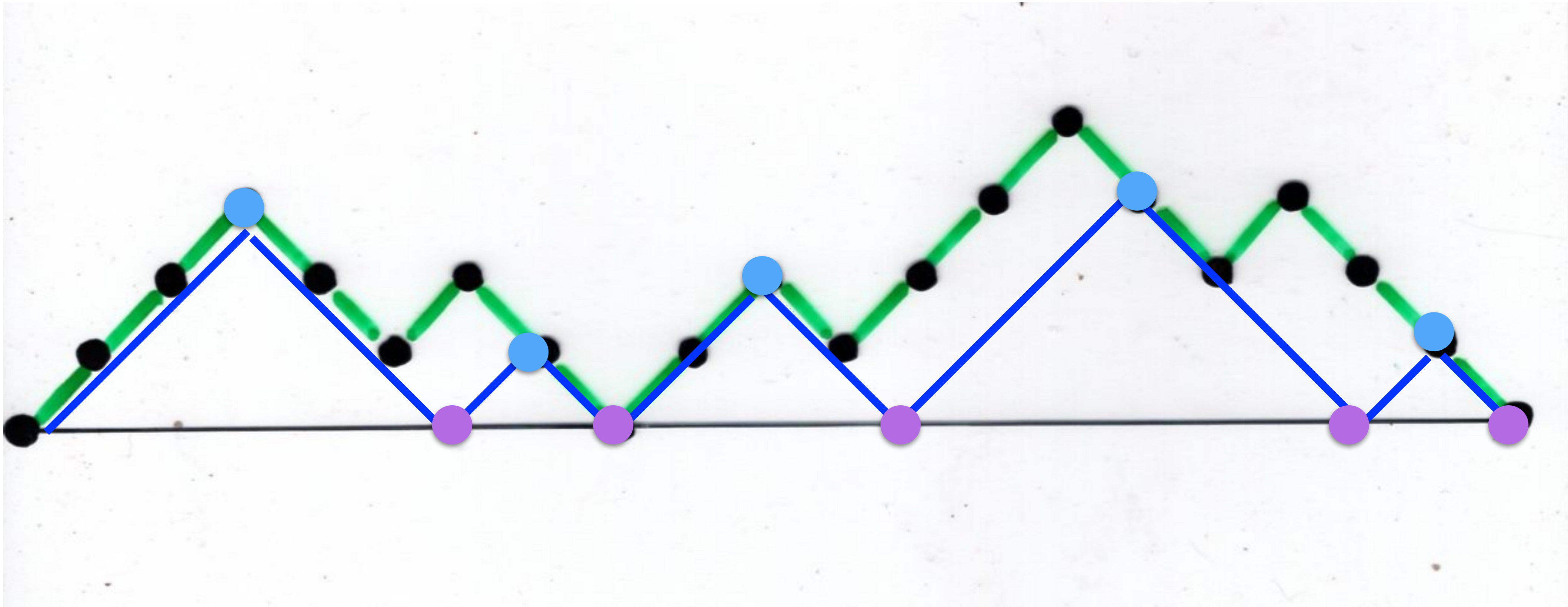
q-Bessel functions

→ see course
on heaps

Complement

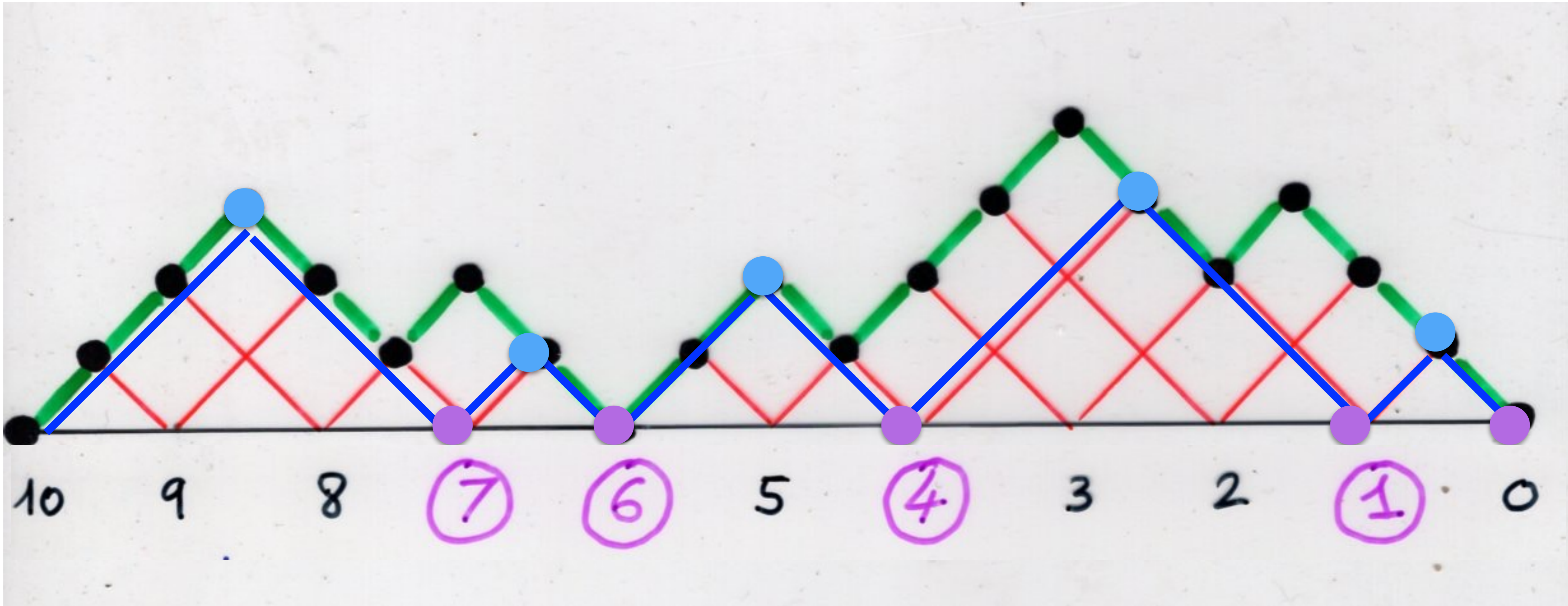
(q,t) - Catalan

bounce



$1 + 4 + 6 + 7 = 18$

bounce



area

bounce

same distribution over Dyck paths

$$\sum_{\substack{\omega \\ \text{Dyck paths}}} q^{\text{area}(\omega)} = \sum_{\substack{\omega \\ \text{Dyck paths}}} q^{\text{bounce}(\omega)}$$

$$C_n(q,t) = \sum_{\substack{\omega \\ \text{Dyck paths} \\ |\omega| = 2n}} q^{\text{area}(\omega)}\, t^{\text{bounce}(\omega)}$$

(q,t)- Catalan

polynomial
symmetric in q, t (!)

J. Haglund (2008)

The (q,t)- Catalan numbers and the space of diagonal harmonics

A. Garsia, F. Bergeron, ...
+ many people

$$C_n(q,t) = \sum_{\substack{\omega \\ \text{Dyck paths} \\ |\omega| = 2n}} q^{\mathrm{area}(\omega)}\, t^{\mathrm{dinv}(\omega)}$$

(q,t)-Catalan

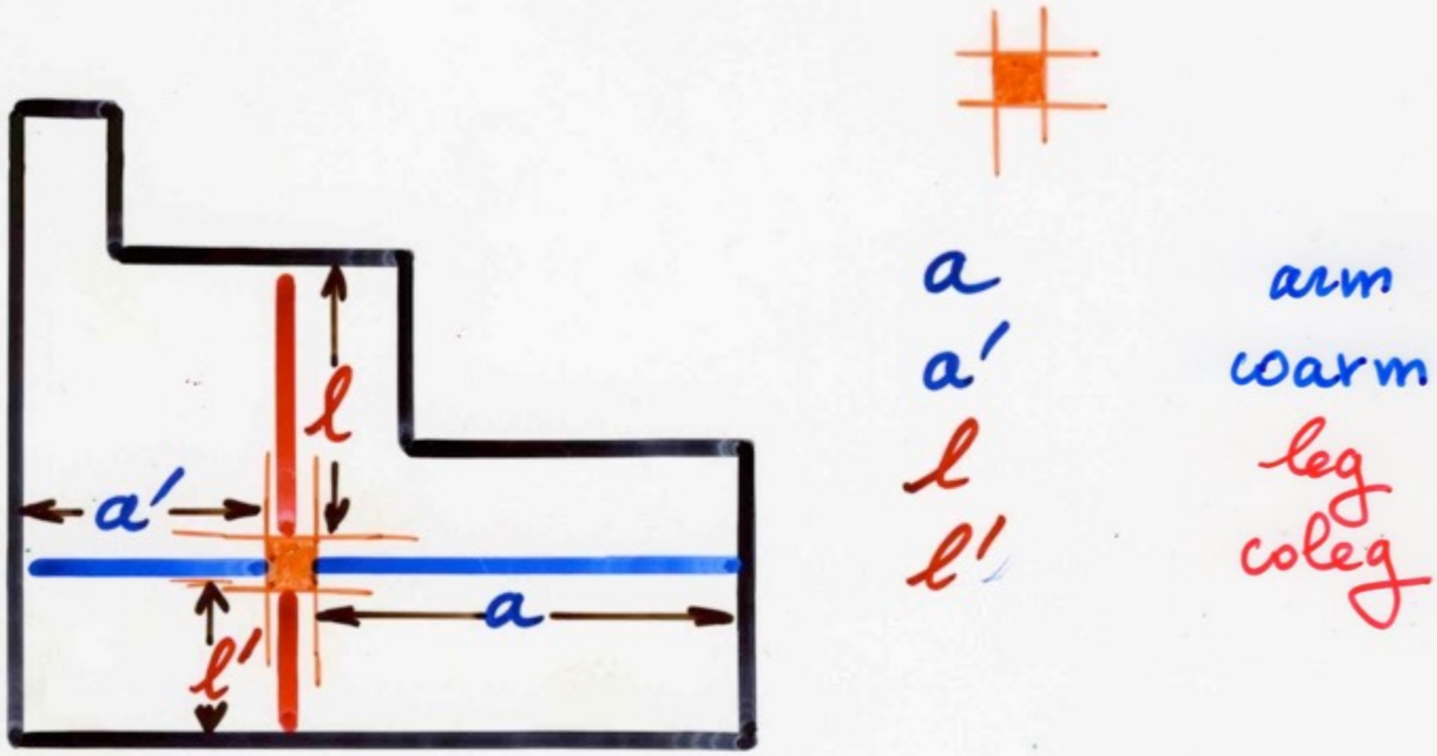
parameter
dinv

this morning
arXiv:1602:01126
Lee, Li, Loehr

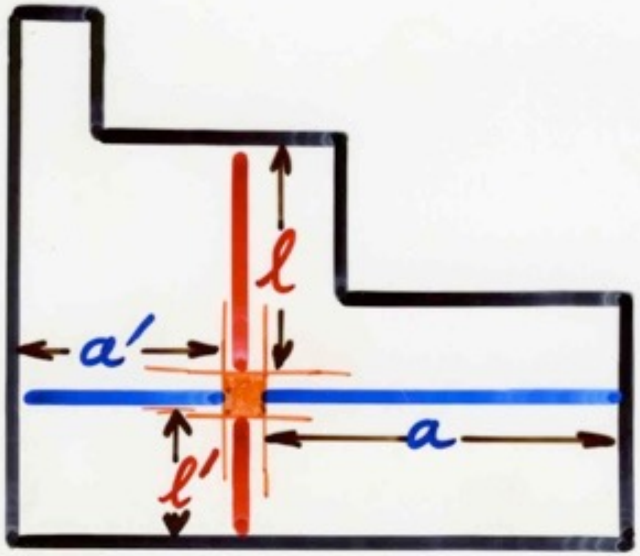
Complement to the complement

Macdonald polynomials

original definition of
(q,t)-Catalan
Garsia, Haiman
(1994)



μ Ferrers diagram

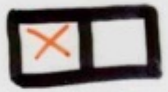


a	arm
a'	coarm
ℓ	leg
ℓ'	coleg

μ Ferrers diagram

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2\sum_{\square} \ell}\, q^{2\sum_{\square} a}\,(1-t)(1-q) \prod_{\square}^{0,0} (1 - q^{a'} t^{\ell'}) \sum_{\square} q^{a'} t^{\ell'}}{\prod_{\square} (q^{a} - t^{\ell+1})(t^{\ell} - q^{a+1})}$$

A. Garsia, M. Haiman (1994)

 $\longrightarrow \dfrac{q^2}{(q-t)}$

$n=2$

$$\frac{t^{2(0+1)}\, q^{0}\, (1-t)(1-q)\big(1-q^{0}t\big)\big(q^{0}t+q^{0}t^{0}\big)}{\big(q^{0}-t\big)\big(q^{0}-t^{2}\big)\big(t^{0}-q\big)\big(t-q\big)}$$

$$= \frac{t^{2}(1-t)(1-q)(1-t)(1+t)}{(1-t)(1-t^{2})(1-q)(t-q)}$$

$$= \frac{t^{2}}{(t-q)}$$

$$\frac{t^2}{(t-q)} + \frac{q^2}{(q-t)} = \frac{q^2-t^2}{q-t}$$

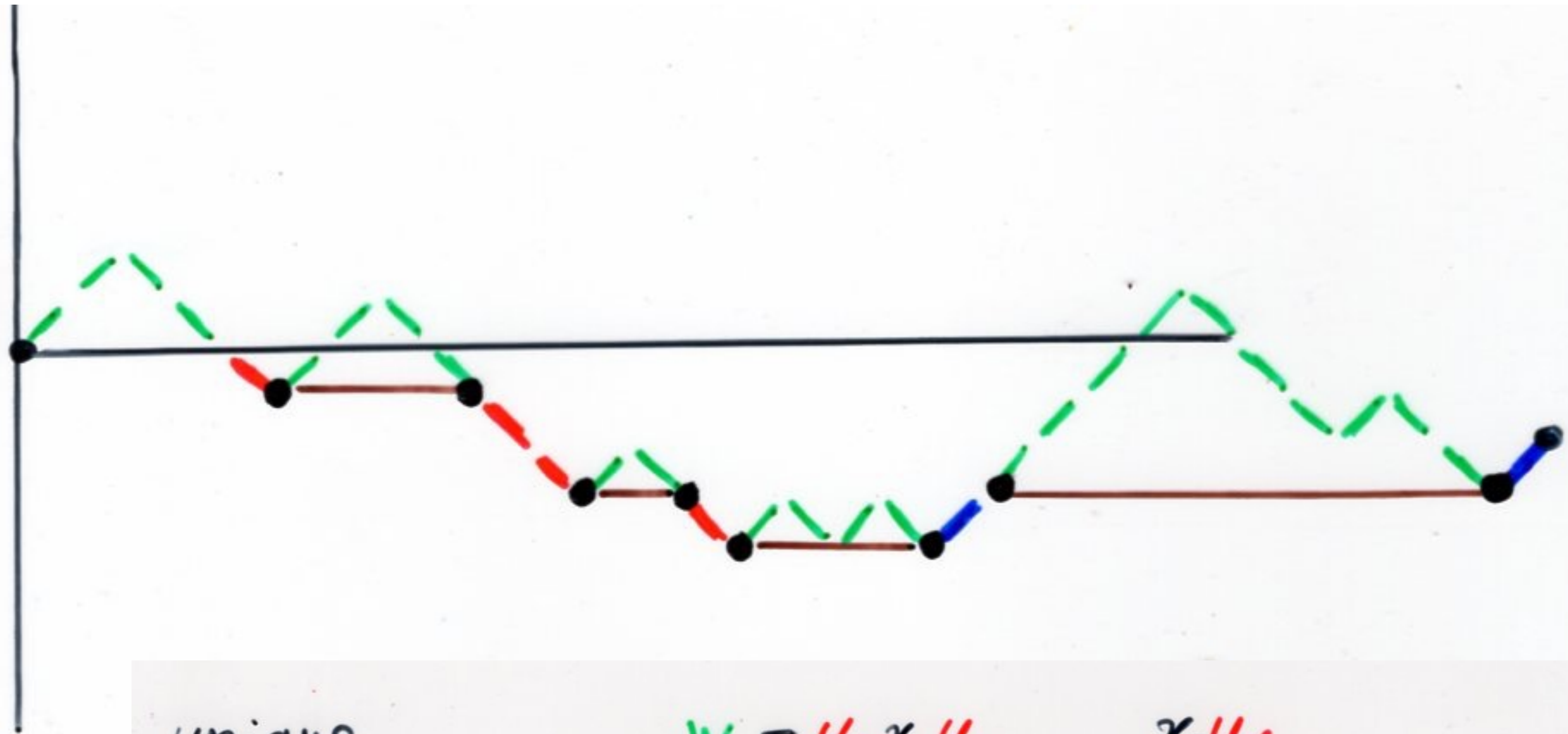
$$= q+t$$

some exercises using

Catalan factorization

and Catalan words

Catalan factorization of a word $w \in \{x, \bar{x}\}^*$

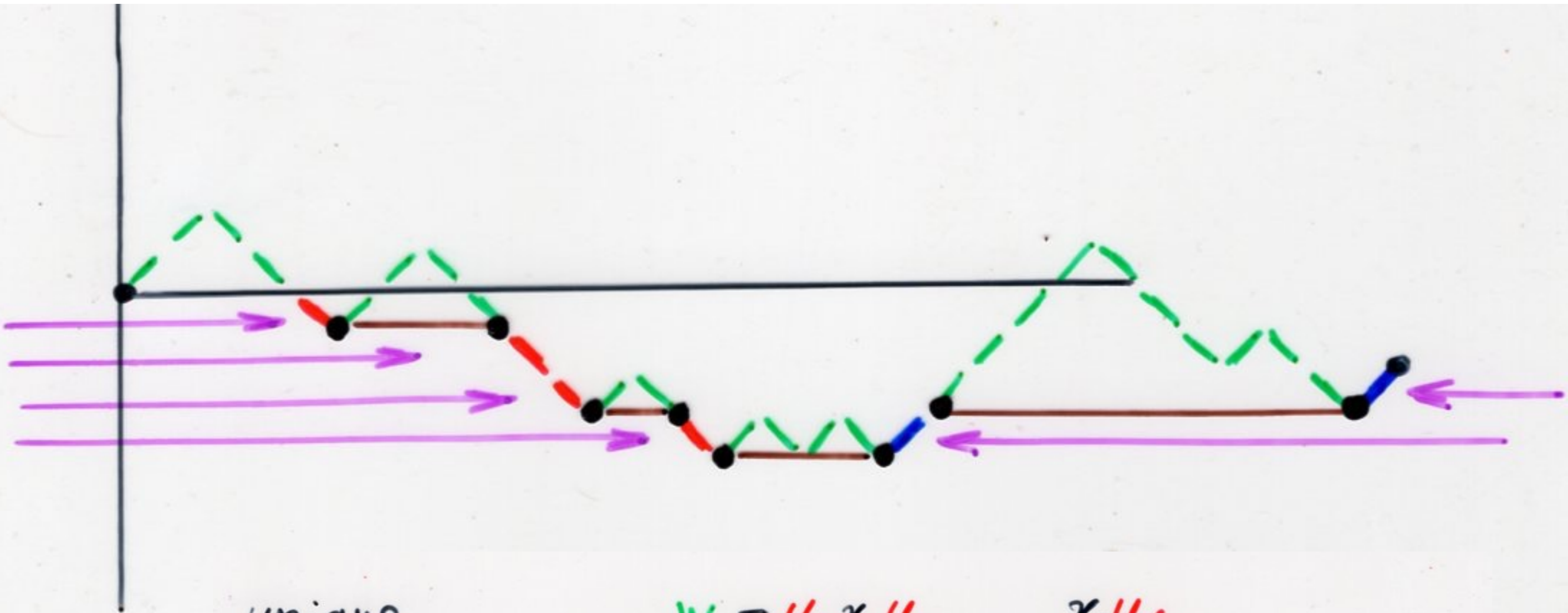


unique factorization

$$w = u_0 x_1 u_1 \quad \cdots \quad x_k u_k$$

$$\begin{cases} \bullet \ u_i \quad \text{Dyck word} \quad i = 0, \ldots, k \\ \bullet \ x_i \in X \quad , \quad 1 \leq i \leq k \\ \bullet \ x_i = x, \ x_j = \bar{x} \implies i > j \end{cases}$$

Catalan factorization of a word $w \in \{x, \bar{x}\}^*$



unique factorization $\quad w = u_0 x_1 u_1 \quad \cdots \quad x_k u_k$

$$\begin{cases} \bullet\ u_i \ \text{ Dyck word } \ i=0,\dots,k \\ \bullet\ x_i \in X, \quad 1 \le i \le k \\ \bullet\ x_i = x,\ x_j = \bar{x} \ \Rightarrow\ i > j \end{cases}$$

Catalan word $w \in \{x, \bar{x}, z\}$

$$\left(\mathcal{D} + \{z\}\right)^*$$

set of Dyck words

product of Dyck words and the letter z.

bijection θ:

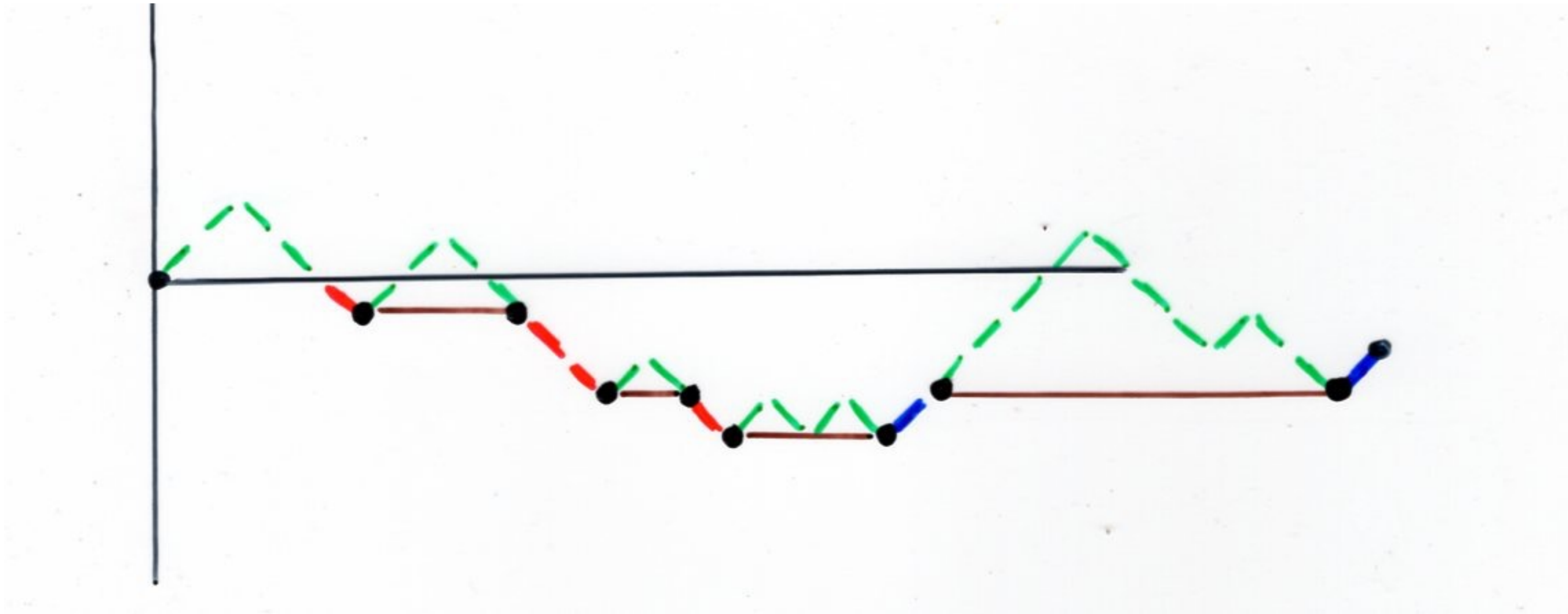
$$w \in \{x, \bar{x}\}^* \longrightarrow \left(p(w)\,,\ \hat{w}\right)$$

$|w| = n$

$0 \leq p(w) \leq |\hat{w}|_z$

Catalan word

$|\hat{w}| = n$



$$w \in \{x, \bar{x}\}^* \longrightarrow (p(w), \hat{w})$$

$|w| = n$

$0 \leqslant p(w) \leqslant |\hat{w}|_z$

Catalan word

$|\hat{w}| = n$

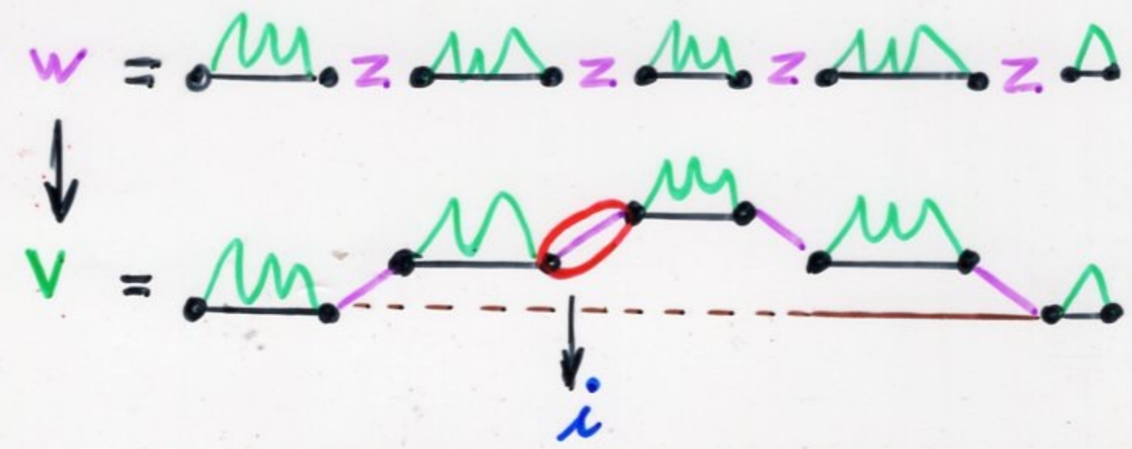
$$\rho : \underset{\substack{\text{Catalan} \\ \text{word} \\ |w|=2n}}{w} \longrightarrow (i, v)$$

$0 \le i \le n$

v: Dyck word, $|v| = 2n$

- if $|w|_z = 0$, then $\rho(w) = (0, w)$

- if $|w|_z \neq 0$



i : is the i^{th} letter x of the word v

$$\rho : \quad w \longrightarrow (i, v)$$

Catalan word
$|w| = 2n$

$0 \le i \le n$

Dyck word
$|v| = 2n$

$$\binom{2n}{n} = (n+1)\,C_n$$

<u>exercise</u> • The number of Catalan words of length n is $\binom{n}{\lfloor n/2 \rfloor}$

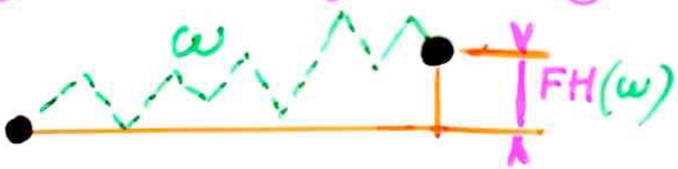
This is also the number of left factors of Dyck words of length n.

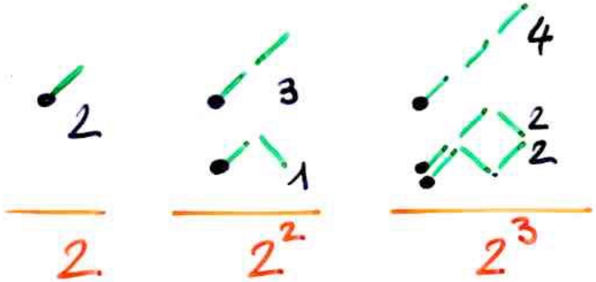
length $2n$: $\binom{2n}{n}$

$2n+1$ $\binom{2n+1}{n}$

ex. average height of final point of left factors of Dyck path

(height starts at 1)

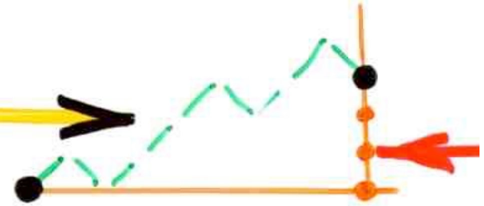




Prove that $$\sum_{\substack{|\omega|=n \\ \text{left factor of Dyck path}}} FH(\omega) = 2^n$$

with a bijection

ω any path $|\omega|=n$ $\longleftrightarrow$ pointed paths



This will imply:

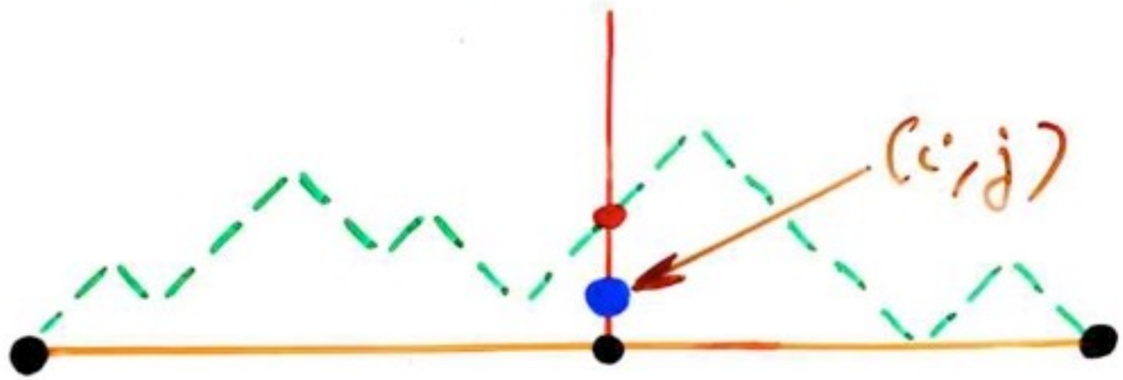
average final height $\sim \sqrt{n}$

exercises

a) give a bijection between the words of $\{x, \bar{x}\}^{2n}$ and

pointed Dyck paths

i.e. the pair $((i,j), \omega)$ where ω is a Dyck path of length $2n$ and (i,j) is a point of $\mathbb{N}\times\mathbb{N}$ "below" ω, as in fig:

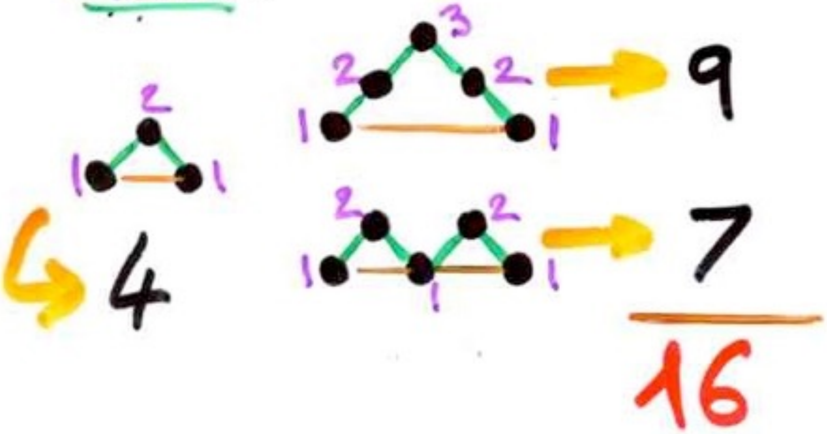


- Thus you have proved

$$\sum_{\substack{|\omega|=2n \\ \text{Dyck paths}}} \text{Area}(\omega) = 4^n$$

where $\text{Area}(\omega)$ = total nb of points of $\mathbb{N}\times\mathbb{N}$ "below" the path.

check:



$\Rightarrow$ average area $\sim n^{3/2}$

b) Using the bijection between

Dyck path $|\omega| = 2n$ $\longleftrightarrow$ complete binary tree $|B| = 2n+1$

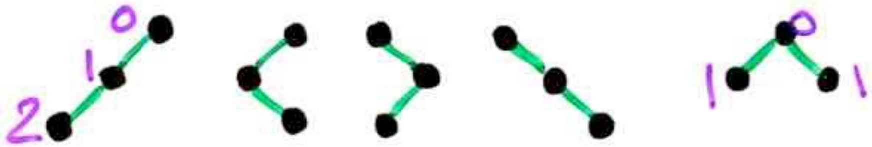
deduce the following fact, classical in computer science :

define the (internal) path length of a binary tree B as the sum of the height of all the vertices s of B.

Then deduce:

$$\text{average path length (for binary tree } |B| = n) = \frac{4^n}{C_n} - (3n+1)$$

<u>check</u>:



path length → 3 3 3 3 2 $\xrightarrow{\text{total}}$ 14

average $= \frac{14}{5}$

$$\frac{4^3}{C_3} - (3\times 3+1) = \frac{64}{5} - 10 = \frac{14}{5} \quad \text{(ouf!)}$$

exercise

- The average length (number of edges) of the left branch of a random binary tree with n vertices is

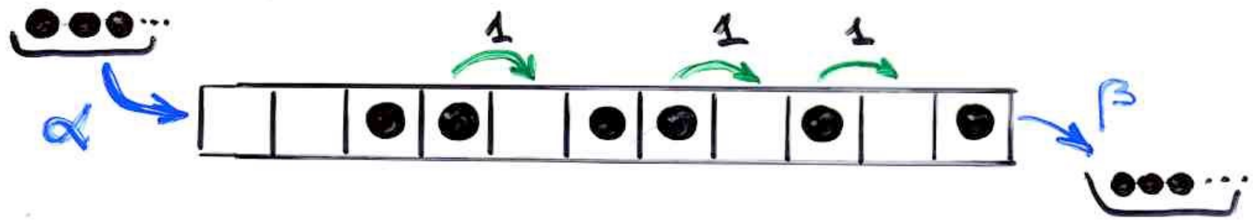
$$2 - \frac{6}{n+2}$$

Complement

The TASEP

TASEP

"totally asymmetric exclusion process"



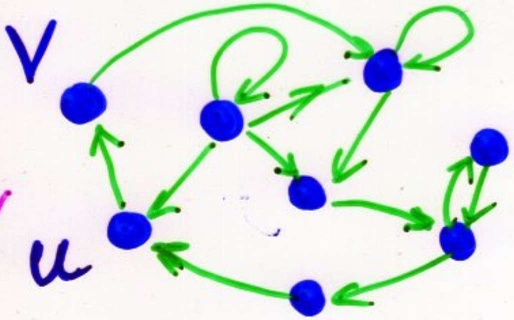
stationnary probabilities

time $\to \infty$

Markov chain

2^n states

$p_{u,v}$



graph

S set of states
(vertices of the graph)

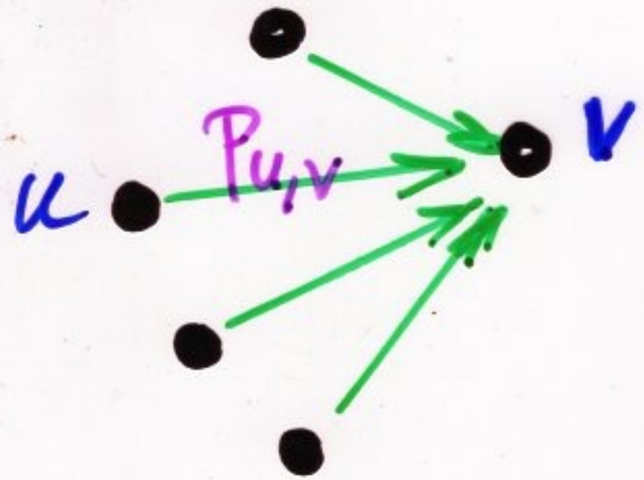
$p_{u,v}$ probability

$$T = (p_{u,v})_{u,v \in S}$$

(stochastic)
transition matrix

time t $\quad V_t = (p_u^t, \dots)_{u \in S}$ probability vector at time t

time $t+1$ $\quad V_{t+1} = V_t T$



$$\underset{\text{time } (t+1)}{p_v^{(t+1)}} = \sum_u \underset{\text{time } t}{p_u^{(t)}} p_{u,v}$$

$$V_t = V_{t+1}$$

$$V = (p_u^\infty, \dots)_{u \in S}$$

$$V = VT$$

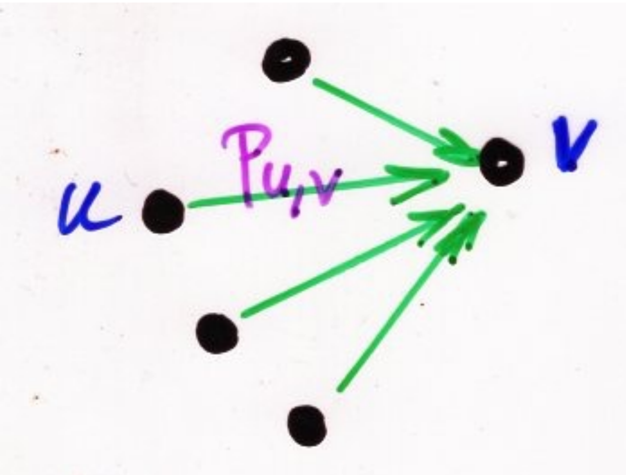
eigenvector of T^T

eigenvalue 1

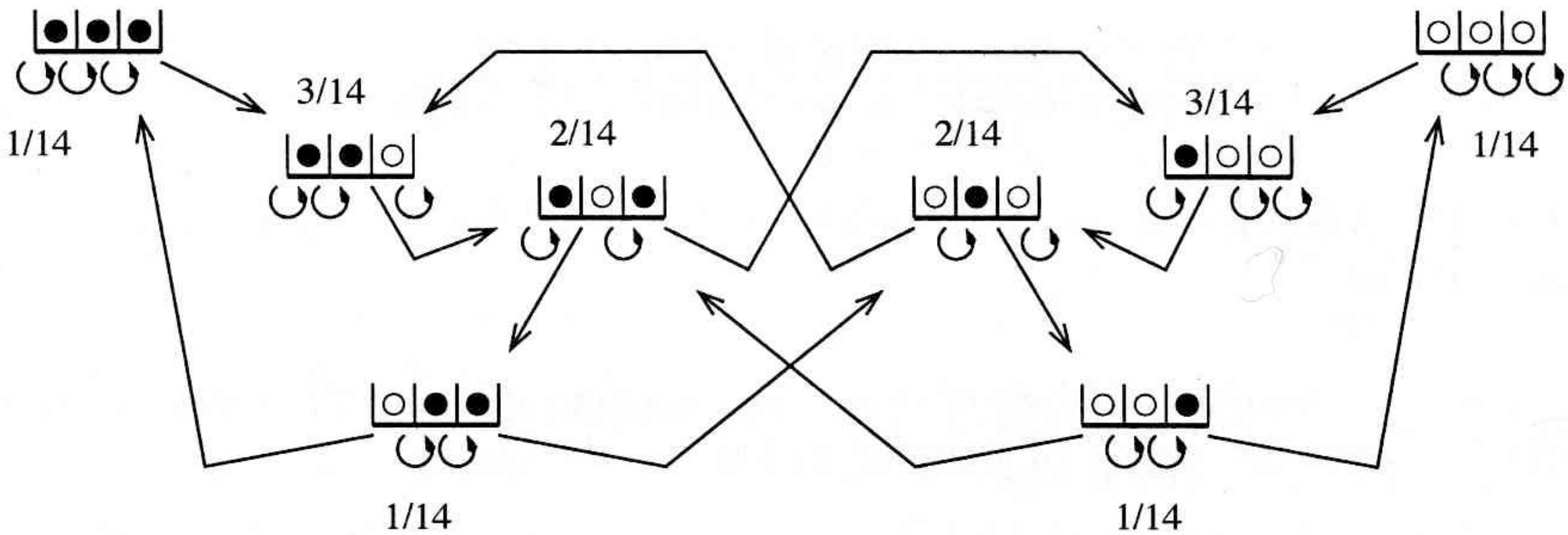
unique

stationnary probabilities

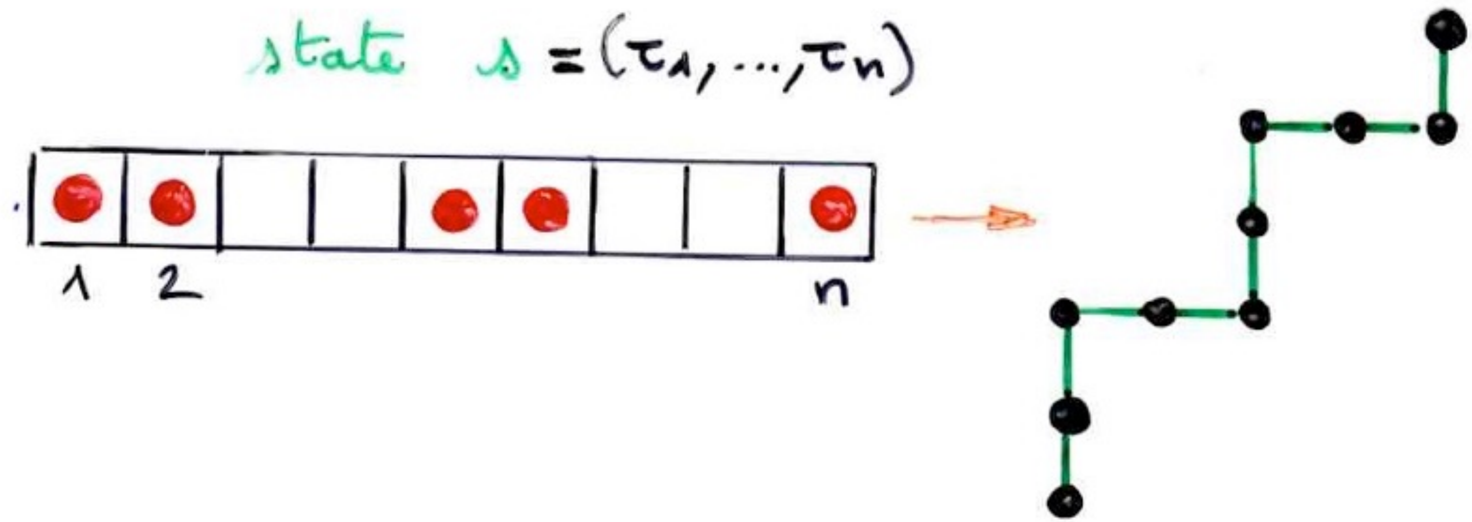
time $\longrightarrow \infty$



$$p_v^\infty = \sum_{u \in S} p_u^\infty \, p_{u,v}$$


1/14
3/14
2/14
2/14
3/14
1/14
1/14
1/14

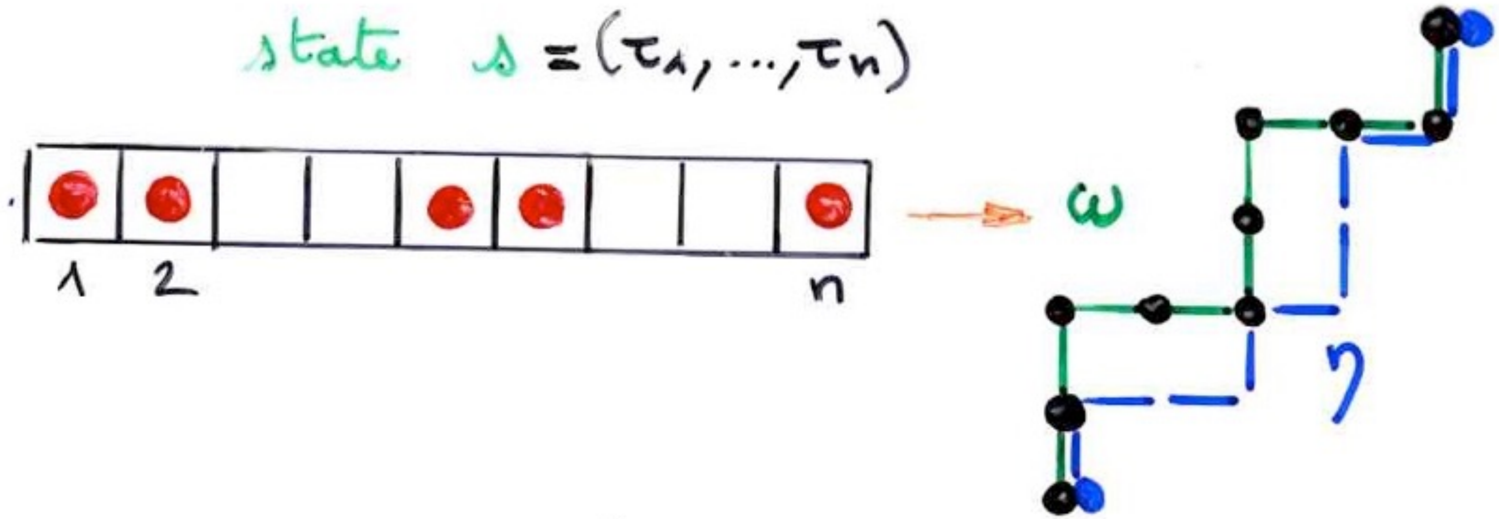
state $s = (\tau_1, \dots, \tau_n)$



$\mathcal{P}_n(s) =$

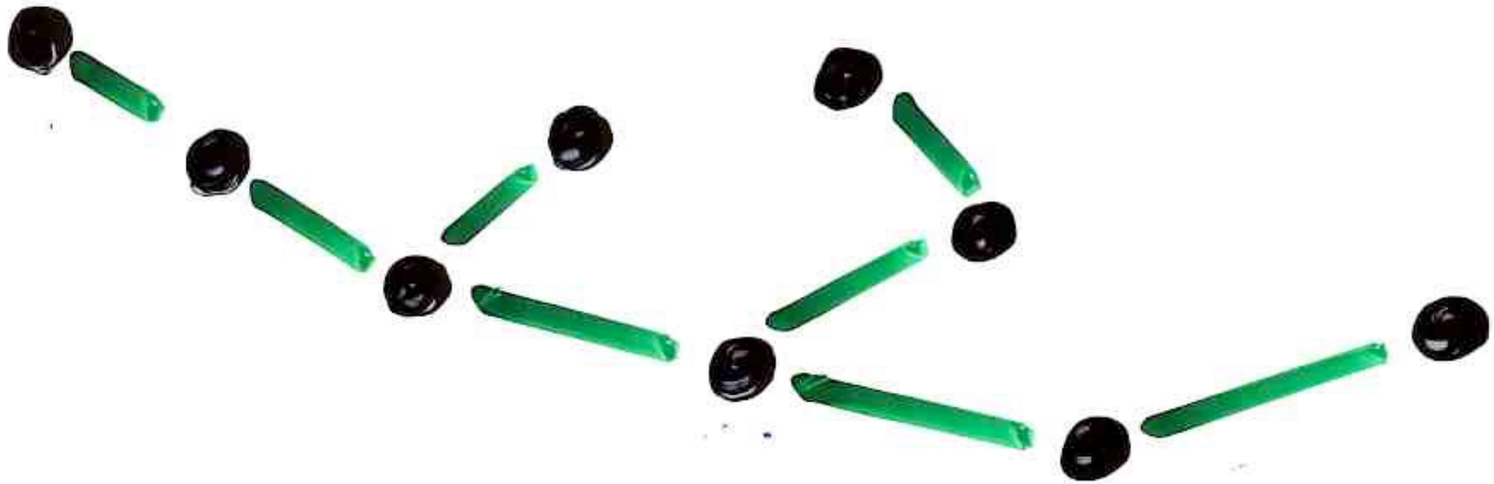
Shapiro, Zeilberger (1982)

state $s = (\tau_1, \dots, \tau_n)$

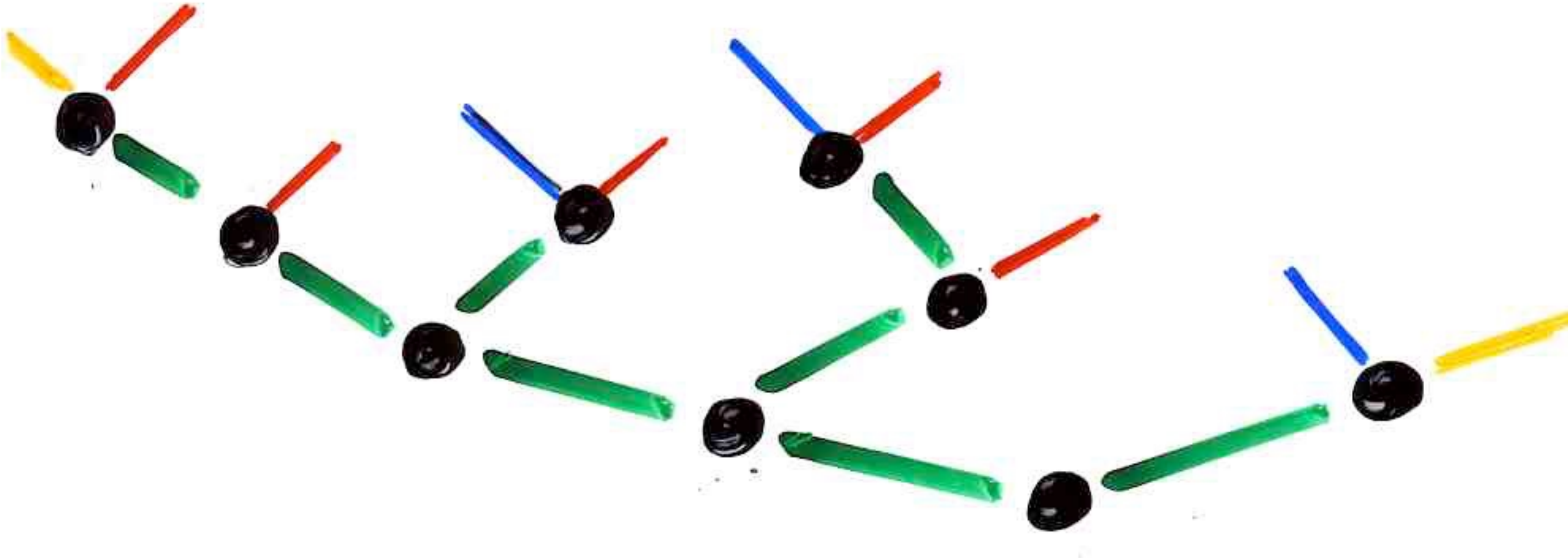


$$\mathbb{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

Shapiro, Zeilberger (1982)



canopy of a binary tree



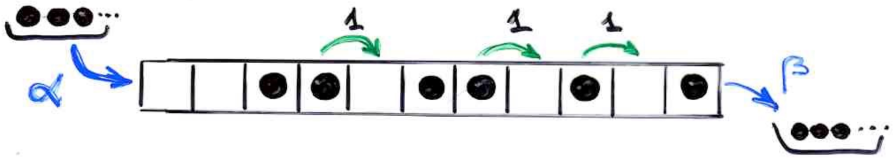
canopy of a binary tree

$c(B) = / \; / \; \backslash \; / \; \backslash \; / \; / \; \backslash$

Loday, Ronco (1998, 2012)

TASEP

"totally asymmetric exclusion process"



stationary probabilities

$$\frac{1}{Z_n}\sum_{\substack{\text{binary trees } T\\ c(T)=w}} \bar{\alpha}^{\,lb(T)}\,\bar{\beta}^{\,rb(T)}$$

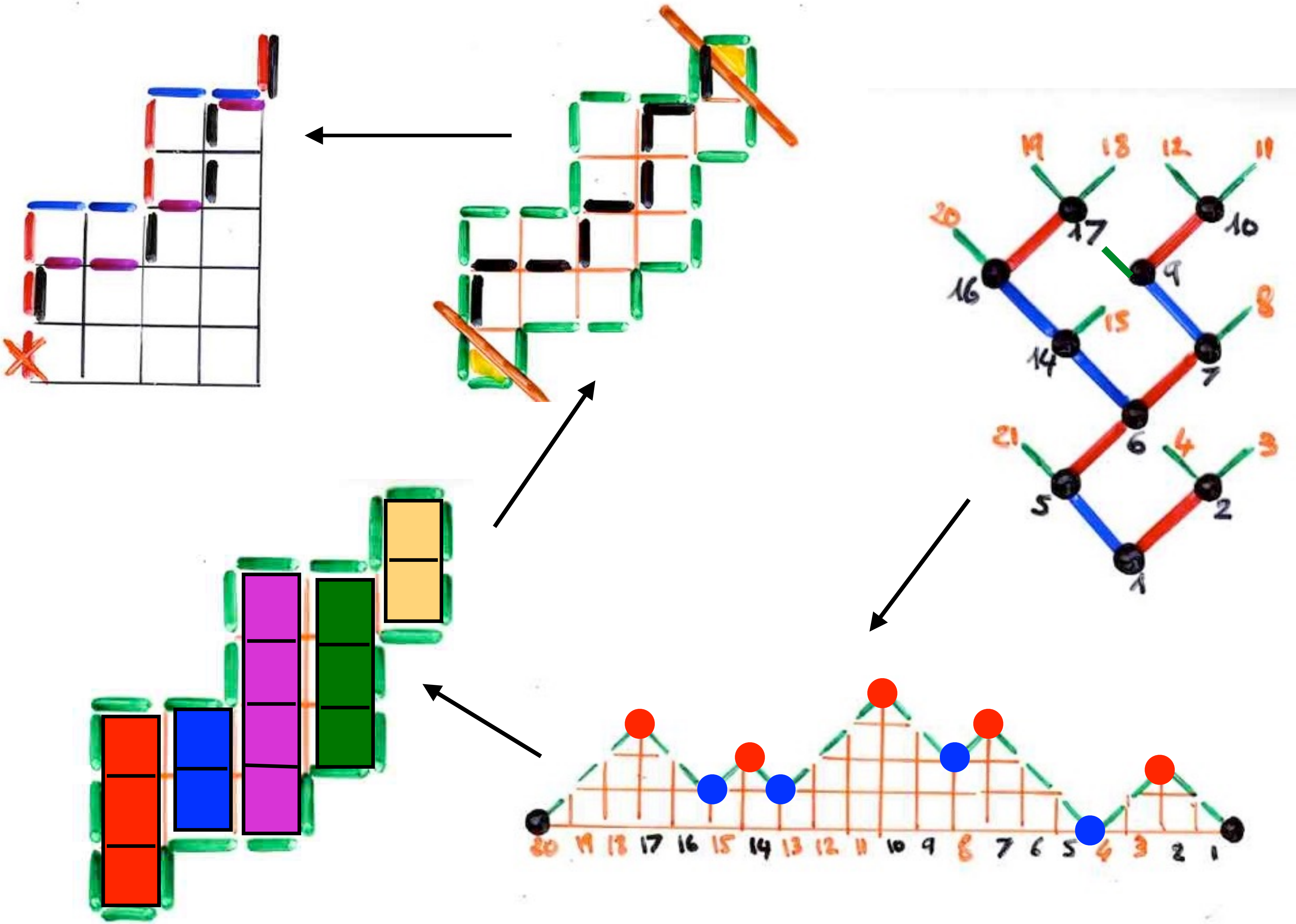
canopy

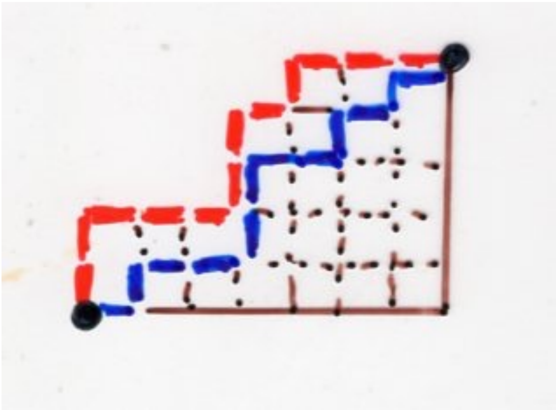
$\bar{\alpha} = \alpha^{-1}$ $\quad \bar{\beta} = \beta^{-1}$

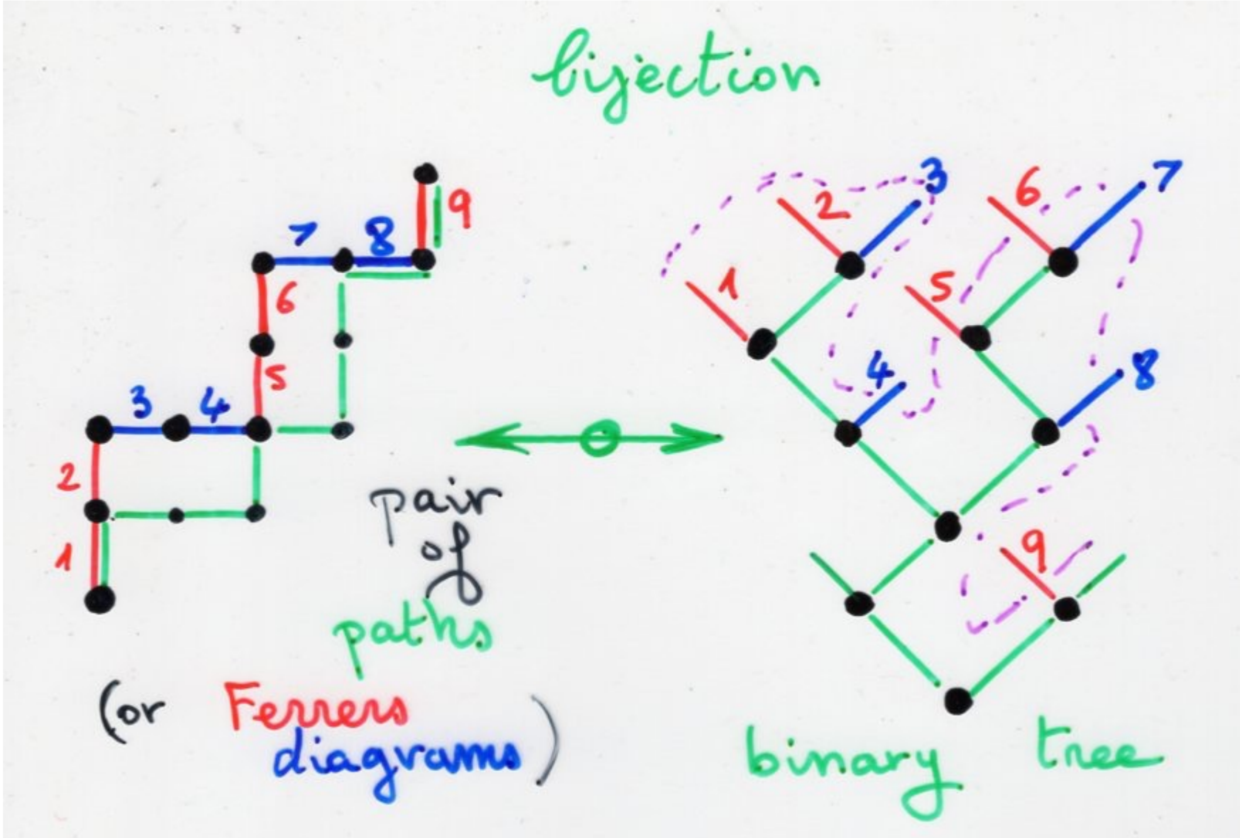
partition function

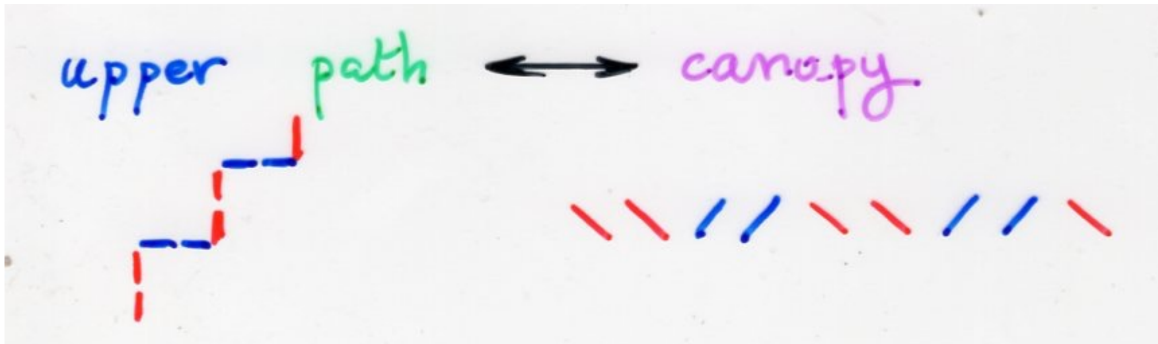
$$Z_n = \sum_{\substack{T \\ \text{binary trees} \\ n \text{ vertices}}} \bar{\alpha}^{\,lb(T)}\,\bar{\beta}^{\,rb(T)}$$

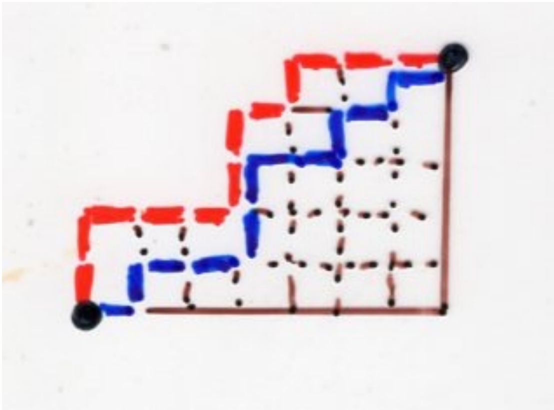
→ see course quadratic algebra in combinatorics

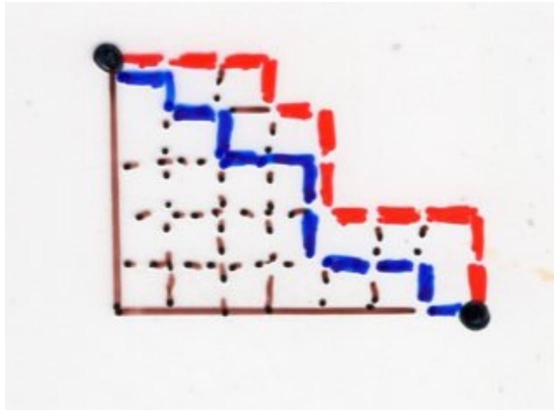

19 18 12 11
20 17 10
16 9 8
15 14 7
6
21 4 3
5 2
1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



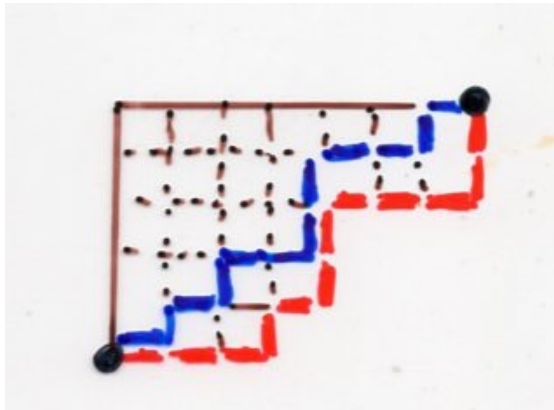

bijection
1
2
3
4
5
6
7
8
9
pair of paths
(or Ferrers diagrams)
binary tree


upper path ⟷ canopy





Young lattice



$$Z_n = \sum_{\substack{T \\ \text{binary trees} \\ n \text{ vertices}}} \bar{\alpha}^{\,lb(T)} \, \bar{\beta}^{\,rb(T)}$$

$$\bar{\alpha} = \alpha^{-1} \qquad \bar{\beta} = \beta^{-1}$$

__exercise__ prove the following formula for Z_n

partition function

$$Z_n = \sum \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$